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MICHAEL HITRIK, JOHANNES SJÖSTRAND AND JOE VIOLA

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RESOLVENT ESTIMATES FOR ELLIPTIC QUADRATIC DIFFERENTIAL OPERATORS

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Sharp resolvent bounds for nonselfadjoint semiclassical elliptic quadratic differential operators are established, in the interior of the range of the associated quadratic symbol.

1. Introduction and statement of result

It is well known that the spectrum of a nonselfadjoint operator does not control its resolvent, and that the latter may become very large even far from the spectrum. Understanding the behavior of the norm of the resolvent of a given nonselfadjoint operator is therefore a natural and basic problem, which has recently received considerable attention, in particular, within the circle of questions around the notion of the pseudospectrum [Trefethen and Embree 2005]. Some general upper bounds on resolvents are provided by abstract operator theory, and, restricting our attention to the setting of semiclassical pseudodifferential operators on $\mathbb{R}^n$, relevant for this note, we recall a rough statement of such bounds, following [Dencker et al. 2004; Markus 1988; Viola 2012]. Assume that $P = p^w(x, hD_x)$ is the semiclassical Weyl quantization on $\mathbb{R}^n$ of a complex-valued smooth symbol p with $\operatorname{Re} p \geq 0$, belonging to a suitable symbol class and satisfying an ellipticity condition at infinity, guaranteeing that the spectrum of P is discrete in a small neighborhood of the origin. Then the norm of the L^2-resolvent of P is bounded from above by a quantity of the form $\mathcal{O}(1)\exp(\mathcal{O}(1)h^{-n})$, provided that $z \in \operatorname{neigh}(0, \mathbb{C})$ is not too close to the spectrum of P. On the other hand, the available lower bounds on the resolvent of P, coming from the pseudospectral considerations, are typically of the form $C_N^{-1}h^{-N}$, $N \in \mathbb{N}$, or $(1/C)e^{1/(Ch)}$, provided that p enjoys some analyticity properties [Dencker et al. 2004]. Therefore, there appears to be a substantial gap between the available upper and lower bounds on the resolvent, especially when $n \geq 2$. The purpose of this note is to address the issue of bridging this gap in the particular case of an elliptic quadratic semiclassical differential operator on $\mathbb{R}^n$, and to establish a sharp upper bound on the norm of its resolvent.

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Let q be a complex-valued quadratic form:

$$q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}, \quad (x, \xi) \mapsto q(x, \xi). \tag{1-1}$$

We shall assume throughout the following discussion that the quadratic form q is elliptic on $\mathbb{R}^{2n}$, in the sense that $q(X) = 0$, $X \in \mathbb{R}^{2n}$, precisely when $X = 0$. In this case, according to Lemma 3.1 of [Sjöstrand 1974], if $n > 1$, there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that $\mathrm{Re}(\lambda q)$ is positive definite. In the case when $n = 1$, the same conclusion holds, provided that the range of q on $\mathbb{R}^2$ is not all of $\mathbb{C}$ [Sjöstrand 1974; Hitrik 2004], which we assume in what follows. After a multiplication of q by λ, we may and do henceforth assume that $\lambda = 1$, so that

$$\mathrm{Re}\, q > 0. \tag{1-2}$$

It follows that the range $\Sigma(q) = q(\mathbb{R}^{2n})$ of q on $\mathbb{R}^{2n}$ is a closed angular sector with a vertex at zero, contained in the union of $\{0\}$ and the open right half-plane.

Associated to the quadratic form q is the semiclassical Weyl quantization $q^w(x, hD_x)$, $0 < h \le 1$, which we view as a closed densely defined operator on $L^2(\mathbb{R}^n)$, equipped with the domain

$$\{u \in L^2(\mathbb{R}^n) : q^w(x, hD_x)u \in L^2(\mathbb{R}^n)\}.$$

The spectrum of $q^w(x, hD_x)$ is discrete, and following [Sjöstrand 1974], we shall now recall its explicit description. See also [Boutet de Monvel 1974]. To that end, let us introduce the Hamilton map F of q,

$$F : \mathbb{C}^{2n} \to \mathbb{C}^{2n},$$

defined by the identity

$$q(X, Y) = \sigma(X, FY), \quad X, Y \in \mathbb{C}^{2n}. \tag{1-3}$$

Here the left-hand side is the polarization of q, viewed as a symmetric bilinear form on $\mathbb{C}^{2n}$, and σ is the complex symplectic form on $\mathbb{C}^{2n}$. We notice that the Hamilton map F is skew-symmetric with respect to σ, and, furthermore,

$$FY = \tfrac{1}{2} H_q(Y), \tag{1-4}$$

where $H_q = q'_\xi \cdot \partial_x - q'_x \cdot \partial_\xi$ is the Hamilton field of q.

The ellipticity condition (1-2) implies that the spectrum of the Hamilton map F avoids the real axis, and, in general, we know from Section 21.5 of [Hörmander 1985] that if λ is an eigenvalue of F, so is $-\lambda$, and the algebraic multiplicities agree. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of F, counted according to their multiplicity, such that $\lambda_j / i \in \Sigma(q)$, $j = 1, \dots, n$. Then the spectrum of the operator $q^w(x, hD_x)$ is given by the eigenvalues of the form

$$h \sum_{j=1}^{n} \frac{\lambda_j}{i}(2\nu_{j,\ell} + 1), \quad \nu_{j,\ell} \in \mathbb{N} \cup \{0\}. \tag{1-5}$$

We notice that $\mathrm{Spec}(q^w(x, hD_x)) \subset \Sigma(q)$, and from [Pravda-Starov 2007] we also know that

$$\mathrm{Spec}(q^w(x, hD_x)) \cap \partial\Sigma(q) = \varnothing,$$

provided that the operator $q^w(x, hD_x)$ is not normal.

Here is the main result of this work.

