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SMOOTHING AND GLOBAL ATTRACTORS FOR THE ZAKHAROV SYSTEM ON THE TORUS

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We consider the Zakharov system with periodic boundary conditions in dimension one. In the first part of the paper, it is shown that for fixed initial data in a Sobolev space, the difference of the nonlinear and the linear evolution is in a smoother space for all times the solution exists. The smoothing index depends on a parameter distinguishing the resonant and nonresonant cases. As a corollary, we obtain polynomial-in-time bounds for the Sobolev norms with regularity above the energy level. In the second part of the paper, we consider the forced and damped Zakharov system and obtain analogous smoothing estimates. As a corollary we prove the existence and smoothness of global attractors in the energy space.

1. Introduction

We study the system of nonlinear partial differential equations, introduced in [Zakharov 1972]. It describes the propagation of Langmuir waves in an ionized plasma. The system with periodic boundary conditions consists of a complex field u (Schrödinger part) and a real field n (wave part) satisfying the equation

$$\begin{cases} iu_t + \alpha u_{xx} = nu, \quad x \in \mathbb{T}, \quad t \in [-T, T], \\ n_{tt} - n_{xx} = (|u|^2)_{xx}, \\ u(x,0) = u_0(x) \in H^{s_0}(\mathbb{T}), \\ n(x,0) = n_0(x) \in H^{s_1}(\mathbb{T}), \quad n_t(x,0) = n_1(x) \in H^{s_1-1}(\mathbb{T}), \end{cases} \tag{1}$$

where $\alpha > 0$ and T is the time of existence of the solutions. The function $u(x,t)$ denotes the slowly varying envelope of the electric field with a prescribed frequency and the function $n(x,t)$ denotes the deviation of the ion density from the equilibrium. Here α is the dispersion coefficient. In the literature (see, e.g., [Takaoka 1999]) it is standard to include the speed of an ion acoustic wave in a plasma as a coefficient β^{-2} in front of n_{tt} where $\beta > 0$. One can scale away this parameter using time and amplitude coefficients of the form $t \to \beta t$, $u \to \sqrt{\beta}u$, and $n \to \beta n$ and reduce the system to (1). Smooth solutions of the Zakharov system obey the conservation laws

$$\|u(t)\|_{L^2(\mathbb{T})} = \|u_0\|_{L^2(\mathbb{T})}$$

and

$$E(u,n,\nu)(t) = \alpha \int_{\mathbb{T}} |\partial_x u|^2\,dx + \frac{1}{2}\int_{\mathbb{T}} n^2\,dx + \frac{1}{2}\int_{\mathbb{T}} \nu^2\,dx + \int_{\mathbb{T}} n|u|^2\,dx = E(u_0,n_0,n_1)$$

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where ν is such that $n_t = \nu_x$ and $\nu_t = (n + |u|^2)_x$. These conservation laws identify $H^1 \times L^2 \times H^{-1}$ as the energy space for the system.

For $\alpha = 1$, Bourgain [1994] proved that the problem is locally well-posed in the energy space using the restricted norm method (see, e.g., [Bourgain 1993]). The solutions are well-posed in the sense of the following definition

Definition 1.1. Let X, Y, Z be Banach spaces. We say that the system of equations (1) is locally well-posed in $H^{s_0}(\mathbb{T}) \times H^{s_1}(\mathbb{T}) \times H^{s_1-1}(\mathbb{T})$, if for given initial data

$$(u_0, n_0, n_1) \in H^{s_0}(\mathbb{T}) \times H^{s_1}(\mathbb{T}) \times H^{s_1-1}(\mathbb{T}),$$

there exists $T = T(\|u_0\|_{H^{s_0}}, \|n_0\|_{H^{s_1}}, \|n_1\|_{H^{s_1-1}}) > 0$ and a unique solution

$$(u, n, n_t) \in \big(X \cap C_t^0 H_x^{s_0}([-T, T] \times \mathbb{T}),\ Y \cap C_t^0 H_x^{s_1}([-T, T] \times \mathbb{T}),\ Z \cap C_t^0 H_x^{s_1-1}([-T, T] \times \mathbb{T})\big).$$

We also demand that there is continuity with respect to the initial data in the appropriate topology. If T can be taken to be arbitrarily large then we say that the problem is globally well-posed.

Thus, the energy solutions exist for all times due to the a priori bounds on the local theory norms. We should note that although the quantity $\int_{\mathbb{T}} n|u|^2\, dx$ has no definite sign it can be controlled using Sobolev inequalities by the H^1 norm of u and the L^2 norm of n. This gives the a priori bound (see [Pecher 2001])

$$\|u(t)\|_{H^1} + \|n(t)\|_{L^2} + \|n_t(t)\|_{H^{-1}} \lesssim \|u(0)\|_{H^1} + \|n(0)\|_{L^2} + \|n_t(0)\|_{H^{-1}}, \quad t \in \mathbb{R} \tag{2}$$

Takaoka [1999] extended the local-in-time theory of Bourgain and proved that when $\frac{1}{\alpha} \in \mathbb{N}$ we have local well-posedness in $H^{s_0} \times H^{s_1} \times H^{s_1-1}$ for $s_1 \geq 0$ and $\max(s_1, \frac{s_1}{2} + \frac{1}{2}) \leq s_0 \leq s_1 + 1$. In the case that $\frac{1}{\alpha} \notin \mathbb{N}$ one has local well-posedness for $s_1 \geq -\frac{1}{2}$, $\max(s_1, \frac{s_1}{2} + \frac{1}{4}) \leq s_0 \leq s_1 + 1$. A recent result [Kishimoto 2011] establishes well-posedness in the case of the higher dimensional torus.

The corresponding Cauchy problem on $\mathbb{R}^d$ has a long history. In this case it is somehow easier to establish the well-posedness of the system due to the dispersive effects of the solution waves. We cite the following papers [Added and Added 1984; 1988, Bejenaru and Herr 2011; Bejenaru et al. 2009; Bourgain and Colliander 1996; Colliander et al. 2008; Ginibre et al. 1997; Kenig et al. 1995; Sulem and Sulem 1979] as a historical summary of the results. It is expected that (see, e.g., [Kishimoto 2011]) the optimal regularity range for local well-posedness is on the line $s_1 = s_0 - \frac{1}{2}$ because the two equations in the Zakharov system equally share the loss of derivative. The Zakharov system is not scale invariant but it can be reduced to a simplified system like in [Ginibre et al. 1997], and one can then define a critical regularity. This is given by the pair $(s_0, s_1) = (\frac{d-3}{2}, \frac{d-4}{2})$, which is also on the line. In dimensions 1 and 2, the lowest regularity for the system to have local solutions has been found to be $(s_0, s_1) = (0, -\frac{1}{2})$ [Ginibre et al. 1997]. It is harder to establish the global ! solutions at this level since there is no conservation law controlling the wave part. This has been done only in one dimension [Colliander et al. 2008].

In the first part of this paper we study the dynamics of the solutions in the periodic case in more detail.[1] We prove that the difference between the nonlinear and the linear evolution for both the Schrödinger

[1] We restrict ourselves to the one-dimensional periodic case because the resonance structure is simpler. The corresponding problem in higher dimensions, $\mathbb{T}^d$ or $\mathbb{R}^d$, appears to be much harder.

and the wave part is in a smoother space than the corresponding initial data, see Theorems 2.3 and 2.4 below. This smoothing property is not apparent if one views the nonlinear evolution as a perturbation of the linear flow and apply standard Picard iteration techniques to absorb the nonlinear terms. The result will follow from a combination of the method of normal forms (through differentiation by parts) inspired by the result in [Babin et al. 2011], and the restricted norm method of Bourgain [1993]. Here the method is applied to a dispersive system of equations where the resonances are harder to control and the coupling nonlinear terms introduce additional difficulties in estimating the first order corrections. As a corollary, in the case $\alpha > 0$, we obtain polynomial-in-time bounds for Sobolev norms above the energy level $(s_0, s_1) = (1, 0)$ by a bootstrapping argument utilizing the a priori bounds and the smoothing estimates, see Corollary 2.5 below. We have applied this method in [Erdoğan and Tzirakis 2012] to obtain similar results for the periodic KdV with a smooth space-time potential. We note that the resonance structure in one-dimensional is easier to handle.

In the second part we study the existence of a global attractor (see the next section for a definition of global attractors and the statement of our result) for the dissipative Zakharov system in the energy space. Our motivation comes from the smoothing estimates that we obtained in the first part of the paper and our work in [Erdoğan and Tzirakis 2011] (also see [Goubet and Molinet 2009] in which the existence of global attractors was obtained as a corollary of a Kato type smoothing estimate). More precisely we consider

$$\begin{cases} iu_t + \alpha u_{xx} + i\gamma u = nu + f, \quad x \in \mathbb{T}, \quad t \in [-T, T], \\ n_{tt} - n_{xx} + \nu n_t = (|u|^2)_{xx} + g, \\ u(x,0) = u_0(x) \in H^1(\mathbb{T}), \\ n(x,0) = n_0(x) \in L^2(\mathbb{T}), \quad n_t(x,0) = n_1(x) \in H^{-1}(\mathbb{T}), \quad f \in H^1(\mathbb{T}), \quad g \in L^2(\mathbb{T}) \end{cases} \tag{3}$$

where $f,\ g$ are time-independent, g is mean-zero, $\int_{\mathbb{T}} g(x)dx = 0$, and the damping coefficients $\nu,\ \gamma > 0$. For simplicity we set $\gamma = \nu$, and $g = 0$. Our calculations apply equally well to the full system and all proofs go through with minor modifications (in particular, one does not need any other a priori estimates).

The problem with Dirichlet boundary conditions has been considered in [Flahaut 1991; Goubet and Moise 1998] in more regular spaces than the energy space. The regularity of the attractor in Gevrey spaces with periodic boundary problem was considered in [Shcherbina 2003].

Notation. To avoid the use of multiple constants, we write $A \lesssim B$ to denote that there is an absolute constant C such that $A \le CB$. We also write $A \sim B$ to denote both $A \lesssim B$ and $B \lesssim A$. We also define $\langle \cdot \rangle = 1 + |\cdot|$.

We define the Fourier sequence of a 2π-periodic L^2 function u as

$$u_k = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx, \quad k \in \mathbb{Z}.$$

With this normalization we have

$$u(x) = \sum_k e^{ikx}u_k \quad \text{and} \quad (uv)_k = u_k * v_k = \sum_{m+n=k} u_n v_m.$$

As usual, for $s<0$, H^s is the completion of L^2 under the norm

$$\|u\|_{H^s} = \|\hat{u}(k)\langle k\rangle^s\|_{\ell^2}.$$

Note that for a mean-zero L^2 function u, $\|u\|_{H^s} \sim \|\hat{u}(k)|k|^s\|_{\ell^2}$. For a sequence u_k, with $u_0=0$, we will use $\|u\|_{H^s}$ notation to denote $\|u_k|k|^s\|_{\ell^2}$. We also define $\dot{H}^s = \{u \in H^s : u \text{ is mean-zero}\}$. For $s=0$ we write $\dot{H}^0 = \dot{L}^2$.

The following function will appear many times in the proofs below.

$$\phi_\beta(k) := \sum_{|n|\leq|k|} \frac{1}{|n|^\beta} \sim \begin{cases} 1 & \text{if } \beta>1, \\ \log(1+\langle k\rangle) & \text{if } \beta=1, \\ \langle k\rangle^{1-\beta} & \text{if } \beta<1. \end{cases}$$

2. Statement of results

Smoothing estimates for the Zakharov system. First note that if n_0 and n_1 are mean-zero then n, n_t remain mean-zero during the evolution since by integrating the wave part of the system (1) we obtain $\partial_t^2 \int_{\mathbb{T}} n(x,t)\,dx = 0$. We will work with this mean-zero assumption in this paper. This is no loss of generality since if $\int_T n_0(x)\,dx = A$ and $\int_T n_1(x)\,dx = B$, then one can consider the new variables $n \to n - A - Bt$ and $u \to e^{i(Bt^2/2+At)}u$, and obtain the same system with mean-zero data.

By considering the operator $d = (-\partial_{xx})^{1/2}$, and writing $n_\pm = n \pm i d^{-1} n_t$, the system (1) can be rewritten as

$$\begin{cases} iu_t + \alpha u_{xx} = \frac{1}{2}(n_+ + n_-)u, \quad x\in\mathbb{T}, \quad t\in[-T,T], \\ (i\partial_t \mp d)n_\pm = \pm d(|u|^2), \\ u(x,0)=u_0(x)\in H^{s_0}(\mathbb{T}), \quad n_\pm(x,0) = n_0(x) \pm i d^{-1}n_1(x) \in H^{s_1}(\mathbb{T}). \end{cases} \tag{4}$$

Note that $d^{-1}n_1(x)$ is well-defined because of the mean-zero assumption, and that $n_+ = \overline{n_-}$.

