ANALYSIS & PDE

Volume 6 No. 1 2013

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ANALYSIS AND PDE
Vol. 6, No. 1, 2013
dx.doi.org/10.2140/apde.2013.6.197

BILINEAR HILBERT TRANSFORMS ALONG CURVES I: THE MONOMIAL CASE

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We establish an $L^2 \times L^2$ to L^1 estimate for the bilinear Hilbert transform along a curve defined by a monomial. Our proof is closely related to multilinear oscillatory integrals.

1. Introduction

Let $d \geq 2$ be a positive integer. We consider the bilinear Hilbert transform along a curve $\Gamma(t) = (t, t^d)$, defined by

$$H_\Gamma(f, g)(x) = \text{p.v.} \int_{\mathbb{R}} f(x-t) g(x-t^d) \frac{dt}{t}, \tag{1-1}$$

where f, g are Schwartz functions on $\mathbb{R}$.

The main theorem we prove in this paper is:

Theorem 1.1. *The bilinear Hilbert transform along the curve $\Gamma(t) = (t, t^d)$ can be extended to a bounded operator from $L^2 \times L^2$ to L^1.*

Remark 1.2. It can be shown, with a little modification of our method, that the bilinear Hilbert transforms along polynomial curves $(t, P(t))$ are bounded from $L^p \times L^q$ to L^r whenever $(1/p, 1/q, 1/r)$ is in the closed convex hull of $\left(\frac{1}{2}, \frac{1}{2}, 1\right)$, $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$ and $\left(0, \frac{1}{2}, \frac{1}{2}\right)$. The condition $d \in \mathbb{N}$ is not necessary. Indeed, d can be any positive real number that is not equal to 1.

This problem is motivated by the Hilbert transform along a curve $\Gamma = (t, \gamma(t))$, defined by

$$H_\Gamma(f)(x_1, x_2) = \text{p.v.} \int_{\mathbb{R}} f\big(x_1 - t, x_2 - \gamma(t)\big) \frac{dt}{t},$$

and the bilinear Hilbert transform, defined by

$$H(f, g)(x) = \text{p.v.} \int_{\mathbb{R}} f(x-t) g(x+t) \frac{dt}{t}.$$

Among various curves, one simple model case is the parabola (t, t^2) in the two-dimensional plane. This work was initiated by Fabes and Rivière [1966] in order to study the regularity of parabolic differential equations. In the last thirty years, considerable work on this type of problem has been done. A nice survey on this type of operators can be found in [Stein and Wainger 1978]. For curves on homogeneous nilpotent

This research was partially supported by the NSF.
MSC2010: primary 42B20, 42B25; secondary 46B70, 47B38.
Keywords: bilinear Hilbert transform along curves.

Lie groups, the L^p estimates were established in [Christ 1985a]. The work for the Hilbert transform along more general curves with certain geometric conditions, such as the "flat" case, can be found in papers by Christ [1985b], Duoandikoetxea and J. L. Rubio de Francia [1986], and Nagel, Vance, Wainger and Weinberg [Nagel et al. 1983]. The general results were established recently in [Christ et al. 1999] for the singular Radon transforms and their maximal analogues over smooth submanifolds of $\mathbb{R}^n$ with some curvature conditions.

In recent years there has been a very active trend of harmonic analysis using time-frequency analysis to deal with multilinear operators. A breakthrough on the bilinear Hilbert transform was made by Lacey and Thiele [1997; 1999]. Following their work, the field of multilinear operators has been actively developed, to the point that some of the most interesting open questions have a strong connection to analysis on nilpotent groups. For instance, the trilinear Hilbert transform

$$\text{p.v.} \int f_1(x+t) f_2(x+2t) f_3(x+3t) \frac{dt}{t}$$

has a hidden quadratic modulation symmetry which must be accounted for in any proposed method of analysis. This nonabelian character is explicit in the work of B. Host and B. Kra [2005], who characterize the characteristic factor of the corresponding ergodic averages

$$N^{-1} \sum_{n=1}^{N} f_1(T^n) f_2(T^{2n}) f_3(T^{3n}) \longrightarrow \prod_{j=1}^{3} \mathbb{E}(f_j \mid \mathcal{N}).$$

Here, $(X, \mathcal{A}, \mu, T)$ is a measure-preserving system, and $\mathcal{N} \subset \mathcal{A}$ is the sigma-field which describes the characteristic factor, related to certain 2-step nilpotent groups. The limit above is in the sense of L^2-norm convergence, and holds for all bounded f_1, f_2, f_3.

The ergodic analogue of the bilinear Hilbert transform along a parabola is the nonconventional bilinear average

$$N^{-1} \sum_{n=1}^{N} f_1(T^n) f_2(T^{n^2}) \longrightarrow \prod_{j=1}^{2} \mathbb{E}(f_j \mid \mathcal{K}_{\text{profinite}}),$$

where $\mathcal{K}_{\text{profinite}} \subset \mathcal{A}$ is the profinite factor, a sub-σ-field of the maximal abelian factor of $(X, \mathcal{A}, \mu, T)$. The proof of the characteristic factor result above, due to Furstenberg [1990], utilizes the characteristic factor for the three-term result. We are indebted to M. Lacey for bringing Furstenberg's theorems to our attention. However, a notable fact is that our proof for the bilinear Hilbert transform along a monomial curve does not have to go through the trilinear Hilbert transform. The proof provided in this article relies heavily on the concept of a "quadratic uniformity", inspired by [Gowers 1998].

Another prominent theme is the relation of the bilinear Hilbert transforms along curves and the multilinear oscillatory integrals. The bilinear Hilbert transforms along curves are closely associated to the multilinear oscillatory integrals of the type

$$\Lambda_\lambda(f_1, f_2, f_3) = \int_{\boldsymbol{B}} f_1(\boldsymbol{x} \cdot \boldsymbol{v}_1) f_2(\boldsymbol{x} \cdot \boldsymbol{v}_2) f_3(\boldsymbol{x} \cdot \boldsymbol{v}_3) e^{i\lambda\varphi(\boldsymbol{x})} \, d\boldsymbol{x}, \tag{1-2}$$

where $\boldsymbol{B}$ is the unit ball in $\mathbb{R}^3$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ are vectors in $\mathbb{R}^3$, and the phase function φ satisfies a nondegenerate condition

$$\left|\prod_{j=1}^{3}(\nabla\cdot\boldsymbol{v}_j^{\perp})\varphi(\boldsymbol{x})\right|\geq 1. \tag{1-3}$$

Here $\boldsymbol{v}_j^{\perp}$ is a unit vector orthogonal to $\boldsymbol{v}_j$, for each j. For a polynomial phase φ with the nondegenerate condition (1-3), it was proved in [Christ et al. 2005] that

$$\left|\Lambda_\lambda(f_1, f_2, f_3)\right|\leq C(1+|\lambda|)^{-\varepsilon}\prod_{j=1}^{3}\|f_j\|_\infty \tag{1-4}$$

holds for some positive number ε. For the particular vectors $\boldsymbol{v}$ and the nondegenerate phase φ encountered in our problem, an estimate similar to (1-4) still holds. However, one of the main difficulties arises from the falsity of L^2 decay estimates for the trilinear form Λ_λ. It is to overcome this difficulty that we introduce the quadratic uniformity, which plays the role of a bridge connecting two spaces L^2 and L^∞.

The method used in this paper essentially works for those curves on nilpotent groups. It is possible to extend Theorem 1.1 to the general setting of nilpotent Lie groups. But we will not pursue this in this article. There are some related questions one can pose. Besides the generalization to the more general curves, it is natural to ask the corresponding problems in higher-dimensional cases and/or in multilinear cases. For instance, in the trilinear case, one can consider

$$T(f_1, f_2, f_3)(x)=\text{p.v.}\int f_1(x+t)f_2(x+p_1(t))f_3(x+p_2(t))\frac{dt}{t}. \tag{1-5}$$

Here p_1, p_2 are polynomials of t. The investigation of such problems will be discussed in subsequent papers.

2. A decomposition

Let ρ_1 be a standard bump function supported on $\left[\frac{1}{2}, 2\right]$, and let

$$\rho(t)=\rho_1(t)\mathbf{1}_{\{t>0\}}-\rho_1(-t)\mathbf{1}_{\{t<0\}}.$$

It is clear that ρ is an odd function. To obtain the L^r estimates for H_Γ, it is sufficient to get L^r estimates for T_Γ defined by $T_\Gamma=\sum_{j\in\mathbb{Z}}T_{\Gamma,j}$, where $T_{\Gamma,j}$ is

$$T_{\Gamma,j}(f, g)(x)=\int f(x-t)g(x-t^d)2^j\rho(2^jt)\,dt. \tag{2-1}$$

Let L be a large positive number (larger than 2^{100}). By Lemma 9.1, we have that if $|j|\leq L$,

$$\left\|T_{\Gamma,j}(f, g)\right\|_r\leq C_L\|f\|_p\|g\|_q$$

for all $p, q>1$ and $1/p+1/q=1/r$, where the operator norm C_L depends on the upper bound L. Hence in the following we only need to consider the case when $|j|>L$. In fact we prove the following theorem.

Theorem 2.1. *Let $T_{\Gamma,j}$ be defined as in* (2-1). *Then the bilinear operator $T_L=\sum_{j\in\mathbb{Z}:|j|>L}T_{\Gamma,j}$ is bounded from $L^2\times L^2$ to L^1.*

Clearly Theorem 1.1 follows from Theorem 2.1 and Lemma 9.1. The rest of the article is devoted to a proof of Theorem 2.1.

We begin the proof of Theorem 2.1 by constructing an appropriate decomposition of the operator $T_{\Gamma,j}$. This is done by an analysis of the bilinear symbol associated with the operator.

Expressing $T_{\Gamma,j}$ in dual frequency variables, we have

$$T_{\Gamma,j}(f,g)(x)=\iint \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\mathfrak{m}_j(\xi,\eta)\,d\xi\,d\eta,$$

where the symbol $\mathfrak{m}_j$ is defined by

$$\mathfrak{m}_j(\xi,\eta)=\int\rho(t)\exp\bigl(-2\pi i(2^{-j}\xi t+2^{-dj}\eta t^d)\bigr)\,dt. \tag{2-2}$$

First we introduce a resolution of the identity. Let Θ be a Schwarz function supported on $(-1,1)$ such that $\Theta(\xi)=1$ if $|\xi|\le\frac{1}{2}$. Set Φ to be a Schwartz function satisfying

$$\widehat{\Phi}(\xi)=\Theta\left(\frac{\xi}{2}\right)-\Theta(\xi)\,.$$

Then Φ is a Schwartz function such that $\widehat{\Phi}$ is supported on $\bigl\{\xi:\frac{1}{2}<|\xi|<2\bigr\}$ and

$$\sum_{m\in\mathbb{Z}}\widehat{\Phi}\left(\frac{\xi}{2^m}\right)=1\quad\text{for all}\quad \xi\in\mathbb{R}\backslash\{0\}, \tag{2-3}$$

and for any $m_0\in\mathbb{Z}$,

$$\widehat{\Phi}_{m_0}(\xi):=\sum_{m=-\infty}^{m_0}\widehat{\Phi}\left(\frac{\xi}{2^m}\right)=\Theta\left(\frac{\xi}{2^{m_0+1}}\right), \tag{2-4}$$

which is a bump function supported on $(-2^{m_0+1},2^{m_0+1})$.

