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MICROLOCAL PROPERTIES OF SCATTERING MATRICES FOR SCHRÖDINGER EQUATIONS ON SCATTERING MANIFOLDS

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Let M be a scattering manifold, i.e., a Riemannian manifold with an asymptotically conic structure, and let H be a Schrödinger operator on M. One can construct a natural time-dependent scattering theory for H with a suitable reference system, and a scattering matrix is defined accordingly. We show here that the scattering matrices are Fourier integral operators associated to a canonical transform on the boundary manifold generated by the geodesic flow. In particular, we learn that the wave front sets are mapped according to the canonical transform. These results are generalizations of a theorem by Melrose and Zworski, but the framework and the proof are quite different. These results may be considered as generalizations or refinements of the classical off-diagonal smoothness of the scattering matrix for two-body quantum scattering on Euclidean spaces.

1. Introduction

Let M be an n-dimensional smooth noncompact manifold such that $M = M_c \cup M_\infty$, where M_c is relatively compact, and M_∞ is diffeomorphic to $\mathbb{R}_+ \times \partial M$, where ∂M is a compact manifold. In the following, we often identify M_∞ with $\mathbb{R}_+ \times \partial M$, and we also suppose $M_c \cap M_\infty \subset (0, 1) \times \partial M$ under this identification.

We recall the construction of the model introduced in [Ito and Nakamura 2010]. Let $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^{n-1}\}$, $U_\alpha \subset \partial M$, be a local coordinate system of ∂M. We take

$$\{\tilde{\varphi}_\alpha = I \otimes \varphi_\alpha : \tilde{U}_\alpha = \mathbb{R}_+ \times U_\alpha \to \mathbb{R} \times \mathbb{R}^{n-1}\}$$

as the local coordinate system for $M_\infty \cong \mathbb{R}_+ \times \partial M$, and we use $(r, \theta) \in \mathbb{R} \times \mathbb{R}^{n-1}$ to represent a point in M_∞.

We suppose ∂M is equipped with a smooth strictly positive density $H = H(\theta)$ and a positive $(2, 0)$-tensor $h = (h^{jk}(\theta))$ on ∂M. We let

$$Q = -\frac{1}{2} \sum_{j,k} H(\theta)^{-1} \frac{\partial}{\partial \theta_j} H(\theta) h^{jk}(\theta) \frac{\partial}{\partial \theta_k} \quad \text{on } \mathcal{H}_b = L^2(\partial M, H(\theta) d\theta).$$

Q is an essentially self-adjoint operator on $\mathcal{H}_b$, and we denote its unique self-adjoint extension by the same symbol Q.

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We let G be a smooth strictly positive density on M such that

$$G(x)\,dx = r^{n-1}H(\theta)\,dr d\theta \quad \text{on } (1,\infty)\times\partial M \subset M_\infty,$$

and we set $\mathcal{H} = L^2(M, G(x)\,dx)$. Let P be a formally self-adjoint second order elliptic operator on M such that

$$P = -\tfrac{1}{2}G^{-1}(\partial_r, \partial_\theta/r)G \begin{pmatrix} a_1 & a_2 \\ {}^t a_2 & a_3 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta/r \end{pmatrix} + V \quad \text{on } M_\infty,$$

where $\left(\begin{smallmatrix} a_1 & a_2 \\ {}^t a_2 & a_3 \end{smallmatrix}\right)$ defines a real-valued smooth tensor and V is a real-valued smooth function. As in [Ito and Nakamura 2010], we introduce the following assumption:

Assumption A. There is $\mu > 0$ such that for any $\ell \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^{n-1}$, there is $C_{\ell\alpha} > 0$ and

$$|\partial_r^\ell \partial_\theta^\alpha (a_1(r,\theta) - 1)| \le C_{\ell\alpha} r^{-1-\mu-\ell}, \qquad |\partial_r^\ell \partial_\theta^\alpha a_2(r,\theta)| \le C_{\ell\alpha} r^{-\mu-\ell},$$
$$|\partial_r^\ell \partial_\theta^\alpha (a_3(r,\theta) - h(\theta))| \le C_{\ell\alpha} r^{-\mu-\ell}, \qquad |\partial_r^\ell \partial_\theta^\alpha V(r,\theta)| \le C_{\ell\alpha} r^{-1-\mu-\ell},$$

in each local coordinate of M_∞ described above.

We may consider P as a *short range* perturbation of $-\frac{1}{2}\partial_r^2 + \frac{1}{r^2}Q$, but we will use different operators to construct a scattering theory. It is known that P is essentially self-adjoint, that $\sigma_{ess}(P) = [0,\infty)$, and that P is absolutely continuous except on a countable discrete spectrum, the only possible accumulation point being 0 (see [Ito and Nakamura 2010] and references therein). We construct a time-dependent scattering theory for H as follows: We set

$$M_f = \mathbb{R}\times\partial M, \quad \mathcal{H}_f = L^2(M_f, H(\theta)\,dr d\theta), \quad P_f = -\frac{1}{2}\frac{\partial^2}{\partial r^2} \quad \text{on } M_f.$$

P_f is the one-dimensional free Schrödinger operator, and it is self-adjoint with $\mathcal{D}(P_f) = H^2(\mathbb{R})\otimes\mathcal{H}_b$. Let $j(r) \in C^\infty(\mathbb{R})$ such that $j(r) = 0$ on $(-\infty, \frac{1}{2}]$ and $j(r) = 1$ on $[1,\infty)$. We define $\mathcal{J} : \mathcal{H}_f \to \mathcal{H}$ by

$$(\mathcal{J}\varphi)(r,\theta) = r^{-(n-1)/2} j(r)\varphi(r,\theta) \quad \text{if } (r,\theta)\in M_\infty,$$

and $(\mathcal{J}\varphi)(x) = 0$ if $x \notin M_\infty$. We define the wave operators by

$$W_\pm = W_\pm(P, P_f, \mathcal{J}) = \operatorname*{s-lim}_{t\to\pm\infty} e^{itP}\mathcal{J}e^{-itP_f}.$$

It is shown in [Ito and Nakamura 2010] that these operators exist and are complete in the following sense. Let $\mathcal{F}$ be the Fourier transform in r, i.e.,

$$(\mathcal{F}\varphi)(\rho,\theta) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ir\rho}\varphi(r,\theta)\,dr \quad \text{for } \varphi \in C_0^\infty(M_f),$$

and extend it to a unitary map in $L^2(M_f)$. If we set

$$\mathcal{H}_{f,\pm} = \{\varphi\in\mathcal{H}_f \mid \operatorname{supp}(\mathcal{F}\varphi) \subset \overline{\mathbb{R}_\pm}\times\partial M\},$$

then $\mathcal{H}_f = \mathcal{H}_{f,+}\oplus\mathcal{H}_{f,-}$. We consider $W_\pm$ as maps from $\mathcal{H}_{f,\pm}$ to $\mathcal{H}$; they are asymptotically complete, i.e., unitary operators from $\mathscr{H}_{f,\pm}$ to $\mathscr{H}_{ac}(P)$ [ibid., Theorem 2]. Then the scattering operator defined by

$$S = W_+^* W_- : \mathscr{H}_{f,-} \to \mathscr{H}_{f,+}$$

is unitary. By the intertwining property ($P_f S = S P_f$), there is $S(\lambda) \in \mathscr{B}(\mathscr{H}_b)$ for $\lambda > 0$ such that

$$(\mathscr{F} S \mathscr{F}^{-1}\varphi)(\rho, \cdot) = S(\rho^2/2)\varphi(-\rho, \cdot) \quad \text{for } \rho > 0, \varphi \in \mathscr{F}\mathscr{H}_{f,-}.$$

$S(\lambda)$ is our scattering matrix, and we study its microlocal properties.

Let

$$q(\theta, \omega) = \frac{1}{2}\sum_{j,k} h^{jk}(\theta)\omega_j\omega_k \quad \text{for } (\theta, \omega) \in T^*\partial M$$

be the classical Hamiltonian associated to Q. We denote the Hamilton flow generated by b by $\exp(tH_b)$ for $t \in \mathbb{R}$.

Theorem 1.1. *Suppose Assumption A holds, and let $u \in \mathscr{H}_b$. Then*

$$WF(S(\lambda)u) = \exp(\pi H_{\sqrt{2q}})WF(u),$$

where $WF(u)$ denotes the wave front set of u.

If $\mu = 1$, then we can show $S(\lambda)$ is a Fourier integral operator (FIO). This is a slight extension of a theorem by Melrose and Zworski [1996].

Theorem 1.2. *Suppose Assumption A holds with $\mu = 1$. Then for each $\lambda > 0$, $S(\lambda)$ is an FIO associated to* $\exp(\pi H_{\sqrt{2q}})$.

If $0 < \mu < 1$, then $S(\lambda)$ is not necessarily an FIO in the usual sense, but we can still show it is an FIO in a generalized sense:

Theorem 1.3. *Suppose Assumption A holds, and let $S(\lambda)$ be the scattering matrix defined as above. Then for each $\lambda > 0$, $S(\lambda)$ is an FIO associated to an asymptotically homogeneous canonical transform in $T^*\partial M$, which is asymptotic to* $\exp(\pi H_{\sqrt{2q}})$ *as $\omega \to \infty$.*

The exact definition of the phrase *an FIO associated to an asymptotically homogeneous canonical transform* is given in [Ito and Nakamura 2012], and we discuss it in Section 6.

Remark 1.4. Since we do not introduce a Riemannian metric, our model looks rather different from the scattering metric defined by Melrose [1994; Melrose and Zworski 1996]. However, as explained in [Ito and Nakamura 2010, Appendix A], the Laplacian on scattering manifolds is a special case of our model. Namely, their model corresponds to the case that $\mu = 1$ and that each a_j has asymptotic expansion in r^{-1} as $r \to \infty$ and $V = 0$.

Theorems 1.1 and 1.2 are essentially corollaries of Theorem 1.3, but they can be proved by a simpler argument than Theorem 1.3. We feel the simpler argument is interesting in itself, and we first prove Theorems 1.1 and 1.2, and then we refine the argument to prove Theorem 1.3 later.

The main idea to prove Theorems 1.1–1.3 is to consider the evolution

$$A(t) = e^{itP_f/h}\mathscr{J}^* e^{-itP/h} a(hr, D_r, \theta, hD_\theta) e^{itP/h}\mathscr{J} e^{-itP_f/h}$$

with some symbol a, and use an argument similar to Egorov's theorem for this time-dependent operator. We use a semiclassical argument, i.e., we consider the asymptotic behavior of the operator as $h \to 0$. We consider $W(t) = e^{itP_f/h}\mathcal{J}^* e^{-itP/h}$ as a time-evolution, and then construct an asymptotic solution for $A(t)$ (with slight modifications) as a solution to a Heisenberg equation. The construction of the asymptotic solution relies on the classical Hamilton flow generated by p, the symbol of P. The dominant part of the symbol p is given by the unperturbed conic Hamiltonian: $p_c = \frac{1}{2}\rho^2 + \frac{1}{r^2}q(\theta, \omega)$. The classical scattering operator for the pair p_c and $p_f = \frac{1}{2}\rho^2$ is explicitly computed, and it is $\exp(\pi H_{\sqrt{2q}})$, which appears in the statement of our main theorems. Thus, one may consider our results as a quantization of the classical mechanical scattering on the scattering manifold. More precisely, we show that the canonical transform appearing in Theorem 1.3 is actually the classical scattering map for the pair p and p_f, which is not necessarily homogeneous, and we need to use the method of FIOs with asymptotically conic Lagrangian manifolds.

As mentioned in the beginning, Theorem 1.2 is slight generalization of the Melrose–Zworski theorem [1996] (see also [Vasy 1998] for a simplification of the theory). They used the theory of Legendre distribution and the notion of scattering wave front sets, whereas we use relatively elementary pseudodifferential operator calculus with somewhat nonstandard symbol classes, and a Beals-type characterization of FIOs. We also note that our proof, as well as the setting, are time-dependent-theoretical, and we investigate the scattering phenomena directly to obtain the properties of the wave operators and scattering operators, whereas the Melrose–Zworski paper relies on the stationary, generalized eigenfunction expansion theory.

Our method is closely related to our previous works on the propagation of singularities for Schrödinger evolution equations [Nakamura 2009a; 2009b; Ito and Nakamura 2009; 2012]. In these works, we considered singularities of solutions, which are described by their high energy behavior, whereas in the scattering phenomena we are concerned with the large r behavior (which in turn is related to the high $|\omega|$ behavior, where ω is the conjugate variable to $\theta \in \partial M$). Thus we are forced to use different symbol classes in the calculus, and the corresponding classical mechanics look slightly different, but the general strategy is essentially the same as in these papers.

If $M = \mathbb{R}^n$ and the Hamiltonian P is a short-range perturbation of the Laplacian $-\frac{1}{2}\triangle$, then the canonical map $\exp(\pi H_{\sqrt{2q}})$ is the antipodal map on T^*S^{n-1}. In this case, the off-diagonal smoothness of the scattering cross-section is well-known (see [Isozaki and Kitada 1986], and Section 9.4 and the references of [Yafaev 2000]), and our result (as well as the Melrose–Zworski theorem) may be considered as its generalizations. For such models, our result implies the scattering matrix is an FIO (associated to a canonical map which is asymptotic to the identity map), and if $\mu = 1$ then it is in fact a pseudodifferential operator. It is also not difficult to show from our argument that the scattering matrix is a pseudodifferential operator with symbol in $S^0_{\mu,0}(S^{n-1})$ if $\mu \in (0, 1)$.

The paper is organized as follows. In Section 2, we discuss Hamilton flows generated by p_c and p, and their scattering theory. In Section 3, we prepare the symbol calculus on the scattering manifolds. In Section 4, we discuss an Egorov-type theorem and the construction of asymptotic solutions, which are sufficient to show Theorems 1.1 and 1.2. We prove Theorems 1.1 and 1.2 in Section 5. In Section 6, we discuss the modification of the argument to show Theorem 1.3. We discuss a local decay estimate necessary in the proof in Appendix A. A Beals-type characterization, or an inverse of Egorov's theorem, is discussed in Appendix B, along with a technical lemma on FIOs used in the proof.

Throughout this paper, we use the following notation: For norm spaces X and Y, the space of bounded linear maps is denoted by $B(X, Y)$, and if $X = Y$, we also write $B(X, X) = B(X)$. More generally, if X and Y are topological linear spaces, the space of continuous linear maps is denoted by $\mathscr{L}(X, Y)$. For a symbol g on T^*X with a manifold X, we denote by $\exp(tH_g)$ the Hamilton flow generated by the Hamilton vector field

$$H_g = \frac{\partial g}{\partial \xi}\cdot\frac{\partial}{\partial x} - \frac{\partial g}{\partial x}\cdot\frac{\partial}{\partial \xi}.$$

We also write $T^*X \setminus 0 = \{(x, \xi) \in T^*X \mid \xi \neq 0\}$.

2. Classical flow and scattering theory

In this section, we consider the classical mechanics, or the Hamilton flow for the Hamiltonian with conic structure on T^*M_∞, where $M_\infty = \mathbb{R}_+ \times \partial M$, and then the Hamilton flow generated by the principal symbol of P.

Exact solutions to the conic Hamilton flow. We set

$$p_c(r, \rho, \theta, \omega) = \frac{1}{2}\rho^2 + \frac{1}{r^2} q(\theta, \omega) \quad \text{and} \quad q(\theta, \omega) = \frac{1}{2}\sum_{j,k} h^{jk}(\theta)\omega_j\omega_k$$

on $T^*M_\infty \cong T^*\mathbb{R}_+ \times T^*\partial M$. We consider

$$(r(t), \rho(t), \theta(t), \omega(t)) = \exp(tH_{p_c})(r_0, \rho_0, \theta_0, \omega_0),$$

with $(r_0, \rho_0, \theta_0, \omega_0) \in T^*\mathbb{R}_+ \times (T^*\partial M \setminus 0)$, that is, with $\omega_0 \neq 0$. It satisfies the Hamilton equation

$$r'(t) = \frac{\partial p_c}{\partial \rho} = \rho(t), \qquad \rho'(t) = -\frac{\partial p_c}{\partial r} = \frac{2}{r(t)^3} q(\theta(t), \omega(t)),$$
$$\theta'(t) = \frac{\partial p_c}{\partial \omega} = \frac{1}{r(t)^2}\frac{\partial q}{\partial \omega}(\theta(t), \omega(t)), \qquad \omega'(t) = \frac{\partial p_c}{\partial \theta} = -\frac{1}{r(t)^2}\frac{\partial q}{\partial \theta}(\theta(t), \omega(t)).$$

The solution has two invariants: total energy $E_0 = p_c(r_0, \rho_0, \theta_0, \omega_0)$ and angular energy $q_0 = q(\theta_0, \omega_0)$. (The conservation of the total energy follows from $\{p_c, p_c\} = 0$, and of the angular energy from $\{q, p_c\} = \frac{1}{2}\{q, \rho^2\} + \{q, \frac{1}{r^2}\}q + \frac{1}{r^2}\{q, q\} = 0$.) Then $(r(t), \rho(t))$ satisfies

$$r'(t) = \rho(t), \qquad \rho'(t) = \frac{2}{r(t)^3} q_0,$$

which is independent of $(\theta(t), \omega(t))$. Noting that $(r^2(t))'' = 4E_0$, we can easily solve this equation to obtain

$$r(t) = \sqrt{2E_0t^2 + 2r_0\rho_0 t + r_0^2}, \qquad \rho(t) = \frac{2E_0 t + r_0\rho_0}{\sqrt{2E_0t^2 + 2r_0\rho_0 t + r_0^2}}, \qquad t \in \mathbb{R}.$$

We now set

$$\tau(t) = \int_0^t \frac{ds}{r(s)^2} = \frac{1}{\sqrt{2q_0}}\left(\tan^{-1}\frac{2E_0t + r_0\rho_0}{\sqrt{2q_0}} - \tan^{-1}\frac{r_0\rho_0}{\sqrt{2q_0}}\right).$$

Then $(\theta(t), \omega(t))$ satisfies

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial q}{\partial\omega}(\theta, \omega), \qquad \frac{\partial\omega}{\partial\tau} = -\frac{\partial q}{\partial\theta}(\theta, \omega),$$

and hence we learn that

$$(\theta(t), \omega(t)) = \exp(\tau(t)H_q)(\theta_0, \omega_0).$$

Moreover, if we set $\sigma(t) = \sqrt{2q_0}\cdot\tau(t)$, then we learn that

$$\frac{\partial\theta}{\partial\sigma} = \frac{1}{\sqrt{2q}}\frac{\partial q}{\partial\omega} = \frac{\partial\sqrt{2q}}{\partial\omega}(\theta, \omega), \qquad \frac{\partial\omega}{\partial\sigma} = -\frac{1}{\sqrt{2q}}\frac{\partial q}{\partial\theta} = -\frac{\partial\sqrt{2q}}{\partial\theta}(\theta, \omega),$$

and hence that

$$(\theta(t), \omega(t)) = \exp\bigl(\sigma(t)H_{\sqrt{2q}}\bigr)(\theta_0, \omega_0).$$

Note that $\exp\bigl(tH_{\sqrt{2q}}\bigr)$ is the geodesic flow on ∂M with respect to the (co)metric $(h^{jk}(\theta))$ on $T^*\partial M$.

Classical mechanical wave operators and a scattering operator for the conic Hamilton flow. Now we consider the asymptotics as $t \to \pm\infty$. We set

$$\begin{aligned} r_\pm &= \lim_{t\to\pm\infty}\tilde r(t) = \lim_{t\to\pm\infty}(r(t) - t\rho(t)) = \pm\frac{r_0\rho_0}{\sqrt{2E_0}}, \\ \rho_\pm &= \lim_{t\to\pm\infty}\rho(t) = \pm\sqrt{2E_0}, \\ (\theta_\pm, \omega_\pm) &= \lim_{t\to\pm\infty}(\theta(t), \omega(t)) = \exp(\sigma_\pm H_{\sqrt{2q}})(\theta_0, \omega_0), \end{aligned}$$

where $\sigma_\pm = \pm\frac{1}{2}\pi - \tan^{-1}(r_0\rho_0/\sqrt{2q_0})$. Note we need a modification only for $r(t)$. $(r_\pm, \rho_\pm, \theta_\pm, \omega_\pm)$ are the scattering data for the trajectory $(r(t), \rho(t), \theta(t), \omega(t))$. We also note the identities

$$E_0 = \frac{1}{2}\rho_0^2 + \frac{1}{r_0^2}q_0 = \frac{1}{2}\rho_\pm^2, \qquad r_0\rho_0 = r_\pm\rho_\pm, \qquad q_0 = q(\theta_\pm, \omega_\pm).$$

Using these, we can solve $(r_0, \rho_0, \theta_0, \omega_0)$ for given $(r_\pm, \rho_\pm, \theta_\pm, \omega_\pm)$ if $\pm\rho_\pm > 0$ and $\omega_\pm \neq 0$:

$$r_0 = \sqrt{r_\pm^2 + 2q_0/\rho_\pm^2}, \qquad \rho_0 = \frac{r_\pm\rho_\pm}{\sqrt{r_\pm^2 + 2q_0/\rho_\pm^2}}, \qquad (\theta_0, \omega_0) = \exp(-\sigma_\pm H_{\sqrt{2q}})(\theta_\pm, \omega_\pm),$$

where $\sigma_\pm = \pm\frac{1}{2}\pi - \tan^{-1}(r_\pm\rho_\pm/\sqrt{2q})$. We define the classical wave operators (for the pair p_c and $p_f := \frac{1}{2}\rho^2$) by

$$w_{c,\pm} : (r_\pm, \rho_\pm, \theta_\pm, \omega_\pm) \mapsto (r_0, \rho_0, \theta_0, \omega_0).$$

We can also write

$$w_{c,\pm}(r_\pm, \rho_\pm, \theta_\pm, \omega_\pm) = \lim_{t\to\pm\infty}\exp(-tH_{p_c})\circ\exp(tH_{p_f})(r_\pm, \rho_\pm, \theta_\pm, \omega_\pm).$$

It is easy to check that $w_{c,\pm}$ are diffeomorphisms from $\mathbb{R}\times\mathbb{R}_\pm\times(T^*\partial M\setminus 0)$ to $\mathbb{R}_+\times\mathbb{R}\times(T^*\partial M\setminus 0)$. Hence the classical scattering operator

$$s_c = w_{c,+}^{-1}\circ w_{c,-} : (r_-, \rho_-, \theta_-, \omega_-) \mapsto (r_+, \rho_+, \theta_+, \omega_+)$$

is a diffeomorphism from $\mathbb{R}\times\mathbb{R}_-\times(T^*\partial M\setminus 0)$ to $\mathbb{R}\times\mathbb{R}_+\times(T^*\partial M\setminus 0)$. We can easily compute s_c explicitly, we have

$$s_c(r,\rho,\theta,\omega)=(-r,-\rho,\exp(\pi H_{\sqrt{2q}})(\theta,\rho)),$$

and this is the classical analogue of the Melrose–Zworski theorem.

We write

$$w_c(t)=\exp(-tH_{p_c})\circ\exp(tH_{p_f})\quad\text{so that } w_{c,\pm}=\lim_{t\to\pm\infty}w_c(t).$$

Let $U\subset\mathbb{R}_+\times\mathbb{R}\times(T^*\partial M\setminus 0)$ be a relatively compact domain. Then the convergence of $w_c(t)^{-1}$ to $w_{c,\pm}^{-1}$ (as $t\to\pm\infty$) is uniform in U, along with all derivatives. Since the limit is a diffeomorphism, its inverse $w(t)$ also has the same property (on $w_c(t)^{-1}U$). In particular, all the derivatives of $w_c(t)^{-1}$ on U are uniformly bounded in t, and all the derivatives of $w_c(t)$ on $w_c(t)^{-1}U$ are uniformly bounded.

We note that it is easy to check that $w_{c,\pm}$ and hence s_c are homogeneous of order one with respect to the (r,ω)-variables, i.e.,

$$w_{c,\pm}^{-1}(\lambda r_0,\rho_0,\theta_0,\lambda\omega_0)=(\lambda r_\pm,\rho_\pm,\theta_\pm,\lambda\omega)\quad\text{for }\lambda>0.$$

This is consistent with the scaling property of $w_c(t)$:

$$w_c^{-1}(\lambda t)(\lambda r_0,\rho_0,\theta_0,\lambda\omega_0)=(\lambda\tilde r(t),\rho(t),\theta(t),\lambda\omega(t))$$

for any $\lambda>0$, $t\in\mathbb{R}$.

Classical flow generated by the scattering metric. Here we discuss the Hamilton flow generated by the symbol of P:

$$p(r,\rho,\theta,\omega)=\frac{1}{2}\left(a_1(r,\theta)\rho^2+\frac{2\rho a_2(r,\theta)\cdot\omega}{r}+\frac{\omega\cdot a_3(r,\theta)\omega}{r^2}\right)+V \tag{2-1}$$

on T^*M_∞.

We let $\Omega_0\Subset T^*\mathbb{R}_+\times(T^*\partial M\setminus 0)$. For $h\in(0,1]$, we set

$$\Omega_0^h=\{(r,\rho,\theta,\omega)\in T^*\mathbb{R}_+\times(T^*\partial M\setminus 0)\mid (hr,\rho,\theta,h\omega)\in\Omega_0\},$$

and we consider the Hamilton flow with initial conditions in Ω_0^h. We show that if h is sufficiently small then the classical (inverse) wave operators exist on Ω_0^h, and they are very close to $w_{c,\pm}^{-1}$, the (inverse) wave operators for the conic metric.

Theorem 2.1. (i) *Let Ω_0 and Ω_0^h as above. Then there is $h_0>0$ such that if $h\in(0,h_0]$, then*

$$w_\pm^*(r,\rho,\theta,\omega):=\lim_{t\to\pm\infty}\exp(-tH_{p_f})\circ\exp(tH_p)(r,\rho,\theta,\omega)$$

exists for $(r,\rho,\theta,\omega)\in\Omega_0^h$, and the convergence holds in the C^∞-topology on Ω_0^h.

(ii) *We write*

$$(r(t),\rho(t),\theta(t),\omega(t))=\exp(tH_p)(r,\rho,\theta,\omega),$$

$$(r_c(t),\rho_c(t),\theta_c(t),\omega_c(t))=\exp(tH_{p_c})(r,\rho,\theta,\omega),$$

for $(r,\rho,\theta,\omega)\in\Omega_0^h$. *Then for any indices* α,β,γ *and* δ, *there is* $C>0$ *such that*

$$|\partial_r^\alpha\partial_\rho^\beta\partial_\theta^\gamma\partial_\omega^\delta(r(t)-r_c(t))|+|\partial_r^\alpha\partial_\rho^\beta\partial_\theta^\gamma\partial_\omega^\delta(\omega(t)-\omega_c(t))|\le Ch^{-1+\mu+|\alpha|+|\delta|},$$
$$|\partial_r^\alpha\partial_\rho^\beta\partial_\theta^\gamma\partial_\omega^\delta(\rho(t)-\rho_c(t))|+|\partial_r^\alpha\partial_\rho^\beta\partial_\theta^\gamma\partial_\omega^\delta(\theta(t)-\theta_c(t))|\le Ch^{\mu+|\alpha|+|\delta|},$$

for $(r,\rho,\theta,\omega)\in\Omega_0^h$, $t\in\mathbb{R}$, $0<h\le h_0$.

(iii) *If we write*

$$w_\pm^*(r,\rho,\theta,\omega)=(r_\pm,\rho_\pm,\theta_\pm,\omega_\pm)\quad\textit{and}\quad w_{c,\pm}^*(r,\rho,\theta,\omega)=(r_{c,\pm},\rho_{c,\pm},\theta_{c,\pm},\omega_{c,\pm}),$$

then

$$|\partial_r^\alpha\partial_\rho^\beta\partial_\theta^\gamma\partial_\omega^\delta(r_\pm-r_{c,\pm})|+|\partial_r^\alpha\partial_\rho^\beta\partial_\theta^\gamma\partial_\omega^\delta(\omega_\pm-\omega_{c,\pm})|\le Ch^{-1+\mu+|\alpha|+|\delta|},$$
$$|\partial_r^\alpha\partial_\rho^\beta\partial_\theta^\gamma\partial_\omega^\delta(\rho_\pm-\rho_{c,\pm})|+|\partial_r^\alpha\partial_\rho^\beta\partial_\theta^\gamma\partial_\omega^\delta(\theta_\pm-\theta_{c,\pm})|\le Ch^{\mu+|\alpha|+|\delta|}$$

for $(r,\rho,\theta,\omega)\in\Omega_0^h$, $0<h\le h_0$.

For $(r_0,\rho_0,\theta_0,\omega_0)\in\Omega_0$, we define $(r^h(t),\rho^h(t),\theta^h(t),\omega^h(t))$ so that

$$(h^{-1}r^h(t),\rho^h(t),\theta^h(t),h^{-1}\omega^h(t))=\exp(h^{-1}tH_p)(h^{-1}r_0,\rho_0,\theta_0,h^{-1}\omega_0).$$

We also set

$$p^h(r,\rho,\theta,\omega)=p(h^{-1}r,\rho,\theta,h^{-1}\omega),\quad (r,\rho,\theta,\omega)\in T^*M_\infty.$$

Then it is easy to check that

$$(r^h(t),\rho^h(t),\theta^h(t),\omega^h(t))=\exp(tH_{p^h})(r_0,\rho_0,\theta_0,\omega_0).$$

On the other hand, if we write

$$p^h(r,\rho,\theta,\omega)=p_c(r,\rho,\theta,\omega)+v^h(r,\rho,\theta,\omega),$$

then we learn by Assumption A that for any indices $\alpha,\beta,\gamma,\delta$,

$$\left|\partial_r^\alpha\partial_\rho^\beta\partial_\theta^\gamma\partial_\omega^\delta v^h(r,\rho,\theta,\omega)\right|\le C_{\alpha\beta\gamma\delta}h^\mu\big(r^{-1}\langle\rho\rangle^2+r^{-1}\langle\rho\rangle\langle\omega\rangle+r^{-2}\langle\omega\rangle^2\big)r^{-\mu-|\alpha|}\langle\rho\rangle^{-|\beta|}\langle\omega\rangle^{-|\delta|}.\tag{2-2}$$

In order to prove Theorem 2.1, it suffices to show:

Theorem 2.2. (i) *There is* $h_0>0$ *such that if* $h\in(0,h_0]$, *then*

$$w_{\pm,h}^*(r_0,\rho_0,\theta_0,\omega_0):=\lim_{t\to\pm\infty}\exp(-tH_{p_f})\circ\exp(tH_{p^h})(r_0,\rho_0,\theta_0,\omega_0)$$

exists for $(r_0,\rho_0,\theta_0,\omega_0)\in\Omega_0$, *and the convergence holds in the* C^∞*-topology.*

(ii) *For any indices* $\alpha,\beta,\gamma,\delta$, *there is* $C>0$ *such that*

$$\left|\partial_{r_0}^\alpha\partial_{\rho_0}^\beta\partial_{\theta_0}^\gamma\partial_{\omega_0}^\delta(r^h(t)-r_c(t))\right|+\left|\partial_{r_0}^\alpha\partial_{\rho_0}^\beta\partial_{\theta_0}^\gamma\partial_{\omega_0}^\delta(\rho^h(t)-\rho_c(t))\right|$$
$$+\left|\partial_{r_0}^\alpha\partial_{\rho_0}^\beta\partial_{\theta_0}^\gamma\partial_{\omega_0}^\delta(\theta^h(t)-\theta_c(t))\right|+\left|\partial_{r_0}^\alpha\partial_{\rho_0}^\beta\partial_{\theta_0}^\gamma\partial_{\omega_0}^\delta(\omega^h(t)-\omega_c(t))\right|\le Ch^\mu$$

for $(r_0,\rho_0,\theta_0,\omega_0)\in\Omega_0$, $h\in(0,h_0]$, $t\in\mathbb{R}$, *where*

$$(r_c(t),\rho_c(t),\theta_c(t),\omega_c(t))=\exp(tH_{p_c})(r_0,\rho_0,\theta_0,\omega_0).$$

(iii) *Writing*

$$(r^h_\pm, \rho^h_\pm, \theta^h_\pm, \omega^h_\pm) = w^*_{\pm,h}(r_0, \rho_0, \theta_0, \omega_0),$$

we have for any indices $\alpha, \beta, \gamma, \delta$ *that*

$$\begin{aligned}\big|\partial^\alpha_{r_0}\partial^\beta_{\rho_0}\partial^\gamma_{\theta_0}\partial^\delta_{\omega_0}(r^h_\pm - r_{c,\pm})\big| + \big|\partial^\alpha_{r_0}\partial^\beta_{\rho_0}\partial^\gamma_{\theta_0}\partial^\delta_{\omega_0}(\rho^h_\pm - \rho_{c,\pm})\big| \\ + \big|\partial^\alpha_{r_0}\partial^\beta_{\rho_0}\partial^\gamma_{\theta_0}\partial^\delta_{\omega_0}(\theta^h_\pm - \theta_{c,\pm})\big| + \big|\partial^\alpha_{r_0}\partial^\beta_{\rho_0}\partial^\gamma_{\theta_0}\partial^\delta_{\omega_0}(\omega^h_\pm - \omega_{c,\pm})\big| \le Ch^\mu.\end{aligned}$$

Proof of Theorem 2.2. The proof is analogous to the arguments in [Nakamura 2009a, Section 2; Ito and Nakamura 2009, Section 2]. We only outline the proof, and we omit the details.

Step 1. By the standard virial-type argument, we learn that there is $R > 0$ such that

$$\frac{d^2}{dt^2}(r^h(t)^2) \ge c > 0 \quad \text{if } r^h(t) \ge R,$$

if $(r_0, \rho_0, \theta_0, \omega_0) \in \Omega_0$. Here we use the fact that $|\rho|$ and $|\omega/r|$ are uniformly bounded by the conservation of energy. On the other hand, since $v^h = O(h^\mu)$, we also learn that $r^h(t) \to r_c(t)$ as $h \downarrow 0$, locally uniformly in t. Thus, if t_0 is large and h is small enough, $r^h(t) \ge R$, and combining this with the above observation, we have

$$|r^h(t)| \ge \sqrt{R + c|t - t_0|^2/2} \quad \text{for } t \ge t_0.$$

Hence we learn

$$c_1\langle t\rangle \le r^h(t) \le c_2\langle t\rangle \quad \text{for } h \in (0, h_0],\ t > 0,$$

with some $h_0, c_1, c_2 > 0$. The case $t < 0$ can be handled similarly.

Step 2. We consider the time evolution of $q_0(t) = q(\theta^h(t), \omega^h(t))$. By the Hamilton equation and (2-2), we have

$$\frac{d}{dt}q_0(t) = -\{p^h, q_0\} = -\{v^h, q_0\} = O(h^\mu r^{-1-\mu}\langle\omega\rangle^2) = O(h^\mu\langle t\rangle^{-1-\mu}(1 + q_0(t))).$$

Here we have used the boundedness of $|\rho(t)|$ and $|\omega(t)/r(t)|$ again. Then by the Duhamel formula, we learn that $q_0(t)$ is uniformly bounded for initial conditions in Ω_0 and $h \in (0, h_0]$. This implies $|\omega^h(t)|$ is also uniformly bounded.

Step 3. Combining these observations with the Hamilton equation, we learn that

$$\left|\frac{d\rho^h(t)}{dt}\right| \le C\langle t\rangle^{-2-\mu}, \quad \left|\frac{d\theta^h(t)}{dt}\right| \le C\langle t\rangle^{-1-\mu}, \quad \left|\frac{d}{dt}(r^h(t) - t\rho^h(t))\right| \le C\langle t\rangle^{-1-\mu}, \quad \left|\frac{d\omega^h(t)}{dt}\right| \le C\langle t\rangle^{-1-\mu},$$

uniformly for $(r_0, \rho_0, \theta_0, \omega_0) \in \Omega_0$, $h \in (0, h_0]$ and $t \in \mathbb{R}$. These imply the existence of $w^*_{\pm,h}$ on Ω_0. We can show the similar estimates for the derivatives, i.e.,

$$\left|\frac{d}{dt}\big(\partial^\alpha_{r_0}\partial^\beta_{\rho_0}\partial^\gamma_{\theta_0}\partial^\delta_{\omega_0}\rho^h(t)\big)\right| \le C\langle t\rangle^{-2-\mu-|\alpha|},$$

and so on. These imply the convergence in C^∞-topology, and we conclude that assertion (i) holds.

Step 4. We set

$$g^h(t) = |r^h(t) - r_c(t)| + |\rho^h(t) - \rho_c(t)| + |\theta^h(t) - \theta_c(t)| + |\omega^h(t) - \omega_c(t)|.$$

Then by the Hamilton equation, (2-2), and the estimates in Steps 1 and 2, we learn that

$$\left|\frac{d}{dt} g^h(t)\right| \le C\langle t\rangle^{-1-\mu} g^h(t) + Ch^\mu \langle t\rangle^{-1-\mu}$$

uniformly for initial conditions in Ω_0 and $h \in (0, h_0]$. Then by using the Duhamel formula and noting that $g^h(0) = 0$, we obtain

$$|g^h(t)| \le Ch^\mu, \quad t \in \mathbb{R}.$$

This is assertion (ii) with $\alpha = \beta = \gamma = \delta = 0$. The derivatives can be estimated similarly by induction. For the details of this argument, we refer to [Craig et al. 1995, Section 2; Nakamura 2009a, Section 2]. Assertion (iii) follows immediately from assertion (ii). □

By the above argument, we also learn that $w^*_{\pm,h}$ are invertible for small h. The inverses are uniformly bounded, and their inverses

$$w_{\pm,h} = (w^*_{\pm,h})^{-1}$$

are well-defined for $h \in (0, h_0]$. It follows that

$$w_\pm = (w^*_\pm)^{-1}$$

is well-defined and diffeomorphic on $w^*_\pm[\Omega^h_0]$ with $h \in (0, h_0]$. Thus we can define the classical scattering operator by

$$s = w^*_+ \circ w_-$$

on $w^*_-[\Omega^h_0]$, with sufficiently small h.

3. Symbol classes and their quantization on scattering manifolds

Here we prepare a pseudodifferential operator calculus which is used extensively in the proof of the main theorems. We refer to [Hörmander 1985; Taylor 1981, Chapter XVIII] for the standard theory of microlocal analysis.

In the following, we employ symbol calculus on T^*M, but we always suppose the symbol is supported in T^*M_∞, and we use a local coordinate system as in Section 1. More specifically, we choose a local coordinate system on ∂M: $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^{n-1}\}$, $U_\alpha \subset \partial M$, and we use the coordinate system $\{1 \otimes \varphi_\alpha : \mathbb{R}_+ \times U_\alpha \to \mathbb{R} \times \mathbb{R}^{n-1}\}$ on M_∞. We also use a similar local coordinate system on M_f, defined by $\{1 \otimes \varphi_\alpha : \mathbb{R} \times U_\alpha \to \mathbb{R} \times \mathbb{R}^{n-1}\}$. We often identify U_α (or $\mathbb{R}_+ \times U_\alpha$, $\mathbb{R} \times U_\alpha$, respectively) with Ran φ_α (or Ran $(1 \otimes \varphi_\alpha)$, respectively).

Symbol classes. We define a metric either on T^*M_∞ or T^*M_f by

$$g_1 = \frac{dr^2}{\langle r\rangle^2} + d\rho^2 + d\theta^2 + \frac{d\omega^2}{\langle\omega\rangle^2},$$

and consider symbols in $S(m, g_1)$ with a weight function m, i.e., $a \in S(m, g_1)$ if and only if for any indices a, β, γ, δ, there is C such that

$$\left|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta a(r, \rho, \theta, \omega)\right| \leq C m(r, \rho, \theta, \omega)\langle r\rangle^{-|\alpha|}\langle\omega\rangle^{-|\delta|}.$$

Later, we will consider the calculus of such symbols on sets $\Omega^h = \{(r, \rho, \theta, \omega) \mid (hr, \rho, \theta, h\omega) \in \Omega\}$, where $\Omega \subset T^*M_\infty$ is some compact set (supported away from $\{\omega = 0\}$) and $h > 0$ is small. In such cases, the symbol satisfies

$$\left|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta a(h; r, \rho, \theta, \omega)\right| \leq C m(h) h^{|\alpha|+|\delta|},$$

and we denote such a (h-dependent) symbol as $a \in S_h(m, g_1^h)$, where m is an h-dependent weight. The corresponding metric is naturally

$$g_1^h = h^2 dr^2 + d\rho^2 + d\theta^2 + h^2 d\omega^2.$$

Weyl quantization. Let $\{\chi_\alpha^2\}$ be a partition of unity on ∂M compatible with our coordinate system $\{\varphi_\alpha, U_\alpha\}$, that is, $\chi_\alpha \in C_0^\infty(U_\alpha)$ and $\sum_\alpha \chi_\alpha(\theta)^2 \equiv 1$ on ∂M. We set $\tilde{\chi}_\alpha(r, \theta) = \chi_\alpha(\theta) j(r) \in C^\infty(M_\infty)$. Let $a \in S(m, g_1)$ be a symbol on T^*M_∞, and let $u \in C_0^\infty(T^*M)$. We denote by $a_{(\alpha)}$ and $G_{(\alpha)}$ the representations of a and G in the local coordinate $(1 \otimes \varphi_\alpha, \mathbb{R} \times U_\alpha)$, respectively. We quantize a by

$$\mathrm{Op}^W(a)u = \sum_\alpha \tilde{\chi}_\alpha G_{(\alpha)}^{-1/2} a_{(\alpha)}^W(r, D_r, \theta, D_\theta) G_{(\alpha)}^{1/2} \tilde{\chi}_\alpha u,$$

where $a_{(\alpha)}^W(r, D_r, \theta, D_\theta)$ denotes the usual Weyl quantization on the Euclidean space $\mathbb{R}^n$, and we use the identification $\mathbb{R}_+ \times U_\alpha \cong \mathbb{R}_+ \times (\mathrm{Ran}\, \varphi_\alpha)$ for each α. (Strictly speaking, we should have written this as

$$\mathrm{Op}^W(a)u = \sum_\alpha \tilde{\chi}_\alpha (\tilde{\varphi}_\alpha)^* \big(G_{(\alpha)}^{-1/2} a_{(\alpha)}^W(r, D_r, \theta, D_\theta) G_{(\alpha)}^{1/2} (\tilde{\varphi}_\alpha)_*(\tilde{\chi}_\alpha u)\big),$$

but we will omit $(\tilde{\varphi}_\alpha)_*$, $(\tilde{\varphi}_\alpha)^*$, ..., when there can be no confusion.) This definition is compatible with the standard definition of pseudodifferential operators on manifolds, but we choose a specific quantization that preserves the asymptotically conic structure of M. Similarly, for a symbol a on T^*M_f, we quantize it by

$$\mathrm{Op}^W(a)u = \sum_\alpha \chi_\alpha H_{(\alpha)}^{-1/2} a_{(\alpha)}^W(r, D_r, \theta, D_\theta) H_{(\alpha)}^{1/2} \chi_\alpha u$$

for $u \in C_0^\infty(M_f)$, where $H_{(\alpha)}$ denotes the representation of H in the local coordinate $(\varphi_\alpha, U_\alpha)$. In this case, the linear structure in r is preserved.

In the above definition, we put weights around the locally defined pseudodifferential operators a_α^W so that $\mathrm{Op}^W(a)$ is symmetric if a is real-valued. Moreover, by virtue of these weights, the symbol corresponding to the operator is unique, including the subprincipal symbol, though we will not take advantage of this fact in this paper.

The above definitions of quantizations also have the convenient property that if we identify a symbol a on T^*M_∞ with a symbol on T^*M_f (by the obvious identification), then we have

$$\mathcal{I}\mathrm{Op}^W(a)\mathcal{I}^* = \mathrm{Op}^W(a) \quad \text{on } \mathcal{H},$$

provided a is supported in $\{r > 1\}$, and we may identify these quantizations by using $\mathcal{I}$. For a symbol supported in $\{r > 1\}$, we may consider $\mathrm{Op}^W(a)$ as an operator from $\mathcal{H}$ to $\mathcal{H}_f$ (or from $\mathcal{H}_f$ to $\mathcal{H}$) also. We define these operators by

$$\mathrm{Op}^W(a)u = \sum_\alpha \chi_\alpha H_{(\alpha)}^{-1/2} a_{(\alpha)}^W(r, D_r, \theta, D_\theta) G_{(\alpha)}^{1/2} \tilde{\chi}_\alpha u$$

for $u \in C_0^\infty(M)$ and

$$\mathrm{Op}^W(a)u = \sum_\alpha \tilde{\chi}_\alpha G_{(\alpha)}^{-1/2} a_{(\alpha)}^W(r, D_r, \theta, D_\theta) H_{(\alpha)}^{1/2} \chi_\alpha u$$

for $u \in C_0^\infty(M_f)$.

If $A = \mathrm{Op}^W(a)$, we denote the (Weyl) symbol of A by $a = \Sigma(A)$.

Hamiltonians. Now we consider properties of our Schrödinger operators and related operators as a preparation for the next section.

We note that, as in the usual Weyl calculus on $\mathbb{R}^n$, if $a(x, \xi) = \sum_{j,k} a_{jk}(x)\xi_j\xi_k$, then

$$\mathrm{Op}^W(a) = \sum_{j,k} D_j a_{jk}(x) D_k - \frac{1}{4}\sum_{j,k}(\partial_j\partial_k a_{jk}(x)).$$

Hence, if we let p be the symbol of P as in (2-1), we have

$$\mathrm{Op}^W(p) = P + f,$$

where $f \in C^\infty(M_f)$ is such that

$$\left|\partial_r^\alpha \partial_\theta^\beta f(r, \theta)\right| \le C_{\alpha\beta}\langle r\rangle^{-2-|\alpha|}$$

for any α, β. Thus, we can include this error term in V and we may consider $P = \mathrm{Op}^W(p)$. On the other hand, it is easy to see $P_f = \mathrm{Op}^W(p_f)$ on $\mathcal{H}_f$, where $p_f = \frac{1}{2}\rho^2$.

4. An Egorov-type theorem

Let $(r_0, \rho_0, \theta_0, \omega_0) \in T^*(\mathbb{R}_+ \times \partial M)$, $\omega_0 \ne 0$, and suppose $a \in C_0^\infty(T^*(\mathbb{R}_+ \times \partial M))$ is supported in a small neighborhood of $(r_0, \rho_0, \theta_0, \omega_0)$ so that a is supported away from $\{\omega = 0\}$. We set

$$a^h(r, \rho, \theta, \omega) = a(h; hr, \rho, \theta, h\omega), \quad h > 0,$$

where a itself may depend on the parameter $h > 0$, but we suppose it is bounded uniformly in the C_0^∞-topology, and supported in the same small neighborhood of $(r_0, \rho_0, \theta_0, \omega_0)$. The notation here is different from that of Section 2. We set

$$A_0 = \mathrm{Op}^W(a^h) \quad \text{on } M.$$

We set $\varepsilon > 0$ so small that

$$\exp(tH_{p_c})(\operatorname{supp} a) \cap \{r \le \varepsilon\langle t\rangle\} = \varnothing$$

for all $t \in \mathbb{R}$. We choose $\eta \in C^\infty(\mathbb{R})$ such that $\eta(r) = 1$ for $r \geq 1$ and $\eta(r) = 0$ for $r \leq 1/2$, and we set

$$Y = \eta\Big(\frac{hr}{\varepsilon\langle t\rangle}\Big).$$

Then we define

$$A(t) = e^{itP_f/h}\mathcal{J}^* Y e^{-itP/h} A_0 e^{itP/h} Y \mathcal{J} e^{-itP_f/h}$$

for $t \in \mathbb{R}$. The purpose of this section is to obtain the symbols of $A(t)$ as a pseudodifferential operator, and to study its behavior as $t \to \pm\infty$.

We compute (formally) that

$$\frac{d}{dt}\big(e^{itP/h} Y \mathcal{J} e^{-itP_f/h}\big) = \frac{i}{h} e^{itP/h} T(t) e^{-itP_f/h},$$

where

$$T(t) = PY\mathcal{J} - Y\mathcal{J}P_f - \frac{h(hr)t}{i\varepsilon\langle t\rangle^3}\eta'\Big(\frac{hr}{\varepsilon\langle t\rangle}\Big)\mathcal{J}.$$

We further rewrite this as

$$\begin{aligned}\frac{d}{dt}\big(e^{itP/h}Y\mathcal{J}e^{-itP_f/h}\big) &= \frac{i}{h}\big(e^{itP/h}Y\mathcal{J}e^{-itP_f/h}\big)\big(e^{itP_f/h}\mathcal{J}^*T(t)e^{-itP_f/h}\big) + \frac{i}{h}e^{itP/h}(1-Y\mathcal{J}\mathcal{J}^*)T(t)e^{-itP_f/h}\\ &= \frac{i}{h}\big(e^{itP/h}Y\mathcal{J}e^{-itP_f/h}\big)L(t) + R_1(t),\end{aligned}$$

where

$$L(t) = e^{itP_f/h}\mathcal{J}^*T(t)e^{-itP_f/h} \quad \text{and} \quad R_1(t) = \frac{i}{h}e^{itP/h}(1-Y\mathcal{J}\mathcal{J}^*)T(t)e^{-itP_f/h}.$$

We now consider the symbols of $T(t)$ and $L(t)$ as pseudodifferential operators. By direct computations, it is easy to see that for any indices α, β, γ, δ,

$$\begin{aligned}&\big|\partial_r^\alpha\partial_\rho^\beta\partial_\theta^\gamma\partial_\omega^\delta\Sigma(T(t))(r,\rho,\theta,\omega)\big|\\ &\qquad\leq C\big(\langle r\rangle^{-1-\mu}\langle\rho\rangle^2 + \langle r\rangle^{-1-\mu}\langle\rho\rangle\langle\omega\rangle + \langle r\rangle^{-2}\langle\omega\rangle^2\big)\langle r\rangle^{-|\alpha|}\langle\rho\rangle^{-|\beta|}\langle\omega\rangle^{-|\delta|}. \quad (4\text{-}1)\end{aligned}$$

Since $T(t)$ is supported in $\{r \geq \varepsilon\langle t\rangle/2h\}$, we may replace $\langle r\rangle$ by $\langle r\rangle + \varepsilon\langle t\rangle/2h$ in the above estimate. We also have

$$\begin{aligned}&\bigg|\partial_r^\alpha\partial_\rho^\beta\partial_\theta^\gamma\partial_\omega^\delta\bigg(\Sigma(T(t)) - Y\frac{q(\theta,\omega)}{r^2}\bigg)\bigg|\\ &\qquad\leq C\big(\langle r\rangle^{-1-\mu}\langle\rho\rangle^2 + \langle r\rangle^{-1-\mu}\langle\rho\rangle\langle\omega\rangle + \langle r\rangle^{-2-\mu}\langle\omega\rangle^2\big)\langle r\rangle^{-|\alpha|}\langle\rho\rangle^{-|\beta|}\langle\omega\rangle^{-|\delta|}.\end{aligned}$$

In particular, we learn that

$$\bigg|\partial_r^\alpha\partial_\rho^\beta\partial_\theta^\gamma\partial_\omega^\delta\bigg(\Sigma(T(t)) - Y\frac{q(\theta,\omega)}{r^2}\bigg)\bigg| \leq C\langle t\rangle^{-1-\mu-|\alpha|}h^{\mu+|\alpha|+|\delta|} \quad (4\text{-}2)$$

on $\exp(tH_{p_c})[\operatorname{supp} a^h]$, where the constant is independent of t and h.

Now we note, by virtue of the Weyl calculus (and our choice of the quantization), that

$$\Sigma(L(t))(r,\rho,\theta,\omega) = \Sigma(\mathcal{J}^*T(t))(r + (t/h)\rho, \rho, \theta, \omega).$$

Hence we have, by (4-1),

$$|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta \Sigma(L(t))| \le C\big(\langle\tilde r\rangle^{-1-\mu}\langle\rho\rangle^2 + \langle\tilde r\rangle^{-1-\mu}\langle\rho\rangle\langle\omega\rangle + \langle\tilde r\rangle^{-2}\langle\omega\rangle^2\big)\langle\tilde r\rangle^{-|\alpha|}\langle\omega\rangle^{-|\delta|},$$

where $\tilde r = r + (t/h)\rho$. Note that we take advantage of the cut-off function Y in this estimate. We also note, along with (4-2), that

$$\left|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta \left(\Sigma(L(t)) - \frac{q(\theta,\omega)}{\tilde r^2}\right)\right| \le C\langle t\rangle^{-1-\mu-|\alpha|} h^{\mu+|\alpha|+|\delta|} \tag{4-3}$$

on $\exp(-tH_{p_f})\circ\exp(tH_{p_c})[\operatorname{supp} a^h] = \operatorname{supp}(a^h\circ w_c(t))$.

We then construct an asymptotic solution to the Heisenberg equation as

$$\frac{d}{dt}B(t) = -\frac{i}{h}[L(t), B(t)], \qquad B(0) = \mathcal{I}^* A_0 \mathcal{I}. \tag{4-4}$$

Lemma 4.1. *There exists $b^h(t; r, \rho, \theta, \omega) \in C_0^\infty(T^*M_f)$ satisfying the following conditions*:

(i) $b^h(0) = a^h$.

(ii) *$b^h(t)$ is supported in $w_c(t/h)^{-1}[\operatorname{supp} a^h]$.*

(iii) *$b^h(t) \in S(1, g_1^h)$, and it is bounded uniformly in $t \in \mathbb{R}$.*

(iv) *$b^h(t) - a^h\circ w_c(t/h) \in S(h^\mu, g_1^h)$, i.e., the principal symbol of $b^h(t)$ is given by $a^h\circ w_c(t/h)$, and the remainder is bounded uniformly in t.*

(v) *If we set $B(t) = \mathrm{Op}^W(b^h(t))$, then*

$$\left\|\frac{d}{dt}B(t) + \frac{i}{h}[L(t), B(t)]\right\| \le C_N\langle t\rangle^{-1-\mu}h^N, \qquad h > 0,$$

for any N.

(vi) *$B(t)$ converges to $B_\pm$ as $t\to\pm\infty$ in $B(\mathcal{H}_f)$, and the symbols $b_\pm^h := \Sigma(B_\pm)$ satisfy*

$$b_\pm^h - a^h\circ w_{c,\pm} \in S(h^\mu, g_1^h).$$

Proof. We follow the standard procedure to construct asymptotic solutions to Heisenberg equations (see [Taylor 1981, Chapter 8; Martinez 2002, Chapter 4]). We let

$$\ell_0(t; r, \rho, \theta, \omega) = \frac{q(\theta,\omega)}{(r+t\rho)^2}$$

be the principal symbol of $L(ht)$. If we set

$$b_0(t) = a\circ w_c(t) = a\circ\exp(-tH_{p_c})\circ\exp(tH_{p_f}),$$

then b_0 satisfies the equation

$$\frac{\partial}{\partial t}b_0(t) = -\{\ell_0(t), b_0(t)\}, \qquad b_0(0) = a.$$

We set $b_0^h(t; r, \rho, \theta, \omega) = b_0(t/h; hr, \rho, \theta, h\omega)$, and we also set $B_0(t) = \mathrm{Op}^W(b_0^h(t))$. We note that

$$|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta b_0^h(t; r, \rho, \theta, \omega)| \le Ch^{|\alpha|+|\delta|}$$

uniformly in t with any $\alpha, \beta, \gamma, \delta$, since $b_0(t)$ converges to $a \circ w_{c,\pm}$ as $t \to \pm\infty$. We write

$$R_0^0(t) = \frac{d}{dt} B_0(t) + \frac{i}{h}[L(t), B_0(t)], \quad r_0^0(t) = \Sigma(R_0^0(t)).$$

Then by (4-3) and the symbol calculus, $r_0^0(t)$ is supported on $w_c(t/h)^{-1}[\operatorname{supp} a^h]$ modulo $O(h^\infty)$-terms, and

$$\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta r_0^0(t) \le C\langle t\rangle^{-1-\mu-|\alpha|} h^{\mu+|\alpha|+|\delta|} \tag{4-5}$$

for any $\alpha, \beta, \gamma, \delta$. We set $\tilde{r}_0^0(t)$ so that

$$\tilde{r}_0^0(t/h; hr, \rho, \theta, h\omega) = r_0^0(t; r, \rho, \theta, \omega),$$

and solve the transport equation

$$\frac{\partial}{\partial t} b_1(t) + \{\ell_0(t), b_1(t)\} = -\tilde{r}_0^0(t), \quad b_1(0) = 0.$$

By (4-5), it is easy to observe that $|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta b_1(t; r, \rho, \theta, \omega)| \le Ch^\mu$ uniformly in t. Moreover, $b_1(t)$ converges to a symbol supported in $w_{c,\pm}^{-1}[\operatorname{supp} a]$ in the C_0^∞-topology as $t \to \pm\infty$. We then set

$$B_1(t) = \operatorname{Op}^W(b_1^h(t)), \quad b_1^h(t; r, \rho, \theta, \omega) = b_1(t/h; hr, \rho, \theta, h\omega).$$

We construct b_j, $j = 1, 2, \ldots$, iteratively, so that $b_j^h \in S(h^{j\mu}, g_1^h)$, and set

$$b^h(t) \sim \sum_{j=0}^{\infty} b_j^h(t), \quad B(t) = \operatorname{Op}^W(b^h(t)).$$

By construction, $b^h(t)$ and $B(t) = \operatorname{Op}^W(b^h(t))$ satisfy the assertion. □

We then observe that $A(t)$ is very close to $B(t)$ constructed as above.

Lemma 4.2. *For any N, there is $C_N > 0$ such that*

$$\|A(t) - B(t)\| \le C_N h^N, \quad t \in \mathbb{R}.$$

In particular, A_+ and A_-, defined by

$$A_\pm := \operatorname*{w-lim}_{t\to\pm\infty} A(t),$$

have the symbols $b_\pm^h$ as pseudodifferential operators.

Proof. We first observe that

$$\begin{aligned}
\|A(t) - B(t)\| &= \|e^{itP_f/h}\mathcal{J}^* Y e^{-itP/h} A_0 e^{itP/h} Y \mathcal{J} e^{-itP_f/h} - B(t)\| \\
&= \|\mathcal{J}^* Y e^{-itP/h} A_0 e^{itP/h} Y \mathcal{J} - e^{-itP_f/h} B(t) e^{itP_f/h}\| \\
&\le \|Y\mathcal{J}\mathcal{J}^* Y e^{-itP/h} A_0 e^{itP/h} Y \mathcal{J}\mathcal{J}^* Y - Y\mathcal{J} e^{-itP_f/h} B(t) e^{itP_f/h} \mathcal{J}^* Y\| \\
&\le \|e^{-itP/h} A_0 e^{itP/h} - Y\mathcal{J} e^{-itP_f/h} B(t) e^{itP_f/h}\mathcal{J}^* Y\| + R_2 \\
&= \|A_0 - \tilde{B}(t)\| + R_2,
\end{aligned}$$

where

$$R_2 = 2\|(1 - Y\mathcal{J}\mathcal{J}^* Y)e^{-itP/h} A_0\| \quad \text{and} \quad \tilde{B}(t) = e^{itP/h} Y\mathcal{J} e^{-itP_f/h} B(t) e^{itP_f/h} \mathcal{J}^* Y e^{-itP/h}.$$

By Corollary A.2, we learn that $R_2 = O(\langle t\rangle^{-N} h^N)$ for any N. We then show $\tilde{B}(t)$ is very close to A_0 uniformly in t. We compute

$$\begin{aligned}
\frac{d}{dt}\tilde{B}(t) &= \big(e^{itP/h} Y\mathcal{J} e^{-itP_f/h}\big)\frac{d}{dt}B(t)\big(e^{itP_f/h}\mathcal{J}^* Y e^{-itP/h}\big) \\
&\quad + \frac{i}{h}\big(e^{itP/h} Y\mathcal{J} e^{-itP_f/h}\big)L(t)B(t)\big(e^{itP_f/h}\mathcal{J}^* Y e^{-itP/h}\big) \\
&\quad - \frac{i}{h}\big(e^{itP/h} Y\mathcal{J} e^{-itP_f/h}\big)B(t)L(t)^*\big(e^{itP_f/h}\mathcal{J}^* Y e^{-itP/h}\big) \\
&\quad + R_1(t)B(t)\big(e^{itP_f/h}\mathcal{J}^* Y e^{-itP/h}\big) - \big(e^{itP/h} Y\mathcal{J} e^{-itP_f/h}\big)B(t)R_1(t)^* \\
&= \big(e^{itP/h} Y\mathcal{J} e^{-itP_f/h}\big)\Big(\frac{d}{dt}B(t) + \frac{i}{h}[L(t), B(t)]\Big)\big(e^{itP_f/h}\mathcal{J}^* Y e^{-itP/h}\big) + R_3(t),
\end{aligned}$$

where

$$\begin{aligned}
R_3(t) &= R_1(t)B(t)\big(e^{itP_f/h}\mathcal{J}^* Y e^{-itP/h}\big) - \big(e^{itP/h} Y\mathcal{J} e^{-itP_f/h}\big)B(t)R_1(t)^* \\
&\quad + \frac{i}{h}\big(e^{itP/h} Y\mathcal{J} e^{-itP_f/h}\big)B(t)(L(t) - L(t)^*)\big(e^{itP_f/h}\mathcal{J}^* Y e^{-itP/h}\big).
\end{aligned}$$

We can show that $\|R_3(t)\| = O(\langle t\rangle^{-N} h^N)$ for any N. For example,

$$\begin{aligned}
\big\|R_1(t)B(t)\big(e^{itP_f/h}\mathcal{J}^* Y e^{-itP/h}\big)\big\| &\le h^{-1}\|(1 - Y\mathcal{J}\mathcal{J}^*)T(t)e^{-itP_f/h}B(t)\| \\
&= h^{-1}\|e^{itP_f/h}(1 - Y\mathcal{J}\mathcal{J}^*)T(t)e^{-itP_f/h}B(t)\|.
\end{aligned}$$

As we have seen already, $e^{itP_f/h}(1 - Y\mathcal{J}\mathcal{J}^*)T(t)e^{-itP_f/h}$ is a pseudodifferential operator, and its support is separated from the support of $b^h(t)$ by a distance not less than $c\langle t\rangle h^{-1}$, for some $c > 0$. Thus their product has a vanishing symbol, and its norm is $O(\langle t\rangle^{-N} h^N)$ with any N. The other terms are estimated similarly. Combining this with Lemma 4.1(v), we learn that

$$\Big\|\frac{d}{dt}\tilde{B}(t)\Big\| \le C_N \langle t\rangle^{-1-\mu} h^N$$

for any N, and hence $\|\widetilde{B}(t) - \widetilde{B}(0)\| \le C_N h^N$. We note that

$$\widetilde{B}(0) = \eta\Big(\frac{hr}{\varepsilon}\Big)\mathcal{J}\mathcal{J}^* A_0 \mathcal{J}\mathcal{J}^* \eta\Big(\frac{hr}{\varepsilon}\Big) = A_0 + O(h^N)$$

by the choice of $\varepsilon > 0$. Combining these facts, we conclude the assertion holds. □

5. Proofs of Theorems 1.1 and 1.2

Let $(r_0, \rho_0, \theta_0, \omega_0) \in T^*(\mathbb{R}_+ \times \partial M)$, and suppose $\omega_0 \neq 0$ as in the last section. Also we let a in $C_0^\infty(T^*(\mathbb{R}_+ \times \partial M))$ be supported in a small neighborhood of $(r_0, \rho_0, \theta_0, \omega_0)$ and we set

$$A_0 = \mathrm{Op}^W(a^h), \quad a^h(r, \rho, \theta, \omega) = a(hr, \rho, \theta, h\omega).$$

Let $\varepsilon > 0$ also as in the last section. Write

$$(r_\pm, \rho_\pm, \theta_\pm, \omega_\pm) = w_{c,\pm}^{-1}(r_0, \rho_0, \theta_0, \omega_0)$$

as in Section 2, and recall that $w_{c,\pm}$ are diffeomorphisms from $\mathbb{R}\times\mathbb{R}_\pm\times(T^*\partial M\setminus 0)$ to $\mathbb{R}_+\times\mathbb{R}\times(T^*\partial M\setminus 0)$. We also note that

$$E_0 = p_c(r_0, \rho_0, \theta_0, \omega_0) = \tfrac{1}{2}\rho_\pm^2 > 0$$

by conservation of energy.

Lemma 5.1. *If $\delta > 2\varepsilon^2$, then*

$$\operatorname*{w-lim}_{t\to\pm\infty} \eta(P_f/\delta)A(t)\eta(P_f/\delta) = \eta(P_f/\delta)W_\pm^* A_0 W_\pm \eta(P_f/\delta).$$

Proof. It is easy to show by the stationary phase method that

$$\operatorname*{s-lim}_{t\to\pm\infty} \Big(1 - \eta\Big(\frac{hr}{\varepsilon\langle t\rangle}\Big)\Big)\mathcal{J} e^{-itP_f/h}\eta(P_f/\delta) = 0$$

(for fixed h), since the stationary points (in ρ) satisfy $hr = t\rho$. This implies that

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itP/h} Y \mathcal{J} e^{-itP_f/h}\eta(P_f/\delta) = W_\pm \eta(P_f/\delta),$$

and the claim follows immediately. $\square$

This implies, combined with Lemmas 4.1 and 4.2:

Lemma 5.2. *Let A_0 as above. Then $W_\pm^* A_0 W_\pm$ are pseudodifferential operators with the symbols $b_\pm^h$ given in Lemma 4.1. In particular, $\Sigma(W_\pm^* A_0 W_\pm)$ are supported in $w_{c,\pm}^{-1}[\operatorname{supp} a^h]$ modulo $O(h^\infty)$-terms, and the principal symbols (modulo $S(h^\mu, g_1^h)$) are given by $a^h \circ w_{c,\pm}$.*

For the moment, we set

$$\rho_0 = 0 \quad \text{and hence} \quad r_\pm = 0.$$

Then we may take $\varepsilon = \sqrt{E_0}$ provided a is supported in a sufficiently small neighborhood of $(r_0, 0, \theta_0, \omega_0)$.

Now let us suppose $(0, \rho_-, \theta_-, \omega_-)$ (with $\omega_- \neq 0$, $\rho_- > 2\varepsilon$) is given, and $(0, \rho_0, \theta_0, \omega_0)$ is defined by $w_{c,-}(0, \rho_-, \theta_-, \omega_-) = (0, \rho_0, \theta_0, \omega_0)$. The converse of Lemma 5.2 is given as follows:

Lemma 5.3. *Let $\tilde a \in C_0^\infty(\mathbb{R}\times\mathbb{R}_-\times(T^*\partial M\setminus 0))$ be supported in a small neighborhood of $(0, \rho_-, \theta_-, \omega_-)$, and let*

$$\tilde A = \mathrm{Op}^W(\tilde a^h), \quad \tilde a^h(r, \rho, \theta, \omega) = \tilde a(hr, \rho, \theta, h\omega).$$

Then there is a symbol a_0^h supported in $w_{c,-}[\operatorname{supp}\tilde a^h]$ such that for any $f \in C_0^\infty(\mathbb{R}_+)$,

$$f(P)A_0 f(P) = W_- f(P_f)\tilde A f(P_f) W_-^*,$$

where $A_0 = \mathrm{Op}^W(a_0^h)$. Moreover, the principal symbol (modulo $S(h^\mu, g_1^h)$) is $\tilde a^h \circ w_{c,-}^{-1}$.

Proof. We set $a^h_{0,0} = \tilde{a}^h \circ w^{-1}_{c,-}$. Then by Lemma 5.2, we have

$$a^h_{-,1} := \Sigma(\tilde{A} - W^*_-\mathrm{Op}^W(a^h_{0,0})W_-) \in S(h^\mu, g^h_1),$$

and it is supported in $\mathrm{supp}[\tilde{a}^h]$ modulo $O(h^\infty)$-terms. Then we set $a_{0,1} = a^h_{-,1} \circ w^{-1}_{c,-}$, and set

$$a^h_{-,2} := \Sigma(\tilde{A} - W^*_-\mathrm{Op}^W(a^h_{0,0} + a^h_{0,1})W_-) \in S(h^{2\mu}, g^h_1).$$

We construct $a^h_{-,j}$, $j = 2, 3, \ldots$, iteratively by

$$a^h_{-,j} := \Sigma(\tilde{A} - W^*_-\mathrm{Op}^W(a^h_{0,0} + \cdots + a^h_{0,j-1})W_-) \in S(h^{j\mu}, g^h_1),$$

$a^h_{0,j} = a^h_{-,j} \circ w^{-1}_{c,-}$, and we set $a^h_0 \sim \sum_{j=0}^\infty a^h_{0,j}$ as an asymptotic sum. Then we have

$$\tilde{A} = W^*_- A_0 W_-$$

modulo $S(h^\infty \langle r\rangle^{-\infty}\langle\omega\rangle^{-\infty}, g_1)$-terms. Since there are no positive eigenvalues [Ito and Skibsted 2011; Melrose and Zworski 1996], we also have $W_\pm f(P_f)W^*_\pm = f(P)$ by virtue of the intertwining property and asymptotic completeness [Ito and Nakamura 2010]. These imply

$$W_- f(P_f)\tilde{A} f(P_f)W^*_- = W_- f(P_f)W^*_- A_0 W_- f(P_f)W^*_- = f(P)A_0 f(P),$$

and this implies the assertion. □

We note Lemma 5.3 naturally holds for $w_{c,+}$ instead of $w_{c,-}$, but we only use the above case. By Lemma 5.3, we learn that

$$Sf(P_f)\tilde{A} f(P_f)S^* = W^*_+ f(P)A_0 f(P)W_+ = f(P_f)(W^*_+ A_0 W_+) f(P_f).$$

By Lemma 5.2, $W^*_+ A_0 W_+$ is a pseudodifferential operator. By choosing $f \in C^\infty_0(\mathbb{R}_+)$ so that $f(\rho^2/2) = 1$ in a neighborhood of the support of $\tilde{a}$, we may omit $f(P_f)$ factors up to negligible terms. Thus, $S\tilde{A}S^*$ is a pseudodifferential operator with a symbol supported in $s_c[\mathrm{supp}\,\tilde{a}^h]$, and the principal symbol is given by $\tilde{a}^h \circ s_c^{-1}$, where $\tilde{a}$ is the symbol given in Lemma 5.3, i.e., $\tilde{a}$ is supported in a small neighborhood of $(0, \rho_-, \theta_-, \omega_-)$.

We note that, by the intertwining property of the scattering operator,

$$e^{-itP_f} S = S e^{-itP_f}, \quad \forall t \in \mathbb{R}.$$

This in turn implies

$$T_\tau S = S T_\tau, \quad \forall \tau \in \mathbb{R}, \text{ where } T_\tau = \exp(-i\tau\sqrt{2P_f}).$$

On the other hand, $\sqrt{2P_f} = \mp i \frac{\partial}{\partial r}$ on $\mathcal{H}_{f,\pm}$, and hence T_τ are translations with respect to r. More precisely, we have

$$T_\tau u_\pm(r, \theta) = u_\pm(r \mp \tau, \theta) \quad \text{for } u_\pm \in \mathcal{H}_{f,\pm}.$$

We learn from these facts that

$$S(T_\tau \tilde{A} T^*_\tau)S^* = T_\tau(S\tilde{A}S^*)T^*_\tau,$$

and the symbols of $T_\tau \tilde{A} T_\tau^*$ and $T_\tau(S\tilde{A}S^*)T_\tau^*$ are given by $\tilde{a}^h(r+\tau, \rho, \theta, \omega)$ and $\Sigma(S\tilde{A}S^*)(r+\tau, \rho, \theta, \omega)$, respectively. Using this observation, we may replace $\tilde{a}$ by a symbol supported in a small neighborhood $(r_-, \rho_-, \theta_-, \omega_-)$ with arbitrary $r_- \in \mathbb{R}$. Thus we have proved:

Lemma 5.4. *Let $a \in C_0^\infty(\mathbb{R}\times\mathbb{R}_-\times(T^*\partial M\setminus 0))$ be supported in a small neighborhood of $(r_-, \rho_-, \theta_-, \omega_-)$ with $|\rho_-| \geq 2\varepsilon$, and let*

$$\tilde{A} = \mathrm{Op}^W(a^h), \quad a^h(r, \rho, \theta, \omega) = a(hr, \rho, \theta, h\omega).$$

Then $S\tilde{A}S^*$ is a pseudodifferential operator with a symbol supported in $s_c[\operatorname{supp} a^h]$ modulo $O(h^\infty)$-terms, and the principal symbol (modulo $S(h^\mu, g_1^h)$) is given by $a^h \circ s_c^{-1}$.

Here we have used the formula

$$s_c(r, \rho, \theta, \omega) = (-r, -\rho, \exp(\pi H_{\sqrt{2q}})(\theta, \omega)).$$

We set $\hat{\mathcal{H}}_{f,\pm} = \mathcal{F}\mathcal{H}_{f,\pm}$. Then $\mathcal{F}S\mathcal{F}^{-1}$ is a unitary map from $\hat{\mathcal{H}}_{f,-}$ to $\hat{\mathcal{H}}_{f,+}$. For notational simplicity, we set

$$\Pi u(r, \theta) = u(-r, \theta) \quad \text{for } u \in \mathcal{H}_{f,\pm},$$

so that $\mathcal{F}(S\Pi)\mathcal{F}^{-1}$ is a unitary map on $\hat{\mathcal{H}}_{f,+}$. By the intertwining property above, $\mathcal{F}(S\Pi)\mathcal{F}^{-1}$ commutes with functions of ρ, and hence is decomposed so that

$$\mathcal{F}(S\Pi)\mathcal{F}^{-1}u(\rho, \omega) = (S(\rho^2/2)u(\rho, \cdot))(\omega) \quad \text{on } \hat{\mathcal{H}}_{f,+} \cong L^2(\mathbb{R}_+; L^2(\partial M)),$$

where $S(\lambda) \in B(L^2(\partial M))$ is the scattering matrix.

Proof of Theorem 1.1. We recall the semiclassical-type characterization of the wave front set: Let $g(\rho, \theta) \in \mathcal{D}'(\mathbb{R}_+ \times \partial M)$, and let $(\rho_0, \theta_0, r_0, \omega_0) \in T^*(\mathbb{R}_+ \times \partial M)$. $(\rho_0, \theta_0, r_0, \omega_0) \notin WF(g)$ if and only if there is $a \in C_0^\infty(T^*(\mathbb{R}_+ \times \partial M))$ such that $a(\rho_0, \theta_0, r_0, \omega_0) \neq 0$ and

$$\|a(\rho, \theta, hD_\rho, hD_\theta)g\| = O(h^\infty) \quad \text{as } h \to +0.$$

We may replace a by an h-dependent symbol with a principal symbol which does not vanish at $(\rho_0, \theta_0, r_0, \omega_0)$.

We fix $\lambda_0 = \rho_0^2/2$ with $\rho_0 > 2\varepsilon$ and consider $S(\lambda)$ where λ is in a small neighborhood of λ_0. Let $u \in L^2(\partial M)$ and let $v \in C_0^\infty(\mathbb{R}_+)$ be supported in a small neighborhood of λ_0. Then it is easy to see that

$$WF(v(\rho)u(\theta)) = \{(\rho, \theta, 0, \omega) \mid \rho \in \operatorname{supp} v, (\theta, \omega) \in WF(u)\}.$$

Then, by Lemma 5.4 and the above characterization of the wave front set, we learn that

$$\begin{aligned} WF(\mathcal{F}(S\Pi)\mathcal{F}^{-1}v(\rho)u(\theta)) &= (1 \otimes \exp(\pi H_{\sqrt{2q}}))WF(v(\rho)u(\theta)) \\ &= \{(\rho, \theta, 0, \omega) \mid \rho \in \operatorname{supp} v, (\theta, \omega) \in \exp(\pi H_{\sqrt{2q}})WF(u)\}; \end{aligned}$$

see [Nakamura 2009b]. By the definition of the scattering matrix, this implies

$$WF(S(\lambda)u) \subset \exp(\pi H_{\sqrt{2q}})WF(u)$$

for $\lambda \in \operatorname{supp} v$. Since this argument works for S^{-1} also, the above inclusion is actually an equality, and we conclude Theorem 1.1. □

Proof of Theorem 1.2. Here we suppose $\mu = 1$. Then by Lemma 5.4 and the Beals-type characterization of FIOs (Theorem B.1), $\mathscr{F}(S\Pi)\mathscr{F}^{-1}$ is an FIO associated to $1 \otimes \exp(\pi H_{\sqrt{2q}})$ on $\{(\rho, \theta, r, \omega) \mid \omega \neq 0\}$. Since $\mathscr{F}(S\Pi)\mathscr{F}^{-1}$ is decomposed as $\{S(\lambda)\}$, this implies $S(\lambda)$ are FIOs on ∂M associated to the canonical transform $\exp(\pi H_{\sqrt{2q}})$ (see Proposition B.4). □

6. Proof of Theorem 1.3

Here we discuss how to generalize the proof of Theorem 1.2 to conclude Theorem 1.3.

We first modify the Egorov-type argument in Section 4. Let $(r_0, \rho_0, \theta_0, \omega_0) \in T^*M_\infty$, $\omega_0 \neq 0$, and let Ω_0 be a small neighborhood of $(r_0, \rho_0, \theta_0, \omega_0)$. We suppose $a \in C_0^\infty(T^*M_\infty)$ is supported in Ω_0, and we consider the behavior of $A(t)$ as in Section 4. We set

$$w^*(t) = \exp(-itH_{p_f}) \circ \exp(tH_p),$$

which is well-defined for $X \in T^*M_\infty$ as long as $\exp(tH_p)(X) \in T^*M_\infty$. By the discussion in the proof of Theorem 2.2, this condition is always satisfied if $X = (r, \rho, \theta, \omega) \in \Omega_0^h$ and h is sufficiently small. We set

$$w(t) = w^*(t)^{-1} = \exp(-tH_p) \circ \exp(tH_{p_f})$$

on the range of $w(t)$. We note that

$$w_\pm^* = \lim_{t\to\pm\infty} w^*(t)$$

on Ω_0^h with sufficiently small h, and that

$$w_\pm = \lim_{t\to\pm\infty} w(t)$$

on $w_\pm^{-1}[\Omega_0^h]$ with sufficiently small h. Convergence of these maps holds in the C^∞-topology.

We replace Lemma 4.1 by the following slightly different statement:

Lemma 6.1. *There exists $b^h(t; r, \rho, \theta, \omega) \in C_0^\infty(T^*M_f)$ satisfying the following conditions:*

(i) $b^h(0) = a^h$.

(ii) *$b^h(t)$ is supported in $w^*(t)[\operatorname{supp} a^h]$.*

(iii) *$b^h(t) \in S(1, g_1^h)$, and it is bounded uniformly in $t \in \mathbb{R}$.*

(iv) *$b^h(t) - a^h \circ w(t) \in S(h, g_1^h)$, i.e., the principal symbol of $b^h(t)$ is given by $a^h \circ w(t)$, and the remainder is bounded uniformly in t.*

(v) *If we set $B(t) = \mathrm{Op}^W(b^h(t))$, then*

$$\left\| \frac{d}{dt}B(t) + \frac{i}{h}[L(t), B(t)] \right\| \leq C_N \langle t\rangle^{-1-\mu} h^N, \quad h > 0,$$

for any N.

(vi) *$B(t)$ converges to $B_\pm$ as $t \to \pm\infty$ in $B(\mathcal{H}_f)$, and the symbols $b_\pm^h := \Sigma(B_\pm)$ satisfy*

$$b_\pm^h - a^h \circ w_\pm \in S(h, g_1^h).$$

We note that $w(t)$ is not homogeneous in the (r, ω)-variables, but very close to a homogeneous map when $|(r, \omega)|$ is very large thanks to Theorem 2.2.

In order to prove Lemma 6.1, we set

$$b_0^h(t) = a^h \circ w(t) = a \circ \exp(-tH_p) \circ \exp(tH_{p_f}),$$

which is supported in $w^*(t)[\Omega_0^h]$. We have $b_0^h(t) \in S(1, g_1^h)$ uniformly in t (for small h) again by Theorem 2.2. Moreover, b_0^h satisfies

$$\frac{\partial}{\partial t} b_0^h(t) = -h^{-1}\{\ell(t), b_0^h(t)\},$$

where $\ell(t) = \Sigma(L(t))$. Hence the first remainder term $r_0^0(t)$ (as defined in Section 4) satisfies

$$|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta r_0^0(t)| \le C\langle t\rangle^{-1-\mu-|\alpha|} h^{1+|\alpha|+|\delta|}$$

for any indices $\alpha, \beta, \gamma, \delta$. Then we construct the asymptotic solution as in the proof of Lemma 4.1 by solving transport equations

$$\frac{\partial}{\partial t} b_j^h(t) + h^{-1}\{\ell(t), b_j^h(t)\} = -r_j^h(t), \quad j = 0, 1, 2, \dots,$$

and we conclude Lemma 6.1. □

Lemma 4.2 holds when the construction of $B(t)$ is replaced by the one above, with no modifications. Lemmas 5.2 and 5.3 hold in the following form. The proofs are the same.

Lemma 6.2. *Let A_0 as above. Then $W_\pm^* A_0 W_\pm$ are pseudodifferential operators with the symbols $b_\pm^h$ given in Lemma 6.1. In particular, $\Sigma(W_\pm^* A_0 W_\pm)$ are supported in $w_\pm^{-1}[\operatorname{supp} a^h]$ modulo $O(h^\infty)$-terms, and the principal symbols (modulo $S(h, g_1^h)$) are given by $a^h \circ w_\pm$.*

Lemma 6.3. *Let $\tilde{a} \in C_0^\infty(\mathbb{R} \times \mathbb{R}_- \times (T^*\partial M \setminus 0))$ be supported in a small neighborhood of $(0, \rho_-, \theta_-, \omega_-)$, and let*

$$\tilde{A} = \operatorname{Op}^W(\tilde{a}^h), \quad \tilde{a}^h(r, \rho, \theta, \omega) = \tilde{a}(hr, \rho, \theta, h\omega).$$

Then $W_- \tilde{A} W_-^*$ is a pseudodifferential operator with a symbol supported in $w_-[\operatorname{supp} \tilde{a}^h]$, and the principal symbol (modulo $S(h, g_1^h)$) is give by $\tilde{a}^h \circ w_-^*$.

Combining these, we learn (as in Section 5) the following assertion.

Lemma 6.4. *Let $a \in C_0^\infty(\mathbb{R} \times \mathbb{R}_- \times (T^*\partial M \setminus 0))$ be supported in a small neighborhood of $(r_-, \rho_-, \theta_-, \omega_-)$ with $|\rho_-| \ge 2\varepsilon$, $\omega_- \ne 0$, and let*

$$\tilde{A} = \operatorname{Op}^W(a^h), \quad a^h(r, \rho, \theta, \omega) = a(hr, \rho, \theta, h\omega).$$

Then $S\tilde{A}S^*$ is a pseudodifferential operator with a symbol supported in $s[\operatorname{supp} a^h]$ modulo $O(h^\infty)$-terms, and the principal symbol (modulo $S(h, g_1^h)$) is given by $a^h \circ s^{-1}$.

In the following, we consider $(r, \rho, \theta, \omega) \in \Omega_0^h$ with some Ω_0 and sufficiently small h, or equivalently, when $|\omega|$ is sufficiently large. By conservation of energy (or equivalently, by invariance under a shift in r), the classical scattering operator has the form

$$s(r, \rho, \theta, \omega) = (-r + g(\rho, \theta, \omega), -\rho, s(\lambda)(\theta, \omega)), \tag{6-1}$$

where $\lambda = \rho^2/2$ and $s(\lambda)$ is a canonical transform on $T^*\partial M$ for each $\lambda > 0$. (We note that without $g(\rho, \theta, \omega)$, the map s is not necessarily canonical.) Moreover, by Theorem 2.1, we have for any indices α, β, γ that

$$|\partial_\rho^\alpha \partial_\theta^\beta \partial_\omega^\gamma g(\rho, \theta, \omega)| \le C h^{-1+\mu+|\gamma|},$$
$$|\partial_\rho^\alpha \partial_\theta^\beta \partial_\omega^\gamma s_1(\rho, \theta, \omega)| \le C h^{\mu+|\gamma|},$$
$$|\partial_\rho^\alpha \partial_\theta^\beta \partial_\omega^\gamma s_2(\rho, \theta, \omega)| \le C h^{-1+\mu+|\gamma|},$$

on Ω_0^h, where Ω_0 is a small neighborhood of $(0, \rho_-, \theta_-, \omega_-)$, and s_1, s_2 are defined by

$$(s_1(\rho, \theta, \omega), s_2(\rho, \theta, \omega)) = s(\lambda)(\theta, \omega) - \exp(\pi H_{\sqrt{2q}})(\theta, \omega),$$

i.e., s_1 denotes the θ-components of the right-hand side terms, and s_2 denotes the ω-components. These estimates imply that s is asymptotically homogeneous (in (r, ω)-variables) in the sense of [Ito and Nakamura 2012, Section 4].

In general, an operator U with distribution kernel u is called an FIO of order m associated to an asymptotically homogeneous canonical transform S if u is a Lagrangian distribution associated to

$$\Sigma_S := \{(x, y, \xi, -\eta) \mid (x, \xi) = S(y, \eta)\},$$

that is, for any $a_1, \dots, a_N \in S^1_{cl}$ such that a_j vanishes on Σ_S for each j, we have that $\mathrm{Op}(a_1) \cdots \mathrm{Op}(a_N) u$ is in $B^{-m-n/2,\infty}_{2,\infty}(\mathbb{R}^{2n})$ [Ito and Nakamura 2012]. The Beals-type characterization of FIOs discussed in Appendix B holds for such FIOs without any change.

By Lemma 6.4 and the analogue of Corollary B.2, we learn that S is an FIO associated to the classical scattering map s. Moreover, by Proposition B.4, we learn that the scattering matrix $S(\lambda)$ is an FIO associated to $s(\lambda)$, where $s(\lambda)$ is defined by (6-1) and it is asymptotic to $\exp(\pi H_{\sqrt{2q}})$. Thus we have proved the following slightly more precise version of Theorem 1.3:

Theorem 6.5. *Suppose Assumption A holds. Then for each $\lambda > 0$, $S(\lambda)$ is an FIO associated to $s(\lambda)$ defined by (6-1). The canonical map $s(\lambda)$ on $T^*\partial M$ is asymptotically homogeneous in ω, asymptotic to* $\exp(\pi H_{\sqrt{2q}})$ *with the error of $O(|\omega|^{1-\mu})$.*

Appendix A: Local decay estimates

Let P be as in Section 1. For a symbol a, we set $a^h(r, \rho, \theta, \omega) = a(hr, \rho, \theta, h\omega)$. Then we have the following:

Theorem A.1. *Let $(r_0, \rho_0, \theta_0, \omega_0) \in T^*M_\infty \cong T^*\mathbb{R}_+ \times T^*\partial M$, and suppose $\omega_0 \neq 0$. We denote the ε-neighborhood of $(r_0, \rho_0, \theta_0, \omega_0)$ by Ω_ε. We suppose $\varepsilon > 0$ so small that $\Omega_{2\varepsilon} \Subset T^*\mathbb{R}_+ \times (T^*\partial M \setminus 0)$.*

If $a \in C_0^\infty(T^*M_\infty)$ is real-valued, and supported in Ω_ε, then there is an h-dependent symbol $b(t)$ in $C_0^\infty(T^*M_\infty)$ for any $t \in \mathbb{R}$ such that:

(i) $|a(r,\rho,\theta,\omega)| \le c_1 b(0;r,\rho,\theta,\omega)$ *with some* $c_1 > 0$.

(ii) $b(t)$ *is supported in* $\Omega(t) := \exp(tH_{p_c})[\Omega_{2\varepsilon}]$ *for* $t \in \mathbb{R}$.

(iii) *For any indices α, β, γ and δ, there is $C_{\alpha\beta\gamma\delta} > 0$ such that*

$$|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta b(t,r,\rho,\theta,\omega)| \le C_{\alpha\beta\gamma\delta}, \quad (r,\rho,\theta,\omega) \in T^*M_\infty, t \in \mathbb{R}.$$

(iv) *There is $R(t) \in B(L^2(M))$ such that $\|R(t)\| \le C_N h^N$ for any N, and*

$$e^{-itP/h}\mathrm{Op}^W(a^h)e^{itP/h} \le c_1 \mathrm{Op}^W(b^h(t)) + R(t)$$

for $t > 0$, and the reverse inequality for $t < 0$. Moreover, $R(t)$ satisfies

$$\|K^N R(t) K^N\|_{B(L^2)} \le C_N h^N, \quad t \in \mathbb{R},$$

for any N, where $K(\cdot) = \langle \mathrm{dist}(\cdot, \Omega^h(t))\rangle$ with

$$\mathrm{supp}[b^h(t)] \subset \Omega^h(t) := \{(r,\rho,\theta,\omega) \mid (hr,\rho,\theta,h\omega) \in \Omega(t)\}.$$

Before proving Theorem A.1, we present a corollary which is needed in Section 4.

Corollary A.2. *Let $\bar\eta \in C^\infty(\mathbb{R})$ be such that $\bar\eta(r) = 0$ if $r > 2$, and $\bar\eta(r) = 1$ if $r \le 1$. We choose $\varepsilon_1 > 0$ so small that*

$$\mathrm{dist}\big(\{(r,\rho,\theta,\omega) \mid |r| \le \varepsilon_1\langle t\rangle\}, \Omega(t)\big) \ge \delta\langle t\rangle$$

with some $\delta > 0$. Then for any N there is $C_N > 0$ such that

$$\left\|\bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right)e^{-itP/h}\mathrm{Op}(a^h)\right\| \le C_N h^N \langle t\rangle^{-N}, \quad t \in \mathbb{R}.$$

We note that if $\varepsilon > 0$ is chosen sufficiently small, then we can find $\varepsilon_1 > 0$ satisfying the property above.

Proof of Corollary A.2. We apply Theorem A.1 with $\tilde a$ such that $\mathrm{Op}^W(\tilde a) = \mathrm{Op}^W(a)\mathrm{Op}^W(a)^*$, which satisfies the same condition. Then we have

$$\begin{aligned}\left\|\bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right)e^{-itP/h}\mathrm{Op}(a^h)\right\|^2 &= \bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right)e^{-itP/h}\mathrm{Op}(\tilde a^h)e^{itP/h}\bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right)\\ &\le c_1\bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right)\mathrm{Op}(b^h(t))\bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right) + \bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right)R(t)\bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right)\\ &\le C_N h^N\langle t\rangle^{-N},\end{aligned}$$

where we used the fact that $\mathrm{supp}[b^h(t)]$ is separated from $\Omega^h(t)$ by a distance not less than $\delta\langle t/h\rangle$. □

Proof of Theorem A.1. The proof is analogous to that of [Nakamura 2009b; Ito 2006; Ito and Nakamura 2009, Section 3], and we only sketch the main steps. We may suppose a is nonnegative without loss of generality. If we set

$$\psi(t) = a \circ \exp(tH_{p_c})^{-1},$$

then it is easy to see that

$$\frac{\partial}{\partial t}\psi = -\{p_c, \psi\}, \quad \psi(0) = a,$$

and this is a good candidate for the principal term of $b(t)$, but ψ does not satisfy the boundedness of the derivatives uniformly in t. We choose $\varphi \in C_0^\infty(\mathbb{R})$ so that

$$\operatorname{supp}\varphi \subset [-1, 1], \quad \varphi(t) \geq 0 \text{ for all } t, \quad \int_{-1}^{1} \varphi(t)\, dt = 1,$$

and moreover, $\pm\varphi'(t) \leq 0$ for $\pm t \geq 0$. We set

$$\varphi_\nu(t) = \varphi(t/\nu), \quad \text{for } \nu > 0,$$

and we denote convolution in t by $\underset{t}{*}$. Then we set

$$b_0(t, \cdot) = \varphi_{\delta\langle t\rangle} \underset{t}{*} \psi = \int \varphi_{\delta\langle t\rangle}(t-s)\psi(s, \cdot)\, ds$$

with sufficiently small $\delta > 0$. Then we have

$$\begin{aligned}\frac{\partial}{\partial t} b_0 = \int \partial_t(\varphi_{\delta\langle t\rangle}(t-s))\psi(s, \cdot)\, ds &= -\int \frac{t(t-s)}{\delta\langle t\rangle^3}\varphi'((t-s)/\delta\langle t\rangle)\psi(s, \cdot)\, ds + \varphi_{\delta\langle t\rangle} \underset{t}{*} (\partial_t\psi) \\ &\geq -\varphi_{\delta\langle t\rangle} \underset{t}{*} \{p_c, \psi\} = -\{p_c, b_0(t, \cdot)\} \end{aligned} \tag{A-1}$$

for $t > 0$, by the conditions on φ. We have the reverse inequality for $t < 0$.

We then show the derivatives of b_0 satisfy the required uniform boundedness. We first note that

$$\tilde{\psi}(t; r, \rho, \theta, \omega) := \psi(t; r + t\rho, \rho, \theta, \omega) \to a \circ w_\pm \quad (t \to \pm\infty)$$

in the C_0^∞-topology, by virtue of the existence of the classical scattering for p_c. Thus we have the representation

$$\psi(t; r, \rho, \theta, \omega) = \tilde{\psi}(t; r - t\rho, \rho, \theta, \omega),$$

with $\tilde{\psi}(t)$ uniformly bounded in $C_0^\infty(T^*M)$. Hence we learn that the derivatives in variables other than ρ are uniformly bounded. Then this property applies also to $b_0(t)$. Let us consider the first derivative of $b_0(t)$ in ρ:

$$\partial_\rho b_0(t) = -\int \varphi_{\delta\langle t\rangle}(t-s)s(\partial_r\tilde{\psi})(s, r - s\rho, \rho, \theta, \omega)\, ds + \int \varphi_{\delta\langle t\rangle}(t-s)(\partial_\rho\tilde{\psi})(s, r - s\rho, \rho, \theta, \omega)\, ds.$$

The second term is clearly uniformly bounded. We note that

$$(\partial_r\tilde{\psi})(s; r - s\rho, \rho, \theta, \omega) = -\frac{1}{\rho}\left\{\frac{\partial}{\partial s}[\tilde{\psi}(s; r - s\rho, \rho, \theta, \omega)] - (\partial_s\tilde{\psi})(s; r - s\rho, \rho, \theta, \omega)\right\},$$

and then by integration by parts we have

$$\begin{aligned}\int \varphi_{\delta\langle t\rangle}(t-s)s(\partial_r\tilde{\psi})(s,r-s\rho,\rho,\theta,\omega)\,ds \\ &= \frac{1}{\rho}\int \frac{\partial}{\partial s}\Big(\varphi\Big(\frac{t-s}{\delta\langle t\rangle}\Big)s\Big)\tilde{\psi}(s,r-s\rho,\rho,\theta,\omega)\,ds + \frac{1}{\rho}\int \varphi\Big(\frac{t-s}{\delta\langle t\rangle}\Big)s(\partial_s\tilde{\psi})(s;r-s\rho,\rho,\theta,\omega)\,ds \\ &= \frac{1}{\rho}\int \varphi\Big(\frac{t-s}{\delta\langle t\rangle}\Big)\tilde{\psi}(s,r-s\rho,\rho,\theta,\omega)\,ds - \frac{1}{\rho}\int \frac{s}{\delta\langle t\rangle}\varphi'\Big(\frac{t-s}{\delta\langle t\rangle}\Big)\tilde{\psi}(s,r-s\rho,\rho,\theta,\omega)\,ds \\ &\qquad + \frac{1}{\rho}\int \varphi\Big(\frac{t-s}{\delta\langle t\rangle}\Big)s(\partial_s\tilde{\psi})(s;r-s\rho,\rho,\theta,\omega)\,ds.\end{aligned}$$

Each term in the last expression is bounded uniformly in t since $s \sim t$, and $\partial_s\tilde{\psi} = O(\langle s\rangle^{-2})$. Repeating this procedure, we can show that all the derivatives of b_0 are uniformly bounded. It is also easy to check that b_0 satisfies the required support property provided a is supported in a sufficiently small neighborhood, and $\delta > 0$ is chosen sufficiently small.

Now by (A-1) and the sharp Gårding inequality, we have

$$\frac{d}{dt}\mathrm{Op}^W(b_0^h(t)) \ge -\frac{i}{h}[P,\mathrm{Op}^W(b_0^h(t))] + \mathrm{Op}(r_1^h(t))$$

with $r_1(t) = O(h^\mu)$. We set $c_j = 7/4 - 2^{-j}$ for $j = 1, 2, \dots$, and set

$$a_j(r,\rho,\theta,\omega) = a\Big(\frac{r}{c_j},\frac{\rho}{c_j},\frac{\theta}{c_j},\frac{\omega}{c_j}\Big), \qquad b_j(t) = \varphi_{\delta\langle t\rangle} \underset{t}{*} (a_j \circ \exp(tH_{p_c})).$$

Then we set

$$b(t) \sim b_0(t) + \sum_{j=1}^{\infty}\mu_j b_j(t),$$

with appropriately chosen constants $\mu_j > 0$ so that

$$\frac{d}{dt}\mathrm{Op}^W(b^h(t)) \ge -\frac{i}{h}[P,\mathrm{Op}^W(b^h(t))] + O(h^\infty),$$

and $b(t)$ satisfies all the required properties. We refer to [Nakamura 2009b; Ito and Nakamura 2009] for the details of the construction. □

Appendix B: Beals-type characterization of Fourier integral operators

In this appendix, we consider operators on $\mathbb{R}^n$, and we discuss Beals-type characterization of FIOs in terms of h-pseudodifferential operators. We use the result for scattering manifolds, but the generalization is straightforward, and we omit it. Most of the arguments here are similar to those of [Ito and Nakamura 2012, Section 2], and we mainly discuss the modifications necessary to show our results.

We let S be a canonical diffeomorphism on $T^*\mathbb{R}^n$, which is also supposed to be homogeneous in the ξ-variable, i.e.,

$$\text{if } (y,\eta) = S(x,\xi), \text{ then } S(x,\lambda\xi) = (y,\lambda\eta) \text{ for } \lambda > 0.$$

We also let $U \in \mathscr{L}(\mathscr{S}, \mathscr{S}')$, and let $u \in \mathscr{D}'(\mathbb{R}^{2n})$ be its distribution kernel. For a symbol $a \in C^\infty(T^*\mathbb{R}^n)$, we write

$$a^h(x,\xi) = a(x, h\xi), \quad \mathrm{Op}^W(a^h) = a^W(x, hD_x),$$

for $h > 0$ as before. For $a \in C_0^\infty(T^*\mathbb{R}^n \setminus 0)$, we define

$$\mathrm{Ad}_S(a^h)U = \mathrm{Op}^W(a^h \circ S^{-1})U - U\mathrm{Op}^W(a^h) \in \mathscr{L}(\mathscr{S}, \mathscr{S}').$$

We note that $\mathrm{Op}^W(a^h \circ S^{-1}) = \mathrm{Op}^W((a \circ S^{-1})^h)$ since S is homogeneous in ξ.

Theorem B.1. *Let $U \in B(L^2_{cpt}(\mathbb{R}^n), L^2_{loc}(\mathbb{R}^n))$. Suppose for any $a_1, a_2, \dots, a_N \in C_0^\infty(T^*\mathbb{R}^n \setminus 0)$, there is $C_N > 0$ such that*

$$\|\mathrm{Ad}_S(a_1^h)\,\mathrm{Ad}_S(a_2^h)\cdots \mathrm{Ad}_S(a_N^h)U\|_{B(L^2)} \le C_N h^N. \tag{B-1}$$

Then U is an FIO of order 0 associated to S.

Corollary B.2. *Let S and U as above. If for any $a \in C_0^\infty(T^*\mathbb{R}^n \setminus 0)$ there is an h-dependent symbol $b \in C_0^\infty(T^*\mathbb{R}^n \setminus 0)$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta b(h; x, \xi)| \le C_{\alpha\beta} h,$$

for any $\alpha, \beta \in \mathbb{Z}_+^n$, $h \in (0,1]$, and

$$\mathrm{Ad}_S(a^h)U = \mathrm{Op}^W(b^h)U + R, \quad \|R\|_{B(L^2)} = O(h^\infty),$$

then U is an FIO of order 0 associated to S.

Proof of Corollary B.2. We show (B-1) follows from the above condition. The cases $N = 0, 1$ are obvious. Let $N = 2$ and we write

$$\mathrm{Ad}_S(a_j^h)U = \mathrm{Op}^W(b_j^h)U + R_j, \quad j = 1, 2.$$

Then we have

$$\begin{aligned}\mathrm{Ad}_S(a_1^h)\,\mathrm{Ad}_S(a_2^h)U &= \mathrm{Op}^W(a_1^h \circ S^{-1})\mathrm{Op}^W(b_2^h)U - \mathrm{Op}^W(b_2^W)U\mathrm{Op}^W(a_1^h) + \mathrm{Ad}_S(a_1^h)R_2 \\ &= \big[\mathrm{Op}^W(a_1^h \circ S^{-1}), \mathrm{Op}^W(b_2^h)\big]U + \mathrm{Op}^W(b_2^h)\mathrm{Op}^W(b_1^h)U + \mathrm{Ad}_S(a_1^h)R_2 + \mathrm{Op}^W(b_2^h)R_1 \\ &= \mathrm{Op}^W(b_{12}^h)U + R_{12},\end{aligned}$$

where $R_{12} = O(h^\infty)$ and $b_{12} \in C_0^\infty(T^*\mathbb{R}^n \setminus 0)$ satisfies

$$\big|\partial_x^\alpha \partial_\xi^\beta b_{12}(h; x, \xi)\big| \le C'_{\alpha\beta} h^2, \quad \text{for any } \alpha, \beta \in \mathbb{Z}_+^n,\ h \in (0,1],$$

and (B-1) for $N = 2$ follows. Iterating this procedure, we obtain (B-1) for any N. □

In order to prove Theorem B.1, we first note the semiclassical-type characterization of Besov spaces. By the standard partition-of-unity argument, it is straightforward to observe that $u \in B^{\sigma,\infty}_{2,loc}(\mathbb{R}^m)$ if and only if for any $(x_0, \xi_0) \in T^*\mathbb{R}^m \setminus 0$ there is $\varphi \in C_0^\infty(T^*\mathbb{R}^m)$ such that $\varphi(x_0, \xi_0) \ne 0$ and

$$\|\mathrm{Op}^W(\varphi^h)u\|_{L^2} \le Ch^\sigma, \quad h > 0.$$

Thus, in turn, we learn that $u \in B^{\sigma,\infty}_{2,loc}(\mathbb{R}^{2n})$ if and only if for any $(x_0, y_0, \xi_0, \eta_0)$, $(\xi_0, \eta_0) \neq (0, 0)$, there are $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_1(x_0, \xi_0) \neq 0$, $\varphi_2(y_0, \eta_0) \neq 0$, and

$$\|\mathrm{Op}^W(\varphi_1^h) U \mathrm{Op}^W(\varphi_2^h)\|_{HS} \leq C h^\sigma, \quad h > 0,$$

where $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt norm in $B(L^2(\mathbb{R}^n))$. Now we choose $\varphi_3 \in C_0^\infty(\mathbb{R}^n)$ so that $\varphi_3 = 1$ in a neighborhood of $\operatorname{supp}\varphi_2$. We note that

$$\|\mathrm{Op}^W(\varphi_3)\|_{HS} = (2\pi)^{-n/2}\biggl(\int_{\mathbb{R}^n} |\varphi_3(x, h\xi)|^2 dx d\xi\biggr)^{1/2} = (2\pi h)^{-n/2}\biggl(\int_{\mathbb{R}^n} |\varphi_3(x, \xi)|^2 dx d\xi\biggr)^{1/2} = Ch^{-n/2}$$

for $h > 0$ with some $C > 0$. Hence we have

$$\begin{aligned}\|\mathrm{Op}^W(\varphi_1^h) U \mathrm{Op}^W(\varphi_2^h)\|_{HS} &\leq \|\mathrm{Op}^W(\varphi_1^h) U \mathrm{Op}^W(\varphi_2^h)\mathrm{Op}^W(\varphi_3^h)\|_{HS} + R\\ &\leq Ch^{-n/2}\|\mathrm{Op}^W(\varphi_1^h) U \mathrm{Op}^W(\varphi_2^h)\|_{B(L^2)} + R,\end{aligned}$$

where

$$R = \|\mathrm{Op}^W(\varphi_1^h) U \mathrm{Op}^W(\varphi_2^h)(1 - \mathrm{Op}^W(\varphi_3^h))\|_{HS} = O(h^\infty)$$

by the symbol calculus. Thus we have proved the following lemma:

Lemma B.3. *If for any $(x_0, y_0, \xi_0, \eta_0) \in T^*\mathbb{R}^{2n}$ with $(\xi_0, \eta_0) \neq (0, 0)$ there are $\varphi_1, \varphi_2 \in C_0^\infty(T^*\mathbb{R}^n)$ such that $\varphi_1(x_0, \xi_0) \neq 0$, $\varphi_2(y_0, \eta_0) \neq 0$ and*

$$\|\mathrm{Op}^W(\varphi_1^h) U \mathrm{Op}^W(\varphi_2^h)\|_{B(L^2)} \leq C, \quad h > 0,$$

then $u \in B^{-n/2,\infty}_{2,loc}(\mathbb{R}^{2n})$.

Proof of Theorem B.1. We modify the proof of Theorem 2.1 in [Ito and Nakamura 2012], to which we refer for further details.

We first note that

$$WF(u) \subset \Lambda_S = \{(x, y, \xi, -\eta) \in T^*\mathbb{R}^{2n} \mid (x, \xi) = S(y, \eta)\}.$$

We note that if $(x_0, y_0, \xi_0, -\eta_0) \notin \Lambda_S$ with $\eta_0 \neq 0$, it is straightforward to show $(x_0, y_0, \xi_0, -\eta_0) \notin WF(u)$. If $\xi_0 \neq 0$, we consider U^* and we can also conclude $(x_0, y_0, \xi_0, -\eta_0) \notin WF(u)$.

Now we let $a_1, a_2, \ldots, a_N \in S^1_{cl}(\mathbb{R}^n)$ and let $(x_0, \xi_0) = S(y_0, \eta_0)$. We may assume a_j are homogeneous of order one in the ξ-variable. By Lemma B.3 and the proof just cited, it suffices to show the following to conclude U is an FIO of order 0 associated to S: There are $\psi_1, \psi_2 \in C_0^\infty(T^*\mathbb{R}^n)$ such that $\psi_1(x_0, \xi_0) \neq 0$, $\psi_2(y_0, \eta_0) \neq 0$ and

$$\|\mathrm{Op}^W(\psi_1^h)[\mathrm{Ad}_S(a_1)\cdots \mathrm{Ad}_S(a_N) U]\mathrm{Op}^W(\psi_2^h)\|_{B(L^2)} \leq C, \quad h \in (0, 1], \tag{B-2}$$

with some $C > 0$.

We set $\Psi_0, \Psi_1 \in C_0^\infty(T^*\mathbb{R}^n)$ so that they are supported in a small neighborhood of (y_0, η_0), $\Psi_j = 1$ on a neighborhood of (y_0, η_0), and $\Psi_0 = 1$ on $\operatorname{supp}\Psi_1$. We then set

$$\varphi_j(x, \xi) = a_j(x, \xi)\Psi_0(x, \xi) \in C_0^\infty(T^*\mathbb{R}^n).$$

We note, since a_j are homogeneous of order one in ξ, that

$$a_j(x,\xi)\Psi_0(x,h\xi) = h^{-1}a_j(x,h\xi)\Psi_0(x,h\xi) = h^{-1}\varphi_j(x,h\xi).$$

We also set

$$\psi_1 = \Psi_1 \circ S^{-1} \quad \text{and} \quad \psi_2 = \Psi_1$$

so $\psi_1(1-\Psi_0\circ S^{-1}) = 0$ and $(1-\Psi_0)\psi_2 = 0$. This implies, in particular, that

$$\psi_1(x,h\xi)(a_j\circ S^{-1})(x,\xi) = h^{-1}\psi_1(x,h\xi)(\varphi_j\circ S^{-1})(x,h\xi),$$
$$a_j(y,\eta)\psi_2(y,h\eta) = h^{-1}\varphi_j(y,h\eta)\psi_2(y,h\eta).$$

Using these, and applying the h-pseudodifferential operator calculus, we learn that

$$\begin{aligned}&\mathrm{Op}^W(\psi_1^h)[\mathrm{Ad}_S(a_1)\cdots\mathrm{Ad}_S(a_N)U]\mathrm{Op}^W(\psi_2^h)\\ &\qquad = h^{-N}\mathrm{Op}^W(\psi_1^h)[\mathrm{Ad}_S(\varphi_1^h)\cdots\mathrm{Ad}_S(\varphi_N^h)U]\mathrm{Op}^W(\psi_2^h) + O(h^\infty),\end{aligned}$$

and this implies the right-hand side is bounded by the assumption of Theorem B.1. Now (B-2) follows from this observation, and we conclude that the assertion hold. □

We note that the conditions and the assertion of Theorem B.1 are microlocal, and hence the theorem is easily extended to a statement in a conic set in $T^*\mathbb{R}^n$. In the next proposition, we use the extended statement on a conic set.

Proposition B.4. *Let $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^k$, and let U be a bounded operator on $L^2(\mathbb{R}^m)$ and let S be a homogeneous canonical diffeomorphism on $T^*\mathbb{R}^m$. Suppose U commutes with multiplication operators in y so that U is decomposed as*

$$U = \int_{\mathbb{R}^k}^{\oplus} \tilde{U}(y)dy \quad \text{on } L^2(\mathbb{R}^m) \cong L^2(\mathbb{R}^k_y, L^2(\mathbb{R}^n_x)),$$

where $\{U(y)\}$ is a family of operators on $L^2(\mathbb{R}^n_x)$. Suppose also that S is decomposed as

$$S : (x,\xi,y,\eta) \mapsto (\tilde{S}(y)(x,\xi), y, \eta + g(x,\xi,y))$$

for $(x,\xi,y,\eta) \in T^*\mathbb{R}^n \cong T^*\mathbb{R}^n_x \times T^*\mathbb{R}^k_y$, where $\{\tilde{S}(y)\}$ is a family of canonical maps on $T^*\mathbb{R}^n_x$. If U is an FIO associated to S on a conic set $\{(x,\xi,t,\eta) \mid \xi \neq 0\}$, then for each $y \in \mathbb{R}^k$, $\tilde{U}(y)$ is an FIO of order 0 associated to $\tilde{S}(y)$.

Remark B.5. The assumption on S actually follows from the properties of U. We include it to introduce the notations.

Proof. Let $a \in C_0^\infty(T^*\mathbb{R}^n \setminus 0)$, and let $\varphi, \psi \in C_0^\infty(\mathbb{R}^k)$ such that $\varphi, \psi \geq 0$ and $\int \psi(\eta)\,d\eta = 1$. We also denote $\psi_z(\eta) = \psi(\eta - z)$ for $z \in \mathbb{R}^k$. We consider

$$A_z = a_z(x,hD_x,y,hD_y) = a(x,hD_x)\varphi(y)\psi_z(hD_y).$$

Since U is an FIO, there is b_z, which is bounded in $C_0^\infty(T^*\mathbb{R}^m)$ uniformly in $h \in (0, 1]$, such that

$$UA_z = B_z U + O(h^\infty), \qquad B_z = b_z(x, hD_x, y, hD_y),$$

with the principal symbol

$$a_z \circ S^{-1} = (a \circ \tilde{S}(y)^{-1})(x, \xi)\varphi(y)\psi\big(\eta - g(\tilde{S}(y)^{-1}(x, \xi), y) - z\big).$$

Since U commutes with $\{e^{iy\cdot z} \mid z \in \mathbb{R}^k\}$ (translations in the η-variable), we learn that

$$b_z(x, \xi, y, \eta) = b_0(x, \xi, y, \eta - z),$$

and the remainder term also satisfies this property. Moreover, these symbols decay rapidly outside $S[\operatorname{supp} a_z]$.

It is also easy to see that

$$\int_{|z|\le R} A_z dz \to a(x, hD_x)\varphi(y) \quad \text{and} \quad \int_{|z|\le R} B_z dz \to \tilde{b}(x, hD_x, y)$$

strongly as $R \to \infty$, where $\tilde{b}(x, \xi, y) = \int_{\mathbb{R}^k} b_0(x, \xi, y, \eta)\, d\eta$. The principal symbol of $\tilde{b}$ is given by $(a \circ \tilde{S}(y)^{-1})(x, \xi)\varphi(y)$. These facts imply that

$$\tilde{U}(y)a(x, hD_x)\varphi(y) = \tilde{b}(x, hD_x, y)\tilde{U}(y) + O(h^\infty),$$

where $\tilde{b}(x, \xi, y) - (a \circ \tilde{S}(y)^{-1})(x, \xi)\varphi(y) = O(h)$. Since $\varphi \in C_0^\infty(\mathbb{R}^k)$ is arbitrary, for a fixed $y \in \mathbb{R}^k$ we may replace $\varphi(y)$ by 1, and we learn $\tilde{U}(y)$ is an FIO of order 0 associated to $\tilde{S}(y)$ by Corollary B.2. □

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LOCAL WELL-POSEDNESS OF THE VISCOUS SURFACE WAVE PROBLEM WITHOUT SURFACE TENSION

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We consider a viscous fluid of finite depth below the air, occupying a three-dimensional domain bounded below by a fixed solid boundary and above by a free moving boundary. The domain is allowed to have a horizontal cross-section that is either periodic or infinite in extent. The fluid dynamics are governed by the gravity-driven incompressible Navier–Stokes equations, and the effect of surface tension is neglected on the free surface. This paper is the first in a series of three on the global well-posedness and decay of the viscous surface wave problem without surface tension. Here we develop a local well-posedness theory for the equations in the framework of the nonlinear energy method, which is based on the natural energy structure of the problem. Our proof involves several novel techniques, including: energy estimates in a "geometric" reformulation of the equations, a well-posedness theory of the linearized Navier–Stokes equations in moving domains, and a time-dependent functional framework, which couples to a Galerkin method with a time-dependent basis.

1. Introduction

Formulation of the equations in Eulerian coordinates. We consider a viscous, incompressible fluid evolving in a moving domain

$$\Omega(t) = \left\{ y \in \Sigma \times \mathbb{R} \;\middle|\; -b(y_1, y_2) < y_3 < \eta(y_1, y_2, t) \right\}.$$

Here we assume that either $\Sigma = \mathbb{R}^2$ or $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$ for $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the usual 1-torus and $L_1, L_2 > 0$ the periodicity lengths. The lower boundary of $\Omega(t)$ is assumed to be rigid and given, but the upper boundary is a free surface that is the graph of the unknown function $\eta : \Sigma \times \mathbb{R}^+ \to \mathbb{R}$. We assume that

$$\begin{cases} 0 < b \in C^\infty(\Sigma) & \text{if } \Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T}), \\ b \in (0, \infty) \text{ is constant} & \text{if } \Sigma = \mathbb{R}^2. \end{cases}$$

For each t, the fluid is described by its velocity and pressure functions $(u, p) : \Omega(t) \to \mathbb{R}^3 \times \mathbb{R}$. We require that (u, p, η) satisfy the gravity-driven incompressible Navier–Stokes equations in $\Omega(t)$ for $t > 0$:

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$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u & \text{in } \Omega(t), \\ \operatorname{div} u = 0 & \text{in } \Omega(t), \\ \partial_t \eta = u_3 - u_1 \partial_{y_1} \eta - u_2 \partial_{y_2} \eta & \text{on } \{y_3 = \eta(y_1, y_2, t)\}, \\ (pI - \mu \mathbb{D}(u))\nu = g\eta\nu & \text{on } \{y_3 = \eta(y_1, y_2, t)\}, \\ u = 0 & \text{on } \{y_3 = -b(y_1, y_2)\}, \end{cases}$$

for ν the outward-pointing unit normal on $\{y_3 = \eta\}$, I the 3×3 identity matrix, $(\mathbb{D}u)_{ij} = \partial_i u_j + \partial_j u_i$ the symmetric gradient of u, $g > 0$ the strength of gravity, and $\mu > 0$ the viscosity. The tensor $(pI - \mu \mathbb{D}(u))$ is known as the viscous stress tensor. The third equation in 1 implies that the free surface is advected with the fluid. Note that in 1, we have shifted the gravitational forcing to the boundary and eliminated the constant atmospheric pressure, p_{atm}, in the usual way by adjusting the actual pressure $\bar{p}$ according to $p = \bar{p} + gy_3 - p_{\mathrm{atm}}$.

The problem is augmented with initial data (u_0, η_0) satisfying certain compatibility conditions, which for brevity we will not write now. We will assume that $\eta_0 > -b$ on Σ. When $\Sigma = (L_1 \mathbb{T}) \times (L_2 \mathbb{T})$, we shall refer to the problem as either the "periodic problem" or the "periodic case", and when $\Sigma = \mathbb{R}^2$, we shall refer to it as either the "nonperiodic problem" or the "infinite case".

Without loss of generality, we may assume that $\mu = g = 1$. Indeed, a standard scaling argument allows us to scale so that $\mu = g = 1$, at the price of multiplying b and the periodicity lengths L_1, L_2 by positive constants and rescaling b. This means that, up to renaming b, L_1, and L_2, we arrive at the above problem with $\mu = g = 1$.

The problem 1 possesses a natural physical energy. For sufficiently regular solutions to both the periodic and nonperiodic problems, we have an energy evolution equation that expresses how the change in physical energy is related to the dissipation:

$$\frac{1}{2} \int_{\Omega(t)} |u(t)|^2 + \frac{1}{2} \int_{\Sigma} |\eta(t)|^2 + \frac{1}{2} \int_0^t \int_{\Omega(s)} |\mathbb{D}u(s)|^2 \, ds = \frac{1}{2} \int_{\Omega(0)} |u_0|^2 + \frac{1}{2} \int_{\Sigma} |\eta_0|^2.$$

The first two integrals constitute the kinetic and potential energies, while the third constitutes the dissipation. The structure of this energy evolution equation is the basis of the energy method that we will use to analyze 1.

Geometric form of the equations. In order to work in a fixed domain, we want to flatten the free surface via a coordinate transformation. We will not use a Lagrangian coordinate transformation, but rather a flattening transformation introduced by Beale [1984]. To this end, we consider the fixed equilibrium domain

$$\Omega := \{ x \in \Sigma \times \mathbb{R} \mid -b(x_1, x_2) < x_3 < 0 \},$$

for which we will write the coordinates as $x \in \Omega$. We will think of Σ as the upper boundary of Ω, and we will write $\Sigma_b := \{x_3 = -b(x_1, x_2)\}$ for the lower boundary. We continue to view η as a function on $\Sigma \times \mathbb{R}^+$. We then define

$$\bar{\eta} := \mathcal{P}\eta = \text{harmonic extension of } \eta \text{ into the lower half-space},$$

where $\mathcal{P}\eta$ is defined by (A-8) when $\Sigma = \mathbb{R}^2$ and by (A-14) when $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$. The harmonic extension $\bar{\eta}$ allows us to flatten the coordinate domain via the mapping

$$\Omega \ni x \mapsto \left(x_1,\ x_2,\ x_3 + \bar{\eta}(x,t)\left(1 + \frac{x_3}{b(x_1,x_2)}\right)\right) =: \Phi(x,t) = (y_1,\ y_2,\ y_3) \in \Omega(t). \tag{1-1}$$

Note that $\Phi(\Sigma,t) = \{y_3 = \eta(y_1,y_2,t)\}$ and $\Phi(\,\cdot\,,t)|_{\Sigma_b} = \mathrm{Id}_{\Sigma_b}$; that is, Φ maps Σ to the free surface and keeps the lower surface fixed. We have

$$\nabla\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A & B & J \end{pmatrix} \quad \text{and} \quad \mathcal{A} := (\nabla\Phi^{-1})^T = \begin{pmatrix} 1 & 0 & -AK \\ 0 & 1 & -BK \\ 0 & 0 & K \end{pmatrix} \tag{1-2}$$

for

$$A = \partial_1\bar{\eta}\tilde{b} - \frac{x_3\bar{\eta}\partial_1 b}{b^2}, \quad B = \partial_2\bar{\eta}\tilde{b} - \frac{x_3\bar{\eta}\partial_2 b}{b^2}, \quad J = 1 + \frac{\bar{\eta}}{b} + \partial_3\bar{\eta}\tilde{b}, \quad K = J^{-1}, \quad \tilde{b} = \frac{1+x_3}{b}. \tag{1-3}$$

Here $J = \det\nabla\Phi$ is the Jacobian of the coordinate transformation. See Lemma A.3 for some properties of $\mathcal{A}$.

If η is sufficiently small (in an appropriate Sobolev space), then the mapping Φ is a diffeomorphism. This allows us to transform the problem to one on the fixed spatial domain Ω for $t \geq 0$. In the new coordinates, the PDE 1 becomes

$$\begin{cases} \partial_t u - \partial_t\bar{\eta}\tilde{b}K\partial_3 u + u\cdot\nabla_{\mathcal{A}} u - \Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} p = 0 & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}} u = 0 & \text{in } \Omega, \\ S_{\mathcal{A}}(p,u)\mathcal{N} = \eta\mathcal{N} & \text{on } \Sigma, \\ \partial_t\eta = u\cdot\mathcal{N} & \text{on } \Sigma, \\ u = 0 & \text{on } \Sigma_b, \\ u(x,0) = u_0(x), \quad \eta(x',0) = \eta_0(x'). \end{cases} \tag{1-4}$$

Here we have written the differential operators $\nabla_{\mathcal{A}}$, $\operatorname{div}_{\mathcal{A}}$, and $\Delta_{\mathcal{A}}$ with their actions given by $(\nabla_{\mathcal{A}} f)_i := \mathcal{A}_{ij}\partial_j f$, $\operatorname{div}_{\mathcal{A}} X := \mathcal{A}_{ij}\partial_j X_i$, and $\Delta_{\mathcal{A}} f = \operatorname{div}_{\mathcal{A}}\nabla_{\mathcal{A}} f$ for appropriate f and X; as for $u\cdot\nabla_{\mathcal{A}} u$, we mean $(u\cdot\nabla_{\mathcal{A}} u)_i := u_j\mathcal{A}_{jk}\partial_k u_i$. We have also written

$$\mathcal{N} := -\partial_1\eta e_1 - \partial_2\eta e_2 + e_3$$

for the nonunit normal to $\{y_3 = \eta(y_1,y_2,t)\}$, and we write $S_{\mathcal{A}}(p,u) = (pI - \mathbb{D}_{\mathcal{A}} u)$ for the stress tensor, where I is the 3×3 identity matrix and $(\mathbb{D}_{\mathcal{A}} u)_{ij} = \mathcal{A}_{ik}\partial_k u_j + \mathcal{A}_{jk}\partial_k u_i$ is the symmetric $\mathcal{A}$-gradient. Note that if we extend $\operatorname{div}_{\mathcal{A}}$ to act on symmetric tensors in the natural way, then $\operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p,u) = \nabla_{\mathcal{A}} p - \Delta_{\mathcal{A}} u$ for vector fields satisfying $\operatorname{div}_{\mathcal{A}} u = 0$.

Recall that $\mathcal{A}$ is determined by η through the relation (1-2). This means that all of the differential operators in (1-4) are connected to η, and hence to the geometry of the free surface. This geometric structure is essential to our analysis, as it allows us to control high-order derivatives that would otherwise be out of reach.

Previous results. Local well-posedness for the problem 1 in a bounded domain, all of whose boundary is free, was proved by Solonnikov [1977]. Local well-posedness for the problem in domains like ours was proved by Beale [1981]. Both of these results employ parabolic regularity theory in a functional framework different from the one we use: Solonnikov worked in Hölder spaces, while Beale worked in L^2-based space-time Sobolev spaces. Abels [2005] extended this local theory to the framework of L^p-based Sobolev spaces. Global well-posedness was proved in the periodic case by Hataya [2009] and discussed in the infinite case by Sylvester [1990] as well as Tani and Tanaka [1995], all within a Beale–Solonnikov functional framework.

If the effect of surface tension is included at the free interface, then the free surface function gains regularity, stabilizing the problem. This led to a proof of small-data global well-posedness by Beale [1984], as well as a proof by Beale and Nishida [1985] that the global solutions with surface tension decay algebraically in time. In the periodic case, Nishida, Teramoto and Yoshihara [Nishida et al. 2004] proved global well-posedness and exponential decay. Bae [2011] proved global well-posedness with surface tension using energy methods rather than a Beale–Solonnikov framework. For a bounded mass of fluid with surface tension, local well-posedness was proved by Coutand and Shkoller [2003].