The local well posedness of the system was established in the framework of $X^{s,b}$ spaces introduced by Bourgain [1993]. Let

$$\|u\|_{X^{s,b}} = \big\|\langle k\rangle^s \langle \tau - \alpha k^2\rangle^b \hat{u}(k,\tau)\big\|_{\ell^2_k L^2_\tau},$$

$$\|n\|_{Y^{s,b}_\pm} = \big\|\langle k\rangle^s \langle \tau \mp |k|\rangle^b \hat{n}(k,\tau)\big\|_{\ell^2_k L^2_\tau}.$$

Here $\pm$ corresponds to the norm of $n_\pm$ in the system (4). As usual we also define the restricted norm

$$\|u\|_{X^{s,b}_T} = \inf_{\substack{\widetilde{u}=u \\ t\in[-T,T]}} \|\widetilde{u}\|_{X^{s,b}}.$$

The norms $Y^{s,b}_{\pm,T}$ are defined accordingly. We also abbreviate $n_\pm(x,0) = n_{\pm,0}$.

Definition 2.1. We say (s_0,s_1) is α-admissible if $s_1 \geq -\frac{1}{2}$ and $\max(s_1, \frac{s_1}{2}+\frac{1}{4}) \leq s_0 \leq s_1+1$ for $\frac{1}{\alpha}\notin\mathbb{N}$, or if $s_1\geq 0$ and $\max(s_1,\frac{s_1}{2}+\frac{1}{2}) \leq s_0 \leq s_1+1$ for $\frac{1}{\alpha}\in\mathbb{N}$.

Takaoka's theorem on local well-posedness can be stated as follows:

Theorem 2.2 [Takaoka 1999]. *Suppose $\alpha \neq 0$ and (s_0, s_1) is α-admissible. Then given initial data $(u_0, n_{+,0}, n_{-,0}) \in H^{s_0} \times H^{s_1} \times H^{s_1}$ there exists*

$$T \gtrsim \left(\|u_0\|_{H^{s_0}} + \|n_{+,0}\|_{H^{s_1}} + \|n_{-,0}\|_{H^{s_1}}\right)^{-\frac{1}{12}+},$$

and a unique solution $(u, n_+, n_-) \in C\left([-T, T] : H^{s_0} \times H^{s_1} \times H^{s_1}\right)$. Moreover, we have

$$\|u\|_{X_T^{s,\frac{1}{2}}} + \|n_{+,0}\|_{Y_{+,T}^{s_1,\frac{1}{2}}} + \|n_{-,0}\|_{Y_{-,T}^{s_1,\frac{1}{2}}} \leq 2\left(\|u_0\|_{H^{s_0}} + \|n_{+,0}\|_{H^{s_1}} + \|n_{-,0}\|_{H^{s_1}}\right).$$

Now, we can state our results on the smoothing estimates:

Theorem 2.3. *Suppose $\frac{1}{\alpha} \notin \mathbb{N}$, and (s_0, s_1) is α-admissible. Consider the solution of* (4) *with initial data $(u_0, n_{+,0}, n_{-,0}) \in H^{s_0} \times H^{s_1} \times H^{s_1}$. Assume that we have a growth bound*

$$\|u(t)\|_{H^{s_0}} + \|n_+(t)\|_{H^{s_1}} + \|n_-(t)\|_{H^{s_1}} \leq C\left(\|u_0\|_{H^{s_0}} + \|n_{+,0}\|_{H^{s_1}} + \|n_{-,0}\|_{H^{s_1}}\right)(1 + |t|)^{\eta(s_0,s_1)}.$$

Then, for any $a_0 \leq \min(1, 2s_0, 1 + 2s_1)$ (the inequality has to be strict if $s_0 - s_1 = 1$) and for any $a_1 \leq \min(1, 2s_0, 2s_0 - s_1)$, we have

$$u(t) - e^{i\alpha t\partial_x^2}u_0 \in C_t^0 H_x^{s_0+a_0}(\mathbb{R} \times \mathbb{T}), \tag{5}$$

$$n_\pm(t) - e^{\mp itd}n_{\pm,0} \in C_t^0 H_x^{s_1+a_1}(\mathbb{R} \times \mathbb{T}). \tag{6}$$

Moreover, for $\beta > 1 + 15\eta(s_0, s_1)$, we have

$$\|u(t) - e^{i\alpha t\partial_x^2}u_0\|_{H^{s_0+a_0}} + \|n_\pm(t) - e^{\mp itd}n_{\pm,0}\|_{H^{s_1+a_1}} \leq C(1 + |t|)^\beta, \tag{7}$$

where $C = C\left(s_0, s_1, a_0, a_1, \|u_0\|_{H^{s_0}}, \|n_{+,0}\|_{H^{s_1}}, \|n_{-,0}\|_{H^{s_1}}\right)$.

Theorem 2.4. *Suppose $\frac{1}{\alpha} \in \mathbb{N}$, and (s_0, s_1) is α-admissible. Assume that we have a growth bound*

$$\|u(t)\|_{H^{s_0}} + \|n_+(t)\|_{H^{s_1}} + \|n_-(t)\|_{H^{s_1}} \leq C\left(\|u_0\|_{H^{s_0}} + \|n_{+,0}\|_{H^{s_1}} + \|n_{-,0}\|_{H^{s_1}}\right)(1 + |t|)^{\alpha(s_0,s_1)}.$$

Then, for any $a_0 \leq \min(1, s_1)$ (the inequality has to be strict if $s_0 - s_1 = 1$ and $s_1 \geq 1$) and for any $a_1 \leq \min(1, 2s_0 - s_1 - 1)$, we have (5), (6) *and* (7).

The growth bound assumption in the theorems above follows from (2) in the case $s_0 = 1$ and $s_1 = 0$. This is used in the corollary below together with a bootstrapping argument to obtain norm growth bounds in all regularity levels above energy. Although the actual growth bounds can be calculated explicitly we won't do so here since we don't believe that the rates are optimal.

Corollary 2.5. *For any $\alpha > 0$, and for any α-admissible (s_0, s_1) with $s_0 \geq 1$, $s_1 \geq 0$, the global solution of* (4) *with $H^{s_0} \times H^{s_1} \times H^{s_1}$ data satisfies the growth bound*

$$\|u(t)\|_{H^{s_0}} + \|n_+(t)\|_{H^{s_1}} + \|n_-(t)\|_{H^{s_1}} \leq C_1(1 + |t|)^{C_2},$$

where C_1 depends on s_0, s_1, and $\|u_0\|_{H^{s_0}} + \|n_{+,0}\|_{H^{s_1}} + \|n_{-,0}\|_{H^{s_1}}$, and C_2 depends on s_0, s_1.

Proof. We drop the $\pm$ signs and work with u and n. First note that because of the energy conservation, $\|u\|_{H^1}$ and $\|n\|_{L^2}$ are bounded for all times. Assume that the claim holds for regularity levels (s_0, s_1). Let (a_0, a_1) be given by Theorem 2.3 or Theorem 2.4. Note that for initial data in $H^{s_0+a_0} \times H^{s_1+a_1}$, applying the theorem with (s_0, s_1) and (a_0, a_1), we have

$$\|u(t) - e^{i\alpha t\partial_x^2} u_0\|_{H^{s_0+a_0}} + \|n_\pm(t) - e^{\mp itd} n_{\pm,0}\|_{H^{s_1+a_1}} \le C(1+|t|)^\beta.$$

Therefore, since the linear groups are unitary, we have

$$\|u(t)\|_{H^{s_0+a_0}} + \|n(t)\|_{H^{s_1+a_1}} \le C(1+|t|)^\beta + \|u_0\|_{H^{s_0+a_0}} + \|n_0\|_{H^{s_1+a_0}}.$$

The statement follows by induction on the regularity.

We note that in the case $\frac{1}{\alpha} \in \mathbb{N}$, $s_0 = 1$, $s_1 = 0$, we have $a_0 = 0$. However, since $a_1 \in [0, 1]$, we obtain the statement for α-admissible $(1, s_1)$, $0 \le s_1 \le 1$. From then on we can take both $a_0 > 0$ and $a_1 > 0$. $\square$

Existence of a global attractor for the dissipative Zakharov system. The problem of global attractors for nonlinear PDEs is concerned with the description of the nonlinear dynamics for a given problem as $t \to \infty$. In particular assuming that one has a well-posed problem for all times we can define the semigroup operator $U(t) : u_0 \in H \to u(t) \in H$ where H is the phase space. We want to describe the long time asymptotics of the solution by an invariant set $X \subset H$ (a global attractor) to which the orbit converges as $t \to \infty$:

$$U(t)X = X, \quad t \in \mathbb{R}_+, \qquad d(u(t), X) \to 0.$$

For dissipative systems there are many results (see, e.g., [Temam 1997]) establishing the existence of a compact set that satisfies the above properties. Dissipativity is characterized by the existence of a bounded absorbing set into which all solutions enter eventually. The candidate for the attractor set is the omega limit set of an absorbing set, B, defined by

$$\omega(B) = \bigcap_{s\ge 0} \overline{\bigcup_{t\ge s} U(t)B},$$

where the closure is taken on H. To state our result we need some definitions from [Temam 1997] (also see [Erdoğan and Tzirakis 2011; Flahaut 1991; Goubet and Moise 1998] for more discussion).

Definition 2.6. We say that a compact subset $\mathcal{A}$ of H is a global attractor for the semigroup $\{U(t)\}_{t\ge 0}$ if $\mathcal{A}$ is invariant under the flow and if for every $u_0 \in H$, $d(U(t)u_0, \mathcal{A}) \to 0$ as $t \to \infty$.

The distance is understood to be the distance of a point to the set $d(x, Y) = \inf_{y\in Y} d(x, y)$.

To state a general theorem for the existence of a global attractor we need one more definition:

Definition 2.7. We say a bounded subset $\mathcal{B}_0$ of H is absorbing if for any bounded $\mathcal{B} \subset H$ there exists $T = T(\mathcal{B})$ such that for all $t \ge T$, $U(t)\mathcal{B} \subset \mathcal{B}_0$.

It is not hard to see that the existence of a global attractor $\mathcal{A}$ for a semigroup $U(t)$ implies the existence of an absorbing set. For the converse we cite the following theorem from [Temam 1997] which gives a general criterion for the existence of a global attractor.

Theorem A. *We assume that H is a metric space and that the operator $U(t)$ is a continuous semigroup from H to itself for all $t \geq 0$. We also assume that there exists an absorbing set $\mathcal{B}_0$. If the semigroup $\{U(t)\}_{t\geq 0}$ is asymptotically compact, i.e., for every bounded sequence x_k in H and every sequence $t_k \to \infty$, $\{U(t_k)x_k\}_k$ is relatively compact in H, then $\omega(\mathcal{B}_0)$ is a global attractor.*

Using Theorem A and a smoothing estimate as above, we will prove the following

Theorem 2.8. *Fix $\alpha > 0$. Consider the dissipative Zakharov system* (3) *on $\mathbb{T} \times [0, \infty)$ with $u_0 \in H^1$ and with mean-zero $n_0 \in L^2$, $n_1 \in H^{-1}$. Then the equation possesses a global attractor in $H^1 \times \dot{L}^2 \times \dot{H}^{-1}$. Moreover, for any $a \in (0, 1)$, the global attractor is a compact subset of $H^{1+a} \times H^a \times H^{-1+a}$, and it is bounded in $H^{1+a} \times H^a \times H^{-1+a}$ by a constant depending only on a, α, γ, and $\|f\|_{H^1}$.*

To prove Theorem 2.8 in the case $\frac{1}{\alpha} \notin \mathbb{N}$ we will demonstrate that the solution decomposes into two parts; a linear one which decays to zero as time goes to infinity and a nonlinear one which always belongs to a smoother space. As a corollary we prove that all solutions are attracted by a ball in $H^{1+a} \times H^a \times H^{-1+a}$, $a \in (0, 1)$, whose radius depends only on a, the H^1 norm of the forcing term and the damping parameter. This implies the existence of a smooth global attractor and provides quantitative information on the size of the attractor set in $H^{1+a} \times H^a \times H^{-1+a}$. In addition it implies that higher order Sobolev norms are bounded for all positive times; see [Erdoğan and Tzirakis 2011]. In the case $\frac{1}{\alpha} \in \mathbb{N}$ the proof is slightly different because of a resonant term.

