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SOME RESULTS ON SCATTERING FOR LOG-SUBCRITICAL AND LOG-SUPERCRITICAL NONLINEAR WAVE EQUATIONS

HSI-WEI SHIH

We consider two problems in the asymptotic behavior of semilinear second order wave equations. First, we consider the $\dot{H}^1_x \times L^2_x$ scattering theory for the energy log-subcritical wave equation

$$\Box u = |u|^4 u g(|u|)$$

in $\mathbb{R}^{1+3}$, where g has logarithmic growth at 0. We discuss the solution with general (respectively spherically symmetric) initial data in the logarithmically weighted (respectively lower regularity) Sobolev space. We also include some observation about scattering in the energy subcritical case. The second problem studied involves the energy log-supercritical wave equation

$$\Box u = |u|^4 u \log^{\alpha}(2+|u|^2) \quad \text{for } 0 < \alpha \leq \tfrac{4}{3}$$

in $\mathbb{R}^{1+3}$. We prove the same results of global existence and $(\dot{H}^1_x \cap \dot{H}^2_x) \times H^1_x$ scattering for this equation with a slightly higher power of the logarithm factor in the nonlinearity than that allowed in previous work by Tao (*J. Hyperbolic Differ. Equ.*, **4**:2 (2007), 259–265).

1. Introduction

Consider the semilinear wave equation

$$\begin{aligned} \Box u := -\partial_t^2 u + \Delta u &= f(u) \quad \text{on } \mathbb{R} \times \mathbb{R}^3, \\ u(0,x) &= u_0(x), \\ \partial_t u(0,x) &= u_1(x), \end{aligned} \tag{1}$$

where f is a complex-valued function. Let the potential function $F : \mathbb{C} \to \mathbb{R}$ be a real-valued function such that

$$2F_{\bar{z}}(z) = f(z), \tag{2}$$

with $F(0) = 0$ and u being the solution to (1) with initial data $u_0 \in \dot{H}^1_x \cap \{\phi : \int_{\mathbb{R}^3} F(\phi)\, dx < \infty\}$ and $u_1 \in L^2_x$. We can easily verify that the equation has conserved energy

$$E(u)(t) := \int_{\mathbb{R}^3} \tfrac{1}{2}|u_t(t,x)|^2 + \tfrac{1}{2}|\nabla u(t,x)|^2 + F(u(t,x))\, dx. \tag{3}$$

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The main goal of the paper is to study the $\dot{H}^1_x \times L^2_x$ scattering theory for log-subcritical wave equations with finite energy initial data, where the energy is defined by (3). In this paper, the term *log-subcritical wave equation* refers to (1) with f defined by

$$f(z) := \begin{cases} |z|^4 z g(|z|), & |z| \neq 0, \\ 0, & |z| = 0, \end{cases} \tag{4}$$

where $g : (0, \infty) \to \mathbb{R}$ is smooth, nonincreasing, and satisfies

$$g(x) := \begin{cases} -\log x, & 0 < x < \frac{1}{3}, \\ \sim 1, & \frac{1}{3} \leq x < 1, \\ 1, & x \geq 1. \end{cases} \tag{5}$$

We also prove global existence in the case of spherical symmetry for *log-supercritical wave equations*, by which we mean equations of the form

$$\Box u = |u|^4 u \log^\alpha(2 + |u|^2) \tag{6}$$

In this paper, we will allow $0 < \alpha \leq \frac{4}{3}$, extending the range $0 < \alpha \leq 1$ allowed in [Tao 2007]. We also assume that the initial data is in the energy space, the set of data for which the energy (3) is finite.

Remark 1.1. We can easily compute that the potential function of log-subcritical wave equations (1), (4), and (5) is

$$F_{\text{sub}}(z) = \begin{cases} -\frac{1}{6}|z|^6\big(\log(|z|) - \frac{1}{6}\big), & 0 < |z| < \frac{1}{3}, \\ \sim \frac{1}{6}|z|^6, & \frac{1}{3} \leq |z| < 1, \\ \frac{1}{6}|z|^6, & |z| \geq 1, \end{cases} \tag{7}$$

and the potential function of the log-supercritical wave equations (6) is

$$F_{\text{sup}}(z) \sim |z|^6 \log^\alpha(2 + |z|^2). \tag{8}$$

We quickly recall some common terminology associated to the scaling properties of (1). Consider $f(z) = |z|^{p-1} z$ and let u be the solution of (1). By scaling, $\lambda^{2/(1-p)} u(t/\lambda, x/\lambda)$ is also a solution with initial data $\lambda^{2/(1-p)} u_0(t_0/\lambda, x/\lambda)$ and $\lambda^{(1+p)/(1-p)} u_1(t_0/\lambda, x/\lambda)$. Hence the scaling of u preserves the homogeneous Sobolev norm $\|u_0\|_{\dot{H}^{s_c}(\mathbb{R}^3)} + \|u_1\|_{\dot{H}^{s_c-1}(\mathbb{R}^3)}$ if

$$s_c := \frac{3}{2} - \frac{2}{p-1}, \qquad \text{or equivalently} \qquad p = 1 + \frac{4}{3 - 2s_c}.$$

Definition 1.2. For $f(z) = |z|^{p-1} z$ and a given value s, we call (1) an $\dot{H}^s_x$-critical (subcritical, supercritical) nonlinear wave equation if p equals (is less than, is greater than) $1 + 4/(3 - 2s)$. In particular, when $s = 1$, we call (1) an energy critical (subcritical, supercritical) nonlinear wave equation if $p = 5$ ($p < 5$, $p > 5$).

The results of global existence and uniqueness for the energy-critical ($\Box u = |u|^4 u$) and energy-subcritical ($\Box u = |u|^{p-1} u$, where $p < 5$) wave equations are already established by [Brenner and von Wahl 1981; Struwe 1988; Grillakis 1990; 1992; Shatah and Struwe 1993; 1994; Kapitanski 1994; Ginibre and Velo 1985]. It is natural to consider the decay of the solution, which we expect to behave linearly as $t \to \pm\infty$. The decay estimate and scattering theory (see section 2 for definition) of critical wave equations are shown in [Bahouri and Shatah 1998]; see also [Bahouri and Gérard 1999; Ginibre and Velo 1989; Nakanishi 1999]. Hidano [2001] (see also [Ginibre and Velo 1987]), by the property of conformal invariance, proved that the solutions for certain subcritical wave equations ($\frac{5}{2} < p \le 3$) scatter in the weighted Sobolev space $\Sigma := X \times Y$, where

$$X := H^1_x(\mathbb{R}^3) \cap \{\phi : \ |x|\nabla\phi \in L^2_x(\mathbb{R}^3)\}, \quad Y := L^2_x(\mathbb{R}^3) \cap \{\phi : \ |x|\phi \in L^2_x(\mathbb{R}^3)\}.$$

However, for energy subcritical equations, the $\dot{H}^1_x \times L^2_x$ scattering theory[1] still remains open. In this paper, we consider the solutions to the log-subcritical wave equations (1), (4), and (5) with finite energy initial data. The global existence result is established in [Grillakis 1990; 1992; Kapitanski 1994; Nakanishi 1999]. We will prove that the solutions with a class of initial data scatter in $\dot{H}^1_x \times L^2_x$. This class of data is contained in logarithmically weighted Sobolev spaces $X_1 \times Y_1$, where

$$\begin{aligned} X_1 &:= \dot{H}^1_x(\mathbb{R}^3) \cap \{\phi : \ \log^\gamma(1+|x|)\nabla\phi \in L^2_x(\mathbb{R}^3)\}, \\ Y_1 &:= L^2_x(\mathbb{R}^3) \cap \{\phi : \ \log^\gamma(1+|x|)\phi \in L^2_x(\mathbb{R}^3)\} \end{aligned} \tag{9}$$

for some $\gamma > \frac{1}{2}$. For initial data in these spaces, we show that the potential energy of the solution decays logarithmically for all large times. After dividing the time interval suitably, this decay helps us to control the key spacetime norm $\|f(u)\|_{L^1_t L^2_x}$. This spacetime bound implies scattering (we will sketch the proof in Section 2; see also [Bahouri and Shatah 1998]). Our proof of the spacetime bound involves establishing a decay rate for certain constant-time norms of the solution and a bootstrap scheme motivated by that in [Tao 2007]. We rely heavily on ideas from [Bahouri and Shatah 1998].

The second part of this paper considers the solution of log-subcritical wave equations with spherically symmetric data. We prove that the solution u with initial data in $X_2 \times Y_2$ scatters in $\dot{H}^1_x \times L^2_x$, where

$$X_2 := \dot{H}^1_x(\mathbb{R}^3) \cap \Big(\bigcup_{\delta>0} \dot{H}^{1-\delta}_x(\mathbb{R}^3)\Big), \quad Y_2 := L^2_x(\mathbb{R}^3) \cap \Big(\bigcup_{\delta>0} \dot{H}^{-\delta}_x(\mathbb{R}^3)\Big). \tag{10}$$

Our proof again uses the ideas from [Tao 2007] and the classical Morawetz inequality; see [Morawetz 1968]. However, we need a slightly sharpened version of the bootstrap argument. We also give remarks for some specific energy subcritical wave equations (see page 15 and following).

The third part of this paper studies global existence for log-supercritical wave equations. The global regularity of energy supercritical wave equations ($\Box u = |u|^{p-1}u$, where $p > 5$) is still open. In [Tao 2007], the author considered the log-supercritical wave equation

$$\Box u = u^5 \log^\alpha(2+u^2) \tag{11}$$

with spherically symmetric initial data and established a global regularity result for $0 < \alpha \le 1$. For general initial data, the same result for loglog-supercritical wave equations

$$\Box u = u^5 \log^c(\log(10+u^2))$$

[1] $\dot{H}^1_x \times L^2_x$ scattering is defined in Definition 2.1.

with $0 < c < \frac{8}{225}$ is obtained in [Roy 2009]. In the present paper, we extend the result in [Tao 2007] to the range $0 < \alpha \leq \frac{4}{3}$, again for spherically symmetric data. This improvement is attained by employing the potential energy bound in place of the kinetic energy bound used in [Tao 2007] for pointwise control.

2. Definitions, notation, and preliminaries

Throughout this paper, we use $M \lesssim N$ to denote the estimate $M \leq CN$ for some absolute constant C (which can vary from line to line).

We use $L_t^q L_x^r$ to denote the spacetime norm

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} := \left(\int_I \left(\int_{\mathbb{R}^3} |u(t,x)|^r \, dx\right)^{q/r} dt\right)^{1/q}$$

with the usual modifications when q or r is equal to infinity.

