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DECAY OF LINEAR WAVES ON HIGHER-DIMENSIONAL SCHWARZSCHILD BLACK HOLES

VOLKER SCHLUE

We consider solutions to the linear wave equation on higher dimensional Schwarzschild black hole spacetimes and prove robust nondegenerate energy decay estimates that are in principle required in a nonlinear stability problem. More precisely, it is shown that for solutions to the wave equation $\Box_g\phi=0$ on the domain of outer communications of the Schwarzschild spacetime manifold $(\mathcal{M}^n_m, g)$ (where $n\geq 3$ is the spatial dimension, and $m>0$ is the mass of the black hole) the associated energy flux $E[\phi](\Sigma_\tau)$ through a foliation of hypersurfaces Σ_τ (terminating at future null infinity and to the future of the bifurcation sphere) decays, $E[\phi](\Sigma_\tau)\leq CD/\tau^2$, where C is a constant depending on n and m, and $D<\infty$ is a suitable higher-order initial energy on Σ_0; moreover we improve the decay rate for the first-order energy to $E[\partial_t\phi](\Sigma^R_\tau)\leq CD_\delta/\tau^{4-2\delta}$ for any $\delta>0$, where Σ^R_τ denotes the hypersurface Σ_τ truncated at an arbitrarily large fixed radius $R<\infty$ provided the higher-order energy D_δ on Σ_0 is finite. We conclude our paper by interpolating between these two results to obtain the pointwise estimate $|\phi|_{\Sigma^R_\tau}\leq CD'_\delta/\tau^{\frac{3}{2}-\delta}$. In this work we follow the new physical-space approach to decay for the wave equation of Dafermos and Rodnianski (2010).

MSC2010: 35L05, 35Q75, 58J45, 83C57.
Keywords: decay, wave equation, Schwarzschild black hole, spacetime, higher dimensions, mathematical general relativity.

1. Introduction

The study of the wave equation on black hole spacetimes has generated considerable interest in recent years. This stems mainly from its role as a model problem for the nonlinear black hole stability problem [Dafermos and Rodnianski 2009a; 2012], and more recent advances in the analysis of linear waves [Dafermos and Rodnianski 2008].

In this paper we study the linear wave equation on higher-dimensional Schwarzschild black holes. The motivation for this problem lies — apart from the above mentioned relation to the nonlinear stability problem (which is expected to be simpler in the higher-dimensional case [Choquet-Bruhat et al. 2006]; for work on the 5-dimensional case under symmetry see also [Dafermos and Holzegel 2006; Holzegel 2010]) — on one hand in the purely mathematical curiosity of dealing with higher dimensions and on the other hand in its interest for theories of high energy physics [Emparan and Reall 2008].

In the philosophy of [Christodoulou and Klainerman 1993] it is understood that the resolution of the nonlinear stability problem requires an understanding of the linear equations in a sufficiently robust setting. In particular, we require a proof of the uniform boundedness and decay of solutions to the linear wave equation based on the method of energy currents, which (ideally) only uses properties of the spacetime that are stable under perturbations, and does not rely heavily on the specifics of the unperturbed metric (for an introduction in the context of black hole spacetimes see [Dafermos and Rodnianski 2008]). Correspondingly in this paper we establish on higher-dimensional Schwarzschild spacetime backgrounds boundedness and decay results analogous to the current state of the art in the $(3+1)$-dimensional case [Luk 2010].

The decay argument presented here departs from earlier work that either makes use of multipliers with weights in the temporal variable (notably [Christodoulou and Klainerman 1990; Blue and Sterbenz 2006; Andersson and Blue 2009; Dafermos and Rodnianski 2009b; Luk 2010]) which in one form or the other are due to Morawetz [1962], or that relies on the exact stationarity of the spacetime (such as [Ching et al. 1995; Tataru 2010; Donninger et al. 2012] based on Fourier analytic methods). Here we follow the new physical-space approach to decay of [Dafermos and Rodnianski 2010], which only uses multipliers with weights in the radial variable. Thus our work — especially the improvement of Section 5C — is of independent interest for the $(3+1)$-dimensional Schwarzschild and Minkowski case and also for a wider class of spacetimes including Kerr black hole exteriors.

1A. *Statement of the theorems.*

We consider solutions to the wave equation

$$\Box_g \phi = 0 \tag{1-1}$$

on higher-dimensional Schwarzschild black hole spacetimes; these backgrounds are a family of $(n+1)$-dimensional Lorentzian manifolds $(\mathcal{M}^n_m, g)$ parametrized by the mass of the black hole $m > 0$ $(n \geq 3)$. They arise as spherically symmetric solutions of the vacuum Einstein equations, the governing equations of general relativity, and are discussed as such in Section 2; for the relevant concepts see also [Dafermos and Rodnianski 2008; Hawking and Ellis 1973].

More precisely, we consider solutions to (1-1) on the domain of outer communications $\mathcal{D}$ of $\mathcal{M}$ — which comprises the exterior up to and including the event horizons of the black hole — with initial data

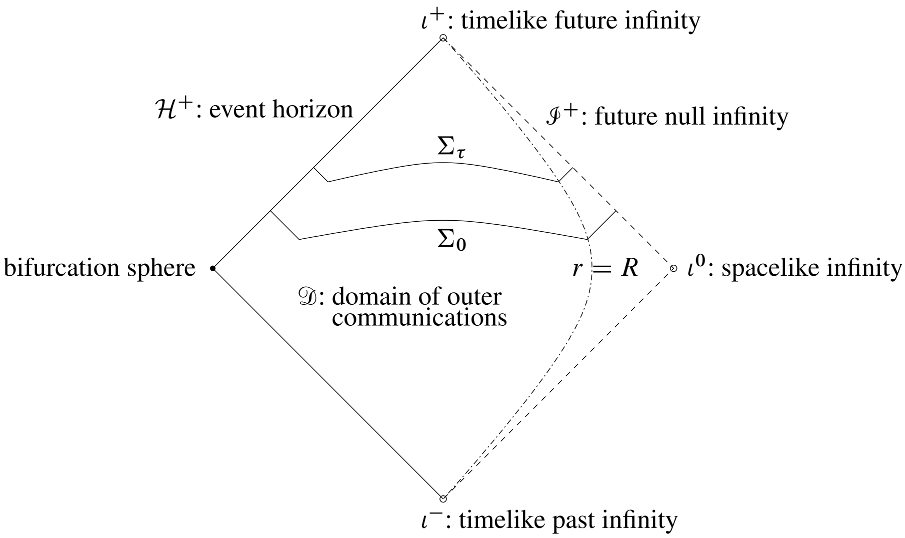

Figure 1. The hypersurface Σ_0 in the domain of outer communications $\mathcal{D}$.

prescribed on a hypersurface Σ_0 consisting of an incoming null segment crossing the event horizon to the future of the bifurcation sphere, a spacelike segment and an outgoing null segment emerging from a larger sphere of radius R terminating at future null infinity; see Figure 1 (the exact parametrization — which is chosen merely for technical reasons — is given in Section 4).

In the exterior of the black hole the metric g takes the classical form in (t, r)-coordinates [Tangherlini 1963]:

$$g = -\left(1 - \frac{2m}{r^{n-2}}\right) \mathrm{d}t^2 + \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} \mathrm{d}r^2 + r^2 \mathring{\gamma}_{n-1}, \tag{1-2}$$

where $r > \sqrt[n-2]{2m}$, $t \in (-\infty, \infty)$, and $\mathring{\gamma}_{n-1}$ denotes the standard metric on the unit $(n-1)$-sphere; however this coordinate system breaks down on the horizon $r = \sqrt[n-2]{2m}$ and we shall for that reason introduce in Section 2 the global geometry of $(\mathcal{M}^n_m, g)$ using a double null foliation, from which we derive an alternative double null coordinate system for the exterior of the black hole:

$$g = -4\left(1 - \frac{2m}{r^{n-2}}\right) \mathrm{d}u^* \, \mathrm{d}v^* + r^2 \mathring{\gamma}_{n-1}, \tag{1-3}$$

the so-called Eddington–Finkelstein coordinates.

In this paper both the conditions on the initial data and the statements on the decay of the solutions are formulated using the concepts of energy and the energy momentum tensor associated to (1-1); in particular (see Section 1B and also Appendix B),

$$T_{\mu\nu}[\phi] = \partial_\mu \phi \, \partial_\nu \phi - \tfrac{1}{2} \, g_{\mu\nu} \, \partial^\alpha \phi \partial_\alpha \phi. \tag{1-4}$$

The corresponding 1-contravariant-1-covariant tensor field fulfills the physical requirement that the linear transformation $-T : T\mathcal{M} \to T\mathcal{M}$ maps the hyperboloid of future-directed unit timelike vectors into the closure of the open future cone at each point. Physically,

$$-T\cdot u\in T_p\mathcal{M}$$

is the energy-momentum density relative to an observer at $p\in\mathcal{M}$ with 4-velocity $u\in T_p\mathcal{M}$, and it is for this reason that we refer to

$$\varepsilon=g(T\cdot u,u)=T(u,u)\geq 0$$

as the energy density at $p\in\mathcal{M}$ relative to the observer with 4-velocity $u\in T_p\mathcal{M}$. One may think of a spacelike hypersurface as a collection of locally simultaneous observers with a 4-velocity given by the normal. The hypersurfaces relative to which we establish energy decay are simply defined by $\Sigma_\tau\doteq\varphi_\tau(\Sigma_0\cap\mathcal{D})$, where φ_τ denotes the 1-parameter group of isometries generated by $\frac{\partial}{\partial t}$. The energy flux through the hypersurface Σ_τ is then given by

$$E[\phi](\Sigma_\tau)\doteq\int_{\Sigma_\tau}\big(J^N[\phi],n_\Sigma\big) \tag{1-5}$$

where $(J^N[\phi],n_\Sigma)\doteq T[\phi](N,n_\Sigma)$, n_Σ is the normal[1] to Σ_τ and N is a timelike φ_τ-invariant future directed vector field which is constructed in Section 3 for the purpose of turning $\varepsilon^N\doteq T(N,N)$ into a nondegenerate energy up to and including the horizon. Note that the energy $E[\phi](\Sigma_\tau)$ in particular bounds a suitably defined $\dot{\mathrm{H}}^1$-norm on Σ_τ.

The classes of solutions to (1-1) to which our results apply are formulated in terms of finite energy conditions on the initial data, for which purpose we list the following quantities:

$$D_2^{(2)}(\tau_0)\doteq\int_{\tau_0+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\sum_{k=0}^{1}r^2\left(\frac{\partial(r^{\frac{n-1}{2}}\partial_t^k\phi)}{\partial v^*}\right)^2\Big|_{u^*=\tau_0} +\int_{\Sigma_{\tau_0}}\left(\sum_{k=0}^{2}J^N[\partial_t^k\phi],n_\Sigma\right), \tag{1-6}$$

$$\begin{aligned}D_5^{(4-\delta)}(\tau_0)\doteq\int_{\tau_0+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Bigg\{&\sum_{k=0}^{1}r^{4-\delta}\left(\frac{\partial^2(r^{\frac{n-1}{2}}\partial_t^k\phi)}{\partial v^{*2}}\right)^2\\&+\sum_{k=0}^{4}r^2\left(\frac{\partial(r^{\frac{n-1}{2}}\partial_t^k\phi)}{\partial v^*}\right)^2+\sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\left(\frac{\partial r^{\frac{n-1}{2}}\Omega_i\partial_t^k\phi}{\partial v^*}\right)^2\Bigg\}\Big|_{u^*=\tau_0}\\&+\int_{\Sigma_{\tau_0}}\left(\sum_{k=0}^{5}J^N[\partial_t^k\phi]+\sum_{k=0}^{4}\sum_{i=1}^{\frac{n(n-1)}{2}}J^N[\Omega_i\partial_t^k\phi],n_\Sigma\right),\end{aligned} \tag{1-7}$$

[1]On spacelike segments of Σ_τ the vector n_Σ is indeed timelike; however, on the null segments of the hypersurfaces Σ_τ the "normal" n_Σ is in fact a null vector, but the notation is kept for convenience; see Appendix A.

$$D^{(4-\delta)}_{7+[\frac{n}{2}]}(\tau_0) \doteq \int_{\tau_0+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \Biggl\{ \sum_{k=0}^{2} \sum_{|\alpha|\leq[\frac{n}{2}]+1} r^{4-\delta} \biggl(\frac{\partial^2 (r^{\frac{n-1}{2}} \Omega^\alpha \partial_t^k \phi)}{\partial v^{*2}} \biggr)^2$$
$$+ \sum_{k=0}^{5} \sum_{|\alpha|\leq[\frac{n}{2}]+1} r^2 \biggl(\frac{\partial r^{\frac{n-1}{2}} \Omega^\alpha \partial_t^k \phi}{\partial v^*} \biggr)^2 + \sum_{k=0}^{4} \sum_{|\alpha|\leq[\frac{n}{2}]+2} r^2 \biggl(\frac{\partial r^{\frac{n-1}{2}} \Omega^\alpha \partial_t^k \phi}{\partial v^*} \biggr)^2 \Biggr\} \Bigg|_{u^*=\tau_0}$$
$$+ \int_{\Sigma_{\tau_0}} \biggl(\sum_{k=0}^{6} \sum_{|\alpha|\leq[\frac{n}{2}]+1} J^N[\Omega^\alpha \partial_t^k \phi] + \sum_{k=0}^{5} \sum_{|\alpha|\leq[\frac{n}{2}]+2} J^N[\Omega^\alpha \partial_t^k \phi], n_\Sigma \biggr). \tag{1-8}$$

Here $\Omega_i : i = 1, \dots, n(n-1)/2$ are the generators of the spherical isometries of the spacetime $\mathcal{M}$, α is a multiindex, and for any radius R we denote by R^* the corresponding Regge–Wheeler radius (2-17). (See also Section 4B.)

Among the propositions on linear waves on higher-dimensional Schwarzschild black hole spacetimes proven in this paper, we wish to highlight the following conclusions[2].

Theorem 1 (energy decay). *Let ϕ be a solution of the wave equation $\Box_g \phi = 0$ on $\mathcal{D} \subset \mathcal{M}^n_m$, where $n \geq 3$ and $m > 0$, with initial data prescribed on Σ_{τ_0} ($\tau_0 > 0$).*

- *If $D \doteq D^{(2)}_2(\tau_0) < \infty$ then there exists a constant $C(n, m)$ such that*

$$E[\phi](\Sigma_\tau) \leq \frac{CD}{\tau^2} \quad (\tau > \tau_0). \tag{1-9}$$

- *Furthermore if for some $0 < \delta < \frac{1}{2}$ and $R > \sqrt[n-2]{8nm/\delta}$ also $D' \doteq D^{(4-\delta)}_5(\tau_0) < \infty$ then there exists a constant $C(n, m, \delta, R)$ such that*

$$E[\partial_t \phi](\Sigma'_\tau) \leq \frac{CD'}{\tau^{4-2\delta}} \quad (\tau > \tau_0), \tag{1-10}$$

where $\Sigma'_\tau \doteq \Sigma_\tau \cap \{r \leq R\}$.

While each of these energy decay statements lend themselves to prove pointwise estimates for ϕ and $\partial_t \phi$ respectively (see Section 6), we would like to emphasize that, using the (refined) integrated local energy decay estimates of Section 4, an interpolation argument allows to improve the pointwise bound on ϕ directly in the interior[3].

Theorem 2 (pointwise decay). *Let ϕ be a solution of the wave equation as in Theorem 1. If for some $0 < \delta < \frac{1}{4}$, $D \doteq D^{(4-\delta)}_{7+[\frac{n}{2}]}(\tau_0) < \infty$ ($\tau_0 > 1$) then there exists a constant $C(n, m, \delta, R)$ such that*

$$r^{\frac{n-2}{2}} |\phi| \big|_{\Sigma'_\tau} \leq \frac{CD}{\tau^{\frac{3}{2}-\delta}} \quad \bigl(\sqrt[n-2]{2m} \leq r < R,\ \tau > \tau_0 \bigr) \tag{1-11}$$

where Σ'_τ and R are as in Theorem 1.

[2] The "redshift" proposition and the "integrated local energy decay" proposition are to be found on page 526 in Section 3 and page 532 in Section 4 respectively.

[3] In this paper we use the term "interior" to refer to a region of finite radius; i.e., the term "interior region" is used interchangeably with "a region of compact r (including the horizon)", and is of course *not* meant to refer to the interior of the black hole, which is not considered in this paper.

Remark (decay rates and method of proof). Theorems 1 and 2 extend the presently known decay results for linear waves on $(3+1)$-dimensional Schwarzschild black holes to higher dimensions $n > 3$; for $(3+1)$-dimensional Schwarzschild black holes, (1-9) was first established in [Dafermos and Rodnianski 2009b], and (1-10), (1-11) more recently in [Luk 2010]. However, both proofs use multipliers with weights in t, [Dafermos and Rodnianski 2009b] by using the conformal Morawetz vector field in the decay argument, and [Luk 2010] by using in addition the scaling vector field. Here we extend (1-9) to higher dimensions $n > 3$ in the spirit of [Dafermos and Rodnianski 2010] only using multipliers with weights in r, and provide a new proof of the improved decay results (1-10) and (1-11) in the $n = 3$-dimensional case in particular.

1B. ***Overview of the proof.*** In this section we give an overview of the work in this paper and present some of the ideas in the proof that lead to Theorem 1; references to previous work are made when useful, but for a more detailed account of previous work on the wave equation on Schwarzschild black hole spacetimes see Section 1.3 in [Dafermos and Rodnianski 2011] and references therein.

Energy identities. Let us recall that the wave equation (1-1) arises from an action principle and that the corresponding energy momentum tensor is conserved. Indeed, here we find (1-4) and by virtue of the wave equation (1-1)

$$\nabla^\mu T_{\mu\nu} = (\Box_g \phi)(\partial_\nu \phi) = 0. \tag{1-12}$$

Moreover, the energy momentum tensor (1-4) satisfies the positivity condition, namely $T(X, Y) \geq 0$ for all future-directed *causal* vectors X, Y at a point.

Now let X be a vector field on $\mathcal{M}$. We define the energy current $J^X[\phi]$ associated to the multiplier X by

$$J^X_\mu[\phi] \doteq T_{\mu\nu}[\phi] X^\nu. \tag{1-13}$$

Then

$$K^X \doteq \nabla^\mu J^X_\mu = {}^{(X)}\pi^{\mu\nu} T_{\mu\nu} \tag{1-14}$$

where we have used that $T_{\mu\nu}$ is conserved and symmetric. Here

$${}^{(X)}\pi(Y, Z) \doteq \tfrac{1}{2}(\mathcal{L}_X g)(Y, Z) = \tfrac{1}{2} g(\nabla_Y X, Z) + \tfrac{1}{2} g(Y, \nabla_Z X) \tag{1-15}$$

is the *deformation tensor* of X.

Remark. If X is a Killing field, i.e., X generates a 1-parameter group of isometries of g, ${}^{(X)}\pi = 0$, then $K^X = 0$; i.e., J^X is conserved.

In the following we shall refer to

$$\int_{\mathcal{R}} K^X \, \mathrm{d}\mu_g = \int_{\partial\mathcal{R}} {}^* J^X \tag{1-16}$$

as the *energy identity for* J^X *(or simply* X*) on* $\mathcal{R}$, where $\mathcal{R} \subset \mathcal{M}$ (this is of course the content of Stokes' theorem, and ${}^*J$ denotes the Hodge-dual of J; see also Appendix B). Moreover we refer to X in (1-16) as the *multiplier vector field*. In this paper we will largely be concerned with the construction of vector fields X, associated currents J^X and their modifications, and the application of (1-16) and various derived energy inequalities to appropriately chosen domains $\mathcal{R} \subset \mathcal{D}$.

The new approach [Dafermos and Rodnianski 2010] to obtaining robust decay estimates requires us to first establish (i) uniform boundedness of energy, (ii) an integrated local energy decay estimate and (iii) good asymptotics towards null infinity.

Redshift effect. The reason (i) is nontrivial as compared to Minkowski space is that the energy corresponding to the multiplier ∂_t degenerates on the horizon (the vector field ∂_t becomes null on the horizon and no control on the angular derivatives is obtained; cf. [ibid. 2008]); it was recognized in [ibid. 2009b], and formulated more generally in [ibid. 2008], that the redshift property of Killing horizons is the key to obtaining an estimate for the *nondegenerate* energy (i.e., an energy with respect to a strictly timelike vector field up to the horizon, which controls all derivatives tangential to the horizons). An explicit construction of a suitable timelike vector field N is given in Section 3 which allows us to state the redshift property in the language of multipliers and energy currents, and a proof of the uniform boundedness of the nondegenerate energy is given (independently of other calculations in this work) in Section 5A.

Integrated local energy decay. Section 4 is devoted to establishing (ii). This is achieved by the use of radial multiplier vector fields of the form $f(r^*)\partial_{r^*}$ (see Section 4A). In Section 4B a construction of a positive definite current for the high angular frequency regime is given using a decomposition on the sphere. In Section 4C a more general construction of a current is given using a commutation with the angular momentum operators. We wish to emphasize that the decay results of Section 5 — albeit with a higher loss of differentiability — could be obtained solely on the basis of the latter current, without the recourse in Section 4B to the Fourier expansion on the sphere. However, the dependence on the initial data is significantly improved by virtue of the integrated local energy decay estimate Proposition 4.1; here (see Section 4D.1) the results of Sections 4B and 4C are combined in order to replace the commutation with the angular momentum operators by a commutation with the vector field ∂_t only. The difficulty in both constructions lies in overcoming the "trapping" obstruction, which is the insight that it is impossible to prove an integrated local energy decay estimate on spacetime regions that contain the photon sphere without losing derivatives (see [Dafermos and Rodnianski 2008]). In the context of the Schwarzschild spacetime the need for vector fields whose associated currents give rise to positive definite spacetime integrals was first recognized and used in [Blue and Soffer 2003; Dafermos and Rodnianski 2009b], and such estimates have since then been extended by many authors [Marzuola et al. 2010; Alinhac 2009].

The p-hierarchy. In Section 5B we use a multiplier of the form $r^p\partial_{v^*}$ that gives rise to a weighted energy inequality which we consequently exploit in a hierarchy of two steps; this approach — which yields the corresponding quadratic decay rate in (1-9) — was pioneered in [Dafermos and Rodnianski 2010] for a large class of spacetimes, including the $(3+1)$-dimensional Schwarzschild and Kerr black hole spacetimes. In Section 5C a further commutation with ∂_{v^*} is carried out, which allows us to extend the hierarchy of commuted weighted energy inequalities to four steps, yielding the corresponding decay rate for the first-order energy. The argument involves dealing with an (arbitrarily small) degeneracy of the first-order energy density at infinity which corresponds to the δ-loss in the decay estimate (1-10). In both cases (iii) is ensured by the imposition of higher-order finite energy conditions on the initial data.

Interpolation. The pointwise decay of Theorem 2 then follows from Theorem 1 and the (refined) integrated local energy decay estimates of Section 4D.2 by a simple interpolation argument given in Section 6.

Final comments. The currents in Sections 4B and 4C and the corresponding integrated local energy decay result already appeared in [Schlue 2010]. Independently a version of integrated local energy decay was subsequently obtained in [Laul and Metcalfe 2012]. In [Schlue 2010] there is also an alternative proof of (1-9) of Theorem 1 using the conformal Morawetz vector field.

2. Global causal geometry of the higher-dimensional Schwarzschild solution

In this section, we give a discussion (in the spirit of Section 3 of [Christodoulou 1995]) of the global geometry of the $(n+1)$-dimensional Schwarzschild black hole spacetime [Tangherlini 1963], the underlying manifold on which the wave equation is studied in this paper.

The $(n+1)$-dimensional Schwarzschild spacetime manifold $\mathcal{M} \doteq \mathcal{M}^n_m$ ($n \geq 3$, $n \in \mathbb{N}$, $m > 0$) is spherically symmetric; i.e., $\mathrm{SO}(n)$ acts by isometry. The group orbits are $(n-1)$-spheres, and the quotient $\mathcal{Q} = \mathcal{M}/\,\mathrm{SO}(n)$ is a 2-dimensional Lorentzian manifold. The metric g on $\mathcal{M}$ assumes the form

$$g = \overset{\mathcal{Q}}{g} + \gamma_r = \overset{\mathcal{Q}}{g} + r^2 \mathring{\gamma}_{n-1} \tag{2-1}$$

where $\overset{\mathcal{Q}}{g}$ is the Lorentzian metric on $\mathcal{Q}$ to be discussed below, $\mathring{\gamma}_{n-1}$ is the standard metric on $\mathbb{S}^{n-1}$, and r is the *area radius* (the area of the $(n-1)$-sphere at $x \in \mathcal{Q}$ is given by $\omega_n r^{n-1}(x)$, where $\omega_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ is the area of the unit $(n-1)$-sphere); or more precisely, in local coordinates $x^a : a = 1, 2$ on $\mathcal{Q}$, and local coordinates $y^A : A = 1, \dots, n-1$ on $\mathbb{S}^{n-1}$,

$$g_{(x,y)} = g_{ab}(x)\,\mathrm{d}x^a\,\mathrm{d}x^b + r^2(x)\big(\mathring{\gamma}_{n-1}\big)_{AB}\,\mathrm{d}y^A\,\mathrm{d}y^B.$$

The Schwarzschild spacetime is a solution of the vacuum Einstein equations, which in other words means that its Ricci curvature vanishes identically. This implies in particular (see derivation in [Schlue 2012]) that the area radius function r satisfies the Hessian equations

$$\nabla_a \partial_b r = \frac{(n-2)}{2r}\big[1 - (\partial^c r)(\partial_c r)\big] g_{ab}, \tag{2-2}$$

as a result of which the *mass function* m on $\mathcal{Q}$ defined[4] by

$$1 - \frac{2m}{r^{n-2}} = g^{ab}\,\partial_a r\,\partial_b r \tag{2-3}$$

is constant; see [ibid.]; we take this parameter m to be positive.

On $\mathcal{Q}$ we choose functions u, v whose level sets are outgoing and incoming null curves, respectively, which are increasing towards the future. These functions define a null system of coordinates, in which the

[4] We choose the normalization of the mass function to be independent of the dimension n; this is motivated by a consideration of the *mass equations* in the presence of matter; see [Schlue 2012].

metric $\overset{Q}{g}$ takes the form

$$\overset{Q}{g} = -\Omega^2 \,\mathrm{d}u\,\mathrm{d}v. \tag{2-4}$$

The Hessian equations (2-2) in null coordinates read

$$\frac{\partial^2 r}{\partial u^2} - \frac{2}{\Omega}\frac{\partial \Omega}{\partial u}\frac{\partial r}{\partial u} = 0, \tag{2-5a}$$

$$\frac{\partial^2 r}{\partial u\,\partial v} + \frac{n-2}{r}\frac{\partial r}{\partial u}\frac{\partial r}{\partial v} = -\frac{n-2}{4r}\Omega^2, \tag{2-5b}$$

$$\frac{\partial^2 r}{\partial v^2} - \frac{2}{\Omega}\frac{\partial \Omega}{\partial v}\frac{\partial r}{\partial v} = 0, \tag{2-5c}$$

and the defining equation for the mass function (2-3) is

$$1 - \frac{2m}{r^{n-2}} = -\frac{4}{\Omega^2}\frac{\partial r}{\partial u}\frac{\partial r}{\partial v}. \tag{2-6}$$

The system (2-5b), (2-6) can be rewritten as the partial differential equation

$$\frac{\partial r^*}{\partial u \partial v} = 0 \tag{2-7}$$

for a new radial function $r^*(r)$ that is related to r by

$$\frac{\mathrm{d}r^*}{\mathrm{d}r} = \frac{1}{1 - \frac{2m}{r^{n-2}}}. \tag{2-8}$$

A solution of (2-7), (2-8) is given by[5]

$$r^* = \frac{1}{(n-2)} \sqrt[n-2]{2m} \log|uv|, \tag{2-9}$$

or

$$|uv| = e^{\frac{(n-2)\,r^*}{\sqrt[n-2]{2m}}} = e^{\frac{(n-2)\,r}{\sqrt[n-2]{2m}}} \exp\left[\int \frac{n-2}{x^{n-2}-1}\,\mathrm{d}x\Big|_{x=\frac{r}{\sqrt[n-2]{2m}}}\right].$$

We find more explicitly, by an elementary integration (see [Schlue 2012]), that

$$uv = \begin{cases} e^{\frac{r}{2m}}\left(1 - \dfrac{r}{2m}\right), & n = 3, \\ e^{\frac{2r}{\sqrt{2m}}}\dfrac{\left(1 - \frac{r}{\sqrt{2m}}\right)}{\left(1 + \frac{r}{\sqrt{2m}}\right)}, & n = 4, \end{cases} \tag{2-10}$$

[5]Here the representation in terms of null coordinates is such that $r^* = -\infty$ is contained in the (u, v) plane *and* the metric is nondegenerate at $r = \sqrt[n-2]{2m}$.

and

$$uv = e^{\frac{(n-2)\,r}{\sqrt[n-2]{2m}}}\left(1-\frac{r}{\sqrt[n-2]{2m}}\right)\begin{cases}1, & n \text{ odd},\\ \left(1+\frac{r}{\sqrt[n-2]{2m}}\right)^{-1}, & n \text{ even},\end{cases}$$
$$\times \prod_{j=1}^{[\frac{n-3}{2}]}\left(\frac{r^2}{(2m)^{\frac{2}{n-2}}}-2\cos\left(\frac{2\pi j}{n-2}\right)\frac{r}{(2m)^{\frac{1}{n-2}}}+1\right)^{\cos(2\pi j\frac{n-3}{n-2})}$$
$$\times \prod_{j=1}^{[\frac{n-3}{2}]}\exp\left[2\sin\left(2\pi j\frac{n-3}{n-2}\right)\arctan\left(\frac{\frac{r}{\sqrt[n-2]{2m}}-\cos\left(\frac{2\pi j}{n-2}\right)}{\sin\left(\frac{2\pi j}{n-2}\right)}\right)\right], \quad n \geq 5. \tag{2-11}$$

Note in particular that the $u = 0$ and $v = 0$ lines are the constant $r = \sqrt[n-2]{2m}$ curves, and that all other curves of constant radius are hyperbolas in the (u, v) plane — timelike for $r > \sqrt[n-2]{2m}$, spacelike for $r < \sqrt[n-2]{2m}$. This outlines the well-known *global* causal geometry of the Schwarzschild solution (see Figure 2).

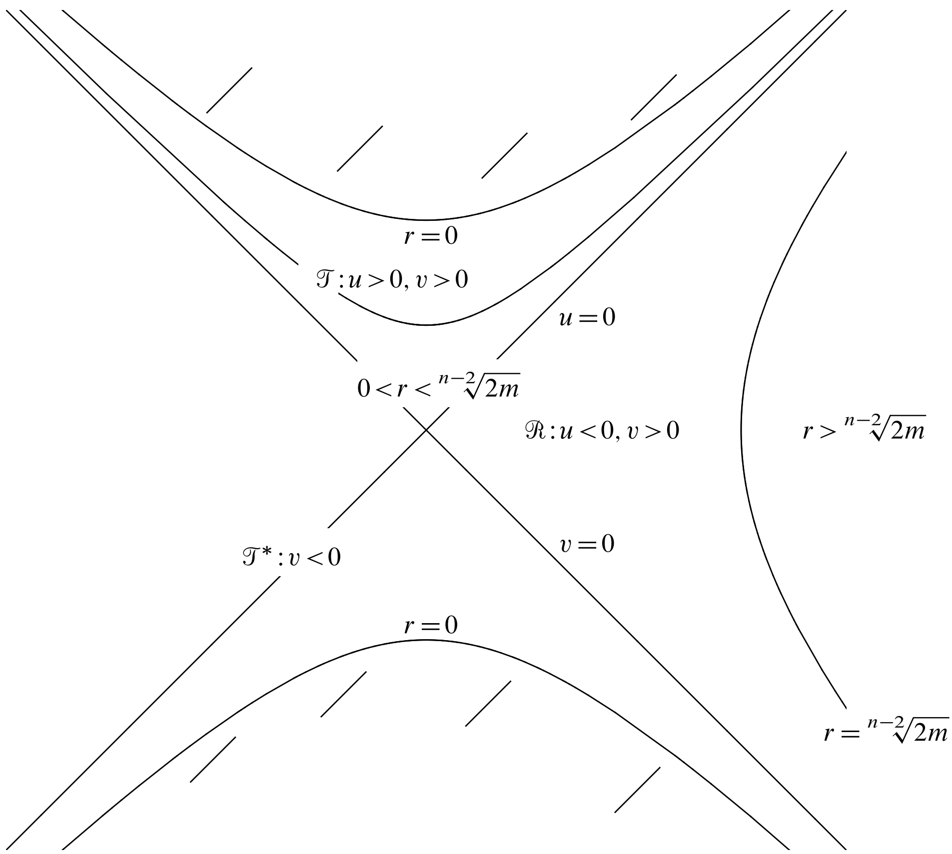

Figure 2. Global causal geometry of the Schwarzschild solution.

It is easy to see [Schlue 2012] that for (2-9) the trapped region, the apparent horizon, the exterior, and the antitrapped regions, respectively, are given by

$$\mathcal{T} \doteq \left\{(u,v)\in\mathcal{Q} : \frac{\partial r}{\partial u} < 0, \frac{\partial r}{\partial v} < 0\right\} = \left\{(u,v)\in\mathcal{Q} : u>0, v>0\right\},$$
$$\mathcal{A} \doteq \left\{(u,v)\in\mathcal{Q} : \frac{\partial r}{\partial u} < 0, \frac{\partial r}{\partial v} = 0\right\} = \left\{(u,v)\in\mathcal{Q} : u=0, v>0\right\},$$
$$\mathcal{R} \doteq \left\{(u,v)\in\mathcal{Q} : \frac{\partial r}{\partial u} < 0, \frac{\partial r}{\partial v} > 0\right\} = \left\{(u,v)\in\mathcal{Q} : u<0, v>0\right\},$$
$$\mathcal{T}^* \doteq \left\{(u,v)\in\mathcal{Q} : \frac{\partial r}{\partial u} > 0\right\} = \left\{(u,v)\in\mathcal{Q} : v<0\right\}.$$

Note this forms a partition of $\mathcal{Q} = \overline{\mathcal{T}\cup\mathcal{A}\cup\mathcal{R}\cup\mathcal{T}^*}$, and that in view of (2-6), $r < \sqrt[n-2]{2m}$ in $\mathcal{T}$, $r = \sqrt[n-2]{2m}$ in $\mathcal{A}$ and $r > \sqrt[n-2]{2m}$ in $\mathcal{R}$. We shall refer to

$$\mathcal{D} \doteq \overline{\mathcal{R}} = \left\{(u,v)\in\mathcal{Q} : u\le 0, v\ge 0\right\} \tag{2-12}$$

as the *domain of outer communications*.

Finally,

$$\Omega^2 = \begin{cases} 4\dfrac{(2m)^3}{r}e^{-\frac{r}{2m}}, & n=3,\\[2mm] \left(\dfrac{2m}{r}\right)^2\left(\dfrac{r}{\sqrt{2m}}+1\right)^2 e^{-\frac{2r}{\sqrt{2m}}}, & n=4,\\[2mm] \left(\dfrac{2}{n-2}\right)^2\dfrac{(2m)^{\frac{n}{n-2}}}{r^{n-2}}\begin{cases}1, & n \text{ odd},\\ \left(\dfrac{r}{\sqrt[n-2]{2m}}+1\right)^2, & n \text{ even},\end{cases} & \\ \quad\times\displaystyle\prod_{j=1}^{[\frac{n-3}{2}]}\left(\frac{r^2}{(2m)^{\frac{2}{n-2}}}-2\cos\left(\frac{2\pi j}{n-2}\right)\frac{r}{\sqrt[n-2]{2m}}+1\right)^{1-\cos(2\pi j\frac{n-3}{n-2})} & \\ \quad\times\displaystyle\prod_{j=1}^{[\frac{n-3}{2}]}\exp\left[-2\sin\left(2\pi j\frac{n-3}{n-2}\right)\arctan\left(\frac{\frac{r}{\sqrt[n-2]{2m}}-\cos(\frac{2\pi j}{n-2})}{\sin(\frac{2\pi j}{n-2})}\right)\right]e^{-\frac{(n-2)r}{\sqrt[n-2]{2m}}}, & n\ge 5.\end{cases} \tag{2-13}$$

One may now also think of r as a function of u, v implicitly defined by (2-10) and (2-11). In $\mathcal{R}$ where $r > \sqrt[n-2]{2m}$ (and $v-u > |u+v|$), r may be complemented by

$$t = \frac{2}{n-2}\sqrt[n-2]{2m}\,\operatorname{arctanh}\left(\frac{u+v}{v-u}\right); \tag{2-14}$$

note

$$\mathrm{d}t = \frac{1}{n-2}\sqrt[n-2]{2m}\left(\frac{1}{v}\,\mathrm{d}v - \frac{1}{u}\,\mathrm{d}u\right) \tag{2-15}$$

and we will denote by $\overline{\Sigma}_t$ the corresponding level sets in $\mathcal{D}$.

We find in these coordinates the classic expression for the Schwarzschild metric in the exterior region:

$$g = -\left(1 - \frac{2m}{r^{n-2}}\right) \mathrm{d}t^2 + \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} \mathrm{d}r^2 + r^2 \mathring{\gamma}_{n-1}. \tag{2-16}$$

In Regge–Wheeler coordinates (t, r^*), where r^* is centered at the photon sphere $r = \sqrt[n-2]{nm}$:

$$r^* = \int_{(nm)^{\frac{1}{n-2}}}^{r} \frac{1}{1 - \frac{2m}{r'^{n-2}}} \, \mathrm{d}r', \tag{2-17}$$

the metric obviously takes the conformally flat form

$$g = \left(1 - \frac{2m}{r^{n-2}}\right)\left(-\mathrm{d}t^2 + \mathrm{d}r^{*2}\right) + r^2 \mathring{\gamma}_{n-1}. \tag{2-18}$$

We shall also use the Eddington–Finkelstein coordinates

$$u^* = \tfrac{1}{2}(t - r^*), \quad v^* = \tfrac{1}{2}(t + r^*), \tag{2-19}$$

which are again double null coordinates:

$$g = -4\left(1 - \frac{2m}{r^{n-2}}\right) \mathrm{d}u^* \, \mathrm{d}v^* + r^2 \mathring{\gamma}_{n-1}. \tag{2-20}$$

The two systems of null coordinates in $\mathcal{R}$ are related by

$$u = -e^{-\frac{(n-2)\,u^*}{\sqrt[n-2]{2m}}}, \quad v = e^{\frac{(n-2)\,v^*}{\sqrt[n-2]{2m}}}. \tag{2-21}$$

3. The redshift effect

In this section we prove a manifestation of the *local redshift effect* in the Schwarzschild geometry of Section 2 in the framework of multiplier vector fields.

Proposition 3.1 (local redshift effect). *Let ϕ be a solution of the wave equation (1-1). Then there exists a φ_t-invariant future-directed timelike smooth vector field N on $\mathcal{D}$, two radii $\sqrt[n-2]{2m} < r_0^{(N)} < r_1^{(N)}$, and a constant $b > 0$ such that*

$$K^N(\phi) \geq b\,\big(J^N(\phi), N\big) \quad \big(\sqrt[n-2]{2m} \leq r < r_0^{(N)}\big) \tag{3-1}$$

and $N = T$ $(r \geq r_1^{(N)})$.

The vector field N will be constructed explicitly with the following vector fields.