We close this section with a discussion of the well-posedness of (3) in $H^1 \times L^2 \times H^{-1}$. We first rewrite the system (when $\gamma = \nu$, $g = 0$) by passing to $n_\pm$ variables as above:

$$\begin{cases} (i\partial_t + \alpha\partial_x^2 + i\gamma)u = \frac{1}{2}(n_+ + n_-)u + f, \quad x \in \mathbb{T}, \quad t \in [-T, T], \\ (i\partial_t \mp d + i\gamma)n_\pm = \pm d(|u|^2), \\ u(x, 0) = u_0(x) \in H^1(\mathbb{T}), \quad n_\pm(x, 0) = n_{\pm,0}(x) = n_0(x) \pm i d^{-1} n_1(x) \in L^2(\mathbb{T}). \end{cases} \tag{8}$$

Theorem 2.9. *Given initial data $(u_0, n_{+,0}, n_{-,0}) \in H^1 \times L^2 \times L^2$ there exists*

$$T = T\big(\|u_0\|_{H^1}, \|n_{+,0}\|_{L^2}, \|n_{-,0}\|_{L^2}, \|f\|_{H^1}, \gamma\big),$$

and a unique solution $(u, n_+, n_-) \in C\big([-T, T] : H^1 \times L^2 \times L^2\big)$ of (8)*. Moreover, we have*

$$\|u\|_{X_T^{1,\frac{1}{2}}} + \|n_{+,0}\|_{Y_{+,T}^{0,\frac{1}{2}}} + \|n_{-,0}\|_{Y_{-,T}^{0,\frac{1}{2}}} \leq 2\big(\|u_0\|_{H^1} + \|n_{+,0}\|_{L^2} + \|n_{-,0}\|_{L^2}\big).$$

This theorem follows by using the a priori estimates of Takaoka [1999]. In the case of forced and damped KdV, this was done in [Erdoğan and Tzirakis 2011, Theorem 2.1, Lemma 2.2]. We should note that the spaces where the contraction argument is done are independent of γ. One can possibly use dissipative variants of Bourgain spaces in the spirit of [Molinet and Ribaud 2002] but we don't need to do so here.

The global well-posedness follows from the following a priori estimate for the system (8) which was obtained in [Flahaut 1991] (recall that $n_\pm = n \pm i d^{-1} n_t$):

$$\|u\|_{H^1} + \|n_+\|_{L^2} + \|n_-\|_{L^2} \leq C_1 + C_2 e^{-C_3 t}, \quad t > 0, \tag{9}$$

where $C_1 = C_1(\alpha, \gamma, \|f\|_{H^1})$, $C_2 = C_2(\alpha, \gamma, \|f\|_{H^1}, \|u_0\|_{H^1}, \|n_{\pm,0}\|_{L^2})$, and $C_3 = C_3(\alpha, \gamma)$. In fact this was proved in [Flahaut 1991] for Dirichlet boundary conditions. In the case of periodic boundary conditions, the proof remains valid. Note that (9) also implies the existence of an absorbing set $\mathcal{B}_0$ in $H^1 \times L^2 \times L^2$ of radius $C_1(\alpha, \gamma, \|f\|_{H^1})$.

3. Proofs of 2.3 and 2.4

In this section we drop the $\pm$ signs and work with one n. We also set $Y = Y_+$.

$$\begin{cases} iu_t + \alpha u_{xx} = nu, \quad x \in \mathbb{T}, \quad t \in [-T, T], \\ (i\partial_t - d)n = d(|u|^2), \\ u(x,0) = u_0(x) \in H^{s_0}(\mathbb{T}), \quad n(x,0) = n_0(x) + id^{-1}n_1(x) \in H^{s_1}(\mathbb{T}). \end{cases} \tag{10}$$

Remark 3.1. We note that since $n_+ = \overline{n_-}$ all of our claims about (10) is also valid for (4). The difference in the proof will arise in the differentiation by parts process and the $X^{s,b}$ estimates. Because of (15), in formulas (16) and (17) there will additional sums in which every term, in the phase and in the multiplier with an $|\cdot|$ sign, will have a $\pm$ sign in front. This change won't alter the proofs for the $X^{s,b}$ estimates, in fact, all the cases we considered will work exactly the same way. Also it won't change the structure of the resonant sets in the case $\frac{1}{\alpha} \in \mathbb{N}$.

We will prove Theorem 2.4 only for $\alpha = 1$. Therefore, below we either have $\frac{1}{\alpha} \notin \mathbb{N}$ or $\alpha = 1$. The case $\alpha \neq 1$, $\frac{1}{\alpha} \in \mathbb{N}$ can be handled by only cosmetic changes in the proof. Writing

$$u(x,t) = \sum_k u_k(t) e^{ikx}, \quad n(x,t) = \sum_{j \neq 0} n_j(t) e^{ijx},$$

we obtain the following system for the Fourier coefficients:

$$\begin{cases} i\partial_t u_k - \alpha k^2 u_k = \sum_{\substack{k_1+k_2=k \\ k_1 \neq 0}} n_{k_1} u_{k_2}, \\ i\partial_t n_j - |j| n_j = |j| \sum_{j_1+j_2=j} u_{j_1} \overline{u_{-j_2}}, \quad j \neq 0 \\ u_k(0) = (u_0)_k, \quad n_j(0) = (n_0)_j + i|j|^{-1}(n_1)_j, \quad j \neq 0. \end{cases} \tag{11}$$

We start with the following proposition, which follows from differentiation by parts.

Proposition 3.2. *The system* (11) *can be written in the following form*:

$$i\partial_t \big[e^{it\alpha k^2} u_k + e^{it\alpha k^2} B_1(n,u)_k \big] = e^{it\alpha k^2} \big[\rho_1(k) + R_1(u)(\hat{k}, t) + R_2(u,n)(\hat{k}, t) \big], \tag{12}$$

$$i\partial_t \big[e^{it|j|} n_j + e^{it|j|} B_2(u)_j \big] = e^{it|j|} \big[\rho_2(j) + R_3(u,n)(\hat{j}, t) + R_4(u,n)(\hat{j}, t) \big], \tag{13}$$

where

$$B_1(n,u)_k = \sum_{\substack{k_1+k_2=k \\ k_1 \neq 0}}^{*} \frac{n_{k_1} u_{k_2}}{\alpha k^2 - \alpha k_2^2 - |k_1|}, \quad B_2(u)_j = |j| \sum_{j_1+j_2=j}^{*} \frac{u_{j_1} \overline{u_{-j_2}}}{|j| - \alpha j_1^2 + \alpha j_2^2},$$

$$R_1(u)(\hat{k},t) = \sum^{*}_{k_1,k_2} \frac{|k_1+k_2| u_{k_1} \overline{u_{-k_2}}\, u_{k-k_1-k_2}}{\alpha k^2 - \alpha(k-k_1-k_2)^2 - |k_1+k_2|},$$
$$R_2(u,n)(\hat{k},t) = \sum^{*}_{k_1,k_2\neq 0} \frac{n_{k_1} n_{k_2} u_{k-k_1-k_2}}{\alpha k^2 - \alpha(k-k_1)^2 - |k_1|},$$
$$R_3(u,n)(\hat{j},t) = |j| \sum^{*}_{j_1\neq 0, j_2} \frac{n_{j_1} u_{j_2} \overline{u_{j_1+j_2-j}}}{|j| - \alpha(j_1+j_2)^2 + \alpha(j-j_1-j_2)^2},$$
$$R_4(u,n)(\hat{j},t) = |j| \sum^{*}_{j_1\neq 0, j_2} \frac{\overline{n_{-j_1}} u_{j_2}\, \overline{u_{j_1+j_2-j}}}{|j| - \alpha j_2^2 + \alpha(j-j_2)^2}.$$

Here, $\sum^{*}$ means that the sum is over all nonresonant terms, i.e., over all indices for which the denominator is not zero. Moreover, the resonant terms ρ_1 and ρ_2 are zero if $\frac{1}{\alpha} \notin \mathbb{N}$. For $\alpha = 1$,

$$\rho_1(k) = n_{2k-\operatorname{sgn}(k)} u_{\operatorname{sgn}(k)-k}, \qquad k \neq 0,$$
$$\rho_2(j) = |j|\, u_{\frac{1}{2}(j+\operatorname{sgn} j)} \overline{u_{\frac{1}{2}(j-\operatorname{sgn} j)}}, \quad j \textit{ odd}.$$

Proof of Proposition 3.2. Changing the variables $m_j = n_j e^{i|j|t}$ and $v_k = u_k e^{i\alpha k^2 t}$ in (11), we obtain

$$\begin{cases} i\partial_t v_k = \sum_{\substack{k_1+k_2=k\\ k_1\neq 0}} e^{it(\alpha k^2 - \alpha k_2^2 - |k_1|)} m_{k_1} v_{k_2}, \\ i\partial_t m_j = |j| \sum_{j_1+j_2=j} e^{it(|j|-\alpha j_1^2 + \alpha j_2^2)} v_{j_1} \overline{v_{-j_2}}, \quad j \neq 0, \\ v_k(0) = (u_0)_k, \quad m_j(0) = (n_0)_j + i|j|^{-1}(n_1)_j, \quad j \neq 0. \end{cases} \tag{14}$$

It is easy to check that if we define m_j^+ and m_j^- accordingly, then

$$\partial_t m_j^- = \overline{\partial_t m_{-j}^+}. \tag{15}$$

Note that the exponents do not vanish if $1/\alpha$ is not an integer. On the other hand if $\alpha = 1$, then the resonant set is

$$(k_1,k_2) = \big(2k - \operatorname{sgn}(k), \operatorname{sgn}(k) - k\big), \quad k \neq 0.$$
$$(j_1,j_2) = \Big(\frac{j+\operatorname{sgn}(j)}{2}, \frac{j-\operatorname{sgn}(j)}{2}\Big), \quad j \text{ odd}.$$

The contribution of the corresponding terms give ρ_1 and ρ_2 in the case $\alpha = 1$. Below, we assume that $\frac{1}{\alpha} \notin \mathbb{N}$.

Differentiating by parts in the v equation we obtain

$$
\begin{aligned}
i\partial_t v_k &= \sum_{\substack{k_1+k_2=k\\ k_1\neq 0}} e^{it(\alpha k^2-\alpha k_2^2-|k_1|)} m_{k_1} v_{k_2} \\
&= \sum_{\substack{k_1+k_2=k\\ k_1\neq 0}} \frac{\partial_t\big(e^{it(\alpha k^2-\alpha k_2^2-|k_1|)} m_{k_1} v_{k_2}\big)}{i(\alpha k^2-\alpha k_2^2-|k_1|)} + i \sum_{\substack{k_1+k_2=k\\ k_1\neq 0}} \frac{e^{it(\alpha k^2-\alpha k_2^2-|k_1|)}\partial_t\big(m_{k_1} v_{k_2}\big)}{\alpha k^2-\alpha k_2^2-|k_1|}.
\end{aligned}
$$

The second sum can be rewritten using the equation as follows:

$$
\sum_{\substack{k_1+k_2+k_3=k\\ k_1+k_2\neq 0}} \frac{e^{it\alpha(k^2-k_1^2+k_2^2-k_3^2)}|k_1+k_2| v_{k_1}\overline{v_{-k_2}} v_{k_3}}{\alpha k^2-\alpha k_3^2-|k_1+k_2|} + \sum_{\substack{k_1+k_2+k_3=k\\ k_1+k_2\neq 0}} \frac{e^{it(\alpha k^2-\alpha k_3^2-|k_1|-|k_2|)} m_{k_1} m_{k_2} v_{k_3}}{\alpha k^2-\alpha(k_2+k_3)^2-|k_1|}. \tag{16}
$$

Now, we differentiate by parts in the m equation:

$$
\begin{aligned}
i\partial_t m_j &= |j| \sum_{j_1+j_2=j} e^{it(|j|-\alpha j_1^2+\alpha j_2^2)} v_{j_1}\overline{v_{-j_2}} \\
&= |j| \sum_{j_1+j_2=j} \frac{\partial_t\big(e^{it(|j|-\alpha j_1^2+\alpha j_2^2)} v_{j_1}\overline{v_{-j_2}}\big)}{i(|j|-\alpha j_1^2+\alpha j_2^2)} + i|j| \sum_{j_1+j_2=j} \frac{e^{it(|j|-\alpha j_1^2+\alpha j_2^2)}\partial_t\big(v_{j_1}\overline{v_{-j_2}}\big)}{|j|-\alpha j_1^2+\alpha j_2^2}.
\end{aligned}
$$

The second sum can be rewritten using the equation as follows:

$$
|j| \sum_{\substack{j_1+j_2+j_3=k\\ j_1\neq 0}} \frac{e^{it(|j|+\alpha j_3^2-\alpha j_2^2-|j_1|)} m_{j_1} v_{j_2}\overline{v_{-j_3}}}{|j|-\alpha(j_1+j_2)^2+\alpha j_3^2} + |j| \sum_{\substack{j_1+j_2+j_3=k\\ j_2\neq 0}} \frac{e^{it(|j|-\alpha j_1^2+\alpha j_3^2+|j_2|)} v_{j_1}\overline{m_{-j_2}}\,\overline{v_{-j_3}}}{|j|-\alpha j_1^2+\alpha(j_2+j_3)^2}. \tag{17}
$$

The statement follows by going back to the variables u and n. □

Integrating (12) and (13) from 0 to t, we obtain

$$
u_k(t)-e^{-it\alpha k^2}u_k(0) = e^{-it\alpha k^2}B_1(n,u)_k(0)-B_1(n,u)_k(t) - i\int_0^t e^{-i\alpha k^2(t-s)}\big[\rho_1(k)+R_1(u)(\hat{k},s)+R_2(u,n)(\hat{k},s)\big]ds. \tag{18}
$$

$$
n_j(t)-e^{-it|j|}n_j(0) = e^{-it|j|}B_2(u)_j(0)-B_2(u)_j(t) - i\int_0^t e^{-i|j|(t-s)}\big[\rho_2(j)+R_3(u,n)(\hat{j},s)+R_4(u,n)(\hat{j},s)\big]ds. \tag{19}
$$

Below we obtain a priori estimates for ρ_1, ρ_2, B_1, and B_2. Before that we state a technical lemma that will be used many times in the proofs.