Definition 2.1. We say that a global solution $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ to (1) scatters in $\dot{H}_x^1 \times L_x^2$ (or $\dot{H}_x^1 \times L_x^2$ scattering) as $t \to +\infty$ $(-\infty)$ if there exists a linear solution v^+ (v^-) with initial data in $\dot{H}_x^1 \times L_x^2$ such that

$$\|u(t,x) - v^+(t,x)\|_{\dot{H}_x^1 \times L_x^2} \to 0 \quad \text{as } t \to +\infty$$
$$(\|u(t,x) - v^-(t,x)\|_{\dot{H}_x^1 \times L_x^2} \to 0 \quad \text{as } t \to -\infty).$$

Remark 2.2. We will sketch here that the *spacetime bound*,

$$\|f(u)\|_{L_t^1 L_x^2([t_0,\infty)\times\mathbb{R}^3)} < \infty \tag{12}$$

for some $t_0 > 0$, of the solution u to (1) implies the $\dot{H}_x^1 \times L_x^2$ scattering (as $t \to \infty$). Let

$$u \in C_t^1(\mathbb{R}, \dot{H}_x^1(\mathbb{R}^3)) \cap C_t^0(\mathbb{R}, L_x^2(\mathbb{R}^3))$$

be the solution to (1) and let v satisfy $\Box v = 0$ with initial data $v_0 \in \dot{H}_x^1(\mathbb{R}^3)$, $v_1 \in L_x^2(\mathbb{R}^3)$ (to be chosen shortly). By Duhamel's formula,

$$u(t,x) = \cos\big(t\sqrt{-\Delta}\big)u_0(x) + \frac{\sin\big(t\sqrt{-\Delta}\big)}{\sqrt{-\Delta}}u_1(x) - \int_0^t \frac{\sin\big((t-\tau)\sqrt{-\Delta}\big)}{\sqrt{-\Delta}} f(u(\tau))\, d\tau \tag{13}$$

and

$$v(t,x) = \cos\big(t\sqrt{-\Delta}\big)v_0(x) + \frac{\sin\big(t\sqrt{-\Delta}\big)}{\sqrt{-\Delta}}v_1(x), \tag{14}$$

where the operators $\cos\big(t\sqrt{-\Delta}\big)$ and $\sin\big(t\sqrt{-\Delta}\big)/\sqrt{-\Delta}$ are defined by

$$\big(\cos\big(t\sqrt{-\Delta}\big)\phi\big)\widehat{\ }(\xi) = \cos(t|\xi|)\hat{\phi}(\xi)$$

and

$$\left(\frac{\sin\big(t\sqrt{-\Delta}\big)}{\sqrt{-\Delta}}\phi\right)\widehat{\ }(\xi) = \frac{\sin(t|\xi|)}{|\xi|}\hat{\phi}(\xi).$$

Hence, that the solution u scatters and asymptotically approaches v in $\dot{H}^1_x \times L^2_x$ means that

$$\left\| \cos\big(t\sqrt{-\Delta}\big)(u_0 - v_0) + \frac{\sin\big(t\sqrt{-\Delta}\big)}{\sqrt{-\Delta}}(u_1 - v_1) - \int_0^t \frac{\sin\big((t-\tau)\sqrt{-\Delta}\big)}{\sqrt{-\Delta}} f(u(\tau))\, d\tau \right\|_{\dot{H}^1_x \times L^2_x} \to 0 \quad (15)$$

as $t \to \infty$. From basic trigonometric identities, we can verify that (15) is implied by

$$\left\| (u_0 - v_0) + \int_0^t \frac{\sin\big(-\tau\sqrt{-\Delta}\big)}{\sqrt{-\Delta}} f(u(\tau))\, d\tau \right\|_{\dot{H}^1_x} \to 0$$

and

$$\left\| (u_1 - v_1) + \int_0^t \cos\big(-\tau\sqrt{-\Delta}\big) f(u(\tau))\, d\tau \right\|_{L^2_x} \to 0$$

as $t \to \infty$. Therefore, if

$$\left(\int_0^t \frac{\sin\big(-\tau\sqrt{-\Delta}\big)}{\sqrt{-\Delta}} f(u(\tau))\, d\tau,\ \int_0^t \cos\big(-\tau\sqrt{-\Delta}\big) f(u(\tau))\, d\tau \right) \quad (16)$$

converges in $\dot{H}^1_x \times L^2_x$ as $t \to \infty$, and we take

$$v_0(x) := u_0(x) - \int_0^\infty \frac{\sin\big(-\tau\sqrt{-\Delta}\big)}{\sqrt{-\Delta}} f(u(\tau))\, d\tau,$$
$$v_1(x) := u_1(x) - \int_0^\infty \cos\big(-\tau\sqrt{-\Delta}\big) f(u(\tau))\, d\tau,$$

in (14), we then have, by (13), (14), and elementary trigonometric formulas,

$$\begin{aligned} \|u - v\|_{\dot{H}^1_x \times L^2_x} &= \Bigg\| -\int_0^t \frac{\sin\big((t-\tau)\sqrt{-\Delta}\big)}{\sqrt{-\Delta}} f(u(\tau))\, d\tau + \int_0^\infty \frac{\cos\big(t\sqrt{-\Delta}\big)\sin\big(-\tau\sqrt{-\Delta}\big)}{\sqrt{-\Delta}} f(u(\tau))\, d\tau \\ &\qquad + \int_0^\infty \frac{\sin\big(t\sqrt{-\Delta}\big)\cos\big(-\tau\sqrt{-\Delta}\big)}{\sqrt{-\Delta}} f(u(\tau))\, d\tau \Bigg\|_{\dot{H}^1_x \times L^2_x} \\ &= \left\| \int_t^\infty \frac{\sin(t-\tau)\sqrt{-\Delta}}{\sqrt{-\Delta}} f(u(\tau))\, d\tau \right\|_{\dot{H}^1_x \times L^2_x}. \end{aligned} \quad (17)$$

It remains to show two things:

(i) Our initial data v_0, v_1 are well-defined, that is, that (16) does indeed converge in $\dot{H}^1_x \times L^2_x$.

(ii) The right side of (17) converges to 0 as $t \to \infty$.

The claim (i) can be shown in several ways, for example, by showing that

$$\lim_{N\to\infty} \left\| \int_N^\infty \frac{\sin\big(-\tau\sqrt{-\Delta}\big)}{\sqrt{-\Delta}} f(u(\tau))\, d\tau \right\|_{\dot{H}^1_x} = 0 \quad \text{and} \quad \lim_{N\to\infty} \left\| \int_N^\infty \cos\big(-\tau\sqrt{-\Delta}\big) f(u(\tau))\, d\tau \right\|_{L^2_x} = 0,$$

where $N \in \mathbb{N}$. These two equalities follow from the dominated convergence theorem once we show that

$$\int_0^\infty \left\| \frac{\sin\big(-\tau\sqrt{-\Delta}\big)}{\sqrt{-\Delta}} f(u(\tau)) \right\|_{\dot{H}^1_x} (\tau)\, d\tau < \infty \quad \text{and} \quad \int_0^\infty \left\| \cos\big(-\tau\sqrt{-\Delta}\big) f(u(\tau)) \right\|_{L^2_x} (\tau)\, d\tau < \infty.$$

But this follows quickly from (12) and the Plancherel theorem.

Claim (ii) has already been established in the discussion of claim (i). This concludes the argument that the finiteness of (12) implies scattering.

Definition 2.3. We say that the pair (q, r) is admissible if $2 \le q, r \le \infty$, $(q, r) \ne (2, \infty)$ and

$$\frac{1}{q} + \frac{1}{r} \le \frac{1}{2}. \tag{18}$$

Theorem 2.4 (Strichartz estimates for wave equation [Strichartz 1977; Kapitanski 1989; Ginibre and Velo 1995; Lindblad and Sogge 1995; Keel and Tao 1998]). *Let I be a time interval and let $u : I \times \mathbb{R}^3 \to \mathbb{C}$ be a Schwartz solution to the wave equation $\square u = G$ with initial data $u(t_0) = u_0$, $\partial_t u(t_0) = u_1$ for some $t_0 \in I$. Then we have the estimates*

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} + \|u\|_{C_t^0 \dot{H}_x^\sigma(I\times\mathbb{R}^3)} + \|\partial_t u\|_{C_t^0 \dot{H}_x^{\sigma-1}(I\times\mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}_x^\sigma(\mathbb{R}^3)} + \|u_1\|_{\dot{H}_x^{\sigma-1}(\mathbb{R}^3)} + \|G\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{R}^3)}, \tag{19}$$

where (q, r) and $(\tilde{q}, \tilde{r})$ are admissible pairs and obey the scaling condition

$$\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - \sigma = \frac{1}{\tilde{q}'} + \frac{3}{\tilde{r}'} - 2, \tag{20}$$

and where $\tilde{q}'$ and $\tilde{r}'$ are conjugate to $\tilde{q}$ and $\tilde{r}$, respectively. In addition, if u is a spherically symmetric solution, we allow $(q, r) = (2, \infty)$.

We define the Strichartz space $S_\sigma(I)$ for any time interval I, as the closure of the Schwartz function on $I \times \mathbb{R}^3$ under the norm

$$\|u\|_{S_\sigma(I)} := \sup_{(q,r)\text{ admissible}} \|u\|_{L_t^q L_x^r(I\times\mathbb{R}^3)}, \tag{21}$$

where (q, r) satisfies (20).

Morawetz inequality [Morawetz 1968]. *Let I be any time interval and $u : I \times \mathbb{R}^3 \to \mathbb{C}$ be the solution to (1) with finite energy E. Let F be the potential function as in (2). Then*

$$\int_I \int_{\mathbb{R}^3} \frac{F(u)}{|x|}\, dx\, dt \lesssim E. \tag{22}$$

Spherically symmetric solutions. In the last part of this section, we assume that u is the spherically symmetric solution to the log-subcritical wave equations (4), (5) (or log-supercritical wave equation (6)) and F is the corresponding potential function. We obtain the following a priori estimate for the solution.

Lemma 2.5 (pointwise estimate for spherically symmetric solution [Ginibre et al. 1992; Tao 2007]). *Let I be any time interval and let $u : I \times \mathbb{R}^3 \to \mathbb{C}$ be the spherically symmetric solution to the log-subcritical wave equations (4), (5) (or log-supercritical wave equation (6)) with finite energy E and vanishing at ∞. Let F be the potential function. Then, for any $t \in I$,*

$$|x|^2 (F(u)^{1/2} |u|)(t, x) \lesssim E \tag{23}$$

Proof. We tackle the log-subcritical case; the proof for the log-supercritical case is similar and easier. Define $\phi(z) := (F(z))^{1/2} z$ and $r := |x|$. From (7), we can compute that, for fixed t,

$$\big|\partial_r(\phi(u(t,x)))\big| \lesssim |u|^3 |\partial_r u|(t,x)\chi_{\{|u|\geq 1/3\}}(x) + |u|^3(-\log|u|)^{1/2}|\partial_r u|(t,x)\chi_{\{|u|<1/3\}}(x),$$

where χ is the characteristic function on $\mathbb{R}^3$. Then, by the fundamental theorem of calculus, Hölder's inequality, and energy conservation,

$$\begin{aligned}
|\phi(u(t,x))| &\lesssim \left| \int_r^\infty \big[|u|^3|\partial_r u|\chi_{\{|u|\geq 1/3\}} + |u|^3(-\log|u|)^{1/2}|\partial_r u|\chi_{\{|u|<1/3\}}\big](t,s)\,ds \right| \\
&\lesssim \left(\int_r^\infty \frac{|u|^6}{s^2} s^2 \chi^2_{\{|u|\geq 1/3\}}\,ds\right)^{1/2}\left(\int_r^\infty \frac{|\partial_r u|^2}{s^2}\chi_{\{|u|\geq 1/3\}} s^2\,ds\right)^{1/2} \\
&\qquad + \left(\int_r^\infty \frac{|u|^6(-\log|u|)}{s^2} s^2 \chi^2_{\{|u|<1/3\}}\,ds\right)^{1/2}\left(\int_r^\infty \frac{|\partial_r u|^2}{s^2}\chi_{\{|u|<1/3\}} s^2\,ds\right)^{1/2} \\
&\lesssim \frac{1}{r^2}\left(\int_{\mathbb{R}^3} F(u)\,dx\right)^{1/2} E^{1/2} \lesssim \frac{1}{r^2} E. \qquad \square
\end{aligned}$$

