ANALYSIS & PDE

Volume 5 No. 5 2012

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ANALYSIS AND PDE
Vol. 5, No. 5, 2012
dx.doi.org/10.2140/apde.2012.5.1105

C^∞ SPECTRAL RIGIDITY OF THE ELLIPSE

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We prove that ellipses are infinitesimally spectrally rigid among C^∞ domains with the symmetries of the ellipse.

An isospectral deformation of a plane domain Ω_0 is a one-parameter family Ω_ϵ of plane domains for which the spectrum of the Euclidean Dirichlet (or Neumann) Laplacian Δ_ϵ is constant (including multiplicities). We say that Ω_ϵ is a C^1 curve of C^∞ plane domains if there exists a C^1 curve of diffeomorphisms φ_ϵ of a neighborhood of $\Omega_0 \subset \mathbb{R}^2$ with $\varphi_0 = \mathrm{id}$ and with $\Omega_\epsilon = \varphi_\epsilon(\Omega_0)$. The infinitesimal generator $X = d\varphi_\epsilon/d\epsilon$ is a vector field in a neighborhood of Ω_0 which restricts to a vector field along $\partial\Omega_0$; we denote by $X_\nu = \dot\rho\nu$ its outer normal component. With no essential loss of generality we may assume that $\varphi_\epsilon|_{\partial\Omega_0}$ is a map of the form

$$x \in \partial\Omega_0 \to x + \rho_\epsilon(x)\nu_x, \tag{1}$$

where $\rho_\epsilon \in C^1([0, \epsilon_0], C^\infty(\partial\Omega_0))$, $\epsilon_0 > 0$ and $\rho_0 = 0$. We put

$$\dot\rho(x) = \delta\rho\,(x) := \frac{d}{d\epsilon}\Big|_{\epsilon=0} \rho_\epsilon(x).$$

An isospectral deformation is said to be trivial if $\Omega_\epsilon \simeq \Omega_0$ (up to isometry) for sufficiently small ϵ. A domain Ω_0 is said to be spectrally rigid if all isospectral deformations Ω_ϵ are trivial. The domain Ω_0 is called infinitesimally spectrally rigid if $\dot\rho = 0$ (up to rigid motions) for all isospectral deformations.

In this article, we use the Hadamard variational formula of the wave trace (apparently for the first time) to study spectral rigidity problems (Theorem 2). Our main application is the infinitesimal spectral rigidity of ellipses among C^1 curves of C^∞ plane domains with the symmetries of an ellipse. We orient the domains so that the symmetry axes are the x-y axes. The symmetry assumption is then that each φ_ϵ is invariant under $(x, y) \to (\pm x, \pm y)$.

Theorem 1. *Suppose that Ω_0 is an ellipse and that Ω_ϵ is a C^1 Dirichlet (or Neumann) isospectral deformation of Ω_0 through C^∞ domains with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. Then $X_\nu = 0$ or equivalently $\dot\rho = 0$.*

As discussed in Sections 0.2 and 3.2, Theorem 1 implies that ellipses admit no isospectral deformations for which the Taylor expansion of ρ_ϵ at $\epsilon = 0$ is nontrivial. A function such as e^{-1/ϵ^2} for which the Taylor series at $\epsilon = 0$ vanishes is called *flat* at $\epsilon = 0$.

The first author is partially supported by NSF grant DMS-0969745 and the second author is partially supported by NSF grant DMS-0904252.
MSC2010: 35PXX.
Keywords: inverse spectral problems, spectral rigidity, isospectral deformations, ellipses.

Corollary 1. *Suppose that Ω_0 is an ellipse and that $\epsilon \to \Omega_\epsilon$ is a C^∞ Dirichlet (or Neumann) isospectral deformation through $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric C^∞ domains. Then ρ_ϵ must be flat at $\epsilon = 0$. In particular, there exist no nontrivial real analytic curves $\epsilon \to \Omega_\epsilon$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric C^∞ domains with the spectrum of an ellipse.*

Spectral rigidity of the ellipse has been expected for a long time and is a kind of model problem in inverse spectral theory. Ellipses are special since their billiard flows and maps are completely integrable. It was conjectured by G. D. Birkhoff that the ellipse is the only convex smooth plane domain with a completely integrable billiard. We cannot assume that the deformed domains Ω_ϵ have this property, although the results of [Siburg 2000; Zelditch 1998] come close to showing that they do. The results are somewhat analogous to the spectral rigidity of flat tori or the sphere in the Riemannian setting.

The main novel step in the proof is the Hadamard variational formula for the wave trace (Theorem 2), which holds for all smooth Euclidean domains $\Omega \subset \mathbb{R}^n$ satisfying standard "cleanliness" assumptions. It is of independent interest and may have applications to spectral rigidity beyond the setting of ellipses. We therefore present the proof in detail. (See also [Golse and Lochak 2003], where a variational formula for the Selberg's trace formula on compact Riemann surfaces is derived.)

The main advance over prior results is that the domains Ω_ϵ are allowed to be C^∞ rather than real analytic. Much less than C^∞ could be assumed for the domains Ω_ϵ, but we do not belabor the point. For real analytic domains a length spectral rigidity result for analytic domains with the symmetries of the ellipse was proved in [Colin de Verdière 1984]. The method does not apply directly to Δ-isospectral deformations of ellipses since the length spectrum of the ellipse may have multiplicities and the full length spectrum might not be a Δ-isospectral invariant. If it were, then Siburg's results would imply that the marked length spectrum is preserved [Siburg 1999; 2000; 2004]. In [Zelditch 2009; 2000] it is shown that analytic domains with one symmetry are spectrally determined if the length of the minimal bouncing ball orbit and one iterate is a Δ-isospectral invariant. The prior results on Δ-isospectral deformations that we are aware of are contained in the articles [Guillemin and Melrose 1979a; Popov and Topalov 2003; 2012] and concern deformations of boundary conditions. To our knowledge, the only prior results on Δ-isospectral deformations of the domain are contained in [Marvizi and Melrose 1982]. Marvizi and Melrose [1982] introduce new spectral invariants and prove certain rigidity results, but they do not apparently settle the case of the ellipse (see also [Amiran 1993; 1996] for further attempts to apply them to the ellipse). It would be desirable to remove the symmetry assumption (to the extent possible), but symmetry seems quite necessary for our argument. Further discussion of prior results can be found in the earlier arXiv posting of this article [Hezari and Zelditch 2010].

0.1. *Theorem on variation of the wave trace.*

We now state a general result on the variation of the wave trace on a domain with boundary under variations of the boundary.

To state the result, we need some notation. We denote by

$$E_B(t) = \cos\big(t\sqrt{-\Delta_B}\,\big) \quad \text{and} \quad S_B(t) = \frac{\sin\big(t\sqrt{-\Delta_B}\,\big)}{\sqrt{-\Delta_B}} \tag{2}$$

the even and odd wave operators of a domain Ω with boundary conditions B. We recall that $E_B(t)$ has a distribution trace as a tempered distribution on $\mathbb{R}$. That is, $E_B(\varphi) = \int_{\mathbb{R}} \varphi(t) E_B(t)\, dt$ is of trace class for any $\varphi \in C_0^\infty(\mathbb{R})$; we refer to [Guillemin and Melrose 1979b; Petkov and Stoyanov 1992] for background.

The Poisson relation of a manifold with boundary gives a precise description of the singularities of this distribution trace in terms of periodic transversal reflecting rays of the billiard flow, or equivalently periodic points of the billiard map. For the definitions of "billiard map", "clean", "transversal reflecting rays", etc., we refer to [Guillemin and Melrose 1979a; 1979b; Petkov and Stoyanov 1992]. A periodic point of the billiard map $\beta : B^*\partial\Omega \to B^*\partial\Omega$ on the unit ball bundle $B^*\partial\Omega = \{(q, \zeta) \in T^*\partial\Omega;\ |\zeta| < 1\}$ of the boundary corresponds to a billiard trajectory, i.e an orbit of the billiard flow Φ^t on $S^*\Omega$. We define the *length* of the periodic orbit of β to be the length of the corresponding billiard trajectory in $S^*\Omega$. Note that the *period* of a periodic point of β is ambiguous since it could refer to this length or to the power of β. We also denote by $\mathrm{Lsp}(\Omega)$ the length spectrum of Ω, that is, the set of lengths of closed billiard trajectories. The perimeter of Ω is denoted by $|\partial\Omega|$.

In the following deformation theorem, the boundary conditions are fixed during the deformation and we therefore do not include them in the notation. We also do not include ϵ in our notation for Δ even though all Laplacians below are associated with Ω_ϵ and hence dependent on ϵ.

Theorem 2. *Let $\Omega_0 \subset \mathbb{R}^n$ be a C^∞ convex Euclidean domain with the property that the fixed point sets of the billiard map are clean. Then, for any C^1 variation of Ω_0 through C^∞ domains Ω_ϵ, the variation of the wave traces $\delta \operatorname{Tr} \cos(t\sqrt{-\Delta}\,)$, with Dirichlet (or Neumann) boundary conditions is a classical conormal distribution for $t \neq m|\partial\Omega_0|$ $(m \in \mathbb{Z})$ with singularities contained in* $\mathrm{Lsp}(\Omega_0)$. *For each $T \in \mathrm{Lsp}(\Omega_0)$ for which the set F_T of periodic points of the billiard map β of length T is a d-dimensional clean fixed point set consisting of transverse reflecting rays, there exist nonzero constants C_Γ independent of $\dot\rho$ such that, near T, the leading order singularity is*

$$\delta \operatorname{Tr} \cos(t\sqrt{-\Delta}\,) \sim \frac{t}{2}\, \Re\biggl\{\biggl(\sum_{\Gamma \subset F_T} C_\Gamma \int_\Gamma \dot\rho\, \gamma_1\, d\mu_\Gamma\biggr)\, (t - T + i0^+)^{-2-(d/2)}\biggr\},$$

modulo lower order singularities. The sum is over the connected components Γ of F_T. Here $\delta = \dfrac{d}{d\epsilon}\Big|_{\epsilon=0}$ and $\gamma_1(q, \zeta) = \sqrt{1 - |\zeta|^2}$.

The function γ_1 on $B^*\partial\Omega$ is defined in (27) and appeared earlier in [Hassell and Zelditch 2004]. The densities $d\mu_\Gamma$ on the fixed point sets of β and its powers are very similar to the canonical densities defined in Lemma 4.2 of [Duistermaat and Guillemin 1975], and further discussed in [Guillemin and Melrose 1979a; Popov and Topalov 2003; 2012]. The constants C_Γ are explicit and depend on the boundary conditions. We suppress the exact formulae since we do not need them, but their definition is reviewed in the course of the proof.

To clarify the dimensional issues, we note that there are four closely related definitions of the set of closed billiard trajectories (or closed orbits of the billiard map). The first is the fixed point set of the billiard flow Φ^T at time T in $T^*\Omega$. The second is the set of unit vectors in the fixed point set. The third is the fixed point set of the billiard flow restricted to $T^*_{\partial\Omega}\Omega$, the set of covectors with foot points at the boundary. The fourth is the set of periodic points of the billiard map β on $B^*\partial\Omega$ of length T, where as above the length is defined by the length of the corresponding billiard trajectory. The dimension d refers to the dimension of the latter which we show by F_T. In the case of the ellipse, for instance, $d = 1$; the periodic points of a given length form invariant curves for β.

To prove Theorem 2, we use the Hadamard variational formula for the Green's kernel to give an exact formula for the wave trace variation (Lemma 1). We then prove that it is a classical conormal distribution and calculate its principal symbol.

It is verified in [Guillemin and Melrose 1979a] that the ellipse satisfies the cleanliness assumptions.

Corollary 2. *For any C^1 variation of an ellipse through C^∞ domains Ω_ϵ, the leading order singularity of the wave trace variation is*

$$\delta \operatorname{Tr} \cos(t\sqrt{-\Delta}) \sim \frac{t}{2} \Re\biggl\{\biggl(\sum_{\Gamma \subset F_T} C_\Gamma \int_\Gamma \dot{\rho}\, \gamma_1 \, d\mu_\Gamma\biggr) (t - T + i0^+)^{-5/2}\biggr\},$$

modulo lower-order singularities, where the sum is over the connected components Γ of the set F_T of periodic points of β (and its powers) of length T.

0.2. ***Flatness issues.*** We now discuss an apparently new flatness issue in isospectral deformations. The rather technical assumption that Ω_ϵ is a C^1 family of C^∞ domains rather than a C^∞ family in the ϵ variable is made to deal with a somewhat neglected and obscure point about isospectral deformations. Isospectral deformations are curves in the "manifold" of domains. The curve might be a nontrivial C^∞ family in ϵ but the first derivative $\dot{\rho}$ might vanish at $\epsilon = 0$. Thus, infinitesimal spectral rigidity is at least apparently weaker than spectral rigidity. We impose the C^1 regularity to allow us to reparametrize the family and show that the first derivative of any C^1 reparametrization must be zero. This is not the primary focus of Theorem 1, but with no additional effort the proof extends to the C^1 case.

