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SCHRÖDINGER OPERATORS AND THE DISTRIBUTION OF RESONANCES IN SECTORS

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The purpose of this paper is to give some refined results about the distribution of resonances in potential scattering. We use techniques and results from one and several complex variables, including properties of functions of completely regular growth. This enables us to find asymptotics for the distribution of resonances in sectors for certain potentials and for certain families of potentials.

1. Introduction

The purpose of this paper is to prove some results about the distribution of resonances in potential scattering. In particular, we study the distribution of resonances in sectors and give asymptotics of the "expected value" of the number of resonances in certain settings.

More precisely, we consider the operator $-\Delta + V$, where $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ and Δ is the (nonpositive) Laplacian. Then, except for a finite number of values of λ, $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$, $\operatorname{Im}\lambda > 0$, is a bounded operator on $L^2(\mathbb{R}^d)$ for λ in the upper half-plane. When d is odd and $\chi \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ satisfies $\chi V = V$, $\chi R_V(\lambda)\chi$ has a meromorphic continuation to the lower half-plane. The poles of $\chi R_V(\lambda)\chi$ are called *resonances*, and are independent of choice of χ satisfying these hypotheses. Resonances are analogous to eigenvalues not only in their appearance as poles of the resolvent, but also because they appear in trace formulas much as eigenvalues do [Bardos et al. 1982; Guillopé and Zworski 1997; Melrose 1982]. Physically, they may be thought of as corresponding to decaying waves.

Let $n_V(r)$ denote the number of resonances of $-\Delta + V$, counted with multiplicity, with norm at most r. When $d = 1$, asymptotics of $n_V(r)$ are known:

$$\lim_{r\to\infty} \frac{n_V(r)}{r} = \frac{2}{\pi} \operatorname{diam}(\operatorname{supp}(V))$$

[Zworski 1987]; see also [Froese 1997; Regge 1958; Simon 2000]. Moreover, "most" of the resonances occur in sectors about the real axis, in the sense that for any $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{\#\{\lambda_j \text{ pole of } R_V(\lambda) : |\arg\lambda_j - \pi| < \epsilon \text{ or } |\arg\lambda_j - 2\pi| < \epsilon\}}{r} = \frac{2}{\pi} \operatorname{diam}(\operatorname{supp}(V))$$

[Froese 1997]. These results are valid for complex-valued V. The case $d = 1$ is exceptional, though: in higher dimensions much less is known.

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Now we turn to $d \geq 3$ odd, where the question is more subtle. If $V \in L^\infty(\mathbb{R}^d)$ has support in $\overline{B}(0,a) = \{x \in \mathbb{R}^d : |x| \leq a\}$, then

$$d \int_0^r \frac{n_V(t) - n_V(0)}{t}\, dt \leq c_d a^d r^d + o(r^d), \tag{1-1}$$

where c_d is defined in (3-5) and depends only on the dimension. Zworski [1989a] showed that such a bound holds, and Stefanov [2006] identified the optimal constant c_d. There are examples for which equality holds in (1-1) [Zworski 1989b; Stefanov 2006]. Lower bounds have proved more elusive. The current best-known general quantitative lower bound is for nontrivial real-valued $V \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$:

$$\limsup_{r\to\infty} \frac{n_V(r)}{r} > 0 \tag{1-2}$$

[Sá Barreto 2001]. On the other hand, there are nontrivial complex-valued potentials V for which $\chi R_V(\lambda)\chi$ has no poles [Christiansen 2006].

We wish to single out the set for which asymptotics actually hold in (1-1). This is the set defined, for $a > 0$, as

$$\mathfrak{M}_a = \bigl\{ V \in L^\infty(\mathbb{R}^d) : \operatorname{supp} V \subset \overline{B}(0,a) \text{ and } n_V(r) = c_d a^d r^d + o(r^d) \text{ as } r \to \infty \bigr\}. \tag{1-3}$$

We remark that it is equivalent to require, as $r \to \infty$, that $n_V(r) = c_d a^d r^d + o(r^d)$ or

$$d \int_0^r t^{-1}\bigl(n_V(t) - n_V(0)\bigr)\, dt = c_d a^d r^d + o(r^d).$$

The set $\mathfrak{M}_a$ contains infinitely many radial potentials. By results from [Zworski 1989b; Stefanov 2006], this set contains any potential of the form $V(x) = v(|x|)$, where $v \in C^2([0,a])$ is real-valued, $v(a) \neq 0$, and $V(x) = 0$ for $|x| > a$. Additionally, it contains infinitely many complex-valued potentials which are isoresonant with these real-valued radial potentials [Christiansen 2008].

We now can state some results. For the first, we set, for $\varphi < \theta$, $n_V(r, \varphi, \theta)$ to be the set of poles of $R_V(\lambda)$, counted with multiplicity, with norm at most r and with argument between φ and θ inclusive.

Proposition 1.1. *Let $V \in \mathfrak{M}_a$. Then, if $0 < \varphi < \theta < \pi$,*

$$n_V(r, \pi+\varphi, \pi+\theta) = \frac{1}{2\pi d}\tilde{s}_d(\varphi,\theta) r^d a^d + o(r^d) \quad \text{as } r \to \infty,$$

where

$$\tilde{s}_d(\varphi,\theta) = h_d'(\theta) - h_d'(\varphi) + d^2 \int_\varphi^\theta h_d(s)\, ds,$$

and $h_d(\theta)$ is as defined in (3-4).

If V is real-valued, then λ_0 is a resonance of $-\Delta + V$ if and only if $-\overline{\lambda_0}$ is a resonance. In this case, for $V \in \mathfrak{M}_a$ and $0 < \theta < \pi$,

$$n_V(r, \pi, \pi+\theta) = \frac{1}{2\pi d}\left[h_d'(\theta) + d^2 \int_0^\theta h_d(s)\, ds\right] a^d r^d + o(r^d). \tag{1-4}$$

Here, as elsewhere in this paper, we are concerned with the behavior as $r \to \infty$. Thus, one should understand that statements of the type $f(r) = g(r) + o(r^p)$ are statements which hold for r sufficiently large.

Corollary 1.4 shows that (1-4) holds for any $V \in \mathfrak{M}_a$. These results show that any potential in $\mathfrak{M}_a$ must have resonances distributed regularly in sectors, as well as being distributed regularly in balls centered at the origin. A result like this proposition and Corollary 1.4 is, for the special potentials of the form $V(x) = v(|x|)$ mentioned earlier, implicit in [Zworski 1989b] and [Stefanov 2006]. Here we derive it as a corollary of some complex-analytic results, and note that it holds for *any* potential $V \in \mathfrak{M}_a$. We note that this proposition could, in fact, follow as a corollary to Theorem 1.3. However, we prefer to give a separate proof that uses standard results for functions of completely regular growth.

In the following theorem, we use the notation $N_V(r) = \int_0^r (1/t)\big(n_V(t) - n_V(0)\big)\,dt$ and $N_V(r,\varphi,\theta) = \int_0^r (1/t)\big(n_V(r,\varphi,\theta) - n_V(0,\varphi,\theta)\big)\,dt$. This theorem shows that there are many potentials for which something close to the optimal upper bound on the resonances is achieved.

Theorem 1.2. *Let $\Omega \subset \mathbb{C}^p$ be an open connected set. Suppose that $V(z) = V(z,x)$ is holomorphic in $z \in \Omega$, that $V(z,x) \in L^\infty(\mathbb{R}^d)$ for each $z \in \Omega$, and that $V(z,x) = 0$ if $|x| > a$. Suppose in addition that for some $z_0 \in \Omega$, $V(z_0) \in \mathfrak{M}_a$. Then there is a pluripolar set $E \subset \Omega$ so that*

$$\limsup_{r\to\infty} \frac{N_{V(z)}(r)}{r^d} = \frac{c_d a^d}{d} \quad \textit{for all } z \in \Omega \setminus E.$$

Moreover, for any $\theta > 0$, $\theta < \pi$, there is a pluripolar set E_θ so that

$$\limsup_{r\to\infty} \frac{N_{V(z)}(r,\pi,\pi+\theta)}{r^d} \ge \lim_{\epsilon\downarrow 0} \frac{a^d}{2\pi d^2} h_d'(\epsilon)$$

for all $z \in \Omega \setminus E_\theta$.

For example, for a family of potentials satisfying the conditions of the theorem, one may take, for $z \in \mathbb{C}$, $V(z) = zV_1 + (1-z)V_0$, where $V_0 \in \mathfrak{M}_a$ and $V_1 \in L^\infty(\mathbb{R}^d)$ have support in $\overline{B}(0,a)$. Since $h_d'(0+) = \lim_{\epsilon\downarrow 0} h_d'(\epsilon) > 0$ (see Lemma 3.3), the second statement of the theorem is meaningful. This result is of particular interest since resonances near the real axis are considered the more physically relevant ones.

We recall the definition of a pluripolar set in Section 2. Here we mention that a pluripolar set is small. A pluripolar set $E \subset \mathbb{C}^p$ has $\mathbb{R}^{2p}$ Lebesgue measure 0, and if $E \subset \mathbb{C}$ is pluripolar, $E \cap \mathbb{R}$ has one-dimensional Lebesgue measure 0 (see, for example, [Lelong and Gruman 1986; Ransford 1995]). Thus the statements of Theorem 1.2 hold for "most" values of $z \in \Omega$.

If we take a weighted average over a family of potentials, a kind of expected value, we are able to find asymptotics analogous to those which hold for a potential in $\mathfrak{M}_a$. In the statement of the next theorem and later in the paper, we use the notation $d\mathscr{L}(z) = d\,\mathrm{Re}\,z_1 d\,\mathrm{Im}\,z_1 \dots d\,\mathrm{Re}\,z_p d\,\mathrm{Im}\,z_p$.