Inserting (23) into (22), we obtain that, for any time interval I,

$$\int_I \int_{\mathbb{R}^3} F^{5/4}(u)|u|^{\frac{1}{2}}\,dx\,dt \leq \int_I \int_{\mathbb{R}^3} \frac{F(u)}{|x|} \cdot \sup_{x\in\mathbb{R}^3} (|x| F^{1/4}(u)|u|^{1/2})\,dx\,dt \lesssim E^{3/2}. \tag{24}$$

This implies

$$\int_I \int_{\{|u|\leq 1/3\}} |u|^8(-\log|u|)^{5/4}\,dx\,dt + \int_I \int_{\{|u|>1/3\}} |u|^8\,dx\,dt \lesssim E^{3/2} \qquad \text{(log-subcritical case)} \tag{25}$$

and

$$\int_I \int_{\mathbb{R}^3} |u|^8 \log^{5\alpha/4}(2+|u|^2)\,dx\,dt \lesssim E^{3/2} \qquad \text{(log-supercritical case)}. \tag{26}$$

3. Log-subcritical wave equations

In this section, we consider the scattering theory for log-subcritical wave equations. We can take advantage of time reversal symmetry, and it suffices to prove that the solution u scatters in $\dot{H}^1_x \times L^2_x$ as $t \to \infty$.

Throughout this section, we use the notation

$$A = \left\{(t,x) \in (0,\infty)\times\mathbb{R}^3 : |u| < \tfrac{1}{3}\right\}, \quad B = \left\{(t,x) \in (0,\infty)\times\mathbb{R}^3 : |u| \geq \tfrac{1}{3}\right\},$$

and for any interval I,

$$A_I = A \cap (I \times \mathbb{R}^3), \quad B_I = B \cap (I \times \mathbb{R}^3). \tag{27}$$

General initial data in log-weighted Sobolev spaces.

Theorem 3.1. *Let $\gamma > \frac{1}{2}$ and let u be the solution to the log-subcritical wave equations (1), (4), and (5) with initial data*

$$u_0(x) \in X_1, \quad u_1(x) \in Y_1, \tag{28}$$

where X_1 and Y_1 are defined by (9). Then u scatters in $\dot{H}^1_x \times L^2_x$.

Proof. We need some decay estimates for the equation with initial data satisfying (28).

Lemma 3.2. *Let γ and u be as in Theorem 3.1. There exists $T = T(\|u_0\|_{X_1}, \|u_1\|_{Y_1}, \gamma) \gg 1$ such that, for $\tau > T$,*

$$\int_{\mathbb{R}^3} F(u(\tau, x))\, dx \lesssim \frac{1}{\log^{2\gamma} \tau}, \tag{29}$$

where $F(z) = F_{\text{sub}}(z)$ is defined by (7).

Proof. We essentially follow the proof of Lemma 2.1 in [Bahouri and Shatah 1998], with some changes. Define

$$e[u](t,x) := \tfrac{1}{2}|\partial_t u(t,x)|^2 + \tfrac{1}{2}|\nabla u(t,x)|^2 + F(u(t,x)).$$

We claim that there exists $C_\gamma = C_\gamma(\|u_0\|_{X_1}, \|u_1\|_{Y_1}, \gamma) \gg 1$ such that for $s > C_\gamma$,

$$\int_{|x|>s} e[u](0,x)\, dx \lesssim \frac{1}{\log^{2\gamma} s}. \tag{30}$$

We prove this claim in the Appendix and continue the proof of this lemma here. Choose T such that $T > \max(C_\gamma^2, \log^{4\gamma} T)$. We aim to show that (29) holds for all $\tau > T$.

Define the truncated forward light cone by

$$K_a^b(c) := \{(t,x) : a \le t \le b,\ |x| \le t + c,\ 0 \le a < b \le \infty\}$$

and the boundary of the truncated cone by

$$M_a^b(c) := \partial K_a^b(c) = \{(t,x) : a \le t \le b,\ |x| = t + c,\ 0 \le a < b \le \infty\}.$$

Fix $\tau > T$ and let $s = \sqrt{\tau} > C_\gamma$. For any $t_1 > 0$, the energy conservation law on the exterior of the truncated forward light cone $K_0^{t_1}(s)$ implies that

$$\int_{|x|>s+t_1} e[u](t_1)\, dx + \frac{1}{\sqrt{2}} \operatorname{flux}(0, t_1, s) = \int_{|x|>s} e[u](0)\, dx \lesssim \frac{1}{\log^{2\gamma} s}, \tag{31}$$

where

$$\operatorname{flux}(a,b,c) := \int_{M_a^b(c)} \left\{ \frac{1}{2}\left| u_t + \frac{x \cdot \nabla u}{|x|} \right|^2 + F(u) \right\} d\sigma.$$

Hence

$$\int_{|x|>s+\tau} F(u(\tau))\, dx \le \int_{|x|>s+\tau} e[u](\tau)\, dx \lesssim \frac{1}{\log^{2\gamma} s} \lesssim \frac{1}{\log^{2\gamma} \tau}, \tag{32}$$

and it suffices to show that

$$\int_{|x|\le s+\tau} F(u(\tau))\, dx \lesssim \frac{1}{\log^{2\gamma} \tau}. \tag{33}$$

Define $w(t,x) = u(t-s, x)$. The bound (33) is equivalent to

$$\int_{|x|\le s+\tau} F(w(s+\tau))\, dx \lesssim \frac{1}{\log^{2\gamma} \tau}.$$

Set $w_t := \partial_t w$. Multiplying the equation $f(w) - \Box w = 0$ by $t w_t + x \cdot \nabla w + w$, we get

$$\partial_t(tQ_0 + w_t w) - \operatorname{div}(tP_0) + R_0 = 0, \tag{34}$$

where

$$\begin{aligned} Q_0 &= e[w] + w_t\Big(\frac{x}{t}\cdot\nabla w\Big),\\ P_0 &= \frac{x}{t}\Big(\frac{w_t^2 - |\nabla w|^2}{2} - F(w)\Big) + \nabla w\Big(w_t + \frac{x}{t}\cdot\nabla w + \frac{w}{t}\Big),\\ R_0 &= |w|^6 g(|w|) - 4F(w), \end{aligned}$$

with g defined by (5). Define the horizontal sections of the forward solid cone by

$$D(t) := \{|x| \in \mathbb{R}^3 : |x| \le t\}.$$

Fix $0 < T_1 < T_2$ and integrate (34) on $K_{T_1}^{T_2}(0)$. By the divergence theorem, we have

$$\int_{D(T_2)}(T_2Q_0+w_tw)\,dx - \int_{D(T_1)}(T_1Q_0+w_tw)\,dx - \frac{1}{\sqrt{2}}\int_{M_{T_1}^{T_2}(0)}\Big(tQ_0+w_tw+tP_0\frac{x}{|x|}\Big)\,d\sigma + \int_{K_{T_1}^{T_2}(0)} R_0\,dx\,dt =: L_1+L_2+L_3+L_4 = 0. \tag{35}$$

Now, following the same steps as in [Bahouri and Shatah 1998], we define $v(y) := w(|y|, y)$. Since L_3 is the integral on $M_{T_1}^{T_2}(0)$, using spherical coordinates, we obtain that

$$L_3 = -\int_{T_1}^{T_2}\int_{S^2} r\Big(v_r + \frac{v}{r}\Big)^2 r^2\,dr\,d\omega + \frac{1}{2}\int_{S^2} T_2^2 v^2(T_2\omega)\,d\omega - \frac{1}{2}\int_{S^2} T_1^2 v^2(T_1)\omega\,d\omega, \tag{36}$$

$$L_1 = \int_{D(T_2)}\Big\{T_2\Big(\frac{|w_t|^2}{2} + \frac{1}{2}\Big(w_r + \frac{1}{r}w\Big)^2 + \frac{1}{2r^2}|\nabla_\omega w|^2 + F(w)\Big) + r\Big(w_r + \frac{1}{r}w\Big)w_t\Big\}\,dx - \frac{1}{2}\int_{S^2} T_2^2 v^2(T_2\omega)\,d\omega, \tag{37}$$

and

$$L_2 = -\int_{D(T_1)}\Big\{T_1\Big(\frac{|w_t|^2}{2} + \tfrac{1}{2}\Big(w_r + \frac{1}{r}w\Big)^2 + \frac{1}{2r^2}|\nabla_\omega w|^2 + F(w)\Big) + r\Big(w_r + \frac{1}{r}w\Big)w_t\Big\}\,dx + \frac{1}{2}\int_{S^2} T_1^2 v^2(T_1\omega)\,d\omega. \tag{38}$$

Since $L_4 \ge 0$, plugging (36), (37) and (38) into (35), we deduce that

$$T_2\int_{D(T_2)} F(w)\,dx \le CT_1E + \int_{T_1}^{T_2}\int_{S^2} T_2\Big(v_r + \frac{v}{r}\Big)^2 r^2\,dr\,d\omega,$$

where C is a constant and E is the energy. Therefore,

$$\int_{D(T_2)} F(w(T_2))\,dx \le C\frac{T_1}{T_2}E + \int_{T_1}^{T_2}\int_{S^2}\Big(v_r + \frac{v}{r}\Big)^2 r^2\,dr\,d\omega. \tag{39}$$

For any $T_1 \geq s$, by (31), the second term in the right-hand side of (39) is controlled by

$$\int_{T_1}^{T_2}\int_{S^2}\Big(v_r+\frac{v}{r}\Big)^2 r^2\,dr\,d\omega \lesssim \int_{M_{T_1}^{T_2}(0)}\Big\{\frac{1}{2}\Big|w_t+\frac{x\cdot\nabla w}{|x|}\Big|^2\Big\}\,d\sigma \lesssim \frac{1}{\log^{2\gamma} s} \lesssim \frac{1}{\log^{2\gamma}\tau}.$$

Now, choosing $T_2=\tau+s$ and $T_1=(\tau+s)/\log^{2\gamma}\tau > \sqrt{\tau}=s$, (39) implies

$$\int_{D(\tau+s)} F(w(\tau+s,x))\,dx \lesssim \frac{1}{\log^{2\gamma}\tau}. \tag{40}$$