This flatness issue does not seem to have arisen before in inverse spectral theory, even when the main conclusions are derived from infinitesimal rigidity. The main reason is that first-order perturbation theory very often requires analytic perturbations (i.e., analyticity in the deformation parameter ϵ), and so most (if not all) prior results on isospectral deformations assume that the deformation is real analytic. But our proof is based on Hadamard's variational formula, which is valid for C^1 perturbations of domains and so we can study this more general situation. Further, the prior spectral rigidity results [Guillemin and Kazhdan 1980] are proved for an open set of domains and metrics and therefore flatness at all points implies triviality of the deformations. We are only deforming the one-parameter family of ellipses and therefore cannot eliminate flat isospectral deformations by that kind of argument. We also note that there could exist continuous but nondifferentiable isospectral deformations.

0.3. ***Pitfalls and complications.*** The route taken in the proof of Theorem 1, and the flatness issues just discussed, reflect certain technical issues that arise in the inverse problem.

First is the issue of multiplicities in the eigenvalue spectrum or in the length spectrum. The multiplicities of the Δ-eigenvalues of the ellipse (for either Dirichlet or Neumann boundary conditions) appear to be almost completely unknown. If a sufficiently large portion of the eigenvalue spectrum were simple (i.e., of multiplicity one), one could simplify the proof of Theorem 1 by working directly with the eigenfunctions and their semiclassical limits (as in the first arXiv posting of this article, [Hezari and Zelditch 2010]).

The dual multiplicity of the length spectrum is also largely unknown for the ellipse. Without length spectral simplicity one cannot work with the wave trace invariants. Our proof relies on the observation in [Guillemin and Melrose 1979a] that the multiplicities have to be one (modulo the symmetry) for periodic orbits that creep close enough to the boundary.

Second is the issue of cleanliness. Theorem 2 and Corollary 2 would apply to any of the deformed domains Ω_ϵ if the fixed points sets were known to be clean. One could then use the conclusion of Corollary 2 to rule out flat isospectral deformations. However, we do not know that the fixed point sets are clean for the deformed domains even though we do know that they have the same wave trace singularities as the ellipse. Equality of the wave traces for isospectral deformations of ellipses shows that the periodic points of β of Ω_ϵ can never be nondegenerate. Hence the deformations are very nongeneric. It is plausible that equality of wave traces forces the sets of periodic points to be clean invariant curves of dimension one. But we do not know how to prove this kind of inverse result at this time.

1. Hadamard variational formula for wave traces

In this section we consider the Dirichlet and Neumann eigenvalue problems for a C^1 one-parameter family of smooth Euclidean domains $\Omega_\epsilon \subset \mathbb{R}^n$,

$$\begin{cases} -\Delta_{B_\epsilon}\Psi_j(\epsilon) = \lambda_j^2(\epsilon)\Psi_j(\epsilon) \ \text{ in } \ \Omega_\epsilon, \\ B_\epsilon \Psi_j(\epsilon) = 0, \end{cases} \tag{3}$$

where the boundary condition B_ϵ could be $B_\epsilon\Psi_j(\epsilon) = \Psi_j(\epsilon)|_{\partial\Omega_\epsilon}$ (Dirichlet) or $\partial_{\nu_\epsilon}\Psi_j(\epsilon)|_{\partial\Omega_\epsilon}$ (Neumann). Here, $\lambda_j^2(\epsilon)$ are the eigenvalues of $-\Delta_{B_\epsilon}$, enumerated in order and with multiplicity, and ∂_{ν_ϵ} is the interior unit normal to Ω_ϵ. We do not assume that $\lambda_j^2(\epsilon)$ and $\Psi_j(\epsilon)$ are C^1 in ϵ.

We will use Hadamard's variational formula for the variation of Green's kernels, and adapt the formula to give the variation of the (regularized) trace of the wave kernel. Our references are [Garabedian 1964; Peetre 1980; Fujiwara et al. 1978; Ozawa 1982; Fujiwara and Ozawa 1978].

To state our main variational Lemma 1 we introduce some notation. We denote by dq the surface measure on the boundary $\partial\Omega$ of a domain Ω, and by $ru = u|_{\partial\Omega}$ the trace operator. We use $S_B^b(t, q', q) \in \mathcal{D}'(\mathbb{R}\times\partial\Omega\times\partial\Omega)$ for the following boundary traces of the Schwartz kernel $S_B(t, x, y) \in \mathcal{D}'(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n)$ of $S_B(t)$ defined in (2):

$$S_B^b(t, q', q) = \begin{cases} r_{q'} r_q \partial_{\nu_{q'}} \partial_{\nu_q} S_D(t, q', q) & \text{(Dirichlet)}, \\ \nabla_{q'}^T \nabla_q^T r_{q'} r_q S_N(t, q', q) + r_{q'} r_q \ \Delta_{q'} \ S_N(t, q', q) & \text{(Neumann)}. \end{cases} \tag{4}$$

Here, the subscripts q', q refer to the variable involved in the differentiating or restricting. According to convenience, we may also indicate this by subscripting with indices 1, 2, referring to the first and second variables in the kernel. For instance,

$$\frac{\partial}{\partial_{\nu_{q'}}} K(q', q) = \frac{\partial}{\partial_{\nu_1}} K(q', q).$$

We may also use the notations ∂_ν and $\partial/\partial\nu$ interchangeably to refer to the inward normal derivative. Here, ∇^T corresponds to tangential differentiation which is the gradient associated to the hypersurface $\partial\Omega$.

Lemma 1. *The variation of the wave trace with boundary conditions B is given by*

$$\delta \operatorname{Tr} E_B(t) = \frac{t}{2} \int_{\partial\Omega_0} S^b_{B_0}(t, q, q)\dot{\rho}(q)\, dq.$$

We summarize by writing

$$\delta \operatorname{Tr} E_B(t) = \frac{t}{2} \operatorname{Tr}_{\partial\Omega_0} \dot{\rho}\; S^b_B.$$

Here, $\delta = \frac{d}{d\epsilon}|_{\epsilon=0}$ and the equality is understood in the sense of distributions; meaning if $\varphi \in C_0^\infty(\mathbb{R})$ then

$$\delta \operatorname{Tr}\Big(\int \varphi(t) E_B(t)\, dt\Big) = \int_{\partial\Omega_0} \Big(\int \frac{t}{2}\varphi(t) S^b_{B_0}(t, q, q)\, dt\Big)\dot{\rho}(q)\, dq.$$

We note that the right hand side is well defined because the kernel of the operator $\int \varphi(t) S_{B_0}(t)\, dt$ is smooth up to the boundary.

We prove the lemma by relating the variation of the wave trace to the known variational formula for the Green's function (resolvent kernel). We now review the latter.

1.1. ***Hadamard variational formula for Green's function.*** Here by the Green's function $G_{B_\epsilon}(\lambda, x, y)$ of Ω_ϵ, with the boundary condition B_ϵ, we mean the integral kernel of the resolvent $R_B(\lambda) = (-\Delta_{B_\epsilon} - \lambda^2)^{-1}$ where $\Im\lambda > 0$. We also define $R_B(\lambda)$ for $\lambda \in \mathbb{R}$ by $R_B(\lambda + i0^+)$ (that the limit exists follows, for example, from Theorem 3.1.11 of [Hörmander 1983]). The variational formula below is valid for both of these resolvents (also for $\Im\lambda < 0$). Since the domains of $G_{B_\epsilon}(\lambda, x, y)$ depend on ϵ we first have to make our definition of δ precise.

Definition. Let $u_\epsilon \in C^1([0, \epsilon_0], \mathcal{D}'(\Omega_\epsilon))$ with $\epsilon_0 > 0$, be a C^1 family of distributions in Ω_ϵ. We use δu_ϵ or $\dot{u}$ to represent the first variation of u_ϵ at $\epsilon = 0$ as a distribution in Ω_0:

$$\delta u_\epsilon = \frac{d}{d\epsilon}\Big|_{\epsilon=0} u_\epsilon.$$

We note that if $\alpha \in C_0^\infty(\Omega_0)$ then for ϵ small $\operatorname{supp}(\alpha) \subset \Omega_\epsilon$, and therefore we can define δu_ϵ by

$$(\delta u_\epsilon)(\alpha) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} (u_\epsilon(\alpha)).$$

However, the problem with this definition is that it defines $\dot{u}$ only in the interior of Ω_0 and not at the boundary even if u_ϵ is defined there. Below we will see another definition of $\dot{u}$, using diffeomorphisms, which resolves this issue.

In the statement of the formulas we will not include ϵ in our notation. In the Dirichlet case, the classical Hadamard variational formula states that, under a C^1 deformation Ω_ϵ,

$$\delta G_D(\lambda, x, y) = \int_{\partial\Omega_0} \frac{\partial}{\partial\nu_2} G_D(\lambda, x, q) \frac{\partial}{\partial\nu_1} G_D(\lambda, q, y)\dot{\rho}(q)\, dq. \tag{5}$$

In the Neumann case,

$$\delta G_N(\lambda, x, y) = \int_{\partial\Omega_0} \nabla_2^T G_N(\lambda, x, q) \cdot \nabla_1^T G_N(\lambda, q, y)\dot{\rho}(q)\, dq - \lambda^2 \int_{\partial\Omega_0} G_N(\lambda, x, q) G_N(\lambda, q, y)\dot{\rho}(q)\, dq. \tag{6}$$

We briefly review the proof of the Hadamard variational formula to clarify the definition of $\delta G_B(\lambda, x, y)$ and of the other kernels. We give the proof for the variation of the resolvent $R_B(\lambda)$ with $\Im\lambda > 0$. From this we can obtain the analogous formula for $\delta R_B(\lambda + i0^+)$ by taking $\Im\lambda \to 0^+$. Following [Peetre 1980], we write the inhomogeneous problem

$$\begin{cases}(-\Delta - \lambda^2)u = f & \text{in } \Omega\,(\lambda \in \mathbb{C},\ \Im\lambda > 0),\\ u = 0\ \ (\text{resp. } \partial_\nu u = 0) & \text{on } \partial\Omega,\end{cases}$$

in terms of the energy integral

$$E(u, v) = \int_\Omega \nabla u \cdot \nabla v\, dx - \lambda^2 \int_\Omega uv\, dx = \int_\Omega v(-\Delta - \lambda^2)u\, dx - \int_{\partial\Omega} v\partial_\nu u\, dq,$$

where ∂_ν is the inward unit normal. The inhomogeneous problem is to solve

$$E(u, v) = \int_\Omega f v\, dx,$$

where v is a smooth test function which vanishes to order 1 (resp. 0) on $\partial\Omega$ for the Dirichlet (resp. Neumann) problem. We denote the energy density by $e(u, v) = \nabla u \cdot \nabla v - \lambda^2 uv$.

We now vary the problems over a one-parameter family of domains. We use one-parameter families of smooth diffeomorphisms φ_ϵ of a neighborhood of $\Omega_0 \subset \mathbb{R}^n$ to define the one-parameter families $\Omega_\epsilon = \varphi_\epsilon(\Omega_0)$ of domains. We assume φ_ϵ to be a C^1 curve of diffeomorphisms with $\varphi_0 = \mathrm{id}$.

The variational derivative of the solution is defined as follows: Let u_ϵ be a C^1 curve of functions in $H^s(\Omega_\epsilon)$. Then $\varphi_\epsilon^* u_\epsilon \in H^s(\Omega_0)$ and $d(\varphi_\epsilon^* u_\epsilon)/d\epsilon$ is a continuous curve in $H^s(\Omega_0)$. Put

$$X = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\varphi_\epsilon \quad \text{and} \quad \theta_X u = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\varphi_\epsilon^* u_\epsilon.$$

Assume that $u_0 \in H^{s+1}(\Omega_0)$. Then $\dot{u}$, defined by

$$\dot{u} = \theta_X u - X u_0,$$

exists in $H^s(\Omega_0)$. This gives a new definition of $\dot{u}$ which has a well-defined restriction to $\partial\Omega_0$ (for $s \geq 1$), and it agrees with $\dot{u}$ defined above in the interior of Ω_0. Further, let v be a test function on Ω_0 and use $\varphi_\epsilon^{-1*} v$ as a test function on Ω_ϵ. Now rewrite the boundary problems as

$$\int_{\Omega_\epsilon} e(u_\epsilon, (\varphi_\epsilon^{-1})^* v)\, dx = \int_{\Omega_\epsilon} f_\epsilon((\varphi_\epsilon^{-1})^* v)\, dx.$$

Changing variables, one pulls back the equation to Ω_0 as

$$\int_{\Omega_0} e_\epsilon(\varphi_\epsilon^* u_\epsilon, v)\varphi_\epsilon^* dx = \int_{\Omega_0} (\varphi_\epsilon^* f_\epsilon) v \varphi_\epsilon^* dx,$$

where

$$e_\epsilon(w, v) := \varphi_\epsilon^*(e(\varphi_\epsilon^{-1*} w, \varphi_\epsilon^{-1*} v)).$$

Then, by the computations of [Peetre 1980, (8) and (10)] we have

$$\int_{\Omega_0} \dot{u}(-\Delta - \lambda^2)v\,dx$$
$$= \int_{\Omega_0} \dot{f} v\,dx + \int_{\partial\Omega_0} f v \dot{\rho}\,dq + \int_{\partial\Omega_0} (\nabla u_0 \cdot \nabla v - \lambda^2 u_0 v)\dot{\rho}\,dq + \begin{cases} \lambda^2 \int_{\partial\Omega_0} u_0 v \dot{\rho} dq & \text{(Dirichlet)}, \\ 0 & \text{(Neumann)}. \end{cases} \tag{7}$$

To obtain (5)–(6), at least formally, one puts

$$u_\epsilon(x) = G_{B_\epsilon}(\lambda, z, x), \quad v(x) = G_{B_0}(\lambda, y, x), \quad f_\epsilon(x) = \delta_z(x)$$

where $z \in \mathring{\Omega}$. Thus $\dot{u}(x) = \delta G_B(\lambda, z, x)$ and $\dot{f} = 0$. Since $z \in \mathring{\Omega}$ we have $z \in \Omega_\epsilon$ for sufficiently small ϵ and one easily verifies that (7) implies (5)–(6). The Green's kernel depends on ϵ as smoothly as the coefficients of operator $\tilde{\Delta}_\epsilon$ on Ω_0 defined by the pulled back energy form.