Lemma 3.3. (a) *If $\kappa \geq \lambda \geq 0$ and $\kappa + \lambda > 1$, then*

$$\sum_n \frac{1}{\langle n-k_1\rangle^{\kappa}\langle n-k_2\rangle^{\lambda}} \lesssim \langle k_1-k_2\rangle^{-\lambda}\phi_\kappa(k_1-k_2).$$

(b) *For $\kappa \in (0,1]$, we have*

$$\int_{\mathbb{R}} \frac{d\tau}{\langle \tau+\rho_1\rangle^{\kappa}\langle \tau+\rho_2\rangle} \lesssim \frac{1}{\langle \rho_1-\rho_2\rangle^{\kappa-}}.$$

(c) *If $\kappa > 1/2$, then*

$$\sum_n \frac{1}{\langle n^2+c_1n+c_2\rangle^{\kappa}} \lesssim 1,$$

where the implicit constant is independent of c_1 and c_2.

We will prove this lemma in the Appendix.

Lemma 3.4. *Under the conditions of Theorem 2.3 and Theorem 2.4, for each t, we have*

$$\begin{aligned}
\|\rho_1(t)\|_{H^s} &\lesssim \|n(t)\|_{H^{s_1}}\|u(t)\|_{H^{s_0}} && \text{if } s \leq s_0+s_1,\\
\|\rho_2(t)\|_{H^s} &\lesssim \|u(t)\|_{H^{s_0}}^2 && \text{if } s \leq 2s_0-1,\\
\|B_1(n,u)(t)\|_{H^s} &\lesssim \|n(t)\|_{H^{s_1}}\|u(t)\|_{H^{s_0}} && \text{if } s \leq 1+s_0+\min(s_1,0),\\
\|B_2(u)(t)\|_{H^s} &\lesssim \|u(t)\|_{H^{s_0}}^2 && \text{if } s \leq \min(2s_0, 1+s_0).
\end{aligned}$$

Proof. The proof for ρ_1 and ρ_2 is immediate from their definition.

To estimate B_1, first note that

$$\left|\alpha k^2 - \alpha k_2^2 - |k_1|\right| = |\alpha|\,|k_1|\left|2k-k_1-\frac{1}{\alpha}\operatorname{sgn}(k_1)\right| \sim \langle k_1\rangle\langle 2k-k_1\rangle.$$

The last equality is immediate in the case $\frac{1}{\alpha} \notin \mathbb{N}$, when $\alpha = 1$, it follows from the nonresonant condition. Therefore we have

$$|B_1(n,u)_k| \lesssim \sum_{k_1\neq 0} \frac{|n_{k_1}|\,|u_{k-k_1}|}{\langle k_1\rangle\langle 2k-k_1\rangle}.$$

We estimate the H^s norm as follows:

$$\|B_1\|_{H^s}^2 \lesssim \Big\|\sum_{k_1\neq 0}\langle k_1\rangle^{2s_1}|n_{k_1}|^2\langle k-k_1\rangle^{2s_0}|u_{k-k_1}|^2\Big\|_{\ell^1_k}\Big\|\sum_{k_1}\frac{\langle k\rangle^{2s}}{\langle k_1\rangle^{2+2s_1}\langle k-k_1\rangle^{2s_0}\langle 2k-k_1\rangle^2}\Big\|_{\ell^\infty_k}$$

The first sum is bounded by $\|n\|_{H^{s_1}}^2\|u\|_{H^{s_0}}^2$ since it is a convolution of two ℓ^1 sequences. To estimate the second sum we distinguish the cases $|k_1| < |k|/2$, $|k_1| > 4|k|$, and $|k_1| \sim |k|$. In the first case, we bound the sum by

$$\sum_{k_1}\frac{\langle k\rangle^{2s-2-2s_0}}{\langle k_1\rangle^{2+2s_1}} \lesssim \langle k\rangle^{2s-2-2s_0},$$

since $2+2s_1>1$. In the second case, we bound the sum by

$$\sum_{|k_1|>4|k|}\frac{\langle k\rangle^{2s}}{\langle k_1\rangle^{4+2s_1+2s_0}}\lesssim\langle k\rangle^{2s-3-2s_1-2s_0}\le\langle k\rangle^{2s-2-2s_0}.$$

In the final case, we have

$$\sum_{|k_1|\sim|k|}\frac{\langle k\rangle^{2s-2-2s_1}}{\langle k-k_1\rangle^{2s_0}\langle 2k-k_1\rangle^2}\lesssim\langle k\rangle^{2s-2-2s_1-2\min(s_0,1)}.$$

In the last inequality we used part (a) of Lemma 3.3.

Combining these cases we see that $B_1\in H^s$ for $s\le 1+\min(s_0,s_1+\min(s_0,1))$. In particular, $B_1\in H^s$ if $s\le 1+s_0+\min(s_1,0)$ which can be seen by distinguishing the cases $s_0\ge 1$ and $s_0<1$ and using the condition $1+s_1\ge s_0$.

Similarly, we estimate

$$|B_2(u)_j|\lesssim\sum_{j_1}\frac{|u_{j_1}|\,|u_{j_1-j}|}{\langle j-2j_1\rangle}.$$

As in the case of B_1, we see that $B_2\in H^s$ if

$$\sup_j\sum_{j_1}\frac{\langle j\rangle^{2s}}{\langle j-2j_1\rangle^2\langle j_1\rangle^{2s_0}\langle j-j_1\rangle^{2s_0}}<\infty.$$

We distinguish the cases $|j_1|<|j|/4$, $|j_1|>2|j|$, and $|j_1|\sim|j|$. In the first case, we bound the sum by

$$\sum_{|j_1|<|j|/4}\frac{\langle j\rangle^{2s-2-2s_0}}{\langle j_1\rangle^{2s_0}}\lesssim\langle j\rangle^{2s-2-2s_0}\phi_{2s_0}(j).$$

In the second case, we bound the sum by

$$\sum_{|j_1|>2|j|}\frac{\langle j\rangle^{2s}}{\langle j_1\rangle^{2+4s_0}}\lesssim\langle j\rangle^{2s-1-4s_0}.$$

In the final case, we have

$$\sum_{|j_1|\sim|j|}\frac{\langle j\rangle^{2s-2s_0}}{\langle j-2j_1\rangle^2\langle j-j_1\rangle^{2s_0}}\lesssim\langle j\rangle^{2s-2s_0-2\min(s_0,1)}.$$

Combining this cases, we see that B_2 is in H^s if $s\le\min(2s_0,1+s_0)$. □

Using the estimates in Lemma 3.4 in the equations (18) and (19) after writing the equations in the x variable, we obtain

$$\begin{aligned}\|u(t)-e^{it\alpha\partial_x^2}u_0\|_{H^{s_0+a_0}}&\lesssim\|n_0\|_{H^{s_1}}\|u_0\|_{H^{s_0}}+\|n(t)\|_{H^{s_1}}\|u(t)\|_{H^{s_0}}\\&+\int_0^t\|n(s)\|_{H^{s_1}}\|u(s)\|_{H^{s_0}}ds+\left\|\int_0^t e^{i\alpha(t-s)\partial_x^2}\big[R_1(u)(s)+R_2(u,n)(s)\big]ds\right\|_{H^{s_0+a_0}},\end{aligned}\tag{20}$$

$$\|n(t)-e^{-itd}n_0\|_{H^{s_1+a_1}} \lesssim \|u_0\|_{H^{s_0}}^2 + \|u(t)\|_{H^{s_0}}^2 + \int_0^t \|u(s)\|_{H^{s_0}}^2\,ds + \left\| \int_0^t e^{-id(t-s)}\big[R_3(u,n)(s)+R_4(u,n)(s)\big]ds \right\|_{H^{s_1+a_1}}, \quad (21)$$

where

$$R_\ell(s) = \sum_k R_\ell(\hat{k},s)e^{ikx}, \quad \ell = 1,2,3,4.$$

Above, the smoothing indexes a_0 and a_1 depend on α as stated in Theorem 2.3 and Theorem 2.4. The dependence arises only from the contribution of the resonant terms ρ_1 and ρ_2.

Note that, with δ as in Theorem 2.2,

$$\left\| \int_0^t e^{i\alpha(t-s)\partial_x^2}\big[R_1(u)(s)+R_2(u,n)(s)\big]ds \right\|_{L^\infty_{t\in[-\delta,\delta]}H_x^{s_0+a_0}} \lesssim \left\| \psi_\delta(t) \int_0^t e^{i\alpha(t-s)\partial_x^2}\big[R_1(u)(s)+R_2(u,n)(s)\big]ds \right\|_{X^{s_0+a_0,b}} \lesssim \|R_1(u)+R_2(u,n)\|_{X_\delta^{s_0+a_0,b-1}}, \quad (22)$$

for $b>1/2$. Here we used the imbedding $X^{s_0+a_0,b} \subset L_t^\infty H_x^{s_0+a_0}$. Similarly,

$$\left\| \int_0^t e^{-id(t-s)}\big[R_3(u,n)(s)+R_4(u,n)(s)\big]ds \right\|_{L^\infty_{t\in[-\delta,\delta]}H_x^{s_1+a_1}} \lesssim \|R_3(u,n)+R_4(u,n)\|_{X_\delta^{s_1+a_1,b-1}}. \quad (23)$$

Remark 3.5. We note that the inequalities (22) and (23) remain valid in the case the linear group is modified with a damping term; see Lemma 3.3 from [Erdoğan and Tzirakis 2011]. It is important to note that we don't need to alter the definition of the $X^{s,b}$ norm.

Proposition 3.6. *Given* $s_1 > -\frac{1}{2}$, $\max(s_1, \frac{s_1}{2}+\frac{1}{4}) \le s_0 \le s_1+1$, *and* $\frac{1}{2} < b < \min(\frac{3}{4}, \frac{s_0+1}{2})$, *we have*

$$\|R_1(u)\|_{X^{s,b-1}} \lesssim \|u\|^3_{X^{s_0,\frac{1}{2}}}, \quad \textit{provided } s \le s_0 + \min(1, 2s_0).$$

We also have

$$\|R_2(u,n)\|_{X^{s,b-1}} \lesssim \|n\|^2_{Y^{s_1,\frac{1}{2}}}\|u\|_{X^{s_0,\frac{1}{2}}},$$

provided $s \le \min(s_0+1+2s_1, s_0+1, 3+2s_1-2b, 3+s_1-2b)$.

Proposition 3.7. *Given* $s_1 > -\frac{1}{2}$, $\max(s_1, \frac{s_1}{2}+\frac{1}{4}) \le s_0 \le s_1+1$, *and* $\frac{1}{2} < b < \frac{3}{4} + \min(0, \frac{s_0+s_1}{2})$, *we have*

$$\|R_3(u,n)\|_{X^{s,b-1}} + \|R_4(u,n)\|_{X^{s,b-1}} \lesssim \|n\|_{Y^{s_1,\frac{1}{2}}}\|u\|^2_{X^{s_0,\frac{1}{2}}},$$

provided $s \le s_1 + \min(1, 2s_0, 2s_0 - s_1)$.