Theorem 1.3. *Suppose the hypotheses of Theorem 1.2 are satisfied. Then for any $\psi \in C_c(\Omega)$,*

$$\int_\Omega \psi(z) n_{V(z)}(r)\,d\mathscr{L}(z) = c_d a^d r^d \int_\Omega \psi(z)\,d\mathscr{L}(z) + o(r^d)$$

as $r \to \infty$. Additionally, we have, for $0 < \varphi < \theta < \pi$,

$$\int_\Omega \psi(z) n_{V(z)}(r,\varphi+\pi,\theta+\pi)\,d\mathscr{L}(z) = \frac{1}{2\pi d}\tilde{s}_d(\varphi,\theta) r^d a^d \int_\Omega \psi(z)\,d\mathscr{L}(z) + o(r^d),$$

where $\tilde{s}_d$ is as defined in Proposition 1.1. Moreover, for $0 < \theta < \pi$,

$$\int_\Omega \psi(z) n_{V(z)}(r,\pi,\theta+\pi)\, d\mathcal{L}(z) = \frac{1}{2\pi d}\left[h_d'(\theta) + d^2\int_0^\theta h_d(s)\,ds\right] a^d r^d \int_\Omega \psi(z)\, d\mathcal{L}(z) + o(r^d).$$

Corollary 1.4. *Let $V \in \mathfrak{M}_a$. For any $0 < \theta < \pi$,*

$$n_V(r,\pi,\theta+\pi) = \frac{1}{2\pi d}\left[h_d'(\theta) + d^2\int_0^\theta h_d(s)\,ds\right] a^d r^d + o(r^d) \tag{1-5}$$

and, for any $0 < \varphi < \pi$,

$$n_V(r,\varphi+\pi,2\pi) = \frac{1}{2\pi d}\left[-h_d'(\varphi) + d^2\int_\varphi^\pi h_d(s)\,ds\right] a^d r^d + o(r^d) \tag{1-6}$$

as $r \to \infty$.

This corollary follows from Theorem 1.3 by taking $V(z)$ equal to the constant (in z) potential V. We could instead give a more direct proof by, essentially, simplifying the proof of Proposition 5.3 and then applying Lemma 5.4.

It is worth noting that the coefficients of r^d in (1-5) and (1-6) are positive, so that in any sector in the lower half-plane which touches the real axis, the number of resonances grows like r^d.

The proofs of the results here are possible because of the optimal upper bounds on

$$\limsup_{r\to\infty} r^{-d} \ln\left|\det S_V(re^{i\theta})\right|,$$

$0 < \theta < \pi$, proved in [Stefanov 2006] (see Theorem 3.2 here). These, combined with some one-dimensional complex analysis, are used to prove Proposition 1.1, and could be used to give a direct proof of Corollary 1.4. The proofs of the other theorems use, in addition to one-dimensional complex analysis, some facts about plurisubharmonic functions. Many of the complex-analytic results which we shall use are recalled in Section 2.

Again, we emphasize that we are concerned here with large r behavior of resonance counting functions, and consequently of other functions as well. Thus, statements of the type $f(r) = g(r) + o(r^p)$ are to be understood as holding for large values of r.

2. Some complex analysis

In this section we recall some definitions and results from complex analysis in one and several variables. We will mostly follow the notation and conventions of [Levin 1964; Lelong and Gruman 1986]. We also prove a result, Proposition 2.2, for which we are unaware of a proof in the literature.

The *upper relative measure* of a set $E \subset \mathbb{R}_+$ is

$$\limsup_{r\to\infty} \frac{\text{meas}(E \cap (0,r))}{r}.$$

A set $E \subset \mathbb{R}_+$ is said to have *zero relative measure* if it has upper relative measure 0.

If f is a function holomorphic in the sector $\varphi < \arg z < \theta$, we shall say f is of order ρ if

$$\limsup_{r\to\infty} \frac{\ln\ln\bigl(\max_{\varphi<\phi<\theta} |f(re^{i\phi})|\bigr)}{\ln r} = \rho.$$

We shall further restrict ourselves to functions of order ρ and *finite type*, so that

$$\limsup_{r\to\infty} \frac{\ln\bigl(\max_{\varphi<\phi<\theta} |f(re^{i\phi})|\bigr)}{r^{\rho}} < \infty.$$

We shall use similar definitions for a function holomorphic in a neighborhood of a closed sector. In this section only, we shall, following [Levin 1964], use the notation h_f for the *indicator function* (or *indicator*) of a function f of order ρ:

$$h_f(\theta) \stackrel{\text{def}}{=} \limsup_{r\to\infty} \bigl(r^{-\rho} \ln |f(re^{i\theta})|\bigr).$$

Suppose f is a function analytic in the angle (θ_1, θ_2) and of order ρ and finite type there. The function f is of *completely regular growth* on some set of rays $R_{\mathfrak{M}}$ ($\mathfrak{M}$ is the set of values of θ) if the function

$$h_{f,r}(\theta) \stackrel{\text{def}}{=} \frac{\ln |f(re^{i\theta})|}{r^{\rho}}$$

converges uniformly to $h_f(\theta)$ for $\theta \in \mathfrak{M}$ when r goes to infinity taking on all positive values except possibly for a set $E_{\mathfrak{M}}$ of zero relative measure. The function f is of *completely regular growth in the angle* (θ_1, θ_2) if it is of completely regular growth on every closed interior angle.

Functions of completely regular growth have zeros that are rather regularly distributed. For a function f holomorphic in $\{z : \theta_1 < \arg z < \theta_2\}$ we define $m_f(r, \varphi, \theta)$, for $\theta_1 < \varphi < \theta < \theta_2$, to be the number of zeros of $f(z)$ in the sector $\varphi \le \arg z \le \theta$, $|z| \le r$.[1]

Theorem 2.1 [Levin 1964, Chapter III, Theorem 3]. *If a holomorphic function $f(z)$ of order d and finite type has completely regular growth within an angle (θ_1, θ_2), then for all values of φ and θ with $\theta_1 < \varphi < \theta < \theta_2$, except possibly for a denumerable set, the following limit will exist*:

$$\lim_{r\to\infty} \frac{m_f(r, \varphi, \theta)}{r^d} = \frac{1}{2\pi d} \tilde{s}_f(\varphi, \theta),$$

where

$$\tilde{s}_f(\varphi, \theta) = \left[h'_f(\theta) - h'_f(\varphi) + d^2 \int_{\varphi}^{\theta} h_f(s)\, ds \right].$$

The exceptional denumerable set can only consist of points for which $h'_f(\theta + 0) \neq h'_f(\theta - 0)$.

In the following proposition, we use the notation $m_f(r)$ to denote the number of zeros of a function f, counted with multiplicity, with norm at most r. It is likely that some of the hypotheses included here could be relaxed. However, when we apply this proposition, f will be the determinant of the scattering matrix, perhaps multiplied by a rational function, and many of these hypotheses are natural in such applications.

[1]More standard notation would be $n(r, \varphi, \theta)$, but we have already defined $n_V(r, \varphi, \theta)$ to be something else.

Let $f(z)$ be a function meromorphic on $\mathbb{C}$. Then $f(z) = g_1(z)/g_2(z)$, with g_1, g_2 entire. The functions g_1 and g_2 are not uniquely determined. However, the order of f can be defined to be

$$\min\{\max(\text{order of } g_1, \text{ order of } g_2)\colon f(z) = g_1(z)/g_2(z) \text{ with } g_1, g_2 \text{ entire}\}.$$

It is possible to define the order of a meromorphic function by using the Nevanlinna characteristic function, yielding the same result.

Proposition 2.2. *Let f be a function meromorphic in the complex plane, with neither zeros nor poles on the real line. Suppose all the zeros of f lie in the open upper half-plane, and all the poles in the open lower half-plane. Furthermore, assume f is of order $d > 1$, h_f is finite for $0 \le \theta \le \pi$, and $h_f(\theta_0) \ne 0$ for some θ_0, $0 < \theta_0 < \pi$. Suppose in addition that*

$$\int_0^r \frac{f'(t)}{f(t)}\, dt = o(r^d) \quad \text{as } r \to \pm\infty, \tag{2-1}$$

and that the number of poles of f with norm at most r is of order at most d. If

$$\liminf_{r\to\infty} \frac{m_f(r)}{r^d} = \frac{d}{2\pi} \int_0^\pi h_f(\theta)\, d\theta,$$

then f is of completely regular growth in the angle $(0, \pi)$.

Before proving the proposition, we note that Govorov [1965; 1967] has studied the issue of completely regular growth of functions holomorphic in an angle. This is discussed in [Levin 1964, Appendix VIII, Section 2]. This is somewhat different than what we consider, since we use the assumption that f is meromorphic and of order d on the plane. Thus Govorov uses different restrictions on the distribution of the zeros of f.

Proof. The proof of this proposition follows in outline the proof of the analogous theorem for entire functions in the plane [Levin 1964, Chapter IV, Theorem 3]. Rather than using Jensen's theorem, though, it uses the equality

$$\int_0^r \frac{m_f(t)}{t}\, dt = \frac{1}{2\pi} \operatorname{Im} \int_0^r \frac{1}{t} \int_{-t}^t \frac{f'(s)}{f(s)}\, ds\, dt + \frac{1}{2\pi} \int_0^\pi \ln |f(re^{i\theta})|\, d\theta \tag{2-2}$$

if $|f(0)| = 1$, which follows using the proof of [Froese 1998, Lemma 6.1].