1.2. ***Proof of Lemma 1.*** Rather than the Green's function, we are interested in the Hadamard variational formula for the wave kernels $E_B(t)$, $S_B(t)$ in (2), or more precisely, for their distribution traces. We will give two proofs for the lemma.

First proof. By the definition of the distribution trace, we only need the variational formula for traces of variations $\delta \int_{\mathbb{R}} e^{i\lambda t} \hat{\psi}(t) E_B(t)\,dt$ of integrals of these kernels against test functions $\hat{\psi}(t)e^{i\lambda t} \in C_0^\infty(\mathbb{R})$, which are simpler because the Schwartz kernels are smooth.

We derive the Hadamard variational formula for wave traces from that of the Green's function by using the identities

$$-i\lambda R_B(\lambda) = \int_0^\infty e^{i\lambda t} E_B(t)\,dt, \quad \frac{d}{dt} S_B(t) = E_B(t). \tag{8}$$

Using integration by parts (and $\Im\lambda > 0$), we get

$$R_B(\lambda) = \int_0^\infty e^{i\lambda t}\, S_B(t)\,dt. \tag{9}$$

We will assume that $\hat{\psi}$ is supported in $\mathbb{R}_+$ since in the wave trace we localize its support to the length of a closed geodesic. Hence by (8),

$$\begin{aligned} \int_{\mathbb{R}} \hat{\psi}(t) e^{i\lambda t}\, E_B(t)\,dt &= \int_{\mathbb{R}} \psi(\mu) \int_0^\infty e^{i(\lambda-\mu)t} E_B(t)\,dt\,d\mu \\ &= -i \int_{\mathbb{R}} \psi(\mu)(\lambda - \mu) R_B(\lambda - \mu)\,d\mu. \end{aligned} \tag{10}$$

This implies that

$$\delta \int_{\mathbb{R}} \hat{\psi}(t) e^{i\lambda t}\, E_B(t)\,dt = -i \int_{\mathbb{R}} \psi(\mu)(\lambda - \mu)\delta R_B(\lambda - \mu)\,d\mu.$$

That we can pass δ under the integral sign can be justified using the dominated convergence theorem and we leave the proof to the reader. In the Dirichlet case, it follows from (10), (5), (8) and (9) that

$$
\begin{aligned}
&\delta \int_{\mathbb{R}} \hat{\psi}(t) e^{i\lambda t}\, E_D(t,x,y)\,dt \\
&\qquad = -i \int_{\mathbb{R}} (\lambda-\mu)\psi(\mu) \int_{\partial\Omega_0} \partial_{\nu_2} G_D(\lambda-\mu,x,q)\partial_{\nu_1} G_D(\lambda-\mu,q,y)\dot{\rho}(q)\,dq\,d\mu \\
&\qquad = \int_{\mathbb{R}}\int_0^\infty e^{i(\lambda-\mu)t}\psi(\mu)\int_{\partial\Omega_0} \partial_{\nu_2} E_D(t,x,q)\partial_{\nu_1} G_D(\lambda-\mu,q,y)\dot{\rho}(q)\,dq\,d\mu\,dt \\
&\qquad = \int_{\mathbb{R}}\int_0^\infty\int_0^\infty e^{i(\lambda-\mu)(t+t')}\psi(\mu)\int_{\partial\Omega_0} \partial_{\nu_2} E_D(t,x,q)\partial_{\nu_1} S_D(t',q,y)\dot{\rho}(q)\,dq\,d\mu\,dt\,dt' \\
&\qquad = \int_0^\infty\int_0^\infty e^{i\lambda(t+t')}\hat{\psi}(t+t')\int_{\partial\Omega_0} \partial_{\nu_2} E_D(t,x,q)\partial_{\nu_1} S_D(t',q,y)\dot{\rho}(q)\,dq\,dt\,dt' \\
&\qquad = \int_0^\infty\int_{\partial\Omega_0} e^{i\lambda\tau}\hat{\psi}(\tau)\biggl(\int_0^\tau \partial_{\nu_2} E_D(\tau-t',x,q)\partial_{\nu_1} S_D(t',q,y)dt'\biggr)\dot{\rho}(q)\,dq\,d\tau.
\end{aligned}
$$

The inner integral is the same if we change the argument of E_D to t' and that of S_D to $\tau - t'$. We then average the two, set $x = y$, integrate over Ω_0 and use the angle addition formula for sin to obtain

$$
\delta\,\mathrm{Tr}\int_{\mathbb{R}} \hat{\psi}(t)e^{i\lambda t}E_D(t)\,dt = \frac{1}{2}\int_{\partial\Omega_0}\int_{\mathbb{R}} t\hat{\psi}(t)e^{i\lambda t}\partial_{\nu_1}\partial_{\nu_2}S_D(t,q,q)\dot{\rho}(q)\,dt\,dq. \tag{11}
$$

The proof in the Neumann case is similar and left to the reader. We notice that in the above argument we have commuted the operations δ and Tr:

$$
\delta\,\mathrm{Tr}\int_{\mathbb{R}} \hat{\psi}(t)e^{i\lambda t}E_D(t)\,dt = \mathrm{Tr}\,\delta\int_{\mathbb{R}} \hat{\psi}(t)e^{i\lambda t}E_D(t)\,dt. \tag{12}
$$

To show this we first put $K_\epsilon(x,y) = \int_{\mathbb{R}} \hat{\psi}(t)e^{i\lambda t}E_D(t,x,y)\,dt$. We then note that $K_\epsilon(x,y)$ is a C^1 curve in $C^\infty(\overline{\Omega}_\epsilon \times \overline{\Omega}_\epsilon)$, in the sense that $(d/d\epsilon)\varphi_\epsilon^* K_\epsilon(x,y)$ is a continuous curve in $C^\infty(\overline{\Omega}_0 \times \overline{\Omega}_0)$. Therefore both traces in (12) are the integrals of their corresponding kernels on the diagonal and hence (12) is equivalent to

$$
\frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_{\Omega_\epsilon} K_\epsilon(x,x)\,dx = \int_{\Omega_0}\frac{d}{d\epsilon}\Big|_{\epsilon=0} K_\epsilon(x,x)\,dx.
$$

However we have to be careful since the domain of integration on the left hand side depends on ϵ and under the variation it contributes an integral along the boundary. More precisely, since $(d/d\epsilon)\varphi_\epsilon^*(K_\epsilon(x,x))$ is a continuous curve in $C^\infty(\overline{\Omega}_0)$ and hence uniformly bounded, by the dominated convergence theorem

$$
\begin{aligned}
\frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_{\Omega_\epsilon} K_\epsilon(x,x)\,dx &= \frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_{\Omega_0}\varphi_\epsilon^*(K_\epsilon(x,x))\varphi_\epsilon^*(dx) \\
&= \int_{\Omega_0}\frac{d}{d\epsilon}\Big|_{\epsilon=0} K_\epsilon(x,x)dx + \int_{\partial\Omega_0} K_0(q,q)\dot{\rho}(q)\,dq.
\end{aligned}
$$

But the second integral is zero in the Dirichlet case because $K_0(q,q) = 0$ for all $q \in \partial\Omega_0$. (Note: this term does not vanish in the Neumann case but it cancels out with a term which appears in the analogous computations). This concludes the first proof of Lemma 1. □

Second proof. This derivation is based on the Hadamard variational formulas for eigenvalues. When $\lambda_j^2(0)$ is a simple eigenvalue (i.e., of multiplicity one), Hadamard's variational formula for Dirichlet eigenvalues of Euclidean domains states that if $\epsilon \to \Omega_\epsilon$ is C^1 then

$$\delta(\lambda_j^2(\epsilon)) = -\int_{\partial\Omega_0} (\partial_\nu \Psi_j(q))^2 \dot{\rho}(q)\, dq,$$

where Ψ_j is an L^2 normalized eigenfunction for the eigenvalue $\lambda_j^2(0)$. See [Garabedian 1964]. However if the eigenvalue $\lambda_j^2(0)$ is multiple with multiplicity $m(\lambda_j(0))$ and if $\{\lambda_{j,k}^2(\epsilon)\}_{k=1}^{m(\lambda_j(0))}$ is the perturbed set of eigenvalues, then we cannot assume that $\lambda_{j,k}^2(\epsilon)$ are C^1 in ϵ (although this is known to be true for symmetric operators on finite-dimensional spaces. See, for example, Theorem II.6.8 of [Kato 1980]). But as we shall see, the sum $\sum_{k=1}^{m(\lambda_j(0))} \lambda_{j,k}^2(\epsilon)$ is C^1 in ϵ and there exists a Hadamard's variational formula for it which can be derived from the one for Green's function. In fact we prove a slightly more general statement. For the sake of convenience we let $\widetilde{R}_{B_\epsilon}(z) = (-\Delta_{B_\epsilon} - z)^{-1}$ where $z \notin \mathrm{Spec}(-\Delta_{B_\epsilon})$ and we use $\widetilde{G}_{B_\epsilon}(z, x, y)$ for its integral kernel. Now let $g(z)$ be a holomorphic function on the right half-plane $\Re(z) > 0$. We will show that

$$\delta \sum_{k=1}^{m(\lambda_j(0))} g(\lambda_{j,k}^2(\epsilon)) = -g'(\lambda_j^2(0)) \sum_{k=1}^{m(\lambda_j(0))} \int_{\partial\Omega_0} (\partial_\nu \Psi_{j,k}(q))^2 \dot{\rho}(q)\, dq, \tag{13}$$

where $\{\Psi_{j,k}\}_{k=1}^{m(\lambda_j(0))}$ is an orthonormal basis for the eigenspace of the multiple eigenvalue $\lambda_j^2(0)$. Lemma 1 follows easily from (13) by putting $g(z) = \cos(t\sqrt{z})$:

$$\begin{aligned}\delta \operatorname{Tr} E_B(t) = \delta \sum \cos(t\lambda_{j,k}) &= -t \sum_j \frac{\sin(t\lambda_j(0))}{2\lambda_j(0)} \left(\sum_{k=1}^{m(\lambda_j(0))} \int_{\partial\Omega_0} (\partial_\nu \Psi_{j,k})^2 \dot{\rho}(q)\, dq \right) \\ &= \frac{t}{2} \int_{\partial\Omega_0} \partial_{\nu_1} \partial_{\nu_2} S_B(t, q, q) \dot{\rho}(q)\, dq.\end{aligned}$$

We have pushed the operation δ under the sum. This can be done because for a test function $\varphi(t)$ the sums

$$\sum \int \cos(t\lambda_{j,k}(\epsilon))\varphi(t)\, dt \quad \text{and} \quad \sum \int \frac{d}{d\epsilon} \cos(t\lambda_{j,k}(\epsilon))\varphi(t) dt$$

are (by Weyl's law) uniformly convergent in ϵ.