T-vector field. Here φ_t is the 1-parameter group of diffeomorphisms generated by the vector field

$$T = \frac{1}{2} \frac{n-2}{\sqrt[n-2]{2m}} \left(v \frac{\partial}{\partial v} - u \frac{\partial}{\partial u}\right); \tag{3-2}$$

note that in $\mathcal{R}$, where $r > \sqrt[n-2]{2m}$ (recall (2-15)),

$$T = \frac{\partial}{\partial t}.$$

T is a Killing vector field:

$$ {}^{(T)}\pi = 0, \tag{3-3} $$

which is timelike in the exterior, spacelike in the interior of the black hole and *null on the horizon*:

$$ g(T,T) = \frac{1}{4}\frac{(n-2)^2}{(2m)^{\frac{2}{n-2}}}\,uv\,\Omega^2 = -\left(1-\frac{2m}{r^{n-2}}\right)\begin{cases} <0, & r > \sqrt[n-2]{2m}, \\ =0, & r = \sqrt[n-2]{2m}, \\ >0, & r < \sqrt[n-2]{2m}. \end{cases} \tag{3-4} $$

In particular,

$$ T|_{\mathcal{H}^+} = \frac{1}{2}\frac{n-2}{\sqrt[n-2]{2m}}\,v\frac{\partial}{\partial v}, \quad T|_{\mathcal{H}^+\cap\mathcal{H}^-} = 0. \tag{3-5} $$

Y-vector field. Let us also define a vector field Y on $\mathcal{H}^+$ conjugate to T:

$$ Y|_{\mathcal{H}^+} = -\frac{2}{\frac{\partial r}{\partial u}}\frac{\partial}{\partial u}. \tag{3-6} $$

Indeed,

$$ g(T,Y)|_{\mathcal{H}^+} = -2 \tag{3-7} $$

because

$$ \Omega^2\big|_{\mathcal{H}^+} = -4\frac{\sqrt[n-2]{2m}}{n-2}\frac{1}{v}\frac{\partial r}{\partial u}. $$

Furthermore, as a consequence of (2-5b),

$$ \frac{\partial^2 r}{\partial u\,\partial v}\bigg|_{\mathcal{H}^+} = -\frac{n-2}{4r}\Omega^2\bigg|_{\mathcal{H}^+} = \frac{1}{v}\frac{\partial r}{\partial u}\bigg|_{\mathcal{H}^+}, $$

and we have

$$ [T,Y]|_{\mathcal{H}^+} = [T,Y]^u\frac{\partial}{\partial u}\bigg|_{\mathcal{H}^+} + [T,Y]^v\frac{\partial}{\partial v}\bigg|_{\mathcal{H}^+} = \frac{n-2}{\sqrt[n-2]{2m}}\frac{1}{\frac{\partial r}{\partial u}}\left[v\frac{1}{\frac{\partial r}{\partial u}}\frac{\partial^2 r}{\partial u\,\partial v} - 1\right]\frac{\partial}{\partial u}\bigg|_{\mathcal{H}^+} = 0. \tag{3-8} $$

E_A-vector fields. We denote by $E_A : A = 1,\dots,n-1$ an orthonormal frame field tangential to the orbits of the spherical isometry:

$$ g(E_A,E_B) = \delta_{AB} = \begin{cases} 1, & A = B, \\ 0, & A \neq B, \end{cases} \tag{3-9a} $$

$$ g(E_A,Y)|_{\mathcal{H}^+} = 0, \quad g(E_A,T) = 0|_{\mathcal{H}^+} \quad (A = 1,\dots,n-1). \tag{3-9b} $$

We can now state that the surface gravity of the event horizon is *positive*; this is essential for the existence of the redshift effect (see more generally [Dafermos and Rodnianski 2008] and also [Aretakis 2011] for work where this is not the case).

Lemma 3.2 (surface gravity). *On $\mathcal{H}^+$,*

$$ \nabla_T T = \kappa_n T \tag{3-10} $$

with

$$\kappa_n = \frac{1}{2}\frac{n-2}{\sqrt[n-2]{2m}} > 0. \tag{3-11}$$

We call κ_n *the* surface gravity.

Note. $T = \kappa_n\left(v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u}\right)$.

Alternatively, κ_n is characterized by

$$\nabla_T Y = -\kappa_n Y \tag{3-12}$$

on $\mathcal{H}^+$. Clearly

$$g(\nabla_T Y, Y) = \tfrac{1}{2} T \cdot g(Y, Y) = 0$$

since Y is null along $\mathcal{H}^+$, and

$$g(\nabla_T Y, T) \overset{(3\text{-}8)}{=} g(\nabla_Y T, T) \overset{(3\text{-}3)}{=} -g(\nabla_T T, Y) = 2\kappa_n;$$

also

$$g(\nabla_T Y, E_A) \overset{(3\text{-}8)}{=} g(\nabla_Y T, E_A) \overset{(3\text{-}3)}{=} -g(\nabla_{E_A} T, Y) = 0 \quad \text{for } A = 1, \dots, n-1,$$

because $\nabla_{E_A} T = 0$. Note, for later use, on $\mathcal{H}^+$,

$$\nabla_{E_A} Y = -\frac{2}{\sqrt[n-2]{2m}} E_A. \tag{3-13}$$

We defined Y on $\mathcal{H}^+$ conjugate to T, $g(T, Y)|_{\mathcal{H}^+} = -2$. Next we extend Y to a neighborhood of the horizon by

$$\nabla_Y Y = -\sigma(Y + T) \quad \left(\sigma > \frac{16}{n-2}(2m)^{\frac{3}{n-2}}\right) \tag{3-14}$$

and then we extend Y to $\mathcal{R}$ by Lie-transport along the integral curves of T:

$$[T, Y] = 0. \tag{3-15}$$

Proposition 3.3 (redshift). *For the future-directed timelike vector field*

$$N = T + Y \tag{3-16}$$

there is a $b > 0$ *such that on* $\mathcal{H}^+$

$$K^N \geq b\,(J^N, N). \tag{3-17}$$

Proof. Let us calculate

$$\begin{aligned} K^Y &= {}^{(Y)}\pi^{\mu\nu} T_{\mu\nu} \\ &= \frac{1}{4}\Big\{{}^{(Y)}\pi(T,T)\,T(Y,Y) + 2\,{}^{(Y)}\pi(T,Y)\,T(Y,T) + {}^{(Y)}\pi(Y,Y)\,T(T,T)\Big\} \\ &\quad - \sum_{A=1}^{n-1}\Big\{{}^{(Y)}\pi(E_A,Y)\,T(E_A,T) + {}^{(Y)}\pi(E_A,T)\,T(E_A,Y)\Big\} + \sum_{A,B=1}^{n-1} {}^{(Y)}\pi(E_A,E_B)\,T(E_A,E_B). \end{aligned}$$

Now, on one hand, on $\mathcal{H}^+$,

$$ {}^{(Y)}\pi(T,T) = 2\kappa_n, \quad {}^{(Y)}\pi(T,Y) = \sigma, \quad {}^{(Y)}\pi(Y,Y) = 2\sigma, $$
$$ {}^{(Y)}\pi(E_A,Y) = 0, \quad {}^{(Y)}\pi(E_A,T) = 0, \quad {}^{(Y)}\pi(E_A,E_B) = -\frac{2}{\sqrt[n-2]{2m}}\delta_{AB}. $$

Thus

$$ K^Y = \tfrac{1}{2}\kappa_n\, T(Y,Y) + \tfrac{1}{2}\sigma\, T(Y+T,T) - \frac{2}{\sqrt[n-2]{2m}} \sum_{A=1}^{n-1} T(E_A,E_A). $$

On the other hand, on $\mathcal{H}^+$,

$$ T(Y,Y) = \left(\frac{2}{\frac{\partial r}{\partial u}} \frac{\partial \phi}{\partial u}\right)^2, \quad T(Y,T) = \left|\nabla\!\!\!/ \phi\right|^2_{r^2\mathring{\gamma}_{n-1}}, \quad T(T,T) = \left(\kappa_n v \frac{\partial\phi}{\partial v}\right)^2, $$

and, on $\mathcal{H}^+$,

$$ T(E_A,E_B) = (E_A\cdot\phi)(E_B\cdot\phi) - \tfrac{1}{2}(2m)^{\frac{2}{n-2}}\delta_{AB}\left|\nabla\!\!\!/\phi\right|_{r^2\mathring{\gamma}_{n-1}} - \tfrac{1}{2}(n-2)(2m)^{\frac{1}{n-2}} \frac{v}{\frac{\partial r}{\partial u}}\delta_{AB}\left(\frac{\partial\phi}{\partial u}\right)\left(\frac{\partial\phi}{\partial v}\right). $$

Using Cauchy's inequality, on $\mathcal{H}^+$,

$$ \begin{aligned} -\frac{2}{\sqrt[n-2]{2m}} \sum_{A=1}^{n-1} T(E_A,E_A) &= (n-3)(2m)^{\frac{1}{n-2}}\left|\nabla\!\!\!/\phi\right|^2_{r^2\mathring{\gamma}_{n-1}} + (n-2)(n-1)\frac{v}{\frac{\partial r}{\partial u}}\left(\frac{\partial\phi}{\partial u}\right)\left(\frac{\partial\phi}{\partial v}\right) \\ &\geq (n-3)(2m)^{\frac{1}{n-2}}T(Y,T) - \tfrac{1}{4}\kappa_n T(Y,Y) - \frac{1}{\kappa_n}\frac{2(n-1)}{(n-2)}(2m)^{\frac{2}{n-2}}\, T(T,T) \\ &\geq -\tfrac{1}{4}\kappa_n T(Y,Y) - \frac{n-1}{\kappa_n^2}(2m)^{\frac{1}{n-2}}T(T,T). \end{aligned} $$

Since we have chosen $\sigma > 2\frac{n-1}{\kappa_n^2}(2m)^{\frac{1}{n-2}}$, K^Y has a sign,

$$ K^Y \geq \tfrac{1}{4}\kappa_n\, T(Y,Y) + \sigma'\, T(Y+T,T) $$

for $0 < \sigma' < \frac{\sigma}{2} - \frac{n-1}{\kappa_n^2}(2m)^{\frac{1}{n-2}}$, or

$$ K^Y \geq b\, T(Y+T, Y+T) $$

for $0 < b < \min\{\frac{\kappa_n}{4}, \frac{\sigma'}{2}\}$. This yields the result

$$ K^N = K^Y \geq b\, T(N,N) = b\,(J^N, N). \qquad \square $$

Finally, we find an explicit expression for Y. Consider the vector field

$$ \hat{Y} = -\frac{2}{\frac{\partial r}{\partial u}}\frac{\partial}{\partial u} $$

on $\mathcal{R} \cup \mathcal{A}$ formally defined by the expression for Y on $\mathcal{H}^+$. In $\mathcal{R}$

$$\hat{Y} = \frac{2}{1 - \frac{2m}{r^{n-2}}} \frac{\partial}{\partial u^*}.$$

$\hat{Y}$ generates geodesics, this being a consequence of the Hessian equations (2-5a),

$$\nabla_{\hat{Y}} \hat{Y} = \left(\frac{2}{\frac{\partial r}{\partial u}} \right)^2 \left[-\frac{1}{\frac{\partial r}{\partial u}} \frac{\partial^2 r}{\partial u^2} + \frac{2}{\Omega} \frac{\partial \Omega}{\partial u} \right] \frac{\partial}{\partial u} = 0,$$

and is Lie-transported by T:

$$[T, \hat{Y}] = \frac{2}{\left(\frac{\partial r}{\partial u}\right)^2} \left(\left[T, \frac{\partial}{\partial u} \right] \cdot r \right) \frac{\partial}{\partial u} - \frac{2}{\frac{\partial r}{\partial u}} \left[T, \frac{\partial}{\partial u} \right] = -\kappa_n \hat{Y} + \kappa_n \hat{Y} = 0$$

because $[T, \frac{\partial}{\partial u}] = \kappa_n \frac{\partial}{\partial u}$. Y as constructed above coincides with

$$Y = \alpha(r)\hat{Y} + \beta(r)T \tag{3-18}$$

where

$$\alpha(r) = 1 + \frac{\sigma}{4\kappa_n}\left(1 - \frac{2m}{r^{n-2}}\right), \quad \beta(r) = \frac{\sigma}{4\kappa_n}\left(1 - \frac{2m}{r^{n-2}}\right).$$

Indeed, on $\mathcal{H}^+$,

$$Y|_{\mathcal{H}^+} = \hat{Y}|_{\mathcal{H}^+} = -\frac{2}{\frac{\partial r}{\partial u}} \frac{\partial}{\partial u}\bigg|_{\mathcal{H}^+}$$

and

$$\nabla_Y Y\big|_{\mathcal{H}^+} = \nabla_{\hat{Y}} Y\big|_{\mathcal{H}^+} = (\hat{Y} \cdot \alpha)\hat{Y}\big|_{\mathcal{H}^+} + \nabla_{\hat{Y}} \hat{Y}\big|_{\mathcal{H}^+} + (\hat{Y} \cdot \beta)T\big|_{\mathcal{H}^+} = -\sigma\,(Y + T)\big|_{\mathcal{H}^+}$$

since

$$\hat{Y} \cdot \alpha\big|_{\mathcal{H}^+} = \frac{\sigma}{4\kappa_n}(n-2)\frac{2m}{r^{n-1}} \hat{Y} \cdot r\bigg|_{\mathcal{H}^+} = -\sigma, \quad \hat{Y} \cdot \beta\big|_{\mathcal{H}^+} = -\sigma$$

and Y remains Lie-transported by T:

$$[T, Y] = (T \cdot \alpha)\hat{Y} + (T \cdot \beta)T + \alpha\,[T, \hat{Y}] + \beta\,[T, T] = 0$$

since

$$T \cdot \alpha = 0 = T \cdot \beta.$$

Thus the vector field Y is given explicitly by

$$Y = \begin{cases} -\frac{2}{\frac{\partial r}{\partial u}} \frac{\partial}{\partial u} & \text{on } \mathcal{H}^+, \\ \left[1 + \frac{\sigma}{4\kappa_n}\left(1 - \frac{2m}{r^{n-2}}\right)\right] \frac{2}{1 - \frac{2m}{r^{n-2}}} \frac{\partial}{\partial u^*} + \frac{\sigma}{4\kappa_n}\left(1 - \frac{2m}{r^{n-2}}\right) \frac{\partial}{\partial t} & \text{in } \mathcal{R}. \end{cases} \tag{3-19}$$

Clearly, by continuity, we can choose two values $\sqrt[n-2]{2m} < r_0^{(N)} < r_1^{(N)} < \infty$ and set

$$N = \begin{cases} T + Y, & \sqrt[n-2]{2m} \leq r \leq r_0^{(N)}, \\ T, & r \geq r_1^{(N)}, \end{cases}$$

with a smooth φ_t-invariant transition of the timelike vector field N in $r_0^{(N)} \leq r \leq r_1^{(N)}$, such that (3-17) extends to the neighborhood $\sqrt[n-2]{2m} < r < r_0^{(N)}$ of the event horizon.

Remark 3.4. For a geometric interpretation of Proposition 3.3 see [Schlue 2012] and also [Dafermos and Rodnianski 2008].

4. Integrated local energy decay

In this section we prove several *integrated local energy decay* statements, i.e., estimates on the energy density of solutions to (1-1) integrated on (bounded) *space-time* regions; this in an essential ingredient for the decay mechanism employed in Section 5.

Let $\mathcal{R}_{r_0,r_1}(t_0, t_1, u_1^*, v_1^*)$ be the region composed of a trapezoid and characteristic rectangles as follows (see Figure 3):

$$\begin{aligned} \mathcal{R}_{r_0,r_1}(t_0, t_1, u_1^*, v_1^*) \doteq \big\{(t,r) : t_0 \leq t \leq t_1,\ r_0 \leq r \leq r_1\big\} \\ \cup \big\{(t,r) : r \leq r_0,\ \tfrac{1}{2}(t - r^*) \leq u_1^*,\ t_0 + r_0^* \leq t + r^* \leq t_1 + r_0^*\big\} \\ \cup \big\{(t,r) : r \geq r_1,\ \tfrac{1}{2}(t + r^*) \leq v_1^*,\ t_0 - r_1^* \leq t - r^* \leq t_1 - r_1^*\big\}. \end{aligned} \tag{4-1}$$

We define

$$\mathcal{R}^{\infty}_{r_0,r_1}(t_0) \doteq \bigcup_{t_1 \geq t_0} \ \bigcup_{u_1^* \geq \frac{1}{2}(t_1 - r_0^*)} \ \bigcup_{v_1^* \geq \frac{1}{2}(t_1 + r_1^*)} \mathcal{R}(t_0, t_1, u_1^*, v_1^*) \tag{4-2}$$

and denote its past boundary by

$$\Sigma_{\tau_0} \doteq \partial^- \mathcal{R}^{\infty}_{r_0,r_1}(t_0), \qquad \tau_0 = \tfrac{1}{2}(t_0 - r_1^*). \tag{4-3}$$

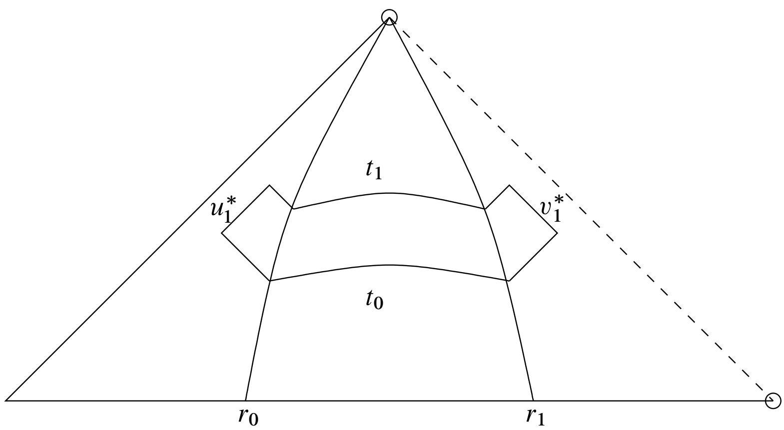

Figure 3. The region $\mathcal{R}_{r_0,r_1}(t_0, t_1, u_1^*, v_1^*)$.

We shall first state the central estimate.

Proposition 4.1 (integrated local energy decay estimate). *There exist* $(2m)^{\frac{1}{n-2}} < r_0 < r_1 < \infty$ *and a constant* $C(n,m)$ *depending only on the dimension* n *and the mass* m, *such that for any given solution* ϕ *of the wave equation* $\Box_g \phi = 0$,

$$\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)} \left\{ \frac{1}{r^n}\left(\frac{\partial\phi}{\partial r^*}\right)^2 + \frac{1}{r^{n+1}}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{r^3}\left(1-\frac{2m}{r^{n-2}}\right)\left|\nabla\!\!\!/\,\phi\right|^2_{r^2\mathring{\gamma}_{n-1}} \right\} \mathrm{d}\mu_g \\ \leq C(n,m)\int_{\Sigma_{\tau_0}} \bigl(J^T(\phi)+J^T(T\cdot\phi),n\bigr) \quad (4\text{-}4)$$

for all $t_0 \geq 0$, *where* $\tau_0 = \frac{1}{2}(t_0 - r_1^*)$.

The degeneracy at infinity can in fact be improved:

Proposition 4.2 (improved integrated local energy decay estimate). *Let* ϕ *be a solution of the wave equation* $\Box_g\phi = 0$. *Then there exists a constant* $C(n,m,\delta)$ *for each* $0 < \delta < 1$ *such that*

$$\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)} \left\{ \frac{1}{r^{1+\delta}}\left(\frac{\partial\phi}{\partial r^*}\right)^2 + \frac{1}{r^{1+\delta}}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{r}\left(1-\frac{2m}{r^{n-2}}\right)\left|\nabla\!\!\!/\,\phi\right|^2_{r^2\mathring{\gamma}_{n-1}} \right\} \mathrm{d}\mu_g \\ \leq C(n,m,\delta)\int_{\Sigma_{\tau_0}} \bigl(J^T(\phi)+J^T(T\cdot\phi),n\bigr) \quad (4\text{-}5)$$

for any $t_0 \geq 0$, *where* $r_0 < r_1$ *are as above, and* $\tau_0 = \frac{1}{2}(t_0 - r_1^*)$.

As a consequence of the redshift effect of Section 3 and the uniform boundedness of the nondegenerate energy (which is proven independently in Section 5A), we can infer in a more geometric formulation:

Corollary 4.3 (nondegenerate integrated local energy decay). *Let* ϕ *be a solution of* (1-1). *Then for any* $R > \sqrt[n-2]{2m}$ *there exists a constant* $C(n,m,R)$ *such that*

$$\int_{\tau'}^{\tau} \mathrm{d}\bar{\tau} \int_{\Sigma'_{\bar{\tau}}} \bigl(J^N(\phi),n\bigr) \leq C(n,m,R)\int_{\Sigma_{\tau'}} \bigl(J^N(\phi)+J^T(T\cdot\phi),n\bigr), \quad (4\text{-}6)$$

for all $\tau' < \tau$, *where* $\Sigma'_\tau \doteq \Sigma_\tau \cap \{r \leq R\}$.

Proof. Let

$$\mathcal{R}'(\tau',\tau) \doteq \mathrm{J}^-(\Sigma'_\tau)\cap \mathrm{J}^+(\Sigma_{\tau'}).$$

In $\mathcal{R}'(\tau',\tau)\cap\{r < r_0^{(N)}\}$ we have by Proposition 3.1

$$\bigl(J^N(\phi),n\bigr) \leq \frac{1}{b}K^N(\phi),$$

and in $\mathcal{R}'(\tau',\tau)\cap\{r \geq r_1^{(N)}\}$ trivially $(J^N(\phi),n) \leq (J^T(\phi),n)$. Therefore using the energy identity for N on $\mathcal{R}'(\tau',\tau)$ the estimate (4-6) follows from Proposition 5.2 and Proposition 4.1. □

In the above, no control is obtained on a spacetime integral of ϕ^2 itself; however, all that is needed for the decay argument of Section 5 is an estimate for the integral of ϕ^2 on timelike boundaries.

Proposition 4.4 (zeroth-order terms on timelike boundaries). *Let ϕ be solution of the wave equation (1-1), and $R > \sqrt[n-2]{8nm}$. Then there is a constant $C(n,m,R)$ such that, for all $\tau' < \tau$,*

$$\int_{2\tau'+R^*}^{2\tau+R^*} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \phi^2\big|_{r=R} \\ \leq C(n,m,R)\int_{2\tau'+R^*}^{2\tau+R^*} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left\{\left(\frac{\partial\phi}{\partial r^*}\right)^2 + |\nabla\!\!\!/\,\phi|^2\right\}\bigg|_{r=R} + C(n,m,R)\int_{\Sigma_{\tau'}} \big(J^T(\phi), n\big). \quad (4\text{-}7)$$

The central result of Proposition 4.1 combines results for two different regimes, that of high angular frequencies and that of low angular frequencies. First we will use radial multiplier vector fields to construct positive definite currents to deal with the former regime, and then a more general current using a commutation with angular momentum operators for the latter.

Remark 4.5. The specific parametrization (4-3) has technical advantages, but Σ_τ can in principle be replaced by a foliation of strictly *spacelike* hypersurfaces terminating at future null infinity and crossing the event horizon to the future of the bifurcation sphere.

4A. *Radial multiplier vector fields.*

A *radial multiplier* is a vector field of the form

$$X = f(r^*)\frac{\partial}{\partial r^*}. \quad (4\text{-}8)$$

We would like the associated current to be positive; however we find in general, as it is shown below:

$$K^X = \frac{f'}{1-\frac{2m}{r^{n-2}}}\left(\frac{\partial\phi}{\partial r^*}\right)^2 + \frac{f}{r}\left(1-\frac{nm}{r^{n-2}}\right)|\nabla\!\!\!/\,\phi|^2_{r^2\mathring{\gamma}_{n-1}} - \frac{1}{2}\left[f' + (n-1)\frac{f}{r}\left(1-\frac{2m}{r^{n-2}}\right)\right]\partial^\alpha\phi\,\partial_\alpha\phi. \quad (4\text{-}9)$$

Note. The prefactor to the angular derivatives vanishes at the *photon sphere* at $r = \sqrt[n-2]{nm}$.

Calculation of the deformation tensor ${}^{(X)}\pi$. It is convenient to work in Eddington–Finkelstein coordinates:

$$X = \frac{1}{2}f(r^*)\frac{\partial}{\partial v^*} - \frac{1}{2}f(r^*)\frac{\partial}{\partial u^*}. \quad (4\text{-}10)$$

We then obtain for the components of the deformation tensor:

$$\begin{gathered} {}^{(X)}\pi_{u^*u^*} = \left(1-\frac{2m}{r^{n-2}}\right)f', \quad {}^{(X)}\pi_{v^*v^*} = \left(1-\frac{2m}{r^{n-2}}\right)f', \\ {}^{(X)}\pi_{u^*v^*} = -\left(1-\frac{2m}{r^{n-2}}\right)\left(f' + (n-2)\frac{2m}{r^{n-1}}f\right), \\ {}^{(X)}\pi_{aA} = 0, \quad {}^{(X)}\pi_{AB} = f\,r\left(1-\frac{2m}{r^{n-2}}\right)(\mathring{\gamma}_{n-1})_{AB}. \end{gathered} \quad (4\text{-}11)$$

The formula (4-9) for K^X is now obtained by writing out (see also Appendix B)

$$K^X = {}^{(X)}\pi^{\alpha\beta}\,T_{\alpha\beta}$$

and rearranging the terms so as to complete $\left(\frac{\partial\phi}{\partial u^*}\right)^2+\left(\frac{\partial\phi}{\partial v^*}\right)^2$ to $\left(\frac{\partial\phi}{\partial r^*}\right)^2$. This rearrangement is also related to the following modification of currents; for observe that, if $\Box\phi=0$,

$$\Box(\phi^2)=2(\partial^\alpha\phi)(\partial_\alpha\phi). \tag{4-12}$$

First modified current. With the notation

$$J_\mu^{X,0}=T_{\mu\nu}X^\nu, \tag{4-13}$$

define the first modified current by

$$J_\mu^{X,1}=J_\mu^{X,0}+\frac{1}{4}\left(f'+(n-1)\frac{f}{r}\left(1-\frac{2m}{r^{n-2}}\right)\right)\partial_\mu(\phi^2)-\frac{1}{4}\partial_\mu\left(f'+(n-1)\frac{f}{r}\left(1-\frac{2m}{r^{n-2}}\right)\right)\phi^2. \tag{4-14}$$

Consequently the divergences are

$$K^{X,0}=\nabla^\mu J_\mu^{X,0}=K^X, \tag{4-15}$$

$$\begin{aligned}K^{X,1}=\nabla^\mu J_\mu^{X,1}&=K^X+\frac{1}{4}\left(f'+(n-1)\frac{f}{r}\left(1-\frac{2m}{r^{n-2}}\right)\right)\Box(\phi^2)-\frac{1}{4}\Box\left(f'+(n-1)\frac{f}{r}\left(1-\frac{2m}{r^{n-2}}\right)\right)\phi^2\\&=\frac{f'}{1-\frac{2m}{r^{n-2}}}\left(\frac{\partial\phi}{\partial r^*}\right)^2+\frac{f}{r}\left(1-\frac{nm}{r^{n-2}}\right)|\nabla\!\!\!/\phi|^2_{r^2\mathring{\gamma}_{n-1}}-\frac{1}{4}\Box\left(f'+(n-1)\frac{f}{r}\left(1-\frac{2m}{r^{n-2}}\right)\right)\phi^2.\end{aligned} \tag{4-16}$$

Since, for any function w,

$$\Box(w)=(g^{-1})^{\mu\nu}\nabla_\mu\partial_\nu w=-\frac{1}{1-\frac{2m}{r^{n-2}}}\partial_{u^*}\partial_{v^*}w-\frac{n-1}{2r}\left(\partial_{u^*}w-\partial_{v^*}w\right)+\Delta\!\!\!/_{r^2\mathring{\gamma}_{n-1}}w, \tag{4-17}$$

a straightforward calculation for

$$w=f'+(n-1)\frac{f}{r}\left(1-\frac{2m}{r^{n-2}}\right) \tag{4-18}$$

shows

$$\begin{aligned}\Box\left(f'+(n-1)\frac{f}{r}\left(1-\frac{2m}{r^{n-2}}\right)\right)&=\frac{1}{1-\frac{2m}{r^{n-2}}}f'''+2(n-1)\frac{f''}{r}+(n-1)\left[(n-3)+(n-1)\frac{2m}{r^{n-2}}\right]\frac{f'}{r^2}\\&\quad+(n-1)\left[\big((n-1)(n-2)-(n-3)\big)\left(\frac{2m}{r^{n-2}}\right)^2-n\frac{2m}{r^{n-2}}-(n-3)\right]\frac{f}{r^3}.\end{aligned} \tag{4-19}$$

Thus we finally obtain

$$\begin{aligned}K^{X,1}&=\frac{f'}{1-\frac{2m}{r^{n-2}}}\left(\frac{\partial\phi}{\partial r^*}\right)^2+\frac{f}{r}\left(1-\frac{nm}{r^{n-2}}\right)|\nabla\!\!\!/\phi|^2_{r^2\mathring{\gamma}_{n-1}}-\frac{1}{4}\frac{f'''}{1-\frac{2m}{r^{n-2}}}\phi^2-\frac{n-1}{2}\frac{f''}{r}\phi^2\\&\quad-\frac{n-1}{4}\left[(n-3)+(n-1)\frac{2m}{r^{n-2}}\right]\frac{f'}{r^2}\phi^2-\frac{n-1}{4}\left[(n-1)^2\left(\frac{2m}{r^{n-2}}\right)^2-n\frac{2m}{r^{n-2}}-(n-3)\right]\frac{f}{r^3}\phi^2.\end{aligned} \tag{4-20}$$

Applications of the first modified current. The proofs of Proposition 4.2 and Proposition 4.4 are applications of this formula, as it appears in the energy identity for $J^{X,1}$ on ${}^R\mathcal{D}_{\tau_1}^{\tau_2}$; see Appendix B.

Proof of Proposition 4.4. Choose $f = 1$ identically. Then

$$K^{X,1} = \frac{1}{r}\left(1-\frac{nm}{r^{n-2}}\right)|\nabla\!\!\!/\phi|^2_{r^2\mathring{\gamma}_{n-1}} + \frac{n-1}{4}\left[(n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\left(\frac{2m}{r^{n-2}}\right)^2\right]\frac{1}{r^3}\phi^2. \quad (4\text{-}21)$$

Since precisely

$$g\left(J^{X,1},\frac{\partial}{\partial r^*}\right) = \frac{1}{4}\left(\frac{\partial\phi}{\partial v^*}\right)^2+\frac{1}{4}\left(\frac{\partial\phi}{\partial u^*}\right)^2-\frac{1}{2}\left(1-\frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/\phi|^2_{r^2\mathring{\gamma}_{n-1}} + \frac{n-1}{2r}\left(1-\frac{2m}{r^{n-2}}\right)\phi\frac{\partial\phi}{\partial r^*}+\frac{n-1}{4r^2}\left[1-(n-1)\frac{2m}{r^{n-2}}\right]\left(1-\frac{2m}{r^{n-2}}\right)\phi^2, \quad (4\text{-}22)$$

we deduce from the energy identity for $J^{\frac{\partial}{\partial r^*},1}$ in ${}^R\mathcal{D}^\tau_{\tau'}$ that

$$\int_{R^*+2\tau'}^{R^*+2\tau}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}r^{n-1}\left\{\frac{1}{4}\left(\frac{\partial\phi}{\partial v^*}\right)^2+\frac{1}{4}\left(\frac{\partial\phi}{\partial u^*}\right)^2+\frac{n-1}{4R^2}\left[\frac{1}{2}-(n-1)\frac{2m}{R^{n-2}}\right]\left(1-\frac{2m}{R^{n-2}}\right)\phi^2\right\}\Big|_{r=R} + \int_{{}^R\mathcal{D}^\tau_{\tau'}}\frac{n-1}{4r}\left[(n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\left(\frac{2m}{r^{n-2}}\right)^2\right]\frac{1}{r^2}\phi^2\,\mathrm{d}\mu_g$$
$$\leq \int_{R^*+2\tau'}^{R^*+2\tau}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,r^{n-1}\left\{\frac{1}{2}\left(1-\frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/\phi|^2_{r^2\mathring{\gamma}_{n-1}}+\frac{n-1}{2}\left(1-\frac{2m}{r^{n-2}}\right)\left(\frac{\partial\phi}{\partial r^*}\right)^2\right\}\Big|_{r=R} + C(n,m)\int_{\Sigma_{\tau'}}\left(J^T(\phi),n\right), \quad (4\text{-}23)$$

where we have used Proposition C.1 for the boundary terms on $\partial\,{}^R\mathcal{D}^\tau_{\tau'}\setminus\{r=R\}$; note that

$$(n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\left(\frac{2m}{r^{n-2}}\right)^2>0 \quad \left(R>\sqrt[n-2]{8nm}\right). \qquad \square$$

Proof of Proposition 4.2. On one hand we need $f' = \mathcal{O}\left(\frac{1}{r^{1+\delta}}\right)$ in view of (4-20), while on the other we already know from the proof of Proposition 4.4 that $f = 1$ generates a positive bulk term for r large enough. We choose

$$f = 1-\left(\frac{R}{r}\right)^\delta \quad (4\text{-}24)$$

(where $R > \sqrt[n-2]{2m}$ is chosen suitably in the last step of the proof) and indeed find

$$K^{X,1} = \delta\frac{R^\delta}{r^{1+\delta}}\left(\frac{\partial\phi}{\partial r^*}\right)^2+\frac{f}{r}\left(1-\frac{nm}{r^{n-2}}\right)|\nabla\!\!\!/\phi|^2_{r^2\mathring{\gamma}_{n-1}}$$
$$+\left\{\frac{n-1}{4}(n-3)\left[1-\left(\frac{R}{r}\right)^\delta(1+\delta)\right]+\frac{1}{4}\left(\frac{R}{r}\right)^\delta\left[2(n-1)-(2+\delta)\right]\delta(1+\delta)\right.$$
$$+\left[\frac{n-1}{4}n\left[1-\left(\frac{R}{r}\right)^\delta\right]-\frac{\delta}{4}\left(\frac{R}{r}\right)^\delta\left[n(n+\delta)-2(1+\delta)^2\right]\right]\frac{2m}{r^{n-2}}$$
$$\left.-\left[\frac{(n-1)^3}{4}\left[1-\left(\frac{R}{r}\right)^\delta\right]-\frac{\delta}{4}\left(\frac{R}{r}\right)^\delta\left[\Big(n-(1+\delta)\Big)(n-1)-\delta^2\right]\right]\left(\frac{2m}{r^{n-2}}\right)^2\right\}\frac{1}{r^3}\phi^2\geq 0 \quad (4\text{-}25)$$

for $r \geq R_1 > R$, $R_1 = R_1(n,m) > \sqrt[n-2]{2m}$ chosen large enough. This gives control on $\frac{\partial\phi}{\partial r^*}$ and the angular derivatives:

$$\int_{^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}} \left\{ \delta \frac{R^\delta}{r^{1+\delta}} \left(\frac{\partial\phi}{\partial r^*}\right)^2 + \frac{f(R_1)}{r}\left(1-\frac{nm}{r^{n-2}}\right) |\nabla\!\!\!/\phi|^2_{r^2\mathring{\gamma}_{n-1}} \right\} \leq \int_{^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}} K^{X,1}.$$

Here and in the following, $\tau_2 > \tau_1 > \frac{1}{2}(t_0 - R^*)$. For $\frac{\partial\phi}{\partial t}$ we use the auxiliary current (see also Appendix C)

$$J_\mu^{\mathrm{aux}} = \frac{1}{2}\left(1-\frac{2m}{r^{n-2}}\right)\delta\frac{R^\delta}{r^{1+\delta}}\partial_\mu(\phi^2)$$

to find easily

$$\int_{^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}} \delta\frac{R^\delta}{r^{1+\delta}}\left(\frac{\partial\phi}{\partial t}\right)^2 \leq \int_{^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}} \left\{ \delta(n+\delta)\frac{R^\delta}{r^{1+\delta}}\left(\frac{\partial\phi}{\partial r^*}\right)^2 \right.$$
$$\left. + \delta\frac{R^\delta}{r^{1+\delta}}\left(1-\frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/\phi|^2_{r^2\mathring{\gamma}_{n-1}} + \delta(n+\delta)\frac{R^\delta}{r^{3+\delta}}\phi^2 + K^{\mathrm{aux}} \right\}.$$

Note that for $r \geq R_1$ in particular

$$\tfrac{1}{4}\big[2(n-1)-(2+\delta)\big]\delta(1+\delta)\frac{R^\delta}{r^{3+\delta}}\phi^2 \leq K^{X,1};$$

hence

$$\int_{^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}} \delta\frac{R^\delta}{r^{1+\delta}}\left\{\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial r^*}\right)^2\right\}$$
$$\leq C(n,m,\delta)\int_{^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}} \{K^{X,1}+K^{\mathrm{aux}}\}$$
$$\leq C(n,m,\delta)\int_{^{R}\mathcal{D}_{\tau_1}^{\tau_2}} \{K^{X,1}+K^{\mathrm{aux}}\} + C(n,m,\delta)\int_{^{R}\mathcal{D}_{\tau_1}^{\tau_2}\cap\{R<r<R_1\}} \left\{\frac{1}{r^{1+\delta}}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{r^3}\phi^2\right\}.$$

By Proposition C.1 (also (B-6)),

$$\int_{\partial^{R}\mathcal{D}_{\tau_1}^{\tau_2}} {}^*J^{X,1} \leq C(n,m,\delta)\int_{\Sigma_{\tau_1}} \big(J^T(\phi),n\big) + C(n,m,\delta)\int_{R^*+2\tau_1}^{R^*+2\tau_2} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-1} \times$$
$$\times \left\{\frac{1}{2}\left(\frac{\partial\phi}{\partial v^*}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial u^*}\right)^2 + \frac{1}{2}\left(1-\frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/\phi|^2_{r^2\mathring{\gamma}_{n-1}} + \frac{1}{r^2}\phi^2\right\}\Big|_{r=R}$$

and by Proposition C.8,

$$\int_{\partial^{R}\mathcal{D}_{\tau_1}^{\tau_2}} {}^*J^{\mathrm{aux}} \leq C(n,m,\delta)\int_{\Sigma_{\tau_1}} \big(J^T(\phi),n\big)$$
$$+ \int_{R^*+2\tau_1}^{R^*+2\tau_2} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-1} \left\{\frac{\delta}{2}\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial u^*}\right)^2 + \left(\frac{\partial\phi}{\partial v^*}\right)^2\right] + \frac{\delta}{2}\frac{R^{2\delta}}{r^{2+2\delta}}\phi^2\right\}_{r=R}.$$

Therefore, by the energy identity for $J^{X,1}$ and J^{aux} on ${}^{R}\mathcal{D}_{\tau_1}^{\tau_2}$,

$$\int_{{}^{R}\mathcal{D}_{\tau_1}^{\tau_2}} \{K^{X,1} + K^{\text{aux}}\} \leq C(n,m,\delta) \int_{\Sigma_{\tau_1}} \big(J^T(\phi), n\big) + C(n,m,\delta) \int_{R^*+2\tau_1}^{R^*+2\tau_2} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-1} \times \\ \times \left\{ \frac{1}{2}\left(\frac{\partial\phi}{\partial v^*}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial u^*}\right)^2 + \frac{1}{2}\left(1-\frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/\,\phi|^2_{r^2\mathring{\gamma}_{n-1}} + \frac{1}{r^2}\phi^2 \right\}\bigg|_{r=R}.$$

Our earlier (4-23) derived from the current $J^{\frac{\partial}{\partial r^*},1}$ now allows us to control the $\frac{\partial\phi}{\partial v^*}$, $\frac{\partial\phi}{\partial u^*}$ derivatives and ϕ^2 on the $r = R$ boundary together with the ϕ^2 term in the region $R \leq r \leq R_1$ in one step:

$$\begin{aligned}
&\int_{{}^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}} \frac{R^\delta}{r^{1+\delta}} \left\{ \left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial r^*}\right)^2 \right\} \\
&\quad \leq C(n,m,\delta) \int_{\Sigma_{\tau_1}} \big(J^T(\phi), n\big) \\
&\qquad + C(n,m,\delta) \int_{R^*+2\tau_1}^{R^*+2\tau_2} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-1} \left\{ \frac{1}{2}\left(1-\frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/\,\phi|^2_{r^2\mathring{\gamma}_{n-1}} + \frac{n-1}{2}\left(\frac{\partial\phi}{\partial r^*}\right)^2 \right\}\bigg|_{r=R} \\
&\qquad + C(n,m,\delta) \int_{{}^{R}\mathcal{D}_{\tau_1}^{\tau_2} \cap \{R<r<R_1\}} \frac{1}{r^{1+\delta}} \left(\frac{\partial\phi}{\partial t}\right)^2.
\end{aligned}$$

With t_0 fixed, we can now choose R by Proposition 4.1 such that

$$\int_{{}^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}} \frac{1}{r^{1+\delta}} \left\{ \left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial r^*}\right)^2 \right\} \leq C(n,m,\delta) \int_{\Sigma_{\tau_1}} \big(J^T(\phi) + J^T(T\cdot\phi), n\big). \qquad \square$$

While it is possible to find simple functions $f \geq 0$ to ensure the positivity of $K^{X,1}$ asymptotically, this is not the case in the entire domain of outer communications; the difficulty is the indefinite sign of (4-20) at the photon sphere $r = \sqrt[n-2]{nm}$, which is a manifestation of the *trapping effect*.