By [Levin 1964, Property (4), Chapter I, Section 12],

$$\liminf_{r\to\infty} \frac{m_f(r)}{r^d} \le \liminf_{r\to\infty} d r^{-d} \int_0^r \frac{m_f(t)}{t}\, dt. \tag{2-3}$$

We note [ibid., Chapter I, Theorem 28] that for any $\epsilon > 0$, there is an $R > 0$ so that

$$r^{-d} \ln |f(re^{i\theta})| \le h_f(\theta) + \epsilon, \quad \text{for} \quad r > R,\ 0 \le \theta \le \pi. \tag{2-4}$$

Using this, (2-2), and our assumptions on the behavior of f on the real axis, we see that

$$\limsup_{r\to\infty} r^{-d} \int_0^r \frac{m_f(t)}{t}\, dt \le \frac{1}{2\pi} \int_0^\pi h_f(\theta)\, d\theta.$$

Combining this with (2-3) and using our assumptions on $m_f(r)$, we get

$$\lim_{r\to\infty} r^{-d}\int_0^r \frac{m_f(t)}{t}\,dt = \frac{1}{2\pi}\int_0^\pi h_f(\theta)\,d\theta.$$

Thus using (2-2) and (2-1) again, we have

$$\lim_{r\to\infty}\int_0^\pi \left[h_f(\theta) - r^{-d}\ln|f(re^{i\theta})|\right]d\theta = 0,$$

and, using (2-4),

$$\lim_{r\to\infty}\int_0^\pi \left|h_f(\theta) - r^{-d}\ln|f(re^{i\theta})|\right|d\theta = 0.$$

Since we have assumed f is of order d, we may write f as the quotient of two entire functions, each of order at most d. Then we may apply [Levin 1964, Chapter 2, Theorem 7] to find that for every $\eta > 0$, there is a set E_η of positive numbers of upper relative measure less than η so that if $r \notin E_\eta$, the family of functions of θ,

$$h_{f,r}(\theta) \stackrel{\text{def}}{=} r^{-d}\ln|f(re^{i\theta})|,$$

is equicontinuous in the angle $0 < \epsilon_0 \le \theta \le \pi - \epsilon_0$.

Now let $\theta_2 > \theta_1$, with $[\theta_1, \theta_2] \subset (0, \pi)$. Given $\eta > 0$ and $\epsilon > 0$ we can, by the above result, find a $\delta > 0$ with $[\theta_1 - \delta, \theta_2 + \delta] \subset (0, \pi)$ and a set E_η of upper relative measure at most η so that if $\theta \in [\theta_1, \theta_2]$, $r \notin E_\eta$, and $|\varphi - \theta| < \delta$, then $|h_{f,r}(\theta) - h_{f,r}(\varphi)| < \epsilon/4$ and $|h_f(\theta) - h_f(\varphi)| < \epsilon/4$. Then for $0 < |k| < \delta$, $r \notin E_\eta$, and $\theta \in [\theta_1, \theta_2]$,

$$\begin{aligned}|h_{f,r}(\theta) - h_f(\theta)| &< \frac{\epsilon}{2} + \frac{1}{k}\int_\theta^{\theta+k} |h_{f,r}(\varphi) - h_f(\varphi)|\,d\varphi \\ &\le \frac{\epsilon}{2} + \frac{1}{k}\int_0^\pi |h_{f,r}(\varphi) - h_f(\varphi)|\,d\varphi.\end{aligned}$$

Since the integral goes to 0 as $r \to \infty$, we have shown that $|h_{f,r}(\theta) - h_f(\theta)| < \epsilon$ for $r > r_\epsilon$, $r \notin E_\eta$. Since $\eta > 0$ and $\epsilon > 0$ are arbitrary, we have, by [Levin 1964, Chapter III, Lemma 1], that f is of completely regular growth in $[\theta_1, \theta_2]$. Since θ_1, θ_2 were arbitrary except that $[\theta_1, \theta_2] \subset (0, \pi)$, we have proved the proposition. □

We shall also need some basics about plurisubharmonic functions and pluripolar sets. We use notation as in [Lelong and Gruman 1986] and direct the reader to this reference for more details.

Let $\Omega \subset \mathbb{C}^p$ be an open connected set. A function $\Psi : \Omega \to [-\infty, \infty)$ is said to be *plurisubharmonic* if $\Psi \not\equiv -\infty$, Ψ is upper semicontinuous, and

$$\Psi(z) \le \frac{1}{2\pi}\int_0^{2\pi} \Psi(z + wre^{i\theta})\,d\theta$$

for all w, r such that $z + uw \in \Omega$ for all $u \in \mathbb{C}$, $|u| \le r$. A classic example of a plurisubharmonic function is $\ln|f(z)|$, where $f(z)$ is holomorphic. A subset $E \subset \Omega \subset \mathbb{C}^p$ is said to be *pluripolar* if there is a function Ψ plurisubharmonic on Ω so that $E \subset \{z : \Psi(z) = -\infty\}$.

For the convenience of the reader, we recall an additional fact from several complex variables that we shall need.

Proposition 2.3 [Lelong and Gruman 1986, Proposition 1.39]. *Let $\{\Psi_q\}$ be a sequence of plurisubharmonic functions uniformly bounded above on compact subsets in an open connected set $\Omega \subset \mathbb{C}^p$, with $\limsup_{q\to\infty} \Psi_q \leq 0$, and suppose that there exist $\xi \in \Omega$ such that $\limsup_{q\to\infty} \Psi_q(\xi) = 0$. Then $A = \{z \in \Omega : \limsup_{q\to\infty} \Psi_q(z) < 0\}$ is pluripolar in Ω.*

3. The functions $s_V(\lambda) = \det S_V(\lambda)$ and $h_d(\theta)$

For $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ and $\chi \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ with $\chi V = V$, we have $\chi R_V(\lambda)\chi = \chi R_0(\lambda)\chi(I + V R_0(\lambda)\chi)^{-1}$. Since for any χ with compact support in $\mathbb{R}^d$, $\|\chi R_0(\lambda)\chi\| \leq c_\chi/|\lambda|$ when $\operatorname{Im}\lambda \geq 0$, we see that $R_V(\lambda)$ can have only finitely many poles in the closed upper half-plane.

For $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$, let $S_V(\lambda)$ be the associated scattering matrix and $s_V(\lambda) = \det S_V(\lambda)$. With at most finitely many exceptions, the poles of $s_V(\lambda)$ coincide with the poles of $R_V(\lambda)$, and the multiplicities agree. Moreover, $s_V(\lambda)s_V(-\lambda) = 1$.

Lemma 3.1 [Christiansen 2005, Lemma 3.1]. *Let $V \in L^\infty_{\text{comp}}(\mathbb{R}^d; \mathbb{C})$. For $\lambda \in \mathbb{R}$, there is a C_V so that*

$$\left|\frac{d}{d\lambda} \ln s_V(\lambda)\right| \leq C_V|\lambda|^{d-2}$$

whenever $|\lambda|$ is sufficiently large.

In fact, if $\operatorname{supp} V \subset \overline{B}(0, a)$, there is a constant $\alpha_d = \alpha_{d,a}$, so that it suffices to take $|\lambda| \geq 2\alpha_d\|V\|_\infty$ for such a bound to hold. We note that for $\lambda \in \mathbb{R}$, $|\lambda| \geq 2\alpha_d\|V\|_\infty$, under these same assumptions on V,

$$\|S_V(\lambda) - I\| \leq C|\lambda|^{-1}. \tag{3-1}$$

This is relatively easy to see from an explicit representation of the scattering matrix; see, for example, the proof of the lemma just stated in [Christiansen 2005]. The constants in the statement of that lemma and in (3-1) can be chosen to depend only on the dimension, $\|V\|_\infty$, and the support of V. We note that it follows from Lemma 3.1, (3-1), and (2-2) that as $r \to \infty$,

$$\int_0^r \frac{n_V(t)}{t}\, dt = \int_0^\pi \ln\left|\det S_V(re^{i\theta})\right| d\theta + O(r^{d-1}). \tag{3-2}$$

Let

$$\rho(z) \overset{\text{def}}{=} \ln \frac{1+\sqrt{1-z^2}}{z} - \sqrt{1-z^2}, \quad 0 < \arg z < \pi. \tag{3-3}$$

This is a function which arises in studying the asymptotics of Bessel functions; see [Olver 1954]. To define the square root which appears here, take the branch cut on the negative real axis and define ρ to be a continuous function in $\{0 < \arg z < \pi\} \cup (0, 1)$ and use the principal branches of the logarithm and the square root when $z \in (0, 1)$.

We use some notation from [Stefanov 2006]. Set, for $0 < \theta < \pi$,

$$h_d(\theta) \overset{\text{def}}{=} \frac{4}{(d-2)!} \int_0^\infty \frac{[-\operatorname{Re}\rho]_+(te^{i\theta})}{t^{d+1}}\, dt \tag{3-4}$$

and set $h_d(0) = 0,\ h_d(\pi) = 0$. Further, define

$$c_d \overset{\text{def}}{=} \frac{d}{2\pi} \int_0^{\pi} h_d(\theta)\, d\theta = \frac{2d}{\pi(d-2)!} \int_{\operatorname{Im} z>0} \frac{[-\operatorname{Re} \rho]_+(z)}{|z|^{d+2}}\, dx\, dy. \tag{3-5}$$

This is the constant c_d that appears in (1-1).

The next result is adapted from [Stefanov 2006, Theorem 5]; the original result covers a much larger class of operators.