We will prove these propositions later on. Using (22), (23) and the propositions above (with $b-1/2$ sufficiently small depending on a_0, a_1, s_0, s_1) in (20) and (21), we see that for $t \in [-\delta, \delta]$, we have

$$\|u(t)-e^{it\alpha\partial_x^2}u_0\|_{H^{s_0+a_0}}+\|n(t)-e^{-itd}n_0\|_{H^{s_1+a_1}} \lesssim \big[\|n_0\|_{H^{s_1}}+\|u_0\|_{H^{s_0}}\big]^2 \\ +\big[\|n(t)\|_{H^{s_1}}+\|u(t)\|_{H^{s_0}}\big]^2+\int_0^t \big[\|n(s)\|_{H^{s_1}}+\|u(s)\|_{H^{s_0}}\big]^2 ds+\big[\|n\|_{Y^{s_1,\frac12}}+\|u\|_{X^{s_0,\frac12}}\big]^3.$$

In the rest of the proof the implicit constants depend on $\|n_0\|_{H^{s_1}}$, $\|u_0\|_{H^{s_0}}$. Fix T large. For $t \le T$, we have the bound (with $\gamma = \gamma(s_0, s_1)$)

$$\|u(t)\|_{H^{s_0}}+\|n(t)\|_{H^{s_1}} \lesssim (1+|t|)^\gamma \lesssim T^\gamma.$$

Thus, with $\delta \sim T^{-12\gamma-}$, we have

$$\|u(j\delta)-e^{i\delta\alpha\partial_x^2}u((j-1)\delta)\|_{H^{s_0+a_0}}+\|n(j\delta)-e^{-i\delta d}n((j-1)\delta)\|_{H^{s_1+a_1}} \lesssim T^{3\gamma},$$

for any $j \le T/\delta \sim T^{1+12\gamma+}$. Here we used the local theory bound

$$\|u\|_{X^{s_0,1/2}_{[(j-1)\delta,\, j\delta]}} \lesssim \|u((j-1)\delta)\|_{H^{s_0}} \lesssim T^\gamma,$$

and similarly for n. Using this we obtain (with $J = T/\delta \sim T^{1+12\gamma+}$)

$$\begin{aligned}\|u(J\delta)-e^{i\alpha J\delta\partial_x^2}u(0)\|_{H^{s_0+a_0}} &\le \sum_{j=1}^{J}\|e^{i(J-j)\delta\alpha\partial_x^2}u(j\delta)-e^{i(J-j+1)\delta\alpha\partial_x^2}u((j-1)\delta)\|_{H^{s_0+a_0}}\\ &=\sum_{j=1}^{J}\|u(j\delta)-e^{i\delta\alpha\partial_x^2}u((j-1)\delta)\|_{H^{s_0+a_0}}\\ &\lesssim JT^{3\gamma}\sim T^{1+15\gamma+}.\end{aligned}$$

The analogous bound follows similarly for the wave part n.

The continuity in $H^{s_0+a_0}\times H^{s_1+a_1}$ follows from dominated convergence theorem, the continuity of u and n in H^{s_0}, H^{s_1}, respectively, and from the embedding $X^{s,b} \subset C_t^0 H_x^s$ (for $b > 1/2$). For details, see [Erdoğan and Tzirakis 2012; Ginibre et al. 1997].

4. Proof of Proposition 3.6

First note that the denominator in the definition of R_1 satisfies

$$\begin{aligned}\big|\alpha k^2-\alpha(k-k_1-k_2)^2-|k_1+k_2|\big| &= |\alpha|\,|k_1+k_2|\,\Big|2k-k-k_1-\frac{1}{\alpha}\operatorname{sgn}(k_1+k_2)\Big|\\ &\sim \langle k_1+k_2\rangle\langle 2k-k_1-k_2\rangle.\end{aligned} \tag{24}$$

The last equality holds trivially if $1/\alpha$ is not an integer. In the case that $1/\alpha$ is an integer it holds since the sum is over the nonresonant terms. Similarly, we shall see that the denominators of R_2, R_3, R_4 are respectively comparable to

$$\langle k_1\rangle\langle 2k-k_1\rangle, \quad \langle j\rangle\langle j-2j_1-2j_2\rangle, \quad \langle j\rangle\langle j-2j_2\rangle, \tag{25}$$

We start with the proof for R_2. We have

$$\|R_2(u,n)\|^2_{X^{s,b-1}} = \left\| \int_{\tau_1,\tau_2} \sum^*_{k_1,k_2\neq 0} \frac{\langle k\rangle^s \hat{n}(k_1,\tau_1)\hat{n}(k_2,\tau_2)\hat{u}(k-k_1-k_2,\tau-\tau_1-\tau_2)}{(\alpha k^2-\alpha(k-k_1)^2-|k_1|)\langle \tau-k^2\rangle^{1-b}} \right\|^2_{\ell^2_k L^2_\tau}.$$

Let

$$f(k,\tau)=|\hat{n}(k,\tau)|\langle k\rangle^{s_1}\langle \tau-|k|\rangle^{\frac{1}{2}}, \quad g(k,\tau)=|\hat{u}(k,\tau)|\langle k\rangle^{s_0}\langle \tau-\alpha k^2\rangle^{\frac{1}{2}}.$$

It suffices to prove that

$$\left\| \int_{\tau_1,\tau_2} \sum^*_{k_1,k_2\neq 0} M(k_1,k_2,k,\tau_1,\tau_2,\tau) f(k_1,\tau_1) f(k_2,\tau_2) g(k-k_1-k_2,\tau-\tau_1-\tau_2) \right\|^2_{\ell^2_k L^2_\tau} \lesssim \|f\|_2^4 \|g\|_2^2,$$

where

$$\begin{aligned} &M(k_1,k_2,k,\tau_1,\tau_2,\tau) \\ &\quad = \frac{\langle k\rangle^s \langle k_1\rangle^{-s_1}\langle k_2\rangle^{-s_1}\langle k-k_1-k_2\rangle^{-s_0}}{(\alpha k^2-\alpha(k-k_1)^2-|k_1|)\langle \tau-\alpha k^2\rangle^{1-b}\langle \tau_1-|k_1|\rangle^{\frac{1}{2}}\langle \tau_2-|k_2|\rangle^{\frac{1}{2}}\langle \tau-\tau_1-\tau_2-\alpha(k-k_1-k_2)^2\rangle^{\frac{1}{2}}}. \end{aligned}$$

By Cauchy–Schwarz in the variables τ_1,τ_2,k_1,k_2, we estimate the norm above by

$$\begin{aligned} &\sup_{k,\tau}\left(\int_{\tau_1,\tau_2} \sum^*_{k_1,k_2\neq 0} M^2(k_1,k_2,k,\tau_1,\tau_2,\tau) \right) \\ &\qquad \times \left\| \int_{\tau_1,\tau_2} \sum_{k_1,k_2\neq 0} f^2(k_1,\tau_1) f^2(k_2,\tau_2) g^2(k-k_1-k_2,\tau-\tau_1-\tau_2) \right\|_{\ell^1_k L^1_\tau}. \end{aligned}$$

Note that the norm above is equal to $\|f^2 * f^2 * g^2\|_{\ell^1_k L^1_\tau}$, which can be estimated by $\|f\|_2^4\|g\|_2^2$ by Young's inequality. Therefore, it suffices to prove that the supremum above is finite.

Using part (b) of Lemma 3.3 in τ_1 and τ_2 integrals, we obtain

$$\begin{aligned} &\sup_{k,\tau}\int_{\tau_1,\tau_2} \sum^*_{k_1,k_2\neq 0} M^2 \\ &\quad \lesssim \sup_{k,\tau} \sum^*_{k_1,k_2\neq 0} \frac{\langle k\rangle^{2s}\langle k_1\rangle^{-2s_1}\langle k_2\rangle^{-2s_1}\langle k-k_1-k_2\rangle^{-2s_0}}{(\alpha k^2-\alpha(k-k_1)^2-|k_1|)^2\langle \tau-\alpha k^2\rangle^{2-2b}\langle \tau-|k_1|-|k_2|-\alpha(k-k_1-k_2)^2\rangle^{1-}} \\ &\quad \lesssim \sup_{k} \sum_{k_1,k_2\neq 0} \frac{\langle k\rangle^{2s}\langle k_1\rangle^{-2s_1}\langle k_2\rangle^{-2s_1}\langle k-k_1-k_2\rangle^{-2s_0}}{\langle k_1\rangle^2\langle 2k-k_1\rangle^2\langle \alpha k^2-|k_1|-|k_2|-\alpha(k-k_1-k_2)^2\rangle^{2-2b}}. \end{aligned}$$

The last line follows by (25) and by the simple fact

$$\langle \tau - n\rangle\langle \tau - m\rangle \gtrsim \langle n-m\rangle. \tag{26}$$

Setting $k_2 = l + k - k_1$, we rewrite the sum as

$$\sup_k \sum_{k_1\geq 0,n} \frac{\langle k\rangle^{2s}\langle l+k-k_1\rangle^{-2s_1}}{\langle k_1\rangle^{2+2s_1}\langle 2k-k_1\rangle^2\langle l\rangle^{2s_0}\langle \alpha(l^2-k^2)+k_1+|k_1-l-k|\rangle^{2-2b}}.$$

Here, without loss of generality (since $(k_1,k_2,k)\to(-k_1,-k_2,-k)$ is a symmetry for the sum), we only considered the case $k_1\geq 0$.

<u>Case (i)</u>: $-1/2 < s_1 < 0$, $0 < \frac{s_1}{2}+\frac{1}{4}\leq s_0\leq s_1+1$. We write the sum as

$$\sum_{\substack{|l|\sim|k|\\ k_1\geq 0}} + \sum_{\substack{|l|\ll|k|\\ 0\leq k_1\leq |l+k|}} + \sum_{\substack{|l|\ll|k|\\ k_1\geq |l+k|}} + \sum_{\substack{|l|\gg|k|\\ k_1\geq |l+k|}} + \sum_{\substack{|l|\gg|k|\\ 0\leq k_1\leq |l+k|}} =: S_1+S_2+S_3+S_4+S_5.$$

In the sum S_1, we have

$$\langle l\rangle\sim\langle k\rangle,\quad \langle l+k-k_1\rangle\lesssim\langle k_1\rangle+\langle 2k-k_1\rangle.$$

Using this, we have

$$S_1\lesssim \sum_{k_1\geq 0,l}\frac{\langle k\rangle^{2s-2s_0}\big(\langle k_1\rangle^{-2s_1}+\langle 2k-k_1\rangle^{-2s_1}\big)}{\langle k_1\rangle^{2+2s_1}\langle 2k-k_1\rangle^2\langle \alpha(l^2-k^2)+k_1+|k_1-l-k|\rangle^{2-2b}}.$$

Summing in l using part (c) of Lemma 3.3 and then summing in k_1 using part (a) of Lemma 3.3, we obtain

$$S_1\lesssim \langle k\rangle^{2s-2s_0-2-4s_1}+\langle k\rangle^{2s-2s_0-2-2s_1}\lesssim\langle k\rangle^{2s-2s_0-2-4s_1}.$$

Note that S_1 is bounded in k for $s\leq s_0+1+2s_1$.

In the case of S_2, we have

$$|l\pm k|\sim|k|,\quad |2k-k_1|\sim|k|,\quad |l+k-k_1|\lesssim|k|.$$

Also note that (since we can assume that $|k|\gg 1$)

$$\big|\alpha(l^2-k^2)+k_1+|k_1-l-k|\big| = \alpha(k^2-l^2)+O(|k|)\sim k^2.$$

Using these, and then summing in k_1, we have

$$S_2\lesssim \sum_{\substack{|l|\ll|k|\\ 0\leq k_1\leq|l+k|}}\frac{\langle k\rangle^{2s-6+4b-2s_1}}{\langle k_1\rangle^{2+2s_1}\langle l\rangle^{2s_0}}\lesssim\langle k\rangle^{2s-6-2s_1+4b}\phi_{2s_0}(k)$$

Note that S_2 is bounded in k if $s<\min(s_0+\frac{5}{2}+s_1-2b, 3+s_1-2b)$, and in particular, if $s\leq\min(s_0+1+2s_1, 3+2s_1-2b)$.

In the case of S_3, we have $k_1 \geq |l+k| \gtrsim |k|$. Using this we estimate

$$S_3 \lesssim \sum_{\substack{|l|\ll|k| \\ k_1\geq|l+k|}} \frac{\langle k\rangle^{2s-2-4s_1}}{\langle 2k-k_1\rangle^2\langle l\rangle^{2s_0}\langle\alpha(l^2-k^2)+2k_1-l-k\rangle^{2-2b}}$$
$$\lesssim \sum_{|l|\ll|k|} \frac{\langle k\rangle^{2s-2-4s_1}}{\langle l\rangle^{2s_0}\langle\alpha(l^2-k^2)+3k-l\rangle^{2-2b}}.$$

The second inequality follows from part (a) of Lemma 3.3. Note that

$$\langle\alpha(l^2-k^2)+3k-l\rangle \sim k^2,$$

since $|l| \ll |k|$. Using this and then summing in l, we have

$$S_3 \lesssim \langle k\rangle^{2s-6-4s_1+4b}\phi_{2s_0}(k).$$

Note that this is also bounded in k if $s \leq \min(s_0+1+2s_1, 3+2s_1-2b)$.