Many authors have also considered one-fluid free boundary problems for inviscid fluids, which are modeled by setting $\mu = 0$ in 1 and replacing the no-slip condition with the no-penetration condition, $u \cdot \nu = 0$ on Σ_b. For this problem, it is often assumed that the fluid is initially curl-free, in which case this condition propagates in time and the fluid is said to be irrotational. The velocity field is then both curl-free and divergence-free for all time, and is therefore the gradient of a function that is harmonic in $\Omega(t)$. This allows for the reformulation of the problem as one only on the free surface, involving the Dirichlet-to-Neumann operator. Local well-posedness in this framework was established by Wu [1997; 1999] and Lannes [2005], an almost-global well-posedness result was then proved by Wu [2009] for the 2D problem, and global well-posedness was proved by Wu [2011] and Germain, Masmoudi and Shatah [Germain et al. 2009] in 3D. Only the irrotational problem has been shown to admit global solutions in the inviscid case. Local well-posedness without the irrotationality assumption was proved with a modified surface formulation by Zhang and Zhang [2008] and with the original formulation in [[Christodoulou and Lindblad 2000; Lindblad 2005; Coutand and Shkoller 2007; Shatah and Zeng 2008]]. Note that in the viscous case, it is known that vorticity is generated at the free surface, even if the fluid is initially irrotational. Therefore it is not possible to use the surface formulation of the problem.

Main result. As mentioned above, the standard method for constructing solutions in the existing literature is based on the parabolic regularity theory pioneered by Beale [1981] for domains like ours and by Solonnikov [1977] for bounded, nonperiodic domains. The advantage of full parabolic regularity is that it enables one to treat viscous surface waves as a perturbation of the “flat surface” problem, which is obtained by setting $\eta = 0$, $\mathscr{A} = I$, $\mathscr{N} = e_3$, etc. in (1-4). The actual problem (1-4) is then rewritten as the flat surface problem with nonlinear forcing terms that correspond to the difference between the two forms of the equations. The key to the existence theory of, say, [Beale 1981], is regularity in H^r with the choice of $r = 3 + \delta$ for $\delta \in \left(0, \frac{1}{2}\right)$. According to the natural energy structure of the problem, 1, one might expect r to naturally be an integer. The extra gain of $\delta > 0$ regularity allows for enough control of the nonlinear forcing terms to produce a local solution to (1-4) from solutions to the flat surface problem and an iteration argument. As recognized early on by Beale himself, a disadvantage of Beale–Solonnikov theory is that the functional framework makes it difficult to extract time decay information.

In a pair of companion papers [Guo and Tice 2013b; 2013a], we prove a priori decay estimates that are developed through a high-regularity energy method. This necessitates using the natural energy structure of the problem, 1, which in turn requires us to use positive integer Sobolev indices for u. The advantage of the natural energy structure is that it produces two distinct types of estimates: roughly speaking, $L^\infty([0, T]; L^2)$ "energy estimates" and $L^2([0, T]; H^1)$ "dissipation estimates". The interplay between the energy and the dissipation naturally leads to time decay information. The disadvantage of the energy structure is that our regularity index r must be an integer, so we cannot use the $\delta > 0$ gain that would allow us to treat the problem (1-4) as a perturbation of the flat surface problem.

The difficulty in proving local well-posedness in the natural energy structure is thus clear. We cannot use solutions to the standard flat surface problem to produce solutions to (1-4) via an iteration argument since the forcing terms cannot be controlled in the iteration. For example, we would have trouble controlling the interaction between the highest-order temporal derivatives of p and $\operatorname{div} u$. Our solution, then, is to abandon the flat surface problem and prove local existence directly, using the geometric structure of (1-4). The geometric structure is crucial since it decreases the derivative count of the forcing terms, which then allows us to close an iteration argument using only the natural energy structure. The essential difficulty is that the geometric structure requires us to solve the Navier–Stokes equations in moving domains. In the presence of such a time-dependent geometric effect, even the construction of local-in-time solutions to the linear Navier–Stokes equations is highly delicate and has to be carried out from the beginning.

Before we state our local existence result, let us mention the issue of compatibility conditions for the initial data (u_0, η_0). We will work in a high-regularity context, essentially with regularity up to $2N$ temporal derivatives for $N \geq 3$ an integer. This requires us to use u_0 and η_0 to construct the initial data $\partial_t^j u(0)$ and $\partial_t^j \eta(0)$ for $j = 1, \dots, 2N$ and $\partial_t^j p(0)$ for $j = 0, \dots, 2N-1$. These other data must then satisfy various conditions (essentially what one gets by applying ∂_t^j to (1-4) and then setting $t = 0$), which in turn require u_0 and η_0 to satisfy $2N$ compatibility conditions. We describe these conditions in detail on pages 338–339 and state them explicitly in (5-22), so for brevity we will not state them here.

In order to state our result, we must explain our notation for Sobolev spaces and norms. We take $H^k(\Omega)$ and $H^k(\Sigma)$ for $k \geq 0$ to be the usual Sobolev spaces. When we write norms, we will suppress the H and Ω or Σ. When we write $\|\partial_t^j u\|_k$ and $\|\partial_t^j p\|_k$, we always mean that the space is $H^k(\Omega)$, and when we write $\|\partial_t^j \eta\|_k$, we always mean that the space is $H^k(\Sigma)$. In the following result, we also refer to the space $\mathscr{X}_T$, which is defined later in (2-4).

Theorem 1.1. *Let $N \geq 3$ be an integer. Assume that u_0 and η_0 satisfy the bounds*

$$\|u_0\|_{4N}^2 + \|\eta_0\|_{4N+1/2}^2 < \infty$$

as well as the $(2N)$-th compatibility conditions (5-22). There exist $0 < \delta_0, T_0 < 1$ such that if

$$0 < T \leq T_0 \min\left\{1, \frac{1}{\|\eta_0\|_{4N+1/2}^2}\right\}$$

and $\|u_0\|_{4N}^2 + \|\eta_0\|_{4N}^2 \le \delta_0$, *then there exists a unique solution* (u, p, η) *to* (1-4) *on the interval* $[0, T]$ *that achieves the initial data. The solution obeys the estimates*

$$\sum_{j=0}^{2N} \sup_{0\le t\le T} \left\|\partial_t^j u\right\|_{4N-2j}^2 + \sum_{j=0}^{2N} \sup_{0\le t\le T} \left\|\partial_t^j \eta\right\|_{4N-2j}^2 + \sum_{j=0}^{2N-1} \sup_{0\le t\le T} \left\|\partial_t^j p\right\|_{4N-2j-1}^2$$
$$+ \int_0^T \left(\sum_{j=0}^{2N} \left\|\partial_t^j u\right\|_{4N-2j+1}^2 + \sum_{j=0}^{2N-1} \left\|\partial_t^j p\right\|_{4N-2j}^2 \right) + \|\partial_t^{2N+1} u\|_{(\mathscr{X}_T)^*}^2$$
$$+ \int_0^T \left(\|\eta\|_{4N+1/2}^2 + \|\partial_t \eta\|_{4N-1/2}^2 + \sum_{j=2}^{2N+1} \left\|\partial_t^j \eta\right\|_{4N-2j+5/2}^2 \right)$$
$$\le C\left(\|u_0\|_{4N}^2 + \|\eta_0\|_{4N}^2 + T\|\eta_0\|_{4N+1/2}^2\right) \quad (1\text{-}5)$$

and

$$\sup_{0\le t\le T} \|\eta\|_{4N+1/2}^2 \le C\left(\|u_0\|_{4N}^2 + (1+T)\|\eta_0\|_{4N+1/2}^2\right)$$

for a universal constant $C > 0$. *The solution is unique among functions that achieve the initial data and for which the sum of the first three sums in* (1-5) *is finite. Moreover*, η *is such that the mapping* $\Phi(\cdot, t)$, *defined by* (1-1), *is a* C^{4N-2} *diffeomorphism for each* $t \in [0, T]$.

Remark 1.2. Since the mapping $\Phi(\cdot, t)$ is a C^{4N-2} diffeomorphism, we may change coordinates to $y \in \Omega(t)$ to produce solutions to 1.

The tools needed for the proof of Theorem 1.1 are developed throughout the rest of the paper, and the theorem is proved starting on page 354. We will sketch here the main ideas of the proof.

Linear $\mathscr{A}$-Navier–Stokes. Our iteration procedure is based on a geometric variant of the linear Navier–Stokes problem. We consider η (and hence $\mathscr{A}$, $\mathscr{N}$, etc.) as given and then solve the linear $\mathscr{A}$-Navier–Stokes equations for (u, p):

$$\begin{cases} \partial_t u - \Delta_{\mathscr{A}} u + \nabla_{\mathscr{A}} p = F^1 & \text{in } \Omega, \\ \operatorname{div}_{\mathscr{A}} u = 0 & \text{in } \Omega, \\ S_{\mathscr{A}}(p, u)\mathscr{N} = F^3 & \text{on } \Sigma, \\ u = 0 & \text{on } \Sigma_b, \end{cases} \quad (1\text{-}6)$$

with initial data u_0. Transforming this problem back to a moving domain $\Omega(t)$ using the mapping Φ defined in (1-1) shows that this problem is essentially equivalent (we have absorbed the correction to the time derivative into F^1, so it does not transform exactly) to solving the linear Navier–Stokes equations in a domain whose upper boundary is given by $\eta(t)$. In other words, we are really solving the usual linear problem in a moving domain.

Pressure as a Lagrange multiplier in time-dependent function spaces. It is well-known (see [Solonnikov and Skadilov 1973; Beale 1981; Coutand and Shkoller 2003; 2007]) that for the usual linear Navier–Stokes equations, the pressure can be viewed as a Lagrange multiplier that arises by restricting the dynamics to the class of vectors satisfying $\operatorname{div} u = 0$. To adapt this idea to the problem (1-6), we must restrict to the class of vectors satisfying $\operatorname{div}_{\mathscr{A}} u = 0$, which is a time-dependent condition since η (and hence $\mathscr{A}$) depends on t. This leads us to build time-dependent variants of the usual Sobolev spaces $H^0 = L^2$ and H^1 so that we can make sense of this time-dependent collection of $\operatorname{div}_{\mathscr{A}}$-free vectors. For the purpose of estimates, we want the time-dependent norms on these spaces to all be comparable to the usual Sobolev norms; this can be achieved through a smallness assumption on η, which we quantify. With the spaces in hand, we then adapt a technique from [Solonnikov and Skadilov 1973] to introduce the pressure as a Lagrange multiplier for $\operatorname{div}_{\mathscr{A}}$-free dynamics.

Elliptic estimates for $\mathscr{A}$-problems. In order to get the regularity we need for solutions to the parabolic problem (1-6), we first need the corresponding elliptic regularity theory. We accomplish this by using (1-1) to transform these elliptic problems back into Eulerian coordinates so that the PDEs transform to ones with constant coefficients. We then apply standard estimates for elliptic equations and systems, proved in [Agmon et al. 1959; 1964], and then transform these estimates on the Eulerian domain back to estimates on Ω. The only problem with this process is that the Eulerian domain has a boundary whose regularity is dictated by η and is phrased in H^k norms rather than C^k norms, which are what appear in [Agmon et al. 1959; 1964]. We get around this problem by using a smoothing operator, a limiting argument, and the smallness of η. Similar elliptic estimates were proved in Lagrangian coordinates for open, bounded domains in [Cheng and Shkoller 2010].

Galerkin method with a time-dependent basis. We construct solutions to (1-6) by using a time-dependent Galerkin method. This requires a countable basis of our space of $\operatorname{div}_{\mathscr{A}}$-free vector fields. Since the requirement $\operatorname{div}_{\mathscr{A}} u = 0$ is time-dependent, any basis of this space must also be time-dependent. For each $t \in [0, T]$, the space we work in (basically H^2 with $\operatorname{div}_{\mathscr{A}} u = 0$) is separable, so the existence of a countable basis is not an issue. The technical difficulty is that, in order for the basis to be useful in the Galerkin method, we must be able to differentiate the basis elements in time, and we must be able to express these time derivatives in terms of finitely many basis elements. Fortunately, due to a clever observation of Beale [1984], we are able construct an explicit time-dependent isomorphism that maps the div-free vector fields to the $\operatorname{div}_{\mathscr{A}}$-free fields. This allows us to construct the desired basis and push through the Galerkin method to produce "pressureless" weak solutions that are restricted to the collection of $\operatorname{div}_{\mathscr{A}}$-free fields. We then use our previous analysis to introduce the pressure as a Lagrange multiplier, which gives a weak solution to (1-6). We also use the Galerkin scheme to get higher regularity, showing that the solution is actually strong. The compatibility conditions serve as necessary conditions for controlling the temporal derivatives of the approximate solutions in the Galerkin scheme. The result of our strong existence theorem then allows us to iteratively deduce higher regularity, given that the forcing terms are more regular and higher-order compatibility conditions are satisfied.

Transport estimates. The problem (1-6) considers η as given and then produces (u, p). The second step in our iteration procedure is to take u as given and then solve $\partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3$ on Σ. This is a standard transport equation, so solving it presents no real obstacle. The difficulty is that in our analysis of (1-6), we need control of $\sup_{0 \le t \le T} \|\eta(t)\|^2_{4N+1/2}$, but owing to the transport structure, the only available estimate is, roughly speaking,

$$\sup_{0\le t\le T} \|\eta(t)\|_{4N+1/2}^2 \le C\exp\left(C\int_0^T \|Du(t)\|_{H^2(\Sigma)}\,dt\right)\left[\|\eta_0\|_{4N+1/2}^2 + T\int_0^T \|u(t)\|_{4N+1}^2\,dt\right].$$

Without knowing a priori that u decays, the right side of this estimate has the potential to grow at the rate of $(1+T)e^{C\sqrt{T}}$. Even if u decays rapidly, the right side can still grow like $(1+T)$. Of course, such a growth in time is disastrous for global stability analysis, but even in our local-existence iteration scheme, a delicate technique is required to accommodate such a growth without breaking the estimates of Theorem 1.1.

Closing the iteration with a two-tier energy scheme. Our iteration scheme then proceeds as described, using η^m to produce (u^{m+1}, p^{m+1}), and then using u^{m+1} to produce η^{m+1}. Iterating in this manner without losing control of our high-order energy estimates is rather delicate, and can only be completed by using sufficiently small initial data. The boundedness of the infinite sequence (u^m, p^m, η^m) in our high-order norms gives weak limits in the usual way, but because of the nature of our iteration scheme, we cannot guarantee a priori that the weak limits constitute a solution to (1-4). Instead of using high-order weak limits, we instead show that the sequence contracts in low-order norms, yielding strong convergence in low norms. We then combine the low-order strong convergence with the high-order weak convergence and an interpolation argument to deduce strong convergence in higher (but not all the way to the highest order) norms, which then suffices for passing to the limit $m\to\infty$ to produce a solution to (1-4).

Utility in the global theory. We believe that our local well-posedness result, Theorem 1.1, is interesting in its own right. It provides an alternative to the standard Beale–Solonnikov framework that is perhaps more natural due to the natural energy structure 1. The new ideas and techniques that we have introduced in order to work in this framework will likely be useful in many other problems.

However, we also need Theorem 1.1 as a crucial component in our global analysis of 1, which we carry out in [Guo and Tice 2013b] in the infinite case and in [Guo and Tice 2013a] in the periodic case. In both cases we develop novel a priori estimates that couple to the local theory to produce global-in-time solutions that decay to equilibrium at an algebraic rate. We call our a priori estimates a two-tier energy method because it couples the boundedness of certain high-regularity norms to the decay of certain low-regularity norms. The local theory we develop here both provides the tools for iteratively achieving global well-posedness and justifies all of the computations used in our two-tier a priori estimates.

Let us now informally state the theorems we prove in [Guo and Tice 2013b; 2013a].

Theorem 1.3. *The problem 1 is globally well-posed for sufficiently small initial data. In the infinite case, the solutions decay at a fixed algebraic rate. In the periodic case, by adjusting the smallness of the initial data, the solutions can be made to decay at arbitrarily fast algebraic rates. In other words, solutions in the periodic case decay almost exponentially.*

Remark 1.4. The reader interested in a unified presentation of the present paper and the global decay results of [Guo and Tice 2013b; 2013a] may consult [Guo and Tice 2010].

Remark 1.5. One can see a glimpse of the utility of our two-tier energy method already in the local theory. Indeed, the contraction argument we use to produce local solutions uses the boundedness of the high norms to close the contraction estimate for the low norms.

Definitions and terminology. We now mention some of the definitions, bits of notation, and conventions that we will use throughout the paper.

Einstein summation and constants. We will employ the Einstein convention of summing over repeated indices for vector and tensor operations. Throughout the paper, $C > 0$ will denote a generic constant that can depend on the parameters of the problem, N, and Ω, but does not depend on the data, etc. We refer to such constants as "universal". They are allowed to change from one inequality to the next. When a constant depends on a quantity z, we will write $C = C(z)$ to indicate this. We will employ the notation $a \lesssim b$ to mean that $a \le Cb$ for a universal constant $C > 0$.

Derivatives and norms. We will write Df for the horizontal gradient of f, that is, $Df = \partial_1 f e_1 + \partial_2 f e_2$, while ∇f will denote the usual full gradient. We write $H^k(\Omega)$ with $k \ge 0$ and $H^s(\Sigma)$ with $s \in \mathbb{R}$ for the usual Sobolev spaces. We will typically write $H^0 = L^2$; the exception to this is where we use $L^2([0, T]; H^k)$ notation to indicate the space of square-integrable functions with values in H^k.

To avoid notational clutter, we will avoid writing $H^k(\Omega)$ or $H^k(\Sigma)$ in our norms and typically write only $\|\cdot\|_k$. Since we will do this for functions defined on both Ω and Σ, this presents some ambiguity. We avoid this by adopting two conventions. First, we assume that functions have natural spaces on which they "live". For example, the functions u, p, and $\bar{\eta}$ live on Ω, while η itself lives on Σ. As we proceed in our analysis, we will introduce various auxiliary functions; the spaces they live on will always be clear from the context. Second, whenever the norm of a function is computed on a space different from the one in which it lives, we will explicitly write the space. This typically arises when computing norms of traces onto Σ of functions that live on Ω.

Plan of the paper. Our proof of Theorem 1.1 employs an iteration that is based on the following linear problem for (u, p), where we think of η (and hence $\mathcal{A}$, $\mathcal{N}$, etc.) as given:

$$\begin{cases} \partial_t u - \Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} p = F^1 & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}} u = 0 & \text{in } \Omega, \\ S_{\mathcal{A}}(p, u)\mathcal{N} = F^3 & \text{on } \Sigma, \\ u = 0 & \text{on } \Sigma_b, \end{cases} \tag{1-7}$$

subject to the initial condition $u(0) = u_0$. Note that the first equation in (1-7) may be rewritten as $\partial_t u + \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p, u) = F^1$.

In Section 2, we develop the machinery of time-dependent function spaces so that we can consider the class of $\operatorname{div}_{\mathcal{A}}$-free vector fields. We use an orthogonal splitting of a space to introduce the pressure as a Lagrange multiplier. In Section 3, we record some elliptic estimates for the $\mathcal{A}$-Stokes problem and the $\mathcal{A}$-Poisson problem. In Section 4, we develop the local existence theory for (1-7) by using a time-dependent Galerkin scheme. We iterate this result to produce high-regularity solutions. In Section 5, we do some preliminary work for the nonlinear problem, constructing initial data, detailing the compatibility conditions, and constructing solutions to the transport equation with high-regularity estimates. In Section 6, we construct solutions to (1-4) through the use of iteration and contraction arguments, completing the proof of Theorem 1.1.

Throughout the paper, we assume that $N \geq 3$ is an integer. We consider both the nonperiodic and periodic cases simultaneously. When different analysis is needed for each case, we will indicate so. Otherwise, the argument we write works in both cases.

2. Functional setting

Time-dependent function spaces. We begin our analysis of (1-7) by introducing some function spaces. We write $H^k(\Omega)$ and $H^k(\Sigma)$ for the usual L^2-based Sobolev spaces of either scalar or vector-valued functions. Define

$$
\begin{aligned}
{}_0H^1(\Omega) &:= \big\{u \in H^1(\Omega) \;\big|\; u|_{\Sigma_b} = 0\big\},\\
{}^0H^1(\Omega) &:= \big\{u \in H^1(\Omega) \;\big|\; u|_{\Sigma} = 0\big\},\\
{}_0H^1_\sigma(\Omega) &:= \big\{u \in {}_0H^1(\Omega) \;\big|\; \operatorname{div} u = 0\big\},
\end{aligned}
$$

with the obvious restriction that the last space is for vector-valued functions only.

For our time-dependent function spaces, we will consider η as given with $\mathcal{A}$, J, etc. determined by η via (1-3); in our subsequent analysis, η will always be sufficiently regular for all terms derived from η to make sense. We define a time-dependent inner-product on $L^2 = H^0$ by introducing

$$
(u, v)_{\mathcal{H}^0} := \int_\Omega (u \cdot v) J(t)
$$

with corresponding norm $\|u\|_{\mathcal{H}^0} := \sqrt{(u, u)_{\mathcal{H}^0}}$. Then we write $\mathcal{H}^0(t) := \{\|u\|_{\mathcal{H}^0} < \infty\}$. Similarly, we define a time-dependent inner-product on ${}_0H^1(\Omega)$ according to

$$
(u, v)_{\mathcal{H}^1} := \int_\Omega (\mathbb{D}_{\mathcal{A}(t)} u : \mathbb{D}_{\mathcal{A}(t)} v) J(t),
$$

and we define the corresponding norm by $\|u\|_{\mathcal{H}^1} = \sqrt{(u, u)_{\mathcal{H}^1}}$. Then we define

$$
\mathcal{H}^1(t) := \big\{u \;\big|\; \|u\|_{\mathcal{H}^1} < \infty,\, u|_{\Sigma_b} = 0\big\} \quad \text{and} \quad \mathcal{X}(t) := \big\{u \in \mathcal{H}^1(t) \;\big|\; \operatorname{div}_{\mathcal{A}(t)} u = 0\big\}. \tag{2-1}
$$

We will also need the orthogonal decomposition $\mathcal{H}^0(t) = \mathcal{Y}(t) \oplus \mathcal{Y}(t)^\perp$, where

$$
\mathcal{Y}(t)^\perp := \{\nabla_{\mathcal{A}(t)} \varphi \mid \varphi \in {}^0H^1(\Omega)\}. \tag{2-2}
$$

A further discussion of the space $\mathcal{Y}(t)$ can be found later in Remark 3.4. In our use of these norms and spaces, we will often drop the (t) when there is no potential for confusion.

Finally, for $T > 0$ and $k = 0, 1$, we define inner products on $L^2([0, T]; H^k(\Omega))$ by

$$
(u, v)_{\mathcal{H}^k_T} := \int_0^T (u(t), v(t))_{\mathcal{H}^k}\, dt. \tag{2-3}
$$

Write $\|u\|_{\mathscr{H}^k_T}$ for the corresponding norms and $\mathscr{H}^k_T$ for the corresponding spaces. We define the subspace of $\operatorname{div}_{\mathscr{A}}$-free vector fields as

$$\mathscr{X}_T := \left\{ u \in \mathscr{H}^1_T \;\middle|\; \operatorname{div}_{\mathscr{A}(t)} u(t) = 0 \text{ for almost every } t \in [0, T] \right\}. \tag{2-4}$$

A priori, we do not know that the spaces $\mathscr{H}^k(t)$ and $\mathscr{H}^k_T$ have the same topology as H^k and $L^2 H^k$, respectively. This can be established under a smallness assumption on η.

Lemma 2.1. *There exists a universal $\varepsilon_0 > 0$ such that if*

$$\sup_{0 \le t \le T} \|\eta(t)\|_3 < \varepsilon_0, \tag{2-5}$$

then

$$\frac{1}{\sqrt{2}} \|u\|_k \le \|u\|_{\mathscr{H}^k} \le \sqrt{2} \|u\|_k \tag{2-6}$$

for $k = 0, 1$ and for all $t \in [0, T]$. As a consequence, for $k = 0, 1$,

$$\frac{1}{\sqrt{2}} \|u\|_{L^2 H^k} \le \|u\|_{\mathscr{H}^k_T} \le \sqrt{2} \|u\|_{L^2 H^k}. \tag{2-7}$$

Proof. Consider $\varepsilon \in \left(0, \frac{1}{2}\right)$ with a precise value to be chosen later. It is straightforward to verify, using the definitions in (1-3) along with Lemma A.8 in the nonperiodic case and Lemma A.10 in the periodic case, that

$$\sup\left\{ \|J - 1\|_{L^\infty}, \|A\|_{L^\infty}, \|B\|_{L^\infty} \right\} \le C \|\eta\|_3. \tag{2-8}$$

Then we may choose $\varepsilon_0 = \varepsilon / C$ such that the right side of (2-8) is bounded by ε. Since $K = 1/J$, this implies that

$$\|K - 1\|_{L^\infty} \le \frac{\varepsilon}{1 - \varepsilon}, \quad \|K\|_{L^\infty} \le \frac{1}{1 - \varepsilon}$$

and

$$\|I - \mathscr{A}\|_{L^\infty} \le \frac{\sqrt{3}\varepsilon}{1 - \varepsilon}, \quad \|\mathscr{A} + I\|_{L^\infty} \le 2\sqrt{3} + \frac{\sqrt{3}\varepsilon}{1 - \varepsilon}.$$

In turn, this implies that

$$\|J\|_{L^\infty} \|I - \mathscr{A}\|_{L^\infty} \|I + \mathscr{A}\|_{L^\infty} \le \frac{3\varepsilon(1+\varepsilon)(2-\varepsilon)}{(1-\varepsilon)^2} := g(\varepsilon). \tag{2-9}$$

Notice that g is a continuous, increasing function on $\left(0, \frac{1}{2}\right)$ such that $g(0) = 0$. With the estimates (2-8) and (2-9) in hand, we can show that if ε is chosen sufficiently small, then (2-6) and (2-7) hold.

In the case $k = 0$, the estimate (2-6) follows directly from the estimate for J in (2-8):

$$\frac{1}{2} \int_\Omega |u|^2 \le (1 - \varepsilon) \int_\Omega |u|^2 \le \int_\Omega J |u|^2 \le (1 + \varepsilon) \int_\Omega |u|^2 \le 2 \int_\Omega |u|^2.$$

To derive (2-6) when $k = 1$, we first rewrite

$$\int_\Omega J |\mathbb{D}_{\mathscr{A}} u|^2 = \int_\Omega J |\mathbb{D} u|^2 + \int_\Omega J (\mathbb{D}_{\mathscr{A}} u + \mathbb{D} u) : (\mathbb{D}_{\mathscr{A}} u - \mathbb{D} u). \tag{2-10}$$

To estimate the last term, we note that $|(\mathbb{D}_{\mathscr{A}} u \pm \mathbb{D} u)| \le 2|\mathscr{A} \pm I||\nabla u|$, which implies that

$$\left| \int_\Omega J(\mathbb{D}_{\mathscr{A}} u + \mathbb{D} u) : (\mathbb{D}_{\mathscr{A}} u - \mathbb{D} u) \right| \le 4\|J\|_{L^\infty} \|I - \mathscr{A}\|_{L^\infty} \|I + \mathscr{A}\|_{L^\infty} \int_\Omega |\nabla u|^2 \le 4C_\Omega g(\varepsilon) \int_\Omega |\mathbb{D} u|^2, \tag{2-11}$$

where C_Ω is the constant in Korn's inequality, Lemma A.13. We may then employ the bounds (2-8) and (2-11) in (2-10) to estimate

$$\int_\Omega |\mathbb{D}_{\mathscr{A}} u|^2 J \ge \int_\Omega J|\mathbb{D} u|^2 - 4C_\Omega g(\varepsilon) \int_\Omega |\mathbb{D} u|^2 \ge \big(1 - \varepsilon - 4C_\Omega g(\varepsilon)\big) \int_\Omega |\mathbb{D} u|^2, \tag{2-12}$$

$$\int_\Omega |\mathbb{D}_{\mathscr{A}} u|^2 J \le \int_\Omega J|\mathbb{D} u|^2 + 4C_\Omega g(\varepsilon) \int_\Omega |\mathbb{D} u|^2 \le \big(1 + \varepsilon + 4C_\Omega g(\varepsilon)\big) \int_\Omega |\mathbb{D} u|^2. \tag{2-13}$$

Then (2-6) with $k = 1$ follows from (2-12)–(2-13) by choosing ε small enough so that $\varepsilon + 4C_\Omega g(\varepsilon) \le \frac{1}{2}$. The estimates (2-7) follow by applying (2-6) for almost every $t \in [0, T]$, squaring, and integrating over $t \in [0, T]$. □

Remark 2.2. Throughout the rest of this paper, we will assume that (2-5) is satisfied, so that (2-6)–(2-7) hold.

Remark 2.3. Because of the bound (2-6) and the usual Korn inequality on Ω, Lemma A.13, we have a corresponding Korn-type inequality in $\mathscr{H}^1(t)$ (defined in (2-1)): $\|u\|_{\mathscr{H}^0} \lesssim \|u\|_{\mathscr{H}^1}$. The standard trace embedding $H^1(\Omega) \hookrightarrow H^{1/2}(\Sigma)$ and (2-6) imply that $\|u\|_{H^{1/2}(\Sigma)} \lesssim \|u\|_{\mathscr{H}^1}$ for all $t \in [0, T]$. Similarly, given $f \in H^{1/2}(\Sigma)$, we may construct an extension $\tilde{f} \in \mathscr{H}^1(t)$ such that $\|f\|_{\mathscr{H}^1} \lesssim \|f\|_{H^{1/2}(\Sigma)}$.

We now prove a result about the differentiability of norms in our time-dependent spaces.

Lemma 2.4. *Suppose that $u \in \mathscr{H}^1_T$, $\partial_t u \in (\mathscr{H}^1_T)^*$, where $\mathscr{H}^1_T$ is defined in (2-3). Then the mapping $t \mapsto \|u(t)\|^2_{\mathscr{H}^0(t)}$ is absolutely continuous, and*

$$\frac{d}{dt}\|u(t)\|^2_{\mathscr{H}^0} = 2\langle \partial_t u(t), u(t)\rangle_{(\mathscr{H}^1)^*} + \int_\Omega |u(t)|^2 \partial_t J(t) \tag{2-14}$$

for almost every $t \in [0, T]$. Moreover, $u \in C^0\big([0, T]; H^0(\Omega)\big)$. If $v \in \mathscr{H}^1_T$, $\partial_t v \in (\mathscr{H}^1_T)^*$ as well, then

$$\frac{d}{dt}(u(t), v(t))_{\mathscr{H}^0} = \langle \partial_t u(t), v(t)\rangle_{(\mathscr{H}^1)^*} + \langle \partial_t v(t), u(t)\rangle_{(\mathscr{H}^1)^*} + \int_\Omega u(t) \cdot v(t) \partial_t J(t). \tag{2-15}$$

A similar result holds for $u \in \mathscr{X}_T$ with $\partial_t u \in (\mathscr{X}_T)^*$.

Proof. In light of Lemma 2.1, the time-dependent spaces $\mathscr{H}^0_T$, $\mathscr{H}^1_T$, $(\mathscr{H}^1_T)^*$ present no obstacle to the usual method of approximation by temporally smooth functions via convolution. This allows us to argue as in Theorem 3 in Section 5.9 of [Evans 2010] to deduce (2-14) and the continuity $u \in C^0([0, T]; H^0(\Omega))$. The equality (2-15) follows by applying (2-14) to $u + v$ and canceling terms by using (2-14) with u and with v. □

Now we want to show the spaces ${}_0H^1(\Omega)$ and ${}_0H^1_\sigma(\Omega)$ are related to the spaces $\mathscr{H}^1(t)$ and $\mathscr{X}(t)$. To this end, we define the matrix

$$M := M(t) = K\nabla\Phi = \begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ AK & BK & 1 \end{pmatrix}, \tag{2-16}$$

where A, B, and K are as defined in (1-3). Note that M is invertible, and $M^{-1} = J\mathscr{A}^T$. Since $J \neq 0$ and $\partial_j(J\mathscr{A}_{ij}) = 0$ for each $i = 1, 2, 3$ (see Lemma A.3),

$$p = \operatorname{div}_{\mathscr{A}} v \iff$$
$$Jp = J\operatorname{div}_{\mathscr{A}} v = J\mathscr{A}_{ij}\partial_j v_i = \partial_j(J\mathscr{A}_{ij}v_i) = \partial_j(J\mathscr{A}^T v)_j = \partial_j(M^{-1}v)_j = \operatorname{div}(M^{-1}v). \tag{2-17}$$

The matrix $M(t)$ induces a linear operator $\mathcal{M}_t : u \mapsto \mathcal{M}_t(u) = M(t)u$ that possesses several nice properties, the most important of which is that div-free vector fields are mapped to $\operatorname{div}_{\mathscr{A}}$-free vector fields. We record these now.

Proposition 2.5. *For each $t \in [0, T]$, $\mathcal{M}_t$ is a bounded, linear isomorphism: from $H^k(\Omega)$ to $H^k(\Omega)$ for $k = 0, 1, 2$; from $L^2(\Omega)$ to $\mathscr{H}^0(t)$; from ${}_0H^1(\Omega)$ to $\mathscr{H}^1(t)$; and from ${}_0H^1_\sigma(\Omega)$ to $\mathscr{X}(t)$. In each case the norms of the operators $\mathcal{M}_t$, $\mathcal{M}_t^{-1}$ are bounded by a constant times $1 + \|\eta(t)\|_{9/2}$.*

The mapping $\mathcal{M}$ given by $\mathcal{M}u(t) := \mathcal{M}_t u(t)$ is a bounded, linear isomorphism: from $L^2([0, T]; H^k(\Omega))$ to $L^2([0, T]; H^k(\Omega))$ for $k = 0, 1, 2$; from $L^2([0, T]; H^0(\Omega))$ to $\mathscr{H}^0_T$; from $L^2([0, T]; {}_0H^1(\Omega))$ to $\mathscr{H}^1_T$; and from $L^2([0, T]; {}_0H^1_\sigma(\Omega))$ to $\mathscr{X}_T$. In each case, the norms of the operators $\mathcal{M}$ and $\mathcal{M}^{-1}$ are bounded by a constant times the sum $1 + \sup_{0\le t\le T}\|\eta(t)\|_{9/2}$.

Proof. For each $t \in [0, T]$, it is easy to see, using Lemma A.8 in the nonperiodic case and Lemma A.10 in the periodic case, that

$$\|\mathcal{M}_t u\|_k \lesssim \|M(t)\|_{C^2}\|u\|_k \lesssim \big(1 + \|\bar\eta(t)\|_{C^3}\big)\|u\|_k \lesssim \big(1 + \|\eta(t)\|_{9/2}\big)\|u\|_k$$

for $k = 0, 1, 2$, which establishes that $\mathcal{M}_t$ is a bounded operator on H^k. Since $M(t)$ is an invertible matrix, $\mathcal{M}_t^{-1}v = M(t)^{-1}v = J\mathscr{A}^T(t)v$, which allows us to argue similarly to see that for $k = 0, 1, 2$, $\|\mathcal{M}_t^{-1}v\|_k \lesssim (1 + \|\eta(t)\|_{9/2})\|v\|_k$. Hence $\mathcal{M}_t$ is an isomorphism of H^k to itself for $k = 0, 1, 2$. With this fact in hand, Lemma 2.1 implies that $\mathcal{M}_t$ is an isomorphism of $H^0(\Omega)$ to $\mathscr{H}^0(t)$ and of ${}_0H^1(\Omega)$ to $\mathscr{H}^1(t)$.

To prove that $\mathcal{M}_t$ is an isomorphism of ${}_0H^1_\sigma(\Omega)$ to $\mathscr{X}(t)$, we must only establish that $\operatorname{div} u = 0$ if and only if $\operatorname{div}_{\mathscr{A}}(Mu) = 0$. To see this, we appeal to (2-17) with $p = 0$ to see that $0 = \operatorname{div}_{\mathscr{A}} v$ if and only if $0 = \operatorname{div}(M^{-1}v)$. Hence, writing $v = Mu$, we see that $\operatorname{div} u = 0$ if and only if $\operatorname{div}_{\mathscr{A}}(Mu) = 0$.

The mapping properties of the operator $\mathcal{M}$ on space-time functions may be established in a similar manner. □

Pressure as a Lagrange multiplier. It is well-known [Solonnikov and Skadilov 1973; Beale 1981; Coutand and Shkoller 2007] that the space ${}_0H^1(\Omega)$ can be orthogonally decomposed as ${}_0H^1(\Omega) = {}_0H^1_\sigma(\Omega) \oplus R(Q)$, where $R(Q)$ is the range of the operator $Q : H^0(\Omega) \to {}_0H^1(\Omega)$, defined by the Riesz representation theorem via the relation

$$\int_\Omega p \operatorname{div} u = \int_\Omega \mathbb{D}(Qp) : \mathbb{D}u \quad \text{for all } u \in {}_0H^1(\Omega).$$

We now wish to establish a similar decomposition for our spaces $\mathscr{X}(t) \subset \mathscr{H}^1(t)$. Unfortunately, the mappings $\mathscr{M}_t$, while isomorphisms, are not isometries, so we cannot use the known result to decompose $\mathscr{H}^1(t)$. Instead, we must adapt the method of [Solonnikov and Skadilov 1973] to our time-dependent context.

For $p \in \mathscr{H}^0(t)$, we define the functional $\mathscr{Q}_t \in (\mathscr{H}^1(t))^*$ by $\mathscr{Q}_t(v) = (p, \operatorname{div}_{\mathscr{A}} v)_{\mathscr{H}^0}$. By the Riesz representation theorem, there exists a unique $Q_t p \in \mathscr{H}^1(t)$ such that $\mathscr{Q}_t(v) = (Q_t p, v)_{\mathscr{H}^1}$ for all $v \in \mathscr{H}^1(t)$. This defines a linear operator $Q_t : \mathscr{H}^0(t) \to \mathscr{H}^1(t)$, which is bounded since we may take $v = Q_t p$ to get the bound

$$\begin{aligned} \|Q_t p\|_{\mathscr{H}^1}^2 = (Q_t p, Q_t p)_{\mathscr{H}^1} = \mathscr{Q}_t(v) = (p, \operatorname{div}_{\mathscr{A}} v)_{\mathscr{H}^0} \\ \le \|p\|_{\mathscr{H}^0} \|\operatorname{div}_{\mathscr{A}} v\|_{\mathscr{H}^0} \le \|p\|_{\mathscr{H}^0} \|v\|_{\mathscr{H}^1} = \|p\|_{\mathscr{H}^0} \|Q_t p\|_{\mathscr{H}^1}, \end{aligned} \tag{2-18}$$

so that $\|Q_t p\|_{\mathscr{H}^1} \le \|p\|_{\mathscr{H}^0}$. In the previous inequality, we have utilized the simple bound $\|\operatorname{div}_{\mathscr{A}} v\|_{\mathscr{H}^0} \le \|v\|_{\mathscr{H}^1}$, which follows from the fact that $\operatorname{div}_{\mathscr{A}} v = \operatorname{tr}(\mathbb{D}_{\mathscr{A}} u)/2$. In a straightforward manner, we may also define a bounded linear operator $Q : \mathscr{H}^0_T \to \mathscr{H}^1_T$ via the relation

$$(p, \operatorname{div}_{\mathscr{A}} v)_{\mathscr{H}^0_T} = (Qp, v)_{\mathscr{H}^1_T} \quad \text{for all } v \in \mathscr{H}^1_T.$$

Arguing as above, we can show that Q satisfies $\|Qp\|_{\mathscr{H}^1_T} \le \|p\|_{\mathscr{H}^0_T}$.

In order to study the range of Q_t in $\mathscr{H}^1(t)$ and of Q in $\mathscr{H}^1_T$, we will first need a lemma on the solvability of the equation $\operatorname{div}_{\mathscr{A}} v = p$.

Lemma 2.6. *Let $p \in \mathscr{H}^0(t)$. Then there exists a $v \in \mathscr{H}^1(t)$ such that $\operatorname{div}_{\mathscr{A}} v = p$ and*

$$\|v\|_{\mathscr{H}^1} \lesssim \big(1 + \|\eta(t)\|_{9/2}\big) \|p\|_{\mathscr{H}^0}.$$

If instead $p \in \mathscr{H}^0_T$, then there exists a $v \in \mathscr{H}^1_T$ such that $\operatorname{div}_{\mathscr{A}} v = p$ for almost every $t \in [0, T]$, and $\|v\|_{\mathscr{H}^1_T} \lesssim (1 + \sup_{0 \le t \le T} \|\eta(t)\|_{9/2}) \|p\|_{\mathscr{H}^0_T}$.

Proof. It is established in the proof of Lemma 3.3 of [Beale 1981] that for any $q \in L^2(\Omega)$, the problem $\operatorname{div} u = q$ admits a solution $u \in {}_0H^1(\Omega)$ such that $\|u\|_1 \lesssim \|q\|_0$. The result in [Beale 1981] concerns the nonperiodic case, but its proof may be easily adapted to the periodic case as well. Choose $q = Jp$ so that

$$\|q\|_0^2 = \int_\Omega |q|^2 = \int_\Omega |p|^2 J^2 \le \|J\|_{L^\infty} \|p\|_{\mathscr{H}^0}^2 \le 2\|p\|_{\mathscr{H}^0}^2.$$

Then by (2-17), we know that $v = M(t)u \in \mathscr{H}^1(t)$ satisfies $\operatorname{div}_{\mathscr{A}} v = p$, and Proposition 2.5 implies that

$$\|v\|_{\mathscr{H}^1} \lesssim \big(1 + \|\eta(t)\|_{9/2}\big)\|u\|_1 \lesssim \big(1 + \|\eta(t)\|_{9/2}\big)\|q\|_0 \lesssim \big(1 + \|\eta(t)\|_{9/2}\big)\|p\|_{\mathscr{H}^0}. \tag{2-19}$$

If $p \in \mathscr{H}^0_T$, then for almost every $t \in [0, T]$, $p(t) \in \mathscr{H}^0(t)$, so we may apply the above analysis to find

$v(t) \in \mathcal{H}^1(t)$ such that $\operatorname{div}_{\mathcal{A}} v(t) = p(t)$ and the bound (2-19) holds with $v = v(t)$ and $p = p(t)$. We may then square both sides and integrate over $t \in [0, T]$ to deduce that

$$\begin{aligned}\|v\|^2_{\mathcal{H}^1_T} &= \int_0^T \|v(t)\|^2_{\mathcal{H}^1}\,dt \lesssim \Bigl(1 + \sup_{0\le t\le T} \|\eta(t)\|^2_{9/2}\Bigr) \int_0^T \|p(t)\|^2_{\mathcal{H}^0}\,dt \\ &\lesssim \Bigl(1 + \sup_{0\le t\le T} \|\eta(t)\|^2_{9/2}\Bigr) \|v\|^2_{\mathcal{H}^0_T}. \end{aligned}$$ □

With this lemma in hand, we can show that the range of Q_t, $R(Q_t)$, is a closed subspace of $\mathcal{H}^1(t)$ and that $R(Q)$ is a closed subspace of $\mathcal{H}^1_T$.

Lemma 2.7. *$R(Q_t)$ is closed in $\mathcal{H}^1(t)$, and $R(Q)$ is closed in $\mathcal{H}^1_T$.*

Proof. For $p \in \mathcal{H}^0(t)$, let $v \in \mathcal{H}^1(t)$ be the solution to $\operatorname{div}_{\mathcal{A}} v = p$ provided by Lemma 2.6. Then

$$\begin{aligned}\|p\|^2_{\mathcal{H}^0} &= (p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} = \mathcal{Q}_t(v) = (Q_t p, v)_{\mathcal{H}^1} \\ &\le \|Q_t p\|_{\mathcal{H}^1} \|v\|_{\mathcal{H}^1} \lesssim \|Q_t p\|_{\mathcal{H}^1} \bigl(1 + \|\eta(t)\|_{9/2}\bigr) \|p\|_{\mathcal{H}^0},\end{aligned}$$

so that we get, using (2-18),

$$\|Q_t p\|_{\mathcal{H}^1} \le \|p\|_{\mathcal{H}^0} \lesssim (1 + \|\eta(t)\|_{9/2}) \|Q_t p\|_{\mathcal{H}^1}.$$

Hence $R(Q_t)$ is closed in $\mathcal{H}^1(t)$. A similar analysis shows that $R(Q)$ is closed in $\mathcal{H}^1_T$. □

Now we can perform the orthogonal decomposition of $\mathcal{H}^1(t)$ and $\mathcal{H}^1_T$, defined by (2-1) and (2-3) respectively.

Lemma 2.8. *We have that $\mathcal{H}^1(t) = \mathcal{X}(t) \oplus R(Q_t)$, that is, $\mathcal{X}(t)^\perp = R(Q_t)$. Also, $\mathcal{H}^1_T = \mathcal{X}_T \oplus R(Q)$, that is, $\mathcal{X}_T^\perp = R(Q)$.*

Proof. By Lemma 2.7, $R(Q_t)$ is a closed subspace of $\mathcal{H}^1(t)$, and so it suffices to prove the equality $R(Q_t)^\perp = \mathcal{X}(t)$.

Let $v \in R(Q_t)^\perp$. Then for all $p \in \mathcal{H}^0(t)$, we know that

$$\int_\Omega p \operatorname{div}_{\mathcal{A}} v J = \mathcal{Q}_t(v) = (Q_t p, v)_{\mathcal{H}^1} = 0,$$

and hence $\operatorname{div}_{\mathcal{A}} v = 0$. This implies that $R(Q_t)^\perp \subseteq \mathcal{X}(t)$.

Now suppose that $v \in \mathcal{X}(t)$. Then $\operatorname{div}_{\mathcal{A}} v = 0$ implies that

$$0 = \int_\Omega p \operatorname{div}_{\mathcal{A}} v J = \mathcal{Q}_t(v) = (Q_t p, v)_{\mathcal{H}^1}$$

for all $p \in \mathcal{H}^0(t)$. Hence $v \in R(Q_t)^\perp$, and we see that $\mathcal{X}(t) \subseteq R(Q_t)^\perp$.

A similar argument shows that $\mathcal{H}^1_T = \mathcal{X}_T \oplus R(Q)$. □

This decomposition will eventually allow us to introduce the pressure function. This will be accomplished by use of the following result.

Proposition 2.9. *If $\Lambda_t \in (\mathscr{H}^1(t))^*$ is such that $\Lambda_t(v) = 0$ for all $v \in \mathscr{X}(t)$, then there exists a unique $p(t) \in \mathscr{H}^0(t)$ such that*

$$(p(t), \operatorname{div}_{\mathscr{A}} v)_{\mathscr{H}^0} = \Lambda_t(v) \quad \textit{for all } v \in \mathscr{H}^1(t)$$

and $\|p(t)\|_{\mathscr{H}^0} \lesssim (1 + \|\eta(t)\|_{9/2}) \|\Lambda_t\|_{(\mathscr{H}^1(t))^*}$.

If $\Lambda \in (\mathscr{H}^1_T)^*$ is such that $\Lambda(v) = 0$ for all $v \in \mathscr{X}_T$, then there exists a unique $p \in \mathscr{H}^0_T$ such that

$$(p, \operatorname{div}_{\mathscr{A}} v)_{\mathscr{H}^0_T} = \Lambda(v) \quad \textit{for all } v \in \mathscr{H}^1_T$$

and $\|p\|_{\mathscr{H}^0_T} \lesssim \bigl(1 + \sup_{0 \le t \le T} \|\eta(t)\|_{9/2}\bigr) \|\Lambda\|_{(\mathscr{H}^1_T)^*}$.

Proof. If $\Lambda_t(v) = 0$ for all $v \in \mathscr{X}(t)$, then the Riesz representation theorem yields the existence of a unique $w \in \mathscr{X}(t)^\perp$ such that $\Lambda_t(v) = (w, v)_{\mathscr{H}^1}$ for all $v \in \mathscr{H}^1(t)$. By Lemma 2.8, $w = Q_t p(t)$ for a unique $p(t) \in \mathscr{H}^0(t)$. Then $\Lambda_t(v) = (Q_t p(t), v)_{\mathscr{H}^1} = (p(t), \operatorname{div}_{\mathscr{A}} v)_{\mathscr{H}^0}$ for all $v \in \mathscr{H}^1(t)$. By Lemma 2.6, we may find $v(t) \in \mathscr{H}^1(t)$ such that $\operatorname{div}_{\mathscr{A}} v(t) = p(t)$ and $\|v(t)\|_{\mathscr{H}^1} \lesssim (1 + \|\eta(t)\|_{9/2}) \|p(t)\|_{\mathscr{H}^0}$. Hence

$$\|p(t)\|^2_{\mathscr{H}^0} = \bigl(p(t), \operatorname{div}_{\mathscr{A}} v(t)\bigr)_{\mathscr{H}^0} = \Lambda_t(v(t)) \le \|\Lambda_t\|_{(\mathscr{H}^1(t))^*} \bigl(1 + \|\eta(t)\|_{9/2}\bigr) \|p(t)\|_{\mathscr{H}^0},$$

and the desired estimate holds. A similar argument proves the result for $\Lambda \in (\mathscr{H}^1_T)^*$ such that $\Lambda(v) = 0$ for all $v \in \mathscr{X}_T$. □

3. Elliptic estimates

Preliminary estimates. In studying the elliptic problems in the rest of this section, we will utilize the fact that the equations can be transformed into constant coefficient equations on the domain $\Omega' = \Phi(\Omega)$, where Φ is defined by (1-1). In order to properly utilize this transformation, we must verify that composition with Φ generates an isomorphism of $H^k(\Omega')$ to $H^k(\Omega)$. This type of result is standard (see the appendix of [Bourguignon and Brezis 1974] for the case of a bounded domain, or of [Beale 1984, Lemma 5.2] and [Sylvester 1990, Lemma 6.2] for the case of $\mathbb{R}^n$), but the precise form we need is not readily available in the literature, so we record it now.

Lemma 3.1. *Let $\Psi : \Omega \to \Omega'$ be a C^1 diffeomorphism satisfying $\Psi \in H^{k+1}_{\mathrm{loc}}(\Omega)$ and $\nabla \Psi - I \in H^k(\Omega)$ for an integer $k \ge 3$, as well as the estimate $\|1 - \det \nabla \Psi\|_{L^\infty} \le \frac{1}{2}$. If $v \in H^m(\Omega')$, then $v \circ \Psi \in H^m(\Omega)$ for $m = 0, 1, \dots, k+1$, and*

$$\|v \circ \Psi\|_{H^m(\Omega)} \lesssim C\bigl(\|\nabla \Psi - I\|_{H^k(\Omega)}\bigr) \|v\|_{H^m(\Omega')} \tag{3-1}$$

for $C(\|\nabla \Psi - I\|_{H^k(\Omega)})$ a constant depending on $\|\nabla \Psi - I\|_{H^k(\Omega)}$. Similarly, for $u \in H^m(\Omega)$, $u \circ \Psi^{-1} \in H^m(\Omega')$ for $m = 0, 1, \dots, k+1$, and

$$\|u \circ \Psi^{-1}\|_{H^m(\Omega')} \lesssim C\bigl(\|\nabla \Psi - I\|_{H^k(\Omega)}\bigr) \|u\|_{H^m(\Omega)}. \tag{3-2}$$

Let $\Sigma' = \Psi(\Sigma)$ denote the upper boundary of Ω'. If $v \in H^{m-1/2}(\Sigma')$ for $m = 1, \dots, k-1$, then $v \circ \Psi \in H^{m-1/2}(\Sigma)$ and

$$\|v \circ \Psi\|_{H^{m-1/2}(\Sigma)} \lesssim C\bigl(\|\nabla \Psi - I\|_{H^k(\Omega)}\bigr) \|v\|_{H^{m-1/2}(\Sigma')}. \tag{3-3}$$

If $u \in H^{m-1/2}(\Sigma)$ *for* $m = 1, \dots, k-1$, *then* $v \circ \Psi^{-1} \in H^{m-1/2}(\Sigma')$ *and*

$$\|u \circ \Psi^{-1}\|_{H^{m-1/2}(\Sigma')} \lesssim C\big(\|\nabla\Psi - I\|_{H^k(\Omega)}\big)\|u\|_{H^{m-1/2}(\Sigma)}. \tag{3-4}$$

Proof. The proof of (3-1)–(3-2) is similar to the proofs of the results in [Bourguignon and Brezis 1974; Beale 1984; Sylvester 1990] mentioned above, so we present only a sketch. We first prove that for $m \in \{0, 1, 2\}$, we have

$$\|v \circ \Psi\|_{H^m(\Omega)} \lesssim C\big(\|\nabla\Psi - I\|_{H^k(\Omega)}\big)\|v\|_{H^m(\Omega')}. \tag{3-5}$$

Such a bound follows easily from the size of k, the Sobolev embeddings, and the bound on $\det \nabla\Psi$. We then proceed inductively for $m = 3, \dots, k+1$. Suppose the bound (3-5) holds for $m = 0, 1, 2, \dots, m_0$ for $2 \le m_0 \le k$. To show that it holds for $m_0 + 1$, we write x for coordinates in Ω and y for coordinates in Ω' and note that

$$\frac{\partial}{\partial x_i}(v \circ \Psi)(x) = \frac{\partial v}{\partial y_j} \circ \Psi(x) \cdot \frac{\partial \Psi_j}{\partial x_i}(x) = \frac{\partial v}{\partial y_i} \circ \Psi(x) + \frac{\partial v}{\partial y_j} \circ \Psi(x) \cdot \Big(\frac{\partial \Psi_j}{\partial x_i}(x) - I_{ij}\Big).$$

By the induction hypothesis, if $v \in H^{m_0+1}$, then

$$\frac{\partial v}{\partial y_j} \circ \Psi \in H^{m_0} \quad \text{for } j = 1, 2, 3,$$

and since we have the multiplicative embedding $H^{m_0} \cdot H^k \hookrightarrow H^{m_0}$ for $m_0 \ge 2$ and $k \ge 3$, we deduce that

$$\frac{\partial}{\partial x_i}(v \circ \Psi) \in H^{m_0} \quad \text{for } i = 1, 2, 3,$$

and hence that $v \circ \Psi \in H^{m_0+1}$. Moreover, an estimate of the form (3-5) holds. By induction, we deduce that (3-1) holds. The result (3-2) follows similarly, utilizing the fact that $\nabla\Psi^{-1}(y) = (\nabla\Psi)^{-1} \circ \Psi^{-1}(y)$.

We now turn to the proof of (3-3)–(3-4). First note that since $\Psi \in H^{k+1}_{\mathrm{loc}}$, the usual Sobolev embeddings imply that Σ' is locally the graph of a $C^{k-1,1/2}$ function. Hence (see [Adams 1975]), there exists a bounded extension operator $E : H^{m-1/2}(\Sigma') \to H^m(\Omega')$ for $m = 1, \dots, k-1$ with the norm of the operator depending on $C(\|\nabla\Psi - I\|_{H^k(\Omega)})$. For $v \in H^{m-1/2}(\Sigma')$, let $V = Ev \in H^m(\Omega')$. By (3-1), we have that $V \circ \Psi \in H^m(\Omega)$, and by the usual trace theory, $v \circ \Psi = V \circ \Psi|_\Sigma \in H^{m-1/2}(\Sigma)$. Moreover,

$$\begin{aligned}\|v \circ \Psi\|_{H^{m-1/2}(\Sigma)} \lesssim \|V \circ \Psi\|_{H^m(\Omega)} &\lesssim C\big(\|\nabla\Psi - I\|_{H^k(\Omega)}\big)\|Ev\|_{H^m(\Omega')} \\ &\lesssim C\big(\|\nabla\Psi - I\|_{H^k(\Omega)}\big)\|v\|_{H^{m-1/2}(\Sigma')},\end{aligned}$$

which is (3-3). The bound (3-4) follows similarly. □

Remark 3.2. It is easy to show, using Lemma A.10 in the periodic case and Lemma A.8 in the nonperiodic case, that if $\|\eta\|^2_{k+1/2}$ is sufficiently small for $k \ge 3$, then the mapping Φ defined by (1-1) is a C^1 diffeomorphism that satisfies the hypotheses of Lemma 3.1.

We will also need the following $H^{-1/2}$ boundary estimates for functions satisfying $u, \operatorname{div}_{\mathcal{A}} u \in \mathcal{H}^0(t)$.

Lemma 3.3. *If $v \in \mathscr{H}^0(t)$ and $\operatorname{div}_{\mathscr{A}} v \in \mathscr{H}^0(t)$, then $v \cdot \mathscr{N} \in H^{-1/2}(\Sigma)$, $v \cdot \nu \in H^{-1/2}(\Sigma_b)$ (with ν the unit normal on Σ_b), and*

$$\|v \cdot \mathscr{N}\|_{H^{-1/2}(\Sigma)} + \|v \cdot \nu\|_{H^{-1/2}(\Sigma_b)} \lesssim \|v\|_{\mathscr{H}^0} + \|\operatorname{div}_{\mathscr{A}} v\|_{\mathscr{H}^0}.$$

Proof. We will only prove the result on Σ; the result on Σ_b may be derived in a similar manner, using the fact that $J\mathscr{A}\nu = \nu$ on Σ_b.

Let $\varphi \in H^{1/2}(\Sigma)$ be a scalar function, and let $\tilde{\varphi} \in {}_0H^1(\Omega)$ be a bounded extension. If we define the vector field $w = \tilde{\varphi} e_1$, then a straightforward computation reveals that

$$2 \int_\Omega |\nabla_{\mathscr{A}} \tilde{\varphi}|^2 J \le \|w\|^2_{\mathscr{H}^1} \quad \text{and} \quad \|w\|^2_{{}_0H^1(\Omega)} \le 4 \int_\Omega |\nabla \tilde{\varphi}|^2,$$

which, when combined with Lemma 2.1, implies that $\|\tilde{\varphi}\|_{\mathscr{H}^0} + \|\nabla_{\mathscr{A}} \tilde{\varphi}\|_{\mathscr{H}^0} \lesssim \|\varphi\|_{H^{1/2}(\Sigma)}$. Then

$$\begin{aligned} \int_\Sigma \varphi v \cdot \mathscr{N} = \int_\Sigma J \mathscr{A}_{ij} v_i \varphi (e_j \cdot e_3) = \int_\Omega \operatorname{div}_{\mathscr{A}}(v \tilde{\varphi}) J = \int_\Omega \tilde{\varphi} \operatorname{div}_{\mathscr{A}} v J + v \cdot \nabla_{\mathscr{A}} \tilde{\varphi} J \\ \le \|\tilde{\varphi}\|_{\mathscr{H}^0} \|\operatorname{div}_{\mathscr{A}} v\|_{\mathscr{H}^0} + \|v\|_{\mathscr{H}^0} \|\nabla_{\mathscr{A}} \tilde{\varphi}\|_{\mathscr{H}^0} \lesssim \|\varphi\|_{H^{1/2}(\Sigma)} \big(\|v\|_{\mathscr{H}^0} + \|\operatorname{div}_{\mathscr{A}} v\|_{\mathscr{H}^0} \big). \end{aligned}$$

The desired bound follows from this inequality by taking the supremum over all φ, so that $\|\varphi\|_{H^{1/2}(\Sigma)} \le 1$. $\square$

Remark 3.4. Recall the space $\mathscr{Y}(t) \subset \mathscr{H}^0(t)$, defined by (2-2). It can be shown that if $v \in \mathscr{Y}(t)$, then $\operatorname{div}_{\mathscr{A}} v = 0$ in the weak sense, so that Lemma 3.3 implies that $v \cdot \mathscr{N} \in H^{-1/2}(\Sigma)$ and $v \cdot \nu \in H^{-1/2}(\Sigma_b)$. Moreover, since the elements of $\mathscr{Y}(t)$ are orthogonal to each $\nabla_{\mathscr{A}} \varphi$ for $\varphi \in {}^0H^1(\Omega)$, we find that $v \cdot \nu = 0$ on Σ_b.

The $\mathscr{A}$-Stokes problem. In order to derive the regularity for our solutions to (1-7), we will first need to study the regularity of the corresponding stationary problem

$$\begin{cases} \operatorname{div}_{\mathscr{A}} S_{\mathscr{A}}(p,u) = F^1 & \text{in } \Omega, \\ \operatorname{div}_{\mathscr{A}} u = F^2 & \text{in } \Omega, \\ S_{\mathscr{A}}(p,u)\mathscr{N} = F^3 & \text{on } \Sigma, \\ u = 0 & \text{on } \Sigma_b. \end{cases} \tag{3-6}$$

In these equations, recall that we have written $S_{\mathscr{A}}(p,u) = (pI - \mathbb{D}_{\mathscr{A}} u)$. Since this problem is stationary, we will temporarily ignore the time dependence of η, $\mathscr{A}$, etc.

We are interested in the regularity theory for strong solutions to (3-6), but before discussing that, we shall mention the weak formulation. Our method of solution is similar to that of [Solonnikov and Skadilov 1973; Beale 1981; Coutand and Shkoller 2007]; we utilize Proposition 2.9 to introduce p after first solving a pressureless problem. Suppose $F^1 \in (\mathscr{H}^1)^*$, $F^2 \in \mathscr{H}^0$, $F^3 \in H^{-1/2}(\Sigma)$. We say $(u,p) \in \mathscr{H}^1 \times \mathscr{H}^0$ is a weak solution to (3-6) if $\operatorname{div}_{\mathscr{A}} u = F^2$ almost everywhere in Ω, and

$$\tfrac{1}{2}(u,v)_{\mathscr{H}^1} - (p, \operatorname{div}_{\mathscr{A}} v)_{\mathscr{H}^0} = \langle F^1, v \rangle_{(\mathscr{H}^1)^*} - \langle F^3, v \rangle_{-1/2} \quad \text{for all } v \in \mathscr{H}^1, \tag{3-7}$$

where $\langle \cdot\,,\cdot \rangle_{(\mathcal{H}^1)^*}$ denotes the dual pairing in $\mathcal{H}^1$ and $\langle \cdot\,,\cdot \rangle_{-1/2}$ denotes the dual pairing between $H^{-1/2}(\Sigma)$ and $H^{1/2}(\Sigma)$.

Proposition 3.5. *Suppose $F^1 \in (\mathcal{H}^1)^*$, $F^2 \in \mathcal{H}^0$, $F^3 \in H^{-1/2}(\Sigma)$. Then there exists a unique weak solution $(u, p) \in \mathcal{H}^1 \times \mathcal{H}^0$ to (3-7).*

Proof. By Lemma 2.6, there exists a $\bar{u} \in \mathcal{H}^1$ such that $\operatorname{div}_{\mathcal{A}} \bar{u} = F^2$. We may then switch unknowns to $w = u - \bar{u}$ so that the weak formulation for w is $\operatorname{div}_{\mathcal{A}} w = 0$ and

$$\tfrac{1}{2}(w, v)_{\mathcal{H}^1} - (p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} = -\tfrac{1}{2}(\bar{u}, v)_{\mathcal{H}^1} + \langle F^1, v \rangle_{(\mathcal{H}^1)^*} - \langle F^3, v \rangle_{-1/2} \quad \text{for all } v \in \mathcal{H}^1. \tag{3-8}$$

To solve for w without p, we restrict the test functions to $v \in \mathcal{X}$ so that the second term on the left vanishes. A straightforward application of the Riesz representation theorem then provides a unique $w \in \mathcal{X}$ satisfying

$$\tfrac{1}{2}(w, v)_{\mathcal{H}^1} = -\tfrac{1}{2}(\bar{u}, v)_{\mathcal{H}^1} + \langle F^1, v \rangle_{(\mathcal{H}^1)^*} - \langle F^3, v \rangle_{-1/2} \quad \text{for all } v \in \mathcal{X}. \tag{3-9}$$

To introduce the pressure, p, we define $\Lambda \in (\mathcal{H}^1)^*$ as the difference between the left and right sides of (3-9). Then $\Lambda(v) = 0$ for all $v \in \mathcal{X}$, so by Proposition 2.9, there exists a unique $p \in \mathcal{H}^0$ satisfying $(p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} = \Lambda(v)$ for all $v \in \mathcal{H}^1$, which is equivalent to (3-8). □

The regularity gain available for solutions to (3-6) is limited by the regularity of the coefficients of the operators $\Delta_{\mathcal{A}}$, $\nabla_{\mathcal{A}}$, $\operatorname{div}_{\mathcal{A}}$, and hence by the regularity of η. In the next result, we establish the strong solvability of (3-6) and present some elliptic estimates, but we do not yet seek the optimal regularity.

Lemma 3.6. *Suppose that $\eta \in H^{k+1/2}(\Sigma)$ for $k \geq 3$ is as small as in Remark 3.2, so that the mapping Φ defined by (1-1) is a C^1 diffeomorphism of Ω to $\Omega' = \Phi(\Omega)$. If $F^1 \in H^0(\Omega)$, $F^2 \in H^1(\Omega)$, and $F^3 \in H^{1/2}(\Sigma)$, then the problem (3-6) admits a unique strong solution $(u, p) \in H^2(\Omega) \times H^1(\Omega)$, that is, (u, p) satisfy (3-6) almost everywhere in Ω, Σ, and Σ_b. Moreover, for $r = 2, \ldots, k-1$, we have the estimate*

$$\|u\|_r + \|p\|_{r-1} \lesssim C(\eta)\big(\|F^1\|_{r-2} + \|F^2\|_{r-1} + \|F^3\|_{r-3/2}\big), \tag{3-10}$$

whenever the right-hand side is finite, where $C(\eta)$ is a constant depending on $\|\eta\|_{k+1/2}$.

Proof. We transform the problem (3-6) to one on $\Omega' = \Phi(\Omega)$ by introducing the unknowns (v, q) according to $u = v \circ \Phi$, $p = q \circ \Phi$. Then (v, q) should be solutions to the usual Stokes problem on $\Omega' = \{-b(y_1, y_2) \leq y_3 \leq \eta(y_1, y_2)\}$ with upper boundary $\Sigma' = \{y_3 = \eta\}$:

$$\begin{cases} \operatorname{div} S(q, v) = G^1 = F^1 \circ \Phi^{-1} & \text{in } \Omega', \\ \operatorname{div} v = G^2 = F^2 \circ \Phi^{-1} & \text{in } \Omega', \\ S(q, v)\mathcal{N} = G^3 = F^3 \circ \Phi^{-1} & \text{on } \Sigma', \\ v = 0 & \text{on } \Sigma_b, \end{cases} \tag{3-11}$$

where we recall that $S(q, v) = (qI - \mathbb{D}v)$. Note that, according to Lemma 3.1, $G^1 \in H^0(\Omega')$, $G^2 \in H^1(\Omega')$, and $G^3 \in H^{1/2}(\Sigma')$. We claim that there exist unique $v \in H^2(\Omega')$, $q \in H^1(\Omega')$, solving problem (3-11) with

$$\|v\|_{H^2(\Omega')} + \|q\|_{H^1(\Omega')} \lesssim C(\eta)\big(\|G^1\|_{H^0(\Omega')} + \|G^2\|_{H^1(\Omega')} + \|G^3\|_{H^{1/2}(\Sigma')}\big), \tag{3-12}$$

for $C(\eta)$ a constant depending on $\|\eta\|_{k+1/2}$. Let us assume for the moment that the claim is true; we first show how (3-10) follows from the claim, and then turn to its proof.

To go from $H^2 \times H^1$ to higher regularity, we appeal to the theory of elliptic systems with complementary boundary conditions, developed in [Agmon et al. 1964]. It is well-known that the Stokes system (3-11) is such an elliptic system. Theorem 10.5 of [Agmon et al. 1964] provides estimates in bounded domains, but we may argue as in Lemma 3.3 of [Beale 1981] to transform the localized estimates into estimates in all of Ω', provided that the boundary Σ' is sufficiently smooth. In order for estimates of the form (3-10) to hold for $r = 2, \dots, k-1$, [Agmon et al. 1964] requires that Σ' be C^{k-1}, which is satisfied since $\eta \in H^{k+1/2}(\Sigma) \hookrightarrow C^{k-1,1/2}(\Sigma)$. Hence, for $r = 2, \dots, k-1$,

$$\|v\|_{H^r(\Omega)'} + \|q\|_{H^{r-1}(\Omega')} \lesssim C(\eta)\big(\|G^1\|_{H^{r-2}(\Omega')} + \|G^2\|_{H^{r-1}(\Omega)'} + \|G^3\|_{H^{r-3/2}(\Sigma')}\big), \tag{3-13}$$

for $C(\eta)$ a constant depending on $\|\eta\|_{k+1/2}$, whenever the right side is finite.

We now transform back to Ω with $u = v \circ \Phi$, $p = q \circ \Phi$. It is readily verified that (u, p) are strong solutions of (3-6). Since Φ satisfies $\nabla \Phi - I \in H^k$, Lemma 3.1 and (3-13) imply that

$$\|u\|_r + \|p\|_{r-1} \lesssim C(\eta)\big(\|F^1\|_{r-2} + \|F^2\|_{r-1} + \|F^3\|_{r-3/2}\big)$$

for $r = 2, \dots, k-1$, whenever the right side is finite. This is (3-10).

We now turn to the proof of the above claim, which employs ideas from [Beale 1981]. To demonstrate the existence of $H^2 \times H^1$ solutions of (3-11), we first consider the special case in which $G^2 = 0$, $G^3 = 0$, and $G^1 \in H^0(\Omega')$ is arbitrary. In this case, we may argue as in Lemma 3.3 of [Beale 1981] (which in turn invokes [Solonnikov and Skadilov 1973]) to deduce the existence of a unique solution to (3-11) satisfying (3-12) with $G^2 = 0$, $G^3 = 0$.

To handle the case of nonvanishing G^2 and G^3, we construct some special auxiliary functions that allow us to reduce to the special case. First, there exists a $v^1 \in H^2(\Omega') \cap {}_0H^1(\Omega')$ such that $\operatorname{div} v^1 = G^2 \in H^1(\Omega')$ and

$$\|v^1\|_{H^2(\Omega')} \lesssim \|G^2\|_{H^1(\Omega')}. \tag{3-14}$$

The existence of v^1 may be established as in Lemma 3.3 and Section 4 of [Beale 1981]. To deal with the boundary term G^3, we first need some projections. For a vector field $X : \Sigma' \to \mathbb{R}^3$, let us write ΠX for the vector field, so that $\Pi X(y)$ is the orthogonal projection of $X(y)$ onto the space of vectors orthogonal to $\mathcal{N}(y)$, and let us write $\Pi^\perp X(y)$ for the orthogonal projection onto the line generated by $\mathcal{N}(y)$. Our second special function is $v^2 \in H^2(\Omega') \cap {}_0H^1_\sigma(\Omega')$ that satisfies $\Pi(-\mathbb{D}v^2\mathcal{N}) = \Pi(G^3 + \mathbb{D}v^1\mathcal{N})$ and

$$\|v^2\|_{H^2(\Omega')} \lesssim C(\eta)\big(\|G^3 + \mathbb{D}v^1\mathcal{N}\|_{H^{1/2}(\Sigma')}\big) \lesssim C(\eta)\big(\|G^2\|_{H^1(\Omega')} + \|G^3\|_{H^{1/2}(\Sigma')}\big). \tag{3-15}$$

The construction of v^2 may be carried out through a simple modification of the proof of Lemma 4.2 in [Beale 1981], working in Sobolev spaces defined on Ω' rather than $\Omega' \times (0, T)$. The third special function is $q^1 \in H^1(\Omega')$ that satisfies $q|_{\Sigma'} = \Pi^\perp(G^3 + \mathbb{D}v^1\mathcal{N})$ and

$$\|q^1\|_{H^1(\Omega')} \lesssim C(\eta)\big(\|G^3 + \mathbb{D}v^1\mathcal{N}\|_{H^{1/2}(\Sigma')}\big) \lesssim C(\eta)\big(\|G^2\|_{H^1(\Omega')} + \|G^3\|_{H^{1/2}(\Sigma')}\big). \tag{3-16}$$

The existence of q^1 follows from the usual trace and extension theory since $G^3 + \mathbb{D}v^1\mathcal{N} \in H^{1/2}(\Sigma')$.

Now, with v^1, v^2 and q^1 in hand, we reduce the solvability of (3-11) with the estimate (3-12) to the special case discussed above. The construction of these special functions guarantees that $w = v - v^1 - v^2$, $Q = q - q^1$ should satisfy

$$\begin{cases} \operatorname{div} S(Q, w) = G^1 + \operatorname{div}(\mathbb{D}v^1 + \mathbb{D}v^2) - \nabla q^2 \in H^0(\Omega') & \text{in } \Omega', \\ \operatorname{div} w = 0 & \text{in } \Omega', \\ S(Q, w)\mathcal{N} = 0 & \text{on } \Sigma', \\ w = 0 & \text{on } \Sigma_b. \end{cases}$$

As above, there exist unique (w, Q) solving this so that

$$\|w\|_{H^2(\Omega')} + \|Q\|_{H^1(\Omega')} \lesssim C(\eta) \big\| G^1 + \operatorname{div}(\mathbb{D}v^1 + \mathbb{D}v^2) - \nabla q^2 \big\|_{H^0(\Omega')}. \tag{3-17}$$

The existence of unique (v, q) solving (3-11) is immediate, and the estimate (3-12) follows by combining (3-17) with (3-14)–(3-16), finishing the proof of the claim. □

It turns out that we can achieve a gain of somewhat more regularity than is mentioned in Lemma 3.6 by making a smallness assumption on η. The smallness allows us to view the problem (3-6) as a perturbation of the Stokes problem on Ω. For this problem there is no constraint to regularity gain since the coefficients are constant and the boundary is smooth. This allows us to shift the constraint of regularity gain to the regularity of η in $H^{k+1/2}$ rather than in C^{k-1}. We note that although we require $\eta \in H^{k+1/2}$, the smallness assumption is written in terms of $\|\eta\|_{k-1/2}$.

Proposition 3.7. *Let $k \geq 4$ be an integer and suppose that $\eta \in H^{k+1/2}$. There exists $\varepsilon_0 > 0$ such that if $\|\eta\|_{k-1/2} \leq \varepsilon_0$, then solutions to (3-6) satisfy*

$$\|u\|_r + \|p\|_{r-1} \leq C\big(\|F^1\|_{r-2} + \|F^2\|_{r-1} + \|F^3\|_{r-3/2}\big) \tag{3-18}$$

for $r = 2, \dots, k$, whenever the right side is finite. Here C is a constant that does not depend on η.

In the case $r = k+1$, solutions to (3-6) satisfy

$$\|u\|_{k+1} + \|p\|_k \leq C\big(\|F^1\|_{k-1} + \|F^2\|_k + \|F^3\|_{k-1/2}\big) + C\|\eta\|_{k+1/2}\big(\|F^1\|_2 + \|F^2\|_3 + \|F^3\|_{5/2}\big). \tag{3-19}$$

Proof. In the case that $\Sigma = \mathbb{R}^2$, we let $\rho \in C_c^\infty(\mathbb{R}^2)$ be such that $\operatorname{supp}(\rho) \subset B(0, 2)$ and $\rho(x) = 1$ for $x \in B(0, 1)$. For $m \in \mathbb{N}$, define η^m by $\mathcal{F}\eta^m(\xi) = \rho(\xi/m)\mathcal{F}\eta(\xi)$, where $\mathcal{F}$ denotes the Fourier transform. Clearly, for each m, $\eta^m \in H^j(\Sigma)$ for all $j \geq 0$, and also $\eta^m \to \eta$ in $H^{k-1/2}(\Sigma)$ (and in $H^{k+1/2}(\Sigma)$ if $\eta \in H^{k+1/2}(\Sigma)$) as $m \to \infty$. In the periodic case, we similarly define η^m by throwing away high frequencies: $\mathcal{F}\eta^m(n) = 0$ for $|n| \geq m$. In this case, η^m has the same convergence properties as before. Let $\mathcal{A}^m$ and $\mathcal{N}^m$ be defined in terms of η^m according to (1-3). Initially, let ε_0 be small enough that η^m is as small as in Remark 3.2. This allows the mapping Φ^m defined by η^m to be a C^1 diffeomorphism.

Consider the problem (3-6) with $\mathcal{A}$ and $\mathcal{N}$ replaced with $\mathcal{A}^m$ and $\mathcal{N}^m$. Since $\eta^m \in H^{k+5/2}(\Sigma)$, we may apply Lemma 3.6 to deduce the existence of a unique pair (u^m, p^m) that solve (3-6) (with $\mathcal{A}^m$, $\mathcal{N}^m$) and that satisfy

$$\|u^m\|_r + \|p^m\|_{r-1} \lesssim C(\|\eta^m\|_{k+5/2})\big(\|F^1\|_{r-2} + \|F^2\|_{r-1} + \|F^3\|_{r-3/2}\big) \tag{3-20}$$

for $r=2,\dots,k+1$, whenever the right-hand side is finite. We rewrite the equations (3-6) as a perturbation of the usual Stokes equations on Ω:

$$\begin{cases} \operatorname{div} S(p^m,u^m) = F^1 + G^{1,m} & \text{in } \Omega, \\ \operatorname{div} u^m = F^2 + G^{2,m} & \text{in } \Omega, \\ S(p^m,u^m)e_3 = F^3 + G^{3,m} & \text{on } \Sigma, \\ u^m = 0 & \text{on } \Sigma_b, \end{cases} \tag{3-21}$$

where

$$\begin{aligned} G^{1,m} &= \operatorname{div}_{I-\mathcal{A}} S_{\mathcal{A}}(p^m,u^m) + \operatorname{div} S_{I-\mathcal{A}}(p^m,u^m), \\ G^{2,m} &= \operatorname{div}_{I-\mathcal{A}} u^m, \\ G^{3,m} &= S(p^m,u^m)(e_3-\mathcal{N}^m) + S_{I-\mathcal{A}}(p^m,u^m)\mathcal{N}^m. \end{aligned}$$

Suppose that $\|\eta^m\|_{k+1/2}\le 1$, which implies that $\|\eta^m\|_{k+1/2}^{\ell}\le \|\eta^m\|_{k+1/2}$ for any $\ell\ge 1$. This fact and a straightforward calculation, using Lemma A.8 in the nonperiodic case and Lemma A.10 in the periodic case, reveal that

$$\begin{aligned} \|G^{1,m}\|_{r-2} &\le C\|\eta^m\|_{k-1/2}\big(\|u^m\|_r + \|p^m\|_{r-1}\big), \\ \|G^{2,m}\|_{r-1} &\le C\|\eta^m\|_{k-1/2}\|u^m\|_r, \end{aligned} \tag{3-22}$$

and

$$\begin{aligned} \|G^{3,m}\|_{H^{r-3/2}(\Sigma)} &\le C\|\eta^m\|_{k-1/2}\big(\|u^m\|_{H^{r-1/2}(\Sigma)} + \|p^m\|_{H^{r-3/2}(\Sigma)}\big) \\ &\le C\|\eta^m\|_{k-1/2}\big(\|u^m\|_r + \|p^m\|_{r-1}\big) \end{aligned} \tag{3-23}$$

for $r=2,\dots,k$ and a constant $C>0$ independent of η and m. In the case $r=k+1$, a minor variant of this argument shows that

$$\begin{aligned} &\|G^{1,m}\|_{k-1} + \|G^{2,m}\|_k + \|G^{3,m}\|_{H^{k-1/2}(\Sigma)} \\ &\qquad \le C\|\eta^m\|_{k-1/2}\big(\|u^m\|_{k+1} + \|p^m\|_k\big) + C\|\eta^m\|_{k+1/2}\|u^m\|_{7/2} \end{aligned} \tag{3-24}$$

for C independent of η and m. The key to this variant is that nowhere in the terms $G^{i,m}$ do there occur products of the highest derivative count of both η^m and u^m (or p^m). Note that the right sides of (3-22), (3-23), and (3-24) are finite by virtue of the estimate (3-20).

Since the boundaries Σ and Σ_b are smooth and the problem (3-21) has constant coefficients, we may argue as in Lemma 3.6, employing the elliptic estimates of [Agmon et al. 1964] as done in Lemma 3.3 of [Beale 1981], to arrive at the estimate

$$\|u^m\|_r + \|p^m\|_{r-1} \le C\big(\|F^1+G^{1,m}\|_{r-2} + \|F^2+G^{2,m}\|_{r-1} + \|F^3+G^{3,m}\|_{r-3/2}\big) \tag{3-25}$$

for $r=2,\dots,k+1$ and for $C>0$ independent of η and m. We may then combine (3-22)–(3-23) with (3-25) to find that, if $\|\eta^m\|_{k-1/2}\le 1$, then

$$\begin{aligned} \|u^m\|_r + \|p^m\|_{r-1} &\le C\big(\|F^1\|_{r-2} + \|F^2\|_{r-1} + \|F^3\|_{r-3/2}\big) \\ &\qquad + C\|\eta^m\|_{k-1/2}\big(\|u^m\|_r + \|p^m\|_{r-1}\big) + \delta_{r,k+1}C\|\eta^m\|_{k+1/2}\|u^m\|_{7/2}. \end{aligned} \tag{3-26}$$

On the right side of (3-26), we have written $\delta_{r,k+1}$ for the quantity that vanishes when $r \neq k+1$ and is unity when $r = k+1$.

We now derive the estimate (3-18). Since $\eta^m \to \eta$ in $H^{k-1/2}$, we may assume that m is sufficiently large that $\|\eta^m\|_{k-1/2} \le 2\|\eta\|_{k-1/2}$. Then if

$$\|\eta\|_{k-1/2} \le \min\left\{\frac{1}{4C}, \frac{1}{2}\right\} := \varepsilon_0$$

for $C > 0$ the constant appearing on the right side of (3-26), the bound (3-26) may be rearranged to get

$$\|u^m\|_r + \|p^m\|_{r-1} \le 2C\big(\|F^1\|_{r-2} + \|F^2\|_{r-1} + \|F^3\|_{r-3/2}\big), \tag{3-27}$$

for $r = 2, \dots, k$ when the right side is finite.

The bound (3-27) implies that the sequence $\{(u^m, p^m)\}$ is uniformly bounded in $H^r \times H^{r-1}$, so up to the extraction of a subsequence, $u^m \rightharpoonup u^0$ weakly in $H^r(\Omega)$ and $p^m \rightharpoonup p^0$ weakly in $H^{r-1}(\Omega)$. Since $\eta^m \to \eta$ in $H^{k-1/2}(\Sigma)$, we also have that $\mathscr{A}^m - \mathscr{A} \to 0$, $J^m - J \to 0$ in $H^{k-1}(\Omega)$, and $\mathscr{N}^m - \mathscr{N} \to 0$ in $H^{k-3/2}(\Sigma)$. We multiply the equation $\operatorname{div}_{\mathscr{A}} u^m = F^2$ by $J^m w$ for $w \in C_c^\infty(\Omega)$ to see that

$$\int_\Omega F^2 w J^m = \int_\Omega \operatorname{div}_{\mathscr{A}^m}(u^m) w J^m = -\int_\Omega u^m \cdot \nabla_{\mathscr{A}^m} w J^m \to -\int_\Omega u^0 \cdot \nabla_{\mathscr{A}} w J = \int_\Omega \operatorname{div}_{\mathscr{A}}(u^0) w J,$$

from which we deduce that $\operatorname{div}_{\mathscr{A}}(u^0) = F^2$. Then we multiply the first equation in (3-6) (with u^m, etc.) by wJ^m for $w \in {}_0H^1(\Omega)$ and integrate by parts to see that

$$\int_\Omega \tfrac{1}{2}\mathbb{D}_{\mathscr{A}^m} u^m : \mathbb{D}_{\mathscr{A}^m} w J^m - p^m \operatorname{div}_{\mathscr{A}^m}(w) J^m = \int_\Omega F^1 \cdot w J^m - \int_\Sigma F^3 \cdot w.$$

Passing to the limit $m \to \infty$, we deduce that

$$\int_\Omega \tfrac{1}{2}\mathbb{D}_{\mathscr{A}} u^0 : \mathbb{D}_{\mathscr{A}} w J - p^0 \operatorname{div}_{\mathscr{A}} w J = \int_\Omega F^1 \cdot w J - \int_\Sigma F^3 \cdot w,$$

which reveals, upon integrating by parts again, that (u^0, p^0) satisfy (3-6). Since (u, p) are the unique solutions to (3-6), we have that $u = u^0$, $p = p^0$. This, weak lower semicontinuity, and the bound (3-27) imply (3-18).

Now we derive the estimate (3-19), supposing that $F^1 \in H^{k-1}$, $F^2 \in H^k$, and $F^3 \in H^{k-1/2}$. The bound (3-27) with $r = 4$ implies that

$$\|u^m\|_4 \le 2C\big(\|F^1\|_2 + \|F^2\|_3 + \|F^3\|_{5/2}\big) < \infty. \tag{3-28}$$

Since $\eta^m \to \eta$ in $H^{k+1/2}$, we are free to assume that m is sufficiently large that $\|\eta^m\|_{k+1/2} \le 2\|\eta\|_{k+1/2}$. Then if $\|\eta\|_{k-1/2} \le \varepsilon_0$, we may use (3-26) and (3-28) to deduce that

$$\begin{aligned}\|u^m\|_{k+1} &+ \|p^m\|_k \\ &\le 2C\big(\|F^1\|_{k-1} + \|F^2\|_k + \|F^3\|_{k-1/2}\big) + 4C\|\eta\|_{k+1/2}\big(\|F^1\|_2 + \|F^2\|_3 + \|F^3\|_{5/2}\big).\end{aligned} \tag{3-29}$$

We may then argue as above to extract weak limits, show that the limits equal u and p, and then deduce that the bound (3-29) holds with u^m and p^m replaced by u and p. This is (3-19). □

The $\mathcal{A}$-Poisson problem. Next we consider the scalar elliptic problem

$$\begin{cases} \Delta_{\mathcal{A}} p = f^1 & \text{in } \Omega, \\ p = f^2 & \text{on } \Sigma, \\ \nabla_{\mathcal{A}} p \cdot \nu = f^3 & \text{on } \Sigma_b, \end{cases} \tag{3-30}$$

where ν is the outward-pointing normal on Σ_b. We will eventually discuss the strong solvability of this problem, but first we consider the weak formulation of the problem. We define a scalar $\mathcal{H}^1$ in a natural way through the norm

$$\|f\|_{\mathcal{H}^1}^2 = \int_\Omega J |\nabla_{\mathcal{A}} f|^2.$$

Note that $\sqrt{2}\|f\|_{\mathcal{H}^1}^2 \le \|f e_1\|_{\mathcal{H}^1} \le 2\|f\|_{\mathcal{H}^1}^2$, where the middle term is the $\mathcal{H}^1$ norm for vectors. Then Lemma 2.1 shows that this scalar norm generates the same topology as the usual scalar H^1 norm.

For the weak formulation, we suppose $f^1 \in ({}^0H^1(\Omega))^*$, $f^2 \in H^{1/2}(\Sigma)$, and $f^3 \in H^{-1/2}(\Sigma_b)$. Let $\bar{p} \in H^1(\Omega)$ be an extension of f^2 such that $\operatorname{supp}(\bar{p}) \subset \{-(\inf b)/2 < x_3 \le 0\}$. We switch unknowns to $q = p - \bar{p}$. Then we can define a weak formulation of (3-30) by finding a $q \in {}^0H^1(\Omega)$ such that

$$(q, \varphi)_{\mathcal{H}^1} = -(\bar{p}, \varphi)_{\mathcal{H}^1} - \langle f^1, \varphi \rangle_* + \langle f^3, \varphi \rangle_{-1/2} \quad \text{for all } \varphi \in {}^0H^1(\Omega), \tag{3-31}$$

where $\langle \cdot, \cdot \rangle_*$ is the dual pairing with ${}^0H^1(\Omega)$ and $\langle \cdot, \cdot \rangle_{-1/2}$ is the dual pairing with $H^{1/2}(\Sigma_b)$. The existence and uniqueness of a solution to (3-31) follow from standard arguments, and the resulting $p = q + \bar{p} \in H^1(\Omega)$ satisfies

$$\|p\|_{\mathcal{H}^1}^2 \lesssim \left(\|f^1\|_{({}^0H^1(\Omega))^*}^2 + \|f^2\|_{H^{1/2}(\Sigma)}^2 + \|f^3\|_{H^{-1/2}(\Sigma_b)}^2 \right). \tag{3-32}$$

In the event that the action of f^1 is given in a more specific fashion, we will rewrite the PDE (3-30) to accommodate the structure of f^1. To make this precise, suppose that the action of f^1 on an element $\varphi \in {}^0H^1(\Omega)$ is given by

$$\langle f^1, \varphi \rangle_* = (g_0, \varphi)_{\mathcal{H}^0} + (G, \nabla_{\mathcal{A}} \varphi)_{\mathcal{H}^0}$$

for $(g_0, G) \in H^0(\Omega; \mathbb{R}) \times H^0(\Omega; \mathbb{R}^3)$ with $\|g_0\|_0^2 + \|G\|_0^2 = \|f^1\|_{({}^0H^1(\Omega))^*}^2$ (standard arguments show that it is always possible to uniquely write f^1 in this way). Then (3-31) may be rewritten as

$$(\nabla_{\mathcal{A}} p + G, \nabla_{\mathcal{A}} \varphi)_{\mathcal{H}^0} = -(g_0, \varphi)_{\mathcal{H}^0} + \langle f^3, \varphi \rangle_{-1/2} \quad \text{for all } \varphi \in {}^0H^1(\Omega).$$

We may take $\varphi \in C_c^\infty(\Omega)$ in this equality and integrate by parts to see that $\operatorname{div}_{\mathcal{A}}(\nabla_{\mathcal{A}} p + G) = g_0 \in \mathcal{H}^0$, which allows us to deduce from Lemma 3.3 that $(\nabla_{\mathcal{A}} p + G) \cdot \nu \in H^{-1/2}(\Sigma_b)$. This serves as motivation for us to say that p is a weak solution to the PDE

$$\begin{cases} \operatorname{div}_{\mathcal{A}}(\nabla_{\mathcal{A}} p + G) = g_0 \in H^0(\Omega), \\ p = f^2 \in H^{1/2}(\Sigma), \\ (\nabla_{\mathcal{A}} p + G) \cdot \nu = f^3 \in H^{-1/2}(\Sigma_b). \end{cases} \tag{3-33}$$

This way of writing the weak solution will be utilized later in Theorem 4.3. Note that when $f^1 \in H^0(\Omega)$, there is no need to make this distinction since then $G = 0$ and $f^1 = g_0$.

Our next result on this problem is the analogue of Lemma 3.6; it establishes the strong solvability of (3-30) and some regularity.

Lemma 3.8. *Suppose that $\eta \in H^{k+1/2}(\Sigma)$ for $k \geq 3$ is as small as in Remark 3.2, so that the mapping Φ defined by (1-1) is a C^1 diffeomorphism of Ω to $\Omega' = \Phi(\Omega)$. If $f^1 \in H^0(\Omega)$, $f^2 \in H^{3/2}(\Sigma)$, and $f^3 \in H^{1/2}(\Sigma_b)$, then the problem (3-30) admits a unique strong solution $p \in H^2(\Omega)$. Moreover, for $r = 2, \dots, k-1$, we have the estimate*

$$\|p\|_r \lesssim C(\eta)\big(\|f^1\|_{r-2} + \|f^2\|_{r-1/2} + \|f^3\|_{r-3/2}\big), \tag{3-34}$$

whenever the right-hand side is finite, where $C(\eta)$ is a constant depending on $\|\eta\|_{k+1/2}$.

Proof. If $f^2 \in H^{r-1/2}(\Sigma)$ for $r = 2, \dots, k-1$, there exists a $\psi \in H^r(\Omega)$ such that $\psi|_\Sigma = f^2$, $\operatorname{supp}(\psi) \subset \{-(\inf b)/2 < x_3 \leq 0\}$, and $\|\psi\|_r \lesssim \|f^2\|_{r-1/2}$. Writing $p = q + \psi$, the problem (3-30) may be rewritten for the unknown q as

$$\begin{cases} \Delta_{\mathscr{A}} q = f^1 + g^1 & \text{in } \Omega, \\ q = 0 & \text{on } \Sigma, \\ \nabla_{\mathscr{A}} q \cdot \nu = f^3 & \text{on } \Sigma_b, \end{cases} \tag{3-35}$$

where $g^1 = -\Delta_{\mathscr{A}} \psi \in H^{r-2}$.

The problem (3-35) may be solved as in Lemma 3.6 by transforming to the domain Ω', where the problem for $Q = q \circ \Phi^{-1}$ becomes $\Delta Q = (f^1 + g^1) \circ \Phi^{-1}$ in Ω' with boundary conditions $Q = 0$ on Σ' and $\nabla Q \cdot \nu = f^3 \circ \Phi^{-1}$ on Σ_b. The existence of a unique solution to this problem is established in the nonperiodic case in Lemma 2.8 of [Beale 1981], and estimates of the form (3-34) for Q hold by virtue of the elliptic estimates in [Agmon et al. 1959], adapted to Ω' as in [Beale 1981]. This method may be adapted easily to the periodic case as well. Then the existence and uniqueness of a solution to (3-30) satisfying (3-34) follows by transforming to $q = Q \circ \Phi$ on Ω for a solution to (3-35) and then applying Lemma 3.1. □

Our next result is the analogue of Proposition 3.7 for the problem (3-30). For our purposes, we only need a regularity gain up to k, and this is less important than the estimate in terms of a constant independent of η. Notice again that the smallness assumption is stated in $H^{k-1/2}$ even though we require $\eta \in H^{k+1/2}$.

Proposition 3.9. *Let $k \geq 4$ be an integer and suppose that $\eta \in H^{k+1/2}$. There exists $\varepsilon_0 > 0$ such that, if $\|\eta\|_{k-1/2} \leq \varepsilon_0$, then solutions to (3-30) satisfy*

$$\|p\|_r \leq C\big(\|f^1\|_{r-2} + \|f^2\|_{r-1/2} + \|f^3\|_{r-3/2}\big) \tag{3-36}$$

for $r = 2, \dots, k$, whenever the right side is finite. Here C is a constant that does not depend on η.

Proof. The proof is similar to that of Proposition 3.7. We smooth η to get η^m and solve (3-30) with $\mathscr{A}$ replaced with $\mathscr{A}^m$. Then we rewrite the problem as a perturbation of the Poisson problem

$$\begin{cases} \Delta p^m = f^1 + g^{1,m} & \text{in } \Omega, \\ p^m = f^2 & \text{on } \Sigma, \\ \nabla p^m \cdot \nu = f^3 + g^{3,m} & \text{on } \Sigma_b. \end{cases}$$

The constants in the elliptic estimates for this problem do not depend on η^m, and we may estimate $g^{i,m}$ in terms of p^m. Then if $\|\eta\|_{k-1/2} \le \varepsilon_0$ for some ε_0 sufficiently small, we can absorb the highest Sobolev norms on the right side of the elliptic estimate into the left side, and we deduce (3-36) for p^m. Then we pass to the limit $m \to \infty$. □

4. Solving the time-dependent problem (1-7)

The weak solution. In our analysis of problem (1-7), we will employ two notions of solution: strong and weak. The meaning of the former is standard, but the latter merits some explanation. The definition of a weak solution to (1-7) is motivated by assuming the existence of a smooth solution to (1-7), multiplying by Jv for $v \in {}_0\mathcal{H}^1_T$, integrating over Ω by parts, and then in time from 0 to T to see that

$$(\partial_t u, v)_{\mathcal{H}^0_T} + \tfrac{1}{2}(u, v)_{{}_0\mathcal{H}^1_T} - (p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0_T} = (F^1, v)_{\mathcal{H}^0_T} - (F^3, v)_{0,\Sigma,T} \tag{4-1}$$

for $(F^3, v)_{0,\Sigma,T} = \int_0^T \int_\Sigma F^3 \cdot v$. If we were to restrict our class of test functions to $v \in \mathcal{X}_T$ (defined by (2-4)), then the term $(p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0_T}$ would vanish above, and we would be left with a "pressureless" formulation of the problem involving only the velocity field. This leads us to define a weak formulation without the pressure.

Suppose that

$$\bar{F} \in (\mathcal{X}_T)^* \quad \text{and} \quad u_0 \in \mathcal{Y}(0),$$

where $\mathcal{Y}(0)$ is defined by (2-2). Then our definition of a weak solution requires that u satisfies

$$\begin{cases} u \in \mathcal{X}_T,\ \partial_t u \in (\mathcal{X}_T)^*, \\ \langle \partial_t u, \psi \rangle_* + \tfrac{1}{2}(u, \psi)_{{}_0\mathcal{H}^1_T} = \langle \bar{F}, \psi \rangle_*, \quad \text{for every } \psi \in \mathcal{X}_T, \\ u(0) = u_0, \end{cases} \tag{4-2}$$

where $\langle \cdot\,, \cdot \rangle_*$ denotes the dual pairing between $(\mathcal{X}_T)^*$ and $\mathcal{X}_T$. Note that the third condition in (4-2) makes sense in light of Lemma 2.4. Our weak formulation requires only that $u \in \mathcal{X}_T$, which means that $\bar{F} \in (\mathcal{X}_T)^*$ is natural. Within the context of problem (1-7), the functional $\bar{F}$ is most naturally of the form appearing on the right side of (4-1), and if $\bar{F}$ admits a representation of this form, we may say that a solution to (4-2) is a weak solution of (1-7).

Since our aim is to construct solutions to (1-7) with high regularity, we will not need to directly construct weak solutions to (4-2). Rather, weak solutions to problems of this type will arise as a byproduct of our construction of strong solutions of (1-7). Hence, for our purposes, it will suffice to ignore the issue of existence and only record a couple results on the properties of weak solutions.

We now record a result on some integral equalities and bounds satisfied by solutions of (4-2).

Lemma 4.1. *Suppose that u is a weak solution of (4-2). Then, for almost every $t \in [0, T]$,*

$$\frac{1}{2}\|u(t)\|^2_{\mathscr{H}^0(t)} + \frac{1}{2}\int_0^t \|u(s)\|^2_{\mathscr{H}^1(s)}\,ds = \frac{1}{2}\|u(0)\|^2_{\mathscr{H}^0(0)} + \langle \bar{F}, u\rangle_{(\mathscr{X}_t)^*} + \frac{1}{2}\int_0^t\int_\Omega |u(s)|^2 \partial_t J(s)\,ds. \tag{4-3}$$

Also

$$\sup_{0\le t\le T}\|u(t)\|^2_{\mathscr{H}^0(t)} + \|u\|^2_{\mathscr{H}^1_T} \lesssim \exp(C_0(\eta)T)\big(\|u(0)\|^2_{\mathscr{H}^0(0)} + \|\bar{F}\|^2_{(\mathscr{X}_T)^*}\big), \tag{4-4}$$

where $C_0(\eta) := \sup_{0\le t\le T}\|\partial_t JK\|_{L^\infty}$.

Proof. The identity (4-3) follows directly from (4-2) and Lemma 2.4 by using the test function $\psi = u\chi_{[0,t]} \in \mathscr{X}_T$, where $\chi_{[0,t]}$ is a temporal indicator function equal to unity on the interval $[0, t]$.

From (4-3) it is straightforward to derive the inequality

$$\tfrac{1}{2}\|u(t)\|^2_{\mathscr{H}^0(t)} + \tfrac{1}{2}\|u\|^2_{\mathscr{H}^1_t} \le \tfrac{1}{2}\|u(0)\|^2_{\mathscr{H}^0(0)} + \|\bar{F}\|_{(\mathscr{X}_t)^*}\|u\|_{\mathscr{H}^1_t} + \frac{C_0(\eta)}{2}\|u\|^2_{\mathscr{H}^0_t}, \tag{4-5}$$

where we have written

$$\|u\|^2_{\mathscr{H}^k_t} = \int_0^t \|u(s)\|^2_{\mathscr{H}^k(s)}\,ds \quad \text{for } k = 0, 1,$$

and similarly defined $\|\bar{F}\|_{(\mathscr{X}_t)^*}$. Inequality (4-5) and Cauchy's inequality then imply that

$$\tfrac{1}{2}\|u(t)\|^2_{\mathscr{H}^0(t)} + \tfrac{1}{4}\|u\|^2_{\mathscr{H}^1_t} \le \tfrac{1}{2}\|u(0)\|^2_{\mathscr{H}^0(0)} + \|\bar{F}\|^2_{(\mathscr{X}_t)^*} + \frac{C_0(\eta)}{2}\|u\|^2_{\mathscr{H}^0_t}. \tag{4-6}$$

Then (4-4) follows from the differential inequality (4-6) and Gronwall's lemma. □

We can now parlay the results of Lemma 4.1 into uniqueness results for weak solutions to (4-2).

Proposition 4.2. *Weak solutions to (4-2) are unique.*

Proof. If u^1 and u^2 are both weak solutions to (4-2), then $w = u^1 - u^2$ is a weak solution with $\bar{F} = 0$ and $w(0) = u^1(0) - u^2(0) = 0$. Then the bound (4-4) of Lemma 4.1 implies that $w = 0$; hence solutions to (4-2) are unique. □

The strong solution. Now we turn to the construction of strong solutions to (1-7). We will assume that the forcing functions satisfy

$$\begin{aligned} F^1 &\in L^2\big([0,T]; H^1(\Omega)\big) \cap C^0\big([0,T]; H^0(\Omega)\big),\\ F^3 &\in L^2\big([0,T]; H^{3/2}(\Sigma)\big) \cap C^0\big([0,T]; H^{1/2}(\Sigma)\big),\\ \partial_t(F^1 - F^3) &\in L^2\big([0,T]; ({}_0H^1(\Omega))^*\big). \end{aligned} \tag{4-7}$$

Here in the last line we mean that the weak time derivative of the functional $v \mapsto (F^1, v)_{\mathscr{H}^0} - (F^3, v)_{0,\Sigma}$ (which is itself in $L^2([0,T]; ({}_0H^1(\Omega))^*)$) is in $L^2([0,T]; ({}_0H^1(\Omega))^*) \hookrightarrow (\mathscr{X}_T)^*$. We also assume the initial velocity $u_0 \in H^2(\Omega) \cap \mathscr{X}(0)$.

The solution that we construct will satisfy (1-7) in the strong sense, but we will also show that $D_t u$ satisfies an equation of the form (1-7) in the weak sense of (4-2). Here we define

$$D_t u := \partial_t u - Ru \quad \text{for } R := \partial_t M M^{-1}, \tag{4-8}$$

with M the matrix defined by (2-16). We employ the operator D_t because it preserves the $\operatorname{div}_{\mathscr{A}}$-free condition. Before turning to the result, we define the quantity

$$\mathfrak{K}(\eta) := \sup_{0\le t\le T}\bigl(\|\eta\|_{9/2}^2 + \|\partial_t\eta\|_{7/2}^2 + \|\partial_t^2\eta\|_{5/2}^2\bigr). \tag{4-9}$$

We also define an orthogonal projection onto the tangent space of the surface $\{x_3 = \eta_0\}$ according to

$$\Pi_0 v = v - (v\cdot\mathcal{N}_0)\mathcal{N}_0|\mathcal{N}_0|^{-2} \tag{4-10}$$

for $\mathcal{N}_0 = (-\partial_1\eta_0, -\partial_2\eta_0, 1)$. By construction, $\Pi_0 v = 0$ if and only if $v \parallel \mathcal{N}_0$.

Theorem 4.3. *Suppose that F^1, F^3 satisfy (4-7), that $u_0 \in H^2(\Omega)\cap\mathscr{X}(0)$, and that $u_0, F^3(0)$ satisfy the compatibility condition*

$$\Pi_0\bigl(F^3(0) + \mathbb{D}_{\mathscr{A}_0}u_0\mathcal{N}_0\bigr) = 0, \quad \textit{where } \mathcal{N}_0 = (-\partial_1\eta_0, -\partial_2\eta_0, 1), \tag{4-11}$$

and Π_0 is the projection defined by (4-10). Further suppose that $\mathfrak{K}(\eta)$ is less than the smaller of ε_0 from Lemma 2.1 and ε_0 from Proposition 3.7 (in particular, this requires $\mathfrak{K}(\eta)\le 1$). Then there exists a unique strong solution (u, p) to (1-7) such that

$$\begin{gathered} u \in \mathscr{X}_T \cap C^0\bigl([0,T]; H^2(\Omega)\bigr)\cap L^2\bigl([0,T]; H^3(\Omega)\bigr),\\ \partial_t u \in C^0\bigl([0,T]; H^0(\Omega)\bigr)\cap L^2\bigl([0,T]; H^1(\Omega)\bigr), \quad D_t u \in \mathscr{X}_T, \quad \partial_t^2 u \in (\mathscr{X}_T)^*,\\ p \in C^0\bigl([0,T]; H^1(\Omega)\bigr)\cap L^2\bigl([0,T]; H^2(\Omega)\bigr). \end{gathered} \tag{4-12}$$

The solution satisfies the estimate

$$\begin{aligned} &\|u\|_{L^\infty H^2}^2 + \|u\|_{L^2H^3}^2 + \|\partial_t u\|_{L^\infty H^0}^2 + \|\partial_t u\|_{L^2H^1}^2 + \|\partial_t^2 u\|_{(\mathscr{X}_T)^*}^2 + \|p\|_{L^\infty H^1}^2 + \|p\|_{L^2H^2}^2\\ &\qquad\lesssim (1+\mathfrak{K}(\eta))\exp\bigl(C(1+\mathfrak{K}(\eta))T\bigr)\Bigl(\|u_0\|_2^2 + \|F^1(0)\|_0^2 + \|F^3(0)\|_{1/2}^2 + \|F^1\|_{L^2H^1}^2\\ &\qquad\qquad + \|F^3\|_{L^2H^{3/2}}^2 + \|\partial_t(F^1 - F^3)\|_{(\mathscr{X}_T)^*}^2\Bigr), \end{aligned} \tag{4-13}$$

where C is a constant independent of η. The initial pressure, $p(0)\in H^1(\Omega)$, is determined in terms of $u_0, F^1(0), F^3(0)$ as the weak solution to

$$\begin{cases} \operatorname{div}_{\mathscr{A}_0}\bigl(\nabla_{\mathscr{A}_0}p(0) - F^1(0)\bigr) = -\operatorname{div}_{\mathscr{A}_0}(R(0)u_0) \in H^0(\Omega),\\ p(0) = (F^3(0) + \mathbb{D}_{\mathscr{A}_0}u_0\mathcal{N}_0)\cdot\mathcal{N}_0|\mathcal{N}_0|^{-2} \in H^{1/2}(\Sigma),\\ \bigl(\nabla_{\mathscr{A}_0}p(0) - F^1(0)\bigr)\cdot\nu = \Delta_{\mathscr{A}_0}u_0\cdot\nu \in H^{-1/2}(\Sigma_b), \end{cases} \tag{4-14}$$

in the sense of (3-33). Also, $D_t u(0) = \partial_t u(0) - R(0)u_0$ satisfies

$$D_t u(0) = \Delta_{\mathscr{A}_0}u_0 - \nabla_{\mathscr{A}_0}p(0) + F^1(0) - R(0)u_0 \in \mathscr{Y}(0), \tag{4-15}$$

where $\mathscr{Y}(0)$ is defined by (2-2).

Moreover, $D_t u$ satisfies

$$\begin{cases} \partial_t(D_t u) - \Delta_{\mathcal{A}}(D_t u) + \nabla_{\mathcal{A}}(\partial_t p) = D_t F^1 + G^1 & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}}(D_t u) = 0 & \text{in } \Omega, \\ S_{\mathcal{A}}(\partial_t p, D_t u)\mathcal{N} = \partial_t F^3 + G^3 & \text{on } \Sigma, \\ D_t u = 0 & \text{on } \Sigma_b, \end{cases} \tag{4-16}$$

in the weak sense of (4-2), where G^1 is defined by

$$G^1 = -(R + \partial_t J K)\Delta_{\mathcal{A}} u - \partial_t R u + (\partial_t J K + R + R^T)\nabla_{\mathcal{A}} p + \operatorname{div}_{\mathcal{A}}\big(\mathbb{D}_{\mathcal{A}}(Ru) - R\mathbb{D}_{\mathcal{A}} u + \mathbb{D}_{\partial_t \mathcal{A}} u\big)$$

(R^T denoting the matrix transpose of R), and G^3 by

$$G^3 = \mathbb{D}_{\mathcal{A}}(Ru)\mathcal{N} - (pI - \mathbb{D}_{\mathcal{A}} u)\partial_t \mathcal{N} + \mathbb{D}_{\partial_t \mathcal{A}} u \mathcal{N}.$$

More precisely, (4-16) holds in the weak sense of (4-2) in that

$$\begin{aligned} \langle \partial_t D_t u, \psi\rangle_* + \tfrac{1}{2}(\partial_t u, \psi)_{\mathcal{H}^1_T} &= \langle \partial_t(F^1 - F^3), \psi\rangle_* - (\partial_t R u + R\partial_t u, \psi)_{\mathcal{H}^0_T} \\ &\quad + (\partial_t J K F^1, \psi)_{\mathcal{H}^0_T} - (\partial_t J K \partial_t u, \psi)_{\mathcal{H}^0_T} - (p, \operatorname{div}_{\mathcal{A}}(R\psi))_{\mathcal{H}^0_T} \\ &\quad - \frac{1}{2}\int_0^T \int_\Omega \big(\partial_t J K \mathbb{D}_{\mathcal{A}} u : \mathbb{D}_{\mathcal{A}}\psi + \mathbb{D}_{\partial_t \mathcal{A}} u : \mathbb{D}_{\mathcal{A}}\psi + \mathbb{D}_{\mathcal{A}} u : \mathbb{D}_{\partial_t \mathcal{A}}\psi\big) J \end{aligned} \tag{4-17}$$

for all $\psi \in \mathcal{X}_T$. Here the inclusions (4-12) guarantee that G^1 and G^3 satisfy the same inclusions as F^1, F^3 listed in (4-7), whereas (4-14) guarantees that the initial data $D_t u(0) \in \mathcal{Y}(0)$.

Finally, let

$$\operatorname{div}_{\mathcal{A}} \partial_t u = -\partial_t \mathcal{A}_{ij}\partial_j u_i := F^2 \in C^0\big([0, T]; H^1(\Omega)\big) \cap L^2\big([0, T]; H^2(\Omega)\big) \cap H^1\big([0, T]; H^0(\Omega)\big).$$

Then for any $0 \le s \le t \le T$, we have the equality

$$\begin{aligned} &\tfrac{1}{2}\|\partial_t u(t)\|^2_{\mathcal{H}^0} - \tfrac{1}{2}\|\partial_t u(s)\|^2_{\mathcal{H}^0} - (p(t), F^2(t))_{\mathcal{H}^0} + (p(s), F^2(s))_{\mathcal{H}^0} + \frac{1}{2}\int_s^t \|\partial_t u\|^2_{\mathcal{H}^1} \\ &= -\frac{1}{2}\int_s^t \int_\Omega \big(\partial_t J K \mathbb{D}_{\mathcal{A}} u : \mathbb{D}_{\mathcal{A}}\partial_t u + \mathbb{D}_{\partial_t \mathcal{A}} u : \mathbb{D}_{\mathcal{A}}\partial_t u + \mathbb{D}_{\mathcal{A}} u : \mathbb{D}_{\partial_t \mathcal{A}}\partial_t u\big) J + \int_s^t \langle \partial_t(F^1 - F^3), \partial_t u\rangle_* \\ &\quad + \int_\Omega \partial_t J F^1 \cdot \partial_t u - \tfrac{1}{2}\partial_t J |\partial_t u|^2 + p\partial_t(J\mathcal{A}_{ij})\partial_j \partial_t u_i - p\partial_t(J F^2). \end{aligned} \tag{4-18}$$

Proof. The result will be established by first solving a pressureless problem and then introducing the pressure via Proposition 2.9. For the pressureless problem, we will make use of the Galerkin method. We divide the proof into several steps.

Step 1: The Galerkin setup. In order to utilize the Galerkin method, we must first construct a countable basis of $H^2(\Omega) \cap \mathcal{X}(t)$ for each $t \in [0, T]$. Since the requirement $\operatorname{div}_{\mathcal{A}} v = 0$ is time-dependent, any basis of this space must also be time-dependent. For each $t \in [0, T]$, the space $H^2(\Omega) \cap \mathcal{X}(t)$ is separable, so the existence of a countable basis is not an issue. The technical difficulty is that, in order for the basis to be useful in the Galerkin method, we must be able to differentiate the basis elements in time, and we must be able to express these time derivatives in terms of finitely many basis elements. Fortunately, it is possible to overcome this difficulty by employing the matrix $M(t)$, defined by (2-16).

Since $H^2(\Omega) \cap {}_0H^1_\sigma(\Omega)$ is separable, it possesses a countable basis $\{w^j\}_{j=1}^\infty$. Note that this basis is not time-dependent. Define $\psi^j = \psi^j(t) := M(t)w^j$ for $M(t)$ defined by (2-16). According to Proposition 2.5, $\psi^j(t) \in H^2(\Omega) \cap \mathscr{X}(t)$, and $\{\psi^j(t)\}_{j=1}^\infty$ is a basis of $H^2(\Omega) \cap \mathscr{X}(t)$ for each $t \in [0, T]$. Moreover,

$$\partial_t \psi^j(t) = \partial_t M(t) w^j = \partial_t M(t) M^{-1}(t) M(t) w^j = \partial_t M(t) M^{-1}(t) \psi^j(t) := R(t)\psi^j(t), \tag{4-19}$$

which allows us to express $\partial_t \psi^j$ in terms of ψ^j. For any integer $m \geq 1$, we define the finite-dimensional space $\mathscr{X}_m(t) := \operatorname{span}\{\psi^1(t), \dots, \psi^m(t)\} \subset H^2(\Omega) \cap \mathscr{X}(t)$, and we write $\mathscr{P}^m_t : H^2(\Omega) \to \mathscr{X}_m(t)$ for the $H^2(\Omega)$ orthogonal projection onto $\mathscr{X}_m(t)$. Clearly, for each $v \in H^2(\Omega) \cap \mathscr{X}(t)$, we have that $\mathscr{P}^m_t v \to v$ as $m \to \infty$.

The next ingredient needed for the Galerkin method is the orthogonal projection onto the tangent space of the surface $\{x_3 = \eta(0)\}$, Π_0, defined by (4-10). This projection will be used to compensate for the fact that our finite-dimensional Galerkin approximation of the initial data u_0 may fail to satisfy the compatibility conditions (4-11).

Step 2: Solving the Galerkin problem. For our Galerkin problem, we will first construct a solution to the pressureless problem as follows. For each $m \geq 1$, we define an approximate solution

$$u^m(t) = a^m_j(t)\psi^j(t), \quad \text{with } a^m_j : [0, T] \to \mathbb{R} \ \text{ for } j = 1, \dots, m,$$

where as usual we use the Einstein convention of summation of the repeated index j. We want to choose the coefficients a^m_j so that

$$(\partial_t u^m, \psi)_{\mathscr{H}^0} + \tfrac{1}{2}(u^m, \psi)_{\mathscr{H}^1} = (F^1, \psi)_{\mathscr{H}^0} - \big(F^3 - \Pi_0(F^3(0) + \mathbb{D}_{\mathscr{A}_0}(\mathscr{P}^m_0 u_0)\mathscr{N}_0), \psi\big)_{0,\Sigma} \tag{4-20}$$

for each $\psi \in \mathscr{X}_m(t)$, where we have written $(\cdot\,,\cdot)_{0,\Sigma}$ for the usual $H^0(\Sigma)$ inner product, and where Π_0 and $\mathscr{P}^m_0$ are defined in the previous step. We supplement Equation (4-20) with the initial condition

$$u^m(0) = \mathscr{P}^m_0 u_0 \in \mathscr{X}_m(0). \tag{4-21}$$

Note that in (4-20), we have added the last projection term to compensate for the fact that $u^m(0)$ may not satisfy the compatibility condition (4-13). Appealing to (4-19), we find that $\partial_t u^m(t) = \dot{a}^m_j(t)\psi^j(t) + R(t)u^m(t)$, and hence (4-20) is equivalent to the system of ODEs for a^m_j given by

$$\begin{aligned} \dot{a}^m_j(\psi^j, \psi^k)_{\mathscr{H}^0} &+ a^m_j\big((R(t)\psi^j, \psi^k)_{\mathscr{H}^0} + \tfrac{1}{2}(\psi^j, \psi^k)_{\mathscr{H}^1}\big) \\ &= (F^1, \psi^k)_{\mathscr{H}^0} - \big(F^3 - \Pi_0(F^3(0) + \mathbb{D}_{\mathscr{A}_0} u^m(0)\mathscr{N}_0), \psi^k\big)_{0,\Sigma} \end{aligned} \tag{4-22}$$

for $j, k = 1, \dots, m$. The $m \times m$ matrix with j, k entry $(\psi^j, \psi^k)_{\mathscr{H}^0}$ is invertible, the coefficients of the linear system (4-22) are $C^1([0, T])$, and the forcing term is $C^0([0, T])$, so the usual well-posedness theory of ODEs guarantees the existence of $a^m_j \in C^1([0, T])$, a unique solution to (4-22) that satisfies the initial conditions induced by (4-21). This, in turn, provides the desired solution, u^m, to (4-20)–(4-21). Since

F^1, F^3 satisfy (4-7), Equation (4-22) may be differentiated in time to see that actually $a_j^m \in C^{1,1}([0,T])$, with a_j^m twice differentiable almost everywhere in $[0,T]$.

Note that throughout the rest of the proof, we use constants C and the symbol $\lesssim$ with the assumption that the constants do not depend on m.

Step 3: Energy estimates for u^m. Since $u^m(t) \in \mathscr{X}_m(t)$, we may use $\psi = u^m$ as a test function in (4-20). Doing so, employing Remark 2.3, and using the fact that Π_0 is an orthogonal projection, we may derive the bound

$$\partial_t \frac{1}{2}\|u^m\|^2_{\mathscr{H}^0} + \frac{1}{2}\|u^m\|^2_{\mathscr{H}^1} \le C\|F^1\|_{\mathscr{H}^0}\|u^m\|_{\mathscr{H}^1} - \frac{1}{2}\int_\Omega |u^m|^2 \partial_t J \\ + C\|u^m\|_{\mathscr{H}^1}\big(\|F^3\|_{H^{1/2}(\Sigma)} + \big\|F^3(0) + \mathbb{D}_{\mathscr{A}_0} u^m(0)\mathscr{N}_0\big\|_{H^0(\Sigma)}\big). \tag{4-23}$$

We may then apply Cauchy's inequality to (4-23) to find that

$$\partial_t \tfrac{1}{2}\|u^m\|^2_{\mathscr{H}^0} + \tfrac{1}{8}\|u^m\|^2_{\mathscr{H}^1} \le C\big\|F^3(0) + \mathbb{D}_{\mathscr{A}_0} u^m(0)\mathscr{N}_0\big\|^2_{H^0(\Sigma)} \\ + C\big(\|F^1\|^2_{\mathscr{H}^0} + \|F^3\|^2_{H^{1/2}(\Sigma)}\big) + C_0(\eta)\tfrac{1}{2}\|u^m\|^2_{\mathscr{H}^0} \tag{4-24}$$

for $C_0(\eta) := 1 + \sup_{0\le t\le T}\|\partial_t J K\|_{L^\infty}$. Note that since $\mathscr{P}_0^m$ is the $H^2(\Omega)$ orthogonal projection, we may use Lemma 2.1 to obtain the bound

$$\|u^m(0)\|_{\mathscr{H}^0} \le 2\|u^m(0)\|_0 \le 2\|u^m(0)\|_2 = 2\|\mathscr{P}_0^m u_0\|_2 \le 2\|u_0\|_2. \tag{4-25}$$

Now we can apply Gronwall's lemma to the differential inequality (4-24) and utilize (4-25) to deduce energy estimates for u^m:

$$\sup_{0\le t\le T}\|u^m\|^2_{\mathscr{H}^0} + \|u^m\|^2_{\mathscr{H}^1_T} \\ \le \sup_{0\le t\le T}\|u^m\|^2_{\mathscr{H}^0} + \int_0^T \exp\big(C_0(\eta)(T-s)\big)\|u^m(s)\|^2_{\mathscr{H}^1}\,ds \\ \lesssim \exp\big(C_0(\eta)T\big)\big(\big\|F^3(0) + \mathbb{D}_{\mathscr{A}_0} u^m(0)\mathscr{N}_0\big\|^2_{H^0(\Sigma)} + \|u_0\|_2^2 + \|F^1\|^2_{\mathscr{H}^0_T} + \|F^3\|^2_{L^2H^{1/2}}\big). \tag{4-26}$$

Step 4: Estimate of $\|\partial_t u^m(0)\|_{\mathscr{H}^0}$. We will eventually derive energy estimates for $\partial_t u^m$ similar to those derived in the previous step for u^m, but first we must be able to estimate $\|\partial_t u^m(0)\|_{\mathscr{H}^0}$. If $u \in H^2(\Omega)\cap \mathscr{X}(t)$, $\psi \in \mathscr{H}^1$, then an integration by parts reveals that

$$\tfrac{1}{2}(u,\psi)_{\mathscr{H}^1} = \int_\Omega -\Delta_{\mathscr{A}} u\cdot\psi J + \int_\Sigma (\mathbb{D}_{\mathscr{A}} u\mathscr{N})\cdot\psi = (-\Delta_{\mathscr{A}} u,\psi)_{\mathscr{H}^0} + (\mathbb{D}_{\mathscr{A}} u\mathscr{N},\psi)_{0,\Sigma}. \tag{4-27}$$

Evaluating (4-20) at $t=0$ and employing (4-27), we find that

$$\big(\partial_t u^m(0),\psi\big)_{\mathscr{H}^0} = \big(\Delta_{\mathscr{A}_0} u^m(0) + F^1(0),\psi\big)_{\mathscr{H}^0} - \big(\Pi_0^\perp(F^3(0) + \mathbb{D}_{\mathscr{A}_0} u^m(0)\mathscr{N}_0),\psi\big)_{0,\Sigma} \tag{4-28}$$

for all $\psi \in \mathscr{X}_m(0)$, where we have written $\Pi_0^\perp = I - \Pi_0$ for the orthogonal projection onto the line generated by $\mathscr{N}_0$.

For $\psi \in \mathscr{X}_m(0)$, we must estimate the last term in (4-28) in terms of $\|\psi\|_{\mathscr{H}^0}$. This is possible due to the appearance of $\Pi_0^\perp$ and Lemma 3.3. Indeed, we know that

$$\Pi_0^\perp\big(F^3(0)+\mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0\big)=\big(F^3(0)\cdot\mathscr{N}_0+\mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0\cdot\mathscr{N}_0\big)\frac{\mathscr{N}_0}{|\mathscr{N}_0|^2},$$

which implies, since $|\mathscr{N}_0|^2\geq 1$ and $\operatorname{div}_{\mathscr{A}_0}\psi=0$, that

$$\begin{aligned}\big|\big(\Pi_0^\perp(F^3(0)+\mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0),\,\psi\big)_{0,\Sigma}\big| &\leq |\mathscr{N}_0|^2\big|\big(\Pi_0^\perp(F^3(0)+\mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0),\,\psi\big)_{0,\Sigma}\big|\\ &=\big|\big(F^3(0)\cdot\mathscr{N}_0+\mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0\cdot\mathscr{N}_0,\,\psi\cdot\mathscr{N}_0\big)_{0,\Sigma}\big|\\ &\leq\|\psi\cdot\mathscr{N}_0\|_{H^{-1/2}(\Sigma)}\big\|\big(F^3(0)+\mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0\big)\cdot\mathscr{N}_0\big\|_{H^{1/2}(\Sigma)}\\ &\lesssim C_1(\eta)\|\psi\|_{\mathscr{H}^0}\big\|F^3(0)+\mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0\big\|_{H^{1/2}(\Sigma)}.\end{aligned}\tag{4-29}$$

In the last inequality, we have used Lemmas 3.3 and A.1, and we have written $C_1(\eta):=\|\mathscr{N}_0\|_{C^1(\Sigma)}$.

By virtue of (4-19), we have that

$$\partial_t u^m(t)-R(t)u^m(t)=\dot{a}_j^m(t)\psi^j(t)\in\mathscr{X}_m(t),\tag{4-30}$$

so that $\psi=\partial_t u^m(0)-R(0)u^m(0)\in\mathscr{X}_m(0)$ is a valid choice of a test function in (4-28). We plug this ψ into (4-28), rearrange, and employ the bound (4-29) to see that

$$\begin{aligned}\|\partial_t u^m(0)\|_{\mathscr{H}^0}^2\leq\|R(0)u^m(0)\|_{\mathscr{H}^0}\|\partial_t u^m(0)\|_{\mathscr{H}^0}+\big\|\partial_t u^m(0)-R(0)u^m(0)\big\|_{\mathscr{H}^0}\big\|\Delta_{\mathscr{A}_0}u^m(0)+F^1(0)\big\|_{\mathscr{H}^0}\\ +CC_1(\eta)\big\|\partial_t u^m(0)-R(0)u^m(0)\big\|_{\mathscr{H}^0}\big\|F^3(0)+\mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0\big\|_{H^{1/2}(\Sigma)}.\end{aligned}\tag{4-31}$$

A simple computation and (4-25) imply that $\|\Delta_{\mathscr{A}_0}u^m(0)\|_{\mathscr{H}^0}\lesssim\|\mathscr{A}_0\|_{C^1}^2\|u_0\|_2$. This allows us to use Cauchy's inequality and (4-25) to derive from (4-31) the bound

$$\|\partial_t u^m(0)\|_{\mathscr{H}^0}^2\lesssim C_2(\eta)\big(\|u_0\|_2^2+\|F^1(0)\|_{\mathscr{H}^0}^2+\big\|F^3(0)+\mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0\big\|_{H^{1/2}(\Sigma)}^2\big)\tag{4-32}$$

for $C_2(\eta):=1+\|R(0)\|_{L^\infty}^2+\|\mathscr{A}_0\|_{C^1}^2+C_1(\eta)^2$. This is our desired estimate of $\|\partial_t u^m(0)\|_{\mathscr{H}^0}$.