Theorem 3.2. *Let $V \in L^\infty(\mathbb{R}^d)$ be supported in $\overline{B}(0, a)$.*

(a) *For any $\theta \in [0, \pi]$,*

$$\ln |s_V(re^{i\theta})| \le h_d(\theta) a^d r^d + o(r^d) \quad \text{as } r \to \infty, \tag{3-6}$$

and the remainder term depends on V and is uniform for $0 < \delta \le \theta \le \pi - \delta$ for any $\delta \in (0, \pi)$.

(b) *For any $\delta > 0$,*

$$\ln |s_V(re^{i\theta})| \le (h_d(\theta) a^d + \delta) r^d + o(r^d) \quad \text{as } r \to \infty$$

uniformly in $\theta \in [0, \pi]$.

We remark that both of these statements are about "large r" behavior, so that the possibility that s_V has a finite number of poles in the upper half-plane does not affect the validity of the statements.

It is important to note several things about the bounds in this theorem. One is that although Stefanov's theorem is stated only for self-adjoint operators (hence V real), it is equally valid when we allow complex-valued potentials. In fact, the proof of (a) in [Stefanov 2006, Theorem 5] uses self-adjointness only to obtain a bound on the resolvent for λ in the upper half-plane. A similar bound is true for the operator $-\Delta + V$ when V is complex-valued. The proof of (b) uses the fact that $\ln |s_V(\lambda)| = 1$ for real V and $\lambda \in \mathbb{R}$. For complex-valued V, the proof in [Stefanov 2006] of (b) can be adapted by using (3-1) and Lemma 3.1 to show that $\big|\ln |s_V(\lambda)|\big| \le C(1 + |\lambda|)^{d-1}$ for $\lambda \in \mathbb{R}$ with $|\lambda| \ge 2\alpha_d \|V\|_\infty$. Here C can be chosen to depend only on d, $\|V\|_\infty$, and the diameter of the support of V.

Likewise, the particulars of the operator enter only through the diameter of the support of the perturbation (for us, the diameter of the support of V, which is $2a$) and the aforementioned bound on the resolvent in the good half-plane $\operatorname{Im} \lambda > 0$. Thus, it is easy to see that the estimates of Theorem 3.2 are uniform in V as long as $\operatorname{supp} V \subset \overline{B}(0, a)$, $\|V\|_\infty \le M$, and $r \ge 2\alpha_d M$.

We note that the upper bound (1-1) on the integrated resonance-counting function holds with the constant c_d defined in (3-5) even if V is complex-valued. This follows from the proof in [Stefanov 2006]. In fact, the proof uses the bounds recalled in Theorem 3.2 and the identity (2-2). Together with the bounds in Lemma 3.1 and (3-1), these prove (1-1), even when V is complex-valued.

We shall want to understand the function $h_d(\theta)$ better. Note that for $0 < \theta \le \pi/2$,

$$h_d\Big(\frac{\pi}{2} + \theta\Big) = h_d\Big(\frac{\pi}{2} - \theta\Big).$$

This can be seen directly using the definition of h_d and ρ.

Lemma 3.3. *The function $h_d(\theta)$, defined in* (3-4), *is C^1 on* $(0,\pi)$. *Moreover*,

$$h_d'(0+) \stackrel{\text{def}}{=} \lim_{\epsilon\downarrow 0} h_d'(\epsilon) = \sqrt{\pi}\,\frac{\Gamma\left(\frac{d-1}{2}\right)}{(d-2)!\,\Gamma\left(1+\frac{d}{2}\right)}.$$

Proof. We note [Olver 1954, Section 4] that $\operatorname{Re}\rho(z) < 0$ if $0 < \arg z < \pi$ and $|z| > |z_0(\arg z)|$, where $z_0(\theta)$ is the unique point in $\mathbb{C}$ with argument θ and which lies on the curve given by

$$\pm(s\coth s - s^2)^{1/2} + i(s^2 - s\tanh s)^{1/2}, \quad 0 \le s \le s_0.$$

Here s_0 is the positive solution of $\coth s = s$. Furthermore, $\operatorname{Re}\rho(z) > 0$ if z is in the upper half-plane but $|z| < |z_0(\arg z)|$. Hence, recalling the definition of h_d, we have

$$h_d(\theta) = \frac{4}{(d-2)!}\int_{|z_0(\theta)|}^{\infty} \frac{[-\operatorname{Re}\rho](te^{i\theta})}{t^{d+1}}\,dt.$$

Using the definition of ρ in (3-3) and the following comments, we see that ρ is in fact a smooth function of z with $0 < \arg z < \pi$, $|z| > 0$. Since $|\rho(z)|/|z| \to 1$ when $|z| \to \infty$ in this region, the integral defining h_d is absolutely convergent. Likewise, since

$$\frac{\partial}{\partial\theta}\rho(te^{i\theta}) = -i\sqrt{1-(te^{i\theta})^2},$$

we have

$$\left|\frac{-\operatorname{Re}\left[\frac{\partial}{\partial\theta}\rho(te^{i\theta})\right]}{t^{d+1}}\right| \le Ct^{-d},$$

and the integral

$$\int_{|z_0(\theta)|}^{\infty} \frac{-\operatorname{Re}\left[\frac{\partial}{\partial\theta}\rho(te^{i\theta})\right]}{t^{d+1}}\,dt$$

converges absolutely. A computation shows that $|z_0|$ is a C^1 function of θ for θ in $(0,\pi)$, and that $\lim_{\epsilon\downarrow 0}(\partial/\partial\theta)|z_0|$ is finite. Thus, using that $\operatorname{Re}\rho(z_0(\theta)) = 0$ and the regularity of the derivative of $|z_0|(\theta)$, we get

$$\frac{d}{d\theta}h_d(\theta) = \frac{4}{(d-2)!}\int_{|z_0(\theta)|}^{\infty} \frac{\operatorname{Re} i\sqrt{1-(te^{i\theta})^2}}{t^{d+1}}\,dt,$$

which is continuous in θ. Thus h_d is C^1 on $(0,\pi)$, we have

$$h_d'(0+) = \frac{4}{(d-2)!}\int_1^{\infty} \frac{\sqrt{t^2-1}}{t^{d+1}}\,dt,$$

and a computation now finishes the proof of the lemma. □

If $d = 3$, we can compute that

$$h_3(\theta) = \tfrac{4}{9}\left(\sin(3\theta) + \operatorname{Re}\frac{(1-z_0^2(\theta))^{3/2}}{|z_0(\theta)|^3}\right),$$

where $z_0(\theta)$ is as in the proof of the lemma. We comment that the $\sin(3\theta)$ term is missing from the first remark following the statement of [Stefanov 2006, Theorem 5].

4. Proof of Proposition 1.1

We can now give the proof of Proposition 1.1, which follows by combining Theorem 2.1, Proposition 2.2, and [Stefanov 2006, Theorem 5].

Recall that $S_V(\lambda)$ is the scattering matrix associated with the operator $-\Delta + V$, and $s_V(\lambda) = \det S_V(\lambda)$. Then s_V has a pole at λ if and only if s_V has a zero at $-\lambda$, and the multiplicities coincide. Moreover, with at most a finite number of exceptions, the poles of $s_V(\lambda)$ coincide, with multiplicity, with the zeros of $R_V(\lambda)$.

If $s_V(\lambda)$ has poles in the closed upper half-plane, it has only finitely many, say $\lambda_1, \ldots, \lambda_m$, where the poles are repeated according to multiplicity. Set

$$f(\lambda) = \prod_{j=1}^{m} \frac{(\lambda - \lambda_j)}{\lambda + \lambda_j} s_V(\lambda).$$

We check that f satisfies the hypotheses of Proposition 2.2. Note that f and $s_V(\lambda)$ have the same order and they have the same indicator function for $0 \le \theta \le \pi$. We know that s_V has order at most d by [Zworski 1997, Theorem 7]. Moreover, for any M chosen large enough that s_V has no zeros or poles bigger than M on the real line, for $r > M$ we have

$$\int_0^r \frac{f'(t)}{f(t)}\, dt = \int_M^r \frac{s_V'(t)}{s_V(t)}\, dt + O(1).$$

Using (3-1) and Lemma 3.1, we see that

$$\int_M^r \frac{s_V'(t)}{s_V(t)}\, dt = O(r^{d-1}) \quad \text{as } r \to \infty,$$

yielding

$$\int_0^r \frac{f'(t)}{f(t)}\, dt = O(r^{d-1}) \quad \text{as } r \to \infty. \tag{4-1}$$

A similar argument gives the same bound for $r \to -\infty$. It remains to check the hypotheses on the indicator function; this is done in the next paragraph.

From [Stefanov 2006, Theorem 5], recalled here in Theorem 3.2, for $0 \le \theta \le \pi$ and large r,

$$r^{-d} \ln |f(re^{i\theta})| \le a^d h_d(\theta) + o(1),$$

where we have some uniformity in θ. Thus, using (2-2) and (4-1), we get

$$\limsup_{r\to\infty} r^{-d} N_V(r) = \limsup_{r\to\infty} r^{-d} \frac{1}{2\pi} \int_0^\pi \ln |f(re^{i\theta})| d\theta \le \frac{a^d}{2\pi} \int_0^\pi h_d(\theta) d\theta.$$

But since $V \in \mathfrak{M}_a$,

$$\lim_{r\to\infty} r^{-d} N_V(r) = \frac{c_d a^d}{d} = \frac{a^d}{2\pi} \int_0^\pi h_d(\theta)\, d\theta,$$

and we see that we must have

$$\limsup_{r\to\infty} r^{-d} \ln |f(re^{i\theta})| = a^d h_d(\theta), \quad \text{for almost every } \theta \in (0, \pi).$$

The left-hand side of the above equation is the value of the indicator function of f at θ. But the indicator function of f is continuous on $(0, \pi)$ [Levin 1964, Section 16, point (a) on p. 54], and so is $h_d(\theta)$. Thus we must have

$$\limsup_{r\to\infty} r^{-d} \ln |f(re^{i\theta})| = a^d h_d(\theta) \quad \text{for } \theta \in (0, \pi).$$

Applying Proposition 2.2 to $f(\lambda)$, we see that $f(\lambda)$ is a function of completely regular growth in the upper half-plane. Since $h_d(\theta)$ is a C^1 function of θ for $\theta \in (0, \pi)$, we get the proposition from Theorem 2.1.

