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HYPOELLIPTICITY AND NONHYPOELLIPTICITY FOR SUMS OF SQUARES OF COMPLEX VECTOR FIELDS

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In this paper we consider a model sum of squares of complex vector fields in the plane, close to Kohn's operator but with a point singularity,

$$P = BB^* + B^*(t^{2l} + x^{2k})B, \quad B = D_x + i x^{q-1} D_t.$$

The characteristic variety of P is the symplectic real analytic manifold $x = \xi = 0$. We show that this operator is C^∞-hypoelliptic and Gevrey hypoelliptic in G^s, the Gevrey space of index s, provided $k < lq$, for every $s \geq lq/(lq-k) = 1 + k/(lq-k)$. We show that in the Gevrey spaces below this index, the operator is not hypoelliptic. Moreover, if $k \geq lq$, the operator is not even hypoelliptic in C^∞. This fact leads to a general negative statement on the hypoellipticity properties of sums of squares of complex vector fields, even when the complex Hörmander condition is satisfied.

1. Introduction

In [Kohn 2005] (and [Bove et al. 2006]; see below) the operator

$$E_{m,k} = L_m \overline{L_m} + \overline{L_m} |z|^{2k} L_m, \quad L_m = \frac{\partial}{\partial z} - i\bar{z}|z|^{2(m-1)} \frac{\partial}{\partial t},$$

was introduced and shown to be hypoelliptic, yet to lose $2 + (k-1)/m$ derivatives in L^2 Sobolev norms. Christ [2005] showed that the addition of one more variable destroys hypoellipticity altogether. In those seminal works, $m = 1$, but Kohn, A. Bove, M. Derridj, and D. S. Tartakoff generalized the results to higher m in [Bove et al. 2006] and elsewhere.

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Subsequently, Bove and Tartakoff [2010] showed that Kohn's operator with an added Oleinik-type singularity, of the form studied in [Bove and Tartakoff 1997],

$$E_{m,k} + |z|^{2(p-1)} D_y^2,$$

is Gevrey s-hypoelliptic for any $s \geq 2m/(p-k)$ (here $2m > p > k$). A related result is that the "real" version, with $X = D_x + i x^{q-1} D_t$, where $D_x = i^{-1}\partial_x$,

$$R_{q,k} + x^{2(p-1)} D_y^2 = XX^* + (x^k X)^*(x^k X) + x^{2(p-1)} D_y^2$$

is sharply Gevrey s-hypoelliptic for any $s \geq q/(p-k)$, where $q > p > k$ and q is an even integer.

In this paper we consider the operator

$$P = BB^* + B^*(t^{2l} + x^{2k})B, \quad B = D_x + i x^{q-1} D_t, \tag{1-1}$$

where k, l and q are positive integers, q even; see [Bove et al. 2010].

Observe that P is a sum of three squares of complex vector fields, but with a small change not altering the results, we might make P a sum of two squares of complex vector fields in two variables, depending on the same parameters: for example, $BB^* + B^*(t^{2l} + x^{2k})^2 B$.

Let us also note that the characteristic variety of P is $\{x = 0, \xi = 0\}$, a codimension-two real analytic symplectic submanifold of $T^*\mathbb{R}^2 \setminus 0$, as in the case of Kohn's operator. Moreover, the Poisson–Treves stratification for P has a single stratum, thus coinciding with the characteristic manifold of P.

We want to analyze the hypoellipticity of P, both in C^∞ and in Gevrey classes. As we shall see, the Gevrey classes play an important role. Here are our results:

Theorem 1.1. *Let P be as in* (1-1), *q even.*

(i) *Suppose that*

$$l > \frac{k}{q}. \tag{1-2}$$

Then P is C^∞-hypoelliptic (*in a neighborhood of the origin*) *with a loss of $2(q-1+k)/q$ derivatives.*

(ii) *Assume that* (1-2) *is satisfied by the parameters l, k and q. Then P is Gevrey s-hypoelliptic for any s, with*

$$s \geq \frac{lq}{lq-k}. \tag{1-3}$$

(iii) *The value in* (1-3) *for the Gevrey hypoellipticity of P is optimal, that is, P is **not** Gevrey s-hypoelliptic for any*

$$1 \leq s < \frac{lq}{lq-k}.$$

(iv) *Assume now that*

$$l \leq \frac{k}{q}. \tag{1-4}$$

Then P is not C^∞-hypoelliptic.

It is worth noting that the operator P satisfies the complex Hörmander condition, that is, the brackets of the fields of length up to $k+q$ generate a two-dimensional complex Lie algebra. Note that in the present case the vector fields involved are B^*, $x^k B$ and $t^l B$, but only the first two enter in the brackets spanning $\mathbb{C}^2$. Actually the third vector field, despite being, as we have said, completely irrelevant in computing the elliptic brackets or the characteristic manifold, proves essential for the hypoellipticity of the operator in the sense that it determines whether the operator turns out hypoelliptic (in some sense) or not. As of now we do not have a thorough understanding of this phenomenon.

Corollary 1.2. *The complex Hörmander condition does not imply C^∞-hypoellipticity for sums of squares of complex vector fields.*

The real Hörmander condition, using as vector fields both the real and the imaginary parts of the vector fields defining P, does not imply C^∞-hypoellipticity either.

This also followed from Christ's theorem [2005], but in this case we are in two variables instead of three. We are not aware of any sufficient condition for C^∞-hypoellipticity of sums of squares of complex vector fields, except the result proved in [Kohn 2005], according to which if the (complex) Lie algebra is generated by the fields and their brackets of length at most 2, then the operator is C^∞-hypoelliptic.

Restricting ourselves to the case q even is no loss of generality, since the operator (1-1) corresponding to an odd integer q is plainly hypoelliptic and actually subelliptic, that is, there is a loss of less than two derivatives. This fact is due to special circumstances, that is, that the operator B^* has a trivial kernel in that case. Actually when q is odd, we have the estimate $\|u\|_{1/q} \le C\|B^*u\|$, $u \in C_0^\infty(\Omega)$, with Ω a subset of $\mathbb{R}^2$ that is open and containing the origin. From the straightforward inequality $|\langle Pu, u\rangle| \ge \|B^*u\|^2$, $u \in C_0^\infty(\Omega)$, we deduce that $\|u\|_{1/q}^2 \le C|\langle Pu, u\rangle|$. The latter estimate can be used to prove the hypoellipticity (subellipticity) of P. We stress that Kohn's original operator, in the complex variable z, automatically has an even q, while in the "real case" the parity of q does matter.

We want to discuss the issue of analytic (Gevrey) hypoellipticity. For sums of squares of real vector fields, there is a conjecture due to F. Treves [1999; Bove and Treves 2004] stating a necessary and sufficient condition for analytic hypoellipticity. To this end, one considers the characteristic set of the operator and "decomposes" it into real analytic strata where the symplectic form has constant rank and where the vector fields as well as their brackets up to a certain length have vanishing symbols, but there exists at least a bracket of length greater by one whose symbol does not vanish. Roughly stated, the conjecture says that if every stratum is a symplectic real analytic manifold, then the operator is analytic hypoelliptic. In the case of the operator $R_{q,k}$ (or $E_{m,k}$), the stratification has just one stratum, coinciding with the characteristic manifold, which is also a symplectic manifold. In [Kohn 2005; Bove et al. 2006] it is proved that the operator is both C^∞ and analytic hypoelliptic.

From Theorem 1.1(iii), however, we deduce the following:

Corollary 1.3. *Treves's conjecture does not carry over to sums of squares of complex vector fields.*

We also want to stress microlocal aspects of the theorem: the characteristic manifold of P is symplectic in $T^*\mathbb{R}^2$ of codimension two, and as such it may be identified with $T^*\mathbb{R} \setminus 0 \sim \{(t, \tau) \mid \tau \neq 0\}$ (leaving aside the origin in the τ variable, i.e., the zero section.)

On the other hand, the operator $P(x, t, D_x, \tau)$, thought of as a differential operator in the x-variable depending on (t, τ) as parameters, for $\tau > 0$ has an eigenvalue of the form $\tau^{2/q}(t^{2l} + a(t, \tau))$, possibly multiplied by a nonzero function of t. Here $a(t, \tau)$ denotes a (nonclassical) symbol of order -1 defined for $\tau > 0$ and such that $a(0, \tau) \sim \tau^{-2k/q}$. Thus we may consider the pseudodifferential operator $\Lambda(t, D_t) = \mathrm{Op}\big(\tau^{2/q}(t^{2l} + a(t, \tau))\big)$ as defined in a microlocal neighborhood of our base point in the characteristic manifold of P. One can show that the hypoellipticity properties of P are shared by Λ; for example, P is C^∞-hypoelliptic if and only if Λ is.

The paper is organized as follows. In Sections 2–4 the operator $\Lambda(t, D_t)$ is computed and its hypoellipticity properties are related to those of P. This is done following ideas of Boutet de Monvel, Helffer and Sjöstrand using a calculus of pseudodifferential operators that degenerate on a symplectic manifold. The sufficient part of the theorem is proved in this way. Since we do not want to encumber an already lengthy paper with too many technical details, we decided to give only a sketchy description of the pseudodifferential calculus, leaving it to the reader to fill in the (classical) proofs.

In order to prove the optimality of the Gevrey index in (1-3), we have to show that the pseudodifferential operator $\Lambda(t, D_t)$ is hypoelliptic in that Gevrey class and not in any better class, that is, not in any class of index closer to 1, the analytic class. We do this in Section 5. This brings in the question of determining the hypoellipticity index for a pseudodifferential operator in one variable. A detailed treatment of the general case is given in [Bove and Mughetti 2013]. In the present case, determining the Gevrey class does not require the detailed construction of a Newton polygon, and things are definitely easier from the technical point of view. This is why we include here the optimality proof for $\Lambda(t, D_t)$.

In Section 6 we prove assertion (iii) of Theorem 1.1. The idea of the proof is to construct a solution of the equation $\Lambda(t, D_t)u = 0$ violating an a priori estimate which is necessary and sufficient for Gevrey hypoellipticity. Such a solution is at first constructed only from a formal point of view. In a second step, we make sure to have estimates allowing us to turn a formal solution into a true solution, albeit of an equation of the form $\Lambda(t, D_t)v = g$, where g, though not zero, is in an optimal Gevrey class $\mathscr{B}^{s_0}$, where these Gevrey classes $\mathscr{B}^s$ are characterized by arbitrarily small constants in the estimates of derivatives.

The proof of assertion (iv) of Theorem 1.1 is done in Section 7 using similar ideas, but one needs less control on the formal solution.

2. The q-pseudodifferential calculus

The idea, attributed by J. Sjöstrand and M. Zworski [2007] to Schur, is essentially a linear algebra remark: assume that the $n \times n$ matrix A has zero in its spectrum with multiplicity one. Then of course A is not invertible, but, denoting by e_0 the zero eigenvector of A, the matrix (in block form)

$$\begin{bmatrix} A & e_0 \\ {}^t e_0 & 0 \end{bmatrix}$$

is invertible as an $(n+1) \times (n+1)$ matrix in $\mathbb{C}^{n+1}$. Here ${}^t e_0$ denotes the row vector e_0.

All we want to do is apply this remark to the operator P whose part BB^* has the same problem as the matrix A, that is, a zero simple eigenvalue. This occurs since q is even. (In the case when q is odd, P is easily seen to be hypoelliptic.)

It is convenient to use self-adjoint derivatives from now on, so the vector field B^* equals $D_x - i x^{q-1} D_t$, where $D_x = i^{-1}\partial_x$. It will also be convenient to write $B(x,\xi,\tau)$ for the symbol of the vector field B, that is, $B(x,\xi,\tau) = \xi + i x^{q-1}\tau$, and analogously for the other vector fields involved. The symbol of P can be written as

$$P(x,t,\xi,\tau) = P_0(x,t,\xi,\tau) + P_{-q}(x,t,\xi,\tau) + P_{-2k}(x,t,\xi,\tau), \tag{2-1}$$

where

$$\begin{aligned} P_0(x,t,\xi,\tau) &= (1+t^{2l})(\xi^2 + x^{2(q-1)}\tau^2) + (-1+t^{2l})(q-1)x^{q-2}\tau, \\ P_{-q}(x,t,\xi,\tau) &= -2lt^{2l-1}x^{q-1}(\xi + i x^{q-1}\tau), \\ P_{-2k}(x,t,\xi,\tau) &= x^{2k}(\xi^2 + x^{2(q-1)}\tau^2) - i2kx^{2k-1}(\xi + i x^{q-1}\tau) + (q-1)x^{2k+q-2}\tau. \end{aligned}$$

It is evident at a glance that the different pieces into which P has been decomposed include terms of different order and vanishing speed. We thus need to say something about the adopted criteria for the above decomposition.

Let μ be a positive number and consider the following canonical dilation in the variables (x,t,ξ,τ):

$$x \to \mu^{-1/q}x, \quad t \to t, \quad \xi \to \mu^{1/q}\xi, \quad \tau \to \mu\tau.$$

It is then evident that P_0 has the homogeneity property

$$P_0\big(\mu^{-1/q}x, t, \mu^{1/q}\xi, \mu\tau\big) = \mu^{2/q} P_0(x,t,\xi,\tau). \tag{2-2}$$

Analogously,

$$P_{-q}\big(\mu^{-1/q}x, t, \mu^{1/q}\xi, \mu\tau\big) = \mu^{2/q-1} P_{-q}(x,t,\xi,\tau) \tag{2-3}$$

and

$$P_{-2k}\big(\mu^{-1/q}x, t, \mu^{1/q}\xi, \mu\tau\big) = \mu^{2/q-(2k)/q} P_{-2k}(x,t,\xi,\tau). \tag{2-4}$$

Now these homogeneity properties help us in identifying some symbol classes suitable for P.

Definition 2.1. Following the ideas of [Boutet de Monvel and Trèves 1974; Boutet de Monvel 1974], we define the class of symbols $S_q^{m,k}(\Omega,\Sigma)$, where Ω is a conic neighborhood of the point $(0,e_2)$ and Σ denotes the characteristic manifold $\{x=0, \xi=0\}$, as the set of all C^∞ functions such that on any conic subset of Ω with compact base,

$$\big|\partial_t^\alpha \partial_\tau^\beta \partial_x^\gamma \partial_\xi^\delta a(x,t,\xi,\tau)\big| \lesssim (1+|\tau|)^{m-\beta-\delta}\left(\frac{|\xi|}{|\tau|} + |x|^{q-1} + \frac{1}{|\tau|^{(q-1)/q}}\right)^{k-\gamma/(q-1)-\delta}. \tag{2-5}$$

We write $S_q^{m,k}$ for $S_q^{m,k}(\mathbb{R}^2\times\mathbb{R}^2,\Sigma)$.

By a straightforward computation (see for example [Boutet de Monvel 1974]), we have $S_q^{m,k} \subset S_q^{m',k'}$ if and only if $m \le m'$ and

$$m - \frac{q-1}{q}k \le m' - \frac{q-1}{q}k'.$$

$S_q^{m,k}$ can be embedded in the Hörmander classes $S_{\rho,\delta}^{m+\frac{q-1}{q}k_-}$, where $k_- = \max\{0, -k\}$ and $\rho = \delta = 1/q \le \frac{1}{2}$. Thus we immediately deduce that

$$P_0 \in S_q^{2,2}, \quad P_{-q} \in S_q^{1,2} \subset S_q^{2,2+\frac{q}{q-1}}, \quad \text{and finally} \quad P_{-2k} \in S_q^{2,2+\frac{2k}{q-1}}.$$

Definition 2.2 [Boutet de Monvel 1974]. With Ω and Σ as specified above, we define the class

$$\mathcal{H}_q^m(\Omega, \Sigma) = \bigcap_{j=1}^{\infty} S_q^{m-j,-\frac{q}{q-1}j}(\Omega, \Sigma).$$

We write $\mathcal{H}_q^m$ for $\mathcal{H}_q^m(\mathbb{R}^2 \times \mathbb{R}^2, \Sigma)$.

Now it is easy to see that P_0, as a differential operator with respect to the variable x, depending on the parameters t, $\tau \ge 1$, has a nonnegative discrete spectrum. Moreover, the dependence on τ of the eigenvalue is particularly simple, because of (2-2). Call $\Lambda_0(t,\tau)$ the lowest eigenvalue of P_0. Then

$$\Lambda_0(t,\tau) = \tau^{2/q}\tilde{\Lambda}_0(t).$$

Moreover, Λ_0 has multiplicity one and $\tilde{\Lambda}_0(0) = 0$, since BB^* has a null eigenvalue with multiplicity one. Denote by $\varphi_0(x,t,\tau)$ the corresponding eigenfunction. Because of (2-2), we have the following properties of φ_0:

(a) For fixed (t,τ), φ_0 is exponentially decreasing with respect to x as $x \to \pm\infty$. In fact, because of (2-2), setting $y = x\tau^{1/q}$, we have $\varphi_0(y,t,\tau) \sim e^{-y^q/q}$.

(b) It is convenient to normalize φ_0 in such a way that $\|\varphi_0(\cdot,t,\tau)\|_{L^2(\mathbb{R}_x)} = 1$. This implies that a factor $\sim\tau^{1/2q}$ appears. Thus we are led to the definition of a Hermite operator (see [Helffer 1977] for more details).

Let $\Sigma_1 = \pi_x\Sigma$ be the space projection of Σ.

Definition 2.3. We write H_q^m for $\mathcal{H}_q^m(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$, the class of all smooth functions in

$$\bigcap_{j=1}^{\infty} S_q^{m-j,-\frac{q}{q-1}j}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1).$$

Here $S_q^{m,k}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$ denotes the set of all smooth functions such that

$$\left|\partial_t^\alpha \partial_\tau^\beta \partial_x^\gamma a(x,t,\tau)\right| \lesssim (1+|\tau|)^{m-\beta}\left(|x|^{q-1} + \frac{1}{|\tau|^{(q-1)/q}}\right)^{k-\frac{\gamma}{q-1}}. \tag{2-6}$$

Define the action of a symbol $a(x,t,\tau)$ in H_q^m as the map

$$a(x,t,D_t)\colon C_0^\infty(\mathbb{R}_t) \longrightarrow C^\infty(\mathbb{R}^2_{x,t})$$

defined by

$$a(x,t,D_t)u(x,t) = (2\pi)^{-1}\int e^{it\tau}a(x,t,\tau)\hat{u}(\tau)\,d\tau.$$

This operator, modulo a regularizing operator (with respect to the variable t, but locally uniform in x), is called a Hermite operator, and we denote by OPH_q^m the corresponding class.

We need also the adjoint of the Hermite operators defined in Definition 2.3.

Definition 2.4. Let $a \in H_q^m$. We define the map

$$a^*(x,t,D_t)\colon C_0^\infty(\mathbb{R}^2_{x,t}) \longrightarrow C^\infty(\mathbb{R}_t)$$

as

$$a^*(x,t,D_t)u(t) = (2\pi)^{-1} \iint e^{it\tau}\overline{a(x,t,\tau)}\hat{u}(x,\tau)\,dx\,d\tau,$$

where $\hat{u}(x,\tau)$ denotes the Fourier transform of u with respect to the variable t. We denote by OPH_q^{*m} the related set of operators.

Lemma 2.5. *Let $a \in H_q^m$ and $b \in S_q^{m,k}$.*

(i) *The formal adjoint $a(x,t,D_t)^*$ belongs to OPH_q^{*m} and its symbol has the asymptotic expansion*

$$\sigma\bigl(a(x,t,D_t)^*\bigr) - \sum_{\alpha=0}^{N-1} \frac{1}{\alpha!}\partial_\tau^\alpha D_t^\alpha \overline{a(x,t,\tau)} \in H_q^{m-N}. \tag{2-7}$$

(ii) *The formal adjoint $(a^*(x,t,D_t))^*$ belongs to OPH_q^m and its symbol has the asymptotic expansion*

$$\sigma\bigl(a^*(x,t,D_t)^*\bigr) - \sum_{\alpha=0}^{N-1} \frac{1}{\alpha!}\partial_\tau^\alpha D_t^\alpha a(x,t,\tau) \in H_q^{m-N}. \tag{2-8}$$

(iii) *The formal adjoint $b(x,t,D_x,D_t)^*$ belongs to $\mathrm{OPS}_q^{m,k}$ and its symbol has the asymptotic expansion*

$$\sigma\bigl(a(x,t,D_x,D_t)^*\bigr) - \sum_{\alpha=0}^{N-1} \frac{1}{\alpha!}\partial_{(\xi,\tau)}^\alpha D_{(x,t)}^\alpha \overline{a(x,t,\xi,\tau)} \in S_q^{m-N,k-Nq/(q-1)}. \tag{2-9}$$

The following is a lemma on compositions involving the two different types of Hermite operators defined above. First we give a definition of "global" homogeneity:

Definition 2.6. We say that a symbol $a(x,t,\xi,\tau)$ is globally homogeneous (abbreviated g.h.) of degree m if for $\lambda \geq 1$, $a(\lambda^{-1/q}x, t, \lambda^{1/q}\xi, \lambda\tau) = \lambda^m a(x,t,\xi,\tau)$. Analogously, we say that a symbol, independent of ξ, of the form $a(x,t,\tau)$ is globally homogeneous of degree m if $a(\lambda^{-1/q}x, t, \lambda\tau) = \lambda^m a(x,t,\tau)$.

Let $f_{-j}(x,t,\xi,\tau) \in S_q^{m,k+j/(q-1)}$, $j \in \mathbb{N}$; then there exists $f(x,t,\xi,\tau) \in S_q^{m,k}$ such that $f \sim \sum_{j\geq 0} f_{-j}$, that is, $f - \sum_{j=0}^{N-1} f_{-j} \in S_q^{m,k+N/(q-1)}$. Thus f is defined modulo a symbol in

$$S_q^{m,\infty} = \bigcap_{h\geq 0} S_q^{m,h}.$$

Analogously, let f_{-j} be globally homogeneous of degree $m - k(q-1)/q - j/q$ and such that for every $\alpha, \beta \geq 0$ satisfies the estimates

$$\left|\partial^{\gamma}_{(t,\tau)} \partial^{\alpha}_{x} \partial^{\beta}_{\xi} f_{-j}(x, t, \xi, \tau)\right| \lesssim \left(|\xi| + |x|^{q-1} + 1\right)^{k-\alpha/(q-1)-\beta}, \quad (x, \xi) \in \mathbb{R}^2, \tag{2-10}$$

for (t, τ) in a compact subset of $\mathbb{R} \times \mathbb{R} \setminus 0$ and every multi-index γ. Then $f_{-j} \in S^{m,k+j/(q-1)}_{q}$.

Accordingly, let $\varphi_{-j}(x, t, \tau) \in H^{m-j/q}_{q}$; then there exists $\varphi(x, t, \tau) \in H^{m}_{q}$ such that $\varphi \sim \sum_{j \geq 0} \varphi_{-j}$, that is, $\varphi - \sum_{j=0}^{N-1} \varphi_{-j} \in H^{m-N/q}_{q}$, so that φ is defined modulo a regularizing symbol (with respect to the t variable).

Similarly, let φ_{-j} be globally homogeneous of degree $m - j/q$ and such that for every $\alpha, l \geq 0$ satisfies the estimates

$$\left|\partial^{\beta}_{(t,\tau)} \partial^{\alpha}_{x} \varphi_{-j}(x, t, \tau)\right| \lesssim \left(|x|^{q-1} + 1\right)^{-l-\alpha/(q-1)}, \quad x \in \mathbb{R}, \tag{2-11}$$

for (t, τ) in a compact subset of $\mathbb{R} \times \mathbb{R} \setminus 0$ and every multi-index β. Then $\varphi_{-j} \in H^{m-j/q}_{q}$.

As a matter of fact, in the construction below we deal with asymptotic series of homogeneous symbols.

Next we give a brief description of the composition of the various types of operator introduced so far.

Lemma 2.7 [Helffer 1977, Formula 2.4.9]. *Let $a \in S^{m,k}_{q}$, $b \in S^{m',k'}_{q}$, with asymptotic globally homogeneous expansions*

$$a \sim \sum_{j \geq 0} a_{-j}, \quad a_{-j} \in S^{m,k+j/(q-1)}_{q}, \text{ g.h. of degree } m - \frac{q-1}{q} k - \frac{j}{q},$$

$$b \sim \sum_{i \geq 0} b_{-i}, \quad b_{-i} \in S^{m',k'+i/(q-1)}_{q}, \text{ g.h. of degree } m' - \frac{q-1}{q} k' - \frac{i}{q}.$$

Then $a \circ b$ is an operator in $\mathrm{OPS}^{m+m',k+k'}_{q}$ with

$$\sigma(a \circ b) - \sum_{s=0}^{N-1} \sum_{q\alpha+i+j=s} \frac{1}{\alpha!} \sigma\left(\partial^{\alpha}_{\tau} a_{-j}(x, t, D_x, \tau) \circ_x D^{\alpha}_{t} b_{-i}(x, t, D_x, \tau)\right) \in S^{m+m'-N,k+k'}_{q}. \tag{2-12}$$

Here $\circ_x$ denotes the composition with respect to the x-variable.

Lemma 2.8 [Boutet de Monvel 1974, Section 5; Helffer 1977, Sections 2.2, 2.3]. *Let $a \in H^{m}_{q}$, $b \in H^{m'}_{q}$ and $\lambda \in S^{m''}_{1,0}(\mathbb{R}_t \times \mathbb{R}_\tau)$ with homogeneous asymptotic expansions*

$$a \sim \sum_{j \geq 0} a_{-j}, \quad a_{-j} \in H^{m-j/q}_{q}, \text{ g.h. of degree } m - \frac{j}{q},$$

$$b \sim \sum_{i \geq 0} b_{-i}, \quad b_{-i} \in H^{m'-i/q}_{q}, \text{ g.h. of degree } m' - \frac{i}{q},$$

$$\lambda \sim \sum_{l \geq 0} \lambda_{-l}, \quad \lambda_{-l} \in S^{m''-l/q}_{1,0}, \text{ homogeneous of degree } m'' - \frac{l}{q}.$$

Then:

(i) *$a\circ b^*$ is an operator in* $\mathrm{OP}\,\mathscr{H}_q^{m+m'-1/q}(\mathbb{R}^2,\Sigma)$ *with*

$$\sigma(a\circ b^*)(x,t,\xi,\tau)-e^{-ix\xi}\sum_{s=0}^{N-1}\sum_{q\alpha+i+j=s}\frac{1}{\alpha!}\partial_\tau^\alpha a_{-j}(x,t,\tau)D_t^\alpha\hat{\bar{b}}_{-i}(\xi,t,\tau)\in\mathscr{H}_q^{m+m'-1/q-N/q},\tag{2-13}$$

where the Fourier transform in $D_t^\alpha\hat{\bar{b}}_{-i}(\xi,t,\tau)$ is taken with respect to the x-variable.

(ii) *$b^*\circ a$ is an operator in* $\mathrm{OPS}_{1,0}^{m+m'-1/q}(\mathbb{R}_t)$ *with*

$$\sigma(b^*\circ a)(t,\tau)-\sum_{s=0}^{N-1}\sum_{q\alpha+j+i=s}\frac{1}{\alpha!}\int\partial_\tau^\alpha\bar{b}_{-i}(x,t,\tau)D_t^\alpha a_{-j}(x,t,\tau)\,dx\in S_{1,0}^{m+m'-1/q-N/q}(\mathbb{R}_t).\tag{2-14}$$

(iii) *$a\circ\lambda$ is an operator in* $\mathrm{OPH}_q^{m+m''}$. *Furthermore, its asymptotic expansion is given by*

$$\sigma(a\circ\lambda)-\sum_{s=0}^{N-1}\sum_{q\alpha+j+l=s}\frac{1}{\alpha!}\partial_\tau^\alpha a_{-j}(x,t,\tau)D_t^\alpha\lambda_{-l}(t,\tau)\in H_q^{m+m''-N/q}.\tag{2-15}$$

Lemma 2.9. *Let $a(x,t,D_x,D_t)$ be an operator in the class $\mathrm{OPS}_q^{m,k}(\mathbb{R}^2,\Sigma)$ and $b(x,t,D_t)\in\mathrm{OPH}_q^{m'}$ with g.h. asymptotic expansions*

$$a\sim\sum_{j\ge0}a_{-j},\quad a_{-j}\in S_q^{m,k+j/(q-1)},\ \text{g.h. of degree } m-\frac{q-1}{q}k-\frac{j}{q},$$
$$b\sim\sum_{i\ge0}b_{-i},\quad b_{-i}\in H_q^{m'-i/(q-1)},\quad \text{g.h. of degree } m'-\frac{i}{q}.$$

Then $a\circ b\in\mathrm{OPH}_q^{m+m'-k(q-1)/q}$ and has a g.h. asymptotic expansion of the form

$$\sigma(a\circ b)-\sum_{s=0}^{N-1}\sum_{ql+i+j=s}\frac{1}{l!}\partial_\tau^l a_{-j}(x,t,D_x,\tau)\big(D_t^l b_{-i}(\cdot,t,\tau)\big)\in H_q^{m+m'-k(q-1)/q-N/q}.\tag{2-16}$$

Lemma 2.10. *Let $a(x,t,D_x,D_t)$ be an operator in $\mathrm{OPS}_q^{m,k}(\mathbb{R}^2,\Sigma)$, let $b^*(x,t,D_t)\in\mathrm{OPH}_q^{*m'}$, and let $\lambda(t,D_t)\in\mathrm{OPS}_{1,0}^{m''}(\mathbb{R}_t)$, with homogeneous asymptotic expansions*

$$a\sim\sum_{j\ge0}a_{-j},\quad a_{-j}\in S_q^{m,k+j/(q-1)},\ \text{g.h. of degree } m-\frac{q-1}{q}k-\frac{j}{q},$$
$$b\sim\sum_{i\ge0}b_{-i},\quad b_{-i}\in H_q^{m'-i/(q-1)},\quad \text{g.h. of degree } m'-\frac{i}{q},$$
$$\lambda\sim\sum_{l\ge0}\lambda_{-l},\quad \lambda_{-l}\in S_{1,0}^{m''-l/q},\qquad \text{homogeneous of degree } m''-\frac{l}{q}.$$

Then

(i) $b^*(x,t,D_t)\circ a(x,t,D_x,D_t)\in \mathrm{OPH}_q^{*m+m'-((q-1)/q)k}$ *with g.h. asymptotic expansion*

$$\sigma(b^*\circ a)-\sum_{s=0}^{N-1}\sum_{ql+i+j=s}\frac{1}{l!}D_t^l\big(\overline{a_{-j}}(x,t,D_x,\tau)\big)^*\big(\partial_\tau^l\overline{b_{-i}}(\,\cdot\,,t,\tau)\big)\in {H_q}^{m+m'-k(q-1)/q-N/q}. \tag{2-17}$$

(ii) $\lambda(t,D_t)\circ b^*(x,t,D_t)\in \mathrm{OPH}_q^{*m'+m''}$ *with asymptotic expansion*

$$\sigma(\lambda\circ b^*)-\sum_{s=0}^{N-1}\sum_{q\alpha+i+l=s}\frac{1}{\alpha!}\partial_\tau^\alpha\lambda_{-l}(t,\tau)D_t^\alpha\overline{b_{-i}}(x,t,\tau)\in H_q^{m'+m''-N/q}. \tag{2-18}$$

The proofs of Lemmas 2.7–2.9 are obtained with the calculus developed by Boutet de Monvel [1974] and Helffer [1977], slightly generalized to handle general q. The proof of Lemma 2.10 is performed taking the adjoint and involves a combinatorial argument; we sketch it here.

Proof. We prove item (i). The proof of (ii) is similar and simpler.

Since

$$b^*(x,t,D_t)\circ a(x,t,D_x,D_t)=\big(a(x,t,D_x,D_t)^*\circ b^*(x,t,D_t)^*\big)^*,$$

using Lemmas 2.5 and 2.7, we first compute

$$\begin{aligned}\sigma\big(a(x,t,D_x,D_t)^*\circ b^*(x,t,D_t)^*\big)\\ &=\sum_{\alpha,l,p,i,j\ge 0}\frac{1}{l!\alpha!p!}\partial_\tau^{\alpha+p}D_t^\alpha\big(a_{-j}(x,t,D_x,\tau)\big)^*\big(\partial_\tau^l D_t^{l+p}b_{-i}(\,\cdot\,,t,\tau)\big)\\ &=\sum_{\gamma\ge 0}\frac{1}{\gamma!}\partial_\tau^\gamma D_t^\gamma\bigg(\sum_{\beta,i,j\ge 0}\frac{1}{\beta!}(-D_t)^\beta\big(a_{-j}(x,t,D_x,\tau)\big)^*\big(\partial_\tau^\beta b_{-i}(\,\cdot\,,t,\tau)\big)\bigg),\end{aligned}$$

where $(-D_t)^\beta\big(a_{-j}(x,t,D_x,\tau)\big)^*$ is the formal adjoint of the operator with symbol $D_t^\beta a_{-j}(x,t,\xi,\tau)$ as an operator in the x-variable, depending on (t,τ) as parameters. Here we used (A-2) in Appendix A. Hence

$$\begin{aligned}\sigma\big(b^*(x,t,D_t)\circ a(x,t,D_x,D_t)\big)\\ &=\sum_{l\ge 0}\frac{1}{l!}\partial_\tau^l D_t^l\cdot\overline{\bigg(\sum_{\gamma\ge 0}\frac{1}{\gamma!}\partial_\tau^\gamma D_t^\gamma\bigg(\sum_{\beta,i,j\ge 0}\frac{(-D_t)^\beta}{\beta!}\big(a_{-j}(x,t,D_x,\tau)\big)^*\big(\partial_\tau^\beta b_{-i}(\,\cdot\,,t,\tau)\big)\bigg)\bigg)}\\ &=\sum_{\beta,i,j\ge 0}\frac{1}{\beta!}D_t^\beta\big(\overline{a_{-j}}(x,t,D_x,\tau)\big)^*\big(\partial_\tau^\beta\overline{b_{-i}}(\,\cdot\,,t,\tau)\big)\\ &=\sum_{s\ge 0}\sum_{q\beta+i+j=s}\frac{1}{\beta!}D_t^\beta\big(\overline{a_{-j}}(x,t,D_x,\tau)\big)^*\big(\partial_\tau^\beta\overline{b_{-i}}(\,\cdot\,,t,\tau)\big),\end{aligned}$$

because of (A-3) in Appendix A. $\square$

3. Computation of the "degenerate eigenvalue"

We are now in a position to start computing the symbol of Λ.

Let us first examine the minimum eigenvalue and the corresponding eigenfunction of $P_0(x,t,D_x,\tau)$ in (2-1), as an operator in the x-variable. It is well known that $P_0(x,t,D_x,\tau)$ has a discrete set of nonnegative, simple eigenvalues depending in a real analytic way on the parameters (t,τ).