Step 5: Energy estimates for $\partial_t u^m$. We now turn to estimates for $\partial_t u^m$ of a similar form to those we already derived for u^m. Suppose for now that $\psi(t)=b_j^m(t)\psi^j$ for $b_j^m\in C^{0,1}([0,T])$, $j=1,\dots,m$; it is easily verified, as in (4-30), that $\partial_t\psi-R(t)\psi\in\mathscr{X}_m(t)$ as well. We now use this ψ in (4-20), temporally differentiate the resulting equation, and then subtract from the result Equation (4-20) with test function $\partial_t\psi-R\psi$; this eliminates the appearance of $\partial_t\psi$ and leaves us with the equality

$$\begin{aligned}\langle\partial_t^2u^m,\psi\rangle_{\mathscr{X}*}+\tfrac12(\partial_t u^m,\psi)_{\mathscr{H}^1}=\big\langle\partial_t(F^1-F^3),\psi\big\rangle_{\mathscr{X}*}-\big(F^3-\Pi_0(F^3(0)+\mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0),\,R\psi\big)_{0,\Sigma}\\ +\big(F^1,(\partial_t JK+R)\psi\big)_{\mathscr{H}^0}-\big(\partial_t u^m,(\partial_t JK+R)\psi\big)_{\mathscr{H}^0}-\tfrac12(u^m,R\psi)_{\mathscr{H}^1}\\ -\frac12\int_\Omega\big(\partial_t JK\mathbb{D}_{\mathscr{A}}u^m:\mathbb{D}_{\mathscr{A}}\psi+\mathbb{D}_{\partial_t\mathscr{A}}u^m:\mathbb{D}_{\mathscr{A}}\psi+\mathbb{D}_{\mathscr{A}}u^m:\mathbb{D}_{\partial_t\mathscr{A}}\psi\big)J.\end{aligned}\tag{4-33}$$

According to (4-30) and the fact that a_j^m is twice differentiable almost everywhere, we may use $\psi=\partial_t u^m(t)-R(t)u^m(t)\in\mathscr{X}_m(t)$ as a test function in (4-33). Plugging in this ψ and arguing as in the previous steps by employing Remark 2.3, Cauchy's inequality, and trace embeddings, we may deduce from (4-33) that

$$\begin{aligned}\partial_t\big(\tfrac{1}{2}\|\partial_t u^m\|^2_{\mathscr{H}^0} - (\partial_t u^m, Ru^m)_{\mathscr{H}^0}\big) + \tfrac{1}{8}\|\partial_t u^m\|^2_{\mathscr{H}^1} \\ \le CC_3(\eta)\|u^m\|^2_{\mathscr{H}^1} + C_0(\eta)\big(\tfrac{1}{2}\|\partial_t u^m\|^2_{\mathscr{H}^0} - (\partial_t u^m, Ru^m)_{\mathscr{H}^0}\big) + C\big(\|F^1\|^2_{\mathscr{H}^0} + \|F^3\|^2_{H^{1/2}(\Sigma)}\big) \\ + C\big\|F^3(0) + \mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0\big\|^2_{H^0(\Sigma)} + C\|\partial_t(F^1 - F^3)\|^2_{\mathscr{X}^*} \end{aligned} \tag{4-34}$$

for $C_0(\eta)$ as defined above and

$$C_3(\eta) := \sup_{0\le t\le T}\big[1 + \|R\|^2_{C^1} + \|\partial_t R\|^2_{L^\infty} + \|\partial_t \mathscr{A}\|^2_{L^\infty} + \big(1 + \|\mathscr{A}\|^2_{L^\infty}\big)\big(1 + \|\partial_t JK\|^2_{L^\infty}\big)\big] \times \sup_{0\le t\le T}\big[1 + \|R\|^2_{C_1}\big].$$

Then (4-34), Gronwall's lemma, and a further application of Cauchy's inequality imply that

$$\begin{aligned}\sup_{0\le t\le T}\|\partial_t u^m\|^2_{\mathscr{H}^0} + \|\partial_t u^m\|^2_{\mathscr{H}^1_T} \\ \lesssim \exp\left(C_0(\eta)T\right)\Big(\|\partial_t u^m(0)\|^2_{\mathscr{H}^0} + C_2(\eta)\|u^m(0)\|^2_{\mathscr{H}^0} \\ + \big\|F^3(0) + \mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0\big\|^2_{H^0(\Sigma)} + \|F^1\|^2_{\mathscr{H}^0_T} + \|F^3\|^2_{L^2H^{1/2}} + \|\partial_t(F^1 - F^3)\|^2_{(\mathscr{X}_T)^*}\Big) \\ + C_3(\eta)\bigg(\sup_{0\le t\le T}\|u^m\|^2_{\mathscr{H}^0} + \int_0^T \exp\big(C_0(\eta)(T-s)\big)\|u^m(s)\|^2_{\mathscr{H}^1}\,ds\bigg).\end{aligned} \tag{4-35}$$

Now we combine (4-35) with the estimates (4-25), (4-26), and (4-32) to deduce our energy estimates for $\partial_t u^m$:

$$\begin{aligned}\sup_{0\le t\le T}\|\partial_t u^m\|^2_{\mathscr{H}^0} + \|\partial_t u^m\|^2_{\mathscr{H}^1_T} \\ \lesssim \big(C_2(\eta) + C_3(\eta)\big)\exp(C_0(\eta)T)\big(\|u_0\|^2_2 + \|F^1(0)\|^2_{\mathscr{H}^0} + \big\|F^3(0) + \mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0\big\|^2_{H^0(\Sigma)}\big) \\ + \exp(C_0(\eta)T)\big[C_3(\eta)\big(\|F^1\|^2_{\mathscr{H}^0_T} + \|F^3\|^2_{L^2H^{1/2}}\big) + \big\|\partial_t(F^1 - F^3)\big\|^2_{(\mathscr{X}_T)^*}\big].\end{aligned} \tag{4-36}$$

Step 6: Improved energy estimate for u^m. We can now improve our energy estimates for u^m by using $\psi = \partial_t u^m(t) - R(t)u^m(t) \in \mathscr{X}_m(t)$ as a test function in (4-20). Plugging this in and rearranging yields the equality

$$\begin{aligned}\partial_t \tfrac{1}{4}\|u^m\|^2_{\mathscr{H}^1} + \|\partial_t u^m\|^2_{\mathscr{H}^0} \\ = (\partial_t u^m, Ru^m)_{\mathscr{H}^0} + \tfrac{1}{2}(u^m, Ru^m)_{\mathscr{H}^1} + (F^1, \partial_t u^m - Ru^m)_{\mathscr{H}^0} \\ -\big(F^3 - \Pi_0(F^3(0) + \mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0), \partial_t u^m - Ru^m\big)_{0,\Sigma} + \frac{1}{2}\int_\Omega\bigg(\mathbb{D}_{\mathscr{A}}u^m : \mathbb{D}_{\partial_t\mathscr{A}}u^m + \partial_t JK\frac{|\mathbb{D}_{\mathscr{A}}u^m|^2}{2}\bigg)J.\end{aligned} \tag{4-37}$$

We may then argue as before to use (4-37) to derive the inequality

$$\begin{aligned}\partial_t \tfrac{1}{4}\|u^m\|^2_{\mathscr{H}^1} + \|\partial_t u^m\|^2_{\mathscr{H}^0} \\ \le C\big\|F^3(0) + \mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0\big\|^2_{H^{1/2}(\Sigma)} + C\big(\|F^1\|^2_{\mathscr{H}^0} + \|F^3\|^2_{H^{1/2}(\Sigma)}\big) + C\big(\|\partial_t u^m\|^2_{\mathscr{H}^1} + C_3(\eta)\|u^m\|^2_{\mathscr{H}^1}\big).\end{aligned} \tag{4-38}$$

We could regard (4-38) as a differential inequality for $\|u^m\|^2_{\mathscr{H}^1}$ and apply Gronwall's lemma as before, but this is not necessary since we already control $\|u^m\|^2_{\mathscr{H}^1_T}$ and $\|\partial_t u^m\|^2_{\mathscr{H}^1_T}$. Indeed, we may simply integrate (4-38) in time to deduce an improved energy estimate for u^m:

$$\begin{aligned}\sup_{0\le t\le T} \|u^m\|^2_{\mathscr{H}^1} + \|\partial_t u^m\|^2_{\mathscr{H}^0_T} \\ \lesssim \big(C_2(\eta)+C_3(\eta)\big)\exp(C_0(\eta)T)\big(\|u_0\|_2^2 + \|F^1(0)\|^2_{\mathscr{H}^0} + \big\|F^3(0)+\mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0\big\|^2_{H^{1/2}(\Sigma)}\big) \\ + \exp(C_0(\eta)T)\big[C_3(\eta)\big(\|F^1\|^2_{\mathscr{H}^0_T} + \|F^3\|^2_{L^2H^{1/2}}\big) + \big\|\partial_t(F^1+F^3)\big\|^2_{(\mathscr{X}_T)^*}\big]. \quad (4\text{-}39)\end{aligned}$$

Step 7: Estimating terms in (4-36), (4-39). In order to use (4-36) and (4-39) as uniform bounds, we must first remove the appearance of $u^m(0)$ on the right side of the estimates. For this we use Lemma A.2, the embedding $H^2(\Omega)\hookrightarrow H^{3/2}(\Sigma)$, and the bound $\|u^m(0)\|_2 \le \|u_0\|_2$ to find that

$$\big\|F^3(0)+\mathbb{D}_{\mathscr{A}_0}u^m(0)\mathscr{N}_0\big\|^2_{H^{1/2}(\Sigma)} \lesssim C_4(\eta)\big(\|F^3(0)\|^2_{H^{1/2}(\Sigma)} + \|u_0\|_2^2\big) \quad (4\text{-}40)$$

for $C_4(\eta) := 1 + \|\mathscr{N}_0\|^2_{C^1(\Sigma)}\|\mathscr{A}_0\|^2_{C^1}$.

We now seek to estimate the constants $C_i(\eta)$, $i=0,\dots,4$ in terms of the quantity $\mathscr{K}(\eta)$. A simple computation shows that

$$C_0(\eta) + \big(C_2(\eta)+C_3(\eta)\big)\big(1+C_4(\eta)\big) \le \sup_{0\le t\le T} \mathfrak{Q}_1\big(\|\bar\eta\|^2_{C^2}, \|\partial_t\bar\eta\|^2_{C^2}, \|\partial_t^2\bar\eta\|^2_{C^1}\big), \quad (4\text{-}41)$$

where $\mathfrak{Q}_1$ is a polynomial in three variables. According to Lemma A.8 in the nonperiodic case and Lemma A.10 in the periodic case, we have the estimate $\|\partial_t^j\bar\eta\|^2_{C^k} \lesssim \|\partial_t^j\eta\|^2_{k+3/2}$ for $j,k\ge 0$. This, (4-41), and the fact that $\mathscr{K}(\eta)\le 1$ then imply that

$$C_0(\eta) + \big(C_2(\eta)+C_3(\eta)\big)\big(1+C_4(\eta)\big) \le \mathfrak{Q}_1\big(\mathscr{K}(\eta),\mathscr{K}(\eta),\mathscr{K}(\eta)\big) \le C\big(1+\mathscr{K}(\eta)\big) \quad (4\text{-}42)$$

for a constant C independent of η.

Step 8: Passing to the limit. We now utilize the energy estimates (4-36) and (4-39) in conjunction with (4-40) to pass to the limit $m\to\infty$. According to these energy estimates and Lemma 2.1, we have that the sequence $\{u^m\}$ is uniformly bounded in $L^\infty H^1$ and $\{\partial_t u^m\}$ is uniformly bounded in $L^\infty H^0 \cap L^2 H^1$. Up to the extraction of a subsequence, we then know that

$$u^m \overset{*}{\rightharpoonup} u \text{ weakly-}* \text{ in } L^\infty H^1, \quad \partial_t u^m \overset{*}{\rightharpoonup} \partial_t u \text{ in } L^\infty H^0, \quad \text{and} \quad \partial_t u^m \rightharpoonup \partial_t u \text{ weakly in } L^2 H^1.$$

By lower semicontinuity and (4-42), the energy estimates imply that the quantity

$$\|u\|^2_{L^\infty H^1} + \|\partial_t u\|^2_{L^\infty H^0} + \|\partial_t u\|^2_{L^2 H^1}$$

is bounded above by the right-hand side of (4-13).

Because of these convergence results, we can integrate (4-33) in time from 0 to T and send $m\to\infty$ to deduce that $\partial_t^2 u^m \rightharpoonup \partial_t^2 u$ weakly in $\mathscr{X}_T^*$, with the action of $\partial_t^2 u$ on an element $\psi\in\mathscr{X}_T$ defined by replacing u^m with u everywhere in (4-33). From the equation resulting from passing to the limit in (4-33), it is straightforward to show that $\|\partial_t^2 u\|^2_{(\mathscr{X}_T)^*}$ is bounded by the right-hand side of (4-13). This bound then shows that $\partial_t u \in C^0 L^2$.

Step 9: The strong solution. Due to the convergence established in the last step, we may pass to the limit in (4-20) for almost every $t \in [0, T]$. Since $u^m(0) \to u_0$ in H^2 and u_0, $F^3(0)$ satisfy the compatibility condition (4-11), we have

$$\left\| \Pi_0(F^3(0) + \mathbb{D}_{\mathscr{A}_0} u^m(0)\mathscr{N}_0) \right\|_{H^{1/2}(\Sigma)} \to 0.$$

In the limit, (4-20) implies that for almost every t,

$$(\partial_t u, \psi)_{\mathscr{H}^0} + \tfrac{1}{2}(u, \psi)_{\mathscr{H}^1} = (F^1, \psi)_{\mathscr{H}^0} - (F^3, \psi)_{0,\Sigma} \quad \text{for every } \psi \in \mathscr{X}(t). \tag{4-43}$$

Now we introduce the pressure. Define the functional $\Lambda_t \in (\mathscr{H}^1(t))^*$ so that $\Lambda_t(v)$ equals the difference between the left and right sides of (4-43), with ψ replaced by $v \in \mathscr{H}^1(t)$. Then $\Lambda_t(v) = 0$ for all $v \in \mathscr{X}(t)$, so by Proposition 2.9, there exists a unique $p(t) \in \mathscr{H}^0(t)$ such that $(p(t), \operatorname{div}_{\mathscr{A}} v)_{\mathscr{H}^0} = \Lambda_t(v)$ for all $v \in \mathscr{H}^1(t)$. This is equivalent to

$$(\partial_t u, v)_{\mathscr{H}^0} + \tfrac{1}{2}(u, v)_{\mathscr{H}^1} - (p, \operatorname{div}_{\mathscr{A}} v)_{\mathscr{H}^0} = (F^1, v)_{\mathscr{H}^0} - (F^3, v)_{0,\Sigma} \quad \text{for every } v \in \mathscr{H}^1(t). \tag{4-44}$$

For almost every $t \in [0, T]$, $(u(t), p(t))$ is the unique weak solution to the elliptic problem (3-6) in the sense of (3-7), with F^1 replaced by $F^1(t) - \partial_t u(t)$, $F^2 = 0$, and F^3 replaced by $F^3(t)$. Since $F^1(t) - \partial_t u(t) \in H^0(\Omega)$ and $F^3(t) \in H^{1/2}(\Sigma)$, Lemma 3.6 implies that this elliptic problem admits a unique strong solution, which must coincide with the weak solution. We may then apply Proposition 3.7 and Lemma 2.1 for the bound

$$\|u(t)\|_r^2 + \|p(t)\|_{r-1}^2 \lesssim \left(\|\partial_t u(t)\|^2_{\mathscr{H}^{r-2}} + \|F^1(t)\|^2_{r-2} + \|F^3(t)\|^2_{H^{r-3/2}(\Sigma)} \right) \tag{4-45}$$

when $r = 2, 3$. When $r = 2$, we take the supremum of (4-45) over $t \in [0, T]$, and when $r = 3$, we integrate over $[0, T]$; the resulting inequalities imply that $u \in L^\infty H^2 \cap L^2 H^3$ and $p \in L^\infty H^1 \cap L^2 H^2$ with estimates as in (4-13). This, in turn, implies that (u, p) is a strong solution to (1-7).

Since we already know that $u \in L^2 H^3$ and $\partial_t u \in L^2 H^1$, Lemma A.4 implies that $u \in C^0 H^2$. Then since $F^1 - \partial_t u \in C^0 H^0$ and $\mathbb{D}_{\mathscr{A}} u \mathscr{N} + F^3 \in C^0 H^{1/2}(\Sigma)$, we know that $\nabla_{\mathscr{A}} p \in C^0 H^0$ and $p \in C^0 H^{1/2}(\Sigma)$ as well, from which we see, via Poincaré's inequality (Lemma A.12), that $p \in C^0 H^1$. With these continuity results established, we can compute $p(0)$ and $\partial_t u(0)$. We start with the Dirichlet condition for $p(0)$ on Σ, the second equation in (4-14). Since $p \in C^0 H^1(\Omega)$, $u \in C^0 H^2(\Omega)$, and $F^3 \in C^0 H^{1/2}(\Sigma)$, the boundary condition $S_{\mathscr{A}}(p, u)\mathscr{N} = F^3$, which holds in $H^{1/2}(\Sigma)$ for each $t > 0$, can be evaluated at $t = 0$. Then the Dirichlet condition for $p(0)$ on Σ in (4-14) is easily deduced by solving $S_{\mathscr{A}_0}(p(0), u_0)\mathscr{N}_0 = F^3(0)$ for $p(0)$.

Now we derive the PDE satisfied by $p(0)$ and compute $\partial_t u(0)$. First note that for any $\varphi \in {}^0H^1(\Omega)$, we may integrate by parts and use the fact that $\operatorname{div}_{\mathscr{A}} D_t u = 0$ in Ω and $D_t u = 0$ on Σ_b to see that

$$(D_t u, \nabla_{\mathscr{A}} \varphi)_{\mathscr{H}^0} = -(\operatorname{div}_{\mathscr{A}} D_t u, \varphi)_{\mathscr{H}^0} + (D_t u \cdot \mathscr{N}, \varphi)_{0,\Sigma} = 0.$$

Then since (u, p) is a strong solution to (1-7), we have that

$$\left(Ru + \nabla_{\mathscr{A}} p - \Delta_{\mathscr{A}} u - F^1, \nabla_{\mathscr{A}} \varphi \right)_{\mathscr{H}^0} = -(D_t u, \nabla_{\mathscr{A}} \varphi)_{\mathscr{H}^0} = 0 \quad \text{for all } \varphi \in {}^0H^1(\Omega). \tag{4-46}$$

By the established continuity properties, we may set $t = 0$ in (4-46), and again integrate by parts to see that

$$\big(\nabla_{\mathscr{A}_0} p(0) - F^1(0), \nabla_{\mathscr{A}_0}\varphi\big)_{\mathscr{H}^0} = -\big(-\operatorname{div}_{\mathscr{A}_0}(R(0)u_0), \varphi\big)_{\mathscr{H}^0} + \langle \Delta_{\mathscr{A}_0} u_0 \cdot \nu, \varphi\rangle_{-1/2}$$

for all $\varphi \in {}^0H^1(\Omega)$. This establishes that $p(0)$ is the weak solution to (4-14). According to (3-32), we then have $p(0) \in H^1(\Omega)$. This and (4-44) allow us to solve for $\partial_t u(0)$ as in (4-15), and then (4-46) implies that $\partial_t u(0) - R(0)u_0 \in \mathscr{Y}(0)$ since then $D_t u(0) \perp \nabla_{\mathscr{A}(0)}\varphi$ for every $\varphi \in {}^0H^1(\Omega)$.

Step 10: The weak solution satisfied by $D_t u = \partial_t u - Ru$. Now we seek to use (4-33) to determine the PDE satisfied by $D_t u$. As mentioned above, we may integrate (4-33) in time from 0 to T and pass to the limit $m \to \infty$. For any $\psi \in \mathscr{X}_T$, we have $R\psi \in \mathscr{H}^1_T$, so that we may replace all of the terms $R\psi$ in the resulting equation by using $v = R\psi$ in (4-44); this yields the equality

$$\begin{aligned}
\langle \partial_t^2 u, \psi\rangle_* &+ \tfrac{1}{2}(\partial_t u, \psi)_{\mathscr{H}^1_T} \\
&= \big\langle \partial_t(F^1 - F^3), \psi\big\rangle_* + (\partial_t J K F^1, \psi)_{\mathscr{H}^0_T} - (\partial_t J K \partial_t u, \psi)_{\mathscr{H}^0_T} - (p, \operatorname{div}_{\mathscr{A}}(R\psi))_{\mathscr{H}^0_T} \\
&\qquad - \frac{1}{2}\int_0^T \int_\Omega \big(\partial_t J K \mathbb{D}_{\mathscr{A}} u : \mathbb{D}_{\mathscr{A}}\psi + \mathbb{D}_{\partial_t \mathscr{A}} u : \mathbb{D}_{\mathscr{A}}\psi + \mathbb{D}_{\mathscr{A}} u : \mathbb{D}_{\partial_t \mathscr{A}}\psi\big) J
\end{aligned} \tag{4-47}$$

for all $\psi \in \mathscr{X}_T$. Equation (4-17) follows directly from (4-47) via

$$\langle \partial_t^2 u, v\rangle_* = \langle \partial_t D_t u, v\rangle_* + (R\partial_t u, v)_{\mathscr{H}^0_T} + (\partial_t R u, v)_{\mathscr{H}^0_T}.$$

To justify that (4-17) implies (4-16), we will now perform some computations.

Lemma A.3 shows that $-R^T\mathscr{N} = \partial_t \mathscr{N}$ on Σ, so that we may integrate by parts for the equality

$$-\big(p, \operatorname{div}_{\mathscr{A}}(Rv)\big)_{\mathscr{H}^0_T} = (R^T\nabla_{\mathscr{A}} p, v)_{\mathscr{H}^0_T} - \langle pR^T\mathscr{N}, v\rangle_{-1/2} = (R^T\nabla_{\mathscr{A}} p, v)_{\mathscr{H}^0_T} - \langle -p\partial_t\mathscr{N}, v\rangle_{-1/2}, \tag{4-48}$$

where R^T is the matrix transpose of R. Another integration by parts yields

$$\begin{aligned}
-\frac{1}{2}\int_0^T \int_\Omega &\big(\partial_t J K \mathbb{D}_{\mathscr{A}} u : \mathbb{D}_{\mathscr{A}} v + \mathbb{D}_{\partial_t \mathscr{A}} u : \mathbb{D}_{\mathscr{A}} v + \mathbb{D}_{\mathscr{A}} u : \mathbb{D}_{\partial_t \mathscr{A}} v\big)\, J \\
&= -\int_0^T \int_\Omega \big(-R\mathbb{D}_{\mathscr{A}} u + \mathbb{D}_{\partial_t \mathscr{A}} u\big) : \nabla_{\mathscr{A}} v J \\
&= \big(\operatorname{div}_{\mathscr{A}}(-R\mathbb{D}_{\mathscr{A}} u + \mathbb{D}_{\partial_t \mathscr{A}} u), v\big)_{\mathscr{H}^0_T} - \langle \mathbb{D}_{\mathscr{A}} u \partial_t \mathscr{N} + \mathbb{D}_{\partial_t \mathscr{A}} u \mathscr{N}, v\rangle_{-1/2}.
\end{aligned} \tag{4-49}$$

We may then combine (4-48)–(4-49) with the fact that $D_t u = \partial_t u - Ru \in \mathscr{X}_T$ to deduce from (4-17) that $D_t u$ is weak solutions of (4-16) in the sense of (4-2) with $D_t u(0) \in \mathscr{Y}(0)$ given by (4-15). Here, the fact that G^1 and G^3 satisfy the same inclusions as F^1 and F^3 listed in (4-7) is easily established from the above bounds on (u, p).

Step 11: Proof of (4-18). Let us now define the functional $J\partial_t u - P \in ({}_0H^1(\Omega))^*$ via

$$\langle J\partial_t u - P, v\rangle := \int_\Omega J\partial_t u \cdot v - pJ\mathscr{A}_{ij}\partial_j v_i \quad \text{for } v \in {}_0H^1(\Omega).$$

By our estimates on (u, p), we clearly have $J\partial_t u - P \in L^2\big([0, T]; ({}_0H^1(\Omega))^*\big)$. Since (u, p) are a strong solution, the equality (4-44) holds also for arbitrary $v \in {}_0H^1(\Omega)$, which is equivalent to

$$\langle J\partial_t u - P, v\rangle = -\tfrac{1}{2}(u, v)_{\mathcal{H}^1} + (F^1, v)_{\mathcal{H}^0} - (F^3, v)_{0,\Sigma}.$$

Then for any $\varphi \in C_c^\infty(\mathbb{R})$, we may compute the weak derivative via

$$\begin{aligned} -\int_0^T \langle J\partial_t u - P, v\rangle \varphi' &= -\int_0^T \bigl(-\tfrac{1}{2}(u, v)_{\mathcal{H}^1} + (F^1, v)_{\mathcal{H}^0} - (F^3, v)_{0,\Sigma}\bigr)\varphi' \\ &= \int_0^T \varphi\bigl(-\tfrac{1}{2}(\partial_t u, v)_{\mathcal{H}^1} + \langle \partial_t(F^1 - F^3), v\rangle_* + \mathcal{U}(v)\bigr), \end{aligned}$$

where we have written

$$\mathcal{U}(v) = (\partial_t JKF^1, v)_{\mathcal{H}^0} - \frac{1}{2}\int_\Omega \bigl(\partial_t JK\mathbb{D}_{\mathcal{A}}u : \mathbb{D}_{\mathcal{A}}v + \mathbb{D}_{\partial_t\mathcal{A}}u : \mathbb{D}_{\mathcal{A}}v + \mathbb{D}_{\mathcal{A}}u : \mathbb{D}_{\partial_t\mathcal{A}}v\bigr) J.$$

Using this, we find that $\partial_t(J\partial_t u - P) \in L^2\bigl([0, T]; ({}_0H^1(\Omega))^*\bigr)$ with

$$\begin{aligned} \bigl\langle \partial_t(J\partial_t u - P), v\bigr\rangle = \bigl\langle \partial_t(F^1 - F^3), v\bigr\rangle_* + \int_\Omega \partial_t JF^1 \cdot v \\ - (\partial_t u, v)_{\mathcal{H}^1} - \frac{1}{2}\int_\Omega \bigl(\partial_t JK\mathbb{D}_{\mathcal{A}}u : \mathbb{D}_{\mathcal{A}}v + \mathbb{D}_{\partial_t\mathcal{A}}u : \mathbb{D}_{\mathcal{A}}v + \mathbb{D}_{\mathcal{A}}u : \mathbb{D}_{\partial_t\mathcal{A}}v\bigr) J. \end{aligned}$$

We may then use this and the inclusions (4-12) in conjunction with Lemma A.16 to deduce (4-18). □

Remark 4.4. Notice that the compatibility condition (4-11) was essential in achieving the $\partial_t u$ estimate of Theorem 4.3.

Higher regularity. In order to state our higher regularity results for the problem (1-7), we must be able to define the forcing terms and initial data for the problem that results from temporally differentiating (1-7) several times. To this end, we first define some mappings. Given F^1, F^3, v, q, we define the vector fields $\mathfrak{G}^0, \mathfrak{G}^1$ on Ω and $\mathfrak{G}^3$ on Σ by

$$\begin{aligned} \mathfrak{G}^0(F^1, v, q) &= \Delta_{\mathcal{A}}v - \nabla_{\mathcal{A}}q + F^1 - Rv, \\ \mathfrak{G}^1(v, q) &= -(R + \partial_t JK)\Delta_{\mathcal{A}}v - \partial_t Rv + (\partial_t JK + R + R^T)\nabla_{\mathcal{A}}q \\ &\qquad + \operatorname{div}_{\mathcal{A}}(\mathbb{D}_{\mathcal{A}}(Rv) - R\mathbb{D}_{\mathcal{A}}v + \mathbb{D}_{\partial_t\mathcal{A}}v), \\ \mathfrak{G}^3(v, q) &= \mathbb{D}_{\mathcal{A}}(Rv)\mathcal{N} - (qI - \mathbb{D}_{\mathcal{A}}v)\partial_t\mathcal{N} + \mathbb{D}_{\partial_t\mathcal{A}}v\mathcal{N}, \end{aligned} \tag{4-50}$$

and we define the functions $\mathfrak{f}^1$ on Ω, $\mathfrak{f}^2$ on Σ, and $\mathfrak{f}^3$ on Σ_b according to

$$\begin{aligned} \mathfrak{f}^1(F^1, v) &= \operatorname{div}_{\mathcal{A}}(F^1 - Rv), \\ \mathfrak{f}^2(F^3, v) &= (F^3 + \mathbb{D}_{\mathcal{A}}v\mathcal{N}) \cdot \mathcal{N}|\mathcal{N}|^{-2}, \\ \mathfrak{f}^3(F^1, v) &= (F^1 + \Delta_{\mathcal{A}}v) \cdot \nu. \end{aligned} \tag{4-51}$$

In the definitions of $\mathfrak{G}^i$ and $\mathfrak{f}^i$, we assume that $\mathcal{A}, \mathcal{N}, R$ (recall that R is defined by (4-8)), etc. are evaluated at the same t as F^1, F^3, v, q. These mappings allow us to define the forcing terms as follows. Write

$F^{1,0} = F^1$ and $F^{3,0} = F^3$. When F^1, F^3, u, and p are sufficiently regular for the following to make sense, we recursively define the vectors

$$\begin{aligned} F^{1,j} &:= D_t F^{1,j-1} + \mathfrak{G}^1(D_t^{j-1} u, \partial_t^{j-1} p) = D_t^j F^1 + \sum_{\ell=0}^{j-1} D_t^\ell \mathfrak{G}^1\big(D_t^{j-\ell-1} u, \partial_t^{j-\ell-1} p\big), \\ F^{3,j} &:= \partial_t F^{3,j-1} + \mathfrak{G}^3(D_t^{j-1} u, \partial_t^{j-1} p) = \partial_t^j F^3 + \sum_{\ell=0}^{j-1} \partial_t^\ell \mathfrak{G}^3\big(D_t^{j-\ell-1} u, \partial_t^{j-\ell-1} p\big) \end{aligned} \tag{4-52}$$

on Ω and Σ, respectively, for $j = 1, \dots, 2N$. These are the forcing terms that appear when we apply j temporal derivatives to (1-7) (see (4-74)).

Now we define various sums of norms of F^1, F^3, and η that will appear in our estimates. Define the quantities

$$\begin{aligned} \mathfrak{F}(F^1, F^3) &:= \sum_{j=0}^{2N-1} \|\partial_t^j F^1\|_{L^2 H^{4N-2j-1}} + \|\partial_t^{2N} F^1\|_{L^2({}_0H^1(\Omega))^*} + \sum_{j=0}^{2N} \|\partial_t^j F^3\|_{L^2 H^{4N-2j-1/2}} \\ &\qquad + \sum_{j=0}^{2N-1} \|\partial_t^j F^1\|_{L^\infty H^{4N-2j-2}} + \|\partial_t^j F^3\|_{L^\infty H^{4N-2j-3/2}}, \\ \mathfrak{F}_0(F^1, F^3) &:= \sum_{j=0}^{2N-1} \|\partial_t^j F^1(0)\|_{4N-2j-2} + \|\partial_t^j F^3(0)\|_{4N-2j-3/2}. \end{aligned} \tag{4-53}$$

For brevity, we will only write $\mathfrak{F}$ for $\mathfrak{F}(F^1, F^3)$ and $\mathfrak{F}_0$ for $\mathfrak{F}_0(F^1, F^3)$ throughout the rest of this section. Lemmas A.4 and 2.4 imply that if $\mathfrak{F} < \infty$, then

$$\partial_t^j F^1 \in C^0\big([0, T]; H^{4N-2j-2}(\Omega)\big) \quad \text{and} \quad \partial_t^j F^3 \in C^0\big([0, T]; H^{4N-2j-3/2}(\Sigma)\big)$$

for $j = 0, \dots, 2N-1$. The same lemmas also imply that the sum of the $L^\infty H^k$ norms in the definition of $\mathfrak{F}$ can be bounded by a constant that depends on T times the sum of the $L^2 H^{k+1}$ norms. To avoid the introduction of a constant that depends on T, we will retain the L^∞ terms. For η, we define

$$\begin{aligned} \mathfrak{D}(\eta) &:= \|\eta\|_{L^2 H^{4N+1/2}} + \|\partial_t \eta\|_{L^2 H^{4N-1/2}} + \sum_{j=2}^{2N+1} \|\partial_t^j \eta\|_{L^2 H^{4N-2j+5/2}}, \\ \bar{\mathfrak{E}}(\eta) &:= \|\eta\|_{4N} + \|\partial_t \eta\|_{4N-1} + \sum_{j=2}^{2N} \|\partial_t^j \eta\|_{4N-2j+3/2}, \\ \mathfrak{E}(\eta) &:= \sum_{j=0}^{2N} \|\partial_t^j \eta\|_{L^\infty H^{4N-2j}}, \quad \text{and} \quad \mathfrak{K}(\eta) := \bar{\mathfrak{E}}(\eta) + \mathfrak{D}(\eta), \end{aligned} \tag{4-54}$$

as well as

$$\mathfrak{E}_0(\eta) := \|\eta(0)\|_{4N}^2 + \|\partial_t \eta(0)\|_{4N-1}^2 + \sum_{j=2}^{2N} \|\partial_t^j \eta(0)\|_{4N-2j+3/2}^2. \tag{4-55}$$

Again, Lemma A.4 implies that $\eta \in C^0([0,T]; H^{4N}(\Sigma))$, $\partial_t \eta \in C^0([0,T]; H^{4N-1}(\Sigma))$, and $\partial_t^j \eta \in C^0([0,T]; H^{4N-2j+3/2}(\Sigma))$ for $j = 2, \dots, 2N$. Throughout the rest of this section, we will assume that $\mathfrak{K}(\eta), \mathfrak{E}_0(\eta) \le 1$, which implies that $\mathfrak{Q}(\mathfrak{K}(\eta)) \lesssim 1 + \mathfrak{K}(\eta)$ and $\mathfrak{Q}(\mathfrak{E}_0(\eta)) \lesssim 1 + \mathfrak{E}_0(\eta)$ for any polynomial $\mathfrak{Q}$. Note that $\mathcal{K}(\eta) \le \mathfrak{E}(\eta) \le \mathfrak{K}(\eta)$, where $\mathcal{K}(\eta)$ is defined by (4-9); also, we have that $\|\eta_0\|_{4N-1/2}^2 \le \mathfrak{E}_0(\eta)$.

We now record an estimate of the $F^{i,j}$ in terms of $\mathfrak{F}$, $\mathfrak{K}(\eta)$ and certain norms of u, p.

Lemma 4.5. *For $m = 1, \dots, 2N-1$ and $j = 1, \dots, m$, the following estimates hold whenever the right-hand sides are finite*:

$$\|F^{1,j}\|_{L^2 H^{2m-2j+1}}^2 + \|F^{3,j}\|_{L^2 H^{2m-2j+3/2}}^2 \lesssim (1+\mathfrak{K}(\eta))\Big(\mathfrak{F} + \sum_{\ell=0}^{j-1} \|\partial_t^\ell u\|_{L^2 H^{2m-2j+3}}^2 + \sum_{\ell=0}^{j-1} \|\partial_t^\ell u\|_{L^\infty H^{2m-2j+2}}^2 + \|\partial_t^\ell p\|_{L^2 H^{2m-2j+2}}^2 + \|\partial_t^\ell p\|_{L^\infty H^{2m-2j+1}}^2\Big), \tag{4-56}$$

$$\|F^{1,j}\|_{L^\infty H^{2m-2j}}^2 + \|F^{3,j}\|_{L^\infty H^{2m-2j+1/2}}^2 \lesssim (1+\mathfrak{K}(\eta))\Big(\mathfrak{F} + \sum_{\ell=0}^{j-1} \|\partial_t^\ell u\|_{L^\infty H^{2m-2j+2}}^2 + \sum_{\ell=0}^{j-1} \|\partial_t^\ell p\|_{L^\infty H^{2m-2j+1}}^2\Big), \tag{4-57}$$

and

$$\big\|\partial_t(F^{1,m} - F^{3,m})\big\|_{L^2({}_0H^1(\Omega))^*}^2 \lesssim (1+\mathfrak{K}(\eta))\Big(\mathfrak{F} + \sum_{\ell=0}^{m-1} \|\partial_t^\ell u\|_{L^\infty H^2}^2 + \|\partial_t^\ell u\|_{L^2 H^3}^2 + \|\partial_t^m u\|_{L^2 H^2}^2 + \|\partial_t^m p\|_{L^2 H^1}^2 + \sum_{\ell=0}^{m-1} \|\partial_t^\ell p\|_{L^\infty H^1}^2 + \|\partial_t^\ell p\|_{L^2 H^2}^2\Big). \tag{4-58}$$

Similarly, for $j = 1, \dots, 2N-1$,

$$\|F^{1,j}(0)\|_{4N-2j-2}^2 + \|F^{3,j}(0)\|_{4N-2j-3/2}^2 \lesssim (1+\mathfrak{E}_0(\eta))\Big(\mathfrak{F}_0 + \sum_{\ell=0}^{j-1} \|\partial_t^\ell u(0)\|_{4N-2\ell}^2 + \|\partial_t^\ell p(0)\|_{4N-2\ell-1}^2\Big). \tag{4-59}$$

Proof. The estimates follow from simple but lengthy computations, invoking standard arguments. For this reason, we present only a sketch of how to derive the estimates (4-56) and (4-58). The estimates (4-57) and (4-59) follow from similar arguments.

To derive the estimate (4-56), we use the definition of $F^{1,j}$, $F^{3,j}$ given by (4-52) and expand all terms using the Leibniz rule and the definition D_t (given in (4-8)) to rewrite $F^{i,j}$ as a sum of products of two terms: one involving products of various derivatives of $\bar{\eta}$, and one linear in derivatives of u, p, F^1, or F^3. For almost every $t \in [0,T]$, we then estimate the norm ($H^{2m-2j+1}$ and $H^{2m-2j+3/2}$, respectively) of the resulting products by using the usual algebraic properties of Sobolev spaces (that is, Lemma A.1) in conjunction with the Sobolev embeddings. The resulting inequalities may then be integrated in time from 0 to T to find an inequality of the form

$$\|F^{1,j}\|^2_{L^2H^{2m-2j+1}} + \|F^{3,j}\|^2_{L^2H^{2m-2j+3/2}} \lesssim \mathfrak{Q}(\bar{\mathfrak{E}}(\eta))(\mathfrak{D}(\eta)Y_\infty + Y_2), \tag{4-60}$$

where $\mathfrak{Q}(\cdot)$ is a polynomial,

$$Y_\infty = \sum_{j=0}^{2N-1} \|\partial_t^j F^1\|^2_{L^\infty H^{4N-2j-2}} + \|\partial_t^j F^3\|^2_{L^\infty H^{4N-2j-3/2}} + \sum_{\ell=0}^{j-1} \|\partial_t^\ell u\|^2_{L^\infty H^{2m-2j+2}} + \|\partial_t^\ell p\|^2_{L^\infty H^{2m-2j+1}},$$

and

$$Y_2 = \sum_{j=0}^{2N-1} \|\partial_t^j F^1\|^2_{L^2H^{4N-2j-1}} + \|\partial_t^{2N} F^1\|^2_{L^2({}_0H^1(\Omega))^*} + \sum_{j=0}^{2N} \|\partial_t^j F^3\|^2_{L^2H^{4N-2j-1/2}} + \sum_{\ell=0}^{j-1} \|\partial_t^\ell u\|^2_{L^2H^{2m-2j+3}} + \|\partial_t^\ell p\|^2_{L^2H^{2m-2j+2}}.$$

Since $\mathfrak{K}(\eta) \le 1$, we know that

$$\mathfrak{Q}(\bar{\mathfrak{E}}(\eta))(1+\mathfrak{D}(\eta)) \lesssim (1+\mathfrak{K}(\eta)),$$

and the bound (4-56) follows immediately from (4-60).

For the estimate (4-58), we first use the trivial bound

$$\left\|\partial_t(F^{1,m} - F^{3,m})\right\|^2_{L^2({}_0H^1(\Omega))^*} \lesssim \|\partial_t F^{1,m}\|_0^2 + \|\partial_t F^{3,m}\|_0^2. \tag{4-61}$$

Then we appeal to (4-52) to note that $\partial_t F^{1,m}$ and $\partial_t F^{3,m}$ involve at most m temporal derivatives of u and p through the appearance of $\mathfrak{G}^1(D_t^m u, \partial_t^m p)$ and $\mathfrak{G}^3(D_t^m u, \partial_t^m p)$. With this observation in hand, we may argue as above to get the bound the right side of (4-61) by the right side of (4-58). □

Next we record an estimate for the difference between $\partial_t v$ and $D_t v$ for a general v. The proof is similar to that of Lemma 4.5, and is thus omitted.

Lemma 4.6. *If $k = 0, \dots, 4N-1$ and v is sufficiently regular, then*

$$\left\|\partial_t v - D_t v\right\|^2_{L^2H^k} \lesssim \bigl(1+\mathfrak{K}(\eta)\bigr)\|v\|^2_{L^2H^k}, \tag{4-62}$$

and if $k = 0, \dots, 4N-2$, then

$$\left\|\partial_t v - D_t v\right\|^2_{L^\infty H^k} \lesssim \bigl(1+\mathfrak{K}(\eta)\bigr)\|v\|^2_{L^\infty H^k}. \tag{4-63}$$

If $m = 1, \dots, 2N-1$, $j = 1, \dots, m$, and v is sufficiently regular, then

$$\left\|\partial_t^j v - D_t^j v\right\|^2_{L^2H^{2m-2j+3}} \lesssim \bigl(1+\mathfrak{K}(\eta)\bigr) \sum_{\ell=0}^{j-1} \bigl(\|\partial_t^\ell v\|^2_{L^2H^{2m-2j+3}} + \|\partial_t^\ell v\|^2_{L^\infty H^{2m-2j+2}}\bigr), \tag{4-64}$$

$$\left\|\partial_t^j v - D_t^j v\right\|^2_{L^\infty H^{2m-2j+2}} \lesssim \bigl(1+\mathfrak{K}(\eta)\bigr) \sum_{\ell=0}^{j-1} \|\partial_t^\ell v\|^2_{L^\infty H^{2m-2j+2}}, \tag{4-65}$$

and

$$\|\partial_t D_t^m v - \partial_t^{m+1} v\|_{L^2H^1}^2 + \|\partial_t^2 D_t^m v - \partial_t^{m+2} v\|_{(\mathscr{X}_T)^*}^2 \lesssim (1+\mathfrak{K}(\eta))\left(\sum_{\ell=0}^{m} \|\partial_t^\ell v\|_{L^2H^1}^2 + \|\partial_t^\ell v\|_{L^\infty H^2}^2 + \|\partial_t^{m+1} u\|_{(\mathscr{X}_T)^*}^2\right). \tag{4-66}$$

Also, if $j=0,\dots,2N$ and v is sufficiently regular, then

$$\|\partial_t^j v(0) - D_t^j v(0)\|_{4N-2j}^2 \lesssim (1+\mathfrak{E}_0(\eta)) \sum_{\ell=0}^{j-1} \|\partial_t^\ell v(0)\|_{4N-2\ell}^2. \tag{4-67}$$

Now we record an estimate for the terms $\mathfrak{G}^0$ and $\mathfrak{f}^i$ (defined in (4-50) and (4-51), respectively) that will be used in computing initial data.

Lemma 4.7. *Suppose that v, q, G^1, G^3 are evaluated at $t=0$ and are sufficiently regular for the right sides of the following estimates to make sense. For $j=0,\dots,2N-1$, we have*

$$\|\mathfrak{G}^0(G^1,v,q)\|_{4N-2j-2}^2 \lesssim (1+\|\eta(0)\|_{4N}^2 + \|\partial_t\eta(0)\|_{4N-1}^2)(\|v\|_{4N-2j}^2 + \|q\|_{4N-2j-1}^2 + \|G^1\|_{4N-2j-2}^2). \tag{4-68}$$

If $j=0,\dots,2N-2$, then

$$\|\mathfrak{f}^1(G^1,v)\|_{4N-2j-3}^2 + \|\mathfrak{f}^2(G^3,v)\|_{4N-2j-3/2}^2 + \|\mathfrak{f}^3(G^1,v)\|_{4N-2j-5/2}^2 \lesssim (1+\|\eta(0)\|_{4N}^2)(\|G^1\|_{4N-2j-2}^2 + \|G^3\|_{4N-2j-3/2}^2 + \|v\|_{4N-2j}^2). \tag{4-69}$$

For $j=2N-1$, if $\operatorname{div}_{\mathscr{A}(0)} v = 0$ in Ω, then

$$\|\mathfrak{f}^2(G^3,v)\|_{1/2}^2 + \|\mathfrak{f}^3(G^1,v)\|_{-1/2}^2 \lesssim (1+\|\eta(0)\|_{4N}^2)(\|G^1\|_2^2 + \|G^3\|_{1/2}^2 + \|v\|_2^2). \tag{4-70}$$

Proof. The proof of the estimates (4-68) and (4-69) as well as the $\mathfrak{f}^2$ estimate in (4-70) can be carried out as in the proof Lemma 4.5. We omit further details. For the $\mathfrak{f}^3$ estimate of (4-70), we note that $\operatorname{div}_{\mathscr{A}(0)} v = 0$ implies that $\operatorname{div}_{\mathscr{A}(0)} \Delta_{\mathscr{A}(0)} v = 0$, so that Lemmas 3.3 and 2.1 provide the bound $\|\Delta_{\mathscr{A}(0)} v\cdot\nu\|_{H^{-1/2}(\Sigma_b)}^2 \lesssim \|\Delta_{\mathscr{A}(0)} v\|_0^2$. We may then argue as in Lemma 4.5 to derive the $\mathfrak{f}^3$ bound. □

Now we assume that $u_0\in H^{4N}(\Omega)$, $\eta_0\in H^{4N+1/2}(\Sigma)$, $\mathfrak{F}_0<\infty$ (see (4-53) for the definition), and that $\|\eta_0\|_{4N-1/2}^2 \le \mathfrak{E}_0(\eta) \le 1$ (defined in (4-55)) is sufficiently small for the hypothesis of Propositions 3.7 and 3.9 to hold when $k=4N$. Note, though, that we do not need $\|\eta_0\|_{4N+1/2}^2$ to be small. We will iteratively construct the initial data $D_t^j u(0)$ for $j=0,\dots,2N$ and $\partial_t^j p(0)$ for $j=0,\dots,2N-1$. To do so, we will first construct all but the highest-order data, and then we will state some compatibility conditions for the data. These are necessary to construct $D_t^{2N}u(0)$ and $\partial_t^{2N-1}p(0)$, and to construct high-regularity solutions in Theorem 4.8.

We now turn to the construction of $D_t^j u(0)$ for $j=0,\dots,2N-1$ and $\partial_t^j p(0)$ for $j=0,\dots,2N-2$, which will employ Lemma 4.7 in conjunction with estimates (4-59) of Lemma 4.5 and (4-67) of Lemma 4.6. For $j=0$, we write $F^{1,0}(0)=F^1(0)\in H^{4N-2}$, $F^{3,0}(0)=F^3(0)\in H^{4N-3/2}$, and $D_t^0 u(0)=u_0\in H^{4N}$.

Suppose now that $F^{1,\ell} \in H^{4N-2\ell-2}$, $F^{3,\ell} \in H^{4N-2\ell-3/2}$, and $D_t^\ell u(0) \in H^{4N-2\ell}$ are given for $0 \le \ell \le j \in [0, 2N-2]$; we will define $\partial_t^j p(0) \in H^{4N-2j-1}$ as well as $D_t^{j+1} u(0) \in H^{4N-2j-2}$, $F^{1,j+1}(0) \in H^{4N-2j-4}$, and $F^{3,j+1}(0) \in H^{4N-2j-7/2}$, which allows us to define all of said data via iteration. By virtue of estimate (4-69), we know that

$$\begin{aligned} f^1 &= \mathfrak{f}^1(F^{1,j}(0), D_t^j u(0)) \in H^{4N-2j-3}, \\ f^2 &= \mathfrak{f}^2(F^{3,j}(0), D_t^j u(0)) \in H^{4N-2j-3/2}, \\ f^3 &= \mathfrak{f}^3(F^{1,j}(0), D_t^j u(0)) \in H^{4N-2j-5/2}. \end{aligned}$$

This allows us to define $\partial_t^j p(0)$ as the solution to (3-30) with this choice of f^1, f^2, f^3, and then Proposition 3.9 with $k = 4N$ and $r = 4N-2j-1 < k$ implies that $\partial_t^j p(0) \in H^{4N-2j-1}$. Now the estimates (4-59), (4-67), and (4-68) allow us to define

$$\begin{aligned} D_t^{j+1} u(0) &:= \mathfrak{G}^0\big(F^{1,j}(0), D_t^j u(0), \partial_t^j p(0)\big) \in H^{4N-2j-2}, \\ F^{1,j+1}(0) &:= D_t F^{1,j}(0) + \mathfrak{G}^1\big(D_t^j u(0), \partial_t^j p(0)\big) \in H^{4N-2j-4}, \\ F^{3,j+1}(0) &:= \partial_t F^{3,j}(0) + \mathfrak{G}^3\big(D_t^j u(0), \partial_t^j p(0)\big) \in H^{4N-2j-7/2}. \end{aligned}$$

Using this analysis, we iteratively construct all of the desired data except for $D_t^{2N} u(0)$ and $\partial_t^{2N-1} p(0)$.

By construction, the initial data $D_t^j u(0)$ and $\partial_t^j p(0)$ are determined in terms of u_0 as well as $\partial_t^\ell F^1(0)$ and $\partial_t^\ell F^3(0)$ for $\ell = 0, \dots, 2N-1$. In order to use these in Theorem 4.3 and to construct $D_t^{2N} u(0)$ and $\partial_t^{2N-1} p(0)$, we must enforce compatibility conditions for $j = 0, \dots, 2N-1$. For such j, we say that the j-th compatibility condition is satisfied if

$$\begin{cases} D_t^j u(0) \in \mathscr{X}(0) \cap H^2(\Omega), \\ \Pi_0\big(F^{3,j}(0) + \mathbb{D}_{\mathscr{A}_0} D_t^j u(0) \mathscr{N}_0\big) = 0. \end{cases} \tag{4-71}$$

The construction of $D_t^j u(0)$ and $\partial_t^j p(0)$ ensures that $D_t^j u(0) \in H^2(\Omega)$ and $\operatorname{div}_{\mathscr{A}_0}(D_t^j u(0)) = 0$, so the condition $D_t^j u(0) \in \mathscr{X}(0) \cap H^2(\Omega)$ may be reduced to the condition $D_t^j u(0)|_{\Sigma_b} = 0$.

It remains only to define $\partial_t^{2N-1} p(0) \in H^1$ and $D_t^{2N} u(0) \in H^0$. According to the $j = 2N-1$ compatibility condition (4-71), $\operatorname{div}_{\mathscr{A}_0} D_t^{2N-1} u(0) = 0$, which means that we can use estimate (4-70) of Lemma 4.7 to see that $f^2 = \mathfrak{f}^2(F^{3,2N-1}(0), D_t^{2N-1} u(0)) \in H^{1/2}$ and $f^3 = \mathfrak{f}^3(F^{1,2N-1}(0), D_t^{2N-1} u(0)) \in H^{-1/2}$. We also see from (4-71) that if we define the quantity $g_0 = -\operatorname{div}_{\mathscr{A}_0}(R(0) D_t^{2N-1} u(0))$, then $g_0 \in H^0$. Then, owing to the fact that $G = -F^{1,2N-1} \in H^0$, we can define $\partial_t^{2N-1} p(0) \in H^1$ as a weak solution to (3-30) in the sense of (3-33) with this choice of f^2, f^3, g_0, and G. Then we define

$$D_t^{2N} u(0) = \mathfrak{G}^0\big(F^{1,2N-1}(0), D_t^{2N-1} u(0), \partial_t^{2N-1} p(0)\big) \in H^0,$$

employing (4-68) for the inclusion in H^0. In fact, the construction of $\partial_t^{2N-1} p(0)$ guarantees that $D_t^{2N} u(0) \in \mathscr{Y}(0)$. Besides providing the inclusions above, the bounds (4-59), (4-69), (4-68) also imply the estimate

$$\sum_{j=0}^{2N} \| D_t^j u(0) \|_{4N-2j}^2 + \sum_{j=0}^{2N-1} \| \partial_t^j p(0) \|_{4N-2j-1}^2 \lesssim (1 + \mathfrak{E}_0(\eta)\big(\|u_0\|_{4N}^2 + \mathfrak{F}_0\big). \tag{4-72}$$

Owing to estimate (4-67), the bound (4-72) also holds, with $\partial_t^j u(0)$ replacing $D_t^j u(0)$ on the left.

Before stating our result on higher regularity for solutions to problem (1-7), we define two quantities associated to (u, p). Write

$$\begin{aligned} \mathfrak{D}(u, p) &:= \sum_{j=0}^{2N} \|\partial_t^j u\|^2_{L^2 H^{4N-2j+1}} + \|\partial_t^{2N+1} u\|^2_{(\mathscr{X}_T)^*} + \sum_{j=0}^{2N-1} \|\partial_t^j p\|^2_{L^2 H^{4N-2j}}, \\ \mathfrak{E}(u, p) &:= \sum_{j=0}^{2N} \|\partial_t^j u\|^2_{L^\infty H^{4N-2j}} + \sum_{j=0}^{2N-1} \|\partial_t^j p\|^2_{L^\infty H^{4N-2j-1}}, \\ \mathfrak{K}(u, p) &:= \mathfrak{E}(u, p) + \mathfrak{D}(u, p). \end{aligned} \tag{4-73}$$

Theorem 4.8. *Suppose that $u_0 \in H^{4N}(\Omega)$, $\eta_0 \in H^{4N+1/2}(\Sigma)$, $\mathfrak{F} < \infty$, and that $\mathfrak{K}(\eta) \le 1$ is sufficiently small that $\mathscr{K}(\eta)$, defined by (4-9), satisfies the hypotheses of Theorem 4.3 and Proposition 3.9. Let $D_t^j u(0) \in H^{4N-2j}(\Omega)$ and $\partial_t^j p(0) \in H^{4N-2j-1}$, for $j = 0, \dots, 2N-1$ along with $D_t^{2N} u(0) \in \mathscr{Y}(0)$, all be determined as above in terms of u_0 and $\partial_t^j F^1(0)$, $\partial_t^j F^3(0)$ for $j = 0, \dots, 2N-1$. Suppose that for $j = 0, \dots, 2N-1$, the initial data satisfy the j-th compatibility condition (4-71).*

There exists a universal constant $T_0 > 0$ such that if $0 < T \le T_0$, then there exists a unique strong solution (u, p) to (1-7) on $[0, T]$ such that

$$\begin{aligned} &\partial_t^j u \in C^0([0, T]; H^{4N-2j}(\Omega)) \cap L^2([0, T]; H^{4N-2j+1}(\Omega)) \quad \text{for } j = 0, \dots, 2N, \\ &\partial_t^j p \in C^0([0, T]; H^{4N-2j-1}(\Omega)) \cap L^2([0, T]; H^{4N-2j}(\Omega)) \quad \text{for } j = 0, \dots, 2N-1, \\ &\partial_t^{2N+1} u \in (\mathscr{X}_T)^*. \end{aligned}$$

The pair $(D_t^j u, \partial_t^j p)$ satisfies the PDE

$$\begin{cases} \partial_t (D_t^j u) - \Delta_{\mathscr{A}} (D_t^j u) + \nabla_{\mathscr{A}} (\partial_t^j p) = F^{1,j} & \text{in } \Omega, \\ \operatorname{div}_{\mathscr{A}} (D_t^j u) = 0 & \text{in } \Omega, \\ S_{\mathscr{A}} (\partial_t^j p, D_t^j u) \mathscr{N} = F^{3,j} & \text{on } \Sigma, \\ D_t^j u = 0 & \text{on } \Sigma_b, \end{cases} \tag{4-74}$$

in the strong sense with initial data $(D_t^j u(0), \partial_t^j p(0))$ for $j = 0, \dots, 2N-1$, and in the weak sense of (4-2) with initial data $D_t^{2N} u(0) \in \mathscr{Y}(0)$ for $j = 2N$. Here the vectors $F^{1,j}$ and $F^{3,j}$ are as defined by (4-52). Moreover, the solution satisfies the estimate

$$\mathfrak{E}(u, p) + \mathfrak{D}(u, p) \lesssim (1 + \mathfrak{E}_0(\eta) + \mathfrak{K}(\eta)) \exp(C(1 + \mathfrak{E}(\eta))T) (\|u_0\|^2_{4N} + \mathfrak{F}_0 + \mathfrak{F}) \tag{4-75}$$

for a constant $C > 0$, independent of η.

Proof. For notational convenience, throughout the proof we write

$$\mathscr{Z} := (1 + \mathfrak{E}_0(\eta) + \mathfrak{K}(\eta)) \exp(C(1 + \mathfrak{E}(\eta))T) (\|u_0\|^2_{4N} + \mathfrak{F}_0 + \mathfrak{F}).$$

Since the 0-th order compatibility condition (4-71) is satisfied and $\mathfrak{K}(\eta)$ is small enough for $\mathscr{K}(\eta)$ to satisfy the hypotheses of Theorem 4.3, we may apply Theorem 4.3. It guarantees the existence of (u, p)

satisfying the inclusions (4-12). The $(D_t^j u, \partial_t^j p)$ are solutions in that (4-74) is satisfied in the strong sense when $j = 0$ and in the weak sense when $j = 1$. Finally, the estimate (4-13) holds, but we may replace its right-hand side by $\mathfrak{X}$ since $\mathcal{K}(\eta) \le \mathfrak{E}(\eta) \le \mathfrak{K}(\eta)$.

For an integer $m \ge 0$, let $\mathbb{P}_m$ denote the proposition asserting the following three statements. First, that $(D_t^j u, \partial_t^j p)$ are solutions to (4-74) in the strong sense for $j = 0, \dots, m$ and in the weak sense for $j = m+1$. Second, that

$$\partial_t^j u \in L^\infty H^{2m-2j+2} \cap L^2 H^{2m-2j+3}$$

for $j = 0, 1, \dots, m+1$, $\partial_t^{m+2} u \in (\mathcal{X}_T)^*$, and

$$\partial_t^j p \in L^\infty H^{2m-2j+1} \cap L^2 H^{2m-2j+2}$$

for $j = 0, 1, \dots, m$. Third, that the estimate

$$\sum_{j=0}^{m+1} \|\partial_t^j u\|_{L^\infty H^{2m-2j+2}}^2 + \|\partial_t^j u\|_{L^2 H^{2m-2j+3}}^2 + \|\partial_t^{m+2} u\|_{(\mathcal{X}_T)^*}^2 + \sum_{j=0}^{m} \|\partial_t^j p\|_{L^\infty H^{2m-2j+1}}^2 + \|\partial_t^j p\|_{L^2 H^{2m-2j+2}}^2 \lesssim \mathfrak{X} \tag{4-76}$$

holds.

The above analysis implies that $\mathbb{P}_0$ holds. We claim that if $\mathbb{P}_m$ holds for some $m = 0, \dots, 2N-2$, then $\mathbb{P}_{m+1}$ also holds. Once the claim is established, a finite induction implies that $\mathbb{P}_m$ holds for all $m = 0, \dots, 2N-1$, which immediately implies all of the conclusions of the theorem. The rest of the proof, which we divide into two steps, is dedicated to the proof of this claim.

Step 1: Applying Theorem 4.3. Suppose that $\mathbb{P}_m$ holds for some $m = 0, \dots, 2N-2$. In order to prove that the first assertion of $\mathbb{P}_{m+1}$ holds, we would like employ Theorem 4.3 to solve problem (1-7), with F^1, F^3 replaced by $F^{1,m+1}$, $F^{3,m+1}$ and with initial data $D_t^{m+1} u(0)$. In order to do so, we must verify three things. First, that the compatibility condition (4-11) is satisfied. This is guaranteed by the fact that $D_t^{m+1} u(0)$ satisfies the $(m+1)$-st order compatibility condition (4-71). Second, we need that $F^{1,m+1} \in L^2 H^1$ and $F^{3,m+1} \in L^2 H^{3/2}$. This follows directly from the estimate (4-56) in Lemma 4.5 and the bound (4-76) provided by $\mathbb{P}_m$. Third, we need that $\partial_t (F^{1,m+1} - F^{3,m+1}) \in L^2({}_0H^1(\Omega))^*$. Appealing to (4-58) in Lemma 4.5, we encounter an obstacle, namely that we can use $\mathbb{P}_m$ to control every term on the right-hand side except for $\|\partial_t^{m+1} u\|_{L^2 H^2}^2 + \|\partial_t^{m+1} p\|_{L^2 H^1}^2$. However, we may trivially estimate

$$\|\partial_t^{m+1} u\|_{L^2 H^2}^2 + \|\partial_t^{m+1} p\|_{L^2 H^1}^2 \le T \big(\|\partial_t^{m+1} u\|_{L^\infty H^2}^2 \|\partial_t^{m+1} p\|_{L^\infty H^1}^2 \big)$$

and note that the term on the right would be controlled via (4-13) by formally applying Theorem 4.3 with forcing terms $F^{1,m+1}$, $F^{3,m+1}$. This suggests that we may employ an iteration argument in conjunction with a small T assumption to get around our obstacle, and indeed this strategy works. Such an iteration argument is fairly standard, so we will only provide a sketch.

First we consider an arbitrary pair (v, q) of sufficient regularity to make sense of

$$F^{1,m+1} = F^{1,m+2}(v, q) \quad \text{and} \quad F^{3,m+1} = F^{1,m+2}(v, q)$$

via (4-52). Note that the forcing terms depend linearly on (v, q). From (4-56) and (4-58) of Lemma 4.5, we have that

$$\|F^{1,m+1}(v,q)\|^2_{L^2H^1} + \|F^{3,m+1}(v,q)\|^2_{L^2H^{3/2}} \lesssim (1+\mathfrak{K}(\eta))\Big(\mathfrak{F} + \sum_{\ell=0}^{m} \|\partial_t^\ell v\|^2_{L^2H^3} + \sum_{\ell=0}^{m} \|\partial_t^\ell v\|^2_{L^\infty H^2} + \|\partial_t^\ell q\|^2_{L^2H^2} + \|\partial_t^\ell q\|^2_{L^\infty H^1}\Big), \tag{4-77}$$

$$\|F^{1,m+1}(v,q)\|^2_{L^\infty H^0} + \|F^{3,m+1}(v,q)\|^2_{L^\infty H^{1/2}} \lesssim (1+\mathfrak{K}(\eta))\Big(\mathfrak{F} + \sum_{\ell=0}^{m} \|\partial_t^\ell v\|^2_{L^\infty H^2} + \sum_{\ell=0}^{m} \|\partial_t^\ell q\|^2_{L^\infty H^1}\Big), \tag{4-78}$$

and

$$\begin{aligned}\|\partial_t(F^{1,m+1}(v,q) - F^{3,m+1}(v,q))\|^2_{L^2({}_0H^1(\Omega))^*} &\lesssim (1+\mathfrak{K}(\eta))\Big(\mathfrak{F} + \sum_{\ell=0}^{m} \|\partial_t^\ell v\|^2_{L^\infty H^2} + \|\partial_t^\ell v\|^2_{L^2H^3} \\ &\quad + \|\partial_t^{m+1} v\|^2_{L^2H^2} + \|\partial_t^{m+1} q\|^2_{L^2H^1} + \sum_{\ell=0}^{m} \|\partial_t^\ell q\|^2_{L^\infty H^1} + \|\partial_t^\ell q\|^2_{L^2H^2}\Big).\end{aligned} \tag{4-79}$$

Now we let u^0 be the extension of the initial data $\partial_t^j u(0)$, $j = 1, \dots, 2N$, given by Lemma A.5, and we similarly let p^0 be the extension of $\partial_t^j p(0)$, $j = 1, \dots, 2N-1$, given by Lemma A.6; by (4-72) and the estimates given in the lemmas, they satisfy

$$\begin{aligned}\sum_{j=0}^{2N} \|\partial_t^j u^0\|^2_{L^2H^{4N-2j+1}} + \|\partial_t^j u^0\|^2_{L^\infty H^{4N-2j}} + \sum_{j=0}^{2N-1} \|\partial_t^j p^0\|^2_{L^2H^{4N-2j}} + \|\partial_t^j p^0\|^2_{L^\infty H^{4N-2j-1}} \\ \lesssim \sum_{j=0}^{2N} \|\partial_t^j u(0)\|^2_{4N-2j} + \sum_{j=0}^{2N-1} \|\partial_t^j p(0)\|^2_{4N-2j-1} \lesssim (1+\mathfrak{E}_0(\eta))(\|u_0\|^2_{4N} + \mathfrak{F}_0).\end{aligned} \tag{4-80}$$

By combining (4-77)–(4-80), we find that $F^{1,m+1}(u^0, p^0)$ and $F^{3,m+1}(u^0, p^0)$ satisfy (4-7). Also, the compatibility condition (4-11) with F^3 replaced by $F^{3,m+1}(u^0, p^0)$ and u_0 replaced by $D_t^{m+1}u(0)$ is satisfied by virtue of (4-71) since u^0 and p^0 achieve the initial data. We are then free to apply Theorem 4.3 to find (v^1, q^1) satisfying the conclusions of the theorem. In particular, if we abbreviate (1-7) as $\mathscr{L}(v,q) = \mathbb{F} = (F^1, F^3)$, then

$$\begin{aligned}\mathscr{L}(v^1, q^1) = \mathbb{F}^{m+1}(u^0, p^0) &:= \big(F^{1,m+1}(u^0, p^0), F^{3,m+2}(u^0, p^0)\big), \\ v^1(0) = D_t^{m+1} u(0), &\quad q^1(0) = \partial_t^{m+1} p(0).\end{aligned}$$

Let us write $\mathfrak{B}(u, p)$ for the left-hand side of (4-13). Then (4-13), (4-59), (4-77), (4-79), and (4-80) imply that

$$\mathfrak{B}(v^1, q^1) \lesssim (1 + \mathfrak{E}_0(\eta) + \mathfrak{K}(\eta)) \exp\big(C(1+\mathfrak{E}(\eta))T\big)\big(\|u_0\|^2_{4N} + \mathfrak{F}_0 + \mathfrak{F}\big) \lesssim \mathfrak{X}.$$

Now, given a pair (v^n, q^n) satisfying $\mathfrak{B}(v^n, q^n) < \infty$, we define a corresponding pair (u^n, p^n) by solving the linear ODEs

$$\begin{cases} D_t^{m+1} u^n = v^n, \\ \partial_t^j u^n(0) = \partial_t^j u(0) \quad \text{for } j = 0, \dots, m, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t^{m+1} p^n = q^n, \\ \partial_t^j p^n(0) = \partial_t^j p(0) \quad \text{for } j = 0, \dots, m. \end{cases} \tag{4-81}$$

Such solutions exist and are unique. Let us define $\mathfrak{R}(v, q)$ by

$$\mathfrak{R}(v, q) = \|\partial_t^{m+1} v\|_{L^2 H^2}^2 + \|\partial_t^{m+1} q\|_{L^2 H^1}^2 + \sum_{\ell=0}^m \|\partial_t^\ell v\|_{L^2 H^3}^2 + \|\partial_t^\ell v\|_{L^\infty H^2}^2 + \|\partial_t^\ell q\|_{L^2 H^2}^2 + \|\partial_t^\ell q\|_{L^\infty H^1}^2.$$

Then the solutions satisfy the estimate

$$\mathfrak{R}(u^n, p^n) \lesssim p(T)\big(1 + \mathfrak{K}(\eta)\big)\left(\sum_{j=0}^m \|\partial_t^j u(0)\|_3^2 + \|\partial_t^j p(0)\|_2^2 + T\mathfrak{B}(v^n, q^n) \right), \tag{4-82}$$

where $p(T)$ is a polynomial in T. Note that the data norm terms on the right side of (4-82) are finite because $m \le 2N - 2$.

We iteratively apply Theorem 4.3 to produce sequences $\{(v^n, q^n)\}_{n=1}^\infty$ and $\{(u^n, p^n)\}_{n=1}^\infty$ satisfying

$$\begin{aligned} \mathscr{L}(v^n, q^n) &= \mathbb{F}^{m+1}(u^{n-1}, p^{n-1}), \\ v^n(0) = D_t^{m+1} u(0), &\quad q^n(0) = \partial_t^{m+1} p(0) \end{aligned} \tag{4-83}$$

and (4-81). Then

$$\begin{aligned} \mathscr{L}(v^{n+1} - v^n, q^{n+1} - q^n) &= \mathbb{F}^{m+1}(u^n - u^{n-1}, p^n - p^{n-1}) \\ (v^{n+1} - v^n)(0) = 0, &\quad (q^{n+1} - q^n)(0) = 0. \end{aligned}$$

Notice that the terms involving F^1 and F^3 cancel in $\mathbb{F}^{m+1}(u^n - u^{n-1}, p^n - p^{n-1})$, so from (4-77) and (4-79), we have that

$$\begin{aligned} &\|F^{1,m+1}(u^n - u^{n-1}, p^n - p^{n-1})\|_{L^2 H^1}^2 + \|F^{3,j}(u^n - u^{n-1}, p^n - p^{n-1})\|_{L^2 H^{3/2}}^2 \\ &\quad + \|\partial_t (F^{1,m+1}(u^n - u^{n-1}, p^n - p^{n-1}) - F^{3,m+1}(u^n - u^{n-1}, p^n - p^{n-1}))\|_{L^2({}_0H^1(\Omega))^*}^2 \\ &\qquad \lesssim \big(1 + \mathfrak{K}(\eta)\big)\mathfrak{R}\big(u^n - u^{n-1}, p^n - p^{n-1}\big). \end{aligned}$$

On the other hand, since every (u^n, p^n) satisfies the same initial conditions, a simple modification of (4-82) implies that

$$\mathfrak{R}(u^n - u^{n-1}, p^n - p^{n-1}) \lesssim (1 + \mathfrak{K}(\eta)) T p(T) \mathfrak{B}(v^n - v^{n-1}, q^n - q^{n-1}).$$

These two estimates, together with the estimate (4-13) of Theorem 4.3, then imply that

$$\begin{aligned} &\mathfrak{B}(v^{n+1} - v^n, q^{n+1} - q^n) \\ &\quad \lesssim \big(1 + \mathfrak{E}_0(\eta) + \mathfrak{K}(\eta)\big) \exp\big(C(1 + \mathfrak{E}(\eta))T\big) T p(T) \mathfrak{B}(v^n - v^{n-1}, q^n - q^{n-1}). \end{aligned} \tag{4-84}$$

Then from (4-84), we find that there exists a universal $T_0 > 0$ such that if $T \le T_0$, then the sequence $\{(v^n, q^n)\}_{n=1}^{\infty}$ converges to (v, q) in the norm $\sqrt{\mathfrak{B}(\,\cdot\,,\,\cdot\,)}$, which in turn implies that $\{(u^n, p^n)\}_{n=1}^{\infty}$ converges to (u, p) in the norm $\sqrt{\mathfrak{R}(\,\cdot\,,\,\cdot\,)}$.

Passing to the limit in (4-81) reveals that $v = D_t^{m+1} u$ and $q = \partial_t^{m+1} p$. We then pass to the limit in (4-83) to see that

$$\mathscr{L}(D_t^{m+1} u, \partial_t^{m+1} p) = \mathbb{F}^{m+1}(u, p).$$

Since $\mathbb{P}_m$ already provides that $(D_t^j u, \partial_t^j p)$ are solutions to (4-74) in the strong sense for $j = 0, \dots, m$, we deduce that the first assertion of $\mathbb{P}_{m+1}$ holds.

Theorem 4.3, together with the estimates (4-77), (4-79), and (4-76), then provides us with the estimate

$$\mathfrak{B}(D_t^{m+1} u, \partial_t^{m+1} p) \lesssim \bigl(1 + \mathfrak{E}_0(\eta) + \mathfrak{K}(\eta)\bigr) \exp\bigl(C(1 + \mathfrak{E}(\eta))T\bigr) \times \bigl(\|u_0\|_{4N}^2 + \mathfrak{F}_0 + \mathfrak{F} + \mathfrak{X} + \|\partial_t^{m+1} u\|_{L^2 H^2}^2 + \|\partial_t^{m+1} p\|_{L^2 H^1}^2\bigr). \tag{4-85}$$

On the other hand, the estimate (4-65) of Lemma 4.6 implies that

$$\begin{aligned}
\|\partial_t^{m+1} u\|_{L^2 H^2}^2 + \|\partial_t^{m+1} p\|_{L^2 H^1}^2 &\le T\bigl(\|\partial_t^{m+1} u\|_{L^\infty H^2}^2 + \|\partial_t^{m+1} p\|_{L^\infty H^1}^2\bigr) \\
&\lesssim T\bigl(\bigl\|\partial_t^{m+1} u - D_t^{m+1} u\bigr\|_{L^\infty H^2}^2 + \|D_t^{m+1} u\|_{L^\infty H^2}^2 + \|\partial_t^{m+1} p\|_{L^\infty H^1}^2\bigr) \\
&\lesssim T\Bigl(\bigl(1 + \mathfrak{K}(\eta)\bigr) \sum_{\ell=0}^{m} \|\partial_t^\ell u\|_{L^\infty H^2}^2 + \mathfrak{B}\bigl(D_t^{m+1} u, \partial_t^{m+1} p\bigr)\Bigr) \\
&\lesssim T\bigl(\mathfrak{X} + \mathfrak{B}(D_t^{m+1} u, \partial_t^{m+1} p)\bigr),
\end{aligned} \tag{4-86}$$

where in the last inequality we have again used (4-76). Chaining together (4-85) and (4-86), we find that we may further restrict the size of the universal constant $T_0 > 0$ such that if $T \le T_0$, then

$$\mathfrak{B}\bigl(D_t^{m+1} u, \partial_t^{m+1} p\bigr) \lesssim \bigl(1 + \mathfrak{E}_0(\eta) + \mathfrak{K}(\eta)\bigr) \exp\bigl(C(1 + \mathfrak{E}(\eta))T\bigr)\bigl(\|u_0\|_{4N}^2 + \mathfrak{F}_0 + \mathfrak{F} + \mathfrak{X}\bigr) \lesssim \mathfrak{X}. \tag{4-87}$$

Step 2: Proving the second and third assertions. It remains to prove the second and third assertions of $\mathbb{P}_{m+1}$; they are intertwined and will be derived simultaneously. The estimates of the u terms in (4-87), together with the estimates (4-64)–(4-66) of Lemma 4.6 and the estimate (4-76), imply that

$$\begin{aligned}
&\|\partial_t^{m+1} u\|_{L^2 H^3}^2 + \|\partial_t^{m+2} u\|_{L^2 H^1}^2 + \|\partial_t^{m+3} u\|_{(\mathscr{X}_T)^*}^2 + \|\partial_t^{m+1} u\|_{L^\infty H^2}^2 + \|\partial_t^{m+2} u\|_{L^\infty H^0}^2 \\
&\qquad \lesssim \bigl(1 + \mathfrak{K}(\eta)\bigr)\Biggl(\sum_{\ell=0}^{m+2} \|\partial_t^\ell u\|_{L^2 H^{2m-2\ell+3}}^2 + \sum_{\ell=0}^{m+1} \|\partial_t^\ell u\|_{L^\infty H^{2m-2\ell+2}}^2\Biggr) + \mathfrak{X} \\
&\qquad \lesssim \bigl(1 + \mathfrak{K}(\eta)\bigr)\mathfrak{X} + \mathfrak{X} \lesssim \mathfrak{X}.
\end{aligned} \tag{4-88}$$

Hence

$$\sum_{j=m+1}^{m+2} \|\partial_t^j u\|_{L^\infty H^{2(m+1)-2j+2}}^2 + \|\partial_t^j u\|_{L^2 H^{2(m+1)-2j+3}}^2 + \|\partial_t^{m+3} u\|_{(\mathscr{X}_T)^*}^2 + \sum_{j=m+1}^{m+1} \|\partial_t^j p\|_{L^\infty H^{2(m+1)-2j+1}}^2 + \|\partial_t^j p\|_{L^2 H^{2(m+1)-2j+2}}^2 \lesssim \mathfrak{X}, \tag{4-89}$$

Thus, in order to derive the estimate (4-76) with m replaced by $m+1$, it suffices to prove that

$$\sum_{j=0}^{m} \|\partial_t^j u\|^2_{L^\infty H^{2(m+1)-2j+2}} + \|\partial_t^j p\|^2_{L^\infty H^{2(m+1)-2j+1}} + \sum_{j=0}^{m} \|\partial_t^j u\|^2_{L^2 H^{2(m+1)-2j+3}} + \|\partial_t^j p\|^2_{L^2 H^{2(m+1)-2j+2}} \lesssim \mathfrak{X}. \tag{4-90}$$

Once (4-90) is established, summing (4-89) and (4-90) implies that (4-76) holds with m replaced by $m+1$, which further implies that the second and third assertions of $\mathbb{P}_{m+1}$ hold, so that then all of $\mathbb{P}_{m+1}$ holds.

In order to prove (4-90), we will use the elliptic regularity of Proposition 3.7 (with $k = 4N$) and an iteration argument. As the first step, we must record estimates for the forcing terms. For these, we combine (4-76) with the estimates (4-56) and (4-57) of Lemma 4.5 to see that

$$\begin{aligned}
&\sum_{j=1}^{m+1}\Big(\|F^{1,j}\|^2_{L^2 H^{2m-2j+3}} + \|F^{3,j}\|^2_{L^2 H^{2m-2j+7/2}} + \|F^{1,j}\|^2_{L^\infty H^{2m-2j+2}} + \|F^{3,j}\|^2_{L^\infty H^{2m-2j+5/2}}\Big)\\
&\lesssim (1+\mathfrak{K}(\eta))\Big(\mathfrak{F} + \sum_{\ell=0}^{m} \|\partial_t^\ell u\|^2_{L^\infty H^{2m-2\ell+2}} + \|\partial_t^\ell u\|^2_{L^2 H^{2m-2\ell+3}} + \sum_{\ell=0}^{m} \|\partial_t^\ell p\|^2_{L^\infty H^{2m-2\ell+1}} + \|\partial_t^\ell p\|^2_{L^2 H^{2m-2\ell+2}}\Big)\\
&\lesssim (1+\mathfrak{K}(\eta))(\mathfrak{F}+\mathfrak{X}) \lesssim \mathfrak{X}.
\end{aligned} \tag{4-91}$$

The last inequality in (4-91) follows from the fact that $\mathfrak{K}(\eta) \le 1$ and the definition of $\mathfrak{X}$.

The estimates of $D_t^{m+1}u$ in (4-87), together with (4-76) and the estimates (4-62) and (4-63) of Lemma 4.6, allow us to deduce that

$$\|\partial_t D_t^m u\|^2_{L^\infty H^2} + \|\partial_t D_t^m u\|^2_{L^2 H^3} \lesssim \mathfrak{X}. \tag{4-92}$$

Since (4-74) is satisfied in the strong sense for $j = m$, we may rearrange to find that for almost every $t \in [0, T]$, $(D_t^m, \partial_t^m p)$ solve the elliptic problem (3-6) with F^1 replaced by $F^{1,m} - \partial_t D_t^m u$, $F^2 = 0$, and F^3 replaced by $F^{3,m}$. We may then apply Proposition 3.7 with $r = 5$ to deduce that the estimate (3-18) holds for almost every $t \in [0, T]$; squaring this estimate and integrating over $[0, T]$ then yields the inequality

$$\begin{aligned}
\|D_t^m u\|^2_{L^2 H^5} + \|\partial_t^m p\|^2_{L^2 H^4} &\lesssim \big\|F^{1,m} - \partial_t D_t^m u\big\|^2_{L^2 H^3} + \|F^{3,m}\|^2_{L^2 H^{7/2}}\\
&\lesssim \|F^{1,m}\|^2_{L^2 H^3} + \|\partial_t D_t^m u\|^2_{L^2 H^3} + \|F^{3,m}\|^2_{L^2 H^{7/2}} \lesssim \mathfrak{X},
\end{aligned} \tag{4-93}$$

where in the last inequality we have used (4-91) and (4-92). Similarly, we may apply Proposition 3.7 with $r = 4$ to deduce

$$\|D_t^m u\|^2_{L^\infty H^4} + \|\partial_t^m p\|^2_{L^\infty H^3} \lesssim \big\|F^{1,m} - \partial_t D_t^m u\big\|^2_{L^\infty H^2} + \|F^{3,m}\|^2_{L^\infty H^{5/2}} \lesssim \mathfrak{X}. \tag{4-94}$$

We may argue as before to deduce from (4-93) and (4-94) that

$$\|\partial_t^m u\|^2_{L^\infty H^4} + \|\partial_t^m u\|^2_{L^2 H^5} \lesssim \mathfrak{X}$$

as well. This argument may be iterated to estimate $\partial_t^j u$, $\partial_t^j p$ for $j = 1, \dots, m$; this yields the estimate

$$\sum_{j=1}^{m} \|\partial_t^j u\|^2_{L^\infty H^{2(m+1)-2j+2}} + \|\partial_t^j p\|^2_{L^\infty H^{2(m+1)-2j+1}} + \sum_{j=1}^{m} \|\partial_t^j u\|^2_{L^2 H^{2(m+1)-2j+3}} + \|\partial_t^j p\|^2_{L^2 H^{2(m+1)-2j+2}} \lesssim \mathfrak{X}. \tag{4-95}$$

We then apply Proposition 3.7 with $r = 2(m+1)+2 \le 4N$ to see that

$$\begin{aligned} \|u\|^2_{L^\infty H^{2(m+1)+2}} + \|p\|^2_{L^\infty H^{2(m+1)+1}} &\lesssim \|F^1 - \partial_t u\|^2_{L^\infty H^{2(m+1)}} + \|F^3\|^2_{L^\infty H^{2(m+1)+1/2}} \\ &\lesssim \|F^1\|^2_{L^\infty H^{2(m+1)}} + \|\partial_t u\|^2_{L^\infty H^{2(m+1)}} + \|F^3\|^2_{L^\infty H^{2(m+1)+1/2}} \lesssim \mathfrak{X}, \end{aligned} \tag{4-96}$$

and then again with $r = 2(m+1)+3 \le 4N+1$ to see that

$$\begin{aligned} \|u\|^2_{L^2 H^{2(m+1)+3}} + \|p\|^2_{L^2 H^{2(m+1)+2}} &\lesssim \|F^1 - \partial_t u\|^2_{L^2 H^{2(m+1)+1}} + \|F^3\|^2_{L^2 H^{2(m+1)+3/2}} + \|\eta\|^2_{L^2 H^{4N+1/2}} \big(\|F^1 - \partial_t u\|^2_{L^\infty H^2} + \|F^3\|^2_{L^\infty H^{5/2}} \big) \\ &\lesssim \|F^1\|^2_{L^2 H^{2(m+1)+1}} + \|\partial_t u\|^2_{L^2 H^{2(m+1)+1}} + \|F^3\|^2_{L^2 H^{2(m+1)+3/2}} + \mathfrak{K}(\eta)(\mathfrak{F} + \mathfrak{X}) \lesssim \mathfrak{X}. \end{aligned} \tag{4-97}$$

Summing (4-95)–(4-97) then gives (4-90), completing the proof. □

5. Preliminaries for the nonlinear problem

Forcing estimates. We want to eventually use our linear theory for the problem (1-7) in order to solve the nonlinear problem (1-4). To do so, we define forcing terms F^1, F^3 to be used in the linear theory that match the terms in (1-4). That is, given u, η, we define

$$F^1(u, \eta) = \partial_t \bar{\eta} \tilde{b} K \partial_3 u - u \cdot \nabla_{\mathcal{A}} u \quad \text{and} \quad F^3(u, \eta) = \eta \mathcal{N} = -\eta D\eta + \eta e_3, \tag{5-1}$$

where $\mathcal{A}$, $\mathcal{N}$, K are determined as usual by η.

We will need to be able to estimate various norms of $F^1(u, \eta)$ and $F^3(u, \eta)$ in terms of the norms of u and η that appear in $\mathfrak{K}(\eta)$, $\mathfrak{E}_0(\eta)$, and $\mathfrak{K}(u, p)$, defined by (4-54), (4-55), and (4-73), respectively. The norms of the F^i terms are contained in $\mathfrak{F}$ and $\mathfrak{F}_0$, as defined by (4-53). We will actually need a slight modification of $\mathfrak{K}(u, p)$, which we define as

$$\mathfrak{K}_{2N}(u) = \sum_{j=0}^{2N} \|\partial_t^j u\|^2_{L^2 H^{4N-2j+1}} + \|\partial_t^j u\|^2_{L^\infty H^{4N-2j}}. \tag{5-2}$$

Our estimates are the content of the following lemma.

Lemma 5.1. *Suppose that $\mathfrak{K}(\eta) \le 1$ and $\mathfrak{K}_{2N}(u) < \infty$. Then*

$$\mathfrak{F}\big(F^1(u, \eta), F^3(u, \eta)\big) \lesssim \big[1 + T + \mathfrak{K}(\eta)\big]\mathfrak{E}(\eta) + \mathfrak{K}(\eta)\big[\mathfrak{K}_{2N}(u) + (\mathfrak{K}_{2N}(u))^2\big] + (\mathfrak{K}_{2N}(u))^2. \tag{5-3}$$

Proof. All terms in the definition of $F^1(u,\eta)$, $F^3(u,\eta)$ are quadratic or higher-order except the term ηe_3 in F^3. Hence we may argue as in the proof of Lemma 4.5 to deduce the bound

$$\mathfrak{F}\big(F^1(u,\eta), F^3(u,\eta)-\eta e_3\big) \lesssim \mathfrak{E}(\eta)\mathfrak{K}(\eta)+\mathfrak{K}(\eta)\big(\mathfrak{K}(\eta)+\mathfrak{K}_{2N}(u)+(\mathfrak{K}_{2N}(u))^2\big)+(\mathfrak{K}_{2N}(u))^2. \tag{5-4}$$

Here the appearance of the term $\mathfrak{E}(\eta)\mathfrak{K}(\eta)$ is due to the term $\eta D\eta$ in F^3, while the appearance of $\mathfrak{K}_{2N}(u)^2$ is due to the term $u\cdot\nabla u$ that appears when we write

$$u\cdot\nabla_{\mathscr{A}}u = u\cdot\nabla u + u\cdot\nabla_{\mathscr{A}-I}u$$

in F^1.

On the other hand, by definition, we have

$$\begin{aligned}\mathfrak{F}(0,\eta e_3) &= \sum_{j=0}^{2N}\|\partial_t^j\eta\|^2_{L^2H^{4N-2j-1/2}}+\sum_{j=0}^{2N-1}\|\partial_t^j\eta\|^2_{L^\infty H^{4N-2j-3/2}}\\ &\lesssim (1+T)\sum_{j=0}^{2N}\|\partial_t^j\eta\|^2_{L^\infty H^{4N-2j}} = (1+T)\mathfrak{E}(\eta).\end{aligned} \tag{5-5}$$

Then, since $\mathfrak{F}(X,Y+Z)\lesssim\mathfrak{F}(X,Y)+\mathfrak{F}(0,Z)$, we may combine (5-4) with (5-5) to deduce (5-3). □

Data estimates. In the construction of the initial data performed after Lemma 4.7, it was assumed that $\partial_t^j\eta(0)$ for $j=0,\dots,2N$ and $\partial_t^jF^1(0)$, $\partial_t^jF^3(0)$ for $j=0,\dots,2N-1$ were all known. Knowledge of the former allowed us to compute $R(0)$, $\mathscr{A}_0$, $\mathscr{N}_0$, etc. along with their temporal derivatives; these quantities then served as coefficients in deriving the initial conditions for (u,p) and their temporal derivatives. Since for the full nonlinear problem the function η is unknown and its evolution is coupled to that of u and p, we must revise the construction of the data to include this coupling, assuming only that u_0 and η_0 are given. This will also reveal the compatibility conditions that must be satisfied by u_0 and η_0 in order to solve the nonlinear problem (1-4). To this end, we first define the quantities

$$\mathscr{E}_0 := \|u_0\|^2_{4N}+\|\eta_0\|^2_{4N}\quad\text{and}\quad \mathscr{F}_0 := \|\eta_0\|^2_{4N+1/2}. \tag{5-6}$$

For our estimates, we must also introduce the quantity

$$\mathfrak{E}_0(u,p)=\sum_{j=0}^{2N}\big\|\partial_t^ju(0)\big\|^2_{4N-2j}+\sum_{j=0}^{2N-1}\big\|\partial_t^jp(0)\big\|^2_{4N-2j-1}. \tag{5-7}$$

We will also need a more exact enumeration of the terms in $\mathfrak{E}_0(u,p)$, $\mathfrak{E}_0(\eta)$, and $\mathfrak{F}_0$ (as defined in (5-7), (4-55), and (4-53), respectively). For $j=0,\dots,2N-1$, we define

$$\mathfrak{F}_0^j\big(F^1(u,\eta),F^3(u,\eta)\big) := \sum_{\ell=0}^{j}\big\|\partial_t^\ell F^1(0)\big\|^2_{4N-2\ell-2}+\big\|\partial_t^\ell F^3(0)\big\|^2_{4N-2\ell-3/2} \tag{5-8}$$

and

$$\mathfrak{E}_0^j(\eta) := \|\eta_0\|^2_{4N}+\big\|\partial_t\eta(0)\big\|^2_{4N-1}+\sum_{\ell=2}^{j}\big\|\partial_t^\ell\eta(0)\big\|^2_{4N-2\ell+3/2}, \tag{5-9}$$

with the sum in (5-9) only including the first term when $j = 0$ and only the first two terms when $j = 1$. For $j = 0$, we write $\mathfrak{E}_0^0(u, p) := \|u_0\|_{4N}^2$, and for $j = 1, \ldots, 2N$ we write

$$\mathfrak{E}_0^j(u, p) := \sum_{\ell=0}^{j} \|\partial_t^\ell u(0)\|_{4N-2j}^2 + \sum_{\ell=0}^{j-1} \|\partial_t^\ell p(0)\|_{4N-2j-1}^2.$$

The following lemma records more refined versions of the estimates (4-59) and (4-67) as well as some other related estimates that are useful in dealing with the initial data.

Lemma 5.2. *For $F^1(u, \eta)$ and $F^3(u, \eta)$ defined by (5-1) and $j = 0, \ldots, 2N - 1$, we have*

$$\mathfrak{F}_0^j\big(F^1(u, \eta), F^3(u, \eta)\big) \le P_j\big(\mathfrak{E}_0^{j+1}(\eta), \mathfrak{E}_0^j(u, p)\big) \tag{5-10}$$

for $P_j(\cdot, \cdot)$ a polynomial such that $P_j(0, 0) = 0$.

For $j = 1, \ldots, 2N - 1$, let $F^{1,j}(0)$ and $F^{3,j}(0)$ be determined by (4-52) and (5-1), using $\partial_t^\ell \eta(0)$, $\partial_t^\ell u(0)$, and $\partial_t^\ell p(0)$ for appropriate values of ℓ. Then

$$\|F^{1,j}(0)\|_{4N-2j-2}^2 + \|F^{3,j}(0)\|_{4N-2j-3/2}^2 \le P_j\big(\mathfrak{E}_0^{j+1}(\eta), \mathfrak{E}_0^j(u, p)\big) \tag{5-11}$$

for $P_j(\cdot, \cdot)$ a polynomial such that $P_j(0, 0) = 0$.

For $j = 0, \ldots, 2N$, we have

$$\big\|\partial_t^j u(0) - D_t^j u(0)\big\|_{4N-2j}^2 \le P_j\big(\mathfrak{E}_0^j(\eta), \mathfrak{E}_0^j(u, p)\big) \tag{5-12}$$

for $P_j(\cdot, \cdot)$ a polynomial such that $P_j(0, 0) = 0$.

For $j = 1, \ldots, 2N - 1$, we have

$$\left\| \sum_{\ell=0}^{j} \binom{j}{\ell} \partial_t^\ell \mathcal{N}(0) \cdot \partial_t^{j-\ell} u(0) \right\|_{H^{4N-2j+3/2}(\Sigma)} \le P_j\big(\mathfrak{E}_0^j(\eta), \mathfrak{E}_0^j(u, p)\big)$$

for $P_j(\cdot, \cdot)$ a polynomial such that $P_j(0, 0) = 0$. Also,

$$\|u_0 \cdot \mathcal{N}_0\|_{H^{4N-1}(\Sigma)}^2 \lesssim \|u_0\|_{4N}^2 (1 + \|\eta_0\|_{4N}^2). \tag{5-13}$$

Proof. These bounds may be derived by arguing as in the proof of Lemma 4.5, so again we omit the details. $\square$

This lemma allows us to modify the construction presented after Lemma 4.7 to construct all of the initial data $\partial_t^j u(0)$, $\partial_t^j \eta(0)$ for $j = 0, \ldots, 2N$ and $\partial_t^j p(0)$ for $j = 0, \ldots, 2N - 1$. Along the way, we will also derive estimates of $\mathfrak{E}_0(u, p) + \mathfrak{E}_0(\eta)$ in terms of $\mathscr{E}_0$ and determine the compatibility conditions for u_0, η_0 necessary for existence of solutions to (1-4).

We assume that u_0, η_0 satisfy $\mathscr{F}_0 < \infty$ and that $\|\eta_0\|_{4N-1/2}^2 \le \mathscr{E}_0 \le 1$ is sufficiently small for the hypothesis of Proposition 3.9 to hold when $k = 4N$. As before, we will iteratively construct the initial data, but this time we will use the estimates in Lemma 5.2.

Step 1. Define $\partial_t \eta(0) = u_0 \cdot \mathcal{N}_0$, where $u_0 \in H^{4N-1/2}(\Sigma)$ when traced onto Σ, and $\mathcal{N}_0$ is determined in terms of η_0. Estimate (5-13) implies that $\|\partial_t \eta(0)\|_{4N-1}^2 \lesssim \mathscr{E}_0$, and hence that $\mathfrak{E}_0^0(u, p) + \mathfrak{E}_0^1(\eta) \lesssim \mathscr{E}_0$. We may use this bound in (5-10) with $j = 0$ to find that

$$\mathfrak{F}_0^0\big(F^1(u, \eta), F^3(u, \eta)\big) \le P_0\big(\mathfrak{E}_0^1(\eta), \mathfrak{E}_0^0(u, p)\big) \le P(\mathscr{E}_0)$$

for a polynomial $P(\cdot)$ such that $P(0) = 0$. Note that in this estimate and in the estimates below, we employ a convention with polynomials of $\mathscr{E}_0$ similar to the one we employ with constants: they are allowed to change from line to line, but they always satisfy $P(0) = 0$.

Step 2: Iterative definition of $\partial_t^j p(0)$, $\partial_t^{j+1} u(0)$, and $\partial_t^{j+2}\eta(0)$, for $0 \le j \le 2N-2$. Now suppose, for given $j \in [0, 2N-2]$, that $\partial_t^\ell u(0)$ is known for $\ell = 0, \dots, j$, $\partial_t^\ell \eta(0)$ is known for $\ell = 0, \dots, j+1$, and $\partial_t^\ell p(0)$ is known for $\ell = 0, \dots, j-1$ (with the understanding that nothing is known of $p(0)$ when $j = 0$), and that

$$\mathfrak{E}_0^j(u, p) + \mathfrak{E}_0^{j+1}(\eta) + \mathfrak{F}_0^j\big(F^1(u, \eta), F^3(u, \eta)\big) \le P(\mathscr{E}_0). \tag{5-14}$$

According to the estimates (5-11) and (5-12), we then know that

$$\|F^{1,j}(0)\|_{4N-2j-2}^2 + \|F^{3,j}(0)\|_{4N-2j-3/2}^2 + \|D_t^j u(0)\|_{4N-2j}^2 \le P(\mathscr{E}_0). \tag{5-15}$$

By virtue of estimates (4-69) and (5-14), we know that

$$\|\mathfrak{f}^1(F^{1,j}(0), D_t^j u(0))\|_{4N-2j-3}^2 + \|\mathfrak{f}^2(F^{3,j}(0), D_t^j u(0))\|_{4N-2j-3/2}^2 + \|\mathfrak{f}^3(F^{1,j}(0), D_t^j u(0))\|_{4N-2j-5/2}^2 \le P(\mathscr{E}_0). \tag{5-16}$$

This allows us to define $\partial_t^j p(0)$ as the solution to (3-30) with f^1, f^2, f^3 given by $\mathfrak{f}^1, \mathfrak{f}^2, \mathfrak{f}^3$. Then Proposition 3.9 with $k = 4N$ and $r = 4N - 2j - 1 < k$ implies that

$$\|\partial_t^j p(0)\|_{4N-2j-1}^2 \le P(\mathscr{E}_0). \tag{5-17}$$

Now the estimates (4-68), (5-14), and (5-15) allow us to define

$$D_t^{j+1} u(0) := \mathfrak{G}^0\big(F^{1,j}(0), D_t^j u(0), \partial_t^j p(0)\big) \in H^{4N-2j-2}, \tag{5-18}$$

and owing to (5-12), we have the estimate

$$\|\partial_t^{j+1} u(0)\|_{4N-2(j+1)}^2 \le P(\mathscr{E}_0). \tag{5-19}$$

Now we define $\partial_t^{j+2}\eta(0) = \sum_{\ell=0}^{j+1} \binom{j}{\ell} \partial_t^\ell \mathcal{N}(0) \cdot \partial_t^{j-\ell} u(0)$. The estimate (5-13), together with (5-14) and (5-19), then imply that

$$\|\partial_t^{j+2}\eta(0)\|_{4N-2(j+2)+3/2}^2 \le P(\mathscr{E}_0). \tag{5-20}$$

We may combine (5-14) with (5-17)–(5-20) to deduce that

$$\mathfrak{E}_0^{j+1}(u, p) + \mathfrak{E}_0^{j+2}(\eta) \le P(\mathscr{E}_0);$$

but then (5-10) implies that $\mathfrak{F}_0^{j+1}(F^1(u,\eta), F^3(u,\eta)) \le P(\mathscr{E}_0)$ as well, and we deduce that the bound (5-14) also holds with j replaced by $j+1$.

Using the above analysis, we may iterate from $j = 0, \dots, 2N-2$ to deduce that

$$\mathfrak{E}_0^{2N-1}(u,p) + \mathfrak{E}_0^{2N}(\eta) + \mathfrak{F}_0^{2N-1}\big(F^1(u,\eta), F^3(u,\eta)\big) \le P(\mathscr{E}_0). \tag{5-21}$$

Step 3: Definition of $\partial_t^{2N-1}p(0)$ and $D_t^{2N}u(0)$. After this iteration, it remains only to define $\partial_t^{2N-1}p(0)$ and $D_t^{2N}u(0)$. In order to do this, we must first impose the compatibility conditions on u_0 and η_0. These are the same as in (4-71), but because now the temporal derivatives of η have been constructed as well, we restate them in a slightly different way. Let $\partial_t^j u(0)$, $F^{1,j}(0)$, $F^{3,j}(0)$ for $j = 0, \dots, 2N-1$, $\partial_t^j \eta(0)$ for $j = 0, \dots, 2N$, and $\partial_t^j p(0)$ for $j = 0, \dots, 2N-2$ be constructed in terms of η_0, u_0 as above. Let Π_0 be the projection defined in terms of η_0 as in (4-10) and D_t be the operator defined by (4-8). We say that u_0, η_0 satisfy the $(2N)$-th order compatibility conditions if

$$\begin{cases} \operatorname{div}_{\mathscr{A}_0}(D_t^j u(0)) = 0 & \text{in } \Omega, \\ D_t^j u(0) = 0 & \text{on } \Sigma_b, \\ \Pi_0\big(F^{3,j}(0) + \mathbb{D}_{\mathscr{A}_0} D_t^j u(0)\mathscr{N}_0\big) = 0 & \text{on } \Sigma, \end{cases} \tag{5-22}$$

for $j = 0, \dots, 2N-1$. Note that if u_0, η_0 satisfy (5-22), then the j-th order compatibility condition (4-71) is satisfied for $j = 0, \dots, 2N-1$.

Now we define $\partial_t^{2N-1}p(0)$ and $D_t^{2N}u(0)$. We use the compatibility conditions (5-22) and argue as above and in the derivation of (4-70) in Lemma 4.7 to estimate

$$\big\|\mathfrak{f}^2(F^{3,2N-1}(0), D_t^{2N-1}u(0))\big\|_{1/2}^2 + \big\|\mathfrak{f}^3(F^{1,2N-1}(0), D_t^{2N-1}u(0))\big\|_{-1/2}^2 \le P(\mathscr{E}_0) \tag{5-23}$$

and

$$\big\|F^{1,2N-1}(0)\big\|_0^2 + \big\|\operatorname{div}_{\mathscr{A}_0}(R(0)D_t^{2N-1}u(0))\big\|_0^2 \le P(\mathscr{E}_0). \tag{5-24}$$

We then define $\partial_t^{2N-1}p(0) \in H^1$ as a weak solution to (3-30) in the sense of (3-33) with this choice of $f^2 = \mathfrak{f}^2$, $f^3 = \mathfrak{f}^3$, $g_0 = -\operatorname{div}_{\mathscr{A}_0}(R(0)D_t^{2N-1}u(0))$, and $G = -F^{1,2N-1}(0)$. The estimate (3-32), when combined with (5-23)–(5-24), allows us to deduce that

$$\big\|\partial_t^{2N-1}p(0)\big\|_1^2 \le P(\mathscr{E}_0). \tag{5-25}$$

Then we set $D_t^{2N}u(0) = \mathfrak{G}^0(F^{1,2N-1}(0), D_t^{2N-1}u(0), \partial_t^{2N-1}p(0))$, using (4-68) to see that $D_t^{2N} \in H^0$. In fact, the construction of $\partial_t^{2N-1}p(0)$ guarantees that $D_t^{2N}u(0) \in \mathscr{Y}(0)$. Arguing as before, we also have the estimate

$$\big\|\partial_t^{2N}u(0)\big\|_0^2 \lesssim P(\mathscr{E}_0). \tag{5-26}$$

This completes the construction of the initial data, but we will record a form of the estimates (5-21), (5-25)–(5-26) in the following proposition.

Proposition 5.3. *Suppose that u_0, η_0 satisfy $\mathscr{F}_0 < \infty$ and that $\mathscr{E}_0 \le 1$ is sufficiently small for the hypothesis of Proposition 3.9 to hold when $k = 4N$. Let $\partial_t^j u(0)$, $\partial_t^j \eta(0)$ for $j = 0, \dots, 2N$ and $\partial_t^j p(0)$*

for $j = 0, \dots, 2N-1$ *be given as above. Then*

$$\mathscr{E}_0 \le \mathfrak{E}_0(u, p) + \mathfrak{E}_0(\eta) \lesssim \mathscr{E}_0. \tag{5-27}$$

Proof. The first inequality in (5-27) is trivial. Summing (5-21) and (5-25)–(5-26) yields the estimate $\mathfrak{E}_0(u, p)+\mathfrak{E}_0(\eta) \le P(\mathscr{E}_0)$ for a polynomial P satisfying $P(0)=0$. Since $\mathscr{E}_0 \le 1$, we have that $P(\mathscr{E}_0) \lesssim \mathscr{E}_0$, and the last inequality in (5-27) follows directly. □

Transport problem. Thus far we have considered solving for (u, p), given η. Now we discuss how to solve for η, given u (more precisely, its trace on Σ). We do so by considering the transport problem

$$\begin{cases} \partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 & \text{in } \Sigma, \\ \eta(0) = \eta_0. \end{cases} \tag{5-28}$$

We now state a well-posedness theory for (5-28) involving the quantities $\mathscr{E}_0$, $\mathscr{F}_0$, $\mathfrak{K}_{2N}(u)$, $\mathfrak{K}(\eta)$ as defined by (5-6), (5-2), (4-54), respectively. We will also need one more quantity, which we write as

$$\mathscr{F}(\eta) := \|\eta\|^2_{L^\infty H^{4N+1/2}}.$$

Theorem 5.4. *Suppose that u_0, η_0 satisfy $\mathscr{F}_0 < \infty$ and that $\mathfrak{E}_0(\eta) \le 1$ is sufficiently small for the hypothesis of Proposition 3.9 to hold when $k = 4N$. Let $\partial_t^j \eta(0)$, $\partial_t^j u(0)$ for $j = 1, \dots, 2N$ be defined in terms of u_0, η_0 as in Section 5 and suppose that u satisfies $\mathfrak{K}_{2N}(u) \le 1$ and achieves the initial conditions $\partial_t^j u(0)$ for $j = 0, \dots, 2N$. Then the problem (5-28) admits a unique solution η that satisfies $\mathscr{F}(\eta) + \mathfrak{K}(\eta) < \infty$ and achieves the initial data $\partial_t^j \eta(0)$ for $j = 0, \dots, 2N$. Moreover, there exists a $0 < \overline{T} \le 1$, depending on N, such that if $0 < T \le \overline{T} \min\{1, 1/\mathscr{F}_0\}$, then we have the estimates*

$$\mathscr{F}(\eta) \lesssim \mathscr{F}_0 + T \mathfrak{K}_{2N}(u), \tag{5-29}$$

$$\mathfrak{E}(\eta) \lesssim \mathscr{E}_0 + T \mathfrak{K}_{2N}(u), \tag{5-30}$$

$$\overline{\mathfrak{E}}(\eta) \lesssim \mathfrak{E}(\eta) + \mathfrak{K}_{2N}(u)(1 + \mathfrak{E}(\eta)), \tag{5-31}$$

$$\mathfrak{D}(\eta) \lesssim \mathscr{E}_0 + T\mathscr{F}_0 + \mathfrak{K}_{2N}(u). \tag{5-32}$$

Proof. The proof proceeds through four steps. We first establish the solvability of problem (5-28), then we establish the $L^\infty H^k$ estimates needed to bound $\mathfrak{E}(\eta)$ and $\overline{\mathfrak{E}}(\eta)$ as in (5-30) and (5-31), and then we handle the $L^2 H^k$ estimates for the terms in $\mathfrak{D}(\eta)$ to derive (5-32). Summing the bounds (5-31) and (5-32) shows that $\mathfrak{K}(\eta) = \overline{\mathfrak{E}}(\eta) + \mathfrak{D}(\eta) < \infty$.

Step 1: Solving the transport equation. The assumptions on u imply, via trace theory, that

$$u \in L^2\big([0, T]; H^{4N+1/2}(\Sigma)\big),$$

which allows us to employ the a priori estimates for solutions of the transport equation derived in [Danchin 2005a] (more specifically, Proposition 2.1 with $p = p_2 = r = 2$, $\sigma = 4N + \frac{1}{2}$). Although the well-posedness of (5-28) is not proved in [Danchin 2005a], it can be deduced from the a priori estimates in a standard way; full details are provided in Theorem 3.3.1 of [Danchin 2005b]. The result is that (5-28) admits a unique solution $\eta \in C^0([0,T]; H^{4N+1/2}(\Sigma))$ with $\eta(0) = \eta_0$ that satisfies the estimate

$$\|\eta\|_{L^\infty H^{4N+1/2}} \le \exp\left(C \int_0^T \|u(t)\|_{H^{4N+1/2}(\Sigma)}\, dt\right)\left(\sqrt{\mathcal{F}_0} + \int_0^T \|u_3(t)\|_{H^{4N+1/2}(\Sigma)}\, dt\right) \tag{5-33}$$

for $C > 0$. By trace theory, we have $\|u(t)\|_{H^{4N+1/2}(\Sigma)} \lesssim \sqrt{\mathfrak{K}_{2N}(u)}$, so that the Cauchy–Schwarz inequality implies $C \int_0^T \|u(t)\|_{H^{4N+1/2}(\Sigma)}\, dt \lesssim \sqrt{T}\sqrt{\mathfrak{K}_{2N}(u)} \lesssim \sqrt{T}$, and hence that

$$\exp\left(C \int_0^T \|u(t)\|_{H^{4N+1/2}(\Sigma)}\, dt\right) \le 2 \tag{5-34}$$

for $T \le \overline{T}$ with $\overline{T} \le 1$ sufficiently small. We deduce from (5-33) and (5-34) that

$$\sqrt{\mathcal{F}(\eta)} \le 2\big(\sqrt{\mathcal{F}_0} + \sqrt{T \mathfrak{K}_{2N}(u)}\,\big), \tag{5-35}$$

from which (5-29) easily follows.

Step 2: Bounding $\mathfrak{E}(\eta)$. Proposition 2.1 of [Danchin 2005a] also implies the a priori estimate

$$\begin{aligned}\|\eta\|_{L^\infty H^{4N}} &\le \exp\left(C \int_0^T \|u(t)\|_{H^{4N+1/2}(\Sigma)}\, dt\right)\left(\|\eta_0\|_{4N} + \int_0^T \|u_3(t)\|_{H^{4N}(\Sigma)}\, dt\right)\\ &\lesssim \big(\sqrt{\mathfrak{E}_0(\eta)} + \sqrt{T\mathfrak{K}_{2N}(u)}\,\big),\end{aligned} \tag{5-36}$$

where we have used the smallness of $\overline{T}$, trace theory, and Cauchy–Schwarz as above. Since $\partial_t \eta$ satisfies $\partial_t \eta = u_3 - D\eta \cdot u$ and $\mathfrak{K}_{2N}(u) < \infty$, we know that $\partial_t \eta$ is temporally differentiable and satisfies

$$\partial_t(\partial_t \eta) + u \cdot D(\partial_t \eta) = \partial_t u_3 - \partial_t u \cdot D\eta$$

with initial condition $\partial_t \eta(0) = u_0 \cdot \mathcal{N}_0$, which matches the initial data constructed in terms of u_0, η_0. We may again apply Proposition 2.1 of [Danchin 2005a] and then use (5-36) to find

$$\begin{aligned}\|\partial_t \eta\|_{L^\infty H^{4N-2}} &\le 2\left(\|\partial_t \eta(0)\|_{4N-2} + \int_0^T \|\partial_t u_3\|_{H^{4N-2}(\Sigma)} + \|\partial_t u \cdot D\eta\|_{H^{4N-2}(\Sigma)}\right)\\ &\lesssim \|\partial_t \eta(0)\|_{4N-2} + \big(1 + \|\eta\|_{L^\infty H^{4N-1}}\big) \int_0^T \|\partial_t u\|_{H^{4N-2}(\Sigma)}\\ &\lesssim \sqrt{\mathfrak{E}_0(\eta)} + \sqrt{T\mathfrak{K}_{2N}(u)}\big(1 + \|\eta\|_{L^\infty H^{4N-1}}\big)\\ &\lesssim \sqrt{\mathfrak{E}_0(\eta)} + \sqrt{T\mathfrak{K}_{2N}(u)}\big(1 + \sqrt{\mathfrak{E}_0(\eta)} + \sqrt{T\mathfrak{K}_{2N}(u)}\,\big)\\ &\lesssim P\big(\sqrt{\mathfrak{E}_0(\eta)}, \sqrt{T\mathfrak{K}_{2N}(u)}\,\big)\end{aligned}$$

for a polynomial $P(\cdot\,,\cdot)$ with $P(0,0) = 0$. A straightforward modification of this argument allows us to iterate to obtain, for $j = 1, \dots, 2N$, the estimate

$$\|\partial_t^j \eta\|_{L^\infty H^{4N-2j}} \le P\big(\sqrt{\mathfrak{E}_0(\eta)}, \sqrt{T\mathfrak{K}_{2N}(u)}\,\big) \tag{5-37}$$

for $P(\cdot\,,\cdot)$ a polynomial with $P(0,0) = 0$. We also find that the initial data $\partial_t^j \eta(0)$ is achieved for $j = 0, \dots, 2N$. Squaring (5-36) and (5-37) and summing, we then deduce that $\mathfrak{E}(\eta) \le P(\mathfrak{E}_0(\eta), T\mathfrak{K}_{2N}(u))$

for another polynomial with $P(0,0)=0$. Since $\mathfrak{E}_0(\eta)\le 1$ and $T\mathfrak{K}_{2N}(u)\le \overline{T}\mathfrak{K}_{2N}(u)\le 1$, we then have

$$\mathfrak{E}(\eta)\lesssim \mathfrak{E}_0(\eta)+T\mathfrak{K}_{2N}(u), \tag{5-38}$$

which yields (5-30) when combined with Proposition 5.3.

Step 3: Bounding $\overline{\mathfrak{E}}(\eta)$. We can improve the estimates for $\partial_t^j\eta$, $j=1,\dots,2N$ by using the equation $\partial_t\eta=u_3-D\eta\cdot u$ directly. Indeed,

$$\|\partial_t\eta\|_{4N-1}^2\lesssim \|u_3\|_{H^{4N-1}(\Sigma)}^2+\|D\eta\cdot u\|_{4N-1}^2\lesssim \|u\|_{4N}^2(1+\|\eta\|_{4N}^2)\lesssim \mathfrak{K}_{2N}(u)(1+\mathfrak{E}(\eta)). \tag{5-39}$$

For higher-order temporal derivatives, we simply apply ∂_t^{j-1} with $j=2,\dots,2N-1$ to $\partial_t\eta=u_3-D\eta\cdot u$ and argue as above to find that

$$\|\partial_t^j\eta\|_{4N-2j+3/2}^2\lesssim \mathfrak{K}_{2N}(u)(1+\mathfrak{E}(\eta)). \tag{5-40}$$

Then (5-31) follows by summing (5-39), (5-40), and the trivial estimate $\|\eta\|_{4N}^2\le\mathfrak{E}(\eta)$.

Step 4: Bounding $\mathfrak{D}(\eta)$. Now we control the terms in $\mathfrak{D}(\eta)$. From (5-35), Cauchy–Schwarz, and the fact that $T\le 1$, we see that

$$\|\eta\|_{L^2H^{4N+1/2}}\le\sqrt{T}\sqrt{\mathscr{F}(\eta)}\le 2\big(\sqrt{T\mathscr{F}_0}+\sqrt{\mathfrak{K}_{2N}(u)}\big). \tag{5-41}$$

We may then use Equation (5-28), trace theory, the fact that $H^{4N-1/2}(\Sigma)$ is an algebra, and estimate (5-41) to get the bound

$$\begin{aligned}\|\partial_t\eta\|_{L^2H^{4N-1/2}}&\lesssim\|u_3\|_{L^2H^{4N-1/2}}+\|u\|_{L^\infty H^{4N-1/2}}\|\eta\|_{L^2H^{4N+1/2}}\\&\lesssim\sqrt{\mathfrak{K}_{2N}(u)}\big(1+\sqrt{T\mathscr{F}_0}+\sqrt{\mathfrak{K}_{2N}(u)}\big)\lesssim P\big(\sqrt{T\mathscr{F}_0},\sqrt{\mathfrak{K}_{2N}(u)}\big)\end{aligned} \tag{5-42}$$

for P a polynomial with $P(0,0)=0$. We argue similarly (employing (5-42) along the way) to find that

$$\begin{aligned}\|\partial_t^2\eta\|_{L^2H^{4N-3/2}}&\lesssim\|\partial_tu_3\|_{L^2H^{4N-1/2}}+\|\eta\|_{L^\infty H^{4N-1/2}}\|\partial_tu\|_{L^2H^{4N-3/2}}+\|\partial_t\eta\|_{L^2H^{4N-1/2}}\|u\|_{L^\infty H^{4N-3/2}}\\&\lesssim\sqrt{\mathfrak{K}_{2N}(u)}\big(1+\|\eta\|_{L^\infty H^{4N-1/2}}+\|\partial_t\eta\|_{L^2H^{4N-1/2}}\big)\\&\lesssim\sqrt{\mathfrak{K}_{2N}(u)}\big(1+\sqrt{\mathfrak{E}(\eta)}+P\big(\sqrt{T\mathscr{F}_0},\sqrt{\mathfrak{K}_{2N}(u)}\big)\big)\\&\lesssim P\big(\sqrt{T\mathscr{F}_0},\sqrt{\mathfrak{K}_{2N}(u)},\sqrt{\mathfrak{E}(\eta)}\big)\end{aligned} \tag{5-43}$$

for a polynomial P with $P(0,0,0)=0$. Iterating this argument for $j=2,\dots,2N+1$ then yields the inequalities

$$\|\partial_t^j\eta\|_{L^2H^{4N-2j+5/2}}\le P\big(\sqrt{T\mathscr{F}_0},\sqrt{\mathfrak{K}_{2N}(u)},\sqrt{\mathfrak{E}(\eta)}\big) \tag{5-44}$$

for a polynomial with $P(0,0,0)=0$. We may then square and sum (5-41)–(5-44) to find that $\mathfrak{D}(\eta)\le P(T\mathscr{F}_0,\mathfrak{K}_{2N}(u),\mathfrak{E}(\eta))$, but then (5-38) and the bound $T\le 1$ imply that $\mathfrak{D}(\eta)\le P(T\mathscr{F}_0,\mathfrak{K}_{2N}(u),\mathfrak{E}_0(\eta))$ for another P. By assumption, $T\mathscr{F}_0\le\overline{T}\le 1$, and $\mathfrak{K}_{2N}(u)$, $\mathfrak{E}_0(\eta)\le 1$ as well; hence

$$\mathfrak{D}(\eta)\lesssim T\mathscr{F}_0+\mathfrak{K}_{2N}(u)+\mathfrak{E}_0(\eta),$$

which provides the estimate (5-32) when combined with Proposition 5.3. □

6. Local well-posedness of the nonlinear problem

Sequence of approximate solutions. In order to construct the solution to (1-4), we will pass to the limit in a sequence of approximate solutions. The construction of this sequence is the content of our next result.

Theorem 6.1. *Assume the initial data are given as on pages 338–339 and satisfy the* $(2N)$*-th compatibility conditions* (5-22). *There exist* $0 < \delta < 1$ *and* $0 < \overline{T} < 1$ *such that if* $\mathscr{E}_0 \le \delta$, $\mathscr{F}_0 < \infty$, *and* $0 < T \le T_0 := \overline{T} \min\{1, 1/\mathscr{F}_0\}$, *then there exists an infinite sequence* $\{(u^m, p^m, \eta^m)\}_{m=1}^{\infty}$ *with the following three properties. First, for* $m \ge 1$ *we have*

$$\begin{cases} \partial_t u^{m+1} - \Delta_{\mathscr{A}^m} u^{m+1} + \nabla_{\mathscr{A}^m} p^{m+1} = \partial_t \bar{\eta}^m \tilde{b} K^m \partial_3 u^m - u^m \cdot \nabla_{\mathscr{A}^m} u^m & \text{in } \Omega, \\ \operatorname{div}_{\mathscr{A}^m} u^{m+1} = 0 & \text{in } \Omega, \\ S_{\mathscr{A}^m}(p^{m+1}, u^{m+1}) \mathscr{N}^m = \eta^m \mathscr{N}^m & \text{on } \Sigma, \\ u^{m+1} = 0 & \text{on } \Sigma_b, \end{cases} \tag{6-1}$$

and

$$\partial_t \eta^{m+1} = u^{m+1} \cdot \mathscr{N}^{m+1} \quad \text{on } \Sigma, \tag{6-2}$$

where $\mathscr{A}^m, \mathscr{N}^m, K^m$ *are given in terms of* η^m. *Second,* (u^m, p^m, η^m) *achieve the initial data for each* $m \ge 1$, *that is,* $\partial_t^j u^m(0) = \partial_t^j u(0)$ *and* $\partial_t^j \eta^m(0) = \partial_t^j \eta(0)$ *for* $j = 0, \dots, 2N$, *while* $\partial_t^j p^m(0) = \partial_t^j p(0)$ *for* $j = 0, \dots, 2N-1$. *Third, for each* $m \ge 1$, *we have the estimates*

$$\mathfrak{K}(\eta^m) + \mathfrak{K}(u^m, p^m) \le C(\mathscr{E}_0 + T\mathscr{F}_0) \quad \text{and} \quad \mathscr{F}(\eta^m) \le C(\mathscr{F}_0 + \mathscr{E}_0 + T\mathscr{F}_0) \tag{6-3}$$

for a universal constant $C > 0$.

Proof. We divide the proof into three steps. First, we construct an initial pair (u^0, η^0) that will be used as a starting point for constructing (u^m, p^m, η^m) for $m \ge 1$. Second, we prove that if (u^m, p^m, η^m) are known and satisfy certain estimates, then we can construct $(u^{m+1}, p^{m+1}, \eta^{m+1})$. Third, we combine the first two steps in an appropriate way to iteratively construct all of the (u^m, p^m, η^m). Throughout the proof, we will need to explicitly enumerate the various constants appearing in estimates where previously we have written $\lesssim$. We do so with $C_1, \dots, C_{10} > 0$.

Before proceeding to the steps, we define some terms and make some assumptions. Let $\delta_1 > 0$ be such that if $\mathfrak{K}(\eta) \le \delta_1$, then the hypotheses of Theorem 4.8 are satisfied. Similarly, let $\delta_2 > 0$ be the constant such that if $\mathfrak{E}_0(\eta) \le \delta_2$, then the hypotheses of Theorem 5.4 are satisfied. We assume that δ is sufficiently small that $\mathscr{E}_0 \le \delta$ satisfies the hypotheses of Proposition 5.3 and that (using the estimate (5-27))

$$\mathfrak{E}_0(\eta) + \mathfrak{E}_0(u, p) \le C_1 \mathscr{E}_0 \le C_1 \delta \le \min\{1, \delta_2\}. \tag{6-4}$$

This allows us to use (5-10) of Lemma 5.2 with $j = 2N - 1$ to get the bound

$$\mathfrak{F}_0\big(F^1(u, \eta), F^2(u, \eta)\big) \le C_2 \mathscr{E}_0. \tag{6-5}$$

Step 1: Seeding the sequence. We begin by extending the initial data $\partial_t^j u(0) \in H^{4N-2j}(\Omega)$ to a time-dependent function u^0 such that $\partial_t^j u^0(0) = \partial_t^j u(0)$. We do so by applying Lemma A.5. Although this produces a u^0 defined on the time interval $[0, \infty)$, we may restrict to $[0, T]$ without increasing any of the space-time norms in $\mathfrak{K}_{2N}(u^0)$. We may combine the estimate of $\mathfrak{K}_{2N}(u^0)$ provided by Lemma A.5 with (6-4) to get the bound

$$\mathfrak{K}_{2N}(u^0) \le C_3 \mathscr{E}_0. \tag{6-6}$$

With u^0 in hand, we define η^0 as the solution to (5-28) with u^0 replacing u. To do so, we apply Theorem 5.4, the hypotheses of which are satisfied by virtue of (6-4) and (6-6) if we further restrict to $C_3\delta \le 1$. Restricting $\overline{T}$ as in the theorem, we find our solution η^0, which satisfies $\partial_t^j \eta^0(0) = \partial_t^j \eta(0)$ as well as the estimates

$$\begin{aligned} \mathscr{F}(\eta^0) &\le C_4\big(\mathscr{F}_0 + T\mathfrak{K}_{2N}(u^0)\big), \\ \mathfrak{E}(\eta^0) &\le C_5\big(\mathscr{E}_0 + T\mathfrak{K}_{2N}(u^0)\big), \\ \mathfrak{D}(\eta^0) &\le C_6\big(\mathscr{E}_0 + T\mathscr{F}_0 + \mathfrak{K}_{2N}(u^0)\big). \end{aligned} \tag{6-7}$$

Step 2: The iteration argument. We claim that there exist $\gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0$ and $0 < \tilde{\delta}, \tilde{T} < 1$ (both depending on the γ_i) such that if $\delta \le \tilde{\delta}$ and $\overline{T} \le \tilde{T}$, then the following property is satisfied. If (u^m, η^m) are known and satisfy the estimates

$$\begin{aligned} \mathfrak{E}(\eta^m) &\le \gamma_1(\mathscr{E}_0 + T\mathscr{F}_0), \quad & \mathfrak{D}(\eta^m) &\le \gamma_2(\mathscr{E}_0 + T\mathscr{F}_0), \\ \mathfrak{K}_{2N}(u^m) &\le \gamma_3(\mathscr{E}_0 + T\mathscr{F}_0), \quad & \mathscr{F}(\eta^m) &\le C_4\mathscr{F}_0 + \gamma_4(\mathscr{E}_0 + T\mathscr{F}_0), \end{aligned} \tag{6-8}$$

then there exists a unique triple $(u^{m+1}, p^{m+1}, \eta^{m+1})$ that achieves the initial data, satisfies (6-1) and (6-2), and obeys the estimates

$$\begin{aligned} \mathfrak{E}(\eta^{m+1}) &\le \gamma_1(\mathscr{E}_0 + T\mathscr{F}_0), \quad & \mathfrak{D}(\eta^{m+1}) &\le \gamma_2(\mathscr{E}_0 + T\mathscr{F}_0), \\ \mathfrak{K}(u^{m+1}, p^{m+1}) &\le \gamma_3(\mathscr{E}_0 + T\mathscr{F}_0), \quad & \mathscr{F}(\eta^{m+1}) &\le C_4\mathscr{F}_0 + \gamma_4(\mathscr{E}_0 + T\mathscr{F}_0). \end{aligned} \tag{6-9}$$

To prove the claim, we will first use η^m to solve for (u^{m+1}, p^{m+1}), and then we will use the resulting u^{m+1} to solve for η^{m+1}. Along the way, we will restrict the size of $\tilde{\delta}$ and $\tilde{T}$ in terms of γ_i, $i = 1, 2, 3, 4$. We will define the γ_i in terms of the C_i, so the $\tilde{\delta}$ and $\tilde{T}$ can be thought of as universal constants. Note that the estimates of (6-9) are stronger than those of (6-8) since $\mathfrak{K}_{2N}(u^{m+1}) \le \mathfrak{K}(u^{m+1}, p^{m+1})$. This asymmetry is useful to us since in Step 1, we have not bothered to construct p^0, so only (u^0, η^0) are available to begin the iterative construction of $\{(u^m, p^m, \eta^m)\}_{m=1}^{\infty}$.