5. Proof of Theorem 1.3

This section proves Theorem 1.3. We begin by outlining the strategy of the proof.

For $0 < \varphi < \theta < 2\pi$, recall the notation $n_V(r, \varphi, \theta)$ for the number of poles of $R_V(\lambda)$ in the sector $\{z : |z| \le r,\ \varphi \le \arg z \le \theta\}$. A representative claim of the theorem is that with $V(z)$, Ω as in the statement of the theorem, $0 < \theta < \pi$,

$$\int_\Omega \psi(z) n_{V(z)}(r, \pi, \theta+\pi)\, d\mathcal{L}(z) = \frac{1}{2\pi d}\left[h_d'(\theta) + d^2 \int_0^\theta h_d(s)\, ds\right] a^d r^d \int_\Omega \psi(z)\, d\mathcal{L}(z) + o(r^d) \tag{5-1}$$

as $r \to \infty$ for any $\psi \in C_c(\Omega)$. We prove this via the intermediate step of showing that (5-1) holds for ψ which is the characteristic function of any suitable ball in Ω (Proposition 5.7). To get (5-1) for $\psi \in C_c(\Omega)$, we cover the support of ψ with the union of a finite number of small disjoint balls and a set of small volume. On each small ball, we can approximate ψ by its value at the center of the ball and apply Proposition 5.7. This and the necessary estimates are done in the proof of the theorem which ends this section.

The proof of Proposition 5.7 is done in a number of steps. We set

$$N_V(r, \varphi, \theta) = \int_0^r \frac{1}{t}\big(n_V(t, \varphi, \theta) - n_V(0, \varphi, \theta)\big)\, dt.$$

Lemma 5.2 gives $\int_0^\theta N_V(r, \pi, \theta'+\pi)\, d\theta'$ as a sum of two integrals involving $\ln |s_V|$ and an error of order r^{d-1}. This follows from an application of one-dimensional complex analysis, Lemma 3.1, and (3-1). Next we consider the function

$$\Psi(z, r, \rho) \stackrel{\text{def}}{=} \frac{1}{\operatorname{vol}(B(z, \rho))} \int_{z' \in B(z,\rho)} \int_0^\theta N_{V(z')}(r, \pi, \theta'+\pi)\, d\theta' d\mathcal{L}(z').$$

Here we use $B(z, \rho)$ to be the ball with center z and radius ρ in $\mathbb{C}^p$. Thus the function Ψ is the average over balls of varying center z. Fix ρ small, and consider this as a function of z and r. Lemma 5.2 is used to show that Ψ is the sum of a function Ψ_1 which is plurisubharmonic in z and a function which is $O(r^{d-1})$. The proof of Proposition 5.3 uses a combination of properties of plurisubharmonic functions and the fact that $r^{-d} N_{V(z')}(r, \pi, \theta'+\pi)$ is not negative and can be (locally) uniformly bounded above for large r to prove an "averaged" in θ and r version of (5-1) for ψ the characteristic function of a ball in Ω

satisfying some conditions. Propositions 5.5 and then 5.7 eliminate the need to average in θ and r, using Lemma 5.4.

The proofs of the other claims of Theorem 1.3 are quite similar; the proof of Proposition 5.6 and the final proof of the theorem indicate the differences.

Now we turn to proving the theorem. We shall need an identity related to both (2-2) and to what Levin [1964, Chapter 3, Section 2] calls a generalized formula of Jensen. We define, following [Levin 1964], for a function f meromorphic in a neighborhood of $\arg z = \theta$ and with $|f(0)| = 1$,

$$J_f^r(\theta) \stackrel{\text{def}}{=} \int_0^r \frac{\ln|f(te^{i\theta})|}{t}\,dt. \tag{5-2}$$

This integral is well-defined even if f has a zero or pole with argument θ.

Lemma 5.1. *Let f be holomorphic in $\varphi \le \arg z \le \theta$, let $|f(0)| = 1$, let f have no zeros with argument φ or θ and with norm less than r, and let $m(r,\varphi,\theta)$ be the number of zeros of f in the sector $\varphi < \arg z < \theta$, $|z| \le r$. Then*

$$\int_0^r \frac{m(t,\varphi,\theta)}{t}\,dt \\ = \frac{1}{2\pi}\int_0^r \frac{\partial}{\partial\theta} J_f^t(\theta)\frac{dt}{t} + \frac{1}{2\pi}\int_0^r \frac{1}{t}\int_0^t \frac{\partial}{\partial s}\arg f(se^{i\varphi})\,ds\,dt + \frac{1}{2\pi}\int_\varphi^\theta \ln|f(re^{i\omega})|\,d\omega. \tag{5-3}$$

Proof. Using the argument principle and the Cauchy–Riemann equations just as in [Levin 1964, Chapter 3, Section 2], we see that

$$2\pi m(r',\varphi,\theta) = \int_0^{r'} \frac{\partial}{\partial t}\arg f(te^{i\varphi})\,dt + \int_0^{r'} \frac{1}{t}\frac{\partial}{\partial\theta}\ln|f(te^{i\theta})|\,dt + r'\int_\varphi^\theta \frac{\partial}{\partial r'}\ln|f(r'e^{i\omega})|\,d\omega$$

when there are no zeros on the boundary of the sector. As in [Levin 1964], by dividing by $2\pi r'$ and integrating from 0 to r in r', we obtain the lemma. □

We note that $|s_V(0)| = 1$, since $s_V(\lambda)s_V(-\lambda) = 1$.

Lemma 5.2. *Suppose $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$. Then for $0 < \theta < \pi$,*

$$\int_0^\theta N_V(r,\pi,\theta'+\pi)\,d\theta' = \frac{1}{2\pi}\int_0^r J_{s_V}^t(\theta)\frac{dt}{t} + \frac{1}{2\pi}\int_0^\theta\int_0^{\theta'} \ln|s_V(re^{i\omega})|\,d\omega\,d\theta' + O(r^{d-1})$$

as $r \to \infty$. The error can be bounded by $c\langle r^{d-1}\rangle$, where the constant depends only on $\|V\|_\infty$, the support of V, and d.

Proof. Recall that with at most a finite number of exceptions, λ_0 is a pole of $R_V(\lambda)$ if and only if $-\lambda_0$ is a zero of $s_V(\lambda)$, and the multiplicities coincide. As in the proof of Proposition 1.1, if $s_V(\lambda)$ has poles $\lambda_1, \dots, \lambda_m$ in the closed upper half-plane, we introduce the function

$$f(\lambda) = \frac{(\lambda-\lambda_1)\dots(\lambda-\lambda_m)}{(\lambda+\lambda_1)\dots(\lambda+\lambda_m)}s_V(\lambda),$$

which is holomorphic in the closed upper half-plane. The poles of s_V in the closed upper half-plane correspond to eigenvalues, and the number of such poles can be bounded by a constant depending on d, $\|V\|_\infty$, and the support of V. Note that f has no zeros on the real line and that s_V and f have all but finitely many of the same zeros. Moreover, $\ln|f(re^{i\theta})| = \ln|s_V(re^{i\theta})| + O(1)$ for $r \to \infty$, $0 \le \theta \le \pi$.

Choose $0 < M < \infty$ so that $s_V(\lambda)$ has no zeros in the upper half-plane with norm greater than or equal to M. This constant M can be chosen to depend only on $\|V\|_\infty$, the support of V, and d. Now, by using the relationship between the poles of $R_V(\lambda)$ and the zeros of $s_V = \det S_V$ and the relationships between f and s_V just mentioned, and applying Lemma 5.1 to f, we see that for $r > M$, $0 < \theta' < \pi$,

$$N_V(t,\pi,\theta'+\pi) = \frac{1}{2\pi}\int_M^r \frac{\partial}{\partial\theta'} J^t_{s_V}(\theta')\frac{dt}{t} + \frac{1}{2\pi}\int_M^r \frac{1}{t}\int_M^t \frac{d}{dt'}\arg s_V(t')\,dt'dt + \frac{1}{2\pi}\int_0^{\theta'} \ln|s_V(re^{i\omega})|\,d\omega + O((\ln r)^2) \quad (5\text{-}4)$$

if f has no zeros with argument θ' and norm not exceeding r. Here we are using that

$$\int_0^M \frac{\partial}{\partial\theta'} J^t_f(\theta')\frac{dt}{t} = O(1)$$

and

$$\int_0^r \frac{1}{t}\int_0^t \frac{d}{dt'}\arg f(t')\,dt'dt = \int_M^r \frac{1}{t}\int_M^t \frac{d}{dt'}\arg f(t')\,dt'dt + \int_M^r \frac{1}{t}\int_0^M \frac{d}{dt'}\arg f(t')\,dt'dt + \int_0^M \frac{1}{t}\int_0^t \frac{d}{dt'}\arg f(t')\,dt'dt.$$

But

$$\int_M^r \frac{1}{t}\int_0^M \frac{d}{dt'}\arg f(t')\,dt'dt = O(\ln r) \quad \text{and} \quad \int_0^M \frac{1}{t}\int_0^t \frac{d}{dt'}\arg f(t')\,dt'dt = O(1).$$

Additionally, for $t \to \infty$,

$$\frac{d}{dt}\arg f(t) = \frac{d}{dt}\arg s_V(t) + O\Big(\frac{1}{t}\Big).$$

These remainders can be bounded using constants depending only on $\|V\|_\infty$, $\operatorname{supp} V$, and d.