It remains to prove (13). Let γ be a circle in $\mathbb{C}$ centered at $\lambda_j^2(0)$ such that no other eigenvalues of $-\Delta_{B_0}$ are in the interior of γ or on γ. We define

$$T_{g,\epsilon} = -\frac{1}{2\pi i} \int_\gamma g(z) \widetilde{R}_{B_\epsilon}(z)\, dz.$$

By the Cauchy integral formula, it is clear that at $\epsilon = 0$ we have $T_{g,0} = g(P_{\lambda_j^2(0)})$ where $P_{\lambda_j^2(0)}$ is the orthogonal projector on the eigenspace of $\lambda_j^2(0)$. Since the eigenvalues $\lambda_{j,k}^2(\epsilon)$ vary continuously in ϵ, for ϵ small these are the only eigenvalues of $-\Delta_{B_\epsilon}$ in γ. Therefore $T_{g,\epsilon}$ is the total projector (the direct sum of projectors) associated with $\{\lambda_{j,k}^2(\epsilon)\}_{k=1}^m$. The operator $T_{g,\epsilon}$ is C^1 in ϵ. See, for example, Theorem II.5.4

of [Kato 1980]. Although this theorem is stated for operators on finite dimensional spaces but the same proof works for our case. It is basically because the resolvent (and so the Green's function) is C^1 in ϵ. We now write

$$\begin{aligned}
\delta \sum_{k=1}^{m(\lambda_j(0))} g(\lambda_{j,k}^2(\epsilon)) &= \delta \operatorname{Tr}(T_{g,\epsilon}) = -\operatorname{Tr} \frac{1}{2\pi i} \int_\gamma g(z)\delta \widetilde{R}_{D_\epsilon}(z)\, dz \\
&= -\int_{\Omega_0}\int_{\partial\Omega_0}\int_\gamma \frac{1}{2\pi i} g(z) \frac{\partial}{\partial \nu_2} G_D(z,x,q) \frac{\partial}{\partial \nu_1} G_D(z,q,x) \dot{\rho}(q)\, dz\, dq\, dx \\
&= -\int_{\Omega_0}\int_{\partial\Omega_0}\int_\gamma \frac{1}{2\pi i} \frac{g(z)}{(\lambda_j^2(0)-z)^2} \sum_{k=1}^{m} (\partial_\nu \Psi_{j,k}(q))^2 (\Psi_{j,k}(x))^2 \dot{\rho}(q)\, dz\, dq\, dx \\
&= -g'(\lambda_j^2(0)) \sum_{k=1}^{m(\lambda_j(0))} \int_{\partial\Omega_0} (\partial_\nu \Psi_{j,k}|_{\partial\Omega_0})^2 \dot{\rho}(q)\, dq.
\end{aligned}$$

We leave it to the reader to show that, on the first line one can commute δ with Tr by means of the dominated convergence theorem.

There exist similar Hadamard variational formulas in the Neumann case. When the eigenvalue is simple, we have

$$\delta(\lambda_j^2) = \int_{\partial\Omega_0} \big(|\nabla_q^T (\Psi_j(q))|^2 - \lambda_j^2(0)(\Psi_j(q))^2 \big) \dot{\rho}(q)\, dq, \tag{14}$$

For a multiple eigenvalue we sum over the expressions over an orthonormal basis of the eigenspace. The result does not depend on a choice of orthonormal basis. Similar computation using (14) follows to show Lemma 1 for the Neumann case. □

2. Proof of Theorem 2

We now study the singularity expansion of $\delta \operatorname{Tr}\cos(t\sqrt{-\Delta_B})$ and prove Theorem 2. At first sight, one could do this in two ways: by taking the variation of the spectral side of the formula, or by taking the variation of the singularity expansion. It seems simpler and clearer to do the former since we do not know how the invariant tori of the integrable elliptical billiard deform under an isospectral deformation. For example, one difficulty in taking the variation of the singularity expansion is that we do not know whether the fixed point set of an isospectral deformation Ω_ϵ of domain Ω_0 (satisfying the conditions of Theorem 2) is necessarily clean. Hence, even though we know that the wave trace of Ω_ϵ has the same type of singularity as the one for Ω_0, but we cannot apply the method of stationary phase and compute the principal term in the singularity expansion of the wave trace of Ω_ϵ.

In this section we will drop the subscript 0 in Ω_0 and we assume Ω is a smooth convex domain.

The variational formula for $\delta \operatorname{Tr}\cos(t\sqrt{-\Delta_B})$ is given in Lemma 1. In the Dirichlet case, by (4),

$$\operatorname{Tr}_{\partial\Omega}\ \dot{\rho}\, S_D^b = \pi_* \,\Delta^* \dot{\rho}\ (r_1 r_2 N_{\nu_1} N_{\nu_2} S_D(t,x,y)), \tag{15}$$

where N_ν is any smooth vector field in Ω extending ν, and where the subscripts indicate the variables on which the operator acts. In the Neumann case by (4),

$$\mathrm{Tr}_{\partial\Omega}\ \dot{\rho}\ S_N^b = \pi_*\ \Delta^* \dot{\rho}\ \big((\nabla_1^T \nabla_2^T r_1 r_2 + r_1 r_2\ \Delta_x) S_N(t, x, y)\big). \tag{16}$$

Here, $\Delta : \partial\Omega \to \partial\Omega \times \partial\Omega$ is the diagonal embedding $q \to (q, q)$ and π_* (the pushforward of the natural projection $\pi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$) is the integration over the fibers with respect to the surface measure dq. The duplication in notation between the Laplacian and the diagonal is regrettable, but both are standard and should not cause confusion. Since $S_B(t, x, y)$ is microlocally a Fourier integral operator near the transversal periodic reflecting rays of F_T, it will follow from (15) that the trace is locally a Fourier integral distribution near $t = T$.

We are assuming that the set of periodic points of the billiard map corresponding to space-time billiard trajectories of length $T \in \mathrm{Lsp}(\Omega)$ is a submanifold F_T of $B^*\partial\Omega$. We thus fix $T \in \mathrm{Lsp}(\Omega)$ consisting only of periodic reflecting rays, that is, we assume $T \neq m|\partial\Omega|$ ($|\partial\Omega|$ being the perimeter) for $m \in \mathbb{Z}$. In order to study the singularity of the boundary trace near a component F_T of the fixed point set, we construct a pseudo-differential cutoff $\chi_T = \chi_T(t, D_t, q, D_q) \in \Psi^0(\mathbb{R} \times \partial\Omega)$ whose complete symbol $\chi_T(t, \tau, q, \zeta)$ has the form $\chi_T(q, \zeta/\tau)$ with $\chi_T(y, \zeta)$ supported in a small neighborhood of the fixed point set $F_T \subset B^*\partial\Omega$, equals one in a smaller neighborhood, and in particular vanishes in a neighborhood of the glancing directions in $S^*\partial\Omega = \partial(B^*\partial\Omega)$. Since the symbol of χ_T is independent of t we will instead use $\chi_T(D_t, q, D_q)$. We may assume that the support of the cutoff is invariant under the billiard map β. Therefore we need to study the operator

$$\pi_*\Delta^*\ \dot{\rho}\ \chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q) S_B^b, \tag{17}$$

and compute its symbol. To do this we first study the operators r and $S_B(t)$ and review their basic properties. Next we study the composition

$$\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q) S_B^b,$$

and compute its symbol. Finally in Lemma 7 we take composition with $\pi_*\Delta^*\ \dot{\rho}$ and calculate the symbol of (17).

2.1. ***FIOs and their symbol.*** We recall that the principal symbol σ_I of a Fourier integral distribution

$$I = \int_{\mathbb{R}^N} e^{i\varphi(x,\theta)} a(x, \theta)\, d\theta, \quad I \in I^m(M, \Lambda_\varphi),$$

of order m is defined in terms of the parametrization

$$\iota_\varphi : C_\varphi = \{(x, \theta) : d_\theta\varphi = 0\} \to (x, d_x\varphi) \in \Lambda_\varphi \subset T^*M$$

of the associated Lagrangian Λ_φ. It is a half-density on Λ_φ given by $\sigma_I = (\iota_\varphi)_*(a_0|d_{C_\varphi}|^{1/2})$, where a_0 is the leading term of the classical symbol $a \in S^{m+\frac{n}{4}-\frac{N}{2}}(M \times \mathbb{R}^N)$, $n = \dim M$ and

$$d_{C_\varphi} := \frac{dc}{|D(c, \varphi'_\theta)/D(x, \theta)|}$$

is the Gelfand–Leray form on C_φ, where c is a system of coordinates on C_φ. For notation and background we refer to [Hörmander 1985b, Chapter XXV]. When $I(x, y) \in I^m(X \times Y, \Lambda)$ is the kernel of an FIO it is very standard to use the symplectic form $\omega_X - \omega_Y$ on $X \times Y$ and define

$$\iota_\varphi : C_\varphi = \{(x, y, \theta) : d_\theta \varphi = 0\} \to (x, d_x\varphi, y, -d_y\varphi) \in \Lambda_\varphi \subset T^*X \times T^*Y.$$

We will call Λ_φ the *canonical relation* of $I(x, y)$.

2.2. ***The restriction operator r as an FIO.*** The restriction r to the boundary lies in $I^{1/4}(\partial\Omega \times \mathbb{R}^n, \Gamma_{\partial\Omega})$, with the canonical relation

$$\Gamma_{\partial\Omega} = \left\{(q, \zeta, q, \xi) \in T^*\partial\Omega \times T^*_{\partial\Omega}\mathbb{R}^n; \xi|_{T_q\partial\Omega} = \zeta\right\}. \tag{18}$$

The adjoint then satisfies $r^* \in I^{1/4}(\mathbb{R}^n \times \partial\Omega, \Gamma^*_{\partial\Omega})$, where

$$\Gamma^*_{\partial\Omega} = \left\{(q, \xi, q, \zeta) \in T^*_{\partial\Omega}\mathbb{R}^n \times T^*\partial\Omega; \xi|_{T_q\partial\Omega} = \zeta\right\}.$$

Here, $T^*_{\partial\Omega}\mathbb{R}^n$ is the set of covectors to $\mathbb{R}^n$ with footpoint on $\partial\Omega$. We parametrize $\Gamma_{\partial\Omega}$ (18) by $T^{*+}_{\partial\Omega}(\Omega)$, the inward pointing covectors, using the Lagrange immersion

$$\iota_{\Gamma_{\partial\Omega}}(q, \xi) = (q, \xi|_{T_q(\partial\Omega)}, q, \xi). \tag{19}$$

To prove these statements, we introduce Fermi normal coordinates (q, x_n) along $\partial\Omega$, that is, $x = \exp_q(x_n\nu_q)$ where ν_q is the interior unit normal at q. Let $\xi = (\zeta, \xi_n) \in T^*_{(q,x_n)}\mathbb{R}^n$ denote the corresponding symplectically dual fiber coordinates. In these coordinates, the kernel of r is given by

$$r(q, (q', x'_n)) = C_n \int_{\mathbb{R}^n} e^{i\langle q-q',\zeta\rangle - ix'_n\xi_n} d\xi_n d\zeta. \tag{20}$$

The phase $\varphi(q, (q', x'_n), (\zeta, \xi_n)) = \langle q - q', \zeta\rangle - x'_n\xi_n$ is nondegenerate and its critical set is $C_\varphi = \{(q, q', x'_n, \xi_n, \zeta); q' = q, x'_n = 0\}$. The Lagrange map $\iota_\varphi : (q, q, 0, \xi_n, \zeta) \to (q, \zeta, q, \zeta, \xi_n)$ embeds $C_\varphi \to T^*\partial\Omega \times T^*\mathbb{R}^n$ and maps onto $\Gamma_{\partial\Omega}$. The adjoint kernel has the form $K^*(x, q) = \overline{K}(q, x)$ and therefore has a similar oscillatory integral representation. It is clear from ((20)) that the order of r as an FIO is $\frac{1}{4}$. Also, in the parametrization (19), the principal symbol of r is $\sigma_r = |dq \wedge d\zeta \wedge d\xi_n|^{1/2}$.

2.3. ***Background on parametrices for $S_B(t)$.*** We first review the Fourier integral description of $E_B(t)$, $S_B(t)$ microlocally near transversal reflecting rays. This is partly for the sake of completeness, but mainly because we need to compute their principal symbols (and related ones) along the boundary. Although the principal symbols are calculated in the interior in [Guillemin and Melrose 1979b, Proposition 5.1; Marvizi and Melrose 1982, Section 6; Petkov and Stoyanov 1992, Section 6], the results do not seem to be stated along the boundary (i.e., the symbols are not calculated at the boundary). The statements we need are contained in Theorem 3.1 of [Chazarain 1976] (and its proof), and we largely follow its presentation.

We need to calculate the canonical relation and principal symbol of the wave group, its derivatives and their restrictions to the boundary. We begin by recalling that the propagation of singularities theorem for the mixed Cauchy–Dirichlet (or Neumann) problem for the wave equation states that the wave front set of the wave kernel satisfies

$$WF(S_B(t,x,y)) \subset \bigcup_{\pm} \Lambda_{\pm},$$

where $\Lambda_{\pm} = \{(t,\tau,x,\xi,y,\eta) : (x,\xi) = \Phi^t(y,\eta),\ \tau = \pm|\eta|_y\} \subset T^*(\mathbb{R}\times\Omega\times\Omega)$ is the graph of the generalized (broken) geodesic flow, that is, the billiard flow Φ^t. For background we refer to [Guillemin and Melrose 1979b; Petkov and Stoyanov 1992; Chazarain 1973; 1976; Hörmander 1985a, Theorem 23.1.4; 1985b, Proposition 29.3.2]. For the application to spectral rigidity, we only need a microlocal description of wave kernels away from the glancing set, that is, in the hyperbolic set microlocally near periodic transversal reflecting rays. In these regions, there exists a microlocal parametrix due to Chazarain [1976], which is more fully analyzed in [Guillemin and Melrose 1979b; Petkov and Stoyanov 1992] and applied to the ellipse in [Guillemin and Melrose 1979a].