In the following our strategy will be to prove nonnegativity of $K^{X,1}$ not pointwise but by using Poincaré inequalities after integration over the spheres (the group orbits of SO(n)). This is achieved in two alternative constructions: in Section 4B with a decomposition into spherical harmonics, and in Section 4C by a commutation with angular momentum operators.

4B. ***High angular frequencies.*** Here we construct a positive definite current for the projection of solutions to the wave equation to eigenspaces corresponding to high angular frequencies in the spherical decomposition. Since by Poincaré's inequality the second term in (4-20) then becomes comparable to the zeroth-order terms, the idea is to choose f such that this term dominates. We evidently need

$$f(r^*) \begin{cases} < 0, & r < \sqrt[n-2]{nm}, \\ = 0, & r = \sqrt[n-2]{nm}, \\ > 0, & r > \sqrt[n-2]{nm}, \end{cases}$$

and since f should also be bounded one may guess that

$$f(r^*) = \arctan\left(\frac{(n-1)\,r^*}{\sqrt[n-2]{nm}}\right)$$

is a good choice; however, while it can ensure positivity at the photon sphere, it fails to do so near the horizon and in the asymptotics. After briefly recalling the spherical decomposition, we will give a more refined construction of f, nonetheless guided by the overall characteristics of this function.

Fourier expansion on the sphere $\mathbb{S}^{n-1}$. We recall the Fourier expansion on the sphere $\mathbb{S}^{n-1}$:

$$\phi = \sum_{l\geq 0} \pi_l \phi, \quad \phi \in \mathrm{L}^2(\mathbb{S}^{n-1}), \tag{4-26}$$

where π_l denotes the orthogonal projection of $\mathrm{L}^2(\mathbb{S}^{n-1})$ onto E_l (see below):

$$\mathring{\Delta}_{n-1}\pi_l\phi = -l(l+n-2)\pi_l\phi. \tag{4-27}$$

In other words, denoting by $E_l \subset \mathrm{L}^2(\mathbb{S}^{n-1})$, $l \geq 0$, the eigenspaces of

$$-\mathring{\Delta}_{n-1} + \left(\frac{n-2}{2}\right)^2$$

corresponding to the eigenvalue $\left(l + \frac{n-2}{2}\right)^2$, then

$$\mathrm{L}^2(\mathbb{S}^{n-1}) = \bigoplus_{l\geq 0} E_l.$$

If we assume $\pi_l\phi = 0$ $(0 \leq l < L)$ for some $L > 0$, then it is easy to show (see, e.g., [Schlue 2012]) that

$$L(L+n-2)\frac{1}{r^2}\int_{S_r} \phi^2 \,\mathrm{d}\mu_{\gamma_r} \leq \int_{S_r} \left|\nabla\!\!\!/\phi\right|^2_{r^2\mathring{\gamma}_{n-1}} \mathrm{d}\mu_{\gamma_r};$$

this is a well known Poincaré-type inequality on the sphere:

Lemma 4.6 (Poincaré inequality). *Let* $\phi \in \mathrm{H}^1(S_r)$, $S_r = (\mathbb{S}^{n-1}, r^2\mathring{\gamma}_{n-1})$, *have vanishing projection to* E_l, $0 \leq l < L$, *for some* $L \in \mathbb{N}$; *i.e.*,

$$\pi_l\phi = 0 \quad (0 \leq l < L).$$

Then

$$\int_{S_r} \left|\nabla\!\!\!/\phi\right|^2 \mathrm{d}\mu_{\gamma_r} \geq L(L+n-2)\frac{1}{r^2}\int_{S_r} \phi^2 \,\mathrm{d}\mu_{\gamma_r}.$$

Construction of the multiplier function for high angular frequencies. The idea is to prescribe the third derivative of f and to find its second and first derivatives by integration with boundary values and parameters that ensure that f remains bounded. Let

$$\alpha = \frac{n-1}{(nm)^{\frac{1}{n-2}}} \tag{4-28}$$

and $\gamma \geq 2$, $\gamma \in \mathbb{N}$. Consider

$$f^{\mathrm{III}}_{\gamma,\alpha}(r^*) = \begin{cases} -1, & |r^*| \leq \dfrac{1}{\gamma\alpha}, \\ 1, & \dfrac{1}{\gamma\alpha} < |r^*| \leq b_{\gamma,\alpha}, \\ \left(\dfrac{b_{\gamma,\alpha}}{r^*}\right)^6, & |r^*| \geq b_{\gamma,\alpha}, \end{cases} \tag{4-29}$$

where

$$b_{\gamma,\alpha} = \frac{5}{6}\frac{2}{\gamma\alpha}. \tag{4-30}$$

Note that $b_{\gamma,\alpha}$ is chosen so that

$$\int_0^\infty f^{\mathrm{III}}_{\gamma,\alpha}(r^*)\,\mathrm{d}r^* = 0. \tag{4-31}$$

Now define

$$f^{\mathrm{II}}_{\gamma,\alpha}(r^*) = \int_0^{r^*} f^{\mathrm{III}}_{\gamma,\alpha}(t)\,\mathrm{d}t. \tag{4-32}$$

Obviously $f^{\mathrm{II}}_{\gamma,\alpha}(-r^*) = -f^{\mathrm{II}}_{\gamma,\alpha}(r^*)$ and, in explicit form,

$$f^{\mathrm{II}}_{\gamma,\alpha}(r^*) = \begin{cases} -r^*, & |r^*| \leq \dfrac{1}{\gamma\alpha}, \\ r^* - \dfrac{2}{\gamma\alpha}, & \dfrac{1}{\gamma\alpha} < r^* \leq b_{\gamma,\alpha}, \\ r^* + \dfrac{2}{\gamma\alpha}, & -b_{\gamma,\alpha} \leq r^* < -\dfrac{1}{\gamma\alpha}, \\ -\dfrac{b^6_{\gamma,\alpha}}{5r^{*5}}, & |r^*| \geq b_{\gamma,\alpha}. \end{cases} \tag{4-33}$$

The functions $f^{\mathrm{II}}_{\gamma,\alpha}$ and $f^{\mathrm{III}}_{\gamma,\alpha}$ are sketched in Figure 4.

Next define

$$f^{\mathrm{I}}_{\gamma,\alpha} = \int_{-\infty}^{r^*} f^{\mathrm{II}}_{\gamma,\alpha}(t)\,\mathrm{d}t. \tag{4-34}$$

Here we find

$$f^{\mathrm{I}}_{\gamma,\alpha}(r^*) = \begin{cases} \dfrac{b^6_{\gamma,\alpha}}{20r^{*4}}, & r^* \leq -b_{\gamma,\alpha}, \\ \dfrac{b^2_{\gamma,\alpha}}{20} + \dfrac{1}{2}(r^{*2} - b^2_{\gamma,\alpha}) + \dfrac{2}{\gamma\alpha}(r^* + b_{\gamma,\alpha}), & -b_{\gamma,\alpha} \leq r^* \leq -\dfrac{1}{\gamma\alpha}, \\ \dfrac{13}{12}\dfrac{1}{(\gamma\alpha)^2} - \dfrac{r^{*2}}{2}, & -\dfrac{1}{\gamma\alpha} \leq r^* \leq 0, \end{cases} \tag{4-35}$$

and $f^{\mathrm{I}}_{\gamma,\alpha}(r^*) = f^{\mathrm{I}}_{\gamma,\alpha}(-r^*)$, as sketched in Figure 5.

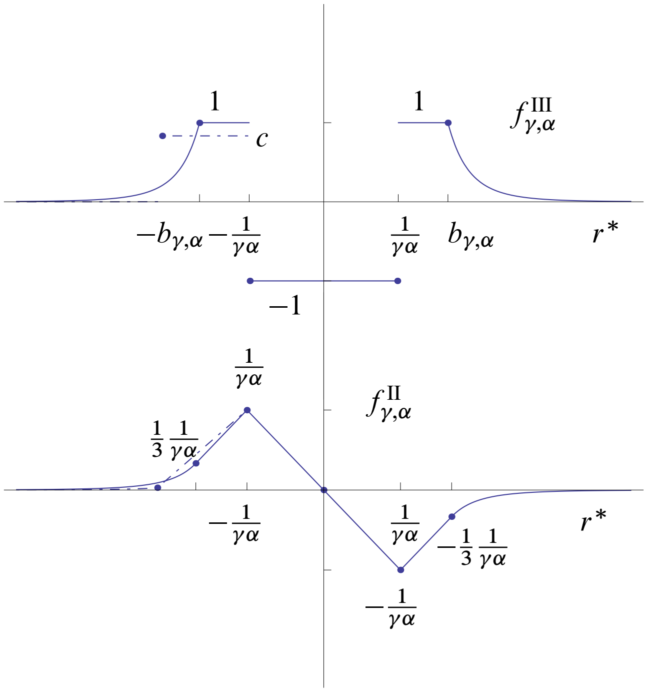

Figure 4. Sketch of the functions $f^{\mathrm{II}}_{\gamma,\alpha}$ and $f^{\mathrm{III}}_{\gamma,\alpha}$, and the adjusted functions (dot-dashed) for $r^* \leq 0$.

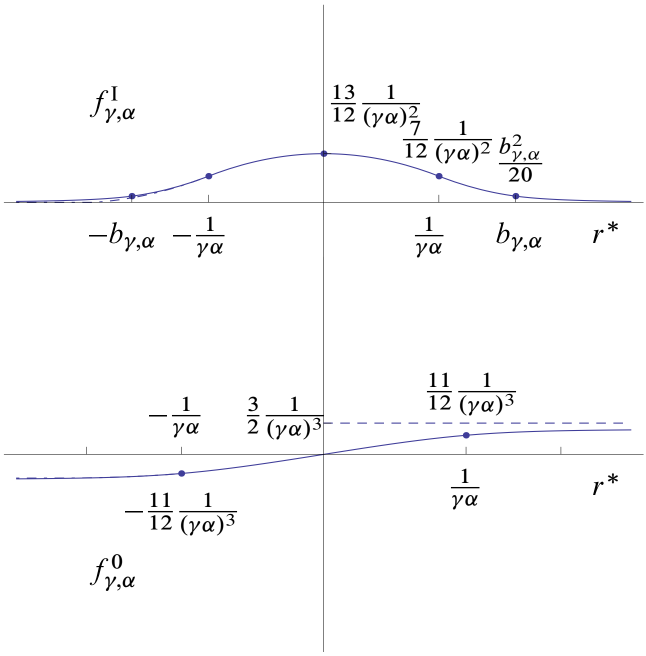

Figure 5. Sketch of the functions $f^{\mathrm{I}}_{\gamma,\alpha}$ and $f^{0}_{\gamma,\alpha}$, and the adjusted functions (dot-dashed) for $r^* \leq 0$.

Finally define

$$f^0_{\gamma,\alpha}(r^*) = \int_0^{r^*} f^{\mathrm{I}}_{\gamma,\alpha}(t)\,\mathrm{d}t. \tag{4-36}$$

Here again $f^0_{\gamma,\alpha}(-r^*) = -f^0_{\gamma,\alpha}(r^*)$ and, in particular,

$$f^0_{\gamma,\alpha}\Big(\frac{1}{\gamma\alpha}\Big) = \int_0^{\frac{1}{\gamma\alpha}} \Big(\frac{13}{12}\frac{1}{(\gamma\alpha)^2} - \frac{t^2}{2}\Big)\,\mathrm{d}t = \frac{11}{12}\frac{1}{(\gamma\alpha)^3}. \tag{4-37}$$

Moreover the calculus yields

$$f(b_{\gamma,\alpha}) > \frac{1}{(\gamma\alpha)^3}, \qquad \lim_{r^*\to\infty} f^0_{\gamma,\alpha}(r^*) < \frac{3}{2}\frac{1}{(\gamma\alpha)^3}. \tag{4-38}$$

The function $f^0_{\gamma,\alpha}$ is sketched in Figure 5. While this function would suffice in the region $r^* \geq -\frac{1}{\gamma\alpha}$ it does not fall-off fast enough as $r^* \to -\infty$.

Lemma 4.7. *With r^* defined by (2-17) we have, for all $n \geq 3$,*

$$\lim_{r^*\to-\infty}\Big(1 - \frac{2m}{r^{n-2}}\Big)(-r^*) = 0.$$

In fact, for all $r^* < 0$,

$$\Big(1 - \frac{2m}{r^{n-2}}\Big) \leq \frac{(2m)^{\frac{1}{n-2}}}{(-r^*)}.$$

Proof. See Appendix B. □

Next we will make an adjustment to f^{III} on $r^* \leq 0$ that introduces faster decay while keeping the area under the graph of f^{III} *and* f^{II} fixed [Schlue 2012]. In other words, there are constants

$$b_{\gamma,\alpha} \leq b \leq \frac{4}{\gamma\alpha}, \qquad \frac{1}{4} \leq c \leq 1 \tag{4-39}$$

such that, if we redefine $f^{\mathrm{III}}_{\gamma,\alpha}$ for $r^* \leq 0$ as

$$f^{\mathrm{III}}_{\gamma,\alpha}(r^*) = \begin{cases} -1, & -\frac{1}{\gamma\alpha} \leq r^* \leq 0, \\ c, & -b \leq r^* \leq \frac{1}{\gamma\alpha}, \\ \Big(1 - \frac{2m}{r^{n-2}}\Big)^6 \Big(\frac{b}{(2m)^{\frac{1}{n-2}}}\Big)^6, & r^* \leq -b, \end{cases} \tag{4-40}$$

then

$$\int_0^{-\infty} f^{\mathrm{III}}_{\gamma,\alpha}(r^*)\,\mathrm{d}r^* = 0, \qquad \int_{-\infty}^0 \int_0^{r^*} f^{\mathrm{III}}_{\gamma,\alpha}(t)\,\mathrm{d}t\,\mathrm{d}r^* = \int_{-\infty}^0 \int_0^{-r^*} (-f^{\mathrm{III}}_{\gamma,\alpha}(t))\,\mathrm{d}t\,\mathrm{d}r^*.$$

The adjusted functions in comparison to the old are also sketched in Figures 4 and 5. Note in particular that, for $r^* \le 0$,

$$f^{\mathrm{II}}(r^*) \le \frac{1}{\gamma\alpha}, \tag{4-41}$$

$$f^{\mathrm{I}}(r^*) \le f^{\mathrm{I}}(-r^*) \le \frac{13}{12}\frac{1}{(\gamma\alpha)^2}, \tag{4-42}$$

and, for $r^* \le -\dfrac{1}{\gamma\alpha}$,

$$\frac{11}{12}\frac{1}{(\gamma\alpha)^3} \le |f^0(r^*)| \le f^0(-r^*) < \frac{3}{2}\frac{1}{(\gamma\alpha)^3}. \tag{4-43}$$

Remark 4.8. In order to deal with smooth functions one could use (e.g., at the level of second derivatives) a convolution with a Gaussian on the scale given by $\gamma\alpha$ (or finer); i.e., one could define

$$f''_{\gamma,\alpha}(r^*) = \frac{\gamma\alpha}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(\gamma\alpha)^2(r^*-t)^2} f^{\mathrm{II}}_{\gamma,\alpha}(t)\,\mathrm{d}t$$

and find $f'''_{\gamma,\alpha} = \frac{\mathrm{d}}{\mathrm{d}r^*} f''_{\gamma,\alpha}$ by differentiation, and $f'_{\gamma,\alpha}$ and $f_{\gamma,\alpha}$ by integration with the boundary values $f'_{\gamma,\alpha}(-\infty) = 0$, $f_{\gamma,\alpha}(0) = 0$ as above. However, we choose not to do so (as it does not give further insight) and work directly with the step-functions, i.e., define

$$f''_{\gamma,\alpha} = f^{\mathrm{III}}_{\gamma,\alpha}.$$

We are now in the position to prove a nonnegativity property of the terms occurring in (4-20), which we will denote by ${}^0K^{X,1}$:

$$K^{X,1} = \frac{f'}{1-\frac{2m}{r^{n-2}}}\left(\frac{\partial\phi}{\partial r^*}\right)^2 + {}^0K^{X,1}. \tag{4-44}$$

Proposition 4.9 (positivity of the current $J^{X_{\gamma,\alpha},1}$). *For $n \ge 3$,*

$$X_{\gamma,\alpha} = f_{\gamma,\alpha}\frac{\partial}{\partial r^*} \quad (\textit{where we choose } \gamma = 12)$$

and $\phi \in \mathrm{H}^1(S)$ satisfy

$$\int_S {}^0K^{X_{\gamma,\alpha},1}\,\mathrm{d}\mu_\gamma \ge 0$$

provided $\pi_l\phi = 0$ for $0 \le l < L$, where $L \ge (6\gamma n)^2$ is fixed.

Proof. By Lemma 4.6,

$$\begin{aligned}\int_S {}^0K^{X_{\gamma,\alpha},1}\,\mathrm{d}\mu_\gamma \ge \int_S \Bigg\{ & L(L+n-2)\frac{f_{\gamma,\alpha}}{r^3}\left(1-\frac{nm}{r^{n-2}}\right) \\ & -\frac{1}{4}\frac{f'''_{\gamma,\alpha}}{1-\frac{2m}{r^{n-2}}} - \frac{n-1}{2}\frac{f''_{\gamma,\alpha}}{r} - \frac{n-1}{4}\left[(n-3)+(n-1)\frac{2m}{r^{n-2}}\right]\frac{f'_{\gamma,\alpha}}{r^2} \\ & -\frac{n-1}{4}\left[(n-1)^2\left(\frac{2m}{r^{n-2}}\right)^2 - n\frac{2m}{r^{n-2}} - (n-3)\right]\frac{f_{\gamma,\alpha}}{r^3}\Bigg\}\phi^2\,\mathrm{d}\mu_\gamma.\end{aligned} \tag{4-45}$$

We consider the five regions

$$-\infty < -\frac{4}{\gamma\alpha} < -\frac{1}{\gamma\alpha} < \frac{1}{\gamma\alpha} < b_{\gamma,\alpha} < \infty.$$

The proofs of the following four lemmas are omitted here; see [Schlue 2012].

Step 1 (near the photon sphere, $|r^*| < \frac{1}{\gamma\alpha}$).

Lemma 4.10. *In the region* $|r^*| < \frac{1}{\gamma\alpha}$ *the corresponding value of* r *lies in the interval*

$$\sqrt[n-2]{\delta n m} < r < \frac{n}{\alpha}$$

where $\delta = \max\{\frac{1}{3}, \frac{4}{3}\frac{2}{n}\}$.

Recalling $f_{\gamma,\alpha}$ and its derivatives, we then find, in the region $|r^*| < \frac{1}{\gamma\alpha}$,

$$\begin{aligned}\int_S {}^0K^{X_{\gamma,\alpha},1}\,\mathrm{d}\mu_\gamma &\geq \int_S \Big\{\frac{1}{4} - \frac{1}{2}\frac{\alpha}{\delta^{\frac{1}{n-2}}}\frac{1}{\gamma\alpha} - \frac{1}{4}\frac{\alpha^2}{\delta^{\frac{2}{n-2}}}\frac{1}{n-1}\Big[(n-3) + \frac{1}{\delta}(n-1)\frac{2}{n}\Big]\frac{13}{12}\frac{1}{(\gamma\alpha)^2} - \frac{1}{4}\frac{\alpha^3}{\delta^{\frac{3}{n-2}}}\frac{2}{\delta n}\frac{3}{2}\frac{1}{(\gamma\alpha)^3}\Big\}\phi^2\,\mathrm{d}\mu_\gamma \\ &\geq \int_S \Big\{\frac{1}{4} - \frac{1}{2}\frac{3}{\gamma} - \frac{3}{4}\frac{13}{12}\Big(\frac{3}{\gamma}\Big)^2 - \frac{3}{4}\Big(\frac{3}{\gamma}\Big)^3\Big\}\phi^2\,\mathrm{d}\mu_\gamma \geq \int_S \frac{1}{4}\frac{1}{8}\phi^2\,\mathrm{d}\mu_\gamma\end{aligned}$$

because $\gamma = 12$.

Step 2 (in the intermediate region, $\frac{1}{\gamma\alpha} \leq r^* \leq \frac{5}{6}\frac{2}{\gamma\alpha}$).

Lemma 4.11. *In the region* $\frac{1}{\gamma\alpha} \leq r^* \leq \frac{5}{6}\frac{2}{\gamma\alpha}$ *we have, for the corresponding value of* r,

$$\Big(1 + \frac{1}{3\gamma(n-1)}\Big)(nm)^{\frac{1}{n-2}} \leq r \leq \frac{n}{\alpha}.$$

Collecting the first term and the last, we find in this region,

$$\begin{aligned}\int_S {}^0K^{X_{\gamma,\alpha},1}\,\mathrm{d}\mu_\gamma \geq \int_S \Big\{&\frac{\alpha^3}{n^3}\frac{11}{12}\frac{1}{(\gamma\alpha)^3}\Big[\Big(1-\frac{nm}{r^{n-2}}\Big)L(L+n-2) - \frac{n-1}{4}(n-1)^2\Big(\frac{2}{n}\Big)^2 + \tfrac{1}{4}(n-1)(n-3)\Big] \\ &-\frac{1}{4}\frac{1}{1-\frac{2}{n}} + \frac{\alpha}{3}\frac{1}{3}\frac{1}{\gamma\alpha} - \frac{1}{4}\alpha^2\frac{1}{n-1}\Big[(n-3)+(n-1)\frac{2}{n}\Big]\frac{1}{(\gamma\alpha)^2}\Big\}\phi^2\,\mathrm{d}\mu_\gamma \\ \geq \int_S \Big\{&\frac{11}{12}\frac{1}{(n\gamma)^3}\Big[\frac{1}{6\gamma(n-1)}L(L+n-2) - (n-1) + \tfrac{1}{4}(n-1)(n-3)\Big] - \frac{3}{4} - \frac{1}{4}\frac{1}{\gamma^2}\Big\}\phi^2\,\mathrm{d}\mu_\gamma \\ \geq \int_S \Big\{&\frac{11}{12}\frac{1}{6}\Big(\frac{(6\gamma n)^2}{\gamma^2 n^2}\Big)^2 - 1\Big\}\phi^2\,\mathrm{d}\mu_\gamma \geq \int_S \phi^2\,\mathrm{d}\mu_\gamma\end{aligned}$$

because $L \geq (6\gamma n)^2$, where we have used that, for $\frac{1}{\gamma\alpha} \leq r^* \leq \frac{5}{6}\frac{2}{\gamma\alpha}$,

$$1 - \frac{nm}{r^{n-2}} \geq \frac{1}{6\gamma(n-1)}.$$

Step 3 (in the asymptotics, $r^* \geq b_{\gamma,\alpha}$). Given the general fact Proposition B.1 we here only need the weaker statement:

Lemma 4.12. *For* $r^* \geq \frac{5}{6}\frac{2}{\gamma\alpha}$,

$$\frac{r}{r^*} \leq 2\gamma n.$$

Here

$$\int_S {}^0K^{X_{\gamma,\alpha},1}\,\mathrm{d}\mu_\gamma \geq \int_S \Bigl\{ \frac{1}{(\gamma\alpha)^3}\Bigl[\frac{1}{6\gamma(n-1)}L(L+n-2)-\frac{3}{2}(n-1)\Bigr]\frac{1}{r^3}-\frac{1}{4}\frac{1}{1-\frac{2}{n}}\Bigl(\frac{5}{6}\frac{2}{\gamma\alpha r^*}\Bigr)^6 - \frac{1}{4}\alpha^2\frac{1}{n-1}\Bigl[(n-3)+(n-1)\frac{2}{n}\Bigr]\frac{1}{20}\frac{\bigl(\frac{5}{6}\frac{2}{\gamma\alpha}\bigr)^6}{r^{*4}}\Bigr\}\phi^2\,\mathrm{d}\mu_\gamma$$

$$\geq \int_S \Bigl[\frac{L^2}{6\gamma^4 n}+\frac{L}{6\gamma^4 n}(n-2)-\frac{3}{2}\frac{1}{\gamma^3}(n-1)-\frac{3}{4}\Bigl(\frac{r}{r^*}\Bigr)^3\Bigl(\frac{5}{6}\frac{2}{\gamma}\Bigr)^6\frac{1}{(\alpha r^*)^3} - \frac{1}{4}\frac{1}{20}\Bigl(\frac{r}{r^*}\Bigr)^3\Bigl(\frac{5}{6}\frac{2}{\gamma}\Bigr)^6\frac{1}{\alpha r^*}\Bigr]\frac{1}{(\alpha r)^3}\phi^2\,\mathrm{d}\mu_\gamma$$

$$\geq \int_S \bigl[(6n)^3-(4n)^3\bigr]\frac{1}{(\alpha r)^3}\phi^2\,\mathrm{d}\mu_\gamma \geq \int_S \Bigl(\frac{n}{\alpha r}\Bigr)^3\phi^2\,\mathrm{d}\mu_\gamma \geq \int_S \Bigl(\frac{(nm)^{\frac{1}{n-2}}}{r}\Bigr)^3\phi^2\,\mathrm{d}\mu_\gamma,$$

where in the third bound we have again used $L \geq (6\gamma n)^2$ and the lemma.

Step 4 (in the intermediate region, $-\frac{4}{\gamma\alpha} \leq r^* \leq -\frac{1}{\gamma\alpha}$). Recall $\gamma = 12$.

Lemma 4.13. *For* $k \leq \gamma$, $k \in \mathbb{N}$,

$$\Bigl(1-\frac{2m}{r^{n-2}}\Bigr)^{-1}\Bigr|_{r^*=-\frac{k}{\gamma\alpha}} \leq 17,$$

and, *consequently*,

$$-\Bigl(1-\frac{nm}{r^{n-2}}\Bigr)\Bigr|_{r^*=-\frac{1}{\gamma\alpha}} \geq \frac{1}{20}\frac{1}{2\gamma}.$$

In the region $-\frac{4}{\gamma\alpha} \leq r^* \leq -\frac{1}{\gamma\alpha}$ we directly apply the lemma to see that

$$\int_S {}^0K^{X_{\gamma,\alpha},1}\,\mathrm{d}\mu_\gamma$$

$$\geq \int_S \Bigl\{ L(L+n-2)\frac{1}{(nm)^{\frac{3}{n-2}}}\frac{11}{12}\frac{1}{(\gamma\alpha)^3}\frac{1}{20}\frac{1}{2\gamma}-\frac{1}{4}17-\frac{n-1}{2}\frac{1}{(2m)^{\frac{1}{n-2}}}\frac{1}{\gamma\alpha} - \frac{n-1}{2}\bigl[(n-3)+(n-1)\bigr]\frac{1}{(2m)^{\frac{2}{n-2}}}\frac{13}{12}\frac{1}{(\gamma\alpha)^2}-\frac{n-1}{4}\bigl[n+(n-3)\bigr]\frac{1}{(2m)^{\frac{3}{n-2}}}2\frac{1}{(\gamma\alpha)^3}\Bigr\}\phi^2\,\mathrm{d}\mu_\gamma$$

$$\geq \int_S \Bigl\{ \frac{1}{(3\gamma)^4}\frac{1}{(n-1)^3}L(L+n-2)-\frac{17}{4}-\frac{3}{2}\frac{1}{2\gamma}-\frac{13}{12}\frac{1}{\gamma^2}\Bigl(\frac{n}{2}\Bigr)^{\frac{2}{n-2}}-\frac{1}{n-1}\frac{1}{\gamma^3}\Bigl(\frac{n}{2}\Bigr)^{\frac{3}{n-2}}\Bigr\}\phi^2\,\mathrm{d}\mu_\gamma$$

$$\geq \int_S \Bigl\{2^4 n-\frac{23}{4}\Bigr\}\phi^2\,\mathrm{d}\mu_\gamma \geq \int_S \phi^2\,\mathrm{d}\mu_\gamma,$$

because $L \geq (6\gamma n)^2$.

Step 5 (near the horizon, $r^* \leq -b$). Finally we see for $r^* \leq -b$, recalling the adjustment to faster fall-off,

$$\begin{aligned}
&\int_S {}^0K^{X_{\gamma,\alpha},1}\,\mathrm{d}\mu_\gamma \\
&\geq \int_S \Big\{ L(L+n-2)\frac{1}{(nm)^{\frac{3}{n-2}}}\frac{11}{12}\frac{1}{(\gamma\alpha)^3}\frac{1}{20}\frac{1}{2\gamma} - \frac{1}{4}\Big(1-\frac{2}{n}\Big)^5 - \frac{n-1}{2}\frac{1}{(2m)^{\frac{1}{n-2}}}\frac{1}{\gamma\alpha} \\
&\qquad - \frac{(n-1)^2}{4}\frac{1}{(2m)^{\frac{1}{n-2}}}\frac{1}{(\gamma\alpha)^2} - \frac{n-1}{4}\big[n+(n-3)\big]\frac{1}{(2m)^{\frac{3}{n-2}}}2\frac{1}{(\gamma\alpha)^3}\Big\}\phi^2\,\mathrm{d}\mu_\gamma \\
&\geq \int_S \Big\{ \frac{1}{(3\gamma)^4}\frac{1}{(n-1)^3}L(L+n-2) - \frac{1}{4} - \frac{1}{2\gamma}\Big(\frac{n}{2}\Big)^{\frac{1}{n-2}} - \frac{1}{(2\gamma)^2}\Big(\frac{n}{2}\Big)^{\frac{2}{n-2}} - \frac{4}{n-1}\frac{1}{(2\gamma)^3}\Big(\frac{n}{2}\Big)^{\frac{3}{n-2}}\Big\}\phi^2\,\mathrm{d}\mu_\gamma \\
&\geq \int_S \{2^4 n - \tfrac{5}{4}\}\phi^2\,\mathrm{d}\mu_\gamma \geq \int_S \phi^2\,\mathrm{d}\mu_\gamma,
\end{aligned}$$

where we have used that here

$$\frac{f'''}{1-\frac{2m}{r^{n-2}}} = \Big(1-\frac{2m}{r^{n-2}}\Big)^5\Big(\frac{b}{(2m)^{\frac{1}{n-2}}}\Big)^6 \leq \Big(1-\frac{2}{n}\Big)^5 \leq 1. \qquad \square$$

In fact, we have shown more, because all lower bounds in Steps 1–5 are minorized by $\frac{1}{4}\frac{1}{8}\frac{(2m)^{\frac{3}{n-2}}}{r^3}$.

Corollary 4.14. *Let ϕ be a solution of the wave equation $\Box_g\phi = 0$ satisfying*

$$\pi_l\phi = 0 \quad (0 \leq l < L)$$

on the standard sphere $S = (\mathbb{S}^{n-1}, r^2\mathring{\gamma}_{n-1})$ for a fixed $L \geq (6\gamma n)^2$. Then

$$\int_S \Big\{ \frac{1}{4}\frac{1}{8}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\phi^2 + \frac{1}{(20\gamma^2)^3}\frac{1}{(n-2)^2(n-1)^6}\Big(1-\frac{2m}{r^{n-2}}\Big)^5\frac{(2m)^{\frac{6}{n-2}}}{r^4}\Big(\frac{\partial\phi}{\partial r^*}\Big)^2\Big\}\,\mathrm{d}\mu_\gamma \leq \int_S K^{X_{\gamma,\alpha},1}\,\mathrm{d}\mu_\gamma.$$

Proof. It remains to be shown that

$$\frac{1}{20}\frac{1}{(4\cdot 5(n-2))^2}\Big(1-\frac{2m}{r^{n-2}}\Big)^6\frac{b_{\gamma,\alpha}^6}{r^4} \leq f'_{\gamma,\alpha}. \tag{$*$}$$

First,

$$\int_{-\infty}^{r^*}\Big(1-\frac{2m}{r^{n-2}}\Big)^6\,\mathrm{d}r^* = \int_{(2m)^{\frac{1}{n-2}}}^{r}\Big(1-\frac{2m}{r^{n-2}}\Big)^5\,\mathrm{d}r,$$

because $\mathrm{d}r^*/\mathrm{d}r = \big(1-\frac{2m}{r^{n-2}}\big)^{-1}$. Now choose $\sqrt[n-2]{2m} < r_0 < r$ so close to r as to satisfy

$$\frac{r-r_0}{r_0} = \frac{1}{2}\frac{1}{5(n-2)}\Big(1-\frac{2m}{r^{n-2}}\Big).$$

Then, by the mean value theorem,

$$\int_{(2m)^{\frac{1}{n-2}}}^{r}\left(1-\frac{2m}{r^{n-2}}\right)^5 \mathrm{d}r \geq \left(1-\frac{2m}{r_0^{n-2}}\right)^5 (r-r_0) \geq \left(1-\frac{2m}{r^{n-2}}\right)^5 \left[1-5(n-2)\frac{1}{1-\frac{2m}{r^{n-2}}}\frac{r-r_0}{r_0}\right](r-r_0)$$
$$\geq \frac{1}{4}\frac{1}{5(n-2)}\left(1-\frac{2m}{r^{n-2}}\right)^6 (2m)^{\frac{1}{n-2}}.$$

We conclude, for $r^* \leq -b$,

$$f'_{\gamma,\alpha}(r^*) = \int_{-\infty}^{r^*}\int_{-\infty}^{s^*}\left(1-\frac{2m}{r^{n-2}}\right)\bigg|_{r^*=s^*}\left(\frac{b}{(2m)^{\frac{1}{n-2}}}\right)^6 \mathrm{d}s^*\,\mathrm{d}r^* \geq \frac{1}{4}\frac{1}{5(n-2)}\int_{-\infty}^{r^*}\left(1-\frac{2m}{r^{n-2}}\right)^6 \mathrm{d}r^* \frac{b^6}{(2m)^{\frac{5}{n-2}}}$$
$$\geq \left(\frac{1}{4}\frac{1}{5(n-2)}\right)^2\left(1-\frac{2m}{r^{n-2}}\right)^6 \frac{b^6}{(2m)^{\frac{4}{n-2}}} \geq \frac{1}{(4\cdot 5(n-2))^2}\left(1-\frac{2m}{r^{n-2}}\right)^6 \frac{b_{\gamma,\alpha}^6}{r^4}.$$

Second, for $r^* \geq 0$,

$$\frac{1}{(4\cdot 5(n-2))^2}\frac{1}{r^4} = \frac{1}{(4\cdot 5(n-2))^2}\left(\frac{r^*}{r}\right)^4 \frac{1}{r^{*4}} \leq \frac{1}{r^{*4}}.$$

Since, thirdly,

$$\frac{b_{\gamma,\alpha}}{r} \leq 1,$$

we have established $(*)$ for the regions $r^* \leq -b$, $r^* \geq b_{\gamma,\alpha}$, $-b \leq r^* \leq b_{\gamma,\alpha}$, respectively. □

Remark 4.15. This estimate of the zeroth-order term ϕ^2 suffices to obtain an estimate for all derivatives using a commutation with the vector field T; see the proof of Proposition 4.1 in Section 4D.1.

4C. *Low angular frequencies and commutation.* While the current constructed in Section 4B required a decomposition into spherical harmonics, we will now altogether avoid a recourse to the Fourier expansion on the sphere. The key to the positivity property was Poincaré's inequality, which states in more generality:

Lemma 4.16 (Poincaré inequality). *Let (S,γ) be a compact Riemannian manifold, and $\phi \in \mathrm{H}^1(S)$ a function on S with mean value*

$$\bar{\phi} = \frac{1}{\int_S \mathrm{d}\mu_\gamma}\int_S \phi\,\mathrm{d}\mu_\gamma.$$

Then

$$\int_S (\phi-\bar{\phi})^2\,\mathrm{d}\mu_\gamma \leq \frac{1}{\lambda_1(S)}\int_S |\nabla\!\!\!/\phi|^2\,\mathrm{d}\mu_\gamma,$$

where $\lambda_1(S)$ is the first nonzero eigenvalue of the negative Laplacian, $-\Delta\!\!\!/ = -\nabla\!\!\!/^a\nabla\!\!\!/_a$, on S ($\nabla\!\!\!/$ denotes covariant differentiation on S).

Now let $(S,\gamma) = (\mathbb{S}^{n-1}, \mathring{\gamma}_{n-1})$. Then we read off from (4-27) here

$$\lambda_1(\mathbb{S}^{n-1}) = n-1. \tag{4-46}$$

Choose a basis of the Lie algebra of SO(n),

$$\Omega_i : i = 1, \dots, \frac{n(n-1)}{2}, \tag{4-47}$$

and apply Lemma 4.16 to the functions $\Omega_i \phi$ of vanishing mean:

$$\int_{\mathbb{S}^{n-1}} \Omega_i \phi \, \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} = 0. \tag{4-48}$$

Then we obtain

$$\int_{\mathbb{S}^{n-1}} |\nabla\!\!\!/ \Omega_i \phi|^2 \, \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \geq (n-1) \int_{\mathbb{S}^{n-1}} (\Omega_i \phi)^2 \, \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \tag{4-49}$$

or, on $(S, \gamma) = (S_r, \gamma_r) = (\mathbb{S}^{n-1}, r^2 \mathring{\gamma}_{n-1})$,

$$\int_{S_r} |\nabla\!\!\!/ \Omega_i \phi|^2 \, \mathrm{d}\mu_{\gamma_r} \geq \frac{n-1}{r^2} \int_{S_r} (\Omega_i \phi)^2 \, \mathrm{d}\mu_{\gamma_r}. \tag{4-50}$$

Also note

$$\sum_{i=1}^{\frac{n(n-1)}{2}} (\Omega_i \phi)^2 = r^2 \big|\nabla\!\!\!/ \phi\big|^2_{r^2 \mathring{\gamma}_{n-1}}. \tag{4-51}$$

Second modified current. Recall we are considering vector fields of the form

$$X = f(r^*) \frac{\partial}{\partial r^*}.$$

Define

$$J^{X,2}_\mu = J^{X,1}_\mu + \frac{f'}{f(1 - \frac{2m}{r^{n-2}})} \beta \, X_\mu \, \phi^2, \tag{4-52}$$

where $\beta = \beta(r^*)$ is a function to be chosen below. Then

$$\begin{aligned}
K^{X,2} &= K^{X,1} + \nabla^\mu \left(\frac{f'}{f(1 - \frac{2m}{r^{n-2}})} \beta \, X_\mu \, \phi^2 \right) \\
&= \frac{f'}{1 - \frac{2m}{r^{n-2}}} \left(\frac{\partial \phi}{\partial r^*} + \beta \phi \right)^2 + \frac{f}{r} \left(1 - \frac{nm}{r^{n-2}} \right) |\nabla\!\!\!/ \phi|^2_{r^2 \mathring{\gamma}_{n-1}} \\
&\qquad - \frac{1}{4} \frac{f'''}{1 - \frac{2m}{r^{n-2}}} \phi^2 + \frac{f''}{1 - \frac{2m}{r^{n-2}}} \left[\beta - \frac{n-1}{2r} \left(1 - \frac{2m}{r^{n-2}} \right) \right] \phi^2 \\
&\qquad - \frac{f'}{1 - \frac{2m}{r^{n-2}}} \left[\beta^2 - \beta' - \frac{n-1}{r} \beta \left(1 - \frac{2m}{r^{n-2}} \right) + \frac{n-1}{4r^2} \left((n-3) + (n-1) \frac{2m}{r^{n-2}} \right) \left(1 - \frac{2m}{r^{n-2}} \right) \right] \phi^2 \\
&\qquad - \frac{n-1}{4} \left[(n-1)^2 \left(\frac{2m}{r^{n-2}} \right) - n \frac{2m}{r^{n-2}} - (n-3) \right] \frac{f}{r^3} \phi^2.
\end{aligned} \tag{4-53}$$

Now choose

$$\beta = \frac{n-1}{2r} \left(1 - \frac{2m}{r^{n-2}} \right) + \delta. \tag{4-54}$$

Then

$$\beta^2-\beta'-\frac{n-1}{r}\beta\Big(1-\frac{2m}{r^{n-2}}\Big)+\frac{n-1}{4r^2}\Big((n-3)+(n-1)\frac{2m}{r^{n-2}}\Big)\Big(1-\frac{2m}{r^{n-2}}\Big)=-\delta'+\delta^2 \tag{4-55}$$

and

$$K^{X,2}=\frac{f'}{1-\frac{2m}{r^{n-2}}}\Big(\frac{\partial\phi}{\partial r^*}+\beta\phi\Big)^2+\frac{f}{r}\Big(1-\frac{nm}{r^{n-2}}\Big)|\nabla\!\!\!/\phi|^2_{r^2\mathring{\gamma}_{n-1}}-\frac{1}{1-\frac{2m}{r^{n-2}}}\{\tfrac14 f'''-\delta f''+(\delta^2-\delta')f'\}\phi^2$$
$$-\frac{n-1}{4}\bigg[(n-1)^2\Big(\frac{2m}{r^{n-2}}\Big)^2-n\frac{2m}{r^{n-2}}-(n-3)\bigg]\frac{f}{r^3}\phi^2. \tag{4-56}$$

Note. Suppose, outside a compact interval $[-\alpha,\alpha]\subset\mathbb{R}$, f' is of the form $f'(r^*)=\frac{1}{r^{*2}}$ $(|r^*|>\alpha)$. Then we could choose $\delta=-\frac{1}{r^*}$ $(|r^*|>\alpha)$ so that $\delta f''=\frac{2}{r^{*4}}\geq 0$ and $-\delta'+\delta^2=0$.