Notice that for fixed value of $r > M$, there are only finitely many values of θ' with s_V having a zero with argument θ' and norm at most r. We integrate (5-4) in θ' from 0 to θ and, as in the proof of Jensen's equality, use the fact that both sides of the equation below are continuous functions of θ, to get

$$\int_0^\theta N_V(r,\pi,\theta'+\pi)\,d\theta' = \frac{1}{2\pi}\int_M^r J^t_{s_V}(\theta)\frac{dt}{t} - \frac{1}{2\pi}\int_M^r J^t_{s_V}(0)\frac{dt}{t} + \frac{\theta}{2\pi}\int_M^r \frac{1}{t}\int_M^t \frac{d}{dt'}\arg s_V(t')\,dt'dt + \frac{1}{2\pi}\int_0^\theta\int_0^{\theta'} \ln|s_V(re^{i\omega})|\,d\omega\,d\theta' + O((\ln r)^2).$$

The bounds of Lemma 3.1 and (3-1) mean that, as $r \to \infty$,

$$\frac{1}{2\pi}\int_M^r J^t_{s_V}(0)\frac{dt}{t} = O(r^{d-1})$$

and

$$\frac{\theta}{2\pi}\int_M^r \frac{1}{t}\int_M^t \frac{d}{dt'}\arg s_V(t')\,dt'dt = O(r^{d-1}),$$

where the bounds can be made uniform in V with support contained in a fixed compact set and $\|V\|_\infty$ bounded. Moreover, we note that $\int_0^M J^t_{s_V}(\theta)(dt/t) = O(1)$. □

We shall need some notation for the results which follow. Let $\Omega \subset \mathbb{C}^{d'}$ be an open set containing a point z_0. For $\rho > 0$ small enough that $B(z_0, \rho) \subset \Omega$, we define Ω_ρ to be the connected component of $\{z \in \Omega : \operatorname{dist}(z, \Omega^c) > \rho\}$ which contains z_0.

Proposition 5.3. *Let V, z_0, Ω satisfy the assumptions of Theorem 1.2, let $\rho > 0$ be small enough that $B(z_0, 2\rho) \subset \Omega$, and let Ω_ρ be as defined above. Then, for $z \in \Omega_{2\rho}$, $0 < \theta < \pi$,*

$$\begin{aligned}\Psi(z, r, \rho) &\overset{\text{def}}{=} \frac{1}{\operatorname{vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_0^\theta N_{V(z')}(r, \pi, \theta' + \pi)\,d\theta' d\mathcal{L}(z')\\ &= \frac{1}{2\pi}a^d r^d\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)\,d\omega\,d\theta'\right) + o(r^d)\end{aligned}$$

as $r \to \infty$.

Proof. First note that since $0 \le dN_{V(z)}(z, \pi, \theta + \pi) \le c_d r^d a^d + o(r^d)$, and the bound is uniform on compact sets of z, we get that holding ρ fixed, $r^{-d}\Psi(\,\cdot\,, r, \rho)$ is a family uniformly continuous in z for z in compact sets of $\overline{\Omega}_{2\rho}$.

We shall use Lemma 5.2. Note that by Stefanov's results recalled in Theorem 3.2, for large r,

$$\frac{1}{2\pi}\int_0^r J^t_{s_{V(z)}}(\theta)\frac{dt}{t} \le \frac{1}{2\pi}\frac{1}{d^2}h_d(\theta)a^d r^d + o(r^d),$$

where the term $o(r^d)$ can be bounded uniformly in z in compact sets of $\overline{\Omega}_\rho$. Recall that this is a statement about large r behavior, and holds even if $s_V(z)$ has poles in the upper half-plane, since it has at most finitely many. By the same argument, for large r,

$$\int_0^\theta\int_0^{\theta'} \ln|s_{V(z)}(re^{i\omega})|\,d\omega\,d\theta' \le \int_0^\theta\int_0^{\theta'} h_d(\omega)\,d\omega\,d\theta' a^d r^d + o(r^d).$$

Using Lemma 5.2, we find that

$$\begin{aligned}\Psi(z, r, \rho) = {} & \frac{1}{2\pi\operatorname{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_0^r J^t_{s_{V(z')}}(\theta)\frac{dt}{t}\,d\mathcal{L}(z')\\ &+ \frac{1}{2\pi\operatorname{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_0^\theta\int_0^{\theta'} \ln|s_{V(z')}(re^{i\omega})|\,d\omega\,d\theta' d\mathcal{L}(z') + O(r^{d-1}).\end{aligned}$$

Let $M = 2\alpha_d \max_{z\in\overline{\Omega_\rho}} \|V(z)\|_\infty$ and set, for $r > M$,

$$\Psi_1(z,r,\rho) = \frac{1}{2\pi \operatorname{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_M^r J^t_{s_{V(z')}}(\theta)\frac{dt}{t}\,d\mathscr{L}(z') \\ + \frac{1}{2\pi \operatorname{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_0^\theta\int_0^{\theta'} \ln|s_{V(z')}(re^{i\omega})|\,d\omega\,d\theta' d\mathscr{L}(z'),$$

and note that

$$\Psi(z,r,\rho) = \Psi_1(z,r,\rho) + O(r^{d-1}).$$

By the bounds above,

$$\Psi_1(z,r,\rho) \le \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)\,d\omega\,d\theta'\right)a^d r^d + o(r^d). \tag{5-5}$$

Using [Lelong and Gruman 1986, Proposition I.14] and the fact that $\ln|s_{V(z)}(\lambda)|$ is a plurisubharmonic function of $z \in \Omega$ when $|\lambda| > 2\alpha_d\|V(z)\|_\infty$ and λ lies in the upper half-plane, we see that $\Psi_1(z,r,\rho)$ is a plurisubharmonic function of $z \in \Omega_{2\rho}$. Since by Proposition 2.2, $s_{V(z_0)}(\lambda)$ is of completely regular growth in $0 < \arg\lambda < \pi$, using Lemma 5.2 and [Levin 1964, Chapter III, Section 2, Lemma 2],

$$\lim_{r\to\infty} r^{-d}\int_0^\theta N_{V(z_0)}(r,\pi,\theta'+\pi)\,d\theta' = \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)\,d\omega\,d\theta'\right)a^d.$$

By the most basic property of plurisubharmonic functions,

$$\Psi_1(z_0,r,\rho) \ge \frac{1}{2\pi}\int_M^r J^t_{s_{V(z_0)}}(\theta)\frac{dt}{t} + \frac{1}{2\pi}\int_0^\theta\int_0^{\theta'} \ln|s_{V(z_0)}(re^{i\omega})|\,d\omega\,d\theta'.$$

But the right-hand side of this equation is $\int_0^\theta N_{V(z_0)}(r,\pi,\theta'+\pi)\,d\theta' + O(r^{d-1})$, so we see that

$$\liminf_{r\to\infty} r^{-d}\Psi_1(z_0,r,\rho) \ge \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)\,d\omega\,d\theta'\right)a^d.$$

Combining this with (5-5), we find

$$\lim_{r\to\infty} r^{-d}\Psi_1(z_0,r,\rho) = \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)\,d\omega\,d\theta'\right)a^d. \tag{5-6}$$

Using this and the upper bound (5-5) on Ψ_1, since Ψ_1 is plurisubharmonic in z, it follows from [Lelong and Gruman 1986, Proposition 1.39] (recalled here in Proposition 2.3) that for any sequence $\{r_j\}$, $r_j \to \infty$, there is a pluripolar set $E \subset \Omega_\rho$ (which may depend on the sequence) so that

$$\limsup_{j\to\infty} r_j^{-d}\Psi_1(z,r_j,\rho) = \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)\,d\omega\,d\theta'\right)a^d$$

for all $z \in \Omega_\rho \setminus E$. Since $\lim_{r\to\infty} r^{-d}\big(\Psi_1(z,r,\rho) - \Psi(z,r,\rho)\big) = 0$, the same conclusion holds for Ψ in place of Ψ_1.

Suppose there is some $z_1 \in \Omega_{2\rho}$ and some sequence $r_j \to \infty$ so that

$$\lim_{j\to\infty} r_j^{-d}\Psi(z_1, r_j, \rho) < \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega)\,d\omega\,d\theta'\right)a^d.$$

Then, using the uniform continuity of $r^{-d}\Psi(z, r, \rho)$ in z, we find there must be an $\epsilon > 0$ so that

$$\limsup_{j\to\infty} r_j^{-d}\Psi(z, r_j, \rho) < \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega)\,d\omega\,d\theta'\right)a^d$$

for all $z \in B(z_1, \epsilon)$. But since $B(z_1, \epsilon)$ is not contained in a pluripolar set, we have a contradiction. Thus

$$\lim_{r\to\infty} r^{-d}\Psi(z, r, \rho) = \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega)\,d\omega\,d\theta'\right)a^d$$

for all $z \in \Omega_{2\rho}$. □

The following lemma will be used to remove the need to average in θ as in Proposition 5.3.