The microlocal parametrices for E_B and S_B are constructed in the ambient space $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$. Since $E_B = dS_B/dt$ it suffices to consider the latter. Then there exists a Fourier integral (Lagrangian) distribution,

$$\tilde{S}_B(t,x,y) = \sum_{j=-\infty}^{\infty} S_j(t,x,y), \quad \text{with } S_j \in I^{-\frac{1}{4}-1}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n, \Gamma^j_{\pm}),$$

which microlocally approximates $S_B(t,x,y)$ modulo a smooth kernel near a transversal reflecting ray. The sum is locally finite hence well-defined. The canonical relation of $\tilde{S}_B$ is contained in a union

$$\Gamma = \bigcup_{\pm,\, j\in\mathbb{Z}} \Gamma^j_{\pm} \subset T^*(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n)$$

of canonical relations $\Gamma^j_{\pm}$ corresponding to the graph of the broken geodesic flow with j reflections. Notice we let $j\in\mathbb{Z}$ which is different from [Chazarain 1976] where j goes from 0 to ∞ and where the two graphs $\Gamma^j_{\pm}$ and $\Gamma^{-j}_{\pm}$ are combined.

We know discuss these graphs more precisely. We first recall some useful notation from [Chazarain 1976] with a slight adjustment. We have two Hamiltonian flows $g^{\pm t}$ corresponding to the Hamiltonians $\pm|\eta|$. For (y,η) in $T^*\Omega$ or (y,η) in $T^*_{\partial\Omega}\mathbb{R}^n$ where η is transversal to $\partial\Omega$ and is pointing inward, we define

$$t^1_{\pm}(y,\eta) = \inf\{t>0 : \pi g^{\pm t}(y,\eta) \in \partial\Omega\},$$
$$t^{-1}_{\pm}(y,\eta) = \sup\{t<0 : \pi g^{\pm t}(y,\eta) \in \partial\Omega\}.$$

In this notation we have $t^{-1}_{\pm} = -t^1_{\mp}$. We define $t^j_{\pm}$ inductively for $j>0$ (resp. $j<0$) to be the time of j-th reflection for the flow $g^{\pm t}$ as t increases (resp. decreases). Then we put

$$\lambda^1_{\pm}(y,\eta) = g^{\pm t^1_{\pm}(y,\eta)}(y,\eta) \in T^*_{\partial\Omega}\mathbb{R}^n,$$
$$\lambda^{-1}_{\pm}(y,\eta) = g^{\pm t^{-1}_{\pm}(y,\eta)}(y,\eta) \in T^*_{\partial\Omega}\mathbb{R}^n.$$

Next we define $\widehat{\lambda^1_{\pm}(y,\eta)}$ to be the reflection of $\lambda^1_{\pm}(y,\eta)$ at the boundary. That is, it has the same foot point y and the same tangential projection as $\lambda^1_{\pm}(y,\eta)$ but opposite normal component. Similarly we define $\widehat{\lambda_\pm^{-1}(y,\eta)}$. Flowing $\widehat{\lambda_\pm^{1}(y,\eta)}$ (resp. $\widehat{\lambda_\pm^{-1}(y,\eta)}$) by $g^{\pm t}$ as t increases (resp. decreases) and continuing the same procedure we get $t_\pm^j(y,\eta)$ and $\lambda_\pm^j(y,\eta)$ for all $j \in \mathbb{Z}$. We also set $T_\pm^j = \sum_{k=1}^j t_\pm^k$ for $j > 0$ and $T_\pm^j = \sum_{k=-1}^j t_\pm^k$ for $j < 0$.

The canonical graph $\Gamma_\pm^j$ can now be written as

$$\Gamma_\pm^j = \begin{cases} \{(t,\tau,g^{\pm t}(y,\eta),y,\eta) : \tau = \pm|\eta|_y\} & j = 0, \\ \{(t,\tau,g^{\pm(t-T_\pm^j(y,\eta))}\widehat{\lambda_\pm^j(y,\eta)},y,\eta) : \tau = \pm|\eta|_y\} & j \in \mathbb{Z},\ j \neq 0. \end{cases} \tag{21}$$

For each $j \in \mathbb{Z}$, $\bigcup_\pm \Gamma_\pm^j$ is the union of two canonical graphs, which we refer to as its *branches* or *components* (see Figure 3.2 of [Guillemin and Melrose 1979b] for an illustration). These two branches arise because

$$S_B(t) = \frac{1}{2i\sqrt{-\Delta_B}}\left(e^{it\sqrt{-\Delta_B}} - e^{-it\sqrt{-\Delta_B}}\right)$$

is the sum of two terms whose canonical relations are respectively the graphs of the forward/backward broken geodesic flow and which correspond to the two halves $\tau > 0$, $\tau < 0$ of the characteristic variety $\tau^2 - |\eta|^2 = 0$ of the wave operator.

2.3.1. *Symbol of $S_B(t,x,y)$ in the interior.* In the boundaryless case of [Duistermaat and Guillemin 1975], the half-density symbol of $e^{it\sqrt{-\Delta_g}}$ is a constant multiple (Maslov factor) of the canonical graph volume half-density $\sigma_{\text{can}} = |dt \wedge dy \wedge d\eta|^{1/2}$ on Γ_+ in the graph parametrization $(t,y,\eta) \to \Gamma_+ = (t,|\eta|_g,g^t(y,\eta),y,\eta)$. In the boundary case for $E_B(t)$ the symbol in the interior is computed in Corollary 4.3 of [Guillemin and Melrose 1979b] as a scalar multiple of the graph half-density. It is a constant multiple of the graph half-density

$$\sigma_{\text{can},\pm} = |dt \wedge dy \wedge d\eta|^{1/2} \tag{22}$$

in the obvious graph parametrization of $\Gamma_\pm^j$ in (21); the constant equals $\frac{1}{2}$ in the Neumann case and $\frac{1}{2}(-1)^j$ in the Dirichlet case. However in [Guillemin and Melrose 1979b] the symbols are not calculated at the boundary.

Remark. We will have four modes of propagation at the boundary: in addition to the two $\pm$ branches corresponding to $\tau > 0$ and $\tau < 0$, at the boundary, the boundary condition requires two modes of propagation corresponding to the two "sides" of $\partial\Omega$. To illustrate this we first discuss a simple model of the upper half space.

2.3.2. *Upper half space; a local model for one reflection.* Let $\mathbb{R}_+^n = \{(x',x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n \geq 0\}$ be the upper half space. Denote by $S_0(t,x,y)$ the kernel of $\sin(t\sqrt{-\Delta})/\sqrt{-\Delta}$ of Euclidean $\mathbb{R}^n$. Then

$$\begin{cases} S_D(t,x,y) = S_0(t,x,y) - S_0(t,x,y^*), \\ S_N(t,x,y) = S_0(t,x,y) + S_0(t,x,y^*), \end{cases}$$

where $y^* \in \mathbb{R}_-^n$ is the reflection of y through the boundary $\mathbb{R}^{n-1} \times \{0\}$. Indeed, $y \to y^*$ is an isometry, so both kernels satisfy $\Box E_B = 0$ (in either the x or y variable) and have the correct initial conditions since $y^* \notin \mathbb{R}_+^n$. Further they satisfy the correct boundary conditions: it is clear that $S_D(t,x,y) = 0$ if

$y \in \mathbb{R}_+^{n-1} \times \{0\}$ since $y^* = y$ for such points. Also, if $x_n = 0$ then $S_D(t, x, y) = 0$ since $S_D(t, x, y)$ is a function of the distance $|x - y|$ and $|x - y| = |x - y^*|$ if $x_n = 0$. Similarly, the normal derivative is $\partial/\partial y_n$, so the normal derivatives cancel for $S_N(t, x, y)$ when $y_n = 0$. Also, $S_0(t, x, y^*) = S_0(t, x^*, y)$ and $S_0(t, x, y) = S_0(t, y, x)$, so the same calculation applies in the x variable. The canonical relation associated to S_N and S_D is the union of the canonical relations of S_0 and of $S_0^* = S_0(t, x, y^*)$. More precisely, by our notation in (21),

$$WF(S_B(t, x, y)) \subset \Gamma_\pm^0 \cup \Gamma_\pm^1 \cup \Gamma_\pm^{-1}.$$

Note that this example is asymmetric in past and future: the forward trajectory may intersect boundary, but then backward one does not. Also, in this example for $j > 1$ and $j < -1$ the graphs $\Gamma_\pm^j$ are empty.

2.3.3. *Symbol of $S_B(t, x, y)$ at the boundary.* Since we want to restrict kernels and symbols to the boundary, we introduce further notation for the subset of the canonical relations lying over boundary points. Following [Chazarain 1976], we denote by

$$A_\pm^0 = \{(0, \tau, y, \eta, y, \eta) : \tau = \pm|\eta|_y\}$$

the subset of $\Gamma_\pm^0$ with $t = 0$. Under the flow $\psi_\pm^t$ of the Hamiltonian $\tau \pm |\xi|_x$ on $\mathbb{R} \times \mathbb{R}^n$, it flows out to the graph $\Gamma_\pm^0$ (denoted by $C_\pm^0$ in [loc. cit., (2.11)]). One then defines $A_\pm^1 \subset \Gamma_\pm^0$ (resp. $A_\pm^{-1} \subset \Gamma_\pm^0$) as the subset lying over $\mathbb{R}_+ \times \partial\Omega \times \Omega$ (resp. $\mathbb{R}_- \times \partial\Omega \times \Omega$). Still following Chazarain, we denote by $\xi \to \widehat{\xi}$ the reflection map for $(q, \xi) \in T_q^*\mathbb{R}^n$, $q \in \partial\Omega$. That is, $\widehat{\xi}$ has the same tangential projection as ξ but opposite normal component. We then have

$$\Gamma_\pm^1 = \bigcup_{t \in \mathbb{R}} \psi_\pm^t \widehat{A}_\pm^1 \quad \text{and} \quad \Gamma_\pm^{-1} = \bigcup_{t \in \mathbb{R}} \psi_\pm^t \widehat{A}_\pm^{-1},$$

as the flow out under the Euclidean space-time geodesic flow of $\widehat{A}_\pm^1$ and $\widehat{A}_\pm^{-1}$. Thus, along the boundary, for $t > 0$ (resp. $t < 0$) $A_\pm^1$ and $\widehat{A}_\pm^1$ (resp. $A_\pm^{-1}$ and $\widehat{A}_\pm^{-1}$) both lie in the canonical relation of $E_B(t)$, $S_B(t)$. In a similar way one defines $A_\pm^2$ to be the subset of $\Gamma_\pm^1$ lying over $\mathbb{R}_+ \times \partial\Omega \times \Omega$ and $\widehat{A}_\pm^2$ to be its reflection. Then also $A_\pm^2 \cup \widehat{A}_\pm^2$ lies in the canonical relation. Similarly one defines $A_\pm^j$ and $\widehat{A}_\pm^j$ for all $j \in \mathbb{Z}$.

Remark. Since we are interested in the singularity of the trace at $t = T > 0$ we will only consider the graphs $\Gamma_\pm^j$ for $j \geq 0$. Regardless of this, because $\delta \operatorname{Tr} E_B(t)$ is even in t it has the same singularity at $t = T$ and $t = -T$.

The symbols of $E_B(t)$ and $S_B(t)$ are half-densities on the associated canonical relations, and therefore are sums of four terms at boundary points, that is, there is a contribution from each of $A_\pm^j$ and $\widehat{A}_\pm^j$. In the interior, there is only a contribution from the $\pm$ components.

The following lemma gives formulas for the principal symbol of S_B (and therefore E_B) on $\Gamma_\pm^j$ and its restriction to $\Gamma_{\partial\Omega} \circ (A_\pm^j \cup \widehat{A}_\pm^j)$.

Lemma 2. *Let $e_\pm$ be the principal symbol of $\tilde{S}_B$ when restricted to $\Gamma_\pm = \bigcup_j \Gamma_\pm^j$. Let σ_r be the principal symbol of the boundary restriction operator r.*

1. *In the interior, on $\Gamma^j_\pm$, up to Maslov factors we have:*

Dirichlet case:
$$e_\pm = \frac{(-1)^j}{2\tau}\sigma_{\text{can},\pm} = \pm\frac{(-1)^j}{2|\eta|}\sigma_{\text{can},\pm}.$$

Neumann case:
$$e_\pm = \frac{1}{2\tau}\sigma_{\text{can},\pm} = \pm\frac{1}{2|\eta|}\sigma_{\text{can},\pm}.$$

2. *At the boundary, on $\Gamma_{\partial\Omega}\circ A^j_\pm = \Gamma_{\partial\Omega}\circ \widehat{A}^j_\pm$ we have:*

Dirichlet case:
$$\sigma_r \circ e_\pm(t^j_\pm, \pm\tau, \widehat{\lambda^j_\pm(y,\eta)}, y, \eta) = -\sigma_r \circ e_\pm(t^j_\pm, \pm\tau, \lambda^j_\pm(y,\eta), y, \eta).$$

Neumann case:
$$\sigma_r \circ e_\pm(t^j_\pm, \pm\tau, \widehat{\lambda^j_\pm(y,\eta)}, y, \eta) = \sigma_r \circ e_\pm(t^j_\pm, \pm\tau, \lambda^j_\pm(y,\eta), y, \eta).$$

Proof. These formulas are obtained from the transport equations in [Chazarain 1976, (b_0')–(e_0'), p. 175]. We now sketch the proof.