Definition of the current $J^{(\alpha)}$. Let $\alpha>0$ and introduce a shifted coordinate

$$x=r^*-\alpha-\sqrt{\alpha}. \tag{4-57}$$

The modification we choose is

$$\delta=-\frac{x}{\alpha^2+x^2} \tag{4-58}$$

so that

$$-\delta'+\delta^2=\frac{\alpha^2}{(\alpha^2+x^2)^2}. \tag{4-59}$$

Let

$$f^a=-\frac{C}{\alpha^2 r^{n-1}}\quad (C>0) \tag{4-60}$$

and

$$(f^b)'=\frac{1}{\alpha^2+x^2},\quad (f^b)(r^*)=\int_0^{r^*}\frac{1}{\alpha^2+x(t^*)^2}\,\mathrm{d}t^*. \tag{4-61}$$

Note that then

$$(f^a)'+(n-1)\frac{f^a}{r}\Big(1-\frac{2m}{r^{n-2}}\Big)=0 \tag{4-62}$$

and

$$\tfrac14 (f^b)'''-\delta(f^b)''+(\delta^2-\delta')(f^b)'=-\frac12\frac{x^2-\alpha^2}{(x^2+\alpha^2)^3}. \tag{4-63}$$

Our current is built from the multiplier vector fields

$$X^a=f^a\frac{\partial}{\partial r^*},\quad X^b=f^b\frac{\partial}{\partial r^*} \tag{4-64}$$

by setting

$$J^{(\alpha)}_\mu(\phi)\doteq J^{X^a,0}_\mu(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^{X^b,2}_\mu(\Omega_i\phi) \tag{4-65}$$

and will be shown to have the property that its divergence

$$K^{(\alpha)} \doteq \nabla^{\mu} J^{(\alpha)}_{\mu} \tag{4-66}$$

is nonnegative upon integration over the spheres.

Proposition 4.17 (positivity of the current $J^{(\alpha)}$). *For $n \geq 3$ and $\phi \in \mathrm{H}^1(S)$,*

$$\int_S K^{(\alpha)} \, \mathrm{d}\mu_{\gamma} \geq 0,$$

provided α is chosen sufficiently large, and $C(n, m, \alpha)$ set to be $(*)$ below.

Proof. In view of (4-62) and (4-63),

$$K^{(\alpha)} \geq \frac{(f^a)'}{1-\frac{2m}{r^{n-2}}}\left(\frac{\partial\phi}{\partial r^*}\right)^2 + \frac{f^a}{r}\left(1-\frac{nm}{r^{n-2}}\right)|\nabla\!\!\!/\,\phi|^2_{r^2\mathring{\gamma}_{n-1}} + \sum_{i=1}^{\frac{n(n-1)}{2}} \frac{f^b}{r}\left(1-\frac{nm}{r^{n-2}}\right)|\nabla\!\!\!/\,\Omega_i\phi|^2_{r^2\mathring{\gamma}_{n-1}}$$
$$+ \sum_{i=1}^{\frac{n(n-1)}{2}} F(\Omega_i\phi)^2 + \sum_{i=1}^{\frac{n(n-1)}{2}} \frac{n-1}{4r^3}\left[(n-3) + n\frac{2m}{r^{n-2}} - (n-1)^2\left(\frac{2m}{r^{n-2}}\right)^2\right] f^b(\Omega_i\phi)^2, \tag{4-67}$$

where

$$F \doteq \frac{1}{2}\frac{1}{1-\frac{2m}{r^{n-2}}}\frac{x^2-\alpha^2}{(x^2+\alpha^2)^3}. \tag{4-68}$$

So, by Poincaré's inequality (4-50) and (4-51),

$$\int_S K^{(\alpha)} \, \mathrm{d}\mu_{\gamma} \geq \int_S \left\{ \frac{C(n-1)}{\alpha^2 r^n}\left(\frac{\partial\phi}{\partial r^*}\right)^2 + \left[(n-1)\frac{f^b}{r}\left(1-\frac{nm}{r^{n-2}}\right) + Fr^2 + \frac{1}{r}H\right]|\nabla\!\!\!/\,\phi|^2_{r^2\mathring{\gamma}_{n-1}} \right\} \mathrm{d}\mu_{\gamma}, \tag{4-69}$$

where

$$H \doteq \frac{n-1}{4}\left[(n-3) + n\frac{2m}{r^{n-2}} - (n-1)^2\left(\frac{2m}{r^{n-2}}\right)^2\right] f^b - \frac{C}{\alpha^2 r^{n-1}}\left(1-\frac{nm}{r^{n-2}}\right). \tag{4-70}$$

Step 1: $H \geq 0$. It is equivalent to show that

$$\breve{H}(r) \doteq r^{n-1} H(r) \frac{r^{n-2}}{2m}$$

is nonnegative. We consider $\breve{H}$ to be a function of

$$\rho \doteq \frac{r^{n-2}}{2m},$$

so

$$\breve{H} = \frac{n-1}{4}(2mr)\left[(n-3)\rho^2 + n\rho - (n-1)^2\right] f^b - \frac{C}{\alpha^2}\left(\rho - \frac{n}{2}\right).$$

Note that

$$r = \sqrt[n-2]{nm} \iff \rho = \frac{n}{2} \iff r^* = 0$$

and

$$\breve{H}\left(\frac{n}{2}\right) = 0.$$

Moreover we choose the constant C such that

$$\left.\frac{\mathrm{d}\breve{H}}{\mathrm{d}\rho}\right|_{\rho=\frac{n}{2}} = 0.$$

Then

$$\frac{\mathrm{d}\breve{H}}{\mathrm{d}\rho} = \frac{n-1}{4}(2mr)\Big[\frac{(n-3)(2n-3)}{n-2}\rho + \frac{n-1}{n-2}n - \frac{(n-1)^2}{n-2}\frac{1}{\rho}\Big]f^b. \\ + \frac{n-1}{4(n-2)}\frac{2mr^2}{\rho-1}\big[(n-3)\rho^2 + n\rho - (n-1)^2\big](f^b)' - \frac{C}{\alpha^2},$$

where we have used

$$\frac{\mathrm{d}r}{\mathrm{d}\rho} = \frac{r}{(n-2)\rho}, \quad \frac{\mathrm{d}r^*}{\mathrm{d}\rho} = \frac{1}{\rho-1}\frac{r}{n-2}.$$

Hence we choose

$$C = \frac{(n-1)^2}{4(n-2)}\frac{(\frac{n}{2})^2-(n-1)}{\frac{n}{2}-1}\,2m\,(nm)^{\frac{2}{n-2}}\,\frac{\alpha^2}{\alpha^2+(\alpha+\sqrt{\alpha})^2}. \tag{$*$}$$

Note that then also

$$\left.\frac{\mathrm{d}H}{\mathrm{d}r}\right|_{r=\sqrt[n-2]{nm}} = 0.$$

Now returning to the expression for $\breve{H}$, let us denote by $1 \leq \rho_0 \leq \frac{n}{2}$ the value of ρ for which

$$(n-3)\rho_0 + n - (n-1)^2\frac{1}{\rho_0} = 0;$$

i.e.,

$$\rho_0 = \frac{2(n-1)^2}{n+\sqrt{n^2+4(n-1)^2(n-3)}}.$$

We divide into the four regions

$$1 < \rho_0 < \frac{n}{2} < \rho^* < \infty,$$

where ρ^* is to be chosen large enough below.

Step 1a (near the horizon, $1 \leq \rho \leq \rho_0$). Clearly $\breve{H} \geq 0$ termwise, because $f^b \leq 0$.

Step 1b (near the photon sphere, $\rho_0 \leq \rho \leq \frac{n}{2}$). We show $H = H(r)$ is convex on $r_0 \leq r \leq \sqrt[n-2]{nm}$, where

$$r_0 = \sqrt[n-2]{\frac{4(n-1)^2m}{n+\sqrt{n^2+4(n-1)^2(n-3)}}}.$$

Differentiating twice yields

$$\begin{aligned}\frac{\mathrm{d}^2 H}{\mathrm{d}r^2} &= \frac{n-1}{4}\frac{1}{\left(1-\frac{2m}{r^{n-2}}\right)^2}(f^b)''\left[(n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\left(\frac{2m}{r^{n-2}}\right)^2\right]\\ &\quad+\frac{n-1}{2}\frac{1}{1-\frac{2m}{r^{n-2}}}(f^b)'(n-2)\left[2(n-1)^2\frac{2m}{r^{n-2}}-n\right]\frac{2m}{r^{n-1}}\\ &\quad-\frac{n-1}{4}\frac{1}{\left(1-\frac{2m}{r^{n-2}}\right)^2}(f^b)'(n-2)\left[(n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\left(\frac{2m}{r^{n-2}}\right)^2\right]\frac{2m}{r^{n-1}}\\ &\quad+\frac{n-1}{4}(f^b)\left[(n-2)(n-1)n\frac{2m}{r^n}-2(2n-3)(n-2)(n-1)^2\left(\frac{2m}{r^{n-1}}\right)^2\right]\\ &\quad-\frac{(n-1)nC}{\alpha^{n-1}r^{n+1}}\left(1-\frac{nm}{r^{n-2}}\right)+3\frac{(n-1)(n-2)C}{\alpha^{n-1}r^n}\frac{nm}{r^{n-1}}.\end{aligned}$$

Since $(f^b)''\geq 0$, we further have in this region the bound

$$\begin{aligned}\frac{\mathrm{d}^2 H}{\mathrm{d}r^2} &\geq \frac{n-1}{2}\frac{1}{1-\frac{2m}{r^{n-2}}}\times\left[2(n-1)^2\frac{2m}{r^{n-2}}-n-\frac{1}{2}\frac{1}{1-\frac{2m}{r_0^{n-2}}}\left((n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\left(\frac{2m}{r^{n-2}}\right)^2\right)\right]\\ &\qquad\times\frac{2m}{r^{n-1}}(n-2)(f^b)'+\frac{n-1}{4}\frac{2m}{r^{n-2}}\left[1-\frac{2(2n-3)(n-1)}{n}\left(\frac{2m}{r^{n-2}}\right)\right]\frac{(f^b)}{r^2}.\end{aligned}$$

Since, for $n\geq 3$,

$$2(n-1)^2\frac{2}{n}-n-\frac{1}{2}\frac{2(n-1)^2}{2(n-1)^2-n-\sqrt{n^2+4(n-1)^2(n-3)}}\left((n-3)+2-\left(2\frac{n-1}{n}\right)^2\right)\geq 1,$$
$$1-\frac{2(2n-3)(n-1)}{n}\frac{2}{n}\leq -1,$$

we finally obtain in this region

$$\frac{\mathrm{d}^2 H}{\mathrm{d}r^2}\geq\frac{(n-1)(n-2)}{2r}\frac{1}{\rho-1}(f^b)'>0.$$

Step 1c (in the intermediate region, $\frac{n}{2}\leq\rho\leq\rho^*$). We show $\breve{H}=\breve{H}(\rho)$ is convex on $\frac{n}{2}\leq\rho\leq\rho^*$ for $r^*(\rho=\rho^*)\leq\alpha$. We have

$$\begin{aligned}\frac{\mathrm{d}^2\breve{H}}{\mathrm{d}\rho^2} &= \frac{(n-1)^2}{4(n-2)^2}\frac{2mr}{\rho^2}\left[(n-3)(2n-3)\rho^2+n\rho+(n-3)(n-1)\right](f^b)\\ &\quad+\frac{(n-1)^2}{4(n-2)^2}\frac{2mr^2}{(\rho-1)^2}\left[3(n-3)\rho^2-3(n-5)\rho+(n-1)(n-5)-n\frac{2n-1}{n-1}+3(n-1)\frac{1}{\rho}\right](f^b)'\\ &\quad+\frac{n-1}{4(n-2)^2}\frac{2mr^3}{(\rho-1)^2}\left[(n-3)\rho^2+n\rho-(n-1)^2\right](f^b)''.\end{aligned}$$

Since, for $\rho \geq \frac{n}{2}$ and $n \geq 3$,

$$3(n-3)\rho(\rho-1)+6\rho+(n-1)(n-5)-n\frac{2n-1}{n-1}+3(n-1)\frac{1}{\rho} \geq 1 \quad \text{and} \quad (n-3)\rho^2+n\rho-(n-1)^2 \geq 0,$$

we have

$$\frac{\mathrm{d}^2\breve{H}}{\mathrm{d}\rho^2} \geq \frac{(n-1)^2}{4(n-2)^2}\frac{2mr^2}{(\rho-1)^2}(f^b)' > 0,$$

because $(f^b) \geq 0$ for $r^* \geq 0$, and $(f^b)'' \geq 0$ for $x \leq 0$.

Step 1d (in the asymptotics, $\rho \geq \rho^*$). We show directly $H(r) > 0$ for $r^* \geq R^* \doteq r^*(\rho = \rho^*)$ and ρ^* chosen large enough. Let $r^* \geq R^*$, $R^* \leq \alpha$. Then

$$f^b \geq \int_0^{R^*} (f^b)' \,\mathrm{d}r^* = \frac{1}{\alpha}\int_{-\left(1+\frac{1}{\sqrt{\alpha}}\right)}^{\frac{R^*-\alpha-\sqrt{\alpha}}{\alpha}} \frac{1}{1+t^{*2}}\,\mathrm{d}t^* \geq \frac{R^*}{5\alpha^2} \tag{4-71}$$

provided $\alpha \geq 1$, and of course

$$f^b \leq \frac{1}{\alpha}\arctan t^*\Big|_{-\left(1+\frac{1}{\sqrt{\alpha}}\right)}^{0} \leq \frac{\pi}{2\alpha}.$$

Thus

$$\begin{aligned} H &= \frac{(n-1)(n-3)}{4} + \left[\frac{(n-1)n}{4}f^b - \frac{C}{\alpha^2 2m}\frac{1}{r}\right]\frac{2m}{r^{n-2}} - \left[\frac{(n-1)^3}{4}f^b - \frac{Cn}{\alpha^2 4m}\frac{1}{r}\right]\left(\frac{2m}{r^{n-2}}\right)^2 \\ &\geq \frac{1}{\alpha^2}\left[\frac{(n-1)n}{4}\frac{R^*}{5} - \frac{C}{2m}\frac{1}{r}\right]\frac{2m}{r^{n-2}} - \frac{(n-1)^3}{4}\frac{\pi}{2\alpha}\left(\frac{2m}{r^{n-2}}\right)^2 > 0 \end{aligned}$$

for R^* (and consequently α) chosen large enough.

Step 2: (4-72). Since $\left(1-\frac{nm}{r^{n-2}}\right)f^b \geq 0$ and $F \geq 0$ for $|x| \geq \alpha$, we need to show

$$(n-1)(f^b)\left(1-\frac{nm}{r^{n-2}}\right) + F\,r^3 \geq 0 \tag{4-72}$$

for

$$-\alpha \leq x \leq \alpha \iff \sqrt{\alpha} \leq r^* \leq \sqrt{\alpha}+2\alpha.$$

In this whole region, in view of Proposition B.1,

$$\lim_{\alpha\to\infty}\frac{r^*}{r} = 1, \quad \lim_{\alpha\to\infty}\left(1-\frac{2m}{r^{n-2}}\right) = \lim_{\alpha\to\infty}\left(1-\frac{nm}{r^{n-2}}\right) = 1.$$

$\boxed{n \geq 4:}$ Since

$$f^b(r^*) \geq \int_{\sqrt{\alpha}}^{r^*} \frac{1}{\alpha^2+x^2}\,\mathrm{d}r^* \geq \frac{x+\alpha}{2\alpha^2}, \tag{4-73}$$

it suffices to show

$$(n-1)\frac{x+\alpha}{2\alpha^2} + \frac{1}{2}\frac{x^2-\alpha^2}{(x^2+\alpha^2)^3}r^3 \geq 0, \tag{4-74}$$

which is implied by

$$\frac{\alpha - x}{n-1}\frac{(x+\alpha+\sqrt{\alpha})^3}{(x^2+\alpha^2)^2} \le 1. \tag{4-75}$$

For $-\alpha \le x \le 0$,

$$(x+\alpha+\sqrt{\alpha})^3 \le \alpha^3\left(1+\frac{1}{\sqrt{\alpha}}\right)^3 \le \frac{4}{3}\alpha^3$$

for α large enough; thus

$$\frac{\alpha - x}{n-1}\frac{(x+\alpha+\sqrt{\alpha})^3}{(x^2+\alpha^2)^2} \le \frac{1}{n-1}\frac{2\alpha}{\alpha^4}\frac{4}{3}\alpha^3 \le \frac{8}{9}. \tag{4-76}$$

For $0 \le x \le \alpha$, we have to show

$$\frac{\alpha}{n-1}\frac{(x+\alpha+\sqrt{\alpha})^3}{(x^2+\alpha^2)^2} \le 1.$$

Since

$$(x+\alpha+\sqrt{\alpha})^3 \le 2^{\frac{3}{2}}\left(1+\frac{1}{\sqrt{\alpha}}\right)^3(x^2+\alpha^2)^{\frac{3}{2}},$$

we have, for α large enough,

$$\frac{\alpha}{n-1}\frac{(x+\alpha+\sqrt{\alpha})^3}{(x^2+\alpha^2)^2} \le \frac{\alpha}{n-1}\frac{2^{\frac{3}{2}}(1+\frac{1}{\sqrt{\alpha}})^3}{(x^2+\alpha^2)^{\frac{1}{2}}} \le \frac{2^{\frac{3}{2}}}{3}\left(1+\frac{1}{\sqrt{\alpha}}\right)^3 < 1. \tag{4-77}$$

$\boxed{n=3:}$ We see that (4-76) and (4-77) fail in the case $n=3$, as a consequence of which also (4-75) fails to hold. In the case $n=3$, we have to use a better approximation of (4-73); see [Dafermos and Rodnianski 2007] for details. Note also that in view of (4-75), the positivity property (4-72) is "easily" satisfied for large values of n, which indicates that there may be yet another simplified proof in higher dimensions. □

Given the *strict* inequalities proven in Step 2 of the proof of Proposition 4.17, for α chosen large enough, we can keep a fraction of the manifestly nonnegative $|\nabla\!\!\!/\, \Omega_i\phi|^2$ term in (4-67). Furthermore we have obtained control on the $|\nabla\!\!\!/\, \phi|^2$ term from (4-69).

Corollary 4.18. *Let $\phi \in \mathrm{H}^2(S)$ be a solution of the wave equation (1-1). Then there exists a constant $C(n,m)$ and a current K such that*

$$\int_S\left\{\frac{1}{r^n}\left(\frac{\partial\phi}{\partial r^*}\right)^2+\frac{1}{r^{n+1}}\left(\frac{\partial\phi}{\partial t}\right)^2+r\left(1-\frac{nm}{r^{n-2}}\right)^2|\nabla\!\!\!/^2\phi|^2_{r^2\mathring{\gamma}_{n-1}}+\frac{r^2}{(1-\frac{2m}{r^{n-2}})(1+r^{*2})^2}|\nabla\!\!\!/\,\phi|^2_{r^2\mathring{\gamma}_{n-1}}\right\}\mathrm{d}\mu_\gamma \le C(n,m)\int_S K\,\mathrm{d}\mu_\gamma. \tag{4-78}$$

Proof. Set $K = K^{(\alpha)} + K^{\text{aux}}$ and choose α large enough.

Here we retrieve the time derivatives with the auxiliary current

$$K^{\text{aux}} = \nabla^\mu J^{\text{aux}}_\mu; \quad J^{\text{aux}} = J^{X^{\text{aux}},0}; \quad X^{\text{aux}} = f^{\text{aux}}\frac{\partial}{\partial r^*},$$

where $f^{\text{aux}} = -\frac{1}{r^n}$ satisfies

$$(f^{\text{aux}})' + (n-1)\frac{f^{\text{aux}}}{r}\Big(1-\frac{2m}{r^{n-2}}\Big) = \frac{1}{r^{n+1}}\Big(1-\frac{2m}{r^{n-2}}\Big);$$

for, in view of (4-9),

$$\frac{1}{r^{n+1}}\Big(\frac{\partial\phi}{\partial t}\Big)^2 \leq 2\,K^{\text{aux}} + 3\,\frac{1}{r^{n+1}}\big|\nabla\!\!\!/\,\phi\big|^2_{r^2\mathring{\gamma}_{n-1}}. \qquad \square$$

4D. *Boundary terms.* In this section we first prove Proposition 4.1 and then a refinement thereof for finite regions, which requires us to estimate the boundary terms of the currents introduced in Sections 4B and 4C.

4D.1. *Proof of Proposition 4.1.* We can now combine our earlier results Corollary 4.14 and Corollary 4.18 to prove the *integrated local energy decay estimate* (4-4); note that there is no restriction on the spherical harmonic number, and that no commutation with angular momentum operators is required.

Proof of Proposition 4.1. Write

$$\phi = \pi_{<L}\phi + \pi_{\geq L}\phi \tag{4-79}$$

with

$$\pi_{<L} = \sum_{l=0}^{L-1}\pi_l\phi, \quad \pi_{\geq L} = \sum_{l=L}^{\infty}\pi_l\phi, \tag{4-80}$$

where $L = (6\gamma n)^2$ is fixed (recall here $\gamma = 12$ from Section 4B).

Step 1 (high spherical harmonics). By Corollary 4.14,

$$\int_{\mathcal{R}(t_0,t_1,u_1^*,v_1^*)} \frac{1}{4}\frac{1}{8}\frac{(2m)^{\frac{3}{n-2}}}{r^3}(\pi_{\geq L}\phi)^2 \leq \int_{\mathcal{R}(t_0,t_1,u_1^*,v_1^*)} K^{X_{\gamma,\alpha},1}(\pi_{\geq L}\phi). \tag{4-81}$$

It remains to estimate the boundary terms of the current $J^{X_{\gamma,\alpha},1}$, and to use this estimate to recover all derivatives using a commutation with the Killing vector field T.

Step 1a (boundary terms). We may assume $|r^*_{0,1}| \geq \frac{4}{\gamma\alpha}$, $r_{0,1}$ entering the definition (4-3). Recalling the properties of $f_{\gamma,\alpha}$ away from the photon sphere, we find

$$\begin{aligned}
&\Big|\Big(J^{X_{\gamma,\alpha},1}(\pi_{\geq L}\phi), \frac{\partial}{\partial v^*}\Big)\Big| \\
&\quad\leq \Big|\Big(J^{X_{\gamma,\alpha}}(\pi_{\geq L}\phi), \frac{\partial}{\partial v^*}\Big)\Big| + \frac{1}{2}\Big|f_{\gamma,\alpha}{}' + (n-1)\frac{f_{\gamma,\alpha}}{r}\Big(1-\frac{2m}{r^{n-2}}\Big)\Big|\big(\pi_{\geq L}\phi\big)\Big(\frac{\partial\pi_{\geq L}}{\partial v^*}\Big) \\
&\qquad + \frac{1}{4}\Big|\Big(f_{\gamma,\alpha}{}' + (n-1)\frac{f_{\gamma,\alpha}}{r}\Big(1-\frac{2m}{r^{n-2}}\Big)\Big)'\Big|\big(\pi_{\geq L}\phi\big)^2 \\
&\quad\leq \frac{n+1}{(\gamma\alpha)^3}\Big(J^T(\pi_{\geq L}\phi), \frac{\partial}{\partial v^*}\Big) + \frac{1}{(\gamma\alpha)^6}\frac{1}{|r^*|^4}\Big(\frac{\partial\pi_{\geq L}\phi}{\partial v^*}\Big)^2 + \frac{1}{(\gamma\alpha)^6}\frac{1}{|r^*|^4}\Big[1+\frac{1}{|r^*|}\Big]\big(\pi_{\geq L}\phi\big)^2 \\
&\qquad + \frac{n-1}{(\gamma\alpha)^3}\Big[n + \frac{4}{(\gamma\alpha)^6}\frac{r}{|r^*|^4}\Big]\frac{1}{2r^2}\Big(1-\frac{2m}{r^{n-2}}\Big)\big(\pi_{\geq L}\phi\big)^2,
\end{aligned}$$

and, by Lemma 4.6,

$$\int_{S_r} \frac{1}{2}\frac{1}{r^2}\Big(1-\frac{2m}{r^{n-2}}\Big)(\pi_{\geq L}\phi)^2 \leq \frac{1}{(6\gamma n)^4}\int_{S_r}\Big(J^T, \frac{\partial}{\partial v^*}\Big);$$

similarly for

$$\Big|\Big(J^{X_{\gamma,\alpha},1}(\pi_{\geq L}\phi), \frac{\partial}{\partial u^*}\Big)\Big|.$$

Since also, by Lemma C.7 and Lemma 4.6,

$$\begin{aligned}
&\int_{\frac{1}{2}(t_0-r_0^*)}^{u_1^*} \mathrm{d}u^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{n-1}\frac{1}{r^{*4}}(\pi_{\geq L}\phi)^2|_{v^*=\frac{1}{2}(t_0+r_0^*)} \\
&\quad\leq \frac{8}{|r_0^*|^4}\frac{(1+|r_0^*|^2)^2}{|r_0^*|^2}\int_{\frac{1}{2}(t_0-r_0^*)}^{\infty}\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{n-1}\Big(\frac{\partial\pi_{\geq L}\phi}{\partial u^*}\Big)^2 \mathrm{d}u^* \\
&\qquad + 2\pi\frac{1+|r_0^*|^2}{|r_0^*|^4}\Big[1+\frac{(nm)^{\frac{2}{n-2}}}{(6\gamma n)^4}\Big]\int_{\frac{1}{2}(t_0-r_0^*)}^{\frac{1}{2}(t_0-r_0^*)+1}\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{n-1}\Big\{\big|\nabla\!\!\!/\,\pi_{\geq L}\phi\big|^2+\Big(\frac{\partial\pi_{\geq L}\phi}{\partial u^*}\Big)^2\Big\}\, \mathrm{d}u^*,
\end{aligned}$$

there is a constant $C(n,m)$ (recall $\gamma = 12$, $\alpha = (n-1)/(nm)^{\frac{1}{n-2}}$) such that

$$\begin{aligned}
&\int_{\frac{1}{2}(t_0-r_0^*)}^{u_1^*} \mathrm{d}u^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-1}\Big|\Big(J^{X_{\gamma,\alpha},1}(\pi_{\geq L}\phi), \frac{\partial}{\partial u^*}\Big)\Big| \\
&\qquad\qquad \leq C(n,m)\int_{\frac{1}{2}(t_0-r_0^*)}^{\infty} \mathrm{d}u^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-1}\Big(J^T(\pi_{\geq L}\phi), \frac{\partial}{\partial u^*}\Big)\Big|_{v^*=\frac{1}{2}(t_0+r_0^*)}.
\end{aligned}$$

To establish

$$\int_{\mathbb{S}^{n-1}}\Big|\Big(J^{X_{\gamma,\alpha},1}(\pi_{\geq L}\phi), \frac{\partial}{\partial t}\Big)\Big| r^{n-1}\,\mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \leq C(n,m)\int_{\mathbb{S}^{n-1}}\Big(J^T(\pi_{\geq L}\phi), \frac{\partial}{\partial t}\Big) r^{n-1}\,\mathrm{d}\mu_{\mathring{\gamma}_{n-1}},$$

note that

$$\begin{aligned}
\Big|\Big(J^{X_{\gamma,\alpha},1}(\pi_{\geq L}\phi), \frac{\partial}{\partial t}\Big)\Big| &\leq \Big|\Big(J^{X_{\gamma,\alpha}}(\pi_{\geq L}\phi), \frac{\partial}{\partial t}\Big)\Big| + \frac{1}{2}\Big|f_{\gamma,\alpha}{}' + (n-1)\frac{f_{\gamma,\alpha}}{r}\Big(1-\frac{2m}{r^{n-2}}\Big)\Big|(\pi_{\geq L}\phi)\Big(\frac{\partial\pi_{\geq L}\phi}{\partial t}\Big) \\
&\leq |f_{\gamma,\alpha}|\Big|T(\pi_{\geq L}\phi)\Big(\frac{\partial}{\partial r^*}, \frac{\partial}{\partial t}\Big)\Big| \\
&\qquad + \frac{1}{2}\Big[\frac{1}{2}r^2|f_{\gamma,\alpha}{}'| + \frac{3}{2}\frac{n-1}{2}\frac{1}{(\gamma\alpha)^3}\Big(1-\frac{2m}{r^{n-2}}\Big)^2\Big]\frac{1}{r^2}(\pi_{\geq L}\phi)^2 \\
&\qquad + \frac{1}{2}\frac{1}{(\gamma\alpha)^2}\Big[1+\frac{3}{2}\frac{n-1}{2}\frac{1}{\gamma\alpha}\Big]\Big(\frac{\partial\pi_{\geq L}\phi}{\partial t}\Big)^2,
\end{aligned}$$

and, by Lemma 4.6,

$$\int_{S_r} \frac{1}{2}\frac{1}{r^2}(\pi_{\geq L}\phi)^2\,\mathrm{d}\mu_\gamma \leq \frac{1}{(6\gamma n)^4}\Big(1-\frac{2m}{r_0^{n-2}}\Big)^{-1}\int_{S_r}\Big(J^T(\pi_{\geq L}\phi), \frac{\partial}{\partial t}\Big)\,\mathrm{d}\mu_\gamma,$$

which suffices in view of the properties of $f_{\gamma,\alpha}$, in particular that there is a constant $r^2|f_{\gamma,\alpha}{}'| \leq C(n,m)$. For the boundary term

$$\int_{\mathbb{S}^{n-1}} \left| \left(J^{X_{\gamma,\alpha},1}(\pi_{\geq L}\phi), \frac{\partial}{\partial v^*} \right) \right|_{u^*=u_1^*} r^{n-1} \, \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}$$
$$\leq \int_{\mathbb{S}^{n-1}} \left\{ \frac{1}{(\gamma\alpha)^3} \left[n+1+\frac{1}{2}\frac{1}{(\gamma\alpha)^2} + \frac{n-1}{(6\gamma n)^4}\left(n + \frac{4(nm)^{\frac{1}{n-2}}}{(\gamma\alpha)^2} \right) \right] \left(J^T(\pi_{\geq L}\phi), \frac{\partial}{\partial v^*} \right) + \frac{1}{(\gamma\alpha)^6}\frac{1}{|r^*|^4}\left(1+\frac{\gamma\alpha}{4}\right)(\pi_{\geq L}\phi)^2 \right\} r^{n-1} \, \mathrm{d}\mu_{\mathring{\gamma}_{n-1}},$$

we find (using the boundedness of ϕ on the horizon; see Section 5A) in the limit $u_1^* \to \infty$ a constant $C(n,m)$ such that

$$\int_{\mathbb{S}^{n-1}} \left| \left(J^{X_{\gamma\alpha},1}(\pi_{\geq L}\phi), \frac{\partial}{\partial v^*} \right) \right| r^{n-1} \, \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \Big|_{u^*=\infty} \leq C(n,m) \int_{\mathbb{S}^{n-1}} \left(J^T(\pi_{\geq L}\phi), \frac{\partial}{\partial v^*} \right) r^{n-1} \, \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \Big|_{u^*=\infty}.$$

We conclude that there is a constant $C(n,m)$ such that

$$\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)} \frac{(2m)^{\frac{3}{n-2}}}{r^3} (\pi_{\geq L}\phi)^2 \leq C(n,m) \int_{\Sigma_{\tau_0}} \left(J^T(\pi_{\geq L}\phi), n \right), \tag{4-82}$$

where $\tau_0 = \frac{1}{2}(t_0 - r_1^*)$ because

$$\Box_g(\pi_{\geq L}\phi) = 0, \quad K^T(\pi_{\geq L}\phi) = 0. \tag{4-83}$$

Step 1b (commutation with T). Since

$$\Box_g\left(T \cdot \pi_{\geq L}\phi\right) = 0, \tag{4-84}$$

we also have

$$\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)} \frac{(2m)^{\frac{3}{n-2}}}{r^3} \left(\frac{\partial \pi_{\geq L}\phi}{\partial t} \right)^2 \leq C(n,m) \int_{\Sigma_{\tau_0}} \left(J^T(T \cdot \pi_{\geq L}\phi), n \right). \tag{4-85}$$

This is enough to control the remaining derivatives, too; for the auxiliary current (C-10) yields

$$K^{\mathrm{aux}} = \phi \, (\partial^\mu h)(\partial_\mu \phi) + h \, \partial^\alpha \phi \, \partial_\alpha \phi, \tag{4-86}$$

which, upon choosing

$$h = \left(1 - \frac{2m}{r^{n-2}} \right) \frac{(2m)^{\frac{3}{n-2}}}{r^3}, \tag{4-87}$$

presents us with

$$K^{\mathrm{aux}} = \phi \frac{\partial h}{\partial r} \frac{\partial \phi}{\partial r^*} - \frac{(2m)^{\frac{3}{n-2}}}{r^3} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{(2m)^{\frac{3}{n-2}}}{r^3} \left(\frac{\partial \phi}{\partial r^*} \right)^2 + \frac{(2m)^{\frac{3}{n-2}}}{r^3} \left(1 - \frac{2m}{r^{n-2}} \right) |\nabla\!\!\!/ \phi|^2_{r^2\mathring{\gamma}_{n-1}}. \tag{4-88}$$

Using Cauchy's inequality for the first term, namely

$$\begin{aligned}\phi\frac{\partial h}{\partial r}\frac{\partial\phi}{\partial r^*}&=(n-2)\phi\frac{2m}{r^{n-1}}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\frac{\partial\phi}{\partial r^*}-3\Big(1-\frac{2m}{r^{n-2}}\Big)\phi\frac{(2m)^{\frac{3}{n-2}}}{r^4}\frac{\partial\phi}{\partial r^*}\\&\geq-\frac{1}{2}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\Big(\frac{\partial\phi}{\partial r^*}\Big)^2-\Big(\frac{n-2}{r}\Big)^2\Big(\frac{2m}{r^{n-2}}\Big)^2\frac{(2m)^{\frac{3}{n-2}}}{r^3}\phi^2-\Big(\frac{3}{r}\Big)^2\Big(1-\frac{2m}{r^{n-2}}\Big)^2\frac{(2m)^{\frac{3}{n-2}}}{r^3}\phi^2\\&\geq-\frac{1}{2}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\Big(\frac{\partial\phi}{\partial r^*}\Big)^2-2\frac{n^2}{(2m)^{\frac{2}{n-2}}}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\phi^2,\end{aligned}\tag{4-89}$$

we obtain the bound

$$\begin{aligned}K^{\mathrm{aux}}\geq\frac{1}{2}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\Big(\frac{\partial\phi}{\partial r^*}\Big)^2+\frac{(2m)^{\frac{3}{n-2}}}{r^3}\Big(1-\frac{2m}{r^{n-2}}\Big)|\nabla\!\!\!/\phi|^2_{r^2\mathring{\gamma}_{n-1}}&\\-\frac{(2m)^{\frac{3}{n-2}}}{r^3}\Big(\frac{\partial\phi}{\partial t}\Big)^2-2\frac{n^2}{(2m)^{\frac{2}{n-2}}}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\phi^2.&\end{aligned}\tag{4-90}$$

Therefore

$$\begin{aligned}\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)}\Big\{\frac{1}{2}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\Big(\frac{\partial\pi_{\geq L}\phi}{\partial r^*}\Big)^2+\frac{(2m)^{\frac{3}{n-2}}}{r^3}\Big(1-\frac{2m}{r^{n-2}}\Big)|\nabla\!\!\!/\pi_{\geq L}\phi|^2_{r^2\mathring{\gamma}_{n-1}}\Big\}&\\\leq\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)}\Big\{K^{\mathrm{aux}}(\pi_{\geq L}\phi)+\frac{(2m)^{\frac{3}{n-2}}}{r^3}\Big(\frac{\partial\pi_{\geq L}\phi}{\partial t}\Big)^2+2\frac{n^2}{(2m)^{\frac{2}{n-2}}}\frac{(2m)^{\frac{3}{n-2}}}{r^3}(\pi_{\geq L}\phi)^2\Big\}.&\end{aligned}\tag{4-91}$$

The boundary terms are controlled using Proposition C.8:

$$\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)}K^{\mathrm{aux}}(\pi_{\geq L}\phi)\leq C(n,m)\int_{\Sigma_{\tau_0}}\big(J^T(\pi_{\geq L}\phi),n\big).\tag{4-92}$$

Hence

$$\begin{aligned}\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\Big\{\Big(\frac{\partial\pi_{\geq L}\phi}{\partial r^*}\Big)^2+\Big(1-\frac{2m}{r^{n-2}}\Big)|\nabla\!\!\!/\pi_{\geq L}\phi|^2_{r^2\mathring{\gamma}_{n-1}}\Big\}&\\\leq C(n,m)\int_{\Sigma_{\tau_0}}\big(J^T(\pi_{\geq L}\phi)+J^T(T\cdot\pi_{\geq L}\phi),n\big).&\end{aligned}\tag{4-93}$$

Step 2 (low spherical harmonics). Now recall the $J^{(\alpha)}$ current (4-65); we will show in a first step that

$$\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)}K^{(\alpha)}(\phi)\leq C(n,m)\int_{\Sigma_{\tau_0}}\Big(J^T(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^T(\Omega_i\phi),n\Big).\tag{4-94}$$

Then in particular, by Corollary 4.18,

$$\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)}\left\{\frac{1}{r^n}\left(\frac{\partial\pi_{<L}\phi}{\partial r^*}\right)^2+\frac{1}{r^{n+1}}\left(\frac{\partial\pi_{<L}\phi}{\partial t}\right)^2+\frac{r^2}{(1-\frac{2m}{r^{n-2}})(1+|r^*|^2)^2}\left|\nabla\!\!\!/\,\pi_{<L}\phi\right|^2_{r^2\mathring{\gamma}_{n-1}}\right\}$$
$$\le C(n,m)\int_{\Sigma_{\tau_0}}\left(J^T(\pi_{<L}\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^T(\Omega_i\cdot\pi_{<L}\phi),n\right). \tag{4-95}$$

But in a second step we will show that in fact there exists a constant $C(n)$ such that

$$\int_{\Sigma_{\tau_0}}\sum_{i=1}^{\frac{n(n-1)}{2}}\left(J^T(\Omega_i\cdot\pi_{<L}\phi),n\right)\le C(n)\int_{\Sigma_{\tau_0}}\left(J^T(\pi_{<L}\phi),n\right). \tag{4-96}$$

Step 2a (boundary terms). The energy identity for $J^{(\alpha)}$ on the domain (4-1) implies, more explicitly,

$$\begin{aligned}\int_{\mathcal{R}(t_0,t_1,u_1^*,v_1^*)}K^{(\alpha)}\le&\int_{\frac12(t_0+r_0^*)}^{\frac12(t_1+r_0^*)}\int_{\mathbb{S}^{n-1}}\left|\left(J^{(\alpha)},\frac{\partial}{\partial v^*}\right)\right|r^{n-1}|_{u^*=u_1^*}\,\mathrm{d}v^*\,\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\\
&+\int_{\frac12(t_1-r_0^*)}^{u_1^*}\int_{\mathbb{S}^{n-1}}\left|\left(J^{(\alpha)},\frac{\partial}{\partial u^*}\right)\right|r^{n-1}|_{v^*=\frac12(t_1+r_0^*)}\,\mathrm{d}u^*\,\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\\
&+\int_{r_0^*}^{r_1^*}\int_{\mathbb{S}^{n-1}}\left|(J^{(\alpha)},T)\right|r^{n-1}|_{t=t_1}\,\mathrm{d}r^*\,\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\\
&+\int_{\frac12(t_1+r_1^*)}^{v_1^*}\int_{\mathbb{S}^{n-1}}\left|\left(J^{(\alpha)},\frac{\partial}{\partial v^*}\right)\right|r^{n-1}|_{u^*=\frac12(t_1-r_1^*)}\,\mathrm{d}v^*\,\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\\
&+\int_{\frac12(t_0-r_1^*)}^{\frac12(t_1-r_1^*)}\int_{\mathbb{S}^{n-1}}\left|\left(J^{(\alpha)},\frac{\partial}{\partial u^*}\right)\right|r^{n-1}|_{v^*=v_1^*}\,\mathrm{d}v^*\,\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\\
&+\int_{\frac12(t_0-r_0^*)}^{u_1^*}\int_{\mathbb{S}^{n-1}}\left|\left(J^{(\alpha)},\frac{\partial}{\partial u^*}\right)\right|r^{n-1}|_{v^*=\frac12(t_0+r_0^*)}\,\mathrm{d}u^*\,\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\\
&+\int_{r_0^*}^{r_1^*}\int_{\mathbb{S}^{n-1}}\left|(J^{(\alpha)},T)\right|r^{n-1}|_{t=t_0}\,\mathrm{d}r^*\,\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\\
&+\int_{\frac12(t_0+r_1^*)}^{v_1^*}\int_{\mathbb{S}^{n-1}}\left|\left(J^{(\alpha)},\frac{\partial}{\partial v^*}\right)\right|r^{n-1}|_{u^*=\frac12(t_0-r_1^*)}\,\mathrm{d}v^*\,\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}.\end{aligned}$$

For the boundary integrals on the t-constant hypersurfaces, we will use (ii) of the following lemma.