P_0 can be written in the form $LL^* + t^{2l}L^*L$, where $L = D_x + ix^{q-1}\tau$. The kernel of L^* is a one-dimensional vector space generated by $\varphi_{0,0}(x,\tau) = c_0\tau^{1/2q}\exp(-(x^q/q)\tau)$, c_0 being a normalization constant such that

$$\|\varphi_{0,0}(\cdot,\tau)\|_{L^2(\mathbb{R}_x)} = 1.$$

We remark that in this case τ is positive. For negative values of τ, the situation is much better since the following proposition holds:

Proposition 3.1 [Boutet de Monvel 1974]. *The localized operator of P in* (1-1), *which is LL^*, is injective in a cone near $\tau < 0$. Hence the operator P is subelliptic.*

Denoting by $\varphi_0(x,t,\tau)$ the eigenfunction of P_0 corresponding to its lowest eigenvalue $\Lambda_0(t,\tau)$, we obtain that $\varphi_0(x,0,\tau) = \varphi_{0,0}(x,\tau)$ and that $\Lambda_0(0,\tau) = 0$. As a consequence, the operator

$$P = BB^* + B^*(t^{2l} + x^{2k})B, \quad B = D_x + ix^{q-1}D_t \tag{3-1}$$

is not "maximally" hypoelliptic, that is, hypoelliptic with a loss of $2-2/q$ derivatives.

Next we give a more precise description of the t-dependence of both the eigenvalue Λ_0 and its corresponding eigenfunction φ_0 of $P_0(x,t,D_x,\tau)$.

It is well known that there exists an $\varepsilon > 0$ small enough that the operator

$$\Pi_0 = \frac{1}{2\pi i}\oint_{|\mu|=\varepsilon} \big(\mu I - P_0(x,t,D_x,\tau)\big)^{-1}\,d\mu$$

is the orthogonal projection onto the eigenspace generated by φ_0. Note that Π_0 depends on the parameters (t,τ). The operator LL^* is thought of as an unbounded operator in $L^2(\mathbb{R}_x)$ with domain

$$B_q^2(\mathbb{R}_x) = \Big\{u \in L^2(\mathbb{R}_x) \;\Big|\; x^\alpha D_x^\beta u \in L^2,\ 0 \le \beta + \frac{\alpha}{q-1} \le 2\Big\}. \tag{3-2}$$

We have

$$(\mu I - P_0)^{-1} = \big(I + t^{2l}\big[-A(I + t^{2l}A)^{-1}\big]\big)(\mu I - LL^*)^{-1},$$

where $A = (LL^* - \mu I)^{-1}L^*L$. Plugging this into the formula defining Π_0, we get

$$\Pi_0 = \frac{1}{2\pi i}\oint_{|\mu|=\varepsilon}(\mu I - LL^*)^{-1}\,d\mu - \frac{1}{2\pi i}t^{2l}\oint_{|\mu|=\varepsilon} A(I + t^{2l}A)^{-1}(\mu I - LL^*)^{-1}\,d\mu.$$

Hence

$$\begin{aligned}\varphi_0 = \Pi_0\varphi_{0,0} &= \varphi_{0,0} - t^{2l}\frac{1}{2\pi i}\oint_{|\mu|=\varepsilon} A(I + t^{2l}A)^{-1}(\mu I - LL^*)^{-1}\varphi_{0,0}\,d\mu \\ &= \varphi_{0,0}(x,\tau) + t^{2l}\tilde{\varphi}_0(x,t,\tau).\end{aligned} \tag{3-3}$$

Since Π_0 is an orthogonal projection, $\|\varphi_0(\cdot, t, \tau)\|_{L^2(\mathbb{R}_x)} = 1$.

As a consequence, since $P_0 = LL^* + t^{2l} L^* L$, we obtain that

$$\Lambda_0(t, \tau) = \langle P_0 \varphi_0, \varphi_0 \rangle = t^{2l} \|L\varphi_{0,0}\|^2 + \mathbb{O}(t^{4l}). \tag{3-4}$$

We point out that $L\varphi_{0,0} \neq 0$. Observe that, in view of (2-2), writing $u_\mu(x) = u(\mu^{-1/q} x)$,

$$\begin{aligned}
\Lambda_0(t, \mu\tau) &= \min_{\substack{u \in B_q^2 \\ \|u\|_{L^2}=1}} \big\langle P_0(x, t, D_x, \mu\tau) u(x), u(x) \big\rangle \\
&= \min_{\substack{u \in B_q^2 \\ \|u\|_{L^2}=1}} \left\langle P_0\big(\mu^{-1/q} x, t, \mu^{1/q} D_x, \mu\tau\big) \frac{u_\mu(x)}{\mu^{1/(2q)}}, \frac{u_\mu(x)}{\mu^{1/(2q)}} \right\rangle \\
&= \mu^{2/q} \min_{\substack{v \in B_q^2 \\ \|v\|_{L^2}=1}} \big\langle P_0(x, t, D_x, \tau) v(x), v(x) \big\rangle \\
&= \mu^{2/q} \Lambda_0(t, \tau).
\end{aligned} \tag{3-5}$$

This shows that Λ_0 is homogeneous of degree $2/q$ with respect to the variable τ.

Since φ_0 is the unique normalized solution of the equation

$$\big(P_0(x, t, D_x, \tau) - \Lambda_0(t, \tau)\big) u(\cdot, t, \tau) = 0,$$

from (2-2) and (3-5) it follows that φ_0 is globally homogeneous of degree $1/(2q)$. Moreover, φ_0 is rapidly decreasing with respect to the x-variable smoothly dependent on (t, τ) in a compact subset of $\mathbb{R}^2 \setminus 0$. Using estimates of the form (2-11), we can conclude that $\varphi_0 \in H_q^{1/(2q)}$.

Let us start now the construction of a right parametrix of the operator

$$\begin{bmatrix} P(x, t, D_x, D_t) & \varphi_0(x, t, D_t) \\ \varphi_0^*(x, t, D_t) & 0 \end{bmatrix}$$

as a map from $C_0^\infty(\mathbb{R}^2_{(x,t)}) \times C_0^\infty(\mathbb{R}_t)$ into $C^\infty(\mathbb{R}^2_{(x,t)}) \times C^\infty(\mathbb{R}_t)$. In particular, we are looking for an operator such that

$$\begin{bmatrix} P(x, t, D_x, D_t) & \varphi_0(x, t, D_t) \\ \varphi_0^*(x, t, D_t) & 0 \end{bmatrix} \circ \begin{bmatrix} F(x, t, D_x, D_t) & \psi(x, t, D_t) \\ \psi^*(x, t, D_t) & -\Lambda(t, D_t) \end{bmatrix} \equiv \begin{bmatrix} \mathrm{Id}_{C_0^\infty(\mathbb{R}^2)} & 0 \\ 0 & \mathrm{Id}_{C_0^\infty(\mathbb{R})} \end{bmatrix}. \tag{3-6}$$

Here ψ and ψ^* denote operators in $\mathrm{OPH}_q^{1/2q}$ and $\mathrm{OPH}_q^{*\,1/2q}$ respectively, and $F \in \mathrm{OPS}_q^{-2,-2}$ and $\Lambda \in \mathrm{OPS}_{1,0}^{2/q}$. Moreover, the sign $\equiv$ means equality modulo a regularizing operator.

From (3-6) we obtain four relations:

$$P(x,t,D_x,D_t)\circ F(x,t,D_x,D_t)+\varphi_0(x,t,D_t)\circ\psi^*(x,t,D_t)\equiv \mathrm{Id}, \tag{3-7}$$
$$P(x,t,D_x,D_t)\circ\psi(x,t,D_t)-\varphi_0(x,t,D_t)\circ\Lambda(t,D_t)\equiv 0, \tag{3-8}$$
$$\varphi_0^*(x,t,D_t)\circ F(x,t,D_x,D_t)\equiv 0, \tag{3-9}$$
$$\varphi_0^*(x,t,D_t)\circ\psi(x,t,D_t)\equiv \mathrm{Id}. \tag{3-10}$$

We are going to find the symbols F, ψ and Λ as asymptotic series of globally homogeneous symbols:

$$F\sim\sum_{j\ge 0}F_{-j},\quad \psi\sim\sum_{j\ge 0}\psi_{-j},\quad \Lambda\sim\sum_{j\ge 0}\Lambda_{-j}, \tag{3-11}$$

where the symbols F_{-j}, ψ_{-j} and Λ_{-j} are globally homogeneous of order $-2/q-j/q$, $1/(2q)-j/q$ and $2/q-j/q$ respectively; see for example Definition 2.6 and (3-5).

From Lemma 2.7, we obtain that

$$\sigma(P\circ F)\sim\sum_{s\ge 0}\sum_{q\alpha+i+j=s}\frac{1}{\alpha!}\sigma\big(\partial_\tau^\alpha P_{-j}(x,t,D_x,\tau)\circ_x D_t^\alpha F_{-i}(x,t,D_x,\tau)\big),$$

where we denote by P_{-j} the globally homogeneous parts of degree $2/q-j/q$ of the symbol of P, so that $P=P_0+P_{-q}+P_{-2k}$. Furthermore, from Lemma 2.8(i), we may write that

$$\sigma(\varphi_0\circ\psi^*)\sim e^{-ix\xi}\sum_{s\ge 0}\sum_{q\alpha+i=s}\frac{1}{\alpha!}\partial_\tau^\alpha\varphi_0(x,t,\tau)D_t^\alpha\hat{\bar{\psi}}_{-i}(\xi,t,\tau).$$

Analogously, Lemmas 2.9 and 2.8(iii) give

$$\sigma(P\circ\psi)\sim\sum_{s\ge 0}\sum_{ql+i+j=s}\frac{1}{l!}\partial_\tau^l P_{-j}(x,t,D_x,\tau)\big(D_t^l\psi_{-i}(\,\cdot\,,t,\tau)\big),$$
$$\sigma(\varphi_0\circ\Lambda)\sim\sum_{s\ge 0}\sum_{q\alpha+l=s}\frac{1}{\alpha!}\partial_\tau^\alpha\varphi_0(x,t,\tau)D_t^\alpha\Lambda_{-l}(t,\tau).$$

Finally, Lemmas 2.10(i) and 2.8(ii) yield

$$\sigma(\varphi_0^*\circ F)\sim\sum_{s\ge 0}\sum_{ql+j=s}\frac{1}{l!}D_t^l\big(\overline{F_{-j}}(x,t,D_x,\tau)\big)^*\big(\partial_\tau^l\overline{\varphi_0}(\,\cdot\,,t,\tau)\big)$$

and

$$\sigma(\varphi_0^*\circ\psi)\sim\sum_{s\ge 0}\sum_{q\alpha+j=s}\frac{1}{\alpha!}\int\partial_\tau^\alpha\bar{\varphi}_0(x,t,\tau)D_t^\alpha\psi_{-j}(x,t,\tau)\,dx.$$

Let us consider the terms globally homogeneous of degree 0. We obtain the relations

$$P_0(x,t,D_x,\tau)\circ_x F_0(x,t,D_x,\tau)+\varphi_0(x,t,\tau)\otimes\psi_0(\,\cdot\,,t,\tau)=\mathrm{Id}, \tag{3-12}$$

$$P_0(x,t,D_x,\tau)\big(\psi_0(\,\cdot\,,t,\tau)\big)-\Lambda_0(t,\tau)\varphi_0(x,t,\tau)=0, \tag{3-13}$$

$$\big(F_0(x,t,D_x,\tau)\big)^*\big(\varphi_0(\,\cdot\,,t,\tau)\big)=0, \tag{3-14}$$

$$\int\bar{\varphi}_0(x,t,\tau)\psi_0(x,t,\tau)\,dx=1. \tag{3-15}$$

Here we denoted by $\varphi_0\otimes\psi_0$ the operator $u=u(x)\mapsto\varphi_0\int\bar{\psi}_0 u\,dx$; $\varphi_0\otimes\psi_0$ must be a globally homogeneous symbol of degree zero.

Conditions (3-13) and (3-15) imply that $\psi_0=\varphi_0$. Moreover, (3-13) yields that

$$\Lambda_0(t,\tau)=\big\langle P_0(x,t,D_x,\tau)\varphi_0(x,t,\tau),\varphi_0(x,t,\tau)\big\rangle_{L^2(\mathbb{R}_x)},$$

coherently with the notation chosen above. Conditions (3-12) and (3-14) are rewritten as

$$P_0(x,t,D_x,\tau)\circ_x F_0(x,t,D_x,\tau)=\mathrm{Id}-\Pi_0,$$
$$F_0(x,t,D_x,\tau)\big(\varphi_0(\,\cdot\,,t,\tau)\big)\in[\varphi_0]^\perp,$$

whence (compare (3-2))

$$F_0(x,t,D_x,\tau)=\begin{cases}\big(P_0(x,t,D_x,\tau)_{|_{[\varphi_0]^\perp\cap B_q^2}}\big)^{-1} & \text{on } [\varphi_0]^\perp,\\ 0 & \text{on } [\varphi_0].\end{cases} \tag{3-16}$$

Since P_0 is q-globally elliptic with respect to (x,ξ) smoothly depending on the parameters (t,τ), one can show that $F_0(x,t,D_x,\tau)$ is actually a pseudodifferential operator whose symbol satisfies (2-10) with $m=k=-2$, $j=0$, and is globally homogeneous of degree $-2/q$.

From now on we assume that $q<2k$ and that $2k$ is not a multiple of q; the complementary cases are analogous.

Because of the fact that $P_{-j}=0$ for $j=1,\dots,q-1$, relations (3-12)–(3-15) are satisfied at degree $-j/q$, $j=1,\dots,q-1$, by choosing $F_{-j}=0$, $\psi_{-j}=0$, $\Lambda_{-j}=0$. Then we must examine homogeneity degree -1 in Equations (3-7)–(3-10). We get

$$P_{-q}\circ_x F_0+P_0\circ_x F_{-q}+\partial_\tau P_0\circ_x D_tF_0+\varphi_0\otimes\psi_{-q}+\partial_\tau\varphi_0\otimes D_t\varphi_0=0, \tag{3-17}$$

$$P_0(\psi_{-q})+P_{-q}(\varphi_0)+\partial_\tau P_0(D_t\varphi_0)-\Lambda_{-q}\varphi_0-D_t\Lambda_0\partial_\tau\varphi_0=0, \tag{3-18}$$

$$(F_{-q})^*(\varphi_0)-(D_tF_0^*)(\partial_\tau\varphi_0)=0, \tag{3-19}$$

$$\langle\psi_{-q},\varphi_0\rangle_{L^2(\mathbb{R}_x)}+\langle D_t\varphi_0,\partial_\tau\varphi_0\rangle_{L^2(\mathbb{R}_x)}=0. \tag{3-20}$$

First we solve with respect to $\psi_{-q}=\langle\psi_{-q},\varphi_0\rangle_{L^2(\mathbb{R}_x)}\varphi_0+\psi_{-q}^\perp\in[\varphi_0]\oplus[\varphi_0]^\perp$. From (3-20), we immediately get that

$$\langle\psi_{-q},\varphi_0\rangle_{L^2(\mathbb{R}_x)}=-\langle D_t\varphi_0,\partial_\tau\varphi_0\rangle_{L^2(\mathbb{R}_x)}. \tag{3-21}$$

Equation (3-18) implies that

$$P_0\big(\langle\psi_{-q},\varphi_0\rangle\varphi_0\big)+P_0(\psi_{-q}^{\perp})=-P_{-q}(\varphi_0)-\partial_\tau P_0(D_t\varphi_0)+\Lambda_{-q}\varphi_0+D_t\Lambda_0\partial_\tau\varphi_0.$$

Thus, using (3-21) we obtain

$$[\varphi_0]^{\perp}\ni P_0(\psi_{-q}^{\perp})=-P_{-q}(\varphi_0)-\partial_\tau P_0(D_t\varphi_0)+\Lambda_{-q}\varphi_0+D_t\Lambda_0\partial_\tau\varphi_0+\langle D_t\varphi_0,\partial_\tau\varphi_0\rangle\Lambda_0\varphi_0,$$

whence

$$\Lambda_{-q}=\big\langle P_{-q}(\varphi_0)+\partial_\tau P_0(D_t\varphi_0)-D_t\Lambda_0\partial_\tau\varphi_0,\varphi_0\big\rangle_{L^2(\mathbb{R}_x)}-\langle D_t\varphi_0,\partial_\tau\varphi_0\rangle\Lambda_0, \tag{3-22}$$

$$\psi_{-q}=-\langle D_t\varphi_0,\partial_\tau\varphi_0\rangle_{L^2(\mathbb{R}_x)}\varphi_0+F_0\big(-P_{-q}(\varphi_0)-\partial_\tau P_0(D_t\varphi_0)+D_t\Lambda_0\partial_\tau\varphi_0\big), \tag{3-23}$$

since, by (3-16), $F_0\varphi_0=0$. From (3-19) we deduce that for every $u\in L^2(\mathbb{R}_x)$,

$$\Pi_0F_{-q}u=\big\langle u,(D_tF_0^*)(\partial_\tau\varphi_0)\big\rangle_{L^2(\mathbb{R}_x)}\varphi_0=\big[\varphi_0\otimes(D_tF_0^*)(\partial_\tau\varphi_0)\big]u.$$

Let $-\omega_{-q}=P_{-q}\circ_x F_0+\partial_\tau P_0\circ_x D_tF_0+\varphi_0\otimes\psi_{-q}+\partial_\tau\varphi_0\otimes D_t\varphi_0$. Then from (3-16), applying F_0 to both sides of (3-17), we obtain that

$$(\mathrm{Id}-\Pi_0)F_{-q}=-F_0\omega_{-q}.$$

Therefore we deduce that

$$F_{-q}=\varphi_0\otimes(D_tF_0^*)(\partial_\tau\varphi_0)-F_0\omega_{-q}. \tag{3-24}$$

Inspecting (3-23) and (3-24), we see that ψ_{-q} is in $H_q^{1/2q-1}$ and is globally homogeneous of degree $1/2q-1$, while F_{-q} is in $S_q^{-2,-2+q/(q-1)}$ and is globally homogeneous of degree $-2/q-1$.

From (3-22) we have that Λ_{-q} is in $S_{1,0}^{2/q-1}$ and is homogeneous of degree $2/q-1$. Moreover, P_{-q} is $\mathbb{O}(t^{2l-1})$, $D_t\varphi_0$ is estimated by t^{2l-1} for $t\to 0$ because of (3-3), $D_t\Lambda_0$ is also $\mathbb{O}(t^{2l-1})$, and $\Lambda_0=\mathbb{O}(t^{2l})$ because of (3-4). We thus obtain that

$$\Lambda_{-q}(t,\tau)=\mathbb{O}(t^{2l-1}). \tag{3-25}$$

This ends the analysis of the terms of degree -1 in (3-6).

From now until the end of the proof we assume that $2l>2k/q$. The complementary case can be obtained analogously.

We iterate this procedure arguing in the same way. We would like to point out that the first homogeneity degree that arises and is not a negative integer is $-2k/q$. (We are availing ourselves of the fact that $2k$ is not a multiple of q. If it is a multiple of q, the above argument applies literally, but we need also the supplementary remark that we are going to make in the sequel.)

At homogeneity degree $-2k/q$ we do not see the derivatives with respect to t or τ of the symbols found at the previous levels, since they would only account for a negative integer degree of homogeneity.

In particular, condition (3-8) for homogeneity degree $-2k/q$ reads as

$$P_0\psi_{-2k}+P_{-2k}\varphi_0-\varphi_0\Lambda_{-2k}=0.$$

Taking the scalar product of the above equation with the eigenfunction φ_0 and recalling that

$$\|\varphi_0(\cdot, t, \tau)\|_{L^2(\mathbb{R}_x)} = 1,$$

we obtain that

$$\Lambda_{-2k}(t, \tau) = \langle P_{-2k}\varphi_0, \varphi_0\rangle_{L^2(\mathbb{R}_x)} + \langle P_0\psi_{-2k}, \varphi_0\rangle_{L^2(\mathbb{R}_x)}. \tag{3-26}$$

Now, because of the structure of P_{-2k}, $\langle P_{-2k}\varphi_0, \varphi_0\rangle_{L^2(\mathbb{R}_x)} > 0$, while the second term on the right, which is equal to $\langle \psi_{-2k}, \varphi_0\rangle \bar{\Lambda}_0$, vanishes for $t = 0$. Thus we deduce that

$$\Lambda_{-2k}(0, \tau) > 0. \tag{3-27}$$

Let j_0 be a positive integer such that

$$j_0 q < 2k < (j_0 + 1)q. \tag{3-28}$$

In the sequel we need some information on the behavior of the symbol $\Lambda_{-(j_0+1)}$. To obtain this, we make a proof by induction.

Suppose that

$$\Lambda_{-j}(t, \tau) = \begin{cases} \mathcal{O}(t^{2l-j/q}) & \text{for } j/q = 0, \dots, j_0, \\ 0 & \text{if } j/q \text{ is not an integer } \le j_0 \end{cases} \tag{3-29}$$

and

$$\psi_{-j}(t, x, \tau) = \begin{cases} \mathcal{O}(t^{2l-j/q}) & \text{for } j/q = 0, \dots, j_0, \\ 0 & \text{if } j/q \text{ is not an integer } \le j_0. \end{cases} \tag{3-30}$$

Let us write the symbols of (3-7)–(3-10) at the homogeneity degree $-(j_0 + 1)$. From (3-8), we have

$$\sum_{\substack{q\alpha+i+j=\\ q(j_0+1)}} \frac{1}{\alpha!} \partial_\tau^\alpha P_{-j}(x, t, D_x, \tau)\big(D_t^\alpha \psi_{-i}(\cdot, t, \tau)\big) - \sum_{\substack{q\alpha+i=\\ q(j_0+1)}} \frac{1}{\alpha!} \partial_\tau^\alpha \varphi_0(x, t, \tau) D_t^\alpha \Lambda_{-i}(t, \tau) = 0.$$

This can be rewritten as

$$P_0(\psi_{-(j_0+1)q}) - \varphi_0\Lambda_{-(j_0+1)q} = -\sum_{\substack{q\alpha+i+j=q(j_0+1)\\ i<q(j_0+1)}} \frac{1}{\alpha!} \partial_\tau^\alpha P_{-j}(D_t^\alpha \psi_{-i}) + \sum_{\substack{q\alpha+i=q(j_0+1)\\ i<q(j_0+1)}} \frac{1}{\alpha!} \partial_\tau^\alpha \varphi_0 D_t^\alpha \Lambda_{-i}. \tag{3-31}$$

Taking the scalar product of (3-31) with φ_0 and using the equalities $\|\varphi_0\|_{L^2(\mathbb{R}_x)} = 1$ and $\Lambda_0(t, \tau) = \mathcal{O}(t^{2l})$ and the self-adjointness of P_0, we at once find, because of the inductive hypothesis, that $\Lambda_{-(j_0+1)q} = \mathcal{O}(t^{2l-(j_0+1)})$.

In order to show that $\psi_{-(j_0+1)q} = \mathcal{O}(t^{2l-(j_0+1)})$, set

$$\psi_{-(j_0+1)q}(x, t, \tau) = \int_{\mathbb{R}} \varphi_0(y, t, \tau)\psi_{-(j_0+1)q}(y, t, \tau)\, dy \cdot \varphi_0(x, t, \tau) + \psi^{\perp}_{-(j_0+1)q}(x, t, \tau), \tag{3-32}$$

where $\psi^{\perp}_{-(j_0+1)q} \in [\varphi_0]^{\perp}$. Let us consider then (3-10). At the homogeneity level $-(j_0+1)$, it can be written as

$$\int_{\mathbb{R}} \varphi_0(y,t,\tau)\psi_{-(j_0+1)q}(y,t,\tau)\,dy = -\sum_{\substack{q\alpha+j=(j_0+1)q\\ \alpha>0}} \frac{1}{\alpha!}\int \partial_\tau^\alpha \varphi_0(y,t,\tau) D_t^\alpha \psi_{-j}(y,t,\tau)\,dy.$$

By (3-30), we conclude that the scalar product in the left-hand side of the above identity is $\mathbb{O}(t^{2l-(j_0+1)})$. Let us now consider (3-31). Applying F_0 to both sides of (3-31) and taking both Equation (3-32) and the inductive hypothesis into account allows us to conclude that

$$\psi^{\perp}_{-(j_0+1)q} = \mathbb{O}(t^{2l-(j_0+1)}).$$

We have thus proved:

Theorem 3.2. *The operator Λ defined in* (3-6) *is a pseudodifferential operator with symbol $\Lambda(t,\tau) \in S^{2/q}_{1,0}(\mathbb{R}_t \times \mathbb{R}_\tau)$. The symbol of Λ has an asymptotic expansion of the form*

$$\Lambda(t,\tau) \sim \sum_{j=0}^{j_0} \Lambda_{-jq}(t,\tau) + \sum_{s\ge 0}\big(\Lambda_{-2k-sq}(t,\tau) + \Lambda_{-(j_0+1)q-sq}(t,\tau)\big). \tag{3-33}$$

Here Λ_{-p} has homogeneity $2/q - p/q$ and

$$\Lambda_{-jq}(t,\tau) = \mathbb{O}(t^{2l-j}) \quad \textit{for} \begin{cases} j = 0,\dots,j_0+1 & \textit{if } 2l > 2k/q,\\ j = 0,\dots,2l-1 & \textit{if } 2l \le 2k/q,\end{cases} \tag{3-34}$$

while

$$\Lambda_{-2k} \textit{ satisfies } (3\text{-}27) \quad \textit{and} \quad t^{-2l}\Lambda_0(t,\tau)_{|t=0} > 0. \tag{3-35}$$

Furthermore, as a consequence of the calculus for real analytic symbols, $\Lambda_{r-sq}(t,\tau)$, with $r=-2k$ or $r=-(j_0+1)q$, satisfies the estimates

$$\big|\partial_t^\beta \partial_\tau^\alpha \Lambda_{r-sq}(t,\tau)\big| \le C_\Lambda^{1+s+\alpha+\beta}\alpha!\,\beta!\,s!\,(1+|\tau|)^{2/q+r/q-s-\alpha}, \tag{3-36}$$

where C_Λ denotes a positive constant depending only on the symbol Λ. (See Section 5 below for more details.) *In particular, $\Lambda(t,\tau)$ is a real analytic symbol in the sense of Boutet de Monvel* [1972].

We point out that the operator $\Lambda(t, D_t)$ defined above, modulo an elliptic factor of order $2/q - 2k/q$, has a form of the type

$$t^{2l} D_t^{2k/q} + 1. \tag{3-37}$$

The latter operator is G^s-hypoelliptic for $s \ge s_0 = lq/(lq-k)$. To get a rough idea of this fact, if $q=1$, let us consider the equation $t^{2l}D_t^{2k}u + u = 0$. The behavior of u can be obtained by WKB, solving $t^{2l}(\varphi')^{2k} + 1 = 0$, which yields $\varphi(t) = \omega t^{-l/k+1}$ and $u \sim e^{i\varphi}$, where ω is a suitable complex constant. This gives $u \in G^{l/(l-k)}$.

4. C^∞-hypoellipticity of P: sufficient part

In this section we prove the C^∞-hypoellipticity of P. This is accomplished by showing that the hypoellipticity of P follows from the hypoellipticity of Λ and proving that Λ is hypoelliptic if condition (1-2) is satisfied. As a matter of fact, the hypoellipticity of P is equivalent to the hypoellipticity of Λ, so that the structure of Λ in Theorem 3.2 may be used to prove assertion (ii) in Theorem 1.1.

We state without proof:

Lemma 4.1. *Let $a \in S_q^{m,k}$ be properly supported with $k \le 0$. Then* $\operatorname{Op} a$ *is continuous from $H^s_{\mathrm{loc}}(\mathbb{R}^2)$ to $H^{s-m+k(q-1)/q}_{\mathrm{loc}}(\mathbb{R}^2)$. Let $\varphi \in H_q^{m+1/2q}$ be properly supported. Then* $\operatorname{Op}\varphi$ *is continuous from $H^s_{\mathrm{loc}}(\mathbb{R})$ to $H^{s-m}_{\mathrm{loc}}(\mathbb{R}^2)$. Moreover, $\varphi^*(x,t,D_t)$ is continuous from $H^s_{\mathrm{loc}}(\mathbb{R}^2)$ to $H^{s-m}_{\mathrm{loc}}(\mathbb{R})$.*

Repeating the argument above for a left parametrix, we can find symbols $F \in S_q^{-2,-2}$, $\psi \in H_q^{1/2q}$ and $\Lambda \in S^{2/q}_{1,0}$ as in (3-11) such that

$$\begin{bmatrix} F(x,t,D_x,D_t) & \psi(x,t,D_t) \\ \psi^*(x,t,D_t) & -\Lambda(t,D_t) \end{bmatrix} \circ \begin{bmatrix} P(x,t,D_x,D_t) & \varphi_0(x,t,D_t) \\ \varphi_0^*(x,t,D_t) & 0 \end{bmatrix} \equiv \begin{bmatrix} \mathrm{Id}_{C_0^\infty(\mathbb{R}^2)} & 0 \\ 0 & \mathrm{Id}_{C_0^\infty(\mathbb{R})} \end{bmatrix}. \tag{4-1}$$

From (4-1) we get the pair of relations

$$F(x,t,D_x,D_t) \circ P(x,t,D_x,D_t) = \mathrm{Id} - \psi(x,t,D_t) \circ \varphi_0^*(x,t,D_t), \tag{4-2}$$

$$\psi^*(x,t,D_t) \circ P(x,t,D_x,D_t) = \Lambda(t,D_t) \circ \varphi_0^*(x,t,D_t). \tag{4-3}$$

Proposition 4.2. *If Λ is hypoelliptic with a loss of $\delta > 0$ derivatives, then P is also hypoelliptic with a loss of derivatives equal to*

$$2\frac{q-1}{q} + \delta.$$

The converse is also true. Furthermore, Λ is C^∞-hypoelliptic if and only if P is C^∞-hypoelliptic.

Proof. Assume that $Pu \in H^s_{\mathrm{loc}}(\mathbb{R}^2)$. From Lemma 4.1 we have

$$FPu \in H^{s+2/q}_{\mathrm{loc}}(\mathbb{R}^2).$$

By (4-2) we have $u - \psi\varphi_0^* u \in H^{s+2/q}_{\mathrm{loc}}(\mathbb{R}^2)$. Again, using Lemma 4.1, $\psi^* Pu \in H^s_{\mathrm{loc}}(\mathbb{R})$, so that by (4-3), $\Lambda\varphi_0^* u \in H^s_{\mathrm{loc}}(\mathbb{R})$. The hypoellipticity of Λ yields then that $\varphi_0^* u \in H^{s+2/q-\delta}_{\mathrm{loc}}(\mathbb{R})$. From Lemma 4.1 we obtain that $\psi\varphi_0^* u \in H^{s+2/q-\delta}_{\mathrm{loc}}(\mathbb{R})$. Thus

$$u = (\mathrm{Id} - \psi\varphi_0^*)u + \psi\varphi_0^* u \in H^{s+2/q-\delta}_{\mathrm{loc}}.$$

This proves the first sentence of the proposition. The proof of the other assertions is similar. □

Next we prove the hypoellipticity of Λ under the assumption that $l > k/q$.

First we want to show that there exists a smooth nonnegative function $M(t,\tau)$ such that

$$M(t,\tau) \le C|\Lambda(t,\tau)|, \quad |\Lambda^{(\alpha)}_{(\beta)}(t,\tau)| \le C_{\alpha,\beta} M(t,\tau)(1+|\tau|)^{-\rho\alpha+\delta\beta}, \tag{4-4}$$

where α, β are nonnegative integers, C, $C_{\alpha,\beta}$ are suitable positive constants, and the inequality holds for t in a compact neighborhood of the origin and $|\tau|$ large. Moreover, ρ and δ are such that $0 \le \delta < \rho \le 1$.

We actually need to check the above estimates for Λ only when τ is positive and large.

Let us choose $\rho = 1$, $\delta = k/lq < 1$ and

$$M(t,\tau) = \tau^{2/q}\big(t^{2l} + \tau^{-2k/q}\big),$$

for $\tau \ge c \ge 1$. It is then evident, from Theorem 3.2, that the first of the conditions in (4-4) is satisfied. The second condition in (4-4) is also straightforward for $\Lambda_0 + \Lambda_{-2k}$, because of (3-27) and (3-4). To verify the second condition in (4-4) for Λ_{-jq}, $q \in \{1, \dots, j_0\}$, we have to use property (3-34) in the statement of Theorem 3.2. Finally, the verification is straightforward for the lower-order parts of the symbol in (3-33). Using Theorem 22.1.3 of [Hörmander 1985], we see that there exists a parametrix for Λ. Moreover, from the proof of the same theorem, we get that the symbol of any parametrix satisfies the same estimates that Λ^{-1} satisfies, that is,

$$\big|D_t^\beta D_\tau^\alpha \Lambda^{-1}(t,\tau)\big| \le C_{\alpha,\beta}\big[\tau^{2/q}(t^{2l} + \tau^{-2k/q})\big]^{-1}(1+\tau)^{-\alpha+(k/lq)\beta} \le C_{\alpha,\beta}(1+\tau)^{2k/q-2/q-\alpha+k/lq\beta},$$

for t in a compact set and $\tau \ge C$. Thus the parametrix obtained from Theorem 22.1.3 of [Hörmander 1985] has a symbol in $S^{2k/q-2/q}_{1,k/lq}$.

Theorem 4.3. *Λ has a parametrix whose symbol belongs to $S^{2k/q-2/q}_{1,k/lq}$ and is hypoelliptic with a loss of $2k/q$ derivatives, that is, $\Lambda u \in H^s_{\mathrm{loc}}$ implies $u \in H^{s+2/q-2k/q}_{\mathrm{loc}}$.*

Theorem 4.3 together with Proposition 4.2 proves assertion (i) of Theorem 1.1.

5. Analytic symbols and Gevrey regularity

The purpose of this section is to prove the second statement in Theorem 1.1. To this end, we need to work with real analytic symbols and their asymptotic expansions.

Let us first define the symbol classes of Section 2 for analytic symbols. Since the coefficients of P are analytic, we are interested only in symbols with real analytic regularity.

Definition 5.1. We define the class of symbols $S^{m,k}_{q,a}(\Omega,\Sigma)$, where Ω is a conic neighborhood of the point $(0, e_2)$ and Σ denotes the characteristic manifold $\{x = 0, \xi = 0\}$, as the set of all C^ω functions such that on any conic subset of Ω with compact base,

$$\begin{aligned}&\big|\partial_t^\alpha \partial_\tau^\beta \partial_x^\gamma \partial_\xi^\delta a(x,t,\xi,\tau)\big| \\ &\qquad \le C^{1+\alpha+\beta+\gamma+\delta}\alpha!\,\beta!\,\gamma!\,\delta!\,(1+|\tau|)^{m-\beta-\delta}\left(\frac{|\xi|}{|\tau|} + |x|^{q-1} + \frac{1}{|\tau|^{(q-1)/q}}\right)^{k-\gamma/(q-1)-\delta}, \qquad (5\text{-}1)\end{aligned}$$

for $|(\xi,\tau)| \ge B(\beta+\delta)$, where $B > 0$ is a suitable constant.

We write $S^{m,k}_q$ for $S^{m,k}_{q,a}(\mathbb{R}^2 \times \mathbb{R}^2, \Sigma)$.