Theorem 1.1. *Let $q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$ be a quadratic form such that* $\operatorname{Re} q$ *is positive definite. Let $\Omega \Subset \mathbb{C}$. There exists $h_0 > 0$, and for every $C > 0$ there exists $A > 0$ such that*

$$\|(q^w(x, hD_x) - z)^{-1}\|_{\mathscr{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))} \le A \exp(Ah^{-1}), \tag{1-6}$$

for all $h \in (0, h_0]$, and all $z \in \Omega$, with $\operatorname{dist}\big(z, \operatorname{Spec}(q^w(x, hD_x))\big) \ge 1/C$. Furthermore, for all $C > 0$, $L \ge 1$, there exists $A > 0$ such that for $h \in (0, h_0]$, we have

$$\|(q^w(x, hD_x) - z)^{-1}\|_{\mathscr{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))} \le A \exp\Big(Ah^{-1} \log \frac{1}{h}\Big), \tag{1-7}$$

if the spectral parameter $z \in \Omega$ is such that

$$\operatorname{dist}\big(z, \operatorname{Spec}(q^w(x, hD_x))\big) \ge h^L/C.$$

Remark 1.2. Assume that the elliptic quadratic form q, with $\operatorname{Re} q > 0$, is such that the Poisson bracket $\{\operatorname{Re} q, \operatorname{Im} q\}$ does not vanish identically, and let $z \in \Sigma(q)^o$, $z \notin \operatorname{Spec}(q^w(x, hD_x))$. Here $\Sigma(q)^o$ is the interior of $\Sigma(q)$. Then it follows from the results of [Dencker et al. 2004; Pravda-Starov 2008] that we have the following lower bound for $(q^w(x, hD_x) - z)^{-1}$, as $h \to 0$:

$$\|(q^w(x, hD_x) - z)^{-1}\|_{\mathscr{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))} \ge \frac{1}{C_0} e^{1/(C_0 h)}, \quad C_0 > 0.$$

It follows that the upper bound (1-6) is of the right order of magnitude, when $z \in \Sigma(q)^o \cap \Omega$, $|z| \sim 1$, avoids a closed cone $\subset \Sigma(q) \cup \{0\}$, containing the spectrum of $q^w(x, hD_x)$.

Remark 1.3. In Section 4, we give a simple example of an elliptic quadratic operator on $\mathbb{R}^2$, for which the associated Hamilton map has a nonvanishing nilpotent part in its Jordan decomposition, and whose resolvent exhibits the superexponential growth given by the right-hand side of (1-7), in the region of the complex spectral plane where $|z| \sim 1$, $\operatorname{dist}\big(z, \operatorname{Spec}(q^w(x, hD_x))\big) \sim h$. On the other hand, sharper resolvent estimates can be obtained when the Hamilton map F of q is diagonalizable. In this case, we shall see in Section 4 that the bound (1-7) improves to the following, when $z \in \Omega$ and $h \in (0, h_0]$:

$$\|(q^w(x, hD_x) - z)^{-1}\|_{\mathscr{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))} \le \frac{Ae^{A/h}}{\operatorname{dist}\big(z, \operatorname{Spec}(q^w(x, hD_x))\big)}. \tag{1-8}$$

Remark 1.4. Let $z_0 \in \operatorname{Spec}(q^w(x, hD_x)) \cap \Omega$ and let

$$\Pi_{z_0} = \frac{1}{2\pi i} \int_{\partial D} (z - q^w(x, hD_x))^{-1} \, dz$$

be the spectral projection of $q^w(x, hD_x)$, associated to the eigenvalue z_0. Here $D \subset \Omega$ is a small open disc centered at z_0, such that the closure $\overline{D}$ avoids the set $\operatorname{Spec}(q^w(x, hD_x)) \backslash \{z_0\}$, and ∂D is its positively oriented boundary. Assume for simplicity that the quadratic form q is such that its Hamilton map is diagonalizable. Then it follows from (1-8) that

$$\Pi_{z_0} = \mathcal{O}(1) \exp(\mathcal{O}(1) h^{-1}) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

In the context of elliptic quadratic differential operators in dimension one, resolvent bounds have been studied in particular, in [Boulton 2002; Davies 2000; Davies and Kuijlaars 2004]. We should also mention the general resolvent estimates in [Dencker et al. 2004; Sjöstrand 2010], valid for h-pseudodifferential operators when the spectral parameter is close to the boundary of the range of the corresponding symbol.

The plan of this note is as follows. In Section 2, we make an essentially well-known reduction of our problem to the setting of a quadratic differential operator acting in a Bargmann space of holomorphic functions, convenient for the subsequent analysis. Section 3 is devoted to suitable a priori elliptic estimates, valid for holomorphic functions vanishing to a high, h-dependent order at the origin. The proof of Theorem 1.1 is completed in Section 4 by some elementary considerations in the space of holomorphic polynomials on $\mathbb{C}^n$, of degree not exceeding $\mathcal{O}(h^{-1})$.

2. The normal form reduction

We shall be concerned here with a quadratic form $q : T^*\mathbb{R}^n \to \mathbb{C}$, such that $\operatorname{Re} q$ is positive definite. Let F be the Hamilton map of q, introduced in (1-3). When $\lambda \in \operatorname{Spec}(F)$, we let

$$V_\lambda = \operatorname{Ker}((F-\lambda)^{2n}) \subset T^*\mathbb{C}^n \tag{2-1}$$

be the generalized eigenspace belonging to the eigenvalue λ. The symplectic form σ is then nondegenerate viewed as a bilinear form on $V_\lambda \times V_{-\lambda}$.

We introduce the stable outgoing manifold for the Hamilton flow of the quadratic form $i^{-1}q$, given by

$$\Lambda^+ := \bigoplus_{\operatorname{Im}\lambda>0} V_\lambda \subset T^*\mathbb{C}^n. \tag{2-2}$$

It is then true that Λ^+ is a complex Lagrangian plane such that q vanishes along Λ^+, and Proposition 3.3 of [Sjöstrand 1974] states that the complex Lagrangian Λ^+ is strictly positive in the sense that

$$\frac{1}{i}\sigma(X, \overline{X}) > 0, \quad 0 \neq X \in \Lambda^+. \tag{2-3}$$

We also define

$$\Lambda^- = \bigoplus_{\operatorname{Im}\lambda<0} V_\lambda \subset T^*\mathbb{C}^n, \tag{2-4}$$

which is a complex Lagrangian plane such that q vanishes along Λ^-, and from the arguments of [Sjöstrand 1974] we also know that Λ^- is strictly negative in the sense that

$$\frac{1}{i}\sigma(X, \overline{X}) < 0, \quad 0 \neq X \in \Lambda^-. \tag{2-5}$$