Combining (32) and (40), the lemma is proved. □

Before we prove Theorem 3.1, let's observe the following fact. Let I be any time interval with length $3<|I|<\infty$. By Hölder's inequality, we have that, for $0<\delta<2$,

$$\begin{aligned}
&\| |u|^4 u(-\log|u|)\|_{L_t^1 L_x^2(I\times\mathbb{R}^3)}\\
&\quad\leq \|u^{3-\delta}(-\log|u|)^{(3-\delta)/6}\|_{L_t^\infty L_x^{6/(3-\delta)}(I\times\mathbb{R}^3)} \|u^2\|_{L_t^{2/(2-\delta)}L_x^{6/\delta}(I\times\mathbb{R}^3)} \|u^\delta(-\log|u|)^{(3+\delta)/6}\|_{L_t^{2/\delta}L_x^\infty(I\times\mathbb{R}^3)}\\
&\quad= \|u(-\log|u|)^{1/6}\|^{3-\delta}_{L_t^\infty L_x^6(I\times\mathbb{R}^3)} \|u\|^2_{L_t^{4/(2-\delta)}L_x^{12/\delta}(I\times\mathbb{R}^3)} \|u^\delta(-\log|u|)^{(3+\delta)/6}\|_{L_t^{2/\delta}L_x^\infty(I\times\mathbb{R}^3)}\\
&\quad\leq \|u(-\log|u|)^{1/6}\|^{3-\delta}_{L_t^\infty L_x^6(I\times\mathbb{R}^3)} \|u\|^2_{L_t^{4/(2-\delta)}L_x^{12/\delta}(I\times\mathbb{R}^3)} \|u^\delta(-\log|u|)^{(3+\delta)/6}\|_{L_t^\infty L_x^\infty(I\times\mathbb{R}^3)}|I|^{\delta/2}.
\end{aligned}$$

If $|u|\leq\frac{1}{3}$, we can estimate that

$$\|u^\delta(-\log|u|)^{(3+\delta)/6}\|_{L_t^\infty L_x^\infty(I\times\mathbb{R}^3)} \lesssim \Big(\frac{1}{\delta}\Big)^{1/2+\delta/6}.$$

Letting $\delta=2/\log|I|$, we obtain

$$\| |u|^4u(-\log|u|)\|_{L_t^1L_x^2(I\times\mathbb{R}^3)} \lesssim \|u(-\log|u|)^{1/6}\|^{3-\delta}_{L_t^\infty L_x^6(I\times\mathbb{R}^3)} \|u\|^2_{L_t^{4/(2-\delta)}L_x^{12/\delta}(I\times\mathbb{R}^3)} \log^{1/2}|I|. \tag{41}$$

To complete the proof of Theorem 3.1, by Remark 2.2, it suffices to show that

$$\|f(u)\|_{L_t^1L_x^2([T,\infty)\times\mathbb{R}^3)}<\infty \quad \text{for some } T<\infty.$$

Let $J=(3^i,\infty)$, where i is sufficiently large and to be determined later. Then

$$\|f(u)\|_{L_t^1L_x^2(J\times\mathbb{R}^3)} \lesssim \||u|^4u(-\log|u|)\|_{L_t^1L_x^2(A_J)} + \||u|^4u\|_{L_t^1L_x^2(B_J)} =: M_1+M_2.$$

Since $(2+\delta, 6(2+\delta)/\delta)$ is an admissible pair satisfying (20) for $\sigma=1$, from Hölder's inequality and Lemma 3.2,

$$M_2 \leq \|u\|^{3-\delta}_{L_t^\infty L_x^6(B_J)} \|u\|^{2+\delta}_{L_t^{2+\delta}L_x^{6(2+\delta)/\delta}(B_J)} \lesssim \frac{1}{(\log(3^i))^{(3-\delta)/3\gamma}}\|u\|^{2+\delta}_{S_1(J)}. \tag{42}$$

On the other hand, define interval J_k by subdividing J according to $J=\bigcup_{k=1}^\infty(3^{2^{k-1}i},3^{2^k i}) =: \bigcup_{k=1}^\infty J_k$. Define $\delta_k := 2/\log|J_k|$. By (41), Lemma 3.2, and the fact that the admissible pairs $(4/(2-\delta_k), 12/\delta_k)$

satisfy (20) for $\sigma = 1$, we have

$$M_1 \le \sum_{k=1}^{\infty} \|u^5(-\log|u|)\|_{L_t^1 L_x^2(J_k\times\mathbb{R}^3)} \lesssim \sum_{k=1}^{\infty} \left(\frac{1}{(\log 3^{2^{k-1}i})^{(3-\delta_k)/3\gamma}} (\log 3^{2^k i})^{1/2} \right) \|u\|^2_{L_t^{4/(2-\delta_k)} L_x^{12/\delta_k}(J_k\times\mathbb{R}^3)}$$
$$\lesssim \left(\sum_{k=1}^{\infty} i^{1/2-(3-\delta_k)/3\gamma} \cdot 2^{(k-1)(1/2-(3-\delta_k)/3)\gamma} \right) \|u\|^2_{S_1(J)}.$$

Since $\gamma > \frac{1}{2}$, we can choose i sufficiently large such that $\left(((3-\delta_k)/3)\gamma - \frac{1}{2}\right) > c > 0$ for all k. Hence

$$M_1 \lesssim i^{-c} \sum_{k=1}^{\infty} 2^{-(k-1)c} \|u\|^2_{S_1(J)}. \tag{43}$$

Combining (42) and (43), for $\epsilon_0 > 0$ sufficiently small, we can choose i sufficiently large such that

$$\|f(u)\|_{L_t^1 L_x^2(J\times\mathbb{R}^3)} \le \epsilon_0 (\|u\|^2_{S_1(J)} + \|u\|^{2+\delta}_{S_1(J)}).$$

By the Strichartz estimate (19), we have

$$\|u\|_{S_1(J)} \le C E^{1/2} + \epsilon_0 (\|u\|^2_{S_1(J)} + \|u\|^{2+\delta}_{S_1(J)}).$$

From a continuity argument, we conclude that

$$\|u\|_{S_1(J)} \le 2CE^{1/2}.$$

This implies that

$$\|f(u)\|_{L_t^1 L_x^2(J\times\mathbb{R}^3)} < \infty. \qquad \square$$

Spherically symmetric initial data in lower regularity Sobolev spaces. In this subsection, we consider the solutions to the log-subcritical wave equations with spherically symmetric initial data. If the finite energy initial data are in any lower regularity Sobolev spaces, we obtain the $\dot H_x^1 \times L_x^2$ scattering. The spirit of the proof follows from [Tao 2007] and a slightly sharpened bootstrap argument in Lemmas 3.5 and 3.6.

Throughout this subsection, for given $\delta > 0$, we denote

$$Z(t) := \|u(t,x)\|_{\dot H_x^{1-\delta}(\mathbb{R}^3)} + \|\partial_t u(t,x)\|_{\dot H_x^{-\delta}(\mathbb{R}^3)}. \tag{44}$$

It is easy to show that $Z(t) > 0$ for any time t.[2]

Theorem 3.3. *Let u be the solution to the log-subcritical wave equations (1), (4), (5) with spherically symmetric initial data*

$$u_0(x) \in X_2, \quad u_1(x) \in Y_2, \tag{45}$$

where X_2 and Y_2 are defined by (10). Then u scatters in $\dot H_x^1 \times L_x^2$.

To prove Theorem 3.3, we need some intermediate lemmas.

[2]If $Z(t_0) = 0$ for some t_0, it is easy to prove that the solution u has energy $E(t_0) = 0$ and, hence, $E(t) = 0$ for any time t, by energy conservation. This implies the solution $u(t,x) \equiv 0$ for all t.

Lemma 3.4. *Let $I=[a,b]$ be any interval where $0\le a<b\le\infty$ and let u be the solution to the log-subcritical wave equations (1), (4), (5) with spherically symmetric initial data*

$$u(a,x)=u_0(x)\in \dot H^1_x\cap \dot H^{1-\delta}_x,\quad \partial_t u(a,x)=u_1(x)\in L^2_x\cap \dot H^{-\delta}_x$$

for some fixed $0<\delta<\frac{1}{2}$. Then there exists $0<\epsilon(\delta)\ll 1$ such that for $0<\epsilon<\epsilon(\delta)$,

$$\|u\|_{S_{1-\delta}(I)}\lesssim Z(a)+(\|u\|^{1+\epsilon/(2\delta)}_{S_{1-\delta}(I)}+\|u\|_{S_{1-\delta}(I)})\big(\||u|(-\log|u|)^{5/32}\|^{4-\epsilon/(2\delta)-\epsilon}_{L^8_{t,x}(A_I)}+\|u\|^4_{L^8_{t,x}(B_I)}\big)\Big(\frac{1}{\epsilon}\Big)^{7/16}, \quad (46)$$

where the constant hidden in (46) is independent of the interval I and ϵ.

Proof. By the Strichartz estimate (19),

$$\|u\|_{S_{1-\delta}(I)}\lesssim Z(a)+\|f(u)\|_{L^{2/(2-\delta)}_t L^{2/(1+\delta)}_x(I\times\mathbb{R}^3)}. \quad (47)$$

Consider that

$$\|f(u)\|_{L^{2/(2-\delta)}_t L^{2/(1+\delta)}_x(I\times\mathbb{R}^3)}\lesssim\|-|u|^4u(\log(|u|))\|_{L^{2/(2-\delta)}_t L^{2/(1+\delta)}_x(A_I)}+\||u|^4u\|_{L^{2/(2-\delta)}_t L^{2/(1+\delta)}_x(B_I)}=:N_1+N_2$$

with A_I and B_I as in (27). By Hölder's inequality,

$$N_2\le\|u\|_{L^{2/(1-\delta)}_t L^{2/\delta}_x(B_I)}\|u\|^4_{L^8_{t,x}(B_I)}\le\|u\|_{S_{1-\delta}(I)}\|u\|^4_{L^8_{t,x}(B_I)}. \quad (48)$$

On the other hand, choosing $\epsilon(\delta)$ sufficiently small such that for $0<\epsilon<\epsilon(\delta)$,

$$0<\frac{1}{p}:=\frac{8\delta+\epsilon-8\delta^2+2\epsilon\delta}{8(2\delta+\epsilon)}\le\frac{1}{2},\quad 0<\frac{1}{q}:=\frac{\delta}{2}+\frac{\epsilon(1-2\delta)}{8(2\delta+\epsilon)}\le\frac{1}{2},\quad \frac{3}{8}\approx\frac{12+5\epsilon/(2\delta)+5\epsilon}{32}<\frac{7}{16}.$$

It is clear that (p,q) is an admissible pair satisfying (20) for $\sigma=1-\delta$. By Hölder's inequality and interpolation theory, we can estimate that

$$\begin{aligned}N_1&\le\||u|^{5-\epsilon}(-\log|u|)^{5(4-\epsilon/(2\delta)-\epsilon)/32}\|_{L^{2/(2-\delta)}_t L^{2/(1+\delta)}_x(A_I)}\||u|^{\epsilon}(-\log|u|)^{(12+5\epsilon/(2\delta)+5\epsilon)/32}\|_{L^\infty_{t,x}(A_I)}\\&\le\|u\|^{1+\epsilon/(2\delta)}_{L^p_tL^q_x(A_I)}\|u(-\log|u|)^{5/32}\|^{4-\epsilon/(2\delta)-\epsilon}_{L^8_{t,x}(A_I)}\||u|^{\epsilon}(-\log|u|)^{(12+5\epsilon/(2\delta)+5\epsilon)/32}\|_{L^\infty_{t,x}(A_I)} \quad (49)\\&\lesssim\|u\|^{1+\epsilon/(2\delta)}_{L^p_tL^q_x(A_I)}\||u|(-\log|u|)^{5/32}\|^{4-\epsilon/(2\delta)-\epsilon}_{L^8_{t,x}(A_I)}\Big(\frac{1}{\epsilon}\Big)^{upnfrac12+5\epsilon/(2\delta)+5\epsilon32}. \quad (50)\end{aligned}$$