From (5-31), (6-8), and the fact that $\mathscr{E}_0 + T_0\mathscr{F}_0 \le 1$, we have that

$$\overline{\mathfrak{E}}(\eta^m) \le C_7\big(\mathfrak{E}(\eta^m) + \mathfrak{K}_{2N}(u^m)(1 + \mathfrak{E}(\eta^m))\big) \le C_7(\gamma_1 + \gamma_3 + \gamma_1\gamma_3)(\mathscr{E}_0 + T_0\mathscr{F}_0). \tag{6-10}$$

We assume initially that $\tilde{T} \le T_0$, the constant appearing in Theorem 4.8. We also assume that

$$\tilde{\delta}, \tilde{T} \le \tfrac{1}{2}\min\left\{\frac{\min\{1, \delta_1\}}{(C_7(\gamma_1 + \gamma_3 + \gamma_1\gamma_3) + \gamma_2)}, \frac{1}{\gamma_3}\right\},$$

so that (6-8) implies that $\mathfrak{K}_{2N}(u^m) \le 1$ and (6-10) implies that

$$\mathfrak{K}(\eta^m) = \overline{\mathfrak{E}}(\eta^m) + \mathfrak{D}(\eta^m) \le \big(C_7(\gamma_1+\gamma_3+\gamma_1\gamma_3)+\gamma_2\big)(\mathscr{E}_0 + T_0\mathscr{F}_0) \le \min\{\delta_1, 1\},$$

the latter of which allows us to use Theorem 4.8 to produce a unique pair (u^{m+1}, p^{m+1}) that achieves the desired initial data and satisfies (6-1). Moreover, from (4-75) and (6-4)–(6-5), we have the estimate

$$\begin{aligned}\mathfrak{K}(u^{m+1}, p^{m+1}) \le C_8\big(1+\mathscr{E}_0+\mathfrak{K}(\eta^m)\big)\exp\big(C_9(1+\mathfrak{E}(\eta^m))T\big)\\ \times\big[(1+C_2)\mathscr{E}_0 + \mathfrak{F}(F^1(u^m,\eta^m), F^3(u^m,\eta^m))\big].\end{aligned} \tag{6-11}$$

Assume that $2\tilde{T}C_9 \le \log 2$; then

$$C_8\big(1+\mathscr{E}_0+\mathfrak{K}(\eta^m)\big)\exp\big(C_9(1+\mathfrak{E}(\eta^m))T\big) \le 3C_8\exp(2C_9\tilde{T}) \le 6C_8. \tag{6-12}$$

On the other hand, we can use our bounds on η^m, u^m in Lemma 5.1 to see that

$$\mathfrak{F}\big(F^1(u^m,\eta^m), F^3(u^m,\eta^m)\big) \le C_{10}\big[3\mathfrak{E}(\eta^m) + 2\mathfrak{K}(\eta^m)\mathfrak{K}_{2N}(u^m) + (\mathfrak{K}_{2N}(u^m))^2\big]. \tag{6-13}$$

Combining (6-11)–(6-13) with (6-8) then shows that

$$\begin{aligned}&\mathfrak{K}(u^{m+1}, p^{m+1})\\ &\le 6C_8\big[(1+C_2)\mathscr{E}_0 + 3C_{10}\gamma_1(\mathscr{E}_0+T\mathscr{F}_0) + 2C_{10}\gamma_3(\gamma_1+\gamma_2)(\mathscr{E}_0+T\mathscr{F}_0)^2 + C_{10}\gamma_3^2(\mathscr{E}_0+T\mathscr{F}_0)^2\big].\end{aligned} \tag{6-14}$$

We have now enumerated all of the constants C_i, $i = 1, \dots, 10$ that we need to define the γ_i, $i = 1, \dots, 4$. We choose the values of the γ_i according to

$$\begin{aligned}\gamma_1 &:= 2C_5, & \gamma_3 &:= 6C_8(3+C_2+3C_{10}\gamma_1)+C_3,\\ \gamma_4 &:= C_4, & \gamma_2 &:= C_6(1+\gamma_3).\end{aligned} \tag{6-15}$$

Notice that even though we have used γ_1 to define γ_3 and γ_3 to define γ_2, all of the γ_i are determined in terms of the constants C_i.

Now we will use the choice of the γ_i in (6-15) to derive the $\mathfrak{K}(u^{m+1}, p^{m+1})$ estimate of (6-9) from (6-14). To do this, we further restrict

$$\tilde{\delta}, \tilde{T} \le \tfrac{1}{2}\min\Big\{\frac{1}{2C_{10}\gamma_3(\gamma_1+\gamma_2)}, \frac{1}{C_{10}\gamma_3^2}\Big\}.$$

Then since $\mathscr{E}_0 + T\mathscr{F}_0 \le \tilde{\delta} + \tilde{T}$, we may use (6-14) to get the bound

$$\mathfrak{K}(u^{m+1}, p^{m+1}) \le 6C_8(3+C_2+3C_{10}\gamma_1)(\mathscr{E}_0+T\mathscr{F}_0) \le \gamma_3(\mathscr{E}_0+T\mathscr{F}_0). \tag{6-16}$$

Now we construct η^{m+1}. Recall that $\tilde{\delta}, \tilde{T} \le 1/(2\gamma_3)$; this and (6-16) yield the bound $\mathfrak{K}_{2N}(u^{m+1}) \le 1$. This estimate then allows us to apply Theorem 5.4 to find η^{m+1} that solves (6-2) and achieves the initial data. Estimates (5-29)–(5-32) of the theorem, together with (6-16) and the bound $T_0\gamma_3 \le \tilde{T}\gamma_3 \le 1$, imply that

$$
\begin{aligned}
\mathscr{F}(\eta^{m+1}) &\le C_4\big(\mathscr{F}_0 + T_0\mathfrak{K}_{2N}(u^{m+1})\big) \le C_4\mathscr{F}_0 + C_4(\mathscr{E}_0 + T\mathscr{F}_0),\\
\mathfrak{E}(\eta^{m+1}) &\le C_5\big(\mathscr{E}_0 + T_0\mathfrak{K}_{2N}(u^{m+1})\big) \le 2C_5(\mathscr{E}_0 + T\mathscr{F}_0),\\
\mathfrak{D}(\eta^{m+1}) &\le C_6\big(\mathscr{E}_0 + T\mathscr{F}_0 + \mathfrak{K}_{2N}(u^{m+1})\big) \le C_6(1+\gamma_3)(\mathscr{E}_0 + T\mathscr{F}_0).
\end{aligned} \tag{6-17}
$$

Using the definitions of the γ_i given in (6-15), we see from (6-17) that the η^{m+1} estimates of (6-9) hold. Then, owing to (6-16), all of the estimates in (6-9) hold, which completes the proof of the claim.

Step 3: Construction of the full sequence. We assume that $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are given by (6-15) and that $\tilde{\delta}$ and $\tilde{T}$ are as small as in Step 2. We assume that $\delta \le \tilde{\delta}$ and $\overline{T} \le \tilde{T}$ in addition to the other restrictions on their size made in Step 1 and before. Returning to (6-6), note that $C_3 \le \gamma_3$, which means that $\mathfrak{K}_{2N}(u^0) \le \gamma_3(\mathscr{E}_0 + T\mathscr{F}_0)$. We can also combine (6-6) and (6-7) and further restrict $\overline{T} \le 1/C_3$ to deduce that

$$
\begin{aligned}
\mathscr{F}(\eta^0) &\le C_4\mathscr{F}_0 + T_0C_3C_4\mathscr{E}_0 \le C_4\mathscr{F}_0 + \gamma_4(\mathscr{E}_0 + T\mathscr{F}_0),\\
\mathfrak{E}(\eta^0) &\le C_5(1+T_0C_3)\mathscr{E}_0 \le 2C_5\mathscr{E}_0 \le \gamma_1(\mathscr{E}_0 + T\mathscr{F}_0),\\
\mathfrak{D}(\eta^0) &\le C_6(\mathscr{E}_0 + T\mathscr{F}_0 + C_3\mathscr{E}_0) \le C_6(1+C_3)(\mathscr{E}_0 + T\mathscr{F}_0) \le \gamma_2(\mathscr{E}_0 + T\mathscr{F}_0).
\end{aligned}
$$

Note that in the last inequality we have used the fact that $C_3 \le \gamma_3$ to bound $C_6(1+C_3) \le C_6(1+\gamma_3) = \gamma_2$. We are then free to use the pair (u^0, η^0) as the starting point in Step 2, which allows us to construct (u^1, p^1, η^1) satisfying the desired PDE and initial conditions, along with the estimates

$$
\begin{aligned}
\mathfrak{E}(\eta^1) &\le \gamma_1(\mathscr{E}_0 + T\mathscr{F}_0), \quad & \mathfrak{D}(\eta^1) &\le \gamma_2(\mathscr{E}_0 + T\mathscr{F}_0),\\
\mathfrak{K}(u^1, p^1) &\le \gamma_3(\mathscr{E}_0 + T\mathscr{F}_0), \quad & \mathscr{F}(\eta^1) &\le C_4\mathscr{F}_0 + \gamma_4(\mathscr{E}_0 + T\mathscr{F}_0).
\end{aligned}
$$

We then iterate from $m = 1, \dots, \infty$, using (u^m, η^m) and Step 2 to produce the next element of the sequence, $(u^{m+1}, p^{m+1}, \eta^{m+1})$, which satisfies (6-9). All of the conclusions of the theorem follow. □

Contraction. Estimates (6-3) of Theorem 6.1 allow us to extract weakly converging subsequences from the sequence $\{(u^m, p^m, \eta^m)\}_{m=1}^{\infty}$. But, given such a convergent subsequence $\{(u^{m_k}, p^{m_k}, \eta^{m_k})\}_{k=1}^{\infty}$, we cannot guarantee that $\{(u^{m_k-1}, p^{m_k-1}, \eta^{m_k-1})\}_{k=1}^{\infty}$ converges to the same limit. This prevents us from simply passing to the limit in (6-1)–(6-2) in order to produce the desired solution to (1-4). We are thus led to study the strong convergence of the sequence, and in particular to consider its contraction in some norm.

We now define the norms in which we will show the sequence contracts. For $T > 0$, we define

$$
\begin{aligned}
\mathfrak{N}(v, q; T) &= \|v\|^2_{L^\infty H^2} + \|v\|^2_{L^2H^3} + \|\partial_t v\|^2_{L^\infty H^0} + \|\partial_t v\|^2_{L^2H^1} + \|q\|^2_{L^\infty H^1} + \|q\|^2_{L^2H^2},\\
\mathfrak{M}(\zeta; T) &= \|\zeta\|^2_{L^\infty H^{5/2}} + \|\partial_t\zeta\|^2_{L^\infty H^{3/2}} + \|\partial_t^2\zeta\|^2_{L^2H^{1/2}},
\end{aligned} \tag{6-18}
$$

where we write L^pH^k for $L^p([0, T]; H^k(\Omega))$ in $\mathfrak{N}$ and $L^p([0, T]; H^k(\Sigma))$ in $\mathfrak{M}$.

The next result provides a comparison of $\mathfrak{N}$ for pairs of solutions to problems of the form (6-1)–(6-2). We will use it later in Theorem 6.3 to show that the sequence of approximate solutions contracts, but we will also use it to prove the uniqueness of solutions to (1-4). To avoid confusion with the sequence $\{(u^m, p^m, \eta^m)\}$, we refer to velocities as v^j, w^j, pressures as q^j, and surface functions as ζ^j for $j = 1, 2$.

Theorem 6.2. *Let w^1, w^2, v^1, v^2, q^1, q^2, and ζ^1, ζ^2 satisfy*

$$\sup\{\mathfrak{E}(\zeta^1), \mathfrak{E}(\zeta^2), \mathfrak{E}(v^1, q^1), \mathfrak{E}(v^2, q^2), \mathfrak{E}(w^1, 0), \mathfrak{E}(w^2, 0)\} \le \varepsilon, \tag{6-19}$$

where the temporal L^∞ norms in $\mathfrak{E}$ are computed over the interval $[0, T]$ with $0 < T$. Suppose that for $j = 1, 2$,

$$\begin{cases} \partial_t v^j - \Delta_{\mathscr{A}^j} v^j + \nabla_{\mathscr{A}^j} q^j = \partial_t \bar{\zeta}^j \tilde{b} K^j \partial_3 w^j - w^j \cdot \nabla_{\mathscr{A}^j} w^j & \text{in } \Omega, \\ \operatorname{div}_{\mathscr{A}^j} v^j = 0 & \text{in } \Omega, \\ S_{\mathscr{A}^j}(q^j, v^j)\mathscr{N}^j = \zeta^j \mathscr{N}^j & \text{on } \Sigma, \\ v^j = 0 & \text{on } \Sigma_b, \\ \partial_t \zeta^j = w^j \cdot \mathscr{N}^j & \text{on } \Sigma, \end{cases} \tag{6-20}$$

where $\mathscr{A}^j$, K^j, $\mathscr{N}^j$ are determined by ζ^j as usual. Further, suppose that $\partial_t^k v^1(0) = \partial_t^k v^2(0)$ for $k = 0, 1$, $\zeta^1(0) = \zeta^2(0)$, and $q^1(0) = q^2(0)$.

Then there exist $\varepsilon_1 > 0$, $T_1 > 0$ such that if $\varepsilon \le \varepsilon_1$ and $0 < T \le T_1$, then

$$\mathfrak{N}(v^1 - v^2, q^1 - q^2; T) \le \tfrac{1}{2}\mathfrak{N}(w^1 - w^2, 0; T) \tag{6-21}$$

and

$$\mathfrak{M}(\zeta^1 - \zeta^2; T) \lesssim \mathfrak{N}(w^1 - w^2, 0; T). \tag{6-22}$$

Proof. The proof proceeds through six steps. First, we define $v = v^1 - v^2$, $w = w^1 - w^2$, $q = q^1 - q^2$, and derive the PDEs satisfied by v, q. We also identify the energy evolution for some norms of $\partial_t v$, $\partial_t q$. Second, we bound various forcing terms that appear in the energy evolution and on the right side of the PDEs for v, q. Third, we prove some bounds for $\partial_t v$, $\partial_t q$, using the energy evolution equation. Fourth, we use elliptic estimates to bound norms of v, q. Fifth, we derive estimates for $\zeta^1 - \zeta^2$ in terms of w. Sixth, we close the estimate to derive the contraction estimates (6-21), (6-22).

Step 1: PDEs and energy evolution for differences. We now derive the PDE satisfied by v, q, which are defined above. We subtract the equations in (6-20) with $j = 2$ from the same equations with $j = 1$. With the help of some simple algebra, we can write the resulting equations in terms of v, q:

$$\begin{cases} \partial_t v + \operatorname{div}_{\mathscr{A}^1} S_{\mathscr{A}^1}(q, v) = \operatorname{div}_{\mathscr{A}^1}(\mathbb{D}_{(\mathscr{A}^1 - \mathscr{A}^2)} v^2) + H^1 & \text{in } \Omega, \\ \operatorname{div}_{\mathscr{A}^1} v = H^2 & \text{in } \Omega, \\ S_{\mathscr{A}^1}(q, v)\mathscr{N}^1 = \mathbb{D}_{(\mathscr{A}^1 - \mathscr{A}^2)} v^2 \mathscr{N}^1 + H^3 & \text{on } \Sigma, \\ v = 0 & \text{on } \Sigma_b, \\ v(t = 0) = 0, \end{cases} \tag{6-23}$$

where H^1, H^2, H^3 are defined by

$$\begin{aligned} H^1 = {} & \operatorname{div}_{(\mathscr{A}^1 - \mathscr{A}^2)}(\mathbb{D}_{\mathscr{A}^2} v^2) - (\mathscr{A}^1 - \mathscr{A}^2)\nabla q^2 + \partial_t \bar{\zeta}^1 \tilde{b} K^1 (\partial_3 w^1 - \partial_3 w^2) + (\partial_t \bar{\zeta}^1 - \partial_t \bar{\zeta}^2)\tilde{b} K^1 \partial_3 w^2 \\ & + \partial_t \bar{\zeta}^1 \tilde{b}(K^1 - K^2)\partial_3 w^2 - (w^1 - w^2) \cdot \nabla_{\mathscr{A}^1} w^1 - w^2 \cdot \nabla_{\mathscr{A}^1}(w^1 - w^2) - w^2 \cdot \nabla_{(\mathscr{A}^1 - \mathscr{A}^2)} w^2, \end{aligned}$$

$$H^2 = -\operatorname{div}_{(\mathscr{A}^1-\mathscr{A}^2)} v^2,$$
$$H^3 = -q^2(\mathcal{N}^1-\mathcal{N}^2)+\mathbb{D}_{\mathscr{A}^1} v^2(\mathcal{N}^1-\mathcal{N}^2)-\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)} v^2(\mathcal{N}^1-\mathcal{N}^2)+(\zeta^1-\zeta^2)\mathcal{N}^1+\zeta^2(\mathcal{N}^1-\mathcal{N}^2).$$

The solutions are sufficiently regular for us to differentiate (6-23) in time, which results in the equations

$$\begin{cases} \partial_t(\partial_t v) + \operatorname{div}_{\mathscr{A}^1} S_{\mathscr{A}^1}(\partial_t q, \partial_t v) = \operatorname{div}_{\mathscr{A}^1}(\mathbb{D}_{(\partial_t \mathscr{A}^1-\partial_t \mathscr{A}^2)} v^2) + \tilde{H}^1 & \text{in } \Omega, \\ \operatorname{div}_{\mathscr{A}^1} \partial_t v = \tilde{H}^2 & \text{in } \Omega, \\ S_{\mathscr{A}^1}(\partial_t q, \partial_t v)\mathcal{N}^1 = \mathbb{D}_{(\partial_t \mathscr{A}^1-\partial_t \mathscr{A}^2)} v^2 \mathcal{N}^1 + \tilde{H}^3 & \text{on } \Sigma, \\ \partial_t v = 0 & \text{on } \Sigma_b, \\ \partial_t v(t=0) = 0, \end{cases} \tag{6-24}$$

where

$$\tilde{H}^1 = \partial_t H^1 + \operatorname{div}_{\partial_t \mathscr{A}^1}(\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)} v^2) + \operatorname{div}_{\mathscr{A}^1}(\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)} \partial_t v^2) + \operatorname{div}_{\partial_t \mathscr{A}^1}(\mathbb{D}_{\mathscr{A}^1} v) + \operatorname{div}_{\mathscr{A}^1}(\mathbb{D}_{\partial_t \mathscr{A}^1} v) - \nabla_{\partial_t \mathscr{A}^1} q, \tag{6-25}$$

$$\tilde{H}^2 = \partial_t H^2 - \operatorname{div}_{\partial_t \mathscr{A}^1} v, \tag{6-26}$$

$$\tilde{H}^3 = \partial_t H^3 + \mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)} \partial_t v^2 \mathcal{N}^1 + \mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)} v^2 \partial_t \mathcal{N}^1 - S_{\mathscr{A}^1}(q, v)\partial_t \mathcal{N}^1 + \mathbb{D}_{\partial_t \mathscr{A}^1} v \mathcal{N}^1. \tag{6-27}$$

Now we multiply (6-24) by $J^1 \partial_t v$, integrate over Ω, and integrate by parts as in the proof of Theorem 4.3 to deduce the evolution equation

$$\partial_t \int_\Omega \frac{|\partial_t v|^2}{2} J^1 + \frac{1}{2}\int_\Omega |\mathbb{D}_{\mathscr{A}^1} \partial_t v|^2 J^1 = \int_\Omega \frac{|\partial_t v|^2}{2}(\partial_t J^1 K^1) J^1 + \int_\Omega J^1 \partial_t q \tilde{H}^2 + \int_\Omega J^1\big(\operatorname{div}_{\mathscr{A}^1}(\mathbb{D}_{(\partial_t \mathscr{A}^1-\partial_t \mathscr{A}^2)} v^2) + \tilde{H}^1\big)\cdot \partial_t v - \int_\Sigma \big(\mathbb{D}_{(\partial_t \mathscr{A}^1-\partial_t \mathscr{A}^2)} v^2 \mathcal{N}^1 + \tilde{H}^3\big)\cdot \partial_t v. \tag{6-28}$$

Another integration by parts reveals that

$$\int_\Omega J^1 \operatorname{div}_{\mathscr{A}^1}(\mathbb{D}_{(\partial_t \mathscr{A}^1-\partial_t \mathscr{A}^2)} v^2)\cdot \partial_t v = -\tfrac{1}{2}\int_\Omega J^1 \mathbb{D}_{(\partial_t \mathscr{A}^1-\partial_t \mathscr{A}^2)} v^2 : \mathbb{D}_{\mathscr{A}^1} \partial_t v + \int_\Sigma \mathbb{D}_{(\partial_t \mathscr{A}^1-\partial_t \mathscr{A}^2)} v^2 \mathcal{N}^1 \cdot \partial_t v. \tag{6-29}$$

We then employ (6-29) to rewrite (6-28), and we integrate in time from 0 to $t < T$; since $\partial_t v(t=0) = 0$, we arrive at the equation

$$\int_\Omega \frac{|\partial_t v|^2}{2} J^1(t) + \frac{1}{2}\int_0^t \int_\Omega |\mathbb{D}_{\mathscr{A}^1} \partial_t v|^2 J^1 = \int_0^t \int_\Omega \frac{|\partial_t v|^2}{2}(\partial_t J^1 K^1) J^1 + \int_0^t \int_\Omega J^1(\tilde{H}^1 \cdot \partial_t v + \tilde{H}^2 \partial_t q) - \frac{1}{2}\int_0^t \int_\Omega J^1 \mathbb{D}_{(\partial_t \mathscr{A}^1-\partial_t \mathscr{A}^2)} v^2 : \mathbb{D}_{\mathscr{A}^1} \partial_t v - \int_0^t \int_\Sigma \tilde{H}^3 \cdot \partial_t v. \tag{6-30}$$

Step 2: Estimates of the forcing terms. In order for Equation (6-30) to be useful, we must be able to estimate the terms that appear on its right. To this end, we now derive estimates for $\tilde{H}^1$, $\tilde{H}^2$, $\partial_t \tilde{H}^2$ in $H^0(\Omega)$ and $\tilde{H}^3$ in $H^{-1/2}(\Sigma)$. We claim that the following estimates hold (here and through the end of this section, we have written $P(\cdot)$ for a polynomial such that $P(0)=0$—possibly a different one each time):

$$\|\tilde{H}^1\|_0 \lesssim P(\sqrt{\varepsilon})\big[\|\zeta^1-\zeta^2\|_{3/2}+\|\partial_t\zeta^1-\partial_t\zeta^2\|_{1/2}+\|\partial_t^2\zeta^1-\partial_t^2\zeta^2\|_0 + \|w^1-w^2\|_1+\|\partial_t w^1-\partial_t w^2\|_1+\|v\|_2+\|q\|_1\big], \tag{6-31}$$

$$\|\tilde{H}^2\|_0 \lesssim P(\sqrt{\varepsilon})\big[\|\zeta^1-\zeta^2\|_{1/2}+\|\partial_t\zeta^1-\partial_t\zeta^2\|_{1/2}+\|v\|_1\big], \tag{6-32}$$

$$\|\partial_t\tilde{H}^2\|_0 \lesssim P(\sqrt{\varepsilon})\big[\|\zeta^1-\zeta^2\|_{1/2}+\|\partial_t\zeta^1-\partial_t\zeta^2\|_{1/2}+\|\partial_t^2\zeta^1-\partial_t^2\zeta^2\|_{1/2}+\|v\|_1+\|\partial_t v\|_1\big], \tag{6-33}$$

$$\|\tilde{H}^3\|_{-1/2} \lesssim P(\sqrt{\varepsilon})\big[\|\zeta^1-\zeta^2\|_{1/2}+\|\partial_t\zeta^1-\partial_t\zeta^2\|_{1/2}+\|v\|_2+\|q\|_1\big]+\|\partial_t\zeta^1-\partial_t\zeta^2\|_{-1/2}. \tag{6-34}$$

According to the definitions (6-25)–(6-27), all of the summands in $\tilde{H}^1$, $\tilde{H}^2$, $\partial_t\tilde{H}^2$ are quadratic, of the form $X\times Y$, where Y is one of v, q, $\partial_t^j\zeta^1-\partial_t^j\zeta^2$ for $j=0,1,2$, or $\partial_t^j w^1-\partial_t^j w^2$ for $j=0,1$. The bounds (6-31)–(6-33) may be established by estimating the products $X\times Y$ with Lemmas A.1, A.7, A.9, A.10, and A.8 and the usual Sobolev and trace embeddings; the appearance of the terms $P(\sqrt{\varepsilon})$ is due to the X terms, whose appropriate Sobolev norm may be bounded above by a polynomial in

$$\sqrt{\sup\big\{\mathfrak{E}(\zeta^1),\,\mathfrak{E}(\zeta^2),\,\mathfrak{E}(v^1,q^1),\,\mathfrak{E}(v^2,q^2),\,\mathfrak{E}(w^1,0),\,\mathfrak{E}(w^2,0)\big\}}\le\sqrt{\varepsilon}. \tag{6-35}$$

The estimate (6-34) follows similarly by using (A-3) of Lemma A.1, except that $\tilde{H}^3$ has a single term, namely $(\partial_t\zeta^1-\partial_t\zeta^2)e_3$, that is not quadratic and that causes the last term on the right side of (6-34) to not be multiplied by $P(\sqrt{\varepsilon})$. The same sort of argument also allows us to deduce the bound

$$\|\mathbb{D}_{(\partial_t\mathscr{A}^1-\partial_t\mathscr{A}^2)}v^2\|_0 \lesssim P(\sqrt{\varepsilon})\big[\|\zeta^1-\zeta^2\|_{1/2}+\|\partial_t\zeta^1-\partial_t\zeta^2\|_{1/2}\big]. \tag{6-36}$$

We will eventually employ an elliptic estimate with (6-23), so we will also need estimates of H^1, H^2, H^3 and the two other terms appearing on the right side of (6-23). The following estimates hold for $r=0,1$:

$$\|H^1\|_r \lesssim P(\sqrt{\varepsilon})\big[\|\zeta^1-\zeta^2\|_{r+1/2}+\|\partial_t\zeta^1-\partial_t\zeta^2\|_{r-1/2}+\|w^1-w^2\|_{r+1}\big], \tag{6-37}$$

$$\|H^2\|_{r+1} \lesssim P(\sqrt{\varepsilon})\|\zeta^1-\zeta^2\|_{r+3/2}, \tag{6-38}$$

$$\|H^3\|_{r+1/2} \lesssim P(\sqrt{\varepsilon})\|\zeta^1-\zeta^2\|_{r+3/2}+\|\zeta^1-\zeta^2\|_{r+1/2}, \tag{6-39}$$

$$\|\operatorname{div}_{\mathscr{A}^1}(\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)}v^2)\|_r \lesssim P(\sqrt{\varepsilon})\|\zeta^1-\zeta^2\|_{r+1/2}, \tag{6-40}$$

$$\|\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)}v^2\mathscr{N}^1\|_{r+1/2} \lesssim P(\sqrt{\varepsilon})\|\zeta^1-\zeta^2\|_{r+3/2}. \tag{6-41}$$

The proof of (6-37)–(6-41) may be carried out in the same manner we used above to prove (6-31)–(6-34).

Step 3: Estimates of $\partial_t v$ from (6-30). Now we employ the estimates of the forcing terms from the previous step in (6-30) in order to deduce estimates for $\partial_t v$. First we note that, owing to (6-35) and Sobolev embeddings, we obtain the bounds

$$\|J^1\|_{L^\infty}+\|K^1\|_{L^\infty}\lesssim 1+P(\sqrt{\varepsilon}) \quad\text{and}\quad \|\partial_t J^1\|_{L^\infty}\lesssim P(\sqrt{\varepsilon}). \tag{6-42}$$

Because of the time derivative on q, the most delicate term in (6-30) is the product $J^1\tilde{H}^2\partial_t q$. To handle it, we integrate by parts in time and use the fact that $q(0)=0$ to see that

$$\begin{aligned}\int_0^t\int_\Omega J^1\tilde{H}^2\partial_t q &= \int_0^t\Big[\partial_t\int_\Omega J^1 q\tilde{H}^2 - \int_\Omega \partial_t J^1 q\tilde{H}^2 + J^1 q\partial_t\tilde{H}^2\Big]\\ &= \int_\Omega J^1 q\tilde{H}^2(t) - J^1 q\tilde{H}^2(0) - \int_0^t\int_\Omega \partial_t J^1 q\tilde{H}^2 + J^1 q\partial_t\tilde{H}^2\\ &= \int_\Omega J^1 q\tilde{H}^2(t) - \int_0^t\int_\Omega \partial_t J^1 q\tilde{H}^2 + J^1 q\partial_t\tilde{H}^2. \end{aligned}\tag{6-43}$$

This, (6-42), and the estimates (6-32) and (6-33) then imply that

$$\begin{aligned}\int_0^t\int_\Omega J^1\tilde{H}^2\partial_t q \lesssim P(\sqrt{\varepsilon})\|q\|_{L^\infty H^0}\Big[\sum_{j=0}^1\big\|\partial_t^j\zeta^1-\partial_t^j\zeta^2\big\|_{L^\infty H^{1/2}}+\|v\|_{L^\infty H^1}\Big]\\ + P(\sqrt{\varepsilon})\int_0^t\|q\|_0\Big[\sum_{j=0}^2\big\|\partial_t^j\zeta^1-\partial_t^j\zeta^2\big\|_{1/2}+\|v\|_1+\|\partial_t v\|_1\Big],\end{aligned}\tag{6-44}$$

where the L^∞ norms are computed over the temporal interval $[0,T]$.

The other terms on the right of (6-30) are not so delicate and may be estimated directly with (6-31), (6-34), and (6-36). Indeed, these estimates together with trace theory and the Poincaré inequality imply

$$\begin{aligned}&\int_0^t\int_\Omega J^1\tilde{H}^1\cdot\partial_t v - \tfrac{1}{2}J^1\mathbb{D}_{(\partial_t\mathscr{A}^1-\partial_t\mathscr{A}^2)}v^2 : \mathbb{D}_{\mathscr{A}^1}\partial_t v - \int_0^t\int_\Sigma \tilde{H}^3\cdot\partial_t v\\ &\quad\le \int_0^t \|J^1\|_{L^\infty}\|\tilde{H}^1\|_0\|\partial_t v\|_0 + \tfrac{1}{2}\|J^1\|_{L^\infty}\big\|\mathbb{D}_{(\partial_t\mathscr{A}^1-\partial_t\mathscr{A}^2)}v^2\big\|_0\|\mathbb{D}_{\mathscr{A}^1}\partial_t v\|_0 + \int_0^t\|\tilde{H}^3\|_{-1/2}\|\partial_t v\|_{H^{1/2}(\Sigma)}\\ &\quad\lesssim \int_0^t\|\partial_t v\|_1\big(P(\sqrt{\varepsilon})\sqrt{\mathfrak{Z}}+\big\|\partial_t\zeta^1-\partial_t\zeta^2\big\|_{-1/2}\big),\end{aligned}\tag{6-45}$$

where we have written

$$\begin{aligned}\mathfrak{Z} := \|\zeta^1-\zeta^2\|_{3/2}^2 + \big\|\partial_t\zeta^1-\partial_t\zeta^2\big\|_{1/2}^2 + \big\|\partial_t^2\zeta^1-\partial_t^2\zeta^2\big\|_{1/2}^2\\ + \|w^1-w^2\|_1^2 + \big\|\partial_t w^1-\partial_t w^2\big\|_1^2 + \|v\|_2^2 + \|q\|_1^2.\end{aligned}\tag{6-46}$$

Also, we may use (6-35) to get the bound

$$\int_0^t\int_\Omega \frac{|\partial_t v|^2}{2}(\partial_t J^1 K^1)J^1 \le C\sqrt{\varepsilon}\int_0^t\int_\Omega\frac{|\partial_t v|^2}{2}J^1\tag{6-47}$$

for some constant $C>0$.

We now combine the estimates (6-44), (6-45), and (6-47) with (6-30), employ Lemma 2.1 to get the bound $\|\partial_t v\|_1/2 \le \|\sqrt{J^1}\mathbb{D}_{\mathscr{A}^1}\partial_t v\|_0$, and utilize Cauchy's inequality to absorb $\int_0^t\|\partial_t v\|_1^2$ into the left side of the resulting inequality; this yields the bound

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\partial_t v|^2 J^1(t) + \frac{1}{8}\int_0^t \|\partial_t v\|_1^2 \\
&\quad\le C\sqrt{\varepsilon}\int_0^t\int_\Omega \frac{|\partial_t v|^2}{2} J^1 \\
&\qquad + P(\sqrt{\varepsilon})\int_0^t \|q\|_0^2 + P(\sqrt{\varepsilon})\|q\|_{L^\infty H^0}\Bigg[\sum_{j=0}^{1}\big\|\partial_t^j\zeta^1-\partial_t^j\zeta^2\big\|_{L^\infty H^{1/2}} + \|v\|_{L^\infty H^1}\Bigg] \\
&\qquad + P(\sqrt{\varepsilon})\int_0^t \|q\|_0\Bigg[\sum_{j=0}^{2}\big\|\partial_t^j\zeta^1-\partial_t^j\zeta^2\big\|_{1/2} + \|v\|_1\Bigg] + \int_0^t\Big[P(\sqrt{\varepsilon})\mathfrak{X} + C\big\|\partial_t\zeta^1-\partial_t\zeta^2\big\|_{-1/2}^2\Big]. \qquad (6\text{-}48)
\end{aligned}$$

This bound can be viewed as a differential inequality of the form

$$x(t)+y(t)\le C\sqrt{\varepsilon}\int_0^t x(s)\,ds + F(t),$$

where x, y, $F\ge 0$, $x(0)=0$, and $F(t)$ is increasing in t. Gronwall's lemma then implies that

$$x(t)+y(t)\le e^{C\sqrt{\varepsilon}t}F(t). \qquad (6\text{-}49)$$

We assume that ε_1 and T_1 are sufficiently small for $e^{C\sqrt{\varepsilon}t}\le e^{C\sqrt{\varepsilon_1}T_1}\le 2$. Then from (6-48), (6-49), and Lemma 2.1, we deduce the bound

$$\begin{aligned}
\|\partial_t v\|_{L^\infty H^0}^2 + \|\partial_t v\|_{L^2 H^1}^2 &\le P(\sqrt{\varepsilon})\|q\|_{L^2H^0}^2 + C\big\|\partial_t\zeta^1-\partial_t\zeta^2\big\|_{L^2H^{-1/2}}^2 + \int_0^T P(\sqrt{\varepsilon})\mathfrak{X} \\
&\quad + P(\sqrt{\varepsilon})\|q\|_{L^\infty H^0}\Bigg[\sum_{j=0}^{1}\big\|\partial_t^j\zeta^1-\partial_t^j\zeta^2\big\|_{L^\infty H^{1/2}} + \|v\|_{L^\infty H^1}\Bigg] \\
&\quad + P(\sqrt{\varepsilon})\|q\|_{L^2H^0}\Bigg[\sum_{j=0}^{2}\big\|\partial_t^j\zeta^1-\partial_t^j\zeta^2\big\|_{L^2H^{1/2}} + \|v\|_{L^2H^1}\Bigg], \qquad (6\text{-}50)
\end{aligned}$$

where again the temporal L^∞ and L^2 norms are computed over $[0, T]$.

Step 4: Elliptic estimates for v and q. In order to close our estimates, we must be able to estimate v and q. This will be accomplished with an elliptic estimate. We combine Proposition 3.7 with the estimates (6-37)–(6-41) to deduce the bound for $r=0, 1$,

$$\begin{aligned}
&\|v\|_{r+2}^2 + \|q\|_{r+1}^2 \\
&\quad\lesssim \|\partial_t v\|_r^2 + \|H^1\|_r^2 + \big\|\mathrm{div}_{\mathscr{A}^1}(\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)}v^2)\big\|_r^2\|H^2\|_{r+1}^2 + \|H^3\|_{r+1/2}^2 + \big\|\mathbb{D}_{(\mathscr{A}^1-\mathscr{A}^2)}v^2\mathscr{N}^1\big\|_{r+1/2}^2 \\
&\quad\lesssim \|\partial_t v\|_r^2 + \|\zeta^1-\zeta^2\|_{r+1/2}^2 + P(\sqrt{\varepsilon})\big[\|\zeta^1-\zeta^2\|_{r+3/2}^2 + \big\|\partial_t\zeta^1-\partial_t\zeta^2\big\|_{r-1/2}^2 + \|w^1-w^2\|_{r+1}^2\big]. \qquad (6\text{-}51)
\end{aligned}$$

We set $r=0$ in (6-51) and then take the supremum in time over $[0, T]$ to find

$$\begin{aligned}
\|v\|_{L^\infty H^2}^2 + \|q\|_{L^\infty H^1}^2 &\lesssim \|\partial_t v\|_{L^\infty H^0}^2 + \|\zeta^1-\zeta^2\|_{L^\infty H^{1/2}}^2 \\
&\quad + P(\sqrt{\varepsilon})\big[\|\zeta^1-\zeta^2\|_{L^\infty H^{3/2}}^2 + \big\|\partial_t\zeta^1-\partial_t\zeta^2\big\|_{L^\infty H^{-1/2}}^2 + \|w^1-w^2\|_{L^\infty H^1}^2\big]. \qquad (6\text{-}52)
\end{aligned}$$

Then we set $r = 1$ in (6-51) and integrate over $[0, T]$ to find

$$\|v\|_{L^2H^3}^2 + \|q\|_{L^2H^2}^2 \lesssim \|\partial_t v\|_{L^2H^1}^2 + \|\zeta^1 - \zeta^2\|_{L^2H^{3/2}}^2 + P(\sqrt{\varepsilon})\big[\|\zeta^1 - \zeta^2\|_{L^2H^{5/2}}^2 + \big\|\partial_t\zeta^1 - \partial_t\zeta^2\big\|_{L^2H^{1/2}}^2 + \|w^1 - w^2\|_{L^2H^2}^2\big]. \tag{6-53}$$

Step 5: Estimates of $\zeta^1 - \zeta^2$. Now we turn to estimating the difference $\zeta^1 - \zeta^2$ in terms of $w^1 - w^2$. We subtract the equations satisfied by ζ^2 from the one for ζ^1 to find that

$$\begin{cases} \partial_t(\zeta^1 - \zeta^2) + w^1 \cdot D(\zeta^1 - \zeta^2) = (w^1 - w^2) \cdot \mathcal{N}^2 & \text{in } \Sigma, \\ (\zeta^1 - \zeta^2)(t = 0) = 0. \end{cases} \tag{6-54}$$

The PDE (6-54) is a transport equation for $\zeta^1 - \zeta^2$, so we can employ Lemma A.11 to estimate

$$\begin{aligned} \|\zeta^1 - \zeta^2\|_{L^\infty H^{5/2}} &\le \exp\left(C\int_0^T \|w^1(r)\|_{H^{7/2}(\Sigma)}\,dr\right)\int_0^T \big\|(w^1 - w^2)\cdot\mathcal{N}^2(r)\big\|_{H^{5/2}(\Sigma)}\,dr \\ &\lesssim e^{C\sqrt{T}\sqrt{\varepsilon}}\big(1 + P(\sqrt{\varepsilon})\big)\int_0^T \big\|(w^1 - w^2)(r)\big\|_3\,dr \\ &\lesssim e^{C\sqrt{T}\sqrt{\varepsilon}}\big(1 + P(\sqrt{\varepsilon})\big)\sqrt{T}\|w^1 - w^2\|_{L^2H^3}. \end{aligned}$$

We can further restrict ε_1 and T_1 so that $e^{C\sqrt{T}\sqrt{\varepsilon}} \le 2$ and $1 + P(\sqrt{\varepsilon}) \le 2$; then

$$\|\zeta^1 - \zeta^2\|_{L^\infty H^{5/2}} \lesssim \sqrt{T}\|w^1 - w^2\|_{L^2H^3}. \tag{6-55}$$

Then we use the first equation in (6-54), trace theory, and the estimate (6-55) to see that

$$\begin{aligned} \big\|\partial_t\zeta^1 - \partial_t\zeta^2\big\|_{L^\infty H^{3/2}} &\le \big\|(w^1 - w^2)\cdot\mathcal{N}^2\big\|_{L^\infty H^{3/2}} + \big\|w^1 \cdot D(\zeta^1 - \zeta^2)\big\|_{L^\infty H^{3/2}} \\ &\lesssim \big(1 + P(\sqrt{\varepsilon})\big)\|w^1 - w^2\|_{L^\infty H^{3/2}(\Sigma)} + P(\sqrt{\varepsilon})\|\zeta^1 - \zeta^2\|_{L^\infty H^{5/2}} \\ &\lesssim \|w^1 - w^2\|_{L^\infty H^2} + P(\sqrt{\varepsilon})\sqrt{T}\|w^1 - w^2\|_{L^2H^3}. \end{aligned} \tag{6-56}$$

Similarly, we differentiate (6-54) in time to find that

$$\begin{aligned} &\big\|\partial_t^2\zeta^1 - \partial_t^2\zeta^2\big\|_{L^2H^{1/2}} \\ &\quad\lesssim \big(1 + P(\sqrt{\varepsilon})\big)\big\|\partial_t w^1 - \partial_t w^2\big\|_{L^2H^1} + P(\sqrt{\varepsilon})\big[\|w^1 - w^2\|_{L^2H^1} + \|\zeta^1 - \zeta^2\|_{L^2H^{3/2}} + \big\|\partial_t\zeta^1 - \partial_t\zeta^2\big\|_{L^2H^{3/2}}\big] \\ &\quad\lesssim \big\|\partial_t w^1 - \partial_t w^2\big\|_{L^2H^1} + P(\sqrt{\varepsilon})\sqrt{T}\big[\|w^1 - w^2\|_{L^\infty H^1} + \|\zeta^1 - \zeta^2\|_{L^\infty H^{3/2}} + \big\|\partial_t\zeta^1 - \partial_t\zeta^2\big\|_{L^\infty H^{3/2}}\big] \\ &\quad\lesssim \big\|\partial_t w^1 - \partial_t w^2\big\|_{L^2H^1} + P(\sqrt{\varepsilon})\sqrt{T}\|w^1 - w^2\|_{L^\infty H^2} + P(\sqrt{\varepsilon})T\|w^1 - w^2\|_{L^2H^3}. \end{aligned} \tag{6-57}$$

Step 6: Synthesis: contraction. We now have all of the ingredients to prove our contraction result. We write

$$\begin{aligned} \mathfrak{N}^v(T) &:= \mathfrak{N}(v^1 - v^2, q^1 - q^2; T), \\ \mathfrak{N}^w(T) &:= \mathfrak{N}(w^1 - w^2, 0; T), \\ \mathfrak{M}(T) &:= \mathfrak{M}(\zeta^1 - \zeta^2; T), \end{aligned} \tag{6-58}$$

where $\mathfrak{M}$ and $\mathfrak{N}$ are defined by (6-18). We will first rewrite the bounds (6-50), (6-52), and (6-53) in terms of these new quantities.

We begin with the right side of (6-50). According to the definition of $\mathfrak{X}$, (6-46), we may bound

$$\|q\|_{L^2H^0}^2 + \int_0^T \mathfrak{X} \lesssim (1+T)\left[\mathfrak{M}(T) + \mathfrak{N}^w(T)\right] + T\mathfrak{N}^v(T). \tag{6-59}$$

Similarly,

$$\|q\|_{L^2H^0}\left[\sum_{j=0}^{2}\|\partial_t^j\zeta^1 - \partial_t^j\zeta^2\|_{L^2H^{1/2}} + \|v\|_{L^2H^1}\right] \lesssim \sqrt{T\mathfrak{N}^v(T)}\left[(1+\sqrt{T})\sqrt{\mathfrak{M}(T)} + \sqrt{T\mathfrak{N}^v(T)}\right], \tag{6-60}$$

$$\left\|\partial_t\zeta^1 - \partial_t\zeta^2\right\|_{L^2H^{-1/2}}^2 \le T\mathfrak{M}(T), \tag{6-61}$$

and

$$\|q\|_{L^\infty H^0}\left[\sum_{j=0}^{1}\|\partial_t^j\zeta^1 - \partial_t^j\zeta^2\|_{L^\infty H^{1/2}} + \|v\|_{L^\infty H^1}\right] \lesssim \sqrt{\mathfrak{N}^v(T)}\left[\sqrt{\mathfrak{M}(T)} + \sqrt{\mathfrak{N}^v(T)}\right]. \tag{6-62}$$

Then, using (6-59)–(6-62) and Cauchy's inequality, we may rewrite (6-50) as

$$\begin{aligned}\|\partial_t v\|_{L^\infty H^0}^2 &+ \|\partial_t v\|_{L^2H^1}^2 \\ &\lesssim \left[T + P(\sqrt{\varepsilon})(1+T)\right]\mathfrak{M}(T) + \left[P(\sqrt{\varepsilon})(1+T)\right]\mathfrak{N}^w(T) + \left[P(\sqrt{\varepsilon})(1+T)\right]\mathfrak{N}^v(T).\end{aligned} \tag{6-63}$$

Now we turn to the elliptic estimates (6-52)–(6-53). The bound (6-52) becomes

$$\|v\|_{L^\infty H^2}^2 + \|q\|_{L^\infty H^1}^2 \lesssim \|\partial_t v\|_{L^\infty H^0}^2 + \|\zeta^1 - \zeta^2\|_{L^\infty H^{1/2}}^2 + P(\sqrt{\varepsilon})\left[\mathfrak{M}(T) + \mathfrak{N}^w(T)\right]. \tag{6-64}$$

Note here that we have kept the term with $\zeta^1 - \zeta^2$ because it does not yet have a small multiplier in front of it. On the other hand, the bound (6-53) becomes

$$\|v\|_{L^2H^3}^2 + \|q\|_{L^2H^2}^2 \lesssim \|\partial_t v\|_{L^2H^1}^2 + T\left(1 + P(\sqrt{\varepsilon})\right)\left[\mathfrak{M}(T) + \mathfrak{N}^w(T)\right]. \tag{6-65}$$

We need not retain the $\zeta^1 - \zeta^2$ term in (6-65) since we can control the square of the temporal L^2 norm by the square of the L^∞ norm to pick up a T factor.

Next we reformulate the bounds (6-55)–(6-57) in a similar fashion. The estimate (6-55) becomes

$$\|\zeta^1 - \zeta^2\|_{L^\infty H^{5/2}}^2 \lesssim T\mathfrak{N}^w(T). \tag{6-66}$$

Similarly, we may sum (6-56) and (6-57) to get the bound

$$\left\|\partial_t\zeta^1 - \partial_t\zeta^2\right\|_{L^\infty H^{3/2}}^2 + \left\|\partial_t\zeta^1 - \partial_t\zeta^2\right\|_{L^2H^{1/2}}^2 \lesssim \left[1 + (T+T^2)P(\sqrt{\varepsilon})\right]\mathfrak{N}^w(T). \tag{6-67}$$

Summing (6-66) and (6-67) yields

$$\mathfrak{M}(T) \lesssim \left[1 + (T+T^2)P(\sqrt{\varepsilon})\right]\mathfrak{N}^w(T). \tag{6-68}$$

The estimate (6-22) directly follows from (6-68) and the definitions (6-58).

We now combine the above to get an estimate for $\mathfrak{N}^v$ from our estimates for v, q. Note that due to (6-66), estimate (6-64) also holds with $\|\zeta^1 - \zeta^2\|_{L^\infty H^{1/2}}^2$ replaced by $T\mathfrak{N}^w(T)$ on the right. We then add this modified version of (6-64) to (6-65), and then add to this a large constant times (6-63). If the constant is chosen to be sufficiently large, we can absorb the appearances of $\partial_t v$ norms on the right side into the left; doing so, we arrive at the bound

$$\mathfrak{N}^v(T) \lesssim \big[T + P(\sqrt{\varepsilon})(1+T)\big]\mathfrak{M}(T) + \big[T + P(\sqrt{\varepsilon})(1+T)\big]\mathfrak{N}^w(T) + \big[P(\sqrt{\varepsilon})(1+T)\big]\mathfrak{N}^v(T). \tag{6-69}$$

This estimate may be combined with (6-68) to see that

$$\mathfrak{N}^v(T) \lesssim \big[1 + (T+T^2)P(\sqrt{\varepsilon})\big]\big[T + P(\sqrt{\varepsilon})(1+T)\big]\mathfrak{N}^w(T) + \big[P(\sqrt{\varepsilon})(1+T)\big]\mathfrak{N}^v(T). \tag{6-70}$$

By further restricting ε_1 and T_1, we may replace (6-70) by $\mathfrak{N}^v(T) \le \frac{1}{4}\mathfrak{N}^w(T) + \frac{1}{2}\mathfrak{N}^v(T)$, which may be rearranged to see that $\mathfrak{N}^v(T) \le \frac{1}{2}\mathfrak{N}^w(T)$, which gives (6-21) after using the definitions of $\mathfrak{N}^w(T)$, $\mathfrak{N}^v(T)$ given in (6-58). □

Local well-posedness: the proof of Theorem 1.1. Now we combine Theorems 6.1 and 6.2 to produce a solution to problem (1-4). Note that Theorem 1.1 follows directly from the following theorem by changing notation.

Theorem 6.3. *Assume that u_0, η_0 satisfy $\mathscr{E}_0, \mathscr{F}_0 < \infty$ and that the initial data $\partial_t^j u(0)$, etc. are as constructed on pages 338–339 and satisfy the $(2N)$-th compatibility conditions (5-22). Then there exist $0 < \delta_0, T_0 < 1$ such that if $\mathscr{E}_0 \le \delta_0$ and $0 < T \le T_0 \min\{1, 1/\mathscr{F}_0\}$, then the following hold. There exists a solution triple (u, p, η) to the problem (1-4) on the time interval $[0, T]$ that achieves the initial data and satisfies*

$$\mathfrak{K}(\eta) + \mathfrak{K}(u, p) \le C(\mathscr{E}_0 + T\mathscr{F}_0) \quad \textit{and} \quad \mathscr{F}(\eta) \le C(\mathscr{F}_0 + \mathscr{E}_0 + T\mathscr{F}_0) \tag{6-71}$$

for a universal constant $C > 0$. The solution is unique among functions that achieve the initial data and satisfy $\mathfrak{E}(\eta) + \mathfrak{E}(u, p) < \infty$. Moreover, η is such that the mapping $\Phi(\,\cdot\,, t)$, defined by (1-1), is a C^{4N-2} diffeomorphism for each $t \in [0, T]$.

Proof. We again divide the proof into several steps. First, we use Theorem 6.1 to construct a sequence of approximate solutions. Then we use Theorem 6.2 to show the sequence converges in the norm $\sqrt{\mathfrak{M}(\eta; T) + \mathfrak{N}(u, p; T)}$, which yields strong convergence of the sequence. Next, we use an interpolation argument to improve the convergence results. These then allow us to pass to the limit in the PDEs to deduce that the limit solves the problem (1-4). Finally, we again use Theorem 6.2 to show that our solution is unique.

We assume throughout the proof that $T_0 \le \min\{T_1, \overline{T}\}$, where $\overline{T}$ is given by Theorem 6.1, and T_1 is given by Theorem 6.2. Let $C > 0$ denote the universal constant in Theorem 6.1. We further assume that $T_0 \le \varepsilon_1/(2C)$, where $\varepsilon_1 > 0$ is the constant from Theorem 6.2.

Step 1: The sequence of approximate solutions. Suppose that $\delta_0 \le \delta$, where δ is given in Theorem 6.1. The hypotheses then allow us to apply Theorem 6.1 to produce the sequence of triples $\{(u^m, p^m, \eta^m)\}_{m=1}^{\infty}$, all elements of which achieve the initial data, satisfy the PDEs (6-1), (6-2), and obey the bounds

$$\sup_{m\ge 1}\big(\mathfrak{K}(\eta^m) + \mathfrak{K}(u^m, p^m)\big) \le C(\mathscr{E}_0 + T\mathscr{F}_0) \quad \text{and} \quad \sup_{m\ge 1}\mathscr{F}(\eta^m) \le C(\mathscr{F}_0 + \mathscr{E}_0 + T\mathscr{F}_0). \tag{6-72}$$

We further assume that δ_0 is small enough for $C\delta_0 \le \varepsilon_1/2$ (with ε_1 again from Theorem 6.2) so that (6-72) implies, in particular, that

$$\sup_{m\ge 1} \max\bigl\{\mathfrak{E}(\eta^m),\, \mathfrak{E}(u^m, p^m)\bigr\} \le C(\mathscr{E}_0 + T\mathscr{F}_0) \le C(\delta_0 + T_0) \le \varepsilon_1. \tag{6-73}$$

The uniform bounds (6-72) allow us to take weak and weak-$*$ limits, up to the extraction of a subsequence:

$$\begin{cases} \partial_t^j u^m \rightharpoonup \partial_t^j u & \text{weakly in } L^2\bigl([0,T]; H^{4N-2j+1}(\Omega)\bigr) \text{ for } j = 0,\dots,2N, \\ \partial_t^{2N+1} u^m \rightharpoonup \partial_t^{2N+1} u & \text{weakly in } (\mathscr{X}_T)^*, \\ \partial_t^j u^m \overset{*}{\rightharpoonup} \partial_t^j u & \text{weakly-}* \text{ in } L^\infty\bigl([0,T]; H^{4N-2j}(\Omega)\bigr) \text{ for } j = 0,\dots,2N, \\ \partial_t^j p^m \rightharpoonup \partial_t^j p & \text{weakly in } L^2\bigl([0,T]; H^{4N-2j}(\Omega)\bigr) \text{ for } j = 0,\dots,2N-1, \\ \partial_t^j p^m \overset{*}{\rightharpoonup} \partial_t^j p & \text{weakly-}* \text{ in } L^\infty\bigl([0,T]; H^{4N-2j-1}(\Omega)\bigr) \text{ for } j = 0,\dots,2N-1 \end{cases}$$

and

$$\begin{cases} \eta^m \rightharpoonup \eta & \text{weakly in } L^2\bigl([0,T]; H^{4N+1/2}(\Sigma)\bigr), \\ \partial_t \eta^m \rightharpoonup \partial_t \eta & \text{weakly in } L^2\bigl([0,T]; H^{4N-1/2}(\Sigma)\bigr), \\ \partial_t^j \eta^m \rightharpoonup \partial_t^j \eta & \text{weakly in } L^2\bigl([0,T]; H^{4N-2j+5/2}(\Sigma)\bigr) \text{ for } j = 2,\dots,2N+1, \\ \eta^m \overset{*}{\rightharpoonup} \eta & \text{weakly-}* \text{ in } L^\infty\bigl([0,T]; H^{4N+1/2}(\Sigma)\bigr), \\ \partial_t^j \eta^m \overset{*}{\rightharpoonup} \partial_t^j \eta & \text{weakly-}* \text{ in } L^\infty\bigl([0,T]; H^{4N-2j}(\Sigma)\bigr) \text{ for } j = 1,\dots,2N. \end{cases}$$

According to the weak and weak-$*$ lower semicontinuity of the norms in $\mathfrak{K}(\eta^m)$, $\mathfrak{K}(u^m, p^m)$, and $\mathscr{F}(\eta^m)$, we find that the limit (u, p, η) satisfies

$$\mathfrak{K}(\eta) + \mathfrak{K}(u,p) \le C(\mathscr{E}_0 + T\mathscr{F}_0) \quad \text{and} \quad \mathscr{F}(\eta) \le C(\mathscr{F}_0 + \mathscr{E}_0 + T\mathscr{F}_0).$$

The collection of triples (v, q, ζ) that achieve the initial data, that is, $\partial_t^j v(0) = \partial_t^j u(0)$, $\partial_t^j \zeta(0) = \partial_t^j \eta(0)$ for $j = 0,\dots,2N$ and $\partial_t^j q(0) = \partial_t^j p(0)$ for $j = 0,\dots,2N-1$, is clearly convex; Lemma A.4 implies that it is also closed with respect to the topology generated by the norm $\sqrt{\mathfrak{D}(\zeta) + \mathfrak{D}(v,q)}$. Therefore, the collection is also closed in the corresponding weak topology. Then, since each (u^m, p^m, η^m) is in this collection, we deduce that the limit (u, p, η) is as well. Hence (u, p, η) achieves the initial data.

Step 2: Contraction. Now we want to improve the weak convergence results of the previous step to strong convergence in the norm $\sqrt{\mathfrak{M}(\eta; T) + \mathfrak{N}(u, p; T)}$, where $\mathfrak{M}$ and $\mathfrak{N}$ are defined by (6-18). For $m \ge 1$, we set $v^1 = u^{m+2}$, $v^2 = u^{m+1}$, $w^1 = u^{m+1}$, $w^2 = u^m$, $q^1 = p^{m+2}$, $q^2 = p^{m+1}$, $\zeta^1 = \eta^{m+1}$, $\zeta^2 = \eta^m$ in Theorem 6.2. Because of (6-1)–(6-2), we have that (6-20) holds; the initial data of w^j, v^j, q^j, ζ^j match for $j = 1, 2$ by construction. Also, (6-73) implies that (6-19) holds, so all of the hypotheses of Theorem 6.2 are satisfied. Then (6-21) and (6-22) imply that

$$\mathfrak{N}\bigl(u^{m+2} - u^{m+1}, p^{m+2} - p^{m+1}; T\bigr) \le \tfrac{1}{2}\mathfrak{N}\bigl(u^{m+1} - u^m, p^{m+1} - p^m; T\bigr) \tag{6-74}$$

and

$$\mathfrak{M}(\eta^{m+1} - \eta^m; T) \lesssim \mathfrak{N}\bigl(u^{m+1} - u^m, p^{m+1} - p^m; T\bigr). \tag{6-75}$$

The bound (6-74) implies that the sequence $\{(u^m, p^m)\}_{m=1}^{\infty}$ is Cauchy in the norm $\sqrt{\mathfrak{N}(\cdot\,,\cdot\,;T)}$, so as $m \to \infty$,

$$\begin{cases} u^m \to u & \text{in } L^\infty\big([0,T]; H^2(\Omega)\big) \cap L^2\big([0,T], H^3(\Omega)\big), \\ \partial_t u^m \to \partial_t u & \text{in } L^\infty\big([0,T]; H^0(\Omega)\big) \cap L^2\big([0,T], H^1(\Omega)\big), \\ p^m \to p & \text{in } L^\infty\big([0,T]; H^1(\Omega)\big) \cap L^2\big([0,T], H^2(\Omega)\big). \end{cases} \tag{6-76}$$

Because of (6-75), we further deduce that the sequence $\{\eta^m\}_{m=1}^{\infty}$ is Cauchy in the norm $\sqrt{\mathfrak{M}(\cdot\,;T)}$, so that, as $m \to \infty$,

$$\begin{cases} \eta^m \to \eta & \text{in } L^\infty\big([0,T]; H^{5/2}(\Sigma)\big), \\ \partial_t \eta^m \to \partial_t \eta & \text{in } L^\infty\big([0,T]; H^{3/2}(\Sigma)\big), \\ \partial_t^2 \eta^m \to \partial_t^2 \eta & \text{in } L^2\big([0,T]; H^{1/2}(\Sigma)\big). \end{cases} \tag{6-77}$$

Step 3: Interpolation for improved strong convergence. Since (u^m, p^m, η^m) obey the bounds (6-72), we can parlay the convergence results (6-76), (6-77) into convergence in better norms by use of interpolation theory. We first interpolate with L^2H^0 norms of temporal derivatives (such estimates take the form

$$\|\partial_t^k f\|_{L^2H^0} \le C(T)\|f\|_{L^2H^0}^{\theta} \big\|\partial_t^j f\big\|_{L^2H^0}^{1-\theta} \tag{6-78}$$

for $j > k \ge 0$ and $\theta = \theta(j,k) \in (0,1)$ and $C(T)$ a constant depending on T), which reveals that

$$\begin{cases} \partial_t^j u^m \to \partial_t^j u & \text{in } L^2\big([0,T]; H^0(\Omega)\big) \text{ for } j = 0, \dots, 2N-1, \\ \partial_t^j p^m \to \partial_t^j p & \text{in } L^2\big([0,T]; H^0(\Omega)\big) \text{ for } j = 0, \dots, 2N-2, \\ \partial_t^j \eta^m \to \partial_t^j \eta & \text{in } L^2\big([0,T]; H^0(\Sigma)\big) \text{ for } j = 0, \dots, 2N. \end{cases} \tag{6-79}$$

Here the range of j is determined by the range of j appearing in $\mathfrak{D}(\eta)$ and $\mathfrak{D}(u,p)$. Then we use spatial interpolation between H^0 and H^k to deduce from (6-79) that

$$\begin{cases} \partial_t^j u^m \to \partial_t^j u & \text{in } L^2\big([0,T]; H^{4N-2j}(\Omega)\big) \text{ for } j = 0, \dots, 2N-1, \\ \partial_t^j p^m \to \partial_t^j p & \text{in } L^2\big([0,T]; H^{4N-2j-1}(\Omega)\big) \text{ for } j = 0, \dots, 2N-2, \\ \eta^m \to \eta & \text{in } L^2\big([0,T]; H^{4N}(\Sigma)\big), \\ \partial_t \eta^m \to \partial_t \eta & \text{in } L^2\big([0,T]; H^{4N-1}(\Sigma)\big), \\ \partial_t^j \eta^m \to \partial_t^j \eta & \text{in } L^2\big([0,T]; H^{4N-2j+2}(\Sigma)\big) \text{ for } j = 2, \dots, 2N. \end{cases} \tag{6-80}$$

Here the Sobolev index is determined by the Sobolev index k in the L^2H^k norms of $\mathfrak{D}(\eta)$ and $\mathfrak{D}(u,p)$. Finally, we use the temporal L^2 convergence of (6-80) to get L^∞ and C^0 convergence by applying Lemma A.4. This yields

$$\begin{cases} \partial_t^j u^m \to \partial_t^j u & \text{in } C^0\big([0,T]; H^{4N-2j-1}(\Omega)\big) \text{ for } j = 0, \dots, 2N-2, \\ \partial_t^j p^m \to \partial_t^j p & \text{in } C^0\big([0,T]; H^{4N-2j-2}(\Omega)\big) \text{ for } j = 0, \dots, 2N-3, \\ \eta^m \to \eta & \text{in } C^0\big([0,T]; H^{4N-1/2}(\Sigma)\big), \\ \partial_t \eta^m \to \partial_t \eta & \text{in } C^0\big([0,T]; H^{4N-3/2}(\Sigma)\big), \\ \partial_t^j \eta^m \to \partial_t^j \eta & \text{in } C^0\big([0,T]; H^{4N-2j+1}(\Sigma)\big) \text{ for } j = 2, \dots, 2N-1. \end{cases} \tag{6-81}$$

Step 4: Passing to the limit in the PDEs. The strong convergence results of (6-81) are more than sufficient for us to pass to the limit in the equations (6-1), (6-2) for each $t \in [0, T]$. Doing so, we find that the limits (u, p, η) are a strong solution to problem (1-4) on the time interval $t \in [0, T]$.

Step 5: Uniqueness. We now turn to the question of uniqueness of our solution (u, p, η). Suppose that (v, q, ζ) is another solution to (1-4) on the time interval $[0, T]$ that achieves the same initial data as (u, p, η) and which satisfies $\mathfrak{E}(\zeta)+\mathfrak{E}(v, q) < \infty$. Since (v, q, ζ) achieve the same data as $(u, p\eta)$, which is small, we may restrict to a temporal subinterval $[0, T_*] \subset [0, T]$ so that $\mathfrak{E}(\zeta) + \mathfrak{E}(v, q) \le \varepsilon_1$, where ε_1 is given in Theorem 6.2 and the norms are computed on $[0, T_*]$. We then set $v^1 = w^1 = u$, $v^2 = w^2 = v$, $q^1 = p$, $q^2 = q$, $\zeta^1 = \eta$, and $\zeta^2 = \zeta$ in Theorem 6.2 to deduce that

$$\mathfrak{N}(u-v, p-q; T_*) \le \tfrac{1}{2}\mathfrak{N}(u-v, p-q; T_*) \quad \text{and} \quad \mathfrak{M}(\eta-\zeta; T_*) \lesssim \mathfrak{N}(u-v, p-q; T_*),$$

which implies that $u = v$, $p = q$, $\eta = \zeta$ on the time interval $[0, T_*]$. This argument can then be iterated in the usual way, repeatedly increasing T_*, to extend the uniqueness to all of the interval $[0, T]$.

Step 6: Diffeomorphism. It is easy to check that the smallness of $\mathfrak{K}(\eta)$ is sufficient to guarantee that the map Φ, given by (1-1), is a C^1 diffeomorphism for each $t \in [0, T]$. The fact that it is in C^{4N-2} follows easily from Lemma A.10 in the periodic case and Lemma A.8 in the infinite case. □

Appendix: Analytic tools

Products in Sobolev spaces. We will need some estimates of the product of functions in Sobolev spaces.

Lemma A.1. *Let U denote either Σ or Ω.*

(1) *Let $0 \le r \le s_1 \le s_2$ be such that $s_1 > n/2$. Let $f \in H^{s_1}(U)$, $g \in H^{s_2}(U)$. Then $fg \in H^r(U)$ and*

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \tag{A-1}$$

(2) *Let $0 \le r \le s_1 \le s_2$ be such that $s_2 > r + n/2$. Let $f \in H^{s_1}(U)$, $g \in H^{s_2}(U)$. Then $fg \in H^r(U)$ and*

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \tag{A-2}$$

(3) *Let $0 \le r \le s_1 \le s_2$ be such that $s_2 > r + n/2$. Let $f \in H^{-r}(\Sigma)$, $g \in H^{s_2}(\Sigma)$. Then $fg \in H^{-s_1}(\Sigma)$ and*

$$\|fg\|_{-s_1} \lesssim \|f\|_{-r} \|g\|_{s_2}. \tag{A-3}$$

Proof. The proofs of (A-1) and (A-2) are standard; the bounds are first proved in $\mathbb{R}^n$ with the Fourier transform, and then the bounds in sufficiently nice subsets of $\mathbb{R}^n$ are deduced by use of an extension operator. To prove (A-3), we argue by duality. For $\varphi \in H^{s_1}$, we use (A-2) bound

$$\int_\Sigma \varphi f g \lesssim \|\varphi g\|_r \|f\|_{-r} \lesssim \|\varphi\|_{s_1} \|g\|_{s_2} \|f\|_{-r},$$

so that taking the supremum over φ with $\|\varphi\|_{s_1} \le 1$, we get (A-3). □

We will also need the following variant.

Lemma A.2. *Suppose that $f \in C^1(\Sigma)$ and $g \in H^{1/2}(\Sigma)$. Then*

$$\|fg\|_{1/2} \lesssim \|f\|_{C^1} \|g\|_{1/2}.$$

Proof. Consider the operator $F : H^k \to H^k$ given by $F(g) = fg$ for $k = 0, 1$. It is a bounded operator for $k = 0, 1$ since

$$\|fg\|_0 \le \|f\|_{C^1} \|g\|_0 \quad \text{and} \quad \|fg\|_1 \lesssim \|f\|_{C^1} \|g\|_1.$$

Then the theory of interpolation of operators implies that F is bounded from $H^{1/2}$ to itself, with operator norm less than a constant times $\sqrt{\|f\|_{C^1}}\sqrt{\|f\|_{C^1}} = \|f\|_{C^1}$, which is the desired result. □

Identities involving $\mathcal{A}$. We now record some useful identities involving $\mathcal{A}$, as defined by (1-3).

Lemma A.3. *The following hold.*

(1) *For each $j = 1, 2, 3$, we have that $\partial_k (J \mathcal{A}_{jk}) = 0$.*

(2) *On Σ, we have that $J\mathcal{A}e_3 = \mathcal{N}$, while on Σ_b, we have that $J\mathcal{A}e_3 = e_3$.*

(3) *Let R be defined by (4-8). Then $R^T \mathcal{N} = -\partial_t \mathcal{N}$ on Σ.*

Proof. The first item may be verified by a simple computation. The first part of the second item holds since $\tilde{b} = 1$ on Σ, which means that

$$J\mathcal{A}e_3 = -Ae_1 - Be_2 + e_3 = -\partial_1 \bar{\eta} e_1 - \partial_2 \bar{\eta} e_2 + e_3 = -\partial_1 \eta e_1 - \partial_2 \eta e_2 + e_3 = \mathcal{N}$$

on Σ. The second part of the third item follows similarly since $\tilde{b} = 0$ on Σ_b. For the third item, we compute $R^T = -K\partial_t J - \partial_t \mathcal{A}\mathcal{A}^{-1}$. Then, using the second item, we find that, on Σ,

$$\begin{aligned} R^T \mathcal{N} &= (-K\partial_t J - \partial_t \mathcal{A}\mathcal{A}^{-1}) J\mathcal{A}e_3 = -\partial_t J \mathcal{A}e_3 - J\partial_t \mathcal{A}e_3 \\ &= \begin{pmatrix} -AJ\partial_t K \\ -BJ\partial_t K \\ \partial_t JK \end{pmatrix} + \begin{pmatrix} \partial_t A + AJ\partial_t K \\ \partial_t B + BJ\partial_t K \\ -J\partial_t K \end{pmatrix} = \begin{pmatrix} \partial_t A \\ \partial_t B \\ 0 \end{pmatrix} = -\partial_t \mathcal{N}. \end{aligned}$$ □

Continuity and temporal derivatives. We will need the following interpolation result, which affords us control of the $L^\infty H^k$ norm of a function f, given that we control f in $L^2 H^{k+m}$ and $\partial_t f$ in $L^2 H^{k-m}$.

Lemma A.4. *Let Γ denote either Σ or Ω. Suppose $\zeta \in L^2([0,T]; H^{s_1}(\Gamma))$ and $\partial_t \zeta \in L^2([0,T]; H^{s_2}(\Gamma))$ for $s_1 \ge s_2 \ge 0$. Let $s = (s_1 + s_2)/2$. Then $\zeta \in C^0([0,T]; H^s(\Gamma))$ (after possibly being redefined on a set of measure 0), and*

$$\|\zeta\|^2_{L^\infty H^s} \lesssim \left(1 + \frac{1}{T}\right) \left(\|\zeta\|^2_{L^2 H^{s_1}} + \|\partial_t \zeta\|^2_{L^2 H^{s_2}} \right). \tag{A-4}$$

Proof. According to the usual theory of extensions and restrictions in Sobolev spaces, it suffices to prove the result with $\Gamma = \mathbb{R}^n$ or $\Gamma = (L_1\mathbb{T}) \times (L_2\mathbb{T}) \times \mathbb{R}^m$ for $n = 2, 3$, $m = 0, 1$. We will prove the result assuming that $\Gamma = \mathbb{R}^n$; the proof in the other case may be derived similarly, replacing integrals in Fourier space with sums, etc. Assume for the moment that ζ is smooth. Writing $\widehat{\ }$ for the Fourier transform, we compute

$$\begin{aligned}\partial_t \|\zeta(t)\|_s^2 &= 2\Re\left(\int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \hat{\zeta}(\xi,t)\overline{\partial_t \hat{\zeta}(\xi,t)}\, d\xi\right) \le 2\int_{\mathbb{R}^n} \langle\xi\rangle^{2s} |\hat{\zeta}(\xi,t)|\, |\partial_t \hat{\zeta}(\xi,t)|\, d\xi \\ &= 2\int_{\mathbb{R}^n} \langle\xi\rangle^{s_1} |\hat{\zeta}(\xi,t)|\, \langle\xi\rangle^{s_2} |\partial_t \hat{\zeta}(\xi,t)|\, d\xi \le \int_{\mathbb{R}^n} \langle\xi\rangle^{2s_1} |\hat{\zeta}(\xi,t)|^2\, d\xi + \int_{\mathbb{R}^n} \langle\xi\rangle^{2s_2} |\partial_t \hat{\zeta}(\xi,t)|^2\, d\xi \\ &= \|\zeta(t)\|_{s_1}^2 + \|\partial_t \zeta(t)\|_{s_2}^2.\end{aligned}$$

Hence for $r, t \in [0, T]$, we have that $\|\zeta(t)\|_s^2 \le \|\zeta(r)\|_s^2 + \|\zeta\|_{L^2 H^{s_1}}^2 + \|\partial_t \zeta\|_{L^2 H^{s_2}}^2$. We can then integrate both sides of this inequality with respect to $r \in [0, T]$ to deduce the bound

$$\sup_{0\le t\le T} \|\zeta(t)\|_s^2 \le \frac{1}{T}\|\zeta\|_{L^2 H^s}^2 + \|\zeta\|_{L^2 H^{s_1}}^2 + \|\partial_t \zeta\|_{L^2 H^{s_2}}^2 \lesssim \left(1 + \frac{1}{T}\right)\left(\|\zeta\|_{L^2 H^{s_1}}^2 + \|\partial_t \zeta\|_{L^2 H^{s_2}}^2\right). \tag{A-5}$$

If ζ is not smooth, we may employ a standard mollification argument (see [Evans 2010, Section 5.9]) in conjunction with (A-5) to deduce that $\zeta \in C^0([0, T]; H^s(\mathbb{R}^n))$ and that (A-4) holds. □

Extension results. In our well-posedness arguments, we need to be able to take the initial data $\partial_t^j u(0)$, $j = 0, \dots, 2N$ and extend it to a function u satisfying $\mathfrak{K}_{2N}(u) \lesssim \mathfrak{E}_0(u, 0)$, defined by (5-2) and (5-7), respectively. This extension is the content of the following lemma.

Lemma A.5. *Suppose that $\partial_t^j u(0) \in H^{4N-2j}(\Omega)$ for $j = 0, \dots, 2N$. Then there exists an extension u, achieving the initial data, so that*

$$\partial_t^j u \in L^2\big([0,\infty); H^{4N-2j+1}(\Omega)\big) \cap L^\infty\big([0,\infty); H^{4N-2j}(\Omega)\big)$$

for $j = 0, \dots, 2N$. Moreover, $\mathfrak{K}_{2N}(u) \lesssim \mathfrak{E}_0(u, 0)$, where in the definition of $\mathfrak{K}_{2N}(u)$ we take $T = \infty$.

Proof. Owing to the usual theory of extensions and restrictions in Sobolev spaces, it suffices to prove the result with Ω replaced by $\mathbb{R}^3$ in the nonperiodic case and $(L_1\mathbb{T}) \times (L_2\mathbb{T}) \times \mathbb{R}$ in the periodic case. The proof in the periodic case can be derived from the nonperiodic proof by trivially changing some integrals over frequencies to sums; thus we present only the proof in $\mathbb{R}^3$.

Let $f_j \in H^{4N-2j}(\mathbb{R}^3)$ denote the spatial extension of $\partial_t^j u(0) \in H^{4N-2j}(\Omega)$. It suffices to construct $F_j(x,t)$ for $j = 0, \dots, 2N$ so that $\partial_t^k F_j(x, 0) = \delta_{j,k} f_j(x)$ ($\delta_{j,k}$ is the Kronecker delta) and

$$\|\partial_t^k F_j\|_{L^2 H^{4N-2k+1}}^2 + \|\partial_t^k F_j\|_{L^\infty H^{4N-2k}}^2 \lesssim \|f_j\|_{4n-2j}^2 \tag{A-6}$$

for $k = 0, \dots, 2N$. Indeed, with such F_j in hand, the sum $F = \sum_{j=0}^{2N} F_j$ is the desired extension. Note that in the norms of (A-6), the symbol $L^p H^m$ denotes $L^p([0,\infty); H^m(\mathbb{R}^3))$.

Let $\varphi_j \in C_c^\infty(\mathbb{R})$ be such that $\varphi_j^{(k)}(0) = \delta_{j,k}$ for $k = 0, \dots, 2N$ (here (k) is the number of derivatives). We then define $\hat{F}_j(\xi, t) = \varphi_j(t\langle\xi\rangle^2)\hat{f}_j(\xi)\langle\xi\rangle^{-2j}$, where $\hat{\ }$ denotes the Fourier transform and $\langle\xi\rangle = \sqrt{1 + |\xi|^2}$.

By construction, $\partial_t^k \hat{F}_j(\xi,t) = \varphi_j^{(k)}(t\langle\xi\rangle^2)\hat{f}_j(\xi)\langle\xi\rangle^{2(k-j)}$, so that $\partial_t^k F(\,\cdot\,,0) = \delta_{j,k} f_j$. We estimate

$$\begin{aligned}\left\|\partial_t^k F_j(\,\cdot\,,t)\right\|_{4N-2k}^2 &= \int_{\mathbb{R}^3} \langle\xi\rangle^{2(4N-2k)} \left|\varphi_j^{(k)}(t\langle\xi\rangle^2)\right|^2 \left|\hat{f}_j(\xi)\right|^2 \langle\xi\rangle^{2(2k-2j)}\,d\xi \\ &= \int_{\mathbb{R}^3} \left|\varphi_j^{(k)}(t\langle\xi\rangle^2)\right|^2 \left|\hat{f}_j(\xi)\right|^2 \langle\xi\rangle^{2(4N-2j)}\,d\xi \le \left\|\varphi_j^{(k)}\right\|_{L^\infty}^2 \left\|f_j\right\|_{4N-2j}^2,\end{aligned}$$

so that $\|\partial_t^k F_j\|_{L^\infty H^{4N-2k}}^2 \lesssim \|f_j\|_{4N-2j}^2$. Similarly,

$$\begin{aligned}\left\|\partial_t^k F_j\right\|_{L^2 H^{4N-2k+1}}^2 &= \int_0^\infty \int_{\mathbb{R}^3} \langle\xi\rangle^{2(4N-2k+1)} \left|\varphi_j^{(k)}(t\langle\xi\rangle^2)\right|^2 \left|\hat{f}_j(\xi)\right|^2 \langle\xi\rangle^{2(2k-2j)}\,d\xi\,dt \\ &= \int_0^\infty \int_{\mathbb{R}^3} \left|\varphi_j^{(k)}(t\langle\xi\rangle^2)\right|^2 \left|\hat{f}_j(\xi)\right|^2 \langle\xi\rangle^{2(4N-2j+1)}\,d\xi\,dt \\ &= \int_{\mathbb{R}^3} \left|\hat{f}_j(\xi)\right|^2 \langle\xi\rangle^{2(4N-2j+1)} \left(\int_0^\infty \left|\varphi_j^{(k)}(t\langle\xi\rangle^2)\right|^2 dt\right) d\xi \\ &= \int_{\mathbb{R}^3} \left|\hat{f}_j(\xi)\right|^2 \langle\xi\rangle^{2(4N-2j+1)} \left(\frac{1}{\langle\xi\rangle^2}\int_0^\infty \left|\varphi_j^{(k)}(r)\right|^2 dr\right) d\xi \\ &= \left\|\varphi_j^{(k)}\right\|_{L^2}^2 \int_{\mathbb{R}^3} \left|\hat{f}_j(\xi)\right|^2 \langle\xi\rangle^{2(4N-2j)}\,d\xi = \left\|\varphi_j^{(k)}\right\|_{L^2}^2 \left\|f_j\right\|_{4N-2j}^2,\end{aligned} \tag{A-7}$$

so that $\|\partial_t^k F_j\|_{L^2 H^{4N-2k+1}}^2 \lesssim \|f_j\|_{4N-2j}^2$. Note that in (A-7), we have used Fubini's theorem to switch the order of integration; this is possible since φ is compactly supported. We then have that F_j satisfies the desired properties, completing the proof. □

A similar result can be proved for the pressure. We omit the proof.

Lemma A.6. *Suppose that* $\partial_t^j p(0) \in H^{4N-2j-1}(\Omega)$ *for* $j = 0, \dots, 2N-1$. *Then there exists an extension* p, *achieving the initial data*, *such that*

$$\partial_t^j p \in L^2\big([0,\infty); H^{4N-2j}(\Omega)\big) \cap L^\infty\big([0,\infty); H^{4N-2j-1}(\Omega)\big)$$

for $j = 0, \dots, 2N-1$. *Moreover*,

$$\sum_{j=0}^{2N-1} \left\|\partial_t^j p\right\|_{L^2 H^{4N-2j}}^2 + \left\|\partial_t^j p\right\|_{L^\infty H^{4N-2j-1}}^2 \lesssim \sum_{j=0}^{2N-1} \left\|\partial_t^j p(0)\right\|_{H^{4N-2j-1}}^2.$$

Poisson integral: nonperiodic case. For a function f, defined on $\Sigma = \mathbb{R}^2$, the Poisson integral in $\mathbb{R}^2 \times (-\infty, 0)$ is defined by

$$\mathcal{P} f(x', x_3) = \int_{\mathbb{R}^2} \hat{f}(\xi) e^{2\pi|\xi|x_3} e^{2\pi i x'\cdot\xi}\,d\xi. \tag{A-8}$$

Although $\mathcal{P} f$ is defined in all of $\mathbb{R}^2 \times (-\infty, 0)$, we will only need bounds on its norm in the restricted domain $\Omega = \mathbb{R}^2 \times (-b, 0)$. This yields a couple improvements of the usual estimates of $\mathcal{P} f$ on the set $\mathbb{R}^2 \times (-\infty, 0)$.

Lemma A.7. *Let $\mathcal{P} f$ be the Poisson integral of a function f that is either in $\dot{H}^q(\Sigma)$ or $\dot{H}^{q-1/2}(\Sigma)$ for $q \in \mathbb{N}$ (here $\dot{H}^s$ is the usual homogeneous Sobolev space of order s). Then*

$$\|\nabla^q \mathcal{P} f\|_0^2 \lesssim \int_{\mathbb{R}^2} |\xi|^{2q} |\hat{f}(\xi)|^2 \Big(\frac{1 - e^{-4\pi b|\xi|}}{|\xi|} \Big) \, d\xi, \tag{A-9}$$

and in particular,

$$\|\nabla^q \mathcal{P} f\|_0^2 \lesssim \|f\|^2_{\dot{H}^{q-1/2}(\Sigma)} \quad \textit{and} \quad \|\nabla^q \mathcal{P} f\|_0^2 \lesssim \|f\|^2_{\dot{H}^{q}(\Sigma)}. \tag{A-10}$$

Proof. Employing Fubini, the horizontal Fourier transform, and Parseval, we may bound

$$\begin{aligned} \|\nabla^q \mathcal{P} f\|_0^2 &\lesssim \int_{\mathbb{R}^2} \int_{-b}^0 |\xi|^{2q} \big|\hat{f}(\xi)\big|^2 e^{4\pi|\xi|x_3} \, dx_3 \, d\xi \\ &\leq \int_{\mathbb{R}^2} |\xi|^{2q} \big|\hat{f}(\xi)\big|^2 \Big(\int_{-b}^0 e^{4\pi|\xi|x_3} \, dx_3 \Big) \, d\xi \lesssim \int_{\mathbb{R}^2} |\xi|^{2q} \big|\hat{f}(\xi)\big|^2 \Big(\frac{1 - e^{-4\pi b|\xi|}}{|\xi|} \Big) \, d\xi. \end{aligned} \tag{A-11}$$

This is (A-9). To deduce (A-10) from (A-9), we simply note that

$$\frac{1 - e^{-4\pi b|\xi|}}{|\xi|} \leq \min\Big\{ 4\pi b, \frac{1}{|\xi|} \Big\}, \tag{A-12}$$

which means we are free to bound the right-hand side of (A-11) by either $\|f\|^2_{\dot{H}^{q-1/2}(\Sigma)}$ or $\|f\|^2_{\dot{H}^{q}(\Sigma)}$. □

We will also need L^∞ estimates.

Lemma A.8. *Let $\mathcal{P} f$ be the Poisson integral of f, defined on Σ. Let $q \in \mathbb{N}$, $s > 1$. Then*

$$\|\nabla^q \mathcal{P} f\|^2_{L^\infty} \lesssim \|D^q f\|^2_s. \tag{A-13}$$

Proof. We use the definition of $\mathcal{P} f$ and the trivial estimate $\exp(2\pi|\xi|x_3) \leq 1$ in Ω to get the bound

$$\|\nabla^q \mathcal{P} f\|_{L^\infty} \lesssim \int_{\mathbb{R}^2} |\xi|^q \big|\hat{f}(\xi)\big| \, d\xi.$$

The estimate (A-13) then follows from this and the easy bound

$$\int_{\mathbb{R}^2} |\xi|^q \big|\hat{f}(\xi)\big| \, d\xi \lesssim \|D^q f\|_s \Big(\int_{\mathbb{R}^2} \langle \xi \rangle^{-2s} d\xi \Big)^{1/2} \lesssim \|D^q f\|_s,$$

which holds when $s > 1$. □

Poisson integral: periodic case. Suppose that $\Sigma = (L_1 \mathbb{T}) \times (L_2 \mathbb{T})$. We define the Poisson integral in $\Omega_- = \Sigma \times (-\infty, 0)$ by

$$\mathcal{P} f(x) = \sum_{n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z})} e^{2\pi i n \cdot x'} e^{2\pi|n|x_3} \hat{f}(n), \tag{A-14}$$

where for $n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z})$, we have written

$$\hat{f}(n) = \int_\Sigma f(x') \frac{e^{-2\pi i n \cdot x'}}{L_1 L_2} \, dx'.$$

It is well known that $\mathcal{P} : H^s(\Sigma) \to H^{s+1/2}(\Omega_-)$ is a bounded linear operator for $s > 0$. We now show how derivatives of $\mathcal{P} f$ can be estimated in the smaller domain Ω.

Lemma A.9. *Let $\mathcal{P} f$ be the Poisson integral of a function f that is either in $\dot{H}^q(\Sigma)$ or $\dot{H}^{q-1/2}(\Sigma)$ for $q \in \mathbb{N}$. Then*

$$\|\nabla^q \mathcal{P} f\|_0^2 \lesssim \|f\|^2_{\dot{H}^{q-1/2}(\Sigma)} \quad \text{and} \quad \|\nabla^q \mathcal{P} f\|_0^2 \lesssim \|f\|^2_{\dot{H}^{q}(\Sigma)}.$$

Proof. Since $\mathcal{P} f$ is defined on $\Sigma \times (-\infty, 0)$, it suffices to prove the estimates on $\tilde{\Omega} := \Sigma \times (-b_+, 0)$ with $b_+ = \sup_{x' \in \Sigma} b$ since $\Omega \subset \tilde{\Omega}$. By Fubini and Parseval,

$$\begin{aligned} \left\|\nabla^q \mathcal{P} f\right\|^2_{H^0(\tilde{\Omega})} &\lesssim \sum_{n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z})} \int_{-b_+}^0 |n|^{2q} \left|\hat{f}(n)\right|^2 e^{4\pi |n| x_3} \, dx_3 \\ &\lesssim \sum_{n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z})} |n|^{2q} \left|\hat{f}(n)\right|^2 \left(\frac{1 - e^{-4\pi b_+ |n|}}{|n|}\right). \end{aligned} \tag{A-15}$$

However,

$$\frac{1 - e^{-4\pi b_+ |n|}}{|n|} \le \min\left\{4\pi b_+, \frac{1}{|n|}\right\},$$

which means we are free to bound the right-hand side of (A-15) by either $\|f\|^2_{\dot{H}^{q-1/2}(\Sigma)}$ or $\|f\|^2_{\dot{H}^{q}(\Sigma)}$. □

We will also need L^∞ estimates.

Lemma A.10. *Let $\mathcal{P} f$ be the Poisson integral of a function f that is in $\dot{H}^{q+s}(\Sigma)$ for $q \ge 1$ an integer and $s > 1$. Then*

$$\|\nabla^q \mathcal{P} f\|^2_{L^\infty} \lesssim \|f\|^2_{\dot{H}^{q+s}}.$$

The same estimate holds for $q = 0$ if f satisfies $\displaystyle\int_\Sigma f = 0$.

Proof. We estimate

$$\|\nabla^q \mathcal{P} f\|_{L^\infty} \lesssim \sum_{n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z})} \left|\hat{f}(n)\right| |n|^q \lesssim \|f\|_{\dot{H}^{q+s}} \Bigg(\sum_{n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z}) \setminus \{0\}} |n|^{-2s} \Bigg)^{1/2} \lesssim \|f\|_{\dot{H}^{q+s}}$$

if $s > 1$. The same estimate works with $q = 0$ if $\hat{f}(0) = 0$. □

Transport estimate. Let Σ be either periodic or nonperiodic. Consider the equation

$$\begin{cases} \partial_t \eta + u \cdot D\eta = g & \text{in } \Sigma \times (0, T), \\ \eta(t = 0) = \eta_0, \end{cases} \tag{A-16}$$

with $T \in (0, \infty]$. We have the following estimate of the transport of regularity for solutions to (A-16), which is a particular case of a more general result proved in [Danchin 2005a]. Note that the result in [Danchin 2005a] is stated for $\Sigma = \mathbb{R}^2$, but the same result holds in the periodic setting $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$, as described in [Danchin 2005b].

Lemma A.11 [Danchin 2005a, Proposition 2.1]. *Let η be a solution to (A-16). Then there is a universal constant $C > 0$ such that for any $0 \le s < 2$,*

$$\sup_{0\le r\le t} \|\eta(r)\|_{H^s} \le \exp\left(C\int_0^t \|Du(r)\|_{H^{3/2}}\,dr\right)\left(\|\eta_0\|_{H^s} + \int_0^t \|g(r)\|_{H^s}\,dr\right).$$

Proof. Use $p = p_2 = 2$, $N = 2$, and $\sigma = s$ in Proposition 2.1 of [Danchin 2005a] along with the embedding $H^{3/2} \hookrightarrow B^1_{2,\infty} \cap L^\infty$. □

Poincaré-type inequalities. Let Σ and Ω be either periodic or nonperiodic.

Lemma A.12. *We have*

$$\|f\|^2_{L^2(\Omega)} \lesssim \|f\|^2_{L^2(\Sigma)} + \|\partial_3 f\|^2_{L^2(\Omega)} \tag{A-17}$$

for all $f \in H^1(\Omega)$. Also, if $f \in W^{1,\infty}(\Omega)$, then

$$\|f\|^2_{L^\infty(\Omega)} \lesssim \|f\|^2_{L^\infty(\Sigma)} + \|\partial_3 f\|^2_{L^\infty(\Omega)}. \tag{A-18}$$

Proof. By density, we may assume that f is smooth. Writing $x = (x', x_3)$ for $x' \in \Sigma$ and $x_3 \in (-b(x'), 0)$, we have

$$\begin{aligned}|f(x',x_3)|^2 &= |f(x',0)|^2 - 2\int_{x_3}^0 f(x',z)\partial_3 f(x',z)\,dz\\ &\le |f(x',0)|^2 + 2\int_{-b(x')}^0 |f(x',z)||\partial_3 f(x',z)|\,dz.\end{aligned}$$

We may integrate this with respect to $x_3 \in (-b(x'), 0)$ to get

$$\int_{-b(x')}^0 |f(x',x_3)|^2\,dx_3 \lesssim |f(x',0)|^2 + 2\int_{-b(x')}^0 |f(x',z)||\partial_3 f(x',z)|\,dz.$$

Now we integrate over $x' \in \Sigma$ to find

$$\int_\Omega |f(x)|^2\,dx \lesssim \|f\|^2_{L^2(\Sigma)} + 2\int_\Omega |f(x)||\partial_3 f(x)|\,dx \le \|f\|^2_{L^2(\Sigma)} + \varepsilon\,\|f\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\,\|\partial_3 f\|^2_{L^2(\Omega)}$$

for any $\varepsilon > 0$. Choosing $\varepsilon > 0$ sufficiently small then yields (A-17). The estimate (A-18) follows similarly, taking suprema rather than integrating. □

We will need a version of Korn's inequality, proved, for instance, in Lemma 2.7 of [Beale 1981].

Lemma A.13. *We have $\|u\|_1 \lesssim \|\mathbb{D}u\|_0$ for all $u \in H^1(\Omega;\mathbb{R}^3)$ such that $u = 0$ on Σ_b.*

We also record the standard Poincaré inequality, which applies for functions taking either vector or scalar values.

Lemma A.14. *We have $\|f\|_0 \lesssim \|f\|_1 \lesssim \|\nabla f\|_0$ for all $f \in H^1(\Omega)$ such that $f = 0$ on Σ_b. Also, $\|f\|_{L^\infty(\Omega)} \lesssim \|f\|_{W^{1,\infty}(\Omega)} \lesssim \|\nabla f\|_{L^\infty(\Omega)}$ for all $f \in W^{1,\infty}(\Omega)$ such that $f = 0$ on Σ_b.*

An elliptic estimate. The proof of the following estimate may be found in [Beale 1981] in the nonperiodic case. The same proof holds in the periodic case with obvious modification.

Lemma A.15. *Suppose* (u, p) *solve*

$$\begin{cases} -\Delta u + \nabla p = \phi \in H^{r-2}(\Omega), \\ \operatorname{div} u = \psi \in H^{r-1}(\Omega), \\ (pI - \mathbb{D}(u))e_3 = \alpha \in H^{r-3/2}(\Sigma), \\ u|_{\Sigma_b} = 0. \end{cases}$$

Then, for $r \geq 2$,

$$\|u\|_{H^r}^2 + \|p\|_{H^{r-1}}^2 \lesssim \|\phi\|_{H^{r-2}}^2 + \|\psi\|_{H^{r-1}}^2 + \|\alpha\|_{H^{r-3/2}}^2 .$$

Integration by parts. Here we record a temporal integration-by-parts equation. We assume throughout that η is sufficiently regular that J and $\mathcal{A}$ are $C^1([0, T]; L^\infty(\Omega))$.

Lemma A.16. *Suppose that*

$$\begin{aligned} &p \in C^0\big([0, T]; H^0(\Omega)\big), \\ &w \in C^0\big([0, T]; H^0(\Omega)\big) \cap L^2\big([0, T]; {}_0H^1(\Omega)\big), \\ &\operatorname{div}_{\mathcal{A}} w = F \in H^1\big((0, T); H^0(\Omega)\big). \end{aligned}$$

Define $P \in C^0\big([0, T]; ({}_0H^1(\Omega))^*\big)$ *via* $\langle P, v\rangle_* = (p, \operatorname{div}_{\mathcal{A}} v)_0$. *Suppose also that*

$$\partial_t(Jw - P) \in L^2\big([0, T]; ({}_0H^1(\Omega))^*\big).$$

Then, for any $0 \leq s \leq t \leq T$, *we have*

$$\tfrac{1}{2}\|w(t)\|_0^2 - \tfrac{1}{2}\|w(s)\|_0^2 - \big(p(t), F(t)\big)_0 + \big(p(s), F(s)\big)_0 = \int_s^t \langle \partial_t(Jw - P), w\rangle_* \int_s^t \int_\Omega -\tfrac{1}{2}\partial_t J|w|^2 + p\partial_t(J\mathcal{A}_{ij})\partial_j w_i - p\partial_t(JF). \quad \text{(A-19)}$$

Proof. Step 1: Mollification. Let $\varphi \in C_c^\infty(\mathbb{R})$ be such that $\varphi(t) = 1$ for $t \in [-2T, 2T]$. We define $\overline{w} \in C_c^0(\mathbb{R}; H^0(\Omega))$ via

$$\overline{w} = \varphi \tilde{w}, \quad \text{where } \tilde{w}(t) := \begin{cases} w(0) & t < 0, \\ w(t) & 0 \leq t \leq T, \\ w(T) & t \geq T. \end{cases}$$

Similarly, we define $\bar{p} \in C_c^0(\mathbb{R}; H^0(\Omega))$ via

$$\bar{p} = \varphi \tilde{p}, \quad \text{where } \tilde{p}(t) := \begin{cases} p(0) & t < 0, \\ p(t) & 0 \leq t \leq T, \\ p(T) & t \geq T. \end{cases}$$

Also let $\overline{F} \in H^1(\mathbb{R}; H^0(\Omega))$ denote a bounded extension of F to all of $\mathbb{R}$ such that $\operatorname{supp}(\overline{F}) \subset [-2T, 2T]$.

Now we let ψ_ε be the usual $1-D$ approximate identity (satisfying $\psi_\varepsilon(x) = \psi(x/\varepsilon)/\varepsilon$ for $\psi \in C_c^\infty(\mathbb{R})$ with $0 \le \psi$, $\operatorname{supp}(\psi) \subset (-1, 1)$, and $\int \psi = 1$) and define

$$\begin{aligned} w_\varepsilon &:= \psi_\varepsilon * \overline{w} \in C_c^\infty(\mathbb{R}; H^0(\Omega)), \\ p_\varepsilon &:= \psi_\varepsilon * \bar{p} \in C_c^\infty(\mathbb{R}; H^0(\Omega)), \\ F_\varepsilon &:= \psi_\varepsilon * \overline{F} \in C_c^\infty(\mathbb{R}; H^0(\Omega)). \end{aligned}$$

Let us define

$$P_\varepsilon \in C^1\big((0, T); ({}_0H^1(\Omega))^*\big)$$

via

$$\langle P_\varepsilon, v\rangle_* = \int_\Omega p_\varepsilon J \operatorname{div}_{\mathcal{A}} v.$$

The usual properties of mollifiers imply that

$$\begin{aligned} p_\varepsilon &\to p \quad \text{in } C^0\big([0, T]; H^0(\Omega)\big), \\ w_\varepsilon &\to w \quad \text{in } C^0\big([0, T]; H^0(\Omega)\big), \\ w_\varepsilon &\to w \quad \text{in } L^2\big([0, T]; {}_0H^1(\Omega)\big), \\ F_\varepsilon &\to F \quad \text{in } H^1\big((0, T); H^0(\Omega)\big), \\ P_\varepsilon &\to P \quad \text{in } C^0\big([0, T]; ({}_0H^1(\Omega))^*\big). \end{aligned} \tag{A-20}$$

Step 2: Computation. Now we define the function

$$f(t) = \tfrac{1}{2}\|w(t)\|_0^2 - (p(t), F(t))_0,$$

which clearly satisfies $f \in C^0([0, T])$. We also define

$$f_\varepsilon(t) = \tfrac{1}{2}\|w_\varepsilon(t)\|_0^2 - (p_\varepsilon(t), F_\varepsilon(t))_0,$$

which satisfies $f_\varepsilon \in C^1([0, T])$.

Note that since $F_\varepsilon \to F$ in $H^1\big((0, T); H^0(\Omega)\big)$, the Sobolev embedding in one dimension implies that

$$F_\varepsilon \to F \quad \text{in } C^0\big([0, T]; H^0(\Omega)\big).$$

From this and the C^0H^0 convergence results for p_ε and w_ε listed in (A-20), we see that

$$f_\varepsilon \to f \quad \text{in } C^0([0, T]). \tag{A-21}$$

Now, since f_ε is C^1, we may let $0 \le s \le t \le T$ and compute

$$\begin{aligned} f_\varepsilon(t) - f_\varepsilon(s) &= \int_s^t \partial_t f_\varepsilon = \int_s^t \int_\Omega \partial_t(J w_\varepsilon) \cdot w_\varepsilon - \tfrac{1}{2}\partial_t J |w_\varepsilon|^2 - \int_\Omega \partial_t p_\varepsilon J F_\varepsilon + p_\varepsilon \partial_t(J F_\varepsilon) \\ &= \int_s^t \int_\Omega \partial_t(J w_\varepsilon) \cdot w_\varepsilon - \tfrac{1}{2}\partial_t J |w_\varepsilon|^2 - \int_s^t \int_\Omega \partial_t p_\varepsilon J \operatorname{div}_{\mathcal{A}} w_\varepsilon + p_\varepsilon \partial_t(J \mathcal{A}_{ij}) \partial_j w_{\varepsilon,i} \\ &\qquad\qquad - p_\varepsilon \partial_t(J\mathcal{A}_{ij}) \partial_j w_{\varepsilon,i} + \partial_t p_\varepsilon J(F_\varepsilon - \operatorname{div}_{\mathcal{A}} w_\varepsilon) + p_\varepsilon \partial_t(J F_\varepsilon) \\ &= \int_s^t \big\langle \partial_t(J w_\varepsilon - P_\varepsilon), w_\varepsilon \big\rangle_* + \int_\Omega -\tfrac{1}{2}\partial_t J |w_\varepsilon|^2 + p_\varepsilon \partial_t(J\mathcal{A}_{ij}) \partial_j w_{\varepsilon,i} \\ &\qquad\qquad - \partial_t p_\varepsilon J(F_\varepsilon - \operatorname{div}_{\mathcal{A}} w_\varepsilon) - p_\varepsilon \partial_t(J F_\varepsilon). \end{aligned} \tag{A-22}$$

Now we send ε to 0. Note that by Lemma A.17, the p_ε convergence listed in (A-20), and an integration by parts in time, we know that

$$\int_s^t \int_\Omega \partial_t p_\varepsilon J(F_\varepsilon - \operatorname{div}_{\mathcal{A}} w_\varepsilon) \to 0.$$

Similarly, from Lemma A.17 and the w_ε convergence listed in (A-20), we have that

$$\int_s^t \big\langle \partial_t(J w_\varepsilon - P_\varepsilon), w_\varepsilon \big\rangle_* \to \int_s^t \big\langle \partial_t(J w - P), w \big\rangle_*.$$

Then from these and (A-20), we can pass to the limit on the right side of (A-22), and from (A-21) we can pass to the limit on the left. We then get (A-19). □

The next lemma contains some of the convergence results used in the proof of the previous lemma.

Lemma A.17. *We have*

$$\begin{aligned} \operatorname{div}_{\mathcal{A}} w_\varepsilon - F_\varepsilon &\to 0 \quad \textit{in } C^0\big([0,T]; H^0(\Omega)\big) \quad \textit{as } \varepsilon \to 0, \\ \partial_t(\operatorname{div}_{\mathcal{A}} w_\varepsilon - F_\varepsilon) &\to 0 \quad \textit{in } L^2\big([0,T]; H^0(\Omega)\big) \quad \textit{as } \varepsilon \to 0. \end{aligned} \tag{A-23}$$

Also,

$$\partial_t(J w_\varepsilon - P_\varepsilon) \to \partial_t(J w - P) \quad \textit{in } L^2\big([0,T]; ({}_0H^1(\Omega))^*\big) \quad \textit{as } \varepsilon \to 0. \tag{A-24}$$

Proof. *Step 1: Proof of* (A-23). We compute

$$\operatorname{div}_{\mathcal{A}} w_\varepsilon(t) - F_\varepsilon(t) = \int_{\mathbb{R}} \frac{1}{\varepsilon} \psi\Big(\frac{t-s}{\varepsilon}\Big)\big(\mathcal{A}_{ij}(t) - \mathcal{A}_{ij}(s)\big)\partial_j w_i(s)\, ds.$$

Then

$$\begin{aligned} \big\| \operatorname{div}_{\mathcal{A}} w_\varepsilon(t) - F_\varepsilon(t) \big\|_0 &\le \|\partial_t \mathcal{A}_{ij}\|_{C^0 L^\infty} \int_{t-\varepsilon}^{t+\varepsilon} \frac{|t-s|}{\varepsilon} \psi\Big(\frac{t-s}{\varepsilon}\Big) \|\partial_j w_i(s)\|_0 \, ds \\ &\lesssim \|\partial_t \mathcal{A}_{ij}\|_{C^0 L^\infty} \int_{t-\varepsilon}^{t+\varepsilon} \psi\Big(\frac{t-s}{\varepsilon}\Big) \|\partial_j w_i(s)\|_0 \, ds \\ &\lesssim \|\partial_t \mathcal{A}_{ij}\|_{C^0 L^\infty} \sqrt{2\varepsilon} \Big(\int_{t-\varepsilon}^{t+\varepsilon} \|\partial_j w_i(s)\|_0^2 \, ds \Big)^{1/2} \lesssim \sqrt{\varepsilon} \|\partial_t \mathcal{A}_{ij}\|_{C^0 L^\infty} \|w\|_{L^2 H^1}. \end{aligned}$$

Hence

$$\sup_{t \in [0,T]} \big\| \operatorname{div}_{\mathcal{A}} w_\varepsilon(t) - F_\varepsilon(t) \big\|_0 \lesssim \sqrt{\varepsilon} \|\partial_t \mathcal{A}_{ij}\|_{C^0 L^\infty} \|w\|_{L^2 H^1} \to 0. \tag{A-25}$$

Next, we handle the time derivative. We write $\partial_t\big(\operatorname{div}_{\mathcal{A}} w_\varepsilon(t) - F_\varepsilon(t)\big) = I + II$, with

$$I := \int_{\mathbb{R}} \frac{1}{\varepsilon} \psi\Big(\frac{t-s}{\varepsilon}\Big) \partial_t \mathcal{A}_{ij}(t) \partial_j w_i(s)\, ds.$$

Clearly

$$I \to \partial_t \mathcal{A}_{ij} \partial_j w_i \quad \text{in } L^2 L^2 \quad \text{as } \varepsilon \to 0.$$

Also,

$$II = \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\varepsilon}\psi'\Big(\frac{t-s}{\varepsilon}\Big)\Big(\frac{\mathscr{A}_{ij}(t)-\mathscr{A}_{ij}(s)}{\varepsilon}\Big)\partial_j w_i(s)\,ds$$
$$= \int_{-1}^{1} \psi'(r)\Big(\frac{\mathscr{A}_{ij}(t)-\mathscr{A}_{ij}(t-\varepsilon r)}{\varepsilon}\Big)\partial_j w_i(t-\varepsilon r)\,dr.$$

Note that

$$\int_{\mathbb{R}} r\psi'(r)\,dr = -\int_{\mathbb{R}} \psi(r)\,dr = -1.$$

Hence, for any $k \in L^2H^0$, we have

$$II(t)-k(t) = \int_{-1}^{1} \psi'(r)\Big[\Big(\frac{\mathscr{A}_{ij}(t)-\mathscr{A}_{ij}(t-\varepsilon r)}{\varepsilon}\Big)\partial_j w_i(t-\varepsilon r) + k(t)r\Big]\,dr.$$

From this we see that if we choose

$$k(t) = -\partial_t\mathscr{A}_{ij}(t)\partial_j w_i(t) \in L^2H^0,$$

then

$$II - k \to 0 \quad \text{in } L^2H^0.$$

Hence

$$\partial_t\big(\operatorname{div}_{\mathscr{A}} w_\varepsilon(t) - F_\varepsilon(t)\big) = I + II \to 0 \quad \text{in } L^2H^0,$$

which together with (A-25) is (A-23).

Step 2: Proof of (A-24). Since $Jw - P \in H^1\big((0,T);\,({}_0H^1(\Omega))^*\big)$, the usual theory of mollifiers shows that

$$\psi_\varepsilon * (Jw-P) \to Jw - P \quad \text{in } H^1\big((0,T);\,({}_0H^1(\Omega))^*\big) \quad \text{as } \varepsilon \to 0.$$

Hence, to prove (A-24) it suffices to prove that

$$\partial_t\big[(Jw_\varepsilon - P_\varepsilon) - \psi_\varepsilon * (Jw-P)\big] \to 0 \quad \text{in } L^2\big([0,T];\,({}_0H^1(\Omega))^*\big) \quad \text{as } \varepsilon \to 0.$$

This convergence may be deduced by modifying the argument used above in Step 1. □

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HYPOELLIPTICITY AND NONHYPOELLIPTICITY FOR SUMS OF SQUARES OF COMPLEX VECTOR FIELDS

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In this paper we consider a model sum of squares of complex vector fields in the plane, close to Kohn's operator but with a point singularity,

$$P = BB^* + B^*(t^{2l} + x^{2k})B, \quad B = D_x + i x^{q-1} D_t.$$

The characteristic variety of P is the symplectic real analytic manifold $x = \xi = 0$. We show that this operator is C^∞-hypoelliptic and Gevrey hypoelliptic in G^s, the Gevrey space of index s, provided $k < lq$, for every $s \geq lq/(lq-k) = 1 + k/(lq-k)$. We show that in the Gevrey spaces below this index, the operator is not hypoelliptic. Moreover, if $k \geq lq$, the operator is not even hypoelliptic in C^∞. This fact leads to a general negative statement on the hypoellipticity properties of sums of squares of complex vector fields, even when the complex Hörmander condition is satisfied.

1. Introduction

In [Kohn 2005] (and [Bove et al. 2006]; see below) the operator

$$E_{m,k} = L_m \overline{L_m} + \overline{L_m} |z|^{2k} L_m, \quad L_m = \frac{\partial}{\partial z} - i\bar{z}|z|^{2(m-1)} \frac{\partial}{\partial t},$$

was introduced and shown to be hypoelliptic, yet to lose $2 + (k-1)/m$ derivatives in L^2 Sobolev norms. Christ [2005] showed that the addition of one more variable destroys hypoellipticity altogether. In those seminal works, $m = 1$, but Kohn, A. Bove, M. Derridj, and D. S. Tartakoff generalized the results to higher m in [Bove et al. 2006] and elsewhere.

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Subsequently, Bove and Tartakoff [2010] showed that Kohn's operator with an added Oleinik-type singularity, of the form studied in [Bove and Tartakoff 1997],

$$E_{m,k} + |z|^{2(p-1)} D_y^2,$$

is Gevrey s-hypoelliptic for any $s \geq 2m/(p-k)$ (here $2m > p > k$). A related result is that the "real" version, with $X = D_x + i x^{q-1} D_t$, where $D_x = i^{-1}\partial_x$,

$$R_{q,k} + x^{2(p-1)} D_y^2 = XX^* + (x^k X)^*(x^k X) + x^{2(p-1)} D_y^2$$

is sharply Gevrey s-hypoelliptic for any $s \geq q/(p-k)$, where $q > p > k$ and q is an even integer.

In this paper we consider the operator

$$P = BB^* + B^*(t^{2l} + x^{2k})B, \quad B = D_x + i x^{q-1} D_t, \tag{1-1}$$

where k, l and q are positive integers, q even; see [Bove et al. 2010].

Observe that P is a sum of three squares of complex vector fields, but with a small change not altering the results, we might make P a sum of two squares of complex vector fields in two variables, depending on the same parameters: for example, $BB^* + B^*(t^{2l} + x^{2k})^2 B$.

Let us also note that the characteristic variety of P is $\{x = 0, \xi = 0\}$, a codimension-two real analytic symplectic submanifold of $T^*\mathbb{R}^2 \setminus 0$, as in the case of Kohn's operator. Moreover, the Poisson–Treves stratification for P has a single stratum, thus coinciding with the characteristic manifold of P.

We want to analyze the hypoellipticity of P, both in C^∞ and in Gevrey classes. As we shall see, the Gevrey classes play an important role. Here are our results:

Theorem 1.1. *Let P be as in (1-1), q even.*

(i) *Suppose that*

$$l > \frac{k}{q}. \tag{1-2}$$

Then P is C^∞-hypoelliptic (in a neighborhood of the origin) with a loss of $2(q-1+k)/q$ derivatives.

(ii) *Assume that (1-2) is satisfied by the parameters l, k and q. Then P is Gevrey s-hypoelliptic for any s, with*

$$s \geq \frac{lq}{lq-k}. \tag{1-3}$$

(iii) *The value in (1-3) for the Gevrey hypoellipticity of P is optimal, that is, P is* ***not*** *Gevrey s-hypoelliptic for any*

$$1 \leq s < \frac{lq}{lq-k}.$$

(iv) *Assume now that*

$$l \leq \frac{k}{q}. \tag{1-4}$$

Then P is not C^∞-hypoelliptic.

It is worth noting that the operator P satisfies the complex Hörmander condition, that is, the brackets of the fields of length up to $k+q$ generate a two-dimensional complex Lie algebra. Note that in the present case the vector fields involved are B^*, $x^k B$ and $t^l B$, but only the first two enter in the brackets spanning $\mathbb{C}^2$. Actually the third vector field, despite being, as we have said, completely irrelevant in computing the elliptic brackets or the characteristic manifold, proves essential for the hypoellipticity of the operator in the sense that it determines whether the operator turns out hypoelliptic (in some sense) or not. As of now we do not have a thorough understanding of this phenomenon.

Corollary 1.2. *The complex Hörmander condition does not imply C^∞-hypoellipticity for sums of squares of complex vector fields.*

The real Hörmander condition, using as vector fields both the real and the imaginary parts of the vector fields defining P, does not imply C^∞-hypoellipticity either.

This also followed from Christ's theorem [2005], but in this case we are in two variables instead of three. We are not aware of any sufficient condition for C^∞-hypoellipticity of sums of squares of complex vector fields, except the result proved in [Kohn 2005], according to which if the (complex) Lie algebra is generated by the fields and their brackets of length at most 2, then the operator is C^∞-hypoelliptic.

Restricting ourselves to the case q even is no loss of generality, since the operator (1-1) corresponding to an odd integer q is plainly hypoelliptic and actually subelliptic, that is, there is a loss of less than two derivatives. This fact is due to special circumstances, that is, that the operator B^* has a trivial kernel in that case. Actually when q is odd, we have the estimate $\|u\|_{1/q} \le C\|B^*u\|$, $u \in C_0^\infty(\Omega)$, with Ω a subset of $\mathbb{R}^2$ that is open and containing the origin. From the straightforward inequality $|\langle Pu, u\rangle| \ge \|B^*u\|^2$, $u \in C_0^\infty(\Omega)$, we deduce that $\|u\|_{1/q}^2 \le C|\langle Pu, u\rangle|$. The latter estimate can be used to prove the hypoellipticity (subellipticity) of P. We stress that Kohn's original operator, in the complex variable z, automatically has an even q, while in the "real case" the parity of q does matter.

We want to discuss the issue of analytic (Gevrey) hypoellipticity. For sums of squares of real vector fields, there is a conjecture due to F. Treves [1999; Bove and Treves 2004] stating a necessary and sufficient condition for analytic hypoellipticity. To this end, one considers the characteristic set of the operator and "decomposes" it into real analytic strata where the symplectic form has constant rank and where the vector fields as well as their brackets up to a certain length have vanishing symbols, but there exists at least a bracket of length greater by one whose symbol does not vanish. Roughly stated, the conjecture says that if every stratum is a symplectic real analytic manifold, then the operator is analytic hypoelliptic. In the case of the operator $R_{q,k}$ (or $E_{m,k}$), the stratification has just one stratum, coinciding with the characteristic manifold, which is also a symplectic manifold. In [Kohn 2005; Bove et al. 2006] it is proved that the operator is both C^∞ and analytic hypoelliptic.

From Theorem 1.1(iii), however, we deduce the following:

Corollary 1.3. *Treves's conjecture does not carry over to sums of squares of complex vector fields.*

We also want to stress microlocal aspects of the theorem: the characteristic manifold of P is symplectic in $T^*\mathbb{R}^2$ of codimension two, and as such it may be identified with $T^*\mathbb{R} \setminus 0 \sim \{(t, \tau) \mid \tau \neq 0\}$ (leaving aside the origin in the τ variable, i.e., the zero section.)

On the other hand, the operator $P(x, t, D_x, \tau)$, thought of as a differential operator in the x-variable depending on (t, τ) as parameters, for $\tau > 0$ has an eigenvalue of the form $\tau^{2/q}(t^{2l} + a(t, \tau))$, possibly multiplied by a nonzero function of t. Here $a(t, \tau)$ denotes a (nonclassical) symbol of order -1 defined for $\tau > 0$ and such that $a(0, \tau) \sim \tau^{-2k/q}$. Thus we may consider the pseudodifferential operator $\Lambda(t, D_t) = \operatorname{Op}\bigl(\tau^{2/q}(t^{2l} + a(t, \tau))\bigr)$ as defined in a microlocal neighborhood of our base point in the characteristic manifold of P. One can show that the hypoellipticity properties of P are shared by Λ; for example, P is C^∞-hypoelliptic if and only if Λ is.

The paper is organized as follows. In Sections 2–4 the operator $\Lambda(t, D_t)$ is computed and its hypoellipticity properties are related to those of P. This is done following ideas of Boutet de Monvel, Helffer and Sjöstrand using a calculus of pseudodifferential operators that degenerate on a symplectic manifold. The sufficient part of the theorem is proved in this way. Since we do not want to encumber an already lengthy paper with too many technical details, we decided to give only a sketchy description of the pseudodifferential calculus, leaving it to the reader to fill in the (classical) proofs.

In order to prove the optimality of the Gevrey index in (1-3), we have to show that the pseudodifferential operator $\Lambda(t, D_t)$ is hypoelliptic in that Gevrey class and not in any better class, that is, not in any class of index closer to 1, the analytic class. We do this in Section 5. This brings in the question of determining the hypoellipticity index for a pseudodifferential operator in one variable. A detailed treatment of the general case is given in [Bove and Mughetti 2013]. In the present case, determining the Gevrey class does not require the detailed construction of a Newton polygon, and things are definitely easier from the technical point of view. This is why we include here the optimality proof for $\Lambda(t, D_t)$.

In Section 6 we prove assertion (iii) of Theorem 1.1. The idea of the proof is to construct a solution of the equation $\Lambda(t, D_t)u = 0$ violating an a priori estimate which is necessary and sufficient for Gevrey hypoellipticity. Such a solution is at first constructed only from a formal point of view. In a second step, we make sure to have estimates allowing us to turn a formal solution into a true solution, albeit of an equation of the form $\Lambda(t, D_t)v = g$, where g, though not zero, is in an optimal Gevrey class $\mathscr{B}^{s_0}$, where these Gevrey classes $\mathscr{B}^s$ are characterized by arbitrarily small constants in the estimates of derivatives.

The proof of assertion (iv) of Theorem 1.1 is done in Section 7 using similar ideas, but one needs less control on the formal solution.

2. The q-pseudodifferential calculus

The idea, attributed by J. Sjöstrand and M. Zworski [2007] to Schur, is essentially a linear algebra remark: assume that the $n \times n$ matrix A has zero in its spectrum with multiplicity one. Then of course A is not invertible, but, denoting by e_0 the zero eigenvector of A, the matrix (in block form)

$$\begin{bmatrix} A & e_0 \\ {}^t e_0 & 0 \end{bmatrix}$$

is invertible as an $(n+1) \times (n+1)$ matrix in $\mathbb{C}^{n+1}$. Here ${}^t e_0$ denotes the row vector e_0.

All we want to do is apply this remark to the operator P whose part BB^* has the same problem as the matrix A, that is, a zero simple eigenvalue. This occurs since q is even. (In the case when q is odd, P is easily seen to be hypoelliptic.)

It is convenient to use self-adjoint derivatives from now on, so the vector field B^* equals $D_x - i x^{q-1} D_t$, where $D_x = i^{-1}\partial_x$. It will also be convenient to write $B(x,\xi,\tau)$ for the symbol of the vector field B, that is, $B(x,\xi,\tau) = \xi + i x^{q-1}\tau$, and analogously for the other vector fields involved. The symbol of P can be written as

$$P(x,t,\xi,\tau) = P_0(x,t,\xi,\tau) + P_{-q}(x,t,\xi,\tau) + P_{-2k}(x,t,\xi,\tau), \tag{2-1}$$

where

$$\begin{aligned} P_0(x,t,\xi,\tau) &= (1+t^{2l})(\xi^2 + x^{2(q-1)}\tau^2) + (-1+t^{2l})(q-1)x^{q-2}\tau, \\ P_{-q}(x,t,\xi,\tau) &= -2l t^{2l-1} x^{q-1}(\xi + i x^{q-1}\tau), \\ P_{-2k}(x,t,\xi,\tau) &= x^{2k}(\xi^2 + x^{2(q-1)}\tau^2) - i2k x^{2k-1}(\xi + i x^{q-1}\tau) + (q-1)x^{2k+q-2}\tau. \end{aligned}$$

It is evident at a glance that the different pieces into which P has been decomposed include terms of different order and vanishing speed. We thus need to say something about the adopted criteria for the above decomposition.

Let μ be a positive number and consider the following canonical dilation in the variables (x,t,ξ,τ):

$$x \to \mu^{-1/q}x, \quad t \to t, \quad \xi \to \mu^{1/q}\xi, \quad \tau \to \mu\tau.$$

It is then evident that P_0 has the homogeneity property

$$P_0\big(\mu^{-1/q}x, t, \mu^{1/q}\xi, \mu\tau\big) = \mu^{2/q} P_0(x,t,\xi,\tau). \tag{2-2}$$

Analogously,

$$P_{-q}\big(\mu^{-1/q}x, t, \mu^{1/q}\xi, \mu\tau\big) = \mu^{2/q-1} P_{-q}(x,t,\xi,\tau) \tag{2-3}$$

and

$$P_{-2k}\big(\mu^{-1/q}x, t, \mu^{1/q}\xi, \mu\tau\big) = \mu^{2/q-(2k)/q} P_{-2k}(x,t,\xi,\tau). \tag{2-4}$$

Now these homogeneity properties help us in identifying some symbol classes suitable for P.

Definition 2.1. Following the ideas of [Boutet de Monvel and Trèves 1974; Boutet de Monvel 1974], we define the class of symbols $S_q^{m,k}(\Omega,\Sigma)$, where Ω is a conic neighborhood of the point $(0,e_2)$ and Σ denotes the characteristic manifold $\{x=0, \xi=0\}$, as the set of all C^∞ functions such that on any conic subset of Ω with compact base,

$$\big|\partial_t^\alpha \partial_\tau^\beta \partial_x^\gamma \partial_\xi^\delta a(x,t,\xi,\tau)\big| \lesssim (1+|\tau|)^{m-\beta-\delta}\left(\frac{|\xi|}{|\tau|} + |x|^{q-1} + \frac{1}{|\tau|^{(q-1)/q}}\right)^{k-\gamma/(q-1)-\delta}. \tag{2-5}$$

We write $S_q^{m,k}$ for $S_q^{m,k}(\mathbb{R}^2\times\mathbb{R}^2,\Sigma)$.

By a straightforward computation (see for example [Boutet de Monvel 1974]), we have $S_q^{m,k} \subset S_q^{m',k'}$ if and only if $m \le m'$ and

$$m - \frac{q-1}{q}k \le m' - \frac{q-1}{q}k'.$$

$S_q^{m,k}$ can be embedded in the Hörmander classes $S_{\rho,\delta}^{m+\frac{q-1}{q}k_-}$, where $k_- = \max\{0, -k\}$ and $\rho = \delta = 1/q \le \frac{1}{2}$. Thus we immediately deduce that

$$P_0 \in S_q^{2,2}, \quad P_{-q} \in S_q^{1,2} \subset S_q^{2,2+\frac{q}{q-1}}, \quad \text{and finally} \quad P_{-2k} \in S_q^{2,2+\frac{2k}{q-1}}.$$

Definition 2.2 [Boutet de Monvel 1974]. With Ω and Σ as specified above, we define the class

$$\mathcal{H}_q^m(\Omega, \Sigma) = \bigcap_{j=1}^{\infty} S_q^{m-j, -\frac{q}{q-1}j}(\Omega, \Sigma).$$

We write $\mathcal{H}_q^m$ for $\mathcal{H}_q^m(\mathbb{R}^2 \times \mathbb{R}^2, \Sigma)$.

Now it is easy to see that P_0, as a differential operator with respect to the variable x, depending on the parameters t, $\tau \ge 1$, has a nonnegative discrete spectrum. Moreover, the dependence on τ of the eigenvalue is particularly simple, because of (2-2). Call $\Lambda_0(t, \tau)$ the lowest eigenvalue of P_0. Then

$$\Lambda_0(t, \tau) = \tau^{2/q} \tilde{\Lambda}_0(t).$$

Moreover, Λ_0 has multiplicity one and $\tilde{\Lambda}_0(0) = 0$, since BB^* has a null eigenvalue with multiplicity one. Denote by $\varphi_0(x, t, \tau)$ the corresponding eigenfunction. Because of (2-2), we have the following properties of φ_0:

(a) For fixed (t, τ), φ_0 is exponentially decreasing with respect to x as $x \to \pm\infty$. In fact, because of (2-2), setting $y = x\tau^{1/q}$, we have $\varphi_0(y, t, \tau) \sim e^{-y^q/q}$.