The complex Lagrangians Λ^+ and Λ^- are transversal, and following [Helffer and Sjöstrand 1984; Sjöstrand 1987], we would like to implement a reduction of the quadratic form q to a normal form by applying a linear complex canonical transformation which reduces Λ^+ to $\{(x,\xi) \in \mathbb{C}^{2n} : \xi = 0\}$ and Λ^- to $\{(x,\xi) \in \mathbb{C}^{2n} : x = 0\}$. We shall then be able to implement the canonical transformation in question by an FBI–Bargmann transform. Let us first simplify q by means of a suitable real linear canonical transformation. When doing so, we observe that the fact that the Lagrangian Λ^- is strictly negative implies that it is of the form

$$\eta = A_- y, \quad y \in \mathbb{C}^n,$$

where the complex symmetric $n \times n$ matrix A_- is such that $\operatorname{Im} A_- < 0$. Here (y, η) are the standard canonical coordinates on $T^*\mathbb{R}^n_y$ that we extend to the complexification $T^*\mathbb{C}^n_y$. Using the real linear canonical transformation $(y, \eta) \mapsto (y, \eta - (\operatorname{Re} A_-)y)$, we reduce Λ^- to the form $\eta = i \operatorname{Im} A_- y$, and by a diagonalization of $\operatorname{Im} A_-$, we obtain the standard form $\eta = -iy$. After this real linear symplectic change of coordinates and the conjugation of the semiclassical Weyl quantization $q^w(x, hD_x)$ of q by means of the corresponding unitary metaplectic operator, we may assume that Λ^- is of the form

$$\eta = -iy, \quad y \in \mathbb{C}^n, \tag{2-6}$$

while the positivity property of the complex Lagrangian Λ^+ is unaffected, so that, in the new real symplectic coordinates, extended to the complexification, Λ^+ is of the form

$$\eta = A_+ y, \quad \operatorname{Im} A_+ > 0. \tag{2-7}$$

Let

$$B = B_+ = (1 - iA_+)^{-1} A_+, \tag{2-8}$$

and notice that the matrix B is symmetric. Let us introduce the FBI–Bargmann transform

$$Tu(x) = Ch^{-3n/4} \int e^{i\varphi(x,y)/h} u(y)\, dy, \quad x \in \mathbb{C}^n, \quad C > 0, \tag{2-9}$$

where

$$\varphi(x, y) = \frac{i}{2}(x-y)^2 - \frac{1}{2}(Bx, x). \tag{2-10}$$

The associated complex linear canonical transformation on $\mathbb{C}^{2n}$

$$\kappa_T : (y, -\varphi'_y(x, y)) \mapsto (x, \varphi'_x(x, y)) \tag{2-11}$$

is of the form

$$\kappa_T : (y, \eta) \mapsto (x, \xi) = (y - i\eta, \eta + iB\eta - By), \tag{2-12}$$

and we see that the image of $\Lambda_- : \eta = -iy$ under κ_T is the fiber $\{(x, \xi) \in \mathbb{C}^{2n} : x = 0\}$, while $\kappa_T(\Lambda^+)$ is given by the equation $\{(x, \xi) \in \mathbb{C}^{2n} : \xi = 0\}$.

We know from [Sjöstrand 1996] that for a suitable choice of $C > 0$ in (2-9), the map T is unitary:

$$T : L^2(\mathbb{R}^n) \to H_{\Phi_0}(\mathbb{C}^n), \tag{2-13}$$

where

$$H_{\Phi_0}(\mathbb{C}^n) = \operatorname{Hol}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n : e^{-2\Phi_0/h} L(dx)),$$

and Φ_0 is a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$, given by

$$\Phi_0(x) = \sup_{y \in \mathbb{R}^n}(-\operatorname{Im} \varphi(x, y)) = \tfrac{1}{2}((\operatorname{Im} x)^2 + \operatorname{Im}(Bx, x)). \tag{2-14}$$

We also recall [Sjöstrand 1996] that the canonical transformation κ_T in (2-11) maps $\mathbb{R}^{2n}$ bijectively onto

$$\Lambda_{\Phi_0} := \left\{ \left(x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right) : x \in \mathbb{C}^n \right\}. \tag{2-15}$$

As explained in Chapter 11 of [Sjöstrand 1982], the strict positivity of $\kappa_T(\Lambda^+) = \{(x, \xi) \in \mathbb{C}^{2n} : \xi = 0\}$ with respect to Λ_{Φ_0} implies that the quadratic weight function Φ_0 is strictly convex, so that

$$\Phi_0(x) \sim |x|^2, \quad x \in \mathbb{C}^n. \tag{2-16}$$

Next we have the exact Egorov property [Sjöstrand 1996],

$$Tq^w(y, hD_y)u = \tilde{q}^w(x, hD_x)Tu, \quad u \in \mathcal{S}(\mathbb{R}^n), \tag{2-17}$$

where $\tilde{q}$ is a quadratic form on $\mathbb{C}^{2n}$ given by $\tilde{q} = q \circ \kappa_T^{-1}$. Therefore it follows that

$$\tilde{q}(x, \xi) = Mx \cdot \xi, \tag{2-18}$$

where M is a complex $n \times n$ matrix. We have

$$H_{\tilde{q}} = Mx \cdot \partial_x - M^t \xi \cdot \partial_\xi,$$

and using (1-4), we see that the corresponding Hamilton map

$$\widetilde{F} = \tfrac{1}{2} \begin{pmatrix} M & 0 \\ 0 & -M^t \end{pmatrix}$$

maps $(x, 0) \in \kappa_T(\Lambda^+)$ to $(1/2)(Mx, 0)$. Now F and $\widetilde{F}$ are isospectral, and we conclude that, with the agreement of algebraic multiplicities, the following holds:

$$\operatorname{Spec}(M) = \operatorname{Spec}(2F) \cap \{\operatorname{Im} \lambda > 0\}. \tag{2-19}$$

Therefore the problem of estimating the norm of the resolvent of $q^w(x, hD_x)$ on $L^2(\mathbb{R}^n)$ is equivalent to controlling the norm of the resolvent of the quadratic operator $\tilde{q}^w(x, hD_x)$, acting in the space $H_{\Phi_0}(\mathbb{C}^n)$, where the quadratic weight Φ_0 enjoys the property (2-16).

In what follows, it will be convenient to reduce the matrix M in (2-18) to its Jordan normal form. To this end, let us notice that we can implement this reduction by considering a complex canonical transformation of the form

$$\kappa_C : \mathbb{C}^{2n} \ni (x, \xi) \mapsto (C^{-1}x, C^t \xi) \in \mathbb{C}^{2n}, \tag{2-20}$$

where C is a suitable invertible complex $n \times n$ matrix. On the operator level, associated to the transformation in (2-20), we have the operator $u(x) \mapsto |\det C| u(Cx)$, which maps the space $H_{\Phi_0}(\mathbb{C}^n)$ unitarily onto the space $H_{\Phi_1}(\mathbb{C}^n)$, where $\Phi_1(x) = \Phi_0(Cx)$ is a strictly plurisubharmonic quadratic weight such that $\kappa_C(\Lambda_{\Phi_0}) = \Lambda_{\Phi_1}$. We notice that the property

$$\Phi_1(x) \sim |x|^2, \quad x \in \mathbb{C}^n \tag{2-21}$$

remains valid.