From (2-3), we can decompose $T_{\Gamma,j}$ into two parts: $T_{\Gamma,j,1}$ and $T_{\Gamma,j,2}$, where $T_{\Gamma,j,1}$ is given by

$$\sum_{m\in\mathbb{Z}}\sum_{\substack{m'\in\mathbb{Z}:\\|m'-m|>10^d}}\iint\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\widehat{\Phi}\left(\frac{2^{-j}\xi}{2^m}\right)\widehat{\Phi}\left(\frac{2^{-dj}\eta}{2^{m'}}\right)\mathfrak{m}_j(\xi,\eta)\,d\xi\,d\eta, \tag{2-5}$$

and $T_{\Gamma,j,2}$ is defined by

$$\sum_{m\in\mathbb{Z}}\sum_{\substack{m'\in\mathbb{Z}:\\|m'-m|\le 10^d}}\iint\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\widehat{\Phi}\left(\frac{2^{-j}\xi}{2^m}\right)\widehat{\Phi}\left(\frac{2^{-dj}\eta}{2^{m'}}\right)\mathfrak{m}_j(\xi,\eta)\,d\xi\,d\eta. \tag{2-6}$$

Define $\mathfrak{m}_d$ by

$$\mathfrak{m}_d(\xi,\eta)=\int\rho(t)\exp\bigl(-2\pi i(\xi t+\eta t^d)\bigr)\,dt. \tag{2-7}$$

Clearly $\mathfrak{m}_j(\xi,\eta)=\mathfrak{m}_d(2^{-j}\xi,2^{-dj}\eta)$. In $T_{\Gamma,j,1}$, the phase function $\phi_{\xi,\eta}(t)=\xi t+\eta t^d$ does not have any critical point in a neighborhood of the support of ρ, and therefore a very rapid decay can be obtained

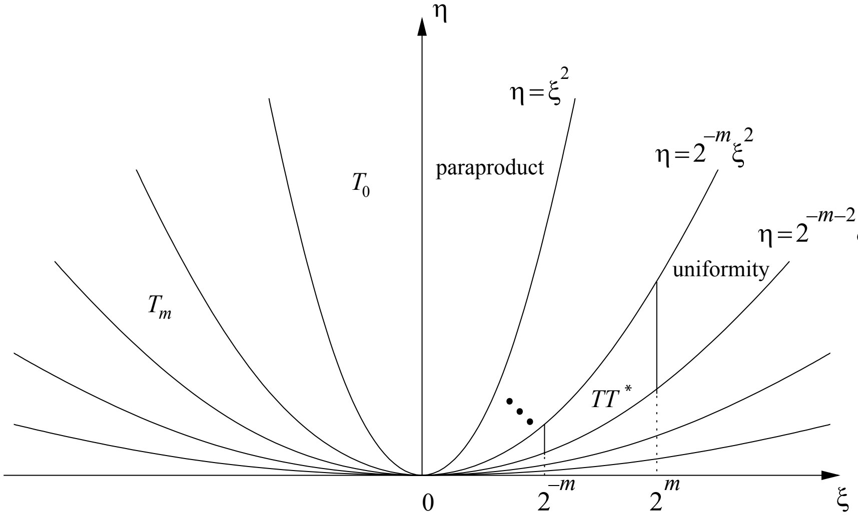

Figure 1. Decomposition of the (ξ, η)-plane for $\sum_m T_m$ when $d = 2$.

by integration by parts so that we can show that $\sum_j T_{\Gamma,j,1}$ is essentially a finite sum of paraproducts (see Section 3). A critical point of the phase function may occur in $T_{\Gamma,j,2}$, and therefore the method of stationary phase must be brought to bear in this case, exploiting in particular the oscillatory term. This case requires the most extensive analysis. Heuristically, the decomposition is made according to the curvature of the curve (t, t^d). For example, for the parabola case, the frequency space is broken into parabolic regions $\{(\xi, \eta) : \eta \sim 2^{-m}\xi^2\}$, as shown in the figure. Naturally, the $2^{-\varepsilon m}$ decay estimate is expected in order to sum up all parabolic regions.

Notice that there are only finitely many m' if m is fixed in (2-6). Without loss of generality, we can assume $m' = m$. Then in order to get the L^r estimates for $\sum_j T_{\Gamma,j,2}$, it suffices to prove the L^r boundedness of $\sum_m T_m$, where the T_m are defined by

$$T_m(f, g)(x) = \sum_{|j|>L} \iint \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\widehat{\Phi}\left(\frac{2^{-j}\xi}{2^m}\right)\widehat{\Phi}\left(\frac{2^{-dj}\eta}{2^m}\right)\mathfrak{m}_j(\xi, \eta)\, d\xi\, d\eta. \tag{2-8}$$

It can be proved that $T_0 = \sum_{m\le 0} T_m$ is equal to $\sum_{m\le 0} O(2^{m/2})\Pi_m$, where Π_m is a paraproduct studied in Theorem 3.1. This can be done by Fourier series and the cancellation condition of ρ, and thus T_0 is essentially a paraproduct. We omit the details, since they are exactly the same as those in Section 3 for the case $\sum_j T_{\Gamma,j,1}$. Therefore, the most difficult term is $\sum_{m\ge 1} T_m$. For this term, we have the following theorem.

Theorem 2.2. *Let T_m be a bilinear operator defined as in* (2-8)*. Then there exists a constant C such that*

$$\left\| \sum_{m\ge 1} T_m(f, g) \right\|_1 \le C\|f\|_2\, \|g\|_2 \tag{2-9}$$

holds for all $f, g \in L^2$.

A delicate analysis is required for proving this theorem. We prove it on page 204. Theorem 2.1 follows from Theorem 2.2 and the boundedness of $\sum_j T_{\Gamma,j,1}$. The rest of the article is organized as follows. In Section 3, the L^r-boundedness will be established for $\sum_j T_{\Gamma,j,1}$. Some crucial bilinear restriction estimates will appear in Section 4 and as a consequence Theorem 2.2 follows. Sections 5–8 are devoted to a proof of the bilinear restriction estimates.

3. Paraproducts and uniform estimates

In this section we prove that $\sum_j T_{\Gamma,j,1}$ is essentially a finite sum of certain paraproducts bounded from $L^p \times L^q$ to L^r.

First let us introduce the paraproduct encountered in our problem. Let $j \in \mathbb{Z}$, L_1, L_2 be positive integers and M_1, M_2 be integers. Then

$$\omega_{1,j} = \left[\frac{2^{L_1 j+M_1}}{2}, 2\cdot 2^{L_1 j+M_1}\right]$$

and

$$\omega_{2,j} = [-2^{L_2 j+M_2}, 2^{L_2 j+M_2}].$$

Let Φ_1 be a Schwartz function whose Fourier transform is a standard bump function supported on a small neighborhood of $\left[\frac{1}{2}, 2\right]$ or $\left[-2, -\frac{1}{2}\right]$, and Φ_2 be a Schwartz function whose Fourier transform is a standard bump function supported on $[-1, 1]$ and $\widehat{\Phi}_2(0) = 1$. For $l \in \{1, 2\}$ and $n_1, n_2 \in \mathbb{Z}$, define Φ_{l,j,n_l} by

$$\widehat{\Phi}_{l,j,n_l}(\xi) = \left(e^{2\pi i n_l(\cdot)}\widehat{\Phi}_l(\cdot)\right)\left(\frac{\xi}{2^{L_l j+M_l}}\right).$$

It is clear that $\widehat{\Phi}_{l,j,n_l}$ is supported on $\omega_{l,j}$. For locally integrable functions f_l, we define $f_{l,j}$ by

$$f_{l,j,n_l}(x) = f_l * \Phi_{l,j,n_l}(x).$$

We now define a paraproduct to be

$$\Pi_{L_1,L_2,M_1,M_2,n_1,n_2}(f_1, f_2)(x) = \sum_{j\in\mathbb{Z}} \prod_{l=1}^{2} f_{l,j,n_l}(x). \tag{3-1}$$

For this paraproduct, we have the following uniform estimates.

Theorem 3.1. *For any $p_1 > 1$, $p_2 > 1$ with $1/p_1 + 1/p_2 = 1/r$, there exists a constant C independent of M_1, M_2, n_1, n_2 such that*

$$\left\| \Pi_{L_1,L_2,M_1,M_2,n_1,n_2}(f_1, f_2)\right\|_r \le C\left(1+|n_1|\right)^{10}\left(1+|n_2|\right)^{10}\|f_1\|_{p_1}\,\|f_2\|_{p_2}, \tag{3-2}$$

for all $f_1 \in L^{p_1}$ and $f_2 \in L^{p_2}$.

The case $r > 1$ can be handled by a telescoping argument. The case $r < 1$ is more complicated and requires a time-frequency analysis. A proof of Theorem 3.1 can be found in [Li 2008]. The constant C in Theorem 3.1 may depend on L_1, L_2. It is easy to see that C is $O(\max\{2^{L_1}, 2^{L_2}\})$. It is possible to get a much better upper bound, such as $O\left(\log(1+\max\{L_2/L_1, L_1/L_2\})\right)$, by tracking the constants carefully in the proof in [Li 2008]. But we do not need the sharp constant in this article. The independence on M_1, M_2 is the most important issue here.

We now return to $\sum_j T_{\Gamma,j,1}$. This sum can be written as $T_{L,1}+T_{L,2}$, where $T_{L,1}$ is a bilinear operator defined by

$$\sum_{|j|>L}\sum_{m\in\mathbb{Z}}\sum_{\substack{m'\in\mathbb{Z}\\ m'<m-10^d}}\iint \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\widehat{\Phi}\Big(\frac{2^{-j}\xi}{2^m}\Big)\widehat{\Phi}\Big(\frac{2^{-dj}\eta}{2^{m'}}\Big)\mathfrak{m}_j(\xi,\eta)\,d\xi\,d\eta,$$

and $T_{L,2}$ is a bilinear operator given by

$$\sum_{|j|>L}\sum_{m'\in\mathbb{Z}}\sum_{\substack{m\in\mathbb{Z}\\ m<m'-10^d}}\iint \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\widehat{\Phi}\Big(\frac{2^{-j}\xi}{2^m}\Big)\widehat{\Phi}\Big(\frac{2^{-dj}\eta}{2^{m'}}\Big)\mathfrak{m}_j(\xi,\eta)\,d\xi\,d\eta.$$

It is standard to verify that $T_{L,1}$ and $T_{L,2}$ are paraproducts as defined in (3-1). Hence the $L^p\times L^q\to L^r$ estimates of these paraproducts follow from Theorem 3.1, for all $p,q>1$ and $1/p+1/q=1/r$.

4. Bilinear Fourier restriction estimates

Let $d\ge 2$, $m\ge 0$, $j\in\mathbb{Z}$. We define a bilinear Fourier restriction operator of f,g by

$$\mathcal{B}_{j,m}(f,g)(x)=2^{-(d-1)j/2}\int_{\mathbb{R}} R_\Phi f\big(2^{-(d-1)j}x-2^m t\big)R_\Phi g(x-2^m t^d)\rho(t)\,dt \quad \text{if } j\ge 0 \tag{4-1}$$

and

$$\mathcal{B}_{j,m}(f,g)(x)=2^{(d-1)j/2}\int_{\mathbb{R}} R_\Phi f(x-2^m t)R_\Phi g(2^{(d-1)j}x-2^m t^d)\rho(t)\,dt \quad \text{if } j<0, \tag{4-2}$$

where $R_\Phi f$ and $R_\Phi g$ are the Fourier (smooth) restrictions of f,g on the support of $\widehat{\Phi}$ respectively. More precisely, $R_\Phi f$, $R_\Phi g$ are given by

$$\widehat{R_\Phi f}(\xi)=\widehat{f}(\xi)\widehat{\Phi}(\xi), \tag{4-3}$$

$$\widehat{R_\Phi g}(\xi)=\widehat{g}(\xi)\widehat{\Phi}(\xi). \tag{4-4}$$

By inserting absolute values throughout and applying the Cauchy–Schwarz inequality, the boundedness of $\mathcal{B}_{j,m}$ from $L^2\times L^2$ to L^1 follows immediately. Moreover, since the Fourier transforms of f,g are restricted on the support of $\widehat{\Phi}$, we actually can improve the estimate. Let us state the improved estimates in the following theorems, which are of independent interest.