(b) It is convenient to normalize φ_0 in such a way that $\|\varphi_0(\cdot, t, \tau)\|_{L^2(\mathbb{R}_x)} = 1$. This implies that a factor $\sim\tau^{1/2q}$ appears. Thus we are led to the definition of a Hermite operator (see [Helffer 1977] for more details).

Let $\Sigma_1 = \pi_x \Sigma$ be the space projection of Σ.

Definition 2.3. We write H_q^m for $\mathcal{H}_q^m(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$, the class of all smooth functions in

$$\bigcap_{j=1}^{\infty} S_q^{m-j, -\frac{q}{q-1}j}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1).$$

Here $S_q^{m,k}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$ denotes the set of all smooth functions such that

$$\left|\partial_t^\alpha \partial_\tau^\beta \partial_x^\gamma a(x, t, \tau)\right| \lesssim (1 + |\tau|)^{m-\beta} \left(|x|^{q-1} + \frac{1}{|\tau|^{(q-1)/q}}\right)^{k - \frac{\gamma}{q-1}}. \tag{2-6}$$

Define the action of a symbol $a(x, t, \tau)$ in H_q^m as the map

$$a(x, t, D_t) \colon C_0^\infty(\mathbb{R}_t) \longrightarrow C^\infty(\mathbb{R}^2_{x,t})$$

defined by

$$a(x, t, D_t)u(x, t) = (2\pi)^{-1} \int e^{it\tau} a(x, t, \tau) \hat{u}(\tau)\, d\tau.$$

This operator, modulo a regularizing operator (with respect to the variable t, but locally uniform in x), is called a Hermite operator, and we denote by OPH_q^m the corresponding class.

We need also the adjoint of the Hermite operators defined in Definition 2.3.

Definition 2.4. Let $a \in H_q^m$. We define the map

$$a^*(x, t, D_t) \colon C_0^\infty(\mathbb{R}^2_{x,t}) \longrightarrow C^\infty(\mathbb{R}_t)$$

as

$$a^*(x, t, D_t)u(t) = (2\pi)^{-1} \iint e^{it\tau} \overline{a(x, t, \tau)} \hat{u}(x, \tau)\, dx\, d\tau,$$

where $\hat{u}(x, \tau)$ denotes the Fourier transform of u with respect to the variable t. We denote by OPH_q^{*m} the related set of operators.

Lemma 2.5. *Let $a \in H_q^m$ and $b \in S_q^{m,k}$.*

(i) *The formal adjoint $a(x, t, D_t)^*$ belongs to OPH_q^{*m} and its symbol has the asymptotic expansion*

$$\sigma\big(a(x, t, D_t)^*\big) - \sum_{\alpha=0}^{N-1} \frac{1}{\alpha!} \partial_\tau^\alpha D_t^\alpha \overline{a(x, t, \tau)} \in H_q^{m-N}. \tag{2-7}$$

(ii) *The formal adjoint $(a^*(x, t, D_t))^*$ belongs to OPH_q^m and its symbol has the asymptotic expansion*

$$\sigma\big(a^*(x, t, D_t)^*\big) - \sum_{\alpha=0}^{N-1} \frac{1}{\alpha!} \partial_\tau^\alpha D_t^\alpha a(x, t, \tau) \in H_q^{m-N}. \tag{2-8}$$

(iii) *The formal adjoint $b(x, t, D_x, D_t)^*$ belongs to $\mathrm{OPS}_q^{m,k}$ and its symbol has the asymptotic expansion*

$$\sigma\big(a(x, t, D_x, D_t)^*\big) - \sum_{\alpha=0}^{N-1} \frac{1}{\alpha!} \partial_{(\xi,\tau)}^\alpha D_{(x,t)}^\alpha \overline{a(x, t, \xi, \tau)} \in S_q^{m-N, k-Nq/(q-1)}. \tag{2-9}$$

The following is a lemma on compositions involving the two different types of Hermite operators defined above. First we give a definition of "global" homogeneity:

Definition 2.6. We say that a symbol $a(x, t, \xi, \tau)$ is globally homogeneous (abbreviated g.h.) of degree m if for $\lambda \geq 1$, $a(\lambda^{-1/q} x, t, \lambda^{1/q}\xi, \lambda\tau) = \lambda^m a(x, t, \xi, \tau)$. Analogously, we say that a symbol, independent of ξ, of the form $a(x, t, \tau)$ is globally homogeneous of degree m if $a(\lambda^{-1/q} x, t, \lambda\tau) = \lambda^m a(x, t, \tau)$.

Let $f_{-j}(x, t, \xi, \tau) \in S_q^{m,k+j/(q-1)}$, $j \in \mathbb{N}$; then there exists $f(x, t, \xi, \tau) \in S_q^{m,k}$ such that $f \sim \sum_{j\geq 0} f_{-j}$, that is, $f - \sum_{j=0}^{N-1} f_{-j} \in S_q^{m,k+N/(q-1)}$. Thus f is defined modulo a symbol in

$$S_q^{m,\infty} = \bigcap_{h\geq 0} S_q^{m,h}.$$

Analogously, let f_{-j} be globally homogeneous of degree $m - k(q-1)/q - j/q$ and such that for every $\alpha, \beta \geq 0$ satisfies the estimates

$$\big|\partial_{(t,\tau)}^\gamma \partial_x^\alpha \partial_\xi^\beta f_{-j}(x, t, \xi, \tau)\big| \lesssim \big(|\xi| + |x|^{q-1} + 1\big)^{k-\alpha/(q-1)-\beta}, \quad (x, \xi) \in \mathbb{R}^2, \tag{2-10}$$

for (t,τ) in a compact subset of $\mathbb{R}\times\mathbb{R}\setminus 0$ and every multi-index γ. Then $f_{-j}\in S_q^{m,k+j/(q-1)}$.

Accordingly, let $\varphi_{-j}(x,t,\tau)\in H_q^{m-j/q}$; then there exists $\varphi(x,t,\tau)\in H_q^m$ such that $\varphi\sim\sum_{j\ge 0}\varphi_{-j}$, that is, $\varphi-\sum_{j=0}^{N-1}\varphi_{-j}\in H_q^{m-N/q}$, so that φ is defined modulo a regularizing symbol (with respect to the t variable).

Similarly, let φ_{-j} be globally homogeneous of degree $m-j/q$ and such that for every α, $l\ge 0$ satisfies the estimates

$$\left|\partial_{(t,\tau)}^{\beta}\partial_x^{\alpha}\varphi_{-j}(x,t,\tau)\right|\lesssim\left(|x|^{q-1}+1\right)^{-l-\alpha/(q-1)},\qquad x\in\mathbb{R},\tag{2-11}$$

for (t,τ) in a compact subset of $\mathbb{R}\times\mathbb{R}\setminus 0$ and every multi-index β. Then $\varphi_{-j}\in H_q^{m-j/q}$.

As a matter of fact, in the construction below we deal with asymptotic series of homogeneous symbols.

Next we give a brief description of the composition of the various types of operator introduced so far.

Lemma 2.7 [Helffer 1977, Formula 2.4.9]. *Let $a\in S_q^{m,k}$, $b\in S_q^{m',k'}$, with asymptotic globally homogeneous expansions*

$$a\sim\sum_{j\ge 0}a_{-j},\quad a_{-j}\in S_q^{m,k+j/(q-1)},\ \text{g.h. of degree } m-\frac{q-1}{q}k-\frac{j}{q},$$
$$b\sim\sum_{i\ge 0}b_{-i},\quad b_{-i}\in S_q^{m',k'+i/(q-1)},\ \text{g.h. of degree } m'-\frac{q-1}{q}k'-\frac{i}{q}.$$

Then $a\circ b$ is an operator in $\mathrm{OPS}_q^{m+m',k+k'}$ with

$$\sigma(a\circ b)-\sum_{s=0}^{N-1}\sum_{q\alpha+i+j=s}\frac{1}{\alpha!}\sigma\left(\partial_\tau^\alpha a_{-j}(x,t,D_x,\tau)\circ_x D_t^\alpha b_{-i}(x,t,D_x,\tau)\right)\in S_q^{m+m'-N,k+k'}.\tag{2-12}$$

Here $\circ_x$ denotes the composition with respect to the x-variable.

Lemma 2.8 [Boutet de Monvel 1974, Section 5; Helffer 1977, Sections 2.2, 2.3]. *Let $a\in H_q^m$, $b\in H_q^{m'}$ and $\lambda\in S_{1,0}^{m''}(\mathbb{R}_t\times\mathbb{R}_\tau)$ with homogeneous asymptotic expansions*

$$a\sim\sum_{j\ge 0}a_{-j},\quad a_{-j}\in H_q^{m-j/q},\ \text{g.h. of degree } m-\frac{j}{q},$$
$$b\sim\sum_{i\ge 0}b_{-i},\quad b_{-i}\in H_q^{m'-i/q},\ \text{g.h. of degree } m'-\frac{i}{q},$$
$$\lambda\sim\sum_{l\ge 0}\lambda_{-l},\quad \lambda_{-l}\in S_{1,0}^{m''-l/q},\ \text{homogeneous of degree } m''-\frac{l}{q}.$$

Then:

(i) *$a\circ b^*$ is an operator in $\mathrm{OP}\,\mathscr{H}_q^{m+m'-1/q}(\mathbb{R}^2,\Sigma)$ with*

$$\sigma(a\circ b^*)(x,t,\xi,\tau)-e^{-ix\xi}\sum_{s=0}^{N-1}\sum_{q\alpha+i+j=s}\frac{1}{\alpha!}\partial_\tau^\alpha a_{-j}(x,t,\tau)D_t^\alpha\hat{\bar{b}}_{-i}(\xi,t,\tau)\in\mathscr{H}_q^{m+m'-1/q-N/q},\tag{2-13}$$

where the Fourier transform in $D_t^\alpha \hat{\bar{b}}_{-i}(\xi, t, \tau)$ *is taken with respect to the* x*-variable.*

(ii) $b^* \circ a$ *is an operator in* $\mathrm{OPS}^{m+m'-1/q}_{1,0}(\mathbb{R}_t)$ *with*

$$\sigma(b^* \circ a)(t,\tau) - \sum_{s=0}^{N-1} \sum_{q\alpha+j+i=s} \frac{1}{\alpha!} \int \partial_\tau^\alpha \bar{b}_{-i}(x,t,\tau) D_t^\alpha a_{-j}(x,t,\tau)\, dx \in S^{m+m'-1/q-N/q}_{1,0}(\mathbb{R}_t). \quad (2\text{-}14)$$

(iii) $a \circ \lambda$ *is an operator in* $\mathrm{OPH}^{m+m''}_q$. *Furthermore, its asymptotic expansion is given by*

$$\sigma(a \circ \lambda) - \sum_{s=0}^{N-1} \sum_{q\alpha+j+l=s} \frac{1}{\alpha!} \partial_\tau^\alpha a_{-j}(x,t,\tau) D_t^\alpha \lambda_{-l}(t,\tau) \in H_q^{m+m''-N/q}. \quad (2\text{-}15)$$

Lemma 2.9. *Let* $a(x,t,D_x,D_t)$ *be an operator in the class* $\mathrm{OPS}^{m,k}_q(\mathbb{R}^2,\Sigma)$ *and* $b(x,t,D_t) \in \mathrm{OPH}^{m'}_q$ *with g.h. asymptotic expansions*

$$a \sim \sum_{j\ge 0} a_{-j}, \quad a_{-j} \in S_q^{m,k+j/(q-1)}, \text{ g.h. of degree } m - \frac{q-1}{q}k - \frac{j}{q},$$
$$b \sim \sum_{i\ge 0} b_{-i}, \quad b_{-i} \in H_q^{m'-i/(q-1)}, \quad \text{g.h. of degree } m' - \frac{i}{q}.$$

Then $a \circ b \in \mathrm{OPH}^{m+m'-k(q-1)/q}_q$ *and has a g.h. asymptotic expansion of the form*

$$\sigma(a \circ b) - \sum_{s=0}^{N-1} \sum_{ql+i+j=s} \frac{1}{l!} \partial_\tau^l a_{-j}(x,t,D_x,\tau)\big(D_t^l b_{-i}(\,\cdot\,,t,\tau)\big) \in H_q^{m+m'-k(q-1)/q-N/q}. \quad (2\text{-}16)$$

Lemma 2.10. *Let* $a(x,t,D_x,D_t)$ *be an operator in* $\mathrm{OPS}^{m,k}_q(\mathbb{R}^2,\Sigma)$, *let* $b^*(x,t,D_t) \in \mathrm{OPH}^{*m'}_q$, *and let* $\lambda(t,D_t) \in \mathrm{OPS}^{m''}_{1,0}(\mathbb{R}_t)$, *with homogeneous asymptotic expansions*

$$a \sim \sum_{j\ge 0} a_{-j}, \quad a_{-j} \in S_q^{m,k+j/(q-1)}, \text{ g.h. of degree } m - \frac{q-1}{q}k - \frac{j}{q},$$
$$b \sim \sum_{i\ge 0} b_{-i}, \quad b_{-i} \in H_q^{m'-i/(q-1)}, \quad \text{g.h. of degree } m' - \frac{i}{q},$$
$$\lambda \sim \sum_{l\ge 0} \lambda_{-l}, \quad \lambda_{-l} \in S_{1,0}^{m''-l/q}, \qquad \text{homogeneous of degree } m'' - \frac{l}{q}.$$

Then

(i) $b^*(x,t,D_t) \circ a(x,t,D_x,D_t) \in \mathrm{OPH}_q^{*m+m'-((q-1)/q)k}$ *with g.h. asymptotic expansion*

$$\sigma(b^* \circ a) - \sum_{s=0}^{N-1} \sum_{ql+i+j=s} \frac{1}{l!} D_t^l \big(\overline{a_{-j}}(x,t,D_x,\tau)\big)^* \big(\partial_\tau^l \overline{b_{-i}}(\,\cdot\,,t,\tau)\big) \in H_q{}^{m+m'-k(q-1)/q-N/q}. \quad (2\text{-}17)$$

(ii) $\lambda(t, D_t) \circ b^*(x,t,D_t) \in \mathrm{OPH}_q^{*m'+m''}$ *with asymptotic expansion*

$$\sigma(\lambda \circ b^*) - \sum_{s=0}^{N-1} \sum_{q\alpha+i+l=s} \frac{1}{\alpha!} \partial_\tau^\alpha \lambda_{-l}(t,\tau) D_t^\alpha \overline{b_{-i}}(x,t,\tau) \in H_q^{m'+m''-N/q}. \tag{2-18}$$

The proofs of Lemmas 2.7–2.9 are obtained with the calculus developed by Boutet de Monvel [1974] and Helffer [1977], slightly generalized to handle general q. The proof of Lemma 2.10 is performed taking the adjoint and involves a combinatorial argument; we sketch it here.

Proof. We prove item (i). The proof of (ii) is similar and simpler.

Since

$$b^*(x,t,D_t) \circ a(x,t,D_x,D_t) = \big(a(x,t,D_x,D_t)^* \circ b^*(x,t,D_t)^*\big)^*,$$

using Lemmas 2.5 and 2.7, we first compute

$$\begin{aligned}
&\sigma\big(a(x,t,D_x,D_t)^* \circ b^*(x,t,D_t)^*\big) \\
&\qquad = \sum_{\alpha,l,p,i,j\geq 0} \frac{1}{l!\,\alpha!\,p!} \partial_\tau^{\alpha+p} D_t^\alpha \big(a_{-j}(x,t,D_x,\tau)\big)^* \big(\partial_\tau^l D_t^{l+p} b_{-i}(\cdot,t,\tau)\big) \\
&\qquad = \sum_{\gamma\geq 0} \frac{1}{\gamma!} \partial_\tau^\gamma D_t^\gamma \bigg(\sum_{\beta,i,j\geq 0} \frac{1}{\beta!} (-D_t)^\beta \big(a_{-j}(x,t,D_x,\tau)\big)^* \big(\partial_\tau^\beta b_{-i}(\cdot,t,\tau)\big)\bigg),
\end{aligned}$$

where $(-D_t)^\beta\big(a_{-j}(x,t,D_x,\tau)\big)^*$ is the formal adjoint of the operator with symbol $D_t^\beta a_{-j}(x,t,\xi,\tau)$ as an operator in the x-variable, depending on (t,τ) as parameters. Here we used (A-2) in Appendix A. Hence

$$\begin{aligned}
&\sigma\big(b^*(x,t,D_t) \circ a(x,t,D_x,D_t)\big) \\
&\qquad = \sum_{l\geq 0} \frac{1}{l!} \partial_\tau^l D_t^l \cdot \overline{\bigg(\sum_{\gamma\geq 0} \frac{1}{\gamma!} \partial_\tau^\gamma D_t^\gamma \bigg(\sum_{\beta,i,j\geq 0} \frac{(-D_t)^\beta}{\beta!} \big(a_{-j}(x,t,D_x,\tau)\big)^* \big(\partial_\tau^\beta b_{-i}(\cdot,t,\tau)\big)\bigg)\bigg)} \\
&\qquad = \sum_{\beta,i,j\geq 0} \frac{1}{\beta!} D_t^\beta \big(\overline{a_{-j}}(x,t,D_x,\tau)\big)^* \big(\partial_\tau^\beta \overline{b_{-i}}(\cdot,t,\tau)\big) \\
&\qquad = \sum_{s\geq 0} \sum_{q\beta+i+j=s} \frac{1}{\beta!} D_t^\beta \big(\overline{a_{-j}}(x,t,D_x,\tau)\big)^* \big(\partial_\tau^\beta \overline{b_{-i}}(\cdot,t,\tau)\big),
\end{aligned}$$

because of (A-3) in Appendix A. □

3. Computation of the "degenerate eigenvalue"

We are now in a position to start computing the symbol of Λ.

Let us first examine the minimum eigenvalue and the corresponding eigenfunction of $P_0(x,t,D_x,\tau)$ in (2-1), as an operator in the x-variable. It is well known that $P_0(x,t,D_x,\tau)$ has a discrete set of nonnegative, simple eigenvalues depending in a real analytic way on the parameters (t,τ).

P_0 can be written in the form $LL^* + t^{2l} L^* L$, where $L = D_x + i x^{q-1}\tau$. The kernel of L^* is a one-dimensional vector space generated by $\varphi_{0,0}(x,\tau) = c_0 \tau^{1/2q} \exp(-(x^q/q)\tau)$, c_0 being a normalization constant such that

$$\|\varphi_{0,0}(\cdot,\tau)\|_{L^2(\mathbb{R}_x)} = 1.$$

We remark that in this case τ is positive. For negative values of τ, the situation is much better since the following proposition holds:

Proposition 3.1 [Boutet de Monvel 1974]. *The localized operator of P in (1-1), which is LL^*, is injective in a cone near $\tau < 0$. Hence the operator P is subelliptic.*

Denoting by $\varphi_0(x,t,\tau)$ the eigenfunction of P_0 corresponding to its lowest eigenvalue $\Lambda_0(t,\tau)$, we obtain that $\varphi_0(x,0,\tau) = \varphi_{0,0}(x,\tau)$ and that $\Lambda_0(0,\tau) = 0$. As a consequence, the operator

$$P = BB^* + B^*(t^{2l} + x^{2k})B, \quad B = D_x + i x^{q-1} D_t \tag{3-1}$$

is not “maximally” hypoelliptic, that is, hypoelliptic with a loss of $2 - 2/q$ derivatives.

Next we give a more precise description of the t-dependence of both the eigenvalue Λ_0 and its corresponding eigenfunction φ_0 of $P_0(x,t,D_x,\tau)$.

It is well known that there exists an $\varepsilon > 0$ small enough that the operator

$$\Pi_0 = \frac{1}{2\pi i} \oint_{|\mu|=\varepsilon} \big(\mu I - P_0(x,t,D_x,\tau)\big)^{-1}\, d\mu$$

is the orthogonal projection onto the eigenspace generated by φ_0. Note that Π_0 depends on the parameters (t,τ). The operator LL^* is thought of as an unbounded operator in $L^2(\mathbb{R}_x)$ with domain

$$B_q^2(\mathbb{R}_x) = \Big\{ u \in L^2(\mathbb{R}_x) \,\Big|\, x^\alpha D_x^\beta u \in L^2,\ 0 \le \beta + \frac{\alpha}{q-1} \le 2 \Big\}. \tag{3-2}$$

We have

$$(\mu I - P_0)^{-1} = \big(I + t^{2l}\big[-A(I + t^{2l}A)^{-1}\big]\big)(\mu I - LL^*)^{-1},$$

where $A = (LL^* - \mu I)^{-1} L^* L$. Plugging this into the formula defining Π_0, we get

$$\Pi_0 = \frac{1}{2\pi i} \oint_{|\mu|=\varepsilon} (\mu I - LL^*)^{-1}\, d\mu - \frac{1}{2\pi i} t^{2l} \oint_{|\mu|=\varepsilon} A(I + t^{2l}A)^{-1}(\mu I - LL^*)^{-1}\, d\mu.$$

Hence

$$\begin{aligned}\varphi_0 = \Pi_0 \varphi_{0,0} &= \varphi_{0,0} - t^{2l} \frac{1}{2\pi i} \oint_{|\mu|=\varepsilon} A(I + t^{2l}A)^{-1}(\mu I - LL^*)^{-1} \varphi_{0,0}\, d\mu \\ &= \varphi_{0,0}(x,\tau) + t^{2l} \tilde{\varphi}_0(x,t,\tau). \end{aligned} \tag{3-3}$$

Since Π_0 is an orthogonal projection, $\|\varphi_0(\cdot,t,\tau)\|_{L^2(\mathbb{R}_x)} = 1$.

As a consequence, since $P_0 = LL^* + t^{2l} L^* L$, we obtain that

$$\Lambda_0(t,\tau) = \langle P_0 \varphi_0, \varphi_0 \rangle = t^{2l} \|L \varphi_{0,0}\|^2 + \mathbb{O}(t^{4l}). \tag{3-4}$$

We point out that $L\varphi_{0,0} \neq 0$. Observe that, in view of (2-2), writing $u_\mu(x) = u(\mu^{-1/q}x)$,

$$\begin{aligned}
\Lambda_0(t,\mu\tau) &= \min_{\substack{u\in B_q^2\\ \|u\|_{L^2}=1}} \langle P_0(x,t,D_x,\mu\tau)u(x),u(x)\rangle \\
&= \min_{\substack{u\in B_q^2\\ \|u\|_{L^2}=1}} \left\langle P_0\big(\mu^{-1/q}x,t,\mu^{1/q}D_x,\mu\tau\big)\frac{u_\mu(x)}{\mu^{1/(2q)}},\frac{u_\mu(x)}{\mu^{1/(2q)}}\right\rangle \\
&= \mu^{2/q} \min_{\substack{v\in B_q^2\\ \|v\|_{L^2}=1}} \langle P_0(x,t,D_x,\tau)v(x),v(x)\rangle \\
&= \mu^{2/q}\Lambda_0(t,\tau). \qquad (3\text{-}5)
\end{aligned}$$

This shows that Λ_0 is homogeneous of degree $2/q$ with respect to the variable τ.

Since φ_0 is the unique normalized solution of the equation

$$\big(P_0(x,t,D_x,\tau)-\Lambda_0(t,\tau)\big)u(\cdot,t,\tau)=0,$$

from (2-2) and (3-5) it follows that φ_0 is globally homogeneous of degree $1/(2q)$. Moreover, φ_0 is rapidly decreasing with respect to the x-variable smoothly dependent on (t,τ) in a compact subset of $\mathbb{R}^2\setminus 0$. Using estimates of the form (2-11), we can conclude that $\varphi_0 \in H_q^{1/(2q)}$.

Let us start now the construction of a right parametrix of the operator

$$\begin{bmatrix} P(x,t,D_x,D_t) & \varphi_0(x,t,D_t) \\ \varphi_0^*(x,t,D_t) & 0 \end{bmatrix}$$

as a map from $C_0^\infty(\mathbb{R}^2_{(x,t)})\times C_0^\infty(\mathbb{R}_t)$ into $C^\infty(\mathbb{R}^2_{(x,t)})\times C^\infty(\mathbb{R}_t)$. In particular, we are looking for an operator such that

$$\begin{bmatrix} P(x,t,D_x,D_t) & \varphi_0(x,t,D_t) \\ \varphi_0^*(x,t,D_t) & 0 \end{bmatrix} \circ \begin{bmatrix} F(x,t,D_x,D_t) & \psi(x,t,D_t) \\ \psi^*(x,t,D_t) & -\Lambda(t,D_t) \end{bmatrix} \equiv \begin{bmatrix} \mathrm{Id}_{C_0^\infty(\mathbb{R}^2)} & 0 \\ 0 & \mathrm{Id}_{C_0^\infty(\mathbb{R})} \end{bmatrix}. \qquad (3\text{-}6)$$

Here ψ and ψ^* denote operators in $\mathrm{OPH}_q^{1/2q}$ and $\mathrm{OPH}_q^{*\,1/2q}$ respectively, and $F \in \mathrm{OPS}_q^{-2,-2}$ and $\Lambda \in \mathrm{OPS}_{1,0}^{2/q}$. Moreover, the sign $\equiv$ means equality modulo a regularizing operator.

From (3-6) we obtain four relations:

$$P(x,t,D_x,D_t)\circ F(x,t,D_x,D_t) + \varphi_0(x,t,D_t)\circ\psi^*(x,t,D_t) \equiv \mathrm{Id}, \qquad (3\text{-}7)$$

$$P(x,t,D_x,D_t)\circ\psi(x,t,D_t) - \varphi_0(x,t,D_t)\circ\Lambda(t,D_t) \equiv 0, \qquad (3\text{-}8)$$

$$\varphi_0^*(x,t,D_t)\circ F(x,t,D_x,D_t) \equiv 0, \qquad (3\text{-}9)$$

$$\varphi_0^*(x,t,D_t)\circ\psi(x,t,D_t) \equiv \mathrm{Id}. \qquad (3\text{-}10)$$

We are going to find the symbols F, ψ and Λ as asymptotic series of globally homogeneous symbols:

$$F \sim \sum_{j\ge 0} F_{-j}, \quad \psi \sim \sum_{j\ge 0}\psi_{-j}, \quad \Lambda \sim \sum_{j\ge 0}\Lambda_{-j}, \qquad (3\text{-}11)$$

where the symbols F_{-j}, ψ_{-j} and Λ_{-j} are globally homogeneous of order $-2/q - j/q$, $1/(2q) - j/q$ and $2/q - j/q$ respectively; see for example Definition 2.6 and (3-5).

From Lemma 2.7, we obtain that

$$\sigma(P \circ F) \sim \sum_{s \geq 0} \sum_{q\alpha+i+j=s} \frac{1}{\alpha!} \sigma\big(\partial_\tau^\alpha P_{-j}(x,t,D_x,\tau) \circ_x D_t^\alpha F_{-i}(x,t,D_x,\tau)\big),$$

where we denote by P_{-j} the globally homogeneous parts of degree $2/q - j/q$ of the symbol of P, so that $P = P_0 + P_{-q} + P_{-2k}$. Furthermore, from Lemma 2.8(i), we may write that

$$\sigma(\varphi_0 \circ \psi^*) \sim e^{-ix\xi} \sum_{s \geq 0} \sum_{q\alpha+i=s} \frac{1}{\alpha!} \partial_\tau^\alpha \varphi_0(x,t,\tau) D_t^\alpha \hat{\bar{\psi}}_{-i}(\xi,t,\tau).$$

Analogously, Lemmas 2.9 and 2.8(iii) give

$$\sigma(P \circ \psi) \sim \sum_{s \geq 0} \sum_{ql+i+j=s} \frac{1}{l!} \partial_\tau^l P_{-j}(x,t,D_x,\tau)\big(D_t^l \psi_{-i}(\cdot,t,\tau)\big),$$

$$\sigma(\varphi_0 \circ \Lambda) \sim \sum_{s \geq 0} \sum_{q\alpha+l=s} \frac{1}{\alpha!} \partial_\tau^\alpha \varphi_0(x,t,\tau) D_t^\alpha \Lambda_{-l}(t,\tau).$$

Finally, Lemmas 2.10(i) and 2.8(ii) yield

$$\sigma(\varphi_0^* \circ F) \sim \sum_{s \geq 0} \sum_{ql+j=s} \frac{1}{l!} D_t^l \big(\overline{F_{-j}}(x,t,D_x,\tau)\big)^* \big(\partial_\tau^l \overline{\varphi_0}(\cdot,t,\tau)\big)$$

and

$$\sigma(\varphi_0^* \circ \psi) \sim \sum_{s \geq 0} \sum_{q\alpha+j=s} \frac{1}{\alpha!} \int \partial_\tau^\alpha \bar{\varphi}_0(x,t,\tau) D_t^\alpha \psi_{-j}(x,t,\tau)\, dx.$$

Let us consider the terms globally homogeneous of degree 0. We obtain the relations

$$P_0(x,t,D_x,\tau) \circ_x F_0(x,t,D_x,\tau) + \varphi_0(x,t,\tau) \otimes \psi_0(\cdot,t,\tau) = \mathrm{Id}, \tag{3-12}$$

$$P_0(x,t,D_x,\tau)\big(\psi_0(\cdot,t,\tau)\big) - \Lambda_0(t,\tau)\varphi_0(x,t,\tau) = 0, \tag{3-13}$$

$$\big(F_0(x,t,D_x,\tau)\big)^* \big(\varphi_0(\cdot,t,\tau)\big) = 0, \tag{3-14}$$

$$\int \bar{\varphi}_0(x,t,\tau)\psi_0(x,t,\tau)\, dx = 1. \tag{3-15}$$

Here we denoted by $\varphi_0 \otimes \psi_0$ the operator $u = u(x) \mapsto \varphi_0 \int \bar{\psi}_0 u\, dx$; $\varphi_0 \otimes \psi_0$ must be a globally homogeneous symbol of degree zero.

Conditions (3-13) and (3-15) imply that $\psi_0 = \varphi_0$. Moreover, (3-13) yields that

$$\Lambda_0(t,\tau) = \big\langle P_0(x,t,D_x,\tau)\varphi_0(x,t,\tau), \varphi_0(x,t,\tau)\big\rangle_{L^2(\mathbb{R}_x)},$$

coherently with the notation chosen above. Conditions (3-12) and (3-14) are rewritten as

$$P_0(x,t,D_x,\tau) \circ_x F_0(x,t,D_x,\tau) = \mathrm{Id} - \Pi_0,$$

$$F_0(x,t,D_x,\tau)\big(\varphi_0(\cdot,t,\tau)\big) \in [\varphi_0]^\perp,$$

whence (compare (3-2))

$$F_0(x,t,D_x,\tau)=\begin{cases}\bigl(P_0(x,t,D_x,\tau)|_{[\varphi_0]^\perp\cap B^2_q}\bigr)^{-1} & \text{on } [\varphi_0]^\perp,\\ 0 & \text{on } [\varphi_0].\end{cases}\tag{3-16}$$

Since P_0 is q-globally elliptic with respect to (x,ξ) smoothly depending on the parameters (t,τ), one can show that $F_0(x,t,D_x,\tau)$ is actually a pseudodifferential operator whose symbol satisfies (2-10) with $m=k=-2$, $j=0$, and is globally homogeneous of degree $-2/q$.

From now on we assume that $q<2k$ and that $2k$ is not a multiple of q; the complementary cases are analogous.

Because of the fact that $P_{-j}=0$ for $j=1,\dots,q-1$, relations (3-12)–(3-15) are satisfied at degree $-j/q$, $j=1,\dots,q-1$, by choosing $F_{-j}=0$, $\psi_{-j}=0$, $\Lambda_{-j}=0$. Then we must examine homogeneity degree -1 in Equations (3-7)–(3-10). We get

$$P_{-q}\circ_x F_0+P_0\circ_x F_{-q}+\partial_\tau P_0\circ_x D_tF_0+\varphi_0\otimes\psi_{-q}+\partial_\tau\varphi_0\otimes D_t\varphi_0=0,\tag{3-17}$$

$$P_0(\psi_{-q})+P_{-q}(\varphi_0)+\partial_\tau P_0(D_t\varphi_0)-\Lambda_{-q}\varphi_0-D_t\Lambda_0\partial_\tau\varphi_0=0,\tag{3-18}$$

$$(F_{-q})^*(\varphi_0)-(D_tF_0^*)(\partial_\tau\varphi_0)=0,\tag{3-19}$$

$$\langle\psi_{-q},\varphi_0\rangle_{L^2(\mathbb{R}_x)}+\langle D_t\varphi_0,\partial_\tau\varphi_0\rangle_{L^2(\mathbb{R}_x)}=0.\tag{3-20}$$

First we solve with respect to $\psi_{-q}=\langle\psi_{-q},\varphi_0\rangle_{L^2(\mathbb{R}_x)}\varphi_0+\psi_{-q}^\perp\in[\varphi_0]\oplus[\varphi_0]^\perp$. From (3-20), we immediately get that

$$\langle\psi_{-q},\varphi_0\rangle_{L^2(\mathbb{R}_x)}=-\langle D_t\varphi_0,\partial_\tau\varphi_0\rangle_{L^2(\mathbb{R}_x)}.\tag{3-21}$$

Equation (3-18) implies that

$$P_0\bigl(\langle\psi_{-q},\varphi_0\rangle\varphi_0\bigr)+P_0(\psi_{-q}^\perp)=-P_{-q}(\varphi_0)-\partial_\tau P_0(D_t\varphi_0)+\Lambda_{-q}\varphi_0+D_t\Lambda_0\partial_\tau\varphi_0.$$

Thus, using (3-21) we obtain

$$[\varphi_0]^\perp\ni P_0(\psi_{-q}^\perp)=-P_{-q}(\varphi_0)-\partial_\tau P_0(D_t\varphi_0)+\Lambda_{-q}\varphi_0+D_t\Lambda_0\partial_\tau\varphi_0+\langle D_t\varphi_0,\partial_\tau\varphi_0\rangle\Lambda_0\varphi_0,$$

whence

$$\Lambda_{-q}=\bigl\langle P_{-q}(\varphi_0)+\partial_\tau P_0(D_t\varphi_0)-D_t\Lambda_0\partial_\tau\varphi_0,\varphi_0\bigr\rangle_{L^2(\mathbb{R}_x)}-\langle D_t\varphi_0,\partial_\tau\varphi_0\rangle\Lambda_0,\tag{3-22}$$

$$\psi_{-q}=-\langle D_t\varphi_0,\partial_\tau\varphi_0\rangle_{L^2(\mathbb{R}_x)}\varphi_0+F_0\bigl(-P_{-q}(\varphi_0)-\partial_\tau P_0(D_t\varphi_0)+D_t\Lambda_0\partial_\tau\varphi_0\bigr),\tag{3-23}$$

since, by (3-16), $F_0\varphi_0=0$. From (3-19) we deduce that for every $u\in L^2(\mathbb{R}_x)$,

$$\Pi_0F_{-q}u=\bigl\langle u,(D_tF_0^*)(\partial_\tau\varphi_0)\bigr\rangle_{L^2(\mathbb{R}_x)}\varphi_0=\bigl[\varphi_0\otimes(D_tF_0^*)(\partial_\tau\varphi_0)\bigr]u.$$

Let $-\omega_{-q}=P_{-q}\circ_x F_0+\partial_\tau P_0\circ_x D_tF_0+\varphi_0\otimes\psi_{-q}+\partial_\tau\varphi_0\otimes D_t\varphi_0$. Then from (3-16), applying F_0 to both sides of (3-17), we obtain that

$$(\mathrm{Id}-\Pi_0)F_{-q}=-F_0\omega_{-q}.$$

Therefore we deduce that

$$F_{-q}=\varphi_0\otimes(D_tF_0^*)(\partial_\tau\varphi_0)-F_0\omega_{-q}.\tag{3-24}$$

Inspecting (3-23) and (3-24), we see that ψ_{-q} is in $H_q^{1/2q-1}$ and is globally homogeneous of degree $1/2q-1$, while F_{-q} is in $S_q^{-2,-2+q/(q-1)}$ and is globally homogeneous of degree $-2/q-1$.

From (3-22) we have that Λ_{-q} is in $S_{1,0}^{2/q-1}$ and is homogeneous of degree $2/q-1$. Moreover, P_{-q} is $\mathbb{O}(t^{2l-1})$, $D_t\varphi_0$ is estimated by t^{2l-1} for $t\to 0$ because of (3-3), $D_t\Lambda_0$ is also $\mathbb{O}(t^{2l-1})$, and $\Lambda_0=\mathbb{O}(t^{2l})$ because of (3-4). We thus obtain that

$$\Lambda_{-q}(t,\tau)=\mathbb{O}(t^{2l-1}). \tag{3-25}$$

This ends the analysis of the terms of degree -1 in (3-6).

From now until the end of the proof we assume that $2l > 2k/q$. The complementary case can be obtained analogously.

We iterate this procedure arguing in the same way. We would like to point out that the first homogeneity degree that arises and is not a negative integer is $-2k/q$. (We are availing ourselves of the fact that $2k$ is not a multiple of q. If it is a multiple of q, the above argument applies literally, but we need also the supplementary remark that we are going to make in the sequel.)

At homogeneity degree $-2k/q$ we do not see the derivatives with respect to t or τ of the symbols found at the previous levels, since they would only account for a negative integer degree of homogeneity.

In particular, condition (3-8) for homogeneity degree $-2k/q$ reads as

$$P_0\psi_{-2k}+P_{-2k}\varphi_0-\varphi_0\Lambda_{-2k}=0.$$

Taking the scalar product of the above equation with the eigenfunction φ_0 and recalling that

$$\|\varphi_0(\,\cdot\,,t,\tau)\|_{L^2(\mathbb{R}_x)}=1,$$

we obtain that

$$\Lambda_{-2k}(t,\tau)=\langle P_{-2k}\varphi_0,\varphi_0\rangle_{L^2(\mathbb{R}_x)}+\langle P_0\psi_{-2k},\varphi_0\rangle_{L^2(\mathbb{R}_x)}. \tag{3-26}$$

Now, because of the structure of P_{-2k}, $\langle P_{-2k}\varphi_0,\varphi_0\rangle_{L^2(\mathbb{R}_x)}>0$, while the second term on the right, which is equal to $\langle\psi_{-2k},\varphi_0\rangle\bar{\Lambda}_0$, vanishes for $t=0$. Thus we deduce that

$$\Lambda_{-2k}(0,\tau)>0. \tag{3-27}$$

Let j_0 be a positive integer such that

$$j_0q<2k<(j_0+1)q. \tag{3-28}$$

In the sequel we need some information on the behavior of the symbol $\Lambda_{-(j_0+1)}$. To obtain this, we make a proof by induction.

Suppose that

$$\Lambda_{-j}(t,\tau)=\begin{cases}\mathbb{O}(t^{2l-j/q}) & \text{for } j/q=0,\dots,j_0,\\ 0 & \text{if } j/q \text{ is not an integer } \le j_0\end{cases} \tag{3-29}$$

and

$$\psi_{-j}(t,x,\tau)=\begin{cases}\mathbb{O}(t^{2l-j/q}) & \text{for } j/q=0,\dots,j_0,\\ 0 & \text{if } j/q \text{ is not an integer } \le j_0.\end{cases} \tag{3-30}$$

Let us write the symbols of (3-7)–(3-10) at the homogeneity degree $-(j_0+1)$. From (3-8), we have

$$\sum_{\substack{q\alpha+i+j=\\ q(j_0+1)}} \frac{1}{\alpha!}\partial_\tau^\alpha P_{-j}(x,t,D_x,\tau)\big(D_t^\alpha \psi_{-i}(\cdot,t,\tau)\big) - \sum_{\substack{q\alpha+i=\\ q(j_0+1)}} \frac{1}{\alpha!}\partial_\tau^\alpha \varphi_0(x,t,\tau) D_t^\alpha \Lambda_{-i}(t,\tau) = 0.$$

This can be rewritten as

$$P_0(\psi_{-(j_0+1)q}) - \varphi_0 \Lambda_{-(j_0+1)q} = - \sum_{\substack{q\alpha+i+j=q(j_0+1)\\ i<q(j_0+1)}} \frac{1}{\alpha!}\partial_\tau^\alpha P_{-j}(D_t^\alpha \psi_{-i}) + \sum_{\substack{q\alpha+i=q(j_0+1)\\ i<q(j_0+1)}} \frac{1}{\alpha!}\partial_\tau^\alpha \varphi_0 D_t^\alpha \Lambda_{-i}. \quad (3\text{-}31)$$

Taking the scalar product of (3-31) with φ_0 and using the equalities $\|\varphi_0\|_{L^2(\mathbb{R}_x)} = 1$ and $\Lambda_0(t,\tau) = \mathbb{O}(t^{2l})$ and the self-adjointness of P_0, we at once find, because of the inductive hypothesis, that $\Lambda_{-(j_0+1)q} = \mathbb{O}(t^{2l-(j_0+1)})$.

In order to show that $\psi_{-(j_0+1)q} = \mathbb{O}(t^{2l-(j_0+1)})$, set

$$\psi_{-(j_0+1)q}(x,t,\tau) = \int_{\mathbb{R}} \varphi_0(y,t,\tau)\psi_{-(j_0+1)q}(y,t,\tau)\,dy \cdot \varphi_0(x,t,\tau) + \psi^{\perp}_{-(j_0+1)q}(x,t,\tau), \quad (3\text{-}32)$$

where $\psi^{\perp}_{-(j_0+1)q} \in [\varphi_0]^{\perp}$. Let us consider then (3-10). At the homogeneity level $-(j_0+1)$, it can be written as

$$\int_{\mathbb{R}} \varphi_0(y,t,\tau)\psi_{-(j_0+1)q}(y,t,\tau)\,dy = - \sum_{\substack{q\alpha+j=(j_0+1)q\\ \alpha>0}} \frac{1}{\alpha!} \int \partial_\tau^\alpha \varphi_0(y,t,\tau) D_t^\alpha \psi_{-j}(y,t,\tau)\,dy.$$

By (3-30), we conclude that the scalar product in the left-hand side of the above identity is $\mathbb{O}(t^{2l-(j_0+1)})$. Let us now consider (3-31). Applying F_0 to both sides of (3-31) and taking both Equation (3-32) and the inductive hypothesis into account allows us to conclude that

$$\psi^{\perp}_{-(j_0+1)q} = \mathbb{O}(t^{2l-(j_0+1)}).$$

We have thus proved:

Theorem 3.2. *The operator Λ defined in (3-6) is a pseudodifferential operator with symbol $\Lambda(t,\tau) \in S^{2/q}_{1,0}(\mathbb{R}_t \times \mathbb{R}_\tau)$. The symbol of Λ has an asymptotic expansion of the form*

$$\Lambda(t,\tau) \sim \sum_{j=0}^{j_0} \Lambda_{-jq}(t,\tau) + \sum_{s\geq 0}\big(\Lambda_{-2k-sq}(t,\tau) + \Lambda_{-(j_0+1)q-sq}(t,\tau)\big). \quad (3\text{-}33)$$

Here Λ_{-p} has homogeneity $2/q - p/q$ and

$$\Lambda_{-jq}(t,\tau) = \mathbb{O}(t^{2l-j}) \quad \textit{for} \begin{cases} j = 0,\dots,j_0+1 & \textit{if } 2l > 2k/q, \\ j = 0,\dots,2l-1 & \textit{if } 2l \leq 2k/q, \end{cases} \quad (3\text{-}34)$$

while

$$\Lambda_{-2k} \text{ satisfies (3-27)} \quad \text{and} \quad t^{-2l}\Lambda_0(t,\tau)_{|_{t=0}} > 0. \quad (3\text{-}35)$$

Furthermore, as a consequence of the calculus for real analytic symbols, $\Lambda_{r-sq}(t,\tau)$, with $r=-2k$ or $r=-(j_0+1)q$, satisfies the estimates

$$\left|\partial_t^\beta \partial_\tau^\alpha \Lambda_{r-sq}(t,\tau)\right| \le C_\Lambda^{1+s+\alpha+\beta} \alpha!\,\beta!\,s!\,(1+|\tau|)^{2/q+r/q-s-\alpha}, \tag{3-36}$$

where C_Λ denotes a positive constant depending only on the symbol Λ. (See Section 5 below for more details.) *In particular, $\Lambda(t,\tau)$ is a real analytic symbol in the sense of Boutet de Monvel [1972].*

We point out that the operator $\Lambda(t, D_t)$ defined above, modulo an elliptic factor of order $2/q-2k/q$, has a form of the type

$$t^{2l} D_t^{2k/q} + 1. \tag{3-37}$$

The latter operator is G^s-hypoelliptic for $s \ge s_0 = lq/(lq-k)$. To get a rough idea of this fact, if $q=1$, let us consider the equation $t^{2l} D_t^{2k} u + u = 0$. The behavior of u can be obtained by WKB, solving $t^{2l}(\varphi')^{2k}+1=0$, which yields $\varphi(t) = \omega t^{-l/k+1}$ and $u \sim e^{i\varphi}$, where ω is a suitable complex constant. This gives $u \in G^{l/(l-k)}$.

4. C^∞-hypoellipticity of P: sufficient part

In this section we prove the C^∞-hypoellipticity of P. This is accomplished by showing that the hypoellipticity of P follows from the hypoellipticity of Λ and proving that Λ is hypoelliptic if condition (1-2) is satisfied. As a matter of fact, the hypoellipticity of P is equivalent to the hypoellipticity of Λ, so that the structure of Λ in Theorem 3.2 may be used to prove assertion (ii) in Theorem 1.1.

We state without proof:

Lemma 4.1. *Let $a \in S_q^{m,k}$ be properly supported with $k \le 0$. Then* $\operatorname{Op} a$ *is continuous from $H^s_{\mathrm{loc}}(\mathbb{R}^2)$ to $H^{s-m+k(q-1)/q}_{\mathrm{loc}}(\mathbb{R}^2)$. Let $\varphi \in H_q^{m+1/2q}$ be properly supported. Then* $\operatorname{Op}\varphi$ *is continuous from $H^s_{\mathrm{loc}}(\mathbb{R})$ to $H^{s-m}_{\mathrm{loc}}(\mathbb{R}^2)$. Moreover, $\varphi^*(x,t,D_t)$ is continuous from $H^s_{\mathrm{loc}}(\mathbb{R}^2)$ to $H^{s-m}_{\mathrm{loc}}(\mathbb{R})$.*

Repeating the argument above for a left parametrix, we can find symbols $F \in S_q^{-2,-2}$, $\psi \in H_q^{1/2q}$ and $\Lambda \in S^{2/q}_{1,0}$ as in (3-11) such that

$$\begin{bmatrix} F(x,t,D_x,D_t) & \psi(x,t,D_t) \\ \psi^*(x,t,D_t) & -\Lambda(t,D_t)\end{bmatrix} \circ \begin{bmatrix} P(x,t,D_x,D_t) & \varphi_0(x,t,D_t) \\ \varphi_0^*(x,t,D_t) & 0\end{bmatrix} \equiv \begin{bmatrix} \mathrm{Id}_{C_0^\infty(\mathbb{R}^2)} & 0 \\ 0 & \mathrm{Id}_{C_0^\infty(\mathbb{R})}\end{bmatrix}. \tag{4-1}$$

From (4-1) we get the pair of relations

$$F(x,t,D_x,D_t)\circ P(x,t,D_x,D_t) = \mathrm{Id} - \psi(x,t,D_t)\circ\varphi_0^*(x,t,D_t), \tag{4-2}$$

$$\psi^*(x,t,D_t)\circ P(x,t,D_x,D_t) = \Lambda(t,D_t)\circ\varphi_0^*(x,t,D_t). \tag{4-3}$$

Proposition 4.2. *If Λ is hypoelliptic with a loss of $\delta > 0$ derivatives, then P is also hypoelliptic with a loss of derivatives equal to*

$$2\frac{q-1}{q} + \delta.$$

The converse is also true. Furthermore, Λ is C^∞-hypoelliptic if and only if P is C^∞-hypoelliptic.

Proof. Assume that $Pu \in H^s_{\mathrm{loc}}(\mathbb{R}^2)$. From Lemma 4.1 we have

$$FPu \in H^{s+2/q}_{\mathrm{loc}}(\mathbb{R}^2).$$

By (4-2) we have $u - \psi\varphi_0^* u \in H^{s+2/q}_{\mathrm{loc}}(\mathbb{R}^2)$. Again, using Lemma 4.1, $\psi^* Pu \in H^s_{\mathrm{loc}}(\mathbb{R})$, so that by (4-3), $\Lambda\varphi_0^* u \in H^s_{\mathrm{loc}}(\mathbb{R})$. The hypoellipticity of Λ yields then that $\varphi_0^* u \in H^{s+2/q-\delta}_{\mathrm{loc}}(\mathbb{R})$. From Lemma 4.1 we obtain that $\psi\varphi_0^* u \in H^{s+2/q-\delta}_{\mathrm{loc}}(\mathbb{R})$. Thus

$$u = (\mathrm{Id} - \psi\varphi_0^*)u + \psi\varphi_0^* u \in H^{s+2/q-\delta}_{\mathrm{loc}}.$$

This proves the first sentence of the proposition. The proof of the other assertions is similar. □

Next we prove the hypoellipticity of Λ under the assumption that $l > k/q$.

First we want to show that there exists a smooth nonnegative function $M(t, \tau)$ such that

$$M(t,\tau) \le C|\Lambda(t,\tau)|, \quad |\Lambda^{(\alpha)}_{(\beta)}(t,\tau)| \le C_{\alpha,\beta} M(t,\tau)(1+|\tau|)^{-\rho\alpha+\delta\beta}, \tag{4-4}$$

where α, β are nonnegative integers, C, $C_{\alpha,\beta}$ are suitable positive constants, and the inequality holds for t in a compact neighborhood of the origin and $|\tau|$ large. Moreover, ρ and δ are such that $0 \le \delta < \rho \le 1$.

We actually need to check the above estimates for Λ only when τ is positive and large.

Let us choose $\rho = 1$, $\delta = k/lq < 1$ and

$$M(t,\tau) = \tau^{2/q}\big(t^{2l} + \tau^{-2k/q}\big),$$

for $\tau \ge c \ge 1$. It is then evident, from Theorem 3.2, that the first of the conditions in (4-4) is satisfied. The second condition in (4-4) is also straightforward for $\Lambda_0 + \Lambda_{-2k}$, because of (3-27) and (3-4). To verify the second condition in (4-4) for Λ_{-jq}, $q \in \{1, \dots, j_0\}$, we have to use property (3-34) in the statement of Theorem 3.2. Finally, the verification is straightforward for the lower-order parts of the symbol in (3-33). Using Theorem 22.1.3 of [Hörmander 1985], we see that there exists a parametrix for Λ. Moreover, from the proof of the same theorem, we get that the symbol of any parametrix satisfies the same estimates that Λ^{-1} satisfies, that is,

$$\big|D_t^\beta D_\tau^\alpha \Lambda^{-1}(t,\tau)\big| \le C_{\alpha,\beta}\big[\tau^{2/q}(t^{2l} + \tau^{-2k/q})\big]^{-1}(1+\tau)^{-\alpha+(k/lq)\beta} \le C_{\alpha,\beta}(1+\tau)^{2k/q-2/q-\alpha+k/lq\beta},$$

for t in a compact set and $\tau \ge C$. Thus the parametrix obtained from Theorem 22.1.3 of [Hörmander 1985] has a symbol in $S^{2k/q-2/q}_{1,k/lq}$.

Theorem 4.3. *Λ has a parametrix whose symbol belongs to $S^{2k/q-2/q}_{1,k/lq}$ and is hypoelliptic with a loss of $2k/q$ derivatives, that is, $\Lambda u \in H^s_{\mathrm{loc}}$ implies $u \in H^{s+2/q-2k/q}_{\mathrm{loc}}$.*

Theorem 4.3 together with Proposition 4.2 proves assertion (i) of Theorem 1.1.

5. Analytic symbols and Gevrey regularity

The purpose of this section is to prove the second statement in Theorem 1.1. To this end, we need to work with real analytic symbols and their asymptotic expansions.

Let us first define the symbol classes of Section 2 for analytic symbols. Since the coefficients of P are analytic, we are interested only in symbols with real analytic regularity.

Definition 5.1. We define the class of symbols $S^{m,k}_{q,a}(\Omega, \Sigma)$, where Ω is a conic neighborhood of the point $(0, e_2)$ and Σ denotes the characteristic manifold $\{x = 0, \xi = 0\}$, as the set of all C^ω functions such that on any conic subset of Ω with compact base,

$$\begin{aligned}\left|\partial_t^\alpha \partial_\tau^\beta \partial_x^\gamma \partial_\xi^\delta a(x,t,\xi,\tau)\right| \\ \le C^{1+\alpha+\beta+\gamma+\delta}\alpha!\,\beta!\,\gamma!\,\delta!\,(1+|\tau|)^{m-\beta-\delta}\left(\frac{|\xi|}{|\tau|}+|x|^{q-1}+\frac{1}{|\tau|^{(q-1)/q}}\right)^{k-\gamma/(q-1)-\delta},\end{aligned} \tag{5-1}$$

for $|(\xi,\tau)| \ge B(\beta+\delta)$, where $B > 0$ is a suitable constant.

We write $S^{m,k}_q$ for $S^{m,k}_{q,a}(\mathbb{R}^2 \times \mathbb{R}^2, \Sigma)$.

Likewise, with the same notations of Definition 2.3, we need the C^ω version of the Hermite symbols:

Definition 5.2. We write $H^m_{q,a}$ for $\mathscr{H}^m_{q,a}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$, the class of all real analytic functions in $\bigcap_{j=1}^\infty S^{m-j,-q/(q-1)j}_{q,a}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$. Here $S^{m,k}_{q,a}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$ is the set of all smooth functions such that

$$\left|\partial_t^\alpha \partial_\tau^\beta \partial_x^\gamma a(x,t,\tau)\right| \lesssim C^{1+\alpha+\beta+\gamma}\alpha!\,\beta!\,\gamma!\,(1+|\tau|)^{m-\beta}\left(|x|^{q-1}+\frac{1}{|\tau|^{(q-1)/q}}\right)^{k-\gamma/(q-1)}, \tag{5-2}$$

for $|\tau| \ge B\beta$, where B denotes a suitable positive constant.

Actually our Hermite operators are better than this and using an easy generalization of Proposition 2.10 in [Grigis and Rothschild 1983], we define the action of a symbol $a(x,t,\tau)$ in $H^m_{q,a}$ as the map

$$a(x,t,D_t)\colon G^s(\mathbb{R}_t) \cap C_0^\infty(\mathbb{R}_t) \longrightarrow G^s(\mathbb{R}^2_{x,t}),$$

for any $s > 1$, defined by

$$a(x,t,D_t)u(x,t) = (2\pi)^{-1}\int e^{it\tau}a(x,t,\tau)\hat{u}(\tau)\,d\tau.$$

Such an operator, modulo a regularizing operator (with respect to the t variable), is called a Hermite operator, and we denote by $\mathrm{OPH}^m_{q,a}$ the corresponding class. When it is clear from the context, to keep the notation simple, we shall omit the subscript a.

The adjoint of a (C^ω) Hermite operator is defined exactly as in Definition 2.4.

Next we define suitable cutoff functions that will be used several times in what follows.

Lemma 5.3. *Let $t > 1$. There exists a family of cutoff functions $\omega_j \in G^t(\mathbb{R}^n_x)$, $0 \le \omega_j(x) \le 1$, for $j = 0, 1, 2, \ldots$, such that:*

(1) *$\omega_j \equiv 0$ if $|x| \le 2R(j+1)^t$, $\omega_j \equiv 1$ if $|x| \ge 4R(j+1)^t$, with R an arbitrary positive constant.*

(2) *There is a suitable constant C_ω, independent of j, α, R, such that*

$$|D^\alpha \omega_j(x)| \le C_\omega^{|\alpha|+1}\left(R(j+1)^{t-1}\right)^{-|\alpha|} \quad \text{if } |\alpha| \le 3j, \tag{5-3}$$

and

$$|D^\alpha \omega_j(x)| \le (RC_\omega)^{|\alpha|+1} \frac{\alpha!^t}{|x|^{|\alpha|}} \quad \textit{for every } \alpha. \tag{5-4}$$

Proof. Pick a function $\psi \in G^t(\mathbb{R}) \cap C_0^\infty(\mathbb{R})$ satisfying $\psi \ge 0$, $\operatorname{supp}\psi \subset \{|x| \le \frac{1}{4}\}$, and $\int \psi(x)\,dx = 1$. Let χ_R denote the characteristic function of the interval $[-2R-r/2, 2R+r/2]$. Set $\psi_a(x) = a^{-1}\psi(x/a)$. Then

$$\varphi_N = \chi_R * \psi_r * \underbrace{\psi_{r/N} * \cdots * \psi_{r/N}}_{N \text{ times}}$$

has support contained in $[-2R-r, 2R+r]$ and is identically equal to 1 on $[-2R, 2R]$. We have, for any α, and for any $\beta \le N$,

$$D^{\alpha+\beta}\varphi_N = \chi_R * D^\alpha \psi_r * \underbrace{D\psi_{r/N} * \cdots * D\psi_{r/N}}_{\beta \text{ times}} * \psi_{r/N} * \cdots * \psi_{r/N}.$$

Whence

$$|D^{\alpha+\beta}\varphi_N| \le (4R+r) C_\psi^{\alpha+1} \alpha!^t r^{-\alpha} \Big(\|D\psi\|_{L^1} \frac{N}{r} \Big)^\beta.$$

Now we define

$$\omega_j(x) = 1 - \varphi_{3j}\Big(\frac{|x|}{(j+1)^t} \Big).$$

Assertion (1) of the lemma and the estimate (5-3) are then a consequence of the definitions and estimates above, once we choose $r = 2R$. Let us now turn to (5-4). We have

$$|D^\alpha \omega_j(x)| \le 6RC_\psi^{\alpha+1} \alpha!^t \frac{1}{[2R(j+1)^t]^\alpha}.$$

On the support of $D^\alpha \omega_j$ we have $|x| \le 4R(j+1)^t$, which implies the conclusion. □

Lemma 5.4. *Let $s > 1$. There exists a family of cutoff functions $\omega_j \in G^s(\mathbb{R}^n_x)$, $0 \le \omega_j(x) \le 1$, $j = 0, 1, 2, \ldots$, such that*:

(1) $\omega_j \equiv 0$ *if* $|x| \le 2R(j+1)$, $\omega_j \equiv 1$ *if* $|x| \ge 4R(j+1)$, *with R an arbitrary positive constant.*

(2) *There is a suitable constant C_ω, independent of j, α, R, such that*

$$|D^\alpha \omega_j(x)| \le C_\omega^{|\alpha|+1} R^{-|\alpha|} \quad \textit{if } |\alpha| \le 3j, \tag{5-5}$$

and

$$|D^\alpha \omega_j(x)| \le (RC_\omega)^{|\alpha|+1} \frac{\alpha!^s}{|x|^{|\alpha|}} \quad \textit{for every } \alpha. \tag{5-6}$$

Proof. The proof is the same as the proof of Lemma 5.3, but the ω_j are defined as

$$\omega_j(x) = 1 - \varphi_{3j}\Big(\frac{|x|}{j+1} \Big). \qquad \square$$

We wish now to define the asymptotic expansion of a symbol in the analytic category.

Let $f_{-j}(x,t,\xi,\tau) \in S^{m,k+j/(q-1)}_{q,a}$, $j \in \mathbb{N} \cup \{0\}$, satisfying an estimate of the form

$$\begin{aligned}&\left|\partial_t^\alpha \partial_\tau^\beta \partial_x^\gamma \partial_\xi^\delta f_{-j}(x,t,\xi,\tau)\right| \\ &\quad \le C^{1+\alpha+\beta+\gamma+\delta+j} \alpha!\,\beta!\,\gamma!\,\delta!\,j!^{1/q}(1+|\tau|)^{m-\beta-\delta}\left(\frac{|\xi|}{|\tau|}+|x|^{q-1}+\frac{1}{|\tau|^{(q-1)/q}}\right)^{k-\gamma/(q-1)-\delta},\end{aligned} \tag{5-7}$$

for $|(\xi,\tau)| \ge B(j+\beta+\delta)$; then there exists $f(x,t,\xi,\tau) \in S^{m,k}_{q,a}$ such that $f \sim \sum_{j\ge 0} f_{-j}$, that is, $f - \sum_{j=0}^{N-1} f_{-j} \in S^{m,k+N/(q-1)}_{q,a}$, and thus f is defined modulo a symbol in $S^{m,\infty}_{q,a} = \bigcap_{h\ge 0} S^{m,h}_{q,a}$.

We point out that the cutoff functions defined in Lemma 5.4 are used to actually sum the formal series $\sum_{j=0}^{\infty} f_j$ to obtain the symbol f.

Let f_{-j} be globally homogeneous of degree $m-k(q-1)/q-j/q$ and such that for every $\alpha,\beta \ge 0$ satisfies the estimates

$$\left|\partial^\gamma_{(t,\tau)} \partial_x^\alpha \partial_\xi^\beta f_{-j}(x,t,\xi,\tau)\right| \le C^{\alpha+\beta+\gamma+j+1}\alpha!\,\beta!\,\gamma!\,j!^{\frac{1}{q}}\left(|\xi|+|x|^{q-1}+1\right)^{k-\frac{\alpha}{q-1}-\beta}, \quad (x,\xi)\in\mathbb{R}^2, \tag{5-8}$$

for (t,τ) in a compact subset of $\mathbb{R}\times\mathbb{R}\setminus 0$ and every multi-index γ. Then $f_{-j} \in S^{m,k+j/(q-1)}_{q,a}$.

Accordingly, let $\varphi_{-j}(x,t,\tau) \in H^{m-j/q}_{q,a}$; then there exists $\varphi(x,t,\tau) \in H^m_{q,a}$ such that $\varphi \sim \sum_{j\ge 0}\varphi_{-j}$, that is, $\varphi - \sum_{j=0}^{N-1}\varphi_{-j} \in H^{m-N/q}_q$, so that φ is defined modulo a symbol analytically regularizing with respect to the t variable.

We again point out that the cutoff functions defined in Lemma 5.4 are used to actually sum the formal series $\sum_{j=0}^{\infty}\varphi_j$ to obtain the symbol φ.

Similarly, let φ_{-j} be globally homogeneous of degree $m-j/q$ and such that for every $\alpha, l \ge 0$ satisfies the estimates

$$\left|\partial^\beta_{(t,\tau)}\partial_x^\alpha \varphi_{-j}(x,t,\tau)\right| \le C^{\alpha+\beta+j+1}\alpha!\,\beta!\,j!^{1/q}\left(|x|^{q-1}+1\right)^{-l-\alpha/(q-1)}, \quad x\in\mathbb{R}, \tag{5-9}$$

for (t,τ) in a compact subset of $\mathbb{R}\times\mathbb{R}\setminus 0$ and every multi-index β. Then $\varphi_{-j} \in H^{m-j/q}_{q,a}$.

Proposition 5.5. *Let F be the operator defined in (3-6). $F \in \mathrm{Op}(S^{2,2}_{q,a})$ and maps functions in G^s into itself. A similar statement holds for the symbols in $H^m_{q,a}$.*

We skip the details of the analytic and Gevrey calculus in these classes of symbols. Suffice it to say that it is a totally standard matter and one may consult [Boutet de Monvel and Krée 1967; Boutet de Monvel 1972].

We explicitly remark that the symbols constructed in (3-6) and (4-1), F, ψ, Λ belong to the (analytic) classes $S^{-2,-2}_{q,a}$, $H^{1/2q}_{q,a}$ and $S^{2/q}_{1,0,a}$ and satisfy better estimates than the above (see [Grigis and Rothschild 1983, Proposition 2.10; Métivier 1981, Section 2]).

We are now ready to prove the second assertion in Theorem 1.1. First we prove:

Proposition 5.6. *The operator P in (1-1) is $G^s(\mathbb{R}^2)$-hypoelliptic if and only if Λ in (3-33) is $G^s(\mathbb{R})$-hypoelliptic.*

Proof. Let us assume first that Λ is G^s-hypoelliptic and that $Pu \in G^s$. Due to (4-2), (4-3) and Proposition 5.5, we have both FPu and $\psi^* Pu \in G^s$. From the latter, we get that $\Lambda\varphi_0^* u \in G^s$, which implies that $\varphi_0^* u \in G^s$, whence $\psi\varphi_0^* u \in G^s$. We thus obtain that $u \in G^s$.

Let us assume first that P is G^s-hypoelliptic and that $\Lambda u \in G^s$. This time we use (3-8) and (3-10). We have $P\psi u = \varphi_0 \Lambda u \in G^s$, which implies that $\psi u \in G^s$. Finally, $u \equiv \varphi_0^* \psi u \in G^s$. □

Next we have only to show that Λ is G^s-hypoelliptic for every $s \geq s_0 = lq/(lq-k)$, in order to prove:

Theorem 5.7. *Let P be as in (1-1). Then P is Gevrey s-hypoelliptic for every $s \geq s_0$, where*

$$s_0 = \frac{lq}{lq-k}.$$

Proof. In order to see that $\Lambda(t, D)$ is Gevrey s-hypoelliptic for every $s \geq s_0$, we are going to show that we can construct a parametrix with symbol in the class $S^{2k/q-2/q}_{1,k/lq,(s)}$, where the latter is defined as the set of all smooth, that is, C^∞, functions $a(t, \tau)$ satisfying the estimates

$$\left|\partial_t^\beta \partial_\tau^\alpha a(t, \tau)\right| \leq C^{1+\alpha+\beta} \alpha!\, \beta!^{s(1-(k/lq))} (1+|\tau|)^{2k/q-2/q-\alpha+(k/lq)\beta},$$

for t in a compact set of the real line, for every α, β, $\tau \in \mathbb{R}$, with $1+|\tau| \geq B\beta^s$; here B and C are suitable positive constants depending only on the symbol a.

As a matter of fact, we do not need symbols exhibiting a Gevrey dependence on the variables: analytic dependence is all we get; nevertheless, the general theory allows Gevrey behavior at no cost. Actually $s_0(1-k/lq) = 1$.

Arguing as in the proof of Proposition 4.2, we choose a weight function

$$M(t, \tau) = \tau^{2/q} \left(t^{2l} + \tau^{-2k/q}\right), \quad \tau \geq 1.$$

We have the estimates

$$M(t, \tau) \leq C|\Lambda(t, \tau)|,$$
$$\left|\partial_t^\beta \partial_\tau^\alpha \Lambda(t, \tau)\right| \leq C^{1+\alpha+\beta} \alpha!\, \beta!^{s(1-k/(lq))} M(t, \tau)(1+|\tau|)^{-\alpha+(k/lq)\beta}.$$

The existence of a parametrix $a(t, \tau)$ for $\Lambda(t, \tau)$, and hence the conclusion, is a standard consequence of the calculus in the Gevrey classes. □

6. Optimality in Gevrey spaces

This section is devoted to the proof of the third assertion of Theorem 1.1. By Proposition 5.6, it is enough to show that $\Lambda(t, D_t)$ is not G^s-hypoelliptic for $1 \leq s < s_0$.

To clarify our technique, let us consider a couple of examples reminiscent of the form (3-37). We stress here the fact that the operators we consider are a much simpler instance of Λ, the operator we are interested in.

Example 1. Consider the operator

$$L(t, \partial_t) = t^2 \partial_t + a + bt,$$

where $a = \frac{i}{4}$, $b = -\frac{1}{2}$. We will show that L is not G^s-hypoelliptic for $1 \le s < 2$. Consider the equation $L(t, \partial_t)u = \frac{i}{4}$. Arguing by contradiction, every solution u is certainly better than G^2-regular.

Let us look for a solution u in the form

$$u(t) = \int_0^{+\infty} e^{i\rho^2 t} e^{-\rho}\, d\rho.$$

One can easily see that this function u is actually a solution of $Lu = 0$. On the other hand,

$$\partial_t^\alpha u(0) = i^\alpha \int_0^{+\infty} e^{-\rho} \rho^{2\alpha}\, d\rho \sim \alpha!^2.$$

The latter estimate contradicts our assumption that u is better than G^2.

Unfortunately, it almost never occurs that the solution has a neat representation of the form above. We are instead forced to represent u as an integral containing both a phase function and an amplitude function. Moreover, the amplitude has to be constructed as a formal series whose convergence must be specifically defined and studied. As a motivation for our technique, we show this on the following, formally slightly different example.

Example 2. Consider the operator

$$L(t, \partial_t) = t^2 \partial_t + \frac{i}{4}.$$

We want to "solve" the equation $L(t, \partial_t)u = 0$. First of all, we look for the solution $u(t)$ in the form

$$u(t) = \int_0^{+\infty} e^{i\rho^2 t} v(\rho)\, d\rho,$$

where v has to be specified.

We proceed formally to find a candidate for v. We have

$$L(t, \partial_t)u = L(t, \partial_t) \int_0^{+\infty} e^{i\rho^2 t} v(\rho)\, d\rho = \frac{i}{4} \int_0^{+\infty} e^{i\rho^2 t} \Big(-\partial_\rho^2 + 1 - \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2} \Big) v(\rho)\, d\rho. \tag{6-1}$$

The operator in parentheses has the form $P_0(\partial_\rho) + \rho^{-1} P_1(\partial_\rho) + \rho^{-2} P_2(\partial_\rho)$, where

$$P_0(\partial_\rho) = -\partial_\rho^2 + 1.$$

In order to put in evidence the phase factor, we write $v(\rho) = e^{-\rho} v_1(\rho)$. As a consequence, we have

$$L(t, \partial_t)u = \frac{i}{4} \int_0^{+\infty} e^{i\rho^2 t} e^{-\rho} \Big(-\partial_\rho^2 + 2\partial_\rho + \frac{1}{\rho} - \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2} \Big) v_1(\rho)\, d\rho.$$

The operator in parentheses still does not have the right form, since the phase factor $e^{-\rho}$ is not enough to guarantee that v_1 has an asymptotic expansion, for large ρ, in decreasing powers of ρ. This, in the end, would give an obstruction to the iterative solution of the "transport" equations. Hence, let us write

$v_1(\rho) = \rho^\lambda \tilde{v}(\rho)$, where both λ and $\tilde{v}$ are to be determined. Bringing the factor ρ^λ to the left and choosing $\lambda = -\frac{1}{2}$ has the effect of canceling the terms of the form $\rho^{-1}\tilde{v}$. We eventually get

$$\begin{aligned} L(t, \partial_t)u &= \frac{i}{4}\int_0^{+\infty} e^{i\rho^2 t} e^{-\rho} \rho^{-1/2}\Big(-\partial_\rho^2 + 2\partial_\rho + \tfrac{3}{4}\frac{1}{\rho^2}\Big)\tilde{v}(\rho)\, d\rho \\ &= \frac{i}{4}\int_0^{+\infty} e^{i\rho^2 t} e^{-\rho} \rho^{-1/2}\Big(P_0(\partial_\rho) + \frac{1}{\rho^2}P_2(\partial_\rho)\Big)\tilde{v}(\rho)\, d\rho. \end{aligned}$$

We write $P_2(\partial_\rho)$ even if P_2 is actually a multiplication operator, to stress the fact that this circumstance is particular to the present example but has no interest in the general case.

The next step is to construct $\tilde{v}$ formally. To do that, we look for $\tilde{v}$ in the form

$$\tilde{v}(\rho) = \sum_{k=0}^{\infty} v_{2k}(\rho),$$

where the v_{2k} are obtained solving the triangular infinite system (transport equations)

$$P_0(\partial_\rho)v_{2k}(\rho) + \frac{1}{\rho^2}P_2 v_{2k-2}(\rho) = 0, \quad k = 0, 1, 2, \dots,$$

with the convention that v_{2k} is identically zero if its subscript is negative.

Choose $v_0(\rho) \equiv 1$. Next we prove:

Minilemma. *If $\rho \geq 1$, we have $|v_{2k}(\rho)| \leq \rho^{-k}$ for $k = 0, 1, 2, \dots$.*

Proof. By induction. It is evident for $k = 0$. Assume that $|v_{2k-2}(\rho)| \leq \rho^{-(k-1)}$. For v_{2k} we have the equation $v_{2k}'' - 2v_{2k}' = (\frac{3}{4})\rho^{-2}v_{2k-2}$. By the inductive assumption, the absolute value of the right-hand side of the equation can be estimated by $\rho^{-(k+1)}$.

Now a solution $y(\rho)$, vanishing at infinity, of the equation $y'' - 2y' = f$ can be written as

$$y(\rho) = \tfrac{1}{2}\int_\rho^{+\infty} f(\sigma)\, d\sigma - \tfrac{1}{2}\int_\rho^{+\infty} e^{-2(\sigma-\rho)} f(\sigma)\, d\sigma.$$

It is now evident that if $|f(\rho)| \leq \rho^{-(k+1)}$, we have that $|y(\rho)| \leq \rho^{-k}$, thus concluding the proof. □

Turning back to our example, we immediately see that the series formally defining $\tilde{v}$ does not converge on the whole positive real axis. To deal with this fact, pick up a C^∞ cutoff function χ such that $\chi \equiv 0$ for $\rho \leq 2$, $\chi \equiv 1$ for $\rho \geq 3$, and $0 \leq \chi \leq 1$. It is then evident that

$$w(\rho) = \chi(\rho)\sum_{k=0}^{\infty} v_{2k}(\rho)$$

is a convergent series defining a smooth bounded function. We have

$$P_0 w + \frac{1}{\rho^2}P_2 w = g,$$

where

$$g = -\chi''\sum_{k=0}^{\infty} v_{2k} - 2\chi'\sum_{k=1}^{\infty} v_{2k}' + 2\chi'\sum_{k=1}^{\infty} v_{2k}.$$

We emphasize that the same argument of the lemma gives us analogous estimates for the derivatives of the v_{2k}, so that there is no problem for the convergence of the series in the expression of g.

Replacing $\tilde{v}$ by w, we see that we have found a function $h(t)$ with

$$h(t) = \int_0^{+\infty} e^{i\rho^2 t} e^{-\rho} \rho^{-1/2} w(\rho)\, d\rho$$

such that

$$L(t, \partial_t) h = \int_0^{+\infty} e^{i\rho^2 t} e^{-\rho} \rho^{-1/2} g(\rho)\, d\rho.$$

We observe now that the function in the right-hand side of the above equality is in fact of class C^ω, since $\operatorname{supp} g \subset [2, 3]$. On the other hand,

$$\partial_t^\alpha h(0) = i^\alpha \int_0^{+\infty} e^{-\rho} \rho^{-1/2+2\alpha} w(\rho)\, d\rho.$$

Since $v_0 \equiv 1$, we see that

$$|\partial_t^\alpha h(0)| \geq \delta^{\alpha+1} \alpha!^2,$$

with δ small and positive; that is, h is not better than G^2 even though the right-hand side is real analytic. This ends the proof that L is G^2-hypoelliptic and not better.

We make a few remarks on this example. First: in general, just one cutoff is not enough to sum the formal series of the v_{2k}'s. A more complex technique is required. Second: solving the transport equations has been possible because there is a "gain" in the decreasing rate of the functions v_{2k}. In general, one also has to control the growth rate of the coefficients of the differential operators defining the operator in parentheses under the integral sign in the second line of (6-1). As a last remark, the conclusion will not follow in general by an easy computation of the derivatives of (the analog of) h. Instead we need to violate an a priori estimate being equivalent to the G^s-hypoellipticity. Such an estimate was proved by Métivier [1980].

6.1. ***Construction of a formal solution.*** We recall from Theorem 3.2 the form of the pseudodifferential operator Λ (the L in Examples 1 and 2 above).

$$\Lambda(t, \tau) \sim \sum_{j=0}^{j_0} \Lambda_{-jq}(t, \tau) + \sum_{s \geq 0} \bigl(\Lambda_{-2k-sq}(t, \tau) + \Lambda_{-(j_0+1)q-sq}(t, \tau) \bigr).$$

In view of Proposition 3.1, we may assume that $\tau > 0$. Then

$$\Lambda(t, \tau) \sim \sum_{j=0}^{j_0} \Lambda_{-jq}(t, 1) \tau^{2/q-j} + \sum_{s \geq 0} \bigl(\Lambda_{-2k-sq}(t, 1) \tau^{2/q-2k/q-s} + \Lambda_{-(j_0+1)q-sq}(t, 1) \tau^{2/q-(j_0+1)-s} \bigr).$$

Multiply on the right by the elliptic factor $\tau^{-2/q+2k/q}$ and keep (3-34) in mind (Theorem 3.2); we then obtain the following expression of the real analytic symbol Λ:

$$\Lambda(t,\tau)\tau^{-2/q+2k/q} \sim \sum_{h=0}^{j_0+1} t^{2l-h} a_h(t)\tau^{2k/q-h} + \tilde{a}_0(t) + \sum_{h=1}^{\infty}\big(\tilde{a}_h(t)\tau^{-h} + b_h(t)\tau^{2k/q-(j_0+1)-h}\big), \quad (6\text{-}2)$$

where

$$\begin{aligned} a_h(t) &= t^{-2l+h}\Lambda_{-hq}(t,1) && \text{for } h = 0,\dots, j_0+1,\\ \tilde{a}_h(t) &= \Lambda_{-2k-hq}(t,1) && \text{for } h \ge 0,\\ b_h(t) &= \Lambda_{-(j_0+1)q-hq}(t,1) && \text{for } h \ge 1. \end{aligned}$$

We point out that a_h, $\tilde{a}_h$, b_h are real analytic functions near the origin.

Moreover, from (3-35) and (3-36) in Theorem 3.2, we have

$$a_0(0),\ \tilde{a}_0(0) > 0, \tag{6-3}$$

and

$$|\partial_t^\alpha \tilde{a}_h(t)| \le C^{1+h+\alpha}\alpha!\,h!\,, \qquad |\partial_t^\alpha b_h(t)| \le C^{1+h+\alpha}\alpha!\,h!\,, \tag{6-4}$$

for t in a (relatively compact) neighborhood of the origin and $h \ge 1$.

In order to simplify the notation, we denote again by $\Lambda(t,\tau)$ the symbol on the left-hand side of (6-2). It will also be useful to employ a more compact notation:

$$\Lambda(t,\tau) \sim \sum_{h=0}^{j_0} t^{2l-h} a_h(t)\tau^{2k/q-h} + \sum_{h=0}^{\infty} c_h(t)\tau^{-h/q}. \tag{6-5}$$

Here we replaced the expansion (6-2), where there is an order scaling by units, with a (more general) expansion exhibiting a scaling by multiples of $1/q$. In particular, (6-3) becomes

$$a_0(0),\ c_0(0) > 0 \tag{6-6}$$

and the estimates (6-4) become

$$|\partial_t^\alpha c_h(t)| \le C^{1+h+\alpha}\alpha!\,h!^{1/q}. \tag{6-7}$$

Furthermore, we shall use in the sequel the equalities

$$c_h(t) \equiv 0, \quad \text{for} \quad h = 1,\ \dots,\ q(j_0+1)-2k-1,\ q(j_0+1)-2k+1,\ \dots,\ q-1, \tag{6-8}$$

and

$$c_{q(j_0+1)-2k}(t) = \mathbb{O}(t^{2l-(j_0+1)}). \tag{6-9}$$

To obtain a formal null solution $\Lambda(t, D_t)$, we expand in power series the coefficients in the expression of Λ in (6-5); actually this is not an approximation, since the coefficients are real analytic functions. Interchanging the summation signs, we have

$$\Lambda(t, D_t) \sim \sum_{n\ge 0}\Big(\sum_{h=0}^{j_0} a_{hn} t^{2l-h+n} D_t^{2k/q-h} + \sum_{j=0}^{\infty} c_{jn} t^n D_t^{-j/q}\Big). \tag{6-10}$$

Here the conditions (6-6)–(6-9) become

$$a_{00},\ c_{00} > 0, \tag{6-11}$$

$$|a_{hn}| \le C_a^{1+n}, \quad |c_{jn}| \le C_a^{1+j+n} j!^{1/q} \quad \text{for } h = 0, \dots, j_0 \text{ and } j, n \ge 0 \tag{6-12}$$

(where C_a denotes a positive constant independent of h, j and n),

$$c_{jn} = 0 \quad \text{for } n \ge 0 \text{ and } j = 1, \dots, q(j_0+1) - 2k - 1, q(j_0+1) - 2k + 1, \dots, q - 1, \tag{6-13}$$

$$c_{q(j_0+1)-2k,n} = 0 \quad \text{for} \quad 0 \le n < 2l - (j_0+1). \tag{6-14}$$

The next step is to formally apply the operator Λ as defined in (6-10) to a function of the form

$$A(u)(t) = \int_0^{+\infty} e^{it\rho^{s_0}} u(\rho)\, d\rho, \tag{6-15}$$

where s_0 has been defined in Theorem 5.7 and u denotes a rapidly decreasing function with support bounded away from the origin. We search for a u such that $\Lambda(t, D_t)A(u)(t) = 0$ formally.

Applying a not necessarily integer power of D_t to $A(u)$ means multiplying u by the corresponding power of ρ. In order to write the contribution due to multiplication by a power of t, we need:

Lemma 6.1.1. *Let s_0 have the same meaning as before. Then*

$$\left(-\partial_\rho \frac{1}{i s_0 \rho^{s_0-1}}\right)^n = \sum_{h=0}^{n} \gamma_{nh} \frac{1}{\rho^{s_0 n - h}} \partial_\rho^h, \tag{6-16}$$

where the γ_{nh} (which now contain s_0) are complex constants satisfying estimates of the form

$$|\gamma_{nh}| \le C_\gamma'^{\,n+h} \frac{n!}{h!} \le C_\gamma^{n+h} (n-h)!. \tag{6-17}$$

Here both C_γ' and C_γ are positive constants depending on s_0 only. In particular, we have $\gamma_{nn} = (i/s_0)^n$, and for convenience set $\gamma_{00} = 1$.

Proof. It is enough to prove the first inequality. Arguing by induction, one easily sees that the coefficients γ_{nh} satisfy the recurrence relations

$$\gamma_{n+1,0} = -\frac{i}{s_0}\big(\gamma_{n0}(s_0(n+1)-1)\big), \quad \gamma_{n+1,n+1} = \frac{i}{s_0}\gamma_{nn}, \quad \gamma_{n+1,h} = \frac{i}{s_0}\big(\gamma_{nh-1} - (s_0(n+1)-h-1)\gamma_{nh}\big).$$

An induction argument allows us to conclude. □

We then have the formula, for $m \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$t^n D_t^m A(u)(t) = \int_0^{+\infty} e^{it\rho^{s_0}} \left(-\partial_\rho \frac{1}{i s_0 \rho^{s_0-1}}\right)^n \rho^{m s_0} u(\rho)\, d\rho.$$

Using this formula repeatedly as well as Lemma 6.1.1, we get

$$\Lambda(t, D_t) A(u)(t) = \int_0^{+\infty} e^{it\rho^{s_0}} P(\rho, D_\rho) u(\rho)\, d\rho, \tag{6-18}$$

where

$$P(\rho,\partial_\rho)=\sum_{n=0}^{\infty}\Bigg\{\sum_{h=0}^{j_0}\sum_{p=0}^{2l-h+n} a_{hn}\gamma_{2l-h+n,p}\frac{1}{\rho^{s_0(2l-h+n)-p}}\partial_\rho^p\rho^{s_0(2k/q-h)} + \sum_{j=0}^{\infty}\sum_{p=0}^{n} c_{jn}\gamma_{np}\frac{1}{\rho^{s_0 n-p}}\partial_\rho^p\rho^{-s_0 j/q}\Bigg\}. \tag{6-19}$$

We use the notation

$$\partial_\rho^p(\rho^\lambda u)=\sum_{\alpha=0}^{p}\binom{p}{\alpha}(\lambda)_{p-\alpha}\rho^{\lambda-p+\alpha}\partial_\rho^\alpha u, \tag{6-20}$$

where $(\lambda)_\beta$ is the Pochhammer symbol, defined by

$$(\lambda)_\beta=\lambda(\lambda-1)\dots(\lambda-\beta+1),\quad (\lambda)_0=1,\quad \lambda\in\mathbb{C}. \tag{6-21}$$

We point out that the following identity is a trivial consequence of the definition of s_0:

$$s_0\frac{2k}{q}-(s_0-1)2l=0. \tag{6-22}$$

Using (6-22) and the preceding identities, we obtain the expression for P

$$P(\rho,\partial_\rho)=\sum_{n=0}^{\infty}\Bigg\{\sum_{h=0}^{j_0}\sum_{p=0}^{2l-h+n}\sum_{\alpha=0}^{p} a_{hn}\gamma_{2l-h+n,p}\binom{p}{\alpha}\cdot\Big(s_0\Big(\frac{2k}{q}-h\Big)\Big)_{p-\alpha}\rho^{-2l-s_0 n+\alpha}\partial_\rho^\alpha + \sum_{j=0}^{\infty}\sum_{p=0}^{n}\sum_{\alpha=0}^{p} c_{jn}\gamma_{np}\binom{p}{\alpha}\Big(-s_0\frac{j}{q}\Big)_{p-\alpha}\rho^{-s_0 n-s_0 j/q+\alpha}\partial_\rho^\alpha\Bigg\}. \tag{6-23}$$

Define now the coefficients

$$A_{h\alpha n}=\sum_{p=\alpha}^{2l-h+n}\gamma_{2l-h+n,p}\binom{p}{\alpha}\Big(s_0\Big(\frac{2k}{q}-h\Big)\Big)_{p-\alpha} \tag{6-24}$$

and

$$B_{j\alpha n}=\sum_{p=\alpha}^{n}\gamma_{np}\binom{p}{\alpha}\Big(-s_0\frac{j}{q}\Big)_{p-\alpha}. \tag{6-25}$$

In particular, $A_{0,2l,0}=\gamma_{2l,2l}=(i/s_0)^{2l}$ and $B_{000}=1$.

Lemma 6.1.2. *For $h\in\{0,\dots,j_o\}$, $n\geq 0$, $\alpha\in\{0,\dots,2l-h+n\}$, we have*

$$|A_{h\alpha n}|\leq C_A^{2l-h+n+1}\frac{(2l-h+n)!}{\alpha!}. \tag{6-26}$$

For $j,n\geq 0$, $\alpha\in\{0,\dots,n\}$, we have

$$|B_{j\alpha n}|\leq C_B^{n+s_0(j/q)+1}\frac{n!}{\alpha!}. \tag{6-27}$$

Proof. Let us first consider the $A_{h\alpha n}$. Since, for $r = 0, \dots, p-\alpha-1$,

$$\left|s_0(2k/q-h)-r\right| = \left|s_0(2k/q-h)-1-(r-1)\right| \le \lfloor s_0(2k/q)\rfloor + r + 1,$$

we have

$$\left|\left(s_0\left(\frac{2k}{q}-h\right)\right)_{p-\alpha}\right| \le \frac{\left(\left\lfloor s_0\frac{2k}{q}\right\rfloor + p-\alpha\right)!}{\left(\left\lfloor s_0\frac{2k}{q}\right\rfloor\right)!} \le C^{p-\alpha}(p-\alpha)!\,,$$

for a convenient positive constant C. We may then write, due to (6-17), that

$$|A_{h\alpha n}| \le \sum_{p=\alpha}^{2l-h+n} C_\gamma^{2l-h+n+p} C^{p-\alpha} \frac{(2l-h+n)!}{p!}\binom{p}{\alpha}(p-\alpha)! \le C_A^{2l-h+n}\frac{(2l-h+n)!}{\alpha!}$$

for a suitable positive constant C_A. This proves the first statement. The second is proved in an analogous way and we omit the details. □

Using the definitions (6-24), (6-25), the operator P in (6-23) can be rewritten as

$$P(\rho,\partial_\rho) = \sum_{n=0}^{\infty}\left\{\sum_{h=0}^{j_0}\sum_{\alpha=0}^{2l-h+n} a_{hn}A_{h\alpha n}\rho^{-2l-s_0n+\alpha}\partial_\rho^\alpha + \sum_{j=0}^{n}\sum_{\alpha=0}^{n} c_{jn}B_{j\alpha n}\rho^{-s_0n-s_0j/q+\alpha}\partial_\rho^\alpha\right\}. \tag{6-28}$$

Setting

$$\tilde{A}_{n\alpha} = \sum_{h=0}^{\min\{j_0,2l+n-\alpha\}} a_{hn}A_{h\alpha n}, \tag{6-29}$$

the above expression of P can be slightly simplified:

$$P(\rho,\partial_\rho) = \sum_{n=0}^{\infty}\left\{\sum_{\alpha=0}^{2l+n}\tilde{A}_{n\alpha}\rho^{-2l-s_0n+\alpha}\partial_\rho^\alpha + \sum_{j=0}^{n}\sum_{\alpha=0}^{n} c_{jn}B_{j\alpha n}\rho^{-s_0n-s_0j/q+\alpha}\partial_\rho^\alpha\right\}. \tag{6-30}$$

Moreover, the estimate of Lemma 6.1.2 carries over to $\tilde{A}_{n\alpha}$:

Lemma 6.1.3. *For $n \ge 0$, $\alpha \in \{0,\dots,2l+n\}$, we have*

$$|\tilde{A}_{n\alpha}| \le C_{\tilde{A}}^{2l+n+1}\frac{(2l+n)!}{\alpha!}. \tag{6-31}$$

For reasons that will become apparent in the sequel, we prefer to write the operator P in a way where the factorial growth of the coefficients is coupled with a corresponding negative power of the variable ρ, that is,

$$P(\rho,\partial_\rho) = \sum_{n=0}^{\infty}\left\{\sum_{\alpha=0}^{2l+n}\frac{\tilde{A}_{n\alpha}}{\rho^{2l+n-\alpha}}\rho^{-(s_0-1)n}\partial_\rho^\alpha + \sum_{j=0}^{n}\sum_{\alpha=0}^{n}\frac{c_{jn}B_{j\alpha n}}{\rho^{n-\alpha+j/q}}\rho^{-(s_0-1)(n+j/q)}\partial_\rho^\alpha\right\}. \tag{6-32}$$

We point out that the powers of ρ in the above expression of P are all negative. However, if we were now to attempt to find a formal solution to $Pu = 0$ by solving iteratively the transport equations obtained by looking for a u in the form $\sum_{k\geq 0} u_k$, we would not be able to conclude that the sequence u_k decreases with respect to ρ in such a way that we can asymptotically sum the series for u. In other words, we wish u to behave as a symbol and we want to compute its asymptotic expansion for large ρ, but for the time being, there is no guarantee that the symbols u_k would have a decreasing order in ρ when k goes to infinity.

A way around this is to introduce a phase function and to write u as $u(\rho) = e^{i\Phi(\rho)} v(\rho)$, in such a way that the negative powers of ρ in the expression of P which are not *negative enough* are canceled by $\Phi(\rho)$. This is what we do in the next step.

Using the Faà di Bruno formula, we have

$$
\begin{aligned}
e^{-i\Phi}\partial_\rho^n e^{i\Phi} = (\partial_\rho + i\Phi_\rho)^n &= e^{-i\Phi} \sum_{h=0}^{n} \binom{n}{h} (\partial_\rho^h e^{i\Phi}) \partial_\rho^{n-h} \\
&= \sum_{h=0}^{n} \sum_{k=1}^{h} i^k \sum_{\substack{k_1,\dots,k_h \\ \sum_i k_i = k \\ \sum_i i k_i = h}} \binom{n}{h} \frac{h!}{k_1!\cdots k_h!} \prod_{p=1}^{h} \Big(\frac{\Phi_\rho^{(p-1)}}{p!} \Big)^{k_p} \partial_\rho^{n-h}.
\end{aligned}
$$

Here $\Phi_\rho^{(k)} = \partial_\rho^{k+1}\Phi$ and $\Phi_\rho = \Phi_\rho^{(0)}$. Plugging this formula into (6-32), we obtain

$$
\begin{aligned}
&e^{-i\Phi(\rho)} P(\rho, \partial_\rho) e^{i\Phi(\rho)} \\
&= \sum_{n=0}^{\infty} \sum_{\alpha=0}^{2l+n} \sum_{h=0}^{\alpha} \sum_{k=1}^{h} i^k \sum_{\substack{k_1,\dots,k_h \\ \sum_i k_i = k \\ \sum_i i k_i = h}} \binom{\alpha}{h} \frac{h!}{k_1!\dots k_h!} \frac{\tilde{A}_{n\alpha}}{\rho^{2l+n-\alpha}} \rho^{-(s_0-1)n} \prod_{p=1}^{h} \Big(\frac{\Phi_\rho^{(p-1)}}{p!} \Big)^{k_p} \partial_\rho^{\alpha-h} \qquad (6\text{-}33) \\
&\quad + \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\alpha=0}^{n} \sum_{h=0}^{\alpha} \sum_{k=1}^{h} i^k \sum_{\substack{k_1,\dots,k_h \\ \sum_i k_i = k \\ \sum_i i k_i = h}} \binom{\alpha}{h} \frac{h!}{k_1!\dots k_h!} \frac{c_{jn} B_{j\alpha n}}{\rho^{n-\alpha+j/q}} \rho^{-(s_0-1)(n+j/q)} \prod_{p=1}^{h} \Big(\frac{\Phi_\rho^{(p-1)}}{p!} \Big)^{k_p} \partial_\rho^{\alpha-h}.
\end{aligned}
$$

Our purpose is to cancel all terms containing powers $\rho^{-\theta}$ with $0 > -\theta \geq -1$ and no derivatives. This is closely connected with the form of the (asymptotic expansion of the) operator Λ and is actually performed by choosing a phase function Φ of the form

$$
\Phi_\rho(\rho) = \sum_{j=0}^{M_0} \varphi_j \rho^{-(s_0-1)j} + \varphi_{-1}\rho^{-1}, \quad M_0 = \Big\lfloor \frac{1}{s_0 - 1} \Big\rfloor. \qquad (6\text{-}34)
$$

Here $\lfloor \dots \rfloor$ denotes the integer part and the φ_j, $j = -1, 0, \dots, M_0$, are complex numbers to be chosen later.

Let us find the terms in both summands in (6-33) where there are no derivatives and the power of ρ is not below -1. To this end, we remark that only Φ_ρ plays a role since, because of (6-34), $\Phi_\rho^{(k)}(\rho) = o(\rho^{-1})$ if $k \geq 1$.

Let us focus first on the first summand in (6-33). The terms with no derivatives correspond to $\alpha = h$. The terms where only first derivatives of Φ appear have $k_1 = k = h$. Moreover, since $2l + n - \alpha$ is an integer, we necessarily must have either $2l + n - \alpha = 0$ and $0 \le n \le M_0$, or $2l + n - \alpha = 1$ and $n = 0$.

Let us consider the second summand in (6-33). Similarly to the preceding case, $\alpha = h$ and $k_1 = k = h$. Moreover, we necessarily have $n = \alpha$. In view of (6-13) and (6-14), either $j = 0$ and $0 \le n \le M_0$, or $j = q(j_0 + 1) - 2k$ and $n = 2l - (j_0 + 1)$ if $j_0 = 0$ (that is, $2k < q$.)

It turns out to be useful to have a notation for the family of indices in both the first and second summands in (6-33) corresponding to terms that do not contribute to the eikonal equation. We call these two families of indices $\mathscr{A}$ and $\mathscr{B}$ respectively. We have

$$\mathscr{A} = \{(n, \alpha, h, k) \mid n > M_0\} \cup \{(n, \alpha, h, k) \mid 0 \le n \le M_0, (\alpha, h, k) \ne (2l+n, 2l+n, 2l+n)\} \\ \cup \{(n, \alpha, h, k) \mid (n, \alpha, h, k) \ne (0, 2l-1, 2l-1, 2l-1)\}, \tag{6-35}$$

$$\mathscr{B} = \{(n, j, \alpha, h, k) \mid n > M_0\} \cup \{(n, j, \alpha, h, k) \mid 0 \le n \le M_0, (j, \alpha, h, k) \ne (0, n, n, n)\} \\ \cup \{(n, j, \alpha, h, k) \mid j = q - 2k, n = 2l-1, (\alpha, h, k) \ne (n, n, n), \text{ if } j_0 = 0\}. \tag{6-36}$$

The terms contributing to the eikonal equation are then

$$\sum_{n=0}^{M_0} i^{2l+n} \tilde{A}_{n,2l+n} \rho^{-(s_0-1)n} \Phi_\rho^{2l+n} + \rho^{-1} i^{2l-1} \tilde{A}_{0,2l-1} \Phi_\rho^{2l-1} \\ + \sum_{n=0}^{M_0} i^n c_{0n} B_{0nn} \rho^{-(s_0-1)n} \Phi_\rho^n + i^n c_{q-2k,2l-1} B_{q-2k,2l-1,2l-1} \rho^{-1} \Phi_\rho^{2l-1}, \tag{6-37}$$

where the last term of the expression above is present only if $j_0 = 0$. Note that there is a kind of "principal part" in the above expression, namely the part not containing negative powers of ρ. This part is obtained by setting $n = 0$. Now by (6-29),

$$i^{2l} \tilde{A}_{0,2l} = i^{2l} a_{00} A_{0,2l,0} = i^{2l} a_{00} (i/s_0)^{2l} = a_{00} s_0^{-2l} > 0,$$

where the next to last equality is due to (6-24) and the positivity is a consequence of (6-11). On the other hand, again by (6-11) and (6-24), $c_{00} B_{000} = c_{00} > 0$.

Lemma 6.1.4. *Consider the equation*

$$\sum_{n=0}^{M_0} \rho^{-(s_0-1)n} \big(a_n \Phi_\rho^{2l+n} + b_n \Phi_\rho^n\big) + \gamma \rho^{-1} \Phi_\rho^{2l-1} = \mathbb{O}(\rho^{-1-\delta}). \tag{6-38}$$

Here a_n, b_n, γ denote complex numbers and $a_0, b_0 > 0$; δ is a positive rational number.

Then there is a function $\Phi_\rho(\rho)$, $\rho > 0$, of the form (6-34), satisfying (6-38) with

$$\delta = (M_0 + 1)(s_0 - 1) - 1 > 0$$

and such that

$$\operatorname{Im} \Phi_\rho(\rho) > 0 \quad \text{modulo } \mathbb{O}(\rho^{-(s_0-1)}). \tag{6-39}$$

Proof. To start with, we remark that the equation

$$a_0 \Phi_\rho^{2l} + b_0 = \mathbb{O}(\rho^{-(s_0-1)})$$

is satisfied by $\Phi_\rho(\rho)$ in (6-34), where $a_0\varphi_0^{2l} + b_0 = 0$. Of course we are always free to choose φ_0 such that $\operatorname{Im}\varphi_0 > 0$. We now argue by induction. Assume that we determined $\varphi_0, \dots, \varphi_{k-1}$ and solved (6-38) modulo $o(\rho^{-(k-1)(s_0-1)})$. Let us compute the coefficient of $\rho^{-k(s_0-1)}$ in (6-38), with $k \le M_0$. First we observe that if α denotes the multi-index $(\alpha_0, \alpha_1, \dots, \alpha_{M_0})$ with $\alpha_r \in \mathbb{Z}^+$ and φ denotes the complex vector $(\varphi_0, \varphi_1, \dots, \varphi_{M_0})$, we have

$$\Phi_\rho^j(\rho) = \sum_{|\alpha|=j} \frac{j!}{\alpha!} \varphi^\alpha \rho^{-(s_0-1)\sum_{p=0}^{M_0} p\alpha_p} \quad \text{modulo } \mathbb{O}(\rho^{-1}).$$

The coefficient of $\rho^{-k(s_0-1)}$ is then given by

$$\sum_{j=0}^{k} \Biggl(a_j \sum_{\substack{|\alpha|=2l+j \\ \sum_p p\alpha_p = k-j}} \frac{(2l+j)!}{\alpha!} \varphi^\alpha + b_j \sum_{\substack{|\alpha|=j \\ \sum_p p\alpha_p = k-j}} \frac{j!}{\alpha!} \varphi^\alpha \Biggr).$$

The constraint on $\sum_p p\alpha_p$ forces the index p to run from 0 to $k-j$, and it is clear that if $j>0$, the first summand above cannot contain φ_k, since $\alpha_{k-j+1} = \cdots = \alpha_{M_0} = 0$. Consider thus the term with $j=0$. Then α_k is zero or one. The first case is similar to the previous cases, so that α_k must be one. Then since $\alpha_1 = \cdots = \alpha_{k-1} = 0$, we see that $\alpha_0 = 2l-1$, thus yielding the coefficient of $\rho^{-k(s_0-1)}$ containing φ_k:

$$2l a_0 \varphi_0^{2l-1} \varphi_k.$$

Arguing analogously, we can see that φ_k is never contained in terms coming from the second summand. This allows us to uniquely determine φ_k, since $a_0, \varphi_0 \ne 0$.

The argument for φ_{-1} is completely similar and we omit it. □

The above lemma gives the existence of the phase function Φ of the form (6-34) such that in the expression of $e^{-i\Phi} P e^{i\Phi}$ there are no terms without derivatives in which ρ has an exponent greater than or equal to -1. We stress that the reason why we need this fact will become apparent when we have to solve the transport equations, which thus far have not played a role.

Thus the operator $e^{-i\Phi} P e^{i\Phi}$ now has the form

$$\begin{aligned}
&e^{-i\Phi(\rho)} P(\rho, \partial_\rho) e^{i\Phi(\rho)} \\
&\quad = \sum_{\substack{n=0 \\ (n,\alpha,h,k)\in\mathcal{A}}}^{\infty} \sum_{\alpha=0}^{2l+n} \sum_{h=0}^{\alpha} \sum_{k=1}^{h} i^k \sum_{\substack{k_1,\dots,k_h \\ \sum_i k_i = k \\ \sum_i i k_i = h}} \binom{\alpha}{h} \frac{h!}{k_1! \cdots k_h!} \frac{\tilde{A}_{n\alpha}}{\rho^{2l+n-\alpha}} \rho^{-(s_0-1)n} \prod_{p=1}^{h} \biggl(\frac{\Phi_\rho^{(p-1)}}{p!} \biggr)^{k_p} \partial_\rho^{\alpha-h} \\
&\qquad + \sum_{\substack{n=0 \\ (n,j,\alpha,h,k)\in\mathcal{B}}}^{\infty} \sum_{j=0}^{\infty} \sum_{\alpha=0}^{n} \sum_{h=0}^{\alpha} \sum_{k=1}^{h} i^k \sum_{\substack{k_1,\dots,k_h \\ \sum_i k_i = k \\ \sum_i i k_i = h}} \binom{\alpha}{h} \frac{h!}{k_1! \cdots k_h!} \frac{c_{jn} B_{j\alpha n}}{\rho^{n-\alpha+j/q}} \rho^{-(s_0-1)(n+j/q)} \\
&\qquad\qquad \cdot \prod_{p=1}^{h} \biggl(\frac{\Phi_\rho^{(p-1)}}{p!} \biggr)^{k_p} \partial_\rho^{\alpha-h} + \mathbb{O}(\rho^{-(1+\delta)}).
\end{aligned} \tag{6-40}$$

Here the last term is a consequence of (6-37) and Lemma 6.1.4, where we defined δ.

Lemma 6.1.5. *Let Φ be as in (6-34) and denote by C_Φ a positive constant such that $|\varphi_j| \le C_\Phi$ for $j = -1, 0, 1, \ldots, M_0$. Then*

$$\prod_{p=1}^{h}\left(\frac{\Phi_\rho^{(p-1)}}{p!}\right)^{k_p} = \rho^{-(h-k)} \sum_{\substack{t_1=1-\delta_{h,k} \\ k-k_1\le t_1+M_0t_2\le kM_0}}^{kM_0} \sum_{t_2=0}^{k} c_{(k_1,\ldots,k_h),t_1,t_2}\,\rho^{-(s_0-1)t_1-t_2}, \tag{6-41}$$

where $k = \sum_{i=1}^{h} k_i$, $h = \sum_{i=1}^{h} i k_i$, and $\delta_{h,k}$ is the usual Kronecker symbol. Moreover, we have the estimate

$$|c_{(k_1,\ldots,k_h),t_1,t_2}| \le \binom{t_1+t_2+k}{k} C_\Phi^k. \tag{6-42}$$

Proof. We argue by induction on h. If $h = 1$, then $k = 1$ and $k_1 = 1$, so that (6-41) is trivial. Assume now that $h > 1$ and suppose that (6-41) holds for every $h' < h$. There are two cases:

Case I. If $k_h \ne 0$, then from $h = \sum_{i=1}^{h} i k_i$, we obtain that $k_h = 1$ and $k_1, \ldots, k_{h-1} = 0$, and hence $k = 1$. Then

$$\begin{aligned}\prod_{p=1}^{h}\left(\frac{\Phi_\rho^{(p-1)}}{p!}\right)^{k_p} &= \frac{1}{h!}\left(\sum_{j=0}^{M_0}\varphi_j\rho^{-(s_0-1)j} + \varphi_1\rho^{-1}\right)^{(h-1)} \\ &= \rho^{-(h-1)}\left(\sum_{t_1=1}^{M_0} c_{(0,\ldots,0,1),t_1,0}\,\rho^{-(s_0-1)t_1} + c_{(0,\ldots,0,1),0,1}\,\rho^{-1}\right),\end{aligned}$$

which proves the statement.

Case II. Suppose $k_h = 0$. Let $s = \min\{j \mid k_j \ne 0\}$ so that $\sum_{i=s}^{h-1} i k_i = h$. Note that

$$s(k_s-1) + (s+1)k_{s+1} + \cdots + (h-1)k_{h-1} = h - s.$$

If $s = 1$, the h-tuple $(k_1-1, k_2, \ldots, k_{h-1}, 0)$ can be thought of as an $(h-1)$-tuple such that

$$k_1 - 1 + 2k_2 + \cdots + (h-1)k_{h-1} = h - 1.$$

On the other hand, if $s > 1$, from $s(k_s-1) + (s+1)k_{s+1} + \cdots + (h-1)k_{h-1} = h - s$ we immediately deduce that $k_{h-a} = 0$ for every $a < s$, so that the h-tuple

$$\big(0, \ldots, 0, k_s - 1, \ldots, k_{h-1}, 0\big) = \big(0, \ldots, 0, k_s - 1, \ldots, k_{h-s}, 0, \ldots, 0\big)$$

can be identified to the $(h-s)$-tuple

$$(k_1, \ldots, k_{s-1}, k_s - 1, \ldots, k_{h-s}),$$

where $k_1 = \cdots = k_{s-1} = 0$ and $s(k_s-1) + \cdots + (h-s)k_{h-s} = h - s$. We are now in a position to apply the inductive hypothesis. Assume, to make things definite, that $s > 1$ (the case $s = 1$ is analogous). Then

$$\prod_{p=1}^{h}\left(\frac{\Phi_\rho^{(p-1)}}{p!}\right)^{k_p}=\prod_{p=1}^{h-s}\left(\frac{\Phi_\rho^{(p-1)}}{p!}\right)^{k_p}=\frac{\Phi_\rho^{(s-1)}}{s!}\prod_{p=s}^{h-s}\left(\frac{\Phi_\rho^{(p-1)}}{p!}\right)^{k_p-\delta_{s,p}}$$
$$=\frac{\Phi_\rho^{(s-1)}}{s!}\rho^{-(h-s-(k-1))}\cdot\sum_{\substack{t_1=1-\delta_{h-s,k-1}\\ k-1\le t_1+M_0t_2\le(k-1)M_0}}^{(k-1)M_0}\sum_{t_2=0}^{k-1}c_{(0,\dots,0,k_s-1,\dots,k_{h-s}),t_1,t_2}\rho^{-(s_0-1)t_1-t_2}.$$

Recall now that

$$\frac{\Phi_\rho^{(s-1)}}{s!}=\rho^{-(s-1)}\left(\sum_{j=1}^{M_0}c_{s,j}\rho^{-(s_0-1)j}+c_{s,-1}\rho^{-1}\right),$$

for certain numbers $c_{s,j}$, $c_{s,-1}$. Note that we can find a positive constant C_Φ such that $|\varphi_j|\le C_\Phi$ for every $j=-1,0,\dots,M_0$, and that then

$$|c_{s,j}|\le C_\Phi,\quad j=-1,1,\dots,M_0.$$

Using the above expression for $\Phi_\rho^{(s-1)}/s!$, we obtain

$$\prod_{p=1}^{h}\left(\frac{\Phi_\rho^{(p-1)}}{p!}\right)^{k_p}=\rho^{-(h-k)}\Bigg[\sum_{j=1}^{M_0}\sum_{\substack{t_1=1-\delta_{h-s,k-1}\\ k-1\le t_1+M_0t_2\le(k-1)M_0}}^{(k-1)M_0}\sum_{t_2=0}^{k-1}c_{(0,\dots,0,k_s-1,\dots,k_{h-s}),t_1,t_2}c_{s,j}\rho^{-(s_0-1)(t_1+j)-t_2}$$
$$+c_{s,-1}\sum_{\substack{t_1=1-\delta_{h-s,k-1}\\ k-1\le t_1+M_0t_2\le(k-1)M_0}}^{(k-1)M_0}\sum_{t_2=0}^{k-1}c_{(0,\dots,0,k_s-1,\dots,k_{h-s}),t_1,t_2}\rho^{-(s_0-1)t_1-t_2}\Bigg].\tag{6-43}$$

Now in the first sum we note that, as far as the powers of ρ are concerned, $k=k-1+1\le t_1+j+t_2M_0\le(k-1)M_0+M_0=kM_0$, while in the second sum above we have $k\le k-1+M_0\le t_1+(t_2+1)M_0\le(k-1)M_0+M_0=kM_0$, where we assume we are in the nontrivial case $M_0\ge1$. This proves the first statement of the lemma. To finish the proof we have to prove estimate (6-42). We again argue by induction and use the expression (6-43) above. Actually the coefficient of $\rho^{-(s_0-1)t_1-t_2}$ coming from the first sum has the form

$$\sum_{t+j=t_1}c_{s,j}c_{(0,\dots,0,k_s-1,\dots,k_{h-s}),t,t_2}.$$

Its absolute value is estimated by

$$\sum_{j=0}^{t_1}C_\Phi^k\binom{j+k-1}{k-1}=C_\Phi^k\binom{t_1+k}{k},$$

where we have used the fact that $|c_{s,j}|\le C_\Phi$. This concludes the proof of the lemma. □

Using Lemma 6.1.5, we are going to make some preparations on the operator $e^{-i\Phi}Pe^{i\Phi}$ in (6-40). First of all, using (6-41), we write it in the rather lengthy form

$$
\begin{aligned}
&e^{-i\Phi(\rho)}P(\rho,\partial_\rho)e^{i\Phi(\rho)}\\
&=\sum_{\substack{n=0\\(n,\alpha,h,k)\in\mathcal{A}}}^{\infty}\sum_{\alpha=0}^{2l+n}\sum_{h=0}^{\alpha}\sum_{k=1}^{h} i^k \sum_{\substack{k_1,\dots,k_h\\ \sum_i k_i=k\\ \sum_i ik_i=h}}\ \sum_{\substack{t_1=1-\delta_{k,h}\\ k-k_1\le t_1+M_0t_2\le kM_0}}^{kM_0}\ \sum_{t_2=0}^{M_0}\binom{\alpha}{h}\frac{h!}{k_1!\dots k_h!}\\
&\qquad\qquad\cdot c_{(k_1,\dots,k_h),t_1,t_2}\frac{\tilde{A}_{n\alpha}}{\rho^{2l+n-\alpha}}\rho^{-(s_0-1)(n+t_1)-(t_2+h-k)}\partial_\rho^{\alpha-h}\\
&\quad+\sum_{\substack{n=0\\(n,j,\alpha,h,k)\in\mathcal{B}}}^{\infty}\sum_{j=0}^{\infty}\sum_{\alpha=0}^{n}\sum_{h=0}^{\alpha}\sum_{k=1}^{h} i^k \sum_{\substack{k_1,\dots,k_h\\ \sum_i k_i=k\\ \sum_i ik_i=h}}\ \sum_{\substack{t_1=1-\delta_{k,h}\\ k-k_1\le t_1+M_0t_2\le kM_0}}^{kM_0}\ \sum_{t_2=0}^{M_0}\binom{\alpha}{h}\frac{h!}{k_1!\cdots k_h!}\\
&\qquad\qquad\cdot c_{(k_1,\dots,k_h),t_1,t_2}\frac{c_{jn}B_{j\alpha n}}{\rho^{n-\alpha+j/q}}\rho^{-(s_0-1)(n+j/q+t_1)-(t_2+h-k)}\partial_\rho^{\alpha-h}+\mathbb{O}(\rho^{-(1+\delta)}).
\end{aligned}\tag{6-44}
$$

Here the last term, $\mathbb{O}(\rho^{-(1+\delta)})$, denotes a finite sum of terms of the form

$$\gamma_k\rho^{-\theta_k},$$

where γ_k is a constant and $\theta_k\ge 1+\delta$.

For every $r\in\mathbb{N}\cup\{0\}$, define the pair of differential operators

$$
\begin{aligned}
Q_{\mathcal{A},r}(\rho,\partial_\rho)=\sum_{\substack{q(n+t_1)=r\\ t_1\le(2l+n)M_0}}\ \sum_{\substack{\alpha=t_1\\(n,\alpha,h,k)\in\mathcal{A}}}^{2l+n}\sum_{h=t_1}^{\alpha}\ \sum_{k=\min\{1,t_1\}}^{h} i^k\sum_{\substack{k_1,\dots,k_h\\ \sum_i k_i=k\\ \sum_i ik_i=h}}\ \sum_{t_2=0}^{M_0}\binom{\alpha}{h}\frac{h!}{k_1!\cdots k_h!}\rho^{-(h-k)}\\
\cdot c_{(k_1,\dots,k_h),t_1,t_2}\frac{\tilde{A}_{n\alpha}}{\rho^{2l+n-\alpha}}\rho^{-t_2}\partial_\rho^{\alpha-h}
\end{aligned}\tag{6-45}
$$

and

$$
\begin{aligned}
Q_{\mathcal{B},r}(\rho,\partial_\rho)=\sum_{\substack{q(n+t_1)+j=r\\ t_1\le nM_0}}\ \sum_{\substack{\alpha=t_1\\(n,\alpha,h,k)\in\mathcal{B}}}^{n}\sum_{h=t_1}^{\alpha}\ \sum_{k=\min\{1,t_1\}}^{h} i^k\sum_{\substack{k_1,\dots,k_h\\ \sum_i k_i=k\\ \sum_i ik_i=h}}\ \sum_{t_2=0}^{M_0}\binom{\alpha}{h}\frac{h!}{k_1!\cdots k_h!}\rho^{-(h-k)}\\
\cdot c_{(k_1,\dots,k_h),t_1,t_2}\frac{c_{j,n}B_{j\alpha n}}{\rho^{n-\alpha+j/q}}\rho^{-t_2}\partial_\rho^{\alpha-h}.
\end{aligned}\tag{6-46}
$$

Then the operator in (6-40) can be rewritten in the simpler form

$$e^{-i\Phi(\rho)}P(\rho,\partial_\rho)e^{i\Phi(\rho)}=\sum_{r=0}^{\infty}\rho^{-(s_0-1)r/q}P_r(\rho,\partial_\rho)+\mathbb{O}(\rho^{-(1+\delta)}),\tag{6-47}$$

where

$$P_r(\rho,\partial_\rho)=Q_{\mathscr{A},r}(\rho,\partial_\rho)+Q_{\mathscr{B},r}(\rho,\partial_\rho) \tag{6-48}$$

is a differential operator of order $2l+\lfloor r/q\rfloor$.

Our next task is to provide growth estimates with respect to r of arbitrary derivatives of the coefficients of the operator P_r in a region where ρ is large. These estimates are essential when one tries to construct a *true* solution from the solution that we have not discussed yet.

Proposition 6.1.6. *Denote by $\alpha_{r,p}(\rho)$ the coefficient of ∂_ρ^p in $P_r(\rho,\partial_\rho)$. Then we may find two positive constants c_1, C_α, such that if $\rho\ge c_1 r^\theta$, with $0<\theta\le 1$, we have*

$$\left|\partial_\rho^t\alpha_{r,p}(\rho)\right|\le C_\alpha^{r+t+1}r!^{1-\theta}\frac{t!}{\rho^t}. \tag{6-49}$$

Proof. First we remark that the coefficient under exam is given by

$$\alpha_{r,p}(\rho)=\begin{cases}\alpha_{r,p,1}(\rho)+\alpha_{r,p,2}(\rho) & \text{if } p\le\lfloor r/q\rfloor,\\ \alpha_{r,p,1}(\rho) & \text{if } \lfloor r/q\rfloor<p\le\lfloor r/q\rfloor+2l,\end{cases}$$

where $\alpha_{r,p,1}(\rho)$ comes from $Q_{\mathscr{A},r}(\rho,\partial_\rho)$ and correspondingly $\alpha_{r,p,2}(\rho)$ comes from $Q_{\mathscr{B},r}(\rho,\partial_\rho)$. Thus

$$P_r(\rho,\partial_\rho)=\sum_{p=0}^{2l+\lfloor r/q\rfloor}\alpha_{r,p}(\rho)\partial_\rho^p. \tag{6-50}$$

The expressions of $\alpha_{r,p,i}(\rho)$ are given by

$$\begin{aligned}\alpha_{r,p,1}(\rho)=\sum_{\substack{q(n+t_1)=r\\ t_1\le(2l+n)M_0}}\ \sum_{\substack{\alpha=\max\{t_1,p\}\\ (n,\alpha,\alpha-p,k)\in\mathscr{A}}}^{2l+n}\ \sum_{k=\min\{1,t_1\}}^{\alpha-p} i^k\sum_{\substack{k_1,\dots,k_{\alpha-p}\\ \sum_i k_i=k\\ \sum_i ik_i=\alpha-p}}\ \sum_{t_2=0}^{M_0}\binom{\alpha}{\alpha-p}\frac{(\alpha-p)!}{k_1!\cdots k_{\alpha-p}!}\rho^{-(\alpha-p-k)}&\\ \cdot c_{(k_1,\dots,k_{\alpha-p}),t_1,t_2}\frac{\tilde{A}_{n\alpha}}{\rho^{2l+n-\alpha}}\rho^{-t_2}&\end{aligned} \tag{6-51}$$

and

$$\begin{aligned}\alpha_{r,p,2}(\rho)=\sum_{\substack{q(n+t_1)+j=r\\ t_1\le nM_0}}\ \sum_{\substack{\alpha=\max\{t_1,p\}\\ (n,\alpha,\alpha-p,k)\in\mathscr{B}}}^{n}\ \sum_{k=\min\{1,t_1\}}^{\alpha-p} i^k\sum_{\substack{k_1,\dots,k_{\alpha-p}\\ \sum_i k_i=k\\ \sum_i ik_i=\alpha-p}}\ \sum_{t_2=0}^{M_0}\binom{\alpha}{\alpha-p}\frac{(\alpha-p)!}{k_1!\cdots k_{\alpha-p}!}\rho^{-(\alpha-p-k)}&\\ \cdot c_{(k_1,\dots,k_{\alpha-p}),t_1,t_2}\frac{c_{j,n}B_{j\alpha n}}{\rho^{n-\alpha+j/q}}\rho^{-t_2}.&\end{aligned} \tag{6-52}$$

We start by estimating (6-51). Differentiating t times the function in (6-51) has the effect of producing in the sum (6-51) the factor

$$(-1)^t\rho^{-t}\prod_{j=0}^{t-1}\bigl(t_2+2l+n-p-k+j\bigr).$$

Hence, using (6-42), (6-31),

$$\left|\partial_\rho^t \alpha_{r,p,1}(\rho)\right| \le \sum_{\substack{q(n+t_1)=r\\ t_1\le(2l+n)M_0}} \sum_{\substack{\alpha=\max\{t_1,p\}\\ (n,\alpha,\alpha-p,k)\in\mathscr{A}}}^{2l+n} \sum_{k=\min\{1,t_1\}}^{\alpha-p} \sum_{\substack{k_1,\dots,k_{\alpha-p}\\ \sum_i k_i=k\\ \sum_i ik_i=\alpha-p}} \sum_{t_2=0}^{M_0} 2^\alpha \frac{(\alpha-p)!}{k_1!\cdots k_{\alpha-p}!}\rho^{-(\alpha-p-k)} C_{\tilde{A}}^{n+1}\frac{(2l+n)!}{\alpha!}\rho^{-(2l+n-\alpha)} \cdot C_\Phi^k \binom{t_1+t_2+k}{k}\binom{t_2+2l+n-p-k+t-1}{t} t!\,\rho^{-t_2-t}.$$

Furthermore, we have

$$\frac{(\alpha-p)!}{k_1!\cdots k_{\alpha-p}!} = \frac{(\alpha-p)!}{k!(\alpha-p-k)!}\,\frac{k!}{k_1!\cdots k_{\alpha-p}!}(\alpha-p-k)! \le 2^{\alpha-p}2^{\alpha-p}(\alpha-p-k)! \le 4^{2l+r/q}(\alpha-p-k)!. \tag{6-53}$$

The number of multi-indices $(k_1,\dots,k_{\alpha-p})$ such that the sum of the components is k is given by

$$\binom{k+\alpha-p-1}{\alpha-p-1} \le 4^{2l+r/q}.$$

If $\rho \ge c_1 r^\theta$, with $0<\theta\le 1$, we may estimate, if $\beta \le c_2 r$,

$$\frac{\beta!}{\rho^\beta} \le \frac{\beta!}{c_3^\beta \beta^{\theta\beta}} \le c_4^\beta \beta!^{1-\theta}, \qquad c_3 = c_1/c_2^\theta,\ c_4 = c_3^{-1}. \tag{6-54}$$

As a consequence, we obtain

$$\left|\partial_\rho^t \alpha_{r,p,1}(\rho)\right| \le \tilde{C}_1^{r+t+1} r!^{1-\theta}\frac{t!}{\rho^t},$$

where $\tilde{C}_1$ is a positive constant depending on the parameters of the problem and on θ.

The function $\partial_\rho^t \alpha_{r,p,2}(\rho)$ is estimated in a completely analogous way, and this proves the assertion. □

Let us now take a closer look at $P_0(\rho,\partial_\rho)$. We may write

$$P_0(\rho,\partial_\rho) = Q_0(\partial_\rho) + \sum_{m=1}^{N}\frac{1}{\rho^m}Q_m(\partial_\rho), \tag{6-55}$$

where the $Q_m(\partial_\rho)$ are differential operators with constant coefficients such that $Q_0(0)=Q_1(0)=0$, all the roots of the equation $Q_0(\lambda)=0$ are such that $\operatorname{Re}\lambda\ge 0$, due to the choice of the phase function Φ, and N is a suitable positive integer.

Let

$$j^*\in\mathbb{N}, \quad j^* = \left\lfloor \frac{q}{s_0-1}\right\rfloor.$$

Consider the order-zero term in the differential polynomial

$$\sum_{r=1}^{j^*} \rho^{-((s_0-1)/q)r} P_r(\rho, \partial_\rho).$$

It is obviously a finite sum involving negative powers of ρ of the form

$$\sum_j f_j \rho^{-\theta_j}, \quad \theta_j > 1, \ f_j \in \mathbb{C}.$$

Define λ by

$$\lambda + 1 = \min\{\theta_j\}.$$

Obviously λ is positive because $\theta_j > 1$. Lastly, set

$$\mu = \min\left\{1, \lambda, \delta, \frac{s_0-1}{q} - \frac{1}{j^*+1}\right\}, \tag{6-56}$$

which is a positive rational number, since

$$\frac{s_0-1}{q} - \frac{1}{j^*+1} > 0.$$

Also recall the definition of δ from Lemma 6.1.4. We are now in a position to define the final form for the operator P. Set

$$\tilde{P}_0(\partial_\rho) = Q_0(\partial_\rho), \tag{6-57}$$

$$\tilde{P}_r(\rho, \partial_\rho) = \rho^{(\mu-(s_0-1)/q)r} P_r(\rho, \partial_\rho), \qquad r \geq j^*+1, \tag{6-58}$$

$$\tilde{P}_r(\rho, \partial_\rho) = \rho^{(\mu-(s_0-1)/q)r} \big(P_r(\rho, \partial_\rho) - P_r(\rho, 0)\big), \quad 2 \leq r \leq j^*. \tag{6-59}$$

Finally, we define $\tilde{P}_1$, including in it both the errors coming from the construction of the phase function and the zero-order terms which have been removed in (6-50) from the definition of $\tilde{P}_r$, $2 \leq r \leq j^*$.

$$\begin{aligned} &\tilde{P}_1(\rho, \partial_\rho) \\ &\quad = \rho^{\mu-(s_0-1)/q} P_1(\rho, \partial_\rho) + \rho^{\mu-1} \sum_{m=1}^{N} \frac{1}{\rho^{m-1}} Q_m(\partial_\rho) + \rho^{\mu-\delta} \sum_{a\geq 0} \gamma_a \frac{1}{\rho^{1+\tilde{\theta}_a}} + \sum_j f_j \rho^{\mu-\theta_j}, \end{aligned} \tag{6-60}$$

where the next to last sum is a finite sum denoting what in (6-47) is $\mathbb{O}(\rho^{-(1+\delta)})$, γ_a are constants, and $\tilde{\theta}_a$ are nonnegative rational numbers.