We summarize the discussion pursued in this section in the following result.

Proposition 2.1. *Let $q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$ be a quadratic form with $\operatorname{Re} q > 0$. The operator*

$$q^w(x, hD_x) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

equipped with the domain

$$\mathcal{D}(q^w(x, hD_x)) = \{u \in L^2(\mathbb{R}^n) : (x^2 + (hD_x)^2)u \in L^2(\mathbb{R}^n)\},$$

is unitarily equivalent to the quadratic operator

$$\tilde{q}^w(x, hD_x) : H_{\Phi_1}(\mathbb{C}^n) \to H_{\Phi_1}(\mathbb{C}^n),$$

with the domain

$$\mathcal{D}(\tilde{q}^w(x, hD_x)) = \{u \in H_{\Phi_1}(\mathbb{C}^n) : (1 + |x|^2)u \in L^2_{\Phi_1}(\mathbb{C}^n)\}.$$

Here

$$\tilde{q}(x, \xi) = Mx \cdot \xi,$$

where M is a complex $n \times n$ block-diagonal matrix, each block being a Jordan matrix. The eigenvalues of M are precisely those of $2F$ in the upper half-plane, and the quadratic weight function $\Phi_1(x)$ satisfies

$$\Phi_1(x) \sim |x|^2, \quad x \in \mathbb{C}^n.$$

We have the ellipticity property

$$\operatorname{Re} \tilde{q}\left(x, \frac{2}{i}\frac{\partial \Phi_1}{\partial x}(x)\right) \sim |x|^2, \quad x \in \mathbb{C}^n. \tag{2-22}$$

Remark 2.2. The normal form reduction described in Proposition 2.1 is close to the corresponding discussion of Section 3 in [Sjöstrand 1974]. Here, for future computations, it will be convenient for us to work in the Bargmann space $H_{\Phi_1}(\mathbb{C}^n)$.

3. An elliptic estimate

Following the reduction of Proposition 2.1, here we concern ourselves with the quadratic operator $\tilde{q}^w(x, hD_x)$, acting on $H_{\Phi_1}(\mathbb{C}^n)$. The purpose of this section is to establish a suitable a priori estimate for holomorphic functions, vanishing to a high, h-dependent order at the origin, instrumental in the proof of Theorem 1.1. The starting point is the following observation, which comes directly from Lemma 4.5 in [Gérard and Sjöstrand 1987], and whose proof we give only for the convenience of the reader.

Lemma 3.1. *Let $u \in \operatorname{Hol}(\mathbb{C}^n)$ and assume that $\partial^\alpha u(0) = 0$, $|\alpha| < N$, and that $0 < C_0 < C_1 < \infty$. Then*

$$\|u\|_{L^\infty(B(0,C_0))} \le \left(N \frac{C_1}{C_1 - C_0}\right)\left(\frac{C_0}{C_1}\right)^N \|u\|_{L^\infty(B(0,C_1))}. \tag{3-1}$$

Here $B(0, C_j) = \{x \in \mathbb{C}^n : |x| \le C_j\}$, $j = 0, 1$.

Proof. By Taylor's formula, we have

$$u(x) = \int_0^1 \frac{(1-t)^{N-1}}{(N-1)!} \left(\frac{d}{dt}\right)^N u(tx)\, dt.$$

We may assume that $|x| = C_0$ and apply Cauchy's inequalities so that

$$\left| \left(\frac{d}{dt}\right)^N u(tx) \right| \le \frac{C_0^N N!}{(C_1 - C_0 t)^N} \|u\|_{L^\infty(B(0,C_1))}.$$

It suffices therefore to remark that the expression

$$N \int_0^1 \frac{(1-t)^{N-1}}{(C_1/C_0 - t)^N}\, dt$$

does not exceed

$$\frac{N}{C_1/C_0 - 1} \left(\frac{C_0}{C_1}\right)^{N-1}. \qquad \square$$

Let $K > 0$ be fixed and assume that $u \in H_{\Phi_1}(\mathbb{C}^n)$ is such that $\partial^\alpha u(0) = 0$, when $|\alpha| < N$. Using Lemma 3.1, we write

$$\begin{aligned} \|u\|^2_{H_{\Phi_1}(B(0,K))} &\le \|u\|^2_{L^2(B(0,K))} \\ &\le \mathcal{O}_K(1)\|u\|^2_{L^\infty(B(0,K))} \le \mathcal{O}_K(1) N^2 e^{-2N} \|u\|^2_{L^\infty(B(0,Ke))} \\ &\le \mathcal{O}_K(1) N^2 e^{-2N} \|u\|^2_{L^2(B(0,(K+1)e))} \le \mathcal{O}_K(1) N^2 e^{-2N} e^{(2/h)C_1(K+1)^2 e^2} \|u\|^2_{H_{\Phi_1}}. \end{aligned} \tag{3-2}$$

In the last inequality we used that $\Phi_1(x) \le C_1|x|^2$ for some $C_1 \ge 1$. It follows that

$$\|u\|_{H_{\Phi_1}(B(0,K))} \le \mathcal{O}_K(1) e^{-1/2h} \|u\|_{H_{\Phi_1}}, \tag{3-3}$$

provided that the integer N satisfies

$$N \ge \frac{2C_1(K+1)^2 e^2 + 1}{h}. \tag{3-4}$$

In what follows, we shall let $N_0 = N_0(K) \in \mathbb{N}$, $N_0 \sim h^{-1}$, be the least integer which satisfies (3-4).