The last factor of (50) comes from maximizing the last factor on the right of (49) using calculus. We note that the constant hidden in the last inequality is independent of ϵ. By (48) and (50), we have

$$\|f(u)\|_{L^{2/(2-\delta)}_t L^{2/(1+\delta)}_x(I\times\mathbb{R}^3)}\lesssim\|u\|^{1+\epsilon/(2\delta)}_{S_{1-\delta}(I)}\||u|(-\log|u|)^{5/32}\|^{4-\epsilon/(2\delta)-\epsilon}_{L^8_{t,x}(A_I)}\Big(\frac{1}{\epsilon}\Big)^{7/16}+\|u\|_{S_{1-\delta}(I)}\|u\|^4_{L^8_{t,x}(B_I)}.$$

From (47),

$$\|u\|_{S_{1-\delta}(I)}\lesssim Z(a)+\|u\|^{1+\epsilon/(2\delta)}_{S_{1-\delta}(I)}\||u|(-\log|u|)^{5/32}\|^{4-\epsilon/(2\delta)-\epsilon}_{L^8_{t,x}(A_I)}\Big(\frac{1}{\epsilon}\Big)^{\frac{7}{16}}+\|u\|_{S_{1-\delta}(I)}\|u\|^4_{L^8_{t,x}(B_I)}\lesssim\text{RHS of (46)}.$$

One can check that all constants hidden in the inequalities above are independent of the interval I and ϵ. Hence, Lemma 3.4 is proved. ☐

Lemma 3.5 (continuity argument). *Let I $(=[a,b])$ and u satisfy the assumptions of Lemma 3.4, C be the constant hidden in (46) and $0<\epsilon(\delta)$ be chosen in Lemma 3.4. Let $\epsilon_0 = 1/(100C)$ and $0<\epsilon<\epsilon(\delta)$ such that $Z(a)^{\epsilon/(2\delta)} \geq \frac{1}{2}$ and $(2C)^{\epsilon/(2\delta)} \leq 2$. We define*

$$Q(I) := \left(\| |u|(-\log|u|)^{5/32}\|^{4-\epsilon/(2\delta)-\epsilon}_{L^8_{t,x}(A_I)} + \|u\|^{4-\epsilon/(2\delta)-\epsilon}_{L^8_{t,x}(B_I)}\right).$$

If $\|u\|_{L^8_{t,x}(B_I)} \leq 1$ and $Q(I) \leq \epsilon_0(\epsilon^{7/16}/(Z(a)^{\epsilon/(2\delta)}))$, we have

$$\|u\|_{S_{1-\delta}(I)} \leq 2CZ(a).$$

Proof. We prove this lemma by contradiction. For $0 \leq t \leq b-a$, from the dominated convergence theorem, we have that the function $\Phi(t) := \|u\|_{S_{1-\delta}([a,a+t])}$ is nondecreasing and continuous in $[0,b-a]$ and $\Phi(0)=0$. By the hypothesis and (46), we have

$$\Phi(t) \leq CZ(a) + \tfrac{1}{100}(\Phi(t)^{1+\epsilon/(2\delta)} + \Phi(t))\left(\frac{1}{Z(a)^{\epsilon/(2\delta)}}\right) \tag{51}$$

for all $t\in[0,b-a]$. Assume for contradiction that there exists $t_0 \in [0,b-a)$ such that $\Phi(t_0) = 2CZ(a)$. If $2CZ(a) < 1$, (51) implies that

$$2CZ(a) = \Phi(t_0) \leq CZ(a) + \frac{1}{50}(2CZ(a))\left(\frac{1}{Z(a)^{\epsilon/(2\delta)}}\right) \leq \frac{11}{10}CZ(a).$$

On the other hand, if $2CZ(a) \geq 1$, (51) implies that

$$2CZ(a) = \Phi(t_0) \leq CZ(a) + \frac{1}{50}(2CZ(a))^{1+\epsilon/(2\delta)}\left(\frac{1}{Z(a)^{\epsilon/(2\delta)}}\right) \leq \frac{11}{10}CZ(a).$$

We get contradictions in both situations, and the lemma is proved. □

Lemma 3.6 (finite division). *Let I $(=[a,b])$ and u satisfy the assumptions of Lemma 3.4 and C be the constant hidden in (46). We denote $Z_i = (2C)^i Z(a)$, where $i = 0,1,2,\ldots$. For any $\epsilon_0 > 0$, we can choose $\epsilon \ll 1$ and finitely many numbers $a = T_0 < T_1 < T_2 < \cdots < T_N < T_{N+1} = b$, where $N = N(\epsilon_0,\epsilon,\delta,E,Z_0,C)$, such that for $I_j := [T_j, T_{j+1}]$,*

$$Q(I_j) = \epsilon_0\left(\frac{\epsilon^{7/16}}{Z_j^{\epsilon/(2\delta)}}\right) \tag{52}$$

for $0 \leq j \leq N-1$ and $Q(I_N) \leq \epsilon_0(\epsilon^{7/16}/Z_N^{\epsilon/(2\delta)})$.

Proof. We observe that

$$\sum_{i=0}^{\infty}\left[\epsilon_0\left(\frac{\epsilon^{7/16}}{Z_i^{\epsilon/(2\delta)}}\right)\right]^{8/(4-\epsilon/(2\delta)-\epsilon)} \gtrsim_{\epsilon_0,Z_0} \left\{\epsilon^{7/(8-\epsilon/(\delta)-2\epsilon)}\sum_{i=0}^{\infty}\frac{1}{(2C)^{8i\epsilon/(8\delta-\epsilon-2\delta\epsilon)}}\right\} \to \infty \quad \text{as } \epsilon \to 0.$$

Therefore, by (25), we can choose ϵ sufficiently small such that

$$3\left(\int\int_A |u|^8(-\log|u|)^{5/4}\,dx\,dt + \int\int_B |u|^8\,dx\,dt\right) < \sum_{i=0}^{K}\left[\epsilon_0\left(\frac{\epsilon^{7/16}}{Z_i^{\epsilon/(2\delta)}}\right)\right]^{8/(4-\epsilon/(2\delta)-\epsilon)} \tag{53}$$

for some $K = K(\epsilon_0,\epsilon,\delta,E,Z_0,C)$.

Fix this ϵ. If $Q(I) < \epsilon_0(\epsilon^{7/16}/(Z_0^{\epsilon/(2\delta)}))$, we say $T_1 = b$ and the lemma is proved. Otherwise, we can choose $0 < T_1 < b$ such that (52) holds for $j = 0$. Again, if $Q([T_1, b) < \epsilon_0(\epsilon^{7/16}/(Z_1^{\epsilon/(2\delta)}))$, we say $T_2 = b$. Otherwise, we can choose $T_1 < T_2 < b$ such that (52) holds for $j = 1$. By continuing this process, we can choose $a < T_1 < T_2 < \cdots$ such that (52) holds for $j = 0, 1, \dots$. It suffices to show that this process will stop in at most $K + 1$ steps. Indeed, assume that there are more than $K + 1$ subintervals satisfying (52). Since

$$Q(I_j)^{8/(4-\epsilon/(2\delta)-\epsilon)} \le 3\left(\iint_{A_{I_j}} |u|^8(-\log|u|)^{5/4}\,dx\,dt + \iint_{B_{I_j}} |u|^8\,dx\,dt\right),$$

for $j = 0, 1, \dots$, by our construction of I_j, we have

$$\begin{aligned}\sum_{j=0}^{K+1} \epsilon_0\left(\frac{\epsilon^{7/16}}{Z_j^{\epsilon/(2\delta)}}\right) &= \sum_{j=0}^{K+1} Q(I_j)^{8/(4-\epsilon/(2\delta)-\epsilon)} \\ &\le \sum_{i=0}^{K+1} 3\left(\iint_{A_{I_j}} |u|^8(-\log|u|)^{5/4}\,dx\,dt + \iint_{B_{I_j}} |u|^8\,dx\,dt\right) \\ &\le 3\left(\iint_A |u|^8(-\log|u|)^{5/4}\,dx\,dt + \iint_B |u|^8\,dx\,dt\right).\end{aligned}$$

This contradicts (53), and the lemma is proved. □

Corollary 3.7. *Let I and u satisfy the assumptions of Lemma 3.4 and C be the constant hidden in (46). If $\|u\|_{L^8_{t,x}(B_I)} \le 1$, $u \in L^{8/(1+2\delta)}_{t,x}(I \times \mathbb{R}^3)$.*

Proof. Let $\epsilon(\delta)$ be chosen in Lemma 3.4 and $0 < \epsilon < \epsilon(\delta)$ satisfy Lemma 3.6, $Z(a)^{\epsilon/(2\delta)} \ge \frac{1}{2}$ and $(2C)^{\epsilon/(2\delta)} \le 2$. Let $\{I_j\}_{j=0}^N$ be the subintervals constructed by Lemma 3.6 such that (52) holds for $0 \le j \le N$.

We claim that

$$\|u\|_{S_{1-\delta}(I_j)} \le 2CZ_j \quad \text{for } 0 \le j \le N, \tag{54}$$

where $Z_j = (2C)^j Z(a)$. Indeed, by Lemma 3.5, (54) holds for $j = 0$. Again, if (54) holds for $j = k - 1$, we have $Z(T_k) \le \|u\|_{S_{1-\delta}(I_{k-1})} \le Z_k$. Since $Z_k^{\epsilon/(2\delta)} \ge Z(a)^{\epsilon/(2\delta)} \ge \frac{1}{2}$, applying Lemma 3.5 on the interval I_k, we obtain (54) for $j = k$. By induction on j, the claim is proved and this implies

$$\|u\|_{L^{8/(1+2\delta)}_{t,x}(I\times\mathbb{R}^3)} \le \sum_{j=0}^{N+1} \|u\|_{S_{1-\delta}(I_j)} \le \sum_{j=0}^{N+1} (2C)^j Z_0 < \infty. \qquad \square$$

Corollary 3.8. *Let u be the solution to the log-subcritical wave equations (1), (4), (5) with spherically symmetric initial data*

$$u(0,x) = u_0(x) \in \dot{H}^1_x \cap \dot{H}^{1-\delta}_x, \quad \partial_t u(0,x) = u_1(x) \in L^2_x \cap \dot{H}^{-\delta}_x$$

for some fixed $0 < \delta < \frac{1}{2}$. Then $u \in L^{8/(1+2\delta)}_{t,x}(\mathbb{R}_+ \times \mathbb{R}^3)$.