Lemma 5.4. *Let $M(r, \theta)$ be a function so that for any fixed positive $r_0 > C_0$, $M(r_0, \theta)$ is a nondecreasing function of θ, and suppose*

$$\lim_{r\to\infty} r^{-d}\int_0^\theta M(r, \theta')\,d\theta' = \alpha(\theta)$$

for $\theta_1 < \theta < \theta_2$. Then if α is differentiable at θ, then

$$\lim_{r\to\infty} r^{-d}M(r, \theta) = \alpha'(\theta).$$

Proof. Let $\epsilon > 0$. Then, since $M(r, \theta)$ is nondecreasing in θ,

$$\int_0^{\theta+\epsilon} M(r, \theta')\,d\theta' - \int_0^\theta M(r, \theta')\,d\theta' \ge \epsilon M(r, \theta),$$

which, under rearrangement, yields

$$r^{-d}M(r, \theta) \le r^{-d}\frac{\int_0^{\theta+\epsilon} M(r, \theta')\,d\theta' - \int_0^\theta M(r, \theta')\,d\theta'}{\epsilon}.$$

Thus

$$\limsup_{r\to\infty} r^{-d}M(r, \theta) \le \frac{\alpha(\theta+\epsilon) - \alpha(\theta)}{\epsilon}.$$

Likewise, we find

$$\liminf_{r\to\infty} r^{-d}M(r, \theta) \ge \frac{\alpha(\theta) - \alpha(\theta-\epsilon)}{\epsilon}.$$

Since both these equalities must hold for all $\epsilon > 0$, the lemma follows from the assumption that α is differentiable at θ. □

The following proposition follows from Proposition 5.3, but is stronger as it does not require averaging in the θ' variables.

Proposition 5.5. *Let V, z_0, Ω satisfy the assumptions of Theorem 1.2, and let $\rho > 0$ and Ω_ρ be as in Proposition 5.3. Then for $z \in \Omega_{2\rho}$, $0 < \theta < \pi$, as $r \to \infty$,*

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} N_{V(z')}(r,\pi,\theta+\pi)\,d\mathcal{L}(z') = \frac{1}{2\pi}a^d r^d\left(\frac{1}{d^2}h'_d(\theta)+\int_0^\theta h_d(\omega)\,d\omega\right)+o(r^d).$$

Proof. This follows from applying Lemmas 5.4 and 3.3 to the results of Proposition 5.3. □

Proposition 5.5 does not give results for the counting function for all the resonances (note that we cannot have $\theta = \pi$). The following fills this gap.

Proposition 5.6. *Let V, z_0, Ω satisfy the assumptions of Theorem 1.2, and let $\rho > 0$ and Ω_ρ be as in Proposition 5.3. Then for $z \in \Omega_{2\rho}$, as $r \to \infty$,*

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} N_{V(z')}(r)\,d\mathcal{L}(z') = \frac{1}{2\pi}a^d r^d\int_0^\theta h_d(\omega)\,d\omega+o(r^d).$$

Proof. The proof of this is very similar to that of Proposition 5.3. In fact, the main difference is the use of (2-2), which together with Lemma 3.1 and (3-1) gives us, by handling possible poles in the upper half-plane using a method similar to the proof of Lemma 5.2,

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} N_{V(z')}(r)\,d\mathcal{L}(z') = \Psi_1(z,r,\rho)+O(r^{d-1}),$$

where

$$\Psi_1(z,r,\rho) = \frac{1}{\mathrm{Vol}(B(z,\rho))}\frac{1}{2\pi}\int_{z'\in B(z,\rho)}\int_0^\pi \ln|s_{V(z')}(re^{i\theta})|\,d\theta\,d\mathcal{L}(z').$$

Using that Ψ_1 is plurisubharmonic in z, the proof now follows just as in Proposition 5.3. □

The following proposition is much like Propositions 5.5 and 5.6, but eliminates the average in the r variable.

Proposition 5.7. *Let V, Ω, z_0 satisfy the conditions of Theorem 1.2, and let ρ and Ω_ρ be as in Proposition 5.3. Then for $0 < \theta < \pi$, $z \in \Omega_{2\rho}$,*

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} n_{V(z')}(r,\pi,\theta+\pi)\,d\mathcal{L}(z') = \frac{a^d r^d}{2\pi}\left(\frac{1}{d}h'_d(\theta)+d\int_0^\theta h_d(\theta)\,d\theta\right)+o(r^d)$$

and

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} n_{V(z')}(r)\,d\mathcal{L}(z') = \frac{d}{2\pi}a^d r^d\int_0^\pi h_d(\theta)\,d\theta+o(r^d)$$

as $r \to \infty$.

Proof. This proof follows from Propositions 5.5 and 5.6, using, in addition, a result like that of [Stefanov 2006, Lemma 1] or Lemma 5.4. □

Proof of Theorem 1.3. Let $M = \max(1 + |\psi(z)|)$, and for $\rho > 0$ small enough that $B(z_0, \rho) \subset \Omega$, set Ω_ρ to be the connected component of $\{z \in \Omega : \operatorname{dist}(z, \Omega^c) > \rho\}$ which contains z_0. Given $\epsilon > 0$, choose $\rho > 0$ such that $B(z_0, 2\rho) \subset \Omega$ and so that

$$\operatorname{vol}\big(\operatorname{supp}\psi \cap (\Omega \setminus \Omega_{2\rho})\big) < \frac{\epsilon}{10Me^d(c_d a^d + 1)}. \tag{5-7}$$

Since ψ is continuous with compact support, we can find a $\delta_1 > 0$, $\delta_1 < \rho$ so that if $|z - z'| < \delta_1$, then

$$|\psi(z) - \psi(z')| < \frac{\epsilon}{10e^d(1 + \operatorname{vol}\operatorname{supp}\psi)(a^d c_d + 1)}.$$

We may find a finite number J of disjoint balls $B(z_j, \epsilon_j)$ so that $\epsilon_j < \delta_1$, $z_j \subset \Omega_{2\rho}$, and

$$\operatorname{vol}\big(\operatorname{supp}\psi \setminus \textstyle\bigcup_1^J B(z_j, \epsilon_j)\big) + \operatorname{vol}\big(\textstyle\bigcup_1^J B(z_j, \epsilon_j) \setminus \operatorname{supp}\psi\big) < \dfrac{\epsilon}{4Me^d(a^d c_d + 1)}.$$

Let $\pi \le \varphi' \le \theta' \le 2\pi$. Now

$$\int \psi(z) n_{V(z)}(r, \varphi', \theta')\, d\mathscr{L}(z) = \sum_{j=1}^J \int_{B(z_j,\epsilon_j)} \psi(z) n_{V(z)}(r, \varphi', \theta')\, d\mathscr{L}(z) + \int_{\operatorname{supp}\psi \setminus (\cup B(z_j,\epsilon_j))} \psi(z) n_{V(z)}(r, \varphi', \theta')\, d\mathscr{L}(z).$$

We will use that the bound (1-1) implies that $n_V(z) \le e^d c_d a^d r^d + o(r^d)$. By our choice of $B(z_j, \epsilon_j)$,

$$\left| \int_{\operatorname{supp}\psi \setminus (\cup B(z_j,\epsilon_j))} \psi(z) n_{V(z)}(r, \varphi', \theta')\, d\mathscr{L}(z) \right| \le \frac{\epsilon}{4}(r^d + o(r^d)).$$

By our choice of δ_1 and the assumption that $\epsilon_j < \delta_1$, we have

$$\left| \sum_{j=1}^J \int_{B(z_j,\epsilon_j)} \psi(z) n_{V(z)}(r, \varphi', \theta')\, d\mathscr{L}(z) - \sum_{j=1}^J \int_{B(z_j,\epsilon_j)} \psi(z_j) n_{V(z)}(r, \varphi', \theta')\, d\mathscr{L}(z) \right| \le \frac{\epsilon}{5}(r^d + o(r^d)).$$

By Proposition 5.7, if $0 < \theta < \pi$,

$$\sum_{j=1}^J \int_{B(z_j,\epsilon_j)} \psi(z_j) n_{V(z)}(r, \pi, \pi + \theta) d\mathscr{L}(z) = \left(\sum_{j=1}^J \psi(z_j) \operatorname{vol}(B(z_j, \epsilon_j)) \right) \frac{1}{2\pi} a^d r^d \left(\frac{1}{d} h_d'(\theta) + d \int_0^\theta h_d(\omega) d\omega \right) + o(r^d),$$

and

$$\sum_{j=1}^J \int_{B(z_j,\epsilon_j)} \psi(z_j) n_{V(z)}(r)\, d\mathscr{L}(z) = \left(\sum_{j=1}^J \psi(z_j) \operatorname{vol}(B(z_j, \epsilon_j)) \right) \frac{d}{2\pi} a^d r^d \int_0^\pi h_d(\omega)\, d\omega + o(r^d).$$

Again using our choice of δ_1, z_j, and ϵ_j, we have

$$\left| \sum_{j=1}^J \psi(z_j) \operatorname{vol}(B(z_j, \epsilon_j)) - \int \psi(z)\, d\mathscr{L}(z) \right| < \frac{2\epsilon}{5(c_d a^d + 1)}.$$

Thus we have shown that given $\epsilon > 0$, if $0 < \theta < \pi$,

$$\left| \int \psi(z) n_{V(z)}(r, \pi, \theta+\pi)\, d\mathcal{L}(z) - \frac{a^d r^d}{2\pi} \int \psi(z) d\mathcal{L}(z) \left(\frac{1}{d} h'_d(\theta) + d \int_0^\theta h_d(\omega)\, d\omega \right) \right| \leq \epsilon r^d + o(r^d) \quad (5\text{-}8)$$

and

$$\left| \int \psi(z) n_{V(z)}(r)\, d\mathcal{L}(z) - c_d a^d r^d \int \psi(z)\, d\mathcal{L}(z) \right| \leq \epsilon r^d + o(r^d). \quad (5\text{-}9)$$

Thus we have proved the first and third statements of the theorem. The second statement of the theorem follows from the other two. □

6. Proof of Theorem 1.2

This proof uses some ideas similar to those used in the proofs of Propositions 5.3 and 5.6. In fact, because the proofs are so similar, we shall only give an outline.