It is now easy to derive an a priori estimate for functions in $H_{\Phi_1}(\mathbb{C}^n)$, which vanish to a high order at the origin. Let $\chi \in C_0^\infty(\mathbb{C}^n)$, $0 \le \chi \le 1$, be such that $\operatorname{supp}(\chi) \subset \{x \in \mathbb{C}^n : |x| \le K\}$, with $\chi(x) = 1$ for $|x| \le K/2$. If $u \in H_{\Phi_1}(\mathbb{C}^n)$ is such that $(1+|x|^2)u \in L^2_{\Phi_1}(\mathbb{C}^n)$, we have the quantization-multiplication formula [Sjöstrand 1990], valid for z in a compact subset of $\mathbb{C}$,

$$\begin{aligned} &((1-\chi)(\tilde{q}^w(x, hD_x) - z)u, u)_{L^2_{\Phi_1}} \\ &\qquad = \int (1-\chi(x)) \left(\tilde{q}\left(x, \frac{2}{i}\frac{\partial \Phi_1}{\partial x}(x)\right) - z \right) |u(x)|^2 e^{-2\Phi_1(x)/h}\, L(dx) + \mathcal{O}(h)\|u\|^2_{H_{\Phi_1}}. \end{aligned}$$

The ellipticity property

$$\operatorname{Re} \tilde{q}\left(x, \frac{2}{i}\frac{\partial \Phi_1}{\partial x}(x)\right) \ge \frac{|x|^2}{C_0}, \quad x \in \mathbb{C}^n, \tag{3-5}$$

valid for some $C_0 > 1$, implies that, on the support of $1-\chi$, we have

$$\mathrm{Re}\left(\tilde{q}\left(x, \frac{2}{i}\frac{\partial \Phi_1}{\partial x}(x)\right) - z\right) \geq \frac{|x|^2}{2C_0},$$

provided that $|z| \leq K^2/8C_0$. Restricting the attention to this range of z's and using the Cauchy–Schwarz inequality, we obtain that

$$\int (1-\chi(x))|u(x)|^2 e^{-2\Phi_1(x)/h} L(dx) \leq \mathcal{O}_K(1)\|(\tilde{q}^w(x, hD_x) - z)u\|_{H_{\Phi_1}} \|u\|_{H_{\Phi_1}} + \mathcal{O}_K(h)\|u\|^2_{H_{\Phi_1}}. \quad (3\text{-}6)$$

If $u \in H_{\Phi_1}(\mathbb{C}^n)$, $(1+|x|^2)u \in L^2_{\Phi_1}(\mathbb{C}^n)$, is such that $\partial^\alpha u(0) = 0$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| < N_0$, an application of (3-3) shows that the left-hand side of (3-6) is of the form

$$\|u\|^2_{H_{\Phi_1}} + \mathcal{O}_K(h^\infty)\|u\|^2_{H_{\Phi_1}}.$$

We may summarize the discussion so far in the following proposition.

Proposition 3.2. *Let $K > 0$ be fixed and assume that $u \in H_{\Phi_1}(\mathbb{C}^n)$, $(1+|x|^2)u \in L^2_{\Phi_1}(\mathbb{C}^n)$, is such that $\partial^\alpha u(0) = 0$, $|\alpha| < N_0$, where $N_0 \sim h^{-1}$ is the least integer such that*

$$N_0 \geq \frac{2C_1(K+1)^2 e^2 + 1}{h}.$$

Here $\Phi_1(x) \leq C_1|x|^2$, $C_1 \geq 1$. Assume also that $|z| \leq K^2/8C_0$, where $C_0 > 1$ is the ellipticity constant in (3-5)*. Then we have the following a priori estimate*, *valid for all $h > 0$ sufficiently small*:

$$\|u\|_{H_{\Phi_1}} \leq \mathcal{O}(1)\|(\tilde{q}^w(x, hD_x) - z)u\|_{H_{\Phi_1}}.$$

We finish this section by discussing norm estimates for the linear continuous projection operator

$$\tau_N : H_{\Phi_1}(\mathbb{C}^n) \to H_{\Phi_1}(\mathbb{C}^n),$$

given by

$$\tau_N u(x) = \sum_{|\alpha|<N} (\alpha!)^{-1}(\partial^\alpha u(0))x^\alpha. \quad (3\text{-}7)$$

As in Proposition 3.2, we shall be concerned with the case when $N \in \mathbb{N}$ satisfies $N \sim h^{-1}$. The projection operator τ_N is highly nonorthogonal — nevertheless, using the strict convexity of the quadratic weight Φ_1, establishing an exponential upper bound on its norm will be quite straightforward, as well as sufficient for our purposes. In the following, we shall use the fact that

$$\frac{1}{C_1}|x|^2 \leq \Phi_1(x) \leq C_1|x|^2, \quad C_1 \geq 1. \quad (3\text{-}8)$$

Notice also that $[\tau_N, \tilde{q}^w(x, hD_x)] = 0$.

Proposition 3.3. *Assume that $N \in \mathbb{N}$ is such that $Nh \leq \mathcal{O}(1)$. There exists a constant $C > 0$ such that*

$$\|\tau_N\|_{\mathcal{L}(H_{\Phi_1}(\mathbb{C}^n), H_{\Phi_1}(\mathbb{C}^n))} \leq Ce^{C/h}. \quad (3\text{-}9)$$

Proof. We first observe that when deriving the bound (3-9), it suffices to restrict the attention to the space of holomorphic polynomials, which is dense in $H_{\Phi_1}(\mathbb{C}^n)$. Indeed, the analysis in [Sjöstrand 1974] tells us that the linear span of the generalized eigenfunctions of the quadratic operator $q^w(x, hD_x)$ is dense in $L^2(\mathbb{R}^n)$, which implies the density of the holomorphic polynomials in $H_{\Phi_1}(\mathbb{C}^n)$. Let

$$u(x) = \sum_{|\alpha|\le N_1} a_\alpha x^\alpha \tag{3-10}$$

for some N_1, where we may assume that $N_1 > N$. We have

$$\tau_N u = \sum_{|\alpha|<N} a_\alpha x^\alpha,$$

and therefore, using (3-8), we see that

$$\|\tau_N u\|^2_{H_{\Phi_1}} \le \|\tau_N u\|^2_{H_{\Phi_\ell}}, \tag{3-11}$$

where $\Phi_\ell(x) = |x|^2/C_1$. When computing the expression in the right-hand side of (3-11), we notice that since Φ_ℓ is radial, we have

$$(x^\alpha, x^\beta)_{H_{\Phi_\ell}} = 0, \quad \alpha \ne \beta,$$

while

$$(x^\alpha, x^\alpha)_{H_{\Phi_\ell}} = \prod_{j=1}^n \int |x_j|^{2\alpha_j} e^{-2|x_j|^2/C_1 h}\, L(dx_j),$$

which is immediately seen to be equal to

$$\left(\frac{C_1 h}{2}\right)^{n+|\alpha|} \pi^n \alpha!.$$

It follows that

$$\|\tau_N u\|^2_{H_{\Phi_1}} \le \sum_{|\alpha|<N} |a_\alpha|^2 \left(\frac{C_1 h}{2}\right)^{n+|\alpha|} \pi^n \alpha!. \tag{3-12}$$