Proof. By (25), we can choose finitely many numbers $0 = S_0 < S_1 < \cdots < S_{M-1} < S_M = \infty$ such that $\|u\|_{L^8_{t,x}(B_{[S_k,S_{k+1}]})} \le 1$ for $0 \le k \le M$. By Corollary 3.7 and energy conservation, we have

$$(u(S_k, x), \partial_t u(S_k, x)) \in (\dot{H}^1_x \cap \dot{H}^{1-\delta}_x) \times (L^2_x \cap \dot{H}^{-\delta}_x)$$

and $\|u\|_{L^{8/1+2\delta}_{t,x}([S_k,S_{k+1}]\times\mathbb{R}^3)} < \infty$ for $0 \le k \le M$. Hence

$$\|u\|_{L^{8/(1+2\delta)}_{t,x}(\mathbb{R}_+\times\mathbb{R}^3)} \le \sum_{k=0}^{M} \|u\|_{L^{8/(1+2\delta)}_{t,x}([S_k,S_{k+1}]\times\mathbb{R}^3)} < \infty. \qquad \square$$

To finish the proof of Theorem 3.3, by Remark 2.2, it suffices to show that $\|f(u)\|_{L^1_t L^2_x((T,\infty)\times\mathbb{R}^3)} < \infty$ for some $0 < T < \infty$. Since the initial data satisfy (45), we can choose some $0 < \delta < \frac{1}{2}$ such that $u_0 \in \dot{H}^1_x(\mathbb{R}^3) \cap \dot{H}^{1-\delta}_x(\mathbb{R}^3)$ and $u_1 \in L^2_x(\mathbb{R}^3) \cap \dot{H}^{-\delta}_x(\mathbb{R}^3)$. Observe that

$$\begin{aligned}
&\|f(u)\|_{L^1_t L^2_x((T,\infty)\times\mathbb{R}^3)} \\
&\quad\lesssim \||u|^5(\log(|u|))\|_{L^1_t L^2_x(A_T)} + \||u|^5\|_{L^1_t L^2_x(B_T)} \\
&\quad\lesssim \|u\|^{4/(1+2\delta)}_{L^8_{t,x}/(1+2\delta)(A_T)} \|u\|_{L^2_t L^\infty_x(A_T)} \|u^{8\delta/(1+2\delta)}(\log(|u|))\|_{L^\infty_{t,x}(A_T)} + \|u\|^4_{L^8_{t,x}(B_T)} \|u\|_{L^2_t L^\infty_x(B_T)} \\
&\quad\lesssim \|u\|_{L^2_t L^\infty_x((T,\infty)\times\mathbb{R}^3)} \left[\left(\frac{1+2\delta}{8\delta}\right) \|u\|^{4/(1+2\delta)}_{L^8_{t,x}/(1+2\delta)(A_T)} + \|u\|^4_{L^8_{t,x}(B_T)}\right],
\end{aligned}$$

where $A_T := A \cap ((T,\infty)\times\mathbb{R}^3)$ and $B_T := B \cap ((T,\infty)\times\mathbb{R}^3)$. The last inequality above is from the fact that $|u^{8\delta/(1+2\delta)}(\log(|u|))| \lesssim (1+2\delta)/(8\delta)$ for $|u| \le \frac{1}{3}$. By Corollary 3.8 and (25), for sufficiently small $\epsilon > 0$, we can choose $T = T(\epsilon)$ sufficiently large such that

$$\left(\frac{1+2\delta}{8\delta}\right) \|u\|^{4/(1+2\delta)}_{L^8_{t,x}/(1+2\delta)(A_T)} + \|u\|^4_{L^8_{t,x}(B_T)} < \epsilon.$$

Hence, by the Strichartz inequality [Klainerman and Machedon 1993],

$$\|u\|_{L^2_t L^\infty_x((T,\infty)\times\mathbb{R}^3)} \le C E^{1/2} + \epsilon C \|u\|_{L^2_t L^\infty_x((T,\infty)\times\mathbb{R}^3)}.$$

Again for $\epsilon < 1/(2C)$, we have $\|u\|_{L^2_t L^\infty_x((T,\infty)\times\mathbb{R}^3)} < 2CE^{1/2}$ and this implies $\|f(u)\|_{L^1_t L^2_x((T,\infty)\times\mathbb{R}^3)} < \infty$.

Energy subcritical nonlinear wave equations with specific spherically symmetric initial data. In the last part of this section, we will discuss an observation, for energy subcritical nonlinear wave equations, inspired by the proof of Theorem 3.3. For given $0 < \delta < \frac{1}{2}$, let $(u_0, u_1) \in (\dot{H}^1_x(\mathbb{R}^3) \cap \dot{H}^{1-\delta}_x(\mathbb{R}^3)) \times (L^2_x(\mathbb{R}^3) \cap \dot{H}^{-\delta}_x(\mathbb{R}^3))$ be spherically symmetric functions. In this subsection, we consider the energy-subcritical nonlinear wave equation

$$\Box u = |u|^{4-\epsilon} u, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \tag{55}$$

where we allow ϵ to depend on the given data (u_0, u_1). That is, we find a relation (R) (see Definition 3.10) among ϵ, the energy E, and $Z(0)$ as in (44), the lower regularity norm of the initial data, for which the solution scatters. We remark that relation (R) holds for data large in both the energy and $\dot{H}^{1-\delta}$ norms provided that ϵ is taken sufficiently small (depending on the size of these norms). In [Lindblad and Sogge

1995], scattering was established in $\dot H^{1-\delta}$ for the $\dot H^{1-\delta}$ critical nonlinear wave equation from small data. Our remarks here are related to that work, for example, relation (R) quantifies the extent to which large data can be allowed. Also, we will prove scattering in $\dot H^1$, rather than $\dot H^{1-\delta}$.

In order to prove that u scatters in $\dot H^1_x \times L^2_x$, It suffices to show that $\|u^{5-\epsilon}\|_{L^1_t L^2_x([T,\infty)\times\mathbb{R}^3)} < \infty$ for some $T < \infty$. By the Strichartz estimate and Hölder's inequality,

$$\begin{aligned}\|u\|_{L^2_t L^\infty_x([T,\infty)\times\mathbb{R}^3)} &\le CE^{1/2} + C\|u^{5-\epsilon}\|_{L^1_t L^2_x([T,\infty)\times\mathbb{R}^3)}\\ &\le CE^{1/2} + C\|u\|_{L^2_t L^\infty_x([T,\infty)\times\mathbb{R}^3)}\|u\|^{4-\epsilon}_{L^{8-2\epsilon}_{t,x}([T,\infty)\times\mathbb{R}^3)}.\end{aligned}$$

Following similar arguments as in the proof of Theorem 3.3, we only need to show that

$$\|u\|_{L^{8-2\epsilon}_{t,x}([T,\infty)\times\mathbb{R}^3)} < \infty \quad \text{for some } T < \infty.$$

Let $\epsilon_0(\delta) := 8\delta/(1+2\delta)$ (so that $\dot H^{1-\delta}$ is the scale invariant norm for (55) with $\epsilon = \epsilon_0(\delta)$). We restrict to the case $0 < \epsilon < \epsilon_0(\delta)$.

In this case, (55) is $\dot H^{1-\delta}$-supercritical nonlinear wave equation. We denote

$$\gamma_\epsilon = \frac{3\epsilon}{16\delta - \frac{5}{2}\delta\epsilon - \frac{5}{4}\epsilon}, \quad \kappa_\epsilon = \frac{8-\frac{5}{4}\epsilon}{4-\gamma_\epsilon-\epsilon}, \quad \frac{1}{\alpha_\epsilon} = \frac{1+2\delta}{8} + \frac{3(1-2\delta)}{8(1+\gamma_\epsilon)}, \quad \frac{1}{\beta_\epsilon} = \frac{1+2\delta}{8} - \frac{1-2\delta}{8(1+\gamma_\epsilon)}.$$

Note that

(i) as $\epsilon \to \epsilon_0(\delta)$, $\gamma_\epsilon \to 4-\epsilon$ and $\kappa_\epsilon \to \infty$;

(ii) $(\alpha_\epsilon, \beta_\epsilon)$ is an admissible pair satisfying (20) for $\sigma = 1-\delta$.

Remark 3.9. Let u be the spherically symmetric solution to the energy-subcritical nonlinear wave equation (55) with energy E. We observe that Lemma 2.5 holds for u. Hence, for any interval $I = [a, b]$ where $0 \le a < b \le \infty$, (24) implies

$$\int_I \int_{\mathbb{R}^3} |u(t,x)|^{8-5\epsilon/4}\, dx\, dt \le C_1 E^{3/2}, \tag{56}$$

where we can choose the constant C_1 to be independent of ϵ. Moreover, by the Strichartz estimate,

$$\|u\|_{S_{1-\delta}(I)} \le CZ(a) + C\|u^{5-\epsilon}\|_{L^2_t/(2-\delta)L^{2/(1+\delta)}_x(I\times\mathbb{R}^3)}. \tag{57}$$

Definition 3.10. Given $0 < \delta < \frac{1}{2}$, let $0 < \epsilon < \epsilon_0(\delta)$, u be the solution to (55) with energy E and lower regularity norm $Z(0) > 0$. We say that the triple $(E, Z(0), \epsilon)$ satisfies the relation (R) if

$$C_1 E^{3/2} \le \left(\frac{1}{2(2C)^{1+\gamma_\epsilon} Z(0)^{\gamma_\epsilon}}\right)^{\kappa_\epsilon} \frac{1}{1-(2C)^{-\gamma_\epsilon\kappa_\epsilon}}$$

Lemma 3.11. *Given* $0 < \delta < \frac{1}{2}$ *and* $0 < \epsilon < \epsilon_0(\delta)$, *let*

$$(u_0, u_1) \in (\dot H^1_x(\mathbb{R}^3) \cap \dot H^{1-\delta}_x(\mathbb{R}^3)) \times (L^2_x(\mathbb{R}^3) \cap \dot H^{-\delta}_x(\mathbb{R}^3))$$

be spherically symmetric functions such that the triple $(E, Z(0), \epsilon)$ *satisfies* (R) *and* u *is the solution to* (55). *Then* $u \in L^{8/(1+2\delta)}_{t,x}(\mathbb{R}_+ \times \mathbb{R}^3)$.

Proof. Since $(E, Z(0), \epsilon)$ satisfies (R), by (56) and an argument similar to that in proof of Lemma 3.6, we can choose finitely many numbers $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = \infty$ such that

$$\|u\|^{4-\gamma_\epsilon-\epsilon}_{L^{8-5\epsilon/4}_{t,x}([T_i,T_{i+1}]\times\mathbb{R}^3)} = \frac{1}{2(2C)^{1+\gamma_\epsilon}((2C)^i Z(0))^{\gamma_\epsilon}} \tag{58}$$

for $0 \le i \le N-1$ and

$$\|u\|^{4-\gamma_\epsilon-\epsilon}_{L^{8-5\epsilon/4}_{t,x}([T_N,T_{N+1}]\times\mathbb{R}^3)} \le \frac{1}{2(2C)^{1+\gamma_\epsilon}((2C)^N Z(0))^{\gamma_\epsilon}}.$$

We claim that

$$Z(T_i) < (2C)^i Z(0) \tag{59}$$

and

$$\|u\|_{S_{1-\delta}([T_i,T_{i+1}])} < (2C)^{i+1} Z(0) \tag{60}$$

for $0 \le i \le N$.