Likewise, with the same notations of Definition 2.3, we need the C^ω version of the Hermite symbols:

Definition 5.2. We write $H^m_{q,a}$ for $\mathscr{H}^m_{q,a}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$, the class of all real analytic functions in $\bigcap_{j=1}^\infty S^{m-j,-q/(q-1)j}_{q,a}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$. Here $S^{m,k}_{q,a}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$ is the set of all smooth functions such that

$$\left|\partial_t^\alpha \partial_\tau^\beta \partial_x^\gamma a(x,t,\tau)\right| \lesssim C^{1+\alpha+\beta+\gamma} \alpha!\, \beta!\, \gamma!\, (1+|\tau|)^{m-\beta} \Big(|x|^{q-1} + \frac{1}{|\tau|^{(q-1)/q}}\Big)^{k-\gamma/(q-1)}, \tag{5-2}$$

for $|\tau| \geq B\beta$, where B denotes a suitable positive constant.

Actually our Hermite operators are better than this and using an easy generalization of Proposition 2.10 in [Grigis and Rothschild 1983], we define the action of a symbol $a(x,t,\tau)$ in $H^m_{q,a}$ as the map

$$a(x,t,D_t)\colon G^s(\mathbb{R}_t) \cap C_0^\infty(\mathbb{R}_t) \longrightarrow G^s(\mathbb{R}^2_{x,t}),$$

for any $s > 1$, defined by

$$a(x,t,D_t)u(x,t) = (2\pi)^{-1} \int e^{it\tau} a(x,t,\tau)\hat{u}(\tau)\, d\tau.$$

Such an operator, modulo a regularizing operator (with respect to the t variable), is called a Hermite operator, and we denote by $\mathrm{OPH}^m_{q,a}$ the corresponding class. When it is clear from the context, to keep the notation simple, we shall omit the subscript a.

The adjoint of a (C^ω) Hermite operator is defined exactly as in Definition 2.4.

Next we define suitable cutoff functions that will be used several times in what follows.

Lemma 5.3. *Let $t > 1$. There exists a family of cutoff functions $\omega_j \in G^t(\mathbb{R}^n_x)$, $0 \leq \omega_j(x) \leq 1$, for $j = 0, 1, 2, \ldots$, such that*:

1. $\omega_j \equiv 0$ *if* $|x| \leq 2R(j+1)^t$, $\omega_j \equiv 1$ *if* $|x| \geq 4R(j+1)^t$, *with R an arbitrary positive constant.*
2. *There is a suitable constant C_ω, independent of j, α, R, such that*

$$|D^\alpha \omega_j(x)| \leq C_\omega^{|\alpha|+1} \big(R(j+1)^{t-1}\big)^{-|\alpha|} \quad \text{if } |\alpha| \leq 3j, \tag{5-3}$$

and

$$|D^\alpha \omega_j(x)| \leq (RC_\omega)^{|\alpha|+1} \frac{\alpha!^t}{|x|^{|\alpha|}} \quad \text{for every } \alpha. \tag{5-4}$$

Proof. Pick a function $\psi \in G^t(\mathbb{R}) \cap C_0^\infty(\mathbb{R})$ satisfying $\psi \geq 0$, $\operatorname{supp}\psi \subset \{|x| \leq \frac{1}{4}\}$, and $\int \psi(x)\, dx = 1$. Let χ_R denote the characteristic function of the interval $[-2R-r/2, 2R+r/2]$. Set $\psi_a(x) = a^{-1}\psi(x/a)$. Then

$$\varphi_N = \chi_R * \psi_r * \underbrace{\psi_{r/N} * \cdots * \psi_{r/N}}_{N \text{ times}}$$

has support contained in $[-2R-r, 2R+r]$ and is identically equal to 1 on $[-2R, 2R]$. We have, for any α, and for any $\beta \leq N$,

$$D^{\alpha+\beta}\varphi_N = \chi_R * D^\alpha \psi_r * \underbrace{D\psi_{r/N} * \cdots * D\psi_{r/N}}_{\beta \text{ times}} * \psi_{r/N} * \cdots * \psi_{r/N}.$$

Whence

$$|D^{\alpha+\beta}\varphi_N| \leq (4R+r) C_\psi^{\alpha+1} \alpha!^t r^{-\alpha} \Big(\|D\psi\|_{L^1} \frac{N}{r}\Big)^\beta.$$

Now we define

$$\omega_j(x) = 1 - \varphi_{3j}\Big(\frac{|x|}{(j+1)^t}\Big).$$

Assertion (1) of the lemma and the estimate (5-3) are then a consequence of the definitions and estimates above, once we choose $r = 2R$. Let us now turn to (5-4). We have

$$|D^\alpha \omega_j(x)| \le 6RC_\psi^{\alpha+1}\alpha!^t \frac{1}{[2R(j+1)^t]^\alpha}.$$

On the support of $D^\alpha\omega_j$ we have $|x| \le 4R(j+1)^t$, which implies the conclusion. □

Lemma 5.4. *Let $s > 1$. There exists a family of cutoff functions $\omega_j \in G^s(\mathbb{R}^n_x)$, $0 \le \omega_j(x) \le 1$, $j = 0, 1, 2, \ldots$, such that*:

(1) $\omega_j \equiv 0$ *if* $|x| \le 2R(j+1)$, $\omega_j \equiv 1$ *if* $|x| \ge 4R(j+1)$, *with R an arbitrary positive constant.*

(2) *There is a suitable constant C_ω, independent of j, α, R, such that*

$$|D^\alpha\omega_j(x)| \le C_\omega^{|\alpha|+1} R^{-|\alpha|} \quad \textit{if } |\alpha| \le 3j, \tag{5-5}$$

and

$$|D^\alpha\omega_j(x)| \le (RC_\omega)^{|\alpha|+1}\frac{\alpha!^s}{|x|^{|\alpha|}} \quad \textit{for every } \alpha. \tag{5-6}$$

Proof. The proof is the same as the proof of Lemma 5.3, but the ω_j are defined as

$$\omega_j(x) = 1 - \varphi_{3j}\left(\frac{|x|}{j+1}\right).$$ □

We wish now to define the asymptotic expansion of a symbol in the analytic category.

Let $f_{-j}(x,t,\xi,\tau) \in S_{q,a}^{m,k+j/(q-1)}$, $j \in \mathbb{N}\cup\{0\}$, satisfying an estimate of the form

$$\begin{aligned}&\left|\partial_t^\alpha\partial_\tau^\beta\partial_x^\gamma\partial_\xi^\delta f_{-j}(x,t,\xi,\tau)\right| \\ &\quad \le C^{1+\alpha+\beta+\gamma+\delta+j}\alpha!\,\beta!\,\gamma!\,\delta!\,j!^{1/q}(1+|\tau|)^{m-\beta-\delta}\left(\frac{|\xi|}{|\tau|}+|x|^{q-1}+\frac{1}{|\tau|^{(q-1)/q}}\right)^{k-\gamma/(q-1)-\delta},\end{aligned} \tag{5-7}$$

for $|(\xi,\tau)| \ge B(j+\beta+\delta)$; then there exists $f(x,t,\xi,\tau) \in S_{q,a}^{m,k}$ such that $f \sim \sum_{j\ge 0} f_{-j}$, that is, $f - \sum_{j=0}^{N-1} f_{-j} \in S_{q,a}^{m,k+N/(q-1)}$, and thus f is defined modulo a symbol in $S_{q,a}^{m,\infty} = \bigcap_{h\ge 0} S_{q,a}^{m,h}$.

We point out that the cutoff functions defined in Lemma 5.4 are used to actually sum the formal series $\sum_{j=0}^\infty f_j$ to obtain the symbol f.

Let f_{-j} be globally homogeneous of degree $m - k(q-1)/q - j/q$ and such that for every $\alpha, \beta \ge 0$ satisfies the estimates

$$\left|\partial_{(t,\tau)}^\gamma\partial_x^\alpha\partial_\xi^\beta f_{-j}(x,t,\xi,\tau)\right| \le C^{\alpha+\beta+\gamma+j+1}\alpha!\,\beta!\,\gamma!\,j!^{\frac{1}{q}}\big(|\xi|+|x|^{q-1}+1\big)^{k-\frac{\alpha}{q-1}-\beta}, \quad (x,\xi)\in\mathbb{R}^2, \tag{5-8}$$

for (t,τ) in a compact subset of $\mathbb{R}\times\mathbb{R}\setminus 0$ and every multi-index γ. Then $f_{-j} \in S_{q,a}^{m,k+j/(q-1)}$.

Accordingly, let $\varphi_{-j}(x,t,\tau) \in H_{q,a}^{m-j/q}$; then there exists $\varphi(x,t,\tau) \in H_{q,a}^m$ such that $\varphi \sim \sum_{j\ge 0}\varphi_{-j}$, that is, $\varphi - \sum_{j=0}^{N-1}\varphi_{-j} \in H_q^{m-N/q}$, so that φ is defined modulo a symbol analytically regularizing with respect to the t variable.

We again point out that the cutoff functions defined in Lemma 5.4 are used to actually sum the formal series $\sum_{j=0}^{\infty} \varphi_j$ to obtain the symbol φ.

Similarly, let φ_{-j} be globally homogeneous of degree $m - j/q$ and such that for every $\alpha, l \geq 0$ satisfies the estimates

$$\left|\partial_{(t,\tau)}^{\beta} \partial_x^{\alpha} \varphi_{-j}(x, t, \tau)\right| \leq C^{\alpha+\beta+j+1} \alpha!\, \beta!\, j!^{1/q} \left(|x|^{q-1} + 1\right)^{-l-\alpha/(q-1)}, \quad x \in \mathbb{R}, \tag{5-9}$$

for (t, τ) in a compact subset of $\mathbb{R} \times \mathbb{R} \setminus 0$ and every multi-index β. Then $\varphi_{-j} \in H_{q,a}^{m-j/q}$.

Proposition 5.5. *Let F be the operator defined in* (3-6). *$F \in \operatorname{Op}(S_{q,a}^{2,2})$ and maps functions in G^s into itself. A similar statement holds for the symbols in $H_{q,a}^m$.*

We skip the details of the analytic and Gevrey calculus in these classes of symbols. Suffice it to say that it is a totally standard matter and one may consult [Boutet de Monvel and Krée 1967; Boutet de Monvel 1972].

We explicitly remark that the symbols constructed in (3-6) and (4-1), F, ψ, Λ belong to the (analytic) classes $S_{q,a}^{-2,-2}$, $H_{q,a}^{1/2q}$ and $S_{1,0,a}^{2/q}$ and satisfy better estimates than the above (see [Grigis and Rothschild 1983, Proposition 2.10; Métivier 1981, Section 2]).

We are now ready to prove the second assertion in Theorem 1.1. First we prove:

Proposition 5.6. *The operator P in* (1-1) *is $G^s(\mathbb{R}^2)$-hypoelliptic if and only if Λ in* (3-33) *is $G^s(\mathbb{R})$-hypoelliptic.*

Proof. Let us assume first that Λ is G^s-hypoelliptic and that $Pu \in G^s$. Due to (4-2), (4-3) and Proposition 5.5, we have both FPu and $\psi^* Pu \in G^s$. From the latter, we get that $\Lambda \varphi_0^* u \in G^s$, which implies that $\varphi_0^* u \in G^s$, whence $\psi \varphi_0^* u \in G^s$. We thus obtain that $u \in G^s$.

Let us assume first that P is G^s-hypoelliptic and that $\Lambda u \in G^s$. This time we use (3-8) and (3-10). We have $P\psi u = \varphi_0 \Lambda u \in G^s$, which implies that $\psi u \in G^s$. Finally, $u \equiv \varphi_0^* \psi u \in G^s$. □

Next we have only to show that Λ is G^s-hypoelliptic for every $s \geq s_0 = lq/(lq - k)$, in order to prove:

Theorem 5.7. *Let P be as in* (1-1). *Then P is Gevrey s-hypoelliptic for every $s \geq s_0$, where*

$$s_0 = \frac{lq}{lq - k}.$$

Proof. In order to see that $\Lambda(t, D)$ is Gevrey s-hypoelliptic for every $s \geq s_0$, we are going to show that we can construct a parametrix with symbol in the class $S_{1,k/lq,(s)}^{2k/q-2/q}$, where the latter is defined as the set of all smooth, that is, C^∞, functions $a(t, \tau)$ satisfying the estimates

$$\left|\partial_t^{\beta} \partial_\tau^{\alpha} a(t, \tau)\right| \leq C^{1+\alpha+\beta} \alpha!\, \beta!^{s(1-(k/lq))} (1 + |\tau|)^{2k/q-2/q-\alpha+(k/lq)\beta},$$

for t in a compact set of the real line, for every α, β, $\tau \in \mathbb{R}$, with $1 + |\tau| \geq B\beta^s$; here B and C are suitable positive constants depending only on the symbol a.

As a matter of fact, we do not need symbols exhibiting a Gevrey dependence on the variables: analytic dependence is all we get; nevertheless, the general theory allows Gevrey behavior at no cost. Actually $s_0(1 - k/lq) = 1$.

Arguing as in the proof of Proposition 4.2, we choose a weight function

$$M(t,\tau) = \tau^{2/q}\left(t^{2l} + \tau^{-2k/q}\right), \quad \tau \geq 1.$$

We have the estimates

$$\begin{aligned} M(t,\tau) &\leq C|\Lambda(t,\tau)|, \\ \left|\partial_t^\beta \partial_\tau^\alpha \Lambda(t,\tau)\right| &\leq C^{1+\alpha+\beta}\alpha!\,\beta!^{s(1-k/(lq))} M(t,\tau)(1+|\tau|)^{-\alpha+(k/lq)\beta}. \end{aligned}$$

The existence of a parametrix $a(t,\tau)$ for $\Lambda(t,\tau)$, and hence the conclusion, is a standard consequence of the calculus in the Gevrey classes. □

6. Optimality in Gevrey spaces

This section is devoted to the proof of the third assertion of Theorem 1.1. By Proposition 5.6, it is enough to show that $\Lambda(t, D_t)$ is not G^s-hypoelliptic for $1 \leq s < s_0$.

To clarify our technique, let us consider a couple of examples reminiscent of the form (3-37). We stress here the fact that the operators we consider are a much simpler instance of Λ, the operator we are interested in.

Example 1. Consider the operator

$$L(t,\partial_t) = t^2\partial_t + a + bt,$$

where $a = \frac{i}{4}$, $b = -\frac{1}{2}$. We will show that L is not G^s-hypoelliptic for $1 \leq s < 2$. Consider the equation $L(t,\partial_t)u = \frac{i}{4}$. Arguing by contradiction, every solution u is certainly better than G^2-regular.

Let us look for a solution u in the form

$$u(t) = \int_0^{+\infty} e^{i\rho^2 t} e^{-\rho}\, d\rho.$$

One can easily see that this function u is actually a solution of $Lu = 0$. On the other hand,

$$\partial_t^\alpha u(0) = i^\alpha \int_0^{+\infty} e^{-\rho}\rho^{2\alpha}\, d\rho \sim \alpha!^2.$$

The latter estimate contradicts our assumption that u is better than G^2.

Unfortunately, it almost never occurs that the solution has a neat representation of the form above. We are instead forced to represent u as an integral containing both a phase function and an amplitude function. Moreover, the amplitude has to be constructed as a formal series whose convergence must be specifically defined and studied. As a motivation for our technique, we show this on the following, formally slightly different example.

Example 2. Consider the operator

$$L(t,\partial_t) = t^2\partial_t + \frac{i}{4}.$$

We want to "solve" the equation $L(t,\partial_t)u=0$. First of all, we look for the solution $u(t)$ in the form

$$u(t)=\int_0^{+\infty} e^{i\rho^2 t} v(\rho)\,d\rho,$$

where v has to be specified.

We proceed formally to find a candidate for v. We have

$$L(t,\partial_t)u=L(t,\partial_t)\int_0^{+\infty} e^{i\rho^2 t} v(\rho)\,d\rho=\frac{i}{4}\int_0^{+\infty} e^{i\rho^2 t}\Big(-\partial_\rho^2+1-\frac{1}{\rho}\partial_\rho+\frac{1}{\rho^2}\Big)v(\rho)\,d\rho. \tag{6-1}$$

The operator in parentheses has the form $P_0(\partial_\rho)+\rho^{-1}P_1(\partial_\rho)+\rho^{-2}P_2(\partial_\rho)$, where

$$P_0(\partial_\rho)=-\partial_\rho^2+1.$$

In order to put in evidence the phase factor, we write $v(\rho)=e^{-\rho}v_1(\rho)$. As a consequence, we have

$$L(t,\partial_t)u=\frac{i}{4}\int_0^{+\infty} e^{i\rho^2 t}e^{-\rho}\Big(-\partial_\rho^2+2\partial_\rho+\frac{1}{\rho}-\frac{1}{\rho}\partial_\rho+\frac{1}{\rho^2}\Big)v_1(\rho)\,d\rho.$$

The operator in parentheses still does not have the right form, since the phase factor $e^{-\rho}$ is not enough to guarantee that v_1 has an asymptotic expansion, for large ρ, in decreasing powers of ρ. This, in the end, would give an obstruction to the iterative solution of the "transport" equations. Hence, let us write $v_1(\rho)=\rho^\lambda\tilde{v}(\rho)$, where both λ and $\tilde{v}$ are to be determined. Bringing the factor ρ^λ to the left and choosing $\lambda=-\frac{1}{2}$ has the effect of canceling the terms of the form $\rho^{-1}\tilde{v}$. We eventually get

$$\begin{aligned}L(t,\partial_t)u&=\frac{i}{4}\int_0^{+\infty} e^{i\rho^2 t}e^{-\rho}\rho^{-1/2}\Big(-\partial_\rho^2+2\partial_\rho+\tfrac{3}{4}\frac{1}{\rho^2}\Big)\tilde{v}(\rho)\,d\rho\\&=\frac{i}{4}\int_0^{+\infty} e^{i\rho^2 t}e^{-\rho}\rho^{-1/2}\Big(P_0(\partial_\rho)+\frac{1}{\rho^2}P_2(\partial_\rho)\Big)\tilde{v}(\rho)\,d\rho.\end{aligned}$$

We write $P_2(\partial_\rho)$ even if P_2 is actually a multiplication operator, to stress the fact that this circumstance is particular to the present example but has no interest in the general case.

The next step is to construct $\tilde{v}$ formally. To do that, we look for $\tilde{v}$ in the form

$$\tilde{v}(\rho)=\sum_{k=0}^{\infty} v_{2k}(\rho),$$

where the v_{2k} are obtained solving the triangular infinite system (transport equations)

$$P_0(\partial_\rho)v_{2k}(\rho)+\frac{1}{\rho^2}P_2v_{2k-2}(\rho)=0,\quad k=0,1,2,\dots,$$

with the convention that v_{2k} is identically zero if its subscript is negative.

Choose $v_0(\rho)\equiv 1$. Next we prove:

Minilemma. *If $\rho\ge 1$, we have $|v_{2k}(\rho)|\le\rho^{-k}$ for $k=0,1,2,\dots$.*

Proof. By induction. It is evident for $k=0$. Assume that $|v_{2k-2}(\rho)| \le \rho^{-(k-1)}$. For v_{2k} we have the equation $v''_{2k} - 2v'_{2k} = (\frac{3}{4})\rho^{-2} v_{2k-2}$. By the inductive assumption, the absolute value of the right-hand side of the equation can be estimated by $\rho^{-(k+1)}$.

Now a solution $y(\rho)$, vanishing at infinity, of the equation $y'' - 2y' = f$ can be written as

$$y(\rho) = \tfrac{1}{2} \int_\rho^{+\infty} f(\sigma)\, d\sigma - \tfrac{1}{2} \int_\rho^{+\infty} e^{-2(\sigma-\rho)} f(\sigma)\, d\sigma.$$

It is now evident that if $|f(\rho)| \le \rho^{-(k+1)}$, we have that $|y(\rho)| \le \rho^{-k}$, thus concluding the proof. □

Turning back to our example, we immediately see that the series formally defining $\tilde{v}$ does not converge on the whole positive real axis. To deal with this fact, pick up a C^∞ cutoff function χ such that $\chi \equiv 0$ for $\rho \le 2$, $\chi \equiv 1$ for $\rho \ge 3$, and $0 \le \chi \le 1$. It is then evident that

$$w(\rho) = \chi(\rho) \sum_{k=0}^{\infty} v_{2k}(\rho)$$

is a convergent series defining a smooth bounded function. We have

$$P_0 w + \frac{1}{\rho^2} P_2 w = g,$$

where

$$g = -\chi'' \sum_{k=0}^{\infty} v_{2k} - 2\chi' \sum_{k=1}^{\infty} v'_{2k} + 2\chi' \sum_{k=1}^{\infty} v_{2k}.$$

We emphasize that the same argument of the lemma gives us analogous estimates for the derivatives of the v_{2k}, so that there is no problem for the convergence of the series in the expression of g.

Replacing $\tilde{v}$ by w, we see that we have found a function $h(t)$ with

$$h(t) = \int_0^{+\infty} e^{i\rho^2 t} e^{-\rho} \rho^{-1/2} w(\rho)\, d\rho$$

such that

$$L(t, \partial_t) h = \int_0^{+\infty} e^{i\rho^2 t} e^{-\rho} \rho^{-1/2} g(\rho)\, d\rho.$$

We observe now that the function in the right-hand side of the above equality is in fact of class C^ω, since $\operatorname{supp} g \subset [2,3]$. On the other hand,

$$\partial_t^\alpha h(0) = i^\alpha \int_0^{+\infty} e^{-\rho} \rho^{-1/2+2\alpha} w(\rho)\, d\rho.$$

Since $v_0 \equiv 1$, we see that

$$|\partial_t^\alpha h(0)| \ge \delta^{\alpha+1} \alpha!^2,$$

with δ small and positive; that is, h is not better than G^2 even though the right-hand side is real analytic. This ends the proof that L is G^2-hypoelliptic and not better.

We make a few remarks on this example. First: in general, just one cutoff is not enough to sum the formal series of the v_{2k}'s. A more complex technique is required. Second: solving the transport equations has been possible because there is a "gain" in the decreasing rate of the functions v_{2k}. In general, one also has to control the growth rate of the coefficients of the differential operators defining the operator in parentheses under the integral sign in the second line of (6-1). As a last remark, the conclusion will not follow in general by an easy computation of the derivatives of (the analog of) h. Instead we need to violate an a priori estimate being equivalent to the G^s-hypoellipticity. Such an estimate was proved by Métivier [1980].

6.1. ***Construction of a formal solution.*** We recall from Theorem 3.2 the form of the pseudodifferential operator Λ (the L in Examples 1 and 2 above).

$$\Lambda(t,\tau) \sim \sum_{j=0}^{j_0} \Lambda_{-jq}(t,\tau) + \sum_{s\geq 0}\bigl(\Lambda_{-2k-sq}(t,\tau) + \Lambda_{-(j_0+1)q-sq}(t,\tau)\bigr).$$

In view of Proposition 3.1, we may assume that $\tau > 0$. Then

$$\Lambda(t,\tau) \sim \sum_{j=0}^{j_0} \Lambda_{-jq}(t,1)\tau^{2/q-j} + \sum_{s\geq 0}\bigl(\Lambda_{-2k-sq}(t,1)\tau^{2/q-2k/q-s} + \Lambda_{-(j_0+1)q-sq}(t,1)\tau^{2/q-(j_0+1)-s}\bigr).$$

Multiply on the right by the elliptic factor $\tau^{-2/q+2k/q}$ and keep (3-34) in mind (Theorem 3.2); we then obtain the following expression of the real analytic symbol Λ:

$$\Lambda(t,\tau)\tau^{-2/q+2k/q} \sim \sum_{h=0}^{j_0+1} t^{2l-h} a_h(t)\tau^{2k/q-h} + \tilde{a}_0(t) + \sum_{h=1}^{\infty}\bigl(\tilde{a}_h(t)\tau^{-h} + b_h(t)\tau^{2k/q-(j_0+1)-h}\bigr), \tag{6-2}$$

where

$$\begin{aligned} a_h(t) &= t^{-2l+h}\Lambda_{-hq}(t,1) && \text{for } h = 0,\dots, j_0+1,\\ \tilde{a}_h(t) &= \Lambda_{-2k-hq}(t,1) && \text{for } h\geq 0,\\ b_h(t) &= \Lambda_{-(j_0+1)q-hq}(t,1) && \text{for } h \geq 1. \end{aligned}$$

We point out that a_h, $\tilde{a}_h$, b_h are real analytic functions near the origin.

Moreover, from (3-35) and (3-36) in Theorem 3.2, we have

$$a_0(0),\ \tilde{a}_0(0) > 0, \tag{6-3}$$

and

$$|\partial_t^\alpha \tilde{a}_h(t)| \leq C^{1+h+\alpha}\alpha!\,h!\,, \quad |\partial_t^\alpha b_h(t)| \leq C^{1+h+\alpha}\alpha!\,h!\,, \tag{6-4}$$

for t in a (relatively compact) neighborhood of the origin and $h \geq 1$.

In order to simplify the notation, we denote again by $\Lambda(t, \tau)$ the symbol on the left-hand side of (6-2). It will also be useful to employ a more compact notation:

$$\Lambda(t,\tau) \sim \sum_{h=0}^{j_0} t^{2l-h} a_h(t) \tau^{2k/q-h} + \sum_{h=0}^{\infty} c_h(t) \tau^{-h/q}. \tag{6-5}$$

Here we replaced the expansion (6-2), where there is an order scaling by units, with a (more general) expansion exhibiting a scaling by multiples of $1/q$. In particular, (6-3) becomes

$$a_0(0),\ c_0(0) > 0 \tag{6-6}$$

and the estimates (6-4) become

$$|\partial_t^\alpha c_h(t)| \le C^{1+h+\alpha} \alpha!\, h!^{1/q}. \tag{6-7}$$

Furthermore, we shall use in the sequel the equalities

$$c_h(t) \equiv 0, \quad \text{for} \quad h = 1,\ \ldots,\ q(j_0+1)-2k-1,\ q(j_0+1)-2k+1,\ \ldots,\ q-1, \tag{6-8}$$

and

$$c_{q(j_0+1)-2k}(t) = \mathbb{O}(t^{2l-(j_0+1)}). \tag{6-9}$$

To obtain a formal null solution $\Lambda(t, D_t)$, we expand in power series the coefficients in the expression of Λ in (6-5); actually this is not an approximation, since the coefficients are real analytic functions. Interchanging the summation signs, we have

$$\Lambda(t, D_t) \sim \sum_{n\ge 0}\biggl(\sum_{h=0}^{j_0} a_{hn} t^{2l-h+n} D_t^{2k/q-h} + \sum_{j=0}^{\infty} c_{jn} t^n D_t^{-j/q}\biggr). \tag{6-10}$$

Here the conditions (6-6)–(6-9) become

$$a_{00},\ c_{00} > 0, \tag{6-11}$$

$$|a_{hn}| \le C_a^{1+n}, \quad |c_{jn}| \le C_a^{1+j+n} j!^{1/q} \quad \text{for } h = 0, \ldots, j_0 \text{ and } j, n \ge 0 \tag{6-12}$$

(where C_a denotes a positive constant independent of h, j and n),

$$c_{jn} = 0 \quad \text{for } n \ge 0 \text{ and } j = 1, \ldots, q(j_0+1)-2k-1, q(j_0+1)-2k+1, \ldots, q-1, \tag{6-13}$$

$$c_{q(j_0+1)-2k,n} = 0 \quad \text{for} \quad 0 \le n < 2l-(j_0+1). \tag{6-14}$$

The next step is to formally apply the operator Λ as defined in (6-10) to a function of the form

$$A(u)(t) = \int_0^{+\infty} e^{it\rho^{s_0}} u(\rho)\, d\rho, \tag{6-15}$$

where s_0 has been defined in Theorem 5.7 and u denotes a rapidly decreasing function with support bounded away from the origin. We search for a u such that $\Lambda(t, D_t)A(u)(t) = 0$ formally.

Applying a not necessarily integer power of D_t to $A(u)$ means multiplying u by the corresponding power of ρ. In order to write the contribution due to multiplication by a power of t, we need:

Lemma 6.1.1. *Let s_0 have the same meaning as before. Then*

$$\left(-\partial_\rho \frac{1}{i s_0 \rho^{s_0-1}}\right)^n = \sum_{h=0}^{n} \gamma_{nh} \frac{1}{\rho^{s_0 n-h}} \partial_\rho^h, \tag{6-16}$$

where the γ_{nh} (which now contain s_0) are complex constants satisfying estimates of the form

$$|\gamma_{nh}| \le C_\gamma'^{n+h} \frac{n!}{h!} \le C_\gamma^{n+h}(n-h)!. \tag{6-17}$$

Here both C'_γ and C_γ are positive constants depending on s_0 only. In particular, we have $\gamma_{nn} = (i/s_0)^n$, and for convenience set $\gamma_{00} = 1$.

Proof. It is enough to prove the first inequality. Arguing by induction, one easily sees that the coefficients γ_{nh} satisfy the recurrence relations

$$\gamma_{n+1,0} = -\frac{i}{s_0}\big(\gamma_{n0}(s_0(n+1)-1)\big), \quad \gamma_{n+1,n+1} = \frac{i}{s_0}\gamma_{nn}, \quad \gamma_{n+1,h} = \frac{i}{s_0}\big(\gamma_{nh-1} - (s_0(n+1)-h-1)\gamma_{nh}\big).$$

An induction argument allows us to conclude. □

We then have the formula, for $m \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$t^n D_t^m A(u)(t) = \int_0^{+\infty} e^{it\rho^{s_0}} \left(-\partial_\rho \frac{1}{i s_0 \rho^{s_0-1}}\right)^n \rho^{m s_0} u(\rho)\, d\rho.$$

Using this formula repeatedly as well as Lemma 6.1.1, we get

$$\Lambda(t, D_t)A(u)(t) = \int_0^{+\infty} e^{it\rho^{s_0}} P(\rho, D_\rho) u(\rho)\, d\rho, \tag{6-18}$$

where

$$P(\rho, \partial_\rho) = \sum_{n=0}^{\infty} \Bigg\{ \sum_{h=0}^{j_0} \sum_{p=0}^{2l-h+n} a_{hn} \gamma_{2l-h+n,p} \frac{1}{\rho^{s_0(2l-h+n)-p}} \partial_\rho^p \rho^{s_0(2k/q-h)} + \sum_{j=0}^{\infty} \sum_{p=0}^{n} c_{jn} \gamma_{np} \frac{1}{\rho^{s_0 n-p}} \partial_\rho^p \rho^{-s_0 j/q} \Bigg\}. \tag{6-19}$$

We use the notation

$$\partial_\rho^p(\rho^\lambda u) = \sum_{\alpha=0}^{p} \binom{p}{\alpha} (\lambda)_{p-\alpha} \rho^{\lambda-p+\alpha} \partial_\rho^\alpha u, \tag{6-20}$$

where $(\lambda)_\beta$ is the Pochhammer symbol, defined by

$$(\lambda)_\beta = \lambda(\lambda-1)\dots(\lambda-\beta+1), \quad (\lambda)_0 = 1, \quad \lambda \in \mathbb{C}. \tag{6-21}$$

We point out that the following identity is a trivial consequence of the definition of s_0:

$$s_0 \frac{2k}{q} - (s_0-1)2l = 0. \tag{6-22}$$

Using (6-22) and the preceding identities, we obtain the expression for P

$$P(\rho,\partial_\rho)=\sum_{n=0}^{\infty}\Bigg\{\sum_{h=0}^{j_0}\sum_{p=0}^{2l-h+n}\sum_{\alpha=0}^{p}a_{hn}\gamma_{2l-h+n,p}\binom{p}{\alpha}\cdot\Big(s_0\Big(\frac{2k}{q}-h\Big)\Big)_{p-\alpha}\rho^{-2l-s_0n+\alpha}\partial_\rho^\alpha \\ +\sum_{j=0}^{\infty}\sum_{p=0}^{n}\sum_{\alpha=0}^{p}c_{jn}\gamma_{np}\binom{p}{\alpha}\Big(-s_0\frac{j}{q}\Big)_{p-\alpha}\rho^{-s_0n-s_0j/q+\alpha}\partial_\rho^\alpha\Bigg\}. \quad (6\text{-}23)$$

Define now the coefficients

$$A_{h\alpha n}=\sum_{p=\alpha}^{2l-h+n}\gamma_{2l-h+n,p}\binom{p}{\alpha}\Big(s_0\Big(\frac{2k}{q}-h\Big)\Big)_{p-\alpha} \quad (6\text{-}24)$$

and

$$B_{j\alpha n}=\sum_{p=\alpha}^{n}\gamma_{np}\binom{p}{\alpha}\Big(-s_0\frac{j}{q}\Big)_{p-\alpha}. \quad (6\text{-}25)$$

In particular, $A_{0,2l,0}=\gamma_{2l,2l}=(i/s_0)^{2l}$ and $B_{000}=1$.

Lemma 6.1.2. *For $h\in\{0,\dots,j_o\}$, $n\geq 0$, $\alpha\in\{0,\dots,2l-h+n\}$, we have*

$$|A_{h\alpha n}|\leq C_A^{2l-h+n+1}\frac{(2l-h+n)!}{\alpha!}. \quad (6\text{-}26)$$

For $j,n\geq 0$, $\alpha\in\{0,\dots,n\}$, we have

$$|B_{j\alpha n}|\leq C_B^{n+s_0(j/q)+1}\frac{n!}{\alpha!}. \quad (6\text{-}27)$$

Proof. Let us first consider the $A_{h\alpha n}$. Since, for $r=0,\dots,p-\alpha-1$,

$$\big|s_0(2k/q-h)-r\big|=\big|s_0(2k/q-h)-1-(r-1)\big|\leq\lfloor s_0(2k/q)\rfloor+r+1,$$

we have

$$\bigg|\Big(s_0\Big(\frac{2k}{q}-h\Big)\Big)_{p-\alpha}\bigg|\leq\frac{\Big(\Big\lfloor s_0\frac{2k}{q}\Big\rfloor+p-\alpha\Big)!}{\Big(\Big\lfloor s_0\frac{2k}{q}\Big\rfloor\Big)!}\leq C^{p-\alpha}(p-\alpha)!\,,$$

for a convenient positive constant C. We may then write, due to (6-17), that

$$|A_{h\alpha n}|\leq\sum_{p=\alpha}^{2l-h+n}C_\gamma^{2l-h+n+p}C^{p-\alpha}\frac{(2l-h+n)!}{p!}\binom{p}{\alpha}(p-\alpha)!\leq C_A^{2l-h+n}\frac{(2l-h+n)!}{\alpha!}$$

for a suitable positive constant C_A. This proves the first statement. The second is proved in an analogous way and we omit the details. □

Using the definitions (6-24), (6-25), the operator P in (6-23) can be rewritten as

$$P(\rho,\partial_\rho)=\sum_{n=0}^{\infty}\Bigg\{\sum_{h=0}^{j_0}\sum_{\alpha=0}^{2l-h+n}a_{hn}A_{h\alpha n}\rho^{-2l-s_0n+\alpha}\partial_\rho^\alpha+\sum_{j=0}^{\infty}\sum_{\alpha=0}^{n}c_{jn}B_{j\alpha n}\rho^{-s_0n-s_0j/q+\alpha}\partial_\rho^\alpha\Bigg\}. \quad (6\text{-}28)$$

Setting

$$\tilde{A}_{n\alpha} = \sum_{h=0}^{\min\{j_0, 2l+n-\alpha\}} a_{hn} A_{h\alpha n}, \tag{6-29}$$

the above expression of P can be slightly simplified:

$$P(\rho, \partial_\rho) = \sum_{n=0}^{\infty} \left\{ \sum_{\alpha=0}^{2l+n} \tilde{A}_{n\alpha} \rho^{-2l-s_0 n+\alpha} \partial_\rho^\alpha + \sum_{j=0}^{\infty} \sum_{\alpha=0}^{n} c_{jn} B_{j\alpha n} \rho^{-s_0 n - s_0 j/q + \alpha} \partial_\rho^\alpha \right\}. \tag{6-30}$$

Moreover, the estimate of Lemma 6.1.2 carries over to $\tilde{A}_{n\alpha}$:

Lemma 6.1.3. *For $n \geq 0$, $\alpha \in \{0, \dots, 2l+n\}$, we have*

$$|\tilde{A}_{n\alpha}| \leq C_{\tilde{A}}^{2l+n+1} \frac{(2l+n)!}{\alpha!}. \tag{6-31}$$

For reasons that will become apparent in the sequel, we prefer to write the operator P in a way where the factorial growth of the coefficients is coupled with a corresponding negative power of the variable ρ, that is,

$$P(\rho, \partial_\rho) = \sum_{n=0}^{\infty} \left\{ \sum_{\alpha=0}^{2l+n} \frac{\tilde{A}_{n\alpha}}{\rho^{2l+n-\alpha}} \rho^{-(s_0-1)n} \partial_\rho^\alpha + \sum_{j=0}^{\infty} \sum_{\alpha=0}^{n} \frac{c_{jn} B_{j\alpha n}}{\rho^{n-\alpha+j/q}} \rho^{-(s_0-1)(n+j/q)} \partial_\rho^\alpha \right\}. \tag{6-32}$$

We point out that the powers of ρ in the above expression of P are all negative. However, if we were now to attempt to find a formal solution to $Pu = 0$ by solving iteratively the transport equations obtained by looking for a u in the form $\sum_{k \geq 0} u_k$, we would not be able to conclude that the sequence u_k decreases with respect to ρ in such a way that we can asymptotically sum the series for u. In other words, we wish u to behave as a symbol and we want to compute its asymptotic expansion for large ρ, but for the time being, there is no guarantee that the symbols u_k would have a decreasing order in ρ when k goes to infinity.