On the other hand, (3-8) also gives that

$$\|u\|^2_{H_{\Phi_1}} \ge \|u\|^2_{H_{\Phi_u}}, \tag{3-13}$$

where $\Phi_u(x) = C_1|x|^2$, and arguing as above, it is straightforward to see that the right-hand side of (3-13) is given by the expression

$$\sum_{|\alpha|\le N_1} |a_\alpha|^2 \left(\frac{h}{2C_1}\right)^{n+|\alpha|} \pi^n \alpha!.$$

We conclude that when $u \in H_{\Phi_1}(\mathbb{C}^n)$ is a holomorphic polynomial of the form (3-10),

$$\|u\|^2_{H_{\Phi_1}} \ge \sum_{|\alpha|<N} |a_\alpha|^2 \left(\frac{h}{2C_1}\right)^{n+|\alpha|} \pi^n \alpha!. \tag{3-14}$$

Combining (3-12), (3-14), and recalling the fact that $Nh \le \mathcal{O}(1)$, we obtain the result of the proposition. $\square$

4. The finite-dimensional analysis and end of the proof

In this section we analyze the resolvent of the quadratic operator $\tilde{q}^w(x, hD_x)$ acting on the finite-dimensional space $\operatorname{Im}\tau_N$, where τ_N is the projection operator introduced in (3-7) and $N \sim h^{-1}$. This will allow us to complete the proof of Theorem 1.1. For $m = 0, 1, \ldots$, define the finite-dimensional subspace $E_m \subset H_{\Phi_1}(\mathbb{C}^n)$ as the linear span of the monomials x^α, with $|\alpha| = m$. We have

$$\operatorname{Im}\tau_N = \bigoplus_{m=0}^{N-1} E_m.$$

We may notice here that

$$\nu_m := \dim E_m = \frac{1}{(n-1)!}(m+1)\cdots(m+n-1), \tag{4-1}$$

and also that each space E_m is invariant under $\tilde{q}^w(x, hD_x)$. We shall equip $\operatorname{Im}\tau_N$ with the basis

$$\varphi_\alpha(x) := (\pi^n \alpha!)^{-1/2} h^{-n/2} (h^{-1/2}x)^\alpha, \quad |\alpha| < N, \tag{4-2}$$

which will be particularly convenient in the following computations, since the normalized monomials φ_α form an orthonormal basis in the weighted space $H_\Phi(\mathbb{C}^n)$, where $\Phi(x) = (1/2)|x|^2$. We have

$$\operatorname{Im}\tau_N \subset H_{\Phi_1}(\mathbb{C}^n) \cap H_\Phi(\mathbb{C}^n),$$

in view of the strict convexity of the weights.

Let us first derive an upper bound on the norm of the inverse of the operator

$$z - \tilde{q}^w(x, hD_x) : E_m \to E_m, \quad 0 \le m < N \sim h^{-1},$$

assuming that E_m has been equipped with the H_Φ-norm. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the Hamilton map F of q in the upper half-plane, repeated according to their algebraic multiplicity. According to Proposition 2.1, we then have

$$\tilde{q}^w(x, hD_x) = \tilde{q}^w_D(x, hD_x) + \tilde{q}^w_N(x, hD_x),$$

where

$$\tilde{q}^w_D(x, hD_x) = \sum_{j=1}^n 2\lambda_j x_j hD_{x_j} + \frac{h}{i}\sum_{j=1}^n \lambda_j, \tag{4-3}$$

is the diagonal part, while

$$\tilde{q}^w_N(x, hD_x) = \sum_{j=1}^{n-1} \gamma_j x_{j+1} hD_{x_j}, \quad \gamma_j \in \{0, 1\}, \tag{4-4}$$

is the nilpotent one. It is also easily seen that the operators $\tilde{q}^w_D(x, hD_x)$ and $\tilde{q}^w_N(x, hD_x)$ commute. It will be important for us to have an estimate of the order of nilpotency of the operator $\tilde{q}^w_N(x, hD_x)$ acting on the space E_m.

Lemma 4.1. *Let $n \geq 2$, $m \geq 1$, and let $E_m(n)$ be the space of homogeneous polynomials of degree m in the variables $x_1, x_2, \ldots, x_n$. The operator*

$$\mathcal{N} := \sum_{j=1}^{n-1} x_{j+1}\partial_{x_j} : E_m(n) \to E_m(n)$$

is nilpotent of order $m(n-1)+1$.

Proof. When $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = m$, let us write

$$S(\alpha) = \sum_{j=1}^{n} j\alpha_j,$$

and notice that $m \leq S(\alpha) \leq nm$. We have

$$\mathcal{N}x^\alpha = \sum_{\substack{|\alpha'|=m \\ S(\alpha')=S(\alpha)+1}} c_{\alpha'} x^{\alpha'},$$

and similarly for powers $\mathcal{N}^p x^\alpha$, but with $S(\alpha') = S(\alpha) + p$. It follows that $\mathcal{N}^{m(n-1)+1}x^\alpha$ must vanish, as

$$S(\alpha') = S(\alpha) + m(n-1) + 1 \geq mn + 1$$

is impossible. We also notice that $\mathcal{N}^{m(n-1)}x_1^m = Cx_n^m \neq 0$, for some $C \neq 0$. □

In what follows, we shall only use that the operator $\tilde{q}_N^w(x, hD_x) : E_m \to E_m$ is nilpotent of order $\mathbb{O}(m)$, with the implicit constant depending on the dimension n only.

It is now straightforward to derive a bound on the norm of the inverse of the operator

$$z - \tilde{q}^w(x, hD_x) : E_m \to E_m,$$

when the space E_m is equipped with the H_Φ-norm. The matrix $\mathfrak{D}(m)$ of the operator $\tilde{q}_D^w(x, hD_x)$ with respect to the basis φ_α, $|\alpha| = m$, is diagonal, with the eigenvalues of $\tilde{q}^w(x, hD_x)$,

$$\mu_\alpha = \frac{h}{i}\sum_{j=1}^{n} \lambda_j(2\alpha_j + 1), \quad |\alpha| = m,$$

along the diagonal. On the other hand, using (4-2), we compute

$$x_{j+1}\partial_{x_j}\varphi_\alpha = \alpha_j^{1/2}(\alpha_{j+1}+1)^{1/2}\varphi_{\alpha-e_j+e_{j+1}}, \quad 1 \leq j \leq n-1,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $e_1, \ldots, e_n$ is the canonical basis in $\mathbb{R}^n$. It follows that