Lemma 4.19 (boundary terms of $J^{(\alpha)}$ current on t-constant hypersurfaces). *On each* $\bar{\Sigma}_t$,

(i) *there exists a constant* $C(n,m,\alpha)$ *such that*

$$\int_{\mathbb{R}}\left|(J^{(\alpha)},T)\right|r^{n-1}\,\mathrm{d}r^*\le C(n,m,\alpha)\int_{\mathbb{R}}\left(J^T(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^T(\Omega_i\phi),T\right)r^{n-1}\,\mathrm{d}r^*;$$

(ii) *for* $r\ge r_0$ *there exists a constant* $C(n,m,\alpha,r_0)$ *such that*

$$|(J^{(\alpha)},T)| \le C(n,m,\alpha,r_0)\biggl(J^T(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i\phi),T\biggr).$$

Proof. Using the definition (4-65),

$$(J^{(\alpha)},T) = f^a\Bigl(\frac{\partial\phi}{\partial t}\Bigr)\Bigl(\frac{\partial\phi}{\partial r^*}\Bigr)+\sum_{i=1}^{\frac{n(n-1)}{2}} f^b\Bigl(\frac{\partial\Omega_i\phi}{\partial t}\Bigr)\Bigl(\frac{\partial\Omega_i\phi}{\partial r^*}\Bigr)+\frac14\sum_{i=1}^{\frac{n(n-1)}{2}}\Bigl((f^b)'+(n-1)\frac{f^b}{r}\Bigl(1-\frac{2m}{r^{n-2}}\Bigr)\Bigr)2(\Omega_i\phi)(\partial_t\Omega_i\phi)$$

because

$$\partial_t\Bigl((f^b)'+(n-1)\frac{f^b}{r}\Bigl(1-\frac{2m}{r^{n-2}}\Bigr)\Bigr)=0$$

and $g(T,\frac{\partial}{\partial r^*})=0$. By Cauchy's inequality,

$$\bigl|(J^{(\alpha)},T)\bigr| \le \frac{C}{\alpha^2 r^{n-1}}\Bigl[\frac12\Bigl(\frac{\partial\phi}{\partial t}\Bigr)^2+\frac12\Bigl(\frac{\partial\phi}{\partial r^*}\Bigr)^2\Bigr]+\sum_{i=1}^{\frac{n(n-1)}{2}}\frac{\pi}{\alpha}\Bigl[\frac12\Bigl(\frac{\partial\Omega_i\phi}{\partial t}\Bigr)^2+\frac12\Bigl(\frac{\partial\Omega_i\phi}{\partial r^*}\Bigr)^2\Bigr]$$
$$+\frac14\sum_{i=1}^{\frac{n(n-1)}{2}}\Bigl(\frac{r}{\alpha^2+x^2}+(n-1)\frac{\pi}{\alpha}\Bigl(1-\frac{2m}{r^{n-2}}\Bigr)\Bigr)\Bigl[\frac{1}{r^2}(\Omega_i\phi)^2+\Bigl(\frac{\partial\Omega_i\phi}{\partial t}\Bigr)^2\Bigr],$$

which proves (ii) in view of

$$\bigl(J^T(\phi),T\bigr)=\frac12\Bigl(\frac{\partial\phi}{\partial t}\Bigr)^2+\frac12\Bigl(\frac{\partial\phi}{\partial r^*}\Bigr)^2+\frac12\Bigl(1-\frac{2m}{r^{n-2}}\Bigr)|\nabla\!\!\!/\,\phi|^2;$$

here we have also used

$$f^b=\int_0^{r^*}\frac{1}{\alpha^2+(t^*-\alpha-\sqrt{\alpha})^2}\,\mathrm{d}t^*=\frac{1}{\alpha}\arctan x\Big|_{\frac{-\alpha-\sqrt{\alpha}}{\alpha}}^{\frac{r^*-\alpha-\sqrt{\alpha}}{\alpha}}\le\frac{\pi}{\alpha}\quad(r^*\ge 0).$$

To establish (i) it is enough to infer

$$\int_{-\infty}^{\infty}\frac{r}{\alpha^2+x^2}|\nabla\!\!\!/\,\phi|^2\,r^{n-1}\,\mathrm{d}r^*=\sum_{i=1}^{\frac{n(n-1)}{2}}\int_{-\infty}^{\infty}\frac{r^{n-2}}{\alpha^2+x^2}(\Omega_i\phi)^2\,\mathrm{d}r^*\le C\sum_{i=1}^{\frac{n(n-1)}{2}}\int_{-\infty}^{\infty}\Bigl(\frac{\partial\Omega_i\phi}{\partial r^*}\Bigr)^2 r^{n-1}\,\mathrm{d}r^*$$
$$\le C\sum_{i=1}^{\frac{n(n-1)}{2}}\int_{-\infty}^{\infty}\bigl(J^T(\Omega_i\phi),T\bigr)\,r^{n-1}\,\mathrm{d}r^*;$$

this is a standard Hardy inequality; cf. proof of Proposition 10.2 in [Dafermos and Rodnianski 2009b]. □

The following lemma will be applied to the boundary terms of the $J^{(\alpha)}$-current on the null hypersurfaces in the region $r\le r_0$.

Lemma 4.20 (boundary terms of the $J^{(\alpha)}$ current on null hypersurfaces). (i) *On any segment of the outgoing null hypersurface $u^* = u_1^* \geq 0$,*

$$\left|\left(J^{(\alpha)}, \frac{\partial}{\partial v^*}\right)\right| \leq C(n,m,\alpha)\left(J^T(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i\phi), \frac{\partial}{\partial v^*}\right) + \epsilon(u_1^*)\left(J^N(\phi), \frac{\partial}{\partial v^*}\right),$$

where $C(n,m,\alpha)$ is a constant, and $\epsilon(u_1^*) \to 0$ as $u_1^* \to \infty$.

(ii) *Let $v_0^* \geq 1$, and $u_0^*(v^*)$ such that $r(u_0^*(v^*), v^*) = r_0$ (in particular $u_0^*(v_0^*) \geq 1$). Then, on the ingoing null hypersurface $v^* = v_0^*$,*

$$\int_{u_0^*}^{\infty}\left|\left(J^{(\alpha)}, \frac{\partial}{\partial u^*}\right)\right| r^{n-1}\,\mathrm{d}u^* \leq C(n,m,\alpha)\int_{u_0^*}^{\infty}\left(J^T(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i\phi), \frac{\partial}{\partial u^*}\right) r^{n-1}\,\mathrm{d}u^*.$$

Proof. Using the definition (4-65) we find

$$\begin{aligned}\left(J^{(\alpha)}, \frac{\partial}{\partial u^*}\right) = f^a\, T(\phi)\left(\frac{\partial}{\partial r^*}, \frac{\partial}{\partial u^*}\right) + \sum_{i=1}^{\frac{n(n-1)}{2}}\Bigg\{ & f^b\, T(\Omega_i\phi)\left(\frac{\partial}{\partial r^*}, \frac{\partial}{\partial u^*}\right) \\ & + \frac{1}{4}\left((f^b)' + (n-1)\frac{f^b}{r}\left(1 - \frac{2m}{r^{n-2}}\right)\right) 2(\Omega_i\phi)\left(\frac{\partial\Omega_i\phi}{\partial u^*}\right) \\ & + \frac{1}{4}\frac{1}{2}\left((f^b)' + (n-1)\frac{f^b}{r}\left(1 - \frac{2m}{r^{n-2}}\right)\right)'(\Omega_i\phi)^2 - (f^b)'\,\beta\,(\Omega_i\phi)^2\Bigg\},\end{aligned}$$

and therefore

$$\begin{aligned}&\left|\left(J^{(\alpha)}, \frac{\partial}{\partial u^*}\right)\right| \\ &\leq \frac{C}{\alpha^2(2m)^{\frac{n-1}{n-2}}}\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial u^*}\right)^2 + \frac{1}{2}\left(1 - \frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/\phi|^2\right] + \sum_{i=1}^{\frac{n(n-1)}{2}}\frac{\pi}{\alpha}\left[\frac{1}{2}\left(\frac{\partial\Omega_i\phi}{\partial u^*}\right)^2 + \frac{1}{2}\left(1 - \frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/\Omega_i\phi|^2\right] \\ &\quad + \sum_{i=1}^{\frac{n(n-1)}{2}}\frac{1}{2}\left(\frac{r}{\alpha^2+x^2} + (n-1)\frac{\pi}{\alpha}\left(1 - \frac{2m}{r^{n-2}}\right)\right)\frac{1}{2}\left(\frac{\partial\Omega_i\phi}{\partial u^*}\right)^2 \\ &\quad + \left(\frac{n-1}{2}\frac{\pi}{\alpha} + \frac{n-1}{4}\frac{r}{\alpha^2+x^2} + \frac{n-1}{4}\frac{\pi}{\alpha}\left(1 - \frac{2m}{r^{n-2}}\right) + \frac{(n-1)(n-2)}{4}\frac{\pi}{\alpha} + (n-1)\frac{r}{\alpha^2+x^2}\right)\frac{1}{2}\left(1 - \frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/\phi|^2 \\ &\quad + \left(\frac{1}{4}\frac{r}{\alpha^2+x^2} + \frac{5}{4}\frac{|x|r^2}{(\alpha^2+x^2)^2}\right)|\nabla\!\!\!/\phi|^2.\end{aligned}$$

Similarly for $\left|\left(J^{(\alpha)}, \frac{\partial}{\partial v^*}\right)\right|$. Clearly, (ii) now follows from

$$\left(J^T, \frac{\partial}{\partial v^*}\right) = \frac{1}{2}\left(\frac{\partial\phi}{\partial v^*}\right)^2 + \frac{1}{2}\left(1 - \frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/\phi|^2$$

and

$$\left(J^N, \frac{\partial}{\partial v^*}\right) = \left[1 + \frac{\sigma}{4\kappa}\left(1 - \frac{2m}{r^{n-2}}\right)\right] T\left(\frac{2}{1 - \frac{2m}{r^{n-2}}}\frac{\partial}{\partial u^*} + \frac{\partial}{\partial t}, \frac{\partial}{\partial v^*}\right) \geq 2|\nabla\!\!\!/ \phi|^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial v^*}\right)^2.$$

In case (i) we only have

$$\left(J^T, \frac{\partial}{\partial u^*}\right) = \frac{1}{2}\left(\frac{\partial\phi}{\partial u^*}\right)^2 + \frac{1}{2}\left(1 - \frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/ \phi|^2, \quad \left(J^N, \frac{\partial}{\partial u^*}\right) \geq \frac{1}{2}\left(1 - \frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/ \phi|^2;$$

but, using the Hardy inequality of Lemma C.7,

$$\begin{aligned}
\int_{u_0^*}^{\infty} \frac{r}{\alpha^2 + x^2}|\nabla\!\!\!/ \phi|^2\, r^{n-1}\, du^* &\leq \sum_{i=1}^{\frac{n(n-1)}{2}} \int_{u_0^*}^{\infty} \frac{1 + u^{*2}}{\alpha^2 + (u^* + \alpha + \sqrt{\alpha} - v^*)^2} r_0^{n-2} \frac{1}{1 + u^{*2}} (\Omega_i \phi)^2\, du^* \\
&\leq 8C(n, m, \alpha)\frac{1 + u_0^{*2}}{u_0^{*2}} \sum_{i=1}^{\frac{n(n-1)}{2}} \int_{u_0^*}^{\infty} \left(\frac{\partial \Omega_i \phi}{\partial u^*}\right)^2 r^{n-1}\, du^* \\
&\quad + 2\pi\, C(n, m, \alpha) \sum_{i=1}^{\frac{n(n-1)}{2}} \int_{u_0^*}^{u_0^*+1} \left\{(\Omega_i \phi)^2 + \left(\frac{\partial \Omega_i \phi}{\partial u^*}\right)^2\right\} du^* \\
&\leq C(n, m, \alpha) \int_{u_0^*}^{\infty} \left(J^T(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i \phi), \frac{\partial}{\partial u^*}\right) r^{n-1}\, du^*.
\end{aligned}$$

Obviously the same bound holds for

$$\int_{u_0^*}^{\infty} \frac{|x| r^2}{(\alpha^2 + x^2)^2} |\nabla\!\!\!/ \phi|^2\, r^{n-1}\, du^*. \qquad \square$$

Step 2b (commutation with Ω_i). Since

$$\left[\Omega_i, \frac{\partial}{\partial t}\right] = 0, \tag{4-97}$$

$$\sum_{i=1}^{\frac{n(n-1)}{2}} \left(\frac{\partial}{\partial t}\Omega_i \cdot \pi_{<L}\phi\right)^2 = r^2 \left|\nabla\!\!\!/ \frac{\partial \pi_{<L}\phi}{\partial t}\right|^2_{r^2 \mathring{\gamma}_{n-1}}, \tag{4-98}$$

and since also

$$\left[\Omega_i, \frac{\partial}{\partial r}\right] = 0, \tag{4-99}$$

we have

$$\sum_{i=1}^{\frac{n(n-1)}{2}} \left(\frac{\partial}{\partial r^*}\Omega_i \cdot \pi_{<L}\phi\right)^2 = r^2 \left|\nabla\!\!\!/ \frac{\partial \pi_{<L}\phi}{\partial r^*}\right|^2. \tag{4-100}$$

Moreover

$$\left[\pi_l, \frac{\partial}{\partial t}\right] = 0, \tag{4-101}$$

so that

$$\int_{S_r} \sum_{i=1}^{\frac{n(n-1)}{2}} \left(\frac{\partial}{\partial t}\Omega_i \cdot \pi_{<L}\phi\right)^2 \mathrm{d}\mu_\gamma = \int_{S_r} r^2 \left|\nabla\!\!\!/ \frac{\partial \pi_{<L}\phi}{\partial t}\right|^2 \mathrm{d}\mu_\gamma \le L(L+n+2) \int_{S_r} \left(\frac{\partial \pi_{<L}\phi}{\partial t}\right)^2 \mathrm{d}\mu_\gamma.$$

Since also

$$\int_{S_r} |\nabla\!\!\!/ \Omega_i \cdot \pi_{<L}\phi|^2_{r^2\mathring{\gamma}_{n-1}} \mathrm{d}\mu_\gamma \le \frac{L(L+n-2)}{r^2} \int_{S_r} (\Omega_i \cdot \pi_{<L}\phi)^2 \mathrm{d}\mu_\gamma,$$

we have

$$\int_{S_r} \sum_{i=1}^{\frac{n(n-1)}{2}} |\nabla\!\!\!/ \Omega_i \cdot \pi_{<L}\phi|^2_{r^2\mathring{\gamma}_{n-1}} \mathrm{d}\mu_\gamma \le L(L+n-2) \int_{S_r} |\nabla\!\!\!/ \pi_{<L}\phi|^2_{r^2\mathring{\gamma}_{n-1}} \mathrm{d}\mu_\gamma.$$

Therefore indeed,

$$\begin{aligned}
&\int_{S_r} \sum_{i=1}^{\frac{n(n-1)}{2}} \left(J^T(\Omega_i \cdot \pi_{<L}\phi), \frac{\partial}{\partial t}\right) \\
&\quad = \frac{1}{2} \sum_{i=1}^{\frac{n(n-1)}{2}} \int_{S_r} \left\{ \left(\frac{\partial \Omega_i \cdot \pi_{<L}\phi}{\partial t}\right)^2 + \left(\frac{\partial \Omega_i \cdot \pi_{<L}\phi}{\partial r^*}\right)^2 + \left(1 - \frac{2m}{r^{n-2}}\right) |\nabla\!\!\!/ \Omega_i \cdot \pi_{<L}\phi|^2_{r^2\mathring{\gamma}_{n-1}} \right\} \mathrm{d}\mu_\gamma \\
&\quad \le \frac{1}{2}(6\gamma n)^2\big((6\gamma n)^2 + n - 2\big) \int_{S_r} \left(J^T(\pi_{<L}\phi), \frac{\partial}{\partial t}\right) \mathrm{d}\mu_\gamma,
\end{aligned}$$

because $L = (6\gamma n)^2$ is fixed; similarly, of course, for $\big(J^T, \frac{\partial}{\partial u^*}\big)$ and $\big(J^T, \frac{\partial}{\partial v^*}\big)$.

We conclude the statement of the proposition with the treatment of the two regimes in Steps 1 and Step 2 above from

$$\begin{aligned}
&\int_{\mathcal{R}^\infty_{r_0,r_1}(t_0)} \left\{ \frac{1}{r^n}\left(\frac{\partial \phi}{\partial r^*}\right)^2 + \frac{1}{r^{n+1}}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{r^3}\left(1 - \frac{2m}{r^{n-2}}\right) |\nabla\!\!\!/ \phi|^2_{r^2\mathring{\gamma}_{n-1}} \right\} \\
&\quad \le 2 \int_{\mathcal{R}^\infty_{r_0,r_1}(t_0)} \left\{ \frac{1}{r^n}\left(\frac{\partial \pi_{<L}\phi}{\partial r^*}\right)^2 + \frac{1}{r^{n+1}}\left(\frac{\partial \pi_{<L}\phi}{\partial t}\right)^2 + \frac{1}{r^2} |\nabla\!\!\!/ \pi_{<L}\phi|^2_{r^2\mathring{\gamma}_{n-1}} \right\} \\
&\qquad + 2 \int_{\mathcal{R}^\infty_{r_0,r_1}(t_0)} \left\{ \frac{1}{r^3}\left(\frac{\partial \pi_{\ge L}\phi}{\partial r^*}\right)^2 + \frac{1}{r^3}\left(\frac{\partial \pi_{\ge L}\phi}{\partial t}\right)^2 + \frac{1}{r^3}\left(1 - \frac{2m}{r^{n-2}}\right) |\nabla\!\!\!/ \pi_{\ge L}\phi|^2_{r^2\mathring{\gamma}_{n-1}} \right\}. \qquad \square
\end{aligned}$$

4D.2. *Refinement for finite regions.* In the proof of Proposition 4.1, neither of the currents used for the high or the low spherical harmonic regime requires the use of Hardy inequalities for the boundary integrals in the asymptotic region; indeed in both cases the zeroth-order terms ϕ^2 can be estimated by the angular derivatives $|\nabla\!\!\!/ \phi|^2$, in the case of the current $J^{X_{\gamma,\alpha},1}$ for high angular frequencies by Poincaré's inequality Lemma 4.6, and in the case of the current $J^{(\alpha)}$ for low angular frequencies as a result of the commutation with Ω_i in (4-65). Therefore we can in fact state a refinement of Proposition 4.1 for finite regions, i.e., an

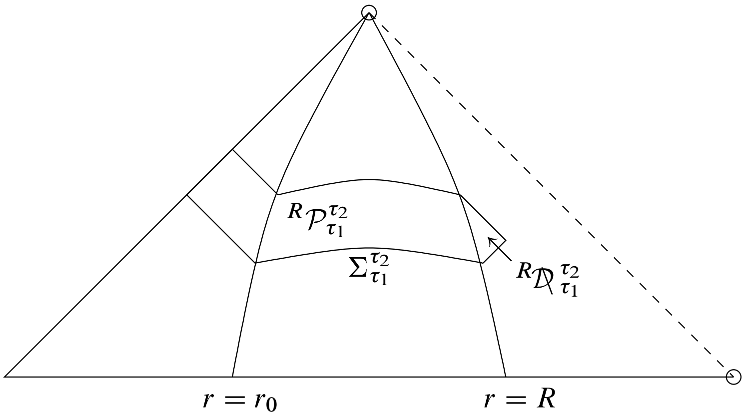

Figure 6. The past boundary $\Sigma_{\tau_1}^{\tau_2}$ of ${}^{R}\mathcal{P}_{\tau_1}^{\tau_2} \cup {}^{R}\mathcal{D}_{\tau_1}^{\tau_2}$.

integrated local energy estimate on bounded domains in terms of the flux through the past boundary of that domain, that will be relevant in Section 5C.

Let

$$ {}^{R}\mathcal{P}_{\tau_1}^{\tau_2} \doteq \mathcal{R}_{r_0,R}^{\infty}(2\tau_1 + R^*, 2\tau_2 + R^*) \cap \{r \leq R\}, \tag{4-102} $$

$$ \begin{aligned} {}^{R}\mathcal{D}_{\tau_1}^{\tau_2} &\doteq \{(u^*, v^*) : \tau_1 \leq u^* \leq \tau_2, v^* - u^* \geq R^*, v^* \leq \tau_2 + R^*\} \\ &= \mathcal{R}_{r_0,R}^{\infty}\big(2\tau_1 + R^*, 2\tau_2 + R^*, \tau_2 + \tfrac{1}{2}(R^* - r_0^*), \tau_2 + R^*\big) \setminus {}^{R}\mathcal{P}_{\tau_1}^{\tau_2}, \end{aligned} \tag{4-103} $$

and denote by $\Sigma_{\tau_1}^{\tau_2}$ the past boundary of ${}^{R}\mathcal{P}_{\tau_1}^{\tau_2} \cup {}^{R}\mathcal{D}_{\tau_1}^{\tau_2}$ (see also Figure 6):

$$ \begin{aligned} \Sigma_{\tau_1}^{\tau_2} \doteq \partial^{-}\big({}^{R}\mathcal{P}_{\tau_1}^{\tau_2} \cup {}^{R}\mathcal{D}_{\tau_1}^{\tau_2}\big) = \big\{(u^*, v^*) :\ & v^* = \tau_1 + \tfrac{1}{2}(R^* + r_0^*), u^* \geq \tau_1 + \tfrac{1}{2}(R^* - r_0^*)\big\} \\ \cup \big\{(u^*, v^*) :\ & u^* + v^* = 2\tau_1 + R^*, r_0^* \leq v^* - u^* \leq R^*\big\} \\ \cup \big\{(u^*, v^*) :\ & u^* = \tau_1, R^* + \tau_1 \leq v^* \leq R^* + \tau_2\big\}. \end{aligned} \tag{4-104} $$

Proposition 4.21 (integrated local energy decay on finite regions). *Let ϕ be a solution of the wave equation $\Box_g \phi = 0$, and $R > \sqrt[n-2]{2m}$. Then there exists a constant $C(n, m, R)$, such that, for any $\tau_2 > \tau_1$,*

$$ \int_{{}^{R}\mathcal{P}_{\tau_1}^{\tau_2}} \left\{ \left(\frac{\partial \phi}{\partial r^*}\right)^2 + \left(\frac{\partial \phi}{\partial t}\right)^2 + \left(1 - \frac{2m}{r^{n-2}}\right) |\nabla\!\!\!/ \phi|^2_{r^2 \mathring{\gamma}_{n-1}} \right\} \mathrm{d}\mu_g \leq C(n, m, R) \int_{\Sigma_{\tau_1}^{\tau_2}} \big(J^T(\phi) + J^T(T \cdot \phi), n\big). \tag{4-105} $$

In view of the remarks above, the proof of Proposition 4.21 is of course identical to the proof of Proposition 4.1 given in Section 4D.1 by replacing the unbounded domain $\mathcal{R}_{r_0,r_1}^{\infty}(2\tau_1 + R^*)$ by the bounded domain ${}^{R}\mathcal{P}_{\tau_1}^{\tau_2} \cup {}^{R}\mathcal{D}_{\tau_1}^{\tau_2}$.

However, this estimate does not include the zeroth-order term, which we have covered separately in Proposition 4.4.

Proposition 4.22 (refinement for zeroth-order terms on timelike boundaries). *Let ϕ be solution of the wave equation (1-1), and $R > \sqrt[n-2]{8nm}$. Then there is a constant $C(n, m, R)$ such that for all $\tau' < \tau$,*

$$\int_{2\tau'+R^*}^{2\tau+R^*} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \phi^2\big|_{r=R} \le C(n, m, R)\left\{\int_{2\tau'+R^*}^{2\tau+R^*} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left\{\left(\frac{\partial\phi}{\partial r^*}\right)^2 + |\nabla\!\!\!/ \phi|^2\right\}\Big|_{r=R} + \int_{\Sigma^{\tau}_{\tau'}} \big(J^T(\phi), n\big) + \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-2}\phi^2\big|_{(\tau', R^*+\tau)}\right\}. \tag{4-106}$$

The proof remains the same as for Proposition 4.4 on page 534 with the exception that we consider the energy identity for $J^{X,1}$ on ${}^R\mathcal{D}\!\!\!/^{\tau}_{\tau'}$ in place of ${}^R\mathcal{D}^{\tau}_{\tau'}$ and use Proposition C.5 instead of Proposition C.1.

5. The decay argument

We will here prove energy decay of the solutions to the wave equation and higher-order energy decay of their time derivatives in the interior based on the integrated local energy decay statements of Section 4, following the new physical-space approach to decay of [Dafermos and Rodnianski 2010].

Remark 5.1. Instead one could use the conformal Morawetz vector field

$$Z = u^{*2}\frac{\partial}{\partial u^*} + v^{*2}\frac{\partial}{\partial v^*}$$

to prove energy decay of solutions to the wave equation with a rate corresponding to the weights in Z; this is done in [Schlue 2010]. Similarly the use of the scaling vector field

$$S = v^*\frac{\partial}{\partial v^*} + u^*\frac{\partial}{\partial u^*}$$

should provide an alternative approach to prove higher-order energy decay [Luk 2010]. Here however, we shall avoid the use of multipliers with weights in t.

5A. *Uniform boundedness.* A preliminary feature of the solutions to the wave equation (1-1) that is necessary to employ the decay mechanism of [Dafermos and Rodnianski 2010] is the uniform boundedness of their (nondegenerate) energy; this is a consequence of the conservation of the degenerate energy associated to the multiplier T, *and the redshift effect* of Section 3, which allows us to control the nondegenerate energy on the horizon.

Let Σ be a (spherically symmetric) spacelike hypersurface in $\mathcal{M}$, $\Sigma' = \Sigma \cap \{r \le R\}$ and $\mathcal{N}$ the outgoing null hypersurface emerging from $\partial\Sigma'$ (Figure 7). Moreover, let

$$\Sigma_\tau = \varphi_\tau\big((\Sigma' \cup \mathcal{N}) \cap \mathcal{D}\big), \quad \Sigma'_\tau = \Sigma_\tau \cap \{r \le R\}, \quad \Sigma^{\tau}_{\tau'} = \Sigma_{\tau'} \cap \mathrm{J}^-(\Sigma'_\tau).$$

Proposition 5.2 (uniform boundedness). *Let ϕ be a solution of the wave equation (1-1) with initial data on Σ_0. Then there exists a constant $C(\Sigma_0)$ such that*

$$\int_{\Sigma'_\tau} \big(J^N(\phi), n\big) \le C \int_{\Sigma^{\tau}_0} \big(J^N(\phi), n\big) \quad (\tau > 0). \tag{5-1}$$

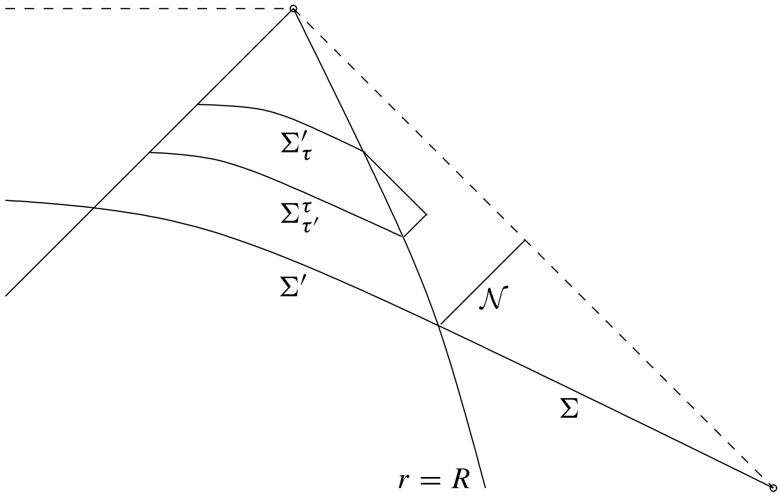

Figure 7. The construction of the surfaces Σ'_τ from Σ.

Proof. One can proceed in analogy to the *local observer's energy estimate* of [Dafermos and Rodnianski 2008]; indeed, from the energy identity for N on the domain $\mathcal{R}(\tau',\tau) = \cup_{\tau'\leq\bar{\tau}\leq\tau}\Sigma^\tau_{\bar{\tau}}$ it follows

$$\int_{\Sigma'_\tau}(J^N,n) + \int_{\mathcal{R}(\tau',\tau)} K^N \leq \int_{\Sigma^\tau_{\tau'}}(J^N,n) \tag{5-2}$$

since $(J^N, n_{\mathcal{H}}) \geq 0$, and $(J^N, n_{\mathcal{N}}) \geq 0$. By Proposition 3.3, namely the redshift effect, K^N is bounded from below by (J^N, n) near the horizon, and from above by (J^T, n) away from the horizon; since also the lapse of the foliation of $\mathcal{R}$ is bounded from above and below we conclude that there are constants $0 < b < B$ only depending on Σ and N such that

$$\begin{aligned}\int_{\Sigma'_\tau}(J^N,n) + b\int_{\tau'}^{\tau}\mathrm{d}\bar{\tau}\int_{\Sigma^\tau_{\bar{\tau}}}(J^N,n) &\leq B\int_{\tau'}^{\tau}\mathrm{d}\bar{\tau}\int_{\Sigma^\tau_{\bar{\tau}}}(J^T,n) + \int_{\Sigma^\tau_{\tau'}}(J^N,n)\\ &\leq B(\tau-\tau')\int_{\Sigma^\tau_{\tau'}}(J^T,n) + \int_{\Sigma^\tau_{\tau'}}(J^N,n),\end{aligned} \tag{5-3}$$

where in the last step we have used the energy identity for T on $\mathcal{R}(\tau',\bar{\tau})$ and $K^T = 0$. Thus the desired energy bound follows from the elementary Lemma 5.3. □

Lemma 5.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative function, $f \geq 0$, such that for all $t_1 \leq t_2$ and two positive constants $0 < c < C$,*

$$f(t_2) + c\int_{t_1}^{t_2} f(t)\,\mathrm{d}t \leq C(t_2 - t_1) + f(t_1).$$

Then

$$f(t_2) \leq f(t_1) + \frac{C}{c} \quad (t_2 \geq t_1).$$

Proof. See, e.g., [Schlue 2012]. □

5B. *Energy decay.* In this section we prove quadratic decay of the nondegenerate energy.

Let

$$\Sigma_{\tau_0} \doteq \partial^-\mathcal{R}^{\infty}_{r_0,R}(t_0), \quad \tau_0 = \tfrac{1}{2}(t_0 - R^*), \tag{5-4}$$

with $R > \sqrt[n-2]{8nm}$, $t_0 > 0$ and $r_0 \doteq r_0^{(N)}$ according to Proposition 3.1.

Proposition 5.4 (energy decay). *Let ϕ be a solution of the wave equation (1-1) with initial data on Σ_{τ_0} satisfying*

$$D \doteq \int_{\tau_0+R^*}^{\infty} \mathrm{d}v \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \sum_{k=0}^{1} r^2 \Big(\frac{\partial r^{\frac{n-1}{2}}\partial_t^k\phi}{\partial v^*}\Big)^2\Big|_{u=\tau_0} + \int_{\Sigma_{\tau_0}} \Big(\sum_{k=0}^{2} J^N(T^k\cdot\phi), n\Big) < \infty. \tag{5-5}$$

Then there exists a constant $C(n,m,R)$ such that

$$\int_{\Sigma_\tau} \big(J^N(\phi), n\big) \leq \frac{CD}{\tau^2} \quad (\tau > \tau_0). \tag{5-6}$$

The proof is based on a *weighted energy inequality*, derived from the energy identity for the current (5-8) on the domain

$${}^R\mathcal{D}^{\tau_2}_{\tau_1} = \big\{(u^*,v^*) : \tau_1 \leq u^* \leq \tau_2, v^* - u^* \geq R^*\big\}. \tag{5-7}$$

Weighted energy identity. Consider the current

$$\overset{r}{J}_\mu(\phi) = T_{\mu\nu}(\psi)V^\nu, \tag{5-8}$$

where

$$\psi = r^{\frac{n-1}{2}}\phi, \tag{5-9}$$

$$V = r^q\frac{\partial}{\partial v^*}, \quad q = p+1-n, \quad p\in\{1,2\}. \tag{5-10}$$

This may also be viewed as the current to the multiplier vector field $r^p\frac{\partial}{\partial v^*}$, modified by the following terms:

$$\begin{aligned}\overset{r}{J}_\mu(\phi) = T_{\mu\nu}(\phi)\, r^p\Big(\frac{\partial}{\partial v^*}\Big)^\nu + \Big(\frac{n-1}{2}\Big)^2 r^{p-2}\Big(1-\frac{2m}{r^{n-2}}\Big)(\partial_\mu r)\phi^2 \\ + \frac{1}{2}\frac{n-1}{2}r^{p-1}(\partial_\mu r)\frac{\partial\phi^2}{\partial v^*} + \frac{1}{2}\frac{n-1}{2}r^{p-1}\Big(1-\frac{2m}{r^{n-2}}\Big)(\partial_\mu\phi^2) \\ - \frac{1}{2}\Big(\frac{n-1}{2}\Big)^2 r^{p-2}\Big(1-\frac{2m}{r^{n-2}}\Big)\Big(\frac{\partial}{\partial v^*}\Big)_\mu\phi^2 - \frac{1}{2}\frac{n-1}{2}\Big(\frac{\partial}{\partial v^*}\Big)_\mu r^{p-1}\frac{\partial\phi^2}{\partial r^*}.\end{aligned}$$

If $\Box_g\phi = 0$ then we calculate

$$\begin{aligned}\Box_g\psi &= -\Big(1-\frac{2m}{r^{n-2}}\Big)^{-1}\partial_{u^*}\partial_{v^*}\psi + \frac{n-1}{r}\frac{\partial\psi}{\partial r^*} + \frac{1}{r^2}\mathring{\Delta}_{n-1}\psi \\ &= \frac{n-1}{2}\Big(\frac{n-3}{2} + \frac{n-1}{2}\frac{2m}{r^{n-2}}\Big)\frac{1}{r^2}\psi + \frac{n-1}{r}\frac{\partial}{\partial r^*}\psi.\end{aligned} \tag{5-11}$$

So the wave equation for ϕ,

$$\Box_g \phi = 0,$$

is equivalent to the following equation for ψ:

$$-\partial_{u^*}\partial_{v^*}\psi + \left(1-\frac{2m}{r^{n-2}}\right)\frac{1}{r^2}\mathring{\Delta}_{n-1}\psi - \frac{n-1}{2}\left(\frac{n-3}{2}+\frac{n-1}{2}\frac{2m}{r^{n-2}}\right)\frac{1}{r^2}\left(1-\frac{2m}{r^{n-2}}\right)\psi = 0. \quad (5\text{-}12)$$

Now,

$$\overset{r}{K}(\phi) = \nabla^\mu \overset{r}{J}_\mu(\phi) = \Box_g(\psi)\, V\cdot\psi + K^V(\psi), \quad (5\text{-}13)$$

where

$$K^V(\psi) = {}^{(V)}\pi^{\mu\nu}\, T_{\mu\nu}(\psi).$$

Since

$$\begin{aligned}
{}^{(V)}\pi_{u^*u^*} &= 2qr^{q-1}\left(1-\frac{2m}{r^{n-2}}\right)^2,\\
{}^{(V)}\pi_{v^*v^*} &= 0,\\
{}^{(V)}\pi_{u^*v^*} &= -\left(1-\frac{2m}{r^{n-2}}\right)r^{q-1}\left[q+(n-q-2)\frac{2m}{r^{n-2}}\right],\\
{}^{(V)}\pi_{aA} &= 0,\\
{}^{(V)}\pi_{AB} &= r^{q-1}\left(1-\frac{2m}{r^{n-2}}\right)g_{AB},
\end{aligned} \quad (5\text{-}14)$$

we find

$$\overset{r}{K}\cdot r^{n-1} = \frac{n-1}{4}\left(\frac{n-3}{2}+\frac{n-1}{2}\frac{2m}{r^{n-2}}\right)\frac{r^p}{r^2}\frac{\partial\psi^2}{\partial v^*} + \frac{p}{2}r^{p-1}\left(\frac{\partial\psi}{\partial v^*}\right)^2 + \frac{1}{2}r^{p-1}\left[(2-p)+(p-n)\frac{2m}{r^{n-2}}\right]|\nabla\!\!\!/\,\psi|^2_{r^2\mathring{\gamma}_{n-1}}. \quad (5\text{-}15)$$

One may integrate the first term by parts to obtain

$$\begin{aligned}
\int_{u^*+R^*}^\infty \mathrm{d}v^*\,\overset{r}{K}\cdot r^{n-1} &= \frac{n-1}{4}\left(\frac{n-3}{2}+\frac{n-1}{2}\frac{2m}{r^{n-2}}\right)\frac{r^p}{r^2}\psi^2\Big|_{u^*+R^*}^\infty\\
&\quad + \int_{u^*+R^*}^\infty \mathrm{d}v^*\Bigg\{\left[\frac{n-1}{4}(2-p)\frac{n-3}{2}+\frac{n-1}{2}(n-p)\frac{2m}{r^{n-2}}\right]\frac{r^p}{r^3}\left(1-\frac{2m}{r^{n-2}}\right)\psi^2\\
&\quad + \frac{p}{2}r^{p-1}\left(\frac{\partial\psi}{\partial v^*}\right)^2 + \frac{1}{2}r^{p-1}\left[2-p+(p-n)\frac{2m}{r^{n-2}}\right]|\nabla\!\!\!/\,\psi|^2_{r^2\mathring{\gamma}_{n-1}}\Bigg\}.
\end{aligned} \quad (5\text{-}16)$$

We can now write down the energy identity for the current $\overset{r}{J}$ (see also Appendix B):

$$\int_{{}^R\mathcal{D}_{\tau_1}^{\tau_2}} \overset{r}{K}\,\mathrm{d}\mu_g = \int_{\partial\,{}^R\mathcal{D}_{\tau_1}^{\tau_2}} {}^*\overset{r}{J}.$$

Dropping the positive zeroth-order terms, we obtain

$$\begin{aligned}
&\int_{\tau_2+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,\frac{1}{2}r^p\Big(\frac{\partial\psi}{\partial v^*}\Big)^2\Big|_{u^*=\tau_2}\\
&\quad+\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Big\{\frac{p}{2}r^{p-1}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\frac{1}{2}r^{p-1}\Big[2-p+(p-n)\frac{2m}{r^{n-2}}\Big]|\nabla\!\!\!/\psi|^2_{r^2\mathring{\gamma}_{n-1}}\Big\}\\
&\quad+\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,\frac{1}{2}r^p|\nabla\!\!\!/\psi|^2_{r^2\mathring{\gamma}_{n-1}}\big|_{v^*\to\infty}\\
&\leq\Big(1-\frac{2m}{R^{n-2}}\Big)^{-1}\Big\{\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,\frac{1}{2}r^p\Big(\frac{\partial\psi}{\partial v^*}\Big)^2\Big|_{u^*=\tau_1}\\
&\qquad\qquad+\int_{2\tau_1+R^*}^{2\tau_2+R^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Big[\frac{1}{4}r^p\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\frac{1}{4}r^p|\nabla\!\!\!/\psi|^2_{r^2\mathring{\gamma}_{n-1}}\\
&\qquad\qquad\qquad+\frac{n-1}{4}\,\frac{1}{2}r^p\Big(\frac{n-3}{2}+\frac{n-1}{2}\,\frac{2m}{R^{n-2}}\Big)\frac{1}{r^2}\psi^2\Big]\Big|_{r=R}\Big\}. \qquad (5\text{-}17)
\end{aligned}$$

Note that the powers of r that appear in the bulk term are 1 less than those that appear in the boundary terms. This allows for a hierarchy of inequalities (5-17) for different values of p, the so-called *p-hierarchy*.