Note that by (2-2), (3-1), and Lemma 3.1, using an argument similar to the proofs of Lemma 5.2 and Proposition 5.3,

$$N_{V(z)}(r) = \Psi(z, r) + o(r^{d-1}),$$

where

$$\Psi(z, r) = \frac{1}{2\pi} \int_0^\pi \ln |s_{V(z)}(re^{i\theta})|\, d\theta$$

is, for fixed (large) r a plurisubharmonic function of $z \in \tilde{\Omega} \Subset \Omega$. Since

$$\limsup_{r\to\infty} r^{-d} \Psi(z, r) \leq \frac{a^d}{2\pi} \int_0^\pi h_d(\theta)\, d\theta$$

and this maximum is achieved at $z = z_0 \in \Omega$, we get the first part of the Theorem by applying [Lelong and Gruman 1986, Proposition 1.39], recalled in Proposition 2.3.

To obtain the second part, note that as in the proof of Proposition 5.3, for $0 < \theta < \pi$,

$$\int_0^\theta N_{V(z)}(r, \pi, \theta'+\pi)\, d\theta' = \Psi_2(z, r, \theta) + o(r^d),$$

where

$$\Psi_2(z, r, \theta) = \frac{1}{2\pi} \int_M^r J^t_{s_{V(z)}}(\theta) \frac{dt}{t} + \frac{1}{2\pi} \int_0^\theta \int_0^{\theta'} \ln |s_{V(z)}(re^{i\omega})|\, d\omega\, d\theta'.$$

Since this is a plurisubharmonic function of $z \in \tilde{\Omega}$, $\tilde{\Omega} \Subset \Omega$, if M is chosen so that $M \geq 2\alpha_d \max_{z\in\overline{\tilde{\Omega}}} \|V\|_\infty$, an argument using Proposition 2.3 as in the proof of Proposition 5.3 shows that there exists a pluripolar set $E_\theta \subset \Omega$ so that

$$2\pi \limsup_{r\to\infty} r^{-d} \Psi_2(z, r, \theta) = a^d \left(\frac{1}{d^2} h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega)\, d\omega\, d\theta' \right)$$

for all $z \in \Omega \setminus E_\theta$. Again, we use that this equality holds when $z = z_0$. Then

$$\limsup_{r\to\infty} r^{-d} \int_0^\theta N_{V(z)}(r, \pi, \pi+\theta')\, d\theta' = \frac{a^d}{2\pi}\left(\frac{h_d(\theta)}{d^2} + \int_0^\theta \int_0^{\theta'} h_d(\omega)\, d\omega\, d\theta'\right) \quad \text{for } z \in \Omega \setminus E_\theta. \tag{6-1}$$

For $0 < \theta < \pi$,

$$\frac{h'_d(\theta)}{d^2} + \int_0^\theta h_d(\omega)\, d\omega$$

is a nondecreasing function of θ. This can be seen by using

$$\lim_{r\to\infty} r^{-d} n_{\tilde{V}}(r, \pi, \pi+\theta) = \frac{1}{2\pi d}\left(h'_d(\theta) + d^2 \int_0^\theta h_d(\omega)\, d\omega\right)$$

for $\tilde{V} \in \mathfrak{M}_1$, and clearly the left-hand side is a nondecreasing function of θ. This, along with the fact that $\lim_{\theta\downarrow 0} h_d(\theta) = 0$, implies that

$$\frac{1}{d^2} h_d(\theta) + \int_0^\theta \int_0^{\theta'} h_d(\omega)\, d\omega\, d\theta' \geq \frac{\theta}{d^2} h'_d(0+)$$

for small $\theta > 0$. Therefore, using (6-1), for $z \in \Omega \setminus E_\theta$,

$$\limsup_{r\to\infty} r^{-d} \int_0^\theta N_{V(z)}(r, \pi, \pi+\theta')\, d\theta' \geq \frac{\theta a^d}{2\pi d^2} h'_d(0+),$$

and so we must have

$$\limsup r^{-d} N_{V(z)}(r, \pi, \pi+\theta) \geq \frac{a^d}{2\pi d^2} h'_d(0+)$$

for the same values of z. □

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References

[Bardos et al. 1982] C. Bardos, J.-C. Guillot, and J. Ralston, "La relation de Poisson pour l'équation des ondes dans un ouvert non borné: application à la théorie de la diffusion", *Comm. Partial Differential Equations* **7**:8 (1982), 905–958. MR 84d:35120 Zbl 0496.35067

[Christiansen 2005] T. Christiansen, "Several complex variables and the distribution of resonances in potential scattering", *Comm. Math. Phys.* **259**:3 (2005), 711–728. MR 2006g:81205 Zbl 1088.81093

[Christiansen 2006] T. Christiansen, "Schrödinger operators with complex-valued potentials and no resonances", *Duke Math. J.* **133**:2 (2006), 313–323. MR 2007h:35246 Zbl 1107.35094

[Christiansen 2008] T. Christiansen, "Isophasal, isopolar, and isospectral Schrödinger operators and elementary complex analysis", *Amer. J. Math.* **130**:1 (2008), 49–58. MR 2009g:35033 Zbl 1140.35005

[Froese 1997] R. Froese, "Asymptotic distribution of resonances in one dimension", *J. Differential Equations* **137**:2 (1997), 251–272. MR 98f:81339 Zbl 0955.35057

[Froese 1998] R. Froese, "Upper bounds for the resonance counting function of Schrödinger operators in odd dimensions", *Canad. J. Math.* **50**:3 (1998), 538–546. MR 99f:35150 Zbl 0918.47005

[Govorov 1965] N. V. Govorov, “On the indicator of functions of non-integral order, analytic and of completely regular growth in a half-plane”, *Dokl. Akad. Nauk SSSR* **162** (1965), 495–498. In Russian; translated in *Sov. Math. Dokl.* **6** (1965), 697–701. MR 33 #7545 Zbl 0156.29604

[Govorov 1967] N. V. Govorov, “On the indicator of functions of integral order, analytic and of completely regular growth in a half-plane”, *Dokl. Akad. Nauk SSSR* **172** (1967), 763–766. In Russian; translated in *Sov. Math. Dokl.* **8** (1967), 159–163. MR 35 #372 Zbl 0167.35701

[Guillopé and Zworski 1997] L. Guillopé and M. Zworski, “Scattering asymptotics for Riemann surfaces”, *Ann. of Math.* (2) **145**:3 (1997), 597–660. MR 98g:58181 Zbl 0898.58054

[Lelong and Gruman 1986] P. Lelong and L. Gruman, *Entire functions of several complex variables*, Grundlehren der Mathematischen Wissenschaften **282**, Springer, Berlin, 1986. MR 87j:32001 Zbl 0583.32001

[Levin 1964] B. J. Levin, *Distribution of zeros of entire functions*, American Mathematical Society, Providence, RI, 1964. MR 28 #217 Zbl 0152.06703

[Melrose 1982] R. Melrose, “Scattering theory and the trace of the wave group”, *J. Funct. Anal.* **45**:1 (1982), 29–40. MR 83j:35128 Zbl 0525.47007

[Olver 1954] F. W. J. Olver, “The asymptotic solution of linear differential equations of the second order for large values of a parameter”, *Philos. Trans. Roy. Soc. London. Ser. A.* **247** (1954), 307–327. MR 16,695c Zbl 0070.30801

[Ransford 1995] T. Ransford, *Potential theory in the complex plane*, London Math. Soc. Student Texts **28**, Cambridge University Press, 1995. MR 96e:31001 Zbl 0828.31001

[Regge 1958] T. Regge, “Analytic properties of the scattering matrix”, *Nuovo Cimento* (10) **8** (1958), 671–679. MR 20 #2203 Zbl 0080.41903

[Sá Barreto 2001] A. Sá Barreto, “Remarks on the distribution of resonances in odd dimensional Euclidean scattering”, *Asymptot. Anal.* **27**:2 (2001), 161–170. MR 2002g:35167 Zbl 1116.35344

[Simon 2000] B. Simon, “Resonances in one dimension and Fredholm determinants”, *J. Funct. Anal.* **178**:2 (2000), 396–420. MR 2001j:34031 Zbl 0977.34075

[Stefanov 2006] P. Stefanov, “Sharp upper bounds on the number of the scattering poles”, *J. Funct. Anal.* **231**:1 (2006), 111–142. MR 2006i:35267 Zbl 1099.35074

[Zworski 1987] M. Zworski, “Distribution of poles for scattering on the real line”, *J. Funct. Anal.* **73**:2 (1987), 277–296. MR 88h:81223 Zbl 0662.34033

[Zworski 1989a] M. Zworski, “Sharp polynomial bounds on the number of scattering poles”, *Duke Math. J.* **59**:2 (1989), 311–323. MR 90h:35190 Zbl 0705.35099

[Zworski 1989b] M. Zworski, “Sharp polynomial bounds on the number of scattering poles of radial potentials”, *J. Funct. Anal.* **82**:2 (1989), 370–403. MR 90d:35233 Zbl 0681.47002

[Zworski 1997] M. Zworski, “Poisson formulae for resonances”, exposé 13 in *Séminaire sur les Équations aux Dérivées Partielles, 1996–1997*, edited by J.-M. Bony et al., École Polytech., Palaiseau, 1997. MR 98j:35036 Zbl 02124115

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