$$\tilde{q}_N^w(x, hD_x)\varphi_\alpha = \sum_{j=1}^{n-1} -ih\gamma_j\alpha_j^{1/2}(\alpha_{j+1}+1)^{1/2}\varphi_{\alpha-e_j+e_{j+1}}, \tag{4-5}$$

and hence the entries $(\mathcal{N}(m)_{\alpha,\beta}) = \big((\tilde{q}_N^w(x, hD_x)\varphi_\beta, \varphi_\alpha)\big)$, $|\alpha| = |\beta| = m$, of the matrix $\mathcal{N}(m) : \mathbb{C}^{\nu_m} \to \mathbb{C}^{\nu_m}$ of $\tilde{q}_N^w(x, hD_x) : E_m \to E_m$ with respect to the basis $\{\varphi_\alpha\}$, are bounded in modulus by

$$h\alpha_j^{1/2}(\alpha_{j+1}+1)^{1/2} \leq h(m+1) \leq \mathbb{O}(1),$$

since $|\alpha| = m$ and m does not exceed $N = \mathcal{O}(h^{-1})$. Furthermore, from (4-5), it follows that the matrix $\mathcal{N}(m)$ has no more than $n-1$ nonzero entries in any column, and a similar reasoning shows that each row of $\mathcal{N}(m)$ also has no more than $n-1$ nonzero entries. Since we have just seen that the entries in $\mathcal{N}(m)$ are $\mathcal{O}(1)$, an application of Schur's lemma shows that that the operator norm of $\mathcal{N}(m)$ on $\mathbb{C}^{\nu_m}$ does not exceed

$$\Bigl(\sup_\beta \sum_\alpha |\mathcal{N}(m)_{\alpha,\beta}|\Bigr)^{1/2} \Bigl(\sup_\alpha \sum_\beta |\mathcal{N}(m)_{\alpha,\beta}|\Bigr)^{1/2} \leq \mathcal{O}(1).$$

Now the inverse of the $\nu_m \times \nu_m$ matrix

$$z - \mathcal{D}(m) - \mathcal{N}(m) : \mathbb{C}^{\nu_m} \to \mathbb{C}^{\nu_m}$$

is given by

$$(z - \mathcal{D}(m))^{-1} \sum_{j=0}^{\infty} \bigl((z - \mathcal{D}(m))^{-1} \mathcal{N}(m)\bigr)^j, \tag{4-6}$$

and according to Lemma 4.1 and the fact that $\bigl[\tilde{q}_D^w(x, hD_x), \tilde{q}_N^w(x, hD_x)\bigr] = 0$, we know that the Neumann series in (4-6) is finite, containing at most $\mathcal{O}(m)$ terms. It follows that

$$(z - \mathcal{D}(m) - \mathcal{N}(m))^{-1} = \frac{\exp(\mathcal{O}(m))}{d(z, \sigma_m)^{\mathcal{O}(m)}} : \mathbb{C}^{\nu_m} \to \mathbb{C}^{\nu_m}, \tag{4-7}$$

where $d(z, \sigma_m) = \inf_{|\alpha|=m} |z - \mu_\alpha|$ is the distance from $z \in \mathbb{C}$ to the set of eigenvalues $\{\mu_\alpha\}$ of $\tilde{q}^w(x, hD_x)$, restricted to E_m.

Using the fact that $\operatorname{Im} \tau_N$ is the orthogonal direct sum of the spaces E_m, $0 \leq m \leq N-1$, we may summarize the discussion so far in the following result.

Proposition 4.2. *Assume that $N \in \mathbb{N}$ is such that $Nh \leq \mathcal{O}(1)$, and let us equip the finite-dimensional space $\operatorname{Im} \tau_N \subset H_{\Phi_1}(\mathbb{C}^n) \cap H_\Phi(\mathbb{C}^n)$ with the H_Φ-norm, where $\Phi(x) = (1/2)|x|^2$. Assume that $z \in \mathbb{C}$ satisfies $\operatorname{dist}(z, \operatorname{Spec}(\tilde{q}^w(x, hD_x))) \geq h^L/C$, for some $C > 0$, $L \geq 1$. Then we have*

$$(z - \tilde{q}^w(x, hD_x))^{-1} = \mathcal{O}(1) \exp\Bigl(\mathcal{O}(1) h^{-1} \log \frac{1}{h}\Bigr) : \operatorname{Im} \tau_N \to \operatorname{Im} \tau_N. \tag{4-8}$$

Assuming that $\operatorname{dist}\bigl(z, \operatorname{Spec}(\tilde{q}^w(x, hD_x))\bigr) \geq 1/C$, the bound (4-8) improves to

$$(z - \tilde{q}^w(x, hD_x))^{-1} = \mathcal{O}(1) \exp(\mathcal{O}(1) h^{-1}) : \operatorname{Im} \tau_N \to \operatorname{Im} \tau_N. \tag{4-9}$$

Remark 4.3. Assume that the quadratic form q is such that the nilpotent part in the Jordan decomposition of the Hamilton map F is trivial. The quadratic operator $\tilde{q}^w(x, hD_x)$ acting on $H_\Phi(\mathbb{C}^n)$ is then normal, and therefore, the estimate (4-8) improves to

$$\|(z - \tilde{q}^w(x, hD_x))^{-1}\|_{\mathcal{L}(\operatorname{Im} \tau_N, \operatorname{Im} \tau_N)} \leq \frac{1}{\operatorname{dist}\bigl(z, \operatorname{Spec}(\tilde{q}^w(x, hD_x))\bigr)}.$$

Example 4.4. Let $n = 2$. Consider the semiclassical Weyl quantization of the elliptic quadratic form

$$\tilde{q}(x, \xi) = 2\lambda \sum_{j=1}^{2} x_j \xi_j + x_2 \xi_1, \quad \lambda = \frac{i}{2},$$

acting on $H_\Phi(\mathbb{C}^2)$. The eigenvalues of $\tilde{q}^w(x,hD_x)$ are of the form $\mu_\alpha = h(|\alpha|+1)$, $|\alpha|\geq 0$, and writing

$$\tilde{q}^w_D(x,hD_x) = 2\lambda\sum_{j=1}^{2} x_j hD_{x_j} + \frac{2\lambda h}{i}, \quad \tilde{q}^w_N(x,hD_x) = x_2 hD_{x_1},$$

we have

$$\tilde{q}^w_D(x,hD_x)\varphi_\alpha = \mu_\alpha\varphi_\alpha,$$

and

$$\tilde{q}^w_N(x,hD_x)\varphi_\alpha = -ih(\alpha_1(\alpha_2+1))^{1/2}\varphi_{\alpha-e_1+e_2}, \tag{4-10}$$

where the φ_α were introduced in (4-2).