The operator P in (6-47) is then written as

$$P_\Phi(\rho, \partial_\rho) \equiv e^{-i\Phi(\rho)} P(\rho, \partial_\rho) e^{i\Phi(\rho)} = \sum_{r=0}^{\infty} \rho^{-\mu r} \tilde{P}_r(\rho, \partial_\rho). \tag{6-61}$$

We explicitly point out that Proposition 6.1.6 holds also for the coefficients of $\tilde{P}_r$. Moreover, the zero-order terms of $\tilde{P}_r$, $2 \leq r \leq j^*$, are zero.

From now on, to keep the notation simple, we forget about the tildes in (6-61).

Finally, we turn to the construction of a formal solution to $P_\Phi u = 0$. Let us look for u in the form

$$u(\rho) = \sum_{p=0}^{\infty} u_p(\rho), \tag{6-62}$$

where the u_p's are the solutions of the differential equations

$$P_0(\partial_\rho)u_0(\rho) = 0, \tag{6-63}$$

$$P_0(\partial_\rho)u_h(\rho) = -\sum_{r=1}^{h} \rho^{-\mu r} P_r(\rho, \partial_\rho)u_{h-r}(\rho), \tag{6-64}$$

for $t \in \mathbb{N}$.

Equation (6-63) is immediately solved by $u_0(\rho) \equiv 1$, because $P_0(0) = 0$.

Lemma 6.1.7. *Let $Q(\partial_\rho)$ be an ordinary differential operator with constant coefficients such that*

$$Q(\partial_\rho) = \prod_{j=1}^{m} (\partial_\rho - \lambda_j)^{m_j}, \tag{6-65}$$

where m_j denotes the multiplicity of the complex characteristic root λ_j and $\operatorname{Re}\lambda_j \geq 0$. *Then the ordinary differential equation $Q(\partial_\rho)u = f$ has a solution of the form*

$$u(\rho) = (E * f)(\rho) = \sum_{j=1}^{m}\sum_{t=1}^{m_j} d_{j,t} \int_\rho^{+\infty} e^{\lambda_j(\rho - w)}(\rho - w)^{t-1} f(w)\, dw, \tag{6-66}$$

where the $d_{j,t}$ are suitable complex constants. In particular, $\partial_\rho^t u = E * \partial_\rho^t f$.

The proof is essentially the classical construction of the fundamental solution E for Q; we omit the details.

Corollary 6.1.8. *In the situation of Lemma 6.1.7, define*

$$\nu = \max\{m_j \mid \operatorname{Re}\lambda_j = 0\},$$

with the understanding that if no characteristic root has zero real part, then $\nu = 0$. Assume further that $f = \mathbb{O}(\rho^{-k})$ for $\rho \to +\infty$, $k - \nu > 1$. Then

$$u(\rho) = \mathbb{O}(\rho^{-(k-\nu)}).$$

Proof. Denote by j_ν one of the indices j where the maximum in the definition of ν is attained. All we have to do is to estimate the integral with $j = j_\nu$ in (6-66):

$$\sum_{t=1}^{m_{j_\nu}} |d_{j_\nu,t}| \left| \int_\rho^{+\infty} e^{i \operatorname{Im}\lambda_{j_\nu}(\rho - w)}(\rho - w)^{t-1} f(w)\, dw \right|.$$

Each summand above gives a contribution of the form

$$|d_{j_\nu,t}| \sum_{\alpha=0}^{t-1} C_f \binom{t-1}{\alpha} \rho^\alpha \int_\rho^{+\infty} w^{t-1-\alpha-k}\, dw,$$

for a suitable positive constant C_f. Note that by assumption, the integral is convergent and can be explicitly evaluated, yielding

$$\sum_{t=1}^{m_{j_\nu}} |d_{j_\nu,t}| \sum_{\alpha=0}^{t-1} C'_f \binom{t-1}{\alpha} \rho^{t-k},$$

for a larger constant C'_f. This concludes the proof of Corollary 6.1.8. □

Lemma 6.1.7 provides a solution of (6-64) iteratively; that is, once we have suitable estimates for u_{h-r}, $r = 0, \dots, h-1$, we can get estimates for u_h.

Proposition 6.1.9. *There exists a sequence of functions u_h, $h \geq 0$, solving (6-64), and positive constants γ, C_u such that if $\rho \geq \gamma h$, then*

$$\left|\partial^t_\rho u_h(\rho)\right| \leq C_u^{h+t+1} \frac{t!}{\rho^{t+\mu h}}. \tag{6-67}$$

Proof. We are going to prove a slightly better estimate of the form

$$\left|\partial^t_\rho u_j(\rho)\right| \leq \tilde{C}_u^{j+t+1} \binom{\sigma j + t - 1}{t} \frac{t!}{\rho^{t+\mu j}}, \qquad \rho \geq \gamma h, \tag{6-68}$$

where $\tilde{C}_u > 0$ is a constant and σ denotes a suitable integer independent of j, t. The important quantity μ was defined in (6-56).

We argue by induction, remarking that there is nothing to prove when $h = 0$. Assume that $h \geq 1$ and that (6-68) holds for $j < h$. Since, by Lemma 6.1.7, $\partial^t_\rho u = E * \partial^t_\rho f$, we have to estimate the t-th derivative of the right-hand side of (6-64). To this end, it is enough to consider just a summand in the right-hand side of (6-64) in the region $\rho \geq \gamma h$:

$$\begin{aligned}\partial^t_\rho\big(\rho^{-\mu r} P_r(\rho, \partial_\rho) u_{h-r}(\rho)\big) &= \sum_{p=0}^{2l+\lfloor r/q \rfloor} \partial^t_\rho\big(\rho^{-\mu r} \alpha_{r,p}(\rho) \partial^p_\rho u_{h-r}(\rho)\big) \\ &= \sum_{p=0}^{2l+\lfloor r/q \rfloor} \sum_{\beta=0}^{t} \binom{t}{\beta} \partial^\beta_\rho\big(\rho^{-\mu r} \alpha_{r,p}(\rho)\big) \partial^{p+t-\beta}_\rho u_{h-r}(\rho).\end{aligned}$$

Before proceeding further, we must distinguish the contributions from terms where $p = 0$ from the other terms.

Let us first consider the terms with $p = 0$. To deal with these, we make a further distinction when $r \geq j^* + 1$ or $r \leq j^*$. We start with $r \geq j^* + 1$. Because of formula (6-58), we have to estimate

$$\begin{aligned}&\sum_{\beta=0}^{t} \binom{t}{\beta} \partial^\beta_\rho\big(\rho^{-(s_0-1)r/q} \alpha_{r,0}(\rho)\big) \partial^{t-\beta}_\rho u_{h-r}(\rho) \\ &\qquad = \sum_{\beta=0}^{t} \sum_{\nu=0}^{\beta} \binom{t}{\beta} \binom{\beta}{\nu} \partial^\nu_\rho\big(\rho^{-(s_0-1)r/q}\big) \partial^{\beta-\nu}_\rho \alpha_{r,0}(\rho) \partial^{t-\beta}_\rho u_{h-r}(\rho).\end{aligned} \tag{6-69}$$

By (6-21), Proposition 6.1.6, and the inductive hypothesis, this quantity is estimated as follows (see (6-20) for the notation):

$$\sum_{\beta=0}^{t}\sum_{\nu=0}^{\beta}\binom{t}{\beta}\binom{\beta}{\nu}\left|\left(-\frac{s_0-1}{q}r\right)_\nu\right|\rho^{-(s_0-1)r/q-\nu}C_\alpha^{r+\beta-\nu+1}\frac{(\beta-\nu)!}{\rho^{\beta-\nu}}$$
$$\cdot\,\tilde{C}_u^{h-r+t-\beta+1}\binom{\sigma(h-r)+t-\beta-1}{t-\beta}\frac{(t-\beta)!}{\rho^{t-\beta+\mu(h-r)}}.$$

The latter quantity can be estimated as

$$\tilde{C}_u^{h-r+t+1}C_\alpha^{r+1}\frac{t!}{\rho^{t+\mu(h-r)+(s_0-1)r/q}}\sum_{\beta=0}^{t}\tilde{C}_u^{-\beta}C_\alpha^{\beta}\binom{\sigma(h-r)+t-\beta-1}{t-\beta}\sum_{\nu=0}^{\beta}\binom{(s_0-1)r/q+\nu-1}{\nu},$$

since without loss of generality we may always choose $C_\alpha>1$. The inner sum is computed exactly:

$$\sum_{\nu=0}^{\beta}\binom{(s_0-1)r/q+\nu-1}{\nu}=\binom{(s_0-1)r/q+\beta}{\beta}.$$

Let us examine the exponent of ρ; it is equal to $t+\mu h+\left(\frac{s_0-1}{q}-\mu\right)r$. On the other hand, if $r\geq j^*+1$, we have

$$\left(\frac{s_0-1}{q}-\mu\right)r=\left(\frac{s_0-1}{q}-\frac{1}{j^*+1}-\mu\right)r+\frac{r}{j^*+1}>1,$$

by the definition of μ. The whole argument here is performed in the case where $(s_0-1)/q$ is not a positive integer. If it is an integer, the argument is analogous, but much simpler and more direct. The above quantity is estimated by

$$\frac{\tilde{C}_u^{h+t+1}\dfrac{C_\alpha^{r+1}}{\tilde{C}_u^{r}}t!}{\rho^{t+\mu h+((s_0-1)/q-\mu)r}}\sum_{\beta=0}^{t}\binom{\sigma(h-r)+t-\beta-1}{t-\beta}\binom{((s_0-1)/q)r+\beta}{\beta}$$
$$\leq\tilde{C}_u^{h+t+1}\frac{C_\alpha^{r+1}}{\tilde{C}_u^{r}}\frac{t!}{\rho^{t+\mu h+1}}\binom{\sigma h-r(\sigma-(s_0-1)/q)+t}{t}$$
$$\leq\tilde{C}_u^{h+t+1}\frac{C_\alpha^{r+1}}{\tilde{C}_u^{r}}\frac{t!}{\rho^{t+\mu h+1}}\binom{\sigma h-1+t}{t}.$$

For the first inequality we chose $\tilde{C}_u>C_\alpha$ and used the identity

$$\sum_{k=0}^{n}\binom{x+k}{k}\binom{y+n-k}{n-k}=\binom{x+y+n+1}{n}, \tag{6-70}$$

for $x,\ y\in\mathbb{R}$. In the second inequality, we chose $\sigma\geq\dfrac{s_0-1}{q}+1$.

As for the terms with $p = 0$ and $1 \le r \le j^*$, there is only the zero-order term of P_1 (see formulas (6-59), (6-60)), for which we have the estimate

$$\left|\partial_\rho^\nu \alpha_{1,0}(\rho)\right| \le C_\alpha^{\nu+2} \frac{\nu!}{\rho^{1+\nu}}.$$

We conclude that the following inequality holds:

$$d \sum_{r=1}^{h} \sum_{p=1}^{2l+\lfloor r/q \rfloor} \sum_{\beta=0}^{t} \binom{t}{\beta} \int_\rho^{+\infty} \partial_w^\beta \big(w^{-\mu r} \alpha_{r,p}(w)\big) \partial_w^{p+t-\beta} u_{h-r}(w)\, dw. \tag{6-71}$$

We use Corollary 6.1.8. Noting that $p + t - \beta \ge 1$, we may integrate by parts, decreasing by one the number of derivatives landing on u_{h-r} and increasing by one the number of derivatives landing on the coefficients. The above quantity then becomes

$$-d \sum_{r=1}^{h} \sum_{p=1}^{2l+\lfloor r/q \rfloor} \sum_{\beta=0}^{t} \binom{t}{\beta} \partial_\rho^\beta \big(\rho^{-\mu r} \alpha_{r,p}(\rho)\big) \partial_\rho^{p+t-\beta-1} u_{h-r}(\rho)$$
$$- d \sum_{r=1}^{h} \sum_{p=1}^{2l+\lfloor r/q \rfloor} \sum_{\beta=0}^{t} \binom{t}{\beta} \cdot \int_\rho^{+\infty} \partial_w^{\beta+1} \big(w^{-\mu r} \alpha_{r,p}(w)\big) \partial_w^{p+t-\beta-1} u_{h-r}(w)\, dw.$$

The above quantities sport the same behavior with respect to the variable ρ, since even though the order of the derivative on the coefficients of the second term is larger by one, the integration, as we shall see, levels that difference. On the other hand, estimating the coefficients is quite analogous, so that we consider only the second term and leave the necessary simple adjustments for the first to the reader.

Now using (6-21) and (6-49) with $\theta = 1$, we get, if $\rho \ge \gamma h$, $\gamma \ge c_1$,

$$\begin{aligned}
\left|\partial_\rho^{\beta+1} (\rho^{-\mu r} \alpha_{r,p}(\rho))\right| &\le \sum_{i=0}^{\beta+1} \binom{\beta+1}{i} (-1)^i (-\mu r)_i \rho^{-\mu r - i} \cdot C_\alpha^{1+r+\beta+1-i} \frac{(\beta+1-i)!}{\rho^{\beta+1-i}} \\
&\le C_\alpha^{1+r+\beta+1} \frac{(\beta+1)!}{\rho^{\mu r+\beta+1}} \sum_{i=0}^{\beta+1} \binom{\mu r + i - 1}{i} \\
&= C_\alpha^{1+r+\beta+1} \binom{\mu r + \beta + 1}{\beta+1} \frac{(\beta+1)!}{\rho^{\mu r+\beta+1}}.
\end{aligned}$$

Hence, by the inductive hypothesis, the second term above is estimated by

$$|d| \sum_{r=1}^{h} \sum_{p=1}^{2l+\lfloor r/q \rfloor} \sum_{\beta=0}^{t} C_\alpha^{1+r+\beta+1} \tilde{C}_u^{h-r+p+t-\beta} \binom{t}{\beta} \binom{\mu r + \beta + 1}{\beta + 1}$$
$$\cdot \binom{\sigma(h-r) + p + t - \beta - 2}{p + t - \beta - 1} \int_\rho^{+\infty} \frac{(\beta+1)!}{w^{\mu r + \beta + 1}} \frac{(p+t-\beta-1)!}{w^{\mu(h-r)+p+t-\beta-1}}\, dw.$$

Now the integral is easily computed, yielding $(\mu h + t + p - 1)^{-1}/\rho^{\mu h+t+p-1}$. Note that since $p \geq 1$ and $h \geq 1$, there is no problem about its convergence. We thus obtain the bound

$$\tilde{C}_u^{h+t+1} \frac{t!}{\rho^{\mu h+t}} |d| \sum_{r=1}^{h} \sum_{p=1}^{2l+\lfloor r/q \rfloor} \sum_{\beta=0}^{t} \frac{C_\alpha^{2+r+\beta} \tilde{C}_u^{p-1}}{\tilde{C}_u^{r+\beta}} \big(\mu h + t + p - 1\big)^{-1} \cdot \frac{(\mu r + \beta + 1) \dots (\mu r + 1)}{\beta!(t-\beta)!} \frac{\big(\sigma(h-r) + p + t - \beta - 2\big)!}{(\sigma(h-r) - 1)!} \frac{1}{\rho^{p-1}}.$$

Since $\rho \geq \gamma h$ and $1 \leq p \leq 2l + \lfloor h/q \rfloor$, we have $\sigma(h-r) + p - 2 \leq \sigma h + 2l + (1/q)h \leq \gamma_1 h$, where γ_1 is a positive constant, $\gamma_1 \geq \sigma + (1/q) + 2l$. We obtain that $\rho^{-1} \leq \gamma^{-1} h^{-1} \leq \gamma^{-1}\gamma_1(\sigma(h-r) + p - 2)^{-1}$. We point out explicitly that $\gamma^{-1}\gamma_1$ can be chosen very small if γ is chosen large enough. Let us denote this constant by δ, where it is understood that δ is small provided the constant γ is chosen large enough. The above expression is then bounded by

$$\tilde{C}_u^{h+t+1} \frac{t!}{\rho^{\mu h+t}} |d| \sum_{r=1}^{h} \sum_{p=1}^{2l+\lfloor r/q \rfloor} \sum_{\beta=0}^{t} \frac{C_\alpha^{2+r+\beta} \tilde{C}_u^{p-1}}{\tilde{C}_u^{r+\beta}} \frac{\mu r + \beta + 1}{\mu h + t + p - 1} \cdot \delta^{p-1} \binom{\mu r + \beta}{\beta} \binom{\sigma(h-r) + t - \beta + p - 2}{t - \beta}.$$

Now $\beta \leq t$ and $p - 1 \geq 0$ imply that the fraction at the end of the top line above is bounded by 2, so that the whole quantity is estimated by

$$\tilde{C}_u^{h+t+1} \frac{t!}{\rho^{\mu h+t}} 2\,|d| \sum_{r=1}^{h} \sum_{p=1}^{2l+\lfloor r/q \rfloor} \frac{C_\alpha^{2+r} \tilde{C}_u^{p-1}}{\tilde{C}_u^{r}} \delta^{p-1} \cdot \sum_{\beta=0}^{t} \frac{C_\alpha^{\beta}}{\tilde{C}_u^{\beta}} \binom{\mu r + \beta}{\beta} \binom{\sigma(h-r) + t - \beta + p - 2}{t - \beta}.$$

Since we already chose $\tilde{C}_u \geq 4C_\alpha^2$, the ratio in the third sum above is less than $\frac{1}{4}$, and the sum over β involving only binomial coefficients is computed by (6-70), yielding

$$\tilde{C}_u^{h+t+1} \frac{t!}{\rho^{\mu h+t}} \frac{|d|}{2} \sum_{r=1}^{h} \sum_{p=1}^{2l+\lfloor r/q \rfloor} \frac{C_\alpha^{2+r} \tilde{C}_u^{p-1}}{\tilde{C}_u^{r}} \delta^{p-1} \cdot \binom{\mu r + \sigma(h-r) + t + p - 1}{t}.$$

Observe now that there is a positive constant $\tilde{c}$ such that $p \leq \tilde{c}r$. Therefore $\mu r + \sigma(h-r) + t + p - 1 \leq \sigma h + t + r(\mu - \sigma + \tilde{c}) - 1 \leq \sigma h + t - 1$, provided σ is chosen large in such a way that $\sigma > \mu + \tilde{c}$. This is always possible and is actually the only constraint on σ. By a well known property of binomial coefficients (with positive real numerators), we then obtain the bound

$$\tilde{C}_u^{h+t+1} \binom{\sigma h + t - 1}{t} \frac{t!}{\rho^{\mu h+t}} \frac{|d|}{2} \sum_{r=1}^{h} \sum_{p=1}^{2l+\lfloor r/q \rfloor} \frac{C_\alpha^{2+r} \tilde{C}_u^{p-1}}{\tilde{C}_u^{r}} \delta^{p-1} = \tilde{C}_u^{h+t+1} \binom{\sigma h + t - 1}{t} \frac{t!}{\rho^{\mu h+t}} \frac{|d|}{2} \sum_{r=1}^{h} \frac{C_\alpha^{2+r}}{\tilde{C}_u^{r}} \sum_{p=1}^{2l+\lfloor r/q \rfloor} \delta^{p-1} \tilde{C}_u^{p-1}.$$

The inner sum is easily evaluated provided, for example, $\delta \le \tilde{C}_u^{-1}/2$. This is always possible and amounts to choosing γ large. The contribution from that sum is thus ≤ 2. As for the outer sum, if we choose $\tilde{C}_u$ in such a way that

$$\tilde{C}_u \ge C_\alpha^3(1+3|d|),$$

which depends only on the problem data, we obtain the final bound

$$\tfrac{1}{3}\tilde{C}_u^{h+t+1}\binom{\sigma h+t-1}{t}\frac{t!}{\rho^{\mu h+t}}.$$

The same bound is obtained for the term without the integral.

This finishes the proof of inequality (6-68). Inequality (6-67) is an easy consequence. □

Proposition 6.1.9 guarantees that we can construct a formal solution to the equation $P(\rho,\partial_\rho)u(\rho)=0$ in (6-18) and thus a formal solution $A(u)$ for

$$\Lambda(t,D_t)A(u)(t)=0. \tag{6-72}$$

In the next subsection we plan to construct from $A(u)$ a true solution; this will only yield a solution of (6-72) with a nonzero right-hand side which will be negligible in an important sense.

6.2. *True solution and the end of the proof.* To establish the notation, we state the result of the previous subsection:

Theorem 6.2.1. *There is a formal solution $A(u)(t)$ of (6-72) of the form*

$$A(u)(t)=\int_0^{+\infty} e^{it\rho^{s_0}}e^{i\Phi(\rho)}u(\rho)\,d\rho, \tag{6-73}$$

satisfying the following conditions:

(1) *The phase function Φ is of the form*

$$\Phi(\rho)=\sum_{j=0}^{M_0}\varphi_j\rho^{1-((s_0-1)/q)j}+\varphi_{-1}\log\rho,\quad M_0=\left\lfloor\frac{1}{s_0-1}\right\rfloor, \tag{6-74}$$

with $\varphi_j\in\mathbb{C}$, $j=-1,0,\dots,M_0$, $\operatorname{Im}\varphi_0>0$.

(2) *The function u has the form $u(\rho)=\sum_{h=0}^{\infty}u_h(\rho)$, where $u_0(\rho)\equiv 1$ and (compare (6-61))*

$$P_0(\partial_\rho)u_h(\rho)+\sum_{r=1}^{h}\rho^{-\mu r}P_r(\rho,\partial_\rho)u_{h-r}(\rho)=0,\quad h=1,2,\dots. \tag{6-75}$$

Moreover, u_h satisfies the estimate (6-67); that is, if $\rho\ge\gamma h$, for γ large enough,

$$\left|\partial_\rho^t u_h(\rho)\right|\le C_u^{h+t+1}\frac{t!}{\rho^{t+\mu h}}. \tag{6-76}$$

As a consequence of the construction, $A(u)$ formally satisfies

$$\begin{aligned}\Lambda(t, D_t)A(u)(t) &= \int_0^{+\infty} e^{it\rho^{s_0}} P(\rho, \partial_\rho)\big(e^{i\Phi(\rho)}u(\rho)\big)\, d\rho \\ &= \int_0^{+\infty} e^{it\rho^{s_0}} e^{i\Phi(\rho)}\big(e^{-i\Phi(\rho)} P(\rho, \partial_\rho) e^{i\Phi(\rho)}\big)u(\rho)\, d\rho \\ &= \int_0^{+\infty} e^{it\rho^{s_0}} e^{i\Phi(\rho)} \sum_{r=0}^{\infty} \rho^{-\mu r} P_r(\rho, \partial_\rho)u(\rho)\, d\rho \\ &= \int_0^{+\infty} e^{it\rho^{s_0}} e^{i\Phi(\rho)} \sum_{h=0}^{\infty}\sum_{r=0}^{h} \rho^{-\mu r} P_r(\rho, \partial_\rho)u_{h-r}(\rho)\, d\rho = 0. \end{aligned} \tag{6-77}$$

Let $\omega_j \in G^s(\mathbb{R})$, $j = 0, 1, 2, \dots$, with $1 < s < s_0$ to be specified later, be the cutoffs introduced in Lemma 5.4, defined in $\mathbb{R}$. We assume from the beginning that the constant $2R$ in Lemma 5.4 is larger than γ, the latter being the constant in the second item of the theorem above. Define

$$v(\rho) = \sum_{h=0}^{\infty} \omega_h(\rho)u_h(\rho). \tag{6-78}$$

Trivially, $v \in G^s(\mathbb{R})$. Moreover:

Lemma 6.2.2. *The function v in (6-78) satisfies the estimate*

$$\begin{cases} \big|\partial_\rho^\alpha v(\rho)\big| \le C_v^{\alpha+1}\dfrac{\alpha!^s}{\rho^\alpha} & \text{for every } \rho \ge 2R, \\ v \equiv 0 & \text{if } \rho \le 2R. \end{cases} \tag{6-79}$$

Proof. Let us start by estimating $\partial_\rho^\beta\omega_h\partial_\rho^{\alpha-\beta}u_h$. For the first factor we have

$$\big|\partial_\rho^\beta\omega_h(\rho)\big| \le (RC_\omega)^{\beta+1}\frac{\beta!^s}{\rho^\beta} \quad \text{for every } \beta.$$

For the second factor, by (6-76) we have, provided $\rho \ge \gamma h$, which is implied by $\rho \in \operatorname{supp}\omega_h$,

$$\big|\partial_\rho^{\alpha-\beta}u_h(\rho)\big| \le C_u^{h+\alpha-\beta+1}\frac{(\alpha-\beta)!}{\rho^{\alpha-\beta+\mu h}} \le C_u^{h+\alpha-\beta+1}\frac{(\alpha-\beta)!}{\rho^{\alpha-\beta}}\big(\gamma(h+1)\big)^{-\mu h}.$$

Putting together the estimates, we obtain

$$\big|\partial_\rho^\alpha v(\rho)\big| \le \tilde{C}_v^{\alpha+1}\frac{\alpha!^s}{\rho^\alpha}\sum_{h=0}^{\infty}\big(\gamma(h+1)\big)^{-\mu h}.$$

This implies the assertion. □

Definition 6.2.3. Let Ω be an open subset of $\mathbb{R}$. We define the class $\mathcal{B}^s(\Omega)$ (of Beurling type functions on Ω) as the set of all smooth functions $u(x)$ defined in Ω and such that for every $\varepsilon > 0$ and for every $K \Subset \Omega$ compact, there exists a positive constant $C = C(\varepsilon, K)$ such that

$$|\partial_x^\alpha u(x)| \le C\varepsilon^\alpha\alpha!^s, \tag{6-80}$$

for every $x \in K$ and every α.

We want to show that $\Lambda(t, D_t)A(v) = g$, where $g \neq 0$ and $g \in \mathscr{B}^{s_0}(\mathbb{R})$. First we show that far from the origin, $A(v)$ has a better regularity than $G^{s_0}(\mathbb{R})$. The following lemma is straightforward:

Lemma 6.2.4. *We have $G^\sigma(\Omega) \subset \mathscr{B}^t(\Omega)$ for every $t > \sigma$.*

Lemma 6.2.5. *Let s be the Gevrey regularity of the cutoff functions ω_j in (6-78). Let $\delta > 0$. Then $A(v) \in \mathscr{B}^\sigma(\{x \mid |x| > \delta\})$, with $s \leq \sigma \leq s_0$.*

Proof. We actually prove that $A(v) \in G^s(\{x \mid |x| > \delta\})$. We have

$$D_t^\alpha A(v)(t) = \int_0^{+\infty} e^{it\rho^{s_0}} \rho^{s_0\alpha} e^{i\Phi(\rho)} v(\rho)\, d\rho.$$

We observe that $(s_0 t\rho^{s_0-1})^{-1} D_\rho e^{i\rho^{s_0}t} = e^{i\rho^{s_0}t}$. Therefore,

$$\begin{aligned} D_t^\alpha A(v)(t) &= \Big(\frac{1}{t}\Big)^j \int_0^{+\infty} e^{it\rho^{s_0}} \Big(-D_\rho \frac{1}{s_0\rho^{s_0-1}}\Big)^j \big(\rho^{s_0\alpha} e^{i\Phi(\rho)} v(\rho)\big)\, d\rho \\ &= \Big(\frac{1}{t}\Big)^j \int_0^{+\infty} e^{it\rho^{s_0}} \sum_{h=0}^{j} \gamma_{jh} \frac{1}{\rho^{s_0 j - h}} \partial_\rho^h \big(\rho^{s_0\alpha} e^{i\Phi(\rho)} v(\rho)\big)\, d\rho, \end{aligned}$$

by Lemma 6.1.1. This quantity is rewritten as

$$\Big(\frac{1}{t}\Big)^j \int_0^{+\infty} e^{it\rho^{s_0}} \sum_{h=0}^{j} \sum_{p+q=0}^{h} \frac{h!}{p!\,q!\,(h-p-q)!} \gamma_{jh} \frac{1}{\rho^{s_0 j-h}} \cdot \partial_\rho^p(\rho^{s_0\alpha}) \partial_\rho^q(v(\rho)) \partial_\rho^{h-p-q}(e^{i\Phi(\rho)})\, d\rho.$$

By the Faà di Bruno formula,

$$\partial_\rho^n e^{i\Phi} = e^{i\Phi} \sum_{k=1}^{n} i^k \sum_{\substack{k_1,\dots,k_n \\ \sum_i k_i = k \\ \sum_i i k_i = n}} \frac{n!}{k_1! \cdots k_n!} \prod_{p=1}^{n} \Big(\frac{\Phi_\rho^{(p-1)}}{p!}\Big)^{k_p},$$

using Lemma 6.1.5 and the estimate (6-42) we obtain for $\rho \geq 2R$, with $\lambda > 0$,

$$|\partial_\rho^n e^{i\Phi}| \leq |e^{i\Phi}| \sum_{k=1}^{n} \sum_{\substack{k_1,\dots,k_n \\ \sum_i k_i = k \\ \sum_i i k_i = n}} \frac{n!}{k_1! \cdots k_n!} C_\Phi'^k \rho^{-(n-k)} \leq C^n e^{-\lambda\rho} \sum_{k=1}^{n} (n-k)!\, \rho^{-(n-k)}, \tag{6-81}$$

where we argued as in (6-53), (6-34) and (6-39). Thus we have if $|t| \geq \delta$,

$$\begin{aligned} \big|D_t^\alpha A(v)(t)\big| \leq \delta^{-j} \int_0^{+\infty} e^{-\lambda\rho} \sum_{h=0}^{j} \sum_{p+q=0}^{h} \sum_{k=1}^{h-p-q} \frac{h!}{p!\,q!\,(h-p-q)!} \\ \cdot C_\gamma^{j+h} (j-h)!\, \frac{1}{\rho^{s_0 j-h}} (s_0\alpha)_p \rho^{s_0\alpha - p} C_v^{1+q} \frac{q!^s}{\rho^q} \cdot C^{h-p-q} (h-p-q-k)!\, \rho^{-(h-p-q-k)}\, d\rho, \end{aligned}$$

by (6-17), (6-21). Choosing $j=\alpha$, we then obtain

$$\begin{aligned}&\bigl|D_t^\alpha A(v)(t)\bigr|\\&\le C_v\Bigl(\frac{CC_\gamma^2C_v}{\delta}\Bigr)^\alpha\sum_{h=0}^{\alpha}\sum_{p+q=0}^{h}\sum_{k=1}^{h-p-q}\frac{h!}{(h-p-q)!}\cdot(\alpha-h)!\binom{s_0\alpha}{p}q!^{s-1}(h-p-q-k)!\int_0^{+\infty}e^{-\lambda\rho}\rho^k\,d\rho.\end{aligned}$$

The integral above is equal to $\lambda^{-(k+1)}k!$, and there is a positive constant C_1 such that $\binom{s_0\alpha}{p}\le C_1^\alpha$. Eventually we get

$$\bigl|D_t^\alpha A(v)(t)\bigr|\le\frac{C_v}{\lambda}\Bigl(\frac{\max\{1,\lambda^{-1}\}CC_1C_\gamma^2C_v}{\delta}\Bigr)^\alpha\alpha!\sum_{h=0}^{\alpha}\sum_{p+q=0}^{h}\binom{\alpha}{h}^{-1}q!^{s-1}\sum_{k=1}^{h-p-q}\binom{h-p-q}{k}^{-1}.$$

We may therefore find a positive constant $\tilde C$ such that $\tilde C^\alpha\ge\alpha^4$ and deduce that

$$\bigl|D_t^\alpha A(v)(t)\bigr|\le\frac{C_v}{\lambda}\Bigl(\frac{\max\{1,\lambda^{-1}\}\tilde CCC_1C_\gamma^2C_v}{\delta}\Bigr)^\alpha\alpha!^s.$$

This proves the statement. □

Next we prove a key result of this section: the regularity of $\Lambda(t,D_t)A(v)$. First of all, we remark that we need to sum the asymptotic expansion (3-33) modulo some reasonably regularizing term. Note also that the symbols in the asymptotic expansion of Λ are real analytic symbols:

$$\Lambda(t,\tau)\sim\sum_{j=0}^{j_0}\Lambda_{-jq}(t,\tau)+\sum_{s\ge0}\bigl(\Lambda_{-2k-sq}(t,\tau)+\Lambda_{-(j_0+1)q-sq}(t,\tau)\bigr).\tag{6-82}$$

We recall that Λ_m in the above expression is (positively) homogeneous with respect to τ of degree $2/q+m/q$. To sum (6-82), we use the cutoff functions constructed in Lemma 5.3; we agree that they are in $G^t(\mathbb{R})$ with $t<s_0$ to be specified later. It is then evident that the error appearing when summing "à la Borel" the asymptotic expansion of Λ will be G^t-regularizing and hence in $\mathscr{B}^{s_0}(\mathbb{R})$.

By (6-2), we may ignore an elliptic factor and rewrite Λ, with a slight difference in the meaning of the coefficients, as

$$\Lambda(t,\tau)\tau^{-2/q+2k/q}\sim\sum_{h=0}^{\infty}a_h(t)\tau^{2k/q-h}+\sum_{h=0}^{\infty}b_h(t)\tau^{-h},\tag{6-83}$$

where without loss of generality $\tau>0$.

We also recall at this time that the first sum above gives rise to the $Q_{\mathscr{A},r}$ in (6-45), while the second contributes to the $Q_{\mathscr{B},r}$ in (6-46). At this point we are not interested in the particular properties of the coefficients, such as, for example, the vanishing of order $2l$ of a_0 and the nonvanishing of b_0 at the origin. These properties have already played their role in the constructions above.

Abusing our notation a bit, we call the operator in (6-83) again Λ.

Proposition 6.2.6. *Let v be the function defined in (6-78) using cutoff functions in $G^{t'}$ and let Λ be the operator defined by the asymptotic expansion in (6-83) using cutoff functions in $G^{t''}$ (see Lemmas 5.3 and*

5.4). *Then, for a suitable choice of t' and t'', we have*

$$\Lambda(t, D_t)A(v)(t) \in \mathcal{B}^{s_0}(\mathbb{R}). \tag{6-84}$$

Proof. It is evident that it will be enough if we argue on just one of the asymptotic expansions in (6-83). At a certain point of the proof though, we have to partially reassemble the operator P_Φ in (6-61), and there we use the argument also for the other expansion. For the sake of simplicity, we argue on the second sum in (6-83).

Due to Lemma 6.2.5, it suffices to show that for every $\varepsilon > 0$, there is a neighborhood of the origin, U_ε, such that $|\partial_t^\alpha(\Lambda A(v))(t)| \le C_\varepsilon \varepsilon^\alpha \alpha!^{s_0}$ for $t \in U_\varepsilon$.

Actually we need to estimate a derivative of $\Lambda(t, D_t)A(v)(t)$, say

$$D_t^\alpha \Lambda(t, D_t)A(v)(t).$$

The latter can be written as

$$D_t^\alpha \sum_{j=0}^{\infty} b_j(t) A\big(\omega_j(\rho^{s_0})\rho^{-js_0}v(\rho)\big),$$

keeping in mind the form of $A(v)$, with v given by (6-78),

$$A(v)(t) = \int_0^{+\infty} e^{i\rho^{s_0}t} e^{i\Phi(\rho)} v(\rho)\, d\rho. \tag{6-85}$$

Let now N be a natural number and consider

$$D_t^\alpha \sum_{j=N}^{\infty} b_j(t) A\big(\omega_j(\rho^{s_0})\rho^{-js_0}v(\rho)\big) = \sum_{j=N}^{\infty}\sum_{p=0}^{\alpha}\binom{\alpha}{p} D_t^p b_j(t) A\big(\omega_j(\rho^{s_0})\rho^{-js_0+s_0(\alpha-p)}v(\rho)\big). \tag{6-86}$$

Applying the definition (6-4) of an analytic symbol as well as the estimates (5-6) for the cutoff functions defining Λ, we have that the latter quantity is estimated by

$$\begin{aligned}\sum_{j=N}^{\infty}\sum_{p=0}^{\alpha}\binom{\alpha}{p} C^{p+j+1} p!\, j!\,(2R)^{-j+\alpha-p}\cdot(j+1)^{t''(-j+\alpha-p)}\big|A(\omega_j(\rho^{s_0})v(\rho))\big| \\ \le C_1 \sum_{j=N}^{\infty}\sum_{p=0}^{\alpha}\binom{\alpha}{p} C^{p+j+1} p!\,(2R)^{-j+\alpha-p}(j+1)^{j+t''(-j+\alpha-p)},\end{aligned}$$

provided $-j+\alpha \le 0$, which is obviously implied by choosing $N \ge \alpha$. In order to handle the power of j above, we make a stronger demand on N, namely,

$$N = \theta_N \alpha \ge \left\lfloor \alpha \frac{t''}{t''-1} \right\rfloor + 1, \tag{6-87}$$

where θ_N is a suitable constant on which we may impose further constraints in the following, independent of α.

Then $j + t''(\alpha - j) \le 0$, and the above sum can be bounded by

$$C_1(2R)^\alpha \alpha! \sum_{j=N}^{\infty} C^{j+1}(2R)^{-j} \sum_{p=0}^{\alpha} \Big(\frac{C}{2R}\Big)^p \le \tilde{C}^{\alpha+1}\alpha!, \tag{6-88}$$

for a suitable positive constant $\tilde{C}$, provided $2R > C$. Thus this part of $\Lambda(t, D_t)A(v)(t)$ exhibits an analytic behavior and therefore belongs to any $\mathscr{B}^s$, with $s > 1$.

Next we must estimate the finite sum

$$D_t^\alpha \sum_{j=0}^{N-1} b_j(t) A\big(\omega_j(\rho^{s_0})\rho^{-js_0} v(\rho)\big)(t),$$

with N defined by (6-87). To do this we write the coefficients b_j as a sum of a polynomial in the variable t and a real analytic function vanishing of high order at $t = 0$ and estimate both contributions. Let us start with the remainder terms in the expansion of b_j.

Thus we have to estimate the sum

$$D_t^\alpha \sum_{j=0}^{N-1} t^M \sum_{i=0}^{\infty} b_{j,i+M} t^i A\big(\omega_j(\rho^{s_0})\rho^{-js_0} v(\rho)\big)(t), \tag{6-89}$$

where M is a large integer to be fixed later. The significant part of the estimate is that where the t-derivatives land on A, since otherwise the derivatives landing on the powers of t give analytic type estimates and hence better estimates:

$$\sum_{j=0}^{N-1} t^M \sum_{i=0}^{\infty} b_{j,i+M} t^i D_t^\alpha A\big(\omega_j(\rho^{s_0})\rho^{-js_0} v(\rho)\big)(t).$$

By (6-4), we have $|b_{j,i+M}| \le C^{i+M+j+1} j!$, so that if $|t| \le \delta$, the absolute value of the above quantity is bounded by

$$\sum_{j=0}^{N-1} \delta^M C^{M+j+1} j! \sum_{i=0}^{\infty} (C\delta)^i \big|A(\omega_j(\rho^{s_0})\rho^{-js_0+s_0\alpha} v(\rho))(t)\big|,$$

since on the support of ω_j, by Lemma 5.4, $\rho^{s_0} \ge R(j+1)^{t'}$, we obtain that $\rho^{-s_0 j} \le R^{-j}(j+1)^{-t'j}$. Furthermore,

$$\begin{aligned}\big|A\big(\omega_j(\rho^{s_0})\rho^{s_0\alpha} v(\rho)\big)(t)\big| &\le \int_0^{+\infty} |e^{i\Phi(\rho)}||v(\rho)|\rho^{s_0\alpha}\, d\rho \\ &\le C_A \int_0^{+\infty} e^{-\lambda\rho}\rho^{s_0\alpha}\, d\rho = C_A \lambda^{-(s_0\alpha+1)}\Gamma(s_0\alpha+1) \le C_A'^{\alpha+1}\alpha!^{s_0}.\end{aligned}$$

Hence (6-89) is bounded by

$$C_A'^{\alpha+1}\alpha!^{s_0} C^{M+1}\delta^M \sum_{j=0}^{N-1} \Big(\frac{C}{R}\Big)^j \sum_{i=0}^{\infty} (C\delta)^i (j+1)^{(1-t')j}.$$

Choose

$$M = \theta_M \alpha, \quad \theta_M \geq 1. \tag{6-90}$$

We may impose further conditions on θ_M provided they depend only on the problem data, that is, θ_M does not depend on α. Moreover, let $R > C$ and $C'_A C^{\theta_M} \delta^{\theta_M} < \varepsilon$, and we have the estimate

$$C'_A C \varepsilon^\alpha \alpha!^{s_0} \sum_{j=0}^{\infty} \Big(\frac{C}{R}\Big)^j \sum_{i=0}^{\infty} (C\delta)^i \leq \tilde{C}_A \varepsilon^\alpha \alpha!^{s_0}. \tag{6-91}$$

This concludes the proof for the term (6-89).

The next step is to estimate the term

$$D_t^\alpha \sum_{j=0}^{N-1} \sum_{i=0}^{M-1} b_{j,i} t^i A\big(\omega_j(\rho^{s_0}) \rho^{-j s_0} v(\rho)\big)(t). \tag{6-92}$$

The latter can be written as

$$D_t^\alpha \sum_{j=0}^{N-1} \sum_{r=0}^{M-1} b_{j,r} \cdot \int_0^{+\infty} e^{i t \rho^{s_0}} \Big(-\partial \frac{1}{i s_0 \rho^{s_0 - 1}}\Big)^r \big(e^{i\Phi(\rho)} \omega_j(\rho^{s_0}) \rho^{-j s_0} v(\rho)\big)\, d\rho.$$

By Lemma 6.1.1, we rewrite the above expression as

$$\sum_{j=0}^{N-1} \sum_{r=0}^{M-1} b_{j,r} \cdot \int_0^{+\infty} e^{i t \rho^{s_0}} \rho^{s_0 \alpha} \sum_{h=0}^{r} \gamma_{rh} \frac{1}{\rho^{s_0 r - h}} \partial_\rho^h \big(e^{i\Phi(\rho)} \omega_j(\rho^{s_0}) \rho^{-j s_0} v(\rho)\big)\, d\rho. \tag{6-93}$$

Let us compute $\partial_\rho^h \big(e^{i\Phi(\rho)} \omega_j(\rho^{s_0}) \rho^{-j s_0} v(\rho)\big)$. This is equal to

$$\sum_{\sum \beta_i = h} \frac{h!}{\beta_1!\, \beta_2!\, \beta_3!\, \beta_4!} \partial_\rho^{\beta_1} e^{i\Phi} \partial_\rho^{\beta_2} \omega_j(\rho^{s_0}) \partial_\rho^{\beta_3} \rho^{-j s_0} \partial_\rho^{\beta_4} v(\rho).$$

By (6-81), (6-79) we have

$$\big|\partial_\rho^{\beta_1} e^{i\Phi(\rho)}\big| \leq C_e^{\beta_1} e^{-\lambda\rho} \sum_{m=1}^{\beta_1} (\beta_1 - m)!\, \rho^{-(\beta_1 - m)}, \qquad \big|\partial_\rho^{\beta_4} v(\rho)\big| \leq C_v^{\beta_4 + 1} \frac{\beta_4!^{t'}}{\rho^{\beta_4}},$$

and finally, using the Faà di Bruno formula,

$$\partial_\rho^{\beta_2} \omega_j(\rho^{s_0}) = \sum_{k=1}^{\beta_2} \omega_j^{(k)}(\rho^{s_0}) \sum_{\substack{\sum k_i = k \\ \sum i k_i = \beta_2}} \frac{\beta_2!}{k_1! \cdots k_{\beta_2}!} \prod_{l=1}^{\beta_2} \bigg(\binom{s_0}{l} \rho^{s_0 - l}\bigg)^{k_l}.$$

By (5-6), arguing as we did to prove (6-53), the absolute value of the above quantity is estimated by

$$\big|\partial_\rho^{\beta_2} \omega_j(\rho^{s_0})\big| \leq C_2^{\beta_2 + 1} \frac{\beta_2!^{t''}}{\rho^{\beta_2}},$$

where C_2 is a suitable positive constant. Let us now consider (6-93). It is natural to consider (6-93) in the two regions $\rho \le 4RN^{t''}$ and $\rho \ge 4RN^{t''}$. We want to estimate (6-93) in the first region. We remark that on the support of ω_j in this region, we have $2R(j+1)^{t''} \le \rho^{s_0} \le 4RN^{t''}$. Thus the absolute value of (6-93) in the latter region is bounded by

$$\sum_{j=0}^{N-1}\sum_{r=0}^{M-1}\sum_{h=0}^{r}\sum_{\sum\beta_i=h} C_e^{\beta_1}\sum_{m=1}^{\beta_1}(\beta_1-m)!\,C_\gamma^{r+h}C_2^{\beta_2+1}C_v^{\beta_4+1}$$
$$\cdot(r-h)!\,\frac{h!}{\beta_1!\,\beta_2!\,\beta_3!\,\beta_4!}\beta_2!^{t''}\binom{s_0j+\beta_3-1}{\beta_3}\beta_3!\,\beta_4!^{t'}|b_{j,r}|$$
$$\cdot\int_{2R(j+1)^{t''}\le\rho^{s_0}\le 4RN^{t''}} e^{-\lambda\rho}\rho^{s_0\alpha}\frac{1}{\rho^{\beta_2}}\frac{1}{\rho^{s_0r-h}}\frac{1}{\rho^{\beta_1-m}}\frac{1}{\rho^{s_0j+\beta_3}}\frac{1}{\rho^{\beta_4}}\,d\rho,$$

which in turn is bounded by

$$\sum_{j=0}^{N-1}\sum_{r=0}^{M-1}\sum_{h=0}^{r}\sum_{\sum\beta_i=h} C_e^{\beta_1}\sum_{m=1}^{\beta_1}(\beta_1-m)!\,C_\gamma^{r+h}C_2^{\beta_2+1}C_v^{\beta_4+1}C_3^{s_0j+\beta_3}4^h$$
$$\cdot(r-h)!\,\beta_2!^{t''}\beta_3!\,\beta_4!^{t'}C_b^{j+r+1}j!\,(4R)^{(\alpha-r)_+}N^{t''(\alpha-r)_+}(2R)^{-j}(j+1)^{-t''j}$$
$$\cdot\int_{2R(j+1)^{t''}\le\rho^{s_0}\le 4RN^{t''}} e^{-\lambda\rho}\rho^m\,d\rho.$$

The integral above is bounded by $\lambda^{-(m+1)}m!$, so that it is clear that there exist positive constants C, C_λ such that the above quantity is bounded by

$$\sum_{j=0}^{N-1}\sum_{r=0}^{M-1}\sum_{h=0}^{r}\sum_{\sum\beta_i=h}(2C_e)^{\beta_1}\beta_1!\,C_\gamma^{r+h}C_2^{\beta_2+1}C_v^{\beta_4+1}C_3^{s_0j+\beta_3}4^hC_\lambda^{\beta_1}C_b^{j+r+1}$$
$$\cdot(4R)^{(\alpha-r)_+}(r-h)!\,\beta_2!^{t''}\beta_3!\,\beta_4!^{t'}N^{t''(\alpha-r)_+}(2R)^{-j}$$
$$\le\sum_{j=0}^{N-1}\Big(\frac{C_3^{s_0}C_b}{2R}\Big)^j\sum_{r=0}^{M-1}C_\gamma^rC_b^{r+1}(4R)^{(\alpha-r)_+}N^{t''(\alpha-r)_+}\sum_{h=0}^{r}L_1^{h+1}h!^{t''}(r-h)!$$
$$\le(4R)^\alpha\sum_{j=0}^{N-1}\Big(\frac{C_3^{s_0}C_b}{2R}\Big)^j\sum_{r=0}^{M-1}C_\gamma^rC_b^{r+1}N^{t''(\alpha-r)_+}L_2^{r+1}r!^{t''}$$
$$\le L_3^{\alpha+1}\alpha!^{t''}.$$

Here we used the fact that $\rho^{s_0}\le 4RN^{t''}$, as well as (6-87). Moreover,

$$N^{t''(\alpha-r)}r!^{t''}\le(\theta_N\alpha)^{t''(\alpha-r)}r^{t''r}\le(\theta_N\alpha)^{t''(\alpha-r)}(\theta_M\alpha)^{t''r}\le\max\{\theta_N,\theta_M\}^{t''\alpha}\alpha^{t''\alpha}.$$

It is also clear in the above deduction that all the constants involved except R depend on the data and are hence fixed; moreover, R can be taken large enough so that $C_3^{s_0}C_b/R<1$. We would like to emphasize at this stage that in performing the above estimate, we assumed that $t'' > t'$. This is no restriction since the only constraint on t' and t'' is that they are positive numbers larger than one.

If

$$1 < t' < t'' < s_0, \tag{6-94}$$

we therefore obtain that the term (6-92) in the region $\rho^{s_0} \le 4RN^{t''}$ gives rise to a function of class $\mathscr{B}^{s_0}$.

We must now discuss the term (6-93) in the complementary region: $\rho^{s_0} \ge 4RN^{t''}$.

$$\sum_{j=0}^{N-1}\sum_{r=0}^{M-1} b_{j,r} \int_{\rho^{s_0}\ge 4RN^{t''}} e^{it\rho^{s_0}} \rho^{s_0\alpha} \cdot \sum_{h=0}^{r} \gamma_{rh} \frac{1}{\rho^{s_0 r-h}} \partial_\rho^h \big(e^{i\Phi(\rho)} \rho^{-js_0} v(\rho)\big)\, d\rho$$
$$= \sum_{j=0}^{N-1}\sum_{r=0}^{M-1} b_{j,r} \int_{\rho^{s_0}\ge 4RN^{t''}} e^{it\rho^{s_0}} \rho^{s_0\alpha} e^{i\Phi(\rho)} \cdot \Big[e^{-i\Phi(\rho)} \sum_{h=0}^{r} \gamma_{rh} \frac{1}{\rho^{s_0 r-h}} \partial_\rho^h \big(e^{i\Phi(\rho)} \rho^{-js_0} v(\rho)\big)\Big] d\rho.$$

The factor in square brackets and its counterpart coming from the first sum in (6-83) yield a differential operator of the form

$$P^{\#}(\rho, \partial_\rho) = \sum_{\nu=0}^{L_\alpha} \rho^{-\mu\nu} P^{\#}_\nu(\rho, \partial_\rho). \tag{6-95}$$

This is obtained by repeating the argument of Section 6.1 that led to (6-61). It is also evident that $L_\alpha = \mathbb{O}(\alpha)$ for α large because of (6-87) and (6-90).

Some of the operators $P^{\#}_\nu$ coincide with the P_ν of (6-61), while the others miss some of the terms due to the fact that we are taking finite sums. Thus we have to estimate

$$\int_{\rho^{s_0}\ge 4RN^{t''}} e^{it\rho^{s_0}} \rho^{s_0\alpha} e^{i\Phi(\rho)} P^{\#}(\rho, \partial_\rho) v(\rho)\, d\rho.$$

Now an inspection of (6-51) and (6-52) immediately suggests that $P^{\#}_\nu = P_\nu$ if $\nu/q \le M-1$ and $\nu \le N-1$.

It is actually useful to have the above relations be satisfied when $\nu \le (s_0/\mu)\alpha$. To do that, it suffices to choose $\theta_N, \theta_M \ge s_0/\mu$ (see (6-87) and (6-90)).

We thus wind up with the following quantity to be estimated:

$$\int_{\rho^{s_0}\ge 4RN^{t''}} e^{it\rho^{s_0}} \rho^{s_0\alpha} e^{i\Phi(\rho)} \sum_{\nu=0}^{(s_0/\mu)\alpha} \rho^{-\mu\nu} P_\nu(\rho, \partial_\rho) v(\rho)\, d\rho$$
$$+ \int_{\rho^{s_0}\ge 4RN^{t''}} e^{it\rho^{s_0}} \rho^{s_0\alpha} e^{i\Phi(\rho)} \sum_{\nu>(s_0/\mu)\alpha}^{L_\alpha} \rho^{-\mu\nu} P^{\#}_\nu(\rho, \partial_\rho) v(\rho)\, d\rho = J_1 + J_2. \tag{6-96}$$

First we want to bound J_2. We have

$$|J_2| \le \int_{\rho^{s_0}\ge 4RN^{t''}} \rho^{s_0\alpha} |e^{i\Phi(\rho)}| \sum_{\nu>(s_0/\mu)\alpha}^{L_\alpha} \rho^{-\mu\nu} \big| P^{\#}_\nu(\rho, \partial_\rho) v(\rho)\big|\, d\rho,$$

where

$$P_\nu^\#(\rho,\partial_\rho)=\sum_{r=0}^{m_\nu}\alpha_{\nu,r}^\#(\rho)\partial_\rho^r,$$

and we explicitly point out that its coefficients satisfy an estimate of the form (6-49)

$$\left|\partial_\rho^t\alpha_{\nu,r}^\#(\rho)\right|\le C_\#^{\nu+t+1}\nu!^{1-\theta}\frac{t!}{\rho^t},$$

where $0\le\theta<1$ and $\rho\ge c_1\nu^\theta$. Consequently, since $\nu\le L_\alpha\le c\alpha\le c/\theta_N N$, we obtain that $\rho\ge 4RN^{t''/s_0}\ge c'\nu^{t''/s_0}$, and hence

$$\left|\alpha_{\nu,r}^\#(\rho)\right|\le C_\#^{\nu+1}\nu!^{1-t''/s_0}.$$

Thus, by (6-79),

$$\rho^{s_0\alpha-\mu\nu}\left|P_\nu^\#(\rho,\partial_\rho)v(\rho)\right|\le\sum_{r=0}^{m_\nu}C_\#^{\nu+1}\nu!^{1-t''/s_0}C_v^{r+1}\frac{r!^{t'}}{\rho^r},$$

since $\rho^{s_0\alpha-\mu\nu}\le 1$. As before, we obtain that $\rho\ge c''r^{t''/s_0}$, $c''>0$ and suitable, because $m_\nu=\mathcal{O}(\nu)$, so that $r!^{t'}\rho^{-r}\le C^{r+1}r!^{t'-t''/s_0}$. The integral has no convergence problem because $|e^{i\Phi(\rho)}|\le e^{-\lambda\rho}$, for a suitable positive constant λ, and eventually we obtain the bound

$$|J_2|\le C_{J_2}^{\alpha+1}\alpha!^{k_1(1-t''/s_0)+k_2(t'-t''/s_0)},\tag{6-97}$$

where k_1, k_2 denote positive constants depending only on the problem data. In the following we denote in this way any constant of this kind, and we shall understand that their meaning may vary depending on the context.

Choosing t' near 1 and t'' near s_0, satisfying (6-94), we see that J_2 gives rise to a function in $\mathcal{B}^{s_0}$.

We are thus left with the term J_1. To estimate it, we have to recall the definition of v in (6-78), where cutoff functions in $G^{t''}$ from Lemma 5.4 have been employed. We have

$$v(\rho)=\sum_{l=0}^{\infty}\omega_l(\rho)u_l(\rho),$$

and without loss of generality we may assume that $\omega_l\equiv 1$ for $\rho\ge 4R(l+1)$ and $\omega_l(\rho)\equiv 0$ for $\rho\le 2R(l+1)$, with the same constant R we used previously. Of course we are free to choose a larger R, if need be. Thus

$$\rho^{s_0\alpha}\sum_{\nu=0}^{(s_0/\mu)\alpha}\rho^{-\mu\nu}P_\nu(\rho,\partial_\rho)v(\rho)=\sum_{k=0}^{\infty}\sum_{\substack{\nu+l=k\\ \nu\le(s_0/\mu)\alpha}}\rho^{-\mu\nu+s_0\alpha}P_\nu(\rho,\partial_\rho)\big(\omega_l(\rho)u_l(\rho)\big).$$

We split this into two parts, according to whether in the above sum the complete expression (6-64) for the transport equation appears or we find only a part of it:

$$
\begin{aligned}
\rho^{s_0\alpha} \sum_{\nu=0}^{\lfloor (s_0/\mu)\alpha \rfloor} \rho^{-\mu\nu} P_\nu(\rho, \partial_\rho) v(\rho) &= \sum_{k=0}^{\lfloor (s_0/\mu)\alpha \rfloor} \sum_{\nu+l=k} \rho^{-\mu\nu+s_0\alpha} P_\nu(\rho,\partial_\rho)\bigl(\omega_l(\rho) u_l(\rho)\bigr) \\
&\qquad + \sum_{k>\lfloor (s_0/\mu)\alpha \rfloor} \sum_{\substack{\nu+l=k \\ \nu \leq \lfloor (s_0/\mu)\alpha \rfloor}} \rho^{-\mu\nu+s_0\alpha} P_\nu(\rho,\partial_\rho)\bigl(\omega_l(\rho) u_l(\rho)\bigr) \\
&= JC_1 + JC_2. \qquad (6\text{-}98)
\end{aligned}
$$

We start by bounding JC_2, which is pretty similar to J_2, studied above. By Proposition 6.1.6, we have

$$
P_\nu(\rho, \partial_\rho) = \sum_{p=0}^{m_\nu} \alpha_{\nu,p}(\rho) \partial_\rho^p, \qquad (6\text{-}99)
$$

where $m_\nu \leq c\nu$ and the coefficients satisfy the estimate

$$
\bigl|\partial_\rho^t \alpha_{\nu,p}(\rho)\bigr| \leq C_\alpha^{\nu+t+1} \nu!^{1-\theta} \frac{t!}{\rho^t}, \qquad (6\text{-}100)
$$

provided $\rho \geq c_1 \nu^\theta$, $0 < \theta \leq 1$. Now

$$
\begin{aligned}
|JC_2| &\leq \sum_{k>\lfloor (s_0/\mu)\alpha \rfloor} \sum_{\substack{\nu+l=k \\ \nu \leq \lfloor (s_0/\mu)\alpha \rfloor}} \sum_{p=0}^{m_\nu} \sum_{\beta=0}^{p} \binom{p}{\beta} \rho^{-\mu\nu+s_0\alpha} \bigl|\alpha_{\nu,p}(\rho)\bigr| \bigl|\partial_\rho^\beta \omega_l(\rho)\bigr| \bigl|\partial_\rho^{p-\beta} u_l(\rho)\bigr| \\
&\leq \sum_{\tilde{k}\geq 0} \sum_{\substack{\nu+l=\tilde{k}+\lfloor (s_0/\mu)\alpha \rfloor+1 \\ \nu \leq \lfloor (s_0/\mu)\alpha \rfloor}} \sum_{p=0}^{m_\nu} \sum_{\beta=0}^{p} \binom{p}{\beta} \rho^{-\mu\tilde{k}-\mu(\lfloor (s_0/\mu)\alpha \rfloor+1)+s_0\alpha} C_\alpha^{\nu+1} \nu!^{1-t''/s_0} \\
&\qquad\qquad \cdot (RC_\omega)^{\beta+1} C_u^{p-\beta+l+1} \frac{p!^{t'}}{\rho^p},
\end{aligned}
$$

where (5-6), (6-49), (6-76) have been used. In particular, (6-49) can be used since $\rho^{s_0} \geq 4RN^{t''} = 4R\theta_N^{t''}\alpha^{t''} \geq 4R\theta_N^{t''}(\mu/s_0)^{t''}\nu^{t''}$, yielding $\theta = t''/s_0$ for R sufficiently large depending on the problem data.

We have $\rho^{-p} \leq \tilde{C}^p p!^{-t''/s_0}$, since $p \leq m_\nu \leq \tilde{c}\nu$. Thus we get

$$
\begin{aligned}
|JC_2| \leq \sum_{\tilde{k}\geq 0} \sum_{\substack{\nu+l=\tilde{k}+\lfloor (s_0/\mu)\alpha \rfloor+1 \\ \nu \leq (s_0/\mu)\alpha}} \sum_{p=0}^{m_\nu} \rho^{-\mu\tilde{k}-\mu(\lfloor (s_0/\mu)\alpha \rfloor+1)+s_0\alpha} C_\alpha^{\nu+1} \nu!^{1-t''/s_0} \\
\cdot (RC_\omega)^{p+1} C_u^{p+l+1} (2\tilde{C})^p p!^{t'-t''/s_0}.
\end{aligned}
$$

We point out that $-\mu(\lfloor (s_0/\mu)\alpha \rfloor + 1) + s_0\alpha < 0$. Moreover, since $m_\nu \leq \tilde{c}\nu$, we may estimate the sum with respect to p, getting

$$
|JC_2| \leq \sum_{\tilde{k}\geq 0} \rho^{-\mu\tilde{k}} \sum_{\substack{\nu+l=\tilde{k}+\lfloor (s_0/\mu)\alpha \rfloor+1 \\ \nu \leq (s_0/\mu)\alpha}} C_\alpha^{\nu+1} R^{\tilde{c}\nu+1} C_u^l C_T^\nu \nu!^{1-t''/s_0+\tilde{c}(t'-t''/s_0)}.
$$

Finally, we want to bound the inner sum, noting that contrary to the sum over $\tilde{k}$, it is a finite sum involving a number of terms proportional to α. Because of the estimate

$$\nu!^{1-t''/s_0+\tilde{c}(t'-t''/s_0)} \leq \left(\left(\frac{s_0}{\mu}\right)\alpha\right)!^{1-t''/s_0+\tilde{c}(t'-t''/s_0)} \leq C^{\alpha+1}\alpha!^{(s_0/\mu)(1-t''/s_0)+(s_0/\mu)\tilde{c}(t'-t''/s_0)},$$

we obtain

$$|JC_2| \leq C^{\alpha+1}\alpha!^{(s_0/\mu)(1-t''/s_0)+(s_0/\mu)\tilde{c}(t'-t''/s_0)} \sum_{\tilde{k}\geq 0} \rho^{-\mu\tilde{k}} C_u^{\tilde{k}}.$$

Since $\rho > 2R$ on the support of v, the above series converges, provided R is large enough. Arguing as for J_2, we conclude that $JC_2 \in \mathscr{B}^{s_0}$.

Consider JC_1 in (6-98). Again we split it into two parts:

$$\begin{aligned}
&\sum_{k=0}^{(s_0/\mu)\alpha} \sum_{\nu+l=k} \rho^{-\mu\nu+s_0\alpha} P_\nu(\rho,\partial_\rho)\big(\omega_l(\rho)u_l(\rho)\big)\\
&\quad = \sum_{k=0}^{(s_0/\mu)\alpha} \sum_{\nu+l=k} \rho^{-\mu\nu+s_0\alpha}\omega_l(\rho)P_\nu(\rho,\partial_\rho)u_l(\rho)\\
&\qquad\qquad + \sum_{k=0}^{(s_0/\mu)\alpha} \sum_{\nu+l=k} \rho^{-\mu\nu+s_0\alpha}\big(P_\nu(\rho,\partial_\rho)(\omega_l(\rho)u_l(\rho)) - \omega_l(\rho)P_\nu(\rho,\partial_\rho)u_l(\rho)\big)\\
&\quad = \sum_{k=0}^{(s_0/\mu)\alpha} (I_{1,k}+I_{2,k}).
\end{aligned} \tag{6-101}$$

Let us consider $I_{1,k}$. Remark that if $\rho \geq 4R(k+1)$, then $\omega_l(\rho) \equiv 1$ for any $l = 0,\dots,k$. Therefore, in this region $I_{1,k} = 0$, due to (6-75). We have only to consider $I_{1,k}$ for $(4R\theta_N^{t''})^{1/s_0}\alpha^{t''/s_0} \leq \rho \leq 4R(k+1)$. In this region — assuming it is not trivially empty — we have

$$\begin{aligned}
|I_{1,k}| &\leq \sum_{\nu+l=k} \rho^{-\mu\nu+s_0\alpha} \sum_{p=0}^{m_\nu} \big|\alpha_{\nu,p}(\rho)\big|\big|\partial_\rho^p u_l(\rho)\big|\big|\omega_l(\rho)\big|\\
&\leq \rho^{-\mu k+s_0\alpha} \sum_{\nu+l=k} C_\alpha^{\nu+1}\nu!^{1-t''/s_0} \sum_{p=0}^{m_\nu} C_u^{p+l+1}\frac{p!}{\rho^p},
\end{aligned}$$

where we applied (6-100) and (6-76), arguing as we did before. As above, $p!\,\rho^{-p} \leq C^{\nu+1}\nu!^{\tilde{c}(1-t''/s_0)}$. Therefore

$$\begin{aligned}
|I_{1,k}| &\leq \rho^{-\mu k+s_0\alpha} \sum_{\nu+l=k} C^{\nu+1}C_u^{l+1}\nu!^{(1+\tilde{c})(1-t''/s_0)}\\
&\leq \rho^{-\mu k+s_0\alpha} \sum_{\nu+l=k} C^{\nu+1}C_u^{l+1}\rho^{(s_0/t'')\nu(1+\tilde{c})(1-t''/s_0)}\\
&\leq \rho^{-\mu k+s_0\alpha}C^{k+1}\rho^{(s_0/t'')k(1+\tilde{c})(1-t''/s_0)}\\
&= C^{k+1}\rho^{s_0\alpha-k[\mu-(s_0/t'')(1+\tilde{c})(1-t''/s_0)]} \leq C^{k+1}\rho^{s_0\alpha-k\tilde{\mu}},
\end{aligned}$$

for some positive $\tilde{\mu}$, choosing t'' close to s_0 as we did before.

Consider now, recalling (6-96),

$$\left|\int_{\rho^{s_0}\geq 4RN^{t''}} e^{it\rho^{s_0}} e^{i\Phi(\rho)} I_{1,k}(\rho)\, d\rho\right| \leq C^{k+1}\int_{(4R)^{1/s_0}N^{t''/s_0}\leq\rho\leq 4R(k+1)} e^{-\lambda\rho}\rho^{s_0\alpha-k\tilde{\mu}}\, d\rho$$
$$\leq C^{k+1}\int_0^{+\infty} e^{-\lambda\rho}\rho^{s_0\alpha-\mu'\rho}\, d\rho \leq C^{\alpha+1}\int_0^{+\infty} e^{-\mu'\rho\log\rho}\rho^{s_0\alpha}\, d\rho.$$

The proof is complete once we show:

Lemma 6.2.7. *Let $\mu>0$. For any $\varepsilon>0$, there is a constant $C_\varepsilon>0$ such that*

$$\int_0^{+\infty} e^{-\mu\rho\log\rho}\rho^{s_0\alpha}\, d\rho \leq C_\varepsilon\varepsilon^\alpha\alpha!^{s_0}. \tag{6-102}$$

Proof. Pick a positive M to be chosen later and write

$$\int_0^{+\infty} e^{-\mu\rho\log\rho}\rho^{s_0\alpha}\, d\rho = \int_0^{M} e^{-\mu\rho\log\rho}\rho^{s_0\alpha}\, d\rho + \int_M^{+\infty} e^{-\mu\rho\log\rho}\rho^{s_0\alpha}\, d\rho = I_1+I_2.$$

Consider I_2. Because $e^{-\mu\rho\log\rho}\leq e^{-\mu\log M\rho}$, we get

$$I_2\leq\int_0^{+\infty} e^{-\mu\rho\log M}\rho^{s_0\alpha} d\rho = \left(\frac{1}{\mu\log M}\right)^{s_0\alpha+1}\alpha!^{s_0}.$$

Choosing $\mu^{-s_0}(\log M)^{-s_0}\leq\varepsilon$, we prove the assertion for I_2.

Consider I_1.

$$I_1\leq e^{\mu/e}\frac{M^{s_0\alpha+1}}{s_0\alpha+1}\leq e^{\mu/e} M\frac{\left(\frac{M^{s_0}}{\varepsilon}\right)^\alpha}{\alpha!^{s_0}}\varepsilon^\alpha\alpha!^{s_0},$$

and this implies the assertion also for I_1. □

Let us now consider $I_{2,k}$. Remark that if $\rho\geq 4R(k+1)$, then $I_{2,k}=0$ due to Lemma 5.4. We have only to consider $I_{2,k}$ for $(4R\theta_N^{t''})^{1/s_0}\alpha^{t''/s_0}\leq\rho\leq 4R(k+1)$. Assuming this region is not trivially empty, we have

$$|I_{2,k}|\leq\sum_{\nu+l=k}\rho^{-\mu\nu+s_0\alpha}\sum_{p=0}^{m_\nu}\sum_{\beta=1}^{p}\binom{p}{\beta}\left|\alpha_{\nu,p}(\rho)\right|\left|\partial_\rho^\beta\omega_l(\rho)\right|\left|\partial_\rho^{p-\beta}u_l(\rho)\right|$$
$$\leq\rho^{-\mu k+s_0\alpha}\sum_{\nu+l=k}C_\alpha^{\nu+1}\nu!^{1-t''/s_0}\sum_{p=0}^{m_\nu}\sum_{\beta=1}^{p}\binom{p}{\beta}(RC_\omega)^{\beta+1}\frac{\beta!^{t'}}{\rho^\beta}C_u^{p-\beta+l+1}\frac{(p-\beta)!}{\rho^{p-\beta}}$$
$$\leq\rho^{-\mu k+s_0\alpha}\sum_{\nu+l=k}C_\alpha^{\nu+1}\nu!^{1-t''/s_0}\sum_{p=0}^{m_\nu}C_u'^{p+l+1}\frac{p!^{t'}}{\rho^p}.$$

As above, $p!^{t'}\rho^{-p}\leq C^{\nu+1}\nu!^{\tilde{c}(t'-t''/s_0)}$, and the argument proceeds as that for $I_{1,k}$.

This completes the proof of Proposition 6.2.6. □

Next we are going to show that if $\Lambda(t, D_t)$ as given by the left-hand side of (6-2) is G^s-hypoelliptic for $s < s_0$, from (6-84), it follows that $A(v)(t) \in \mathcal{B}^{s_0}(\mathbb{R})$.

To this end, we recall the following result. For its proof we refer to Appendix B.

Theorem 6.2.8 [Métivier 1980, Theorem 3.1]. *Let Ω be an open set of $\mathbb{R}$ containing the origin. Assume that there is an open subset $U \Subset \Omega$, a compact subset K of Ω, and a bounded operator $R\colon L^2(U) \to L^2(K)$ such that $(PRu)_{|U} = u_{|U}$.*

The operator Λ is Gevrey s-hypoelliptic at the origin if and only if:

(i) *For any neighborhood ω of the origin, there exists a neighborhood $\omega'' \Subset \omega$ such that*

$$(\Lambda u)_{|\omega} \in H^k(\omega) \quad \textit{implies} \quad u_{|\omega''} \in H^k(\omega'').$$

(ii) *For any neighborhoods of the origin $\omega^{iv} \Subset \omega''' \Subset \omega''$, there are positive constants C, L such that*

$$\|u\|_{k,\omega^{iv}} \le CL^k\big([\![\Lambda(\varphi u)]\!]_{s,k,\omega'''} + k!^s \|u\|_{0,\omega''}\big), \tag{6-103}$$

where $\|u\|_{k,\omega}$ denotes the usual Sobolev norm of order k on the open set ω and

$$[\![u]\!]_{s,k,\omega} = \sum_{\alpha=0}^{k} k^{s(k-\alpha)} \|D^\alpha u\|_{0,\omega}. \tag{6-104}$$

Moreover, $\varphi \in C_0^\infty(\omega'')$, $\varphi \equiv 1$ in a neighborhood of ω''' and C, L are independent of k and u.

By Theorem 4.3 and (6-2), we obtain that the operator Λ has a parametrix whose symbol belongs to $S^0_{1,k/lq}$ (recall that $k/lq < 1$, by assumption). See also Theorem 3.4 of [Kumano-go 1982]. Moreover, by Remark B.111., we have $(P(\varphi u))_{|\omega'''} \in \mathcal{B}^{s_0}(\omega''')$ if and only if $(Pu)_{|\omega'''}$ has the same regularity.

Therefore, Theorem 6.2.8 can be applied to Λ, provided we are on a small enough neighborhood of the origin. To keep the notation simple, we denote by ω' the neighborhood of the origin where the solution has regularity $\mathcal{B}^{s_0}$.

Lemma 6.2.9. *If $A(v) \in \mathcal{B}^{s_0}(\omega')$, then for every $\varepsilon > 0$ there exists $C_{\varepsilon,\omega'} > 0$ such that*

$$\big|\mathcal{F}(A(v))(\tau)\big| \le C_{\varepsilon,\omega'} e^{-(1/(2\varepsilon)^{1/s_0})|\tau|^{1/s_0}}. \tag{6-105}$$

Here $\mathcal{F}(A(v))$ denotes the Fourier transform of $A(v)$.

Proof. First we point out that $A(v) \in \mathcal{S}(\mathbb{R})$, due to the fact that the phase factor $e^{i\Phi(\rho)}$ is rapidly decreasing for $\rho \to +\infty$.

There exists a $\delta > 0$, $[-\delta, \delta] \subset \omega'$ such that for every $\varepsilon > 0$ there is $C_{1,\varepsilon} > 0$ for which, for every α,

$$|D_t^\alpha A(v)(t)| \le C_{1,\varepsilon} \varepsilon^\alpha \alpha!^{s_0}, \quad |t| \le \delta. \tag{6-106}$$

An argument quite similar to that of the proof of Lemma 6.2.5 gives that, for $|t| \ge \delta$,

$$\big|D_t^\alpha A(v)(t)\big| \le \frac{1}{|t|^\alpha} C^{\alpha+1} \alpha!^{t'} \le \frac{1}{|t|^\alpha} C_{2,\varepsilon} \varepsilon^\alpha \alpha!^{s_0}. \tag{6-107}$$

For the Fourier transform of $A(v)$, we obtain

$$\mathcal{F}(A(v))(\tau) = \frac{1}{\tau^\alpha} \int e^{-it\tau} D_t^\alpha A(v)\, dt.$$

We split the latter integral into two parts, I_1, I_2, for the regions $|t| \le \delta$ and $|t| \ge \delta$ respectively.

By (6-106),

$$|I_1| \le 2\delta \frac{1}{|\tau|^\alpha} C_{1,\varepsilon} \varepsilon^\alpha \alpha!^{s_0}.$$

By (6-107), for $\alpha \ge 2$,

$$|I_2| \le \frac{1}{|\tau|^\alpha} C_{2,\varepsilon} \varepsilon^\alpha \alpha!^{s_0} \int_{|t|\ge\delta} |t|^{-\alpha}\, dt = \frac{1}{|\tau|^\alpha} C_{3,\varepsilon} \varepsilon^\alpha \alpha!^{s_0}.$$

Therefore, overall, we get

$$\left|\mathcal{F}(A(v))(\tau)\right| \le \frac{1}{|\tau|^\alpha} C_{4,\varepsilon} \varepsilon^\alpha \alpha!^{s_0},$$

for any α and τ large. Hence

$$\left|\mathcal{F}(A(v))(\tau)\right|^{1/s_0} \frac{\left(\left(\frac{|\tau|}{2\varepsilon}\right)^{1/s_0}\right)^\alpha}{\alpha!} \le C_{4,\varepsilon}^{1/s_0} \left(\frac{1}{2^{1/s_0}}\right)^\alpha.$$

Summing in α from 0 to ∞, we prove the assertion. □

We state the following proposition, leaving the proof to the reader:

Proposition 6.2.10. *Let ω', ω be as in Theorem 6.2.8. If $\Lambda u \in \mathcal{B}^{s_0}(\omega)$, then $u \in \mathcal{B}^{s_0}(\omega'')$.*

Corollary 6.2.11. *Let $A(v)$ be given by (6-85). Then Proposition 6.2.6 implies that $A(v) \in \mathcal{B}^{s_0}(\omega')$.*

Proof of the corollary. Let $\varphi \in C_0^\infty(\mathbb{R}) \cap G^s(\mathbb{R})$, $\varphi \equiv 1$ near the origin. Arguing as in the proof of Lemma 6.2.9, we may show that

$$\left|\widehat{(1-\varphi)A(v)}(\xi)\right| \le Ce^{-|\xi|^{1/s}/C},$$

for a certain positive constant C, whence $\Lambda((1-\varphi)A(v)) \in G^s$. Therefore, Proposition 6.2.6 implies that $\Lambda(\varphi A(v)) \in \mathcal{B}^{s_0}$. From Proposition 6.2.10, it follows that $\varphi A(v) \in \mathcal{B}^{s_0}$, whence the statement. □

Let us now prove that Corollary 6.2.11 implies a contradiction, which in turn yields that Λ is Gevrey s_0-hypoelliptic and not better.

The construction of $A(v)$ shows that the conclusion of Lemma 6.2.9 is violated:

Lemma 6.2.12. *There exist positive constants λ, C_λ such that for τ positive and large,*

$$\left|\mathcal{F}(A(v))(\tau)\right| \ge C_\lambda e^{-\lambda\tau^{1/s_0}}. \tag{6-108}$$

Proof. Since v in $A(v)$ (see (6-79)) has support in $[2R, +\infty[$, we have

$$A(v)(t) = \frac{1}{s_0} \int_{\mathbb{R}} e^{it\tau} e^{i\Phi(\tau^{1/s_0})} v(\tau^{1/s_0}) \tau^{(1/s_0)-1} \chi(\tau)\, d\tau,$$

where $\chi(\tau) \equiv 1$ if $\tau \geq (2R)^{s_0}$ and $\chi(\tau) \equiv 0$ if $\tau \leq R^{s_0}$. From the Fourier transform inversion formula, we obtain that

$$\mathcal{F}(A(v))(\tau) = \frac{2\pi}{s_0} e^{i\Phi(\tau^{1/s_0})} v(\tau^{1/s_0}) \tau^{(1/s_0)-1},$$

for $\tau \geq 2R$. Since, due to the construction performed in Section 6.1, we have for τ large

$$\Phi(\tau^{1/s_0}) = \varphi \tau^{1/s_0}(1 + o(1))$$

with $\operatorname{Im} \varphi > 0$, and

$$v(\tau^{1/s_0}) = 1 + o(1),$$

we conclude, for a suitable $\lambda > 0$, that

$$\left|\mathcal{F}(A(v))(\tau)\right| \geq C_\lambda e^{-\lambda \tau^{1/s_0}}. \qquad \square$$

Thus the inequalities

$$C_\lambda e^{-\lambda \tau^{1/s_0}} \leq \left|\mathcal{F}(A(v))(\tau)\right| \leq C_{\varepsilon,\omega'} e^{-\frac{1}{(2\varepsilon)^{1/s_0}} \tau^{1/s_0}}$$

give a contradiction, provided ε is small and τ is large enough.

This proves assertion (iii) of Theorem 1.1.

7. Non-C^∞-hypoellipticity

The purpose of this section is to prove assertion (iv) of Theorem 1.1. Because of Proposition 4.2, we have to show that Λ in (3-33) is not C^∞-hypoelliptic if $l \leq k/q$.

The method of proof is analogous to that used in the previous section, but much simpler. Multiplying Λ in (3-33) by an elliptic operator, we have to consider the symbol

$$\tau^{2k/q-2/q} \Lambda(t,\tau) \sim \sum_{j=0}^{\infty} a_j(t) \tau^{2k/q-j} + \sum_{s=0}^{\infty} b_s(t) \tau^{-s}, \tag{7-1}$$

where $\tau > 0$, a_j, b_s are real analytic and defined in a neighborhood of the origin and

$$a_j(t) = t^{2l-j} \tilde{a}_j(t) \quad \text{for } j = 0, \ldots, 2l-1, \quad \text{with each } \tilde{a}_j \in C^\omega. \tag{7-2}$$

We rename Λ the operator whose symbol is given by the left-hand side of (7-1).

First we look for a formal solution of the form

$$A(u)(t) = \int_0^\infty e^{it\rho} u(\rho)\, d\rho \tag{7-3}$$

of the equation $\Lambda(t, D_t) A(u) = 0$. In order to do so, we replace the coefficients a_j, b_s by their power series

$$\Lambda(t, D_t) = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} a_{j,n} t^{n+(2l-j)_+} D_t^{2k/q-j} + \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} b_{j,n} t^n D_t^{-j},$$

where $(m)_+ = \max\{m, 0\}$. Taking both t and D_t into the integral sign, we formally obtain

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\int_0^{+\infty} e^{it\rho}\rho^{2k/q}\left[\sum_{\alpha=0}^{n+(2l-j)_+} C_{n,j,\alpha}\rho^{-j-n-(2l-j)_++\alpha}\partial^\alpha u + \sum_{\alpha=0}^{n} C'_{n,j,\alpha}\rho^{-j-n-2k/q+\alpha}\partial^\alpha u\right] d\rho,$$

where $C_{n,j,\alpha}$, $C'_{n,j,\alpha}$ are constants. We organize the expression in brackets according to its homogeneity: making the dilation $\rho \mapsto \lambda\rho$ a generic monomial, $\rho^\alpha\partial^\beta$ has homogeneity $\alpha-\beta$. The principal part then has homogeneity $-2l$, forgetting about the factor $\rho^{2k/q}$ in front, and is obtained from the first sum above when $n = 0$ and $j = 0, \dots, 2l$. We are assuming here that $2l < 2k/q$, which is the generic case. If $2l = 2k/q$, the second sum above contributes the term $(j, n, \alpha) = (0, 0, 0)$ to the principal part.

Denote by $P_{-2l}(\rho, \partial_\rho)$ the principal part so obtained. It has the form

$$P_{-2l}(\rho, \partial_\rho) = \sum_{\alpha=0}^{2l} \gamma_\alpha \rho^{\alpha-2l}\partial_\rho^\alpha. \tag{7-4}$$

As for the terms of lower homogeneity, we note that they are homogeneous of degree either $-2l-r$ or $-2k/q-r$. We may gather the terms of equal homogeneity into differential polynomials. To keep the notation simple, we write the quantity in brackets as

$$\sum_{r=0}^{\infty} P_{-2l-r/q}(\rho, \partial_\rho)u,$$

where $P_{-2l-r/q}(\rho, \partial_\rho)$ is a finite linear combination of homogeneous monomials of degree $-2l-r/q$.

We look for a u of the form

$$u(\rho) = \sum_{s=0}^{\infty} u_s(\rho) \tag{7-5}$$

such that

$$\sum_{r=0}^{k} P_{-2l-r/q}u_{k-r} = 0, \quad k = 0, 1, \dots. \tag{7-6}$$

Let us start with u_0; it solves the equation

$$P_{-2l}(\rho, \partial_\rho)u_0(\rho) = 0,$$

where P_{-2l} is given by (7-4). The latter is a Fuchs type equation and we choose

$$u_0(\rho) = \rho^\lambda, \tag{7-7}$$

where λ denotes the solution of the indicial equation associated to (7-4), that is,

$$\sum_{\alpha=0}^{2l} \gamma_\alpha \lambda(\lambda-1)\dots(\lambda-\alpha+1) = 0, \tag{7-8}$$

such that

$$\operatorname{Re}\lambda = \min\{\operatorname{Re}\mu \mid \mu \text{ is a solution of (7-8)}\}. \tag{7-9}$$

Let us next consider the second *transport* equation in (7-6), corresponding to $k = 1$.

$$P_{-2l}(\rho, \partial_\rho)u_1(\rho) = -P_{-2l-1/q}(\rho, \partial_\rho)u_0(\rho).$$

Since the differential operators $P_{-2l-j/q}(\rho, \partial_\rho)$ have homogeneity $-2l - j/q$, when applied to the function ρ^λ they give a function proportional to $\rho^{\lambda-2l-j/q}$. Therefore the above equation has the form

$$P_{-2l}(\rho, \partial_\rho)u_1(\rho) = \text{const}\, \rho^{\lambda-2l-1/q}.$$

Our purpose is to obtain a function u_1 having a better growth rate compared to u_0 when $\rho \to +\infty$, that is, such that $u_1(\rho) = \mathbb{O}(\rho^\mu)$, with $\operatorname{Re}\mu < \operatorname{Re}\lambda$. If $\operatorname{Re}\lambda$ has the minimality property (7-9), we see at once that the exponent in the right-hand side of the above differential equation cannot be a root of the indicial equation (7-8); thus we can rule out logarithmic factors. Again, keeping in mind the homogeneity-preserving property of the operators $P_{-2l-j/q}$, we conclude that

$$u_1(\rho) = c_1 \rho^{\lambda-1/q}.$$

We iterate this argument and solve the triangular system (7-6), thus obtaining:

Proposition 7.1. *There is a $\lambda \in \mathbb{C}$, satisfying both (7-8) and (7-9), such that for every $s = 0, 1, \dots,$ the system (7-6) has a solution u_s of the form*

$$u_s(\rho) = c_s \rho^{\lambda-s/q}. \tag{7-10}$$

Turning the formal solution (7-5) into a function is easy in the present case: let $\chi \in C^\infty(\mathbb{R})$, $\chi \equiv 0$ for $\rho \le R$, $R > 0$, and $\chi \equiv 1$ if $\rho \ge 2R$. Define

$$v(\rho) = \sum_{s=0}^{\infty} \chi(\varepsilon_s \rho)u_s(\rho), \tag{7-11}$$

where $(\varepsilon_s)_{s\in\mathbb{N}}$ denotes a sequence of positive numbers such that $\varepsilon_s \to 0+$ in a convenient way.

We need to make sense of $A(v)$ defined as in (7-3). First of all, we note that there is no problem near $\rho = 0$, since $0 \notin \operatorname{supp}(v)$ (we may always suppose that $\varepsilon_s \le 1$.) If $\operatorname{Re}\lambda < -1$, λ defined by (7-8), (7-9), $A(v)$ is in $C(\mathbb{R})$. If $\operatorname{Re}\lambda \ge -1$, then the integral $A(v)$ in (7-3) has to be interpreted as an oscillating integral, and then it always defines a distribution of finite order to which a pseudodifferential operator can be applied.

We want to show that $\Lambda(t, D_t)A(v)(t) \in C^\infty(\mathbb{R})$.

Proposition 7.2. *Let $A(v)$ be defined as in (7-3), with v given by (7-11). Then*

$$\Lambda(t, D_t)A(v)(t) \in C^\infty(\mathbb{R}). \tag{7-12}$$

Proof. Actually, all we have to show is that $\Lambda A(v)$ is smooth in a neighborhood of the origin, since away from the origin, $A(v)$ is smooth.

We start arguing on just one of the two asymptotic expansions that build Λ, for example, the second sum in (7-1). The argument for the other is completely analogous and we have to use both sums only when (7-6) is needed. This is exactly what was done in the proof of Proposition 6.2.6.

Modulo a smoothing operator, we may assume that the symbol of the operator Λ has the form

$$\Lambda(t,\tau) \sim \sum_{j=0}^{\infty} b_j(t)\chi(\varepsilon_j\tau)\tau^{-j}.$$

Then

$$\Lambda(t,D_t)A(v)(t) = \sum_{j=0}^{\infty} b_j(t)\int_0^{+\infty} e^{it\rho}\chi(\varepsilon_j\rho)\rho^{-j}v(\rho)\,d\rho.$$

Let us consider $D_t^\alpha \Lambda A(v)$ and show that this is a continuous function for every α. Denote by $N \in \mathbb{N}$ a number to be selected later; then we consider

$$D_t^\alpha\Lambda(t,D_t)A(v)(t) = D_t^\alpha\left\{\left(\sum_{j=0}^{N-1}+\sum_{j=N}^{\infty}\right)b_j(t)\int_0^{+\infty} e^{it\rho}\chi(\varepsilon_j\rho)\rho^{-j}v(\rho)\,d\rho\right\} = I_1+I_2. \tag{7-13}$$

Consider I_2 and let $N > \operatorname{Re}\lambda+\alpha+1$. Then $|\chi(\varepsilon_j\rho)\rho^{-j+\alpha}v(\rho)| = \mathbb{O}(\rho^{\operatorname{Re}\lambda-N+\alpha})$, and therefore $I_2 \in C(\mathbb{R})$. Let us now turn to I_1. Let $M \in \mathbb{N}$ and write

$$\begin{aligned} I_1 &= D_t^\alpha\sum_{j=0}^{N-1}\left(\sum_{n=0}^{M-1} b_{j,n}t^n\int_0^{+\infty} e^{it\rho}\chi(\varepsilon_j\rho)\rho^{-j}v(\rho)\,d\rho + t^M\sum_{n=0}^{\infty} b_{j,M+n}t^n\int_0^{+\infty} e^{it\rho}\chi(\varepsilon_j\rho)\rho^{-j}v(\rho)\,d\rho\right)\\ &= I_{11}+I_{12}. \end{aligned}$$

Consider first I_{12}. We have

$$\begin{aligned} I_{12} &= \sum_{j=0}^{N-1}\sum_{n=0}^{\infty}\sum_{\beta=0}^{\alpha} b_{j,M+n}\binom{\alpha}{\beta}\big(D_t^{\alpha-\beta}t^n\big)\int_0^{+\infty} e^{it\rho}\rho^\beta(-D_\rho)^M\big(\chi(\varepsilon_j\rho)\rho^{-j}v(\rho)\big)\,d\rho\\ &\equiv \sum_{j=0}^{N-1}\sum_{\beta=0}^{\alpha}\left(\sum_{n=0}^{\infty} b_{j,M+n}\binom{\alpha}{\beta}\big(D_t^{\alpha-\beta}t^n\big)\right)\int_0^{+\infty} e^{it\rho}\rho^\beta\chi(\varepsilon_j\rho)(-D_\rho)^M\big(\rho^{-j}v(\rho)\big)\,d\rho, \end{aligned}$$

where the last equality is modulo smooth terms because when the derivative with respect to ρ lands on the cutoff function χ, it produces a compact support function of ρ. Moreover, the sum over n on the last line (in big parentheses) is a real analytic function. The integrand function above is $\mathbb{O}(\rho^{\operatorname{Re}\lambda-j-M+\beta})$, so that if $\operatorname{Re}\lambda+\alpha-M<-1$, we obtain that I_{12} is a continuous function. Note that both N and M so far satisfy the same condition.

Consider I_{11}.

$$\begin{aligned} I_{11} &= D_t^\alpha\sum_{j=0}^{N-1}\left(\sum_{n=0}^{M-1} b_{j,n}t^n\int_0^{+\infty} e^{it\rho}\chi(\varepsilon_j\rho)\rho^{-j}v(\rho)\,d\rho\right)\\ &= D_t^\alpha\sum_{j=0}^{N-1}\sum_{n=0}^{M-1} b_{j,n}\int_0^{+\infty} e^{it\rho}(-D_\rho)^n\big(\chi(\varepsilon_j\rho)\rho^{-j}v(\rho)\big)\,d\rho\\ &= D_t^\alpha\sum_{j=0}^{N-1}\sum_{n=0}^{M-1} b_{j,n}\left(\int_0^{2R/(\varepsilon_{N-1})}+\int_{2R/(\varepsilon_{N-1})}^{+\infty}\right)e^{it\rho}(-D_\rho)^n\big(\chi(\varepsilon_j\rho)\rho^{-j}v(\rho)\big)\,d\rho \end{aligned}$$

$$\equiv D_t^\alpha \sum_{j=0}^{N-1}\sum_{n=0}^{M-1} b_{j,n}\int_{2R/(\varepsilon_{N-1})}^{+\infty} e^{it\rho}(-D_\rho)^n\big(\rho^{-j}v(\rho)\big)\,d\rho,$$

modulo smooth functions. By (7-11), we may write

$$v=\Big(\sum_{s=0}^{N-1}+\sum_{s=N}^{\infty}\Big)\chi(\varepsilon_s\rho)u_s$$

and note that the second sum contributes a $\mathbb{O}(\rho^{\operatorname{Re}\lambda-N/q-j-n+\alpha})$ to the integral. Therefore, if

$$\operatorname{Re}\lambda-\frac{N}{q}+\alpha<-1,$$

we have a continuous function. As for the first sum, on the domain of integration, the cutoff is identically equal to one; thus

$$I_{11}\equiv D_t^\alpha \sum_{s=0}^{N-1}\sum_{j=0}^{N-1}\sum_{n=0}^{M-1} b_{j,n}\int_{2R/(\varepsilon_{N-1})}^{+\infty} e^{it\rho}(-D_\rho)^n\big(\rho^{-j}u_s(\rho)\big)\,d\rho.$$

The same analysis can be applied to the first sum in (7-1), so that eventually we get

$$D_t^\alpha \sum_{s=0}^{N-1}\sum_{j=0}^{N-1}\sum_{n=0}^{M-1}\int_{\frac{2R}{\varepsilon_{N-1}}}^{+\infty} e^{it\rho}\rho^{\frac{2k}{q}}\Bigg[\sum_{\alpha=0}^{n+(2l-j)_+} C_{n,j,\alpha}\rho^{\alpha-j-n-(2l-j)_+}\partial^\alpha u+\sum_{\alpha=0}^{n} C'_{n,j,\alpha}\rho^{\alpha-j-n-\frac{2k}{q}}\partial^\alpha u\Bigg]d\rho$$
$$=D_t^\alpha \sum_{s=0}^{N-1}\sum_{r=0}^{\tilde r(N)}\int_{\frac{2R}{\varepsilon_{N-1}}}^{+\infty} e^{it\rho}\rho^{\frac{2k}{q}}\tilde P_{-2l-r/q}(\rho,\partial_\rho)u_s(\rho)\,d\rho,$$

where we have set $M=N$ and $\tilde r(N)\le N$ is a suitable increasing integer function of N, and where the $\tilde P_{-2l-r/q}$ are differential polynomials homogeneous of degree $-2l-r/q$. We see that there exists a number $r(N)\in\mathbb{N}$ such that $r(N)<\tilde r(N)$, $r(N)\to\infty$ for $N\to\infty$, and (see (7-6))

$$\tilde P_{-2l-r/q}(\rho,\partial_\rho)=P_{-2l-r/q}(\rho,\partial_\rho),$$

if $r<r(N)$. Then the above expression can be written as

$$D_t^\alpha \sum_{\substack{s=0\\ r+s<r(N)}}^{N-1}\sum_{r=0}^{\tilde r(N)}\int_{\frac{2R}{\varepsilon_{N-1}}}^{+\infty} e^{it\rho}\rho^{\frac{2k}{q}}P_{-2l-\frac{r}{q}}(\rho,\partial_\rho)u_s(\rho)\,d\rho+D_t^\alpha \sum_{\substack{s=0\\ r(N)\le r+s}}^{N-1}\sum_{r=0}^{\tilde r(N)}\int_{\frac{2R}{\varepsilon_{N-1}}}^{+\infty} e^{it\rho}\rho^{\frac{2k}{q}}\tilde P_{-2l-\frac{r}{q}}(\rho,\partial_\rho)u_s(\rho)\,d\rho$$
$$=D_t^\alpha \sum_{\substack{s=0\\ r(N)\le r+s}}^{N-1}\sum_{r=0}^{\tilde r(N)}\int_{\frac{2R}{\varepsilon_{N-1}}}^{+\infty} e^{it\rho}\rho^{\frac{2k}{q}}\tilde P_{-2l-\frac{r}{q}}(\rho,\partial_\rho)u_s(\rho)\,d\rho,$$

because of (7-6). Taking the t-derivative under the integral sign, we see immediately that the integrand is $\mathbb{O}(\rho^{(2k/q)+\operatorname{Re}\lambda+\alpha-2l-r(N)/q})$. If N is large enough, the assertion is then proved. □

Proposition 7.3. *$A(v)$ is not smooth near the origin.*

Proof. By Proposition 7.1, $v = \mathcal{O}(\rho^{\operatorname{Re}\lambda})$, so that v is a microlocally elliptic symbol of order $\operatorname{Re}\lambda$. Hence, $A(v)$ cannot be smooth. $\square$

Propositions 7.2 and 7.3 prove statement (iv) of Theorem 1.1.

Appendix A: The adjoint of a product

We prove here a well-known formula for the adjoint of a product of two pseudodifferential operators using just symbolic calculus. Let a, b be symbols in $S^0_{1,0}(\mathbb{R}_t)$. We want to show that

$$(a\,\#\,b)^* = b^*\,\#\,a^*, \tag{A-1}$$

where # denotes the usual symbolic composition law (a higher-dimensional extension involves just a more cumbersome notation.)

We may write

$$(a\,\#\,b)^* = \sum_{l,\alpha\ge 0} \frac{(-1)^\alpha}{\alpha!\,l!}\,\partial_\tau^l D_t^l\big(\partial_\tau^\alpha \bar a D_t^\alpha \bar b\big) = \sum_{l,\alpha\ge 0}\sum_{r,s\le l} \frac{(-1)^\alpha}{\alpha!l!}\binom{l}{r}\binom{l}{s}\partial_\tau^{\alpha+r} D_t^{l-s}\bar a \partial_\tau^{l-r} D_t^{\alpha+s}\bar b.$$

Let us change the summation indices according to the following prescription: $j = \alpha + r$, $\beta + j = l - s$, $i = \alpha + s$, so that $l - r = i + \beta$, and we may rewrite the last equality in the above formula as

$$(a\,\#\,b)^* = \sum_{i,j,\beta\ge 0}\sum_{s\le i} \frac{(-1)^{i-s}}{(i-s)!(\beta+j+s)!}\binom{\beta+j+s}{j-i+s}\binom{\beta+j+s}{s}\partial_\tau^{i+\beta} D_t^i \bar b \partial_\tau^j D_t^{\beta+j}\bar a.$$

Let us examine the s-summation; we claim that

$$\sum_{s=0}^{i} \frac{(-1)^{i-s}}{(i-s)!}\,\frac{1}{(\beta+i)!(j-i+s)!}\binom{\beta+j=s}{s} = \frac{1}{\beta!\,i!\,j!}.$$

This is actually equivalent to

$$\sum_{s=0}^{i}(-1)^{i-s}\binom{i}{s}\binom{\beta+j+s}{\beta+i} = \binom{\beta+j}{j}.$$

Setting $i - s = \nu \in \{0, 1, \ldots, i\}$, the above relation is written as

$$\sum_{\nu=0}^{i}(-1)^{\nu}\binom{i}{\nu}\binom{\beta+i+j-\nu}{\beta+i} = \binom{\beta+j}{j},$$

and this is precisely identity (12.15) in [Feller 1957, Chapter II].

Thus we may conclude that

$$(a\,\#\,b)^* = \sum_{i,j,\beta} \frac{1}{\beta!\,i!\,j!}\partial_\tau^{i+\beta} D_t^i \bar b \partial_\tau^j D_t^{j+\beta}\bar a = \sum_{\beta\ge 0}\frac{1}{\beta!}\partial_\tau^\beta\Big(\sum_{i\ge 0}\frac{1}{i!}\partial_\tau^i D_t^i \bar b\Big) D_t^\beta\Big(\sum_{j\ge 0}\frac{1}{j!}\partial_\tau^j D_t^j \bar a\Big) = b^*\,\#\,a^*.$$

This proves (A-1).

As a byproduct of the above argument, we get the identity

$$\sum_{i,j,\beta} \frac{1}{\beta!\,i!\,j!} \partial_\tau^{i+\beta} D_t^i \bar{b} \partial_\tau^j D_t^{j+\beta} \bar{a} = \sum_{l,\alpha\geq 0} \frac{(-1)^\alpha}{\alpha!\,l!} \partial_\tau^l D_t^l \big(\partial_\tau^\alpha \bar{a} D_t^\alpha \bar{b}\big), \tag{A-2}$$

which is the purpose of this appendix.

We would like to point out that the relation $(a^*)^* = a$ rests on the identity

$$\sum_{l\geq 0} \frac{1}{l!} \partial_\tau^l D_t^l \overline{\left(\sum_{\alpha\geq 0} \frac{1}{\alpha!} \partial_\tau^\alpha D_t^\alpha \bar{a} \right)} = \sum_{s\geq 0} \frac{1}{s!} \left(\sum_{l+\alpha=s} \frac{s!}{l!\alpha!} (-1)^\alpha \right) \partial_\tau^s D_t^s a = \sum_{s\geq 0} \frac{1}{s!} (1-1)^s \partial_\tau^s D_t^s a = a. \tag{A-3}$$

Appendix B: Proof of Theorem 6.2.8

We include in this section the proof of Theorem 6.2.8 for pseudodifferential operators in the Gevrey case, which is the case needed in our argument. Métivier [1980] gives the proof of the same theorem in the analytic category for differential operators, and states that its extension to the pseudodifferential case has no major difficulties. We argue along the same lines.

Since pseudodifferential operators are involved in an essential way, we first recall the definition of hypoellipticity; even though the material is well known, it is useful to state it here for future reference.

When we use a pseudodifferential operator, or its symbol, we mean either a pseudodifferential operator in the C^∞ or in the Gevrey category. In the latter case, although the symbols involved may be analytic functions, the cut off functions will of course be in Gevrey classes (see also Lemmas 5.3 and 5.4 for the construction of some cutoff functions.)

Definition B.1. Let $P(x, D_x)$ denote a properly supported pseudodifferential operator acting on the distributions. We say that P is hypoelliptic at the point x_0 if and only if there exists an open set Ω, $x_0 \in \Omega$, such that for every open set $V \Subset \Omega$ and for every $u \in \mathscr{D}'(\Omega)$, we have

$$(Pu)_{|_V} \in C^\infty \Rightarrow u_{|_V} \in C^\infty$$

or

$$(Pu)_{|_V} \in G^s \Rightarrow u_{|_V} \in G^s,$$

for $s > 1$.

It is well known that (not properly supported) pseudodifferential operators can be extended as operators from $\mathscr{E}'(\Omega) \to \mathscr{D}'(\Omega)$. Thus we may also give the following definition:

Definition B.2. Let $P(x, D_x)$ denote a pseudodifferential operator, which we suppose defined in $\mathbb{R}^n$ and not properly supported, acting on distributions. We say that P is hypoelliptic at the point $x_0 \in \mathbb{R}^n$ if and only if there exists an open set Ω containing x_0 and such that for every open set $V \Subset \Omega$ and for every $u \in \mathscr{E}'(\Omega)$, we have

$$(Pu)_{|_V} \in C^\infty \Rightarrow u_{|_V} \in C^\infty$$

or

$$(Pu)_{|_V} \in G^s \Rightarrow u_{|_V} \in G^s,$$

for $s > 1$.

Proposition B.3. *Let P denote a properly supported pseudodifferential operator. Then Definition B.2 is equivalent to Definition B.1.*

Proof. Let us show first that B.2 implies B.1. Let Ω be the open set from Definition B.2 and let $u \in \mathscr{D}'(\Omega)$. We want to show that for every $V \Subset \Omega$, if, for example, $(Pu)_{|_V} \in C^\infty$, then $u_{|_V} \in C^\infty$. The assertion in the Gevrey category will have a completely analogous proof.

Let $\bar{x} \in V$ and $\varphi \in C_0^\infty(V)$ such that $\varphi \equiv 1$ on $V_1 \Subset V$, $\bar{x} \in V_1$. Since $(Pu)_{|_V} \in C^\infty$, we have

$$Pu = P(\varphi u) + P((1-\varphi)u) \in C^\infty.$$

Since P is properly supported, we have $P = P_1 + R_P$, where $R_P\colon \mathscr{D}'(\Omega) \to C^\infty(\Omega)$ is a regularizing operator and P_1 enlarges support by a fixed quantity, that is, $\operatorname{supp}(Pf) \subset (\operatorname{supp} f)_\delta$ for a certain positive δ, where if $A \subset \mathbb{R}^n$, $A_\delta = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, A) \leq \delta\}$.

Now

$$C^\infty \ni (Pu)_{|_{V_1}} = (P(\varphi u))_{|_{V_1}} + \big(P((1-\varphi)u)\big)_{|_{V_1}} = (P(\varphi u))_{|_{V_1}} + \big(P_1((1-\varphi)u)\big)_{|_{V_1}} + \big(R_P((1-\varphi)u)\big)_{|_{V_1}}.$$

The third term is obviously smooth and the second term vanishes if $\operatorname{dist}(V_1, \complement V) > \delta$.

Therefore $(P(\varphi u))_{|_{V_1}} \in C^\infty$ implies, by Definition B.2, that $\varphi u \in C^\infty(V_1)$ or, since $\varphi \equiv 1$ on V_1, that $u \in C^\infty(V_1)$. Since the choice of the point $\bar{x}$ is arbitrary, we obtain that $u \in C^\infty(V)$, and hence the conclusion in Definition B.1.

The converse implication is easier. Assume that $Pu \in C^\infty(V)$, with $u \in \mathscr{E}'(\Omega)$. Again $(Pu)_{|_V} = (P_1 u)_{|_V} + (R_P u)_{|_V}$, where $R_P\colon \mathscr{E}'(\Omega) \to C^\infty(\Omega)$. Thus $(Pu)_{|_V} \in C^\infty$ implies that $(P_1 u)_{|_V} \in C^\infty$, so that, by Definition B.1, $u_{|_V} \in C^\infty$. This proves the proposition. □

The next proposition shows that, in order to prove that a pseudodifferential operator is hypoelliptic, it is enough to show that the corresponding properly supported operator is hypoelliptic according to Definition B.2.

Proposition B.4. *Let P denote a pseudodifferential operator. Then P is hypoelliptic (G^s-hypoelliptic, $s > 1$) at the point x_0 if and only if P_1 is hypoelliptic (resp. G^s-hypoelliptic, $s > 1$) at x_0 according to Definition B.2. Here we denote by P_1 a properly supported operator such that $P = P_1 + R_P$, with $R_P\colon \mathscr{E}'(\Omega) \to C^\infty(\Omega)$.*

Proof. Assume that P is hypoelliptic at x_0 and let Ω be the open neighborhood of x_0 from Definition B.1. We assume that for every $V \Subset \Omega$, $x_0 \in V$, $(P_1 u)_{|_V} \in C^\infty$ with $u \in \mathscr{E}'(\Omega)$. As we did above, we point out that $(Pu)_{|_V} = (P_1 u)_{|_V} + (R_P u)_{|_V} \in C^\infty$, and this implies that $u_{|_V} \in C^\infty$.

The converse statement has a completely analogous proof.

Again we remark that the proof in the Gevrey category is exactly the same. □

We now turn to proving Theorem 6.2.8. We start by recalling without proof a couple of facts about cutoff functions. This is also useful to establish the notation.

Lemma B.5. *There is a positive constant γ_0, depending only on n, the dimension of the ambient space, such that for every pair of open subsets $\omega' \Subset \omega \Subset \mathbb{R}^n$, there is a sequence of functions $(\chi_k)_{k \in \mathbb{N}}$ in $\mathscr{D}(\omega)$,*

$\chi_k{}_{|_{\omega'}} \equiv 1$, *and such that for every* $k \in \mathbb{N}$, *for every multi-index* $\alpha \in \mathbb{N}^n$ *with* $|\alpha| \le k$, *we have*

$$\|D^\alpha \chi_k\|_\infty \le \left(\frac{\gamma_0 k}{r}\right)^{|\alpha|}, \tag{B-1}$$

where $r = \operatorname{dist}(\omega', \complement\omega) > 0$.

Lemma B.6. *Let* ω', ω, χ_k *be as in the previous lemma and satisfying* (B-1). *Then there is a positive constant* γ *such that for every* $k \in \mathbb{N}$ *and for every* $u \in H^k(\omega)$, *we have*

$$|||\chi_k u|||_{k,\mathbb{R}^n} \le \gamma^k |||u|||_{k,\omega}, \tag{B-2}$$

where the three bar norm, defined right after Theorem 6.2.8, has the meaning

$$|||u|||_{k,\omega} = \sum_{\alpha=0}^{k} k^{k-\alpha} \|D^\alpha u\|_{0,\omega}. \tag{B-3}$$

Lemma B.7. *Let* Ω *denote a neighborhood of* $x_0 \in \mathbb{R}^n$ *and let* B *be a Banach space continuously injecting into* $L^2(\Omega)$. *Assume that* $x_0 \notin \operatorname{sing\,supp}_s u$ *for every* $u \in B$, *where* $\operatorname{sing\,supp}_s u$ *denotes the Gevrey* s*-singular support of* u, *and* $s > 1$. *Then there are neighborhoods* $\omega' \Subset \omega \Subset \Omega$ *of* x_0, *functions* χ_k *satisfying* (B-1), *and positive constants* γ *and* C *such that for every* $k \in \mathbb{N}$ *and every* $u \in B$,

$$|\xi|^k \widehat{\chi_k u} \in L^2(\mathbb{R}^n),$$

or, in different terms,

$$\| |\xi|^k \widehat{\chi_k u} \|_{0,\mathbb{R}^n} \le C(\gamma k^s)^k \|u\|_B.$$

Proof. For $\omega \Subset \mathbb{R}^n$ and $L > 0$, let us denote by $\mathfrak{g}^s_L(\bar\omega)$ the Banach space of all Gevrey s-functions on $\bar\omega$ such that

$$\|u\|_{\mathfrak{g}^s_L(\bar\omega)} = \sup_\alpha \frac{\|D^\alpha u\|_{0,\omega}}{\alpha!^s L^{|\alpha|}} < +\infty. \tag{B-4}$$

Then the space of all functions being Gevrey s at the point x_0 can be written as

$$\operatorname{ind\,lim}_{N\to\infty} \mathfrak{g}^s_N(\overline{B(x_0, N^{-1})}).$$

Using Theorem 1 on p. 147 of [Grothendieck 1973], we can see that there exist a neighborhood ω of the point x_0 and a constant $L > 0$ such that the map $u \mapsto u_{|_\omega}$ is continuous from B to $\mathfrak{g}^s_L(\bar\omega)$. Denote by C its norm.

Let $\omega' \Subset \omega$ and let χ_k be functions as in Lemma B.5. We therefore have

$$\left\| D^\alpha(\chi_k u)\right\|_{0,\mathbb{R}^n} \le \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \left(\frac{\gamma_0 k}{r}\right)^{|\beta|} (\alpha-\beta)!^s L^{|\alpha-\beta|} \|u_{|_\omega}\|_{\mathfrak{g}^s_L(\bar\omega)}.$$

For $|\alpha| \le k$ we may estimate $(\alpha-\beta)!^s \le k^{s|\alpha-\beta|}$, so that

$$\left\| D^\alpha(\chi_k u)\right\|_{0,\mathbb{R}^n} \le C\left(L + \frac{\gamma_0}{r}\right)^{|\alpha|} k^{s|\alpha|} \|u\|_B$$

and

$$\big\| |\xi|^k \widehat{\chi_k u} \big\|_{0,\mathbb{R}^n} \le n^{k/2} C \Big(L + \frac{\gamma_0}{r} \Big)^k k^{sk} \|u\|_B,$$

which is the statement of the lemma. □

Next we remark that there is a constant $\gamma_1 \ge 1$ such that for every $k \in \mathbb{N}$ and every $u \in H^k(\mathbb{R}^n)$, we have

$$\gamma_1^k \|\|u\|\|_{k,\mathbb{R}^n}^2 \le \int_{\mathbb{R}^n} (k + |\xi|)^{2k} |\hat{u}(\xi)|^2 \, d\xi \le \gamma_1^k \|\|u\|\|_{k,\mathbb{R}^n}^2. \tag{B-5}$$

Now define, for $s > 1$,

$$G_{(s)} = \big\{ u \in L^2(\mathbb{R}^n) \,\big|\, e^{|\xi|^{1/s}} \hat{u}(\xi) \in L^2(\mathbb{R}^n) \big\}. \tag{B-6}$$

Once we equip $G_{(s)}$ with the norm $\|u\|_{G_{(s)}} = \|e^{|\xi|^{1/s}} \hat{u}\|_{0,\mathbb{R}^n}$, we see that $G_{(s)}$ is a Hilbert space of $G^s(\mathbb{R}^n)$ functions.

Lemma B.8. *Let k be an integer ≥ 1 and for $j = 0, 1, 2, \dots$, let $N_j = k2^j$. Then every function $u \in H^k(\mathbb{R}^n)$ can be written as the series*

$$u = \sum_{j=0}^{\infty} u_j,$$

where $u_j \in G_{(s)}$, and such that

$$\sum_{j=0}^{\infty} N_j^{2ks} \big(\|u_j\|_{0,\mathbb{R}^n}^2 + e^{-2N_j} \|u_j\|_{G_{(s)}}^2 \big) \le 2(2\gamma_1)^k \|\|u\|\|_{k,\mathbb{R}^n}^2. \tag{B-7}$$

Proof. For $j = 0, 1, \dots$ (and setting $N_{-1} = 0$), we have

$$u_j(x) = (2\pi)^{-n} \int_{N_{j-1} \le |\xi|^{1/s} < N_j} e^{i\langle x, \xi \rangle} \hat{u}(\xi) \, d\xi.$$

If $|\xi| \le N_j^s$, then $e^{|\xi|^{1/s}} \le e^{N_j}$, so that $\|u_j\|_{G_{(s)}} \le e^{N_j} \|u_j\|_{0,\mathbb{R}^n}$. Furthermore, when $|\xi| \ge N_{j-1}^s$, we have $N_j^s \le 2^s(|\xi| + k)$ and

$$\sum_{j=0}^{\infty} N_j^{2ks} \|u_j\|_{0,\mathbb{R}^n}^2 \le \int_{\mathbb{R}^n} \big(2^s(|\xi| + k) \big)^{2k} |\hat{u}(\xi)|^2 \, d\xi,$$

which allows us to conclude. □

Next we prove an inverse of the preceding lemma, but on a neighborhood of the point x_0.

As above, let Ω be a neighborhood of x_0 and let B be a Banach space of functions of class G^s at x_0 such that the injection from B to $L^2(\Omega)$ is continuous.

Lemma B.9. *There is a neighborhood $\omega' \Subset \Omega$ of x_0 and a positive constant C such that for every $k \ge 1$ and every sequence u_j, $j = 0, 1, 2, \dots$, $u_j \in B$, satisfying*

$$\sum_{j=0}^{\infty} N_j^{2ks} \big(\|u_j\|_{0,\Omega}^2 + e^{-2N_j} \|u_j\|_B^2 \big) = \Phi_k^2(u_j) < +\infty,$$

the series

$$u = \sum_{j=0}^{\infty} u_j \tag{B-8}$$

converges in $L^2(\Omega)$ *and* $u_{|_{\omega'}} \in H^k(\omega')$, *with the inequality*

$$\|u_{|_{\omega'}}\|_{k,\omega'} \le C^{k+1}\Phi_k(u_j).$$

Proof. The convergence of the series (B-8) in $L^2(\Omega)$ is a direct consequence of the assumption that $\Phi_k(u_j) < +\infty$.

Applying Lemma B.7, we obtain neighborhoods $\omega' \Subset \omega \Subset \Omega$ of the point x_0, positive constants γ, C_0, and a sequence of cutoff functions $\chi_N \in \mathcal{D}(\omega)$, $\chi_N \equiv 1$ on ω', such that for every N and every function $u \in B$ we have the estimate

$$\left\| \left(\frac{|\xi|}{\gamma N^s}\right)^N \widehat{\chi_N u} \right\|_{0,\mathbb{R}^n} \le C_0 \|u\|_B. \tag{B-9}$$

Define the functions

$$\theta(j,\xi) = e^{-N_j}\left(\frac{|\xi|}{\gamma N_j^s}\right)^{N_j} \tag{B-10}$$

and

$$g_j(\xi) = \bigl(1+\theta(j,\xi)\bigr)\widehat{\chi_{N_j} u_j(\xi)}. \tag{B-11}$$

Both (B-9) and (B-11) yield

$$\|g_j\|_{0,\mathbb{R}^n} \le \|u_j\|_{0,\Omega} + C_0 e^{-N_j}\|u_j\|_B,$$

so that

$$\sum_{j=0}^{\infty} N_j^{2ks}\|g_j\|^2_{0,\mathbb{R}^n} \le 2(1+C_0^2)\Phi_k^2(u_j) < +\infty. \tag{B-12}$$

Let us now define

$$v = \sum_{j=0}^{\infty} \chi_{N_j} u_j.$$

Of course $v \in L^2(\Omega)$ and, by definition, v coincides with u on ω'. Therefore it is enough to show that $v \in H^k(\mathbb{R}^n)$ and that the estimate

$$\|v\|_{k,\mathbb{R}^n} \le C^{k+1}\Phi_k(u_j)$$

holds. Actually one already has the estimate

$$\|v\|_{0,\mathbb{R}^n} \le \sum_{j=0}^{\infty} \|u_j\|_{0,\mathbb{R}^n} \le 2\Phi_k(u_j).$$

We only have to show then that $|\xi|^k \hat{v} \in L^2(\mathbb{R}^n)$ and that the estimate

$$\bigl\| |\xi|^k \hat{v} \bigr\|_{0,\mathbb{R}^n} \le C^{k+1}\Phi_k(u_j) \tag{B-13}$$

holds, where the constant C is independent of k.

To this end, using (B-11), we write

$$|\xi|^k \hat{v}(\xi) = \sum_{j=0}^{\infty} \big(1+\theta(j,\xi)\big)^{-1} g_j(\xi)|\xi|^k.$$

We have

$$|\xi|^{2k} |\hat{v}(\xi)|^2 \le \Big(\sum_{j=0}^{\infty} |g_j(\xi)|^2 N_j^{2ks}\Big)\theta(\xi),$$

where

$$\theta(\xi) = \sum_{j=0}^{\infty} \Big(\frac{|\xi|}{N_j^s}\Big)^{2k} \big(1+\theta(j,\xi)\big)^{-2} = \sum_{j=0}^{\infty} \Psi_j(\xi).$$

Because of (B-12), it suffices to prove that

$$\|\theta(\xi)\|_\infty \le C^{k+1}. \tag{B-14}$$

We argue in two different cases. The first region is $\gamma e^2 N_j^s \le |\xi|$. Then

$$\Psi_j(\xi) \le \gamma^{2k} \left(\frac{|\xi|}{\gamma N_j^s}\right)^{2k-2N_j} e^{2N_j} \le (\gamma e^2)^{2k} e^{-2N_j}.$$

As a consequence,

$$\sum_{\gamma e^2 N_j^s \le |\xi|} \Psi_j(\xi) \le C_1 (\gamma e^2)^{2k}.$$

If now $\gamma e^2 N_j^s \ge |\xi|$, let $j_0 = \min\{j \mid \gamma e^2 N_j^s \ge |\xi|\}$ for a fixed ξ. We have

$$\Psi_j(\xi) \le \gamma^{2k} \left(\frac{|\xi|}{\gamma N_j^s}\right)^{2k} = \gamma^{2k} \left(\frac{|\xi|}{\gamma N_{j_0}^s}\right)^{2k} \left(\frac{N_{j_0}}{N_j}\right)^{2ks} \le (\gamma e^2)^{2k} \left(\frac{1}{2^{j-j_0}}\right)^{2ks}.$$

Therefore

$$\sum_{\gamma e^2 N_j^s \ge |\xi|} \Psi_j(\xi) \le (\gamma e^2)^{2k} \left(\sum_{j=0}^{\infty} 2^{-j}\right)^{2ks} = (\gamma e^2 2^s)^{2k}.$$

This proves the lemma. □

We now want to prove the following theorem in a Gevrey pseudodifferential setting. Define

$$[\![u]\!]_{s,k,\omega} = \sum_{\alpha=0}^{k} k^{s(k-\alpha)} \|D^\alpha u\|_{0,\omega}. \tag{B-15}$$

Note that $[\![u]\!]_{1,k,\omega} = |\!|\!| u |\!|\!|_{k,\omega}$.

Theorem B.10 [Métivier 1980, Theorem 3.1]. *Let $P(x, D)$ be a real analytic pseudodifferential operator. Let $x_0 \in \mathbb{R}^n$ and let Ω denote a neighborhood of x_0. Let $x_0 \in U \Subset \Omega$ be an open set.*

Assume that there is a bounded operator $R: L^2(U) \to L^2(K)$, where K is a suitable compact subset of Ω, such that $(PRu)_{|U} = u_{|U}$. Here $L^2(K)$ denotes the set of all functions in $L^2(\Omega)$ whose support is contained in K.

The operator P is Gevrey s-hypoelliptic at x_0 if and only if:

(i) *For any neighborhood ω of x_0, there exists a neighborhood $\omega'' \Subset \omega$ such that $Pu_{|\omega} \in H^k(\omega)$ implies $u_{|\omega''} \in H^k(\omega'')$.*

(ii) *For any neighborhoods of x_0 $\omega^{iv} \Subset \omega''' \Subset \omega''$, there are positive constants C, L such that*

$$\|u\|_{k,\omega^{iv}} \le CL^k \left([\![P(\varphi u)]\!]_{s,k,\omega'''} + k!^s \|u\|_{0,\omega''} \right) \tag{B-16}$$

for any $u \in \mathscr{E}'(\Omega)$ where $\varphi \in C_0^\infty(\omega'')$, $\varphi \equiv 1$ on a neighborhood of ω''', and C, L are independent of k and u. Here $\|u\|_{k,\omega}$ denotes the usual Sobolev norm of order k on the open set ω.

Remark B.11.

1. Since P has an analytic symbol,

$$\text{sing supp}_\omega \left[(P(\varphi u))_{|\overline{\omega'''}} - (Pu)_{|\overline{\omega'''}} \right] = \varnothing.$$

2. It is not difficult to show that the operator Λ of Section 6 has a local right inverse as in the statement by using Theorem 3.4 of [Kumano-go 1982] and Theorem 4.3.

3. For the limited purpose of this paper, a weaker result would have been enough. We are allowed to have the constants C, L depending on u but not on k. This is much easier to prove and we do not need for this Lemma B.12.

Proof. If (i) and (ii) hold, then clearly P is Gevrey s-hypoelliptic at x_0.

Conversely, assume that P is G^s-hypoelliptic at the point x_0. First we prove (i).

Let $\omega \subset \Omega$ be an open neighborhood of x_0. We choose an open subset $\omega_1 \Subset \omega$, $x_0 \in \omega_1$, and cutoff functions $\chi_k \in C_0^\infty(\omega)$, $k \in \mathbb{N}$, as in Lemma B.5 such that (B-1) is satisfied and $\chi_k \equiv 1$ on ω_1.

Let $u \in \mathscr{E}'(\Omega)$ and assume that $(Pu)_{|\omega} \in H^k(\omega)$, $k \in \mathbb{N}$.

Set

$$f = \chi_k Pu.$$

Clearly f is defined on the whole of $\mathbb{R}^n$, and more precisely $f \in H^k(\mathbb{R}^n)$. Applying Lemma B.6 to the function f, we obtain that

$$|\!|\!| f |\!|\!|_{k,\mathbb{R}^n} \le \gamma^k |\!|\!| Pu |\!|\!|_{k,\omega}, \tag{B-17}$$

for a suitable positive constant γ independent of k.

Furthermore, applying Lemma B.8 to the same function f, we write

$$f = \sum_{j=0}^{\infty} f_j,$$

with $f_j \in G_{(s)}$ (see (B-6) for a definition of $G_{(s)}$), and the following inequality holds:

$$\sum_{j=0}^{\infty} N_j^{2ks}\big(\|f_j\|^2_{0,\mathbb{R}^n} + e^{-2N_j}\|f_j\|^2_{G_{(s)}}\big) \le 2(2\gamma_1)^k |||f|||^2_{k,\mathbb{R}^n}. \tag{B-18}$$

Denote by $\tilde{G}_{(s)}$ the space of all restrictions to U of the functions in $G_{(s)}$ compactly supported in U, and let $B_{(s)} = R\tilde{G}_{(s)}$ equipped with the norm defined by $\|R(g_{|U})\|_{B_{(s)}} = \|g\|_{G_{(s)}}$. Fix an open neighborhood $U' \Subset U$ of x_0 and choose a Gevrey cutoff function $\psi \in C_0^\infty(U)$, $0 \le \psi \le 1$, of Gevrey order s', $1 < s' < s$, such that $\psi_{|U'} \equiv 1$. Set

$$v_j = R(\psi f_{j\,|_U}). \tag{B-19}$$

Clearly $v_j \in B_{(s)}$ and v_j is a function of class G^s near the point x_0. In fact $(Pv_j)_{|U} = (PR(\psi f_{j\,|_U}))_{|U} = (\psi f_j)_{|_U}$. The latter is a Gevrey function of order s and, since P is G^s-hypoelliptic, we conclude. We have the inequality

$$\sum_{j=0}^{\infty} N_j^{2ks}\big(\|v_j\|^2_{0,\Omega} + e^{-2N_j}\|v_j\|^2_{B_{(s)}}\big) \le 2(C_\psi + \|R\|^2)(2\gamma_1)^k |||f|||^2_{k,\mathbb{R}^n}. \tag{B-20}$$

Here C_ψ is a positive constant only depending on ψ and $\|R\|$ denotes the norm of the operator R as an operator from $L^2(U)$ into $L^2(K)$.

Using (B-20) and Lemma B.9, we obtain that the series $\sum_{j=0}^\infty v_j$ converges in $L^2(K)$. Denote by v its sum. From the same lemma, we also get that there is an open set $\omega' \Subset \omega$ such that

$$v_{|_{\omega'}} \in H^k(\omega')$$

and

$$\|v\|_{k,\omega'} \le C^{k+1} |||f|||_{k,\mathbb{R}^n}, \tag{B-21}$$

for a suitable positive constant C. Observe that we may, possibly shrinking ω' as necessary, suppose that $\omega' \subset \omega_1$. Consider now the function $(P(u-v))_{|_{\omega'}}$. Due to that choice of ω', we evidently have $(Pu)_{|_{\omega'}} = f_{|_{\omega'}}$. Then remark that, since

$$v = \sum_{j=0}^{\infty} R(\psi f_{j\,|_U}) = R\Big(\sum_{j=0}^{\infty} \psi f_{j\,|_U}\Big) = R(\psi f_{|_U}),$$

we have that P can be applied to v and $(Pv)_{|_{U'}} = f_{|_{U'}}$. Possibly shrinking the open set ω' so that $\omega' \subset U'$, we have in particular that

$$\big(P(u-v)\big)_{|_{\omega'}} = 0. \tag{B-22}$$

Note that because of the hypoellipticity assumption for P, we deduce that $u - v \in G^s(\omega')$. Furthermore, taking $\omega'' \Subset \omega'$, we obtain that $u - v \in G^s(\overline{\omega''})$. This proves assertion (i).