The transport equations for the symbols of E_B, S_B determine how they propagate along broken geodesics. As in the boundaryless case, the principal symbol has a zero Lie derivative, $\mathscr{L}_{H_{\tau+|\xi|}}\sigma_E = 0$, in the interior along geodesics. The important point for us is the rule by which they are reflected at the boundary. Let σ_B be the principal symbol of the boundary restriction operator B defined in (3) ($B = r$ under Dirichlet and $B = rN$ under Neumann boundary conditions) and let σ_0 be the principal symbol of the restriction operator to $t = 0$. Then:

$$\begin{aligned}
&(b_0) : (d^2/dt^2 - \Delta_B)\tilde{S}_B \sim 0 \Longrightarrow (b_0') : \mathscr{L}_{\psi^t_\pm} e_\pm = 0;\\
&(c_0) : \tilde{S_B}|_{t=0} \sim 0 \qquad\qquad\quad\;\, \Longrightarrow (c_0') : \sigma_0 \circ e_+(0,\tau,y,\eta,y,\eta) + \sigma_0\circ e_-(0,-\tau,y,\eta,y,\eta) = 0;\\
&(d_0) : \frac{d}{dt}\Big|_{t=0}\tilde{S_B} \sim \delta(x-y) \;\; \Longrightarrow (d_0') : \tau\big(\sigma_0\circ e_+(0,\tau,y,\eta,y,\eta) - \sigma_0\circ e_-(0,-\tau,y,\eta,y,\eta)\big) = \sigma_I;\\
&(e_0) : B\tilde{S}_B \sim 0 \qquad\qquad\qquad\;\; \Longrightarrow (e_0') : \sigma_B\circ e_\pm = \sigma_B\circ\big(e_\pm|_{A^j_\pm}\big) + \sigma_B\circ\big(e_\pm|_{\widehat{A}^j_\pm}\big) = 0.
\end{aligned}\tag{23}$$

Here σ_I is the principal symbol of the identity operator. The implication $(b_0) \Longrightarrow (b_0')$ follows, for example, from Theorem 5.3.1 of [Duistermaat and Hörmander 1972]. The other implications are obvious. From (c_0') and (d_0') we get

$$(\sigma_0\circ e_\pm)(y,\eta,y,\eta) = \frac{(-1)^j}{2\tau}\sigma_I \quad \text{on} \quad T^*\Omega.$$

But by (b_0'), the symbol $e_\pm$ is invariant under the flow $\psi^t_\pm$ and therefore the first part of the lemma follows but only on $\Gamma^0_\pm$. The second part of the lemma follows from (e_0'). The first term of (e_0') is known from the previous transport equations. Hence (e_0') determines the "reflected symbol" at the j-th impact time and impact point. In the Dirichlet case, B is just r the restriction to the boundary and so the reflected principal symbol is simply the opposite of the direct principal symbol. In the Neumann case, B is the product of the symbol $\langle\lambda^1_\pm(y,\eta), \nu_y\rangle$ of the inward normal derivative times restriction r. The reflected symbol thus equals the direct symbol since the sign is canceled by the sign of the $\widehat{\langle\lambda^1_\pm(y,\eta),\nu_y\rangle} = -\langle\lambda^1_\pm(y,\eta),\nu_y\rangle$ factor. Thus, the volume half-density is propagated unchanged in the Neumann case and has a sign change at each impact point in the Dirichlet case. Thus on $\Gamma^j_\pm$ and after j reflections, the Dirichlet wave group symbol is $(-1)^j$ times $1/2\tau$ times the graph half-density and the Neumann symbol is $1/2\tau$ times the graph half-density. □

2.4. $\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S^b_B(t, q', q)$ *is a Fourier integral operator.*

Lemma 3. *We have*

$$\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S^b_B(t, q', q) \in I^{(1/2)+1-(1/4)}(\mathbb{R}\times\partial\Omega\times\partial\Omega, \Gamma_{\partial,\pm}).$$

Here, $\Gamma_{\partial,\pm} = \bigcup_{j\in\mathbb{Z}} \Gamma^j_{\partial,\pm}$, *with*

$$\Gamma^j_{\partial,\pm} := \big\{(t,\tau,q',\zeta',q,\zeta) \in T^*(\mathbb{R}\times\partial\Omega\times\partial\Omega) : \exists\, \xi' \in T^*_{q'}\mathbb{R}^n, \xi \in T^*_q\mathbb{R}^n : \\ (t,\tau,q',\xi',q,\xi) \in \Gamma^j_\pm,\ \ \xi'|_{T_{q'}\partial\Omega} = \zeta',\ \xi|_{T_q\partial\Omega} = \zeta\big\}.$$

Proof. We only show the proof in the Dirichlet case. The Neumann case is very similar. The kernel $\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S^b_D(t, q', q)$ for fixed t is the Schwartz kernel of the composition

$$\chi_T \circ (r\,N) \circ S_D(t) \circ (N^*\,r^*) \circ \chi^*_T : L^2(\partial\Omega) \to L^2(\partial\Omega), \tag{24}$$

where r^* is the adjoint of $r : H^{1/2}(\overline{\Omega}) \to L^2(\partial\Omega)$.

To prove the lemma, we use that r is a Fourier integral operator with a folding canonical relation, and that the composition (24) is transversal away from the tangential directions to $\partial\Omega$, where $S_B(t)$ fails to be a Fourier integral operator. The cutoff χ_T removes the part of the canonical relation near the fold locus and near the normal directions $N^*\partial\Omega$ (where the composition $(r\,N)\circ S_D(t)\circ(N^*\,r^*)$ fails to be well-behaved as an FIO), hence the composition is a standard Fourier integral operator.

By the results cited above in [Chazarain 1976; Guillemin and Melrose 1979b; Petkov and Stoyanov 1992; Marvizi and Melrose 1982], microlocally away from the gliding directions, the wave operator $S_B(t)$ is a Fourier integral operator associated to the canonical relations $\Gamma^j_\pm$. Since $\Gamma^j_\pm$ is a union of graphs of canonical transformations, its composition (away from the normal bundle $N^*\partial\Omega$), with the canonical relation of $r^D := rN$ is automatically transversal. The further composition with the canonical relation of r^{D*} is also transversal. Hence, the composition is a Fourier integral operator with the composed wave front relation and the orders add. Taking into account that we have two boundary derivatives, we need to add $\frac{1}{2}$ to the order.

To determine the composite relation, we note that

$$\begin{aligned} &\Phi_\pm : \mathbb{R}\times T^*_{\partial\Omega}\mathbb{R}^n \to T^*\mathbb{R}\times T^*\Omega\times T^*_{\partial\Omega}\mathbb{R}^n,\\ &\Phi_\pm(t,q,\zeta,\xi_n) := (t, \pm|\zeta+\xi_n|, \Phi^t(q,\zeta,\xi_n), q, \zeta, \xi_n) \end{aligned} \tag{25}$$

parametrizes the graph of the (space-time) billiard flow with initial condition on $T^*_{\partial\Omega}\mathbb{R}^n$. Here, $\zeta \in T^*\partial\Omega$ and $\xi_n \in N^*_+\partial\Omega$, the inward pointing (co)normal bundle. $\Phi_\pm$ is a homogeneous folding map with folds along $\mathbb{R} \times T^*\partial\Omega$ (see, e.g., [Hörmander 1985a] for background). It follows that $S_D(t) \circ (N^* r^*) \chi_T^*$ is a Fourier integral operator of order one associated to the canonical relation

$$\left\{ (t, \pm|\xi|, \Phi^t(q, \xi), q, \xi|_{T^*\partial\Omega} \right\} \subset T^*(\mathbb{R} \times \Omega \times \partial\Omega),$$

and is a local canonical graph away from the fold singularity along $T^*\partial\Omega$. Composing on the left by the restriction relation produces a Fourier integral operator with the stated canonical relation. The two normal derivatives N of course do not change the relation. □

2.5. ***Symbol of*** $\boldsymbol{\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q) S_B^b(t, q', q)}$***.*** The next step is to compute the principal symbols of the operators in Lemma 3.

To state the result, we need some further notation. We denote points of $T^*_{\partial\Omega}\mathbb{R}^n$ by $(q, 0, \zeta, \xi_n)$ as above, and put $\tau = \sqrt{|\zeta|^2 + \xi_n^2}$. We note that ξ_n is determined by (q, ζ, τ) by $\xi_n = \sqrt{\tau^2 - |\zeta|^2}$, since it is inward pointing. The coordinates q, ζ are symplectic, so the symplectic form on $T^*\partial\Omega$ is $d\sigma = dq \wedge d\zeta$. Also, below when we write $|\beta^j(q, \zeta/\tau)|$ we mean the norm of the fiber component of $\beta^j(q, \zeta/\tau)$ or when we write $\tau\beta^j(q, \zeta/\tau)$ we mean that τ is multiplied in the fiber component only. We now relate the graph of the billiard flow (25) with initial and terminal point on the boundary to the billiard map (after j reflections) by the formula

$$\Phi^{T_j}(q, 0, \zeta, \xi_n) = \left(\tau\beta^j\left(q, \frac{\zeta}{\tau}\right), \xi_n'(q, \zeta, \xi_n) \right), \tag{26}$$

where $\xi_n' = \tau\sqrt{1 - \left|\beta^j(q, \zeta/\tau)\right|^2}$. We also put

$$\gamma(q, \zeta, \tau) = \sqrt{1 - \frac{|\zeta|^2}{\tau^2}} \quad \text{and} \quad \gamma_1(q, \zeta) = \sqrt{1 - |\zeta|^2}. \tag{27}$$

It is the homogeneous (of degree zero) analogue of the function denoted by γ in [Hassell and Zelditch 2004].

Further, we parametrize the canonical relation $\Gamma^j_{\partial,+}$ of Lemma 3 using the billiard map β and its powers. We define the j-th return time $T^j(q, \xi)$ of the billiard trajectory in a codirection $(q, \xi) \in T^*_q\Omega$ to be the length the j-link billiard trajectory starting at (q, ξ) and ending at a point $\Phi^{T^j(q,\xi)}(q, \xi) \in T^*_{\partial\Omega}\Omega$. It is the same as $T^j_+(q, \xi)$. Then we define

$$\begin{aligned} &\iota_{\partial,j,+} : \mathbb{R}_+ \times T^*\partial\Omega \to T^*(\mathbb{R} \times \partial\Omega \times \partial\Omega), \\ &\iota_{\partial,j,+}(\tau, q, \zeta) = \left(T^j(q, \xi(q, \zeta, \tau)), \tau, \left(\tau\beta^j\left(q, \frac{\zeta}{\tau}\right) \right), q, \zeta \right), \end{aligned} \tag{28}$$

where

$$\xi(q, \zeta, \tau) = \zeta + \xi_n \nu_q, \quad |\zeta|^2 + |\xi_n|^2 = \tau^2.$$

The map (28) parametrizes $\Gamma^j_{\partial,+}$ of Lemma 3.

Proposition 4. *In the coordinates* $(\tau, q, \zeta) \in \mathbb{R}_+ \times T^*\partial\Omega$ *of* (28), *the principal symbol of*

$$\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q) S_B^b(t, q', q)$$

on $\Gamma^j_{\partial,+}$ *is as follows*:

- *in the Dirichlet case*:

$$\sigma_{j,+}(q,\zeta,\tau)=C^D_{j,+}\chi_T\Big(q,\frac{\zeta}{\tau}\Big)\chi_T\Big(\beta^j\Big(q,\frac{\zeta}{\tau}\Big)\Big)\gamma^{1/2}(q,\zeta,\tau)\gamma^{1/2}\Big(\tau\beta^j\Big(q,\frac{\zeta}{\tau}\Big),\tau\Big)\tau\,|dq\wedge d\zeta\wedge d\tau|^{1/2};$$

- *in the Neumann case*:

$$\sigma_{j,+}(q,\zeta,\tau)=C^N_{j,+}\chi_T\Big(q,\frac{\zeta}{\tau}\Big)\chi_T\Big(\beta^j\Big(q,\frac{\zeta}{\tau}\Big)\Big)\gamma^{-1/2}(q,\zeta,\tau)\gamma^{-1/2}\Big(\tau\beta^j\Big(q,\frac{\zeta}{\tau}\Big),\tau\Big)\times\Big(\Big\langle\zeta,\beta^j\Big(q,\frac{\zeta}{\tau}\Big)\Big\rangle-\tau\Big)\,|dq\wedge d\zeta\wedge d\tau|^{1/2},\quad(29)$$

where the $C^B_{j,+}$ *are certain constants* (*Maslov factors*).

Proof. We only show the computations in the Dirichlet case. The Neumann case is very similar and uses (4) which will produce an additional factor of $\tau\langle\zeta,\beta^j(q,\zeta/\tau)\rangle-\tau^2$.