Theorem 4.1. *Let $d\ge 2$ and $\mathcal{B}_{j,m}$ be defined as in* (4-1) *and* (4-2). *If $L\le |j|\le m/(d-1)$, then there exists a constant C independent of j,m such that*

$$\big\|\mathcal{B}_{j,m}(f,g)\big\|_1\le C\,2^{\frac{(d-1)|j|-m}{8}}\|f\|_2\,\|g\|_2 \quad \textit{for all } f,g\in L^2. \tag{4-5}$$

Theorem 4.2. *Let $d\ge 2$ and $\mathcal{B}_{j,m}$ be defined as in* (4-1) *and* (4-2). *If $|j|\ge m/(d-1)$, then there exist a positive number ε_0 and a constant C independent of j,m such that*

$$\big\|\mathcal{B}_{j,m}(f,g)\big\|_1\le C\max\big\{2^{\frac{m-(d-1)|j|}{3}},2^{-\varepsilon_0 m}\big\}\|f\|_2\,\|g\|_2 \quad \textit{for all } f,g\in L^2. \tag{4-6}$$

The positive number ε_0 in Theorem 4.2 can be chosen to be $1/(8d)$. Theorem 4.1 can be proved by a TT^* method. However, the TT^* method fails when $|j| > m/(d-1)$. To obtain Theorem 4.2, we will employ a method related to the uniformity of functions.

Now we can see that Theorem 2.2 is a consequence of Theorems 4.1 and 4.2.

Proof of Theorem 2.2. Define a bilinear operator $T_{j,m}$ to be

$$T_{j,m}(f,g)(x) = \iint \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\widehat{\Phi}\left(\frac{2^{-j}\xi}{2^m}\right)\widehat{\Phi}\left(\frac{2^{-dj}\eta}{2^m}\right)\mathfrak{m}_j(\xi,\eta)\,d\xi\,d\eta. \tag{4-7}$$

Let $\gamma_{j,m}$ be defined by

$$\gamma_{j,m} = \begin{cases} 2^{\frac{(d-1)|j|-m}{8}} & \text{if } |j| \le \dfrac{m}{d-1}, \\ \max\left\{2^{\frac{m-(d-1)|j|}{3}}, 2^{-\varepsilon_0 m}\right\} & \text{if } |j| \ge \dfrac{m}{d-1}. \end{cases} \tag{4-8}$$

A rescaling argument and Theorems 4.1 and 4.2 yield

$$\left\|T_{j,m}(f,g)\right\|_1 \le C\gamma_{j,m}\|f\|_2\,\|g\|_2. \tag{4-9}$$

Since $\sum_m T_m = \sum_m \sum_{j:|j|\ge L} T_{j,m}$, we obtain

$$\left\|\sum_{m\ge 1} T_m(f,g)\right\|_1 \le C\sum_{m\ge 1}\sum_{j:|j|\ge L}\gamma_{j,m}\|f_{j,m}\|_2\,\|g_{j,m}\|_2, \tag{4-10}$$

where

$$\widehat{f_{j,m}}(\xi) = \widehat{f}(\xi)\widehat{\Phi}\left(\frac{\xi}{2^{j+m}}\right),$$

$$\widehat{g_{j,m}}(\eta) = \widehat{g}(\eta)\widehat{\Phi}\left(\frac{\eta}{2^{dj+m}}\right).$$

Clearly the right-hand side of (4-10) is bounded by $C\|f\|_2\|g\|_2$. Therefore, we finish the proof of Theorem 2.2.

Since t is localized, it is sufficient to consider $\tilde{\mathcal{B}}_{j,m,n}$ given by

$$\tilde{\mathcal{B}}_{j,m} = \mathcal{B}_{j,m}\mathbf{1}_I^*. \tag{4-11}$$

Here I is an interval whose size is $2^{(d-1)|j|+m}$ and $\mathbf{1}_I^* = \mathbf{1}_I * \phi_k$, where $\phi_k(x)$ equals $2^{-k}\phi(2^{-k}x)$ for a given nonnegative Schwartz function ϕ whose Fourier transform is a standard bump function on $\left[-\frac{1}{2}, \frac{1}{2}\right]$. In what follows, we still use $\mathcal{B}_{j,m}$ to denote the localized operator $\tilde{\mathcal{B}}_{j,m}$.

Trilinear forms. Let f_1, f_2, f_3 be measurable functions supported on $\frac{1}{16} \le |\xi| \le \frac{39}{16}$. Define a trilinear form $\Lambda_{j,m}(f_1, f_2, f_3)$ by

$$\Lambda_{j,m}(f_1,f_2,f_3) := \left\langle\mathcal{B}_{j,m}(\check{f}_1,\check{f}_2),\check{f}_3\right\rangle. \tag{4-12}$$

Theorems 4.1 and 4.2 can be reduced to the following theorems respectively.

Theorem 4.3. *Let $d \geq 2$ and $\Lambda_{j,m}(f_1, f_2, f_3)$ be defined as in* (4-12). *If $|j| \leq m/(d-1)$, then there exists a constant C independent of j, m such that*

$$\left|\Lambda_{j,m}(f_1, f_2, f_3)\right| \leq C 2^{\frac{-(d-1)|j|-m}{2}} 2^{-\frac{m-(d-1)|j|}{6}} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2 \tag{4-13}$$

for all $f_1, f_2, f_3 \in L^2$.

Theorem 4.4. *Let $d \geq 2$ and $\Lambda_{j,m}(f_1, f_2, f_3)$ be defined as in* (4-12). *If $|j| \geq m/(d-1)$, then there exist a positive number ε_0 and a constant C independent of j, m such that*

$$\left|\Lambda_{j,m}(f_1, f_2, f_3)\right| \leq C \max\left\{2^{\frac{-(d-1)|j|+m}{2}}, 2^{-\varepsilon_0 m}\right\} \|f_1\|_2 \|f_2\|_2 \|\widehat{f_3}\|_\infty \tag{4-14}$$

holds for all $f_1, f_2 \in L^2$ and $\widehat{f_3} \in L^\infty$ such that f_1, f_2, f_3 are supported on $\frac{1}{16} \leq |\xi| \leq \frac{39}{16}$.

A proof of Theorem 4.3 will be provided in Section 5, and a proof of Theorem 4.4 will be given in Section 7.

5. Stationary phases and trilinear oscillatory integrals

In this section we provide a proof of Theorem 4.3 by utilizing essentially a TT^* method. In this case, one cannot reduce the problem to the standard paraproduct problem because the critical points of the phase function may occur in a neighborhood of $\frac{1}{2} \leq |t| \leq 2$, say $\frac{1}{4} \leq |t| \leq \frac{5}{2}$, which provides a stationary phase for the Fourier integral $\mathfrak{m}_d$. This stationary phase gives a highly oscillatory factor in the integral. We expect a suitable decay from the highly oscillatory factor.

Let $\Lambda_{j,m}(f_1, f_2, f_3) = \langle \mathcal{B}_{j,m}(\check{f}_1, \check{f}_2), \check{f}_3 \rangle$. To prove Theorem 4.3, it suffices to prove the following L^2 estimate for the trilinear form $\Lambda_{j,m}(f_1, f_2, f_3)$:

$$\left|\Lambda_{j,m}(f_1, f_2, f_3)\right| \leq C 2^{\frac{-(d-1)|j|-m}{2}} 2^{-\frac{m-(d-1)|j|}{6}} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2 \tag{5-1}$$

holds for all $f_1, f_2, f_3 \in L^2$. Clearly $\Lambda_{j,m}(f_1, f_2, f_3)$ can be expressed as

$$2^{-(d-1)j/2} \iint f_1(\xi)\widehat{\Phi}(\xi) f_2(\eta)\widehat{\Phi}(\eta) f_3\big(2^{-(d-1)j}\xi + \eta\big)\mathfrak{m}_d(2^m\xi, 2^m\eta)\, d\xi\, d\eta$$

if $j > 0$, and as

$$2^{(d-1)j/2} \iint f_1(\xi)\widehat{\Phi}(\xi) f_2(\eta)\widehat{\Phi}(\eta) f_3\big(\xi + 2^{(d-1)j}\eta\big)\mathfrak{m}_d(2^m\xi, 2^m\eta)\, d\xi\, d\eta$$

if $j \leq 0$,

Whenever $\xi, \eta \in \operatorname{supp} \widehat{\Phi}$, the second-order derivative of the phase function $\phi_{m,\xi,\eta}(t) = 2^m(\xi t + \eta t^d)$ is comparable to 2^m. We only need to focus on the worst situation, when there is a critical point of the phase function in a small neighborhood of $\operatorname{supp} \rho$. Thus the method of stationary phase yields

$$\mathfrak{m}_d(2^m\xi, 2^m\eta) \sim 2^{-m/2} \exp\big(i c_d 2^m \xi^{d/(d-1)} \eta^{-1/(d-1)}\big), \tag{5-2}$$

where c_d is a constant depending only on d (see [Sogge 1993; Stein 1993]). Henceforth we reduce Theorem 4.3 to the following lemma.

Proposition 5.1. *Let $\Lambda^*_{j,m}$ be defined by*

$$\Lambda^*_{j,m}(f_1, f_2, f_3) = \iint f_1(\xi)\widehat{\Phi}(\xi) f_2(\eta)\widehat{\Phi}(\eta) f_3\big(2^{-(d-1)j}\xi + \eta\big) \exp\big(i c_d 2^m \xi^{\frac{d}{d-1}} \eta^{-\frac{1}{d-1}}\big)\, d\xi\, d\eta \tag{5-3}$$

if $j > 0$, and by

$$\Lambda^*_{j,m}(f_1, f_2, f_3) = \iint f_1(\xi)\widehat{\Phi}(\xi) f_2(\eta)\widehat{\Phi}(\eta) f_3\big(\xi + 2^{(d-1)j}\eta\big) \exp(i c_d 2^m \xi^{\frac{d}{d-1}} \eta^{-\frac{1}{d-1}}\big)\, d\xi\, d\eta. \tag{5-4}$$

if $j \le 0$. There exists a positive constant C such that

$$\big|\Lambda^*_{j,m}(f_1, f_2, f_3)\big| \le C 2^{-\frac{m-(d-1)|j|}{6}} \|f_1\|_2 \,\|f_2\|_2 \,\|f_3\|_2 \tag{5-5}$$

holds for all $f_1, f_2, f_3 \in L^2$.