Proof of Proposition 5.4. In a first step the decay of the solutions at future null infinity will be deduced from the weighted energy inequality, and in a second step the continuation to the event horizon will be inferred from the redshift effect.

Step 1. The p-hierarchy consists of two steps which exploits (5-17) first with $p=2$, then with $p=1$; but in a zeroth step we need to obtain control on the angular derivatives from (5-17) with $p=1$:

Since

$$1-(n-1)\frac{2m}{r^{n-2}}>\frac{1}{2}\quad(r>R),$$

we have from the weighted energy inequality for $p=1$ on the domain ${}^{r_0'}\mathcal{D}_{\tau'}^{\tau}$ for $\tau>\tau'\geq\tau_0\doteq\frac{1}{2}(t_0-R^*)$,

$$\begin{aligned}
&\int_{\tau'}^{\tau}\mathrm{d}u^*\int_{u^*+r_0'^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,\frac{1}{4}|\nabla\!\!\!/\psi|^2_{r^2\mathring{\gamma}_{n-1}}\\
&\qquad\leq\Big(1-\frac{2m}{R^{n-2}}\Big)^{-1}\int_{\tau'+r_0'^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,\frac{1}{2}r\Big(\frac{\partial\psi}{\partial v^*}\Big)^2\Big|_{u^*=\tau'}\\
&\qquad\qquad+C(n,m,R)\Big(1-\frac{2m}{R^{n-2}}\Big)^{-1}\int_{\Sigma_{\tau_0}}\big(J^T(\phi)+J^T(T\cdot\phi),n\big); \qquad (5\text{-}18)
\end{aligned}$$

here we have estimated the boundary integrals as follows.

Choose $r_0' \in (R^*, R^*+1)$ such that

$$\int_{R^*}^{R^*+1} \mathrm{d}r^* \int_{t_0+(r^*-R^*)}^{\infty} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left(1-\frac{2m}{r^{n-2}}\right) r^{n-1} \times \left\{ \frac{1}{r^n}\left(\frac{\partial\phi}{\partial r^*}\right)^2 + \frac{1}{r^{n+1}}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{r^3}\left(1-\frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/\,\phi|^2_{r^2\mathring{\gamma}_{n-1}} \right\}$$
$$= \int_{t_0+(r_0'^*-R^*)}^{\infty} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left(1-\frac{2m}{r_0'^{\,n-2}}\right) r_0'^{\,n-1} \times \times \left\{ \frac{1}{r_0'^{\,n}}\left(\frac{\partial\phi}{\partial r^*}\right)^2 + \frac{1}{r_0'^{\,n+1}}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{r_0'^{\,3}}\left(1-\frac{2m}{r_0'^{\,n-2}}\right)|\nabla\!\!\!/\,\phi|^2_{r^2\mathring{\gamma}_{n-1}} \right\};$$

then

$$\int_{2\tau'+r_0'^*}^{2\tau+r_0'^*} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left[\frac{1}{4} r^p \left(\frac{\partial\psi}{\partial v^*}\right)^2 + \frac{1}{4} r^p |\nabla\!\!\!/\,\psi|^2_{r^2\mathring{\gamma}_{n-1}} + \frac{n-1}{4}\frac{1}{2} r^p \left(\frac{n-3}{2} + \frac{n-1}{2}\frac{2m}{r^{n-2}}\right)\frac{1}{r^2}\psi^2 \right]\Bigg|_{r=r_0'}$$
$$\le \int_{2\tau'+r_0'^*}^{2\tau+r_0'^*} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r_0'^{\,p-2} \left[\frac{1}{2}\left(\frac{n-1}{2}\right)^2 \phi^2 + \frac{1}{2} r_0'^{\,2} \left(\frac{\partial\phi}{\partial v^*}\right)^2 + \frac{1}{4} r_0'^{\,2} |\nabla\!\!\!/\,\phi|^2_{r^2\mathring{\gamma}_{n-1}} + \frac{n-1}{4}\frac{1}{2}\left(\frac{n-3}{2} + \frac{n-1}{2}\frac{2m}{R^{n-2}}\right)\phi^2 \right]\Bigg|_{r=r_0'} r^{n-1}$$
$$\le C(n,m,R) \int_{\Sigma_{\tau_0}} (J^T(\phi) + J^T(T\cdot\phi), n),$$

because, by (4-23),

$$\int_{r_0'^*+2\tau'}^{r_0'^*+2\tau} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{n-1} \left[\frac{1}{4}\left(\frac{\partial\phi}{\partial v^*}\right)^2 + \frac{n-1}{(4r_0')^2}\left(1-\frac{2m}{r^{n-2}}\right)\phi^2 \right]\Bigg|_{r=r_0'}$$
$$\le \int_{r_0'^*+2\tau'}^{r_0'^*+2\tau} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{n-1} \left[\frac{1}{2}\left(1-\frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/\,\phi|^2_{r^2\mathring{\gamma}_{n-1}} + \frac{n-1}{2}\left(1-\frac{2m}{r^{n-2}}\right)\left(\frac{\partial\phi}{\partial r^*}\right)^2 \right]\Bigg|_{r=r_0'} + C(n,m,R) \int_{\Sigma_{\tau_0}} (J^T(\phi), n),$$

and by Proposition 4.1 (and the choice of r_0'),

$$\int_{t_0+(r_0'^*-R^*)}^{\infty} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{n-1} \left[\frac{1}{r_0'^{\,n}}\left(\frac{\partial\phi}{\partial r^*}\right)^2 + \frac{1}{r_0'^{\,3}}\left(1-\frac{2m}{r_0'^{\,n-2}}\right)|\nabla\!\!\!/\,\phi|^2_{r^2\mathring{\gamma}_{n-1}} \right]\Bigg|_{r=r_0'} \le C(n,m) \int_{\Sigma_{\tau_0}} \big(J^T(\phi) + J^T(T\cdot\phi), n\big).$$

Note that for the use of (4-23) that, with our choice of R,

$$(n-3) + n\frac{2m}{r^{n-2}} - (n-1)\left(\frac{2m}{r^{n-2}}\right)^2 > 0 \quad (r > R).$$

$\boxed{p=2:}$ For $p=2$, (5-17) reads

$$\int_{\tau'}^{\tau} \mathrm{d}u^* \int_{u^*+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \Big[r\Big(\frac{\partial\psi}{\partial v^*}\Big)^2 - \frac{1}{2} r(n-2)\frac{2m}{r^{n-2}} |\nabla\!\!\!/ \psi|^2_{r^2\mathring{\gamma}_{n-1}} \Big]$$
$$\leq \Big(1-\frac{2m}{R^{n-2}}\Big)^{-1} \int_{\tau'+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \frac{1}{2} r^2 \Big(\frac{\partial\psi}{\partial v^*}\Big)^2 \Big|_{u^*=\tau'} + C(n,m,R) \int_{\Sigma_{\tau_0}} \big(J^T(\phi) + J^T(T\cdot\phi), n\big).$$

Thus, with the previous estimate (5-18),

$$\int_{\tau'}^{\tau} \mathrm{d}u^* \int_{u^*+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r \Big(\frac{\partial\psi}{\partial v^*}\Big)^2$$
$$\leq C(n,m,R)^{-1} \Big\{ \int_{\tau'+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \frac{1}{2} r^2 \Big(\frac{\partial\psi}{\partial v^*}\Big)^2 \Big|_{u^*=\tau'} + \int_{\Sigma_{\tau_0}} \big(J^T(\phi)+J^T(T\cdot\phi), n\big) \Big\}. \tag{5-19}$$

Let us define

$$\tau_{j+1} = 2\tau_j \quad (j \in \mathbb{N}_0), \qquad \tau_0 = \tfrac{1}{2}(t_0 - R^*).$$

Then there is a sequence $(\tau_j')_{j\in\mathbb{N}_0}$ with $\tau_j' \in (\tau_j, \tau_{j+1})$ $(j \in \mathbb{N}_0)$ such that

$$\int_{\tau_j'+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r \Big(\frac{\partial\psi}{\partial v^*}\Big)^2 \Big|_{u^*=\tau_j'}$$
$$\leq \frac{1}{\tau_j} C(n,m,R) \Big[\int_{\tau_j+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^2 \Big(\frac{\partial\psi}{\partial v^*}\Big)^2 \Big|_{u^*=\tau_j} + \int_{\Sigma_{\tau_0}} \big(J^T(\phi) + J^T(T\cdot\phi), n\big) \Big],$$

and again by (5-17),

$$\int_{\tau_j+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \frac{1}{2} r^2 \Big(\frac{\partial\psi}{\partial v^*}\Big)^2 \Big|_{u^*=\tau_j}$$
$$\leq C(n,m,R) \Big[\int_{\tau_0+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^2 \Big(\frac{\partial\psi}{\partial v^*}\Big)^2 \Big|_{u^*=\tau_0} + \int_{\Sigma_{\tau_0}} \big(J^T(\phi) + J^T(T\cdot\phi), n\big) \Big].$$

Since $\frac{1}{\tau_j} \leq \frac{1}{\tau_j'} \frac{\tau_{j+1}}{\tau_j} = \frac{2}{\tau_j'}$, we have

$$\int_{\tau_j'+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r \Big(\frac{\partial\psi}{\partial v^*}\Big)^2 \Big|_{u^*=\tau_j'}$$
$$\leq \frac{C(n,m,R)}{\tau_j'} \Big[\int_{\tau_0+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^2 \Big(\frac{\partial\psi}{\partial v^*}\Big)^2 \Big|_{u^*=\tau_0} + \int_{\Sigma_{\tau_0}} \big(J^T(\phi)+J^T(T\cdot\phi), n\big) \Big]. \tag{5-20}$$

$\boxed{p=1:}$ In order to deal with the timelike boundary integrals analogously to the above choose

$$r_j''^* \in (r_0'^*, r_0'^* + 1)$$

such that

$$\int_{2\tau'_j+r''^*_j}^{2\tau'_{j+1}+r''^*_j} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{n-1}\left[\frac{1}{r^n}\left(\frac{\partial\phi}{\partial r^*}\right)^2+\frac{1}{r^{n+1}}\left(\frac{\partial\phi}{\partial t}\right)^2+\frac{1}{r^3}\left(1-\frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/\,\phi|^2_{r^2\mathring{\gamma}_{n-1}}\right]\Bigg|_{r=r''_j} \leq C(n,m)\int_{\Sigma_{\tau'_j}}\big(J^T(\phi)+J^T(T\cdot\phi),n\big).$$

Then, proceeding as before,

$$\int_{2\tau'_j+r''^*_j}^{2\tau'_{j+1}+r''^*_j} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left[\frac{1}{4}r\left(\frac{\partial\psi}{\partial v^*}\right)^2+\frac{1}{4}r|\nabla\!\!\!/\,\psi|^2_{r^2\mathring{\gamma}_{n-1}}+\frac{n-1}{4}\frac{1}{2}r\left(\frac{n-3}{2}+\frac{n-1}{2}\frac{2m}{r^{n-2}}\right)\frac{1}{r^2}\psi^2\right]\Bigg|_{r=r''_j} \leq C(n,m,R)\int_{\Sigma_{\tau'_j}}\big(J^T(\phi)+J^T(T\cdot\phi),n\big). \tag{5-21}$$

Now apply (5-17) to the region ${}^{r''_j}\mathcal{D}_{\tau'_j}^{\tau'_{j+1}}$ to obtain

$$\int_{\tau'_j}^{\tau'_{j+1}} \mathrm{d}u^* \int_{u^*+r''^*_j}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \times\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial v^*}\right)^2+\frac{1}{4}|\nabla\!\!\!/\,\psi|^2_{r^2\mathring{\gamma}_{n-1}}\right]$$
$$\leq\left(1-\frac{2m}{R^{n-2}}\right)^{-1}\int_{\tau'_j+r''^*_j}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\frac{1}{2}r\left(\frac{\partial\psi}{\partial v^*}\right)^2\Bigg|_{u^*=\tau'_j}+C(n,m,R)\int_{\Sigma_{\tau'_j}}\big(J^T(\phi)+J^T(T\cdot\phi),n\big).$$

By virtue of the result (5-20) from the case $p=2$, this yields

$$\int_{\tau'_j}^{\tau'_{j+1}} \mathrm{d}u^* \int_{u^*+r''^*_j}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial v^*}\right)^2+\frac{1}{4}|\nabla\!\!\!/\,\psi|^2_{r^2\mathring{\gamma}_{n-1}}\right]$$
$$\leq\frac{C(n,m,R)}{\tau'_j}\left[\int_{\tau_0+r'^*_0}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,r^2\left(\frac{\partial\psi}{\partial v^*}\right)^2\Bigg|_{u^*=\tau_0}+\int_{\Sigma_{\tau_0}}\big(J^T(\phi)+J^T(T\cdot\phi),n\big)\right] +C(n,m,R)\int_{\Sigma_{\tau'_j}}\big(J^T(\phi)+J^T(T\cdot\phi),n\big). \tag{5-22}$$

Step 2. Our aim is to prove decay for the *nondegenerate* energy. Let us first find an estimate for

$$\int_{\tau'_j}^{\tau'_{j+1}}\mathrm{d}\tau\int_{\Sigma_\tau}\big(J^N(\phi),n\big)=\int_{\tau'_j}^{\tau'_{j+1}}\mathrm{d}\tau\int_{\Sigma_\tau\cap\{r\leq r''_j\}}\big(J^N(\phi),n\big)+\int_{\tau'_j}^{\tau'_{j+1}}\mathrm{d}\tau\int_{\Sigma_\tau\cap\{r\geq r''_j\}}\big(J^T(\phi),n\big).$$

The estimate of the first term is exactly the content of Corollary 4.3, and for the second term

$$\int_{\tau'_j}^{\tau'_{j+1}}\mathrm{d}\tau\int_{\Sigma'_\tau\cap\{r\geq r''_j\}}\big(J^T(\phi),n\big) =\int_{\tau'_j}^{\tau'_{j+1}}\mathrm{d}u^*\int_{u^*+r''^*_j}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,r^{n-1}\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial v^*}\right)^2+\frac{1}{2}\left(1-\frac{2m}{r^{n-2}}\right)|\nabla\!\!\!/\,\phi|^2_{r^2\mathring{\gamma}_{n-1}}\right]$$

we can use (5-22) once we have turned it into an estimate for the derivatives of ϕ. Note that

$$\int_{u^*+r_0'^*}^{\infty}\mathrm{d}v^*\Big(\frac{\partial\psi}{\partial v^*}\Big)^2=\int_{u^*+r_0'^*}^{\infty}\mathrm{d}v^*\Big[\frac{n-1}{2}\Big(1-\frac{2m}{r^{n-2}}\Big)r^{\frac{n-3}{2}}\frac{\partial}{\partial v^*}\big(r^{\frac{n-1}{2}}\phi^2\big)+r^{n-1}\Big(\frac{\partial\phi}{\partial v^*}\Big)^2\Big]$$
$$=-\frac{n-1}{2}\Big(1-\frac{2m}{r^{n-2}}\Big)r^{n-2}\phi^2\big|_{v^*=u^*+r_0'^*}$$
$$+\int_{u^*+r_0'^*}^{\infty}\mathrm{d}v^*\Big\{-\frac{n-1}{2}\Big(1-\frac{2m}{r^{n-2}}\Big)\Big[(n-2)\frac{2m}{r^n}+\frac{n-3}{2}\Big(1-\frac{2m}{r^{n-2}}\Big)\frac{1}{r^2}\Big]\phi^2+\Big(\frac{\partial\phi}{\partial v^*}\Big)^2\Big\}r^{n-1}$$

and, by Lemma C.2,

$$\int_{u^*+r_0'^*}^{\infty}\mathrm{d}v^*\frac{1}{r^2}\phi^2\,r^{n-1}\le C(n,m)\int_{u^*+r_0'^*}^{\infty}\mathrm{d}v^*\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+C(n,m)\,r^{n-1}\phi^2|_{(u^*,v^*=u^*+r_0'^*)}.$$

Thus

$$\int_{u^*+r_0'^*}^{\infty}\mathrm{d}v^*\Big(\frac{\partial\phi}{\partial v^*}\Big)^2r^{n-1}\le C(n,m,R)\Big[\phi^2|_{(u^*,u^*+r_0'^*)}+\int_{u^*+r_0'^*}^{\infty}\mathrm{d}v^*\Big(\frac{\partial\psi}{\partial v^*}\Big)^2\Big],$$

and finally, in view of (5-21),

$$\int_{\tau_j'}^{\tau_{j+1}'}\mathrm{d}u^*\int_{u^*+r_j''^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Big(\frac{\partial\phi}{\partial v^*}\Big)^2r^{n-1}$$
$$\le C(n,m,R)\int_{\tau_j'}^{\tau_{j+1}'}\mathrm{d}u^*\int_{u^*+r_j''^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+C(n,m,R)\int_{\Sigma_{\tau_j'}}\big(J^T(\phi)+J^T(T\cdot\phi),n\big).\tag{5-23}$$

Therefore, putting the estimates for the two terms back together,

$$\int_{\tau_j'}^{\tau_{j+1}'}\mathrm{d}\tau\int_{\Sigma_\tau}\big(J^N(\phi),n\big)$$
$$\le C(n,m,R)\int_{\tau_j'}^{\tau_{j+1}'}\mathrm{d}u^*\int_{u^*+r_j''^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Big\{\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+|\nabla\!\!\!/\,\psi|^2_{r^2\mathring{\gamma}_{n-1}}\Big\}$$
$$+C(n,m)\int_{\Sigma_{\tau_j'}}\big(J^N(\phi)+J^T(T\cdot\phi),n\big)$$
$$\le\frac{C(n,m,R)}{\tau_j'}\Big[\int_{\tau_0+r_0'^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,r^2\Big(\frac{\partial\psi}{\partial v^*}\Big)^2\Big|_{u^*=\tau_0}+\int_{\Sigma_{\tau_0}}\big(J^T(\phi)+J^T(T\cdot\phi),n\big)\Big]$$
$$+C(n,m,R)\int_{\Sigma_{\tau_j'}}\big(J^N(\phi)+J^T(T\cdot\phi),n\big),\tag{5-24}$$

where we have now used (5-22). The same inequality holds for τ_{j+2}' in place of τ_{j+1}', by adding the inequalities corresponding to the intervals $[\tau_j',\tau_{j+1}']$ and $[\tau_{j+1}',\tau_{j+2}']$ and using Proposition 5.2 for the last term. So there is a sequence

$$(\tau_j'')_{j\in\mathbb{N}}, \quad \tau_j'' \in (\tau_j', \tau_{j+2}')$$

such that

$$\int_{\tau_j'}^{\tau_{j+2}'} \mathrm{d}\tau \int_{\Sigma_\tau} \big(J^N(\phi), n\big) \geq \tau_{j+1} \int_{\Sigma_{\tau_j''}} \big(J^N(\phi), n\big),$$

and since $\dfrac{1}{\tau_{j+1}} \leq \dfrac{1}{\tau_j''}\dfrac{\tau_{j+3}}{\tau_{j+1}} = \dfrac{4}{\tau_j''}$, we have

$$\int_{\Sigma_{\tau_j''}} \big(J^N(\phi), n\big) \leq \frac{4}{\tau_j''} \int_{\tau_j}^{\tau_{j+2}'} \mathrm{d}\tau \int_{\Sigma_\tau'} \big(J^N(\phi), n\big). \tag{5-25}$$

Now for any given $\tau > \tau_0$ we may choose

$$j^* = \max\{j \in \mathbb{N} : \tau_j'' \leq \tau\}$$

so that, by (5-1),

$$\int_{\Sigma_\tau} \big(J^N(\phi), n\big) \leq C \int_{\Sigma_{\tau_{j^*}''}} \big(J^N(\phi), n\big)$$

with $\dfrac{\tau}{\tau_{j^*}''} \leq \dfrac{\tau_{j^*+1}''}{\tau_{j^*}''} \leq 2^4$. In particular we may estimate the last integral in (5-24),

$$\int_{\Sigma_{\tau_j'}} \big(J^N(\phi) + J^T(T\cdot\phi), n\big) \leq \frac{C}{\tau_j'} \int_{\tau_{j-1}'}^{\tau_{j+1}'} \mathrm{d}\tau \int_{\Sigma_\tau} \big(J^N(\phi) + J^N(T\cdot\phi), n\big),$$

to see that in fact we have

$$\int_{\tau_j'}^{\tau_{j+2}'} \mathrm{d}\tau \int_{\Sigma_\tau} \big(J^N(\phi), n\big) \leq \frac{C(n,m,R)}{\tau_j'} \bigg[\int_{\tau_0+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \bigg\{ r^2 \bigg(\frac{\partial r^{\frac{n-1}{2}}\phi}{\partial v^*}\bigg)^2 + r^2 \bigg(\frac{\partial r^{\frac{n-1}{2}} \frac{\partial\phi}{\partial t}}{\partial v^*}\bigg) \bigg\} + \int_{\Sigma_{\tau_0}} \big(J^N(\phi) + J^N(T\cdot\phi) + J^T(T^2\phi), n\big)\bigg]. \tag{5-26}$$

Again, with the sequence $(\tau_j{}'')_{j\in\mathbb{N}}$,

$$\int_{\Sigma_{\tau_j''}} \big(J^N(\phi), n\big) \leq \frac{1}{\tau_{j+1}} \int_{\tau_j'}^{\tau_{j+2}'} \mathrm{d}\tau \int_{\Sigma_\tau} \big(J^N(\phi), n\big) \tag{5-27}$$

and since $\dfrac{1}{\tau_{j+1}}\dfrac{1}{\tau_j'} \leq \dfrac{2^5}{\tau_j''^2}$ we obtain by virtue of Proposition 5.2 our final result:

$$\int_{\Sigma_\tau} \big(J^N(\phi), n\big) \leq \frac{C(n,m,R)}{\tau^2} \bigg[\int_{\tau_0+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \bigg\{ r^2 \bigg(\frac{\partial r^{\frac{n-1}{2}}\phi}{\partial v^*}\bigg)^2 + r^2 \bigg(\frac{\partial r^{\frac{n-1}{2}} \frac{\partial\phi}{\partial t}}{\partial v^*}\bigg)^2 \bigg\} + \int_{\Sigma_{\tau_0}} \big(J^N(\phi) + J^N(T\cdot\phi) + J^T(T^2\cdot\phi), n\big)\bigg]. \qquad \square$$

5C. ***Improved interior decay of the first-order energy.*** In this section we prove an energy estimate for the first-order energy which improves the decay rate as compared to Proposition 5.4 in a bounded radial region.

Remark 5.5. The argument largely depends on the asymptotic properties of the spacetime, and is similar and slightly easier in Minkowski space [Schlue 2012].

Proposition 5.6 (improved interior first-order energy decay). *Let* $0 < \delta < \frac{1}{2}$, $R > \sqrt[n-2]{\frac{8nm}{\delta}}$, *and let* ϕ *be a solution of the wave equation* (1-1) *with initial data on* Σ_{τ_1} $(\tau_1 > 0)$ *satisfying*

$$D \doteq \int_{\tau_1+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \Biggl\{ \sum_{k=0}^{1} r^{4-\delta} \Bigl(\frac{\partial (T^k \cdot \chi)}{\partial v^*} \Bigr)^2 + \sum_{k=0}^{4} r^2 \Bigl(\frac{\partial (T^k \cdot \psi)}{\partial v^*} \Bigr)^2 + \sum_{k=0}^{3} \sum_{i=1}^{\frac{n(n-1)}{2}} r^2 \Bigl(\frac{\partial T^k \Omega_i \psi}{\partial v^*} \Bigr)^2 \Biggr\} \Bigg|_{u^*=\tau_1} + \int_{\Sigma_{\tau_1}} \Bigl(\sum_{k=0}^{5} J^N(T^k \cdot \phi) + \sum_{k=0}^{4} \sum_{i=1}^{\frac{n(n-1)}{2}} J^N(T^k \Omega_i \phi), n \Bigr) < \infty. \tag{5-28}$$

Then there exists a constant $C(n, m, \delta, R)$ *such that*

$$\int_{\Sigma'_\tau} \bigl(J^N(T \cdot \phi), n \bigr) \leq \frac{CD}{\tau^{4-2\delta}} \quad (\tau > \tau_1), \tag{5-29}$$

where $\Sigma'_\tau = \Sigma_\tau \cap \{r \leq R\}$.

In addition to the weighted energy identity arising from the multiplier $r^p \frac{\partial}{\partial v^*}$ that was used to prove Proposition 5.4, we will here also use a commutation with $\frac{\partial}{\partial v^*}$ to obtain the energy decay for $\frac{\partial \phi}{\partial t}$ of Proposition 5.6.

Weighted energy and commutation. Consider the current

$$\overset{v}{J}_\mu(\phi) \doteq T_{\mu\nu}(\chi) V^\nu, \tag{5-30}$$

where now

$$\chi \doteq \partial_{v^*} \psi = \frac{\partial (r^{\frac{n-1}{2}} \phi)}{\partial v^*}, \quad V = r^q \frac{\partial}{\partial v^*}, \quad q = p - (n-1), \quad 2 < p < 4, \quad \delta = 4 - p. \tag{5-31}$$

Notation. To make the dependence on p explicit, we define

$$\overset{v}{K}_p(\phi) \doteq \nabla^\mu \overset{v}{J}_\mu(\phi). \tag{5-32}$$

The error terms for $\overset{v}{K}$ arise from the fact that χ is not a solution of (1-1); here, similarly to (5-11), we find

$$\Box_g \chi = -\frac{n-1}{2r^3}\left\{(n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\left(\frac{2m}{r^{n-2}}\right)^2\right\}\psi + \frac{n-1}{4r^2}\left[(n-3)+(n-1)\frac{2m}{r^{n-2}}\right]\chi + \frac{1}{r}\left[2-n\frac{2m}{r^{n-2}}\right]\not{\Delta}\psi + \frac{n-1}{r}\frac{\partial\chi}{\partial r^*}. \tag{5-33}$$

Hence

$$\overset{v}{K}_p(\phi) = \Box(\chi)\, V\cdot\chi + K^V(\chi)$$

$$= \frac{1}{2}p\, r^{q-1}\left(\frac{\partial\chi}{\partial v^*}\right)^2 + \frac{1}{2}\left[(2-p)-(n-p)\frac{2m}{r^{n-2}}\right] r^{q-1}\left|\not{\nabla}\chi\right|^2 \tag{5-34}$$

$$-\frac{n-1}{2}r^{q-3}\left[(n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\left(\frac{2m}{r^{n-2}}\right)^2\right]\psi\frac{\partial^2\psi}{\partial v^{*2}} \tag{5-35}$$

$$+\frac{n-1}{8}r^{q-2}\left[(n-3)+(n-1)\frac{2m}{r^{n-2}}\right]\frac{\partial\chi^2}{\partial v^*} \tag{5-36}$$

$$+r^{q-1}\left[2-n\frac{2m}{r^{n-2}}\right](\not{\Delta}\psi)\left(\frac{\partial\chi}{\partial v^*}\right), \tag{5-37}$$

which is not positive definite. However, we have

$$\frac{1}{4}p\, r^{p-1}\left(\frac{\partial\chi}{\partial v^*}\right)^2 \le \overset{v}{K}_p(\phi)\cdot r^{n-1} + \frac{1}{2}\left[(p-2)+(n-p)\frac{2m}{r^{n-2}}\right] r^{p-1}\left|\not{\nabla}\chi\right|^2 + \frac{(n-1)^2(n-2)^2}{2} r^{(p-2)-1}\frac{1}{r^2}\psi^2 + \frac{4}{p}r^{(p-2)-1}r^2(\not{\Delta}\psi)^2 - \frac{n-1}{8}r^{p-2}\left[(n-3)+(n-1)\frac{2m}{r^{n-2}}\right]\frac{\partial\chi^2}{\partial v^*}, \tag{5-38}$$

where we have used that

$$n-2 > n-3+n\frac{2m}{r^{n-2}}-(n-1)^2\left(\frac{2m}{r^{n-2}}\right)^2 \ge n-3 \tag{5-39}$$

(is decreasing) on $r > \sqrt[n-2]{4nm}$. The key insight here is that we are able to control all other terms on the right-hand side of (5-38) by the current $\overset{r}{J}$ of Section 5B with $p-2$ in the role of p; i.e.,

$$\int_{\tau_1}^{\tau_2} du^* \int_{u^*+R^*}^{\infty} dv^* \int_{\mathbb{S}^{n-1}} d\mu_{\mathring{\gamma}_{n-1}}\, r^{p-1}\left(\frac{\partial\chi}{\partial v^*}\right)^2$$
$$\le C(n,m,\delta,p,R)\int_{{}^R\mathcal{D}_{\tau_1}^{\tau_2}}\left\{\overset{v}{K}_p(\phi)+\overset{r}{K}_{p-2}(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}\overset{r}{K}_{p-2}(\Omega_i\phi)\right\}$$
$$+C(n,m,\delta,p,R)\int_{2\tau_1+R^*}^{2\tau_2+R^*} dt\int_{\mathbb{S}^{n-1}} d\mu_{\mathring{\gamma}_{n-1}}\left\{\psi^2+\left(\frac{\partial\psi}{\partial v^*}\right)^2+\sum_{i=1}^{\frac{n(n-1)}{2}}(\Omega_i\psi)^2+\sum_{i=1}^{\frac{n(n-1)}{2}}\left|\not{\nabla}\Omega_i\psi\right|^2\right\}\Bigg|_{r=R}. \tag{5-40}$$

Indeed, the first term $|\nabla\!\!\!/\,\partial_{v^*}\psi|^2$ can be integrated by parts twice (such that we can absorb the resulting $\partial_{v^*}\chi$ term in the left-hand side):

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{p-1}\big|\nabla\!\!\!/\,\chi\big|^2\\
&\quad=-\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{p-1}\,\triangle\!\!\!/\,\partial_{v^*}\psi\cdot\partial_{v^*}\psi\\
&\quad=-\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{p-1}\,\triangle\!\!\!/\,\psi\,\frac{\partial\psi}{\partial v^*}\bigg|_{u^*+R^*}^{\infty}\\
&\qquad+\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\bigg\{(p-1)r^{p-2}\Big(1-\frac{2m}{r^{n-2}}\Big)(\triangle\!\!\!/\,\psi)\Big(\frac{\partial\psi}{\partial v^*}\Big)\\
&\qquad\qquad+r^{p-1}(\triangle\!\!\!/\,\psi)\frac{\partial\chi}{\partial v^*}+r^{p-1}\frac{2}{r}\Big(1-\frac{2m}{r^{n-2}}\Big)(\triangle\!\!\!/\,\psi)\frac{\partial\psi}{\partial v^*}\bigg\}\\
&\quad\leq\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{p-1}\,(\triangle\!\!\!/\,\psi)\Big(\frac{\partial\psi}{\partial v^*}\Big)\bigg|_{v^*=u^*+R^*}\\
&\qquad+\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\bigg\{\Big(p-1+\frac{n-2}{p}+2\Big)r^{(p-2)-1}(\triangle\!\!\!/\,\psi)^2r^2\\
&\qquad\qquad+(p-1+2)r^{(p-2)-1}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\frac{1}{2}\frac{p}{4}\frac{2}{n-2}r^{p-1}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2\bigg\}.
\end{aligned}\tag{5-41}$$

The second term in (5-38) is controlled by the Hardy inequality

$$\begin{aligned}
&\frac{1}{2}\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\, r^{(p-2)-1}\frac{1}{r^2}\psi^2\\
&\quad\leq\frac{1}{4-p}\frac{1}{R^{4-p}}\frac{1}{1-\frac{2m}{R^{n-2}}}\psi^2\big|_{(u^*,u^*+R^*)}+\frac{2}{(4-p)^2(1-\frac{2m}{R^{n-2}})^2}\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\, r^{(p-2)-1}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2,
\end{aligned}\tag{5-42}$$

and the third term simply by the following commutation with Ω_i:

Lemma 5.7. *For any function $\phi\in\mathrm{H}^2(S_r)$ we have $\triangle\!\!\!/_{r^2\mathring{\gamma}_{n-1}}\phi\in\mathrm{L}^2(S_r)$, and there exists a constant $C>0$ such that*

$$\int_{\mathbb{S}^{n-1}}(\triangle\!\!\!/\,\psi)^2r^2\,\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\leq C\int_{\mathbb{S}^{n-1}}\bigg\{\sum_{i=1}^{\frac{n(n-1)}{2}}\big|\nabla\!\!\!/(\Omega_i\psi)\big|^2+\big|\nabla\!\!\!/\,\psi\big|^2\bigg\}\,\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}.\tag{5-43}$$

The last term in (5-38) we can rearrange as follows:

$$\begin{aligned}
&-\frac{n-1}{8}r^{p-2}\Big[n-3+(n-1)\frac{2m}{r^{n-2}}\Big]\frac{\partial\chi^2}{\partial v^*}\\
&\quad=-\frac{\partial}{\partial v^*}\bigg\{\frac{n-1}{8}r^{p-2}\Big[(n-3)+(n-1)\frac{2m}{r^{n-2}}\Big]\Big(\frac{\partial\psi}{\partial v^*}\Big)^2\bigg\}\\
&\qquad+\frac{n-1}{8}r^{(p-2)-1}\Big[(p-2)(n-3)+(n-1)\big((p-2)+(n-2)\big)\frac{2m}{r^{n-2}}\Big]\Big(1-\frac{2m}{r^{n-2}}\Big)\Big(\frac{\partial\psi}{\partial v^*}\Big)^2.
\end{aligned}\tag{5-44}$$

Therefore (see also Appendix B),

$$
\begin{aligned}
&\int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{u^*+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \frac{1}{8} p\, r^{p-1} \left(\frac{\partial \chi}{\partial v^*}\right)^2 \\
&\quad \le \frac{1}{2} \frac{1}{1-\frac{2m}{R^{n-2}}} \int_{{}^R\mathcal{D}_{\tau_1}^{\tau_2}} \overset{v}{K}_p\,(\phi)\,\mathrm{d}\mu_g \\
&\qquad + C(n,p,\delta,R) \int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left\{ r^{p-2} (\not\Delta \psi)^2 r^2 + r^{p-2} \left(\frac{\partial \psi}{\partial v^*}\right)^2 + \psi^2 \right\}\Big|_{v^*=u^*+R^*} \\
&\qquad + C(p,n,\delta,R) \int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{u^*+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \\
&\qquad\qquad \times \left\{ r^{(p-2)-1} \sum_{i=1}^{\frac{n(n-1)}{2}} \left|\not\nabla \Omega_i \psi\right|^2 + r^{(p-2)-1} \left|\not\nabla \psi\right|^2 + r^{(p-2)-1} \left(\frac{\partial \psi}{\partial v^*}\right)^2 \right\} \\
&\qquad - \frac{n-1}{8} \int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{u^*+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{p-2} \left[n+3+(n-1)\frac{2m}{r^{n-2}} \right] \frac{\partial \chi^2}{\partial v^*}.
\end{aligned}
\tag{5-45}
$$

Now, recall (5-15), and note that

$$
\delta - \big(n-(2-\delta)\big)\frac{2m}{r^{n-2}} > \frac{\delta}{2} \quad \left(r > \sqrt[n-2]{\frac{4nm}{\delta}} \right),
\tag{5-46}
$$

to see that

$$
\begin{aligned}
&r^{(p-2)-1} \sum_{i=1}^{\frac{n(n-1)}{2}} \left|\not\nabla r^{\frac{n-1}{2}} \Omega_i \phi\right|^2 \\
&\quad \le \frac{4}{\delta} \sum_{i=1}^{\frac{n(n-1)}{2}} \overset{r}{K}_{p-2}\,(\Omega_i \phi) r^{n-1} - \frac{4}{\delta} \sum_{i=1}^{\frac{n(n-1)}{2}} \frac{n-1}{4} \left[\frac{n-3}{2} + \frac{n-1}{2} \frac{2m}{r^{n-2}} \right] r^{(p-2)-2} \frac{\partial \big(r^{\frac{n-1}{2}} \Omega_i \phi\big)^2}{\partial v^*}.
\end{aligned}
\tag{5-47}
$$

So

$$
\begin{aligned}
&\int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{u^*+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, p\, r^{p-1} \left(\frac{\partial \chi}{\partial v^*}\right)^2 \\
&\quad \le C(n,m,\delta,p,R) \int_{{}^R\mathcal{D}_{\tau_1}^{\tau_2}} \left\{ \overset{v}{K}_p\,(\phi) + \overset{r}{K}_{p-2}\,(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} \overset{r}{K}_{p-2}\,(\Omega_i \phi) \right\} \\
&\qquad - C(n,m,\delta,p) \int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{u^*+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \frac{n-1}{8} \left[n-3+(n-1)\frac{2m}{r^{n-2}} \right] \\
&\qquad\qquad \times \left\{ \frac{r^{p-2}}{r^2} \frac{\partial \psi^2}{\partial v^*} + \frac{r^{p-2}}{r^2} \sum_{i=1}^{\frac{n(n-1)}{2}} \frac{\partial \big(\Omega_i \psi\big)^2}{\partial v^*} + \frac{r^p}{r^2} \frac{\partial \chi^2}{\partial v^*} \right\} \\
&\qquad + C(n,m,\delta,p,R) \int_{2\tau_1+R^*}^{2\tau_2+R^*} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left\{ \sum_{i=1}^{\frac{n(n-1)}{2}} \left|\not\nabla \Omega_i \phi\right|^2 + \left|\not\nabla \phi\right|^2 + \left(\frac{\partial \psi}{\partial v^*}\right)^2 + \psi^2 \right\}\Big|_{r=R},
\end{aligned}
\tag{5-48}
$$

which, upon integration by parts, yields (5-40); note that the $\partial_{v^*}\psi^2$ and $\partial_{v^*}(\Omega_i\psi)^2$ terms generate boundary terms at infinity and zeroth-order bulk terms *with the right sign* by (5-16), while the $\partial_{v^*}\chi^2$ is reduced to a $(\partial_{v^*}\psi)^2$ term by (5-44).

By virtue of Stokes' theorem (B-5) and in view of (B-6), we conclude that

$$\begin{aligned}
&\int_{\tau_2+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left\{r^p\left(\frac{\partial\chi}{\partial v^*}\right)^2+r^{p-2}\left(\frac{\partial\psi}{\partial v^*}\right)^2+\sum_{i=1}^{\frac{n(n-1)}{2}}r^{p-2}\left(\frac{\partial\Omega_i\psi}{\partial v^*}\right)^2\right\}\Bigg|_{u^*=\tau_2}\\
&\qquad\qquad+\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,r^{p-1}\left(\frac{\partial\chi}{\partial v^*}\right)^2\\
&\leq C(n,m,\delta,p,R)\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\\
&\qquad\qquad\times\left\{r^p\left(\frac{\partial\chi}{\partial v^*}\right)^2+r^{p-2}\left(\frac{\partial\psi}{\partial v^*}\right)^2+\sum_{i=1}^{\frac{n(n-1)}{2}}r^{p-2}\left(\frac{\partial\Omega_i\psi}{\partial v^*}\right)^2\right\}\Bigg|_{u^*=\tau_1}\\
&\quad+C(n,m,\delta,p,R)\int_{2\tau_1+R^*}^{2\tau_2+R^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left\{\psi^2+\left(\frac{\partial\psi}{\partial v^*}\right)^2+\left(\frac{\partial^2\psi}{\partial v^{*2}}\right)^2\right.\\
&\qquad\qquad+\sum_{i=1}^{\frac{n(n-1)}{2}}\left[(\Omega_i\psi)^2+\left(\frac{\partial\Omega_i\psi}{\partial v^*}\right)^2\right]+\left|\nabla\!\!\!/\,\chi\right|^2+(\nabla\!\!\!/\,\psi)^2\\
&\qquad\qquad\left.+\sum_{i=1}^{\frac{n(n-1)}{2}}\left|\nabla\!\!\!/\,\Omega_i\psi\right|^2\right\}\Bigg|_{r=R}.
\end{aligned}\tag{5-49}$$

Proof of Proposition 5.6. We shall use this *weighted energy inequality* for χ to proceed in a hierarchy of four steps.