By Lemma 2, the principal symbol of $S_B(t)$ consists of four pieces at the boundary, one for each mode $A^j_\pm$, $\widehat{A}^j_\pm$. The symbol for the $-$ mode of propagation is equal to that for the $+$ mode of propagation under the time reversal map $\xi\to-\xi$. Further by part 2 of Lemma 2, the symbol at the boundary (adjusted by taking normal derivatives in the Dirichlet case) is invariant under the reflection map $\xi\to\hat{\xi}$ at the boundary due to the boundary conditions. Hence we only calculate the A^j_+ component and use the invariance properties to calculate the symbol on the other components.

We therefore assume that the symbol of S_B is $1/2\tau$ times the graph half-density $|dt\wedge dx\wedge d\xi|^{1/2}$ on Γ^j_+. We need to compose this graph half-density on the left by the symbol $\xi_n\,|dq\wedge d\zeta\wedge d\xi_n|^{1/2}$ of $r^D=r\,N$, and on the right by the symbol $\xi'_n\,|dq'\wedge d\zeta'\wedge d\xi'_n|^{1/2}$ of the adjoint $r^{D*}=N^*r^*$. Therefore we compute the restriction of the Γ^j_+ component onto $\Gamma^j_{\partial,+}$ and we remember to multiply the symbol by $\xi_n\xi'_n=\tau^2\gamma(q,\zeta,\tau)\gamma(\tau\beta^j(q,\frac{\zeta}{\tau}),\tau))$ and also by $1/2\tau$ at the end.

It is simplest to use symbol algebra and pullback formulae to calculate it [Duistermaat and Guillemin 1975]. One can also try to compute the symbol of this composition directly by using the oscillatory integral representations of these operators but that computation is more complicated. The composition is equivalent to the pullback of the symbol under the pullback

$$\Gamma^j_\partial=(i_{\partial\Omega}\times i_{\partial\Omega})^*\Gamma^j,\tag{30}$$

of the canonical relation of the S_B by the canonical inclusion map

$$i_{\partial\Omega}\times i_{\partial\Omega}:\mathbb{R}\times\partial\Omega\times\partial\Omega\to\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n.$$

We recall that a map $f:X\to Y$ is transversal to $W\subset T^*Y$ if $df^*\eta\neq0$ for any $\eta\in W$. If $f:X\to Y$ is smooth and $\Gamma\subset T^*Y$ is Lagrangian, and if f and $\pi:T^*Y\to Y$ are transverse then $f^*\Gamma$ is Lagrangian. Since

$$(i_{\partial\Omega}\times i_{\partial\Omega})^*(t,\tau,\Phi^t(q,\xi),q,\xi)=(t,\tau,\Phi^t(q,\xi)|_{T\partial\Omega},q,\xi|_{T\partial\Omega})$$

at a point over $(i_{\partial\Omega}\times i_{\partial\Omega})(t,q',q)$, and since $\tau=|\xi|\neq0$, it is clear that $i_{\partial\Omega}\times i_{\partial\Omega}$ is transversal to π.

We now claim that on the pullback of Γ^j, using the parametrization (28),

$$(i_{\partial\Omega} \times i_{\partial\Omega})^* |dt \wedge dx \wedge d\xi|^{1/2} = \gamma^{-1/2}(q, \zeta, \tau)\gamma^{-1/2}\Big(\tau\beta^j\Big(q, \frac{\zeta}{\tau}\Big), \tau\Big)\, |dq \wedge d\zeta \wedge d\tau|^{1/2}, \tag{31}$$

where γ is defined in (27). To see this, we use the pullback diagram

$$\begin{tikzcd}
\Gamma^j \arrow[d, "i"'] & F \arrow[l, "\pi"'] \arrow[r, "\alpha"] \arrow[d, "\pi"] & (i_{\partial\Omega} \times i_{\partial\Omega})^*\Gamma^j \subset T^*(\mathbb{R} \times \partial\Omega \times \partial\Omega) \\
T^*(\mathbb{R} \times \mathbb{R}^n \times \Omega) & \mathcal{N}^*(\mathrm{graph}(i_{\partial\Omega} \times i_{\partial\Omega})) \arrow[l, "\pi"'] &
\end{tikzcd}$$

Here, F is the fiber product, $\mathcal{N}^*\,\mathrm{graph}(i_{\partial\Omega} \times i_{\partial\Omega})$ is the conormal bundle to the graph, and the map $\alpha : F \to (i_{\partial\Omega} \times i_{\partial\Omega})^*\Gamma^j$ is the natural projection to the composition [Duistermaat and Guillemin 1975]. Since the composition is transversal, $D\alpha$ is an isomorphism [loc. cit.]. The graph of $i_{\partial\Omega} \times i_{\partial\Omega}$ is the set $\{(t, q, q', t, q, q') : (t, q, q') \in \mathbb{R} \times \partial\Omega \times \partial\Omega\}$ and its conormal bundle is (in the Fermi normal coordinates),

$$\begin{aligned}\mathcal{N}^*(\mathrm{graph}(i_{\partial\Omega} \times i_{\partial\Omega})) = \big\{&(t, \tau, q, \zeta, q', \zeta', t, -\tau, q, -\zeta+\xi_n, q', -\zeta'+\xi_n'),\ (q, \zeta, \xi_n),\ (q', \zeta', \xi_n') \in T^*_{\partial\Omega}\mathbb{R}^n\big\} \\ &\subset T^*(\mathbb{R} \times \partial\Omega \times \partial\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n).\end{aligned}$$

The half-density produced by the pullback diagram takes the exterior tensor product of the canonical half-density

$$\big|dt \wedge d\tau \wedge dq \wedge d\zeta \wedge d\xi_n \wedge d\xi_n' \wedge dq' \wedge d\zeta'\big|^{1/2}$$

on $\mathcal{N}^*(\mathrm{graph}(i_{\partial\Omega} \times i_{\partial\Omega}))$ and

$$|dt' \wedge dx' \wedge d\xi'|^{1/2} \quad \text{on } \Gamma^j \subset T^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$$

at a point of the fiber product (where the $T^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ components are equal) and divides by the canonical half-density

$$\big|dt' \wedge d\tau' \wedge dq' \wedge d\zeta' \wedge dx_n' \wedge d\xi_n' \wedge dx' \wedge d\xi'\big|^{1/2}$$

on the common $T^*\mathbb{R} \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ component.

Since $\tau' = \tau$, the factors of $|dt' \wedge d\tau' \wedge dq' \wedge d\zeta' \wedge d\xi_n' \wedge dx' \wedge d\xi'|^{1/2}$ cancel in the quotient half-density, leaving the half-density

$$\frac{|dt \wedge dq \wedge d\zeta \wedge d\xi_n|^{1/2}}{|dx_n'|^{1/2}}$$

on the composite. The numerator is a half-density on $\mathbb{R} \times T^*_{\partial\Omega}\mathbb{R}^n$. We write it more intrinsically in the following lemma. Note that it explains the first of our two γ factors.

Lemma 5. *Let $\Phi = \Phi_+$ be the parametrization* (25). *Then*

$$|dt \wedge dq \wedge d\zeta \wedge d\xi_n|^{1/2} = \left|\frac{\xi_n}{\sqrt{|\zeta|^2 + \xi_n^2}}\right|^{-1/2} \big|\Phi^*\Omega_{T^*\mathbb{R}^n}\big|^{1/2}$$

as half-densities on $\mathbb{R} \times T^*_{\partial\Omega}\mathbb{R}^n$.

Proof. We have

$$\begin{aligned}\frac{\Phi^*\Omega_{T^*\mathbb{R}^n}}{dt\wedge dq\wedge d\zeta\wedge d\xi_n} &= \Omega_{T^*\mathbb{R}^n}\Big(\frac{d}{dt}\Phi^t(q,\zeta,\xi_n), d\Phi^t\frac{\partial}{\partial q_j}, d\Phi^t\frac{\partial}{\partial \zeta_j}, d\Phi^t\frac{\partial}{\partial \xi_n}\Big)\\ &= \Omega_{T^*\mathbb{R}^n}\Big(H_g, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial \zeta_j}, \frac{\partial}{\partial \xi_n}\Big)\\ &= \frac{\xi_n}{\sqrt{|\zeta|^2+\xi_n^2}}\Omega_{T^*\mathbb{R}^n}\Big(\frac{\partial}{\partial x_n}, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial \zeta_j}, \frac{\partial}{\partial \xi_n}\Big) = \frac{\xi_n}{\sqrt{|\zeta|^2+\xi_n^2}},\end{aligned}$$

since

$$\frac{d}{dt}\Phi^t(q,\eta,\xi_n) = H_g = \frac{\xi_n}{\sqrt{|\zeta|^2+\xi_n^2}}\frac{\partial}{\partial x_n}+\cdots$$

is the Hamilton vector field of $g=\sqrt{g^2}$, $g^2=\xi_n^2+(g')^2$ where $\cdots$ represent vector fields in the span of $\partial/\partial q_j,\ \partial/\partial\zeta_j,\ \partial/\partial\xi_n$. Finally, we use that $d\Phi^t$ is a symplectic linear map and that q, x_n, ζ, ξ_n are symplectic coordinates. Note that we have evaluated the symplectic volume form at the domain point, not the image point. $\square$

Next we consider the points in the image of Φ on $\mathbb{R}\times T^*_{\partial\Omega}\mathbb{R}^n$ where $x_n'=0$ and take the quotient by $|dx_n'|^{1/2}$, resulting in a half-density on Γ_∂^j. The next result explains the origin of the second γ factor.

Lemma 6. *In the subset $\Gamma_\partial^j\subset\Phi(\mathbb{R}\times T^*_{\partial\Omega}\mathbb{R}^n)$ where $x_n'=0$ and where $t=T^j$, we have (in the parametrizing coordinates* (28)),

$$\frac{|dt\wedge dq\wedge d\zeta\wedge d\xi_n|^{1/2}}{|dx_n'|^{1/2}} = \big|((\beta^j)^*\gamma^{-1})\,dq\wedge d\eta\wedge d\tau\big|^{1/2}.$$

Proof. By Lemma 5, it suffices to rewrite

$$|dx_n'|^{-1/2}\,\big|\Phi^*\Omega_{T^*\mathbb{R}^n}\big|^{1/2}$$

in the coordinates (τ,q,η) of $\iota_{\partial,j,+}$ in (28). We observe that $x_n'=\Phi^*x_n$. Hence

$$|dx_n'|^{-1/2}\big|\Phi^*\Omega_{T^*\mathbb{R}^n}\big|^{1/2} = \left|\Phi^*\frac{\Omega_{T^*\mathbb{R}^n}}{|dx_n|}\right|^{1/2} = \big|((\beta^j)^*\gamma^{-1})\,dq\wedge d\zeta\wedge d\tau\big|^{1/2}.$$

In the last equality, we have used (26), the equality $\dfrac{\Omega_{T^*\mathbb{R}^n}}{|dx_n|}=|dq\wedge d\zeta\wedge d\xi_n|$, and the fact that β is symplectic. Indeed, by (26),

$$\begin{aligned}\Phi^*(dq\wedge d\zeta\wedge d\xi_n) &= \Big(\tau(\beta^j)^*\Big(dq\wedge d\frac{\zeta}{\tau}\Big)\wedge\Phi^* d\xi_n\Big)\\ &= \Big(\tau(\beta^j)^*\Big(dq\wedge d\frac{\zeta}{\tau}\Big)\wedge\Phi^* d\sqrt{\tau^2-|\zeta|^2}\Big)\\ &= dq\wedge d\zeta\wedge\Phi^*\frac{\tau\,d\tau}{\sqrt{\tau^2-|\zeta|^2}} = \big((\beta^j)^*\gamma^{-1}\big)\,dq\wedge d\zeta\wedge d\tau.\end{aligned}$$

Note that $\tau(\beta^j)^*(dq\wedge d\frac{\zeta}{\tau}) = dq\wedge d\zeta|_{\beta^j(q,\zeta)}$. $\square$

Combining Lemma 6 with Lemma 5 completes the proof of (31) and Proposition 4. $\square$

2.6. ***Trace along the boundary: composition with $\pi_*\Delta^*$.*** We now take the trace along the boundary of this operator. Analogously to [Duistermaat and Guillemin 1975; Guillemin and Melrose 1979a; Marvizi and Melrose 1982], we define $\Delta : \mathbb{R} \times \partial\Omega \to \mathbb{R} \times \partial\Omega \times \partial\Omega$ to be the diagonal embedding and π_* to be integration over $\partial\Omega$.

Lemma 7. *If the fixed point sets of period T of β^k are clean for all k and form a submanifold F_T of $B^*\partial\Omega$ of dimension d (with connected components Γ), then*

$$\pi_*\Delta^*\dot{\rho}\,\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S_B^b(t, q', q) \in I^{(d/2)+(1/2)+1-(1/4)}(\mathbb{R},\ T_T^*\mathbb{R}),$$

where

$$T_T^*\mathbb{R} = \bigcup\nolimits_\pm \Lambda_{T,\pm} = \bigcup\nolimits_\pm \{(T, \pm\tau) : \tau \in \mathbb{R}_+\},$$

and its principal symbol on $\Lambda_{T,\pm}$ is given by

$$c^\pm \tau^{(d+2)/2}\sqrt{d\tau},$$

where

$$c^\pm = \sum_{\Gamma\subset F_T} C_\Gamma^\pm \int_\Gamma \dot{\rho}\,\gamma_1\,d\mu_\Gamma$$

and $c^- = \overline{c^+}$ the complex conjugate of c^+.