Proof. Without loss of generality, we assume that $\widehat{\Phi}$ is supported on $\big[\frac{1}{2}, 2\big]$ or $\big[-2, -\frac{1}{2}\big]$. And we only give a proof for the case $j > 0$, since a similar argument yields the case $j \le 0$. Let $\phi_{d,m}$ be a phase function defined by

$$\phi_{d,m}(\xi, \eta) = c_d \xi^{d/(d-1)} \eta^{-1/(d-1)},$$

and let $b_1 = 1 - 2^{-(d-1)j}$ and $b_2 = 2^{-(d-1)j}$. Changing variables $\xi \mapsto \xi - \eta$ and $\eta \mapsto b_1\xi + b_2\eta$, we have that $\Lambda^*_{j,m}(f_1, f_2, f_3)$ equals

$$\iint f_1(\xi - \eta) f_2(b_1\xi + b_2\eta) f_3(\xi)\widehat{\Phi}(\xi - \eta)\widehat{\Phi}(b_1\xi + b_2\eta) e^{i 2^m \phi_{d,m}(\xi-\eta, b_1\xi+b_2\eta)}\, d\xi\, d\eta.$$

Thus, by Cauchy–Schwarz, we dominate $|\Lambda^*_{j,m}(f_1, f_2, f_3)|$ by

$$\big\|\boldsymbol{T}_{d,j,m}(f_1, f_2)\big\|_2 \|f_3\|_2,$$

where $\boldsymbol{T}_{d,j,m}$ is defined by

$$\boldsymbol{T}_{d,j,m}(f_1, f_2)(\xi) = \int f_1(\xi - \eta) f_2(b_1\xi + b_2\eta)\widehat{\Phi}(\xi - \eta)\widehat{\Phi}(b_1\xi + b_2\eta) e^{i 2^m \phi_{d,m}(\xi-\eta, b_1\xi+b_2\eta)}\, d\eta.$$

It is easy to see that $\big\|\boldsymbol{T}_{d,j,m}(f_1, f_2)\big\|_2^2$ equals

$$\int \bigg(\iint F(\xi, \eta_1, \eta_2) G(\xi, \eta_1, \eta_2) e^{i 2^m (\phi_{d,m}(\xi-\eta_1, b_1\xi+b_2\eta_1) - \phi_{d,m}(\xi-\eta_2, b_1\xi+b_2\eta_2))} d\eta_1\, d\eta_2 \bigg)\, d\xi,$$

where

$$F(\xi, \eta_1, \eta_2) = (f_1\widehat{\Phi})(\xi - \eta_1)\overline{(f_1\widehat{\Phi})(\xi - \eta_2)},$$
$$G(\xi, \eta_1, \eta_2) = (f_2\widehat{\Phi})(b_1\xi + b_2\eta_1)\overline{(f_2\widehat{\Phi})(b_1\xi + b_2\eta_2)}.$$

Changing variables $\eta_1 \mapsto \eta$ and $\eta_2 \mapsto \eta + \tau$, we see that $\big\|\boldsymbol{T}_{d,j,m}(f_1, f_2)\big\|_2^2$ equals

$$\int \bigg(\iint F_\tau(\xi - \eta) G_\tau(b_1\xi + b_2\eta) e^{i 2^m (\phi_{d,m}(\xi-\eta, b_1\xi+b_2\eta) - \phi_{d,m}(\xi-\eta-\tau, b_1\xi+b_2(\eta+\tau)))}\, d\xi\, d\eta \bigg)\, d\tau,$$

where

$$F_\tau(\cdot) = (f_1\widehat{\Phi})(\cdot)\overline{(f_1\widehat{\Phi})(\cdot - \tau)},$$
$$G_\tau(\cdot) = (f_2\widehat{\Phi})(\cdot)\overline{(f_2\widehat{\Phi})(\cdot + b_2\tau)}.$$

Changing coordinates to $(u, v) = (\xi - \eta, b_1\xi + b_2\eta)$, the inner integral becomes

$$\iint F_\tau(u)G_\tau(v)\exp\bigl(i\,2^m\,\tilde{\boldsymbol{Q}}_\tau(u, v)\bigr)\,du\,dv, \tag{5-6}$$

where $\tilde{\boldsymbol{Q}}_\tau$ is defined by

$$\tilde{\boldsymbol{Q}}_\tau(u, v) = \phi_{d,m}(u, v) - \phi_{d,m}(u - \tau, v + b_2\tau).$$

When j is large enough, the mean value theorem yields

$$\bigl|\partial_u\partial_v\,\tilde{\boldsymbol{Q}}_\tau(u, v)\bigr| \ge C\tau, \tag{5-7}$$

if $u, v, u - \tau, v + b_2\tau \in \operatorname{supp}\widehat{\Phi}$.

A well-known theorem of Hörmander on the nondegenerate phase [Hörmander 1973; Phong and Stein 1994] gives for (5-6) the estimate

$$C\min\bigl\{1, 2^{-m/2}|\tau|^{-1/2}\bigr\}\|F_\tau\|_2\,\|G_\tau\|_2.$$

Hence, by the Cauchy–Schwarz inequality, $\bigl\|\boldsymbol{T}_{d,j,m}(f_1, f_2)\bigr\|_2^2$ is bounded by

$$\tau_0\|f_1\|_2^2\,\|f_2\|_2^2 + C\int_{\tau_0<|\tau|<10}\min\bigl\{1, 2^{-m/2}|\tau|^{-1/2}\bigr\}\|F_\tau\|_2\,\|G_\tau\|_2\,d\tau$$

for any $\tau_0 > 0$. By one more use of the Cauchy–Schwarz inequality, $\bigl\|\boldsymbol{T}_{d,j,m}(f_1, f_2)\bigr\|_2^2$ is dominated by $\bigl(\tau_0 + C\tau_0^{-1/2}2^{-m/2}2^{(d-1)j/2}\bigr)\|f_1\|_2^2\,\|f_2\|_2^2$ for any $\tau_0 > 0$. Thus we have

$$\bigl|\Lambda^*_{j,m}(f_1, f_2, f_3)\bigr| \le C\,2^{\frac{(d-1)j-m}{6}}\|f_1\|_2\,\|f_2\|_2\,\|f_3\|_2. \tag{5-8}$$

This completes the proof of Proposition 5.1. □

It is easy to see that

$$\bigl|\Lambda^*_{j,m}(f_1, f_2, f_3)\bigr| \le C\,2^{-\varepsilon m}\|f_1\|_2\,\|f_2\|_2\,\|f_3\|_2 \tag{5-9}$$

fails for all $|j| \ge m/(d-1)$. Indeed, let us only consider the case $j > m/(d-1)$. Assume that (5-9) holds for all $j > m/(d-1)$. Let $j \to \infty$; then (5-9) implies

$$\bigl|\Lambda^*_m(f_1, f_2, f_3)\bigr| \le C\,2^{-\varepsilon m}\|f_1\|_2\,\|f_2\|_2\,\|f_3\|_2, \tag{5-10}$$

where

$$\Lambda^*_m(f_1, f_2, f_3) = \iint f_1(\xi)\widehat{\Phi}(\xi)f_2(\eta)\widehat{\Phi}(\eta)f_3(\eta)\exp\bigl(ic_d2^m\xi^{d/(d-1)}\eta^{-1/(d-1)}\bigr)\,d\xi\,d\eta.$$

Simply taking $f_2 = f_3$, we obtain

$$\sup_{\eta\sim1}\left|\int f_1(\xi)\widehat{\Phi}(\xi)\exp\bigl(ic_d2^m\xi^{d/(d-1)}\eta^{-1/(d-1)}\bigr)\,d\xi\right| \le C\,2^{-\varepsilon m}\|f_1\|_2. \tag{5-11}$$

This clearly cannot be true, and hence we get a contradiction. Therefore, (5-9) does not hold for $j > m/(d-1)$. Hence the TT^* method cannot work for the case $|j| > m/(d-1)$. In the following sections, we have to introduce a concept of uniformity and employ a "quadratic" Fourier analysis.

6. Uniformity

We introduce a concept related to a notion of uniformity employed by Gowers [1998]. A similar uniformity was utilized in [Christ et al. 2005]. Let $\sigma \in (0, 1]$, let $\mathfrak{Q}$ be a collection of some real-valued measurable functions, and fix a bounded interval $\boldsymbol{I}$ in $\mathbb{R}$.

Definition 6.1. A function $f \in L^2(\boldsymbol{I})$ is σ-uniform in $\mathfrak{Q}$ if

$$\left|\int_{\boldsymbol{I}} f(\xi)e^{-iq(\xi)}d\xi\right| \leq \sigma \|f\|_{L^2(\boldsymbol{I})} \tag{6-1}$$

for all $q \in \mathfrak{Q}$. Otherwise, f is said to be σ-nonuniform in $\mathfrak{Q}$.

Theorem 6.2. *Let L be a bounded sublinear functional from $L^2(\boldsymbol{I})$ to $\mathbb{C}$, let $\boldsymbol{S}_\sigma$ be the set of all functions that are σ-uniform in $\mathfrak{Q}$, and let*

$$U_\sigma = \sup_{f \in \boldsymbol{S}_\sigma} \frac{|L(f)|}{\|f\|_{L^2(\boldsymbol{I})}}. \tag{6-2}$$

Then, for all functions in $L^2(\boldsymbol{I})$,

$$|L(f)| \leq \max\{U_\sigma, 2\sigma^{-1}Q\}\|f\|_{L^2(\boldsymbol{I})}, \tag{6-3}$$

where

$$Q = \sup_{q \in \mathfrak{Q}} |L(e^{iq})|. \tag{6-4}$$

Proof. Clearly the complement $\boldsymbol{S}_\sigma^c$ is a set of all functions that are σ-nonuniform in $\mathfrak{Q}$. Let us set

$$A := \sup_{f \in L^2(\boldsymbol{I})} \frac{|L(f)|}{\|f\|_{L^2(\boldsymbol{I})}} \quad \text{and} \quad A_1 := \sup_{f \in \boldsymbol{S}_\sigma^c} \frac{|L(f)|}{\|f\|_{L^2(\boldsymbol{I})}}.$$

Clearly $A = \max\{A_1, U_\sigma\}$. In order to obtain (6-3), it suffices to prove that if $U_\sigma < A_1$, then

$$A_1 \leq 2\sigma^{-1}Q. \tag{6-5}$$

For any $\varepsilon > 0$, there exists a function $f \in \boldsymbol{S}_\sigma^c$ such that

$$(A_1 - \varepsilon)\|f\|_{L^2(\boldsymbol{I})} \leq |L(f)|. \tag{6-6}$$

Let $\langle \cdot, \cdot \rangle_{\boldsymbol{I}}$ be an inner product on $L^2(\boldsymbol{I})$ defined by

$$\langle f, g \rangle_{\boldsymbol{I}} = \int_{\boldsymbol{I}} f(x)\overline{g(x)}\,dx,$$

for all $f, g \in L^2(\boldsymbol{I})$. Since f is σ-nonuniform in $\mathfrak{Q}$, there exists a function q in $\mathfrak{Q}$ such that

$$\left|\langle f, e^{iq} \rangle_{\boldsymbol{I}}\right| \geq \sigma \|f\|_{L^2(\boldsymbol{I})}. \tag{6-7}$$

There exists $g \in L^2(\boldsymbol{I})$ (depending on f) such that $g \perp e^{iq}$, $\|g\|_{L^2(\boldsymbol{I})} = 1$, and

$$f = \langle f, g\rangle_{\boldsymbol{I}}\, g + \frac{\langle f, e^{iq}\rangle_{\boldsymbol{I}}}{|\boldsymbol{I}|} e^{iq}. \tag{6-8}$$

Sublinearity of L and the triangle inequality then yield

$$\big|L(f)\big| \le \big|\langle f, g\rangle_{\boldsymbol{I}}\big|\big|L(g)\big| + |\boldsymbol{I}|^{-1}\big|\langle f, e^{iq}\rangle_{\boldsymbol{I}}\big|\big|L(e^{iq})\big|. \tag{6-9}$$

Notice that $A = A_1$ if $U_\sigma < A_1$ and

$$\langle f, f\rangle_{\boldsymbol{I}} = \big|\langle f, g\rangle_{\boldsymbol{I}}\big|^2 + |\boldsymbol{I}|^{-1}\big|\langle f, e^{iq}\rangle_{\boldsymbol{I}}\big|^2. \tag{6-10}$$