Next we prove (ii). In order to do that, we need to further shrink the neighborhoods of x_0 involved, in such a way that we already know that in that neighborhood u belongs to H^k and is compactly supported. Actually we proved that $u_{|_{\omega''}} \in H^k(\omega'')$. Let $\omega''' \Subset \omega''$ and choose cutoff functions $\tilde{\chi}_k \in C_0^\infty(\omega''')$, such that $\tilde{\chi}_k \equiv 1$ in $\omega_1''' \Subset \omega'''$. Let $\tilde{f} = \tilde{\chi}_k Pu$. Let also $\varphi \in C_0^\infty(\omega'')$, $\varphi \equiv 1$ on $\tilde{\omega}''' \Supset \omega'''$. Note that $\varphi u \in H^k(\mathbb{R}^n)$ and its support is contained in ω''. Due to the pseudolocality property of P, we have

$$\text{sing supp}_\omega \big((P(\varphi u))_{|_{\tilde{\omega}'''}} - (Pu)_{|_{\tilde{\omega}'''}}\big) = \varnothing,$$

and in particular we have $(P(\varphi u))_{|_{\omega'''}} \in H^k(\omega''')$. This implies in turn that $\tilde{\chi}_k P(\varphi u) \in H^k(\mathbb{R}^n)$.

Arguing as above, and possibly enlarging the compact set $K \subset \Omega$, we obtain that $(P(\varphi u - \tilde{v}))_{|_{\omega'''}} = 0$ and $\varphi u - \tilde{v} \in L^2(K) \cap G^s(\omega''')$. Recall that here $L^2(K)$ denotes the set of all functions in $L^2(\Omega)$ whose support is contained in K.

Lemma B.12. *Let X denote the space of all the functions $u \in G^s(\omega''') \cap L^2(K)$ such that $(Pu)_{|_{\omega'''}} = 0$. Equipped with the $L^2(\Omega)$ norm, X becomes a Banach space. Then for every $\omega^{iv} \Subset \omega'''$, there exists a constant $C_2 > 0$, such that for any multi-index α,*

$$\sup_{\omega^{iv}} |\partial^\alpha u(x)| \le C_2^{|\alpha|} \alpha!^s \|u\|_{0,K}, \tag{B-23}$$

for every $u \in X$.

Applying the lemma, we immediately get that for any $\omega^{iv} \Subset \omega'''$ and for any $k \in \mathbb{N}$, we have

$$\|\varphi u - \tilde{v}\|_{k,\omega^{iv}} \le C_2^{k+1} k!^s \|\varphi u - \tilde{v}\|_{0,K}. \tag{B-24}$$

On the other hand, we also have

$$\|\tilde{v}\|_{0,\omega^{iv}} \le \|R\| \|\tilde{f}\|_{0,\mathbb{R}^n} \le \|R\| \|P(\varphi u)\|_{0,\omega'''}$$

and

$$k!^s \|P(\varphi u)\|_{0,\omega'''} \le [\![P(\varphi u)]\!]_{s,k,\omega'''},$$

as well as

$$|\!|\!| u |\!|\!|_{k,\omega'''} \le [\![u]\!]_{s,k,\omega'''}, \quad s \ge 1.$$

Thus

$$\begin{aligned}
\|\varphi u\|_{k,\omega^{iv}} &\le \|\varphi u - \tilde{v}\|_{k,\omega^{iv}} + \|\tilde{v}\|_{k,\omega^{iv}} \\
&\le C_2^{k+1} k!^s \|\varphi u - \tilde{v}\|_{0,K} + C^{k+1} |\!|\!| \tilde{f} |\!|\!|_{k,\mathbb{R}^n} \\
&\le C^{k+1}\gamma^k |\!|\!| P(\varphi u) |\!|\!|_{k,\omega'''} + C_2^{k+1} k!^s \big(\|\tilde{v}\|_{0,K} + \|\varphi u\|_{0,K}\big) \\
&\le C^{k+1}\gamma^k |\!|\!| P(\varphi u) |\!|\!|_{k,\omega'''} + C_2^{k+1} k!^s \|R\| \|P(\varphi u)\|_{0,\omega'''} + C_2^{k+1} k!^s \|\varphi u\|_{0,K} \\
&\le C^{k+1}\gamma^k |\!|\!| P(\varphi u) |\!|\!|_{k,\omega'''} + C_2^{k+1} \|R\| [\![P(\varphi u)]\!]_{s,k,\omega'''} + C_2^{k+1} k!^s \|u\|_{0,\omega''} \\
&\le C_3 L^k \big([\![P(\varphi u)]\!]_{s,k,\omega'''} + k!^s \|u\|_{0,\omega''}\big).
\end{aligned}$$

This proves the theorem. $\square$

Proof of Lemma B.12. It is an application of the Baire category theorem. For $j \in \mathbb{N}$ and for a certain $\omega^{iv} \Subset \omega'''$, define

$$X_j = \{u \in X \mid |\partial^\alpha u(x)| \le j^{|\alpha|+1}\alpha!^s, \text{ for all } \alpha \text{ and all } x \in \overline{\omega^{iv}}\}.$$

Trivially,

$$X = \bigcup_{j=1}^{\infty} X_j.$$

Next we show that the sets X_j are closed with respect to the $L^2(\Omega)$ topology of X. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X_j converging to $u_0 \in X$. As a consequence, the derivatives $\partial^\alpha u_n$ are uniformly bounded in $\overline{\omega^{iv}}$ so that the functions $\partial^\beta u_n$ are equicontinuous if $|\beta| < |\alpha|$. Applying the Arzelà–Ascoli theorem, we obtain that for any $l \in \mathbb{N}$, there exists a subsequence $u_{n_r,l}$ converging in $C^{(l)}(\overline{\omega^{iv}})$ to an element $u_{(l)} \in C^{(l)}(\overline{\omega^{iv}})$. Hence $u_0 = u_{(l)}$ in $\overline{\omega^{iv}}$ and

$$|\partial^\alpha u_0(x)| \le j^{|\alpha|+1}\alpha!^s, \text{ for all } |\alpha| \le l \text{ and all } x \in \overline{\omega^{iv}}.$$

This implies $u_0 \in X_j$. By the Baire category theorem, there are an index j_0, a number $\varepsilon > 0$, and a function $\tilde{u} \in X_{j_0}$ such that

$$B = \{u \in X \mid \|u - \tilde{u}\|_{0,K} \le \varepsilon\} \subset X_{j_0}, \tag{B-25}$$

where we wrote $\|\ \|_{0,K}$ since the support of u, $\tilde{u}$ is contained in K. Now for every $u \in X$, let

$$v = \delta \frac{u}{\|u\|_{0,K}} + \tilde{u} \in B, \quad \text{if } |\delta| < \varepsilon.$$

Thus

$$u = \frac{\|u\|_{0,K}}{\delta}(v - \tilde{u})$$

and

$$\left|\partial^\alpha u(x)\right| \le \frac{\|u\|_{0,K}}{\delta}\left(|\partial^\alpha v| + |\partial^\alpha \tilde{u}|\right) \le R^{|\alpha|+1}\alpha!^s \|u\|_{0,K}.$$

This proves the lemma. □

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POROUS MEDIA: THE MUSKAT PROBLEM IN THREE DIMENSIONS

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The Muskat problem involves filtration of two incompressible fluids through a porous medium. We consider the problem in three dimensions, discussing the relevance of the Rayleigh–Taylor condition and the topology of the initial interface, in order to prove the local existence of solutions in Sobolev spaces.

1. Introduction

The Muskat problem [Muskat and Wickoff 1937; Bear 1972] involves filtration of two incompressible fluids through a porous medium, characterized by a positive constant κ quantifying its porosity and permeability. The two fluids, having velocity fields v^1 and v^2, occupy disjoint regions D^1 and $D^2 = \mathbb{R}^3 - D^1$, with a common boundary (interface) given by the surface $S = \partial D^1 = \partial D^2$. Naturally, those domains change with time, as does the interface. We denote by p^j $(j = 1, 2)$ the corresponding pressures, and we will also assume that the dynamical viscosities μ^j and the densities ρ^j are constants with $\mu^1 \neq \mu^2$, $\rho^1 \neq \rho^2$.

Conservation of mass in this setting is given by the equation $\nabla \cdot v = 0$ (in the distribution sense), where $v = v^1 \chi_{D^1} + v^2 \chi_{D^2}$.

The momentum equation, obtained experimentally by Darcy [1856] (see also [Bear 1972]), is

$$\frac{\mu^j}{\kappa} v^j = -\nabla p^j - (0, 0, \rho^j g), \qquad j = 1, 2,$$

where g is the acceleration due to gravity.

One can find in the literature several attempts to derive Darcy's law from Navier–Stokes [Tartar 1980; Sánchez-Palencia and Zaoui 1987] through the process of homogenization under the hypothesis of a periodic, or almost periodic, porosity. In any case, the presence of the porous medium justifies the elimination of the inertial terms in the motion, leaving friction (viscosity) and gravity as the only relevant forces, to which one has to add pressure as it appears in the formulation of Darcy's law. There are three scales involved in the analysis: the macroscopic or bulk mass, the microscopic size of the fluid particle, and the mesoscopic scale corresponding to the pores. In the references above, one finds descriptions of the velocity v as an average over the mesoscopic cells of the fluid particle velocities. Taking into account that each cell contains a solid part where the particle velocity vanishes, it is then natural to get the viscous

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forces associated to that average velocity, which is a scaled approximation of the laplacian term appearing in the Navier–Stokes equation.

In this paper, we shall consider the case of a homogeneous and isotropic porous material. Porosity is the fraction of the volume occupied by pores or void space. But it is important to distinguish between two kinds of pores — the kind that forms a continuous interconnected phase within the medium, and the kind that is isolated — because non-interconnected pores cannot contribute to fluid transport. Permeability is the term used to describe the conductivity of a porous medium with respect to a newtonian fluid, and it depends upon the properties of the medium and the fluid. Darcy's law indicates this dependence, allowing us to define the notion of specific permeability κ and its units. In the case of an anisotropic material, κ will be a symmetric and positive definite tensor, and the methods of our proof can be modified to get local existence; but for a nonhomogeneous medium, the properties of the tensor $\kappa(x)$ will have to be specified in a very precise manner in order to allow an interesting theory.

The Muskat problem and related problems [Saffman and Taylor 1958] have been studied recently [Constantin and Pugh 1993; Siegel et al. 2004; Córdoba and Gancedo 2007; 2009; Córdoba et al. 2011]. The first natural question is about the evolution of the system (existence of solutions), at least for a short time $t > 0$, and the persistence of smoothness of the interface $S(t)$ if we begin with a smooth enough surface at time $t = 0$. One can easily deduce from this formulation that in the event of smooth evolution, both pressures can be taken to be equal at the interface:

$$p^1|_{S(t)} = p^2|_{S(t)}.$$

Therefore, we look at the case without surface tension (see [Escher and Simonett 1997], where the regularizing effect of surface tension is considered). The normal component of the velocity fields must also agree at the free boundary; that is, if ν^j is the unit normal to S pointing into D^j, we have

$$(v^1 - v^2) \cdot \nu^j = 0 \quad \text{at } S(t), \qquad j = 1, 2$$

(note that $\nu^2 = -\nu^1$). Furthermore, the vorticity will be concentrated at the interface, having the form

$$\mathrm{curl}(v) = \omega(z)\, dS(z),$$

where ω is tangent to S at the point z and $dS(z)$ is surface measure.

This paper extends to the three-dimensional case the results obtained in [Córdoba et al. 2011] for the case of two dimensions, by proving local existence in the scale of Sobolev spaces of the initial value problem if the Rayleigh–Taylor (R-T) condition is initially satisfied (see [Saffman and Taylor 1958], where this issue is studied from a physical point of view). In our case, that condition amounts to the positivity of the function

$$\sigma = (\nabla p^2 - \nabla p^1) \cdot (\nu^2 - \nu^1)$$

at the interface S. The R-T property also appears in other fluid interface problems, such as water waves [Cordoba et al. 2009].

Together with that hypothesis, one also assumes that the initial surface S is connected and simply connected. In the presence of a global parametrization $X : \mathbb{R}^2 \to S$, the preservation of that character will be controlled by the gauge

$$F(X)(\alpha,\beta)=\frac{|\alpha-\beta|}{|X(\alpha)-X(\beta)|},\qquad \|F(X)\|_{L^\infty}=\sup_{\alpha\neq\beta}\frac{|\alpha-\beta|}{|X(\alpha)-X(\beta)|}<\infty.$$

Section 2 of this paper contains the derivation of the evolution equations for the interface S. In Section 3, we prove the existence of global isothermal parametrization as a consequence of the Koebe–Poincaré uniformization theorem of Riemann surfaces in the geometric scenarios considered in our work, namely, double periodicity in the horizontal variables and asymptotic flatness. Let us add that given the nonlocal character of the operator involved, to obtain a global isothermal parametrization is an important step in the proof, whose main components are sketched in Section 4.

In closing our system (Section 2), we need to control the norm of the inverse operator $(I+\lambda\mathfrak{D})^{-1}$, where $\mathfrak{D}$ is the double-layer potential and $|\lambda|\leq 1$. It is well-known from Fredholm's theory that those operators are bounded on $L^2(S)$. However, since the surface $S=S(t)$ is moving, a precise control of its norm is needed in order to proceed with our proof. That is the purpose of Section 5, where the estimates for the double-layer potential are revisited.

In Sections 6 and 7, we develop the energy estimates needed to conclude local existence. Let us mention that at a crucial point (more precisely, just at that step where the positivity of $\sigma(\alpha,t)$ (R-T) plays its role), we use the pointwise estimate $\theta(x)\Lambda\theta(x)\geq\frac{1}{2}\Lambda\theta^2(x)$ of [Córdoba and Córdoba 2003], with $\Lambda=\sqrt{-\Delta}$.

In the strategy of our proof, it is crucial to analyze the evolution of both quantities σ and F (Section 8) at the same time as the interface X and vorticity ω. There are several publications (see, for example, [Ambrose 2007]) where different authors have treated these problems assuming that the Rayleigh–Taylor condition is preserved during the evolution. Under such a hypothesis the proof can be considerably simplified, especially if one also assumes the appropriate bounds for the resolvent of the double-layer potential with respect to a moving domain, or the existence of global isothermal coordinates, etc. It is our purpose to carefully go over such items, which are responsible for the more delicate and intricate parts of this paper.

2. The contour equation

We consider the following evolution problem for the active scalars $\rho=\rho(x,t)$ and $\mu=\mu(x,t)$, with $x\in\mathbb{R}^3$ and $t\geq 0$:

$$\rho_t+v\cdot\nabla\rho=0,$$
$$\mu_t+v\cdot\nabla\mu=0,$$

with a velocity $v=(v_1,v_2,v_3)$ satisfying the momentum equation

$$\mu v=-\nabla p-(0,0,\ \rho) \tag{2-1}$$

and the incompressibility condition $\nabla\cdot v=0$, where, without loss of generality, we have prescribed the values $\kappa=\mathrm{g}=1$.

The vector (μ,ρ) is defined by

$$(\mu,\rho)(x_1,x_2,x_3,t)=\begin{cases}(\mu^1,\rho^1) & x\in D^1(t),\\ (\mu^2,\rho^2) & x\in D^2(t)=\mathbb{R}^3\setminus D^1(t),\end{cases}$$

where $\mu^1\neq\mu^2$ and $\rho^1\neq\rho^2$. Darcy's law (2-1) implies that the fluid is irrotational in the interior of each domain D^j, and because of the jump of densities and viscosities on the free boundary, we may assume a velocity field such that

$$\operatorname{curl} v=\omega(\alpha,t)\delta(x-X(\alpha,t)),$$

where $\partial D^j(t)=\{X(\alpha,t)\in\mathbb{R}^3:\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2\}$; that is,

$$\langle\operatorname{curl} v,\varphi\rangle=\int_{\mathbb{R}^2}\omega(\alpha,t)\cdot\varphi(X(\alpha,t))\,d\alpha, \tag{2-2}$$

for any $\varphi:\mathbb{R}^3\to\mathbb{R}^3$ vector field in $C_c^\infty(\mathbb{R}^3)$.

The incompressibility hypothesis ($\langle\nabla\cdot v,\varphi\rangle\equiv-\langle v,\nabla\varphi\rangle=0$, for any $\varphi\in C_c^\infty(\mathbb{R}^3)$), yields

$$v^1(X(\alpha,t),t)\cdot N(\alpha,t)=v^2(X(\alpha,t),t)\cdot N(\alpha,t),$$

with $N(\alpha,t)=\partial_{\alpha_1}X(\alpha,t)\wedge\partial_{\alpha_2}X(\alpha,t)$, and Equation (2-2) gives us the identity

$$\omega(\alpha,t)=\big(v^2(X(\alpha,t),t)-v^1(X(\alpha,t),t)\big)\wedge N(\alpha,t).$$

Defining the potential ϕ by $v(x,t)=\nabla\phi(x,t)$ for $x\in\mathbb{R}^2\setminus\partial D^j(t)$, we get

$$\begin{aligned}\Omega(\alpha,t)&=\phi^2(X(\alpha,t),t)-\phi^1(X(\alpha,t),t),\\ \partial_{\alpha_1}\Omega(\alpha,t)&=\big(v^2(X(\alpha,t),t)-v^1(X(\alpha,t),t)\big)\cdot\partial_{\alpha_1}X,\\ \partial_{\alpha_2}\Omega(\alpha,t)&=\big(v^2(X(\alpha,t),t)-v^1(X(\alpha,t),t)\big)\cdot\partial_{\alpha_2}X.\end{aligned}$$

Then one has the equality

$$\omega(\alpha,t)=\big(v^2(X(\alpha,t),t)-v^1(X(\alpha,t),t)\big)\wedge\big(\partial_{\alpha_1}X(\alpha,t)\wedge\partial_{\alpha_2}X(\alpha,t)\big),$$

and therefore

$$\omega(\alpha,t)=\partial_{\alpha_2}\Omega(\alpha,t)\partial_{\alpha_1}X(\alpha,t)-\partial_{\alpha_1}\Omega(\alpha,t)\partial_{\alpha_2}X(\alpha,t), \tag{2-3}$$

implying that $\nabla\cdot\operatorname{curl} v=0$ in a weak sense.

Using the law of Biot–Savart, we have for x not lying in the free surface ($x\neq X(\alpha,t)$) the following expression for the velocity:

$$v(x,t)=-\frac{1}{4\pi}\int_{\mathbb{R}^2}\frac{x-X(\beta,t)}{|x-X(\beta,t)|^3}\wedge\omega(\beta)\,d\beta.$$

It follows that

$$X_t(\alpha)=\mathrm{BR}(X,\omega)(\alpha,t)+C_1(\alpha)\partial_{\alpha_1}X(\alpha)+C_2(\alpha)\partial_{\alpha_2}X(\alpha), \tag{2-4}$$

where BR is the well-known Birkhoff–Rott integral:

$$\mathrm{BR}(X,\omega)(\alpha,t)=-\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{X(\alpha)-X(\beta)}{|X(\alpha)-X(\beta)|^3}\wedge\omega(\beta)\,d\beta. \tag{2-5}$$

Next we will close the system using Darcy's law. Since

$$\nabla\phi = v(x,t) - \Omega(\alpha,t)N(\alpha,t)\delta(x - X(\alpha,t)),$$

we have

$$\langle\Delta\phi,\varphi\rangle = -\langle\nabla\phi,\nabla\varphi\rangle = \int_{\mathbb{R}^2}\Omega(\alpha,t)N(\alpha,t)\cdot\nabla\varphi(X(\alpha,t))\,d\alpha,$$

and taking $\varphi(y) = -1/(4\pi|x-y|)$, one obtains ϕ in terms of the double layer potential:

$$\phi(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^2}\frac{x - X(\alpha)}{|x - X(\alpha)|^3}\cdot N(\alpha)\Omega(\alpha)\,d\alpha.$$

Darcy's law yields

$$\Delta p(x,t) = -\operatorname{div}(\mu(x,t)v(x,t)) - \partial_{x_3}\rho(x,t),$$

that is,

$$\Delta p(x,t) = P(\alpha,t)\delta(x - X(\alpha,t)),$$

where $P(\alpha,t)$ is given by

$$P(\alpha,t) = (\mu^2 - \mu^1)v(X(\alpha,t),t)\cdot N(\alpha,t) + (\rho^2 - \rho^1)N_3(\alpha,t),$$

implying the continuity of the pressure at the free boundary.

Next, if $x \neq X(\alpha,t)$, i.e., x is not placed at the interface, we can write Darcy's law in the form

$$\mu\phi(x,t) = -p(x,t) - \rho x_3,$$

and taking limits in both domains D^j, we get at S the equality

$$\big(\mu^2\phi^2(X(\alpha,t),t) - \mu^1\phi^1(X(\alpha,t),t)\big) = -(\rho^2 - \rho^1)X_3(\alpha,t).$$

Then the formula for the double-layer potential gives

$$\frac{\mu^2+\mu^1}{2}\Omega(\alpha,t) - (\mu^2 - \mu^1)\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{X(\alpha)-X(\beta)}{|X(\alpha)-X(\beta)|^3}\cdot N(\beta)\Omega(\beta)\,d\beta = -(\rho^2 - \rho^1)X_3(\alpha,t),$$

that is,

$$\Omega(\alpha,t) - A_\mu\mathfrak{D}(\Omega)(\alpha,t) = -2A_\rho X_3(\alpha,t), \tag{2-6}$$

where

$$\mathfrak{D}(\Omega)(\alpha) = \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{X(\alpha)-X(\beta)}{|X(\alpha)-X(\beta)|^3}\cdot N(\beta)\Omega(\beta)\,d\beta, \quad A_\mu = \frac{\mu^2-\mu^1}{\mu^2+\mu^1}, \quad A_\rho = \frac{\rho^2-\rho^1}{\mu^2+\mu^1}. \tag{2-7}$$

The evolution equations are then given by (2-3)–(2-7), where the functions C_1 and C_2 will be chosen in the next section.

Furthermore, taking limits, we get from Darcy's law the following two formulas:

$$\partial_{\alpha_1}\Omega(\alpha,t) + 2A_\mu\,\mathrm{BR}(X,\omega)(\alpha,t)\cdot\partial_{\alpha_1}X(\alpha,t) = -2A_\rho\partial_{\alpha_1}X_3(\alpha,t), \tag{2-8}$$

$$\partial_{\alpha_2}\Omega(\alpha,t) + 2A_\mu\,\mathrm{BR}(X,\omega)(\alpha,t)\cdot\partial_{\alpha_2}X(\alpha,t) = -2A_\rho\partial_{\alpha_2}X_3(\alpha,t). \tag{2-9}$$

3. Isothermal parametrization: choosing the tangential terms

Although the normal component of the velocity vector field is the relevant one in the evolution of the interface, it is however very important to choose an adequate parametrization in order to uncover and handle properly the cancellations contained in the equations of motion. Fortunately for our task, we can rely upon the ideas of H. Lewy [1951], and many other authors, who discovered the convenience of using isothermal coordinates in different PDEs for understanding how a minimal surface leaves an obstacle and also in several fluid mechanical problems.

Let us recall that an isothermal parametrization must satisfy

$$|X_{\alpha_1}(\alpha,t)|^2 = |X_{\alpha_2}(\alpha,t)|^2, \qquad X_{\alpha_1}(\alpha,t)\cdot X_{\alpha_2}(\alpha,t) = 0,$$

for $t \geq 0$.

Next we define

$$\begin{aligned} C_1(\alpha) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\alpha_1-\beta_1}{|\alpha-\beta|^2}\,\frac{\mathrm{BR}_{\beta_2}\cdot X_{\beta_2} - \mathrm{BR}_{\beta_1}\cdot X_{\beta_1}}{|X_{\beta_2}|^2}\,d\beta - \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\alpha_2-\beta_2}{|\alpha-\beta|^2}\,\frac{\mathrm{BR}_{\beta_1}\cdot X_{\beta_2} + \mathrm{BR}_{\beta_2}\cdot X_{\beta_1}}{|X_{\beta_1}|^2}\,d\beta \end{aligned} \tag{3-1}$$

and

$$\begin{aligned} C_2(\alpha) &= -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\alpha_2-\beta_2}{|\alpha-\beta|^2}\,\frac{\mathrm{BR}_{\beta_2}\cdot X_{\beta_2} - \mathrm{BR}_{\beta_1}\cdot X_{\beta_1}}{|X_{\beta_2}|^2}\,d\beta - \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\alpha_1-\beta_1}{|\alpha-\beta|^2}\,\frac{\mathrm{BR}_{\beta_1}\cdot X_{\beta_2} + \mathrm{BR}_{\beta_2}\cdot X_{\beta_1}}{|X_{\beta_1}|^2}\,d\beta. \end{aligned} \tag{3-2}$$

That is, $X_t = \mathrm{BR} + C_1 X_{\alpha_1} + C_2 X_{\alpha_2}$ and

$$\begin{aligned} X_{\alpha_1 t} &= \mathrm{BR}_{\alpha_1} + C_1 X_{\alpha_1\alpha_1} + C_2 X_{\alpha_1\alpha_2} + C_{1\alpha_1} X_{\alpha_1} + C_{2\alpha_1} X_{\alpha_2},\\ X_{\alpha_2 t} &= \mathrm{BR}_{\alpha_2} + C_1 X_{\alpha_1\alpha_2} + C_2 X_{\alpha_2\alpha_2} + C_{1\alpha_2} X_{\alpha_1} + C_{2\alpha_2} X_{\alpha_2}. \end{aligned}$$

Writing $f = \big(|X_{\alpha_1}|^2 - |X_{\alpha_2}|^2\big)/2$ and $g = X_{\alpha_1}\cdot X_{\alpha_2}$, we have

$$f_t = \big(\mathrm{BR}_{\alpha_1}\cdot X_{\alpha_1} - \mathrm{BR}_{\alpha_2}\cdot X_{\alpha_2}\big) + C_1 f_{\alpha_1} + C_2 f_{\alpha_2} + (C_{2\alpha_1} - C_{1\alpha_2})g + 2C_{1\alpha_1} f + (C_{1\alpha_1} - C_{2\alpha_2})|X_{\alpha_2}|^2.$$

The expressions for C_1 and C_2 yield the vanishing of the sum of the first and the last terms in the identity above. Therefore, we get

$$f_t = C_1 f_{\alpha_1} + C_2 f_{\alpha_2} + (C_{2\alpha_1} - C_{1\alpha_2})g + 2C_{1\alpha_1} f. \tag{3-3}$$

Similarly, we have

$$g_t = \big(\mathrm{BR}_{\alpha_2}\cdot X_{\alpha_1} + \mathrm{BR}_{\alpha_1}\cdot X_{\alpha_2}\big) + C_1 g_{\alpha_1} + C_2 g_{\alpha_2} + (C_{1\alpha_1} + C_{2\alpha_2})g - 2C_{2\alpha_1} f + (C_{1\alpha_2} + C_{2\alpha_1})|X_{\alpha_1}|^2$$

and

$$g_t = C_1 g_{\alpha_1} + C_2 g_{\alpha_2} + (C_{1\alpha_1} + C_{2\alpha_2})g - 2C_{2\alpha_1} f. \tag{3-4}$$

The linear character of equations (3-3) and (3-4) allows us to conclude that if there is a solution of the system $X_t = \mathrm{BR} + C_1 X_{\alpha_1} + C_2 X_{\alpha_2}$ and we start with isothermal coordinates at time $t = 0$, then they will continue to be isothermal so long as the evolution equations provide us with a smooth enough interface.

The fact that one can always prescribe such coordinates at time $t = 0$ follows from the following argument: in the double periodic setting we have a C^2 simply connected surface, homeomorphic to the euclidean plane $\mathbb{R}^2$, which, by the Riemann–Koebe–Poincaré uniformization theorem, is conformally equivalent to either the Riemann sphere, the plane, or the unit disc. The sphere is easily eliminated by compactness, but we can also rule out the unit disc because the assumption of double periodicity in the horizontal variables implies the existence of a discrete abelian subgroup of rank two in the group of conformal transformations, and that cannot happen in the case of the unit disc.

Therefore, we have an orientation-preserving conformal (isothermal) equivalence

$$\phi : \mathbb{R}^2 \longrightarrow S.$$

Since S is invariant under translations $\tau_\nu(x) = x + 2\pi\nu$, where $\nu \in \mathbb{Z}^2 \times \{0\}$, it follows that $f_\nu(z) = \phi^{-1} \circ \tau_\nu \circ \phi(z)$ must be a diffeoholomorphism of $\mathbb{C} = \mathbb{R}^2$, and therefore it has to be of the form

$$f_\nu(z) = a_\nu z + b_\nu,$$

for certain $a_\nu, b_\nu \in \mathbb{C}$. Clearly, the family f_ν is generated by $f_1 = f_{(1,0,0)}$, $f_2 = f_{(0,1,0)}$. Let

$$f_1(z) = a_1 z + b_1, \qquad f_2(z) = a_2 z + b_2.$$

We claim that $a_1 = a_2 = 1$. Suppose that $|a_1| < 1$; then we get $f_1^n(z) = a_1^n z + b_1(1 + a_1 + \cdots + a_1^{n-1})$, a sequence converging to $b_1/(1 - a_1)$, contradicting the discrete character of the group action. On the other hand, if $|a_1| > 1$, then since

$$f_1^{-1}(z) = f_{(-1,0,0)}(z) = \frac{z}{a_1} - \frac{b_1}{a_1},$$

we get a contradiction with the sequence $f_1^{-n}(z)$. Therefore, we must have $a_1 = e^{2\pi i\theta}$ for some $0 \le \theta < 1$. Assume that $0 < \theta < 1$; then

$$f_1^{(n)}(z) = e^{2\pi i n\theta} z + b_1\big(1 + e^{2\pi i\theta} + \cdots + e^{2\pi i(n-1)\theta}\big) = e^{2\pi i n\theta} z + b_1 \frac{1 - e^{2\pi i n\theta}}{1 - e^{2\pi i\theta}},$$

so the sequence $f^n(z)$ is bounded and satisfies $|f^n(z)| \le |z| + |b_1|/\sin \pi\theta$. Therefore it contains a converging subsequence, again contradicting discreteness. It follows that $f_1(z) = z + b_1$ and, similarly, $f_2(z) = z + b_2$, which leads easily to the double periodicity of the isothermal parametrization ϕ.

In the asymptotically flat case, we start with an orientable simply connected surface S that, outside a ball B in $\mathbb{R}^3$, is the graph of a C^2-function $x_3 = \varphi(x_1, x_2)$ such that $|D^\alpha \varphi(x)| = o(|x|^{-N})$ for every N and $|\alpha| \le 2$. In particular, the normal vector $\nu(x) = (-\nabla\varphi, 1)/\sqrt{1 + |\nabla\varphi|^2}$ is roughly vertical and $1/\sqrt{1 + |\nabla\varphi|^2}$ is close to 1 for $|x|$ big enough.

Then one can find isothermal coordinates whose first fundamental form $\lambda(\alpha, \beta)(d\alpha^2 + d\beta^2)$ converges asymptotically to the identity.

Again by the uniformization theorem, S must be conformally equivalent to either $\mathbb{C}$ or the unit disc. But since outside B, the surface S is conformally equivalent to $\mathbb{C} - B \cap \{x_3 = 0\}$, it cannot be also conformally equivalent to $D - K$, for any regular compact set K contained in the unit disc D, because the harmonic measure of the ideal boundary is 1 in the case of D and 0 for $\mathbb{R}^2$.

4. Main theorem and outline of the proof

The proof of local existence requires the following:

(1) A connected and simply connected surface $S = S(t)$ parametrized by isothermal coordinates

$$X : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \qquad X = X(\alpha, t),$$

with normal vector $N(\alpha, t) = X_{\alpha_1} \wedge X_{\alpha_2}$ and gauge

$$F(X)(\alpha, \beta) = \frac{|\beta|}{|X(\alpha) - X(\alpha-\beta)|},$$

such that $\|F(X)\|_{L^\infty} < \infty$ and $\||N|^{-1}\|_{L^\infty} < \infty$.

(2) The positivity of

$$\begin{aligned} \sigma(\alpha, t) &= -\big(\nabla p^2(X(\alpha, t), t) - \nabla p^1(X(\alpha, t), t)\big) \cdot N(\alpha, t) \\ &= (\mu^2 - \mu^1)\,\mathrm{BR}(X, \omega)(\alpha, t) \cdot N(\alpha, t) + (\rho^2 - \rho^1) N_3(\alpha, t), \end{aligned} \tag{4-1}$$

where the last equality is a consequence of Darcy's law after taking limits in both domains D^j. This is the Rayleigh–Taylor condition to be imposed at time $t = 0$, it being a part of the problem to prove that it remains true as time passes.

(3) The estimates on the norm of $(I - \lambda\mathscr{D})^{-1}$, $|\lambda| < 1$, $\mathscr{D}$ = double-layer potential (see Section 5), allowing us to obtain the inequalities

$$\begin{aligned} \|\Omega\|_{H^{k+1}} &\le P\big(\|X\|_{k+1}^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big), \\ \|\omega\|_{H^k} &\le P\big(\|X\|_{k+1}^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big), \end{aligned}$$

for $k \ge 3$, where P is a polynomial function and the norm $\|\cdot\|_k$ is given by

$$\|X\|_k = \|X_1 - \alpha_1\|_{L^3} + \|X_2 - \alpha_2\|_{L^3} + \|X_3\|_{L^2} + \|\nabla(X - (\alpha, 0))\|_{H^{k-1}}^2,$$

as in (7-1) below, and $\|\cdot\|_{H^j}$ denotes the norm in the Sobolev space H^j.

(4) A control of the Birkhoff–Rott integral $\mathrm{BR}(X, \omega)$:

$$\|\mathrm{BR}(X, \omega)\|_{H^k} \le P\big(\|X\|_{k+1}^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big),$$

for $k \ge 3$.

(5) Energy estimates: the properties of isothermal parametrizations help us to reorganize the terms in such a way that

$$\begin{aligned} \frac{d}{dt}\|X\|_k^2(t) &\le P\big(\|X\|_k^2(t) + \|F(X)\|_{L^\infty}^2(t) + \||N|^{-1}\|_{L^\infty}(t)\big) \\ &\quad - \sum_{i=1,2} \frac{2^{3/2}}{(\mu_1+\mu_2)} \int_{\mathbb{R}^2} \frac{\sigma(\alpha, t)}{|\nabla X(\alpha, t)|^3}\, \partial_{\alpha_i}^k X(\alpha, t) \cdot \Lambda(\partial_{\alpha_i}^k X)(\alpha, t)\, d\alpha, \end{aligned}$$

where $k \geq 4$, $|\nabla X(\alpha)|^3 = \big(|\partial_{\alpha_1} X(\alpha)|^2 + |\partial_{\alpha_2} X(\alpha)|^2\big)^{3/2}$, and $\Lambda = (-\Delta)^{1/2} = R_1(\partial_{\alpha_1}) + R_2(\partial_{\alpha_2})$. Then the pointwise inequality

$$\theta \Lambda(\theta) - \tfrac{1}{2}\Lambda(\theta^2) \geq 0,$$

together with the condition $\sigma > 0$, allows us to get rid of the dangerous terms in the inequality above (those involving $(k+1)$-derivatives of X) to obtain the estimate

$$\frac{d}{dt}\|X\|_k^2(t) \leq P\big(\|X\|_k^2(t) + \|F(X)\|_{L^\infty}^2(t) + \||N|^{-1}\|_{L^\infty}(t)\big).$$

(6) Finally, we need to control the evolution of $\|F(X)\|_{L^\infty}(t)$ and $\inf(t) = \inf_{\alpha \in \mathbb{R}^2} \sigma(\alpha, t)$, which is obtained via the estimates

$$\frac{d}{dt}\|F(X)\|_{L^\infty}^2(t) \leq P\big(\|X\|_4^2(t) + \|F(X)\|_{L^\infty}^2(t) + \||N|^{-1}\|_{L^\infty}(t)\big),$$
$$\frac{d}{dt}\frac{1}{\inf(t)} \leq \frac{1}{\inf(t)^2} P\big(\|X\|_4^2(t) + \|F(X)\|_{L^\infty}^2(t) + \||N|^{-1}\|_{L^\infty}(t)\big).$$

(7) All those facts together yield the inequality

$$\frac{d}{dt}E(t) \leq CP(E(t))$$

for the energy

$$E(t) = \|X\|_k^2(t) + \|F(X)\|_{L^\infty}^2(t) + \||N|^{-1}\|_{L^\infty}(t) + \inf(t)^{-1},$$

where $k \geq 4$, C is a universal constant, and P has polynomial growth (depending upon k).

At this point it is not difficult to prove the existence of a solution, locally in time, so long as the initial data $X(0)$ is in the appropriate Sobolev space of order $k \geq 4$, and the Rayleigh–Taylor and no-self-intersection conditions ($\sigma_0 > c > 0$, $\|F(X(0))\|_{L^\infty} < \infty$) are satisfied.

The main theorem presented in this paper is the following:

Theorem 4.1. *Let $X(0)$ with $\|X(0)\|_k < \infty$ for $k \geq 4$, $\|F(X(0))\|_{L^\infty} < \infty$, $\||N(\alpha, 0)|^{-1}\|_{L^\infty} < \infty$, and*

$$\sigma(\alpha, 0) = -\big(\nabla p^2(X(0), 0) - \nabla p^1(X(0), 0)\big) \cdot N(\alpha, 0) > 0.$$

Then there exists a time $\tau > 0$ such that there is a solution to (2-3), (2-4), (2-6) in $C([0, \tau]; H^k)$ with $X(\alpha, 0) = X(0)$.

Finally, let us point out that since our existence proof is based upon energy inequalities, an extra argument is needed to prove uniqueness. Nevertheless, that task is much easier than proving existence. (The interested reader may consult [Córdoba et al. $\geq$ 2013], where the details of the proof have been written out for some important cases, such as Muskat and SQG patches.)

Let us remark that, at the end, we have to work with a coupled system involving the evolution of the surface X, the "vorticity density" ω, the Rayleigh–Taylor condition σ, the non-self-intersecting character of S quantified by the gauge $F(X)$, and the tangential parts $C_1 X_{\alpha_1} + C_2 X_{\alpha_2}$ of the velocity field.

Remark. This paper is a continuation of [Córdoba et al. 2011], where the two-dimensional case was considered. Many of the needed estimates can be obtained following exactly the same methods that were used in [Córdoba et al. 2011] for the lower-dimensional case. Therefore, in order to simplify our presentation, we shall avoid here many details which were carefully proven there. This is especially the case in Section 6 (control of the Birkhoff–Rott integral) and Section 8 (energy estimates), and also for the approximation schemes which are identical to those developed in [Córdoba et al. 2011]. Therefore, in the following, we shall focus our attention on the more innovative parts of the proof, namely the evolution of the Rayleigh–Taylor condition, the non-self-intersecting property of the free boundary, and the needed estimates for double-layer potentials.

5. Inverting the operator: the single- and double-layer potentials revisited

In this proof, we need to consider the properties of single- and double-layer potentials, which are well-known characters in finding solutions to the Dirichlet and Neumann problems in domains D of $\mathbb{R}^n$.

For our purposes, these domains will be of three different types, namely: bounded, periodic in the "horizontal" variables, and asymptotically flat. We shall also assume that their boundaries are smooth enough (say C^2) and do not present self-intersections. Therefore, one has tangent balls at every point of the boundary, one completely contained in D and the other in D^c. We shall denote by $\nu(x)$ the unit inner normal at the point $x \in \partial D$; then under our hypothesis we have that, for $r > 0$ small enough, the parallel surfaces $\partial D_r = \{x + r\nu(x) \mid x \in \partial D\}$ are also C^2 surfaces with curvatures controlled by those of ∂D. Furthermore, the vector field ν can be extended smoothly up to a collar neighborhood of ∂D, allowing us to write the formula

$$\Delta u(x) = \frac{\partial^2 u}{\partial \nu^2}(x) - h(x)\frac{\partial u}{\partial \nu}(x) + \Delta_s u(x),$$

where Δ denotes the ordinary laplacian in $\mathbb{R}^n$, Δ_s is the Laplace–Beltrami operator in ∂D, $h(x)$ is the mean curvature of ∂D at the point x, and u is any C^2-function defined in a neighborhood of ∂D.

For convenience, we will use the notation $D_1 = D$, $D_2 = D^c$, $S = \partial D_j$, and $\nu_j(x)$ (for $j = 1, 2$) the inner normal at $x \in S$ pointing inside D_j. Let dS be the surface measure in S induced by Lebesgue measure in ambient space. Given integrable functions φ, ψ on S, we call

$$V(x) = c_n \int_S \psi(y) \frac{1}{\|x-y\|^{n-2}}\, dS(y)$$

the single-layer potential of ψ, and we call

$$W(x) = c_n \int_S \varphi(y) \frac{\partial}{\partial \nu_x}\left(\frac{1}{\|x-y\|^{n-2}}\right) dS(y)$$

the double-layer potential of φ. In both cases, c_n is a normalizing constant chosen so that $\dfrac{c_n}{\|x\|^{n-2}}$ is a fundamental solution of Δ in $\mathbb{R}^n$, $n \geq 3$.

For $x \in S$ and $j = 1, 2$, denote by $W_j(x)$ and $V_j(x)$ the corresponding limits of the potentials in D_j. We have

$$W_1(x) = \frac{1}{2}\left(\varphi(x) - \int_S \varphi(y)K(x, y)\, d\sigma(y)\right) = \frac{1}{2}(\varphi(x) - \mathcal{D}\varphi(x)),$$

$$W_2(x) = \frac{1}{2}\left(\varphi(x) + \int_S \varphi(y)K(x, y)\, d\sigma(y)\right) = \frac{1}{2}(\varphi(x) + \mathcal{D}\varphi(x)),$$

$$\frac{\partial V}{\partial \nu_1}(x) = -\frac{1}{2}\left(\psi(x) + \int_S \psi(y)K(y, x)\, d\sigma(y)\right) = -\frac{1}{2}(\psi(x) + \mathcal{D}^*\psi(x)),$$

$$\frac{\partial V}{\partial \nu_2}(x) = -\frac{1}{2}\left(\psi(x) - \int_S \psi(y)K(y, x)\, d\sigma(y)\right) = -\frac{1}{2}(\psi(x) - \mathcal{D}^*\psi(x)),$$

where

$$K(x, y) = 2c_n \frac{\partial}{\partial \nu_y}\left(\frac{1}{\|x-y\|^{n-2}}\right) = \tilde{c}_n \frac{\langle x - y, \nu(y)\rangle}{|x - y|^n}.$$

It is well-known that in the scenarios considered above, the boundary operators $\mathcal{D}$ (and $\mathcal{D}^*$) are smoothing of order -1, and therefore compact. Furthermore, all their eigenvalues are real numbers having absolute value strictly less than 1. Therefore, by the standard Fredholm theory, the operators $I - \lambda\mathcal{D}$, $I - \lambda\mathcal{D}^*$ are invertible when $|\lambda| \le 1$. However, in our case, the domains are moving, and the evolution of their common boundary S involves the inverse operators, making it necessary to estimate their norms in terms of the geometry and smoothness of S.

Although there is a vast literature about single- and double-layer potentials, we have not been able to point out a precise statement giving the information needed for our results. Therefore, in this section, we provide arguments to prove that the norms of such inverse operators grow at most polynomially: $P(|||S|||)$, where $|||S|||$ is just $\|S\|_{C^2}$ plus a term of chord-arc type controlling the non-self-intersecting character of the boundary. The term has the form $r(S)^{-1}$, where $r(S)$ is the sup over all the positive r such that S admits tangent balls of radius r in both domains D_j:

$$|||S||| = \|S\|_{C^2} + (r(S))^{-1}.$$

We shall write $P(|||S|||)$ to denote $\le C(|||S|||^p)$ for certain positive constants C, p which are independent of the characters whose evolution is being controlled, but the size of both constants may change during the proof and we shall make no effort to obtain their best values.

We will consider the case of bounded domains in $\mathbb{R}^n$, $n \ge 3$, because the needed modifications when $n = 2$, namely taking $\log |x|$ as fundamental solution for the laplacian, as well as the changes for the periodic or asymptotically flat domains, are left to the reader.

Let $\mathcal{D}$ and $\mathcal{D}^*$ be the potential defined above, with kernel

$$K(x, y) = c_n \frac{\partial}{\partial \nu(y)} \frac{1}{\|x-y\|^{n-2}} = c_n \frac{\langle x - y, \nu(y)\rangle}{|x - y|^n}$$

and $K(y, x)$ respectively. In the study of the inverse operators $(I - \lambda\mathcal{D})^{-1}$, $|\lambda| \le 1$, it is convenient to consider first the particular values $\lambda = \pm 1$.

Proposition 5.1. *The following estimate holds, where P is a polynomial function*:

$$\|(I \pm \mathfrak{D})^{-1}\|_{L^2(S)} = P(|||S|||).$$

Since the boundedness of $(I \pm \mathfrak{D})^{-1}$ in $L^2(S)$ is well-known from the general theory, we can simplify the proof, considering only functions $f \in L^2(S)$ whose support lies inside a region of S where the normal $\nu(x)$ is close enough to a fixed direction. Then for a general f, an appropriate partition of unity would allow us to add the local estimates, so long as the number of pieces is controlled by $|||S|||$. We shall use the following observation, whose proof is immediate.

Lemma 5.2 (Rellich). *Let u be a harmonic function and h a smooth vector field in the domain D; then we have*

(i) $\operatorname{div}(|\nabla u|^2 h) = 2\operatorname{div}((\nabla u \cdot h)\nabla u) + O(|\nabla u|^2|\nabla h|)$,

(ii) $\int_{\partial D}\langle \nu, h\rangle |\nabla u|^2 d\sigma = 2\int_{\partial D}(\partial u/\partial \nu)(\nabla u \cdot h) d\sigma + O\big(\int_D |\nabla u|^2|\nabla h|\big)$.

Given a function $f \in C^1(S)$, we may define $\nabla_\tau f$, choosing at each point $x \in S$ an orthonormal basis $\{e_1, \dots, e_{n-1}\}$ of the tangent space $T_x(S)$ (we can consider also $\nabla_\tau f$ to be the gradient naturally associated to the induced Riemannian metric by the ambient space). In both ways, although different, we have that $|\nabla_\tau f| \equiv \Lambda_\tau f$ is an elliptic pseudodifferential operator of order 1 in S. Solving the Dirichlet problem $\Delta u = 0$ in D, $u|_S = f$, we obtain the operator $D_\nu \equiv (\partial u/\partial \nu)|_S$, which is also a pseudodifferential operator of order 1 in S.

Lemma 5.3. *Let $f \in L^2(S)$ having support on the region $\frac{1}{2} \le \langle \nu(x), \eta\rangle \le 1$ (for a fixed unit vector η); then we have*

$$\int_S |D_\nu f|^2 d\sigma \simeq \int_S |\nabla_\tau f|^2 d\sigma,$$

where the constants involved in the stated equivalence $\simeq$ are $P(|||S|||)$.

Proof. Let u be harmonic in D so that $u|_S = f$. Under our hypothesis about f, and since $|\nabla u|^2 = |D_\nu u|^2 + |\nabla_\tau u|^2$ and $\nabla_\tau u$ is a local operator ($\operatorname{supp}_S(\nabla_\tau f) \subset \operatorname{supp}(f)$), Lemma 5.2 yields:

$$\tfrac{1}{2}\int_S |\nabla_\tau f|^2 d\sigma \le \int_S \langle \nu(x), \eta\rangle |\nabla_\tau u|^2 d\sigma \le 3\int_S |D_\nu u|^2 d\sigma + 2\int_S |\nabla_\tau u||D_\nu u| d\sigma,$$

from which we easily obtain

$$\int_S |\nabla_\tau f|^2 d\sigma \le P(|||S|||)\int_S |D_\nu f|^2 d\sigma.$$

To get the opposite inequality we proceed as before, but since $D_\nu f$ is not local, an extra argument is needed to control the contribution of the region outside $\operatorname{supp}(f)$. Let us introduce surface discs $B_r(x) = \{y \in S \mid \|x-y\| \le r\}$, $x \in S$, $0 \le r \le |||S|||^{-1}$ and domains $\Delta_r(x) = \{y + \rho\nu(x) \mid y \in B_r(x), \rho \le r\}$. Given $R = \frac{1}{2}|||S|||^{-1}$, there exists a fixed unit vector η so that $\frac{1}{2} \le \langle \nu(y), \eta\rangle \le 1$ for every $y \in B_R(x)$, and also a smooth vector field h such that $h|_{\Delta_R(x)} \equiv \eta$, $\operatorname{supp}(h) \subset \Delta_{2R}(x)$, and $\frac{1}{2}|h(x)| \le \langle h(x), \nu(x)\rangle$, $\|\nabla h\|^2 \le P(|||S|||)\|h\|$.

In order to obtain the estimate

$$\int_S |D_\nu f|^2 d\sigma \le P(\|\|S\|\|)\int_S |\nabla_\tau f|^2 d\sigma,$$

we may assume, without loss of generality, that $\mathrm{supp}(f) \subset B_R(x)$, for some $x \in S$, and then prove that

$$\int_{B_R(y_0)} |D_\nu f|^2 d\sigma \le P(\|\|S\|\|)\int_S |\nabla_\tau f|^2 d\sigma$$

uniformly on $y_0 \in S$.

With the vector field h defined above in $\Delta_{2R}(y)$, let us apply Rellich's estimate to get

$$\int_S |D_\nu f|^2 \langle h, \nu(x)\rangle \, d\sigma(x) = \int_S \langle \nu, h\rangle |\nabla_\tau f|^2 d\sigma - 2\int_S D_\nu f \nabla_\tau f \cdot h \, d\sigma + O\Big(\int_D |\nabla u|^2 |\nabla h|\Big),$$

where u satisfies $\Delta u = 0$ in D, $u|_S = f$. We get easily

$$\int_{B_R(y_0)} |D_\nu f|^2 \langle h, \nu(x)\rangle \, d\sigma(x) = O\Big(\int_S |\nabla_\tau f|^2 d\sigma + \int_D |\nabla u|^2 |\nabla h| \, dx\Big).$$

Then the proof will be finished if we can show that

$$\int_D |\nabla u|^2 |\nabla h| \, dx \le P(\|\|S\|\|)\int_S |\nabla_\tau f|^2 d\sigma.$$

To see this, let us consider the parallel surfaces $S_r = \{x + r\nu(x) \mid x \in S\}$ $(0 \le r \le \|\|S\|\|)$ and observe that

$$\int_{S_r} u^2 d\sigma_r \simeq \int_S u^2(x + r\nu(x)) \, d\sigma$$

and

$$\begin{aligned}\int_S \big[u^2(x + r\nu(x)) - u^2(x)\big] \, d\sigma(x) &= \int_S \int_0^r \nabla u^2(x + t\nu(x)) \cdot \nu(x) \, dt \, d\sigma \\ &= 2\int_{L_r} u(y)\nabla u(y) \cdot \nu(y) \le 2\Big(\int_{L_r} u^2(y)\Big)^{1/2}\Big(\int_{L_r} |\nabla u|^2(y)\Big)^{1/2},\end{aligned}$$

where $L_r = \{x + \rho\nu(x) \mid x \in S, 0 \le \rho \le r\}$.

Let $\mathscr{X}$ be a smooth cut-off function. Taking

$$F(x + r\nu(x)) = f(x)\mathscr{X}(x),$$

as a comparison function, Dirichlet's principle and Poincaré's inequality give us the estimate

$$\int_D |\nabla u|^2 \le \int_D |\nabla F|^2 \le C\Big(\int_S |\nabla_\tau f|^2 + \int_S |f|^2\Big) = O\Big(\int_S |\nabla_\tau f|^2 d\sigma\Big).$$

Therefore

$$\int_{S_r} u^2 d\sigma_r \simeq \int_S u^2(x + r\nu(x)) \, d\sigma \le \int_S f^2(x) \, d\sigma + \Big(\int_{L_r} u^2(y)\Big)^{1/2}\Big(\int_S |\nabla_\tau f|^2\Big)^{1/2}.$$

Integration in r in the range $0 \le r \le R = |||S|||^{-1}$ yields

$$\int_{L_r} u^2 dx \le R\Big(\int_S f^2(x)\, d\sigma + \Big(\int_{L_r} u^2(y)\Big)^{1/2}\Big(\int_S |\nabla_\tau f|^2\Big)^{1/2}\Big).$$

That is,

$$\int_{L_r} u^2 dx \le CR \int_S |\nabla_\tau f|^2\, d\sigma.$$

To conclude, let us observe that

$$\begin{aligned}\int_D |\nabla u|^2 |\nabla h| &= \frac{1}{2}\int_D \Delta u^2 |\nabla h| = \frac{1}{2}\int_D \big(\Delta u^2 |\nabla h| - u^2 \Delta(|\nabla h|)\big) + \frac{1}{2}\int_D u^2(|\nabla h|)\\ &= \frac{1}{2}\int_S u\frac{\partial u}{\partial \nu}\cdot |\nabla h|\, d\sigma - \frac{1}{2}\int_S f^2 \frac{(|\nabla h|)}{\partial \nu}\, d\sigma + \frac{1}{2}\int_D u^2 \nabla |h|\\ &\le \Big(\int_S f^2 d\sigma\Big)^{1/2}\Big(\int \Big|\frac{\partial u}{\partial \nu}\Big|^2 |\nabla h|^2 d\sigma\Big)^{1/2} + C\int_S f^2 d\sigma + C\int_{L_R} u^2. \qquad \square\end{aligned}$$

Proof of Proposition 5.1. As before, let $f \in C^1(S)$, $\mathrm{supp}(f) \subset \mathrm{U}_0$, and let u be its single-layer potential:

$$u(x) = c_n \int_S \frac{f(y)}{\|x-y\|^{n-2}}\, dS(y).$$

Taking derivatives on each domain D_j with respect to the normal direction and evaluating at S, we get

$$\frac{\partial u}{\partial \nu_1} = -\tfrac{1}{2}(f(x) + \mathfrak{D}^* f(x)), \qquad \frac{\partial v}{\partial \nu_2} = -\tfrac{1}{2}(f(x) - \mathfrak{D}^* f(x)).$$

By Lemma 5.3, we know that

$$\int_S \Big|\frac{\partial v}{\partial \nu_1}\Big|^2 d\sigma \simeq \int_S |\nabla_\tau v|^2 d\sigma \simeq \int_S \Big|\frac{\partial v}{\partial \nu_2}\Big|^2 d\sigma,$$

where the constants involved in the equivalences are all controlled by above by $P(|||S|||)$ and below by $1/P(|||S|||)$.

Since $\partial v/\partial \nu_1 + \partial v/\partial \nu_2 = -f$, these estimates imply that

$$\min\big(\|f - \mathfrak{D}^* f\|_2, \|f + \mathfrak{D}^* f\|_2\big) \ge \frac{1}{P(|||S|||)},$$

that is, $\|(I \pm \mathfrak{D})^{-1}\| = P(|||S|||)$. Then using an appropriate partition of unity, that estimate extends to a general $f \in L^2(S)$. $\square$

Next we shall consider Sobolev spaces $H^s(S)$, $0 \le s \le 1$, defined in the usual manner throughout local coordinate charts. We have also the elliptic pseudodifferential operator $\Lambda^s = (-\Delta)^{s/2}$ in such a way that

$$\|f\|_{H^s(S)} \simeq \|f\|_{L^2} + \|\Lambda^s f\|_{L^2}.$$

Then $H^{-s}(S) \equiv (H^s(S))^*$ allows us to consider the negative case by duality, under the pairing

$$\int_S \phi\psi\, d\sigma, \quad \phi \in H^{-s}, \ \psi \in H^s,$$

and we have

$$\|\phi\|_{H^{-s}} = \sup_{\|\psi\|_{H^s}=1} \int_S \phi\psi \, d\sigma.$$

Since both $\mathfrak{D}$ and $\mathfrak{D}^*$ are compact and smoothing operators of degree -1, the commutators $[\Lambda^s, \mathfrak{D}]$, $[\Lambda^s, \mathfrak{D}^*]$ are then bounded in $L^2(S)$ $(0 \le s \le 1)$ with norms controlled by $|||S|||$, allowing us to extend Proposition 5.1 to the chain of Sobolev spaces:

Corollary 5.4. *The norm of the operators* $(I \pm \mathfrak{D})^{-1}$, $(I \pm \mathfrak{D}^*)^{-1}$ *in the space* $H^s(S)$, $-1 \le s \le 1$, *is bounded by* $P(|||S|||)$.

Estimates for $\boldsymbol{(I + \lambda\mathfrak{D})^{-1}}$, $\boldsymbol{|\lambda| \le 1}$. With the same notation used before, we have

$$\frac{1-\lambda}{2}\frac{\partial V}{\partial \nu_1} + \frac{1+\lambda}{2}\frac{\partial V}{\partial \nu_2} = -\tfrac{1}{2}(\phi(x) - \lambda\mathfrak{D}^*\phi(x)) \quad \text{and} \quad \frac{1+\lambda}{2}\frac{\partial V}{\partial \nu_1} + \frac{1-\lambda}{2}\frac{\partial V}{\partial \nu_2} = -\tfrac{1}{2}(\phi(x) + \lambda\mathfrak{D}^*\phi(x)),$$

where

$$V(x) = c_n \int_S \frac{\phi(y)}{\|x-y\|^{n-2}} \, dS(y).$$

Then the identity $\phi - \lambda\mathfrak{D}^*\phi = 0$ yields

$$0 = (1-\lambda)\int_{\partial D_1} V\frac{\partial V}{\partial \nu_1} \, dS + (1+\lambda)\int_{\partial D_2} V\frac{\partial V}{\partial \nu_2} \, dS = (1-\lambda)\int_{D_1} |\nabla V|^2 + (1+\lambda)\int_{D_2} |\nabla V|^2,$$

which implies $\phi \equiv 0$. Similarly for $\phi + \lambda\mathfrak{D}^*\phi = 0$, $-1 \le \lambda \le 1$.

Remark. This observation can be improved applying the following fact (whose proof we skip because it will not be used in our theorem):

$$\int_{D_1} |\nabla u|^2 \simeq \int_{D_2} |\nabla u|^2,$$

where, again, the $\simeq$ is controlled by $P(|||S|||)$. In particular, it implies that the spectral radius of the operators $\mathfrak{D}$, $\mathfrak{D}^*$ is less than $1 - (P(|||S|||))^{-1}$.

Theorem 5.5. *The operator norms* $\|(I + \lambda\mathfrak{D})^{-1}\|_{H^s(S)}$, $\|(I + \lambda\mathfrak{D}^*)^{-1}\|_{H^s(S)}$, $|s| \le 1$, $|\lambda| \le 1$, *are* $P(|||S|||)$ (*growth at most polynomially with* $|||S|||$).

Proof. The identity $(I - \mathfrak{D})^{-1}(I - \lambda\mathfrak{D}) = I + (1-\lambda)(I - \mathfrak{D})^{-1}\mathfrak{D}$ shows that the conclusion of the theorem follows easily when $|1-\lambda| \le 1/P(|||S|||)$, and similarly when $|1+\lambda| \le 1/P(|||S|||)$.

Therefore, without loss of generality, we may assume that

$$1 - |\lambda| \ge \frac{1}{P(|||S|||)}.$$

Assume now that $\phi \in H^{-1/2}(S)$ satisfies $\|\phi\|_{H^{-1/2}} = 1$ and

$$\|\phi - \lambda\mathfrak{D}^*\phi\|_{H^{-1/2}} \le \frac{1}{P(|||S|||)}.$$

Then the single-layer potential

$$V(x) = c_n \int_S \frac{\phi(y)}{\|x-y\|^{n-2}} \, dS(y)$$

satisfies the inequality

$$\left|\int_S V(\phi-\lambda\mathfrak{D}^*\phi)\,dS\right| \le \frac{1}{P(|||S|||)}.$$

On the other hand, one has

$$\int_S V(\phi-\lambda\mathfrak{D}^*\phi)\,dS = (1-\lambda)\int_{D_1}|\nabla V|^2 + (1+\lambda)\int_{D_2}|\nabla V|^2,$$

implying the estimate

$$\int_S V(\phi+\lambda\mathfrak{D}^*\phi)\,dS = (1+\lambda)\int_{D_1}|\nabla V|^2 + (1-\lambda)\int_{D_2}|\nabla V|^2 \le \frac{1}{P(|||S|||)}.$$

Adding both inequalities together, we would obtain

$$\int_S V\phi\,d\sigma \le \frac{1}{P(|||S|||)},$$

which is impossible because of the following:

Lemma 5.6. *If V is the single-layer potential of ϕ, then*

$$\int_S V(x)\phi(x)\,dS(x) = \int_S\int_S \frac{\phi(x)\phi(y)}{\|x-y\|^{n-2}}\,dS(x)\,dS(y) \ge \frac{1}{P(|||S|||)}\|\phi\|^2_{H^{-1/2}(S)}.$$

Let us first observe that

$$\int_S\int_S \frac{\phi(x)\phi(y)}{\|x-y\|^{n-2}}\,d\sigma(x)\,d\sigma(y) = \int_{\mathbb{R}^n}\frac{1}{|\xi|^2}\left|\widehat{\phi\,d\sigma(\xi)}\right|^2 d\xi \ge 0,$$

where $\widehat{\phi\,dS}$ denotes the Fourier transform of the measure $\phi\,dS$ supported on S. This implies that

$$\langle\phi,\psi\rangle = \int_S\int_S \frac{\phi(x)\phi(y)}{\|x-y\|^{n-2}}\,dS(x)\,dS(y)$$

is an inner product satisfying

$$|\langle\phi,\psi\rangle| \le \langle\phi,\phi\rangle^{1/2}\langle\psi,\psi\rangle^{1/2},$$

and we wish to show that

$$\langle\phi,\phi\rangle \backsimeq \|\phi\|^2_{H^{-1/2}(S)},$$

where $\backsimeq$ denotes equivalence modulo a factor $P(|||S|||)$. To see this, observe first that given $\phi\in H^{-1/2}(S)$, its single-layer potential $u|S$ belongs to the space $H^{1/2}(S)$, satisfying

$$\|u\|_{H^{1/2}(S)} \le P(|||S|||)\|\phi\|_{H^{-1/2}(S)},$$

which can be proved easily using local coordinates. As a consequence, we have

$$\int_S\int_S \frac{\phi(x)\phi(y)}{\|x-y\|^{n-2}}\,dS(x)\,dS(y) \le P(|||S|||)\|\phi\|^2_{H^{-1/2}(S)}.$$

In the opposite direction, since $H^{-s} = (H^s)^*$, we have

$$\|\phi\|_{H^{-s}} = \sup_{f\in H^s} \int_S \phi(x) f(x)\, d\sigma(x).$$

Let us assume, for the moment, that given $f \in H^s$, there exists $g \in H^{s-1}$ such that

$$f(x) = c_n \int_S \frac{g(y)}{\|x-y\|^{n-2}}\, dS(y) \quad \text{and} \quad \|f\|_{H^s} \backsimeq \|g\|_{H^{s-1}}.$$

Then

$$\|\phi\|_{H^{-s}} \backsimeq \sup_{\|g\|_{H^{s-1}}=1} \langle \phi, g\rangle,$$

and taking $s = \frac{1}{2}$, $s-1 = -\frac{1}{2}$, we get

$$\|\phi\|_{H^{-1/2}} \le P(\|\!|S|\!\|)\langle\phi,\phi\rangle^{1/2}\langle g,g\rangle^{1/2} \le P(\|\!|S|\!\|)\langle\phi,\phi\rangle^{1/2}\|g\|_{H^{-1/2}} \le P(\|\!|S|\!\|)\langle\phi,\phi\rangle^{1/2}.$$

To close our argument, it remains to solve the equation

$$f(x) = c_n \int_S \frac{g(y)}{\|x-y\|^{n-2}}\, dS(y),$$

that is, to prove that given $f \in H^s$, there exists $g \in H^{s-1}$ satisfying the this equation.

To see that, let us consider the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } D_1, \\ u|_S = f \end{cases}$$

and the equation

$$-2\frac{\partial u}{\partial \nu_1} = g - \mathfrak{D}^* g,$$

that is, $g = (I - \mathfrak{D}^*)^{-1}(-2\partial u/\partial \nu_1)$. Then we claim that such g verifies the identity

$$f(x) = c_n \int_S \frac{g(y)}{\|x-y\|^{n-2}}\, dS(y).$$

This is because the function

$$V(x) = c_n \int_S \frac{g(y)}{\|x-y\|^{n-2}}\, dS(y)$$

is harmonic in D_1 and satisfies

$$-2\frac{\partial V}{\partial \nu_1} = g - \mathfrak{D}^* g = -2\frac{\partial u}{\partial \nu_1},$$

which implies that $V = u$ in D_1, and therefore, taking limits up to the boundary, we obtain

$$f(x) = c_n \int_S \frac{g(y)}{\|x-y\|^{n-2}}\, dS(y).$$

To finish the proof of Theorem 5.5, let us consider, for every $0 \le \tau \le 1$, the identity

$$(I - \lambda\mathfrak{D})^{-1}\Lambda^\tau = \Lambda^\tau (I - \lambda\mathfrak{D})^{-1} + (I - \lambda\mathfrak{D})^{-1} C_\tau (I - \lambda\mathfrak{D})^{-1},$$

where the commutator $C_\tau = [\mathfrak{D}\Lambda^\tau - \Lambda^\tau\mathfrak{D}]$ is a pseudodifferential operator of order $\tau - 2$ whose bounds are controlled by $|||S|||$. Then

$$\begin{aligned}\|(I-\lambda\mathfrak{D})^{-1}f\|_{H^s} &\le \|(I-\lambda\mathfrak{D})^{-1}f\|_{H^{-1/2}} + \|\Lambda^{s+1/2}(I-\lambda\mathfrak{D})^{-1}f\|_{H^{-1/2}}\\ &\lesssim \|f\|_{H^{-1/2}} + \|(I-\lambda\mathfrak{D})^{-1}\Lambda^{s+1/2}f\|_{H^{-1/2}}\\ &\lesssim \|f\|_{L^2} + \|\Lambda^{s+1/2}f\|_{H^{-1/2}} \le P(|||S|||)\|f\|_{H^s}. \qquad \square\end{aligned}$$

Remark 5.7. In the particular case of the sphere $S = S^{n-1}$ $(n \ge 2)$, the estimate of Lemma 5.6 becomes an identity:

$$\int_{S^{n-1}}\int_{S^{n-1}} \frac{\phi(x)\phi(y)}{\|x-y\|^{n-2}}\, dS(x)\, dS(y) = c_n\|\phi\|^2_{H^{-1/2}(S^{n-1})}$$

for $n \ge 3$, and

$$-\int_{S^1}\int_{S^1} \log\|x-y\|\phi(x)\phi(y)\, dS(x)\, dS(y) = c_2\|\phi\|^2_{H^{-1/2}(S^1)}$$

for $n = 2$.

Proof. We present the details when $n \ge 3$. The case $n = 2$ follows similarly. Let $\phi(x) = \sum a_k Y_k(x)$, where Y_k is a spherical harmonic of degree k, normalized so that $\|Y_k\|_{L^2(S^{n-1})} = 1$; then we have

$$|a_0|^2 + \sum_{k\ge 1} \frac{|a_k|^2}{2k+n-2} = \|\phi\|^2_{H^{-1/2}(S)} < \infty.$$

Claim: if $k \ne j$, then

$$\int_{S^{n-1}}\int_{S^{n-1}} \frac{Y_k(x)Y_j(y)}{\|x-y\|^{n-2}}\, dS(x)\, dS(y) = 0.$$

Taking the Fourier transform and using Plancherel, we get

$$\int_{S^{n-1}}\int_{S^{n-1}} \frac{Y_k(x)Y_j(y)}{\|x-y\|^{n-2}}\, dS(x)\, dS(y) = \int_{\mathbb{R}^n} \frac{1}{|\xi|^2}\widehat{Y_k\, dS}(\xi)\overline{\widehat{Y_j\, dS}(\xi)}\, d\xi.$$

But it turns out that

$$\widehat{Y_k\, dS}(\xi) = 2\pi i^{-k}|\xi|^{(n-2)/2} J_{(n+2k-2)/2}(|\xi|)Y_k\Big(\frac{\xi}{|\xi|}\Big),$$

where J_ν designates Bessel's function of order ν, implying the claim.

Therefore our estimate diagonalizes:

$$\int_{\mathbb{R}^n} \frac{1}{|\xi|^2}|\widehat{Y_k\, dS}(\xi)|^2 d\xi = c\int_0^\infty \frac{1}{r}|J_{k+(n-2)/2}(r)|^2 dr,$$

and the well-known identity for Bessel's functions

$$\int_0^\infty \frac{J_\mu^2(r)}{r}\, dr = \frac{1}{2\mu}$$

allows us to finish the proof. $\square$

Estimates for Ω and ω. In the following, we shall consider asymptotically flat domains, leaving to the reader the details of the periodic case. Since we have controlled the norms of the operator relating Ω and X, we are in a position to obtain the inequality

$$\|\Omega\|_{H^k} \leq P\big(\|X\|_k^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big), \tag{5-1}$$

for $k \geq 4$, with P a polynomial function. Then Sobolev's embedding implies

$$\|\omega\|_{H^k} \leq P\big(\|X\|_{k+1}^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big), \tag{5-2}$$

for $k \geq 3$. We will present the proof of (5-1) when $k = 4$, because the case $k > 4$ can be obtained with the same method.

Theorem 5.5 applied to (2-6) yields

$$\|\Omega\|_{H^1} = \|(I - A_\mu \mathfrak{D})^{-1}(-2A_\rho X_3)\|_{H^1} \leq C\|(I - A_\mu \mathfrak{D})^{-1}\|_{H^1}\, \|X_3\|_{H^1} \leq P(\|\!|S|\!\|)\|X_3\|_{H^1},$$

implying that

$$\|\Omega\|_{H^1} \leq P\big(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big).$$

Next we will show that

$$\|\partial_{\alpha_1}^2 \Omega\|_{L^2} \leq P\big(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big)\|\Omega\|_{H^1}, \tag{5-3}$$

which together with the estimate for $\|\Omega\|_{H^1}$ above, will allow us to control $\partial_{\alpha_1}^2 \Omega$ in terms of the free boundary.

In order to do that, we start with formula (2-8) to get $\partial_{\alpha_1}^2 \Omega = I_1 + I_2 + I_3 + I_4 - 2A_\rho \partial_{\alpha_1}^2 X_3$, where

$$I_1 = \frac{A_\mu}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)d\beta \cdot \partial_{\alpha_1}^2 X(\alpha),$$

$$I_2 = \frac{A_\mu}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial_{\alpha_1} X(\alpha) - \partial_{\alpha_1} X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

$$I_3 = -\frac{3A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} A(\alpha,\beta)\frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^5} \wedge \omega(\alpha-\beta)d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

with $A(\alpha,\beta) = (X(\alpha) - X(\alpha-\beta)) \cdot (\partial_{\alpha_1} X(\alpha) - \partial_{\alpha_1} X(\alpha-\beta))$, and

$$I_4 = \frac{A_\mu}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \partial_{\alpha_1}\omega(\alpha-\beta)d\beta \cdot \partial_{\alpha_1} X(\alpha).$$

Our next objective is to introduce the operators $\mathcal{T}_k$ (A-5) defined in the Appendix in the analysis of the integrals I_j. Formula (2-3) gives us $\omega = \partial_{\alpha_2}(\Omega\partial_{\alpha_1} X) - \partial_{\alpha_1}(\Omega\partial_{\alpha_2} X)$, and from standard Sobolev's estimates we get

$$\|I_j\|_{L^2} \leq P\big(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big)\|\Omega\|_{H^1}, \qquad j = 1, 2,$$

and similarly with I_3.

Regarding

$$I_4=\int_{|\beta|>1}d\beta+\int_{|\beta|<1}d\beta=J_1+J_2,$$

we integrate by parts in J_1 to obtain

$$J_1=\frac{A_\mu}{2\pi}\int_{|\beta|>1}\partial_{\beta_1}\Big(\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\Big)\wedge\omega(\alpha-\beta)d\beta\cdot\partial_{\alpha_1}X(\alpha)$$
$$-\frac{A_\mu}{2\pi}\int_{|\beta|=1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\omega(\alpha-\beta)\,dl(\beta)\cdot\partial_{\alpha_1}X(\alpha).$$

From this last expression, it is easy to deduce the inequality

$$J_1\le C\|F(X)\|^3_{L^\infty}\|X-(\alpha,0)\|^2_{C^1}\Big(\int_{|\beta|>1}\frac{|\omega(\alpha-\beta)|}{|\beta|^3}d\beta+\int_{|\beta|=1}|\omega(\alpha-\beta)|\,dl(\beta),\Big)$$

providing us with an appropriate control (see the Appendix for more details).

Next let us consider $J_2=K_1+K_2+K_3+K_4$, where

$$K_1=\frac{A_\mu}{2\pi}\,\mathrm{PV}\!\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_2}\Omega(\alpha-\beta)\partial^2_{\alpha_1}X(\alpha-\beta)d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

$$K_2=\frac{A_\mu}{2\pi}\,\mathrm{PV}\!\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_1}\partial_{\alpha_2}\Omega(\alpha-\beta)\partial_{\alpha_1}X(\alpha-\beta)d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

$$K_3=-\frac{A_\mu}{2\pi}\,\mathrm{PV}\!\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_1}\Omega(\alpha-\beta)\partial_{\alpha_1}\partial_{\alpha_2}X(\alpha-\beta)d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

$$K_4=-\frac{A_\mu}{2\pi}\,\mathrm{PV}\!\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial^2_{\alpha_1}\Omega(\alpha-\beta)\partial_{\alpha_2}X(\alpha-\beta)d\beta\cdot\partial_{\alpha_1}X(\alpha).$$

Then the terms K_1 and K_3 are handled with the same approach used for I_2 — see (A-13) in the Appendix — and we rewrite K_2 in the form

$$K_2=\frac{A_\mu}{2\pi}\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_1}\partial_{\alpha_2}\Omega(\alpha-\beta)(\partial_{\alpha_1}X(\alpha-\beta)-\partial_{\alpha_1}X(\alpha))d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

to show that it can be estimated via an integration by parts in the variable β_1, using the identity

$$\partial_{\alpha_1}\partial_{\alpha_2}\Omega(\alpha-\beta)=-\partial_{\beta_1}(\partial_{\alpha_2}\Omega(\alpha-\beta))$$

and the fact that the kernel in the integral K_2 has degree -1.

It remains to deal with K_4: to do that, let us consider $K_4=L_1+L_2$, where

$$L_1=\frac{A_\mu}{2\pi}\,\mathrm{PV}\!\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial^2_{\alpha_1}\Omega(\alpha-\beta)\big(\partial_{\alpha_2}X(\alpha)-\partial_{\alpha_2}X(\alpha-\beta)\big)d\beta\cdot\partial_{\alpha_1}X(\alpha)$$

and

$$L_2=\frac{A_\mu}{2\pi}\,\mathrm{PV}\!\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\,\partial^2_{\alpha_1}\Omega(\alpha-\beta)d\beta\cdot N(\alpha).$$

The term L_1 can be controlled like K_2, and L_2 can be rewritten in the form

$$L_2 = \frac{A_\mu}{2\pi} \mathrm{PV}\int_{|\beta|<1} \left(\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} - \frac{\nabla X(\alpha)\cdot\beta}{|\nabla X(\alpha)\cdot\beta|^3} \right) \partial^2_{\alpha_1}\Omega(\alpha-\beta)\, d\beta \cdot N(\alpha),$$

showing that it can be estimated as we did with $\mathcal{T}_4$ (A-8), that is, we obtain (5-3). Similarly, Equation (2-9) yields

$$\|\partial^2_{\alpha_2}\Omega\|_{L^2} \le P\big(\|X\|_4^2 + \|F(X)\|^2_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)\|\Omega\|_{H^1},$$

and then the inequality $2\|\partial_{\alpha_1}\partial_{\alpha_2}\Omega\|_{L^2} \le \|\partial^2_{\alpha_1}\Omega\|_{L^2} + \|\partial^2_{\alpha_2}\Omega\|_{L^2}$ gives us the desired control upon $\|\Omega\|_{H^2}$.

Next we will show that

$$\|\partial^3_{\alpha_1}\Omega\|_{L^2} \le P\big(\|X\|_4^2 + \|F(X)\|^2_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)\|\Omega\|_{H^2}, \tag{5-4}$$

allowing us to use the estimates for $\|\Omega\|_{H^2}$ above. In order to do that, we start with formula (2-8), to get $\partial^3_{\alpha_1}\Omega = \partial_{\alpha_1} I_1 + \partial_{\alpha_1} I_2 + \partial_{\alpha_1} I_3 + \partial_{\alpha_1} I_4 - 2A_\rho \partial^3_{\alpha_1} X_3$, where the most singular terms are given by

$$J_3 = \frac{A_\mu}{2\pi} \mathrm{PV}\int_{\mathbb{R}^2} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)\, d\beta \cdot \partial^3_{\alpha_1} X(\alpha),$$

$$J_4 = \frac{A_\mu}{2\pi} \mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^2_{\alpha_1}X(\alpha)-\partial^2_{\alpha_1}X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

$$J_5 = -\frac{3A_\mu}{4\pi} \mathrm{PV}\int_{\mathbb{R}^2} B(\alpha,\beta)\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^5} \wedge \omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

with $B(\alpha,\beta) = (X(\alpha)-X(\alpha-\beta))\cdot(\partial^2_{\alpha_1}X(\alpha)-\partial^2_{\alpha_1}X(\alpha-\beta))$, and

$$J_6 = \frac{A_\mu}{2\pi} \mathrm{PV}\int_{\mathbb{R}^2} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial^2_{\alpha_1}\omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

and where the remainder terms can be estimated with the same method used before.

Now we write

$$J_3 = \frac{A_\mu}{2\pi}\mathcal{T}_1\big(\partial_{\alpha_2}(\Omega\partial_{\alpha_1}X) - \partial_{\alpha_1}(\Omega\partial_{\alpha_2}X)\big)\cdot\partial^3_{\alpha_1}X$$

to obtain

$$\|J_3\|_{L^2} \le C\big\|\mathcal{T}_1\big(\partial_{\alpha_2}(\Omega\partial_{\alpha_1}X) - \partial_{\alpha_1}(\Omega\partial_{\alpha_2}X)\big)\big\|_{L^4}\|\partial^3_{\alpha_1}X\|_{L^4}.$$

Next observe that in the proof of estimate (A-9), one can replace L^2 by L^p for $1<p<\infty$ [Stein 1993]. In particular, we have

$$\|J_3\|_{L^2} \le P\big(\|X-(\alpha,0)\|_{C^{1,\delta}} + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)\big(\|\Omega\partial_{\alpha_1}X\|_{L^4} + \|\Omega\partial_{\alpha_2}X\|_{L^4} + \|\omega\|_{L^4}\big)\|\partial^3_{\alpha_1}X\|_{L^4},$$

and then Sobolev's embedding in dimension two, $\|g\|_{L^4} \le C\|g\|_{H^1}$, yields the desired control. Regarding J_4, we follow the approach taken before for $\mathcal{T}_3$, but now using the L^4 norm. That is, we split

$$J_4 = \int_{|\beta|>1} d\beta + \int_{|\beta|<1} d\beta = K_5 + K_6,$$

and since
$$K_5 \le \|X-(\alpha,0)\|_{C^2}^2 \|F(X)\|_{L^\infty}^3 \int_{|\beta|>1} \frac{|\omega(\alpha-\beta)|}{|\beta|^3}\, d\beta,$$
that term can be estimated as above.

Next we introduce the splitting $K_6 = L_3 + L_4$, where
$$L_3 = \frac{A_\mu}{2\pi} \int_{|\beta|<1} \big(\partial_{\alpha_1}^2 X(\alpha) - \partial_{\alpha_1}^2 X(\alpha-\beta)\big)\Big[\frac{1}{|X(\alpha)-X(\alpha-\beta)|^3} - \frac{1}{|\nabla X(\alpha)\cdot\beta|^3}\Big] \wedge \omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$
$$L_4 = \frac{A_\mu}{2\pi} \mathrm{PV}\int_{|\beta|<1} \frac{\partial_{\alpha_1}^2 X(\alpha) - \partial_{\alpha_1}^2 X(\alpha-\beta)}{|\nabla X(\alpha)\cdot\beta|^3} \wedge \omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha).$$
We have
$$L_3 \le C\|X-(\alpha,0)\|_{C^{2,\delta}}^3 \big(\|F(X)\|_{L^\infty}^4 + \|X-(\alpha,0)\|_{C^1}^4 \||N|^{-1}\|_{L^\infty}^4\big) \int_{|\beta|<1} \frac{|\omega(\alpha-\beta)|}{|\beta|^{2-\delta}}\, d\beta$$
(see the Appendix for more details), giving us the appropriate estimate. Regarding L_4, we use identity (A-16), which, after a careful integration by parts, yields
$$L_4 = \frac{A_\mu}{2\pi} \mathrm{PV}\int_{|\beta|<1} \frac{\beta\cdot\nabla_\beta\big((\partial_{\alpha_1}^2 X(\alpha) - \partial_{\alpha_1}^2 X(\alpha-\beta)) \wedge \omega(\alpha-\beta)\cdot \partial_{\alpha_1} X(\alpha)\big)}{|\nabla X(\alpha)\cdot\beta|^3}\, d\beta$$
$$- \frac{A_\mu}{2\pi} \int_{|\beta|=1} \frac{|\beta|\big(\partial_{\alpha_1}^2 X(\alpha) - \partial_{\alpha_1}^2 X(\alpha-\beta)\big) \wedge \omega(\alpha-\beta)\cdot \partial_{\alpha_1} X(\alpha)}{|\nabla X(\alpha)\cdot\beta|^3}\, dl(\beta),$$
helping us to prove the inequality
$$\|L_4\|_{L^2} \le P\big(\|X-(\alpha,0)\|_{C^2} + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)(\|\partial_{\alpha_1}^3 X\|_{L^4}\|\omega\|_{L^4} + \|\omega\|_{L^2}).$$
Clearly, J_5 can be approached with the same method used for J_4. Regarding the term J_6, we have to decompose further: first, its most singular terms, which are given by
$$L_5 = \frac{A_\mu}{2\pi} \mathrm{PV}\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial_{\alpha_2}\Omega(\alpha-\beta)\partial_{\alpha_1}^3 X(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$
$$L_6 = \frac{A_\mu}{2\pi} \mathrm{PV}\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial_{\alpha_1}^2\partial_{\alpha_2}\Omega(\alpha-\beta)\partial_{\alpha_1} X(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$
$$L_7 = -\frac{A_\mu}{2\pi} \mathrm{PV}\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial_{\alpha_1}\Omega(\alpha-\beta)\partial_{\alpha_1}^2\partial_{\alpha_2} X(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$
$$L_8 = -\frac{A_\mu}{2\pi} \mathrm{PV}\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial_{\alpha_1}^3\Omega(\alpha-\beta)\partial_{\alpha_2} X(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha).$$
Second, let us observe that the remainder is easy to handle: the terms L_5 and L_7 can be estimated as we did with K_1 and K_3, using the L^4 norm and, finally, L_6 and L_8 are like K_2 and K_4, respectively. Putting all these facts together, we obtain (5-4).

Similarly to the case of lower derivatives, Equation (2-9) yields

$$\|\Omega\|_{H^3} \le P\big(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big)\|\Omega\|_{H^2}.$$

To finish, it remains to show the corresponding inequality for derivatives of fourth order:

$$\|\Omega\|_{H^4} \le P\big(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big)\|\Omega\|_{H^3}. \tag{5-5}$$

Identity (2-8) allows us to point out the most singular terms in $\partial_{\alpha_1}^4\Omega$:

$$M_1 = \frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\omega(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}^4 X(\alpha),$$

$$M_2 = \frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}^3 X(\alpha)-\partial_{\alpha_1}^3 X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\omega(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1} X(\alpha),$$

$$M_3 = -\frac{3A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} C(\alpha,\beta)\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^5}\wedge\omega(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1} X(\alpha),$$

with $C(\alpha,\beta) = (X(\alpha)-X(\alpha-\beta))\cdot(\partial_{\alpha_1}^3 X(\alpha)-\partial_{\alpha_1}^3 X(\alpha-\beta))$, and

$$M_4 = \frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_1}^3\omega(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1} X(\alpha).$$

Then, in order to estimate M_1, we start with $\|M_1\|_{L^2} \le CK\|\partial_{\alpha_1}^4 X\|_{L^2}$, where

$$K = \sup_\alpha \left|\mathrm{PV}\int_{\mathbb{R}^2} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\omega(\alpha-\beta)\,d\beta\right|.$$

Following [Córdoba and Gancedo 2007], we have

$$K \le O_1+O_2+O_3+O_4+O_5,$$

where

$$O_1 = \sup_\alpha\left|\mathrm{PV}\int_{|\beta|>1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\omega(\alpha-\beta)\,d\beta\right|,$$

$$O_2 = \sup_\alpha\left|\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)-\nabla X(\alpha)\cdot\beta}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\omega(\alpha-\beta)\,d\beta\right|,$$

$$O_3 = \sup_\alpha\left|\int_{|\beta|<1} \nabla X(\alpha)\cdot\beta\left[\frac{1}{|X(\alpha)-X(\alpha-\beta)|^3}-\frac{1}{|\nabla X(\alpha)\cdot\beta|^3}\right]\wedge\omega(\alpha-\beta)\,d\beta\right|,$$

$$O_4 = \sup_\alpha\left|\int_{|\beta|<1} \frac{\nabla X(\alpha)\cdot\beta}{|\nabla X(\alpha)\cdot\beta|^3}\wedge\big(\omega(\alpha-\beta)-\omega(\alpha)\big)\,d\beta\right|,$$

$$O_5 = \sup_\alpha\left|\mathrm{PV}\int_{|\beta|<1} \frac{\nabla X(\alpha)\cdot\beta}{|\nabla X(\alpha)\cdot\beta|^3}\wedge\omega(\alpha)\,d\beta\right|.$$

An integration by parts in O_1 yields

$$\begin{aligned}O_1 &\le C\|\nabla X\|^2_{L^\infty}\|F(X)\|^3_{L^\infty}\sup_\alpha\Big(\int_{|\beta|>1}\frac{|\Omega(\alpha-\beta)|}{|\beta|^3}\,d\beta+\int_{|\beta|=1}|\Omega(\alpha-\beta)|\,dl(\beta)\Big)\\ &\le C\|\nabla X\|^2_{L^\infty}\|F(X)\|^3_{L^\infty}\|\Omega\|_{L^\infty},\end{aligned}$$

and Sobolev's embedding allows us to conclude.

Regarding O_2, we have

$$O_2 \le \|X-(\alpha,0)\|_{C^{2,\delta}}\|F(X)\|^3_{L^\infty}\|\omega\|_{L^\infty}\Big|\int_{|\beta|<1}|\beta|^{2-\delta}d\beta\Big|,$$

and the estimate $\|\omega\|_{C^\delta}\le C\|\omega\|_{H^2}$, for $0<\delta<1$, gives the desired control. Using (A-15) and some straightforward algebraic manipulations, we get a similar inequality for O_3. Next, we have

$$O_4 \le C\|X-(\alpha,0)\|^4_{C^1}\||N|^{-1}\|^3_{L^\infty}\|\omega\|_{C^\delta}\Big|\int_{|\beta|<1}|\beta|^{2-\delta}d\beta\Big|,$$

giving us also the same estimate. Furthermore, it is easy to prove that $O_5=0$.

Next we consider the term M_2 with the splitting $M_2=Q_1+Q_2+Q_3$, where

$$\begin{aligned}Q_1&=\frac{A_\mu}{2\pi}\int_{|\beta|>1}\frac{\partial^3_{\alpha_1}X(\alpha)-\partial^3_{\alpha_1}X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\omega(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),\\ Q_2&=\frac{A_\mu}{2\pi}\int_{|\beta|<1}\frac{\partial^3_{\alpha_1}X(\alpha)-\partial^3_{\alpha_1}X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge(\omega(\alpha-\beta)-\omega(\alpha))\,d\beta\cdot\partial_{\alpha_1}X(\alpha),\\ Q_3&=\frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{|\beta|<1}\frac{\partial^3_{\alpha_1}X(\alpha)-\partial^3_{\alpha_1}X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\,d\beta\wedge\omega(\alpha)\cdot\partial_{\alpha_1}X(\alpha).\end{aligned}$$

The term Q_1 can be estimated as before; regarding Q_2, we can use the identity

$$\partial^3_{\alpha_1}X(\alpha)-\partial^3_{\alpha_1}X(\alpha-\beta)=\int_0^1\nabla\partial^3_{\alpha_1}X(\alpha+(s-1)\beta)\,ds\cdot\beta,$$

and the control of Q_3 can be approached as we did with the operator in (A-7). Similarly with M_3, while M_4 is analogous to J_6, and all these observations together allow us to obtain (5-5).

6. Controlling the Birkhoff–Rott integral

Here we consider estimates for the Birkhoff–Rott integral along a non-self-intersecting surface. Let us assume that $\nabla(X(\alpha)-(\alpha,0))\in H^k(\mathbb{R}^2)$ for $k\ge 3$, and that both $F(X)$ and $|N|^{-1}$ are in L^∞, where

$$F(X)(\alpha,\beta)=|\beta|\big/|X(\alpha)-X(\alpha-\beta)|\quad\text{and}\quad N(\alpha)=\partial_{\alpha_1}X(\alpha)\wedge\partial_{\alpha_2}X(\alpha).$$

The main purpose of this section is to prove the estimate

$$\|\,\mathrm{BR}(X,\omega)\|_{H^{k-1}}\le P\big(\|X\|^2_k+\|F(X)\|^2_{L^\infty}+\||N|^{-1}\|_{L^\infty}\big),\tag{6-1}$$

for $k \geq 4$. Here we shall show it when $k = 4$, because the other cases, $k > 4$, follow by similar arguments. We rewrite BR in the following manner:

$$\mathrm{BR}(X, \omega)(\alpha, t) = -\frac{1}{4\pi} \mathrm{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\beta)}{|X(\alpha) - X(\beta)|^3} \wedge (\partial_{\beta_2}(\Omega \partial_{\beta_1} X) - \partial_{\beta_1}(\Omega \partial_{\beta_2} X))(\beta)\, d\beta,$$

which, together with the estimates about Ω in Section 5 and also about the operator $\mathcal{T}_1$ in the Appendix, yields

$$\|\mathrm{BR}(X, \omega)\|_{L^2} \leq P\big(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big).$$

To estimate derivatives of order 3, we consider $\partial_{\alpha_i}^3(\mathrm{BR}(X, \omega))$, and observe that the most dangerous terms are given by

$$I_1 = -\frac{1}{4\pi} \mathrm{PV} \int_{\mathbb{R}^2} \frac{(\partial_{\alpha_i}^3 X(\alpha) - \partial_{\alpha_i}^3 X(\alpha - \beta)) \wedge \omega(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3}\, d\beta,$$

$$I_2 = \frac{3}{4\pi} \mathrm{PV} \int_{\mathbb{R}^2} (X(\alpha) - X(\alpha - \beta)) \wedge \omega(\alpha - \beta) \frac{(X(\alpha) - X(\alpha - \beta)) \cdot (\partial_{\alpha_i}^3 X(\alpha) - \partial_{\alpha_i}^3 X(\alpha - \beta))}{|X(\alpha) - X(\alpha - \beta)|^5}\, d\beta,$$

$$I_3 = -\frac{1}{4\pi} \mathrm{PV} \int_{\mathbb{R}^2} \frac{(X(\alpha) - X(\alpha - \beta)) \wedge (\partial_{\alpha_i}^3 \omega)(\alpha - \beta)}{|X(\alpha) - X(\alpha - \beta)|^3}\, d\beta.$$

In the Appendix, we find all the ingredients needed to estimate these terms I_j, while the remainder in $\partial_{\alpha_i}^3(\mathrm{BR}(X, \omega))$ is easily bounded: in I_3 we can recognize an operator with the form of $\mathcal{T}_1$ in (A-5), so the estimate for ω in Section 5 gives the desired control for I_3. Regarding I_1, we may use the splitting $I_1 = J_1 + J_2$, where

$$J_1 = \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{(\partial_{\alpha_i}^3 X(\alpha) - \partial_{\alpha_i}^3 X(\alpha - \beta)) \wedge (\omega(\alpha) - \omega(\alpha - \beta))}{|X(\alpha) - X(\alpha - \beta)|^3}\, d\beta,$$

$$J_2 = \frac{\omega(\alpha)}{4\pi} \wedge \mathrm{PV} \int_{\mathbb{R}^2} \frac{(\partial_{\alpha_i}^3 X(\alpha) - \partial_{\alpha_i}^3 X(\alpha - \beta))}{|X(\alpha) - X(\alpha - \beta)|^3}\, d\beta.$$

Then the identity $\partial_{\alpha_i}^3 X(\alpha) - \partial_{\alpha_i}^3 X(\alpha - \beta) = \beta \cdot \int_0^1 \nabla \partial_{\alpha_i}^3 X(\alpha + (s-1)\beta)\, ds$ allows us to find in J_1 a kernel of degree -1 which we know how to handle (see the Appendix). One uses the estimate for $\mathcal{T}_3$ (A-7) to deal with J_2, and we proceed similarly to control I_2.

7. In search of the Rayleigh–Taylor condition

As was pointed out in Section 4 (outline of the proof), our approach is based on energy estimates, and a crucial step is to characterize those terms involving higher derivatives which are controlled because they have the appropriate sign. In our terminology, they constitute the Rayleigh–Taylor condition, which is supposed to hold at time $T = 0$, it being an important part of the proof to show that it prevails under the evolution.

Let us introduce the notation

$$|||X|||_k^2 = \|X\|_k^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty},$$

where

$$\|X\|_k = \|X_1-\alpha_1\|_{L^3} + \|X_2-\alpha_2\|_{L^3} + \|X_3\|_{L^2} + \|\nabla(X-(\alpha,0))\|^2_{H^{k-1}} \tag{7-1}$$

and

$$\|\nabla(X-(\alpha,0))\|^2_{H^{k-1}} = \|\nabla(X-(\alpha,0))\|^2_{L^2} + \|\partial^k_{\alpha_1}(X-(\alpha,0))\|^2_{L^2} + \|\partial^k_{\alpha_2}(X-(\alpha,0))\|^2_{L^2}.$$

In order to justify the formula

$$\frac{d}{dt}\|X\|^2_k(t) \le -\sum_{i=1,2}\frac{2^{3/2}}{(\mu_1+\mu_2)}\int_{\mathbb{R}^2}\frac{\sigma(\alpha,t)}{|\nabla X(\alpha,t)|^3}\,\partial^k_{\alpha_i}X(\alpha,t)\cdot\Lambda(\partial^k_{\alpha_i}X)(\alpha,t)\,d\alpha + P(\||X\||_k(t)),$$

(here $k \ge 4$, although for the sake of simplicity we will present the explicit computations when $k = 4$, leaving the other cases as an exercise for the interested reader), it will be convenient to make use of the following tools, which give us different kinds of cancellations, and which constitute our particular bestiary of formulas for this paper.

From the definition of the isothermal parametrization, we have the identities

$$|\partial_{\alpha_1}X|^2 = |\partial_{\alpha_2}X|^2, \tag{7-2}$$

$$\partial_{\alpha_1}X\cdot\partial_{\alpha_2}X = 0, \tag{7-3}$$

which yield

$$\tfrac{1}{2}\Delta(|\partial_{\alpha_1}X|^2) = |\partial_{\alpha_1}\partial_{\alpha_2}X|^2 - \partial^2_{\alpha_1}X\cdot\partial^2_{\alpha_2}X, \tag{7-4}$$

$$\partial^4_{\alpha_1}X\cdot\partial_{\alpha_1}X = -3\partial^3_{\alpha_1}X\cdot\partial^2_{\alpha_1}X + (\partial^2_{\alpha_1}\Delta^{-1}\partial_{\alpha_1})\big(|\partial_{\alpha_1}\partial_{\alpha_2}X|^2 - \partial^2_{\alpha_1}X\cdot\partial^2_{\alpha_2}X\big), \tag{7-5}$$

$$\partial^4_{\alpha_2}X\cdot\partial_{\alpha_2}X = -3\partial^3_{\alpha_2}X\cdot\partial^2_{\alpha_2}X + (\partial^2_{\alpha_2}\Delta^{-1}\partial_{\alpha_2})\big(|\partial_{\alpha_1}\partial_{\alpha_2}X|^2 - \partial^2_{\alpha_1}X\cdot\partial^2_{\alpha_2}X\big). \tag{7-6}$$

Using (7-3) and (7-4), we obtain

$$\partial^4_{\alpha_1}X\cdot\partial_{\alpha_2}X = -2\partial^3_{\alpha_1}X\cdot\partial_{\alpha_1}\partial_{\alpha_2}X - \partial^2_{\alpha_1}\partial_{\alpha_2}X\cdot\partial^2_{\alpha_1}X - (\partial_{\alpha_1}\partial_{\alpha_2}\Delta^{-1}\partial_{\alpha_1})\big(|\partial_{\alpha_1}\partial_{\alpha_2}X|^2 - \partial^2_{\alpha_1}X\cdot\partial^2_{\alpha_2}X\big), \tag{7-7}$$

$$\partial^4_{\alpha_2}X\cdot\partial_{\alpha_1}X = -2\partial^3_{\alpha_2}X\cdot\partial_{\alpha_1}\partial_{\alpha_2}X - \partial^2_{\alpha_2}\partial_{\alpha_1}X\cdot\partial^2_{\alpha_2}X - (\partial_{\alpha_1}\partial_{\alpha_2}\Delta^{-1}\partial_{\alpha_2})\big(|\partial_{\alpha_1}\partial_{\alpha_2}X|^2 - \partial^2_{\alpha_1}X\cdot\partial^2_{\alpha_2}X\big). \tag{7-8}$$

And Sobolev inequalities imply that if $\nabla(X-(\alpha,0)) \in H^3$, then $\partial^4_{\alpha_i}X\cdot\partial_{\alpha_j}X \in H^3$ for $i, j = 1, 2$.

With the help of the estimates above, we may now determine σ. There is a part that may be considered a mere "algebraic" manipulation to detect the relevant characters and, in so doing, we disregard many terms because they are of lower order in the sense of Sobolev spaces. At the end, we shall present how to deal with those lower-order terms — if not for the whole collection of them, at least for the ones that we may consider to be the most "dangerous" characters. Here it is convenient to recommend to the reader our previous works [Córdoba and Gancedo 2007; Córdoba et al. 2011], where similar estimates were carried out.

Low-order norms. Since $X_i(\alpha) \to \alpha_i$ for $i = 1, 2$ at infinity, let us consider the evolution of the L^3 norm. That is,

$$\frac{1}{3}\frac{d}{dt}\|X_1-\alpha_1\|^3_{L^3}(t) = \int_{\mathbb{R}^2}|X_1-\alpha_1|(X_1-\alpha_1)X_{1t}\,d\alpha = I_1+I_2+I_3,$$

where

$$I_1 = \int_{\mathbb{R}^2} |X_1 - \alpha_1|(X_1 - \alpha_1)\,\mathrm{BR}_1\,d\alpha,$$
$$I_2 = \int_{\mathbb{R}^2} |X_1 - \alpha_1|(X_1 - \alpha_1)C_1 \partial_{\alpha_1} X_1 d\alpha,$$
$$I_3 = \int_{\mathbb{R}^2} |X_1 - \alpha_1|(X_1 - \alpha_1)C_2 \partial_{\alpha_2} X_1 d\alpha.$$

Then we have

$$I_1 \le \|X_1 - \alpha_1\|_{L^3}^2 \|\,\mathrm{BR}\,\|_{L^3} \le C\big(\|X_1 - \alpha_1\|_{L^3}^3 + \|\,\mathrm{BR}\,\|_{L^\infty} \|\,\mathrm{BR}\,\|_{L^2}^2\big),$$

and Sobolev estimates, together with (6-1), yield the appropriate control in terms of $P(|||X|||_k)$.

Next, since $\partial_{\alpha_1} X_1 \to 1$ as $\alpha \to \infty$, we have

$$I_2 \le \|\partial_{\alpha_1} X_1\|_{L^\infty} \|X_1 - \alpha_1\|_{L^3}^2 \|C_1\|_{L^3},$$

and it remains to get control of C_1. Using (3-1), we introduce the splitting $C_1 = \sum_{j=1}^4 C_1^j$, where

$$C_1^1(\alpha) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1}{|\alpha-\beta|^2}\,\mathrm{BR}_{\beta_2} \cdot \frac{X_{\beta_2}}{|X_{\beta_2}|^2}\,d\beta, \qquad C_1^2(\alpha) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1}{|\alpha-\beta|^2}\,\mathrm{BR}_{\beta_1} \cdot \frac{X_{\beta_1}}{|X_{\beta_2}|^2}\,d\beta,$$
$$C_1^3(\alpha) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\alpha_2 - \beta_2}{|\alpha-\beta|^2}\,\mathrm{BR}_{\beta_1} \cdot \frac{X_{\beta_2}}{|X_{\beta_1}|^2}\,d\beta, \qquad C_1^4(\alpha) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1}{|\alpha-\beta|^2}\,\mathrm{BR}_{\beta_2} \cdot \frac{X_{\beta_1}}{|X_{\beta_1}|^2}\,d\beta.$$

We shall show how to control C_1^1, because the estimates for the other terms follow by similar arguments. Integrating by parts, one obtains $C_1^1 = D_1 + D_2$, where

$$D_1 = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1}{|\alpha-\beta|^2}\,\mathrm{BR} \cdot \partial_{\beta_2}\Big(\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\Big)\,d\beta, \qquad D_2 = -\frac{1}{\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)}{|\alpha-\beta|^4}\,\mathrm{BR} \cdot \frac{X_{\beta_2}}{|X_{\beta_2}|^2}\,d\beta.$$

Regarding D_1, we write $D_1 = E_1 + E_2$, where

$$E_1 = \frac{-1}{2\pi}\int_{|\beta|<1} \frac{\beta_1}{|\beta|^2}\,\mathrm{BR}(\alpha-\beta) \cdot \partial_{\beta_2}\Big(\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\Big)(\alpha-\beta)\,d\beta,$$
$$E_2 = \frac{-1}{2\pi}\int_{|\beta|>1} \frac{\beta_1}{|\beta|^2}\,\mathrm{BR}(\alpha-\beta) \cdot \partial_{\beta_2}\Big(\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\Big)(\alpha-\beta)\,d\beta.$$

The Minkowski and Young inequalities yield, respectively,

$$\|E_1\|_{L^3} \le C\Big\|\mathrm{BR} \cdot \partial_{\beta_2}\Big(\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\Big)\Big\|_{L^3} \le P(|||X|||_4),$$
$$\|E_2\|_{L^3} \le C\Big\|\mathrm{BR} \cdot \partial_{\beta_2}\Big(\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\Big)\Big\|_{L^1} \le C\|\,\mathrm{BR}\,\|_{L^2}\Big\|\partial_{\beta_2}\Big(\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\Big)\Big\|_{L^2} \le P(|||X|||_4),$$

and the desired control is achieved. In the term D_2, we have a double Riesz transform, and the standard

Calderón–Zygmund theory yields

$$\|D_2\|_{L^3} \leq C\left\|\mathrm{BR}\cdot\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\right\|_{L^3} \leq C\||X_{\beta_2}|^{-1}\|_{L^\infty}\|\,\mathrm{BR}\,\|_{L^3} \leq P(\|\!|X|\!\|_4).$$

The estimate for I_3 follows on a similar path, and the case of the second coordinate is also identical:

$$\frac{1}{3}\frac{d}{dt}\|X_2-\alpha_2\|^3_{L^3}(t) \leq P(\|\!|X|\!\|_4).$$

Regarding the third coordinate, we have stronger decay because of the asymptotic flatness hypothesis:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|X_3\|^2_{L^2}(t) &= \int_{\mathbb{R}^2} X_3\,\mathrm{BR}_3\;d\alpha + \int_{\mathbb{R}^2} X_3 C_1\partial_{\alpha_1}X_3\,d\alpha + \int_{\mathbb{R}^2} X_3 C_2\partial_{\alpha_2}X_3\,d\alpha\\ &= \int_{\mathbb{R}^2} X_3\,\mathrm{BR}_3\,d\alpha - \frac{1}{2}\int_{\mathbb{R}^2}(\partial_{\alpha_1}C_1+\partial_{\alpha_2}C_2)|X_3|^2\,d\alpha,\end{aligned}$$

and therefore the use of Sobolev's embedding in the formulas for C_1 (3-1) and C_2 (3-2), together with the estimates for BR (6-1), allows us to obtain:

$$\frac{1}{2}\frac{d}{dt}\|X_3\|^2_{L^2}(t) \leq P(\|\!|X|\!\|_4).$$

Once we have control of higher-order derivatives, we can use the estimates of the Appendix to get

$$\frac{1}{2}\frac{d}{dt}\big\|\nabla(X-(\alpha,0))\big\|^2_{L^2}(t) \leq P(\|\!|X|\!\|_4).$$

Higher-order norms. Let us now consider

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^4_{\alpha_1}X\|^2_{L^2}(t)\\ &\quad= \int_{\mathbb{R}^2}\partial^4_{\alpha_1}X\cdot\partial^4_{\alpha_1}\mathrm{BR}(X,\omega)\,d\alpha + \int_{\mathbb{R}^2}\partial^4_{\alpha_1}X\cdot\partial^4_{\alpha_1}(C_1\partial_{\alpha_1}X)\,d\alpha + \int_{\mathbb{R}^2}\partial^4_{\alpha_1}X\cdot\partial^4_{\alpha_1}(C_2\partial_{\alpha_2}X)\,d\alpha\\ &\quad= I_1+I_2+I_3.\end{aligned}\tag{7-9}$$

The higher-order terms in I_2 and I_3 are given by

$$J_1 = \int_{\mathbb{R}^2} C_1\partial^4_{\alpha_1}X\cdot\partial^5_{\alpha_1}X\,d\alpha, \qquad J_2 = \int_{\mathbb{R}^2}\partial^4_{\alpha_1}X\cdot\partial_{\alpha_1}X\partial^4_{\alpha_1}C_1\,d\alpha,$$

$$J_3 = \int_{\mathbb{R}^2} C_2\partial^4_{\alpha_1}X\cdot\partial^4_{\alpha_1}\partial_{\alpha_2}X\,d\alpha, \qquad J_4 = \int_{\mathbb{R}^2}\partial^4_{\alpha_1}X\cdot\partial_{\alpha_2}X\partial^4_{\alpha_1}C_2\,d\alpha.$$

Integration by parts yields

$$J_1+J_3 = -\frac{1}{2}\int_{\mathbb{R}^2}(\partial_{\alpha_1}C_1+\partial_{\alpha_2}C_2)|\partial^4_{\alpha_1}X|^2 d\alpha,$$

and therefore

$$J_1+J_3 \leq \tfrac{1}{2}\big(\|\partial_{\alpha_1}C_1\|_{L^\infty}+\|\partial_{\alpha_2}C_2\|_{L^\infty}\big)\|\partial^4_{\alpha_1}X\|^2_{L^2} \leq P(\|\!|X|\!\|_4).$$

Then in J_2 we use (7-5) to get

$$J_2 = -\int_{\mathbb{R}^2}\partial_{\alpha_1}(\partial^4_{\alpha_1}X\cdot\partial_{\alpha_1}X)\partial^3_{\alpha_1}C_1\,d\alpha \leq \|\partial_{\alpha_1}(\partial^4_{\alpha_1}X\cdot\partial_{\alpha_1}X)\|_{L^2}\|\partial^3_{\alpha_1}C_1\|_{L^2}.$$

In J_4, we use (7-7) to obtain

$$J_4 = -\int_{\mathbb{R}^2} \partial_{\alpha_1}(\partial^4_{\alpha_1} X \cdot \partial_{\alpha_2} X)\partial^3_{\alpha_1} C_2\, d\alpha \le \|\partial_{\alpha_1}(\partial^4_{\alpha_1} X \cdot \partial_{\alpha_2} X)\|_{L^2}\|\partial^3_{\alpha_1} C_2\|_{L^2}.$$

From formulas (3-1), (3-2), one realizes that C_1 and C_2 are at the same level as Birkhoff–Rott (2-5), and therefore, we can use the estimates for BR (6-1) to control $\|\partial^3_{\alpha_1} C_i\|_{L^2}$, $i = 1, 2$. Then formulas (7-5) and (7-7) indicate how to estimate $\|\partial_{\alpha_1}(\partial^4_{\alpha_1} X \cdot \partial_{\alpha_i} X)\|_{L^2}$, $i = 1, 2$. That is, we have

$$J_2 + J_4 \le P(|||X|||_4).$$

In I_1, the most singular terms are given by

$$\begin{aligned} J_5 &= -\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha)\cdot \frac{(\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta))\wedge \omega(\beta)}{|X(\alpha)-X(\beta)|^3}\, d\alpha\, d\beta,\\ J_6 &= \frac{3}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha)\cdot (X(\alpha)-X(\beta))\wedge\omega(\beta)\frac{(X(\alpha)-X(\beta))\cdot(\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta))}{|X(\alpha)-X(\beta)|^5}\, d\alpha\, d\beta,\\ J_7 &= -\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha)\cdot \frac{(X(\alpha)-X(\beta))\wedge(\partial^4_{\alpha_1}\omega)(\beta)}{|X(\alpha)-X(\beta)|^3}\, d\alpha\, d\beta. \end{aligned} \tag{7-10}$$

Let us consider now the splitting $J_5 = K_1 + K_2$:

$$\begin{aligned} K_1 &= -\frac{1}{8\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha)\wedge(\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta))\cdot\frac{\omega(\beta)+\omega(\alpha)}{|X(\alpha)-X(\beta)|^3}\, d\alpha\, d\beta,\\ K_2 &= \frac{1}{8\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha)\wedge(\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta))\cdot\frac{\omega(\alpha)-\omega(\beta)}{|X(\alpha)-X(\beta)|^3}\, d\alpha\, d\beta, \end{aligned}$$

Next we exchange α and β in K_1 to get

$$\begin{aligned} K_1 &= \frac{1}{8\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\beta)\wedge(\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta))\cdot\frac{\omega(\beta)+\omega(\alpha)}{|X(\alpha)-X(\beta)|^3}\, d\alpha\, d\beta,\\ &= -\frac{1}{16\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta))\wedge(\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta))\cdot\frac{\omega(\beta)+\omega(\alpha)}{|X(\alpha)-X(\beta)|^3}\, d\alpha\, d\beta, \end{aligned}$$

and therefore we can conclude that $K_1 = 0$. In K_2 we find a singular integral with a kernel of degree -2:

$$K_2 = -\frac{1}{8\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha)\cdot\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\beta)\wedge\frac{\omega(\alpha)-\omega(\beta)}{|X(\alpha)-X(\beta)|^3}\, d\beta\, d\alpha,$$

and as is proved in the Appendix, we have

$$K_2 \le P(|||X|||_4).$$

Let us now decompose $J_6 = K_3 + K^1_4 + K^2_4 + K^1_5 + K^2_5$, where

$$K_3 = \frac{3}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha)\cdot(X(\alpha)-X(\beta))\wedge\omega(\beta)\frac{A(\alpha,\beta)\cdot(\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta))}{|X(\alpha)-X(\beta)|^5}\, d\alpha\, d\beta,$$

with $A(\alpha,\beta)=X(\alpha)-X(\beta)-\nabla X(\alpha)(\alpha-\beta)$,

$$K_4^i=-\frac{3}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4 X(\alpha)\cdot(X(\alpha)-X(\beta))\wedge\omega(\beta)\frac{(\alpha_i-\beta_i)(\partial_{\alpha_i}X(\alpha)-\partial_{\alpha_i}X(\beta))\cdot\partial_{\alpha_1}^4 X(\beta)}{|X(\alpha)-X(\beta)|^5}\,d\alpha\,d\beta$$

$$K_5^i=\frac{3}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4 X(\alpha)\cdot(X(\alpha)-X(\beta))\wedge\omega(\beta)\times\frac{(\alpha_i-\beta_i)(\partial_{\alpha_i}X(\alpha)\cdot\partial_{\alpha_1}^4 X(\alpha)-\partial_{\alpha_i}X(\beta)\cdot\partial_{\alpha_1}^4 X(\beta))}{|X(\alpha)-X(\beta)|^5}\,d\alpha\,d\beta.$$

In K_3 and K_4^i we find kernels of degree -2, and as shown in the Appendix, they behave as a Riesz transform acting on $\partial_{\alpha_1}^4 X$. In K_5^i the kernels have degree -3 and act as a Λ operator on $\partial_{\alpha_i}X\cdot\partial_{\alpha_1}^4 X$. Then using formulas (7-5) and (7-7), we get finally the desired estimate.

We will find the R-T condition in J_7. Let us take $J_7=K_6+K_7$, where

$$K_6=-\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4 X(\alpha)\cdot\int_{\mathbb{R}^2}\Big(\frac{(X(\alpha)-X(\beta))}{|X(\alpha)-X(\beta)|^3}-\frac{\nabla X(\alpha)(\alpha-\beta)}{|\nabla X(\alpha)(\alpha-\beta)|^3}\Big)\wedge(\partial_{\alpha_1}^4\omega)(\beta)\,d\beta\,d\alpha,$$

$$K_7=-\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4 X(\alpha)\cdot\int_{\mathbb{R}^2}\frac{\nabla X(\alpha)(\alpha-\beta)}{|\nabla X(\alpha)(\alpha-\beta)|^3}\wedge(\partial_{\alpha_1}^4\omega)(\beta)\,d\beta\,d\alpha.$$

The term K_6 is controlled by (A-8) in the Appendix. Using (7-2) and (7-3), we get

$$K_7=-\tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial_{\alpha_1}^4 X(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}\cdot\big(\partial_{\alpha_1}X(\alpha)\wedge R_1(\partial_{\alpha_1}^4\omega)(\alpha)+\partial_{\alpha_2}X(\alpha)\wedge R_2(\partial_{\alpha_1}^4\omega)(\alpha)\big)\,d\alpha.$$

Formula (2-3) helps us to detect the most singular terms inside K_7, which will be denoted by L_i, $i=1,\dots,8$, and are the following:

$$L_1=-\tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4 X(\alpha)\cdot\frac{\partial_{\alpha_1}X(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}\wedge R_1(\partial_{\alpha_1}^4\partial_{\alpha_2}\Omega\partial_{\alpha_1}X)(\alpha)\,d\alpha,$$

$$L_2=-\tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4 X(\alpha)\cdot\frac{\partial_{\alpha_1}X(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}\wedge R_1(\partial_{\alpha_2}\Omega\partial_{\alpha_1}^5 X)(\alpha)\,d\alpha,$$

$$L_3=\tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4 X(\alpha)\cdot\frac{\partial_{\alpha_1}X(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}\wedge R_1(\partial_{\alpha_1}^5\Omega\partial_{\alpha_2}X)(\alpha)\,d\alpha,$$

$$L_4=\tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4 X(\alpha)\cdot\frac{\partial_{\alpha_1}X(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}\wedge R_1(\partial_{\alpha_1}\Omega\partial_{\alpha_1}^4\partial_{\alpha_2}X)(\alpha)\,d\alpha,$$

$$L_5=-\tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4 X(\alpha)\cdot\frac{\partial_{\alpha_2}X(\alpha)}{|\partial_{\alpha_2}X(\alpha)|^3}\wedge R_2(\partial_{\alpha_1}^4\partial_{\alpha_2}\Omega\partial_{\alpha_1}X)(\alpha)\,d\alpha,$$

$$L_6=-\tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4 X(\alpha)\cdot\frac{\partial_{\alpha_2}X(\alpha)}{|\partial_{\alpha_2}X(\alpha)|^3}\wedge R_2(\partial_{\alpha_2}\Omega\partial_{\alpha_1}^5 X)(\alpha)\,d\alpha,$$

$$L_7=\tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4 X(\alpha)\cdot\frac{\partial_{\alpha_2}X(\alpha)}{|\partial_{\alpha_2}X(\alpha)|^3}\wedge R_2(\partial_{\alpha_1}^5\Omega\partial_{\alpha_2}X)(\alpha)\,d\alpha,$$

$$L_8=\tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4 X(\alpha)\cdot\frac{\partial_{\alpha_2}X(\alpha)}{|\partial_{\alpha_2}X(\alpha)|^3}\wedge R_2(\partial_{\alpha_1}\Omega\partial_{\alpha_1}^4\partial_{\alpha_2}X)(\alpha)\,d\alpha.$$

In L_1 we get a kernel of degree -1 of the form

$$L_1 = \tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha)\cdot \int_{\mathbb{R}^2} \frac{\alpha_1-\beta_1}{|\alpha-\beta|^3}\,\frac{\partial_{\alpha_1}X(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}\wedge(\partial_{\alpha_1}X(\alpha)-\partial_{\alpha_1}X(\beta))\partial^4_{\alpha_1}\partial_{\alpha_2}\Omega(\beta)\,d\beta\,d\alpha,$$

which can be estimated integrating by parts throughout $\partial^4_{\alpha_1}\partial_{\alpha_2}\Omega$; the term L_7 also follows in a similar manner. In order to estimate L_2, L_4, L_6 and L_8, we realize that they can be written like (A-3) in the Appendix plus commutators of the form (A-1). Next we have to deal with L_3 and L_5: with L_3, we proceed as follows:

$$L_3 \le \tilde{L}_3 + \|\partial_{\alpha_1}|X|^{-2}\|_{L^\infty}\|\partial^4_{\alpha_1}X\|_{L^2}\big\|R_1(\partial^5_{\alpha_1}\Omega\partial_{\alpha_2}X)-R_1(\partial^5_{\alpha_1}\Omega)\partial_{\alpha_2}X\big\|_{L^2},$$

where $\tilde{L}_3$ is given by

$$\tilde{L}_3 = \tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2}\partial^4_{\alpha_1}X(\alpha)\cdot\frac{N(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}\Omega)(\alpha)\,d\alpha, \tag{7-11}$$

and the commutator estimates (A-1) show that it only remains to control $\tilde{L}_3$. We now use formula (2-8) to get $\tilde{L}_3 = M_1 + M_2$, where

$$M_1 = -A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\partial^4_{\alpha_1}X(\alpha)\cdot\frac{N(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_3)(\alpha)\,d\alpha$$

and

$$M_2 = -A_\mu\,\mathrm{PV}\int_{\mathbb{R}^2}\partial^4_{\alpha_1}X(\alpha)\cdot\frac{N(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}(R_1\partial_{\alpha_1})\big(\partial^3_{\alpha_1}(\mathrm{BR}(X,\omega)\cdot\partial_{\alpha_1}X)\big)(\alpha)\,d\alpha.$$

Then we write $M_1 = O_1 + O_2 + O_3$, where

$$\begin{aligned}
O_1 &= -A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X_1}{|\partial_{\alpha_1}X|^3}(\partial_{\alpha_1}X_2\partial_{\alpha_2}X_3-\partial_{\alpha_1}X_3\partial_{\alpha_2}X_2)(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_3)\,d\alpha,\\
O_2 &= -A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X_2}{|\partial_{\alpha_1}X|^3}(\partial_{\alpha_1}X_3\partial_{\alpha_2}X_1-\partial_{\alpha_1}X_1\partial_{\alpha_2}X_3)(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_3)\,d\alpha,\\
O_3 &= -A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{N_3}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X_3(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_3)\,d\alpha.
\end{aligned}\tag{7-12}$$

Next we consider $O_1 = P_1 + P_2 + P_3$, with

$$\begin{aligned}
P_1 &= -A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X_1}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_1}X_2(R_1\partial_{\alpha_1})(\partial_{\alpha_2}X_3\partial^4_{\alpha_1}X_3)\,d\alpha,\\
P_2 &= A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X_1}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_2}X_2(R_1\partial_{\alpha_1})(\partial_{\alpha_1}X_3\partial^4_{\alpha_1}X_3)\,d\alpha,\\
P_3 &= A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X_1}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_1}X_2\big[(R_1\partial_{\alpha_1})(\partial_{\alpha_2}X_3\partial^4_{\alpha_1}X_3)-\partial_{\alpha_2}X_3(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_3)\big]\,d\alpha\\
&\quad + A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X_1}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_2}X_2\big[\partial_{\alpha_1}X_3(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_3)-(R_1\partial_{\alpha_1})(\partial_{\alpha_1}X_3\partial^4_{\alpha_1}X_3)\big]\,d\alpha,
\end{aligned}$$

and the commutator estimate allows us to control the term P_3.

Now we use (7-7) to write $P_1 = Q_1 + Q_2 + Q_3$:

$$Q_1 = A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_2 (R_1\partial_{\alpha_1})(\partial_{\alpha_2} X_1 \partial^4_{\alpha_1} X_1)\, d\alpha,$$

$$Q_2 = A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_2 (R_1\partial_{\alpha_1})(\partial_{\alpha_2} X_2 \partial^4_{\alpha_1} X_2)\, d\alpha,$$

$$Q_3 = A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_2 (R_1\partial_{\alpha_1})(\text{lower-order terms})\, d\alpha.$$

The term Q_3 is easily estimated. Regarding P_2, equality (7-5) allows us to write $P_2 = Q_4 + Q_5 + Q_6$, where

$$Q_4 = -A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_2 (R_1\partial_{\alpha_1})(\partial_{\alpha_1} X_1 \partial^4_{\alpha_1} X_1)\, d\alpha,$$

$$Q_5 = -A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_2 (R_1\partial_{\alpha_1})(\partial_{\alpha_1} X_2 \partial^4_{\alpha_1} X_2)\, d\alpha,$$

$$Q_6 = -A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_2 (R_1\partial_{\alpha_1})(\text{lower-order terms})\, d\alpha.$$

Let us recall the identity $P_1 + P_2 = (Q_4 + Q_1) + (Q_2 + Q_5) + (Q_3 + Q_6)$, where Q_3 and Q_6 are easily estimated. With respect to $Q_2 + Q_5$, we have

$$\begin{aligned} Q_2 + Q_5 = A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_2 \big[(R_1\partial_{\alpha_1})(\partial_{\alpha_2} X_2 \partial^4_{\alpha_1} X_2) - \partial_{\alpha_2} X_2 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_2)\big]\, d\alpha \\ + A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_2 \big[\partial_{\alpha_1} X_2 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_2) - (R_1\partial_{\alpha_1})(\partial_{\alpha_1} X_2 \partial^4_{\alpha_1} X_2)\big]\, d\alpha, \end{aligned}$$

and again the commutator estimates yield the desired control.

Next we have

$$\begin{aligned} Q_4 + Q_1 = A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_2 \big[\partial_{\alpha_1} X_1 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1) - (R_1\partial_{\alpha_1})(\partial_{\alpha_1} X_1 \partial^4_{\alpha_1} X_1)\big]\, d\alpha \\ + A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_2 \big[(R_1\partial_{\alpha_1})(\partial_{\alpha_2} X_1 \partial^4_{\alpha_1} X_1) - \partial_{\alpha_2} X_1 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\big]\, d\alpha \\ - A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{N_3}{|\partial_{\alpha_1} X|^3}\, \partial^4_{\alpha_1} X_1 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\, d\alpha. \end{aligned}$$

The first two integrals above are easily handled, allowing us to get

$$O_1 = P_1 + P_2 + P_3 \le P(|\!|\!| X |\!|\!|_4) - A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{N_3}{|\partial_{\alpha_1} X|^3}\, \partial^4_{\alpha_1} X_1 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\, d\alpha. \tag{7-13}$$

For the term O_2, we proceed in a similar manner, first checking that $O_2 = P_4 + P_5 + P_6$:

$$P_4 = A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_1 (R_1 \partial_{\alpha_1})(\partial_{\alpha_2} X_3 \partial^4_{\alpha_1} X_3)\, d\alpha,$$

$$P_5 = -A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_1 (R_1 \partial_{\alpha_1})(\partial_{\alpha_1} X_3 \partial^4_{\alpha_1} X_3)\, d\alpha,$$

$$P_6 = A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_1 \big[\partial_{\alpha_2} X_3 (R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_3) - (R_1 \partial_{\alpha_1})(\partial_{\alpha_2} X_3 \partial^4_{\alpha_1} X_3)\big]\, d\alpha$$
$$+ A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_1 \big[(R_1 \partial_{\alpha_1})(\partial_{\alpha_1} X_3 \partial^4_{\alpha_1} X_3) - \partial_{\alpha_1} X_3 (R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_3)\big]\, d\alpha.$$

We control P_6 as before. Regarding P_4, we use (7-7) to write it in the form $P_4 = S_1 + S_2 + S_3$, where

$$S_1 = -A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_1 (R_1 \partial_{\alpha_1})(\partial_{\alpha_2} X_1 \partial^4_{\alpha_1} X_1)\, d\alpha,$$

$$S_2 = -A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_1 (R_1 \partial_{\alpha_1})(\partial_{\alpha_2} X_2 \partial^4_{\alpha_1} X_2)\, d\alpha,$$

$$S_3 = -A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_1 (R_1 \partial_{\alpha_1})(\text{lower-order terms})\, d\alpha.$$

The identity (7-5) allows us to write $P_5 = S_4 + S_5 + S_6$, where

$$S_4 = A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_1 (R_1 \partial_{\alpha_1})(\partial_{\alpha_1} X_1 \partial^4_{\alpha_1} X_1)\, d\alpha,$$

$$S_5 = A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_1 (R_1 \partial_{\alpha_1})(\partial_{\alpha_1} X_2 \partial^4_{\alpha_1} X_2)\, d\alpha,$$

$$S_6 = A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_1 (R_1 \partial_{\alpha_1})(\text{lower-order terms})\, d\alpha.$$

Next, we reorganize the sum in the form

$$P_4 + P_6 = (S_1 + S_4) + (S_2 + S_5) + (S_3 + S_6),$$

where the term $S_3 + S_6$ can be easily estimated. Regarding $S_1 + S_4$, we have

$$S_1 + S_4 = A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_1 \big[\partial_{\alpha_2} X_1 (R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_1) - (R_1 \partial_{\alpha_1})(\partial_{\alpha_2} X_1 \partial^4_{\alpha_1} X_1)\big]\, d\alpha$$
$$+ A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_1 \big[(R_1 \partial_{\alpha_1})(\partial_{\alpha_1} X_1 \partial^4_{\alpha_1} X_1) - \partial_{\alpha_1} X_1 (R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\big]\, d\alpha,$$

and the commutator estimates give us precise control.