$\boxed{p=4-\delta:}$ Let $\tau_1>0$ and $\tau_{j+1}=2\tau_j$ ($j\in\mathbb{N}$). In a first step we use (5-49) with $p=4-\delta$ and (5-17) with $p=2$ as an estimate for the spacetime integral of $\partial_{v^*}\chi$, $\partial_{v^*}\psi$, and $\partial_{v^*}(\Omega_j\psi)$ on ${}^R\mathcal{D}_{\tau_j}^{\tau_j+1}$, and in a second step as an estimate for the corresponding integral on the future boundary of ${}^R\mathcal{D}_{\tau_1}^{\tau_j}$:

$$\begin{aligned}
&\int_{\tau_j}^{\tau_{j+1}}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left\{r^{3-\delta}\left(\frac{\partial\chi}{\partial v^*}\right)^2+r\left(\frac{\partial\psi}{\partial v^*}\right)^2+\sum_{i=1}^{\frac{n(n-1)}{2}}r\left(\frac{\partial\Omega_i\psi}{\partial v^*}\right)^2\right\}\\
&\leq C(n,m,\delta,R)\int_{\tau_j+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left\{r^{4-\delta}\left(\frac{\partial\chi}{\partial v^*}\right)^2+r^2\left(\frac{\partial\psi}{\partial v^*}\right)^2+\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\left(\frac{\partial\Omega_j\psi}{\partial v^*}\right)^2\right\}\Bigg|_{u^*=\tau_j}\\
&\quad+C(n,m,\delta,R)\int_{2\tau_j+R^*}^{2\tau_{j+1}+R^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left\{\psi^2+\left(\frac{\partial\psi}{\partial v^*}\right)^2+\left(\frac{\partial^2\psi}{\partial v^{*2}}\right)^2+|\nabla\!\!\!/\,\psi|^2+\left|\nabla\!\!\!/\,\frac{\partial\psi}{\partial v^*}\right|^2\right.\\
&\qquad\qquad\left.+\sum_{i=1}^{\frac{n(n-1)}{2}}\left[(\Omega_i\psi)^2+\left(\frac{\partial\Omega_i\psi}{\partial v^*}\right)^2+\left|\nabla\!\!\!/\,\Omega_i\psi\right|^2\right]\right\}\Bigg|_{r=R}
\end{aligned}$$

$$\leq C(n,m,\delta,R)\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left\{r^{4-\delta}\left(\frac{\partial\chi}{\partial v^*}\right)^2+r^2\left(\frac{\partial\psi}{\partial v^*}\right)^2+\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\left(\frac{\partial\Omega_i\psi}{\partial v^*}\right)^2\right\}\Big|_{u^*=\tau_1}$$

$$+C(n,m,\delta,R)\int_{2\tau_1+R^*}^{2\tau_{j+1}+R^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left\{\psi^2+\left(\frac{\partial\psi}{\partial v^*}\right)^2+\left(\frac{\partial^2\psi}{\partial v^{*2}}\right)^2+(\nabla\!\!\!/\psi)^2+\left|\nabla\!\!\!/\frac{\partial\psi}{\partial v^*}\right|^2\right.$$

$$\left.+\sum_{i=1}^{\frac{n(n-1)}{2}}\left[(\Omega_i\psi)^2+\left(\frac{\partial\Omega_i\psi}{\partial v^*}\right)^2+\left|\nabla\!\!\!/\Omega_i\psi\right|^2\right]\right\}\Big|_{r=R}. \tag{5-50}$$

Thus by the mean value theorem of integration we obtain a sequence $\tau_j'\in(\tau_j,\tau_{j+1})$ $(j\in\mathbb{N})$ such that the corresponding integral from the left-hand side on $u^*=\tau_j'$ is bounded by τ_j^{-1} times the right-hand side of (5-50).

$\boxed{p=3-\delta:}$ Next we shall use (5-49) with $p=3-\delta$ on ${}^{R_j}\mathcal{D}_{\tau_{2j-1}'}^{\tau_{2j+1}'}$ (with $R_j^*\in(R^*,R^*+1)$ $(j\in\mathbb{N})$ chosen appropriately below). However, the quantity we are actually interested in is not $\partial_{v^*}\chi$, but rather

$$\left(\frac{\partial r^{\frac{n-1}{2}}T\cdot\phi}{\partial v^*}\right)^2=\left(\frac{\partial T\cdot r^{\frac{n-1}{2}}\phi}{\partial v^*}\right)^2=\left(\frac{1}{2}\frac{\partial^2\psi}{\partial v^{*2}}+\frac{1}{2}\frac{\partial^2\psi}{\partial u^*\partial v^*}\right)^2$$

$$\overset{(5\text{-}12)}{=}\left(\frac{1}{2}\frac{\partial^2\psi}{\partial v^{*2}}+\frac{1}{2}\left(1-\frac{2m}{r^{n-2}}\right)\Delta\!\!\!/\psi-\frac{1}{2}\frac{n-1}{2}\left(\frac{n-3}{2}+\frac{n-1}{2}\frac{2m}{r^{n-2}}\right)\frac{1}{r^2}\left(1-\frac{2m}{r^{n-2}}\right)\psi\right)^2$$

$$\leq\left(\frac{\partial^2\psi}{\partial v^{*2}}\right)^2+(\Delta\!\!\!/\psi)^2+\frac{n-1}{4}2(n-2)\frac{1}{r^4}\psi^2. \tag{5-51}$$

Using the simple Hardy inequality

$$\frac{1}{2}\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\,r^{2-\delta}\frac{1}{r^4}\psi^2$$

$$\leq\frac{1}{1-\frac{2m}{R^{n-2}}}\frac{1}{r^{1+\delta}}\psi^2(u^*,u^*+R^*)+\frac{2}{\left(1-\frac{2m}{R^{n-2}}\right)^2}\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\frac{1}{r^\delta}\left(\frac{\partial\psi}{\partial v^*}\right)^2 \tag{5-52}$$

and again the commutation introduced in Lemma 5.7, we obtain

$$\int_{\tau_{2j-1}'}^{\tau_{2j+1}'}\mathrm{d}u^*\int_{u^*+R_j{}^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,r^{2-\delta}\left(\frac{\partial r^{\frac{n-1}{2}}T\cdot\phi}{\partial v^*}\right)^2$$

$$\leq\int_{\tau_{2j-1}'}^{\tau_{2j+1}'}\mathrm{d}u^*\int_{u^*+R_j{}^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left\{r^{2-\delta}\left(\frac{\partial\chi}{\partial v^*}\right)^2\right.$$

$$\left.+\frac{C}{r^\delta}\sum_{i=1}^{\frac{n(n-1)}{2}}\left|\nabla\!\!\!/\Omega_i\psi\right|^2+\frac{C}{r^\delta}(\nabla\!\!\!/\psi)^2+\frac{(n-1)(n-2)}{2}\frac{2}{\left(1-\frac{2m}{R^{n-2}}\right)^2}r^{-\delta}\left(\frac{\partial\psi}{\partial v^*}\right)^2\right\}$$

$$+\frac{1}{1-\frac{2m}{R^{n-2}}}\int_{2\tau_{2j-1}'+R_j^*}^{2\tau_{2j+1}'+R_j^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left\{\frac{1}{r^{1+\delta}}\psi^2\right\}\Big|_{r=R_j}$$

$$\leq C(n,m,\delta)\int_{\tau'_{2j-1}}^{\tau'_{2j+1}}\mathrm{d}u^*\int_{u^*+R_j{}^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Bigl\{r^{2-\delta}\Bigl(\frac{\partial\chi}{\partial v^*}\Bigr)^2 + \overset{r}{K}_{1-\delta}(\phi)r^{n-1}+\sum_{i=1}^{\frac{n(n-1)}{2}}\overset{r}{K}_{1-\delta}(\Omega_i\phi)r^{n-1}\Bigr\}$$
$$+C(n,m,\delta)\int_{2\tau'_{2j-1}+R_j^*}^{2\tau'_{2j+1}+R_j^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Bigl\{\psi^2+\sum_{i=1}^{\frac{n(n-1)}{2}}(\Omega_i\psi)^2\Bigr\}\Bigr|_{r=R_j}, \tag{5-53}$$

where in the last step we have again used (5-16). Furthermore, by now applying (5-49) with $p=3-\delta$,

$$\int_{\tau'_{2j-1}}^{\tau'_{2j+1}}\mathrm{d}u^*\int_{u^*+R_j{}^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,r^{2-\delta}\Bigl(\frac{\partial r^{\frac{n-1}{2}}T\cdot\phi}{\partial v^*}\Bigr)^2$$
$$\leq C(n,m,\delta,R)\int_{\tau'_{2j-1}+R_j^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Bigl\{r^{3-\delta}\Bigl(\frac{\partial\chi}{\partial v^*}\Bigr)^2+r\Bigl(\frac{\partial\psi}{\partial v^*}\Bigr)^2+\sum_{i=1}^{\frac{n(n-1)}{2}}r\Bigl(\frac{\partial\Omega_i\psi}{\partial v^*}\Bigr)^2\Bigr\}\Bigr|_{u^*=\tau'_{2j-1}}$$
$$+C(n,m,\delta,R)\int_{2\tau'_{2j-1}+R_j^*}^{2\tau'_{2j+1}+R_j^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Bigl\{\psi^2+\Bigl(\frac{\partial\psi}{\partial v^*}\Bigr)^2+\Bigl(\frac{\partial^2\psi}{\partial v^{*2}}\Bigr)^2+(\nabla\!\!\!/\psi)^2+\Bigl|\nabla\!\!\!/\frac{\partial\psi}{\partial v^*}\Bigr|^2$$
$$+\sum_{i=1}^{\frac{n(n-1)}{2}}\Bigl[(\Omega_i\psi)^2+\Bigl(\frac{\partial\Omega_i\psi}{\partial v^*}\Bigr)^2+\bigl|\nabla\!\!\!/\Omega_i\psi\bigr|^2\Bigr]\Bigr\}\Bigr|_{r=R_j}, \tag{5-54}$$

we obtain a sequence $\tau''_j\in(\tau'_{2j-1},\tau'_{2j+1})$ $(j\in\mathbb{N})$ such that, in view of the previous step,

$$\int_{\tau''_j+R_j^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Bigl\{r^{2-\delta}\Bigl(\frac{\partial(r^{\frac{n-1}{2}}T\cdot\phi)}{\partial v^*}\Bigr)^2\Bigr\}\Bigr|_{u^*=\tau''_j}$$
$$\leq\frac{C(n,m,\delta,R)}{\tau_{2j}\tau_{2j-1}}\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Bigl\{r^{4-\delta}\Bigl(\frac{\partial\chi}{\partial v^*}\Bigr)^2+r^2\Bigl(\frac{\partial\psi}{\partial v^*}\Bigr)^2+r^2\sum_{i=1}^{\frac{n(n-1)}{2}}\Bigl(\frac{\partial\Omega_i\psi}{\partial v^*}\Bigr)^2\Bigr\}\Bigr|_{u^*=\tau_1}$$
$$+\frac{C(n,m,\delta,R)}{\tau_{2j}\,\tau_{2j-1}}\int_{2\tau_1+R^*}^{2\tau_{2j+1}+R^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Bigl\{\psi^2+\Bigl(\frac{\partial\psi}{\partial v^*}\Bigr)^2+\Bigl(\frac{\partial^2\psi}{\partial v^{*2}}\Bigr)^2+(\nabla\!\!\!/\psi)^2+\Bigl|\nabla\!\!\!/\frac{\partial\psi}{\partial v^*}\Bigr|^2$$
$$+\sum_{i=1}^{\frac{n(n-1)}{2}}\Bigl[(\Omega_i\psi)^2+\Bigl(\frac{\partial\Omega_i\psi}{\partial v^*}\Bigr)^2+\bigl|\nabla\!\!\!/\Omega_i\psi\bigr|^2\Bigr]\Bigr\}\Bigr|_{r=R}$$
$$+C(n,m,\delta,R)\int_{2\tau'_{2j-1}+R_j^*}^{2\tau'_{2j+1}+R_j^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Bigl\{\psi^2+\Bigl(\frac{\partial\psi}{\partial v^*}\Bigr)^2+\Bigl(\frac{\partial^2\psi}{\partial v^{*2}}\Bigr)^2+(\nabla\!\!\!/\psi)^2+\Bigl|\nabla\!\!\!/\frac{\partial\psi}{\partial v^*}\Bigr|^2$$
$$+\sum_{i=1}^{\frac{n(n-1)}{2}}\Bigl[(\Omega_i\psi)^2+\Bigl(\frac{\partial\Omega_i\psi}{\partial v^*}\Bigr)^2+\bigl|\nabla\!\!\!/\Omega_i\psi\bigr|^2\Bigr]\Bigr\}\Bigr|_{r=R_j}. \tag{5-55}$$

Now, by writing out the derivatives of $\psi = r^{\frac{n-1}{2}}\phi$, and using (5-12), we calculate that

$$\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left\{ \psi^2 + \left(\frac{\partial\psi}{\partial v^*}\right)^2 + \left(\frac{\partial^2\psi}{\partial v^{*2}}\right)^2 + |\nabla\!\!\!/\psi|^2 + \left|\nabla\!\!\!/\frac{\partial\psi}{\partial v^*}\right|^2 + \sum_{i=1}^{\frac{n(n-1)}{2}} \left[(\Omega_i\psi)^2 + \left(\frac{\partial\Omega_i\psi}{\partial v^*}\right)^2 + |\nabla\!\!\!/\Omega_i\psi|^2 \right] \right\}\bigg|_{r=R}$$
$$\le C(R) \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left\{ \phi^2 + \left(\frac{\partial\phi}{\partial v^*}\right)^2 + \left(\frac{\partial\phi}{\partial u^*}\right)^2 + \left(\frac{\partial T\cdot\phi}{\partial v^*}\right)^2 + |\nabla\!\!\!/\phi|^2 + \sum_{i=1}^{\frac{n(n-1)}{2}} \left[|\nabla\!\!\!/\Omega_i\phi|^2 + \left(\frac{\partial\Omega_i\phi}{\partial v^*}\right)^2 \right] \right\}\bigg|_{r=R}; \quad (5\text{-}56)$$

by applying Proposition 4.1 first to the domain ${}^{r_1}\mathcal{D}_{\tau_1}^{\tau_{2j+1}} \subset \mathcal{R}^{\infty}_{r_0,r_1}(2\tau_1 + r_1^*)$ where $r_1 > \sqrt[n-2]{\frac{4nm}{\delta}}$ to fix the radius R, and then to the domain

$${}^{r(r^*=R^*+1)}\mathcal{D}_{\tau'_{2j-1}}^{\tau'_{2j+1}} \setminus {}^{R}\mathcal{D}_{\tau'_{2j-1}}^{\tau'_{2j+1}} \subset \mathcal{R}^{\infty}_{r_0,R}(2\tau'_{2j-1} + R^*)$$

to fix the radii R_j ($j \in \mathbb{N}$) by using the mean value theorem for the integration in r^*, this yields (see also Appendix B)

$$\int_{\tau''_j+R^*_j}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left\{ r^{2-\delta} \left(\frac{\partial r^{\frac{n-1}{2}} T\cdot\phi}{\partial v^*} \right)^2 \right\}\bigg|_{u^*=\tau''_j}$$
$$\le \frac{C(n,m,\delta,R)}{(\tau''_j)^2} \Bigg\{ \int_{\tau_1+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left\{ r^{4-\delta}\left(\frac{\partial\chi}{\partial v^*}\right)^2 + r^2\left(\frac{\partial\psi}{\partial v^*}\right)^2 + r^2 \sum_{i=1}^{\frac{n(n-1)}{2}} \left(\frac{\partial\Omega_i\psi}{\partial v^*}\right)^2 \right\}\bigg|_{u^*=\tau_1}$$
$$+ \int_{\Sigma_{\tau_1}} \Bigg(J^T(\phi) + J^T(T\cdot\phi) + J^T(T^2\cdot\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} \left[J^T(\Omega_i\phi) + J^T(T\cdot\Omega_i\phi) \right], n \Bigg) \Bigg\}$$
$$+ C(n,m,\delta,R) \int_{\Sigma_{\tau_{2j-1}}} \Bigg(J^T(\phi) + J^T(T\cdot\phi) + J^T(T^2\cdot\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} \left[J^T(\Omega_i\phi) + J^T(T\cdot\Omega_i\phi) \right], n \Bigg). \quad (5\text{-}57)$$

Therefore, by Proposition 5.4,

$$\int_{\tau''_j+R^*_j}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left\{ r^{2-\delta} \left(\frac{\partial r^{\frac{n-1}{2}} T\cdot\phi}{\partial v^*} \right)^2 \right\}\bigg|_{u^*=\tau''_j}$$
$$\le \frac{C(n,m,\delta,R)}{(\tau''_j)^2} \Bigg\{ \int_{\tau_1+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left\{ r^{4-\delta}\left(\frac{\partial\chi}{\partial v^*}\right)^2 + \sum_{k=0}^{3} r^2 \left(\frac{\partial T^k\cdot\psi}{\partial v^*}\right)^2 + \sum_{k=0}^{2} \sum_{i=1}^{\frac{n(n-1)}{2}} r^2 \left(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\right)^2 \right\}\bigg|_{u^*=\tau_1}$$
$$+ \int_{\Sigma_{\tau_1}} \Bigg(\sum_{k=0}^{4} J^N(T^k\cdot\phi) + \sum_{k=0}^{3} \sum_{i=1}^{\frac{n(n-1)}{2}} J^N(T^k\Omega_i\phi), n \Bigg) \Bigg\}. \quad (5\text{-}58)$$

Remark 5.8. This statement should be compared to the assumptions of Proposition 5.4 (5-5), from which all that one can deduce with (5-17) is

$$\int_{\tau+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left\{r^2\left(\frac{\partial(r^{\frac{n-1}{2}}T\cdot\phi)}{\partial v^*}\right)^2\right\}\Big|_{u^*=\tau}<\infty\quad(\tau>\tau_0). \tag{5-59}$$

We shall now proceed along the lines of the proof of Proposition 5.4 in Section 5B, just that we have (5-58) as a starting point for the solution $T\cdot\phi$ of (1-1) (and (5-6)); however, as opposed to Proposition 5.4 the hierarchy does not descend from $p=2$ but $p<2$, which introduces a degeneracy in the last step, and requires the refinement of Proposition 4.1 to Proposition 4.21, and Proposition 4.4 to Proposition 4.22; see Section 4D.2.

Lemma 5.9 (pointwise decay under special assumptions). *Let ϕ be a solution of the wave equation (1-1), with initial data on Σ_{τ_1} $(\tau_1>0)$ satisfying*

$$D\doteq\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left\{r^{4-\delta}\left(\frac{\partial\chi}{\partial v^*}\right)^2+\sum_{k=0}^{3}r^2\left(\frac{\partial T^k\psi}{\partial v^*}\right)^2+\sum_{k=0}^{2}\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\left(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\right)^2\right\}\Big|_{u^*=\tau_1}$$
$$+\int_{\Sigma_{\tau_1}}\left(\sum_{k=0}^{4}J^N(T^k\cdot\phi)+\sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}}J^N(T^k\Omega_i\phi),n\right)<\infty$$

for some $\delta>0$ and

$$\int_{\tau'+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\times r^{2-\delta}\left(\frac{\partial T\cdot\psi}{\partial v^*}\right)^2\Big|_{u^*=\tau'}\le\frac{C(n,m,\delta,R)D}{\tau'^2} \tag{$*$}$$

for some $\tau'>\tau_1$. Then there is a constant $C(n,m,\delta,R)$ such that, for all $\tau>\tau'$,

$$\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,r^{n-1-\frac{\delta}{2}}(T\cdot\phi)^2\big|_{(u^*=\tau',v^*=R^*+\tau)}\le\frac{CD}{\tau'^2}.$$

Remark 5.10. Note the gain in powers of r in comparison to the boundary term arising in Proposition 4.22.

Proof. First, integrating from infinity,

$$(T\cdot\phi)(\tau',R^*+\tau')=-\int_{\tau'+R^*}^{\infty}\frac{\partial(T\cdot\phi)}{\partial v^*}\,\mathrm{d}v^*,$$

and then, by Cauchy's inequality,

$$\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}(T\cdot\phi)^2(\tau',R+\tau')$$
$$\le\int_{R^*+\tau'}^{\infty}\frac{1}{r^{n-1}}\,\mathrm{d}v^*\times\int_{R+\tau'}^{\infty}\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left(\frac{\partial(T\cdot\phi)}{\partial v^*}\right)^2r^{n-1}\,\mathrm{d}v^*$$
$$\le\frac{1}{2}\left(1-\frac{2m}{r^{n-2}|_{(u^*=\tau',v^*=R^*+\tau')}}\right)^{-1}\frac{1}{n-2}\frac{1}{r^{n-2}}C(m,n)\int_{R^*+\tau'}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,r^{n-1}\left(\frac{\partial(T\cdot\phi)}{\partial v^*}\right)^2.$$

Therefore, by Proposition 5.4,

$$\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \big(r^{n-2}(T\cdot\phi)^2\big)(\tau', R^*+\tau') \leq \frac{C(n,m)}{1-\frac{2m}{R^{n-2}}} \int_{\Sigma_{\tau'}} \big(J^T(T\cdot\phi), n\big) \leq \frac{C(n,m,R)}{\tau'^2} D. \qquad (**)$$

Now

$$\begin{aligned} r^{n-1} &\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} (T\cdot\phi)^2(\tau', R^*+\tau) \\ &= \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \big(r^{n-1}(T\cdot\phi)^2\big)(\tau', R^*+\tau') + \int_{R^*+\tau'}^{R^*+\tau} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} 2T\cdot\psi \frac{\partial T\cdot\psi}{\partial v^*} \\ &\leq R^{n-1} \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} (T\cdot\phi)^2(\tau', R^*+\tau') \\ &\quad + 2r^{\frac{\delta}{2}}\big|_{\substack{(u^*=\tau', \\ v^*=R^*+\tau)}} \sqrt{\int_{R^*+\tau'}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \frac{1}{r^2}(T\cdot\phi)^2 r^{n-1}} \sqrt{\int_{R^*+\tau'}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{2-\delta}\Big(\frac{\partial T\cdot\psi}{\partial v^*}\Big)^2}, \end{aligned}$$

which proves the pointwise estimate of the lemma in view of the Hardy inequality of Lemma C.2, Proposition 5.4, the assumption (∗) and (∗∗). ☐

$\boxed{p = 2-\delta:}$ By the weighted energy inequality with $p = 2-\delta$ and $r^{\frac{n-1}{2}} T\cdot\phi$ in the role of ψ (see (5-16) in particular),

$$\begin{aligned} \int_{\tau''_{2j-1}}^{\tau''_{2j+1}} \mathrm{d}u^* &\int_{u^*+R'^{\,*}_j}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{1-\delta}\Big(\frac{\partial T\cdot\psi}{\partial v^*}\Big)^2 \\ &\leq C(n,m) \int_{\tau''_{2j-1}}^{\tau''_{2j+1}} \mathrm{d}u^* \int_{u^*+R'^{\,*}_j}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \overset{r}{K}_{2-\delta}(T\cdot\phi) \\ &\qquad + C(n,m) \int_{2\tau''_{2j-1}+R'^{\,*}_j}^{2\tau''_{2j+1}+R'^{\,*}_j} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \{(T\cdot\psi)^2\}\big|_{r=R'_j} \\ &\leq C(n,m,R) \int_{\tau''_{2j-1}+R'^{\,*}_j}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \Big\{ r^{2-\delta}\Big(\frac{\partial T\cdot\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau''_{2j-1}} \\ &\qquad + C(n,m) \int_{2\tau''_{2j-1}+R'^{\,*}_j}^{2\tau''_{2j+1}+R'^{\,*}_j} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \Big\{ (T\cdot\psi)^2 + \Big(\frac{\partial T\cdot\psi}{\partial v^*}\Big)^2 + |\nabla\!\!\!/ T\cdot\psi|^2 \Big\}\Big|_{r=R'_j}, \end{aligned} \qquad (5\text{-}60)$$

where we choose $R'^{\,*}_j \in (R^*+1, R^*+2)$ such that Proposition 4.1 applied to the domain

$$^{r(r^*=R^*+2)}\mathcal{D}_{\tau''_{2j-1}}^{\tau''_{2j+1}} \setminus {}^{r(r^*=R^*+1)}\mathcal{D}_{\tau''_{2j-1}}^{\tau''_{2j+1}}$$

yields an estimate for the integral on the timelike boundary above in terms of the first- and second-order energies on $\Sigma_{\tau''_{2j-1}}$, which in turn decays by Proposition 5.4. Therefore there exists a sequence

$\tau_j''' \in (\tau_{2j-1}'', \tau_{2j+1}'')$ ($j \in \mathbb{N}$) such that

$$\int_{\tau_j'''+R_j'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left\{ r^{1-\delta}\left(\frac{\partial T\cdot\psi}{\partial v^*}\right)^2 \right\}\Big|_{u^*=\tau_j'''}$$
$$\leq \frac{C(n,m,\delta,R)}{(\tau_j''')^3}\left\{ \int_{\tau_1+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left\{ r^{4-\delta}\left(\frac{\partial\chi}{\partial v^*}\right)^2 + \sum_{k=0}^{3} r^2\left(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\right)^2 + \sum_{k=0}^{2}\sum_{i=1}^{\frac{n(n-1)}{2}} r^2\left(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\right)^2 \right\}\Big|_{u^*=\tau_1} + \int_{\Sigma_{\tau_1}} \left(\sum_{k=0}^{4} J^N(T^k\cdot\phi) + \sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}} J^N(T^k\Omega_i\phi), n \right) \right\}. \tag{5-61}$$

$\boxed{p = 1-\delta:}$ Since, by integrating by parts,

$$\int_{u^*+R^*}^{\infty} \mathrm{d}v^* \frac{1}{r^\delta}\left(\frac{\partial\psi}{\partial v^*}\right)^2$$
$$= \int_{u^*+R^*}^{\infty} \mathrm{d}v^* \frac{1}{r^\delta}\left\{ \frac{n-1}{2r} r^{\frac{n-1}{2}}\left(1-\frac{2m}{r^{n-2}}\right)\frac{\partial\left(r^{\frac{n-1}{2}}\phi^2\right)}{\partial v^*} + r^{n-1}\left(\frac{\partial\phi}{\partial v^*}\right)^2 \right\}$$
$$= \frac{1}{r^\delta}\frac{n-1}{2r}\left(1-\frac{2m}{r^{n-2}}\right)\psi^2\Big|_{u^*+R^*}^{\infty}$$
$$+ \int_{u^*+R^*}^{\infty} \mathrm{d}v^* \left\{ \frac{\delta}{r^{1+\delta}}\frac{n-1}{2r}\left(1-\frac{2m}{r^{n-2}}\right)^2\psi^2 + \frac{1}{r^\delta}\frac{n-1}{2r^2}\left(1-\frac{2m}{r^{n-2}}\right)\psi^2\left[(n-2)+\left(1-\frac{2m}{r^{n-2}}\right)\frac{n-3}{2}\right] + \frac{1}{r^\delta}\left(\frac{\partial\phi}{\partial v^*}\right)^2 r^{n-1} \right\}, \tag{5-62}$$

we have by (5-15) that also (with $R_j''^* \in (R^*+2, R^*+3)$)

$$\int_{\tau_{2j-1}'''}^{\tau_{2j+1}'''} \mathrm{d}u^* \int_{u^*+R_j''^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \frac{1}{r^\delta}\left\{ \left(\frac{\partial T\cdot\phi}{\partial v^*}\right)^2 + \left|\nabla\!\!\!/ T\cdot\phi\right|^2 \right\} r^{n-1}$$
$$\leq C(n,m)\left\{ \int_{\tau_{2j-1}'''}^{\tau_{2j+1}'''} \mathrm{d}u^* \int_{u^*+R_j''^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \overset{r}{K}_{1-\delta}(T\cdot\phi)\cdot r^{n-1} + \int_{2\tau_{2j-1}'''+R_j''^*}^{2\tau_{2j+1}'''+R_j''^*} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \left\{(T\cdot\psi)^2\right\}\big|_{r=R_j'} \right\}. \tag{5-63}$$

By virtue of Stokes' theorem (B-5), (B-6) and our previous result (5-61), we obtain

$$
\begin{aligned}
&\int_{\tau'''_{2j-1}}^{\tau'''_{2j+1}} \mathrm{d}u^* \int_{u^*+R''^*_j}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \frac{1}{r^\delta}\Big(J^T(T\cdot\phi), \frac{\partial}{\partial v^*}\Big) r^{n-1} \\
&\le C(n,m)\bigg\{ \int_{\tau'''_{2j-1}+R''^*_j}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \Big\{ r^{1-\delta}\Big(\frac{\partial(T\cdot\psi)}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau'''_{2j-1}} \\
&\qquad + \int_{2\tau'''_{2j-1}+R''^*_j}^{2\tau'''_{2j+1}+R''^*_j} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \Big\{ \Big(\frac{\partial(T\cdot\psi)}{\partial v^*}\Big)^2 + \big|\nabla\!\!\!/ T\cdot\psi\big|^2 + (T\cdot\psi)^2\Big\}\Big|_{r=R''_j}\bigg\} \\
&\le \frac{C(n,m,\delta,R)}{(\tau''_j)^3}\bigg\{ \int_{\tau_1+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \Big\{ r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2 + \sum_{k=0}^{3} r^2\Big(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Big)^2 \\
&\qquad + \sum_{k=0}^{2}\sum_{i=1}^{\frac{n(n-1)}{2}} r^2\Big(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1} \\
&\qquad + \int_{\Sigma_{\tau_1}}\Big(\sum_{k=0}^{4} J^N(T^k\cdot\phi) + \sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}} J^N(T^k\Omega_i\phi), n\Big)\bigg\} \\
&\quad + C(n,m,R)\bigg\{ \int_{\Sigma_{\tau''_{2(2j-1)-1}}^{\tau'''_{2j+1}}} \big(J^T(T\cdot\phi) + J^T(T^2\cdot\phi), n\big) \\
&\qquad + \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-2}(T\cdot\phi)^2\big|_{(u^*=\tau''_{2(2j-1)-1},\ v^*=R''^*_j+\tau'''_{2j+1})}\bigg\},
\end{aligned}
\tag{5-64}
$$

where in the last inequality we have used Proposition 4.22, and then chosen R''_j ($j \in \mathbb{N}$) suitably by Proposition 4.21; furthermore the inequality still holds if we add the integral of the nondegenerate energy on ${}^{R''_j}\mathcal{P}_{\tau'''_{2j-1}}^{\tau'''_{2j+1}}$ on the left-hand side and replace J^T by J^N in the first term of the integral on $\Sigma_{\tau''_{2(2j-1)-1}}^{\tau'''_{2j+1}}$ on the right-hand side. The last two terms on the right-hand side of (5-64) in fact decay with almost the same rate as the first; for first note here that we could have used Proposition 4.4 and Corollary 4.3 instead, and then employed Proposition 5.4 to obtain in any case that

$$
\begin{aligned}
&\int_{\tau'''_{2j-1}}^{\tau'''_{2j+1}} \mathrm{d}\tau \int_{\Sigma_\tau} \frac{1}{r^\delta}\big(J^N(T\cdot\phi), n\big) \\
&\quad \le \frac{C(n,m,\delta,R)}{(\tau'''_{2j-1})^2}\bigg\{ \int_{\tau_1+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \Big\{ r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2 + \sum_{k=0}^{3} r^2\Big(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Big)^2 \\
&\qquad + \sum_{k=0}^{2}\sum_{i=1}^{\frac{n(n-1)}{2}} r^2\Big(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1} \\
&\qquad + \int_{\Sigma_{\tau_1}}\Big(\sum_{k=0}^{4} J^N(T^k\cdot\phi) + \sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}} J^N(T^k\Omega_i\phi), n\Big)\bigg\}.
\end{aligned}
\tag{5-65}
$$

It then follows that there exists a sequence $\tau''''_j \in (\tau'''_{2j-1}, \tau'''_{2j+1})$ such that

$$\begin{aligned}
&\int_{\Sigma_{\tau_j''''}^{\tau'''_{2(j+2)+1}}} \big(J^N(T\cdot\phi),n\big)\\
&\quad\leq r^\delta\big|_{\substack{(u^*=\tau_j'''',\\ v^*={R''_{j+2}}^*+\tau'''_{2(j+2)+1})}} \int_{\Sigma_{\tau_j''''}} \frac{1}{r^\delta}\big(J^N(T\cdot\phi),n\big)\\
&\quad\leq \frac{C(n,m,\delta,R)}{(\tau_j'''')^{3-\delta}}\Bigg\{\int_{\tau_1+R^*}^\infty \mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Bigg\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2+\sum_{k=0}^3 r^2\Big(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Big)^2\\
&\qquad\qquad+\sum_{k=0}^2\sum_{i=1}^{\frac{n(n-1)}{2}} r^2\Big(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Big)^2\Bigg\}\Bigg|_{u^*=\tau_1}\\
&\qquad\qquad+\int_{\Sigma_{\tau_1}}\Bigg(\sum_{k=0}^4 J^N(T^k\cdot\phi)+\sum_{k=0}^3\sum_{i=1}^{\frac{n(n-1)}{2}} J^N(T^k\Omega_i\phi),n\Bigg)\Bigg\}, \qquad (5\text{-}66)
\end{aligned}$$

because $\tau_j''''(\tau'''_{2(j+2)+1}-\tau_j'''')^{-1}\leq 1$. Secondly, the assumptions of Lemma 5.9 are satisfied in view of (5-58) on $u^*=\tau_j''$ ($j\in\mathbb{N}$), which yields

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-2}(T\cdot\phi)^2\big|_{(u^*=\tau''_{2(2j-1)-1},\,v^*={R''_j}^*+\tau'''_{2j+1})}\\
&\quad\leq \frac{C(n,m,\delta,R)}{(\tau'''_{2j-1})^{3-\frac{\delta}{2}}}\Bigg\{\int_{\tau_1+R^*}^\infty \mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Bigg\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2+\sum_{k=0}^3 r^2\Big(\frac{\partial T^k\cdot\psi}{\partial v^*}\Big)^2\\
&\qquad\qquad+\sum_{k=0}^2\sum_{i=1}^{\frac{n(n-1)}{2}} r^2\Big(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Big)^2\Bigg\}\Bigg|_{u^*=\tau_1}\\
&\qquad\qquad+\int_{\Sigma_{\tau_1}}\Bigg(\sum_{k=0}^4 J^N(T^k\cdot\phi)+\sum_{k=0}^3 J^N(T^k\Omega_i\phi),n\Bigg)\Bigg\}, \qquad (5\text{-}67)
\end{aligned}$$

because also

$$\tau'''_{2j-1}(\tau'''_{2j+1}-\tau''_{2(2j-1)-1})^{-1}\leq C.$$

We shall now return to (5-64) — and its extension, which includes the nondegenerate energy on ${}^{R''_j}\mathcal{P}_{\tau'''_{2j-1}}^{\tau'''_{2j+1}}$ — to find that, after inserting (5-66) and using Proposition 5.2,

$$\begin{aligned}
&\int_{\Sigma_{\tau''_{2(2j-1)-1}}^{\tau'''_{2j+1}}} \big(J^N(T\cdot\phi)+J^T(T^2\cdot\phi),n\big)\\
&\quad\leq C\int_{\Sigma_{\tau'''_{(2j-1)-1}}^{\tau'''_{2j+1}}} \big(J^N(T\cdot\phi)+J^T(T^2\cdot\phi),n\big)\\
&\quad\leq C\int_{\Sigma_{\tau''''_{j-2}}^{\tau'''_{2j+1}}} \big(J^N(T\cdot\phi)+J^T(T^2\cdot\phi),n\big)
\end{aligned}$$

$$\leq \frac{C(n,m,\delta,R)}{(\tau''''_{j-2})^{3-\delta}}\biggl\{\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\biggl\{r^{4-\delta}\Bigl(\frac{\partial\chi}{\partial v^*}\Bigr)^2+r^{4-\delta}\Bigl(\frac{\partial(T\cdot\chi)}{\partial v^*}\Bigr)^2+\sum_{k=0}^{4}r^2\Bigl(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Bigr)^2 + \sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\Bigl(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Bigr)^2\biggr\}\bigg|_{u^*=\tau_1} + \int_{\Sigma_{\tau_1}}\Bigl(\sum_{k=0}^{5}J^N(T^k\cdot\phi)+\sum_{k=0}^{4}\sum_{i=1}^{\frac{n(n-1)}{2}}J^N(T^k\Omega_i\phi),n\Bigr)\biggr\}, \tag{5-68}$$

and using (5-67), that there exists (another) sequence $\tau'''''_j\in(\tau'''_{2j-1},\tau'''_{2j+1})$ ($j\in\mathbb{N}$) such that

$$\int_{\Sigma_{\tau'''''_j}}\frac{1}{r^\delta}(J^N(T\cdot\phi),n) \leq \frac{C(n,m,\delta,R)}{(\tau'''''_j)^{4-\delta}}\biggl\{\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\biggl\{\sum_{k=0}^{1}r^{4-\delta}\Bigl(\frac{\partial(T^k\cdot\chi)}{\partial v^*}\Bigr)^2+\sum_{k=0}^{4}r^2\Bigl(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Bigr)^2 + \sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\Bigl(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Bigr)^2\biggr\}\bigg|_{u^*=\tau_1} + \int_{\Sigma_{\tau_1}}\Bigl(\sum_{k=0}^{5}J^N(T^k\cdot\phi)+\sum_{k=0}^{4}\sum_{i=1}^{\frac{n(n-1)}{2}}J^N(T^k\Omega_i\phi),n\Bigr)\biggr\}. \tag{5-69}$$

So for any $\tau>\tau_1$ we can choose $j\in\mathbb{N}$ such that $\tau\in(\tau'''_{2j-1},\tau'''_{2j+1})$ to obtain finally by Proposition 5.2 that

$$\int_{\Sigma_\tau\cap\{r\leq R\}}(J^N(T\cdot\phi),n) \leq \int_{\Sigma_{\tau'''''_{j-1}}^{\tau'''_{2j+1}}}(J^N(T\cdot\phi),n)\leq r^\delta\big|_{(u^*=\tau'''''_{j-1},\, v^*=R^*+\tau'''_{2j+1})}\int_{\Sigma_{\tau'''''_{j-1}}}\frac{1}{r^\delta}(J^N(T\cdot\phi),n) \leq \frac{C(n,m,\delta,R)}{\tau^{4-2\delta}}\biggl\{\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\biggl\{\sum_{k=0}^{1}r^{4-\delta}\Bigl(\frac{\partial(T^k\cdot\chi)}{\partial v^*}\Bigr)^2+\sum_{k=0}^{4}r^2\Bigl(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Bigr)^2 + \sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\Bigl(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Bigr)^2\biggr\}\bigg|_{u^*=\tau_1} + \int_{\Sigma_{\tau_1}}\Bigl(\sum_{k=0}^{5}J^N(T^k\cdot\phi)+\sum_{k=0}^{4}\sum_{i=1}^{\frac{n(n-1)}{2}}J^N(T^k\Omega_i\phi),n\Bigr)\biggr\}. \qquad\square \tag{5-70}$$

Remark 5.11. For the removal of the restriction to dyadic sequences in the last step of the proof, (5-69) and (5-70), we could have equally obtained a decay estimate for the energy flux through $\Sigma_\tau\cap\{r^*\leq R^*+\tau^k\}$ (with $k\in\mathbb{N}$) by replacing $\Sigma_{\tau'''''_{j-1}}^{\tau'''_{2j+1}}$ by $\Sigma_{\tau'''''_{j-1}}^{\tau'''''_{j-1}+\tau^k}$ in the first estimate in (5-70); if $\delta>0$ for a chosen $k\in\mathbb{N}$

is restricted to $\delta < (1+k)^{-1}$ we then still obtain a decay rate of $\tau^{4-(1+k)\delta}$ for the energy flux through $\Sigma_\tau \cap \{r^* \le R^* + \tau^k\}$.