A way around this is to introduce a phase function and to write u as $u(\rho) = e^{i\Phi(\rho)} v(\rho)$, in such a way that the negative powers of ρ in the expression of P which are not *negative enough* are canceled by $\Phi(\rho)$. This is what we do in the next step.

Using the Faà di Bruno formula, we have

$$\begin{aligned} e^{-i\Phi} \partial_\rho^n e^{i\Phi} = (\partial_\rho + i\Phi_\rho)^n &= e^{-i\Phi} \sum_{h=0}^{n} \binom{n}{h} (\partial_\rho^h e^{i\Phi}) \partial_\rho^{n-h} \\ &= \sum_{h=0}^{n} \sum_{k=1}^{h} i^k \sum_{\substack{k_1, \dots, k_h \\ \sum_i k_i = k \\ \sum_i i k_i = h}} \binom{n}{h} \frac{h!}{k_1! \cdots k_h!} \prod_{p=1}^{h} \left(\frac{\Phi_\rho^{(p-1)}}{p!} \right)^{k_p} \partial_\rho^{n-h}. \end{aligned}$$

Here $\Phi_\rho^{(k)} = \partial_\rho^{k+1}\Phi$ and $\Phi_\rho = \Phi_\rho^{(0)}$. Plugging this formula into (6-32), we obtain

$$
\begin{aligned}
e^{-i\Phi(\rho)} & P(\rho, \partial_\rho) e^{i\Phi(\rho)} \\
&= \sum_{n=0}^{\infty} \sum_{\alpha=0}^{2l+n} \sum_{h=0}^{\alpha} \sum_{k=1}^{h} i^k \sum_{\substack{k_1,\dots,k_h \\ \sum_i k_i = k \\ \sum_i i k_i = h}} \binom{\alpha}{h} \frac{h!}{k_1! \dots k_h!} \frac{\tilde{A}_{n\alpha}}{\rho^{2l+n-\alpha}} \rho^{-(s_0-1)n} \prod_{p=1}^{h} \Bigl(\frac{\Phi_\rho^{(p-1)}}{p!} \Bigr)^{k_p} \partial_\rho^{\alpha-h} \qquad (6\text{-}33) \\
&\quad + \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\alpha=0}^{n} \sum_{h=0}^{\alpha} \sum_{k=1}^{h} i^k \sum_{\substack{k_1,\dots,k_h \\ \sum_i k_i = k \\ \sum_i i k_i = h}} \binom{\alpha}{h} \frac{h!}{k_1! \dots k_h!} \frac{c_{jn} B_{j\alpha n}}{\rho^{n-\alpha+j/q}} \rho^{-(s_0-1)(n+j/q)} \prod_{p=1}^{h} \Bigl(\frac{\Phi_\rho^{(p-1)}}{p!} \Bigr)^{k_p} \partial_\rho^{\alpha-h}.
\end{aligned}
$$

Our purpose is to cancel all terms containing powers $\rho^{-\theta}$ with $0 > -\theta \geq -1$ and no derivatives. This is closely connected with the form of the (asymptotic expansion of the) operator Λ and is actually performed by choosing a phase function Φ of the form

$$
\Phi_\rho(\rho) = \sum_{j=0}^{M_0} \varphi_j \rho^{-(s_0-1)j} + \varphi_{-1}\rho^{-1}, \quad M_0 = \Bigl\lfloor \frac{1}{s_0-1} \Bigr\rfloor. \qquad (6\text{-}34)
$$

Here $\lfloor \dots \rfloor$ denotes the integer part and the φ_j, $j = -1, 0, \dots, M_0$, are complex numbers to be chosen later.

Let us find the terms in both summands in (6-33) where there are no derivatives and the power of ρ is not below -1. To this end, we remark that only Φ_ρ plays a role since, because of (6-34), $\Phi_\rho^{(k)}(\rho) = o(\rho^{-1})$ if $k \geq 1$.

Let us focus first on the first summand in (6-33). The terms with no derivatives correspond to $\alpha = h$. The terms where only first derivatives of Φ appear have $k_1 = k = h$. Moreover, since $2l + n - \alpha$ is an integer, we necessarily must have either $2l + n - \alpha = 0$ and $0 \leq n \leq M_0$, or $2l + n - \alpha = 1$ and $n = 0$.

Let us consider the second summand in (6-33). Similarly to the preceding case, $\alpha = h$ and $k_1 = k = h$. Moreover, we necessarily have $n = \alpha$. In view of (6-13) and (6-14), either $j = 0$ and $0 \leq n \leq M_0$, or $j = q(j_0 + 1) - 2k$ and $n = 2l - (j_0 + 1)$ if $j_0 = 0$ (that is, $2k < q$.)

It turns out to be useful to have a notation for the family of indices in both the first and second summands in (6-33) corresponding to terms that do not contribute to the eikonal equation. We call these two families of indices $\mathscr{A}$ and $\mathscr{B}$ respectively. We have

$$
\begin{aligned}
\mathscr{A} = \bigl\{(n,\alpha,h,k) \bigm| n > M_0\bigr\} &\cup \bigl\{(n,\alpha,h,k) \bigm| 0 \leq n \leq M_0, (\alpha,h,k) \neq (2l+n, 2l+n, 2l+n)\bigr\} \\
&\cup \bigl\{(n,\alpha,h,k) \bigm| (n,\alpha,h,k) \neq (0, 2l-1, 2l-1, 2l-1)\bigr\}, \qquad (6\text{-}35)
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{B} = \bigl\{(n,j,\alpha,h,k) \bigm| n > M_0\bigr\} &\cup \bigl\{(n,j,\alpha,h,k) \bigm| 0 \leq n \leq M_0, (j,\alpha,h,k) \neq (0,n,n,n)\bigr\} \\
&\cup \bigl\{(n,j,\alpha,h,k) \bigm| j = q-2k, n = 2l-1, (\alpha,h,k) \neq (n,n,n), \text{ if } j_0 = 0\bigr\}. \qquad (6\text{-}36)
\end{aligned}
$$

The terms contributing to the eikonal equation are then

$$\sum_{n=0}^{M_0} i^{2l+n} \tilde{A}_{n,2l+n} \rho^{-(s_0-1)n} \Phi_\rho^{2l+n} + \rho^{-1} i^{2l-1} \tilde{A}_{0,2l-1} \Phi_\rho^{2l-1} \\ + \sum_{n=0}^{M_0} i^n c_{0n} B_{0nn} \rho^{-(s_0-1)n} \Phi_\rho^n + i^n c_{q-2k,2l-1} B_{q-2k,2l-1,2l-1} \rho^{-1} \Phi_\rho^{2l-1}, \quad (6\text{-}37)$$

where the last term of the expression above is present only if $j_0 = 0$. Note that there is a kind of "principal part" in the above expression, namely the part not containing negative powers of ρ. This part is obtained by setting $n = 0$. Now by (6-29),

$$i^{2l} \tilde{A}_{0,2l} = i^{2l} a_{00} A_{0,2l,0} = i^{2l} a_{00} (i/s_0)^{2l} = a_{00} s_0^{-2l} > 0,$$

where the next to last equality is due to (6-24) and the positivity is a consequence of (6-11). On the other hand, again by (6-11) and (6-24), $c_{00} B_{000} = c_{00} > 0$.

Lemma 6.1.4. *Consider the equation*

$$\sum_{n=0}^{M_0} \rho^{-(s_0-1)n} \big(a_n \Phi_\rho^{2l+n} + b_n \Phi_\rho^n\big) + \gamma \rho^{-1} \Phi_\rho^{2l-1} = \mathbb{O}(\rho^{-1-\delta}). \quad (6\text{-}38)$$

Here a_n, b_n, γ denote complex numbers and a_0, $b_0 > 0$; δ is a positive rational number.

Then there is a function $\Phi_\rho(\rho)$, $\rho > 0$, of the form (6-34), *satisfying* (6-38) *with*

$$\delta = (M_0 + 1)(s_0 - 1) - 1 > 0$$

and such that

$$\operatorname{Im} \Phi_\rho(\rho) > 0 \quad \text{modulo } \mathbb{O}(\rho^{-(s_0-1)}). \quad (6\text{-}39)$$

Proof. To start with, we remark that the equation

$$a_0 \Phi_\rho^{2l} + b_0 = \mathbb{O}(\rho^{-(s_0-1)})$$

is satisfied by $\Phi_\rho(\rho)$ in (6-34), where $a_0 \varphi_0^{2l} + b_0 = 0$. Of course we are always free to choose φ_0 such that $\operatorname{Im} \varphi_0 > 0$. We now argue by induction. Assume that we determined $\varphi_0, \dots, \varphi_{k-1}$ and solved (6-38) modulo $o(\rho^{-(k-1)(s_0-1)})$. Let us compute the coefficient of $\rho^{-k(s_0-1)}$ in (6-38), with $k \le M_0$. First we observe that if α denotes the multi-index $(\alpha_0, \alpha_1, \dots, \alpha_{M_0})$ with $\alpha_r \in \mathbb{Z}^+$ and φ denotes the complex vector $(\varphi_0, \varphi_1, \dots, \varphi_{M_0})$, we have

$$\Phi_\rho^j(\rho) = \sum_{|\alpha|=j} \frac{j!}{\alpha!} \varphi^\alpha \rho^{-(s_0-1)\sum_{p=0}^{M_0} p\alpha_p} \quad \text{modulo } \mathbb{O}(\rho^{-1}).$$

The coefficient of $\rho^{-k(s_0-1)}$ is then given by

$$\sum_{j=0}^{k} \Bigg(a_j \sum_{\substack{|\alpha|=2l+j \\ \sum_p p\alpha_p = k-j}} \frac{(2l+j)!}{\alpha!} \varphi^\alpha + b_j \sum_{\substack{|\alpha|=j \\ \sum_p p\alpha_p = k-j}} \frac{j!}{\alpha!} \varphi^\alpha \Bigg).$$

The constraint on $\sum_p p\alpha_p$ forces the index p to run from 0 to $k-j$, and it is clear that if $j>0$, the first summand above cannot contain φ_k, since $\alpha_{k-j+1}=\cdots=\alpha_{M_0}=0$. Consider thus the term with $j=0$. Then α_k is zero or one. The first case is similar to the previous cases, so that α_k must be one. Then since $\alpha_1=\cdots=\alpha_{k-1}=0$, we see that $\alpha_0=2l-1$, thus yielding the coefficient of $\rho^{-k(s_0-1)}$ containing φ_k:

$$2la_0\varphi_0^{2l-1}\varphi_k.$$

Arguing analogously, we can see that φ_k is never contained in terms coming from the second summand. This allows us to uniquely determine φ_k, since $a_0,\varphi_0\neq 0$.

The argument for φ_{-1} is completely similar and we omit it. □

The above lemma gives the existence of the phase function Φ of the form (6-34) such that in the expression of $e^{-i\Phi}Pe^{i\Phi}$ there are no terms without derivatives in which ρ has an exponent greater than or equal to -1. We stress that the reason why we need this fact will become apparent when we have to solve the transport equations, which thus far have not played a role.

Thus the operator $e^{-i\Phi}Pe^{i\Phi}$ now has the form

$$\begin{aligned}
&e^{-i\Phi(\rho)}P(\rho,\partial_\rho)e^{i\Phi(\rho)}\\
&\quad=\sum_{\substack{n=0\\(n,\alpha,h,k)\in\mathscr{A}}}^{\infty}\sum_{\alpha=0}^{2l+n}\sum_{h=0}^{\alpha}\sum_{k=1}^{h} i^k\sum_{\substack{k_1,\dots,k_h\\ \sum_i k_i=k\\ \sum_i ik_i=h}}\binom{\alpha}{h}\frac{h!}{k_1!\cdots k_h!}\frac{\tilde{A}_{n\alpha}}{\rho^{2l+n-\alpha}}\rho^{-(s_0-1)n}\prod_{p=1}^{h}\Bigl(\frac{\Phi_\rho^{(p-1)}}{p!}\Bigr)^{k_p}\partial_\rho^{\alpha-h}\\
&\qquad+\sum_{\substack{n=0\\(n,j,\alpha,h,k)\in\mathscr{B}}}^{\infty}\sum_{j=0}^{\infty}\sum_{\alpha=0}^{n}\sum_{h=0}^{\alpha}\sum_{k=1}^{h} i^k\sum_{\substack{k_1,\dots,k_h\\ \sum_i k_i=k\\ \sum_i ik_i=h}}\binom{\alpha}{h}\frac{h!}{k_1!\cdots k_h!}\frac{c_{jn}B_{j\alpha n}}{\rho^{n-\alpha+j/q}}\rho^{-(s_0-1)(n+j/q)}\\
&\qquad\qquad\cdot\prod_{p=1}^{h}\Bigl(\frac{\Phi_\rho^{(p-1)}}{p!}\Bigr)^{k_p}\partial_\rho^{\alpha-h}+\mathbb{O}(\rho^{-(1+\delta)}).
\end{aligned}\tag{6-40}$$

Here the last term is a consequence of (6-37) and Lemma 6.1.4, where we defined δ.

Lemma 6.1.5. *Let Φ be as in* (6-34) *and denote by C_Φ a positive constant such that $|\varphi_j|\le C_\Phi$ for $j=-1,0,1,\dots,M_0$. Then*

$$\prod_{p=1}^{h}\Bigl(\frac{\Phi_\rho^{(p-1)}}{p!}\Bigr)^{k_p}=\rho^{-(h-k)}\sum_{\substack{t_1=1-\delta_{h,k}\\ k-k_1\le t_1+M_0t_2\le kM_0}}^{kM_0}\sum_{t_2=0}^{k}c_{(k_1,\dots,k_h),t_1,t_2}\,\rho^{-(s_0-1)t_1-t_2},\tag{6-41}$$

where $k=\sum_{i=1}^{h}k_i$, $h=\sum_{i=1}^{h}ik_i$, and $\delta_{h,k}$ is the usual Kronecker symbol. Moreover, we have the estimate

$$|c_{(k_1,\dots,k_h),t_1,t_2}|\le\binom{t_1+t_2+k}{k}C_\Phi^k.\tag{6-42}$$

Proof. We argue by induction on h. If $h=1$, then $k=1$ and $k_1=1$, so that (6-41) is trivial. Assume now that $h>1$ and suppose that (6-41) holds for every $h'<h$. There are two cases:

Case I. If $k_h \neq 0$, then from $h = \sum_{i=1}^{h} i k_i$, we obtain that $k_h = 1$ and $k_1, \dots, k_{h-1} = 0$, and hence $k = 1$. Then

$$\prod_{p=1}^{h} \Big(\frac{\Phi_\rho^{(p-1)}}{p!} \Big)^{k_p} = \frac{1}{h!} \Big(\sum_{j=0}^{M_0} \varphi_j \rho^{-(s_0-1)j} + \varphi_1 \rho^{-1} \Big)^{(h-1)}$$
$$= \rho^{-(h-1)} \Big(\sum_{t_1=1}^{M_0} c_{(0,\dots,0,1),t_1,0} \rho^{-(s_0-1)t_1} + c_{(0,\dots,0,1),0,1} \rho^{-1} \Big),$$

which proves the statement.

Case II. Suppose $k_h = 0$. Let $s = \min\{j \mid k_j \neq 0\}$ so that $\sum_{i=s}^{h-1} i k_i = h$. Note that

$$s(k_s - 1) + (s+1)k_{s+1} + \cdots + (h-1)k_{h-1} = h - s.$$

If $s = 1$, the h-tuple $(k_1 - 1, k_2, \dots, k_{h-1}, 0)$ can be thought of as an $(h-1)$-tuple such that

$$k_1 - 1 + 2k_2 + \cdots + (h-1)k_{h-1} = h - 1.$$

On the other hand, if $s > 1$, from $s(k_s - 1) + (s+1)k_{s+1} + \cdots + (h-1)k_{h-1} = h - s$ we immediately deduce that $k_{h-a} = 0$ for every $a < s$, so that the h-tuple

$$\big(0, \dots, 0, k_s - 1, \dots, k_{h-1}, 0\big) = \big(0, \dots, 0, k_s - 1, \dots, k_{h-s}, 0, \dots, 0\big)$$

can be identified to the $(h-s)$-tuple

$$(k_1, \dots, k_{s-1}, k_s - 1, \dots, k_{h-s}),$$

where $k_1 = \cdots = k_{s-1} = 0$ and $s(k_s - 1) + \cdots + (h-s)k_{h-s} = h - s$. We are now in a position to apply the inductive hypothesis. Assume, to make things definite, that $s > 1$ (the case $s = 1$ is analogous). Then

$$\prod_{p=1}^{h} \Big(\frac{\Phi_\rho^{(p-1)}}{p!} \Big)^{k_p} = \prod_{p=1}^{h-s} \Big(\frac{\Phi_\rho^{(p-1)}}{p!} \Big)^{k_p} = \frac{\Phi_\rho^{(s-1)}}{s!} \prod_{p=s}^{h-s} \Big(\frac{\Phi_\rho^{(p-1)}}{p!} \Big)^{k_p - \delta_{s,p}}$$
$$= \frac{\Phi_\rho^{(s-1)}}{s!} \rho^{-(h-s-(k-1))} \cdot \sum_{\substack{t_1 = 1 - \delta_{h-s,k-1} \\ k-1 \le t_1 + M_0 t_2 \le (k-1)M_0}}^{(k-1)M_0} \sum_{t_2=0}^{k-1} c_{(0,\dots,0,k_s-1,\dots,k_{h-s}),t_1,t_2} \rho^{-(s_0-1)t_1 - t_2}.$$

Recall now that

$$\frac{\Phi_\rho^{(s-1)}}{s!} = \rho^{-(s-1)} \Big(\sum_{j=1}^{M_0} c_{s,j} \rho^{-(s_0-1)j} + c_{s,-1} \rho^{-1} \Big),$$

for certain numbers $c_{s,j}$, $c_{s,-1}$. Note that we can find a positive constant C_Φ such that $|\varphi_j| \le C_\Phi$ for every $j = -1, 0, \dots, M_0$, and that then

$$|c_{s,j}| \le C_\Phi, \quad j = -1, 1, \dots, M_0.$$

Using the above expression for $\Phi_\rho^{(s-1)}/s!$, we obtain

$$\prod_{p=1}^{h}\left(\frac{\Phi_\rho^{(p-1)}}{p!}\right)^{k_p}=\rho^{-(h-k)}\Bigg[\sum_{j=1}^{M_0}\sum_{\substack{t_1=1-\delta_{h-s,k-1}\\ k-1\le t_1+M_0t_2\le (k-1)M_0}}^{(k-1)M_0}\sum_{t_2=0}^{k-1} c_{(0,\dots,0,k_s-1,\dots,k_{h-s}),t_1,t_2}c_{s,j}\,\rho^{-(s_0-1)(t_1+j)-t_2}$$
$$+c_{s,-1}\sum_{\substack{t_1=1-\delta_{h-s,k-1}\\ k-1\le t_1+M_0t_2\le (k-1)M_0}}^{(k-1)M_0}\sum_{t_2=0}^{k-1} c_{(0,\dots,0,k_s-1,\dots,k_{h-s}),t_1,t_2}\,\rho^{-(s_0-1)t_1-t_2}\Bigg]. \quad (6\text{-}43)$$

Now in the first sum we note that, as far as the powers of ρ are concerned, $k=k-1+1\le t_1+j+t_2M_0\le (k-1)M_0+M_0=kM_0$, while in the second sum above we have $k\le k-1+M_0\le t_1+(t_2+1)M_0\le (k-1)M_0+M_0=kM_0$, where we assume we are in the nontrivial case $M_0\ge 1$. This proves the first statement of the lemma. To finish the proof we have to prove estimate (6-42). We again argue by induction and use the expression (6-43) above. Actually the coefficient of $\rho^{-(s_0-1)t_1-t_2}$ coming from the first sum has the form

$$\sum_{t+j=t_1} c_{s,j}\,c_{(0,\dots,0,k_s-1,\dots,k_{h-s}),t,t_2}.$$

Its absolute value is estimated by

$$\sum_{j=0}^{t_1} C_\Phi^k\binom{j+k-1}{k-1}=C_\Phi^k\binom{t_1+k}{k},$$

where we have used the fact that $|c_{s,j}|\le C_\Phi$. This concludes the proof of the lemma. □

Using Lemma 6.1.5, we are going to make some preparations on the operator $e^{-i\Phi}Pe^{i\Phi}$ in (6-40). First of all, using (6-41), we write it in the rather lengthy form

$$e^{-i\Phi(\rho)}P(\rho,\partial_\rho)e^{i\Phi(\rho)}$$
$$=\sum_{\substack{n=0\\ (n,\alpha,h,k)\in\mathcal{A}}}^{\infty}\sum_{\alpha=0}^{2l+n}\sum_{h=0}^{\alpha}\sum_{k=1}^{h} i^k\sum_{\substack{k_1,\dots,k_h\\ \sum_i k_i=k\\ \sum_i ik_i=h}}\;\sum_{\substack{t_1=1-\delta_{k,h}\\ k-k_1\le t_1+M_0t_2\le kM_0}}^{kM_0}\sum_{t_2=0}^{M_0}\binom{\alpha}{h}\frac{h!}{k_1!\dots k_h!}$$
$$\cdot\, c_{(k_1,\dots,k_h),t_1,t_2}\frac{\tilde{A}_{n\alpha}}{\rho^{2l+n-\alpha}}\rho^{-(s_0-1)(n+t_1)-(t_2+h-k)}\partial_\rho^{\alpha-h}$$
$$+\sum_{\substack{n=0\\ (n,j,\alpha,h,k)\in\mathcal{B}}}^{\infty}\sum_{j=0}^{\infty}\sum_{\alpha=0}^{n}\sum_{h=0}^{\alpha}\sum_{k=1}^{h} i^k\sum_{\substack{k_1,\dots,k_h\\ \sum_i k_i=k\\ \sum_i ik_i=h}}\;\sum_{\substack{t_1=1-\delta_{k,h}\\ k-k_1\le t_1+M_0t_2\le kM_0}}^{kM_0}\sum_{t_2=0}^{M_0}\binom{\alpha}{h}\frac{h!}{k_1!\cdots k_h!}$$
$$\cdot\, c_{(k_1,\dots,k_h),t_1,t_2}\frac{c_{jn}B_{j\alpha n}}{\rho^{n-\alpha+j/q}}\rho^{-(s_0-1)(n+j/q+t_1)-(t_2+h-k)}\partial_\rho^{\alpha-h}+\mathbb{O}(\rho^{-(1+\delta)}). \quad (6\text{-}44)$$

Here the last term, $\mathbb{O}(\rho^{-(1+\delta)})$, denotes a finite sum of terms of the form

$$\gamma_k \rho^{-\theta_k},$$

where γ_k is a constant and $\theta_k \geq 1+\delta$.

For every $r \in \mathbb{N} \cup \{0\}$, define the pair of differential operators

$$Q_{\mathcal{A},r}(\rho, \partial_\rho) = \sum_{\substack{q(n+t_1)=r \\ t_1 \leq (2l+n)M_0}} \sum_{\substack{\alpha=t_1 \\ (n,\alpha,h,k)\in\mathcal{A}}}^{2l+n} \sum_{h=t_1}^{\alpha} \sum_{k=\min\{1,t_1\}}^{h} i^k \sum_{\substack{k_1,\dots,k_h \\ \sum_i k_i = k \\ \sum_i i k_i = h}} \sum_{t_2=0}^{M_0} \binom{\alpha}{h} \frac{h!}{k_1! \cdots k_h!} \rho^{-(h-k)} \cdot c_{(k_1,\dots,k_h),t_1,t_2} \frac{\tilde{A}_{n\alpha}}{\rho^{2l+n-\alpha}} \rho^{-t_2} \partial_\rho^{\alpha-h} \tag{6-45}$$

and

$$Q_{\mathcal{B},r}(\rho, \partial_\rho) = \sum_{\substack{q(n+t_1)+j=r \\ t_1 \leq nM_0}} \sum_{\substack{\alpha=t_1 \\ (n,\alpha,h,k)\in\mathcal{B}}}^{n} \sum_{h=t_1}^{\alpha} \sum_{k=\min\{1,t_1\}}^{h} i^k \sum_{\substack{k_1,\dots,k_h \\ \sum_i k_i = k \\ \sum_i i k_i = h}} \sum_{t_2=0}^{M_0} \binom{\alpha}{h} \frac{h!}{k_1! \cdots k_h!} \rho^{-(h-k)} \cdot c_{(k_1,\dots,k_h),t_1,t_2} \frac{c_{j,n} B_{j\alpha n}}{\rho^{n-\alpha+j/q}} \rho^{-t_2} \partial_\rho^{\alpha-h}. \tag{6-46}$$

Then the operator in (6-40) can be rewritten in the simpler form

$$e^{-i\Phi(\rho)} P(\rho, \partial_\rho) e^{i\Phi(\rho)} = \sum_{r=0}^{\infty} \rho^{-(s_0-1)r/q} P_r(\rho, \partial_\rho) + \mathbb{O}(\rho^{-(1+\delta)}), \tag{6-47}$$

where

$$P_r(\rho, \partial_\rho) = Q_{\mathcal{A},r}(\rho, \partial_\rho) + Q_{\mathcal{B},r}(\rho, \partial_\rho) \tag{6-48}$$

is a differential operator of order $2l + \lfloor r/q \rfloor$.

Our next task is to provide growth estimates with respect to r of arbitrary derivatives of the coefficients of the operator P_r in a region where ρ is large. These estimates are essential when one tries to construct a *true* solution from the solution that we have not discussed yet.

Proposition 6.1.6. *Denote by $\alpha_{r,p}(\rho)$ the coefficient of ∂_ρ^p in $P_r(\rho, \partial_\rho)$. Then we may find two positive constants c_1, C_α, such that if $\rho \geq c_1 r^\theta$, with $0 < \theta \leq 1$, we have*

$$\left|\partial_\rho^t \alpha_{r,p}(\rho)\right| \leq C_\alpha^{r+t+1} r!^{1-\theta} \frac{t!}{\rho^t}. \tag{6-49}$$

Proof. First we remark that the coefficient under exam is given by

$$\alpha_{r,p}(\rho) = \begin{cases} \alpha_{r,p,1}(\rho) + \alpha_{r,p,2}(\rho) & \text{if } p \leq \lfloor r/q \rfloor, \\ \alpha_{r,p,1}(\rho) & \text{if } \lfloor r/q \rfloor < p \leq \lfloor r/q \rfloor + 2l, \end{cases}$$

where $\alpha_{r,p,1}(\rho)$ comes from $Q_{\mathcal{A},r}(\rho, \partial_\rho)$ and correspondingly $\alpha_{r,p,2}(\rho)$ comes from $Q_{\mathcal{B},r}(\rho, \partial_\rho)$. Thus

$$P_r(\rho, \partial_\rho) = \sum_{p=0}^{2l+\lfloor r/q \rfloor} \alpha_{r,p}(\rho) \partial_\rho^p. \tag{6-50}$$

The expressions of $\alpha_{r,p,i}(\rho)$ are given by

$$\alpha_{r,p,1}(\rho)=\sum_{\substack{q(n+t_1)=r\\ t_1\le(2l+n)M_0}}\ \sum_{\substack{\alpha=\max\{t_1,p\}\\ (n,\alpha,\alpha-p,k)\in\mathscr{A}}}^{2l+n}\ \sum_{k=\min\{1,t_1\}}^{\alpha-p} i^k \sum_{\substack{k_1,\dots,k_{\alpha-p}\\ \sum_i k_i=k\\ \sum_i ik_i=\alpha-p}}\ \sum_{t_2=0}^{M_0}\binom{\alpha}{\alpha-p}\frac{(\alpha-p)!}{k_1!\cdots k_{\alpha-p}!}\rho^{-(\alpha-p-k)} \cdot c_{(k_1,\dots,k_{\alpha-p}),t_1,t_2}\frac{\tilde{A}_{n\alpha}}{\rho^{2l+n-\alpha}}\rho^{-t_2} \tag{6-51}$$

and

$$\alpha_{r,p,2}(\rho)=\sum_{\substack{q(n+t_1)+j=r\\ t_1\le nM_0}}\ \sum_{\substack{\alpha=\max\{t_1,p\}\\ (n,\alpha,\alpha-p,k)\in\mathscr{B}}}^{n}\ \sum_{k=\min\{1,t_1\}}^{\alpha-p} i^k \sum_{\substack{k_1,\dots,k_{\alpha-p}\\ \sum_i k_i=k\\ \sum_i ik_i=\alpha-p}}\ \sum_{t_2=0}^{M_0}\binom{\alpha}{\alpha-p}\frac{(\alpha-p)!}{k_1!\cdots k_{\alpha-p}!}\rho^{-(\alpha-p-k)} \cdot c_{(k_1,\dots,k_{\alpha-p}),t_1,t_2}\frac{c_{j,n}B_{j\alpha n}}{\rho^{n-\alpha+j/q}}\rho^{-t_2}. \tag{6-52}$$

We start by estimating (6-51). Differentiating t times the function in (6-51) has the effect of producing in the sum (6-51) the factor

$$(-1)^t\rho^{-t}\prod_{j=0}^{t-1}\big(t_2+2l+n-p-k+j\big).$$

Hence, using (6-42), (6-31),

$$\begin{aligned}\big|\partial_\rho^t\alpha_{r,p,1}(\rho)\big| \le \sum_{\substack{q(n+t_1)=r\\ t_1\le(2l+n)M_0}}\ \sum_{\substack{\alpha=\max\{t_1,p\}\\ (n,\alpha,\alpha-p,k)\in\mathscr{A}}}^{2l+n}\ \sum_{k=\min\{1,t_1\}}^{\alpha-p}\ \sum_{\substack{k_1,\dots,k_{\alpha-p}\\ \sum_i k_i=k\\ \sum_i ik_i=\alpha-p}}\\ \sum_{t_2=0}^{M_0}2^\alpha\frac{(\alpha-p)!}{k_1!\cdots k_{\alpha-p}!}\rho^{-(\alpha-p-k)}C_{\tilde{A}}^{n+1}\frac{(2l+n)!}{\alpha!}\rho^{-(2l+n-\alpha)}\\ \cdot C_\Phi^k\binom{t_1+t_2+k}{k}\binom{t_2+2l+n-p-k+t-1}{t}t!\,\rho^{-t_2-t}.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\frac{(\alpha-p)!}{k_1!\cdots k_{\alpha-p}!}&=\frac{(\alpha-p)!}{k!(\alpha-p-k)!}\ \frac{k!}{k_1!\cdots k_{\alpha-p}!}(\alpha-p-k)!\\ &\le 2^{\alpha-p}2^{\alpha-p}(\alpha-p-k)!\le 4^{2l+r/q}(\alpha-p-k)!.\end{aligned} \tag{6-53}$$

The number of multi-indices $(k_1,\dots,k_{\alpha-p})$ such that the sum of the components is k is given by

$$\binom{k+\alpha-p-1}{\alpha-p-1}\le 4^{2l+r/q}.$$

If $\rho\ge c_1r^\theta$, with $0<\theta\le 1$, we may estimate, if $\beta\le c_2r$,

$$\frac{\beta!}{\rho^\beta}\le\frac{\beta!}{c_3^\beta\beta^{\theta\beta}}\le c_4^\beta\beta!^{1-\theta},\quad c_3=c_1/c_2^\theta,\ c_4=c_3^{-1}. \tag{6-54}$$

As a consequence, we obtain

$$\left|\partial_\rho^t \alpha_{r,p,1}(\rho)\right| \leq \tilde{C}_1^{r+t+1} r!^{1-\theta} \frac{t!}{\rho^t},$$

where $\tilde{C}_1$ is a positive constant depending on the parameters of the problem and on θ.

The function $\partial_\rho^t \alpha_{r,p,2}(\rho)$ is estimated in a completely analogous way, and this proves the assertion. □

Let us now take a closer look at $P_0(\rho, \partial_\rho)$. We may write

$$P_0(\rho, \partial_\rho) = Q_0(\partial_\rho) + \sum_{m=1}^{N} \frac{1}{\rho^m} Q_m(\partial_\rho), \tag{6-55}$$

where the $Q_m(\partial_\rho)$ are differential operators with constant coefficients such that $Q_0(0) = Q_1(0) = 0$, all the roots of the equation $Q_0(\lambda) = 0$ are such that $\operatorname{Re}\lambda \geq 0$, due to the choice of the phase function Φ, and N is a suitable positive integer.