Observe that (59) is clearly true for $i = 0$ and $Z(T_i) \le \|u\|_{S_{1-\delta}([T_{i-1},T_i])}$ for $1 \le i \le N$. Hence it suffices to show that (60) holds and then (59) is automatically true.

A similar proof to that of Lemma 3.5 applies here. Assume (60) is true for $i \le j-1$. We aim to prove (60) for $i = j$. (Note that (59) follows from our assumption when $i = j$.) Let $\phi(t) = \|u\|_{S_{1-\delta}([T_j,T_j+t])}$. Then ϕ is a continuous and nondecreasing function on $[0, T_{j+1} - T_j]$ and $\phi(0) = 0$. Assume for contradiction that there exists $t_0 \in [0, T_{j+1} - T_j]$ such that $\phi(t_0) = (2C)^{j+1} Z(0)$. By Hölder's inequality, (57), (58), and (59), we have

$$\begin{aligned}
(2C)^{j+1} Z(0) = \phi(t_0) &\le C Z(T_j) + C \|u^{5-\epsilon}\|_{L^2_t/(2-\delta) L^{2/(1+\delta)}_x([T_j,T_j+t_0]\times\mathbb{R}^3)} \\
&\le C Z(T_j) + C \|u\|^{1+\gamma_\epsilon}_{L^{\alpha_\epsilon}_t L^{\beta_\epsilon}_x([T_j,T_j+t_0]\times\mathbb{R}^3)} \|u\|^{4-\gamma_\epsilon-\epsilon}_{L^{8-5\epsilon/4}_{t,x}([T_j,T_j+t_0]\times\mathbb{R}^3)} \\
&\le C Z(T_j) + C \|u\|^{1+\gamma_\epsilon}_{S_{1-\delta}([T_j,T_j+t_0])} \|u\|^{4-\gamma_\epsilon-\epsilon}_{L^{8-5\epsilon/4}_{t,x}([T_j,T_j+t_0]\times\mathbb{R}^3)} \\
&\le C(2C)^j Z(0) + \frac{1}{4[(2C)^{j+1} Z(0)]^{\gamma_\epsilon}} \|u\|^{1+\gamma_\epsilon}_{S_{1-\delta}([T_j,T_j+t_0])} \\
&< \tfrac{1}{2}(2C)^{j+1} Z(0) + \frac{1}{4[(2C)^{j+1} Z(0)]^{\gamma_\epsilon}} \times [(2C)^{j+1} Z(0)]^{1+\gamma_\epsilon} \\
&= \tfrac{3}{4}(2C)^{j+1} Z(0).
\end{aligned}$$

The contradiction implies that (60) holds for $i = j$. By an inductive argument on i, the claim is proved. To finish proving this lemma, we have

$$\|u\|_{L^{8/(1+2\delta)}_{t,x}(\mathbb{R}_+\times\mathbb{R}^3)} \le \sum_{i=0}^{N+1} \|u\|_{S_{1-\delta}([T_i,T_{i+1}])} \le \sum_{i=0}^{N+1} (2C)^i Z(0) < \infty \qquad \square$$

Corollary 3.12. *Let δ, ϵ, u_0, u_1 and u satisfy the assumptions of Lemma 3.11. Then u scatters in $\dot H^1_x \times L^2_x$.*

Proof. By the above discussion, it suffices to show

$$\|u\|_{L^{8-2\epsilon}_{t,x}([T,\infty)\times\mathbb{R}^3)} < \infty$$

for some $T < \infty$, since $0 < \epsilon < \epsilon_0(\delta)$ is equivalent to $8/(1+2\delta) < 8-2\epsilon$. The proof of $L^{8-2\epsilon}_{t,x}$ spacetime bound is straightforward by (56), Lemma 3.11, and interpolation theory. $\square$

4. Log-supercritical wave equation

For spherically symmetric log-supercritical nonlinear wave equation (1), (6) with finite energy E, we observe that the potential energy bound provides slightly better pointwise control, (26), of the solution than the one from the kinetic energy bound[3]; see [Ginibre et al. 1992; Tao 2007]. In this section, we consider a slightly more supercritical wave equation than the equation in [Tao 2007] and prove the same global regularity result by using (26).

Theorem 4.1. *Define*

$$\widetilde{H}^2_x(\mathbb{R}^3) := \dot{H}^1_x(\mathbb{R}^3) \cap \dot{H}^2_x(\mathbb{R}^3).$$

Let $0 < \alpha \leq \frac{4}{3}$ and (u_0, u_1) be smooth, compactly supported, and spherically symmetric initial data with energy E. Then there exists a global smooth solution to

$$\square u = |u|^4 u \log^\alpha(2+|u|^2), \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x). \tag{61}$$

Furthermore, we have the universal bound of $\widetilde{H}^2_x \times H^1_x$ norm, which depends on both the energy E and $\widetilde{H}^2_x \times H^1_x$ norm of the initial data, of the solution u; this implies that the solution u scatters in $\widetilde{H}^2_x(\mathbb{R}^3) \times H^1_x(\mathbb{R}^3)$.[4]

Remark 4.2. This theorem was proved in [Tao 2007] for $\alpha = 1$, and it is easy to get the same result for $\alpha < 1$ from that argument. We take advantage of (26) to extend the range of α up to $\frac{4}{3}$. In the remainder of this section, we will essentially follow Tao's argument to prove Theorem 4.1 using (26) and sketch the proof of $\widetilde{H}^2_x \times H^1_x$ scattering. We will skip the argument providing an explicit $\widetilde{H}^2_x \times H^1_x$ universal bound here; see [Tao 2007] for details.

We will use a well-known global continuation result (for a proof see [Sogge 1995], for example).

Theorem 4.3 (classical existence theory). *Let $u : [0,T] \times \mathbb{R}^3 \to \mathbb{C}$ be a classical solution*[5] *to (61) satisfying*

$$\|u\|_{L^\infty_t L^\infty_x([0,T]\times\mathbb{R}^3)} < \infty.$$

Then there is $\delta > 0$ such that one can extend the solution u to $[0, T+\delta] \times \mathbb{R}^3$.

Proof of Theorem 4.1. By time reversal symmetry, it suffices to consider the global existence and scattering theory of u on $\mathbb{R}_+ \times \mathbb{R}^3$.

---

[3] The kinetic energy bound can only provide $\int_I \int_{\mathbb{R}^3} |u|^8 \log^\alpha(2+|u|^2)\, dx\, dt \lesssim E^{3/2}$.

[4] The definition of $\widetilde{H}^2_x \times H^1_x$ scattering for the solution u is similar to Definition 2.1, but the $\dot{H}^1_x \times L^2_x$-norm is replaced by the $\widetilde{H}^2_x \times H^1_x$-norm.

[5] We call u a classical solution to (1) if u solves (1) and is smooth and compactly supported for each time.

By the Sobolev embedding theorem, for a classical solution u to (61) on $[0, T] \times \mathbb{R}^3$, we have

$$\|u\|_{L_t^\infty L_x^\infty([0,T]\times\mathbb{R}^3)} \lesssim \sum_{j=1}^{2} \|\nabla_x^j u\|_{L_t^\infty L_x^2([0,T]\times\mathbb{R}^3)}. \tag{62}$$

Hence, applying classical existence theory (Theorem 4.3), in order to show global existence, it suffices to prove that for any fixed $0 < T \le \infty$, we have

$$\sum_{j=1}^{2} \|\nabla_x^j u\|_{L_t^\infty L_x^2([0,T]\times\mathbb{R}^3)} < \infty,$$

provided that u is the classical solution to (61) on $[0, T] \times \mathbb{R}^3$.

Let $I = [a, b] \subseteq [0, T]$ be any interval. We define

$$M_I := \int_I \int_{\mathbb{R}^3} |u(t,x)|^8 \log^{5\alpha/4}(2 + |u(t,x)|^2)\, dx\, dt,$$
$$N_I := \sum_{j=0}^{1} \|\nabla_x^j u\|_{L_t^2 L_x^\infty(I\times\mathbb{R}^3)} + \|\nabla_{t,x}\nabla_x^j u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)},$$
$$D_I := \|\nabla_{t,x} u(a)\|_{H_x^1(\mathbb{R}^3)}.$$

In addition, we set $D = \|\nabla_{t,x} u(0)\|_{H_x^1(\mathbb{R}^3)}$.

From the Strichartz inequality, Hölder's inequality, and (62), we have

$$\begin{aligned}
N_I &\le C\|\nabla_{t,x} u(a)\|_{H_x^1(\mathbb{R}^3)} + C\sum_{j=0}^{1} \|\nabla_x^j(|u|^4 u \log^\alpha(2+|u|^2))\|_{L_t^1 L_x^2(I\times\mathbb{R}^3)}\\
&\le CD_I + C\sum_{j=0}^{1} \|\,|u|^4 |\nabla_x^j u| \log(2+|u|^2)\|_{L_t^1 L_x^2(I\times\mathbb{R}^3)}\\
&\le CD_I + C\|\,|u|^4 \log^{5\alpha/8}(2+|u|^2)\|_{L_t^2 L_x^2(I\times\mathbb{R}^3)} \left(\sum_{j=0}^{1} \|\nabla_x^j u\|_{L_t^2 L_x^\infty(I\times\mathbb{R}^3)} \|\log^{3\alpha/8}(2+|u|^2)\|_{L_t^\infty L_x^\infty(I\times\mathbb{R}^3)}\right)\\
&\le CD_I + C\|u \log^{5\alpha/32}(2+|u|^2)\|_{L_t^8 L_x^8(I\times\mathbb{R}^3)}^4 \left(\sum_{j=0}^{1} \|\nabla_x^j u\|_{L_t^2 L_x^\infty(I\times\mathbb{R}^3)} \|\log(2+|u|^2)\|_{L_t^\infty L_x^\infty(I\times\mathbb{R}^3)}^{3\alpha/8}\right)\\
&\le CD_I + CM_I^{1/2} N_I \log^{3\alpha/8}(2 + \|u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}^2)\\
&\le CD_I + CM_I^{1/2} N_I \log^{1/2}(2 + N_I^2).
\end{aligned}$$

From the result in [Tao 2007, Corollary 3.2], for any $\epsilon_0 > 0$,

$$\sum_{i=0}^{k} \frac{\epsilon_0}{\log(2 + (2C)^i D)} \to \infty \quad \text{as } k \to \infty.$$

Hence, for any fixed ϵ_0, the finiteness of $M_{[0,T]}$ from (26) implies that we can choose finitely many numbers $0 = T_0 < T_1 < \cdots < T_K < T_{K+1} = T$, with K depending on D, E, and ϵ_0, such that

$$M_i := \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}^3} |u(t,x)|^8 \log^{5\alpha/4}(2 + |u(t,x)|^2)\, dx\, dt = \frac{\epsilon_0}{\log(2 + (2C)^i D)}$$

for $0 \le i \le K-1$ and $M_K \le \epsilon_0 / \log(2 + (2C)^K D)$.