Then from (6-6) and (6-9), we have

$$(A_1 - \varepsilon)\|f\|_{L^2(\boldsymbol{I})} \le A_1\|f\|_{L^2(\boldsymbol{I})}\sqrt{1 - \frac{\big|\langle f, e^{iq}\rangle_{\boldsymbol{I}}\big|^2}{|\boldsymbol{I}|\langle f, f\rangle_{\boldsymbol{I}}}} + |\boldsymbol{I}|^{-1}\big|\langle f, e^{iq}\rangle_{\boldsymbol{I}}\big|Q. \tag{6-11}$$

Applying the elementary inequality $\sqrt{1-x} \le 1 - x/2$ if $0 \le x \le 1$, we then get

$$A_1 \le \frac{2\|f\|_{L^2(\boldsymbol{I})}}{\big|\langle f, e^{iq}\rangle_{\boldsymbol{I}}\big|} Q + \varepsilon|\boldsymbol{I}|\frac{2\|f\|^2_{L^2(\boldsymbol{I})}}{\big|\langle f, e^{iq}\rangle_{\boldsymbol{I}}\big|^2}. \tag{6-12}$$

From (6-7), we have

$$A_1 \le 2\sigma^{-1}Q + 2\varepsilon|\boldsymbol{I}|\sigma^{-2}. \tag{6-13}$$

Now let $\varepsilon \to 0$, and we then obtain (6-5). Therefore we complete the proof. □

7. Estimates of the trilinear forms

We now start to prove Theorem 4.4, and we only present the details for the case $j > 0$, since the other case can be done similarly. Without loss of generality, in the following sections we assume that f_i is supported on $\boldsymbol{I}_i$ for $i \in \{1, 2, 3\}$, where $\boldsymbol{I}_i$ is either $\big[\frac{1}{16}, \frac{39}{16}\big]$ or $\big[-\frac{39}{16}, -\frac{1}{16}\big]$. Let $\mathcal{Q}_1$ be a set of some functions defined by

$$\mathcal{Q}_1 = \big\{a\xi^{d/d-1} + b\xi : 2^{m-100} \le |a| \le 2^{m+100} \text{ and } a, b \in \mathbb{R}\big\}. \tag{7-1}$$

Proposition 7.1. *Let $f_1\widehat{\Phi_1}$ be σ-uniform in $\mathcal{Q}_1$, and let $j > 0$ and $\Lambda_{j,m}(f_1, f_2, f_3)$ be defined as in* (4-12). *Then there exists a constant C independent of j, m, n, f_1 such that*

$$\big|\Lambda_{j,m}(f_1, f_2, f_3)\big| \le C 2^{-\frac{(d-1)j}{2} - \frac{m}{2}} \max\big\{2^{-100m}, 2^{\frac{-(d-1)j+m}{2}}, \sigma\big\} \prod_{i=1}^{3} \|f_i\|_{L^2(\boldsymbol{I}_i)} \tag{7-2}$$

holds for all $f_2 \in L^2(\boldsymbol{I}_2)$ and $f_3 \in L^2(\boldsymbol{I}_3)$.

Proof. Since $\mathcal{B}_{j,m}$ is supported in an interval with size $2^{(d-1)j+m}$, without loss of generality, we may assume that it is restricted to the interval $I_0 = [0, 2^{(d-1)j+m}]$. Let $\mathbf{1}_{m,l} = \mathbf{1}_{\boldsymbol{I}_{m,l}}$, where $\boldsymbol{I}_{m,l} = [2^m l, 2^m(l+1)]$. Also let $\mathcal{B}_{j,m,l}$ be a bilinear operator defined by

$$\mathcal{B}_{j,m,l}(f, g)(x) = \mathcal{B}_{j,m}(f, g)(x)\mathbf{1}_{m,l}(x),$$

for all f, g. Decompose $\Lambda_{j,m}(f_1, f_2, f_3)$ into $\sum_l \Lambda_{j,m,l}$, where

$$\Lambda_{j,m,l}(f_1, f_2, f_3) = \langle \mathcal{B}_{j,m,l}(\check{f}_1, \check{f}_2), \check{f}_3 \rangle.$$

Let $\alpha_{m,l}$ be a fixed point in the interval $\boldsymbol{I}_{m,l}$. And set $F_{\Phi_1,j,m,l}(x,t)$ to be

$$F_{\Phi_1,j,m,l}(x,t) := R_{\Phi_1}\check{f}_1\big(2^{-(d-1)j}x - 2^m t\big) - R_{\Phi_1}\check{f}_1\big(2^{-(d-1)j}\alpha_{m,l} - 2^m t\big).$$

Split $\mathcal{B}_{j,m,l}(\check{f}_1, \check{f}_2)$ into two terms:

$$\mathcal{B}^{(1)}_{j,m,l}(\check{f}_1, \check{f}_2) + \mathcal{B}^{(2)}_{j,m,l}(\check{f}_1, \check{f}_2),$$

where $\mathcal{B}^{(1)}_{j,m,l}(\check{f}_1, \check{f}_2)$ is equal to

$$2^{-(d-1)j/2}\int_{\mathbb{R}} F_{\Phi_1,j,m,l}(x,t) R_{\Phi_1}\check{f}_2(x - 2^m t^d)\rho(t)\,dt\big(\mathbf{1}^*_{I_0}(x)\mathbf{1}_{m,l}(x)\big),$$

and $\mathcal{B}^{(2)}_{j,m,l}(\check{f}_1, \check{f}_2)$ equals

$$2^{-(d-1)j/2}\int_{\mathbb{R}} R_{\Phi_1}\check{f}_1\big(2^{-(d-1)j}\alpha_{m,l} - 2^m t\big) R_{\Phi_1}\check{f}_2(x - 2^m t^d)\rho(t)\,dt\big(\mathbf{1}^*_{I_0}(x)\mathbf{1}_{m,l}(x)\big).$$

For $i = 1, 2$, let $\Lambda^{(i)}_{j,m}(f_1, f_2, f_3)$ denote

$$\sum_l \langle \mathcal{B}^{(i)}_{j,m,l}(\check{f}_1, \check{f}_2), \check{f}_3 \rangle.$$

We now start to prove that

$$\big|\Lambda^{(1)}_{j,m}(f_1, f_2, f_3)\big| \le 2^{-(d-1)j/2} 2^{-(d-1)j+m} \|\check{f}_1\|_\infty \|\check{f}_2\|_2 \|\check{f}_3\|_2. \tag{7-3}$$

The mean value theorem and the smoothness of Φ_1 yield that for $x \in \boldsymbol{I}_{m,l}$,

$$\big|F_{\Phi_1,j,m,l}(x,t)\big| \le C\|\check{f}_1\|_\infty 2^{-(d-1)j}\big|x - \alpha_{m,l}\big| \le C\,2^{-(d-1)j+m}\|\check{f}_1\|_\infty. \tag{7-4}$$

Because $|t| \sim 1$ when $t \in \operatorname{supp}\rho$, $\mathcal{B}^{(1)}_{j,m,l}(\check{f}_1, \check{f}_2)$ can be written as

$$2^{-(d-1)j/2}\int_{\mathbb{R}} F_{\Phi_1,j,m,l}(x,t)\sum_{l_0}\big(\mathbf{1}_{m,l+l_0}R_{\Phi_1}\check{f}_2\big)(x - 2^m t^d)\rho(t)\,dt\big(\mathbf{1}^*_{I_0}(x)\mathbf{1}_{m,l}(x)\big), \tag{7-5}$$

where l_0 is an integer between -10 and 10. Taking absolute values throughout and applying (7-4) plus the Cauchy–Schwarz inequality, we then estimate $\big|\Lambda^{(1)}_{j,m}(f_1, f_2, f_3)\big|$ by

$$C\,2^{-(d-1)j/2}2^{-(d-1)j+m}\|\check{f}_1\|_\infty \sum_{l_0=-10}^{10}\sum_l \big\|\mathbf{1}_{m,l+l_0}R_{\Phi_1}\check{f}_2\big\|_2 \big\|\mathbf{1}_{m,l}\check{f}_3\big\|_2,$$

which clearly gives (7-3) by one more use of the Cauchy–Schwarz inequality.

We now prove that

$$\big|\Lambda^{(1)}_{j,m}(f_1, f_2, f_3)\big| \le 2^{-(d-1)j/2} 2^{-m} \|\check{f}_1\|_1 \|\check{f}_2\|_2 \|\check{f}_3\|_2. \tag{7-6}$$

From (7-5), we get that $\Lambda^{(1)}_{j,m}(f_1, f_2, f_3)$ equals

$$2^{-(d-1)j/2}\sum_{l_0=-10}^{10}\sum_{l}\Lambda_{j,m,l_0,l,1}(f_1, f_2, f_3)-\Lambda_{j,m,l_0,l,2}(f_1, f_2, f_3),$$

where $\Lambda_{j,m,l_0,l,1}(f_1, f_2, f_3)$ is equal to

$$\int_{\mathbb{R}^2} R_{\Phi_1}\check{f}_1\big(2^{-(d-1)j}x-2^m t\big)\big(\mathbf{1}_{m,l+l_0}R_{\Phi_1}\check{f}_2\big)(x-2^m t^d)\rho(t)\big(\mathbf{1}^*_{I_0}\mathbf{1}_{m,l}\check{f}_3\big)(x)\,dt\,dx$$

and $\Lambda_{j,m,l_0,l,2}(f_1, f_2, f_3)$ equals

$$\int_{\mathbb{R}^2} R_{\Phi_1}\check{f}_1\big(2^{-(d-1)j}\alpha_{m,l}-2^m t\big)\big(\mathbf{1}_{m,l+l_0}R_{\Phi_1}\check{f}_2\big)(x-2^m t^d)\rho(t)\big(\mathbf{1}^*_{I_0}\mathbf{1}_{m,l}\check{f}_3\big)(x)\,dt\,dx.$$

The Cauchy–Schwarz inequality yields that

$$\big|\Lambda_{j,m,l_0,l,2}(f_1, f_2, f_3)\big|\le C2^{-m}\|\check{f}_1\|_1\big\|\mathbf{1}_{m,l+l_0}R_{\Phi_1}\check{f}_2\big\|_2\|\mathbf{1}_{m,l}\check{f}_3\|_2. \tag{7-7}$$

In order to obtain a similar estimate for $\Lambda_{j,m,l_0,l,1}(f_1, f_2, f_3)$, we change variables by $u=2^{-(d-1)j}x-2^m t$ and $v=x-2^m t^d$ to express $\Lambda_{j,m,l_0,l,1}(f_1, f_2, f_3)$ as

$$\iint R_{\Phi_1}\check{f}_1(u)\big(\mathbf{1}_{m,l+l_0}R_{\Phi_1}\check{f}_2\big)(v)\rho(t(u,v))\big(\mathbf{1}^*_{I_0}\mathbf{1}_{m,l}\check{f}_3\big)(x(u,v))\frac{du\,dv}{J(u,v)},$$

where $J(u,v)$ is the Jacobian $\partial(u,v)/\partial(x,t)$. It is easy to see that the Jacobian $\partial(u,v)/\partial(x,t)\sim 2^m$. As for $\Lambda_{j,m,l_0,l,1}$, we dominate the previous integral by

$$C2^{-m}\int\big|R_{\Phi_1}\check{f}_1(u)\big|\big\|\mathbf{1}_{m,l+l_0}R_{\Phi_1}\check{f}_2\big\|_2\bigg(\int\Big|\big(\mathbf{1}_{m,l}\check{f}_3\big)(x(u,v))\rho(t(u,v))\Big|^2dv\bigg)^{1/2}du.$$