In the case of S_4, we have $k_1 \gg |k|$. Therefore

$$S_4 \lesssim \sum_{|l|,k_1\gg|k|} \frac{\langle k\rangle^{2s-2s_0}}{\langle k_1\rangle^{4+4s_1}\langle\alpha(l^2-k^2)+2k_1-l-k\rangle^{2-2b}} \lesssim \sum_{k_1\gg|k|} \frac{\langle k\rangle^{2s-2s_0}}{\langle k_1\rangle^{4+4s_1}} \lesssim \langle k\rangle^{2s-2s_0-3-4s_1}.$$

We used part (c) of Lemma 3.3 in the second inequality.

In the case of S_5, we have $|l+k-k_1| \lesssim |l|$ and

$$\big|\alpha(l^2-k^2)+k_1+|k_1-l-k|\big| = \alpha(k^2-l^2)+O(|l|) \sim l^2.$$

Thus, we estimate using part (a) of Lemma 3.3

$$S_5 \lesssim \sum_{|l|\gg|k|,k_1} \frac{\langle k\rangle^{2s}}{\langle k_1\rangle^{2+2s_1}\langle 2k-k_1\rangle^2\langle l\rangle^{2s_0+2s_1+4-4b}} \lesssim \langle k\rangle^{2s-2s_0-5-4s_1+4b}.$$

Note that to sum in l we need $2s_0+2s_1+4-4b>1$, which holds under the conditions of the proposition.

Case (ii): $0 \leq s_1$, $\max(s_1, \frac{s_1}{2}+\frac{1}{4}) \leq s_0 \leq s_1+1$. We write the sum as

$$\sum_{k_1\geq 0,\,|l|\gtrsim|k|} + \sum_{|l|\ll|k|,\,0\leq k_1\ll k^2} + \sum_{|l|\ll|k|,\,k_1\gtrsim k^2} =: S_1+S_2+S_3.$$

In the case of S_1 we have

$$S_1 \lesssim \sum_{k_1\geq 0,\,|l|\gtrsim|k|} \frac{\langle k\rangle^{2s-2s_0}}{\langle k_1\rangle^{2+2s_1}\langle 2k-k_1\rangle^2\langle\alpha(l^2-k^2)+k_1+|k_1-l-k|\rangle^{2-2b}} \lesssim \langle k\rangle^{2s-2s_0-2}.$$

We obtained the second inequality by first summing in l using part (c) of Lemma 3.3, and then in k_1 using part (a) of the Lemma. Thus S_1 is bounded in k if $s \leq s_0+1$.

In the case of S_2, we have

$$\langle \alpha(l^2-k^2)+k_1+|k_1-l-k|\rangle \gtrsim k^2, \text{ and } \langle k_1\rangle\langle l+k-k_1\rangle \gtrsim \langle l+k\rangle \gtrsim \langle k\rangle.$$

Therefore,

$$S_2 \lesssim \langle k\rangle^{2s-4+4b-2s_1} \sum_{|l|\ll|k|,\, 0\le k_1\ll k^2} \frac{1}{\langle k_1\rangle^2\langle 2k-k_1\rangle^2\langle l\rangle^{2s_0}} \lesssim \langle k\rangle^{2s-6+4b-2s_1}\phi_{2s_0}(k).$$

Note that S_2 is bounded in k if $s\le \min(s_0+1, s_1+3-2b)$.

Finally we estimate S_3 as follows

$$\begin{aligned} S_3 &\lesssim \sum_{|l|\ll|k|,\, k_1\gtrsim k^2} \frac{\langle k\rangle^{2s}}{\langle k_1\rangle^{4+4s_1}\langle \alpha(l^2-k^2)+k_1+|k_1-l-k|\rangle^{2-2b}} \\ &\lesssim \langle k\rangle^{2s-6-8s_1}\sum_l \frac{1}{\langle \alpha(l^2-k^2)+k_1+|k_1-l-k|\rangle^{2-2b}} \lesssim \langle k\rangle^{2s-6-8s_1}. \end{aligned}$$

In the last inequality we used part (c) of Lemma 3.3. Note that this term is bounded in k if $s\le s_0+1$.

We now consider R_1. By using Cauchy–Schwarz, the convolution structure, and then integrating in τ_1,τ_2 as in the previous case, it suffices to prove that

$$\sup_k \sum_{k_1,k_2}^{*} \frac{\langle k\rangle^{2s}\langle k_1\rangle^{-2s_0}\langle k_2\rangle^{-2s_0}\langle k-k_1-k_2\rangle^{-2s_0}|k_1+k_2|^2}{(\alpha k^2-\alpha(k-k_1-k_2)^2-|k_1+k_2|)^2\langle k^2-k_1^2+k_2^2-(k-k_1-k_2)^2\rangle^{2-2b}} < \infty.$$

Recalling (24), and using

$$\langle k^2-k_1^2+k_2^2-(k-k_1-k_2)^2\rangle \sim \langle (k_1+k_2)(k-k_1)\rangle,$$

it suffices to prove that

$$\sup_k \sum_{k_1,k_2}^{*} \frac{\langle k\rangle^{2s}\langle k_1\rangle^{-2s_0}\langle k_2\rangle^{-2s_0}\langle k-k_1-k_2\rangle^{-2s_0}}{\langle 2k-k_1-k_2\rangle^2\langle (k_1+k_2)(k-k_1)\rangle^{2-2b}} < \infty.$$

Note that the contribution of the case $k_1=k$ is

$$\lesssim \sum_{k_2} \frac{\langle k\rangle^{2s-2s_0}}{\langle k-k_2\rangle^2\langle k_2\rangle^{4s_0}} \lesssim \langle k\rangle^{2s-2s_0-\min(2,4s_0)},$$

so it satisfies the claim. For $k_1\neq k$ (since we also have $k_1+k_2\neq 0$ by nonresonant condition), we have $\langle (k_1+k_2)(k-k_1)\rangle \sim \langle k_1+k_2\rangle\langle k-k_1\rangle$. Also letting $l=k_1+k_2$ it suffices to consider the sum

$$\begin{aligned} \sum_{k_1,l} \frac{\langle k\rangle^{2s}}{\langle 2k-l\rangle^2\langle k-l\rangle^{2s_0}\langle l\rangle^{2-2b}\langle l-k_1\rangle^{2s_0}\langle k_1\rangle^{2s_0}\langle k-k_1\rangle^{2-2b}} &= \sum_{\substack{k_1\\ |l-2k|>|k|/2}} + \sum_{\substack{k_1\\ |l-2k|\le|k|/2}} \\ &=: S_1+S_2. \end{aligned}$$

We have

$$S_1 \lesssim \langle k\rangle^{2s-2} \sum_{l,k_1} \frac{1}{\langle k-l\rangle^{2s_0}\langle l\rangle^{2-2b}\langle l-k_1\rangle^{2s_0}\langle k_1\rangle^{2s_0}\langle k-k_1\rangle^{2-2b}}.$$

Using $\max(\langle k-l\rangle^{2s_0}, \langle l-k_1\rangle^{2s_0}) \gtrsim \langle k-k_1\rangle^{2s_0}$ and part (a) of Lemma 3.3 (recall that $2s_0+2-2b>1$), we have

$$\begin{aligned} S_1 &\lesssim \langle k\rangle^{2s-2} \sum_{l,k_1} \frac{1}{\langle l\rangle^{2-2b} \min\big(\langle k-l\rangle^{2s_0}, \langle l-k_1\rangle^{2s_0}\big)\langle k_1\rangle^{2s_0}\langle k-k_1\rangle^{2s_0+2-2b}} \\ &\lesssim \langle k\rangle^{2s-2} \sum_{k_1} \frac{1}{\langle k_1\rangle^{2s_0}\langle k-k_1\rangle^{2s_0+2-2b}} \lesssim \langle k\rangle^{2s-2-2s_0}. \end{aligned}$$

In the case of S_2 we have $\langle l\rangle, \langle k-l\rangle \gtrsim \langle k\rangle$, and hence

$$S_2 \lesssim \langle k\rangle^{2s-2s_0-2+2b} \sum_{\substack{k_1 \\ |l-2k|\le|k|/2}} \frac{1}{\langle 2k-l\rangle^{2}\langle l-k_1\rangle^{2s_0}\langle k_1\rangle^{2s_0}\langle k-k_1\rangle^{2-2b}}.$$

Note that $\max(\langle l-k_1\rangle^{2s_0}, \langle k_1\rangle^{2s_0}) \gtrsim \langle l\rangle^{2s_0} \ge \langle k\rangle^{2s_0}$. Thus,

$$S_2 \lesssim \langle k\rangle^{2s-4s_0-2+2b} \sum_{\substack{k_1 \\ |l-2k|\le|k|/2}} \frac{1}{\langle 2k-l\rangle^{2} \min(\langle l-k_1\rangle^{2s_0}, \langle k_1\rangle^{2s_0})\langle k-k_1\rangle^{2-2b}}.$$

Using part (a) of Lemma 3.3 (noting that $|l-k| \gtrsim |k|$ and that $\langle k\rangle^{-\lambda}\phi_\kappa(k) = \langle k\rangle^{-\lambda}\phi_\kappa(k)$ if $0<\kappa,\lambda<1$), we obtain

$$S_2 \lesssim \langle k\rangle^{2s-4s_0-2+2b} \sum_{l} \frac{1}{\langle 2k-l\rangle^{2}} \langle k\rangle^{-2+2b}\phi_{2s_0}(k) \lesssim \langle k\rangle^{2s-4s_0-4+4b}\phi_{2s_0}(k).$$

Note that S_2 is bounded in k if $s \le s_0 + \min(1, 2s_0)$.

5. Proof of Proposition 3.7

We first consider R_3. By using Cauchy–Schwarz, the convolution structure, and then integrating in τ_1, τ_2 as in the proof of the previous proposition, it suffices to prove that

$$\sup_j \sum_{j_1\neq 0, j_2}^{*} \frac{\langle j\rangle^{2s}|j|^2\langle j_1\rangle^{-2s_1}\langle j_2\rangle^{-2s_0}\langle j-j_1-j_2\rangle^{-2s_0}}{\big||j|-\alpha(j_1+j_2)^2+\alpha(j-j_1-j_2)^2\big|^2\langle |j|-|j_1|+\alpha(j-j_1-j_2)^2-\alpha j_2^2\rangle^{2-2b}} < \infty.$$

Recalling (25), it suffices to prove that

$$\sum_{j_1\neq 0, j_2} \frac{\langle j\rangle^{2s}\langle j_1\rangle^{-2s_1}\langle j_2\rangle^{-2s_0}\langle j-j_1-j_2\rangle^{-2s_0}}{\langle j-2j_1-2j_2\rangle^{2}\langle |j|-|j_1|+\alpha(j-j_1-j_2)^2-\alpha j_2^2\rangle^{2-2b}}$$

is bounded in j. Letting $l = j - j_1 - j_2$ and $m = j_2$, we rewrite the sum as

$$\sum_{m,l} \frac{\langle j\rangle^{2s}\langle j-l-m\rangle^{-2s_1}}{\langle 2l-j\rangle^2\langle m\rangle^{2s_0}\langle l\rangle^{2s_0}\langle \alpha l^2-\alpha m^2+|j|-|j-l-m|+\rangle^{2-2b}}. \tag{27}$$

We note that a similar argument gives us the following sum for R_4:

$$\sum_{m,l} \frac{\langle j\rangle^{2s}\langle j-l-m\rangle^{-2s_1}}{\langle 2l-j\rangle^2\langle m\rangle^{2s_0}\langle l\rangle^{2s_0}\langle \alpha l^2-\alpha m^2-|j|-|j-l-m|+\rangle^{2-2b}}. \tag{28}$$

We note that, by symmetry, if we can prove that

$$\sum_{m,l} \frac{\langle j\rangle^{2s}\langle j-l-m\rangle^{-2s_1}}{\langle 2l-j\rangle^2\langle m\rangle^{2s_0}\langle l\rangle^{2s_0}\langle \alpha l^2-\alpha m^2+j-|j-l-m|+\rangle^{2-2b}} \tag{29}$$

is bounded in $j \neq 0$, then the boundedness of (27) and (28) follow.

Case (i): $-\frac{1}{2} < s_1 < 0$. We rewrite (27) as

$$\sum_{|l|\sim|m|\lesssim|j|} + \sum_{|l|\sim|m|\gg|j|} + \sum_{\substack{|l|\ll|m|\\|j|\geq|m+l|}} + \sum_{\substack{|l|\gg|m|\\|j|\geq|m+l|}} + \sum_{\substack{|l|\ll|m|\\|j|\leq|m+l|}} + \sum_{\substack{|l|\gg|m|\\|j|\leq|m+l|}} =: S_1+S_2+S_3+S_4+S_5+S_6.$$

For S_1 we have

$$S_1 \lesssim \sum_{|l|\sim|m|\lesssim|j|} \frac{\langle j\rangle^{2s-2s_1}}{\langle 2l-j\rangle^2\langle l\rangle^{4s_0}\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle^{2-2b}} \lesssim \langle j\rangle^{2s-2s_1-\min(2,4s_0)}.$$

In the second inequality we first summed in m using part (c) of Lemma 3.3, and then in n using part (a) of the lemma.