Choosing $\epsilon_0 = 1/(100C)^2$, by iteration and continuity arguments, we claim that $N_{[T_i, T_{i+1}]} < (2C)^{i+1} D$ for $0 \le i \le K$.[6] Indeed, assume that this claim is false for some $i = j$. Then there exists $t_0 \in (T_j, T_{j+1})$ such that $N_{[T_j, t_0]} = (2C)^{j+1} D$. We have

$$\begin{aligned}(2C)^{j+1} D &\le C(2C)^j D + C M_j^{1/2} N_{[T_j,t_0]} \log^{1/2}(2 + N_{[T_j,t_0]}^2) \\ &\le \tfrac{1}{2}(2C)^{j+1} D + \frac{\log^{1/2}(2+(2C)^{j+1} D)}{100 \log^{1/2}(2+(2C)^j D)} \times (2C)^{j+1} D \\ &\le \tfrac{3}{4}(2C)^{j+1} D.\end{aligned}$$

Thus the claim is proved by contradiction. This implies

$$\sum_{j=1}^{2} \|\nabla_x^j u\|_{L_t^\infty L_x^2([0,T]\times\mathbb{R}^3)} \le N_{[0,T]} \le \sum_{i=0}^{K} N_{[T_i,T_{i+1}]} < \sum_{i=0}^{K} (2C)^{i+1} D < \infty.$$

The universal bound only depends on D and E[7], indicating the global existence.

Now we sketch the proof of $\widetilde{H}_x^2 \times H_x^1$ scattering. From a similar argument as the one discussed in Remark 2.2, in order to prove $\widetilde{H}_x^2(\mathbb{R}^3) \times H_x^1(\mathbb{R}^3)$ scattering, it suffices to show that

$$\| |u|^4 u \log^\alpha(2 + |u|^2) \|_{L_t^1 H_x^1(\mathbb{R}_+ \times \mathbb{R}^3)} < \infty. \tag{63}$$

By the above discussion, the universal bound is independent of T. Hence we have $N_{\mathbb{R}_+} < \infty$. By Hölder's inequality,

$$\| |u|^4 u \log^\alpha(2 + |u|^2) \|_{L_t^1 H_x^1(\mathbb{R}_+ \times \mathbb{R}^3)} \lesssim M_{\mathbb{R}_+}^{1/2} N_{\mathbb{R}_+} \log^{1/2}(2 + N_{\mathbb{R}_+}^2) < \infty. \qquad \square$$

Appendix: Proof of (30)

Since (u_0, u_1) lies in $X_1 \times Y_1$, defined in (9), we have

$$\|u_0\|_{X_1}^2 \ge \int_{\mathbb{R}^3} |\nabla u_0|^2 \log^{2\gamma}(1 + |x|)\, dx \gtrsim (\log^{2\gamma} s) \int_{|x|>s} |\nabla u_0|^2\, dx.$$

Hence

$$\int_{|x|>s} |\nabla u_0|^2\, dx \lesssim \frac{\|u_0\|_{X_1}^2}{\log^{2\gamma} s}. \tag{64}$$

Similarly,

$$\int_{|x|>s} |u_1|^2\, dx \lesssim \frac{\|u_1\|_{Y_1}^2}{\log^{2\gamma} s}. \tag{65}$$

[6] See the similar arguments in Lemma 3.5 and Corollary 3.8 or Proposition 3.1 in [Tao 2007].

[7] In fact, from corollary 3.2 in [Tao 2007], we have $N_{\mathbb{R}_+} \lesssim (2 + D)^{(2+D)^{O(E)}}$.

Now, consider

$$\begin{aligned}\int_{|x|>s} F(u_0(x))\,dx &= \int_{\{|x|>s\}\cap\{|u_0|<1/3\}} F(u_0(x))\,dx + \int_{\{|x|>s\}\cap\{|u_0|\geq 1/3\}} F(u_0(x))\,dx \\ &\lesssim \int_{\{|x|>s\}\cap\{|u_0|<1/3\}} |u_0|^6(-\log|u_0|)\,dx + \int_{\{|x|>s\}\cap\{|u_0|\geq 1/3\}} |u_0|^6\,dx =: I + II.\end{aligned}$$

Let

$$\begin{aligned}I = &\int_{\{|x|>s\}\cap\{|u_0(x)|<1/|x|^{2/3}\}} |u_0|^6(-\log|u_0|)\,dx + \int_{\{|x|>s\}\cap\{1/|x|^{2/3}\leq|u_0(x)|\leq 1/3\}} |u_0|^6(-\log|u_0|)\,dx \\ &=: I_1 + I_2.\end{aligned}$$

When s is sufficiently large,

$$\begin{aligned}I_1 &\lesssim \int_{\{|x|>s\}\cap\{|u_0|<1/|x|^{2/3}\}} |u_0|^{11/2}\Big(\sup_{|u_0|<s^{-2/3}} |u_0|^{1/2}(-\log|u_0|)\Big)\,dx \\ &\lesssim \int_{|x|>s} |x|^{-11/3}\,dx \lesssim s^{-2/3} \lesssim \frac{1}{\log^{2\gamma} s}.\end{aligned} \tag{66}$$

Now we aim to prove that $I_2 + II \lesssim \frac{1}{\log^{2\gamma} s}$ for s sufficiently large. For $\alpha \in \mathbb{R}$, define

$$Q(\alpha) := \int_{\mathbb{R}^3} \left|\frac{u_0 \log^\alpha(2+|x|)}{2+|x|}\right|^2 dx.$$

We claim that

$$Q(\alpha) \leq C(\|u_0\|_{X_1}, E, \alpha) \quad \text{for } \alpha \leq \gamma, \tag{67}$$

where E is the energy. Indeed, if $\alpha \leq 0$, by Hölder's inequality and Hardy's inequality,

$$\begin{aligned}Q(\alpha) &= \int_{|x|<3}\left|\frac{u_0 \log^\alpha(2+|x|)}{2+|x|}\right|^2 dx + \int_{|x|\geq 3}\left|\frac{u_0 \log^\alpha(2+|x|)}{2+|x|}\right|^2 dx \\ &\lesssim_\alpha \int_{|x|<3} |u_0|^2\,dx + \int_{\mathbb{R}^3}\left|\frac{u_0}{|x|}\right|^2 dx \lesssim \|u_0\|^2_{\dot{H}^1_x(\mathbb{R}^3)} + \Big(\int_{\mathbb{R}^3} F(u_0)\,dx\Big)^{1/3} \leq C(E,\alpha).\end{aligned} \tag{68}$$

Again, if $0 < \alpha \leq \gamma$,

$$\begin{aligned}Q(\alpha) &= \int_{\mathbb{R}^3}\left|\frac{u_0 \log^\alpha(2+|x|)}{2+|x|}\right|^2 dx \lesssim_\alpha \int_{|x|<3} |u_0|^2\,dx + \int_{\mathbb{R}^3}\left|\frac{u_0 \log^\alpha(2+|x|)}{|x|}\right|^2 dx \\ &\lesssim \Big(\int_{|x|<3} |u_0|^6\,dx\Big)^{1/3} + \int_{\mathbb{R}^3} |\nabla(u_0 \log^\alpha(2+|x|))|^2\,dx \\ &\lesssim_\alpha \Big(\int_{\mathbb{R}^3} F(u)\,dx\Big)^{1/3} + \int_{\mathbb{R}^3} |\nabla u_0 \log^\alpha(2+|x|)|^2\,dx + \int_{\mathbb{R}^3}\left|\frac{u_0 \log^{\alpha-1}(2+|x|)}{2+|x|}\right|^2 dx \\ &\lesssim E^{1/3} + \int_{|x|<3} |\nabla u_0|^2\,dx + \int_{|x|\geq 3} |\nabla u_0 \log^\gamma(1+|x|)|^2\,dx + Q(\alpha-1) \\ &\lesssim E^{1/3} + E + \|u_0\|_{X_1} + Q(\alpha-1).\end{aligned}$$

By an inductive argument and (68), the claim is proved.

Fix $s \gg 1$. Let χ be the smooth radial function which equals 1 on $\{|x| > s\}$, 0 on $\{|x| < s/2\}$, $0 \le \chi \le 1$ and $|\nabla \chi| \lesssim 1/s$. Then we have $|\nabla \chi| \lesssim 1/|x|$. By the Sobolev embedding theorem and Hardy's inequality,

$$\begin{aligned}
&\log^{6\gamma} s \int_{|x|>s} |u_0|^6 \, dx \\
&\le \int_{|x|>s} |u_0|^6 \log^{6\gamma}(|x|) \, dx \le \int_{\mathbb{R}^3} (\chi |u_0| \log^{\gamma}(2+|x|))^6 \, dx \lesssim \left(\int_{\mathbb{R}^3} \left| \nabla \big(\chi u_0 \log^{\gamma}(2+|x|) \big) \right|^2 dx \right)^3 \\
&\lesssim_{\gamma} \left(\int_{\mathbb{R}^3} |\nabla \chi u_0 \log^{\gamma}(2+|x|)|^2 \, dx + \int_{\mathbb{R}^3} |\chi \nabla u_0 \log^{\gamma}(2+|x|)|^2 \, dx + \int_{\mathbb{R}^3} \left| \frac{\chi u_0 \log^{\gamma-1}(2+|x|)}{2+|x|} \right|^2 dx \right)^3 \\
&=: (J_1 + J_2 + J_3)^3.
\end{aligned}$$

We can compute that

$$\begin{aligned}
&J_2 \lesssim_{\gamma} \int_{\mathbb{R}^3} \left| \nabla u_0 \log^{\gamma}(1+|x|) \right|^2 dx + \int_{|x|<3} |\nabla u_0|^2 \, dx \le \|u_0\|_{X_1}^2 + E, \\
&J_3 \lesssim C(\|u_0\|_{X_1}, E, \gamma), \qquad \text{by (67)}.
\end{aligned}$$

Since $\nabla \chi \lesssim 1/|x|$,

$$\begin{aligned}
J_1 &\lesssim \int_{|x|>s/2} \left| \frac{u_0 \log^{\gamma}(2+|x|)}{|x|} \right|^2 dx \lesssim \int_{|x|>s/2} \left| \frac{u_0 \log^{\gamma}(2+|x|)}{2+|x|} \right|^2 dx \\
&\lesssim C(\|u_0\|_{X_1}, E, \gamma).
\end{aligned}$$

Hence $\log^{6\gamma} s \int_{|x|>s} |u_0|^6 \, dx \le C(\|u_0\|_{X_1}, E, \gamma)$ for sufficiently large s. Then we deduce

$$II \le \int_{|x|>s} |u_0|^6 \, dx \lesssim \frac{1}{\log^{6\gamma} s} \le \frac{1}{\log^{2\gamma} s}. \tag{69}$$

Similarly,

$$\begin{aligned}
\log^{6\gamma-1} s \int_{\{|x|>s\} \cap \{1/|x|^{2/3} \le |u_0| \le 1/3\}} |u_0|^6 (-\log |u_0|) \, dx &\lesssim \log^{6\gamma-1} s \int_{|x|>s} |u_0|^6 \log(|x|) \, dx \\
&\lesssim \int_{|x|>s} |u_0|^6 \log^{6\gamma}(|x|) \, dx \lesssim C(\|u_0\|_{X_1}, E, \gamma).
\end{aligned}$$

Therefore,

$$I_2 \lesssim \frac{1}{\log^{6\gamma-1} s} \le \frac{1}{\log^{2\gamma} s}. \tag{70}$$

Combining (64), (65), (66), (69), and (70), we obtain (30).

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