Notice that $|\partial x/\partial v|\sim 1$ whenever $t\in\operatorname{supp}\rho$. We then estimate

$$\big|\Lambda_{j,m,l_0,l,1}(f_1, f_2, f_3)\big|\le C2^{-m}\|\check{f}_1\|_1\big\|\mathbf{1}_{m,l+l_0}R_{\Phi_1}\check{f}_2\big\|_2\,\|\mathbf{1}_{m,l}\check{f}_3\|_2; \tag{7-8}$$

(7-6) follows from (7-7) and (7-8). An interpolation of (7-3) and (7-6) then yields

$$\big|\Lambda^{(1)}_{j,m}(f_1, f_2, f_3)\big|\le C2^{-\frac{(d-1)j}{2}-\frac{m}{2}}2^{\frac{-(d-1)j+m}{2}}\prod_{i=1}^{3}\|f_i\|_{L^2(\boldsymbol{I}_i)}. \tag{7-9}$$

We now turn to prove that

$$\big|\Lambda^{(2)}_{j,m}(f_1, f_2, f_3)\big|\le C_N 2^{-\frac{(d-1)j}{2}-\frac{m}{2}}\max\{2^{-100m},\sigma\}\prod_{i=1}^{3}\|f_i\|_{L^2(\boldsymbol{I}_i)}. \tag{7-10}$$

In dual frequency variables, $\Lambda^{(2)}_{j,m}(f_1, f_2, f_3)$ can be expressed as

$$\sum_{l_0=-10}^{10}\sum_{l}2^{-\frac{(d-1)j}{2}}\iint f_1(\xi)\widehat{\Phi}_1(\xi)\exp(2\pi i\,2^{-(d-1)j}\alpha_{m,l}\xi)\widehat{F_{2,m,l_0,l}}(\eta)\mathfrak{m}(\xi,\eta)\widehat{F_{3,m,l}}(\eta)\,d\xi\,d\eta,$$

where

$$\mathfrak{m}(\xi,\eta)=\int\rho(t)exp\big(-2\pi i(2^m\xi t+2^m\eta t^d)\big)\,dt, \tag{7-11}$$

$$F_{2,m,l_0,l}=\mathbf{1}_{m,l+l_0}R_{\Phi_1}\check{f}_2, \quad \text{and} \quad F_{3,m,l}=\mathbf{1}^*_{I_0}\mathbf{1}_{m,l}\check{f}_3.$$

If η is not in a small neighborhood of $\widehat{\Phi}_1$, then there is no critical point of the phase function $\phi_{\xi,\eta}(t)=\xi t+\eta t^d$ occurring in a small neighborhood of $\operatorname{supp}\rho$. Integration by parts gives a rapid decay $O(2^{-Nm})$ for $\mathfrak{m}$. Thus in this case, we dominate $\big|\Lambda^{(2)}_{j,m,n}(f_1,f_2,f_3)\big|$ by

$$C_N2^{-Nm}\prod_{i=1}^3\|f_i\|_{L^2(\boldsymbol{I}_i)}, \tag{7-12}$$

for any positive integer N. We now only need to consider the worst case, when there is a critical point of the phase function $\phi_{\xi,\eta}(t)=\xi t+\eta t^d$ in a small neighborhood of $\operatorname{supp}\rho$. In this case, η must be in a small neighborhood of $\widehat{\Phi}_1$, and the stationary phase method gives

$$\mathfrak{m}(\xi,\eta)\sim 2^{-m/2}\exp\big(2\pi i c_d2^m\eta^{-1/(d-1)}\xi^{d/(d-1)}\big), \tag{7-13}$$

where c_d is a constant depending on d only. Thus the principal term of $\Lambda^{(2)}_{j,m}(f_1,f_2,f_3)$ is

$$\sum_{l_0=-10}^{10}\sum_l 2^{-\frac{(d-1)j}{2}-\frac{m}{2}}\iint f_1(\xi)\widehat{\Phi}_1(\xi)e^{i\phi_{d,m,\eta}(\xi)}\widehat{F_{2,m,l_0,l}}(\eta)\widehat{\Phi}_2(\eta)\widehat{F_{3,m,l}}(\eta)\,d\xi\,d\eta,$$

where $\widehat{\Phi}_2$ is a Schwartz function supported on a small neighborhood of $\widehat{\Phi}_1$, and

$$\phi_{d,m,\eta}(\xi)=2\pi c_d2^m\eta^{-1/(d-1)}\xi^{d/(d-1)}+2\pi\,2^{-(d-1)j}\alpha_{m,l}\xi.$$

The key point is that the integral in the previous expression can be viewed as an inner product of $F_{3,m,l}$ and $\mathcal{M}F_{2,m,l_0,l}$, where $\mathcal{M}$ is a multiplier operator defined by

$$\widehat{\mathcal{M}f}(\eta)=\mathfrak{m}_{d,j,m}(\eta)\widehat{f}(\eta).$$

Here the multiplier $\mathfrak{m}_{d,j,m}$ is given by

$$\mathfrak{m}_{d,j,m}(\eta)=\int f_1(\xi)\widehat{\Phi}_1(\xi)e^{i\phi_{d,m,\eta}(\xi)}d\xi. \tag{7-14}$$

Observe that $\phi_{d,m,\eta}(\xi)+b\xi$ is in $\mathfrak{Q}_1$ for any $b\in\mathbb{R}$ and $\eta\in\operatorname{supp}\widehat{\Phi}_2$. Thus σ-uniformity in $\mathfrak{Q}_1$ of f_1 yields

$$\|\mathfrak{m}_{d,j,m}\|_\infty\le C\sigma\|f_1\|_{L^2(\boldsymbol{I}_1)}. \tag{7-15}$$

And henceforth we dominate $\Lambda^{(2)}_{j,m}(f_1,f_2,f_3)$ by

$$\sum_{l_0=-10}^{10}\sum_l 2^{-\frac{(d-1)j}{2}-\frac{m}{2}}\sigma\|f_1\|_{L^2(\boldsymbol{I}_1)}\big\|F_{2,m,l_0,l}\big\|_2\,\|F_{3,m,l}\|_2,$$

which clearly is bounded by

$$2^{-\frac{(d-1)j}{2}-\frac{m}{2}}\sigma\prod_{i=1}^3\|f_i\|_{L^2(\boldsymbol{I}_i)}. \tag{7-16}$$

Now (7-10) follows from (7-12) and (7-16). Combining (7-9) and (7-10), we finish the proof. □

Corollary 7.2. *Let $\Lambda_{j,m}(f_1, f_2, f_3)$ be defined as in* (4-12). *Then there exists a constant C independent of j, m, n such that*

$$\left|\Lambda_{j,m}(f_1, f_2, f_3)\right| \le C \max\left\{2^{-100m}, 2^{\frac{-(d-1)j+m}{2}}, \sigma\right\} \|f_1\|_{L^2(\boldsymbol{I}_1)} \|f_2\|_{L^2(\boldsymbol{I}_1)} \|\widehat{f_3}\|_\infty \tag{7-17}$$

holds for all $f_1 \in L^2(\boldsymbol{I}_1)$ which are σ-uniform in $\mathfrak{Q}_1$, $f_2 \in L^2(\boldsymbol{I}_2)$ and $\widehat{f_3} \in L^\infty$.

Proof. Since there is a smooth restriction factor $\mathbf{1}^*_{I_0}$ in the definition of $\mathcal{B}_{j,m}$, the right-hand side of (7-2) can be sharpened to

$$C 2^{-\frac{(d-1)j}{2} - \frac{m}{2}} \max\left\{2^{-100m}, 2^{\frac{-(d-1)j+m}{2}}, \sigma\right\} \|f_1\|_{L^2(\boldsymbol{I}_1)} \|f_2\|_{L^2(\boldsymbol{I}_2)} \left\|\mathbf{1}^{**}_{(d-1)j+m,n} \check{f_3}\right\|_2, \tag{7-18}$$

which is clearly bounded by

$$C \max\left\{2^{-100m}, 2^{\frac{-(d-1)j+m}{2}}, \sigma\right\} \|f_1\|_{L^2(\boldsymbol{I}_1)} \|f_2\|_{L^2(\boldsymbol{I}_1)} \|\widehat{f_3}\|_\infty. \qquad \square$$

Proposition 7.3. *Let $\Lambda_{j,m}(f_1, f_2, f_3)$ be defined as in* (4-12). *Then there exists a constant C independent of j, m, n such that*

$$\left|\Lambda_{j,m}(e^{iq_1}, f_2, f_3)\right| \le C 2^{-\mathfrak{D}(d-1)m/2} \|f_2\|_{L^2(\boldsymbol{I}_2)} \|\widehat{f_3}\|_\infty \tag{7-19}$$

holds for all $q_1 \in \mathfrak{Q}_1$, $f_2 \in L^2(\boldsymbol{I}_2)$ and $\widehat{f_3} \in L^\infty$, where $\mathfrak{D}(d-1)$ is the positive constant defined in (8-3).

A proof of Proposition 7.3 will be provided in Section 8.

Proof of Theorem 4.4. Corollary 7.2, Proposition 7.3 and Theorem 6.2 yield that $\left|\Lambda_{j,m}(f_1, f_2, f_3)\right|$ is dominated by

$$C\left(\max\left\{2^{-100m}, 2^{\frac{-(d-1)j+m}{2}}, \sigma\right\} + \frac{2^{-\mathfrak{D}(d-1)m/2}}{\sigma}\right) \|f_1\|_{L^2(\boldsymbol{I}_1)} \|f_2\|_{L^2(\boldsymbol{I}_1)} \|\widehat{f_3}\|_\infty \tag{7-20}$$

for all $f_1 \in L^2(\boldsymbol{I}_1)$, $f_2 \in L^2(\boldsymbol{I}_2)$ and $\widehat{f_3} \in L^\infty$. Take σ to be $2^{-\mathfrak{D}(d-1)m/4}$; then we have

$$\left|\Lambda_{j,m}(f_1, f_2, f_3)\right| \le C \max\left\{2^{\frac{-(d-1)j+m}{2}}, 2^{-\mathfrak{D}(d-1)m/4}\right\} \|f_1\|_{L^2(\boldsymbol{I}_1)} \|f_2\|_{L^2(\boldsymbol{I}_1)} \|\widehat{f_3}\|_\infty. \tag{7-21}$$

This gives the desired estimate for the case $j > 0$. Similarly, for $j \neq 0$, we have

$$\left|\Lambda_{j,m}(f_1, f_2, f_3)\right| \le C \max\left\{2^{\frac{(d-1)j+m}{2}}, 2^{-m/8}\right\} \|f_1\|_{L^2(\boldsymbol{I}_1)} \|f_2\|_{L^2(\boldsymbol{I}_1)} \|\widehat{f_3}\|_\infty. \tag{7-22}$$

Combining (7-21) and (7-22) proves Theorem 4.4. □

8. Proof of Proposition 7.3

Lemma 8.1. *Let $l \ge 1$. Let $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ be fixed bounded intervals, and let $\varphi : \boldsymbol{I}_1 \times \boldsymbol{I}_2 : \mathbb{R}$ satisfy*

$$\left|\partial_x^l \partial_y \varphi(x, y)\right| \ge 1 \quad \textit{for all } (x, y) \in \boldsymbol{I}_1 \times \boldsymbol{I}_2. \tag{8-1}$$

Assume an additional condition holds in the case $l = 1$:

$$\left|\partial_x^2 \partial_y \varphi(x, y)\right| \neq 0 \quad \textit{for all } (x, y) \in \boldsymbol{I}_1 \times \boldsymbol{I}_2. \tag{8-2}$$

Set

$$\mathfrak{D}(l)=\begin{cases}1/(2l) & \text{if } l\ge 2,\\ 1/(2+\varepsilon) & \text{if } l=1,\end{cases}\tag{8-3}$$

for some $\varepsilon>0$. Then there exists a constant depending on the length of $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ but independent of φ, λ and the locations of $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ such that

$$\left|\iint_{\boldsymbol{I}_1\times\boldsymbol{I}_2} e^{i\lambda\varphi(x,y)} f(x)g(y)\,dx\,dy\right| \le C(1+|\lambda|)^{-\mathfrak{D}(l)}\|f\|_2\,\|g\|_2,\quad \text{for all } f,g\in L^2.\tag{8-4}$$

This lemma is related to a two-dimensional van der Corput lemma proved in [Carbery et al. 1999]. The case $l\ge 2$ was proved in [Carbery et al. 1999], and a proof of the case $l=1$ can be found in [Phong and Stein 1994]. The estimates on $\mathfrak{D}(l)$ in (8-3) are not sharp. With some additional convexity conditions on the phase function φ, one can improve $\mathfrak{D}(l)$ to $1/(l+1)$ (see [Carbery et al. 1999] for some such improvements). But in this article we do not need to pursue the sharp estimates.