For S_2 we have

$$S_2 \lesssim \sum_{|l|\sim|m|\gg|j|} \frac{\langle j\rangle^{2s}}{\langle l\rangle^{2+4s_0+2s_1}\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle^{2-2b}} \lesssim \langle j\rangle^{2s-2s_1-4s_0-1}.$$

Again, we first summed in m using part (c) of Lemma 3.3.

In the case of S_3 we have $|l| \ll |m| \lesssim |j|$, and hence

$$\begin{aligned} S_3 &\lesssim \sum_{|l|\ll|m|\lesssim|j|} \frac{\langle j\rangle^{2s-2s_1-2}}{\langle l\rangle^{4s_0}\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle^{2-2b}} \\ &\lesssim \sum_{|l|\lesssim|j|} \frac{\langle j\rangle^{2s-2s_1-2}}{\langle l\rangle^{4s_0}} \lesssim \langle j\rangle^{2s-2s_1-2}\phi_{4s_0}(j) \lesssim \langle j\rangle^{2s-2s_1-\min(2,4s_0)}. \end{aligned}$$

In the case of S_4 we have

$$\langle 2l-j\rangle + \langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle \gtrsim l^2.$$

Since $\langle 2l-j\rangle \gtrsim l^2$ implies that $\langle 2l-j\rangle \gtrsim \langle j\rangle$, we have

$$\frac{1}{\langle 2l-j\rangle^2\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle^{2-2b}} \lesssim \frac{1}{\langle j\rangle^2\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle^{2-2b}}+\frac{1}{\langle 2l-j\rangle^2\langle l\rangle^{4-4b}}.$$

Therefore we estimate

$$S_4 \lesssim \sum_{|m|\ll|l|\lesssim|j|}\frac{\langle j\rangle^{2s-2s_1-2}}{\langle m\rangle^{4s_0}\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle^{2-2b}}+\sum_{|m|\ll|l|\lesssim|j|}\frac{\langle j\rangle^{2s-2s_1}}{\langle 2l-j\rangle^2\langle l\rangle^{2s_0+4-4b}\langle m\rangle^{2s_0}}.$$

The first sum can be estimated as in S_3 switching the roles of l and m. To estimate the second, we first sum in l using part (a) of Lemma 3.3, and then in m to obtain

$$\lesssim \langle j\rangle^{2s-2s_1-\min(2,2s_0+4-4b)}\phi_{2s_0}(j)\lesssim \langle j\rangle^{2s-2s_1-\min(2,4s_0)}.$$

In the case of S_5, we have

$$\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle \sim \langle m\rangle^2, \quad |m|\gtrsim |j|.$$

Therefore, noting that $2s_0+2s_1+4-4b>1$, we have

$$S_5 \lesssim \sum_{|l|\ll|m|}\frac{\langle j\rangle^{2s}}{\langle 2l-j\rangle^2\langle l\rangle^{2s_0}\langle m\rangle^{2s_0+2s_1+4-4b}} \lesssim \sum_{l}\frac{\langle j\rangle^{2s}}{\langle 2l-j\rangle^2\langle l\rangle^{4s_0+2s_1+3-4b}} \lesssim \langle j\rangle^{2s-\min(2,4s_0+2s_1+3-4b)}.$$

In the case of S_6, we have

$$\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle \sim \langle l\rangle^2, \quad |l|\gtrsim |j|. \tag{30}$$

Therefore,

$$S_6 \lesssim \sum_{|m|\ll|l|\gtrsim|j|}\frac{\langle j\rangle^{2s}}{\langle 2l-j\rangle^2\langle l\rangle^{2s_0+2s_1+4-4b}\langle m\rangle^{2s_0}} \lesssim \sum_{|l|\gtrsim|j|}\frac{\langle j\rangle^{2s}\phi_{2s_0}(l)}{\langle 2l-j\rangle^2\langle l\rangle^{2s_0+2s_1+4-4b}} \lesssim \langle j\rangle^{2s-2s_0-2s_1-4+4b}\phi_{2s_0}(j).$$

In the last inequality we used $|l|\gtrsim|j|$ and then summed in l.

Case (ii): $s_1\geq 0$. We rewrite (27) as

$$\sum_{|l|\lesssim|m|}+\sum_{|m|\ll|l|\ll|j|}+\sum_{|m|\ll|l|\gtrsim|j|}=: S_1+S_2+S_3.$$

In the case of S_1, we have $|j|\leq|j-l-m|+|m+l|\lesssim|j-l-m|+|m|$, and hence

$$\langle j-l-m\rangle\langle m\rangle \gtrsim \langle j\rangle.$$

Using this and noting that $s_0 \geq s_1$, we have

$$S_1 \lesssim \sum_{|l| \lesssim |m|} \frac{\langle j \rangle^{2s-2s_1}}{\langle 2l-j \rangle^2 \langle l \rangle^{4s_0-2s_1} \langle j - |j-l-m| + \alpha l^2 - \alpha m^2 \rangle^{2-2b}} \lesssim \langle j \rangle^{2s-2s_1-\min(2,4s_0-2s_1)}.$$

In the last inequality we summed in m using part (c) of Lemma 3.3 and then in l using part (a) of the lemma.

In the case of S_2 we have

$$S_2 \lesssim \sum_{|m| \ll |l| \ll |j|} \frac{\langle j \rangle^{2s-2-2s_1}}{\langle m \rangle^{4s_0} \langle j - |j-l-m| + \alpha l^2 - \alpha m^2 \rangle^{2-2b}} \lesssim \langle j \rangle^{2s-2-2s_1} \phi_{4s_0}(j).$$

Note that in the case of S_3 we have (30). Therefore

$$S_3 \lesssim \sum_{|m| \ll |l| \gtrsim |j|} \frac{\langle j \rangle^{2s}}{\langle 2l-j \rangle^2 \langle l \rangle^{2s_0+4-4b} \langle m \rangle^{2s_0} \langle j-l-m \rangle^{2s_1}}.$$

If $s_0 + s_1 > 1/2$, we sum in m and then in n using part (a) of Lemma 3.3 to obtain

$$S_3 \lesssim \sum_{|l| \gtrsim |j|} \frac{\langle j \rangle^{2s-2s_0-4+4b}}{\langle 2l-j \rangle^2 \langle j-l \rangle^{2s_1+\min(0,2s_0-1)-}} \lesssim \langle j \rangle^{2s-2s_0-4+4b-\min(2,2s_1,2s_1+2s_0-1)+}.$$

If $s_0 + s_1 \in (0, 1/2]$, we have

$$S_3 \lesssim \sum_{|l| \gtrsim |j|} \frac{\langle j \rangle^{2s} \langle l \rangle^{1-2s_0-2s_1+}}{\langle 2l-j \rangle^2 \langle l \rangle^{2s_0+4-4b}} \lesssim \langle j \rangle^{2s-4s_0-2s_1-3+4b+}.$$

Note that each term above is bounded in j if $s \leq s_1 + \min(1, 2s_0 - s_1)$.

6. Existence of global attractor

In this section we prove Theorem 2.8. As in the previous sections we drop the $\pm$ signs and work with the system

$$\begin{cases} (i\partial_t + \alpha \partial_x^2 + i\gamma)u = nu + f, \quad x \in \mathbb{T}, \quad t \in [-T, T], \\ (i\partial_t - d + i\gamma)n = d(|u|^2), \\ u(x,0) = u_0(x) \in H^1(\mathbb{T}), \quad n(x,0) = n_0(x) \in \dot{L}^2(\mathbb{T}). \end{cases} \tag{31}$$

We start with a smoothing estimate for (31) that implies the existence of a global attractor:

Theorem 6.1. *Consider the solution of* (31) *with initial data* $(u_0, n_0) \in H^1 \times \dot{L}^2$. *Then, for* $\frac{1}{\alpha} \notin \mathbb{N}$, *and for any* $a < 1$, *we have*

$$u(t) - e^{i\alpha t \partial_x^2 - \gamma t} u_0 \in C_t^0 H_x^{1+a}([0,\infty) \times \mathbb{T}) \quad \text{and} \quad n(t) - e^{-itd-\gamma t} n_0 \in C_t^0 H_x^a([0,\infty) \times \mathbb{T}). \tag{32}$$

Moreover,

$$\|u(t) - e^{i\alpha t \partial_x^2 - \gamma t} u_0\|_{H^{1+a}} + \|n(t) - e^{-itd-\gamma t} n_0\|_{H^a} \leq C\big(a, \alpha, \gamma, \|f\|_{H_1}, \|u_0\|_{H^1}, \|n_0\|_{L^2}\big). \tag{33}$$

In the case $\alpha = 1$ we have, for any $a < 1$,

$$\left\| u(t) - e^{it\partial_x^2 - \gamma t} u_0 + i \int_0^t e^{(i\partial_x^2 - \gamma)(t-t')} \rho_1 \, dt' \right\|_{H^{1+a}} + \left\| n(t) - e^{-itd - \gamma t} n_0 \right\|_{H^a} \leq C\big(a, \gamma, \|f\|_{H_1}, \|u_0\|_{H^1}, \|n_0\|_{L^2}\big), \quad (34)$$

where ρ_1 is as in Proposition 3.2. The analogous continuity statements as in (32) *are also valid.*

Proof. Writing

$$u(x,t) = \sum_k u_k(t) e^{ikx}, \quad n(x,t) = \sum_{j \neq 0} n_j(t) e^{ijx}, \quad f(x) = \sum_k f_k e^{ikx}$$

we obtain the following system for the Fourier coefficients:

$$\begin{cases} i\partial_t u_k + (i\gamma - \alpha k^2) u_k = \sum_{\substack{k_1 + k_2 = k \\ k_1 \neq 0}} n_{k_1} u_{k_2} + f_k, \\ i\partial_t n_j + (i\gamma - |j|) n_j = |j| \sum_{j_1 + j_2 = j} u_{j_1} \overline{u_{-j_2}}. \end{cases} \quad (35)$$

We have the following proposition, which follows from differentiation by parts as in Proposition 3.2 by using the change of variables $m_j = n_j e^{i|j|t + \gamma t}$, and $v_k = u_k e^{i\alpha k^2 t + \gamma t}$.

Proposition 6.2. *The system* (35) *can be written in the form*

$$i\partial_t\big[e^{it\alpha k^2 + \gamma t} u_k\big] + i e^{-\gamma t} \partial_t\big[e^{it\alpha k^2 + 2\gamma t} B_1(n,u)_k\big] = e^{it\alpha k^2 + \gamma t}\big[\rho_1(k) + f_k + B_1(n,f) + R_1(u)(\hat{k},t) + R_2(u,n)(\hat{k},t)\big], \quad (36)$$

$$i\partial_t\big[e^{it|j| + \gamma t} n_j\big] + i e^{-\gamma t} \partial_t\big[e^{it|j| + 2\gamma t} B_2(u)_j\big] = e^{it|j| + \gamma t}\big[\rho_2(j) + B_2(f,u) + B_2(u,f) + R_3(u,n)(\hat{j},t) + R_4(u,n)(\hat{j},t)\big]. \quad (37)$$

where B_i, ρ_i, $i = 1, 2$, and R_j, $j = 1, 2, 3, 4$ are as in Proposition 3.2.