Let

$$j^* \in \mathbb{N}, \quad j^* = \left\lfloor \frac{q}{s_0 - 1} \right\rfloor.$$

Consider the order-zero term in the differential polynomial

$$\sum_{r=1}^{j^*} \rho^{-((s_0-1)/q)r} P_r(\rho, \partial_\rho).$$

It is obviously a finite sum involving negative powers of ρ of the form

$$\sum_j f_j \rho^{-\theta_j}, \quad \theta_j > 1, \ f_j \in \mathbb{C}.$$

Define λ by

$$\lambda + 1 = \min\{\theta_j\}.$$

Obviously λ is positive because $\theta_j > 1$. Lastly, set

$$\mu = \min\left\{1, \lambda, \delta, \frac{s_0 - 1}{q} - \frac{1}{j^* + 1}\right\}, \tag{6-56}$$

which is a positive rational number, since

$$\frac{s_0 - 1}{q} - \frac{1}{j^* + 1} > 0.$$

Also recall the definition of δ from Lemma 6.1.4. We are now in a position to define the final form for the operator P. Set

$$\tilde{P}_0(\partial_\rho) = Q_0(\partial_\rho), \tag{6-57}$$

$$\tilde{P}_r(\rho, \partial_\rho) = \rho^{(\mu-(s_0-1)/q)r} P_r(\rho, \partial_\rho), \qquad r \geq j^* + 1, \tag{6-58}$$

$$\tilde{P}_r(\rho, \partial_\rho) = \rho^{(\mu-(s_0-1)/q)r} \big(P_r(\rho, \partial_\rho) - P_r(\rho, 0)\big), \quad 2 \leq r \leq j^*. \tag{6-59}$$

Finally, we define $\tilde{P}_1$, including in it both the errors coming from the construction of the phase function and the zero-order terms which have been removed in (6-50) from the definition of $\tilde{P}_r$, $2 \leq r \leq j^*$.

$$\tilde{P}_1(\rho, \partial_\rho) = \rho^{\mu-(s_0-1)/q} P_1(\rho, \partial_\rho) + \rho^{\mu-1} \sum_{m=1}^{N} \frac{1}{\rho^{m-1}} Q_m(\partial_\rho) + \rho^{\mu-\delta} \sum_{a \geq 0} \gamma_a \frac{1}{\rho^{1+\tilde{\theta}_a}} + \sum_j f_j \rho^{\mu-\theta_j}, \quad (6\text{-}60)$$

where the next to last sum is a finite sum denoting what in (6-47) is $\mathbb{O}(\rho^{-(1+\delta)})$, γ_a are constants, and $\tilde{\theta}_a$ are nonnegative rational numbers.

The operator P in (6-47) is then written as

$$P_\Phi(\rho, \partial_\rho) \equiv e^{-i\Phi(\rho)} P(\rho, \partial_\rho) e^{i\Phi(\rho)} = \sum_{r=0}^{\infty} \rho^{-\mu r} \tilde{P}_r(\rho, \partial_\rho). \quad (6\text{-}61)$$

We explicitly point out that Proposition 6.1.6 holds also for the coefficients of $\tilde{P}_r$. Moreover, the zero-order terms of $\tilde{P}_r$, $2 \leq r \leq j^*$, are zero.

From now on, to keep the notation simple, we forget about the tildes in (6-61).

Finally, we turn to the construction of a formal solution to $P_\Phi u = 0$. Let us look for u in the form

$$u(\rho) = \sum_{p=0}^{\infty} u_p(\rho), \quad (6\text{-}62)$$

where the u_p's are the solutions of the differential equations

$$P_0(\partial_\rho) u_0(\rho) = 0, \quad (6\text{-}63)$$

$$P_0(\partial_\rho) u_h(\rho) = -\sum_{r=1}^{h} \rho^{-\mu r} P_r(\rho, \partial_\rho) u_{h-r}(\rho), \quad (6\text{-}64)$$

for $t \in \mathbb{N}$.

Equation (6-63) is immediately solved by $u_0(\rho) \equiv 1$, because $P_0(0) = 0$.

Lemma 6.1.7. *Let $Q(\partial_\rho)$ be an ordinary differential operator with constant coefficients such that*

$$Q(\partial_\rho) = \prod_{j=1}^{m} (\partial_\rho - \lambda_j)^{m_j}, \quad (6\text{-}65)$$

where m_j denotes the multiplicity of the complex characteristic root λ_j and $\operatorname{Re} \lambda_j \geq 0$. *Then the ordinary differential equation $Q(\partial_\rho) u = f$ has a solution of the form*

$$u(\rho) = (E * f)(\rho) = \sum_{j=1}^{m} \sum_{t=1}^{m_j} d_{j,t} \int_{\rho}^{+\infty} e^{\lambda_j(\rho-w)} (\rho - w)^{t-1} f(w)\, dw, \quad (6\text{-}66)$$

where the $d_{j,t}$ are suitable complex constants. In particular, $\partial_\rho^t u = E * \partial_\rho^t f$.

The proof is essentially the classical construction of the fundamental solution E for Q; we omit the details.

Corollary 6.1.8. *In the situation of Lemma 6.1.7, define*

$$\nu = \max\{m_j \mid \operatorname{Re}\lambda_j = 0\},$$

with the understanding that if no characteristic root has zero real part, then $\nu = 0$. *Assume further that* $f = \mathbb{O}(\rho^{-k})$ *for* $\rho \to +\infty$, $k - \nu > 1$. *Then*

$$u(\rho) = \mathbb{O}(\rho^{-(k-\nu)}).$$

Proof. Denote by j_ν one of the indices j where the maximum in the definition of ν is attained. All we have to do is to estimate the integral with $j = j_\nu$ in (6-66):

$$\sum_{t=1}^{m_{j_\nu}} |d_{j_\nu,t}| \left| \int_\rho^{+\infty} e^{i \operatorname{Im}\lambda_{j_\nu}(\rho - w)} (\rho - w)^{t-1} f(w)\, dw \right|.$$

Each summand above gives a contribution of the form

$$|d_{j_\nu,t}| \sum_{\alpha=0}^{t-1} C_f \binom{t-1}{\alpha} \rho^\alpha \int_\rho^{+\infty} w^{t-1-\alpha-k}\, dw,$$

for a suitable positive constant C_f. Note that by assumption, the integral is convergent and can be explicitly evaluated, yielding

$$\sum_{t=1}^{m_{j_\nu}} |d_{j_\nu,t}| \sum_{\alpha=0}^{t-1} C'_f \binom{t-1}{\alpha} \rho^{t-k},$$

for a larger constant C'_f. This concludes the proof of Corollary 6.1.8. □

Lemma 6.1.7 provides a solution of (6-64) iteratively; that is, once we have suitable estimates for u_{h-r}, $r = 0, \dots, h-1$, we can get estimates for u_h.

Proposition 6.1.9. *There exists a sequence of functions* u_h, $h \ge 0$, *solving* (6-64), *and positive constants* γ, C_u *such that if* $\rho \ge \gamma h$, *then*

$$\left|\partial_\rho^t u_h(\rho)\right| \le C_u^{h+t+1} \frac{t!}{\rho^{t+\mu h}}. \tag{6-67}$$

Proof. We are going to prove a slightly better estimate of the form

$$\left|\partial_\rho^t u_j(\rho)\right| \le \tilde{C}_u^{j+t+1} \binom{\sigma j + t - 1}{t} \frac{t!}{\rho^{t+\mu j}}, \quad \rho \ge \gamma h, \tag{6-68}$$

where $\tilde{C}_u > 0$ is a constant and σ denotes a suitable integer independent of j, t. The important quantity μ was defined in (6-56).

We argue by induction, remarking that there is nothing to prove when $h = 0$. Assume that $h \ge 1$ and that (6-68) holds for $j < h$. Since, by Lemma 6.1.7, $\partial_\rho^t u = E * \partial_\rho^t f$, we have to estimate the t-th derivative of the right-hand side of (6-64). To this end, it is enough to consider just a summand in the right-hand side of (6-64) in the region $\rho \geq \gamma h$:

$$\begin{aligned}\partial_\rho^t\big(\rho^{-\mu r} P_r(\rho, \partial_\rho) u_{h-r}(\rho)\big) &= \sum_{p=0}^{2l+\lfloor r/q\rfloor} \partial_\rho^t\big(\rho^{-\mu r}\alpha_{r,p}(\rho)\partial_\rho^p u_{h-r}(\rho)\big)\\ &= \sum_{p=0}^{2l+\lfloor r/q\rfloor}\sum_{\beta=0}^{t}\binom{t}{\beta}\partial_\rho^\beta\big(\rho^{-\mu r}\alpha_{r,p}(\rho)\big)\partial_\rho^{p+t-\beta}u_{h-r}(\rho).\end{aligned}$$

Before proceeding further, we must distinguish the contributions from terms where $p = 0$ from the other terms.

Let us first consider the terms with $p = 0$. To deal with these, we make a further distinction when $r \geq j^* + 1$ or $r \leq j^*$. We start with $r \geq j^* + 1$. Because of formula (6-58), we have to estimate

$$\begin{aligned}&\sum_{\beta=0}^{t}\binom{t}{\beta}\partial_\rho^\beta\big(\rho^{-(s_0-1)r/q}\alpha_{r,0}(\rho)\big)\partial_\rho^{t-\beta}u_{h-r}(\rho)\\ &\qquad = \sum_{\beta=0}^{t}\sum_{\nu=0}^{\beta}\binom{t}{\beta}\binom{\beta}{\nu}\partial_\rho^\nu\big(\rho^{-(s_0-1)r/q}\big)\partial_\rho^{\beta-\nu}\alpha_{r,0}(\rho)\partial_\rho^{t-\beta}u_{h-r}(\rho). \qquad (6\text{-}69)\end{aligned}$$

By (6-21), Proposition 6.1.6, and the inductive hypothesis, this quantity is estimated as follows (see (6-20) for the notation):

$$\begin{aligned}&\sum_{\beta=0}^{t}\sum_{\nu=0}^{\beta}\binom{t}{\beta}\binom{\beta}{\nu}\left|\left(-\frac{s_0-1}{q}r\right)_\nu\right|\rho^{-(s_0-1)r/q-\nu}C_\alpha^{r+\beta-\nu+1}\frac{(\beta-\nu)!}{\rho^{\beta-\nu}}\\ &\qquad\cdot\tilde{C}_u^{h-r+t-\beta+1}\binom{\sigma(h-r)+t-\beta-1}{t-\beta}\frac{(t-\beta)!}{\rho^{t-\beta+\mu(h-r)}}.\end{aligned}$$

The latter quantity can be estimated as

$$\tilde{C}_u^{h-r+t+1}C_\alpha^{r+1}\frac{t!}{\rho^{t+\mu(h-r)+(s_0-1)r/q}}\sum_{\beta=0}^{t}\tilde{C}_u^{-\beta}C_\alpha^{\beta}\binom{\sigma(h-r)+t-\beta-1}{t-\beta}\sum_{\nu=0}^{\beta}\binom{(s_0-1)r/q+\nu-1}{\nu},$$

since without loss of generality we may always choose $C_\alpha > 1$. The inner sum is computed exactly:

$$\sum_{\nu=0}^{\beta}\binom{(s_0-1)r/q+\nu-1}{\nu} = \binom{(s_0-1)r/q+\beta}{\beta}.$$

Let us examine the exponent of ρ; it is equal to $t+\mu h+\left(\frac{s_0-1}{q}-\mu\right)r$. On the other hand, if $r \geq j^*+1$, we have

$$\left(\frac{s_0-1}{q}-\mu\right)r = \left(\frac{s_0-1}{q}-\frac{1}{j^*+1}-\mu\right)r+\frac{r}{j^*+1} > 1,$$

by the definition of μ. The whole argument here is performed in the case where $(s_0-1)/q$ is not a positive integer. If it is an integer, the argument is analogous, but much simpler and more direct. The above quantity is estimated by

$$\frac{\tilde{C}_u^{h+t+1}\dfrac{C_\alpha^{r+1}}{\tilde{C}_u^r}t!}{\rho^{t+\mu h+((s_0-1)/q-\mu)r}}\sum_{\beta=0}^{t}\binom{\sigma(h-r)+t-\beta-1}{t-\beta}\binom{((s_0-1)/q)r+\beta}{\beta}$$
$$\le \tilde{C}_u^{h+t+1}\frac{C_\alpha^{r+1}}{\tilde{C}_u^r}\frac{t!}{\rho^{t+\mu h+1}}\binom{\sigma h-r(\sigma-(s_0-1)/q)+t}{t}$$
$$\le \tilde{C}_u^{h+t+1}\frac{C_\alpha^{r+1}}{\tilde{C}_u^r}\frac{t!}{\rho^{t+\mu h+1}}\binom{\sigma h-1+t}{t}.$$

For the first inequality we chose $\tilde{C}_u > C_\alpha$ and used the identity

$$\sum_{k=0}^{n}\binom{x+k}{k}\binom{y+n-k}{n-k}=\binom{x+y+n+1}{n}, \tag{6-70}$$

for $x, y \in \mathbb{R}$. In the second inequality, we chose $\sigma \ge \dfrac{s_0-1}{q}+1$.

As for the terms with $p=0$ and $1\le r\le j^*$, there is only the zero-order term of P_1 (see formulas (6-59), (6-60)), for which we have the estimate

$$\left|\partial_\rho^\nu \alpha_{1,0}(\rho)\right| \le C_\alpha^{\nu+2}\frac{\nu!}{\rho^{1+\nu}}.$$

We conclude that the following inequality holds:

$$d\sum_{r=1}^{h}\sum_{p=1}^{2l+\lfloor r/q\rfloor}\sum_{\beta=0}^{t}\binom{t}{\beta}\int_\rho^{+\infty}\partial_w^\beta\big(w^{-\mu r}\alpha_{r,p}(w)\big)\partial_w^{p+t-\beta}u_{h-r}(w)\,dw. \tag{6-71}$$

We use Corollary 6.1.8. Noting that $p+t-\beta\ge 1$, we may integrate by parts, decreasing by one the number of derivatives landing on u_{h-r} and increasing by one the number of derivatives landing on the coefficients. The above quantity then becomes

$$-d\sum_{r=1}^{h}\sum_{p=1}^{2l+\lfloor r/q\rfloor}\sum_{\beta=0}^{t}\binom{t}{\beta}\partial_\rho^\beta\big(\rho^{-\mu r}\alpha_{r,p}(\rho)\big)\partial_\rho^{p+t-\beta-1}u_{h-r}(\rho)$$
$$-d\sum_{r=1}^{h}\sum_{p=1}^{2l+\lfloor r/q\rfloor}\sum_{\beta=0}^{t}\binom{t}{\beta}\cdot\int_\rho^{+\infty}\partial_w^{\beta+1}\big(w^{-\mu r}\alpha_{r,p}(w)\big)\partial_w^{p+t-\beta-1}u_{h-r}(w)\,dw.$$

The above quantities sport the same behavior with respect to the variable ρ, since even though the order of the derivative on the coefficients of the second term is larger by one, the integration, as we shall see, levels that difference. On the other hand, estimating the coefficients is quite analogous, so that we consider only the second term and leave the necessary simple adjustments for the first to the reader.

Now using (6-21) and (6-49) with $\theta = 1$, we get, if $\rho \geq \gamma h$, $\gamma \geq c_1$,

$$\begin{aligned}\left|\partial_\rho^{\beta+1}(\rho^{-\mu r}\alpha_{r,p}(\rho))\right| &\leq \sum_{i=0}^{\beta+1}\binom{\beta+1}{i}(-1)^i(-\mu r)_i\rho^{-\mu r-i}\cdot C_\alpha^{1+r+\beta+1-i}\frac{(\beta+1-i)!}{\rho^{\beta+1-i}}\\ &\leq C_\alpha^{1+r+\beta+1}\frac{(\beta+1)!}{\rho^{\mu r+\beta+1}}\sum_{i=0}^{\beta+1}\binom{\mu r+i-1}{i}\\ &= C_\alpha^{1+r+\beta+1}\binom{\mu r+\beta+1}{\beta+1}\frac{(\beta+1)!}{\rho^{\mu r+\beta+1}}.\end{aligned}$$

Hence, by the inductive hypothesis, the second term above is estimated by

$$|d|\sum_{r=1}^{h}\sum_{p=1}^{2l+\lfloor r/q\rfloor}\sum_{\beta=0}^{t}C_\alpha^{1+r+\beta+1}\tilde{C}_u^{h-r+p+t-\beta}\binom{t}{\beta}\binom{\mu r+\beta+1}{\beta+1}\\ \cdot\binom{\sigma(h-r)+p+t-\beta-2}{p+t-\beta-1}\int_\rho^{+\infty}\frac{(\beta+1)!}{w^{\mu r+\beta+1}}\frac{(p+t-\beta-1)!}{w^{\mu(h-r)+p+t-\beta-1}}\,dw.$$

Now the integral is easily computed, yielding $(\mu h+t+p-1)^{-1}/\rho^{\mu h+t+p-1}$. Note that since $p \geq 1$ and $h \geq 1$, there is no problem about its convergence. We thus obtain the bound

$$\tilde{C}_u^{h+t+1}\frac{t!}{\rho^{\mu h+t}}|d|\sum_{r=1}^{h}\sum_{p=1}^{2l+\lfloor r/q\rfloor}\sum_{\beta=0}^{t}\frac{C_\alpha^{2+r+\beta}\tilde{C}_u^{p-1}}{\tilde{C}_u^{r+\beta}}\big(\mu h+t+p-1\big)^{-1}\\ \cdot\frac{(\mu r+\beta+1)\dots(\mu r+1)}{\beta!(t-\beta)!}\frac{\big(\sigma(h-r)+p+t-\beta-2\big)!}{(\sigma(h-r)-1)!}\frac{1}{\rho^{p-1}}.$$

Since $\rho \geq \gamma h$ and $1 \leq p \leq 2l+\lfloor h/q\rfloor$, we have $\sigma(h-r)+p-2 \leq \sigma h+2l+(1/q)h \leq \gamma_1 h$, where γ_1 is a positive constant, $\gamma_1 \geq \sigma+(1/q)+2l$. We obtain that $\rho^{-1} \leq \gamma^{-1}h^{-1} \leq \gamma^{-1}\gamma_1(\sigma(h-r)+p-2)^{-1}$. We point out explicitly that $\gamma^{-1}\gamma_1$ can be chosen very small if γ is chosen large enough. Let us denote this constant by δ, where it is understood that δ is small provided the constant γ is chosen large enough. The above expression is then bounded by

$$\tilde{C}_u^{h+t+1}\frac{t!}{\rho^{\mu h+t}}|d|\sum_{r=1}^{h}\sum_{p=1}^{2l+\lfloor r/q\rfloor}\sum_{\beta=0}^{t}\frac{C_\alpha^{2+r+\beta}\tilde{C}_u^{p-1}}{\tilde{C}_u^{r+\beta}}\frac{\mu r+\beta+1}{\mu h+t+p-1}\\ \cdot\delta^{p-1}\binom{\mu r+\beta}{\beta}\binom{\sigma(h-r)+t-\beta+p-2}{t-\beta}.$$

Now $\beta \leq t$ and $p-1 \geq 0$ imply that the fraction at the end of the top line above is bounded by 2, so that the whole quantity is estimated by

$$\tilde{C}_u^{h+t+1}\frac{t!}{\rho^{\mu h+t}}2\,|d|\sum_{r=1}^{h}\sum_{p=1}^{2l+\lfloor r/q\rfloor}\frac{C_\alpha^{2+r}\tilde{C}_u^{p-1}}{\tilde{C}_u^{r}}\delta^{p-1}\cdot\sum_{\beta=0}^{t}\frac{C_\alpha^\beta}{\tilde{C}_u^\beta}\binom{\mu r+\beta}{\beta}\binom{\sigma(h-r)+t-\beta+p-2}{t-\beta}.$$

Since we already chose $\tilde{C}_u \ge 4C_\alpha^2$, the ratio in the third sum above is less than $\frac{1}{4}$, and the sum over β involving only binomial coefficients is computed by (6-70), yielding

$$\tilde{C}_u^{h+t+1}\frac{t!}{\rho^{\mu h+t}}\frac{|d|}{2}\sum_{r=1}^{h}\sum_{p=1}^{2l+\lfloor r/q\rfloor}\frac{C_\alpha^{2+r}\tilde{C}_u^{p-1}}{\tilde{C}_u^r}\delta^{p-1}\cdot\binom{\mu r+\sigma(h-r)+t+p-1}{t}.$$

Observe now that there is a positive constant $\tilde{c}$ such that $p \le \tilde{c}r$. Therefore $\mu r+\sigma(h-r)+t+p-1 \le \sigma h+t+r(\mu-\sigma+\tilde{c})-1 \le \sigma h+t-1$, provided σ is chosen large in such a way that $\sigma > \mu+\tilde{c}$. This is always possible and is actually the only constraint on σ. By a well known property of binomial coefficients (with positive real numerators), we then obtain the bound

$$\begin{aligned}\tilde{C}_u^{h+t+1}\binom{\sigma h+t-1}{t}\frac{t!}{\rho^{\mu h+t}}\frac{|d|}{2}\sum_{r=1}^{h}\sum_{p=1}^{2l+\lfloor r/q\rfloor}\frac{C_\alpha^{2+r}\tilde{C}_u^{p-1}}{\tilde{C}_u^r}\delta^{p-1}&\\ =\tilde{C}_u^{h+t+1}\binom{\sigma h+t-1}{t}\frac{t!}{\rho^{\mu h+t}}\frac{|d|}{2}\sum_{r=1}^{h}\frac{C_\alpha^{2+r}}{\tilde{C}_u^r}\sum_{p=1}^{2l+\lfloor r/q\rfloor}\delta^{p-1}\tilde{C}_u^{p-1}.&\end{aligned}$$

The inner sum is easily evaluated provided, for example, $\delta \le \tilde{C}_u^{-1}/2$. This is always possible and amounts to choosing γ large. The contribution from that sum is thus ≤ 2. As for the outer sum, if we choose $\tilde{C}_u$ in such a way that

$$\tilde{C}_u \ge C_\alpha^3(1+3|d|),$$

which depends only on the problem data, we obtain the final bound

$$\tfrac{1}{3}\tilde{C}_u^{h+t+1}\binom{\sigma h+t-1}{t}\frac{t!}{\rho^{\mu h+t}}.$$

The same bound is obtained for the term without the integral.

This finishes the proof of inequality (6-68). Inequality (6-67) is an easy consequence. □

Proposition 6.1.9 guarantees that we can construct a formal solution to the equation $P(\rho,\partial_\rho)u(\rho)=0$ in (6-18) and thus a formal solution $A(u)$ for

$$\Lambda(t,D_t)A(u)(t)=0. \tag{6-72}$$

In the next subsection we plan to construct from $A(u)$ a true solution; this will only yield a solution of (6-72) with a nonzero right-hand side which will be negligible in an important sense.

6.2. *True solution and the end of the proof.* To establish the notation, we state the result of the previous subsection:

Theorem 6.2.1. *There is a formal solution $A(u)(t)$ of* (6-72) *of the form*

$$A(u)(t)=\int_0^{+\infty}e^{it\rho^{s_0}}e^{i\Phi(\rho)}u(\rho)\,d\rho, \tag{6-73}$$

satisfying the following conditions:

(1) *The phase function* Φ *is of the form*

$$\Phi(\rho)=\sum_{j=0}^{M_0}\varphi_j\rho^{1-((s_0-1)/q)j}+\varphi_{-1}\log\rho,\qquad M_0=\left\lfloor\frac{1}{s_0-1}\right\rfloor,\tag{6-74}$$

with $\varphi_j\in\mathbb{C}$, $j=-1,0,\dots,M_0$, $\operatorname{Im}\varphi_0>0$.

(2) *The function* u *has the form* $u(\rho)=\sum_{h=0}^{\infty}u_h(\rho)$, *where* $u_0(\rho)\equiv 1$ *and* (*compare* (6-61))

$$P_0(\partial_\rho)u_h(\rho)+\sum_{r=1}^{h}\rho^{-\mu r}P_r(\rho,\partial_\rho)u_{h-r}(\rho)=0,\quad h=1,2,\dots.\tag{6-75}$$

Moreover, u_h *satisfies the estimate* (6-67); *that is*, *if* $\rho\ge\gamma h$, *for* γ *large enough*,

$$\left|\partial_\rho^t u_h(\rho)\right|\le C_u^{h+t+1}\frac{t!}{\rho^{t+\mu h}}.\tag{6-76}$$

As a consequence of the construction, $A(u)$ *formally satisfies*

$$\begin{aligned}\Lambda(t,D_t)A(u)(t)&=\int_0^{+\infty}e^{it\rho^{s_0}}P(\rho,\partial_\rho)\big(e^{i\Phi(\rho)}u(\rho)\big)\,d\rho\\&=\int_0^{+\infty}e^{it\rho^{s_0}}e^{i\Phi(\rho)}\big(e^{-i\Phi(\rho)}P(\rho,\partial_\rho)e^{i\Phi(\rho)}\big)u(\rho)\,d\rho\\&=\int_0^{+\infty}e^{it\rho^{s_0}}e^{i\Phi(\rho)}\sum_{r=0}^{\infty}\rho^{-\mu r}P_r(\rho,\partial_\rho)u(\rho)\,d\rho\\&=\int_0^{+\infty}e^{it\rho^{s_0}}e^{i\Phi(\rho)}\sum_{h=0}^{\infty}\sum_{r=0}^{h}\rho^{-\mu r}P_r(\rho,\partial_\rho)u_{h-r}(\rho)\,d\rho=0.\end{aligned}\tag{6-77}$$

Let $\omega_j\in G^s(\mathbb{R})$, $j=0,1,2,\dots$, with $1<s<s_0$ to be specified later, be the cutoffs introduced in Lemma 5.4, defined in $\mathbb{R}$. We assume from the beginning that the constant $2R$ in Lemma 5.4 is larger than γ, the latter being the constant in the second item of the theorem above. Define

$$v(\rho)=\sum_{h=0}^{\infty}\omega_h(\rho)u_h(\rho).\tag{6-78}$$

Trivially, $v\in G^s(\mathbb{R})$. Moreover:

Lemma 6.2.2. *The function* v *in* (6-78) *satisfies the estimate*

$$\begin{cases}\left|\partial_\rho^\alpha v(\rho)\right|\le C_v^{\alpha+1}\dfrac{\alpha!^s}{\rho^\alpha} & \textit{for every } \rho\ge 2R,\\ v\equiv 0 & \textit{if } \rho\le 2R.\end{cases}\tag{6-79}$$

Proof. Let us start by estimating $\partial_\rho^\beta\omega_h\partial_\rho^{\alpha-\beta}u_h$. For the first factor we have

$$\left|\partial_\rho^\beta\omega_h(\rho)\right|\le(RC_\omega)^{\beta+1}\frac{\beta!^s}{\rho^\beta}\quad\text{for every }\beta.$$

For the second factor, by (6-76) we have, provided $\rho \geq \gamma h$, which is implied by $\rho \in \operatorname{supp} \omega_h$,

$$\left|\partial_\rho^{\alpha-\beta} u_h(\rho)\right| \leq C_u^{h+\alpha-\beta+1} \frac{(\alpha-\beta)!}{\rho^{\alpha-\beta+\mu h}} \leq C_u^{h+\alpha-\beta+1} \frac{(\alpha-\beta)!}{\rho^{\alpha-\beta}} \big(\gamma(h+1)\big)^{-\mu h}.$$

Putting together the estimates, we obtain

$$\left|\partial_\rho^\alpha v(\rho)\right| \leq \tilde{C}_v^{\alpha+1} \frac{\alpha!^s}{\rho^\alpha} \sum_{h=0}^{\infty} \big(\gamma(h+1)\big)^{-\mu h}.$$

This implies the assertion. □

Definition 6.2.3. Let Ω be an open subset of $\mathbb{R}$. We define the class $\mathcal{B}^s(\Omega)$ (of Beurling type functions on Ω) as the set of all smooth functions $u(x)$ defined in Ω and such that for every $\varepsilon > 0$ and for every $K \Subset \Omega$ compact, there exists a positive constant $C = C(\varepsilon, K)$ such that

$$|\partial_x^\alpha u(x)| \leq C \varepsilon^\alpha \alpha!^s, \tag{6-80}$$

for every $x \in K$ and every α.

We want to show that $\Lambda(t, D_t) A(v) = g$, where $g \neq 0$ and $g \in \mathcal{B}^{s_0}(\mathbb{R})$. First we show that far from the origin, $A(v)$ has a better regularity than $G^{s_0}(\mathbb{R})$. The following lemma is straightforward:

Lemma 6.2.4. *We have $G^\sigma(\Omega) \subset \mathcal{B}^t(\Omega)$ for every $t > \sigma$.*

Lemma 6.2.5. *Let s be the Gevrey regularity of the cutoff functions ω_j in* (6-78). *Let $\delta > 0$. Then $A(v) \in \mathcal{B}^\sigma(\{x \mid |x| > \delta\})$, with $s \leq \sigma \leq s_0$.*

Proof. We actually prove that $A(v) \in G^s(\{x \mid |x| > \delta\})$. We have

$$D_t^\alpha A(v)(t) = \int_0^{+\infty} e^{it\rho^{s_0}} \rho^{s_0 \alpha} e^{i\Phi(\rho)} v(\rho)\, d\rho.$$

We observe that $(s_0 t \rho^{s_0-1})^{-1} D_\rho e^{i\rho^{s_0} t} = e^{i\rho^{s_0} t}$. Therefore,

$$\begin{aligned} D_t^\alpha A(v)(t) &= \Big(\frac{1}{t}\Big)^j \int_0^{+\infty} e^{it\rho^{s_0}} \Big(-D_\rho \frac{1}{s_0 \rho^{s_0-1}}\Big)^j \big(\rho^{s_0\alpha} e^{i\Phi(\rho)} v(\rho)\big)\, d\rho \\ &= \Big(\frac{1}{t}\Big)^j \int_0^{+\infty} e^{it\rho^{s_0}} \sum_{h=0}^{j} \gamma_{jh} \frac{1}{\rho^{s_0 j - h}} \partial_\rho^h \big(\rho^{s_0\alpha} e^{i\Phi(\rho)} v(\rho)\big)\, d\rho, \end{aligned}$$

by Lemma 6.1.1. This quantity is rewritten as

$$\Big(\frac{1}{t}\Big)^j \int_0^{+\infty} e^{it\rho^{s_0}} \sum_{h=0}^{j} \sum_{p+q=0}^{h} \frac{h!}{p!\,q!\,(h-p-q)!} \gamma_{jh} \frac{1}{\rho^{s_0 j-h}} \cdot \partial_\rho^p(\rho^{s_0\alpha}) \partial_\rho^q(v(\rho)) \partial_\rho^{h-p-q}(e^{i\Phi(\rho)})\, d\rho.$$

By the Faà di Bruno formula,

$$\partial_\rho^n e^{i\Phi} = e^{i\Phi} \sum_{k=1}^{n} i^k \sum_{\substack{k_1,\dots,k_n \\ \sum_i k_i = k \\ \sum_i i k_i = n}} \frac{n!}{k_1! \cdots k_n!} \prod_{p=1}^{n} \Big(\frac{\Phi_\rho^{(p-1)}}{p!}\Big)^{k_p},$$

using Lemma 6.1.5 and the estimate (6-42) we obtain for $\rho \geq 2R$, with $\lambda > 0$,

$$|\partial_\rho^n e^{i\Phi}| \leq |e^{i\Phi}| \sum_{k=1}^{n} \sum_{\substack{k_1,\dots,k_n \\ \sum_i k_i = k \\ \sum_i i k_i = n}} \frac{n!}{k_1!\cdots k_n!} C_\Phi'^k \rho^{-(n-k)} \leq C^n e^{-\lambda\rho} \sum_{k=1}^{n} (n-k)!\,\rho^{-(n-k)}, \tag{6-81}$$

where we argued as in (6-53), (6-34) and (6-39). Thus we have if $|t| \geq \delta$,

$$\begin{aligned}\left|D_t^\alpha A(v)(t)\right| \leq \delta^{-j} \int_0^{+\infty} e^{-\lambda\rho} \sum_{h=0}^{j} \sum_{p+q=0}^{h} \sum_{k=1}^{h-p-q} \frac{h!}{p!\,q!\,(h-p-q)!} \\ \cdot C_\gamma^{j+h} (j-h)! \frac{1}{\rho^{s_0 j - h}} (s_0\alpha)_p \rho^{s_0\alpha - p} C_v^{1+q} \frac{q!^s}{\rho^q} \cdot C^{h-p-q} (h-p-q-k)!\,\rho^{-(h-p-q-k)}\, d\rho,\end{aligned}$$

by (6-17), (6-21). Choosing $j = \alpha$, we then obtain

$$\begin{aligned}&\left|D_t^\alpha A(v)(t)\right| \\ &\leq C_v \Big(\frac{C C_\gamma^2 C_v}{\delta}\Big)^\alpha \sum_{h=0}^{\alpha} \sum_{p+q=0}^{h} \sum_{k=1}^{h-p-q} \frac{h!}{(h-p-q)!} \cdot (\alpha-h)! \binom{s_0\alpha}{p} q!^{s-1} (h-p-q-k)! \int_0^{+\infty} e^{-\lambda\rho} \rho^k \, d\rho.\end{aligned}$$

The integral above is equal to $\lambda^{-(k+1)} k!$, and there is a positive constant C_1 such that $\binom{s_0\alpha}{p} \leq C_1^\alpha$. Eventually we get

$$\left|D_t^\alpha A(v)(t)\right| \leq \frac{C_v}{\lambda} \Big(\frac{\max\{1,\lambda^{-1}\} C C_1 C_\gamma^2 C_v}{\delta}\Big)^\alpha \alpha! \sum_{h=0}^{\alpha} \sum_{p+q=0}^{h} \binom{\alpha}{h}^{-1} q!^{s-1} \sum_{k=1}^{h-p-q} \binom{h-p-q}{k}^{-1}.$$

We may therefore find a positive constant $\tilde{C}$ such that $\tilde{C}^\alpha \geq \alpha^4$ and deduce that

$$\left|D_t^\alpha A(v)(t)\right| \leq \frac{C_v}{\lambda} \Big(\frac{\max\{1,\lambda^{-1}\} \tilde{C} C C_1 C_\gamma^2 C_v}{\delta}\Big)^\alpha \alpha!^s.$$

This proves the statement. ☐

Next we prove a key result of this section: the regularity of $\Lambda(t, D_t)A(v)$. First of all, we remark that we need to sum the asymptotic expansion (3-33) modulo some reasonably regularizing term. Note also that the symbols in the asymptotic expansion of Λ are real analytic symbols:

$$\Lambda(t,\tau) \sim \sum_{j=0}^{j_0} \Lambda_{-jq}(t,\tau) + \sum_{s\geq 0} \big(\Lambda_{-2k-sq}(t,\tau) + \Lambda_{-(j_0+1)q-sq}(t,\tau)\big). \tag{6-82}$$

We recall that Λ_m in the above expression is (positively) homogeneous with respect to τ of degree $2/q + m/q$. To sum (6-82), we use the cutoff functions constructed in Lemma 5.3; we agree that they are in $G^t(\mathbb{R})$ with $t < s_0$ to be specified later. It is then evident that the error appearing when summing “à la Borel” the asymptotic expansion of Λ will be G^t-regularizing and hence in $\mathcal{B}^{s_0}(\mathbb{R})$.