Lemma 8.2. *Let $c,\tau\in\mathbb{R}$ and φ be a function defined by*

$$\varphi_c(x,y)=\left(x-y^{1/d}+c\right)^d.\tag{8-5}$$

Define $\boldsymbol{Q}_{c,j.\tau}(x,y)$ by

$$\boldsymbol{Q}_{c,j,\tau}(x,y)=\varphi_c(x,y)-\varphi_c\left(x+2^{-(d-1)j}\tau,\,y+\tau\right).\tag{8-6}$$

Then there exists a constant C_d depending only on d such that

$$\left|\partial_x^{d-1}\partial_y\boldsymbol{Q}_{c,j,\tau}(x,y)\right|\ge C_d|\tau|\tag{8-7}$$

for all y such that $y+\tau\in[2^{-100},2^{100}]$. Moreover, if $d=2$,

$$\left|\partial_x\partial_y^2\boldsymbol{Q}_{c,j,\tau}(x,y)\right|\ge C_d|\tau|\tag{8-8}$$

for all y such that $y+\tau\in[2^{-100},2^{100}]$.

Proof. A direct computation yields

$$\partial_x^{d-1}\partial_y\boldsymbol{Q}_{c,j,\tau}(x,y)=C_d\left((y+\tau)^{(1/d)-1}-y^{(1/d)-1}\right).\tag{8-9}$$

Hence the desired estimate (8-7) follows immediately from the mean value theorem. The bound (8-8) can be obtained similarly. □

Lemma 8.3. *Let $\boldsymbol{I}$ be a fixed interval of length 1, and let θ be a bump function supported on $\left[\frac{1}{100},2\right]$ (or $\left[-2,-\frac{1}{100}\right]$). Suppose that $\phi_{d,j,m}$ is a phase function defined by*

$$\phi_{d,j,m}(x,y)=C_{d,j,m}2^m\left(x-y^{1/d}+c_{j,m}\right)^d,\tag{8-10}$$

where $C_{d,j,m}$, $c_{j,m}$ are constants independent of x, y such that $2^{-200}\le|C_{d,j,m}|\le 2^{200}$. Let $\Lambda_{d,j,m,\boldsymbol{I}}$ be a bilinear form defined by

$$\Lambda_{d,j,m,\boldsymbol{I}}(f,g)=\iint e^{i\phi_{d,j,m}(x,t)}f\left(x-2^{-(d-1)j}t\right)g(x)\mathbf{1}_{\boldsymbol{I}}(x)\theta(t)\,dx\,dt.\tag{8-11}$$

Then we have

$$\left|\Lambda_{d,j,m,\boldsymbol{I}}(f,g)\right| \le C_d 2^{-\mathfrak{D}(d-1)m/2}\|f\|_2\,\|g\|_\infty \tag{8-12}$$

for all $f \in L^2$ *and* $g \in L^\infty$, *where* C_d *is a constant depending only on* d.

Proof. The bilinear form $\Lambda_{d,j,m,\boldsymbol{I}}(f,g)$ equals $\langle \boldsymbol{T}_{d,j,m,\boldsymbol{I}}(g), f\rangle$, where $\boldsymbol{T}_{d,j,m,\boldsymbol{I}}$ is defined by

$$\boldsymbol{T}_{d,j,m,\boldsymbol{I}}\,g(x) = \int \exp\bigl(i\phi_{d,j,m}(x+2^{-(d-1)j}t,t)\bigr)(g\mathbf{1}_{\boldsymbol{I}})\bigl(x+2^{-(d-1)j}t\bigr)\theta(t)\,dt. \tag{8-13}$$

By a change of variables, $\left\|\boldsymbol{T}_{d,j,m,\boldsymbol{I}}\,g\right\|_2^2$ can be expressed as

$$\int\left(\iint e^{i\Phi_{d,j,m,\tau}(x,t)}G_\tau\bigl(x+2^{-(d-1)j}t\bigr)\Theta_\tau(t)\,dx\,dt\right)d\tau,$$

where

$$\begin{aligned}
\Phi_{d,j,m,\tau}(x,t) &= \phi_{d,j,m}\bigl(x+2^{-(d-1)j}t,t\bigr) - \phi_{d,j,m}\bigl(x+2^{-(d-1)j}t+2^{-(d-1)j}\tau,t+\tau\bigr),\\
G_\tau(x) &= (\mathbf{1}_{\boldsymbol{I}}g)(x)\overline{(\mathbf{1}_{\boldsymbol{I}}g)\bigl(x+2^{-(d-1)j}\tau\bigr)},\\
\Theta_\tau(t) &= \theta(t)\theta(t+\tau).
\end{aligned}$$

Changing coordinates $(x,t)\mapsto(u,v)$ by $u = x+2^{-(d-1)j}t$ and $v=t$, we write the inner double integral in the previous integral as

$$\iint \exp\bigl(iC_{d,j,m}2^m\boldsymbol{Q}_{c_{j,m},j,\tau}(u,v)\bigr)G_\tau(u)\Theta_\tau(v)\,du\,dv,$$

where $\boldsymbol{Q}_{c_{j,m},j,\tau}$ is defined as in (8-6). From (8-7), (8-8) and Lemma 8.1, we then estimate $\left\|\boldsymbol{T}_{d,j,m,\boldsymbol{I}}\,g\right\|_2^2$ by

$$C_d\int_{-10}^{10}\min\bigl\{1,2^{-\mathfrak{D}(d-1)m}\tau^{-\mathfrak{D}(d-1)}\bigr\}\|G_\tau\|_2\,\|\Theta_\tau\|_2\,d\tau,$$

which clearly is bounded by

$$C_d 2^{-\mathfrak{D}(d-1)m}\|g\|_\infty^2.$$

Hence (8-12) follows and therefore we complete the proof. □

We now turn to the proof of Proposition 7.3. For simplicity, we assume ρ is supported on $\bigl[\frac{1}{8},2\bigr]$. For any function $q_1 = a\xi^{d/(d-1)}+b\xi\in\mathfrak{Q}_1$, we have

$$R_{\Phi_1}(\check{e^{iq_1}})(x) = \int\widehat{\Phi}_1(\xi)\exp(ia\xi^{d/(d-1)})\exp(i(x+b)\xi)\,d\xi, \tag{8-14}$$

where $|a|\sim 2^m$. The stationary phase method yields that the principal part of (8-14) is

$$\mathcal{P}(q_1)(x) = C_d|a|^{-1/2}\exp(ic_1a^{-(d-1)}(x+b)^d)\widehat{\Phi}_1\bigl(c_2a^{-(d-1)}(x+b)^{d-1}\bigr), \tag{8-15}$$

where C_d, c_1, c_2 are constants depending only on d. Thus to obtain Proposition 7.3, it suffices to prove that there exists a constant C such that

$$\bigl|\tilde{\Lambda}_{j,m}(e^{iq_1},f_2,f_3)\bigr| \le C\,2^{-\frac{\mathfrak{D}(d-1)m}{2}}\|\check{f}_2\|_2\,\|\check{f}_3\|_\infty \tag{8-16}$$

holds for all $q_1 \in \mathfrak{Q}_1$, $\check{f}_2 \in L^2$, and $\check{f}_3 \in L^\infty$, where $\tilde{\Lambda}_{j,m,n}(e^{iq_1}, f_2, f_3)$ is defined to be

$$2^{-(d-1)j/2} \iint \mathcal{P}(q_1)\bigl(2^{-(d-1)j}x - 2^m t\bigr)\check{f}_2\bigl(x - 2^m t^d\bigr)\bigl(\mathbf{1}^*_{I_0}\check{f}_3\bigr)(x)\rho(t)\,dt\,dx.$$

Observe that $\widehat{\Phi}_1$ is supported essentially in a bounded interval away from 0. Thus we can restrict the variable x in a bounded interval $\boldsymbol{I}_{d,j,m}$ whose length is comparable to $2^{(d-1)j+m}$ and reduce the problem to showing that

$$\bigl|\Lambda_{j,m,\boldsymbol{I}_{d,j,m}}(f_2, f_3)\bigr| \le C 2^{-\frac{\mathfrak{D}(d-1)m}{2}} \|\check{f}_2\|_2 \|\check{f}_3\|_\infty \tag{8-17}$$

for an absolute constant C and all $\check{f}_2 \in L^2$, $\check{f}_3 \in L^\infty$, where $\Lambda_{j,m,n,\boldsymbol{I}_{d,j,m}}(f_2, f_3)$ is equal to

$$2^{-\frac{(d-1)j}{2}-\frac{m}{2}} \iint \mathcal{P}_{d,j,m}(2^{-(d-1)j}x - 2^m t)\check{f}_2(x - 2^m t^d)(\mathbf{1}_{\boldsymbol{I}_{d,j,m}}\check{f}_3)(x)\rho(t)\,dt\,dx. \tag{8-18}$$

Here

$$\mathcal{P}_{d,j,m}(x) = \exp(ic_1 a^{-(d-1)}(x+b)^d)\widehat{\Phi}_1\bigl(c_2 a^{-(d-1)}(x+b)^{d-1}\bigr). \tag{8-19}$$

Let $\boldsymbol{I}$ be an interval of length 1. A rescaling argument then reduces (8-17) to an estimate of a bilinear form $\Lambda_{j,m,n,\boldsymbol{I}}$ associated to $\boldsymbol{I}$, that is,

$$\bigl|\Lambda_{j,m,\boldsymbol{I}}(f, g)\bigr| \le C 2^{-\frac{\mathfrak{D}(d-1)m}{2}} \|f\|_2 \|g\|_\infty, \tag{8-20}$$

where $\Lambda_{j,m,\boldsymbol{I}}(f, g)$ is defined by

$$\iint \mathcal{P}_{d,j,m}\bigl(2^m x - 2^m t\bigr) f\bigl(x - 2^{-(d-1)j}t^d\bigr) g(x)\mathbf{1}_{\boldsymbol{I}}(x)\rho(t)\,dt\,dx.$$

Notice that

$$\mathcal{P}_{d,j,m}(2^m x - 2^m t) = \exp(iC_{d,j,m}2^m(x - t + c_{j,m})^d)\widehat{\Phi}_1\bigl(C_d C_{d,m}(x - t + c_m)^{d-1}\bigr), \tag{8-21}$$

where $C_{d,j,m}$, $C_{d,m}$, $c_{j,m}$, c_m, C_d are constants such that $|C_{d,j,m}|, |C_{d,m}| \in [2^{-100}, 2^{100}]$. Clearly

$$\widehat{\Phi_1}\bigl(C_d C_{d,m}(x - t + c_m)^{d-1}\bigr)$$

can be dropped by utilizing Fourier series since $\widehat{\Phi}_1$ is a Schwartz function, because $x \in \boldsymbol{I}$, $t \in \operatorname{supp}\rho$ are restricted in bounded intervals. Then (8-20) can be reduced to Lemma 8.3 by a change of variable $t^d \mapsto t$. This proves Proposition 7.3.