Integrating (36) from 0 to t, we obtain

$$\begin{aligned} u_k(t) - e^{-it\alpha k^2 - \gamma t} u_k(0) &= -B_1(n,u)_k + e^{-it\alpha k^2 - \gamma t} B_1(n_0, u_0)_k \\ &\quad + \int_0^t e^{-(i\alpha k^2 + \gamma)(t-t')} \Big[-\gamma B_1(n,u)_k - i\rho_1(k) - i f_k - i B_1(n,f)_k \Big] dt' \\ &\quad - i \int_0^t e^{-(i\alpha k^2 + \gamma)(t-t')} \big[R_1(u)(\hat{k},t') + R_2(u,n)(\hat{k},t')\big] dt'. \end{aligned}$$

First note that (identifying the function with its Fourier sequence) we have

$$\left\| \int_0^t e^{-(i\alpha k^2 + \gamma)(t-t')} f_k \, dt' \right\|_{H^{1+a}} = \left\| \frac{f_k}{i\alpha k^2 + \gamma} \big(1 - e^{-it\alpha k^2 - \gamma t}\big) \right\|_{H^{1+a}} \lesssim \|f_k\|_{H^{a-1}}. \quad (38)$$

In the case $\frac{1}{\alpha} \notin \mathbb{N}$, using (38), the estimates in Lemma 3.4 and Proposition 3.6 (see Remark 3.5) as above, and also using the growth bound in (9), we obtain for any $a < 1$

$$\|u(t)-e^{i\alpha\partial_x^2 t-\gamma t}u_0\|_{H^{1+a}} \lesssim \|f\|_{H^{a-1}} + \big[\|f\|_{H^1}+\|n(0)\|_{L^2}+\|u(0)\|_{H^1}\big]^2 + \big[\|u\|_{X_\delta^{1,\frac12}}+\|n\|_{Y_\delta^{1,\frac12}}\big]^3.$$

Using the local theory (Theorem 2.9) bound for $X_\delta^{1,\frac12}, Y_\delta^{1,\frac12}$ norms for a $\delta=\delta(\|n_0\|_{L^2},\|u_0\|_{H^1},\|f\|_{H^1})$, we obtain for $t<\delta$

$$\|u(t)-e^{i\alpha\partial_x^2 t-\gamma t}u_0\|_{H^{1+a}} \lesssim C(a,\gamma,\|f\|_{H^1},\|n_0\|_{L^2}+\|u_0\|_{H^1}).$$

In the rest of the proof the implicit constants depend on $a,\gamma,\|f\|_{H^1},\|n_0\|_{L^2}+\|u_0\|_{H^1}$. Fix t large, and δ as above. We have

$$\big\|u(j\delta)-e^{i\alpha\partial_x^2\delta-\gamma\delta}u((j-1)\delta)\big\|_{H^{1+a}} \lesssim 1,$$

for any $j\le t/\delta$. Using this we obtain (with $J=t/\delta$)

$$\begin{aligned}
\|u(J\delta)-e^{J\delta(i\alpha\partial_x^2-\gamma)}u(0)\|_{H^{1+a}} &\le \sum_{j=1}^{J}\big\|e^{(J-j)\delta(i\alpha\partial_x^2-\gamma)}u(j\delta)-e^{(J-j+1)\delta(i\alpha\partial_x^2-\gamma)}u((j-1)\delta)\big\|_{H^{1+a}}\\
&=\sum_{j=1}^{J}e^{-(J-j)\delta\gamma}\|u(j\delta)-e^{\delta(i\alpha\partial_x^2-\gamma)}u((j-1)\delta)\|_{H^{1+a}}\\
&\lesssim \sum_{j=1}^{J}e^{-(J-j)\delta\gamma}\lesssim \frac{1}{1-e^{-\delta\gamma}}.
\end{aligned}$$

In the case $\alpha=1$, we have to separate the resonant term in this argument. We have the following inequality for $t<\delta$

$$\left\|u(t)-e^{i\alpha\partial_x^2 t-\gamma t}u_0+i\int_0^t e^{(i\alpha\partial_x^2-\gamma)(t-t')}\rho_1\,dt'\right\|_{H^{1+a}} \lesssim C(a,\gamma,\|f\|_{H^1},\|n_0\|_{L^2}+\|u_0\|_{H^1}).$$

Accordingly we have

$$\begin{aligned}
&\left\|u(J\delta)-e^{J\delta(i\alpha\partial_x^2-\gamma)}u(0)+\int_0^{J\delta} e^{(i\alpha\partial_x^2-\gamma)(J\delta-t')}\rho_1\,dt'\right\|_{H^{1+a}}\\
&\quad\le\sum_{j=1}^{J}\left\|e^{(J-j)\delta(i\alpha\partial_x^2-\gamma)}\left(u(j\delta)-e^{\delta(i\alpha\partial_x^2-\gamma)}u((j-1)\delta)+i\int_{(j-1)\delta}^{j\delta} e^{(i\alpha\partial_x^2-\gamma)(j\delta-t')}\rho_1\,dt'\right)\right\|_{H^{1+a}}\\
&\quad=\sum_{j=1}^{J}e^{-(J-j)\delta\gamma}\left\|u(j\delta)-e^{\delta(i\alpha\partial_x^2-\gamma)}u((j-1)\delta)+i\int_{(j-1)\delta}^{j\delta} e^{(i\alpha\partial_x^2-\gamma)(j\delta-t')}\rho_1\,dt'\right\|_{H^{1+a}}\\
&\quad\lesssim\sum_{j=1}^{J}e^{-(J-j)\delta\gamma}\lesssim\frac{1}{1-e^{-\delta\gamma}}.
\end{aligned}$$

The corresponding inequalities for the wave part follow similarly. The only difference is that we don't need to separate the resonant term, since $\rho_2\in H^1$ by Lemma 3.4.

This completes the proof of the global bound stated in Theorem 6.1. Finally the continuity in $H^1 \times \dot{L}^2$ follows as in [Erdoğan and Tzirakis 2012]. We omit the details. □

Proof of Theorem 2.8. We follow the strategy we outlined in [Erdoğan and Tzirakis 2011]. We start with the case $\frac{1}{\alpha} \notin \mathbb{N}$. First of all note that the existence of an absorbing set, $\mathcal{B}_0 \subset H^1 \times \dot{L}^2$, is immediate from (9). Second, we need to verify the asymptotic compactness of the propagator U_t. It suffices to prove that for any sequence $t_r \to \infty$ and for any sequence $(u_{0,r}, n_{0,r})$ in $\mathcal{B}_0$, the sequence $U_{t_r}(u_{0,r}, n_{0,r})$ has a convergent subsequence in $H^1 \times \dot{L}^2$.

To see this note that by Theorem 6.1, (if $(u_0, n_0) \in \mathcal{B}_0$)

$$U_t\big(u_0, n_0\big) = \big(e^{i\alpha t \partial_x^2 - \gamma t} u_0, e^{-itd - \gamma t} n_0\big) + N_t\big(u_0, n_0\big)$$

where $N_t\big(u_0, n_0\big)$ is in a ball in $H^{1+a} \times H^a$ with radius depending on $a \in (0,1), \alpha, \gamma$, and $\|f\|_{H^1}$. By Rellich's theorem, $\{N_t\big(u_0, n_0\big) : t > 0, (u_0, n_0) \in \mathcal{B}_0\}$ is precompact in $H^1 \times \dot{L}^2$. Since

$$\big\|\big(e^{i\alpha t \partial_x^2 - \gamma t} u_0, e^{-itd - \gamma t} n_0\big)\big\|_{H^1 \times \dot{L}^2} \lesssim e^{-\gamma t} \to 0, \qquad \text{as } t \to \infty,$$

uniformly on $\mathcal{B}_0$, we conclude that $\{U_{t_r}\big(u_{0,r}, n_{0,r}\big) : r \in \mathbb{N}\}$ is precompact in $H^1 \times \dot{L}^2$. Thus, U_t is asymptotically compact. This and Theorem A imply the existence of a global attractor $\mathcal{A} \subset H^1 \times \dot{L}^2$.

We now prove that the attractor set $\mathcal{A}$ is a compact subset of $H^{1+a} \times H^a$ for any $a \in (0,1)$. By Rellich's theorem, it suffices to prove that for any $a \in (0,1)$, there exists a closed ball $B_a \subset H^{1+a} \times H^a$ of radius $C(a, \alpha, \gamma, \|f\|_{H^1})$ such that $\mathcal{A} \subset B_a$. By definition

$$\mathcal{A} = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} U_t \mathcal{B}_0} =: \bigcap_{\tau \geq 0} V_\tau.$$

By Theorem 6.1 and the discussion above, V_τ is contained in a δ_τ neighborhood, N_τ, of a ball B_a in $H^1 \times \dot{L}^2$ whose radius depends only on $a, \alpha, \gamma, \|f\|_{H^1}$, and where $\delta_\tau \to 0$ as τ tends to infinity. Since B_a is a compact subset of $H^1 \times \dot{L}^2$, we have

$$\mathcal{A} = \bigcap_{\tau \geq 0} V_\tau \subset \bigcap_{\tau > 0} N_\tau = B_a.$$

Now consider the case $\frac{1}{\alpha} \in \mathbb{N}$. For simplicity, we take $\alpha = 1$. We have to be slightly more careful in this case because of the contribution of the resonant term, ρ_1, which is does not belong to H^{1+a} for any $a > 0$. Recall that, by Theorem 6.1, for $(u_0, n_0) \in \mathcal{B}_0$

$$U_t\big(u_0, n_0\big) = \big(e^{i\alpha t \partial_x^2 - \gamma t} u_0, e^{-itd - \gamma t} n_0\big) + N_t\big(u_0, n_0\big) + i\bigg(\int_0^t e^{(i\partial_x^2 - \gamma)(t-t')} \rho_1 \, dt', 0\bigg), \tag{39}$$

where $N_t\big(u_0, n_0\big)$ is in a ball in $H^{1+a} \times H^a$ with radius depending on $a \in (0,1), \gamma$, and $\|f\|_{H^1}$. Recall from Proposition 3.2, that the Fourier coefficients of ρ_1 are

$$(\rho_1)_k = \rho_1(n, u)_k = n_{2k - \operatorname{sgn}(k)} u_{\operatorname{sgn}(k) - k}, \quad k \neq 0.$$

In light of the proof of the case $\frac{1}{\alpha} \notin \mathbb{N}$ above, it suffices to consider the contribution of the resonant term under the assumption that $(u_0, n_0) \in \mathcal{B}_0$. Using (39), we write

$$\rho_1\big(n(t'), u(t')\big) = \rho_1\big(e^{-it'd-\gamma t'} n_0, u(t')\big) + \rho_1\big(N_{t'}(n_0), u(t')\big). \tag{40}$$

Now note that, by Lemma 3.4, we have

$$\big\|\rho_1(n,u)\big\|_{H^{1+a}} \lesssim \|n\|_{H^a} \|u\|_{H^1}.$$

Using this with $a = 0$, we see that the contribution of the first summand in (40) to the resonant term in (39) satisfies

$$\begin{aligned}\bigg\| \int_0^t e^{(i\partial_x^2-\gamma)(t-t')} \rho_1\big(e^{-it'd-\gamma t'} n_0, u(t')\big) dt' \bigg\|_{H^1} &\lesssim \int_0^t e^{-\gamma(t-t')} \|e^{-it'd-\gamma t'} n_0\|_{L^2} \|u(t')\|_{H^1} dt' \\ &\leq t e^{-\gamma t} C(a, \gamma, \|f\|_{H^1}),\end{aligned}$$

which goes to zero uniformly in $\mathcal{B}_0$. Similarly, the contribution of the second summand in (40) to the resonant term in (39) satisfies

$$\begin{aligned}\bigg\| \int_0^t e^{(i\partial_x^2-\gamma)(t-t')} \rho_1\big(N_{t'}(n_0), u(t')\big) dt' \bigg\|_{H^{1+a}} &\lesssim \int_0^t e^{-\gamma(t-t')} \|N_{t'}(n_0)\|_{H^a} \|u(t')\|_{H^1} dt' \\ &\leq C(a, \gamma, \|f\|_{H^1}).\end{aligned}$$

The rest of the proof is same as the case $\frac{1}{\alpha} \notin \mathbb{N}$. □

Appendix

We prove Lemma 3.3. Note that, with $m = k_2 - k_1$, we can rewrite the sum in part (a) as

$$\sum_n \frac{1}{\langle n\rangle^\kappa \langle n-m\rangle^\lambda}.$$

For $|n| < |m|/2$, we estimate the sum by

$$\sum_{|n|<|m|/2} \frac{1}{\langle n\rangle^\kappa \langle m\rangle^\lambda} \leq \langle m\rangle^{-\lambda} \phi_\kappa(m).$$

For $|n| > 2|m|$, we estimate by

$$\sum_{|n|>2|m|} \frac{1}{\langle n\rangle^{\kappa+\lambda}} \lesssim \langle m\rangle^{1-\kappa-\lambda} \lesssim \langle m\rangle^{-\lambda} \phi_\kappa(m).$$

Finally for $|n| \sim |m|$, we estimate by

$$\sum_{|n|\sim|m|} \frac{1}{\langle m\rangle^\kappa \langle n-m\rangle^\lambda} \lesssim \langle m\rangle^{-\kappa} \phi_\lambda(m) \lesssim \langle m\rangle^{-\lambda} \phi_\kappa(m).$$

The last inequality follows from the definition of ϕ_κ and the hypothesis $\kappa \geq \lambda$.

Part (b) follows from part (a). To obtain part (c), write

$$|n^2 + c_1 n + c_2| = |(n + z_1)(n + z_2)| \geq |n + x_1|\,|n + x_2|$$

where x_i is the real part of z_i. The contribution of the terms $|n+x_1| < 1$ or $|n+x_2| < 1$ is $\lesssim 1$. Therefore, we estimate the sum in part (c) by

$$\lesssim 1 + \sum_n \frac{1}{\langle n + x_1\rangle^\kappa \langle n + x_2\rangle^\kappa} \lesssim 1$$

by part (a).

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