By (6-2), we may ignore an elliptic factor and rewrite Λ, with a slight difference in the meaning of the coefficients, as

$$\Lambda(t,\tau)\tau^{-2/q+2k/q} \sim \sum_{h=0}^{\infty} a_h(t)\tau^{2k/q-h} + \sum_{h=0}^{\infty} b_h(t)\tau^{-h}, \tag{6-83}$$

where without loss of generality $\tau > 0$.

We also recall at this time that the first sum above gives rise to the $Q_{\mathcal{A},r}$ in (6-45), while the second contributes to the $Q_{\mathcal{B},r}$ in (6-46). At this point we are not interested in the particular properties of the coefficients, such as, for example, the vanishing of order $2l$ of a_0 and the nonvanishing of b_0 at the origin. These properties have already played their role in the constructions above.

Abusing our notation a bit, we call the operator in (6-83) again Λ.

Proposition 6.2.6. *Let v be the function defined in* (6-78) *using cutoff functions in $G^{t'}$ and let Λ be the operator defined by the asymptotic expansion in* (6-83) *using cutoff functions in $G^{t''}$* (*see Lemmas 5.3 and 5.4*). *Then, for a suitable choice of t' and t'', we have*

$$\Lambda(t, D_t)A(v)(t) \in \mathcal{B}^{s_0}(\mathbb{R}). \tag{6-84}$$

Proof. It is evident that it will be enough if we argue on just one of the asymptotic expansions in (6-83). At a certain point of the proof though, we have to partially reassemble the operator P_Φ in (6-61), and there we use the argument also for the other expansion. For the sake of simplicity, we argue on the second sum in (6-83).

Due to Lemma 6.2.5, it suffices to show that for every $\varepsilon > 0$, there is a neighborhood of the origin, U_ε, such that $|\partial_t^\alpha(\Lambda A(v))(t)| \le C_\varepsilon \varepsilon^\alpha \alpha!^{s_0}$ for $t \in U_\varepsilon$.

Actually we need to estimate a derivative of $\Lambda(t, D_t)A(v)(t)$, say

$$D_t^\alpha \Lambda(t, D_t)A(v)(t).$$

The latter can be written as

$$D_t^\alpha \sum_{j=0}^{\infty} b_j(t)A\big(\omega_j(\rho^{s_0})\rho^{-js_0}v(\rho)\big),$$

keeping in mind the form of $A(v)$, with v given by (6-78),

$$A(v)(t) = \int_0^{+\infty} e^{i\rho^{s_0}t} e^{i\Phi(\rho)} v(\rho)\, d\rho. \tag{6-85}$$

Let now N be a natural number and consider

$$D_t^\alpha \sum_{j=N}^{\infty} b_j(t)A\big(\omega_j(\rho^{s_0})\rho^{-js_0}v(\rho)\big) = \sum_{j=N}^{\infty}\sum_{p=0}^{\alpha}\binom{\alpha}{p} D_t^p b_j(t)A\big(\omega_j(\rho^{s_0})\rho^{-js_0+s_0(\alpha-p)}v(\rho)\big). \tag{6-86}$$

Applying the definition (6-4) of an analytic symbol as well as the estimates (5-6) for the cutoff functions defining Λ, we have that the latter quantity is estimated by

$$\sum_{j=N}^{\infty}\sum_{p=0}^{\alpha}\binom{\alpha}{p}C^{p+j+1}p!\,j!\,(2R)^{-j+\alpha-p}\cdot(j+1)^{t''(-j+\alpha-p)}\big|A(\omega_j(\rho^{s_0})v(\rho))\big|$$
$$\leq C_1\sum_{j=N}^{\infty}\sum_{p=0}^{\alpha}\binom{\alpha}{p}C^{p+j+1}p!\,(2R)^{-j+\alpha-p}(j+1)^{j+t''(-j+\alpha-p)},$$

provided $-j+\alpha\leq 0$, which is obviously implied by choosing $N\geq\alpha$. In order to handle the power of j above, we make a stronger demand on N, namely,

$$N=\theta_N\alpha\geq\Big\lfloor\alpha\frac{t''}{t''-1}\Big\rfloor+1, \tag{6-87}$$

where θ_N is a suitable constant on which we may impose further constraints in the following, independent of α.

Then $j+t''(\alpha-j)\leq 0$, and the above sum can be bounded by

$$C_1(2R)^{\alpha}\alpha!\sum_{j=N}^{\infty}C^{j+1}(2R)^{-j}\sum_{p=0}^{\alpha}\Big(\frac{C}{2R}\Big)^{p}\leq\tilde{C}^{\alpha+1}\alpha!\,, \tag{6-88}$$

for a suitable positive constant $\tilde{C}$, provided $2R>C$. Thus this part of $\Lambda(t,D_t)A(v)(t)$ exhibits an analytic behavior and therefore belongs to any $\mathscr{B}^s$, with $s>1$.

Next we must estimate the finite sum

$$D_t^{\alpha}\sum_{j=0}^{N-1}b_j(t)A\big(\omega_j(\rho^{s_0})\rho^{-js_0}v(\rho)\big)(t),$$

with N defined by (6-87). To do this we write the coefficients b_j as a sum of a polynomial in the variable t and a real analytic function vanishing of high order at $t=0$ and estimate both contributions. Let us start with the remainder terms in the expansion of b_j.

Thus we have to estimate the sum

$$D_t^{\alpha}\sum_{j=0}^{N-1}t^{M}\sum_{i=0}^{\infty}b_{j,i+M}t^{i}A\big(\omega_j(\rho^{s_0})\rho^{-js_0}v(\rho)\big)(t), \tag{6-89}$$

where M is a large integer to be fixed later. The significant part of the estimate is that where the t-derivatives land on A, since otherwise the derivatives landing on the powers of t give analytic type estimates and hence better estimates:

$$\sum_{j=0}^{N-1}t^{M}\sum_{i=0}^{\infty}b_{j,i+M}t^{i}D_t^{\alpha}A\big(\omega_j(\rho^{s_0})\rho^{-js_0}v(\rho)\big)(t).$$

By (6-4), we have $|b_{j,i+M}| \le C^{i+M+j+1} j!$, so that if $|t| \le \delta$, the absolute value of the above quantity is bounded by

$$\sum_{j=0}^{N-1} \delta^M C^{M+j+1} j! \sum_{i=0}^{\infty} (C\delta)^i \left| A(\omega_j(\rho^{s_0})\rho^{-js_0+s_0\alpha} v(\rho))(t) \right|,$$

since on the support of ω_j, by Lemma 5.4, $\rho^{s_0} \ge R(j+1)^{t'}$, we obtain that $\rho^{-s_0 j} \le R^{-j}(j+1)^{-t'j}$. Furthermore,

$$\begin{aligned} \left| A\big(\omega_j(\rho^{s_0})\rho^{s_0\alpha} v(\rho)\big)(t) \right| &\le \int_0^{+\infty} |e^{i\Phi(\rho)}||v(\rho)|\rho^{s_0\alpha}\, d\rho \\ &\le C_A \int_0^{+\infty} e^{-\lambda\rho}\rho^{s_0\alpha}\, d\rho = C_A \lambda^{-(s_0\alpha+1)}\Gamma(s_0\alpha+1) \le C_A'^{\alpha+1}\alpha!^{s_0}. \end{aligned}$$

Hence (6-89) is bounded by

$$C_A'^{\alpha+1}\alpha!^{s_0} C^{M+1}\delta^M \sum_{j=0}^{N-1} \left(\frac{C}{R}\right)^j \sum_{i=0}^{\infty} (C\delta)^i (j+1)^{(1-t')j}.$$

Choose

$$M = \theta_M \alpha, \quad \theta_M \ge 1. \tag{6-90}$$

We may impose further conditions on θ_M provided they depend only on the problem data, that is, θ_M does not depend on α. Moreover, let $R > C$ and $C_A' C^{\theta_M}\delta^{\theta_M} < \varepsilon$, and we have the estimate

$$C_A' C \varepsilon^\alpha \alpha!^{s_0} \sum_{j=0}^{\infty} \left(\frac{C}{R}\right)^j \sum_{i=0}^{\infty} (C\delta)^i \le \tilde{C}_A \varepsilon^\alpha \alpha!^{s_0}. \tag{6-91}$$

This concludes the proof for the term (6-89).

The next step is to estimate the term

$$D_t^\alpha \sum_{j=0}^{N-1} \sum_{i=0}^{M-1} b_{j,i} t^i A\big(\omega_j(\rho^{s_0})\rho^{-js_0} v(\rho)\big)(t). \tag{6-92}$$

The latter can be written as

$$D_t^\alpha \sum_{j=0}^{N-1} \sum_{r=0}^{M-1} b_{j,r} \cdot \int_0^{+\infty} e^{it\rho^{s_0}} \left(-\partial \frac{1}{i s_0 \rho^{s_0-1}} \right)^r \big(e^{i\Phi(\rho)} \omega_j(\rho^{s_0})\rho^{-js_0} v(\rho) \big)\, d\rho.$$

By Lemma 6.1.1, we rewrite the above expression as

$$\sum_{j=0}^{N-1} \sum_{r=0}^{M-1} b_{j,r} \cdot \int_0^{+\infty} e^{it\rho^{s_0}} \rho^{s_0\alpha} \sum_{h=0}^{r} \gamma_{rh} \frac{1}{\rho^{s_0 r-h}} \partial_\rho^h \big(e^{i\Phi(\rho)} \omega_j(\rho^{s_0})\rho^{-js_0} v(\rho) \big)\, d\rho. \tag{6-93}$$

Let us compute $\partial_\rho^h \big(e^{i\Phi(\rho)} \omega_j(\rho^{s_0})\rho^{-js_0} v(\rho) \big)$. This is equal to

$$\sum_{\sum \beta_i = h} \frac{h!}{\beta_1!\,\beta_2!\,\beta_3!\,\beta_4!} \partial_\rho^{\beta_1} e^{i\Phi} \partial_\rho^{\beta_2} \omega_j(\rho^{s_0}) \partial_\rho^{\beta_3} \rho^{-js_0} \partial_\rho^{\beta_4} v(\rho).$$

By (6-81), (6-79) we have

$$\left|\partial_\rho^{\beta_1} e^{i\Phi(\rho)}\right| \le C_e^{\beta_1} e^{-\lambda\rho} \sum_{m=1}^{\beta_1} (\beta_1 - m)!\, \rho^{-(\beta_1 - m)}, \quad \left|\partial_\rho^{\beta_4} v(\rho)\right| \le C_v^{\beta_4+1} \frac{\beta_4!^{t'}}{\rho^{\beta_4}},$$

and finally, using the Faà di Bruno formula,

$$\partial_\rho^{\beta_2} \omega_j(\rho^{s_0}) = \sum_{k=1}^{\beta_2} \omega_j^{(k)}(\rho^{s_0}) \sum_{\substack{\sum k_i = k \\ \sum i k_i = \beta_2}} \frac{\beta_2!}{k_1! \cdots k_{\beta_2}!} \prod_{l=1}^{\beta_2} \left(\binom{s_0}{l} \rho^{s_0 - l} \right)^{k_l}.$$

By (5-6), arguing as we did to prove (6-53), the absolute value of the above quantity is estimated by

$$\left|\partial_\rho^{\beta_2} \omega_j(\rho^{s_0})\right| \le C_2^{\beta_2+1} \frac{\beta_2!^{t''}}{\rho^{\beta_2}},$$

where C_2 is a suitable positive constant. Let us now consider (6-93). It is natural to consider (6-93) in the two regions $\rho \le 4RN^{t''}$ and $\rho \ge 4RN^{t''}$. We want to estimate (6-93) in the first region. We remark that on the support of ω_j in this region, we have $2R(j+1)^{t''} \le \rho^{s_0} \le 4RN^{t''}$. Thus the absolute value of (6-93) in the latter region is bounded by

$$\begin{aligned} \sum_{j=0}^{N-1} \sum_{r=0}^{M-1} \sum_{h=0}^{r} \sum_{\sum \beta_i = h} C_e^{\beta_1} \sum_{m=1}^{\beta_1} (\beta_1 - m)!\, C_\gamma^{r+h} C_2^{\beta_2+1} C_v^{\beta_4+1} \\ \cdot (r-h)!\, \frac{h!}{\beta_1!\,\beta_2!\,\beta_3!\,\beta_4!} \beta_2!^{t''} \binom{s_0 j + \beta_3 - 1}{\beta_3} \beta_3!\, \beta_4!^{t'} |b_{j,r}| \\ \cdot \int_{2R(j+1)^{t''} \le \rho^{s_0} \le 4RN^{t''}} e^{-\lambda\rho} \rho^{s_0\alpha} \frac{1}{\rho^{\beta_2}} \frac{1}{\rho^{s_0 r - h}} \frac{1}{\rho^{\beta_1 - m}} \frac{1}{\rho^{s_0 j + \beta_3}} \frac{1}{\rho^{\beta_4}}\, d\rho, \end{aligned}$$

which in turn is bounded by

$$\begin{aligned} \sum_{j=0}^{N-1} \sum_{r=0}^{M-1} \sum_{h=0}^{r} \sum_{\sum \beta_i = h} C_e^{\beta_1} \sum_{m=1}^{\beta_1} (\beta_1 - m)!\, C_\gamma^{r+h} C_2^{\beta_2+1} C_v^{\beta_4+1} C_3^{s_0 j + \beta_3} 4^h \\ \cdot (r-h)!\, \beta_2!^{t''} \beta_3!\, \beta_4!^{t'} C_b^{j+r+1} j!\, (4R)^{(\alpha - r)_+} N^{t''(\alpha - r)_+} (2R)^{-j} (j+1)^{-t'' j} \\ \cdot \int_{2R(j+1)^{t''} \le \rho^{s_0} \le 4RN^{t''}} e^{-\lambda\rho} \rho^m\, d\rho. \end{aligned}$$

The integral above is bounded by $\lambda^{-(m+1)} m!$, so that it is clear that there exist positive constants C, C_λ such that the above quantity is bounded by

$$
\begin{aligned}
\sum_{j=0}^{N-1}\sum_{r=0}^{M-1}\sum_{h=0}^{r}\sum_{\sum\beta_i=h} &(2C_e)^{\beta_1}\beta_1!\, C_\gamma^{r+h} C_2^{\beta_2+1} C_v^{\beta_4+1} C_3^{s_0 j+\beta_3} 4^h C_\lambda^{\beta_1} C_b^{j+r+1} \\
&\cdot (4R)^{(\alpha-r)_+}(r-h)!\, \beta_2!^{t''}\beta_3!\, \beta_4!^{t'} N^{t''(\alpha-r)_+}(2R)^{-j} \\
&\le \sum_{j=0}^{N-1}\Big(\frac{C_3^{s_0}C_b}{2R}\Big)^j \sum_{r=0}^{M-1} C_\gamma^r C_b^{r+1}(4R)^{(\alpha-r)_+} N^{t''(\alpha-r)_+}\sum_{h=0}^{r} L_1^{h+1} h!^{t''}(r-h)! \\
&\le (4R)^\alpha \sum_{j=0}^{N-1}\Big(\frac{C_3^{s_0}C_b}{2R}\Big)^j \sum_{r=0}^{M-1} C_\gamma^r C_b^{r+1} N^{t''(\alpha-r)_+} L_2^{r+1} r!^{t''} \\
&\le L_3^{\alpha+1}\alpha!^{t''}.
\end{aligned}
$$

Here we used the fact that $\rho^{s_0} \le 4RN^{t''}$, as well as (6-87). Moreover,

$$N^{t''(\alpha-r)} r!^{t''} \le (\theta_N\alpha)^{t''(\alpha-r)} r^{t''r} \le (\theta_N\alpha)^{t''(\alpha-r)}(\theta_M\alpha)^{t''r} \le \max\{\theta_N,\theta_M\}^{t''\alpha}\alpha^{t''\alpha}.$$

It is also clear in the above deduction that all the constants involved except R depend on the data and are hence fixed; moreover, R can be taken large enough so that $C_3^{s_0}C_b/R < 1$. We would like to emphasize at this stage that in performing the above estimate, we assumed that $t'' > t'$. This is no restriction since the only constraint on t' and t'' is that they are positive numbers larger than one.

If

$$1 < t' < t'' < s_0, \tag{6-94}$$

we therefore obtain that the term (6-92) in the region $\rho^{s_0} \le 4RN^{t''}$ gives rise to a function of class $\mathcal{B}^{s_0}$.

We must now discuss the term (6-93) in the complementary region: $\rho^{s_0} \ge 4RN^{t''}$.

$$
\begin{aligned}
&\sum_{j=0}^{N-1}\sum_{r=0}^{M-1} b_{j,r}\int_{\rho^{s_0}\ge 4RN^{t''}} e^{it\rho^{s_0}}\rho^{s_0\alpha}\cdot\sum_{h=0}^{r}\gamma_{rh}\frac{1}{\rho^{s_0 r-h}}\partial_\rho^h\big(e^{i\Phi(\rho)}\rho^{-js_0}v(\rho)\big)\,d\rho \\
&\quad=\sum_{j=0}^{N-1}\sum_{r=0}^{M-1} b_{j,r}\int_{\rho^{s_0}\ge 4RN^{t''}} e^{it\rho^{s_0}}\rho^{s_0\alpha}e^{i\Phi(\rho)}\cdot\Big[e^{-i\Phi(\rho)}\sum_{h=0}^{r}\gamma_{rh}\frac{1}{\rho^{s_0 r-h}}\partial_\rho^h\big(e^{i\Phi(\rho)}\rho^{-js_0}v(\rho)\big)\Big]\,d\rho.
\end{aligned}
$$

The factor in square brackets and its counterpart coming from the first sum in (6-83) yield a differential operator of the form

$$P^\#(\rho,\partial_\rho) = \sum_{\nu=0}^{L_\alpha}\rho^{-\mu\nu}P_\nu^\#(\rho,\partial_\rho). \tag{6-95}$$

This is obtained by repeating the argument of Section 6.1 that led to (6-61). It is also evident that $L_\alpha = \mathbb{O}(\alpha)$ for α large because of (6-87) and (6-90).

Some of the operators $P^{\#}_\nu$ coincide with the P_ν of (6-61), while the others miss some of the terms due to the fact that we are taking finite sums. Thus we have to estimate

$$\int_{\rho^{s_0}\geq 4RN^{t''}} e^{it\rho^{s_0}}\rho^{s_0\alpha}e^{i\Phi(\rho)}P^{\#}(\rho,\partial_\rho)v(\rho)\,d\rho.$$

Now an inspection of (6-51) and (6-52) immediately suggests that $P^{\#}_\nu = P_\nu$ if $\nu/q \leq M-1$ and $\nu \leq N-1$.

It is actually useful to have the above relations be satisfied when $\nu \leq (s_0/\mu)\alpha$. To do that, it suffices to choose θ_N, $\theta_M \geq s_0/\mu$ (see (6-87) and (6-90)).

We thus wind up with the following quantity to be estimated:

$$\int_{\rho^{s_0}\geq 4RN^{t''}} e^{it\rho^{s_0}}\rho^{s_0\alpha}e^{i\Phi(\rho)}\sum_{\nu=0}^{(s_0/\mu)\alpha}\rho^{-\mu\nu}P_\nu(\rho,\partial_\rho)v(\rho)\,d\rho \\ + \int_{\rho^{s_0}\geq 4RN^{t''}} e^{it\rho^{s_0}}\rho^{s_0\alpha}e^{i\Phi(\rho)}\sum_{\nu>(s_0/\mu)\alpha}^{L_\alpha}\rho^{-\mu\nu}P^{\#}_\nu(\rho,\partial_\rho)v(\rho)\,d\rho = J_1+J_2. \tag{6-96}$$

First we want to bound J_2. We have

$$|J_2| \leq \int_{\rho^{s_0}\geq 4RN^{t''}} \rho^{s_0\alpha}|e^{i\Phi(\rho)}|\sum_{\nu>(s_0/\mu)\alpha}^{L_\alpha}\rho^{-\mu\nu}\big|P^{\#}_\nu(\rho,\partial_\rho)v(\rho)\big|\,d\rho,$$

where

$$P^{\#}_\nu(\rho,\partial_\rho)=\sum_{r=0}^{m_\nu}\alpha^{\#}_{\nu,r}(\rho)\partial^r_\rho,$$

and we explicitly point out that its coefficients satisfy an estimate of the form (6-49)

$$\big|\partial^t_\rho\alpha^{\#}_{\nu,r}(\rho)\big| \leq C_{\#}^{\nu+t+1}\nu!^{1-\theta}\frac{t!}{\rho^t},$$

where $0\leq\theta<1$ and $\rho\geq c_1\nu^\theta$. Consequently, since $\nu \leq L_\alpha \leq c\alpha \leq c/\theta_N N$, we obtain that $\rho \geq 4RN^{t''/s_0} \geq c'\nu^{t''/s_0}$, and hence

$$\big|\alpha^{\#}_{\nu,r}(\rho)\big| \leq C_{\#}^{\nu+1}\nu!^{1-t''/s_0}.$$

Thus, by (6-79),

$$\rho^{s_0\alpha-\mu\nu}\big|P^{\#}_\nu(\rho,\partial_\rho)v(\rho)\big| \leq \sum_{r=0}^{m_\nu}C_{\#}^{\nu+1}\nu!^{1-t''/s_0}C_v^{r+1}\frac{r!^{t'}}{\rho^r},$$

since $\rho^{s_0\alpha-\mu\nu}\leq 1$. As before, we obtain that $\rho\geq c''r^{t''/s_0}$, $c''>0$ and suitable, because $m_\nu=\mathcal{O}(\nu)$, so that $r!^{t'}\rho^{-r}\leq C^{r+1}r!^{t'-t''/s_0}$. The integral has no convergence problem because $|e^{i\Phi(\rho)}|\leq e^{-\lambda\rho}$, for a suitable positive constant λ, and eventually we obtain the bound

$$|J_2| \leq C_{J_2}^{\alpha+1}\alpha!^{k_1(1-t''/s_0)+k_2(t'-t''/s_0)}, \tag{6-97}$$

where k_1, k_2 denote positive constants depending only on the problem data. In the following we denote in this way any constant of this kind, and we shall understand that their meaning may vary depending on the context.

Choosing t' near 1 and t'' near s_0, satisfying (6-94), we see that J_2 gives rise to a function in $\mathcal{B}^{s_0}$.

We are thus left with the term J_1. To estimate it, we have to recall the definition of v in (6-78), where cutoff functions in $G^{t''}$ from Lemma 5.4 have been employed. We have

$$v(\rho) = \sum_{l=0}^{\infty} \omega_l(\rho) u_l(\rho),$$

and without loss of generality we may assume that $\omega_l \equiv 1$ for $\rho \geq 4R(l+1)$ and $\omega_l(\rho) \equiv 0$ for $\rho \leq 2R(l+1)$, with the same constant R we used previously. Of course we are free to choose a larger R, if need be. Thus

$$\rho^{s_0\alpha} \sum_{\nu=0}^{(s_0/\mu)\alpha} \rho^{-\mu\nu} P_\nu(\rho, \partial_\rho) v(\rho) = \sum_{k=0}^{\infty} \sum_{\substack{\nu+l=k \\ \nu \leq (s_0/\mu)\alpha}} \rho^{-\mu\nu+s_0\alpha} P_\nu(\rho, \partial_\rho)\big(\omega_l(\rho) u_l(\rho)\big).$$

We split this into two parts, according to whether in the above sum the complete expression (6-64) for the transport equation appears or we find only a part of it:

$$\begin{aligned} \rho^{s_0\alpha} \sum_{\nu=0}^{\lfloor (s_0/\mu)\alpha \rfloor} \rho^{-\mu\nu} P_\nu(\rho, \partial_\rho) v(\rho) = & \sum_{k=0}^{\lfloor (s_0/\mu)\alpha \rfloor} \sum_{\nu+l=k} \rho^{-\mu\nu+s_0\alpha} P_\nu(\rho, \partial_\rho)\big(\omega_l(\rho) u_l(\rho)\big) \\ & + \sum_{k>\lfloor (s_0/\mu)\alpha \rfloor} \sum_{\substack{\nu+l=k \\ \nu \leq \lfloor (s_0/\mu)\alpha \rfloor}} \rho^{-\mu\nu+s_0\alpha} P_\nu(\rho, \partial_\rho)\big(\omega_l(\rho) u_l(\rho)\big) \\ = {} & JC_1 + JC_2. \end{aligned} \tag{6-98}$$

We start by bounding JC_2, which is pretty similar to J_2, studied above. By Proposition 6.1.6, we have

$$P_\nu(\rho, \partial_\rho) = \sum_{p=0}^{m_\nu} \alpha_{\nu,p}(\rho) \partial_\rho^p, \tag{6-99}$$

where $m_\nu \leq c\nu$ and the coefficients satisfy the estimate

$$\big|\partial_\rho^t \alpha_{\nu,p}(\rho)\big| \leq C_\alpha^{\nu+t+1} \nu!^{1-\theta} \frac{t!}{\rho^t}, \tag{6-100}$$

provided $\rho \geq c_1 \nu^{\theta}$, $0 < \theta \leq 1$. Now

$$\begin{aligned} |JC_2| &\leq \sum_{k > \lfloor (s_0/\mu)\alpha \rfloor} \sum_{\substack{\nu+l=k \\ \nu \leq \lfloor (s_0/\mu)\alpha \rfloor}} \sum_{p=0}^{m_\nu} \sum_{\beta=0}^{p} \binom{p}{\beta} \rho^{-\mu\nu + s_0\alpha} \left|\alpha_{\nu,p}(\rho)\right| \left|\partial_\rho^\beta \omega_l(\rho)\right| \left|\partial_\rho^{p-\beta} u_l(\rho)\right| \\ &\leq \sum_{\tilde{k} \geq 0} \sum_{\substack{\nu+l=\tilde{k}+\lfloor (s_0/\mu)\alpha \rfloor+1 \\ \nu \leq \lfloor (s_0/\mu)\alpha \rfloor}} \sum_{p=0}^{m_\nu} \sum_{\beta=0}^{p} \binom{p}{\beta} \rho^{-\mu\tilde{k} - \mu(\lfloor (s_0/\mu)\alpha \rfloor + 1) + s_0\alpha} C_\alpha^{\nu+1} \nu!^{1-t''/s_0} \\ &\qquad\qquad \cdot (RC_\omega)^{\beta+1} C_u^{p-\beta+l+1} \frac{p!^{t'}}{\rho^p}, \end{aligned}$$

where (5-6), (6-49), (6-76) have been used. In particular, (6-49) can be used since $\rho^{s_0} \geq 4RN^{t''} = 4R\theta_N^{t''}\alpha^{t''} \geq 4R\theta_N^{t''}(\mu/s_0)^{t''}\nu^{t''}$, yielding $\theta = t''/s_0$ for R sufficiently large depending on the problem data.

We have $\rho^{-p} \leq \tilde{C}^p p!^{-t''/s_0}$, since $p \leq m_\nu \leq \tilde{c}\nu$. Thus we get

$$\begin{aligned} |JC_2| \leq \sum_{\tilde{k} \geq 0} \sum_{\substack{\nu+l=\tilde{k}+\lfloor (s_0/\mu)\alpha \rfloor+1 \\ \nu \leq (s_0/\mu)\alpha}} \sum_{p=0}^{m_\nu} &\rho^{-\mu\tilde{k} - \mu(\lfloor (s_0/\mu)\alpha \rfloor + 1) + s_0\alpha} C_\alpha^{\nu+1} \nu!^{1-t''/s_0} \\ &\cdot (RC_\omega)^{p+1} C_u^{p+l+1} (2\tilde{C})^p p!^{t'-t''/s_0}. \end{aligned}$$

We point out that $-\mu(\lfloor (s_0/\mu)\alpha \rfloor + 1) + s_0\alpha < 0$. Moreover, since $m_\nu \leq \tilde{c}\nu$, we may estimate the sum with respect to p, getting

$$|JC_2| \leq \sum_{\tilde{k} \geq 0} \rho^{-\mu\tilde{k}} \sum_{\substack{\nu+l=\tilde{k}+\lfloor (s_0/\mu)\alpha \rfloor+1 \\ \nu \leq (s_0/\mu)\alpha}} C_\alpha^{\nu+1} R^{\tilde{c}\nu+1} C_u^l C_T^\nu \nu!^{1-t''/s_0+\tilde{c}(t'-t''/s_0)}.$$

Finally, we want to bound the inner sum, noting that contrary to the sum over $\tilde{k}$, it is a finite sum involving a number of terms proportional to α. Because of the estimate

$$\nu!^{1-t''/s_0+\tilde{c}(t'-t''/s_0)} \leq \left(\left(\frac{s_0}{\mu}\right)\alpha\right)!^{1-t''/s_0+\tilde{c}(t'-t''/s_0)} \leq C^{\alpha+1}\alpha!^{(s_0/\mu)(1-t''/s_0)+(s_0/\mu)\tilde{c}(t'-t''/s_0)},$$

we obtain

$$|JC_2| \leq C^{\alpha+1}\alpha!^{(s_0/\mu)(1-t''/s_0)+(s_0/\mu)\tilde{c}(t'-t''/s_0)} \sum_{\tilde{k} \geq 0} \rho^{-\mu\tilde{k}} C_u^{\tilde{k}}.$$

Since $\rho > 2R$ on the support of v, the above series converges, provided R is large enough. Arguing as for J_2, we conclude that $JC_2 \in \mathcal{B}^{s_0}$.

Consider JC_1 in (6-98). Again we split it into two parts:

$$\begin{aligned}\sum_{k=0}^{(s_0/\mu)\alpha}\sum_{\nu+l=k}\rho^{-\mu\nu+s_0\alpha}P_\nu(\rho,\partial_\rho)\big(\omega_l(\rho)u_l(\rho)\big)\\
&=\sum_{k=0}^{(s_0/\mu)\alpha}\sum_{\nu+l=k}\rho^{-\mu\nu+s_0\alpha}\omega_l(\rho)P_\nu(\rho,\partial_\rho)u_l(\rho)\\
&\qquad+\sum_{k=0}^{(s_0/\mu)\alpha}\sum_{\nu+l=k}\rho^{-\mu\nu+s_0\alpha}\big(P_\nu(\rho,\partial_\rho)(\omega_l(\rho)u_l(\rho))-\omega_l(\rho)P_\nu(\rho,\partial_\rho)u_l(\rho)\big)\\
&=\sum_{k=0}^{(s_0/\mu)\alpha}(I_{1,k}+I_{2,k}).\end{aligned}\tag{6-101}$$

Let us consider $I_{1,k}$. Remark that if $\rho\ge 4R(k+1)$, then $\omega_l(\rho)\equiv 1$ for any $l=0,\dots,k$. Therefore, in this region $I_{1,k}=0$, due to (6-75). We have only to consider $I_{1,k}$ for $(4R\theta_N^{t''})^{1/s_0}\alpha^{t''/s_0}\le\rho\le 4R(k+1)$. In this region — assuming it is not trivially empty — we have

$$\begin{aligned}|I_{1,k}|&\le\sum_{\nu+l=k}\rho^{-\mu\nu+s_0\alpha}\sum_{p=0}^{m_\nu}\big|\alpha_{\nu,p}(\rho)\big|\big|\partial_\rho^p u_l(\rho)\big|\big|\omega_l(\rho)\big|\\
&\le\rho^{-\mu k+s_0\alpha}\sum_{\nu+l=k}C_\alpha^{\nu+1}\nu!^{1-t''/s_0}\sum_{p=0}^{m_\nu}C_u^{p+l+1}\frac{p!}{\rho^p},\end{aligned}$$

where we applied (6-100) and (6-76), arguing as we did before. As above, $p!\,\rho^{-p}\le C^{\nu+1}\nu!^{\tilde c(1-t''/s_0)}$. Therefore

$$\begin{aligned}|I_{1,k}|\le\rho^{-\mu k+s_0\alpha}\sum_{\nu+l=k}C^{\nu+1}C_u^{l+1}\nu!^{(1+\tilde c)(1-t''/s_0)}\\
&\le\rho^{-\mu k+s_0\alpha}\sum_{\nu+l=k}C^{\nu+1}C_u^{l+1}\rho^{(s_0/t'')\nu(1+\tilde c)(1-t''/s_0)}\\
&\le\rho^{-\mu k+s_0\alpha}C^{k+1}\rho^{(s_0/t'')k(1+\tilde c)(1-t''/s_0)}\\
&=C^{k+1}\rho^{s_0\alpha-k[\mu-(s_0/t'')(1+\tilde c)(1-t''/s_0)]}\le C^{k+1}\rho^{s_0\alpha-k\tilde\mu},\end{aligned}$$

for some positive $\tilde\mu$, choosing t'' close to s_0 as we did before.

Consider now, recalling (6-96),

$$\begin{aligned}\left|\int_{\rho^{s_0}\ge 4RN^{t''}}e^{it\rho^{s_0}}e^{i\Phi(\rho)}I_{1,k}(\rho)\,d\rho\right|&\le C^{k+1}\int_{(4R)^{1/s_0}N^{t''/s_0}\le\rho\le 4R(k+1)}e^{-\lambda\rho}\rho^{s_0\alpha-k\tilde\mu}\,d\rho\\
&\le C^{k+1}\int_0^{+\infty}e^{-\lambda\rho}\rho^{s_0\alpha-\mu'\rho}\,d\rho\le C^{\alpha+1}\int_0^{+\infty}e^{-\mu'\rho\log\rho}\rho^{s_0\alpha}\,d\rho.\end{aligned}$$

The proof is complete once we show:

Lemma 6.2.7. *Let $\mu > 0$. For any $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that*

$$\int_0^{+\infty} e^{-\mu\rho\log\rho}\rho^{s_0\alpha}\,d\rho \le C_\varepsilon \varepsilon^\alpha \alpha!^{s_0}. \tag{6-102}$$

Proof. Pick a positive M to be chosen later and write

$$\int_0^{+\infty} e^{-\mu\rho\log\rho}\rho^{s_0\alpha}\,d\rho = \int_0^{M} e^{-\mu\rho\log\rho}\rho^{s_0\alpha}\,d\rho + \int_M^{+\infty} e^{-\mu\rho\log\rho}\rho^{s_0\alpha}\,d\rho = I_1 + I_2.$$

Consider I_2. Because $e^{-\mu\rho\log\rho} \le e^{-\mu\log M\rho}$, we get

$$I_2 \le \int_0^{+\infty} e^{-\mu\rho\log M}\rho^{s_0\alpha} d\rho = \left(\frac{1}{\mu\log M}\right)^{s_0\alpha+1} \alpha!^{s_0}.$$

Choosing $\mu^{-s_0}(\log M)^{-s_0} \le \varepsilon$, we prove the assertion for I_2.