Let $|\alpha| = m$, and let us write, following (4-6),

$$(\tilde{q}^w(x,hD_x)-z)^{-1}\varphi_\alpha = (\mu_\alpha - z)^{-1}\sum_{j=0}^{m}(\mu_\alpha - z)^{-j}(\tilde{q}^w_N(x,hD_x))^j\varphi_\alpha. \tag{4-11}$$

It is then natural to take $\alpha = (m,0)$, and using (4-10), a straightforward computation shows that, for $0\leq j\leq m$,

$$(\tilde{q}^w_N(x,hD_x))^j\varphi_{(m,0)} = (-ih)^j\sqrt{\frac{j!\,m!}{(m-j)!}}\,\varphi_{(m-j,j)}.$$

Let $z=1$ and take $m = h^{-1}\in\mathbb{N}$ so that $\mu_\alpha - z = h$. By Parseval's formula,

$$\|(\tilde{q}^w(x,hD_x)-z)^{-1}\varphi_{(m,0)}\|^2_{H_\Phi} = \sum_{j=0}^{m} h^{-2}h^{-2j}h^{2j}\frac{j!\,m!}{(m-j)!}, \tag{4-12}$$

and the right-hand side can be estimated from below simply by discarding all terms except when $j=m$. An application of Stirling's formula shows that

$$\|(\tilde{q}^w(x,hD_x)-z)^{-1}\varphi_{(m,0)}\|_{H_\Phi} \geq m! \geq \exp\Big(\frac{1}{2h}\log\frac{1}{h}\Big),$$

for all $h>0$ sufficiently small, and therefore, we see that the result of Proposition 4.2 cannot be improved. Let us finally notice that, as can be checked directly, the quadratic operator $\tilde{q}^w(x,hD_x)$ acting on $H_\Phi(\mathbb{C}^2)$ is unitarily equivalent, via an FBI-Bargmann transform, to the quadratic operator

$$q(x,hD_x): L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n),$$

of the form

$$q(x,hD_x) = q_0(x,hD_x) - \frac{i}{2}a_2^*a_1,$$

where

$$q_0(x,hD_x) = -\tfrac{1}{2}h^2\Delta + \tfrac{1}{2}x^2 = \tfrac{1}{2}(a_1^*a_1 + a_2^*a_2) + h$$

is the semiclassical harmonic oscillator, while

$$a_j^* = x_j - h\partial_{x_j}, \quad a_j = x_j + h\partial_{x_j}, \quad j=1,2$$

are the creation and annihilation operators, respectively. See also [Caliceti et al. 2007].

We shall now complete the proof of Theorem 1.1 in a straightforward manner, combining our earlier computations and estimates. Elementary considerations analogous to those used in the proof of Proposition 3.3 show that for some constant $C > 0$, we have, when $u \in \operatorname{Im} \tau_N$,

$$\|u\|_{H_{\Phi_1}} \leq Ce^{C/h}\|u\|_{H_\Phi}, \quad \|u\|_{H_\Phi} \leq Ce^{C/h}\|u\|_{H_{\Phi_1}}. \tag{4-13}$$

Here we recall that $N \sim h^{-1}$. It follows therefore that the result of Proposition 4.2,

$$(z - \tilde{q}^w(x, hD_x))^{-1} = \mathbb{O}(1)\exp\left(\mathbb{O}(1)h^{-1}\log\frac{1}{h}\right) : \operatorname{Im}\tau_N \to \operatorname{Im}\tau_N, \tag{4-14}$$

also holds when the space $\operatorname{Im}\tau_N \subset H_{\Phi_1}(\mathbb{C}^n) \cap H_\Phi(\mathbb{C}^n)$ is equipped with the H_{Φ_1}-norm, at the expense of an $\mathbb{O}(1)$ loss in the exponent. The same conclusion holds for the bound (4-9).

Let $\Omega \Subset \mathbb{C}$ and assume that $z \in \Omega \subset\subset \mathbb{C}$ is such that $\operatorname{dist}\big(z, \operatorname{Spec}(\tilde{q}^w(x, hD_x))\big) \geq h^L/C$ for some $L \geq 1$ and $C > 0$ fixed. Then, according to Proposition 3.2, there exists $N_0 \in \mathbb{N}$, $N_0 \sim h^{-1}$, such that if $u \in H_{\Phi_1}(\mathbb{C}^n)$ is such that $(1 + |x|^2)u \in L^2_{\Phi_1}(\mathbb{C}^n)$, then, using that $[\tilde{q}^w(x, hD_x), \tau_{N_0}] = 0$, we get, for all $h > 0$ small enough,

$$\begin{aligned}\|(1-\tau_{N_0})u\|_{H_{\Phi_1}} &\leq \mathbb{O}(1)\|(\tilde{q}^w(x, hD_x) - z)(1-\tau_{N_0})u\|_{H_{\Phi_1}} \\ &\leq \mathbb{O}(1)\exp(\mathbb{O}(1)h^{-1})\|(\tilde{q}^w(x, hD_x) - z)u\|_{H_{\Phi_1}}.\end{aligned} \tag{4-15}$$

Here we also used Proposition 3.3. On the other hand, the bound (4-14) and Proposition 3.3 show that

$$\begin{aligned}\|\tau_{N_0}u\|_{H_{\Phi_1}} &\leq \mathbb{O}(1)\exp\left(\mathbb{O}(1)h^{-1}\log\frac{1}{h}\right)\|\tau_{N_0}(\tilde{q}^w(x, hD_x) - z)u\|_{H_{\Phi_1}} \\ &\leq \mathbb{O}(1)\exp\left(\mathbb{O}(1)h^{-1}\log\frac{1}{h}\right)\|(\tilde{q}^w(x, hD_x) - z)u\|_{H_{\Phi_1}}.\end{aligned} \tag{4-16}$$

Combining (4-15) and (4-16), we obtain the bound (1-7). The estimate (1-6) follows in a similar way, and hence, the proof of Theorem 1.1 is complete. □

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MICHAEL HITRIK: hitrik@math.ucla.edu
Department of Mathematics, UCLA, Los Angeles 90095-1555, United States

JOHANNES SJÖSTRAND: johannes.sjostrand@u-bourgogne.fr
IMB, Université de Bourgogne, 9, Av. A. Savary, BP 47870, 21078 Dijon, France

and

UMR 5584 CNRS

JOE VIOLA: jviola@maths.lth.se
Mathematical Sciences, Lund University, Box 118, SE-22100 Lund, Sweden

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