Let us consider now

$$
\begin{aligned}
S_2+S_5 = A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1}X_2}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_1}X_1\big[\partial_{\alpha_2}X_2(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_2)-(R_1\partial_{\alpha_1})(\partial_{\alpha_2}X_2\partial^4_{\alpha_1}X_2)\big]\,d\alpha\\
+A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1}X_2}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_2}X_1\big[(R_1\partial_{\alpha_1})(\partial_{\alpha_1}X_2\partial^4_{\alpha_1}X_2)-\partial_{\alpha_1}X_2(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_2)\big]\,d\alpha\\
-A_\rho \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{N_3}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X_2(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_2)\,d\alpha.
\end{aligned}
$$

Here again the commutator estimates control the first two integrals above, allowing us to conclude that

$$
O_2=P_4+P_5+P_6\le P(|||X|||_4)-A_\rho\,\mathrm{PV}\!\int_{\mathbb{R}^2}\frac{N_3}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X_2(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_2)\,d\alpha. \tag{7-14}
$$

Furthermore, inequalities (7-13), (7-14) and (7-12) yield

$$
M_1=O_1+O_2+O_3\le P(|||X|||_4)-A_\rho\,\mathrm{PV}\!\int_{\mathbb{R}^2}\frac{N_3}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X\cdot(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X)\,d\alpha, \tag{7-15}
$$

and at this point we begin to recognize the Rayleigh–Taylor condition in the nonintegrable terms. Let us return now to the term M_2, which can be written in the form

$$
M_2=A_\mu\,\mathrm{PV}\!\int_{\mathbb{R}^2}R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\,\partial^4_{\alpha_1}(\mathrm{BR}(X,\omega)\cdot\partial_{\alpha_1}X))\,d\alpha, \tag{7-16}
$$

and whose most dangerous components are given by

$$
O_4=-\frac{A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2}R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X(\alpha)-\partial^4_{\alpha_1}X(\beta)}{|X(\alpha)-X(\beta)|^3}\wedge\omega(\beta)\cdot\partial_{\alpha_1}X(\alpha)\,d\alpha,
$$

$$
O_5=\frac{3A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2}R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}B(\alpha,\beta)(X(\alpha)-X(\beta))\wedge\omega(\beta)\cdot\partial_{\alpha_1}X(\alpha)\,d\alpha,
$$

with

$$
B(\alpha,\beta)=\frac{(X(\alpha)-X(\beta))\cdot(\partial^4_{\alpha_1}X(\alpha)-\partial^4_{\alpha_2}X(\beta))}{|X(\alpha)-X(\beta)|^5},
$$

$$
O_6=-\frac{A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2}R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{X(\alpha)-X(\beta)}{|X(\alpha)-X(\beta)|^3}\wedge\partial^4_{\alpha_1}\omega(\beta)\cdot\partial_{\alpha_1}X(\alpha)\,d\alpha,
$$

and

$$
O_7=A_\mu\,\mathrm{PV}\!\int_{\mathbb{R}^2}R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\partial_{\alpha_1}(\mathrm{BR}(X,\omega)\cdot\partial^4_{\alpha_1}X)(\alpha)\,d\alpha.
$$

The remainder terms are less singular and can be estimated with the same methods used before.

To deal with O_4, we decompose it further as $O_4 = P_7 + P_8$:

$$P_7 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X(\alpha)-\partial^4_{\alpha_1}X(\beta)}{|X(\alpha)-X(\beta)|^3}\cdot\omega(\beta)\wedge(\partial_{\alpha_1}X(\beta)-\partial_{\alpha_1}X(\alpha))\,d\beta\,d\alpha,$$

$$P_8 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X(\alpha)-\partial^4_{\alpha_1}X(\beta)}{|X(\alpha)-X(\beta)|^3}\cdot N(\beta)\partial_{\alpha_1}\Omega(\beta)\,d\beta\,d\alpha,$$

where in P_8, we have used formula (2-3) to get

$$\omega\wedge\partial_{\alpha_1}X = N\partial_{\alpha_1}\Omega.$$

In the integral (with respect to β) of P_7, we have a kernel of degree -2 applied to 4 derivatives, which can be estimated easily. Next let us consider $P_8 = Q_7 + Q_8 + Q_9$, where

$$Q_7 = -\frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\partial^4_{\alpha_1}X(\alpha)\cdot\int_{\mathbb{R}^2}\frac{N(\alpha)\partial_{\alpha_1}\Omega(\alpha)-N(\beta)\partial_{\alpha_1}\Omega(\beta)}{|X(\alpha)-X(\beta)|^3}\,d\beta\,d\alpha,$$

$$Q_8 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\big((\partial_{\alpha_1}\Omega N\cdot\partial^4_{\alpha_1}X)(\alpha)-(\partial_{\alpha_1}\Omega N\cdot\partial^4_{\alpha_1}X)(\beta)\big)C(\alpha,\beta)\,d\beta\,d\alpha,$$

and

$$C(\alpha,\beta) = \frac{1}{|X(\alpha)-X(\beta)|^3}-\frac{1}{|\nabla X(\alpha)(\alpha-\beta)|^3},$$

$$Q_9 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\frac{1}{|\partial_{\alpha_1}X(\alpha)|^3}\Lambda(\partial_{\alpha_1}\Omega N\cdot\partial^4_{\alpha_1}X)(\alpha)\,d\alpha.$$

In Q_7, we have

$$Q_7 \le \Big\|R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\Big\|_{L^2}\|\partial^4_{\alpha_1}X\|_{L^2}\sup_\alpha\Big|\int_{\mathbb{R}^2}\frac{N(\alpha)\partial_{\alpha_1}\Omega(\alpha)-N(\beta)\partial_{\alpha_1}\Omega(\beta)}{|X(\alpha)-X(\beta)|^3}\,d\beta\Big|,$$

giving us the appropriate control, which can be also obtained in Q_8 because the corresponding kernel has degree -2. Regarding Q_9, we have the expression

$$Q_9 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\Big[\frac{1}{|\partial_{\alpha_1}X|^3}\Lambda(\partial_{\alpha_1}\Omega N\cdot\partial^4_{\alpha_1}X)-\Lambda\Big(\frac{\partial_{\alpha_1}\Omega N\cdot\partial^4_{\alpha_1}X}{|\partial_{\alpha_1}X|^3}\Big)\Big]\,d\alpha$$
$$+\frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\Lambda\Big(\partial_{\alpha_1}\Omega\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\,d\alpha.$$

Then we use (A-2) to control the first integral above, and since $\Lambda = R_1\partial_{\alpha_1}+R_2\partial_{\alpha_2}$ by (A-4), we can also take care of the second term.

With O_5, one proceeds as we did with J_6 (7-10) to get the desired estimate.

Next, we use (2-3) to catch the most singular terms in O_6, which are given by

$$S_7 = -\frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{(X(\alpha)-X(\beta))\wedge\partial_{\alpha_1}X(\beta)\cdot\partial_{\alpha_1}X(\alpha)}{|X(\alpha)-X(\beta)|^3}\,\partial^4_{\alpha_1}\partial_{\alpha_2}\Omega(\beta)\,d\alpha,$$

$$S_8 = -\frac{A_\mu}{8\pi^2}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{(X(\alpha)-X(\beta))\wedge\partial_{\alpha_1}X(\alpha)}{|X(\alpha)-X(\beta)|^3}\cdot\partial_{\alpha_2}\Omega(\beta)\partial^5_{\alpha_1}X(\beta)\,d\alpha,$$

$$S_9 = \frac{A_\mu}{8\pi^2}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{(X(\alpha)-X(\beta))\wedge\partial_{\alpha_2}X(\beta)\cdot\partial_{\alpha_1}X(\alpha)}{|X(\alpha)-X(\beta)|^3}\,\partial^5_{\alpha_1}\Omega(\beta)\,d\alpha,$$

$$S_{10} = \frac{A_\mu}{8\pi^2}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{(X(\alpha)-X(\beta))\wedge\partial_{\alpha_1}X(\alpha)}{|X(\alpha)-X(\beta)|^3}\cdot\partial_{\alpha_1}\Omega(\beta)\partial^4_{\alpha_1}\partial_{\alpha_2}X(\beta)\,d\alpha.$$

One may write

$$S_7 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{(X(\alpha)-X(\beta))\wedge(\partial_{\alpha_1}X(\alpha)-\partial_{\alpha_1}X(\beta))\cdot\partial_{\alpha_1}X(\beta)}{|X(\alpha)-X(\beta)|^3}\,\partial^4_{\alpha_1}\partial_{\alpha_2}\Omega(\beta)\,d\alpha,$$

expressing the fact that we have a kernel of degree -1 applied to $\partial^4_{\alpha_1}\partial_{\alpha_2}\Omega$, and therefore an integration by parts gives us the desired control, as before. To treat S_8, we further decompose $S_8 = T_1 + T_2$:

$$T_1 = -\frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2} D(\alpha,\beta)\cdot\partial_{\alpha_2}\Omega(\beta)\partial^5_{\alpha_1}X(\beta)\,d\alpha,$$

where

$$D(\alpha,\beta) = \Big(\frac{(X(\alpha)-X(\beta))}{|X(\alpha)-X(\beta)|^3} - \frac{\nabla X(\alpha)(\alpha-\beta)}{|\nabla X(\alpha)(\alpha-\beta)|^3}\Big)\wedge\partial_{\alpha_1}X(\alpha)$$

and

$$T_2 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\frac{N(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}\cdot R_2(\partial_{\alpha_2}\Omega\partial^5_{\alpha_1}X)(\alpha)\,d\alpha.$$

In T_1, we use the estimate for the operator (A-8). The term T_2 reads as follows:

$$\begin{aligned} T_2 = &-\frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\frac{N}{|\partial_{\alpha_1}X|^3}\cdot R_2(\partial_{\alpha_2}\partial_{\alpha_1}\Omega\partial^4_{\alpha_1}X)\,d\alpha \\ &+\frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\Big[\frac{N}{|\partial_{\alpha_1}X|^3}\cdot(R_2\partial_{\alpha_1})(\partial_{\alpha_2}\Omega\partial^4_{\alpha_1}X)-(R_2\partial_{\alpha_1})(\partial_{\alpha_2}\Omega\frac{N\cdot\partial^4_{\alpha_1}X}{|\partial_{\alpha_1}X|^3})\Big]\,d\alpha \\ &-\frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(R_2\partial_{\alpha_1})\Big(\partial_{\alpha_2}\Omega\frac{N\cdot\partial^4_{\alpha_1}X}{|\partial_{\alpha_1}X|^3}\Big)\,d\alpha. \end{aligned}$$

The first integral above is easy to estimate, while for the second one we use (A-1), and (A-4) for the third.

For the next term, one has $S_9 = T_3 + T_4$, where

$$T_3 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{(X(\alpha)-X(\beta))\cdot\partial_{\alpha_2}X(\beta)\wedge(\partial_{\alpha_1}X(\alpha)-\partial_{\alpha_1}X(\beta))}{|X(\alpha)-X(\beta)|^3}\,\partial^5_{\alpha_1}\Omega(\beta)\,d\alpha,$$

$$T_4 = -A_\mu\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\mathfrak{D}(\partial^5_{\alpha_1}\Omega)\,d\alpha,$$

Proceeding as before, we get bounds for T_3, and the double-layer potential estimates help us to control T_4.

For S_{10}, one can adapt exactly the same approach used for S_8. Finally, we have to deal with O_7, which is given by

$$O_7 = -A_\mu \,\mathrm{PV}\int_{\mathbb{R}^2} \mathrm{BR}(X,\omega)\cdot \partial^4_{\alpha_1} X (R_1\partial_{\alpha_1})\left(\frac{\partial^4_{\alpha_1} X\cdot N}{|\partial_{\alpha_1} X|^3}\right) d\alpha,$$

after an integration by parts. Let us introduce the splitting $O_7 = \sum_{j,k=1}^3 U_j^k$, where

$$U_j^k = -A_\mu \,\mathrm{PV}\int_{\mathbb{R}^2} \mathrm{BR}_j(X,\omega)\partial^4_{\alpha_1} X_j (R_1\partial_{\alpha_1})\left(\frac{\partial^4_{\alpha_1} X_k N_k}{|\partial_{\alpha_1} X|^3}\right) d\alpha.$$

Then the commutator estimates allow us to write $U_j^k = V_j^k + \text{lower order terms}$, where

$$V_j^k = -A_\mu \,\mathrm{PV}\int_{\mathbb{R}^2} \mathrm{BR}_j(X,\omega)\partial^4_{\alpha_1} X_j \frac{N_k}{|\partial_{\alpha_1} X|^3} (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_k)\, d\alpha.$$

Using (7-5) and (7-7), one has

$$N_1\partial^4_{\alpha_1} X_2 = N_2\partial^4_{\alpha_1} X_1 + \text{lower-order terms},$$

so that V_2^1 becomes

$$V_2^1 = -A_\mu \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\mathrm{BR}_2(X,\omega)N_2}{|\partial_{\alpha_1} X|^3}\, \partial^4_{\alpha_1} X_1 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\, d\alpha - A_\mu \,\mathrm{PV}\int_{\mathbb{R}^2} f (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\, d\alpha,$$

where f is at the level of $\partial^3_{\alpha_i} X$. Integration by parts in the last integral allows us to conclude that

$$V_2^1 \le -A_\mu \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\mathrm{BR}_2(X,\omega)N_2}{|\partial_{\alpha_1} X|^3}\, \partial^4_{\alpha_1} X_1 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\, d\alpha + P(|||X|||_4).$$

With the help of (7-5) and (7-7), we also get

$$N_1\partial^4_{\alpha_1} X_3 = N_3\partial^4_{\alpha_1} X_1 + \text{lower-order terms},$$

and therefore

$$V_3^1 \le -A_\mu \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\mathrm{BR}_3(X,\omega)N_3}{|\partial_{\alpha_1} X|^3}\, \partial^4_{\alpha_1} X_1 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\, d\alpha + P(|||X|||_4).$$

Using the two inequalities above, we obtain

$$V_1^1 + V_2^1 + V_3^1 \le -A_\mu \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\mathrm{BR}(X,\omega)\cdot N}{|\partial_{\alpha_1} X|^3}\, \partial^4_{\alpha_1} X_1 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\, d\alpha + P(|||X|||_4). \tag{7-17}$$

Next, let us observe that

$$N_2\partial^4_{\alpha_1} X_1 = N_1\partial^4_{\alpha_1} X_2 + \text{lower-order terms}, \qquad N_2\partial^4_{\alpha_1} X_3 = N_3\partial^4_{\alpha_1} X_2 + \text{lower-order terms},$$

which implies the estimate

$$V_1^2 + V_2^2 + V_3^2 \le -A_\mu \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\mathrm{BR}(X,\omega)\cdot N}{|\partial_{\alpha_1} X|^3}\, \partial^4_{\alpha_1} X_2 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_2)\, d\alpha + P(|||X|||_4). \tag{7-18}$$

Regarding V_1^3 and V_2^3, the identities

$$N_3\partial_{\alpha_1}^4 X_1 = N_1\partial_{\alpha_1}^4 X_3 + \text{lower-order terms}, \qquad N_3\partial_{\alpha_1}^4 X_3 = N_2\partial_{\alpha_1}^4 X_3 + \text{lower-order terms}$$

yield

$$V_1^3+V_2^3+V_3^3 \le -A_\mu\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\mathrm{BR}(X,\omega)\cdot N}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_1}^4 X_3(R_1\partial_{\alpha_1})(\partial_{\alpha_1}^4 X_3)\,d\alpha + P(|||X|||_4). \tag{7-19}$$

Finally (7-17), (7-18) and (7-19) imply

$$\sum_{j,k=1}^{3} V_j^k \le -A_\mu\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\mathrm{BR}(X,\omega)\cdot N}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_1}^4 X\cdot(R_1\partial_{\alpha_1})(\partial_{\alpha_1}^4 X)\,d\alpha + P(|||X|||_4).$$

Now we put together the estimates (7-16)–(7-19) to conclude that

$$M_2 \le -A_\mu\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\mathrm{BR}(X,\omega)\cdot N}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_1}^4 X\cdot(R_1\partial_{\alpha_1})(\partial_{\alpha_1}^4 X)\,d\alpha + P(|||X|||_4),$$

and taking into account (7-15), we obtain

$$\tilde{L}_3 = M_1+M_2 \le -\frac{1}{\mu_2+\mu_1}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\sigma}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_1}^4 X\cdot(R_1\partial_{\alpha_1})(\partial_{\alpha_1}^4 X)\,d\alpha + P(|||X|||_4). \tag{7-20}$$

Finally, we have to work with L_5, which can be written in the following manner:

$$L_5 = \tilde{L}_5 - \tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4 X\cdot\frac{\partial_{\alpha_2}X}{|\partial_{\alpha_2}X|^3}\wedge\big[R_2(\partial_{\alpha_1}^4\partial_{\alpha_2}\Omega\partial_{\alpha_1}X) - R_2(\partial_{\alpha_1}^4\partial_{\alpha_2}\Omega)\partial_{\alpha_1}X\big]\,d\alpha,$$

where

$$\tilde{L}_5 = \tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4 X\cdot\frac{N}{|\partial_{\alpha_2}X|^3}(R_2\partial_{\alpha_2})(\partial_{\alpha_1}^4\Omega)\,d\alpha.$$

Using the commutator estimate, once more, it remains only to consider $\tilde{L}_5$, but let us point out that replacing the operator $R_1\partial_{\alpha_1}$ by $R_2\partial_{\alpha_2}$, the term $\tilde{L}_3$ (7-11) becomes $\tilde{L}_5$. Therefore, proceeding exactly as we did before, one obtains the inequality

$$\tilde{L}_5 \le -\frac{1}{\mu_2+\mu_1}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\sigma}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_1}^4 X\cdot(R_2\partial_{\alpha_2})(\partial_{\alpha_1}^4 X)\,d\alpha + P(|||X|||_4). \tag{7-21}$$

Introducing now the identity $\Lambda = (R_1\partial_{\alpha_1})+(R_2\partial_{\alpha_2})$ in (7-20) and (7-21), we get

$$\tilde{L}_3+\tilde{L}_5 \le -\frac{1}{\mu_2+\mu_1}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\sigma}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_1}^4 X\cdot\Lambda(\partial_{\alpha_1}^4 X)\,d\alpha + P(|||X|||_4).$$

Finally, all the estimates so far obtained, beginning with (7-9), allow us to write

$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1}^4 X\|_{L^2}^2(t) \le -\frac{1}{\mu_2+\mu_1}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\sigma}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_1}^4 X\cdot\Lambda(\partial_{\alpha_1}^4 X)\,d\alpha + P(|||X|||_4). \tag{7-22}$$

In a similar manner, now using equations (2-9), (7-6) and (7-8) instead of (2-8), (7-5) and (7-7) respectively, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_2}^4 X\|_{L^2}^2(t) \le -\frac{1}{\mu_2+\mu_1}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\sigma}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_2}^4 X\cdot\Lambda(\partial_{\alpha_2}^4 X)\,d\alpha + P(|||X|||_4). \tag{7-23}$$

The two inequalities (7-22) and (7-23) are the main purpose of this section.

8. Estimates for the evolution of $\|F(X)\|_{L^\infty}$ and R-T

In this section we analyze the evolution of the no-self-intersection condition of the free surface as well as the Rayleigh–Taylor property, but in order to do that, we shall need precise bounds for both ∇X_t and Ω_t.

We shall estimate $\|\nabla X_t\|_{H^k}$ by means of equality (2-4) to get

$$\|\nabla X_t\|_{H^k} \le P\big(\|X\|_{k+2}^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big), \tag{8-1}$$

for $k \ge 2$. In fact

$$\|\nabla X_t\|_{H^k} \le \|\nabla \operatorname{BR}(X,\omega)\|_{H^k} + \|\nabla(C_1\partial_{\alpha_1}X + C_2\partial_{\alpha_2}X)\|_{H^k},$$

and with the help of (6-1), we can handle both terms on the right.

Next we shall consider the norms $\|\Omega_t\|_{H^k}$ to obtain the inequality

$$\|\Omega_t\|_{H^k} \le P\big(\|X\|_{k+1}^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big), \tag{8-2}$$

for $k \ge 3$. To do that, let us take a time derivative in the identity (2-6) to get

$$\Omega_t(\alpha,t) - A_\mu\mathfrak{D}(\Omega_t)(\alpha,t) = A_\mu I_1(\alpha,t) - 2A_\rho\partial_t X_3(\alpha,t),$$

which yields

$$\|\Omega_t\|_{H^1} \le C\|(I - A_\mu\mathfrak{D})^{-1}\|_{H^1}\big(\|I_1\|_{H^1} + \|\partial_t X_3\|_{H^1}\big),$$

and since we have control of $\|(I - A_\mu\mathfrak{D})^{-1}\|_{H^1}$ and $\|\partial_t X_3\|_{H^1}$, it only remains to estimate $\|I_1\|_{H^1}$. For that purpose, let us consider the splitting $I_1 = J_1 + J_2 + J_3$, where

$$J_1 = \frac{1}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{X_t(\alpha) - X_t(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}\cdot N(\alpha-\beta)\Omega(\alpha-\beta)\,d\beta,$$

$$J_2 = -\frac{3}{4\pi}\int_{\mathbb{R}^2}\big(X(\alpha) - X(\alpha-\beta)\big)\cdot\big(X_t(\alpha) - X_t(\alpha-\beta)\big)\frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^5}\cdot N(\alpha-\beta)\Omega(\alpha-\beta)\,d\beta,$$

$$J_3 = \frac{1}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}\cdot N_t(\alpha-\beta)\Omega(\alpha-\beta)\,d\beta.$$

Proceeding as we did with the operator $\mathcal{T}_2$ (A-6) (with X_t instead of $\partial_{\alpha_j}X_k$), one gets

$$\|J_1\|_{L^2} + \|J_2\|_{L^2} \le P\big(\|X\|_4 + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big).$$

Regarding J_3, we split further:

$$J_3 = \frac{1}{2\pi}\int_{|\beta|>1} d\beta + \frac{1}{2\pi}\int_{|\beta|<1} d\beta = K_1 + K_2.$$

Since

$$|K_1(\alpha)| \le \|F(X)\|_{L^\infty}^2 \int_{|\beta|>1} \frac{|N_t(\alpha-\beta)||\Omega(\alpha-\beta)|}{2\pi|\beta|^2}\,d\beta,$$

Young's inequality yields

$$\|K_1\|_{L^2} \le \|F(X)\|^2_{L^\infty}\|N_t\Omega\|_{L^1} \le C\|F(X)\|^2_{L^\infty}\|N_t\|_{L^2}\|\Omega\|_{L^2},$$

and since we know that $\|N_t\|_{L^2} \le \|\nabla X\|_{L^\infty}\|\nabla X_t\|_{L^2}$, estimate (8-1) allows us to handle the terms K_1. The estimate for K_2 is similar to the one obtained for I_2 (A-13) in the Appendix.

Next we consider the most singular terms in $\partial_{\alpha_1} I_1$, which are given by

$$\begin{aligned}
J_4 &= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}X_t(\alpha)-\partial_{\alpha_1}X_t(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\cdot N(\alpha-\beta)\Omega(\alpha-\beta)\,d\beta,\\
J_5 &= -\frac{3}{4\pi}\int_{\mathbb{R}^2}\big(X(\alpha)-X(\alpha-\beta)\big)\\
&\qquad\cdot\big(\partial_{\alpha_1}X_t(\alpha)-\partial_{\alpha_1}X_t(\alpha-\beta)\big)\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^5}\cdot N(\alpha-\beta)\Omega(\alpha-\beta)\,d\beta,\\
J_6 &= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\cdot \partial_{\alpha_1}N_t(\alpha-\beta)\Omega(\alpha-\beta)\,d\beta,
\end{aligned}$$

because the remainder terms are easier to handle. Let us write $J_4 = K_3 + K_4$, where

$$\begin{aligned}
K_3 &= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}X_t(\alpha)-\partial_{\alpha_1}X_t(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\cdot\big(N(\alpha-\beta)\Omega(\alpha-\beta)-N(\alpha)\Omega(\alpha)\big)\,d\beta,\\
K_4 &= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}X_t(\alpha)-\partial_{\alpha_1}X_t(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\cdot N(\alpha)\Omega(\alpha)\,d\beta.
\end{aligned}$$

In K_3, the identity $\partial_{\alpha_1}X_t(\alpha)-\partial_{\alpha_1}X_t(\alpha-\beta) = \int_0^1 \nabla\partial_{\alpha_1}X_t(\alpha+(s-1)\beta)\,ds\cdot\beta$ together with (8-1) gives us the desired control. Regarding K_4, we may observe its similarity with $\mathcal{T}_3$ (A-7), so that an application to (8-1) yields the appropriate bound; J_5 can be treated in a similar manner, and J_6 is analogous to J_3. By symmetry, one could get the same estimate for $\partial_{\alpha_2} I_1$, so that finally

$$\|\Omega_t\|_{H^1} \le P\big(\|X\|_4^2 + \|F(X)\|^2_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big). \tag{8-3}$$

Next, we will show how to deal with $\|\Omega_t\|_{H^2}$. Using Equation (2-8), one gets

$$\partial^2_{\alpha_1}\Omega_t = -2A_\mu\partial_{\alpha_1}\partial_t(\mathrm{BR}(X,\omega)\cdot\partial_{\alpha_1}X) - 2A_\rho\partial^2_{\alpha_1}\partial_t X_3,$$

and with the help of (8-1), the last term above is properly controlled. To continue, we shall consider the most singular remainder terms. Namely, in $-\partial_{\alpha_1}\partial_t(\mathrm{BR}(X,\omega)\cdot\partial_{\alpha_1}X)$, we have

$$\begin{aligned}
L_1 &= -\mathrm{BR}(X,\omega)\cdot\partial^2_{\alpha_1}X_t,\\
L_2 &= \frac{1}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}X_t(\alpha)-\partial_{\alpha_1}X_t(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\omega(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),\\
L_3 &= -\frac{3}{8\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} A(\alpha,\beta)\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^5}\wedge\omega(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),
\end{aligned}$$

where $A(\alpha,\beta) = \big(X(\alpha)-X(\alpha-\beta)\big)\cdot\big(\partial_{\alpha_1}X_t(\alpha)-\partial_{\alpha_1}X_t(\alpha-\beta)\big)$,

$$L_4 = \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_1}\omega_t(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha).$$

Let us observe that $\|L_1\|_{L^2}\le\|\mathrm{BR}(X,\omega)\|_{L^\infty}\|\partial^2_{\alpha_1}X_t\|_{L^2}$, where both quantities have been appropriately controlled before. In L_2 and L_3, we have kernels of degree -2, and therefore operators analogous to $\mathcal{T}_3$ (A-7) acting on $\partial_{\alpha_1}X_t$. Therefore, using (8-1), its control follows easily. In L_4, we use the decomposition

$$L_4 = \frac{1}{2\pi}\,\mathrm{PV}\int_{|\beta|>1}d\beta+\frac{1}{2\pi}\,\mathrm{PV}\int_{|\beta|<1}d\beta = M_1+M_2.$$

Thus, an integration by parts yields

$$\|M_1\|_{L^2}\le C\|F(X)\|^3_{L^\infty}\|\nabla X\|^2_{L^\infty}\|w_t\|_{L^2}.$$

Formula (2-3), together with estimates (8-1) and (8-3), provides the appropriated bound.

Next, let us expand (2-3) to obtain the most singular terms in M_2, which are given by the integrals

$$O_1 = -\frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_2}\Omega(\alpha-\beta)\partial^2_{\alpha_1}X_t(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

$$O_2 = -\frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_1}\partial_{\alpha_2}\Omega_t(\alpha-\beta)\partial_{\alpha_1}X(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

$$O_3 = \frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_1}\Omega(\alpha-\beta)\partial_{\alpha_1}\partial_{\alpha_2}X_t(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

$$O_4 = \frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial^2_{\alpha_1}\Omega_t(\alpha-\beta)\partial_{\alpha_2}X(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha).$$

Estimate (8-1) help us with the terms O_1 and O_3, which can be treated with the same approach used for I_2 (A-13) in the Appendix. Let us write O_2 as

$$O_2 = \frac{A_\mu}{2\pi}\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_1}\partial_{\alpha_2}\Omega_t(\alpha-\beta)\big(\partial_{\alpha_1}X(\alpha)-\partial_{\alpha_1}X(\alpha-\beta)\big)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

which can be estimated integrating by parts in the variable β_1 using the identity

$$\partial_{\alpha_1}\partial_{\alpha_2}\Omega_t(\alpha-\beta) = -\partial_{\beta_1}(\partial_{\alpha_2}\Omega_t(\alpha-\beta)).$$

Let us point out that the kernel in the integral O_2 has degree -1, and therefore one can use (8-3) to control it. It remains to deal with O_4, which is decomposed in the form $O_4 = P_1+P_2$, where

$$P_1 = \frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial^2_{\alpha_1}\Omega_t(\alpha-\beta)\big(\partial_{\alpha_2}X(\alpha-\beta)-\partial_{\alpha_2}X(\alpha)\big)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

$$P_2 = -\frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\partial^2_{\alpha_1}\Omega_t(\alpha-\beta)\,d\beta\cdot N(\alpha).$$

P_1 is estimated like O_2. We rewrite P_2 as follows:

$$P_2=-\frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{|\beta|<1}\left(\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}-\frac{\nabla X(\alpha)\cdot\beta}{|\nabla X(\alpha)\cdot\beta|^3}\right)\partial^2_{\alpha_1}\Omega(\alpha-\beta)\,d\beta\cdot N(\alpha),$$

and this expression shows that the above integral can be estimated like $\mathcal{T}_4$ (A-8).

Using (8-3), we obtain

$$\|\partial^2_{\alpha_1}\Omega_t\|_{L^2}\le P\big(\|X\|_4^2+\|F(X)\|^2_{L^\infty}+\||N|^{-1}\|_{L^\infty}\big),$$

and the identity

$$\partial^2_{\alpha_2}\Omega_t=-2A_\mu\partial_{\alpha_2}\partial_t(\mathrm{BR}(X,\omega)\cdot\partial_{\alpha_2}X)-2A_\rho\partial^2_{\alpha_2}\partial_tX_3$$

yields

$$\|\partial^2_{\alpha_2}\Omega_t\|_{L^2}\le P\big(\|X\|_4^2+\|F(X)\|^2_{L^\infty}+\||N|^{-1}\|_{L^\infty}\big),$$

that is,

$$\|\Omega_t\|_{H^2}\le P\big(\|X\|_4^2+\|F(X)\|^2_{L^\infty}+\||N|^{-1}\|_{L^\infty}\big). \tag{8-4}$$

Next we consider third-order derivatives:

$$\partial^3_{\alpha_1}\Omega_t=-2A_\mu\partial^2_{\alpha_1}\partial_t(\mathrm{BR}(X,\omega)\cdot\partial_{\alpha_1}X)-2A_\rho\partial^3_{\alpha_1}\partial_tX_3.$$

Since (8-1) gives us control of the last term, we will concentrate on the other one, which is of a much more difficult character. In particular, for $-\partial^2_{\alpha_1}\partial_t(\mathrm{BR}(X,\omega)\cdot\partial_{\alpha_1}X)$, the most singular components are given by

$$L_5=-\,\mathrm{BR}(X,\omega)\cdot\partial^3_{\alpha_1}X_t,$$

$$L_6=\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial^2_{\alpha_1}X_t(\alpha)-\partial^2_{\alpha_1}X_t(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\omega(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

$$L_7=-\frac{3}{8\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}B(\alpha,\beta)\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^5}\wedge\omega(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

where $B(\alpha,\beta)=\big(X(\alpha)-X(\alpha-\beta)\big)\cdot\big(\partial^2_{\alpha_1}X_t(\alpha)-\partial^2_{\alpha_1}X_t(\alpha-\beta)\big)$,

$$L_8=\frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial^2_{\alpha_1}\omega_t(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha).$$

Inequalities (8-1) and (8-4) show how to handle L_i, $i=5,\dots,8$ as L_j, $j=1,\dots,4$ respectively, and then a similar approach for $\partial^3_{\alpha_2}\Omega_t$ allows us to get finally (8-2) for $k=3$. The cases $k>3$ are similar to deal with.

Our next goal is to obtain estimates for the evolution of $\|F(X)\|_{L^\infty}$ and R-T. Regarding the quantity $F(X)$, we have

$$\begin{aligned}\frac{d}{dt}F(X)(\alpha,\beta,t)&=-\frac{|\beta|\big(X(\alpha,t)-X(\alpha-\beta,t)\big)\cdot\big(X_t(\alpha,t)-X_t(\alpha-\beta,t)\big)}{|X(\alpha,t)-X(\alpha-\beta,t)|^3}\\&\le(F(X)(\alpha,\beta,t))^2\|\nabla X_t\|_{L^\infty}(t).\end{aligned}\tag{8-5}$$

Then Sobolev inequalities in $\|\nabla X_t\|_{L^\infty}(t)$, together with (8-1), yield

$$\frac{d}{dt}F(X)(\alpha,\beta,t) \le F(X)(\alpha,\beta,t)P\big(\|X\|_4^2(t)+\|F(X)\|_{L^\infty}^2(t)+\||N|^{-1}\|_{L^\infty}(t)\big),$$

and an integration in time gives us

$$F(X)(\alpha,\beta,t+h) \le F(X)(\alpha,\beta,t)\exp\bigg(\int_t^{t+h} P(s)\,ds\bigg),$$

for $h>0$, where

$$P(s)=P\big(\|X\|_4^2(s)+\|F(X)\|_{L^\infty}^2(s)+\||N|^{-1}\|_{L^\infty}(s)\big).$$

Hence

$$\|F(X)\|_{L^\infty}(t+h) \le \|F(X)\|_{L^\infty}(t)\exp\bigg(\int_t^{t+h} P(s)\,ds\bigg).$$

This inequality, applied to the limit

$$\frac{d}{dt}\|F(X)\|_{L^\infty}(t)=\lim_{h\to 0^+}\frac{\|F(X)\|_{L^\infty}(t+h)-\|F(X)\|_{L^\infty}(t)}{h},$$

allows us to get

$$\frac{d}{dt}\|F(X)\|_{L^\infty}(t) \le \|F(X)\|_{L^\infty}(t)P\big(\|X\|_4^2+\|F(X)\|_{L^\infty}^2+\||N|^{-1}\|_{L^\infty}\big).$$

Next we search for an a priori estimate for the evolution of the infimum of the difference of the gradients of the pressure in the normal direction to the interface. Let us recall the formula

$$\sigma(\alpha,t)=(\mu^2-\mu^1)\,\mathrm{BR}(X,\omega)(\alpha,t)\cdot N(\alpha,t)+(\rho^2-\rho^1)N_3(\alpha,t)$$

to obtain

$$\frac{d}{dt}\bigg(\frac{1}{\sigma(\alpha,t)}\bigg)=-\frac{\sigma_t(\alpha,t)}{\sigma^2(\alpha,t)},$$

with $\sigma_t(\alpha,t)=I_1+I_2$, where

$$I_1=\big((\mu^2-\mu^1)\mathrm{BR}(X,\omega)(\alpha,t)+(\rho^2-\rho^1)(0,0,1)\big)\cdot N_t(\alpha,t),$$
$$I_2=(\mu^2-\mu^1)\mathrm{BR}_t(X,\omega)(\alpha,t)\cdot N(\alpha,t).$$

First we deal with $\|I_1\|_{L^\infty}$ using the estimates (8-1) for ∇X_t, and then we focus our attention on I_2 using the splitting $I_2=J_1+J_2+J_3$, where

$$J_1=-\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{X_t(\alpha)-X_t(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\omega(\alpha-\beta)\,d\beta,$$

$$J_2=\frac{3}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\big(X(\alpha)-X(\alpha-\beta)\big)\wedge\omega(\alpha-\beta)\frac{(X(\alpha)-X(\alpha-\beta))\cdot(X_t(\alpha)-X_t(\alpha-\beta))}{|X(\alpha)-X(\alpha-\beta)|^5}\,d\beta,$$

$$J_3=-\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\omega_t(\alpha-\beta)\,d\beta.$$

The terms J_1 and J_2 are similar and can be treated with the same method. Let us consider $J_1 = K_1 + K_2 + K_3 + K_4$, where

$$K_1 = -\frac{1}{4\pi}\int_{|\beta|>1} \frac{X_t(\alpha) - X_t(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)\, d\beta,$$

$$K_2 = \frac{1}{4\pi}\int_{|\beta|<1} \frac{X_t(\alpha) - X_t(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \big(\omega(\alpha) - \omega(\alpha-\beta)\big)\, d\beta,$$

$$K_3 = -\frac{1}{4\pi}\int_{|\beta|<1} \left[\frac{1}{|X(\alpha) - X(\alpha-\beta)|^3} - \frac{1}{|\nabla X(\alpha)\cdot\beta|^3}\right]\big(X_t(\alpha) - X_t(\alpha-\beta)\big) \wedge \omega(\alpha)\, d\beta,$$

$$K_4 = -\frac{1}{4\pi}\,\mathrm{PV}\int_{|\beta|<1} \frac{X_t(\alpha) - X_t(\alpha-\beta)}{|\nabla X(\alpha)\cdot\beta|^3} \wedge \omega(\alpha)\, d\beta.$$

First we have

$$\|K_1\|_{L^\infty} \le C\|F(X)\|^3_{L^\infty}\|\nabla X_t\|_{L^\infty}\|\omega\|_{L^2}\left(\int_{|\beta|>1}|\beta|^{-4}d\beta\right)^{1/2},$$

giving us an appropriate control. Next, we get

$$\|K_2\|_{L^\infty} \le C\|F(X)\|^3_{L^\infty}\|\nabla X_t\|_{L^\infty}\|\nabla\omega\|_{L^\infty}\int_{|\beta|<1}|\beta|^{-1}d\beta,$$

and an analogous estimate for K_3. Therefore, Sobolev's embedding helps us to obtain the desired control. Regarding K_4, we have

$$K_4 = -\frac{1}{4\pi}\int_{|\beta|<1} \frac{X_t(\alpha) - X_t(\alpha-\beta) - \nabla X_t(\alpha)\cdot\beta}{|\nabla X(\alpha)\cdot\beta|^3} \wedge \omega(\alpha)\, d\beta.$$

Inequality (A-15) yields

$$\|K_4\|_{L^\infty} \le C\|\nabla X\|^3_{L^\infty}\||N|^{-1}\|^3_{L^\infty}\|\omega\|_{L^\infty}\|\nabla X_t\|_{C^\delta}\int_{|\beta|<1}|\beta|^{-2+\delta}\, d\beta,$$

and the control $\|\nabla X_t\|_{C^\delta}$ follows again by (8-1) and Sobolev's embedding. Next let us continue with $J_3 = K_5 + K_6$, where

$$K_5 = -\frac{1}{4\pi}\,\mathrm{PV}\int_{|\beta|>1} \frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \big(\partial_{\beta_1}((\Omega\partial_{\alpha_2}X)_t(\alpha-\beta)) - \partial_{\beta_2}((\Omega\partial_{\alpha_1}X)_t(\alpha-\beta))\big)\, d\beta,$$

$$K_6 = -\frac{1}{4\pi}\,\mathrm{PV}\int_{|\beta|<1} \frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \omega_t(\alpha-\beta)\, d\beta.$$

Integration by parts yields

$$\|K_5\|_{L^\infty} \le C\|F(X)\|^3_{L^\infty}\|\nabla X\|_{L^\infty}\big(\|\Omega\|_{L^\infty}\|\nabla X_t\|_{L^\infty} + \|\Omega_t\|_{L^\infty}\|\nabla X\|_{L^\infty}\big),$$

where $4\pi C = \int_{|\beta|>1}|\beta|^{-3}d\beta + \int_{|\beta|=1} dl(\beta)$, and we may use (8-2) to estimate $\|\Omega_t\|_{L^\infty}$. With K_6, we introduce a similar splitting to obtain

$$\|K_6\|_{L^\infty} \le P\big(\|X - (\alpha, 0)\|_{C^2} + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)\|\omega_t\|_{C^\delta}.$$

Then it remains to estimate $\|\omega_t\|_{C^\delta}$, for which purpose we use formula (2-3) and inequalities (8-1), (8-2). Therefore, we have the estimate

$$\frac{d}{dt}\left(\frac{1}{\sigma(\alpha,t)}\right) \le \frac{1}{\sigma^2(\alpha,t)} P\big(\|X\|_4(t) + \|F(X)\|_{L^\infty}(t) + \||N|^{-1}\|_{L^\infty}(t)\big),$$

and proceeding similarly as we did for $F(X)$, we finally get

$$\frac{d}{dt}\|\sigma^{-1}\|_{L^\infty}(t) \le \|\sigma^{-1}\|^2_{L^\infty}(t) P\big(\|X\|_4(t) + \|F(X)\|_{L^\infty}(t) + \||N|^{-1}\|_{L^\infty}(t)\big).$$

Remark 8.1. Having obtained the a priori bounds of the preceding sections, we are in position to successfully implement the same approximation scheme developed in [Córdoba et al. 2011] to conclude local existence.

Appendix

Here we prove first some helpful inequalities regarding commutators of the Riesz transform (R_j, $j = 1, 2$) with several differential operators. Next we analyze the singular integral operators associated to the non-self-intersecting surface which appears throughout the paper. But the main goal of this section is to simplify the presentation of the main result.

Lemma A.1. *Consider $f \in L^2(\mathbb{R}^2)$ and $g \in C^{1,\delta}(\mathbb{R}^2)$, with $0 < \delta < 1$. Then for any $k, l = 1, 2$, we have the estimate*

$$\big\|(R_k\partial_{\alpha_l})(gf) - g(R_k\partial_{\alpha_l})(f)\big\|_{L^2} \le C\|g\|_{C^{1,\delta}}\|f\|_{L^2}. \tag{A-1}$$

An application of these inequalities to the operator $\Lambda = (R_1\partial_{\alpha_1}) + (R_2\partial_{\alpha_2})$ yields

$$\|\Lambda(gf) - g\Lambda(f)\|_{L^2} \le C\|g\|_{C^{1,\delta}}\|f\|_{L^2}. \tag{A-2}$$

For vector fields, we have:

Lemma A.2. *Consider $f, g : \mathbb{R}^2 \to \mathbb{R}^3$ vector fields, where $f \in L^2(\mathbb{R}^2)$ and $g \in C^{1,\delta}(\mathbb{R}^2)$, with $0 < \delta < 1$. Then for any $k, l = 1, 2$, the following inequality holds*:

$$\left|\int_{\mathbb{R}^2}(g \wedge f)\cdot(R_k\partial_{\alpha_l})(f)d\alpha\right| \le C\|g\|_{C^{1,\delta}}\|f\|^2_{L^2}. \tag{A-3}$$

Proof. Denoting by I the integral above, and since the operator $R_k\partial_{\alpha_l}$ is self-adjoint, we may write

$$I = \int_{\mathbb{R}^2} f_1\big[(R_k\partial_{\alpha_l})(g_2 f_3) - g_2(R_k\partial_{\alpha_l})(f_3)\big]\,d\alpha + \int_{\mathbb{R}^2} f_2\big[(R_k\partial_{\alpha_l})(g_3 f_1) - g_3(R_k\partial_{\alpha_l})(f_1)\big]\,d\alpha + \int_{\mathbb{R}^2} f_3\big[(R_k\partial_{\alpha_l})(g_1 f_2) - g_1(R_k\partial_{\alpha_l})(f_2)\big]\,d\alpha.$$

Then estimate (A-1) yields (A-3). □

Lemma A.3. *Consider $f \in L^2(\mathbb{R}^2)$ and $g \in C^{1,\delta}(\mathbb{R}^2)$, with $0 < \delta < 1$. Then for any $j, k, l = 1, 2$, the following inequality holds*:

$$\left|\int_{\mathbb{R}^2} R_j(f)(R_k\partial_{\alpha_l})(gf)\,d\alpha\right| \le C\|g\|_{C^{1,\delta}}\|f\|^2_{L^2}. \tag{A-4}$$

Proof. Let J be the integral to be bounded; then we have

$$J = \int_{\mathbb{R}^2} R_j(f)\big[(R_k\partial_{\alpha_l})(gf) - g(R_k\partial_{\alpha_l})(f)\big]\,d\alpha \\ - \int_{\mathbb{R}^2}\big[R_j(fg) - gR_j(f)\big](R_k\partial_{\alpha_l})(f)\,d\alpha + \int_{\mathbb{R}^2} R_j(fg)(R_k\partial_{\alpha_l})(f)\,d\alpha.$$

Since $R_j^* = -R_j$ and $R_k\partial_{\alpha_l}$ is self-adjoint, we get

$$J = \frac{1}{2}\int_{\mathbb{R}^2} R_j(f)\big[(R_k\partial_{\alpha_l})(gf) - g(R_k\partial_{\alpha_l})(f)\big]\,d\alpha - \frac{1}{2}\int_{\mathbb{R}^2}\big[R_j(fg) - gR_j(f)\big](R_k\partial_{\alpha_l})(f)\,d\alpha.$$

An integration by parts in the second integral above yields

$$J = \frac{1}{2}\int_{\mathbb{R}^2} R_j(f)\big[(R_k\partial_{\alpha_l})(gf) - g(R_k\partial_{\alpha_l})(f)\big]\,d\alpha \\ + \frac{1}{2}\int_{\mathbb{R}^2}\big[(R_j\partial_{\alpha_l})(fg) - g(R_j\partial_{\alpha_l})(f)\big](R_k)(f)\,d\alpha - \frac{1}{2}\int_{\mathbb{R}^2}(\partial_{\alpha_l}g)R_j(f)R_k(f)\,d\alpha,$$

allowing us to conclude the proof. □

Lemma A.4. *Let us define, for any $j = 1, 2$ and $k = 1, 2, 3$, the following operators*:

$$\mathcal{T}_1(\partial_{\alpha_j} f)(\alpha) = \mathrm{PV}\int_{\mathbb{R}^2} \frac{X_k(\alpha) - X_k(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}\,\partial_{\alpha_j} f(\alpha-\beta)\,d\beta, \tag{A-5}$$

$$\mathcal{T}_2(f)(\alpha) = \mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial_{\alpha_j} X_k(\alpha) - \partial_{\alpha_j} X_k(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}\, f(\alpha-\beta)\,d\beta, \tag{A-6}$$

$$\mathcal{T}_3(f)(\alpha) = \mathrm{PV}\int_{\mathbb{R}^2} \frac{f(\alpha) - f(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}\,d\beta, \tag{A-7}$$

$$\mathcal{T}_4(\partial_{\alpha_j} f)(\alpha) = \mathrm{PV}\int_{\mathbb{R}^2} \left(\frac{(X(\alpha) - X(\beta))}{|X(\alpha) - X(\beta)|^3} - \frac{\nabla X(\alpha)\cdot(\alpha-\beta)}{|\nabla X(\alpha)\cdot(\alpha-\beta)|^3}\right)\partial_{\alpha_j} f(\beta)\,d\beta\,d\alpha, \tag{A-8}$$

where $\nabla X(\alpha)\cdot\beta = \partial_{\alpha_1}X(\alpha)\beta_1 + \partial_{\alpha_2}X(\alpha)\beta_2$. Assume that $X(\alpha) - (\alpha, 0) \in C^{2,\delta}(\mathbb{R}^2)$, and that both $F(X)$ and $|N|^{-1}$ are in L^∞, where

$$F(X)(\alpha,\beta) = |\beta|\big/|X(\alpha) - X(\alpha-\beta)| \quad \textit{and} \quad N(\alpha) = \partial_{\alpha_1}X(\alpha)\wedge\partial_{\alpha_2}X(\alpha).$$

Then the following estimates hold:

$$\|\mathcal{T}_1(\partial_{\alpha_j} f)\|_{L^2} \le P\big(\|X - (\alpha,0)\|_{C^{1,\delta}} + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)(\|f\|_{L^2} + \|\partial_{\alpha_j} f\|_{L^2}), \tag{A-9}$$

$$\|\mathcal{T}_2(f)\|_{L^2} \le P\big(\|X - (\alpha,0)\|_{C^{2,\delta}} + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)\|f\|_{L^2}, \tag{A-10}$$

$$\|\mathcal{T}_3(f)\|_{L^2} \le P\big(\|X - (\alpha,0)\|_{C^{2,\delta}} + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)\|f\|_{H^1}, \tag{A-11}$$

$$\|\mathcal{T}_4(f)\|_{L^2} \le P\big(\|X - (\alpha,0)\|_{C^{2,\delta}} + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)\|f\|_{L^2}, \tag{A-12}$$

with P a polynomial function.

Proof. To estimate the first set of operators, we first consider the splitting

$$\mathcal{T}_1(\partial_{\alpha_j} f) = \mathrm{PV}\int_{|\beta|>1} d\beta + \mathrm{PV}\int_{|\beta|<1} d\beta = I_1 + I_2, \tag{A-13}$$

and an integration by parts allows us to write $I_1 = J_1 + J_2 + J_3$, where

$$\begin{aligned} J_1 &= \int_{|\beta|>1} -\frac{\partial_{\alpha_j} X_k(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} f(\alpha-\beta)\,d\beta, \\ J_2 &= 3\int_{|\beta|>1} \frac{\big(X_k(\alpha)-X_k(\alpha-\beta)\big)\big(X(\alpha)-X(\alpha-\beta)\big)\cdot\partial_{\alpha_j} X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^5} f(\alpha-\beta)\,d\beta, \\ J_3 &= \int_{|\beta|=1} \frac{X_k(\alpha)-X_k(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} f(\alpha-\beta)\,dl(\beta). \end{aligned}$$

The above decomposition shows that

$$|I_1| \le C\|X-(\alpha,0)\|_{C^1}\|F(X)\|^3_{L^\infty}\bigg(\int_{|\beta|>1} \frac{|f(\alpha-\beta)|}{|\beta|^3}\,d\beta + \int_{|\beta|=1} |f(\alpha-\beta)|\,dl(\beta),\bigg)$$

and then Minkowski's inequality gives the desired control.

Regarding I_2, we write $I_2 = J_4 + J_5 + J_6$, with

$$\begin{aligned} J_4 &= \int_{|\beta|<1} \frac{X_k(\alpha)-X_k(\alpha-\beta)-\nabla X_k(\alpha)\cdot\beta}{|X(\alpha)-X(\alpha-\beta)|^3}\,\partial_{\alpha_j} f(\alpha-\beta)\,d\beta, \\ J_5 &= \nabla X_k(\alpha)\cdot\int_{|\beta|<1} \beta\bigg[\frac{1}{|X(\alpha)-X(\alpha-\beta)|^3} - \frac{1}{|\nabla X(\alpha)\cdot\beta|^3}\bigg]\partial_{\alpha_j} f(\alpha-\beta)\,d\beta, \\ J_6 &= \nabla X_k(\alpha)\cdot\mathrm{PV}\int_{|\beta|<1} \frac{\beta}{|\nabla X(\alpha)\cdot\beta|^3}\,\partial_{\alpha_j} f(\alpha-\beta)\,d\beta. \end{aligned}$$

It is easy to see that

$$J_4 \le \|X-(\alpha,0)\|_{C^{1,\delta}}\|F(X)\|^3_{L^\infty}\int_{|\beta|<1} \frac{|\partial_{\alpha_j} f(\alpha-\beta)|}{|\beta|^{2-\delta}}\,d\beta, \tag{A-14}$$

and therefore that term can also be estimated with the use of Minkowski's inequality.

Some elementary algebraic manipulations allow us to get

$$J_5 \le C\|X-(\alpha,0)\|^2_{C^{1,\delta}}\int_{|\beta|<1}\bigg((F(X)(\alpha,\beta))^4 + \frac{|\beta|^4}{|\nabla X(\alpha)\cdot\beta|^4}\bigg)\frac{|\partial_{\alpha_j} f(\alpha-\beta)|}{|\beta|^{2-\delta}}\,d\beta,$$

and then the inequality

$$\frac{|\beta|}{|\nabla X(\alpha)\cdot\beta|} \le 2\|\nabla X\|_{L^\infty}\||N|^{-1}\|_{L^\infty} \tag{A-15}$$

yields for J_5 the same estimate (A-14).

The term J_6 can be written as

$$J_6 = \nabla X_k(\alpha)\cdot\mathrm{PV}\int_{|\beta|<1} \frac{\Sigma(\alpha,\beta)}{|\beta|^2}\,\partial_{\alpha_j} f(\alpha-\beta)\,d\beta,$$

where

(i) $\Sigma(\alpha,\lambda\beta)=\Sigma(\alpha,\beta)$ for all $\lambda>0$,

(ii) $\Sigma(\alpha,-\beta)=-\Sigma(\alpha,\beta)$,

(iii) $\sup_\alpha|\Sigma(\alpha,\beta)|\le 8\|\nabla X\|^3_{L^\infty}\||N|^{-1}\|^3_{L^\infty}$,

as a consequence of (A-15).

Here we have a singular integral operator with odd kernel [Córdoba and Gancedo 2007; Stein 1993], and therefore a bounded linear map on $L^2(\mathbb{R}^2)$, giving us

$$\|J_6\|_{L^2}\le C\|\nabla X\|^4_{L^\infty}\||N|^{-1}\|^3_{L^\infty}\|\partial_{\alpha_j}f\|_{L^2}.$$

For the family of operators $\mathcal{T}_2(f)(\alpha)$, we use the splitting $\mathcal{T}_2(f)=I_3+I_4$, where

$$I_3=\int_{|\beta|>1}\frac{\partial_{\alpha_j}X_k(\alpha)-\partial_{\alpha_j}X_k(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}f(\alpha-\beta)\,d\beta.$$

We easily get

$$I_3\le 2\|X-(\alpha,0)\|_{C^1}\|F(X)\|^3_{L^\infty}\int_{|\beta|>1}\frac{|f(\alpha-\beta)|}{|\beta|^3}\,d\beta,$$

while for I_4, we proceed with the same method used with I_2, now replacing $X_k(\alpha)$ by $\partial_{\alpha_j}X_k(\alpha)$ and $\partial_{\alpha_j}f(\alpha-\beta)$ by $f(\alpha-\beta)$.

Next we shall show that the operator $\mathcal{T}_3$ behaves like $\Lambda=(-\Delta)^{1/2}$. To do that, we split it as I_5+I_6, where

$$I_5=\int_{|\beta|>1}\frac{f(\alpha)-f(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\,d\beta$$

can be easily estimated by

$$I_5\le\|F(X)\|^3_{L^\infty}\left(2\pi|f(\alpha)|+\int_{|\beta|>1}\frac{|f(\alpha-\beta)|}{|\beta|^3}\,d\beta\right).$$

The other term is written in the form $I_6=J_7+J_8$, where

$$J_7=\int_{|\beta|<1}\left(\frac{1}{|X(\alpha)-X(\alpha-\beta)|^3}-\frac{1}{|\nabla X(\alpha)\cdot\beta|^3}\right)(f(\alpha)-f(\alpha-\beta))\,d\beta.$$

The identity

$$f(\alpha)-f(\alpha-\beta)=\beta\cdot\int_0^1\nabla f(\alpha+(s-1)\beta)\,ds$$

allows us to treat J_7 as we did with J_5. To estimate J_8, the equality

$$\frac{1}{|\nabla X(\alpha)\cdot\beta|^3}=-\partial_{\beta_1}\left(\frac{\beta_1}{|\nabla X(\alpha)\cdot\beta|^3}\right)-\partial_{\beta_2}\left(\frac{\beta_2}{|\nabla X(\alpha)\cdot\beta|^3}\right)\tag{A-16}$$

will be very useful. After a careful integration by parts, it yields

$$J_8=\mathrm{PV}\int_{|\beta|<1}\frac{\nabla f(\alpha-\beta)\cdot\beta}{|\nabla X(\alpha)\cdot\beta|^3}\,d\beta-\int_{|\beta|=1}\frac{(f(\alpha)-f(\alpha-\beta))|\beta|}{|\nabla X(\alpha)\cdot\beta|^3}\,dl(\beta).$$

The principal value in J_8 is treated with the same method used for J_6, and since the integral on the circle is inoffensive, so long as $|N|^{-1}$ is in L^∞, the estimate for $\mathcal{T}_3$ follows.

For the remaining operator, one integrates by parts to get $\mathcal{T}_4 = I_7 + I_8$, where

$$I_7 = \mathrm{PV}\int_{\mathbb{R}^2} P_1(\alpha,\beta) f(\alpha-\beta)\, d\beta, \qquad I_8 = \mathrm{PV}\int_{\mathbb{R}^2} P_2(\alpha,\beta) f(\alpha-\beta)\, d\beta,$$

with

$$P_1(\alpha,\beta) = \frac{\partial_{\alpha_j} X(\alpha)}{|\nabla X(\alpha)\cdot\beta|^3} - \frac{\partial_{\alpha_j} X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}$$

and

$$P_2(\alpha,\beta) = 3\,\frac{\big(X(\alpha)-X(\alpha-\beta)\big)\big(X(\alpha)-X(\alpha-\beta)\big)\cdot\partial_{\alpha_j} X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^5} - 3\,\frac{\nabla X(\alpha)\cdot\beta\big((\nabla X(\alpha)\cdot\beta)\cdot\partial_{\alpha_j} X(\alpha)\big)}{|\nabla X(\alpha)\cdot\beta|^5}.$$

Next we will show how to treat I_7, because the estimate for I_8 follows similarly. For P_1 we introduce the decomposition $P_1 = Q_1 + Q_2$, where

$$Q_1 = \partial_{\alpha_j} X(\alpha)\left(\frac{1}{|\nabla X(\alpha)\cdot\beta|^3} - \frac{1}{|X(\alpha)-X(\alpha-\beta)|^3}\right), \qquad Q_2 = \frac{\partial_{\alpha_j} X(\alpha) - \partial_{\alpha_j} X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}.$$

Since the kernel Q_2 has already appeared in the operator $\mathcal{T}_1$, it only remains to control J_9, which is given by

$$J_9 = \partial_{\alpha_j} X(\alpha)\, \mathrm{PV}\int_{\mathbb{R}^2} Q_1(\alpha,\beta) f(\alpha-\beta)\, d\beta.$$

The decomposition

$$J_9 = \partial_{\alpha_j} X(\alpha)\int_{|\beta|>1} d\beta + \partial_{\alpha_j} X(\alpha)\,\mathrm{PV}\int_{|\beta|<1} d\beta = K_1 + K_2$$

shows that the term K_1 trivializes. Regarding K_2, let us write

$$Q_1 = \frac{\big(|A|^4 + |B|^2|A|^2 + |B|^4\big)(A+B)\cdot(A-B)}{|A|^3|B|^3(|A|^3 + |B|^3)},$$

where

$$A(\alpha,\beta) = X(\alpha) - X(\alpha-\beta), \qquad B(\alpha,\beta) = \nabla X(\alpha)\cdot\beta.$$

This formula shows that inside Q_1 lies a kernel of degree -2. Then let us take $Q_1 = S_1 + S_2$, where

$$S_2 = \frac{3|B|^4 B\cdot(A-B)}{|B|^9} = \frac{3B\cdot(A-B)}{|B|^5}.$$

Next we check that the kernel S_1 has degree -1, and is therefore easy to handle. Finally, we have to consider the kernel S_2 appearing in the integral

$$L = 3\partial_{\alpha_j} X(\alpha)\,\mathrm{PV}\int_{|\beta|<1} \frac{(\nabla X(\alpha)\cdot\beta)\cdot\big(X(\alpha) - X(\alpha-\beta) - \nabla X(\alpha)\cdot\beta\big)}{|\nabla X(\alpha)\cdot\beta|^5} f(\alpha-\beta)\, d\beta.$$

To do that, we introduce a further decomposition $L = M_1 + M_2$, with

$$M_1 = 3\partial_{\alpha_j} X(\alpha) \int_{|\beta|<1} \frac{(\nabla X(\alpha)\cdot\beta)\cdot\big(X(\alpha) - X(\alpha-\beta) - \nabla X(\alpha)\cdot\beta - \frac{1}{2}\beta\cdot\nabla^2 X(\alpha)\cdot\beta\big)}{|\nabla X(\alpha)\cdot\beta|^5} f(\alpha-\beta)\,d\beta$$

and

$$M_2 = \tfrac{3}{2}\partial_{\alpha_j} X(\alpha)\,\mathrm{PV}\int_{|\beta|<1} \frac{(\nabla X(\alpha)\cdot\beta)\cdot(\beta\cdot\nabla^2 X(\alpha)\cdot\beta)}{|\nabla X(\alpha)\cdot\beta|^5} f(\alpha-\beta)\,d\beta,$$

where $\frac{1}{2}\beta\cdot\nabla^2 X(\alpha)\cdot\beta$ is the second-order term in the Taylor expansion of X. It is now easy to check that

$$M_1 \le C\|\nabla X\|^5_{L^\infty}\|X-(\alpha,0)\|_{C^{2,\delta}}\||N|^{-1}\|^4_{L^\infty}\int_{|\beta|<1}\frac{|f(\alpha-\beta)|}{|\beta|^{2-\delta}}\,d\beta.$$

Then we also check that M_2 is controlled like J_6 through the estimate

$$\|M_2\|_{L^2} \le C\|\nabla X\|^5_{L^\infty}\|\nabla^2 X\|_{L^\infty}\||N|^{-1}\|^4_{L^\infty}\|f\|_{L^2},$$

which allows us to finish the proof. □

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EMBEDDINGS OF INFINITELY CONNECTED PLANAR DOMAINS INTO $\mathbb{C}^2$

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We prove that every circled domain in the Riemann sphere admits a proper holomorphic embedding into the affine plane $\mathbb{C}^2$.

1. Introduction

It has been a longstanding open problem whether every open (noncompact) Riemann surface, in particular, every domain in the complex plane $\mathbb{C}$, admits a proper holomorphic embedding into $\mathbb{C}^2$. (By a *domain* we understand a connected open set.) Equivalently:

Is every open Riemann surface biholomorphic to a smoothly embedded, topologically closed complex curve in $\mathbb{C}^2$?

Every open Riemann surface properly embeds in $\mathbb{C}^3$ and immerses in $\mathbb{C}^2$, but there is no constructive method of removing self-intersections of an immersed curve in $\mathbb{C}^2$. For a history of this subject see [Forstnerič and Wold 2009; Forstnerič 2011, §8.9–§8.10].

In this paper we prove the following general result in this direction.

Theorem 1.1. *Every domain in the Riemann sphere* $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ *with at most countably many boundary components, none of which are points, admits a proper holomorphic embedding into* $\mathbb{C}^2$.

By the uniformization theorem of He and Schramm [1993], every domain in Theorem 1.1 is conformally equivalent to a *circled domain*, that is, a domain whose complement is a union of pairwise disjoint closed round discs.

We prove the same embedding theorem also for generalized circled domains whose complementary components are discs and points (punctures), provided that all but finitely many of the punctures belong to the cluster set of the nonpoint boundary components (see Theorem 5.1). In particular, every domain in $\mathbb{C}$ or $\mathbb{P}^1$ with at most countably many boundary components, at most finitely many of which are isolated points, admits a proper holomorphic embedding into $\mathbb{C}^2$ (see Corollary 5.2 and Example 5.3).

For *finitely connected planar domains* without isolated boundary points, Theorem 1.1 was proved by Globevnik and Stensønes [1995]. More recently it was shown by the authors in [Forstnerič and Wold 2009] that for every embedded complex curve $\overline{C} \subset \mathbb{C}^2$, with smooth boundary bC consisting of finitely

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many Jordan curves, the interior $C = \overline{C} \setminus bC$ admits a proper holomorphic embedding into $\mathbb{C}^2$. This result was extended to some infinitely connected Riemann surfaces by I. Majcen [2009] under a nontrivial additional assumption on the accumulation set of the boundary curves. (These results can also be found in [Forstnerič 2011, Chapter 8].) Here we do not impose any restrictions whatsoever.

Our proofs of Theorems 1.1 and 5.1 are rather involved both from the analytic as well as the combinatorial point of view, something that seems inevitable in this notoriously difficult classical problem. Theorem 1.1 is proved in Section 4 after we develop the technical tools in Section 2 and Section 3. The main idea is to successively push the boundary components of an embedded complex curve in $\mathbb{C}^2$ to infinity by using holomorphic automorphisms of the ambient space, thereby ensuring that no self-intersections appear in the process, while at the same time controlling the convergence of the sequence of automorphisms in the interior of the curve. We employ the most advanced available analytic tools developed in recent years, sharpening further several of them. A novel part is our inductive scheme of dealing with an infinite sequence of boundary components, clustering them together into suitable subsets to which the analytic methods can be applied.

For simplicity of exposition we limit ourselves to domains in the Riemann sphere, although it seems likely that minor modifications yield similar results for domains in complex tori. Indeed, any punctured torus admits a proper holomorphic embedding in $\mathbb{C}^2$, and the uniformization theory of He and Schramm [1993] applies in this case as well. For infinitely connected domains in Riemann surfaces of genus > 1 the main problem is to find a suitable initial embedding of the uniformized surface into $\mathbb{C}^2$. One of the difficulties in working with nonuniformized boundary components is indicated in Remark 2.3; another one can be seen in the last part of proof of Lemma 3.1, which is a key ingredient in our construction.

Casting a view to the future, what is now needed to approach the general embedding problem is some progress on embedding *punctured* Riemann surfaces into $\mathbb{C}^2$. It is plausible that a method for answering the following question in the affirmative would lead to a complete solution to the embedding problem for planar domains with countably many boundary components.

Question 1.2. Assume that $f : \overline{\mathbb{D}} \to \mathbb{C}^2$ is a holomorphic embedding, $K \subset \mathbb{C}^2 \setminus f(b\mathbb{D})$ is a compact polynomially convex set, $C \subset \mathbb{D}$ is a compact set with $f^{-1}(K) \subset \mathring{C}$, and $a \in \mathbb{D} \setminus C$ is a point. Is f uniformly approximable on C by proper holomorphic embeddings $g : \overline{\mathbb{D}} \setminus \{a\} \hookrightarrow \mathbb{C}^2$ satisfying

$$g^{-1}(g(\overline{\mathbb{D}} \setminus \{a\}) \cap K) \subset \mathring{C}?$$

In another direction, one can ask to what extent does Theorem 1.1 hold for domains in $\mathbb{P}^1$ with uncountably many boundary components. A quintessential example of this type is a Cantor set, i.e., a compact, totally disconnected, perfect set. Recently Orevkov [2008] constructed an example of a Cantor set K in $\mathbb{C}$ whose complement $\mathbb{C} \setminus K$ embeds properly holomorphically in $\mathbb{C}^2$. (See also [Forstnerič 2011, Theorem 8.10.4]). His method, using compositions of rational shears of $\mathbb{C}^2$, does not seem to apply to a specific Cantor set. The methods explained in this paper offer some hope for future developments as indicated by Theorem 5.1 and Example 5.3 below.

The problem of embedding an open Riemann surface in $\mathbb{C}^2$ is purely complex analytic, and there are no topological obstructions. Indeed, Alarcón and López [2013] recently proved that every open Riemann surface X contains a domain $\Omega \subset X$, homeotopic to X, which embeds properly holomorphically in $\mathbb{C}^2$. In particular, every open orientable surface admits a smooth proper embedding in $\mathbb{C}^2$ whose image is a complex curve.

2. Preliminaries

In this and the following section we prepare the technical tools that will be used in the proof. The main result of this section, Theorem 2.8, gives holomorphic embeddings of bordered Riemann surfaces into $\mathbb{C}^2$ with exposed wedges at finitely many boundary points.

We begin by introducing the notation. Let $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. We denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disc and by $\mathbb{D}_r = \{|z| < r\}$ the disc of radius r centered at the origin. Let (z_1, z_2) be complex coordinates on $\mathbb{C}^2$, and let $\pi_i : \mathbb{C}^2 \to \mathbb{C}$ denote the coordinate projection $\pi_i(z_1, z_2) = z_i$ for $i = 1, 2$. We denote by $\mathbb{B}_r$ and $\overline{\mathbb{B}}_r$ the open and the closed ball in $\mathbb{C}^2$, respectively, of radius r and centered at the origin. Let $\operatorname{Aut}\mathbb{C}^2$ denote the group of all holomorphic automorphisms of $\mathbb{C}^2$. By Id we denote the identity map; its domain will always be clear from the context. We denote by $\widehat{L}$ the polynomial hull of a compact set $L \subset \mathbb{C}^n$.

Definition 2.1. A domain $\Omega \subset \mathbb{P}^1$ is said to be a *circled domain* if the complement $\mathbb{P}^1 \setminus \Omega \neq \varnothing$ is a union of pairwise disjoint closed round discs $\overline{\triangle}_j \subset \mathbb{P}^1$ of positive radii.

Clearly a circled domain has at most countably many complementary discs. Mapping one of them onto $\mathbb{P}^1 \setminus \mathbb{D}$ by an automorphism of $\mathbb{P}^1$ (a fractional linear map) we see that a circled domain can be thought of as being contained in the unit disc $\mathbb{D}$.

The next lemma, and the remark following it, will serve to cluster together certain complementary components into finitely many discs; this will enable the use of holomorphic automorphisms for pushing these components towards infinity in the inductive process.

Lemma 2.2. *Let $\Omega \subset \mathbb{P}^1$ be a domain, let $K \subset \mathbb{P}^1 \setminus \Omega$ be a closed set that is a union of complementary connected components of Ω, and let $U \subset \mathbb{P}^1$ be an open set containing K. Then there exist finitely many pairwise disjoint, smoothly bounded discs $\overline{D}_j \subset U$ $(j = 1, \dots, m)$ such that*

$$K \subset \bigcup_{j=1}^{m} D_j, \qquad bD_j \cap (\mathbb{P}^1 \setminus \Omega) = \varnothing \textit{ for } j = 1, \dots, m.$$

Proof. Let $K_j \subset \mathring{K}_{j+1} \subset K_{j+1}$ be an exhaustion of Ω by smoothly bounded connected compact sets K_j. Then $\mathbb{P}^1 \setminus K_j$ is the union of finitely many discs $\mathcal{U}_j = \{U_1^j, \dots, U_{m(j)}^j\}$ for each j. Clearly $\mathcal{U}_j$ is a cover of K, and we claim that if j is large enough then $\mathcal{U}_j$ contains a subcover whose union is relatively compact in U. Otherwise there would exist a sequence of discs $U_{k(j)}^j \supset U_{k(j+1)}^{j+1}$ such that $U_{k(j)}^j \cap K \neq \varnothing$ and $U_{k(j)}^j \cap (\mathbb{P}^1 \setminus U) \neq \varnothing$ for each j; but then $\bigcap_{j=1}^{\infty} U_{k(j)}^j$ would be a connected complementary component of Ω that is contained in K and intersects $\mathbb{P}^1 \setminus U$, a contradiction. Hence for j large enough the discs $D_1, \dots, D_m$ in $\mathcal{U}_j$ satisfy the stated properties. $\square$

Remark 2.3. When applying Lemma 2.2 to prove Theorem 1.1, it will be crucial that if $\Omega \subset \mathbb{P}^1$ is a circled domain with complementary discs $\overline{\triangle}_j$, and if $C \subset \mathbb{P}^1$ is any compact set, then the union of all discs $\overline{\triangle}_j$ intersecting C is a closed set that is a union of complementary connected components of Ω. The proof is elementary and is left to the reader. However, this fails in general if discs are replaced by more general connected closed sets. This is one of the reasons why our proof of Theorem 1.1 does not apply (at least not directly) to domains in compact Riemann surfaces of genus > 1. □

Definition 2.4. Let $0 < \theta < 2\pi$. A domain $\Omega \subset \mathbb{C}$ is an (open) *θ-wedge with vertex $a \in b\Omega$* if there exist a $\mathscr{C}^2$ map of the form

$$\varphi(\zeta) = a + \lambda\zeta + O(|\zeta|^2), \quad \lambda \neq 0,$$

in a neighborhood of the origin $0 \in \mathbb{C}$, and for every sufficiently small $\epsilon > 0$ a neighborhood $U_\epsilon \subset \mathbb{C}$ of the point a such that

$$U_\epsilon \cap \Omega = \varphi\big(\{\zeta \in \mathbb{C}^* : 0 < \arg(\zeta) < \theta,\ 0 < |\zeta| < \epsilon\}\big).$$

The closure of an open wedge will be called a *closed wedge*.

If Ω is a domain in a Riemann surface Y, we apply the same definition of a θ-wedge in a local holomorphic coordinate near the point $a \in b\Omega \subset Y$. In particular, if $\Omega \subset \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ and $a = \infty \in b\Omega$, we apply the definition in the local chart $z \to 1/z$ on $\mathbb{P}^1$ mapping ∞ to 0.

Given a nonempty subset E of $\mathbb{C}^2$ and a linear projection $\pi : \mathbb{C}^2 \to \mathbb{C}$, a point $p \in E$ is said to be *π-exposed*, and E is said to be *π-exposed at the point p*, if

$$E \cap \pi^{-1}(\pi(p)) = \{p\}. \tag{2-1}$$

Recall that a *bordered Riemann surface* is a compact one-dimensional complex manifold, $\bar{X}$, with boundary bX consisting of finitely many Jordan curves. The interior X of a bordered Riemann surface is biholomorphic to a relatively compact, smoothly bounded domain in a Riemann surface Y.

We shall use the following notion of an *exposed θ-wedge*.

Definition 2.5. Let X be a bordered Riemann surface, embedded as a smoothly bounded relatively compact domain in a Riemann surface Y. Pick a point $a \in bX$ and a number $\theta \in (0, 2\pi)$. An injective continuous map $f : \bar{X} \hookrightarrow \mathbb{C}^2$ is said to be a *holomorphic embedding with a π_1-exposed θ-wedge at $f(a)$* if f is holomorphic in X, and there exists an open neighborhood U of a in Y such that

(i) the domain $\Omega = (\pi_1 \circ f)(U \cap X) \subset \mathbb{C}$ is a θ-wedge with vertex $\pi_1(f(a))$ (see Definition 2.4),

(ii) $f(\overline{U \cap X})$ is a smooth graph over $\overline{\Omega}$ that is holomorphic over Ω, and

(iii) $\pi_1^{-1}(\overline{\Omega}) \cap f(\bar{X}) = f(\overline{U \cap X})$.

If the domain $\Omega \subset \mathbb{C}$ is instead smooth near the point $\pi_1(f(a)) \in b\Omega$, we say that f is a holomorphic embedding that is π_1-exposed at $f(a)$.

Remark 2.6 (on terminology). We shall consider embeddings $f : \bar{X} \hookrightarrow \mathbb{C}^2$ that are holomorphic in the interior X and smooth on $\bar{X}$, except at finitely many boundary points where $f(X)$ has (exposed) wedges in the sense of the above definition. Any such map will be called a *holomorphic embedding with corners*. We shall use embeddings with corners of a particular type: If X is a smoothly bounded, relatively compact domain in a Riemann surface Y, we will construct holomorphic embeddings $\widetilde{f}\colon Y \hookrightarrow \mathbb{C}^2$ and injective continuous maps $\varphi\colon \overline{X} \to Y$, holomorphic on X and smooth at all but finitely many boundary points $a_j \in bX$, such that

$$f := \widetilde{f} \circ \varphi\colon \overline{X} \hookrightarrow \mathbb{C}^2 \ \text{ is an embedding with corners at the points } a_j. \tag{2-2}$$

In the sequel we will refer to such maps simply as *being of the form* (2-2). The precise choice of the Riemann surface Y will not be important, and we will allow Y to shrink around $\overline{X}$ without saying it every time. □

The following lemma shows how to create wedges at smooth boundary points of a domain in a Riemann surface.

Lemma 2.7. *Let $X \Subset Y$ be Riemann surfaces, and assume that bX is smooth outside a finite set of points. Let $a_1, \dots, a_m \in bX$, $b_1, \dots, b_k \in \overline{X}$ be distinct points, with bX smooth near the points a_j, and let $\theta_j \in (0, 2\pi)$ for $j = 1, \dots, m$. Then there exists a sequence of injective continuous maps $\varphi_i : \overline{X} \to Y$, holomorphic on X and smooth on $\overline{X} \setminus \{a_1, \dots, a_m\}$, satisfying the following properties*:

(1) $\varphi_i \to \mathrm{Id}$ *uniformly on $\overline{X}$ as $i \to \infty$.*

(2) $\varphi_i(a_j) = a_j$ *and $\varphi_i(X)$ is a θ_j-wedge with vertex a_j $(j = 1, \dots, m)$.*

(3) $\varphi_i(x) = b_j + o(\mathrm{dist}(x, b_j)^2)$ *as $x \to b_j$ $(j = 1, \dots, k)$.*

Proof. The proof is similar to that of Lemma 8.8.3 in [Forstnerič 2011, p. 366], and it will help the reader to consult Figure 8.1 on p. 367 in that reference.

By enlarging the domain X slightly away from the points a_j we may assume that X is smoothly bounded. For simplicity of notation we explain the proof in the case when there is only one such point $a = a_1$; the same procedure can be performed simultaneously at finitely many points.

Choose a smoothly bounded disc D in Y such that $a \in bD$, $\overline{D}$ does not contain any of the points b_j, and $\overline{U} \cap \overline{X} \setminus \{a\} \subset D$ holds for some small open neighborhood U of the point a in Y. (The disc D is obtained by pushing the boundary of X slightly out near a and then rounding off.) We also choose a compact Cartan pair $(A, B) \subset Y$ with $\overline{X} \subset (A \cup B)^\circ$ and $C := A \cap B \subset D$. (For the notion of a Cartan pair see [Forstnerič 2011, Definition 5.7.1].) The set A is chosen such that it contains a neighborhood of a, and B contains $\overline{X} \setminus U'$ for a small neighborhood $U' \subset U$ of the point a.

The Riemann mapping theorem furnishes a sequence of injective continuous maps $\psi_i : \overline{D} \to Y$ that are holomorphic in D and smooth on $\overline{D} \setminus \{a\}$ such that $\psi_i(a) = a$, $\psi_i(D)$ is a θ_1-wedge with vertex a (see Definition 2.4), and the sets $\psi_i(\overline{D})$ converge to $\overline{D}$ as $i \to \infty$. We may assume that $\psi_i \to \mathrm{Id}$ uniformly on $\overline{D}$ (see [Goluzin 1969, Theorem 2, p. 59]). This implies that $\psi_i(C) \subset D$ for all sufficiently large $i \in \mathbb{N}$.

By Theorem 8.7.2 in [Forstnerič 2011, p. 359] there exist an integer $i_0 \in \mathbb{N}$ and sequences of injective holomorphic maps $f_i : A \to Y$ and $g_i : B \to Y$ $(i \geq i_0)$, both converging to the identity map and tangent to the identity to second order at those points a and b_j which are contained in their respective domains, such that

$$\psi_i \circ f_i = g_i \text{ holds on } C.$$

The sequence of maps $\varphi_i \colon \bar{X} \to Y$, defined by

$$\varphi_i = \psi_i \circ f_i \text{ on } A \cap \bar{X} \quad \text{and} \quad \varphi_i = g_i \text{ on } \bar{X} \cap B$$

then satisfies the conclusion of the lemma. Injectivity of φ_i on $\bar{X}$ for sufficiently large index i can be seen exactly as in the proof of [Forstnerič 2011, Lemma 8.8.3] (see bottom of p. 359 in the cited source). □

Using Lemma 2.7 we obtain the following version of the main tool introduced in [Forstnerič and Wold 2009] for exposing boundary points of bordered Riemann surfaces. (See also Theorem 8.9.10 and Figure 8.2 in [Forstnerič 2011, pp. 372–373].) The main novelty here is that we create *exposed points with wedges*.

Theorem 2.8. *Let $\bar{X}$ be a smoothly bounded domain in a Riemann surface Y, $f \colon \bar{X} \hookrightarrow \mathbb{C}^2$ a holomorphic embedding with corners of the form (2-2), and $a_1, \dots, a_m \in bX$, $b_1, \dots, b_k \in \bar{X}$ distinct points such that f is smooth near the points a_j. Let $\gamma_j \colon [0, 1] \to \mathbb{C}^2$ $(j = 1, \dots, m)$ be smooth embedded arcs with pairwise disjoint images satisfying the following properties*:

- *$\gamma_j([0, 1]) \cap f(\bar{X}) = \gamma_j(0) = f(a_j)$ for $j = 1, \dots, m$.*
- *The image $E := f(\bar{X}) \cup \bigcup_{j=1}^m \gamma_j([0, 1])$ is π_1-exposed at the point $\gamma_j(1)$ for $j = 1, \dots, m$ (see (2-1)).*

Given an open set $V \subset \mathbb{C}^2$ containing $\bigcup_{j=1}^m \gamma_j([0, 1])$, an open set $U \subset Y$ containing the points a_j and satisfying $f(\overline{U \cap X}) \subset V$, and numbers $0 < \theta_j < 2\pi$ $(j = 1, \dots, m)$ and $\epsilon > 0$, there exists a holomorphic embedding with corners $g \colon \bar{X} \hookrightarrow \mathbb{C}^2$ of the form (2-2) satisfying the following properties:

(1) $\|g - f\|_{\bar{X} \setminus U} < \epsilon$.

(2) $g(\overline{U \cap X}) \subset V$.

(3) *$g(x) = f(x) + o(\operatorname{dist}(x, b_j)^2)$ as $x \to b_j$ $(j = 1, \dots, k)$.*

(4) *$g(a_j) = \gamma_j(1)$ and $g(\bar{X})$ is π_1-exposed with a θ_j-wedge at $g(a_j)$ for every $j = 1, \dots, m$.*

(5) *g is smooth near all points $x \in bX \setminus \{a_1, \dots, a_m\}$ at which f is smooth.*

If for some $j \in \{1, \dots, k\}$ we have that $b_j \in bX$ and $f(X)$ is a wedge at the point $f(b_j)$, then property (3) ensures that $g(X)$ remains a wedge with the same angle at $f(b_j) = g(b_j)$. In addition, property (4) ensures that $g(X)$ is an exposed wedge at each of the points $g(a_j)$.

Proof. Since f is of the form (2-2), we write $f = \tilde{f} \circ \varphi$ where $\tilde{f} \colon Y \hookrightarrow \mathbb{C}^2$ is a holomorphic embedding. Set $X' = \varphi(X) \Subset Y$. Lemma 2.7, applied to the domain X' and the points $a'_j = \varphi(a_j) \in bX'$, $b'_j = \varphi(b_j) \in \overline{X'}$, gives an injective continuous map $\psi \colon \overline{X'} \to Y$ close to the identity map, with ψ holomorphic on X' and smooth on $\bar{X}' \setminus \{a'_1, \dots, a'_m\}$, such that

(2′) $\psi(a'_j) = a'_j$ and $\psi(X')$ is a θ_j-wedge with vertex a'_j $(j = 1, \dots, m)$, and

(3′) $\psi(x) = b'_j + o(\operatorname{dist}(x, b'_j)^2)$ as $x \to b'_j$ $(j = 1, \dots, k)$.

(The map ψ is one of the maps φ_i in Lemma 2.7, and the properties (2′), (3′) correspond to (2), (3) in that lemma, respectively.)

Set $\widetilde{\varphi} = \psi \circ \varphi\colon \bar{X} \to Y$; this is an embedding with the analogous properties as φ, but with additional θ_j-wedges at the points $a'_j \in bX'$. The embedding with corners $\widetilde{f} \circ \widetilde{\varphi}\colon \bar{X} \hookrightarrow \mathbb{C}^2$ then satisfies properties (1)–(3) and (5) (for the map g) in Theorem 2.8.

In order to achieve also condition (4) we apply Theorem 8.9.10 in [Forstnerič 2011] and the proof thereof. (The original source for this result is [Forstnerič and Wold 2009, Theorem 4.2].) We recall the main idea and refer to the cited works for the details. By pushing the boundary bX' slightly outward away from the points a'_j we obtain a smoothly bounded domain $Z \Subset Y$ such that $\bar{X}' \subset Z \cup \{a'_1, \dots, a'_m\}$. We attach to $\bar{Z}$ short pairwise disjoint embedded arcs $\Gamma_j \subset Y$ intersecting $\bar{Z}$ only at the points a'_j. By Mergelyan's theorem we can change the embedding $\widetilde{f}$ so that it maps the arc Γ_j approximately onto the arc γ_j for each $j = 1, \dots, m$, taking the other endpoint c_j of Γ_j to the exposed endpoint $\gamma_j(1) \in \mathbb{C}^2$ of γ_j and remaining close to the initial embedding on $\bar{Z}$. At each point $a'_j \in bZ$ we choose a small smoothly bounded disc $D_j \subset Y$ with the same properties as in the proof of Lemma 2.7; in particular, $a'_j \in bD_j$ and D_j contains $\bar{Z} \setminus \{a'_j\}$ near the point a'_j. By the Riemann mapping theorem we find for each $j \in \{1, \dots, m\}$ a holomorphic map $h_j\colon \bar{D}_j \to Y$ stretching $\bar{D}_j$ to contain the arc Γ_j, mapping a'_j to the other endpoint c_j of Γ_j and remaining close to the identity except very near the point a'_j. We then glue the maps h_j to an approximation of the identity map on the rest of the domain $\bar{Z}$, using again Theorem 8.7.2 in [Forstnerič 2011, p. 359]. This gives an injective holomorphic map $h\colon \widetilde{Y} \hookrightarrow Y$ in an open neighborhood $\widetilde{Y}$ of $\bar{Z}$ such that $h|_{\bar{Z}}$ is close to the identity, except very near the points $a'_j \in bZ$. The holomorphic embedding $\widetilde{g} := \widetilde{f} \circ h\colon \widetilde{Y} \hookrightarrow \mathbb{C}^2$ is then close to $\widetilde{f}$ on $\bar{Z}$, except near the points a'_j. By the construction, $\widetilde{g}(a'_j)$ is a π_1-exposed point of $\widetilde{g}(\bar{Z})$ for $j = 1, \dots, m$. The embedding with corners $g = \widetilde{g} \circ \widetilde{\varphi}\colon \bar{X} \hookrightarrow \mathbb{C}^2$ is then of the form (2-2) and satisfies properties (1)–(5) in Theorem 2.8. □

3. The main lemma

In this section we prove the following key lemma that will be used in the proof of Theorem 1.1. It is similar in spirit to Lemma 1 in [Wold 2006, p. 4] (see also [Forstnerič 2011, Lemma 4.14.4, p. 150]), but with improvements needed to deal with the more complicated situation at hand.

Lemma 3.1. *Let $\Omega = \mathbb{P}^1 \setminus \bigcup_{j=0}^{\infty} \overline{\triangle}_j$ be a circled domain, and let $\Omega' = \mathbb{P}^1 \setminus \bigcup_{j=0}^{k} \overline{\triangle}_j$ for some $k \in \mathbb{N}$. Pick a point $c_j \in b\triangle_j$ for $j = 0, 1, \dots, k$. Assume that $f\colon \overline{\Omega'} \hookrightarrow \mathbb{C}^2$ is a holomorphic embedding with a π_1-exposed θ_j-wedge at each point $f(c_j)$ and $\theta_0 + \cdots + \theta_k < 2\pi$. Let g be a rational shear map of the form*

$$g(z_1, z_2) = \left(z_1, z_2 + \sum_{j=0}^{k} \frac{\beta_j}{z_1 - \pi_1(f(c_j))}\right).$$

Assume that there exist open neighborhoods $U_j \subset \mathbb{P}^1$ of the points c_j such that $(\pi_2 \circ g \circ f)(U_j) \subset \mathbb{P}^1$ are θ_j-wedges whose closures only intersect at their common vertex $\infty \in \mathbb{P}^1$. (This can be arranged by a suitable choice of the arguments of the numbers β_j, while at the same time keeping $|\beta_j| > 0$ arbitrarily *small.) Given a compact polynomially convex set* $K \subset \mathbb{C}^2$ *with*

$$K \cap (g \circ f)\Big(b\Omega' \cup \Big(\bigcup_{i=k+1}^{\infty} \overline{\triangle}_i\Big)\Big) = \varnothing$$

and numbers $N \in \mathbb{N}$ *and* $\epsilon > 0$, *there exists a* $\psi \in \operatorname{Aut}\mathbb{C}^2$ *such that*

(1) $(\psi \circ g \circ f)(b\Omega' \cup (\bigcup_{i=k+1}^{\infty} \overline{\triangle}_i)) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_N$, *and*

(2) $\|\psi - \mathrm{Id}\,\|_K < \epsilon$.

Proof. We may assume that $\triangle_0 = \mathbb{P}^1 \setminus \overline{\mathbb{D}}$, so $\Omega = \mathbb{D} \setminus \bigcup_{j=1}^{\infty} \overline{\triangle}_j$. By increasing the number $N \in \mathbb{N}$ we may also assume that $K \subset \mathbb{B}_N$.

Set $X = (g \circ f)(\Omega')$, $\gamma_j = (g \circ f)(b\triangle_j \setminus \{c_j\})$ $(j = 0, \dots, k)$, and $\gamma = \bigcup_{j=0}^{k} \gamma_j$. Then $\overline{X}$ is an embedded bordered Riemann surface in $\mathbb{C}^2$ whose boundary $bX = \gamma$ consists of pairwise disjoint properly embedded real curves γ_j diffeomorphic to $\mathbb{R}$, and the second coordinate projection $\pi_2 \colon \overline{X} \to \mathbb{C}$ is proper. Let $\triangle'_i = (g \circ f)(\triangle_i) \subset X$ for $i = k+1, k+2, \dots$; then

$$X \setminus \bigcup_{i=k+1}^{\infty} \overline{\triangle}'_i = (g \circ f)(\Omega).$$

To prove the lemma we must find an automorphism $\psi \in \operatorname{Aut}\mathbb{C}^2$ sending the boundary curves $bX = \gamma$ and all the discs $\overline{\triangle}'_i$ for $i > k$ out of the ball $\overline{\mathbb{B}}_N$, while at the same time approximating the identity map on the compact set K. We seek ψ of the form

$$\psi = \phi_1 \circ \phi_2, \qquad \text{where } \phi_1, \phi_2 \in \operatorname{Aut}\mathbb{C}^2.$$

We begin by constructing ϕ_1.

The conditions on f and g ensure that for any sufficiently large disc $D \subset \mathbb{C}$ centered at the origin the projection $\pi_2 \colon \overline{X} \setminus \pi_2^{-1}(D) \to \mathbb{C} \setminus D$ is injective and maps $\overline{X} \setminus \pi_2^{-1}(D)$ onto the union of $k+1$ pairwise disjoint wedges with the common vertex at ∞; furthermore, the closed set

$$\overline{D} \cup \pi_2\Big(\gamma \cup \bigcup_{i=k+1}^{\infty} \overline{\triangle}'_i\Big) \subset \mathbb{C} \tag{3-1}$$

can be exhausted by polynomially convex compact sets. To see this, note that if $V'_j \subset V_j$ are small round discs in $\mathbb{C}$ centered at the point c_j such that $V_j \subset U_j$ for $j = 0, 1, \dots, k$, where the neighborhoods U_j satisfy the hypotheses of the lemma, then the sets

$$(bV_j \setminus \triangle_j) \cup \big(b\triangle_j \cap (\overline{V}_j \setminus V'_j)\big) \cup \Big(\bigcup_{i=k+1}^{\infty} \overline{\triangle}_i \cap (\overline{V}_j \setminus V'_j)\Big) \subset \mathbb{C}$$

are polynomially convex, and the map $\pi_2 \circ g \circ f \colon \bigcup_{j=0}^{k} \overline{V}_j \cap \Omega' \to \mathbb{C}$ is an injection onto a union of wedges such that the closures of any two of them intersect only at their common vertex at ∞. An exhaustion of the set in (3-1) by polynomially convex compact sets is constructed by letting the radii of the discs V'_j go to 0.

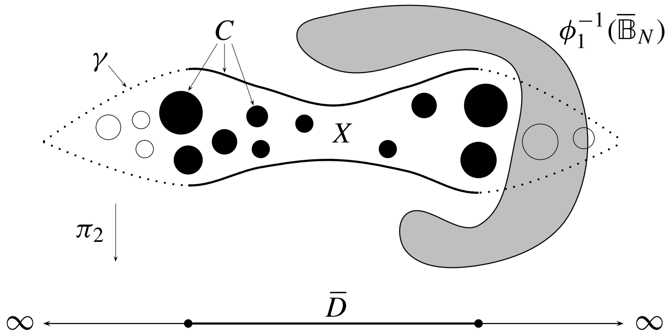

Figure 1. The set C.

Let $J = \{i \in \mathbb{N} : i \geq k+1,\ \pi_2(\overline{\triangle}'_i) \cap \overline{D} \neq \varnothing\}$. Consider the compact set

$$C := \big[\gamma \cap \pi_2^{-1}(\overline{D})\big] \cup \Big[\bigcup_{i \in J} \overline{\triangle}'_i\Big] \subset \overline{X}.$$

(Figure 1 shows C with bold lines and black discs.) We claim that C is polynomially convex. Clearly C is holomorphically convex in $\overline{X}$ since its complement is connected. Furthermore, $\overline{X}$ can be exhausted by compact smoothly bounded subdomains $X_j \subset \overline{X}$ such that each boundary component of X_j intersects the boundary of X. (It suffices to take the intersection of $\overline{X}$ with a sufficiently large ball and smoothen the corners.) Then $\widehat{X}_j \setminus X_j$ is either empty or a pure one-dimensional complex subvariety of $\mathbb{C}^2 \setminus X_j$ (see [Stolzenberg 1966]), the latter being impossible since the variety would have to be unbounded. Hence every such set X_j is polynomially convex, and by choosing it large enough to contain C we see that C is polynomially convex.

We will construct ϕ_1 as a composition $\phi_1 = \sigma_2 \circ \sigma_1 \in \operatorname{Aut}\mathbb{C}^2$ that is close to the identity on K and satisfies $\phi_1(C) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_N$; equivalently, $C \cap \phi_1^{-1}(\overline{\mathbb{B}}_N) = \varnothing$.

By [Wold 2006, Lemma 1] (see also [Forstnerič 2011, Corollary 4.14.5]) there exists $\sigma_1 \in \operatorname{Aut}\mathbb{C}^2$ that is close to the identity on K and satisfies $\sigma_1(\gamma) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_N$.

Let K' be the union of all discs $\overline{\triangle}_i$ $(i \in J)$ whose images $\overline{\triangle}'_i$ satisfy

$$\sigma_1(\overline{\triangle}'_i) \cap \overline{\mathbb{B}}_N \neq \varnothing.$$

Since $\sigma_1(\gamma) \cap \overline{\mathbb{B}}_N = \varnothing$, the set $(\sigma_1 \circ g \circ f)^{-1}(\overline{\mathbb{B}}_N) \subset \Omega'$ is compact, and hence K' is also compact (see Remark 2.3). Lemma 2.2 gives pairwise disjoint smoothly bounded discs $D_1, \ldots, D_m$ in Ω' whose union $\bigcup_{j=1}^m D_j$ contains K' and whose closures $\overline{D}_j$ avoid $b\Omega' \cup (g \circ f)^{-1}(K)$. Set $D'_j = (g \circ f)(D_j) \subset X$ for $j = 1, \ldots, m$. The set

$$L := K \cup \Big(C \setminus \bigcup_{j=1}^m \overline{D}'_j\Big) \subset \mathbb{C}^2$$

is then polynomially convex (argue as above for the set C, using the fact that K is disjoint from C). The union of discs $E_0 := \bigcup_{j=1}^m \sigma_1(\overline{D}'_j)$ is polynomially convex and disjoint from $\sigma_1(L)$, so it can be moved out of the ball $\overline{\mathbb{B}}_N$ by a holomorphic isotopy in the complement of the polynomially convex set

$\sigma_1(L)$. (It suffices to first contract each disc $\sigma_1(\overline{D}'_j)$ into a small ball around one of its points and then move these small balls out of the set $\sigma_1(L)$ along pairwise disjoint arcs.) Furthermore, letting $E_t \subset \mathbb{C}^2$ ($t \in [0, 1]$) denote the trace of E_0 under this isotopy, we can ensure that for every t the union $E_t \cup \sigma_1(L)$ is polynomially convex. The Andersén–Lempert theory (see [Forstnerič 2011, Theorem 4.12.1]) now furnishes an automorphism $\sigma_2 \in \operatorname{Aut}\mathbb{C}^2$ that is close to the identity on the set $\sigma_1(L)$ and satisfies

$$(\sigma_2 \circ \sigma_1)\Big(\bigcup_{j=1}^{m} \overline{D}'_j\Big) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_N.$$

The automorphism $\phi_1 = \sigma_2 \circ \sigma_1 \in \operatorname{Aut}\mathbb{C}^2$ is then close to the identity map on K, and $\phi_1(C) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_N$.

Next we shall find a shear automorphism $\phi_2 \in \operatorname{Aut}\mathbb{C}^2$ of the form

$$\phi_2(z_1, z_2) = (z_1 + h(z_2), z_2) \tag{3-2}$$

that is close to the identity on $\mathbb{C} \times (\pi_2(C) \cup \overline{D})$ and satisfies

$$\phi_2\Big(\gamma \cup \Big(\bigcup_{i=k+1}^{\infty} \overline{\triangle}'_i\Big)\Big) \cap \phi_1^{-1}(\overline{\mathbb{B}}_N) = \varnothing.$$

The automorphism $\psi = \phi_1 \circ \phi_2 \in \operatorname{Aut}\mathbb{C}^2$ will then satisfy Lemma 3.1.

Choose a large number $R > 0$ such that

$$\pi_1(\phi_1^{-1}(\overline{\mathbb{B}}_N)) \subset \mathbb{D}_R \quad \text{and} \quad \pi_2(\phi_1^{-1}(\overline{\mathbb{B}}_N)) \cup \overline{D} \subset \mathbb{D}_R.$$

We shall find ϕ_2 as a composition $\phi_2 = \tau_2 \circ \tau_1$ of two shears of the same type (3-2). The values of the function $h \in \mathscr{O}(\mathbb{C})$ in (3-2) on $\mathbb{C} \setminus \mathbb{D}_R$ are unimportant since $\phi_1^{-1}(\overline{\mathbb{B}}_N)$ projects into $\mathbb{D}_R$.

Recall that the projection $\pi_2 \colon \bar{X} \setminus \pi_2^{-1}(D) \to \mathbb{C} \setminus D$ maps $\bar{X} \setminus \pi_2^{-1}(D)$ bijectively onto a union of pairwise disjoint closed wedges with the common vertex at ∞ (see Figure 2 below). Hence the geometry of subsets of $\bar{X} \setminus \pi_2^{-1}(D)$ is the same as the geometry of their π_2-projections in $\mathbb{C} \setminus D$, an observation that will be tacitly used in the sequel.

By [Wold 2006, Lemma 1] there is an entire function $h_1 \in \mathscr{O}(\mathbb{C})$ that is small on the set $\overline{D} \cup \pi_2(C)$ and takes suitable values on the projected curves $\pi_2(\gamma) \setminus D$ so that the shear $\tau_1(z_1, z_2) = (z_1 + h_1(z_2), z_2)$ satisfies

$$\tau_1(\gamma \cup C) \cap \phi_1^{-1}(\overline{\mathbb{B}}_N) = \varnothing.$$

Set $\widetilde{J} = \{i \in \mathbb{N} : i \ge k+1,\ \pi_2(\overline{\triangle}'_i) \cap \overline{\mathbb{D}}_R \ne \varnothing\}$. Consider the compact set

$$\widetilde{C} := \big[\gamma \cap \pi_2^{-1}(\overline{\mathbb{D}}_R)\big] \cup \Big[\bigcup_{i \in \widetilde{J}} \overline{\triangle}'_i\Big] \subset \bar{X}.$$

Let K'' be the union of all discs $\overline{\triangle}_i$ ($i \in \widetilde{J}$) whose images $\overline{\triangle}'_i = (g \circ f)(\overline{\triangle}_i)$ satisfy the condition

$$\tau_1(\overline{\triangle}'_i) \cap \phi_1^{-1}(\overline{\mathbb{B}}_N) \ne \varnothing.$$

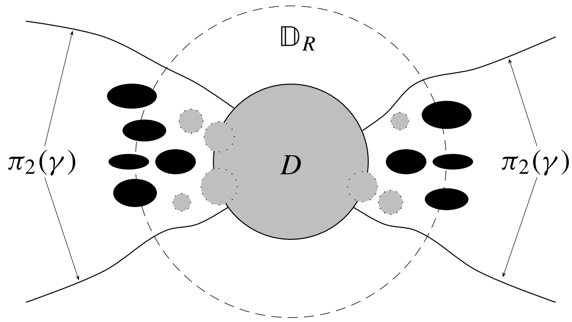

Figure 2. Geometry in the z_2-plane.

Our choices of ϕ_1 and τ_1 imply that for every disc $\overline{\triangle}_i \subset K''$ the projection $\pi_2(\overline{\triangle}_i')$ intersects the disc $\overline{\mathbb{D}}_R$ and avoids the set $\pi_2(C) \cup \overline{D}$. Remark 2.3 shows that K'' is compact. Using Lemma 2.2 we find smoothly bounded discs $B_1, \dots, B_l \subset \Omega'$ with pairwise disjoint closures whose union $\bigcup_{j=1}^{l} B_j$ contains K'' and is disjoint from $b\Omega' \cup (g \circ f)^{-1}(C)$, and whose boundaries bB_j belong to Ω. (Hence every disc $\overline{\triangle}_i$ for $i > k$ is either completely contained in $\bigcup_{j=1}^{l} \overline{B}_j$ or else is disjoint from it.) It follows that the set

$$\widetilde{L} := \bigcup_{j=1}^{l} (\pi_2 \circ g \circ f)(\overline{B}_j) \subset \mathbb{C}$$

is a disjoint union of discs contained in $\mathbb{C} \setminus (\overline{D} \cup \pi_2(\gamma))$. Hence the sets $\widetilde{L}$ and $\pi_2(\widetilde{C}) \setminus \widetilde{L}$ are polynomially convex, and so is their union. (Figure 2 shows $\widetilde{L}$ as the union of black ellipses, while $\pi_2(\widetilde{C}) \setminus \widetilde{L}$ is shown in gray.)

Let $h_2 \in \mathcal{O}(\mathbb{C})$ be such that $|h_2| > R$ on $\widetilde{L}$ and $|h_2|$ is small on $\pi_2(\widetilde{C}) \setminus \widetilde{L}$. Let $\tau_2(z_1, z_2) = (z_1 + h_2(z_2), z_2)$ and $\phi_2 = \tau_2 \circ \tau_1$. The automorphism $\psi = \phi_1 \circ \phi_2 \in \operatorname{Aut} \mathbb{C}^2$ then clearly satisfies Lemma 3.1.

Note that $\phi_2(z_1, z_2) = (z_1 + h(z_2), z_2)$ with $h = h_1 + h_2$, so it is possible to boil down the construction of ϕ_2 to one step. $\square$

4. Proof of Theorem 1.1

We are now ready to prove Theorem 1.1. The construction is similar to the proof of Majcen's theorem [2009] as given in [Forstnerič 2011, §8.10], but the induction scheme is altered and improved at several key points.

Every holomorphic embedding with corners will be assumed to be of the form (2-2).

Let $\Omega \subset \mathbb{P}^1$ be a domain with countably many complementary components, none of which are points. (We assume that there are infinitely many components, for otherwise the result is due to Globevnik and Stensønes [1995]. Our proof also applies in the latter case, but it could be made much simpler.) By the uniformization theorem of He and Schramm [1993] we may assume that Ω is a circled domain. By mapping one of the complementary discs in $\mathbb{P}^1 \setminus \Omega$ onto the complement $\mathbb{P}^1 \setminus \mathbb{D}$ of the unit disc $\mathbb{D}$ we may further assume that $\Omega = \mathbb{D} \setminus \bigcup_{j=1}^{\infty} \overline{\triangle}_j$, where $\overline{\triangle}_j$ are pairwise disjoint closed discs in $\mathbb{D}$.

We construct a proper holomorphic embedding $\Omega \hookrightarrow \mathbb{C}^2$ by induction.

Choose an exhaustion $\varnothing = K_0 \subset K_1 \subset K_2 \subset \ldots \subset \bigcup_{j=1}^{\infty} K_j = \Omega$ of Ω by compact, connected, $\mathcal{O}(\Omega)$-convex sets with smooth boundaries, satisfying $K_j \subset \mathring{K}_{j+1}$ for $j = 0, 1, 2, \ldots$. These conditions imply that for each index $j \in \mathbb{N}$ the set $\widehat{K}_j \setminus K_j \subset \mathbb{D}$ is a union of finitely many open discs, i.e., sets homeomorphic to the standard disc.

We begin the induction at $n = 0$. Set $\Gamma_0 = b\,\mathbb{D}$, $m_0 = k_0 = 0$. Pick a point $c_0 \in \Gamma_0$ and a number $\epsilon_0 > 0$. At the n-th step of the construction we shall obtain the following data:

- Integers $m_n, k_n \in \mathbb{N}$.
- A number ϵ_n such that $0 < \epsilon_n < \frac{1}{2}\epsilon_{n-1}$ (the last inequality is void for $n = 0$).
- Circles $\Gamma_j = b\triangle_{i(j)}$ $(j = 1, \ldots, k_n)$ from the family $\{b\triangle_i\}_{i\in\mathbb{N}}$, at least one in each connected component of $\widehat{K}_{m_n} \setminus K_{m_n}$.
- The domain $\Omega_n = \mathbb{D} \setminus \bigcup_{j=1}^{k_n} \overline{\triangle}_{i(j)}$ with boundary $b\Omega_n = \bigcup_{j=0}^{k_n} \Gamma_j$.
- Points $c_j \in \Gamma_j$ for $j = 0, \ldots, k_n$.
- Numbers $\theta_j > 0$ $(j = 0, \ldots, k_n)$ with $\sum_{j=0}^{k_n} \theta_j < 2\pi$.
- A holomorphic embedding with corners $f_n : \overline{\Omega}_n \hookrightarrow \mathbb{C}^2$ such that the points $c_0, \ldots, c_{k_n}$ are π_1-exposed with θ_j-wedges (see Definition 2.5) and f_n is smooth near $b\Omega_n \setminus \{c_0, \ldots, c_{k_n}\}$.
- A rational shear with poles at the exposed points $f_n(c_j)$ of $f_n(b\Omega_n)$,
$$g_n(z_1, z_2) = \left(z_1, z_2 + \sum_{j=0}^{k_n} \frac{\beta_j}{z_1 - \pi_1(f_n(c_j))}\right),$$
such that $(\pi_2 \circ g_n \circ f_n)(\Omega_n) \subset \mathbb{C}$ is a union of θ_j-wedges whose closures intersect only at their common vertex $\infty \in \mathbb{P}^1$.
- An automorphism ϕ_n of $\mathbb{C}^2$.

In addition, setting
$$F_{n-1} = \Phi_{n-1} \circ g_n \circ f_n, \qquad \Phi_n = \phi_n \circ \Phi_{n-1} = \phi_n \circ \phi_{n-1} \ldots \circ \phi_1,$$
the following conditions hold:
$$|g_n \circ f_n(x) - g_{n-1} \circ f_{n-1}(x)| < \epsilon_n, \qquad x \in K_{m_n}. \tag{4-1}$$
$$|\Phi_{n-1} \circ g_n \circ f_n(x) - \Phi_{n-1} \circ g_{n-1} \circ f_{n-1}(x)| < \epsilon_n, \quad x \in K_{m_n}. \tag{4-2}$$
$$\overline{\mathbb{B}}_{n-1} \cap F_{n-1}(\Omega_n) \subset F_{n-1}(\mathring{K}_{m_n}). \tag{4-3}$$
$$|\phi_n(z) - z| < \epsilon_n, \qquad z \in \overline{\mathbb{B}}_{n-1} \cup F_{n-1}(K_{m_n}). \tag{4-4}$$
$$|\Phi_n \circ g_n \circ f_n(x)| > n, \qquad x \in b\Omega_n \cup (\Omega_n \setminus \Omega). \tag{4-5}$$

Remark 4.1. Setting $J_n = \mathbb{N} \setminus \{i(j) : j = 1, \ldots, k_n\}$, we have
$$\Omega_n = \Omega \cup \bigcup_{j\in J_n} \overline{\triangle}_j, \quad \Omega_n \setminus \Omega = \bigcup_{j\in J_n} \overline{\triangle}_j.$$

Clearly $\mathbb{D} \supset \Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega$, but the intersection $\bigcap_{j=1}^{\infty} \Omega_j$ need not equal Ω. That is, the set of all circles Γ_j that get opened up in the course of the construction may be a proper subset of the family $\{b\triangle_i\}_{i\in\mathbb{Z}_+}$ of all boundary circles of Ω. The only reason for opening a boundary circle contained in $\widehat{K}_{m_n} \setminus K_{m_n}$ is to ensure that the image of K_{m_n} in $\mathbb{C}^2$ becomes polynomially convex; see (4-7) below. □

We begin the induction at $n = 0$ by choosing an embedding $f_0(\zeta) = (\tau_0(\zeta), 0)$ of $\overline{\mathbb{D}}$ in $\mathbb{C} \times \{0\} \subset \mathbb{C}^2$ with a θ_0-wedge at the point $c_0 \in \Gamma_0 = b\mathbb{D}$ (see Theorem 2.8). We also choose a shear

$$g_0(z_1, z_2) = \left(z_1,\ z_2 + \frac{\beta_0}{z_1 - \pi_1 \circ f_0(c_0)}\right)$$

sending the exposed point $\pi_1 \circ f_0(c_0) = \tau_0(c_0)$ to infinity. Let $\phi_0 = \Phi_0 = \Phi_{-1} = \mathrm{Id}$. Conditions (4-1)–(4-4) are then vacuous for $n = 0$ (recall that $K_0 = \varnothing$), and (4-5) is satisfied after a small translation of the embedding $g_0 \circ f_0 \colon \overline{\mathbb{D}} \setminus \{c_0\} \hookrightarrow \mathbb{C}^2$ which removes the image off the origin.

We now explain the inductive step $n \to n+1$. By (4-5) there exists an integer $m_{n+1} > m_n$ such that

$$\overline{\mathbb{B}}_n \cap \big(\Phi_n \circ g_n \circ f_n(\Omega_n)\big) \subset \Phi_n \circ g_n \circ f_n(\mathring{K}_{m_{n+1}}). \tag{4-6}$$

By the inductive hypothesis the polynomial hull $\widehat{K}_{m_{n+1}}$ contains the boundary circles $\Gamma_j \subset b\Omega$ for $1 \le j \le k_n$. (This is vacuous if $n = 0$.) In each of the (finitely many) connected components of $\widehat{K}_{m_{n+1}} \setminus K_{m_{n+1}}$ that does not contain any of the above circles we pick another boundary circle of Ω (such exists since the set $K_{m_{n+1}}$ is $\mathscr{O}(\Omega)$-convex); we label these additional curves $\Gamma_{k_n+1}, \ldots, \Gamma_{k_{n+1}}$. As before, we have $\Gamma_j = b\triangle_{i(j)}$ for some index $i(j)$. Let

$$\Omega_{n+1} = \mathbb{D} \setminus \bigcup_{j=1}^{k_{n+1}} \overline{\triangle}_{i(j)}.$$

Setting $J_{n+1} = \mathbb{N} \setminus \{i(j) : j = 1, \ldots, k_{n+1}\}$, we have that

$$\Omega_{n+1} = \Omega \cup \bigcup_{j \in J_{n+1}} \overline{\triangle}_j.$$

Each of these additional curves will now be opened up. Pick a point $c_j \in \Gamma_j$ for each $j = k_n + 1, \ldots, k_{n+1}$ and positive numbers $\theta_{k_n+1}, \ldots, \theta_{k_{n+1}}$ such that $\sum_{j=0}^{k_{n+1}} \theta_j < 2\pi$. Also choose a number $\epsilon_{n+1} \in (0, \epsilon_n/2)$ such that any holomorphic map $h \colon \Omega \to \mathbb{C}^2$ satisfying $\|h - g_n \circ f_n\|_{K_{m_{n+1}}} < 2\epsilon_{n+1}$ is an embedding on K_{m_n}. Theorem 2.8 furnishes a holomorphic embedding $f_{n+1} \colon \overline{\Omega}_{n+1} \hookrightarrow \mathbb{C}^2$ with corners such that f_{n+1} agrees with f_n to the second order at each of the points $c_0, \ldots, c_{k_n}$, it additionally makes the boundary points $c_{k_n+1}, \ldots, c_{k_{n+1}}$ π_1-exposed with θ_j-wedges, and it approximates f_n as closely as desired outside of small neighborhoods of these points. The image $f_{n+1}(\overline{\Omega}_{n+1})$ stays as close as desired to the union of $f_n(\overline{\Omega}_{n+1})$ with the family of arcs that were attached to this set in order to expose the points $c_{k_n+1}, \ldots, c_{k_{n+1}}$. In particular, we ensure that none of the complex lines $z_1 = \pi_1 \circ f_{n+1}(c_j)$ for $j = k_n + 1, \ldots, k_{n+1}$ intersect the set $\Phi_n^{-1}(\overline{\mathbb{B}}_n)$. The rational shear

$$g_{n+1}(z_1, z_2) = g_n(z_1, z_2) + \left(0,\ \sum_{j=k_n+1}^{k_{n+1}} \frac{\beta_j}{z_1 - \pi_1(f_{n+1}(c_j))}\right)$$

sends the exposed points $f_{n+1}(c_0), \dots, f_{n+1}(c_{k_{n+1}})$ to infinity. A suitable choice of the arguments of $\beta_j \in \mathbb{C}^*$ for $j = k_n + 1, \dots, k_{n+1}$ ensures that, in a neighborhood of infinity, $(\pi_2 \circ g_{n+1} \circ f_{n+1})(\overline{\Omega}_{n+1})$ is a union of pairwise disjoint θ_j-wedges with the common vertex at $\infty \in \mathbb{P}^1$; at the same time the absolute values $|\beta_j| > 0$ can be chosen arbitrarily small in order to obtain good approximation of g_n by g_{n+1}.

Set $F_n = \Phi_n \circ g_{n+1} \circ f_{n+1}$. If the approximations of f_n, g_n by f_{n+1}, g_{n+1}, respectively, were close enough, then the conditions (4-1)–(4-3) hold with n replaced by $n + 1$.

Since every connected component of $\widehat{K}_{m_{n+1}} \setminus K_{m_{n+1}}$ contains at least one of the points $c_1, \dots, c_{m_{n+1}}$ which F_n sends to infinity, the set $F_n(K_{m_{n+1}}) \subset \mathbb{C}^2$ is polynomially convex. (See [Wold 2006, Proposition 3.1] for the details of this argument.) From (4-6) we also infer that $\overline{\mathbb{B}}_n \cap F_n(\Omega_{n+1}) \subset F_n(\mathring{K}_{m_{n+1}})$ provided that the approximations were close enough. It follows that the set

$$L_n := \overline{\mathbb{B}}_n \cup F_n(K_{m_{n+1}}) \subset \mathbb{C}^2 \tag{4-7}$$

is polynomially convex.

Now comes the last, and the main step in the induction: We use Lemma 3.1 to find an automorphism $\phi_{n+1} \in \operatorname{Aut}\mathbb{C}^2$ which satisfies conditions (4-4) and (4-5) with n replaced by $n + 1$. We look for ϕ_{n+1} of the form

$$\phi_{n+1} = \Phi_n \circ \psi \circ \Phi_n^{-1}, \quad \psi \in \operatorname{Aut}\mathbb{C}^2.$$

(Therefore $\Phi_{n+1} = \phi_{n+1} \circ \Phi_n = \Phi_n \circ \psi$.) Pick a small constant $\delta > 0$ such that for any pair of points $z, z' \in \mathbb{C}^2$, with $z \in \Phi_n^{-1}(L_n)$ and $|z - z'| < \delta$, we have $|\Phi_n(z) - \Phi_n(z')| < \epsilon_{n+1}$. (Such δ exists by continuity of Φ_n.) We also pick a large constant $R > 0$ such that $|\Phi_n(z)| > n + 1$ for all $z \in \mathbb{C}^2$ with $|z| > R$. (Equivalently, $\Phi_n^{-1}(\overline{\mathbb{B}}_n) \subset \overline{\mathbb{B}}_R$.) Since the set $\Phi_n^{-1}(L_n)$ is polynomially convex, Lemma 3.1 furnishes an automorphism $\psi \in \operatorname{Aut}\mathbb{C}^2$ satisfying the following two conditions:

(4.4′) $|\psi(z) - z| < \delta$ for $z \in \Phi_n^{-1}(L_n)$.

(4.5′) $|\psi(z)| > R$ for $z \in g_{n+1} \circ f_{n+1}(b\Omega_{n+1} \cup \bigcup_{j \in J_{n+1}} \overline{\triangle}_j)$.

By (4-3) (applied with $n + 1$) the two sets appearing in these conditions are disjoint. It is now immediate that ϕ_{n+1} satisfies conditions (4-4), (4-5).

This completes the induction step, so the induction may proceed.

We now conclude the proof. By (4-1) and the choice of the numbers $\epsilon_n > 0$ we see that the limit map $G = \lim_{n\to\infty} g_n \circ f_n \colon \Omega \to \mathbb{C}^2$ is a holomorphic embedding. Condition (4-4) implies that the sequence $\Phi_n \in \operatorname{Aut}\mathbb{C}^2$ converges on the domain $O = \bigcup_{n=2}^{\infty} \Phi_n^{-1}(\overline{\mathbb{B}}_{n-1}) \subset \mathbb{C}^2$ to a Fatou–Bieberbach map $\Phi = \lim_{n\to\infty} \Phi_n \colon O \to \mathbb{C}^2$, i.e., a biholomorphic map of O onto $\mathbb{C}^2$ (see [Forstnerič 2011, Corollary 4.4.2]). Conditions (4-2) and (4-4) show that the sequence Φ_n converges on $G(\Omega)$, so $G(\Omega) \subset O$. From (4-3) and (4-5) we see that G embeds Ω properly into O. Hence the map

$$F = \Phi \circ G = \lim_{n\to\infty} \Phi_n \circ g_n \circ f_n \colon \Omega \hookrightarrow \mathbb{C}^2$$

is a proper holomorphic embedding of Ω into $\mathbb{C}^2$. □

Remark 4.2. If we choose an initial holomorphic embedding $f_0 \colon \overline{\mathbb{D}} \hookrightarrow \mathbb{C}^2$, a compact set $K = K_0 \subset \Omega$ and a number $\epsilon > 0$, then the above construction is easily modified to yield a proper holomorphic embedding $F \colon \Omega \hookrightarrow \mathbb{C}^2$ satisfying $\|F - f\|_K < \epsilon$. Furthermore, we can choose F to agree with f at finitely many points of Ω. All these additions are standard.

5. Domains with punctures

Theorem 1.1 can be extended to domains Ω in $\mathbb{P}^1$ with certain boundary punctures. By a *puncture* we mean a connected component of $\mathbb{P}^1 \setminus \Omega$ that is a point. We say that a domain $\Omega \subset \mathbb{P}^1$ is a *generalized circled domain* if each complementary component is either a round disc or a puncture. By [He and Schramm 1993], any domain in $\mathbb{P}^1$ with at most countably many boundary components is conformally equivalent to a generalized circled domain.

Our main result in this direction is the following.

Theorem 5.1. *Let Ω be a generalized circled domain in $\mathbb{P}^1$. If all but finitely many punctures in the complement $K := \mathbb{P}^1 \setminus \Omega$ are limit points of discs in K, then Ω embeds properly holomorphically into $\mathbb{C}^2$.*

Corollary 5.2. *If Ω is a circled domain in $\mathbb{C}$ or in $\mathbb{P}^1$ and $p_1, \dots, p_l \in \Omega$ is an arbitrary finite set of points in Ω, then the domain $\Omega \setminus \{p_1, \dots, p_l\}$ admits a proper holomorphic embedding into $\mathbb{C}^2$.*

By He and Schramm, Corollary 5.2 also holds for $\Omega \setminus \{p_1, \dots, p_l\}$, where $\Omega \subset \mathbb{P}^1$ is a domain as in Theorem 1.1.

Proof of Theorem 5.1. We make the following modifications to the proof of Theorem 1.1. We may assume as before that Ω is contained in the unit disc $\mathbb{D}$, with $\Gamma_0 = b\mathbb{D}$ being one of its boundary components. Let $f_0 \colon \Omega \hookrightarrow \mathbb{C}^2$ be the embedding $\zeta \mapsto (\zeta, 0)$. Assume that $p_1, \dots, p_l \in b\Omega$ are the finitely many punctures which do not belong to the cluster set of $\bigcup_i \overline{\Delta}_i$. A rational shear $g_0(z_1, z_2) = (z_1, z_2 + \sum_{j=1}^{l} \beta_j/(z_1 - p_j))$ sends the points $p_1, \dots, p_l$ to infinity. We then apply the rest of the proof exactly as before, ensuring at each step of the inductive construction that the embedding with corners $f_n \colon \overline{\Omega}_n \hookrightarrow \mathbb{C}^2$ agrees with f_0 at the points $p_1, \dots, p_l$ and leaves these points π_1-exposed, and the shear g_n has poles at these points. The coordinate projection $\pi_2 \colon \overline{X}_n = g_n \circ f_n(\overline{\Omega}_n) \to \mathbb{C}$ is no longer injective near infinity due to the poles of g_n at the points $p_1, \dots, p_l$. However, since the discs $\overline{\Delta}_i$ do not accumulate on any of the points $p_1, \dots, p_l$, the discs $(g_n \circ f_n)(\overline{\Delta}_i) \subset X_n$ which approach infinity are still mapped bijectively to a finite union of pairwise disjoint wedges at ∞, and the additional sheets of the projection $\pi_2 \colon \overline{X}_n \to \mathbb{C}$ are irrelevant for the construction of the automorphism, which removes the discs and the boundary curves of X_n out of a given ball in $\mathbb{C}^2$.

The remaining punctures p_λ in $b\Omega$ (a possibly uncountable set) can be treated in the same way as the complementary discs. Indeed, since each of these points is a limit point of the sequence of discs Δ_i, every connected component of the set $\widehat{K}_m \setminus K_m$ (where K_m is a sequence of compacts exhausting the domain Ω, see Section 4) that contains one of these punctures p_λ also contains a disc $\overline{\Delta}_i$. By exposing a boundary point of Δ_i and removing it to infinity by a rational shear we thus ensure that the image of p_λ does not belong to the polynomial hull of the image of K_m in $\mathbb{C}^2$. (See Remark 4.1.) The conclusion of Remark 2.3 is still valid, and hence the arguments in the proof of Theorem 1.1 concerning moving compact sets by automorphisms of $\mathbb{C}^2$ still apply without any changes. □

Example 5.3. Assume that $E \subset \mathbb{P}^1$ is any compact totally disconnected set. (In particular, E could be a Cantor set). Then we may choose a sequence of pairwise disjoint closed round discs $\overline{\triangle}_j \subset \mathbb{P}^1 \setminus E$ such that each point of E is a cluster point of the sequence $\{\triangle_j\}$ and such that $\Omega := \mathbb{P}^1 \setminus (E \cup (\bigcup_j \overline{\triangle}_j))$ is a domain. Then Ω embeds properly in $\mathbb{C}^2$.

There exists a Cantor set in $\mathbb{P}^1$ whose complement embeds properly holomorphically into $\mathbb{C}^2$ [Orevkov 2008], but it is an open problem whether this holds for each Cantor set. □

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