Proof. The calculation of the principal symbol of the trace of a Fourier integral operator in [Duistermaat and Guillemin 1975] is valid for the boundary restriction of the wave kernel, since it only uses that it is $\pi_*\Delta^*$ composed with a Fourier integral kernel with a known symbol and canonical relation. Hence we follow the proof closely and refer there for further details.

As in [Guillemin and Melrose 1979a], the composition of $\pi_*\Delta^*$ with

$$\dot{\rho}\chi_T(D_t, q', D_{q'})\chi_T(D_t, q, D_q)S_B^b(t, q, q') \tag{32}$$

is clean if and only if the fixed point set of β^k corresponding to periodic orbits of period T is clean. When the fixed point set has dimension d in the ball bundle $B^*\partial\Omega$, composition with $\pi_*\Delta^*$ adds $d/2$ to the order [Duistermaat and Guillemin 1975, (6.6)]. Combining with Lemma 3, we obtain the order

$$\frac{d}{2} + \frac{1}{2} + 1 - \frac{1}{4}.$$

Hence under the cleanliness assumption, it follows that $\delta\,\mathrm{Tr}\cos t\sqrt{-\Delta_B}$ is a Lagrangian distribution on $\mathbb{R}$ with singularities at $t \in \mathrm{Lsp}(\Omega)$. As discussed in [loc. cit.] for the upper/lower half lines $\Lambda_{T,\pm}$ in $T_T^*\mathbb{R}$, $I^{\frac{d}{2}+\frac{5}{4}}(\mathbb{R}, \Lambda_{T,\pm})$ consists of multiples of the distribution

$$\int_0^\infty \tau^{(d+2)/2} e^{\pm i\tau(t-T)}d\tau = (t - T \pm i0)^{-(d+4)/2}.$$

The principal symbol of this Fourier integral distribution is $\tau^{(d+2)/2}\sqrt{d\tau}$. Therefore to conclude the Lemma we only need to compute the coefficients of this symbol in the trace.

This coefficient is computed in a universal way from the principal symbol of (32) computed from Proposition 4. Following the proof in [loc. cit.], the coefficient of $\tau^{(d+2)/2}\sqrt{d\tau}$ is

$$c^{\pm} = \sum_{\Gamma \subset F_T} C_{\Gamma}^{\pm} \int_{\Gamma} \dot{\rho}\, \gamma_1 \, d\mu_{\Gamma},$$

where F_T is the fixed point set of β (and its powers) in $B^*\partial\Omega$. The sum is over the connected components Γ of F_T. Here, $d\mu_\Gamma$ is the restriction to Γ of a density $d\mu$ on F_T which is the pushforward (under the natural projection map) of the canonical density defined on the fixed point set of Φ^T on $S^*_{\partial\Omega}\Omega$. This canonical density is defined in Lemma 4.2 of [Duistermaat and Guillemin 1975]. We note that the distribution $c^+(t-T+i0)^{-(d+4)/2} + c^-(t-T-i0)^{-(d+4)/2}$ is real only if $c^- = \bar{c^+}$. This completes the proof of the lemma. □

The lemma also completes the proof of the Theorem 2.

Remark. As a check on the order, we note that for the wave trace in the interior and for nondegenerate closed trajectories, the singularities are of order $(t-T+i0)^{-1}$. When the periodic orbits are degenerate and the unit vectors in the fixed point sets have dimension d, the singularity increases to order

$$(t-T+i0)^{-1-\frac{d}{2}}.$$

If we formally take the variation of the wave trace, the singularity should increase to order

$$(t-T+i0)^{-1-\frac{d}{2}-1}.$$

In comparison, the boundary trace in the Dirichlet case involves two extra derivatives of the wave kernel and composition with $(-\Delta)^{-1/2}$. Compared to the interior trace, this adds one net derivative and order to the trace singularity. We claim that the restriction to the boundary does not further change the order compared to the interior trace. This can be seen by considering the method of stationary phase for oscillatory integrals with Bott–Morse phase functions, whose nondegenerate critical manifolds are transverse to the boundary. If we restrict the integral to the boundary, we do not change the number of phase variables in the integral, but we simultaneously decrease the number of variables by one and the dimension of the fixed point set by one. The number of nondegenerate directions stays the same. It follows that the singularity order of the variational trace goes up by one overall unit compared to the interior trace, consistently with the formal variational calculation.

3. Case of the ellipse and the proof of Theorem 1

In this section we let Ω_0 be an ellipse. In this case, the fixed point sets are clean fixed point sets for Φ^t in $T^*\Omega_0$ and for β in $B^*\partial\Omega_0$ [Guillemin and Melrose 1979a, Proposition 4.3]. In fact the fixed point sets F_T of β in $B^*\partial\Omega_0$ form a one dimensional manifold. Thus $d = 1$ and Corollary 2 follows.

As is well-known, both the billiard flow and billiard map of the ellipse are completely integrable. In particular, except for certain exceptional trajectories, the periodic points of period T form a Lagrangian tori in $S^*\Omega_0$, and the homogeneous extensions of the Lagrangian tori are cones in $T^*\Omega_0$. The exceptions are the two bouncing ball orbits through the major/minor axes and the trajectories which intersect the foci or glide along the boundary. The fixed point sets of Φ^T intersect the coball bundle $B^*\partial\Omega_0$ of the boundary in the fixed point sets of the billiard map $\beta : B^*\partial\Omega_0 \to B^*\partial\Omega_0$ (for background we refer to [Petkov and Stoyanov 1992; Guillemin and Melrose 1979a; 1979b; Hassell and Zelditch 2004; Toth and Zelditch 2012] for instance). Except for the exceptional orbits, the fixed point sets are real analytic curves. For the bouncing ball rays, the associated fixed point sets are nondegenerate fixed points of β.

Since the final step of the proof uses results of [Guillemin and Melrose 1979a], we briefly review the description of the billiard map of the ellipse $\Omega_0 := x^2/a + y^2/b = 1$ (with $a > b > 0$) in that article. In the interior, there exist for each $0 < Z \le b$ a caustic set given by a confocal ellipse

$$\frac{x^2}{E+Z} + \frac{y^2}{Z} = 1,$$

where $E = a - b$, or for $-E < Z < 0$ by a confocal hyperbola. Let (q, ζ) be in $B^*\partial\Omega_0$ and let (q, ξ) in $S^*\Omega_0$ be the unique inward unit normal to boundary that projects to (q, ζ). The line segment $(q, r\xi)$ will be tangent to a unique confocal ellipse or hyperbola (unless it intersects the foci). We then define the function $Z(q, \zeta)$ on $B^*\partial\Omega_0$ to be the corresponding Z. Then Z is a β-invariant function and its level sets $\{Z = c\}$ are the invariant curves of β. The invariant Leray form on the level set is denoted by du_Z [loc. cit., (2.17)]; thus the symplectic form of $B^*\partial\Omega_0$ is $dq \wedge d\zeta = dZ \wedge du_Z$. A level set has a rotation number and the periodic points live in the level sets with rational rotation number. As it is explained in [loc. cit., p. 143] the Leray form du_Z restricted to a connected component Γ of F_T is a constant multiple of the canonical density $d\mu_\Gamma$.

As mentioned in the introduction, the well-known obstruction to using trace formula calculations such as in Theorem 2 is multiplicity in the length spectrum, that is, existence of several connected components of F_T. A higher dimensional component is not itself a problem, but there could exist cancellations among terms coming from components with different Morse indices, since the coefficients C_Γ are complex. This problem arose earlier in the spectral theory of the ellipse in [loc. cit.]. The key Proposition 4.3 there shows that there is a sufficiently large set of lengths T for which F_T has one component up to $(q, \zeta) \to (q, -\zeta)$ symmetry. Since it is crucial here as well, we state the relevant part:

Proposition 8 [Guillemin and Melrose 1979a, Proposition 4.3]. *Let $T_0 = |\partial\Omega_0|$. Then for every interval $(mT_0 - \epsilon,\ mT_0)$, for $m = 1, 2, 3, \dots$, there exist infinitely many periods $T \in \mathrm{Lsp}(\Omega_0)$ for which F_T is the union of two invariant curves which are mapped to each other by $(q, \zeta) \to (q, -\zeta)$.*

Since for an isospectral deformation $\delta \operatorname{Tr} \cos(t\sqrt{-\Delta}) = 0$, we obtain from Theorem 2:

Corollary 9. *Suppose we have an isospectral deformation of an ellipse Ω_0 with velocity $\dot\rho$. Then for each T in Proposition 8 for which F_T is the union of two invariant curves Γ_1 and Γ_2 which are mapped to each other by $(q, \zeta) \to (q, -\zeta)$ we have*

$$\int_{\Gamma_j} \dot\rho\, \gamma_1\, du_Z = 0, \quad j = 1, 2.$$

Proof. From Theorem 2 we get

$$\Re\left\{\left(\sum_{j=1}^{2} C_{\Gamma_j}\int_{\Gamma_j}\dot\rho\,\gamma_1\,d\mu_{\Gamma_j}\right)(t-T+i0)^{-2-(d/2)}\right\}=0.$$

Since $\dot\rho$ and γ_1 are invariant under the time reversal map $(q,\zeta)\to(q,-\zeta)$, the two integrals are identical. Also by directly looking at the stationary phase calculations it can be shown that the Maslov coefficients C_{Γ_1} and C_{Γ_2} are also the same. Thus the corollary follows. □

3.1. *Abel transform.* The remainder of the proof of Theorem 1 is identical to that of Theorem 4.5 of [Guillemin and Melrose 1979a] (see also [Popov and Topalov 2003]). For the sake of completeness, we sketch the proof.

Proposition 10. *The only $\mathbb{Z}_2\times\mathbb{Z}_2$ invariant function $\dot\rho$ satisfying the equations of Corollary 9 is $\dot\rho=0$.*

Proof. First, we may assume $\dot\rho=0$ at the endpoints of the major/minor axes, since the deformation preserves the $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry and we may assume that the deformed bouncing ball orbits will not move and are aligned with the original ones. Thus $\dot\rho(\pm\sqrt{a})=\dot\rho(\pm\sqrt{b})=0$.

The Leray measure may be explicitly evaluated [Guillemin and Melrose 1979a, eq. 2.18]. By a change of variables with Jacobian J, and using the symmetric properties of $\dot\rho$, the integrals become

$$A(Z)=\int_b^a \frac{\dot\rho(t)\,\gamma_1\,J(t)\,dt}{\sqrt{t-(b-Z)}}, \tag{33}$$

for an infinite sequence of Z accumulating at b. The function $A(Z)$ is smooth in Z for Z near b. It vanishes infinitely often in each interval $(b-\epsilon,b)$, hence is flat at b. The k-th Taylor coefficient at b is

$$A^{(k)}(b)=\int_b^a \dot\rho(t)\,\gamma_1\,J(t)t^{-k-(1/2)}dt=0. \tag{34}$$

Since the functions t^{-k} span a dense subset of $C[b,a]$, it follows that $\dot\rho\equiv 0$. □

3.2. *Infinitesimal rigidity and flatness.* We now show that infinitesimal rigidity implies flatness and prove Corollary 1. As mentioned, the Hadamard variational formula is valid for any C^1 parametrization $\Omega_{\alpha(\epsilon)}$ of the domains Ω_ϵ. For each one we have $\delta\rho_{\alpha(\epsilon)}(x)\equiv 0$.

Assume $\rho_\epsilon(x)$ is not flat at $\epsilon=0$ and let ϵ^k be the first nonvanishing term in the Taylor expansion of $\rho_\epsilon(x)$ at $\epsilon=0$. Then

$$\rho_\epsilon(x)=\epsilon^k\frac{\rho^{(k)}(x)}{k!}+\epsilon^{k+1}\frac{\rho^{(k+1)}(x)}{(k+1)!}+\cdots. \tag{35}$$

We then reparametrize the family by $\epsilon\to\alpha(\epsilon):=\epsilon^{1/k}$ so that

$$\rho_{\alpha(\epsilon)}(x)=\frac{\rho^{(k)}(x)}{k!}\epsilon+O(\epsilon^{1+1/k}).$$

By Hadamard's variational formulae we get $\delta\rho_{\alpha(\epsilon)}(x)=\rho^{(k)}(x)\equiv 0$, a contradiction.

Acknowledgement

The authors would like to thank the anonymous referee for useful comments and remarks.

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Received 4 May 2011. Revised 11 Jul 2012. Accepted 6 Aug 2012.

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APDE peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

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A NON-PROFIT CORPORATION
Typeset in LaTeX

ANALYSIS & PDE

Volume 5 No. 5 2012

2157-5045(2012)5:5;1-F