Consider I_1.

$$I_1 \le e^{\mu/e}\frac{M^{s_0\alpha+1}}{s_0\alpha+1} \le e^{\mu/e} M \frac{\left(\frac{M^{s_0}}{\varepsilon}\right)^\alpha}{\alpha!^{s_0}} \varepsilon^\alpha \alpha!^{s_0},$$

and this implies the assertion also for I_1. $\square$

Let us now consider $I_{2,k}$. Remark that if $\rho \ge 4R(k+1)$, then $I_{2,k} = 0$ due to Lemma 5.4. We have only to consider $I_{2,k}$ for $(4R\theta_N^{t''})^{1/s_0}\alpha^{t''/s_0} \le \rho \le 4R(k+1)$. Assuming this region is not trivially empty, we have

$$\begin{aligned}
|I_{2,k}| &\le \sum_{\nu+l=k} \rho^{-\mu\nu+s_0\alpha} \sum_{p=0}^{m_\nu}\sum_{\beta=1}^{p} \binom{p}{\beta} |\alpha_{\nu,p}(\rho)| |\partial_\rho^\beta \omega_l(\rho)| |\partial_\rho^{p-\beta} u_l(\rho)| \\
&\le \rho^{-\mu k+s_0\alpha} \sum_{\nu+l=k} C_\alpha^{\nu+1}\nu!^{1-t''/s_0} \sum_{p=0}^{m_\nu}\sum_{\beta=1}^{p} \binom{p}{\beta} (RC_\omega)^{\beta+1} \frac{\beta!^{t'}}{\rho^\beta} C_u^{p-\beta+l+1} \frac{(p-\beta)!}{\rho^{p-\beta}} \\
&\le \rho^{-\mu k+s_0\alpha} \sum_{\nu+l=k} C_\alpha^{\nu+1}\nu!^{1-t''/s_0} \sum_{p=0}^{m_\nu} C_u'^{p+l+1} \frac{p!^{t'}}{\rho^p}.
\end{aligned}$$

As above, $p!^{t'}\rho^{-p} \le C^{\nu+1}\nu!^{\tilde{c}(t'-t''/s_0)}$, and the argument proceeds as that for $I_{1,k}$.

This completes the proof of Proposition 6.2.6. $\square$

Next we are going to show that if $\Lambda(t, D_t)$ as given by the left-hand side of (6-2) is G^s-hypoelliptic for $s < s_0$, from (6-84), it follows that $A(v)(t) \in \mathcal{B}^{s_0}(\mathbb{R})$.

To this end, we recall the following result. For its proof we refer to Appendix B.

Theorem 6.2.8 [Métivier 1980, Theorem 3.1]. *Let Ω be an open set of $\mathbb{R}$ containing the origin. Assume that there is an open subset $U \Subset \Omega$, a compact subset K of Ω, and a bounded operator $R\colon L^2(U) \to L^2(K)$ such that $(PRu)_{|U} = u_{|U}$.*

The operator Λ is Gevrey s-hypoelliptic at the origin if and only if:

(i) *For any neighborhood ω of the origin, there exists a neighborhood $\omega'' \Subset \omega$ such that*

$$(\Lambda u)_{|\omega} \in H^k(\omega) \quad \textit{implies} \quad u_{|\omega''} \in H^k(\omega'').$$

(ii) *For any neighborhoods of the origin $\omega^{iv} \Subset \omega''' \Subset \omega''$, there are positive constants C, L such that*

$$\|u\|_{k,\omega^{iv}} \le CL^k\big([\![\Lambda(\varphi u)]\!]_{s,k,\omega'''} + k!^s \|u\|_{0,\omega''}\big), \tag{6-103}$$

where $\|u\|_{k,\omega}$ denotes the usual Sobolev norm of order k on the open set ω and

$$[\![u]\!]_{s,k,\omega} = \sum_{\alpha=0}^{k} k^{s(k-\alpha)} \|D^\alpha u\|_{0,\omega}. \tag{6-104}$$

Moreover, $\varphi \in C_0^\infty(\omega'')$, $\varphi \equiv 1$ in a neighborhood of ω''' and C, L are independent of k and u.

By Theorem 4.3 and (6-2), we obtain that the operator Λ has a parametrix whose symbol belongs to $S^0_{1,k/lq}$ (recall that $k/lq < 1$, by assumption). See also Theorem 3.4 of [Kumano-go 1982]. Moreover, by Remark B.111., we have $(P(\varphi u))_{|\omega'''} \in \mathcal{B}^{s_0}(\omega''')$ if and only if $(Pu)_{|\omega'''}$ has the same regularity.

Therefore, Theorem 6.2.8 can be applied to Λ, provided we are on a small enough neighborhood of the origin. To keep the notation simple, we denote by ω' the neighborhood of the origin where the solution has regularity $\mathcal{B}^{s_0}$.

Lemma 6.2.9. *If $A(v) \in \mathcal{B}^{s_0}(\omega')$, then for every $\varepsilon > 0$ there exists $C_{\varepsilon,\omega'} > 0$ such that*

$$\big|\mathcal{F}(A(v))(\tau)\big| \le C_{\varepsilon,\omega'} e^{-(1/(2\varepsilon)^{1/s_0})|\tau|^{1/s_0}}. \tag{6-105}$$

Here $\mathcal{F}(A(v))$ denotes the Fourier transform of $A(v)$.

Proof. First we point out that $A(v) \in \mathcal{S}(\mathbb{R})$, due to the fact that the phase factor $e^{i\Phi(\rho)}$ is rapidly decreasing for $\rho \to +\infty$.

There exists a $\delta > 0$, $[-\delta, \delta] \subset \omega'$ such that for every $\varepsilon > 0$ there is $C_{1,\varepsilon} > 0$ for which, for every α,

$$|D_t^\alpha A(v)(t)| \le C_{1,\varepsilon} \varepsilon^\alpha \alpha!^{s_0}, \quad |t| \le \delta. \tag{6-106}$$

An argument quite similar to that of the proof of Lemma 6.2.5 gives that, for $|t| \ge \delta$,

$$\big|D_t^\alpha A(v)(t)\big| \le \frac{1}{|t|^\alpha} C^{\alpha+1} \alpha!^{t'} \le \frac{1}{|t|^\alpha} C_{2,\varepsilon} \varepsilon^\alpha \alpha!^{s_0}. \tag{6-107}$$

For the Fourier transform of $A(v)$, we obtain

$$\mathcal{F}(A(v))(\tau) = \frac{1}{\tau^\alpha} \int e^{-it\tau} D_t^\alpha A(v)\, dt.$$

We split the latter integral into two parts, I_1, I_2, for the regions $|t| \le \delta$ and $|t| \ge \delta$ respectively.

By (6-106),

$$|I_1| \le 2\delta \frac{1}{|\tau|^\alpha} C_{1,\varepsilon} \varepsilon^\alpha \alpha!^{s_0}.$$

By (6-107), for $\alpha \ge 2$,

$$|I_2| \le \frac{1}{|\tau|^\alpha} C_{2,\varepsilon} \varepsilon^\alpha \alpha!^{s_0} \int_{|t|\ge\delta} |t|^{-\alpha}\, dt = \frac{1}{|\tau|^\alpha} C_{3,\varepsilon} \varepsilon^\alpha \alpha!^{s_0}.$$

Therefore, overall, we get

$$\left|\mathcal{F}(A(v))(\tau)\right| \le \frac{1}{|\tau|^\alpha} C_{4,\varepsilon}\varepsilon^\alpha \alpha!^{s_0},$$

for any α and τ large. Hence

$$\left|\mathcal{F}(A(v))(\tau)\right|^{1/s_0} \frac{\left(\left(\frac{|\tau|}{2\varepsilon}\right)^{1/s_0}\right)^\alpha}{\alpha!} \le C_{4,\varepsilon}^{1/s_0}\left(\frac{1}{2^{1/s_0}}\right)^\alpha.$$

Summing in α from 0 to ∞, we prove the assertion. □

We state the following proposition, leaving the proof to the reader:

Proposition 6.2.10. *Let ω', ω be as in Theorem 6.2.8. If $\Lambda u \in \mathcal{B}^{s_0}(\omega)$, then $u \in \mathcal{B}^{s_0}(\omega'')$.*

Corollary 6.2.11. *Let $A(v)$ be given by* (6-85). *Then Proposition 6.2.6 implies that $A(v) \in \mathcal{B}^{s_0}(\omega')$.*

Proof of the corollary. Let $\varphi \in C_0^\infty(\mathbb{R}) \cap G^s(\mathbb{R})$, $\varphi \equiv 1$ near the origin. Arguing as in the proof of Lemma 6.2.9, we may show that

$$\left|\widehat{(1-\varphi)A(v)}(\xi)\right| \le Ce^{-|\xi|^{1/s}/C},$$

for a certain positive constant C, whence $\Lambda((1-\varphi)A(v)) \in G^s$. Therefore, Proposition 6.2.6 implies that $\Lambda(\varphi A(v)) \in \mathcal{B}^{s_0}$. From Proposition 6.2.10, it follows that $\varphi A(v) \in \mathcal{B}^{s_0}$, whence the statement. □

Let us now prove that Corollary 6.2.11 implies a contradiction, which in turn yields that Λ is Gevrey s_0-hypoelliptic and not better.

The construction of $A(v)$ shows that the conclusion of Lemma 6.2.9 is violated:

Lemma 6.2.12. *There exist positive constants λ, C_λ such that for τ positive and large,*

$$\left|\mathcal{F}(A(v))(\tau)\right| \ge C_\lambda e^{-\lambda\tau^{1/s_0}}. \tag{6-108}$$

Proof. Since v in $A(v)$ (see (6-79)) has support in $[2R, +\infty[$, we have

$$A(v)(t) = \frac{1}{s_0}\int_{\mathbb{R}} e^{it\tau} e^{i\Phi(\tau^{1/s_0})} v(\tau^{1/s_0})\tau^{(1/s_0)-1}\chi(\tau)\,d\tau,$$

where $\chi(\tau) \equiv 1$ if $\tau \ge (2R)^{s_0}$ and $\chi(\tau) \equiv 0$ if $\tau \le R^{s_0}$. From the Fourier transform inversion formula, we obtain that

$$\mathcal{F}(A(v))(\tau) = \frac{2\pi}{s_0} e^{i\Phi(\tau^{1/s_0})} v(\tau^{1/s_0})\tau^{(1/s_0)-1},$$

for $\tau \ge 2R$. Since, due to the construction performed in Section 6.1, we have for τ large

$$\Phi(\tau^{1/s_0}) = \varphi\tau^{1/s_0}(1 + o(1))$$

with $\operatorname{Im}\varphi > 0$, and

$$v(\tau^{1/s_0}) = 1 + o(1),$$

we conclude, for a suitable $\lambda > 0$, that

$$\left|\mathcal{F}(A(v))(\tau)\right| \ge C_\lambda e^{-\lambda\tau^{1/s_0}}.$$

□

Thus the inequalities

$$C_\lambda e^{-\lambda\tau^{1/s_0}} \le \left|\mathcal{F}(A(v))(\tau)\right| \le C_{\varepsilon,\omega'} e^{-\frac{1}{(2\varepsilon)^{1/s_0}}\tau^{1/s_0}}$$

give a contradiction, provided ε is small and τ is large enough.

This proves assertion (iii) of Theorem 1.1.

7. Non-C^∞-hypoellipticity

The purpose of this section is to prove assertion (iv) of Theorem 1.1. Because of Proposition 4.2, we have to show that Λ in (3-33) is not C^∞-hypoelliptic if $l \le k/q$.

The method of proof is analogous to that used in the previous section, but much simpler. Multiplying Λ in (3-33) by an elliptic operator, we have to consider the symbol

$$\tau^{2k/q-2/q}\Lambda(t,\tau) \sim \sum_{j=0}^{\infty} a_j(t)\tau^{2k/q-j} + \sum_{s=0}^{\infty} b_s(t)\tau^{-s}, \tag{7-1}$$

where $\tau > 0$, a_j, b_s are real analytic and defined in a neighborhood of the origin and

$$a_j(t) = t^{2l-j}\tilde{a}_j(t) \quad \text{for } j = 0, \dots, 2l-1, \quad \text{with each } \tilde{a}_j \in C^\omega. \tag{7-2}$$

We rename Λ the operator whose symbol is given by the left-hand side of (7-1).

First we look for a formal solution of the form

$$A(u)(t) = \int_0^\infty e^{it\rho}u(\rho)\,d\rho \tag{7-3}$$

of the equation $\Lambda(t, D_t)A(u) = 0$. In order to do so, we replace the coefficients a_j, b_s by their power series

$$\Lambda(t, D_t) = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} a_{j,n}t^{n+(2l-j)_+}D_t^{2k/q-j} + \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} b_{j,n}t^n D_t^{-j},$$

where $(m)_+ = \max\{m, 0\}$. Taking both t and D_t into the integral sign, we formally obtain

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\int_0^{+\infty} e^{it\rho}\rho^{2k/q}\left[\sum_{\alpha=0}^{n+(2l-j)_+} C_{n,j,\alpha}\rho^{-j-n-(2l-j)_++\alpha}\partial^\alpha u + \sum_{\alpha=0}^{n} C'_{n,j,\alpha}\rho^{-j-n-2k/q+\alpha}\partial^\alpha u\right]d\rho,$$

where $C_{n,j,\alpha}$, $C'_{n,j,\alpha}$ are constants. We organize the expression in brackets according to its homogeneity: making the dilation $\rho \mapsto \lambda\rho$ a generic monomial, $\rho^\alpha\partial^\beta$ has homogeneity $\alpha-\beta$. The principal part then has homogeneity $-2l$, forgetting about the factor $\rho^{2k/q}$ in front, and is obtained from the first sum above when $n = 0$ and $j = 0, \dots, 2l$. We are assuming here that $2l < 2k/q$, which is the generic case. If $2l = 2k/q$, the second sum above contributes the term $(j, n, \alpha) = (0, 0, 0)$ to the principal part.

Denote by $P_{-2l}(\rho, \partial_\rho)$ the principal part so obtained. It has the form

$$P_{-2l}(\rho, \partial_\rho) = \sum_{\alpha=0}^{2l}\gamma_\alpha\rho^{\alpha-2l}\partial_\rho^\alpha. \tag{7-4}$$

As for the terms of lower homogeneity, we note that they are homogeneous of degree either $-2l-r$ or $-2k/q-r$. We may gather the terms of equal homogeneity into differential polynomials. To keep the notation simple, we write the quantity in brackets as

$$\sum_{r=0}^{\infty} P_{-2l-r/q}(\rho, \partial_\rho)u,$$

where $P_{-2l-r/q}(\rho, \partial_\rho)$ is a finite linear combination of homogeneous monomials of degree $-2l-r/q$.

We look for a u of the form

$$u(\rho) = \sum_{s=0}^{\infty} u_s(\rho) \tag{7-5}$$

such that

$$\sum_{r=0}^{k} P_{-2l-r/q} u_{k-r} = 0, \quad k = 0, 1, \dots. \tag{7-6}$$

Let us start with u_0; it solves the equation

$$P_{-2l}(\rho, \partial_\rho)u_0(\rho) = 0,$$

where P_{-2l} is given by (7-4). The latter is a Fuchs type equation and we choose

$$u_0(\rho) = \rho^\lambda, \tag{7-7}$$

where λ denotes the solution of the indicial equation associated to (7-4), that is,

$$\sum_{\alpha=0}^{2l} \gamma_\alpha \lambda(\lambda-1)\dots(\lambda-\alpha+1) = 0, \tag{7-8}$$

such that

$$\operatorname{Re}\lambda = \min\{\operatorname{Re}\mu \mid \mu \text{ is a solution of (7-8)}\}. \tag{7-9}$$

Let us next consider the second *transport* equation in (7-6), corresponding to $k=1$.

$$P_{-2l}(\rho, \partial_\rho)u_1(\rho) = -P_{-2l-1/q}(\rho, \partial_\rho)u_0(\rho).$$

Since the differential operators $P_{-2l-j/q}(\rho, \partial_\rho)$ have homogeneity $-2l-j/q$, when applied to the function ρ^λ they give a function proportional to $\rho^{\lambda-2l-j/q}$. Therefore the above equation has the form

$$P_{-2l}(\rho, \partial_\rho)u_1(\rho) = \operatorname{const}\rho^{\lambda-2l-1/q}.$$

Our purpose is to obtain a function u_1 having a better growth rate compared to u_0 when $\rho \to +\infty$, that is, such that $u_1(\rho) = \mathbb{O}(\rho^\mu)$, with $\operatorname{Re}\mu < \operatorname{Re}\lambda$. If $\operatorname{Re}\lambda$ has the minimality property (7-9), we see at once that the exponent in the right-hand side of the above differential equation cannot be a root of the indicial equation (7-8); thus we can rule out logarithmic factors. Again, keeping in mind the homogeneity-preserving property of the operators $P_{-2l-j/q}$, we conclude that

$$u_1(\rho) = c_1\rho^{\lambda-1/q}.$$

We iterate this argument and solve the triangular system (7-6), thus obtaining:

Proposition 7.1. *There is a* $\lambda \in \mathbb{C}$, *satisfying both* (7-8) *and* (7-9), *such that for every* $s = 0, 1, \dots,$ *the system* (7-6) *has a solution* u_s *of the form*

$$u_s(\rho) = c_s \rho^{\lambda - s/q}. \tag{7-10}$$

Turning the formal solution (7-5) into a function is easy in the present case: let $\chi \in C^\infty(\mathbb{R})$, $\chi \equiv 0$ for $\rho \le R$, $R > 0$, and $\chi \equiv 1$ if $\rho \ge 2R$. Define

$$v(\rho) = \sum_{s=0}^{\infty} \chi(\varepsilon_s \rho) u_s(\rho), \tag{7-11}$$

where $(\varepsilon_s)_{s \in \mathbb{N}}$ denotes a sequence of positive numbers such that $\varepsilon_s \to 0+$ in a convenient way.

We need to make sense of $A(v)$ defined as in (7-3). First of all, we note that there is no problem near $\rho = 0$, since $0 \notin \operatorname{supp}(v)$ (we may always suppose that $\varepsilon_s \le 1$.) If $\operatorname{Re} \lambda < -1$, λ defined by (7-8), (7-9), $A(v)$ is in $C(\mathbb{R})$. If $\operatorname{Re} \lambda \ge -1$, then the integral $A(v)$ in (7-3) has to be interpreted as an oscillating integral, and then it always defines a distribution of finite order to which a pseudodifferential operator can be applied.

We want to show that $\Lambda(t, D_t) A(v)(t) \in C^\infty(\mathbb{R})$.

Proposition 7.2. *Let* $A(v)$ *be defined as in* (7-3), *with* v *given by* (7-11). *Then*

$$\Lambda(t, D_t) A(v)(t) \in C^\infty(\mathbb{R}). \tag{7-12}$$

Proof. Actually, all we have to show is that $\Lambda A(v)$ is smooth in a neighborhood of the origin, since away from the origin, $A(v)$ is smooth.

We start arguing on just one of the two asymptotic expansions that build Λ, for example, the second sum in (7-1). The argument for the other is completely analogous and we have to use both sums only when (7-6) is needed. This is exactly what was done in the proof of Proposition 6.2.6.

Modulo a smoothing operator, we may assume that the symbol of the operator Λ has the form

$$\Lambda(t, \tau) \sim \sum_{j=0}^{\infty} b_j(t) \chi(\varepsilon_j \tau) \tau^{-j}.$$

Then

$$\Lambda(t, D_t) A(v)(t) = \sum_{j=0}^{\infty} b_j(t) \int_0^{+\infty} e^{it\rho} \chi(\varepsilon_j \rho) \rho^{-j} v(\rho)\, d\rho.$$

Let us consider $D_t^\alpha \Lambda A(v)$ and show that this is a continuous function for every α. Denote by $N \in \mathbb{N}$ a number to be selected later; then we consider

$$D_t^\alpha \Lambda(t, D_t) A(v)(t) = D_t^\alpha \left\{ \left(\sum_{j=0}^{N-1} + \sum_{j=N}^{\infty} \right) b_j(t) \int_0^{+\infty} e^{it\rho} \chi(\varepsilon_j \rho) \rho^{-j} v(\rho)\, d\rho \right\} = I_1 + I_2. \tag{7-13}$$

Consider I_2 and let $N > \operatorname{Re}\lambda + \alpha + 1$. Then $|\chi(\varepsilon_j\rho)\rho^{-j+\alpha}v(\rho)| = \mathbb{O}(\rho^{\operatorname{Re}\lambda - N+\alpha})$, and therefore $I_2 \in C(\mathbb{R})$. Let us now turn to I_1. Let $M \in \mathbb{N}$ and write

$$I_1 = D_t^\alpha \sum_{j=0}^{N-1}\Big(\sum_{n=0}^{M-1} b_{j,n}t^n \int_0^{+\infty} e^{it\rho}\chi(\varepsilon_j\rho)\rho^{-j}v(\rho)\,d\rho + t^M \sum_{n=0}^{\infty} b_{j,M+n}t^n \int_0^{+\infty} e^{it\rho}\chi(\varepsilon_j\rho)\rho^{-j}v(\rho)\,d\rho\Big)$$
$$= I_{11} + I_{12}.$$

Consider first I_{12}. We have

$$\begin{aligned} I_{12} &= \sum_{j=0}^{N-1}\sum_{n=0}^{\infty}\sum_{\beta=0}^{\alpha} b_{j,M+n}\binom{\alpha}{\beta}\big(D_t^{\alpha-\beta}t^n\big)\int_0^{+\infty} e^{it\rho}\rho^\beta(-D_\rho)^M\big(\chi(\varepsilon_j\rho)\rho^{-j}v(\rho)\big)\,d\rho \\ &\equiv \sum_{j=0}^{N-1}\sum_{\beta=0}^{\alpha}\Big(\sum_{n=0}^{\infty} b_{j,M+n}\binom{\alpha}{\beta}\big(D_t^{\alpha-\beta}t^n\big)\Big)\int_0^{+\infty} e^{it\rho}\rho^\beta\chi(\varepsilon_j\rho)(-D_\rho)^M\big(\rho^{-j}v(\rho)\big)\,d\rho, \end{aligned}$$

where the last equality is modulo smooth terms because when the derivative with respect to ρ lands on the cutoff function χ, it produces a compact support function of ρ. Moreover, the sum over n on the last line (in big parentheses) is a real analytic function. The integrand function above is $\mathbb{O}(\rho^{\operatorname{Re}\lambda-j-M+\beta})$, so that if $\operatorname{Re}\lambda+\alpha-M<-1$, we obtain that I_{12} is a continuous function. Note that both N and M so far satisfy the same condition.

Consider I_{11}.

$$\begin{aligned} I_{11} &= D_t^\alpha \sum_{j=0}^{N-1}\Big(\sum_{n=0}^{M-1} b_{j,n}t^n\int_0^{+\infty} e^{it\rho}\chi(\varepsilon_j\rho)\rho^{-j}v(\rho)\,d\rho\Big) \\ &= D_t^\alpha \sum_{j=0}^{N-1}\sum_{n=0}^{M-1} b_{j,n}\int_0^{+\infty} e^{it\rho}(-D_\rho)^n\big(\chi(\varepsilon_j\rho)\rho^{-j}v(\rho)\big)\,d\rho \\ &= D_t^\alpha \sum_{j=0}^{N-1}\sum_{n=0}^{M-1} b_{j,n}\Big(\int_0^{2R/(\varepsilon_{N-1})} + \int_{2R/(\varepsilon_{N-1})}^{+\infty}\Big) e^{it\rho}(-D_\rho)^n\big(\chi(\varepsilon_j\rho)\rho^{-j}v(\rho)\big)\,d\rho \\ &\equiv D_t^\alpha \sum_{j=0}^{N-1}\sum_{n=0}^{M-1} b_{j,n}\int_{2R/(\varepsilon_{N-1})}^{+\infty} e^{it\rho}(-D_\rho)^n\big(\rho^{-j}v(\rho)\big)\,d\rho, \end{aligned}$$

modulo smooth functions. By (7-11), we may write

$$v = \Big(\sum_{s=0}^{N-1} + \sum_{s=N}^{\infty}\Big)\chi(\varepsilon_s\rho)u_s$$

and note that the second sum contributes a $\mathbb{O}(\rho^{\operatorname{Re}\lambda - N/q - j - n+\alpha})$ to the integral. Therefore, if

$$\operatorname{Re}\lambda - \frac{N}{q} + \alpha < -1,$$

we have a continuous function. As for the first sum, on the domain of integration, the cutoff is identically equal to one; thus

$$I_{11} \equiv D_t^\alpha \sum_{s=0}^{N-1} \sum_{j=0}^{N-1} \sum_{n=0}^{M-1} b_{j,n} \int_{2R/(\varepsilon_{N-1})}^{+\infty} e^{it\rho}(-D_\rho)^n \big(\rho^{-j} u_s(\rho)\big)\, d\rho.$$

The same analysis can be applied to the first sum in (7-1), so that eventually we get

$$D_t^\alpha \sum_{s=0}^{N-1} \sum_{j=0}^{N-1} \sum_{n=0}^{M-1} \int_{\frac{2R}{\varepsilon_{N-1}}}^{+\infty} e^{it\rho} \rho^{\frac{2k}{q}} \left[\sum_{\alpha=0}^{n+(2l-j)_+} C_{n,j,\alpha} \rho^{\alpha-j-n-(2l-j)_+} \partial^\alpha u + \sum_{\alpha=0}^{n} C'_{n,j,\alpha} \rho^{\alpha-j-n-\frac{2k}{q}} \partial^\alpha u \right] d\rho$$
$$= D_t^\alpha \sum_{s=0}^{N-1} \sum_{r=0}^{\tilde r(N)} \int_{\frac{2R}{\varepsilon_{N-1}}}^{+\infty} e^{it\rho} \rho^{\frac{2k}{q}} \tilde P_{-2l-r/q}(\rho, \partial_\rho) u_s(\rho)\, d\rho,$$

where we have set $M = N$ and $\tilde r(N) \le N$ is a suitable increasing integer function of N, and where the $\tilde P_{-2l-r/q}$ are differential polynomials homogeneous of degree $-2l - r/q$. We see that there exists a number $r(N) \in \mathbb{N}$ such that $r(N) < \tilde r(N)$, $r(N) \to \infty$ for $N \to \infty$, and (see (7-6))

$$\tilde P_{-2l-r/q}(\rho, \partial_\rho) = P_{-2l-r/q}(\rho, \partial_\rho),$$

if $r < r(N)$. Then the above expression can be written as

$$D_t^\alpha \sum_{\substack{s=0\\ r+s<r(N)}}^{N-1} \sum_{r=0}^{\tilde r(N)} \int_{\frac{2R}{\varepsilon_{N-1}}}^{+\infty} e^{it\rho} \rho^{\frac{2k}{q}} P_{-2l-\frac{r}{q}}(\rho, \partial_\rho) u_s(\rho)\, d\rho + D_t^\alpha \sum_{\substack{s=0\\ r(N)\le r+s}}^{N-1} \sum_{r=0}^{\tilde r(N)} \int_{\frac{2R}{\varepsilon_{N-1}}}^{+\infty} e^{it\rho} \rho^{\frac{2k}{q}} \tilde P_{-2l-\frac{r}{q}}(\rho, \partial_\rho) u_s(\rho)\, d\rho$$
$$= D_t^\alpha \sum_{\substack{s=0\\ r(N)\le r+s}}^{N-1} \sum_{r=0}^{\tilde r(N)} \int_{\frac{2R}{\varepsilon_{N-1}}}^{+\infty} e^{it\rho} \rho^{\frac{2k}{q}} \tilde P_{-2l-\frac{r}{q}}(\rho, \partial_\rho) u_s(\rho)\, d\rho,$$

because of (7-6). Taking the t-derivative under the integral sign, we see immediately that the integrand is $\mathbb{O}(\rho^{(2k/q)+\operatorname{Re}\lambda+\alpha-2l-r(N)/q})$. If N is large enough, the assertion is then proved. □

Proposition 7.3. *$A(v)$ is not smooth near the origin.*

Proof. By Proposition 7.1, $v = \mathbb{O}(\rho^{\operatorname{Re}\lambda})$, so that v is a microlocally elliptic symbol of order $\operatorname{Re}\lambda$. Hence, $A(v)$ cannot be smooth. □

Propositions 7.2 and 7.3 prove statement (iv) of Theorem 1.1.

Appendix A: The adjoint of a product

We prove here a well-known formula for the adjoint of a product of two pseudodifferential operators using just symbolic calculus. Let a, b be symbols in $S^0_{1,0}(\mathbb{R}_t)$. We want to show that

$$(a \# b)^* = b^* \# a^*, \tag{A-1}$$

where # denotes the usual symbolic composition law (a higher-dimensional extension involves just a more cumbersome notation.)

We may write

$$(a\,\#\,b)^* = \sum_{l,\alpha\geq 0} \frac{(-1)^\alpha}{\alpha!\,l!}\,\partial_\tau^l D_t^l\big(\partial_\tau^\alpha \bar{a} D_t^\alpha \bar{b}\big) = \sum_{l,\alpha\geq 0}\sum_{r,s\leq l} \frac{(-1)^\alpha}{\alpha!l!}\binom{l}{r}\binom{l}{s}\partial_\tau^{\alpha+r} D_t^{l-s}\bar{a}\partial_\tau^{l-r} D_t^{\alpha+s}\bar{b}.$$

Let us change the summation indices according to the following prescription: $j=\alpha+r$, $\beta+j=l-s$, $i=\alpha+s$, so that $l-r=i+\beta$, and we may rewrite the last equality in the above formula as

$$(a\,\#\,b)^* = \sum_{i,j,\beta\geq 0}\sum_{s\leq i} \frac{(-1)^{i-s}}{(i-s)!(\beta+j+s)!}\binom{\beta+j+s}{j-i+s}\binom{\beta+j+s}{s}\partial_\tau^{i+\beta} D_t^i \bar{b}\partial_\tau^j D_t^{\beta+j}\bar{a}.$$

Let us examine the s-summation; we claim that

$$\sum_{s=0}^{i} \frac{(-1)^{i-s}}{(i-s)!}\,\frac{1}{(\beta+i)!(j-i+s)!}\binom{\beta+j=s}{s} = \frac{1}{\beta!i!\,j!}.$$

This is actually equivalent to

$$\sum_{s=0}^{i}(-1)^{i-s}\binom{i}{s}\binom{\beta+j+s}{\beta+i} = \binom{\beta+j}{j}.$$

Setting $i-s=\nu\in\{0,1,\dots,i\}$, the above relation is written as

$$\sum_{\nu=0}^{i}(-1)^{\nu}\binom{i}{\nu}\binom{\beta+i+j-\nu}{\beta+i} = \binom{\beta+j}{j},$$

and this is precisely identity (12.15) in [Feller 1957, Chapter II].

Thus we may conclude that

$$(a\,\#\,b)^* = \sum_{i,j,\beta} \frac{1}{\beta!i!\,j!}\partial_\tau^{i+\beta} D_t^i\bar{b}\partial_\tau^j D_t^{j+\beta}\bar{a} = \sum_{\beta\geq 0}\frac{1}{\beta!}\partial_\tau^\beta\Big(\sum_{i\geq 0}\frac{1}{i!}\partial_\tau^i D_t^i\bar{b}\Big)D_t^\beta\Big(\sum_{j\geq 0}\frac{1}{j!}\partial_\tau^j D_t^j\bar{a}\Big) = b^*\,\#\,a^*.$$

This proves (A-1).

As a byproduct of the above argument, we get the identity

$$\sum_{i,j,\beta} \frac{1}{\beta!i!\,j!}\partial_\tau^{i+\beta} D_t^i\bar{b}\partial_\tau^j D_t^{j+\beta}\bar{a} = \sum_{l,\alpha\geq 0}\frac{(-1)^\alpha}{\alpha!\,l!}\partial_\tau^l D_t^l\big(\partial_\tau^\alpha\bar{a}D_t^\alpha\bar{b}\big), \tag{A-2}$$

which is the purpose of this appendix.

We would like to point out that the relation $(a^*)^*=a$ rests on the identity

$$\sum_{l\geq 0}\frac{1}{l!}\partial_\tau^l D_t^l\overline{\Big(\sum_{\alpha\geq 0}\frac{1}{\alpha!}\partial_\tau^\alpha D_t^\alpha\bar{a}\Big)} = \sum_{s\geq 0}\frac{1}{s!}\Big(\sum_{l+\alpha=s}\frac{s!}{l!\alpha!}(-1)^\alpha\Big)\partial_\tau^s D_t^s a = \sum_{s\geq 0}\frac{1}{s!}(1-1)^s\partial_\tau^s D_t^s a = a. \tag{A-3}$$

Appendix B: Proof of Theorem 6.2.8

We include in this section the proof of Theorem 6.2.8 for pseudodifferential operators in the Gevrey case, which is the case needed in our argument. Métivier [1980] gives the proof of the same theorem in the analytic category for differential operators, and states that its extension to the pseudodifferential case has no major difficulties. We argue along the same lines.

Since pseudodifferential operators are involved in an essential way, we first recall the definition of hypoellipticity; even though the material is well known, it is useful to state it here for future reference.

When we use a pseudodifferential operator, or its symbol, we mean either a pseudodifferential operator in the C^∞ or in the Gevrey category. In the latter case, although the symbols involved may be analytic functions, the cut off functions will of course be in Gevrey classes (see also Lemmas 5.3 and 5.4 for the construction of some cutoff functions.)

Definition B.1. Let $P(x, D_x)$ denote a properly supported pseudodifferential operator acting on the distributions. We say that P is hypoelliptic at the point x_0 if and only if there exists an open set Ω, $x_0 \in \Omega$, such that for every open set $V \Subset \Omega$ and for every $u \in \mathscr{D}'(\Omega)$, we have

$$(Pu)_{|_V} \in C^\infty \Rightarrow u_{|_V} \in C^\infty$$

or

$$(Pu)_{|_V} \in G^s \Rightarrow u_{|_V} \in G^s,$$

for $s > 1$.

It is well known that (not properly supported) pseudodifferential operators can be extended as operators from $\mathscr{E}'(\Omega) \to \mathscr{D}'(\Omega)$. Thus we may also give the following definition:

Definition B.2. Let $P(x, D_x)$ denote a pseudodifferential operator, which we suppose defined in $\mathbb{R}^n$ and not properly supported, acting on distributions. We say that P is hypoelliptic at the point $x_0 \in \mathbb{R}^n$ if and only if there exists an open set Ω containing x_0 and such that for every open set $V \Subset \Omega$ and for every $u \in \mathscr{E}'(\Omega)$, we have

$$(Pu)_{|_V} \in C^\infty \Rightarrow u_{|_V} \in C^\infty$$

or

$$(Pu)_{|_V} \in G^s \Rightarrow u_{|_V} \in G^s,$$

for $s > 1$.

Proposition B.3. *Let P denote a properly supported pseudodifferential operator. Then Definition B.2 is equivalent to Definition B.1.*

Proof. Let us show first that B.2 implies B.1. Let Ω be the open set from Definition B.2 and let $u \in \mathscr{D}'(\Omega)$. We want to show that for every $V \Subset \Omega$, if, for example, $(Pu)_{|_V} \in C^\infty$, then $u_{|_V} \in C^\infty$. The assertion in the Gevrey category will have a completely analogous proof.

Let $\bar{x} \in V$ and $\varphi \in C_0^\infty(V)$ such that $\varphi \equiv 1$ on $V_1 \Subset V$, $\bar{x} \in V_1$. Since $(Pu)_{|_V} \in C^\infty$, we have

$$Pu = P(\varphi u) + P((1-\varphi)u) \in C^\infty.$$

Since P is properly supported, we have $P = P_1 + R_P$, where $R_P \colon \mathscr{D}'(\Omega) \to C^\infty(\Omega)$ is a regularizing operator and P_1 enlarges support by a fixed quantity, that is, $\operatorname{supp}(Pf) \subset (\operatorname{supp} f)_\delta$ for a certain positive δ, where if $A \subset \mathbb{R}^n$, $A_\delta = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, A) \le \delta\}$.