9. Appendix

In this appendix, we consider a simple bilinear operator associated to a polynomial curve without singularity. A counterexample is given to indicate that the range of $(1/p, 1/q, 1/r)$ must depend on the degree of the polynomial when the linear term does not vanish. Let ρ be a Schwartz function supported in the union of two intervals $\bigl[-2, -\frac{1}{2}\bigr]$ and $\bigl[\frac{1}{2}, 2\bigr]$.

Lemma 9.1. *Let P be a real polynomial with degree $d \ge 2$. And let $2 \le n \le d$. Suppose that the n-th order derivative of P, $P^{(n)}$, does not vanish. Let $T(f,g)(x) = \int f(x-t)g(x-P(t))\rho(t)\,dt$. Then T is bounded from $L^p \times L^q$ to L^r for $p, q > 1$, $r > (n-1)/n$ and $1/p + 1/q = 1/r$.*

Proof. We may without loss of generality restrict x, and hence likewise the supports of f, g, to fixed bounded intervals whose sizes depend on the coefficients of the polynomial P. This is possible because of the restriction $|t| \le 2$ in the integral. Let us restrict x in a bounded interval I_P. It is obvious that T is bounded uniformly from $L^\infty \times L^\infty$ to L^∞ and from $L^p \times L^{p'}$ to L^1 for $1 \le p \le \infty$ and $1/p + 1/p' = 1$. When $P'(t) \ne 1$ in $\frac{1}{2} \le |t| \le 2$, the boundedness from $L^1 \times L^1$ to L^1 can be obtained immediately by changing variable $u = x - t$ and $v = x - P(t)$, since the Jacobian $\partial(u, v)/\partial(x, t) = 1 - P'(t)$. Thus T is bounded from $L^1 \times L^1$ to $L^{1/2}$, since x is restricted to a bounded interval I_P, and then the lemma follows by interpolation. When there is a real solution in $\frac{1}{2} \le |t| \le 2$ to the equation $P'(t) = 1$, the trouble happens at a neighborhood of t_0, where $t_0 \in \{t : \frac{1}{2} \le |t| \le 2\}$ is the real solution to $P'(t) = 1$. There are at most $d - 1$ real solutions to the equation $P'(t) - 1 = 0$. Thus we only need to consider a small neighborhood containing only one real solution t_0 to $P'(t) = 1$. Let $I(t_0)$ be a small neighborhood of t_0 which contains only one real solution to $P'(t) - 1 = 0$. We should prove that

$$\int_{I_P} \left| \int_{I(t_0)} f(x - t) g(x - P(t)) \rho(t)\, dt \right|^r dx \le C_P \|f\|_p^r \|g\|_q^r, \tag{9-1}$$

for $p > 1$, $q > 1$ and $r > (n-1)/n$ with $1/p + 1/q = 1/r$. Let ρ_0 be a suitable bump function supported in $\frac{1}{2} \le |t| \le 2$ such that $\sum_j \rho_0(2^j t) = 1$. To get (9-1), it suffices to prove that there is a positive ε such that

$$\int_{I_P} \left| \int_{I(t_0)} f(x - t) g(x - P(t)) \rho(t) \rho_0(2^j (t - t_0))\, dt \right|^r dx \le C 2^{-\varepsilon j} \|f\|_p^r \|g\|_q^r, \tag{9-2}$$

for all large j, $p > 1$, $q > 1$ and $r > (n-1)/n$ with $1/p + 1/q = 1/r$, since (9-1) follows by summing for all possible $j \ge 1$. By a translation argument, we need to show that

$$\int_{I_P} \left| \int f(x - t) g(x - P_1(t)) \rho_0(2^j t)\, dt \right|^r dx \le C 2^{-\varepsilon j} \|f\|_p^r \|g\|_q^r, \tag{9-3}$$

for all large j, $p > 1$, $q > 1$ and $r > (n-1)/n$ with $1/p + 1/q = 1/r$, where P_1 is a polynomial of degree d defined by $P_1(t) = P(t + t_0) - P(t_0)$. It is clear that $P_1'(0) = 1$ and $P_1^{(n)} \ne 0$. When $|t| \le 2^{-j+1}$, $|P_1(t)| \le C_P 2^{-j}$ for some constant $C_P \ge 1$ depending on the coefficients of P. Let $I_P = [a_P, b_P]$ and A_N be defined by

$$A_N = \left[a_P + N C_P 2^{-j}, a_P + (N+1) C_P 2^{-j}\right] \quad \text{for } N = -1, \dots, \frac{(b_P - a_P) \cdot 2^j}{C_P}.$$

Notice that for a fixed $x \in I_P$, $x - t$, $x - P_1(t)$ is in $A_{N-1} \cup A_N \cup A_{N+1}$ for some N. So we can restrict x in one of the A_N. Now let $T_N(f, g)(x) = 1_{A_N}(x) \int f(x - t) g(x - P_1(t)) \rho_0(2^j t)\, dt$. Due to the restriction of x, we only need to show that

$$\left\| T_N(f, g) \right\|_r^r \le C 2^{-\varepsilon j} \|f_N\|_p^r \|g_N\|_q^r \tag{9-4}$$

for all large $j \ge 1$, $p > 1$, $q > 1$ and $r > (n-1)/n$ with $1/p + 1/q = 1/r$, where $f_N = f 1_{A_N}$, $g_N = g 1_{A_N}$ and C is independent of N.

By inserting absolute values throughout, we get that T_N maps $L^p \times L^q$ to L^r with a bound $C 2^{-j}$ uniform in N, whenever $(1/p, 1/q, 1/r)$ belongs to the closed convex hull of the points $(1, 0, 1)$, $(0, 1, 1)$

and $(0, 0, 0)$. Observe that $P_1'(t) = 1 + \sum_{k=2}^{d-1} \big(P_1^{(k)}(0)/(k-1)!\big) t^{k-1}$ since $P_1'(0) = 1$. By $P_1^{(n)}(0) \neq 0$ and applying the Cauchy–Schwarz inequality, we obtain, for all j large enough,

$$\begin{aligned}\int \big|T_N(f, g)(x)\big|^{1/2} dx &\leq C_P 2^{-j/2} \|T_N(f, g)\|_1^{1/2} \\ &\leq C_P 2^{-j/2} 2^{(n-1)j/2} \|f\|_1^{1/2} \|g\|_1^{1/2} = C_P 2^{(n-2)j/2} \|f\|_1^{1/2}\, \|g\|_1^{1/2}.\end{aligned}$$

Hence, an interpolation then yields a bound $C2^{-\varepsilon j}$ for all triples of reciprocal exponents within the convex hull of $\big(1, 1/(n-1), n/(n-1)\big)$, $\big(1/(n-1), 1, n/(n-1)\big)$, $(1, 0, 1)$, $(0, 1, 1)$ and $(0, 0, 0)$. This finishes the proof of (9-4). Therefore we complete the proof of Lemma 9.1 □

Notice that if P is a monomial t^d, then the lower bound for r in Lemma 9.1 can be improved to $\frac{1}{2}$. This is because $P_1(t) = P(t + t_0) - P(t_0) = (t + t_0)^d - t_0^d$ has nonvanishing $P_1^{(2)}(0)$ when $\frac{1}{2} \leq |t_0| \leq 1$. We now give a counterexample to indicate that the lower bound $(n-1)/n$ for r is sharp in Lemma 9.1.

Proposition 9.2. *Let d, n be integers such that $d \geq 2$ and $2 \leq n \leq d$. There is a real polynomial Q of degree $d \geq 2$ whose n-th order derivative does not vanish such that T_Q is unbounded from $L^p \times L^q$ to L^r for all $p, q > 1$ and $r < (n-1)/n$ with $1/p + 1/q = 1/r$, where T_Q is the bilinear operator defined by $T_Q(f, g)(x) = \int f(x-t) g(x - Q(t)) \rho(t)\, dt$.*

Proof. Let A be a very large number. We define $Q(t)$ by

$$Q(t) = \frac{1}{Ad!}(t-1)^d + \frac{1}{An!}(t-1)^n + (t-1). \tag{9-5}$$

It is sufficient to prove that if T_Q is bounded from $L^p \times L^q$ to L^r for some $p, q > 1$ and $1/r = 1/p + 1/q$, then $r \geq (n-1)/n$. Suppose there is a constant C such that $\big\|T_Q(f, g)\big\|_r \leq C \|f\|_p \|g\|_q$ for all $f \in L^p$ and $g \in L^q$. Let δ be a small positive number, and let $f_\delta = 1_{[0, 2^n\delta]}$ and $g_\delta = 1_{[1-\delta, 1]}$. Let D_1 be the intersection point of the curves $x = Q(t) + 1$ and $x = t + 2^n\delta$ in the tx-plane with $t > 1$, and let D_2 be the intersection point of the curves $x = Q(t) + 1 - \delta$ and $x = t$ in the tx-plane with $t > 1$. Let $D_1 = (t_1, x_1)$ and $D_2 = (t_2, x_2)$. Then

$$\begin{aligned}1 + 2^{1-1/n}(An!)^{1/n}\delta^{1/n} \leq t_1 \leq 1 + 2(An!)^{1/n}\delta^{1/n} \quad \text{and} \\ 1 + 2^{-1/n}(An!)^{1/n}\delta^{1/n} \leq t_2 \leq 1 + (An!)^{1/n}\delta^{1/n}.\end{aligned}$$

Thus we have

$$\begin{aligned}1 + 2^{1-1/n}(An!)^{1/n}\delta^{1/n} + 2^n\delta \leq x_1 \leq 1 + 2(An!)^{1/n}\delta^{1/n} + 2^n\delta \quad \text{and} \\ 1 + 2^{-1/n}(An!)^{1/n}\delta^{1/n} \leq x_2 \leq 1 + (An!)^{1/n}\delta^{1/n}.\end{aligned}$$

When A is large and δ is small, any horizontal line between line $x = x_1$ and line $x = x_2$ has a line segment of length $\delta/2$ staying within the region bounded by curves $x = t$, $x = Q(x) + 1 - \delta$, $x = t + 2^n\delta$ and $x = Q(t) + 1$. Hence, we have

$$\big\|T_Q(f_\delta, g_\delta)\big\|_r^r \geq \frac{\left(\frac{\delta}{2}\right)^r (An!)^{1/n}\delta^{1/n}}{100}. \tag{9-6}$$

By the boundedness of T_Q, we have

$$\left\| T_Q(f_\delta, g_\delta)\right\|_r^r \leq C^r(2^n\delta)^{r/p}\delta^{r/q} = C^r 2^{nr/p}\delta.$$

By (9-6), we have

$$\delta^r \leq \frac{1002^{r+nr/p}C^r}{(An!)^{1/n}}\delta^{\frac{n-1}{n}}. \tag{9-7}$$

Since A can be chosen to be a very large number and δ can be very small, (9-7) implies $r \geq (n-1)/n$, which completes the proof of Proposition 9.2. □

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Received 13 Oct 2011. Accepted 18 Nov 2011.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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ANALYSIS & PDE

Volume 6 No. 1 2013

2157-5045(2013)6:1;1-H