Now

$$C^\infty \ni (Pu)_{|V_1} = (P(\varphi u))_{|V_1} + \big(P((1-\varphi)u)\big)_{|V_1} = (P(\varphi u))_{|V_1} + \big(P_1((1-\varphi)u)\big)_{|V_1} + \big(R_P((1-\varphi)u)\big)_{|V_1}.$$

The third term is obviously smooth and the second term vanishes if $\operatorname{dist}(V_1, \complement V) > \delta$.

Therefore $(P(\varphi u))_{|V_1} \in C^\infty$ implies, by Definition B.2, that $\varphi u \in C^\infty(V_1)$ or, since $\varphi \equiv 1$ on V_1, that $u \in C^\infty(V_1)$. Since the choice of the point $\bar{x}$ is arbitrary, we obtain that $u \in C^\infty(V)$, and hence the conclusion in Definition B.1.

The converse implication is easier. Assume that $Pu \in C^\infty(V)$, with $u \in \mathscr{E}'(\Omega)$. Again $(Pu)_{|V} = (P_1 u)_{|V} + (R_P u)_{|V}$, where $R_P \colon \mathscr{E}'(\Omega) \to C^\infty(\Omega)$. Thus $(Pu)_{|V} \in C^\infty$ implies that $(P_1 u)_{|V} \in C^\infty$, so that, by Definition B.1, $u_{|V} \in C^\infty$. This proves the proposition. □

The next proposition shows that, in order to prove that a pseudodifferential operator is hypoelliptic, it is enough to show that the corresponding properly supported operator is hypoelliptic according to Definition B.2.

Proposition B.4. *Let P denote a pseudodifferential operator. Then P is hypoelliptic (G^s-hypoelliptic, $s > 1$) at the point x_0 if and only if P_1 is hypoelliptic (resp. G^s-hypoelliptic, $s > 1$) at x_0 according to Definition B.2. Here we denote by P_1 a properly supported operator such that $P = P_1 + R_P$, with $R_P \colon \mathscr{E}'(\Omega) \to C^\infty(\Omega)$.*

Proof. Assume that P is hypoelliptic at x_0 and let Ω be the open neighborhood of x_0 from Definition B.1. We assume that for every $V \Subset \Omega$, $x_0 \in V$, $(P_1 u)_{|V} \in C^\infty$ with $u \in \mathscr{E}'(\Omega)$. As we did above, we point out that $(Pu)_{|V} = (P_1 u)_{|V} + (R_P u)_{|V} \in C^\infty$, and this implies that $u_{|V} \in C^\infty$.

The converse statement has a completely analogous proof.

Again we remark that the proof in the Gevrey category is exactly the same. □

We now turn to proving Theorem 6.2.8. We start by recalling without proof a couple of facts about cutoff functions. This is also useful to establish the notation.

Lemma B.5. *There is a positive constant γ_0, depending only on n, the dimension of the ambient space, such that for every pair of open subsets $\omega' \Subset \omega \Subset \mathbb{R}^n$, there is a sequence of functions $(\chi_k)_{k \in \mathbb{N}}$ in $\mathscr{D}(\omega)$, ${\chi_k}_{|\omega'} \equiv 1$, and such that for every $k \in \mathbb{N}$, for every multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| \le k$, we have*

$$\|D^\alpha \chi_k\|_\infty \le \left(\frac{\gamma_0 k}{r}\right)^{|\alpha|}, \tag{B-1}$$

where $r = \operatorname{dist}(\omega', \complement\omega) > 0$.

Lemma B.6. *Let ω', ω, χ_k be as in the previous lemma and satisfying* (B-1). *Then there is a positive constant γ such that for every $k \in \mathbb{N}$ and for every $u \in H^k(\omega)$, we have*

$$|||\chi_k u|||_{k,\mathbb{R}^n} \le \gamma^k |||u|||_{k,\omega}, \tag{B-2}$$

where the three bar norm, defined right after Theorem 6.2.8, has the meaning

$$|||u|||_{k,\omega} = \sum_{\alpha=0}^{k} k^{k-\alpha} \| D^\alpha u\|_{0,\omega}. \tag{B-3}$$

Lemma B.7. *Let Ω denote a neighborhood of $x_0 \in \mathbb{R}^n$ and let B be a Banach space continuously injecting into $L^2(\Omega)$. Assume that $x_0 \notin \operatorname{sing\,supp}_s u$ for every $u \in B$, where $\operatorname{sing\,supp}_s u$ denotes the Gevrey s-singular support of u, and $s > 1$. Then there are neighborhoods $\omega' \Subset \omega \Subset \Omega$ of x_0, functions χ_k satisfying* (B-1), *and positive constants γ and C such that for every $k \in \mathbb{N}$ and every $u \in B$,*

$$|\xi|^k \widehat{\chi_k u} \in L^2(\mathbb{R}^n),$$

or, in different terms,

$$\| |\xi|^k \widehat{\chi_k u} \|_{0,\mathbb{R}^n} \le C(\gamma k^s)^k \|u\|_B.$$

Proof. For $\omega \Subset \mathbb{R}^n$ and $L > 0$, let us denote by $\mathfrak{g}^s_L(\bar\omega)$ the Banach space of all Gevrey s-functions on $\bar\omega$ such that

$$\|u\|_{\mathfrak{g}^s_L(\bar\omega)} = \sup_\alpha \frac{\|D^\alpha u\|_{0,\omega}}{\alpha!^s L^{|\alpha|}} < +\infty. \tag{B-4}$$

Then the space of all functions being Gevrey s at the point x_0 can be written as

$$\operatorname{ind\,lim}_{N\to\infty} \mathfrak{g}^s_N(\overline{B(x_0, N^{-1})}).$$

Using Theorem 1 on p. 147 of [Grothendieck 1973], we can see that there exist a neighborhood ω of the point x_0 and a constant $L > 0$ such that the map $u \mapsto u_{|\omega}$ is continuous from B to $\mathfrak{g}^s_L(\bar\omega)$. Denote by C its norm.

Let $\omega' \Subset \omega$ and let χ_k be functions as in Lemma B.5. We therefore have

$$\big\| D^\alpha(\chi_k u)\big\|_{0,\mathbb{R}^n} \le \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \Big(\frac{\gamma_0 k}{r}\Big)^{|\beta|} (\alpha-\beta)!^s L^{|\alpha-\beta|} \|u_{|\omega}\|_{\mathfrak{g}^s_L(\bar\omega)}.$$

For $|\alpha| \le k$ we may estimate $(\alpha-\beta)!^s \le k^{s|\alpha-\beta|}$, so that

$$\big\| D^\alpha(\chi_k u)\big\|_{0,\mathbb{R}^n} \le C\Big(L + \frac{\gamma_0}{r}\Big)^{|\alpha|} k^{s|\alpha|} \|u\|_B$$

and

$$\big\| |\xi|^k \widehat{\chi_k u}\big\|_{0,\mathbb{R}^n} \le n^{k/2} C\Big(L + \frac{\gamma_0}{r}\Big)^k k^{sk} \|u\|_B,$$

which is the statement of the lemma. □

Next we remark that there is a constant $\gamma_1 \ge 1$ such that for every $k \in \mathbb{N}$ and every $u \in H^k(\mathbb{R}^n)$, we have

$$\gamma_1^k |||u|||^2_{k,\mathbb{R}^n} \le \int_{\mathbb{R}^n} (k+|\xi|)^{2k} |\hat u(\xi)|^2 \, d\xi \le \gamma_1^k |||u|||^2_{k,\mathbb{R}^n}. \tag{B-5}$$

Now define, for $s > 1$,

$$G_{(s)} = \big\{ u \in L^2(\mathbb{R}^n) \;\big|\; e^{|\xi|^{1/s}} \hat u(\xi) \in L^2(\mathbb{R}^n) \big\}. \tag{B-6}$$

Once we equip $G_{(s)}$ with the norm $\|u\|_{G_{(s)}} = \|e^{|\xi|^{1/s}}\hat{u}\|_{0,\mathbb{R}^n}$, we see that $G_{(s)}$ is a Hilbert space of $G^s(\mathbb{R}^n)$ functions.

Lemma B.8. *Let k be an integer ≥ 1 and for $j = 0, 1, 2, \ldots$, let $N_j = k2^j$. Then every function $u \in H^k(\mathbb{R}^n)$ can be written as the series*

$$u = \sum_{j=0}^{\infty} u_j,$$

where $u_j \in G_{(s)}$, and such that

$$\sum_{j=0}^{\infty} N_j^{2ks}\big(\|u_j\|_{0,\mathbb{R}^n}^2 + e^{-2N_j}\|u_j\|_{G_{(s)}}^2\big) \leq 2(2\gamma_1)^k \|\|u\|\|_{k,\mathbb{R}^n}^2. \tag{B-7}$$

Proof. For $j = 0, 1, \ldots$ (and setting $N_{-1} = 0$), we have

$$u_j(x) = (2\pi)^{-n}\int_{N_{j-1}\leq|\xi|^{1/s}<N_j} e^{i\langle x,\xi\rangle}\hat{u}(\xi)\,d\xi.$$

If $|\xi| \leq N_j^s$, then $e^{|\xi|^{1/s}} \leq e^{N_j}$, so that $\|u_j\|_{G_{(s)}} \leq e^{N_j}\|u_j\|_{0,\mathbb{R}^n}$. Furthermore, when $|\xi| \geq N_{j-1}^s$, we have $N_j^s \leq 2^s(|\xi| + k)$ and

$$\sum_{j=0}^{\infty} N_j^{2ks}\|u_j\|_{0,\mathbb{R}^n}^2 \leq \int_{\mathbb{R}^n}\big(2^s(|\xi|+k)\big)^{2k}|\hat{u}(\xi)|^2\,d\xi,$$

which allows us to conclude. □

Next we prove an inverse of the preceding lemma, but on a neighborhood of the point x_0.

As above, let Ω be a neighborhood of x_0 and let B be a Banach space of functions of class G^s at x_0 such that the injection from B to $L^2(\Omega)$ is continuous.

Lemma B.9. *There is a neighborhood $\omega' \Subset \Omega$ of x_0 and a positive constant C such that for every $k \geq 1$ and every sequence u_j, $j = 0, 1, 2, \ldots$, $u_j \in B$, satisfying*

$$\sum_{j=0}^{\infty} N_j^{2ks}\big(\|u_j\|_{0,\Omega}^2 + e^{-2N_j}\|u_j\|_B^2\big) = \Phi_k^2(u_j) < +\infty,$$

the series

$$u = \sum_{j=0}^{\infty} u_j \tag{B-8}$$

converges in $L^2(\Omega)$ and $u_{|_{\omega'}} \in H^k(\omega')$, with the inequality

$$\|u_{|_{\omega'}}\|_{k,\omega'} \leq C^{k+1}\Phi_k(u_j).$$

Proof. The convergence of the series (B-8) in $L^2(\Omega)$ is a direct consequence of the assumption that $\Phi_k(u_j) < +\infty$.

Applying Lemma B.7, we obtain neighborhoods $\omega' \Subset \omega \Subset \Omega$ of the point x_0, positive constants γ, C_0, and a sequence of cutoff functions $\chi_N \in \mathscr{D}(\omega)$, $\chi_N \equiv 1$ on ω', such that for every N and every function $u \in B$ we have the estimate

$$\left\| \Big(\frac{|\xi|}{\gamma N^s}\Big)^N \widehat{\chi_N u} \right\|_{0,\mathbb{R}^n} \leq C_0 \|u\|_B. \tag{B-9}$$

Define the functions

$$\theta(j,\xi) = e^{-N_j} \Big(\frac{|\xi|}{\gamma N_j^s}\Big)^{N_j} \tag{B-10}$$

and

$$g_j(\xi) = \big(1+\theta(j,\xi)\big)\widehat{\chi_{N_j} u_j(\xi)}. \tag{B-11}$$

Both (B-9) and (B-11) yield

$$\|g_j\|_{0,\mathbb{R}^n} \leq \|u_j\|_{0,\Omega} + C_0 e^{-N_j} \|u_j\|_B,$$

so that

$$\sum_{j=0}^{\infty} N_j^{2ks} \|g_j\|^2_{0,\mathbb{R}^n} \leq 2(1+C_0^2)\Phi_k^2(u_j) < +\infty. \tag{B-12}$$

Let us now define

$$v = \sum_{j=0}^{\infty} \chi_{N_j} u_j.$$

Of course $v \in L^2(\Omega)$ and, by definition, v coincides with u on ω'. Therefore it is enough to show that $v \in H^k(\mathbb{R}^n)$ and that the estimate

$$\|v\|_{k,\mathbb{R}^n} \leq C^{k+1}\Phi_k(u_j)$$

holds. Actually one already has the estimate

$$\|v\|_{0,\mathbb{R}^n} \leq \sum_{j=0}^{\infty} \|u_j\|_{0,\mathbb{R}^n} \leq 2\Phi_k(u_j).$$

We only have to show then that $|\xi|^k \hat{v} \in L^2(\mathbb{R}^n)$ and that the estimate

$$\big\| |\xi|^k \hat{v} \big\|_{0,\mathbb{R}^n} \leq C^{k+1}\Phi_k(u_j) \tag{B-13}$$

holds, where the constant C is independent of k.

To this end, using (B-11), we write

$$|\xi|^k \hat{v}(\xi) = \sum_{j=0}^{\infty} \big(1+\theta(j,\xi)\big)^{-1} g_j(\xi)|\xi|^k.$$

We have

$$|\xi|^{2k} |\hat{v}(\xi)|^2 \leq \Big(\sum_{j=0}^{\infty} |g_j(\xi)|^2 N_j^{2ks}\Big)\theta(\xi),$$

where

$$\theta(\xi) = \sum_{j=0}^{\infty} \Big(\frac{|\xi|}{N_j^s}\Big)^{2k} \big(1+\theta(j,\xi)\big)^{-2} = \sum_{j=0}^{\infty} \Psi_j(\xi).$$

Because of (B-12), it suffices to prove that

$$\|\theta(\xi)\|_\infty \le C^{k+1}. \tag{B-14}$$

We argue in two different cases. The first region is $\gamma e^2 N_j^s \le |\xi|$. Then

$$\Psi_j(\xi) \le \gamma^{2k} \bigg(\frac{|\xi|}{\gamma N_j^s}\bigg)^{2k-2N_j} e^{2N_j} \le (\gamma e^2)^{2k} e^{-2N_j}.$$

As a consequence,

$$\sum_{\gamma e^2 N_j^s \le |\xi|} \Psi_j(\xi) \le C_1 (\gamma e^2)^{2k}.$$

If now $\gamma e^2 N_j^s \ge |\xi|$, let $j_0 = \min\{j \mid \gamma e^2 N_j^s \ge |\xi|\}$ for a fixed ξ. We have

$$\Psi_j(\xi) \le \gamma^{2k} \bigg(\frac{|\xi|}{\gamma N_j^s}\bigg)^{2k} = \gamma^{2k} \bigg(\frac{|\xi|}{\gamma N_{j_0}^s}\bigg)^{2k} \bigg(\frac{N_{j_0}}{N_j}\bigg)^{2ks} \le (\gamma e^2)^{2k} \bigg(\frac{1}{2^{j-j_0}}\bigg)^{2ks}.$$

Therefore

$$\sum_{\gamma e^2 N_j^s \ge |\xi|} \Psi_j(\xi) \le (\gamma e^2)^{2k} \bigg(\sum_{j=0}^{\infty} 2^{-j}\bigg)^{2ks} = (\gamma e^2 2^s)^{2k}.$$

This proves the lemma. □

We now want to prove the following theorem in a Gevrey pseudodifferential setting. Define

$$[\![u]\!]_{s,k,\omega} = \sum_{\alpha=0}^{k} k^{s(k-\alpha)} \|D^\alpha u\|_{0,\omega}. \tag{B-15}$$

Note that $[\![u]\!]_{1,k,\omega} = |\!|\!| u |\!|\!|_{k,\omega}$.

Theorem B.10 [Métivier 1980, Theorem 3.1]. *Let $P(x,D)$ be a real analytic pseudodifferential operator. Let $x_0 \in \mathbb{R}^n$ and let Ω denote a neighborhood of x_0. Let $x_0 \in U \Subset \Omega$ be an open set.*

Assume that there is a bounded operator $R\colon L^2(U) \to L^2(K)$, where K is a suitable compact subset of Ω, such that $(PRu)_{|_U} = u_{|_U}$. Here $L^2(K)$ denotes the set of all functions in $L^2(\Omega)$ whose support is contained in K.

The operator P is Gevrey s-hypoelliptic at x_0 if and only if:

(i) *For any neighborhood ω of x_0, there exists a neighborhood $\omega'' \Subset \omega$ such that $Pu_{|_\omega} \in H^k(\omega)$ implies $u_{|_{\omega''}} \in H^k(\omega'')$.*

(ii) *For any neighborhoods of x_0 $\omega^{iv} \Subset \omega''' \Subset \omega''$, there are positive constants C, L such that*

$$\|u\|_{k,\omega^{iv}} \le CL^k \big([\![P(\varphi u)]\!]_{s,k,\omega'''} + k!^s \|u\|_{0,\omega''}\big) \tag{B-16}$$

for any $u \in \mathscr{E}'(\Omega)$ where $\varphi \in C_0^\infty(\omega'')$, $\varphi \equiv 1$ on a neighborhood of ω''', and C, L are independent of k and u. Here $\|u\|_{k,\omega}$ denotes the usual Sobolev norm of order k on the open set ω.

Remark B.11.

1. Since P has an analytic symbol,
$$\text{sing supp}_\omega\big[(P(\varphi u))_{|_{\overline{\omega'''}}} - (Pu)_{|_{\overline{\omega'''}}}\big] = \varnothing.$$
2. It is not difficult to show that the operator Λ of Section 6 has a local right inverse as in the statement by using Theorem 3.4 of [Kumano-go 1982] and Theorem 4.3.
3. For the limited purpose of this paper, a weaker result would have been enough. We are allowed to have the constants C, L depending on u but not on k. This is much easier to prove and we do not need for this Lemma B.12.

Proof. If (i) and (ii) hold, then clearly P is Gevrey s-hypoelliptic at x_0.

Conversely, assume that P is G^s-hypoelliptic at the point x_0. First we prove (i).

Let $\omega \subset \Omega$ be an open neighborhood of x_0. We choose an open subset $\omega_1 \Subset \omega$, $x_0 \in \omega_1$, and cutoff functions $\chi_k \in C_0^\infty(\omega)$, $k \in \mathbb{N}$, as in Lemma B.5 such that (B-1) is satisfied and $\chi_k \equiv 1$ on ω_1.

Let $u \in \mathscr{E}'(\Omega)$ and assume that $(Pu)_{|_\omega} \in H^k(\omega)$, $k \in \mathbb{N}$.

Set
$$f = \chi_k Pu.$$

Clearly f is defined on the whole of $\mathbb{R}^n$, and more precisely $f \in H^k(\mathbb{R}^n)$. Applying Lemma B.6 to the function f, we obtain that
$$|||f|||_{k,\mathbb{R}^n} \le \gamma^k |||Pu|||_{k,\omega}, \tag{B-17}$$
for a suitable positive constant γ independent of k.

Furthermore, applying Lemma B.8 to the same function f, we write
$$f = \sum_{j=0}^\infty f_j,$$
with $f_j \in G_{(s)}$ (see (B-6) for a definition of $G_{(s)}$), and the following inequality holds:
$$\sum_{j=0}^\infty N_j^{2ks}\big(\|f_j\|^2_{0,\mathbb{R}^n} + e^{-2N_j}\|f_j\|^2_{G_{(s)}}\big) \le 2(2\gamma_1)^k |||f|||^2_{k,\mathbb{R}^n}. \tag{B-18}$$

Denote by $\tilde{G}_{(s)}$ the space of all restrictions to U of the functions in $G_{(s)}$ compactly supported in U, and let $B_{(s)} = R\tilde{G}_{(s)}$ equipped with the norm defined by $\|R(g_{|U})\|_{B_{(s)}} = \|g\|_{G_{(s)}}$. Fix an open neighborhood $U' \Subset U$ of x_0 and choose a Gevrey cutoff function $\psi \in C_0^\infty(U)$, $0 \le \psi \le 1$, of Gevrey order s', $1 < s' < s$, such that $\psi_{|U'} \equiv 1$. Set
$$v_j = R(\psi f_{j\,|_U}). \tag{B-19}$$

Clearly $v_j \in B_{(s)}$ and v_j is a function of class G^s near the point x_0. In fact $(Pv_j)_{|U} = (PR(\psi f_{j\,|U}))_{|U} = (\psi f_j)_{|U}$. The latter is a Gevrey function of order s and, since P is G^s-hypoelliptic, we conclude. We have the inequality

$$\sum_{j=0}^{\infty} N_j^{2ks}\big(\|v_j\|_{0,\Omega}^2 + e^{-2N_j}\|v_j\|_{B_{(s)}}^2\big) \le 2(C_\psi + \|R\|^2)(2\gamma_1)^k \,|||f|||_{k,\mathbb{R}^n}^2. \tag{B-20}$$

Here C_ψ is a positive constant only depending on ψ and $\|R\|$ denotes the norm of the operator R as an operator from $L^2(U)$ into $L^2(K)$.

Using (B-20) and Lemma B.9, we obtain that the series $\sum_{j=0}^{\infty} v_j$ converges in $L^2(K)$. Denote by v its sum. From the same lemma, we also get that there is an open set $\omega' \Subset \omega$ such that

$$v_{|\omega'} \in H^k(\omega')$$

and

$$\|v\|_{k,\omega'} \le C^{k+1} |||f|||_{k,\mathbb{R}^n}, \tag{B-21}$$

for a suitable positive constant C. Observe that we may, possibly shrinking ω' as necessary, suppose that $\omega' \subset \omega_1$. Consider now the function $(P(u-v))_{|\omega'}$. Due to that choice of ω', we evidently have $(Pu)_{|\omega'} = f_{|\omega'}$. Then remark that, since

$$v = \sum_{j=0}^{\infty} R(\psi f_{j\,|U}) = R\bigg(\sum_{j=0}^{\infty} \psi f_{j\,|U}\bigg) = R(\psi f_{|U}),$$

we have that P can be applied to v and $(Pv)_{|U'} = f_{|U'}$. Possibly shrinking the open set ω' so that $\omega' \subset U'$, we have in particular that

$$\big(P(u-v)\big)_{|\omega'} = 0. \tag{B-22}$$

Note that because of the hypoellipticity assumption for P, we deduce that $u - v \in G^s(\omega')$. Furthermore, taking $\omega'' \Subset \omega'$, we obtain that $u - v \in G^s(\overline{\omega''})$. This proves assertion (i).

Next we prove (ii). In order to do that, we need to further shrink the neighborhoods of x_0 involved, in such a way that we already know that in that neighborhood u belongs to H^k and is compactly supported. Actually we proved that $u_{|\omega''} \in H^k(\omega'')$. Let $\omega''' \Subset \omega''$ and choose cutoff functions $\tilde{\chi}_k \in C_0^\infty(\omega''')$, such that $\tilde{\chi}_k \equiv 1$ in $\omega_1''' \Subset \omega'''$. Let $\tilde{f} = \tilde{\chi}_k Pu$. Let also $\varphi \in C_0^\infty(\omega'')$, $\varphi \equiv 1$ on $\tilde{\omega}''' \Supset \omega'''$. Note that $\varphi u \in H^k(\mathbb{R}^n)$ and its support is contained in ω''. Due to the pseudolocality property of P, we have

$$\text{sing supp}_\omega \big((P(\varphi u))_{|\tilde{\omega}'''} - (Pu)_{|\tilde{\omega}'''}\big) = \varnothing,$$

and in particular we have $(P(\varphi u))_{|\omega'''} \in H^k(\omega''')$. This implies in turn that $\tilde{\chi}_k P(\varphi u) \in H^k(\mathbb{R}^n)$.

Arguing as above, and possibly enlarging the compact set $K \subset \Omega$, we obtain that $(P(\varphi u - \tilde{v}))_{|\omega'''} = 0$ and $\varphi u - \tilde{v} \in L^2(K) \cap G^s(\omega''')$. Recall that here $L^2(K)$ denotes the set of all functions in $L^2(\Omega)$ whose support is contained in K.

Lemma B.12. *Let X denote the space of all the functions $u \in G^s(\omega''') \cap L^2(K)$ such that $(Pu)_{|_{\omega'''}} = 0$. Equipped with the $L^2(\Omega)$ norm, X becomes a Banach space. Then for every $\omega^{iv} \Subset \omega'''$, there exists a constant $C_2 > 0$, such that for any multi-index α,*

$$\sup_{\omega^{iv}} |\partial^\alpha u(x)| \le C_2^{|\alpha|} \alpha!^s \|u\|_{0,K}, \tag{B-23}$$

for every $u \in X$.

Applying the lemma, we immediately get that for any $\omega^{iv} \Subset \omega'''$ and for any $k \in \mathbb{N}$, we have

$$\|\varphi u - \tilde{v}\|_{k,\omega^{iv}} \le C_2^{k+1} k!^s \|\varphi u - \tilde{v}\|_{0,K}. \tag{B-24}$$

On the other hand, we also have

$$\|\tilde{v}\|_{0,\omega^{iv}} \le \|R\| \|\tilde{f}\|_{0,\mathbb{R}^n} \le \|R\| \|P(\varphi u)\|_{0,\omega'''}$$

and

$$k!^s \|P(\varphi u)\|_{0,\omega'''} \le [\![P(\varphi u)]\!]_{s,k,\omega'''},$$

as well as

$$|\!|\!| u |\!|\!|_{k,\omega'''} \le [\![u]\!]_{s,k,\omega'''}, \quad s \ge 1.$$

Thus

$$\begin{aligned}
\|\varphi u\|_{k,\omega^{iv}} &\le \|\varphi u - \tilde{v}\|_{k,\omega^{iv}} + \|\tilde{v}\|_{k,\omega^{iv}} \\
&\le C_2^{k+1} k!^s \|\varphi u - \tilde{v}\|_{0,K} + C^{k+1} |\!|\!| \tilde{f} |\!|\!|_{k,\mathbb{R}^n} \\
&\le C^{k+1} \gamma^k |\!|\!| P(\varphi u) |\!|\!|_{k,\omega'''} + C_2^{k+1} k!^s \big(\|\tilde{v}\|_{0,K} + \|\varphi u\|_{0,K} \big) \\
&\le C^{k+1} \gamma^k |\!|\!| P(\varphi u) |\!|\!|_{k,\omega'''} + C_2^{k+1} k!^s \|R\| \|P(\varphi u)\|_{0,\omega'''} + C_2^{k+1} k!^s \|\varphi u\|_{0,K} \\
&\le C^{k+1} \gamma^k |\!|\!| P(\varphi u) |\!|\!|_{k,\omega'''} + C_2^{k+1} \|R\| [\![P(\varphi u)]\!]_{s,k,\omega'''} + C_2^{k+1} k!^s \|u\|_{0,\omega''} \\
&\le C_3 L^k \big([\![P(\varphi u)]\!]_{s,k,\omega'''} + k!^s \|u\|_{0,\omega''} \big).
\end{aligned}$$

This proves the theorem. □

Proof of Lemma B.12. It is an application of the Baire category theorem. For $j \in \mathbb{N}$ and for a certain $\omega^{iv} \Subset \omega'''$, define

$$X_j = \big\{ u \in X \;\big|\; |\partial^\alpha u(x)| \le j^{|\alpha|+1} \alpha!^s, \text{ for all } \alpha \text{ and all } x \in \overline{\omega^{iv}} \big\}.$$

Trivially,

$$X = \bigcup_{j=1}^{\infty} X_j.$$

Next we show that the sets X_j are closed with respect to the $L^2(\Omega)$ topology of X. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X_j converging to $u_0 \in X$. As a consequence, the derivatives $\partial^\alpha u_n$ are uniformly bounded in $\overline{\omega^{iv}}$ so that the functions $\partial^\beta u_n$ are equicontinuous if $|\beta| < |\alpha|$. Applying the Arzelà–Ascoli theorem, we obtain that for any $l \in \mathbb{N}$, there exists a subsequence $u_{n_r,l}$ converging in $C^{(l)}(\overline{\omega^{iv}})$ to an element $u_{(l)} \in C^{(l)}(\overline{\omega^{iv}})$. Hence $u_0 = u_{(l)}$ in $\overline{\omega^{iv}}$ and

$$|\partial^\alpha u_0(x)| \le j^{|\alpha|+1}\alpha!^s, \text{ for all } |\alpha| \le l \text{ and all } x \in \overline{\omega^{iv}}.$$

This implies $u_0 \in X_j$. By the Baire category theorem, there are an index j_0, a number $\varepsilon > 0$, and a function $\tilde{u} \in X_{j_0}$ such that

$$B = \{u \in X \mid \|u - \tilde{u}\|_{0,K} \le \varepsilon\} \subset X_{j_0}, \tag{B-25}$$

where we wrote $\| \ \|_{0,K}$ since the support of u, $\tilde{u}$ is contained in K. Now for every $u \in X$, let

$$v = \delta \frac{u}{\|u\|_{0,K}} + \tilde{u} \in B, \quad \text{if } |\delta| < \varepsilon.$$

Thus

$$u = \frac{\|u\|_{0,K}}{\delta}(v - \tilde{u})$$

and

$$\left|\partial^\alpha u(x)\right| \le \frac{\|u\|_{0,K}}{\delta}\left(|\partial^\alpha v| + |\partial^\alpha \tilde{u}|\right) \le R^{|\alpha|+1}\alpha!^s \|u\|_{0,K}.$$

This proves the lemma. □

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References

[Boutet de Monvel 1972] L. Boutet de Monvel, "Opérateurs pseudo-différentiels analytiques et opérateurs d'ordre infini", *Ann. Inst. Fourier (Grenoble)* **22**:3 (1972), 229–268. MR 49 #5939 Zbl 0235.47029

[Boutet de Monvel 1974] L. Boutet de Monvel, "Hypoelliptic operators with double characteristics and related pseudo-differential operators", *Comm. Pure Appl. Math.* **27** (1974), 585–639. MR 51 #6498 Zbl 0294.35020

[Boutet de Monvel and Krée 1967] L. Boutet de Monvel and P. Krée, "Pseudo-differential operators and Gevrey classes", *Ann. Inst. Fourier (Grenoble)* **17**:1 (1967), 295–323. MR 37 #1760 Zbl 0195.14403

[Bove and Mughetti 2013] A. Bove and M. Mughetti, "Analytic and Gevrey hypoellipticity for a class of pseudodifferential operators in one variable", *J. Differential Equations* **255**:4 (2013), 728–758. MR 3056589

[Bove and Tartakoff 1997] A. Bove and D. Tartakoff, "Optimal non-isotropic Gevrey exponents for sums of squares of vector fields", *Comm. Partial Differential Equations* **22**:7-8 (1997), 1263–1282. MR 98f:35026 Zbl 0921.35043

[Bove and Tartakoff 2010] A. Bove and D. S. Tartakoff, "Gevrey hypoellipticity for non-subelliptic operators", *Pure Appl. Math. Q.* **6**:3 (2010), 663–675. MR 2011e:35068 Zbl 1213.35188

[Bove and Treves 2004] A. Bove and F. Treves, "On the Gevrey hypo-ellipticity of sums of squares of vector fields", *Ann. Inst. Fourier (Grenoble)* **54**:5 (2004), 1443–1475. MR 2005k:35053 Zbl 1073.35067

[Bove et al. 2006] A. Bove, M. Derridj, J. J. Kohn, and D. S. Tartakoff, "Sums of squares of complex vector fields and (analytic-) hypoellipticity", *Math. Res. Lett.* **13**:5 (2006), 683–701. MR 2007k:35051 Zbl 1220.35021

[Bove et al. 2010] A. Bove, M. Mughetti, and D. S. Tartakoff, "Gevrey hypoellipticity for an interesting variant of Kohn's operator", pp. 51–73 in *Complex analysis*, edited by P. Ebenfelt et al., Birkhäuser, Basel, 2010. MR 2012k:35067 Zbl 1201.35092

[Christ 2005] M. Christ, "A remark on sums of squares of complex vector fields", preprint, 2005. arXiv 0503506

[Feller 1957] W. Feller, *An introduction to probability theory and its applications*, vol. 1, 2nd ed., Wiley, New York, 1957. MR 35 #1048 Zbl 0138.10207

[Grigis and Rothschild 1983] A. Grigis and L. P. Rothschild, "A criterion for analytic hypoellipticity of a class of differential operators with polynomial coefficients", *Ann. of Math.* (2) **118**:3 (1983), 443–460. MR 85f:35054 Zbl 0541.35017

[Grothendieck 1973] A. Grothendieck, *Topological vector spaces*, Gordon and Breach, New York, 1973. MR 51 #8772 Zbl 0275.46001

[Helffer 1977] B. Helffer, "Sur l'hypoellipticité des opérateurs pseudodifferentiels à caractéristiques multiples (perte de $3/2$ dérivées)", *Bull. Soc. Math. France Suppl. Mém.* **51-52** (1977), 13–61. MR 57 #855 Zbl 0374.35012

[Hörmander 1985] L. Hörmander, *The analysis of linear partial differential operators, III: Pseudodifferential operators*, Grundlehren Math. Wiss. **274**, Springer, Berlin, 1985. MR 87d:35002a Zbl 0601.35001

[Kohn 2005] J. J. Kohn, "Hypoellipticity and loss of derivatives", *Ann. of Math.* (2) **162**:2 (2005), 943–986. MR 2006k:35036 Zbl 1107.35044

[Kumano-go 1982] H. Kumano-go, *Pseudodifferential operators*, MIT Press, Cambridge, MA, 1982. MR 84c:35113 MR 489:35003

[Métivier 1980] G. Métivier, "Une classe d'opérateurs non hypoelliptiques analytiques", *Indiana Univ. Math. J.* **29**:6 (1980), 823–860. MR 82a:35029 Zbl 0455.35041

[Métivier 1981] G. Métivier, "Analytic hypoellipticity for operators with multiple characteristics", *Comm. Partial Differential Equations* **6**:1 (1981), 1–90. MR 82g:35030 Zbl 0455.35040

[Boutet de Monvel and Trèves 1974] L. Boutet de Monvel and F. Trèves, "On a class of pseudodifferential operators with double characteristics", *Invent. Math.* **24** (1974), 1–34. MR 50 #5550 Zbl 0281.35083

[Sjöstrand and Zworski 2007] J. Sjöstrand and M. Zworski, "Elementary linear algebra for advanced spectral problems", *Ann. Inst. Fourier* (*Grenoble*) **57**:7 (2007), 2095–2141. MR 2009m:35351 Zbl 1140.15009

[Treves 1999] F. Treves, "Symplectic geometry and analytic hypo-ellipticity", pp. 201–219 in *Differential equations: La Pietra 1996* (Florence, 1996), edited by M. Giaquinta et al., Proc. Sympos. Pure Math. **65**, American Mathematical Society, Providence, RI, 1999. MR 2000b:35031 Zbl 0938.35038

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