6. Pointwise bounds

In this section we first prove pointwise estimates on $|\phi|$ and $|\partial_t \phi|$ separately based on the energy decay results Propositions 5.4 and 5.6. Then we give the interpolation argument to improve the pointwise decay on $|\phi|$. As we shall see in view of the nondegenerate energy estimates of Section 5 we may restrict ourselves in the first place to a radial region away from the horizon. Recall the definition (4-3) of Σ_τ ($r_1 \doteq R > \sqrt[n-2]{8nm}$).

Proposition 6.1 (pointwise decay). (i) *Let ϕ be a solution of the wave equation (1-1), with initial data on Σ_{τ_0} ($\tau_0 > 0$) such that*

$$D \doteq \int_{\tau_0+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \sum_{k=0}^{[\frac{n}{2}]+1} r^2 \Bigl(\frac{\partial T^k \cdot \psi}{\partial v^*}\Bigr)^2 \Big|_{u^*=\tau_0} + \int_{\Sigma_{\tau_0}} \Bigl(\sum_{k=0}^{[\frac{n}{2}]+2} J^N(T^k \cdot \phi), n \Bigr) < \infty. \quad (6\text{-}1)$$

Then there is a constant $C(n,m)$ such that, for $r_0 < r < R$,

$$|\phi(t,r)| \le \frac{C(n,m)\sqrt{D}}{\tau} \quad \bigl(\tau = \tfrac{1}{2}(t - R^*) > \tau_0\bigr). \quad (6\text{-}2)$$

(ii) *If, moreover, the initial data satisfies*

$$\begin{aligned} D \doteq \int_{\tau_0+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \Biggl\{ \sum_{k=0}^{[\frac{n}{2}]+1} r^{4-\delta} \Bigl(\frac{\partial (T^k \cdot \chi)}{\partial v^*}\Bigr)^2 \\ + \sum_{k=0}^{[\frac{n}{2}]+4} r^2 \Bigl(\frac{\partial (T^k \cdot \psi)}{\partial v^*}\Bigr)^2 + \sum_{k=0}^{[\frac{n}{2}]+3} \sum_{i=1}^{\frac{n(n-1)}{2}} r^2 \Bigl(\frac{\partial T^k \Omega_i \psi}{\partial v^*}\Bigr)^2 \Biggr\} \Big|_{u^*=\tau_0} \\ + \int_{\Sigma_{\tau_0}} \Bigl(\sum_{k=0}^{[\frac{n}{2}]+5} J^N(T^k \cdot \phi) + \sum_{k=0}^{[\frac{n}{2}]+4} \sum_{i=1}^{\frac{n(n-1)}{2}} J^N(T^k \Omega_i \phi), n \Bigr) < \infty \quad (6\text{-}3) \end{aligned}$$

for some $0 < \delta < \frac{1}{4}$, and $R > \sqrt[n-2]{\frac{8nm}{\delta}}$, then there is a constant $C(n,m,\delta,R)$ such that for $r_0 < r < R$,

$$|\partial_t \phi(t,r)| \le \frac{C\sqrt{D}}{\tau^{2-2\delta}} \quad \bigl(\tau = \tfrac{1}{2}(t - R^*) > \tau_0\bigr). \quad (6\text{-}4)$$

The pointwise bounds are obtained from the energy estimates of Section 5 using Sobolev inequalities and elliptic estimates; the former provide the link between pointwise and integral quantities, and the latter allow for the expression of these integral quantities in terms of higher-order energies.

Sobolev embedding. By the extension theorem applied to the Sobolev embedding $\mathrm{H}^s(\mathbb{R}^n) \subset \mathrm{L}^\infty(\mathbb{R}^n)$ ($s > \frac{n}{2}$) we have, for $r_0 < \overline{r} < R$,

$$|\phi(\bar{t},\bar{r})|^2 \leq C(n) \int_{r_0^*}^{R^*} \mathrm{d}r^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \Big\{ \phi^2 + \sum_{|\alpha|\geq 1}^{|\alpha|\leq [\frac{n}{2}]+1} \big|\overline{\nabla}^\alpha \phi\big|^2 \Big\} r^{n-1} \Big|_{t=\bar{t}}, \tag{6-5}$$

where $\overline{\nabla}$ denote the tangential derivatives to the hypersurface $\overline{\Sigma}_t$, and α denotes a multiindex of order n.

Elliptic estimates. Note that for any solution ϕ of the wave equation we have

$$T^2 \cdot \phi = \frac{\partial^2 \phi}{\partial r^{*2}} + \Big(1 - \frac{2m}{r^{n-2}}\Big) \frac{n-1}{r} \frac{\partial \phi}{\partial r^*} + \Big(1 - \frac{2m}{r^{n-2}}\Big) \mathbin{\Delta\!\!\!\!/}_{r^2 \mathring{\gamma}_{n-1}} \phi \doteq L \cdot \phi, \tag{6-6}$$

where the operator

$$L = \Big(1 - \frac{2m}{r^{n-2}}\Big) \bar{g}^{ij} \overline{\nabla}_i \partial_j \tag{6-7}$$

is clearly elliptic. (Here $\bar{g}_t = g|_{\overline{\Sigma}_t}$ denotes the restriction of g to the spacelike hypersurfaces $\overline{\Sigma}_t$, a Riemannian metric on $\overline{\Sigma}_t$, and $i, j = 1, \ldots, n$.) In view of the standard higher-order interior elliptic regularity estimate

$$\|\phi\|_{\mathrm{H}^{m+2}(\widehat{\Sigma}_t)} \leq C\big(\|L \cdot \phi\|_{\mathrm{H}^m(\widehat{\Sigma}_t)} + \|\phi\|_{\mathrm{L}^2(\widehat{\Sigma}_t)}\big), \quad \widehat{\Sigma}_t \doteq \overline{\Sigma}_t \cap \{r_0 < r < R\}, \tag{6-8}$$

we conclude with (6-5) that, in the case where $[\frac{n}{2}] + 1$ is even,

$$|\phi|^2 \leq C(n,m) \int_{r_0^*}^{R^*} \mathrm{d}r^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \sum_{l=0}^{[\frac{n}{2}]+1} \big(T^l \cdot \phi\big)^2 r^{n-1}; \tag{6-9}$$

in general we have:

Lemma 6.2 (pointwise estimate in terms of higher-order energies). *Let ϕ be a solution of the wave equation (1-1), and $n \geq 3$. Then there exists a constant $C(n,m)$ such that, for all $r_0 < r < R$,*

$$|\phi(t,r)|^2 \leq C(n,m) \Big[\|\phi\|^2_{\mathrm{L}^2(\widehat{\Sigma}_t)} + \int_{\widehat{\Sigma}_t} \sum_{l=0}^{[\frac{n}{2}]} \big(J^T(T^l \cdot \phi), n\big) \Big]. \tag{6-10}$$

Proof of Proposition 6.1. In view of the Lemma 6.2 and the energy decay estimates of Section 5 it remains to control the zeroth order term $\|\phi\|_{\mathrm{L}^2(\widehat{\Sigma}_t)}$; we multiply the integrand by $(\frac{R}{r})^2 \geq 1$ and extend the integral to $u^* = \tau = \frac{1}{2}(t - R^*)$, $v^* \geq \frac{1}{2}(t + R^*)$.

(i) By Lemma C.2 we can then estimate $\|\phi\|^2_{\mathrm{L}^2(\widehat{\Sigma}_t)}$ by the energy flux through $\Sigma_{\tau=\frac{1}{2}(t-R^*)}$, and apply Proposition 5.4 to the higher-order energies of Lemma 6.2.

(ii) Here we extend the integral only to $\tau + R^* \leq v^* \leq \tau + R^* + \tau^3$ and apply Lemma C.4 to obtain

$$\int_{r_0^*}^{R^*} \mathrm{d}r^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} (\partial_t \phi)^2 r^{n-1} \leq C(n,m) R^2 \int_{\Sigma_\tau \cap \{r^* \leq R^* + \tau^3\}} \big(J^T(\partial_t \phi), n\big) + C(n,m) \frac{R^2}{r} \int_{\mathbb{S}^{n-1}} r^{n-1} (\partial_t \phi)^2 \big|_{(u^*=\tau, v^*=\tau+R^*+\tau^3)}. \tag{6-11}$$

As in the proof of Lemma 5.9 we obtain by integrating from infinity and Cauchy's inequality that

$$\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-2}(\partial_t\phi)^2(\tau,\tau+R^*) \leq \frac{C(n,m)}{1-\frac{2m}{R^{n-2}}}\int_{\Sigma_\tau}\big(J^T(\partial_t\phi),n\big), \tag{6-12}$$

which decays by Proposition 5.4 with a rate τ^{-2}. Moreover, as in the proof of Lemma 5.9,

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} &\mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-1}(\partial_t\phi)^2\big|_{(u^*=\tau,v^*=\tau+R^*+\tau^3)}\\ &= \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-1}(\partial_t\phi)^2\big|_{(u^*=\tau,v^*=\tau+R^*)} + \int_{\tau+R^*}^{\tau+R^*+\tau^3}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}} 2\partial_t\psi\frac{\partial\partial_t\psi}{\partial v^*}\bigg|_{u^*=\tau}\end{aligned} \tag{6-13}$$

and

$$\begin{aligned}\int_{\tau+R^*}^{\tau+R^*+\tau^3}&\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \partial_t\psi\frac{\partial\partial_t\psi}{\partial v^*}\bigg|_{u^*=\tau}\\ &\leq \sqrt{\int_{\tau+R^*}^{\infty}\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\frac{1}{r^2}(\partial_t\phi)^2 r^{n-1}} \times \sqrt{\int_{\tau+R^*}^{\infty}\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^2\bigg(\frac{\partial r^{\frac{n-1}{2}}\partial_t\phi}{\partial v^*}\bigg)^2},\end{aligned} \tag{6-14}$$

the first factor decaying with a rate τ^{-1} by Lemma C.2 and Proposition 5.4, and the second factor bounded by the weighted energy inequality for $r^{\frac{n-1}{2}}\partial_t\phi$ in place of ψ with $p=2$. Therefore

$$\int_{\mathbb{S}^{n-1}} r^{n-1}(\partial_t\phi)^2\big|_{(u^*=\tau,v^*=\tau+R^*+\tau^3)} \leq \frac{C(n,m)}{1-\frac{2m}{R^{n-2}}}\frac{D}{\tau}. \tag{6-15}$$

By virtue of Proposition 5.6 (compare in particular Remark 5.11 on page 587), the first term on the right-hand side of (6-11) decays with a rate of $\tau^{4-4\delta}$, and this is matched by the second term in view of the prefactor $r^{-1}=(R^*+\tau^3)^{-1}$, which is the result of our choice of powers of τ in the extension of the integral Lemma 6.2 applied to the solution $\partial_t\phi$ of (1-1) then yields the pointwise decay result (6-4) after having applied Proposition 5.6 to the higher-order energies on the right-hand side of (6-10). □

Interpolation. We shall now interpolate between the results (i) and (ii) of Proposition 6.1 to improve the pointwise estimate for $|\phi|$. Our argument can in some sense be compared to the proof of improved decay in [Luk 2010]. The basic observation underlying this argument is that, for $r_0<r<R$ and $t_1>t_0$,

$$\begin{aligned}r^{n-2}\phi^2(r,t_1) &= r^{n-2}\phi^2(r,t_0) + \int_{t_0}^{t_1} 2\phi(t,r)\frac{\partial\phi}{\partial t}(t,r)r^{n-2}\,\mathrm{d}t\\ &\leq r^{n-2}\phi^2(r,t_0) + \frac{1}{t_0^{1-2\delta}}\int_{t_0}^{t_1}\phi^2(t,r)r^{n-2}\,\mathrm{d}t + t_0^{1-2\delta}\int_{t_0}^{t_1}\bigg(\frac{\partial\phi}{\partial t}\bigg)^2(t,r)r^{n-2}\,\mathrm{d}t.\end{aligned} \tag{6-16}$$

Moreover, as a consequence of Lemma 6.3,

$$r^{n-2}\phi^2(t,r) \leq R^{n-2}\phi^2(t,R) + \bigg(1-\frac{2m}{r_0^{n-2}}\bigg)^{-1}\int_{r^*}^{R^*}\bigg(\frac{\partial\phi}{\partial r^*}\bigg)^2 r^{n-1}\,\mathrm{d}r^*, \tag{6-17}$$

we obtain an estimate for the timelike integrals in terms of the corresponding integrals at $r = R$ and spacetime integrals, using the Sobolev inequality on the sphere:

$$\int_{t_0}^{t_1} r^{n-2}\phi^2(t,r)\,\mathrm{d}t \le \int_{t_0}^{t_1}\mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \sum_{|\alpha|\le[\frac{n}{2}]+1} R^{n-2}\big(\Omega^\alpha\phi\big)^2(t,R)$$
$$+\left(1-\frac{2m}{r_0^{n-2}}\right)^{-1}\int_{t_0}^{t_1}\mathrm{d}t\int_{r^*}^{R^*}\mathrm{d}r^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-1}\sum_{|\alpha|\le[\frac{n}{2}]+1}\left(\frac{\partial\Omega^\alpha\phi}{\partial r^*}\right)^2(t,r). \quad (6\text{-}18)$$

Lemma 6.3. *Let $a < b \in \mathbb{R}$ and $\phi \in \mathrm{C}^1([a,b])$. Then, for all $n \ge 3$,*

$$a^{n-2}\phi^2(a) \le b^{n-2}\phi^2(b) + \int_a^b \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 x^{n-1}\,\mathrm{d}x. \quad (6\text{-}19)$$

Proof. Since, by integration by parts,

$$\int_a^b 2\phi(x)\frac{\mathrm{d}\phi}{\mathrm{d}x}(x)x^{n-2}\,\mathrm{d}x = 2\phi^2(x)x^{n-2}\big|_a^b - \int_a^b 2\phi(x)\frac{\mathrm{d}\phi}{\mathrm{d}x}(x)x^{n-2}\,\mathrm{d}x - \int_a^b 2\phi^2(x)(n-2)x^{n-3}\,\mathrm{d}x,$$

it clearly follows, with Cauchy's inequality,

$$a^{n-2}\phi^2(a) \le b^{n-2}\phi^2(b) + \int_a^b\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 x^{n-1}\,\mathrm{d}x + \big[1-(n-2)\big]\int_a^b \frac{1}{x^2}\phi^2(x)x^{n-1}\,\mathrm{d}x. \qquad \square$$

Proposition 6.4 (improved interior pointwise decay). *Let ϕ be a solution of the wave equation (1-1), with initial data on Σ_{τ_0} ($\tau_0 > 1$) satisfying*

$$D \doteq \int_{\tau_0+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\left\{\sum_{k=0}^{2}\sum_{|\alpha|\le[\frac{n}{2}]+1} r^{4-\delta}\left(\frac{\partial(T^k\cdot\Omega^\alpha\chi)}{\partial v^*}\right)^2\right.$$
$$\left.+\sum_{k=0}^{5}\sum_{|\alpha|\le[\frac{n}{2}]+1} r^2\left(\frac{\partial T^k\Omega^\alpha\psi}{\partial v^*}\right)^2+\sum_{k=0}^{4}\sum_{|\alpha|\le[\frac{n}{2}]+2} r^2\left(\frac{\partial T^k\Omega^\alpha\psi}{\partial v^*}\right)^2\right\}\Bigg|_{u^*=\tau_0}$$
$$+\int_{\Sigma_{\tau_0}}\left(\sum_{k=0}^{6}\sum_{|\alpha|\le[\frac{n}{2}]+1} J^N(T^k\Omega^\alpha\phi)+\sum_{k=0}^{5}\sum_{|\alpha|\le[\frac{n}{2}]+2} J^N(T^k\Omega^\alpha\phi),n\right)<\infty \quad (6\text{-}20)$$

for some $0 < \delta < \frac{1}{4}$, where $R > \sqrt[n-2]{\frac{8nm}{\delta}}$, $n \ge 3$. Then there exists a constant $C(n,m,\delta,R)$ such that, for $\sqrt[n-2]{2m} < r_0 < r < R$,

$$r^{\frac{n-2}{2}}|\phi|(t,r) \le \frac{CD}{t^{\frac{3}{2}-\delta}}. \quad (6\text{-}21)$$

Proof. Let $\bar{t}_0 = 2(\tau_0+\tau_0)+R^*$ and $\bar{t}_1 = \bar{t}_0 + 2\tau_0$. Then by (6-18), Proposition 4.4 and Proposition 4.1,

$$\int_{\bar{t}_0}^{\bar{t}_1}\phi^2(t,r)r^{n-2}\,\mathrm{d}t \le C(n,m,R)\int_{\Sigma_{2\tau_0}}\left(\sum_{k=0}^{1}\sum_{|\alpha|\le[\frac{n}{2}]+1} J^T[T^k\Omega^\alpha\phi],n\right); \quad (6\text{-}22)$$

hence by Proposition 5.4 there exists $t_0' \in (\bar{t}_0, \bar{t}_1)$ such that

$$r^{n-2}\phi^2(t_0', r) \leq \frac{C(n,m,R)D}{\bar{t}_0^3}. \tag{6-23}$$

Now set $\tau_0' = \frac{1}{2}(t_0' - R^*)$ and $\tau_j' = 2\tau_{j-1}'$ $(j \in \mathbb{N})$, and $t_j' = 2\tau_j' + R^*$ $(j \in \mathbb{N})$; note that $t_{j+1}' - t_j' = \frac{1}{2}(t_j' - R^*)$. Now consider (6-16) with $t_1 = t_{j+1}'$, $t_0 = t_j'$; since by (6-18), together with Propositions 4.1 and 4.4,

$$\int_{t_j'}^{t_{j+1}'} r^{n-2}\phi^2(t,r)\,\mathrm{d}t \leq C(n,m,R)\int_{\Sigma_{\tau_j'}}\Bigl(\sum_{k=0}^{1}\sum_{|\alpha|\leq[\frac{n}{2}]+1} J^T[T^k\Omega^\alpha\phi], n\Bigr), \tag{6-24}$$

and by Propositions 4.21 and 4.22,

$$\begin{aligned}\int_{t_j'}^{t_{j+1}'} r^{n-2}(\partial_t\phi)^2(t,r)\,\mathrm{d}t \leq C(n,m,R)\biggl\{&\int_{\Sigma_{\tau_j'}\cap\{r^*\leq R^*+(\tau_j')^3\}}\Bigl(\sum_{k=1}^{2}\sum_{|\alpha|\leq[\frac{n}{2}]+1} J^T[T^k\Omega^\alpha\phi], n\Bigr)\\ &+\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\sum_{|\alpha|\leq[\frac{n}{2}]+1} r^{n-2}(\Omega^\alpha\partial_t\phi)^2\big|_{(u^*=\tau_j', v^*=R^*+\tau_j'+(\tau_j')^3)}\biggr\},\end{aligned} \tag{6-25}$$

which decays with the rate $\tau^{4-4\delta}$ as is shown in the proof of Proposition 6.1(ii), we obtain

$$\begin{aligned} r^{n-2}\phi^2(r, t_{j+1}') &\leq r^{n-2}\phi^2(r,t_j') + \frac{C(n,m,R)}{(t_j')^{1-2\delta}}\frac{D}{(\tau_j')^2} + C(n,m,\delta,R)(t_j')^{1-2\delta}\frac{D}{(\tau_j')^{4-4\delta}}\\ &\leq r^{n-2}\phi^2(r,t_j') + \frac{C(n,m,\delta,R)D}{(t_j')^{3-2\delta}}. \end{aligned} \tag{6-26}$$

In fact, by induction on $j \in \mathbb{N}$ using (6-23) for $j = 0$, we have shown

$$r^{n-2}\phi^2(r,t_j') \leq \frac{C(n,m,\delta,R)D}{(t_j')^{3-2\delta}} \quad (j \in \mathbb{N}\cup\{0\}). \tag{6-27}$$

Finally for any $t \geq t_0'$ we may choose $j \in \mathbb{N}\cup\{0\}$ such that $t \in (t_j', t_{j+1}')$ and conclude the proof by applying (6-27) and (6-26), which holds with t in place of t_{j+1}'. □

Extension to the horizon. Note that for $\sqrt[n-2]{2m} \leq r < r_0$, the same interpolation (6-16) by integration along lines of constant radius $r < r_0$ can be carried out. However, on the right-hand sides of (6-17) and (6-18) a new term results from the integration on $v^* = \frac{1}{2}(t_0 + r_0^*)$ from the radius $r < r_0$ to $r = r_0$; but we infer from the explicit construction (3-19) that the resulting integrand

$$\Bigl(\frac{2}{1-\frac{2m}{r^{n-2}}}\frac{\partial\phi}{\partial u^*}\Bigr)^2 \leq T[\phi](Y,Y) \leq \bigl(J^N[\phi], N\bigr) \tag{6-28}$$

is controlled by Corollary 4.3, and the proof of Proposition 6.4 above extends to that of Theorem 2 by replacing J^T by J^N on the right-hand sides of (6-22), (6-24) and (6-25).

Appendix A: Notation

Contraction. We sum over repeated indices. Also we use interchangeably

$$g(V,N) \doteq (V,N) \doteq V_\mu N^\mu, \quad J\cdot N \doteq (J,N) \doteq J_\mu N^\mu, \tag{A-1}$$

where V, N are vector fields, and J is a 1-form.

Integration. Let $\mathcal{D}$ in $\mathcal{M}$ be a domain bounded by two homologous hypersurfaces, Σ_1 and Σ_2 being its past and future boundary, respectively. We then write $\int_{\Sigma_1}(J,n)$ for the boundary terms on Σ_1 arising from a general current J in the expression $\int_{\partial\mathcal{D}} {}^*J$. If $\mathcal{S}\subset\Sigma_1$ is spacelike, then $(J,n)=g(J,n)$ is in fact the inner product of J with the timelike normal n to Σ_1; e.g., on constant t-slices $\overline{\Sigma}_t$ (see Section 2) we have $n=(1-\frac{2m}{r^{n-2}})^{-\frac{1}{2}}\frac{\partial}{\partial t}$. If $\mathcal{U}\subset\Sigma_1$ is an outgoing null segment then $\int_{\mathcal{U}}(J,n)$ denotes an integral of the form $\int \mathrm{d}v \int_{\mathrm{S}} \mathrm{d}\mu_\gamma\, g(J,\frac{\partial}{\partial v})$; e.g., on the outgoing null segments of the hypersurfaces Σ_τ (see Section 4), we have

$$\int_{\Sigma_\tau\cap\{r\geq R\}}(J,n) \doteq \int_{\tau+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{n-1}\left(J,\frac{\partial}{\partial v^*}\right). \tag{A-2}$$

The volume form is usually omitted:

$$\int_{\mathcal{D}} f \doteq \int_{\mathcal{D}} f\,\mathrm{d}\mu_g \quad (\mathcal{D}\subset\mathcal{M}).$$

Appendix B: Formulas for reference

In this appendix we summarize a few formulas for reference.

The wave equation. The d'Alembert operator in (1-1) can we written out in any coordinate system according to

$$\Box_g\phi = (g^{-1})^{\mu\nu}\nabla_\mu\partial_\nu\phi, \tag{B-1}$$

where ∇ denotes the covariant derivative of the Levi-Civita connection of g.

Components of the energy momentum tensor. The components of the energy momentum tensor

$$T_{\mu\nu}(\phi) = \partial_\mu\phi\,\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}\,\partial^\alpha\phi\,\partial_\alpha\phi$$

tangential to $\mathcal{Q}$ are given in (u^*,v^*)-coordinates by

$$T_{u^*u^*} = \left(\frac{\partial\phi}{\partial u^*}\right)^2, \quad T_{v^*v^*} = \left(\frac{\partial\phi}{\partial v^*}\right)^2, \quad T_{u^*v^*} = \left(1-\frac{2m}{r^{n-2}}\right)\left|\nabla\!\!\!/\phi\right|^2_{r^2\mathring{\gamma}_{n-1}}. \tag{B-2}$$

We also refer to (B-2) as the *null decomposition* of the energy momentum tensor. Note here that

$$\partial^\alpha\phi\,\partial_\alpha\phi = -\frac{1}{1-\frac{2m}{r^{n-2}}}\left(\frac{\partial\phi}{\partial u^*}\right)\left(\frac{\partial\phi}{\partial v^*}\right) + \left|\nabla\!\!\!/\phi\right|^2_{r^2\mathring{\gamma}_{n-1}},$$

$$\frac{1}{r^2}\mathring{\gamma}_{n-1}^{AB}T_{AB} = \left|\nabla\!\!\!/\phi\right|^2_{r^2\mathring{\gamma}_{n-1}} - \tfrac{1}{2}(n-1)\partial^\alpha\phi\,\partial_\alpha\phi.$$

Integration. A typical domain of integration that we use is

$$ {}^{R}\mathcal{D}_{\tau_1}^{\tau_2} = \left\{ (u^*, v^*) : \tau_1 \leq u^* \leq \tau_2, v^* - u^* \geq R^* \right\}. \tag{B-3} $$

In local coordinates we have, by calculating the volume form from (2-20), that

$$ \int_{{}^{R}\mathcal{D}_{\tau_1}^{\tau_2}} \mathrm{d}\mu_g = \int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{u^*+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, 2\left(1 - \frac{2m}{r^{n-2}}\right) r^{n-1}. \tag{B-4} $$

For a general current J the energy identity on this domain reads

$$ \int_{{}^{R}\mathcal{D}_{\tau_1}^{\tau_2}} K^X \, \mathrm{d}\mu_g = \int_{\partial {}^{R}\mathcal{D}_{\tau_1}^{\tau_2}} {}^*J, \tag{B-5} $$

where the right-hand side is given more explicitly by

$$ \begin{aligned} &\int_{\partial {}^{R}\mathcal{D}_{\tau_1}^{\tau_2}} {}^*J \\ &= -\int_{R^*+\tau_2}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{n-1} g\left(J, \frac{\partial}{\partial v^*}\right)\Big|_{u^*=\tau_2} - \int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{n-1} g\left(J, \frac{\partial}{\partial u^*}\right)\Big|_{v^*\to\infty} \\ &\quad + \int_{R^*+\tau_1}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{n-1} g\left(J, \frac{\partial}{\partial v^*}\right)\Big|_{u^*=\tau_1} - \int_{R^*+2\tau_1}^{R^*+2\tau_2} \mathrm{d}t \int_{\mathbb{S}^{n-1}} r^{n-1} g\left(J, \frac{\partial}{\partial r^*}\right)\Big|_{r=R}. \end{aligned} \tag{B-6} $$

Radial functions. In this appendix we summarize some statements on the relation between r and

$$ r^* = \int_{(nm)^{\frac{1}{n-2}}}^{r} \frac{1}{1 - \frac{2m}{r^{n-2}}} \, \mathrm{d}r. \tag{B-7} $$

The proofs are omitted here, but can be found in [Schlue 2012].

Proposition B.1. *For all $n \geq 3$,*

$$ \lim_{\frac{r}{\sqrt[n-2]{2m}} \to \infty} \frac{r^*}{r} = 1. $$

While this fact concerns the region $r^* \geq 0$ and is essentially due to $\lim_{x\to\infty} \frac{\log x}{x} = 0$, the next concerns $r^* \leq 0$ and is similarly due to $\lim_{x\to 0} x \log x = 0$.

Proposition B.2. *For all $n \geq 3$,*

$$ \lim_{\frac{r}{\sqrt[n-2]{2m}} \to 1} \left(1 - \frac{2m}{r^{n-2}}\right)(-r^*) = 0. $$

In fact we have:

Proposition B.3. *For $r^* < 0$,*

$$ \left(1 - \frac{2m}{r^{n-2}}\right) \leq \frac{(2m)^{\frac{1}{n-2}}}{(-r^*)}. $$

This being an upper bound on $(-r^*)$, we will also need a lower bound:

Proposition B.4. *For $r^* \le 0$,*

$$(-r^*) \ge \frac{(2m)^{\frac{1}{n-2}}}{n-2} \log\left(\frac{(\frac{n}{2})^{\frac{1}{n-2}} - 1}{(\frac{n}{2})^{\frac{1}{n-2}} + 1} \frac{\frac{r}{\sqrt[n-2]{2m}} + 1}{\frac{r}{\sqrt[n-2]{2m}} - 1} \right).$$

Dyadic sequences. In our argument, Section 5C in particular, we construct a hierarchy of *dyadic sequences*, beginning with a sequence of real numbers $(\tau_j)_{j\in\mathbb{N}}$ where $\tau_1 > 0$ and $\tau_{j+1} = 2\tau_j$ ($j \in \mathbb{N}$). We then obtain (by the mean value theorem of integration) a sequence $(\tau_j')_{j\in\mathbb{N}}$ with τ_j' in the interval (τ_j, τ_{j+1}) of length τ_j for all $j \in \mathbb{N}$. We then built up on these values another sequence $(\tau_j'')_{j\in\mathbb{N}}$ which takes values (as selected by the mean value theorem) in the intervals $(\tau_{2j-1}', \tau_{2j+1}') \ni \tau_j''$; note that their length is at least $\tau_{2j+1}' - \tau_{2j-1}' \ge \tau_{2j+1} - \tau_{2j} = \tau_{2j}$. In the same fashion the sequence $(\tau_j''')_{j\in\mathbb{N}}$ is built upon $(\tau_j'')_{j\in\mathbb{N}}$, etc.

Appendix C: Boundary integrals and Hardy inequalities

In this appendix we prove appropriate Hardy inequalities that are needed in our argument to estimate boundary terms that typically arise in the energy identities.

X-type currents. Let $X = f(r^*)\frac{\partial}{\partial r^*}$ and recall the modification (4-14).

Proposition C.1 (boundary terms near null infinity). *Let $f = \mathcal{O}(1)$, $f' = \mathcal{O}(\frac{1}{r})$, and $f'' = \mathcal{O}(\frac{1}{r^2})$. Then there exists a constant $C(n, m)$ such that*

$$\int_{\partial^R\mathcal{D}_{\tau_1}^{\tau_2}\setminus\{r=R\}} {}^*J^{X,1} \le C(n,m) \int_{\Sigma_{\tau_1}} \big(J^T(\phi), n\big). \tag{C-1}$$

Proof. For the boundary integrals on the null segments $u^* = \tau_1, \tau_2$ we find

$$\left| \int_{R^*+\tau_i}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} g\left(J^{X,1}, \frac{\partial}{\partial v^*}\right) r^{n-1} \right|$$
$$\le C(n) \int_{R^*+\tau_i}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\, r^{n-1} \left\{ \left(\frac{\partial\phi}{\partial v^*}\right)^2 + |\nabla\!\!\!/\phi|^2 + \left[\frac{|f|}{r^2} + \frac{|f'|}{r} + |f'|^2 + |f''| \right]\phi^2 \right\}, \tag{C-2}$$

and, in view of the Hardy inequality Lemma C.2,

$$\int_{R^*+\tau_i}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \frac{1}{r^2}\phi^2\, r^{n-1}\big|_{u^*=\tau_i} \le C(n,m) \int_{\Sigma_{\tau_i}} \big(J^T(\phi), n\big); \tag{C-3}$$

note that the corresponding zero order terms vanish at future null infinity; cf. Remark C.3. Then (C-1) follows from the energy identity for T on ${}^R\mathcal{D}_{\tau_1}^{\tau_2}$. □

Lemma C.2 (Hardy inequality). *Let $\phi \in \mathrm{C}^1([a,\infty))$, $a > 0$, with $|\phi(a)| < \infty$ and*

$$\lim_{x\to\infty} x^{\frac{n-2}{2}}\phi(x) = 0. \tag{C-4}$$

Then a constant $C(n) > 0$ exists such that

$$\int_a^\infty \frac{1}{x^2}\phi^2(x)\, x^{n-1}\,\mathrm{d}x \le C(n) \int_a^\infty \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 x^{n-1}\,\mathrm{d}x. \tag{C-5}$$

Proof. This is a consequence of the Cauchy–Schwarz inequality; after integration by parts

$$\int_a^\infty \frac{1}{x^2}\phi^2(x)\, x^{n-1}\,\mathrm{d}x = \int_a^\infty g'(x)\phi^2(x)\,\mathrm{d}x$$

with

$$g(x) = \int_a^x y^{n-3}\,\mathrm{d}y. \qquad \square$$

Remark C.3. The conditions of Lemma C.2 on ϕ are in fact satisfied for any solution of the wave equation (1-1). By a density argument we may assume without loss of generality that the initial data is compactly supported. Then for a fixed τ, and v^* large enough, $\phi(\tau, v^*) = 0$, and for $u^* \geq \tau$,

$$\phi(u^*, v^*) = \int_\tau^{u^*} \frac{\partial \phi}{\partial u^*}\,\mathrm{d}u^*.$$

Thus

$$\phi(u^*, v^*) \leq \left(\int_\tau^{u^*} \left(\frac{\partial\phi}{\partial u^*}\right)^2 r^{n-1}\,\mathrm{d}u^*\right)^{\frac{1}{2}} \left(\int_\tau^{u^*} \frac{1}{r^{n-1}}\,\mathrm{d}u^*\right)^{\frac{1}{2}}.$$

On one hand,

$$\int_\tau^{u^*}\int_{\mathbb{S}^{n-1}} \left(\frac{\partial\phi}{\partial u^*}\right)^2 r^{n-1}\,\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\,\mathrm{d}u^* \leq \int_{\Sigma_\tau} \left(J^T(\phi), n\right) < \infty,$$

whereas on the other hand,

$$\begin{aligned}\int_\tau^{u^*} \frac{1}{r^{n-1}}\,\mathrm{d}u^* &= \frac{1}{n-2}\int_\tau^{u^*} \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} \frac{\partial}{\partial u^*}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}u^* \\ &\leq \frac{1}{n-2}\left(1 - \frac{2m}{R^{n-2}}\right)^{-1}\left(1 - \left(\frac{r(u^*,v^*)}{r(\tau,v^*)}\right)^{n-2}\right)\frac{1}{r^{n-2}},\end{aligned}$$

if we restrict $u^* \geq \tau$ to $r(u^*, v^*) \geq R$. Hence

$$\lim_{v^*\to\infty} r^{\frac{n-2}{2}}\phi = 0.$$

Instead of (C-5), which requires (C-4), one can prove the corresponding Hardy inequality for finite intervals:

Lemma C.4 (Hardy inequality for finite intervals). *Let* $0 < a < b$, *and* $\phi \in \mathrm{C}^1((a, b))$. *Then*

$$\frac{1}{2}\int_a^b \frac{1}{x^2}\phi^2(x)x^{n-1}\,\mathrm{d}x \leq \frac{1}{n-2}b^{n-2}\phi^2(b) + 2\left(\frac{2}{n-2}\right)^2 \int_a^b \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 x^{n-1}\,\mathrm{d}x. \qquad \text{(C-6)}$$

Proof. Let

$$g(x) = \int_a^x y^{n-3}\,\mathrm{d}y = \frac{1}{n-2}y^{n-2}\Big|_a^x.$$

Then, by integration by parts and using Cauchy's inequality,

$$\begin{aligned}\int_a^b \frac{1}{x^2}\phi^2(x)x^{n-1}\,\mathrm{d}x &= g\,\phi^2\big|_a^b - \int_a^b g(x)2\phi(x)\frac{\mathrm{d}\phi}{\mathrm{d}x}\,\mathrm{d}x \\ &\leq g(b)\phi^2(b) + 2\epsilon\int_a^b \frac{1}{x^2}\phi^2(x)x^{n-1}\,\mathrm{d}x + \frac{1}{2\epsilon}\int_a^b \frac{g(x)^2}{x^{n-3}}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2\mathrm{d}x,\end{aligned}$$

where $\epsilon > 0$; (C-6) follows for $\epsilon = \frac{1}{4}$ because

$$g(b) \leq \frac{1}{n-2} b^{n-2}, \quad \frac{g(x)^2}{x^{n-3}} \leq \frac{2}{n-2}\Bigl(1 + \Bigl(\frac{a}{x}\Bigr)^{2(n-2)}\Bigr) x^{n-1}. \qquad \square$$

Recall the domain (4-103); by using Lemma C.4 instead of Lemma C.2 we can prove the following refinement of Proposition C.1 to bounded domains:

Proposition C.5 (boundary terms on bounded domains). *Let $f = \mathcal{O}(1)$, $f' = \mathcal{O}(\frac{1}{r})$, and $f'' = \mathcal{O}(\frac{1}{r^2})$. Then there exists a constant $C(n,m)$ such that*

$$\int_{\partial^R \mathfrak{D}_{\tau_1}^{\tau_2} \setminus \{r=R\}} {}^*J^{X,1} \leq C(n,m) \Bigl\{ \int_{\Sigma_{\tau_1}^{\tau_2}} \bigl(J^T(\phi), n\bigr) + \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-2} \phi^2 \bigr|_{(u^*=\tau_1, v^*=R^*+\tau_2)} \Bigr\}. \tag{C-7}$$

Recall the domain (4-2).

Proposition C.6 (boundary terms near the event horizon). *Let $f = \mathcal{O}(1)$, $f' = \mathcal{O}\bigl(\frac{1}{|r^*|^4}\bigr)$, and $f'' = \mathcal{O}\bigl(\frac{1}{|r^*|^5}\bigr)$, and*

$$\pi_l \phi = 0 \quad (0 \leq l < L),$$

for some $L \in \mathbb{N}$. Then there exists a constant $C(n,m,L)$ such that

$$\int_{\partial \mathcal{R}_{r_0, r_1}^{\infty}(t_0)} {}^*J^{X,1} \leq C(n,m,L) \int_{\Sigma_{\tau_0}} \bigl(J^T(\phi), n\bigr), \tag{C-8}$$

where $\tau_0 = \frac{1}{2}(t_0 - r_1^*)$.

The proof is given in Section 4D.1 in the special case $f = f_{\gamma,\alpha}$ using the following lemma.

Lemma C.7 (Hardy inequality). *Let $a > 0$, $\phi \in \mathrm{C}^1([a,\infty))$ with*

$$\lim_{x\to\infty} |\phi(x)| < \infty.$$

Then

$$\int_a^\infty \frac{1}{1+x^2} \phi^2(x)\, \mathrm{d}x \leq 8 \frac{1+a^2}{a^2} \int_a^\infty \Bigl(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Bigr)^2 \mathrm{d}x + 2\pi \int_a^{a+1} \Bigl\{ \phi^2 + \Bigl(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Bigr)^2 \Bigr\} \mathrm{d}x. \tag{C-9}$$

Proof. Let us first assume that $\phi(a) = 0$. Define

$$g(x) = -\int_x^\infty \frac{1}{1+y^2}\, \mathrm{d}y.$$

Then

$$\begin{aligned} \int_a^\infty \frac{1}{1+x^2} \phi^2(x)\, \mathrm{d}x &= \int_a^\infty g'(x) \phi^2(x)\, \mathrm{d}x = g(x)\phi^2(x)\bigr|_a^\infty - 2 \int_a^\infty g(x)\phi(x) \frac{\mathrm{d}\phi}{\mathrm{d}x}\, \mathrm{d}x \\ &\leq 2 \Bigl(\int_a^\infty \frac{g(x)^2}{g'(x)} \Bigl(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Bigr)^2 \mathrm{d}x \Bigr)^{\frac{1}{2}} \Bigl(\int_a^\infty g'(x)\phi^2(x)\, \mathrm{d}x \Bigr)^{\frac{1}{2}}. \end{aligned}$$

Since $|g(x)| \leq \frac{1}{x}$ we have

$$\frac{g(x)^2}{g'(x)} \leq \frac{1+x^2}{x^2} \leq \frac{1+a^2}{a^2},$$

and therefore

$$\int_a^\infty \frac{1}{1+x^2}\phi^2(x)\,\mathrm{d}x \leq 4\int_a^\infty \frac{g(x)^2}{g'(x)}\Big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big)^2\,\mathrm{d}x \leq 4\frac{1+a^2}{a^2}\int_a^\infty \Big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big)^2\,\mathrm{d}x.$$

Without the assumption $\phi(a)=0$ this applied to the function $\phi(x)-\phi(a)$ yields

$$\begin{aligned}\int_a^\infty \frac{1}{1+x^2}\phi^2(x)\,\mathrm{d}x &\leq 2\int_a^\infty \frac{1}{1+x^2}\big(\phi(x)-\phi(a)\big)^2\,\mathrm{d}x + 2\int_a^\infty \frac{1}{1+x^2}\phi(a)^2\,\mathrm{d}x\\ &\leq 8\frac{1+a^2}{a^2}\int_a^\infty \Big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big)^2\,\mathrm{d}x + \pi\phi(a)^2.\end{aligned}$$

We conclude the proof with the following pointwise bound: on one hand, for some $a' \in (a, a+1)$,

$$\int_a^{a+1} \phi(x)^2\,\mathrm{d}x = \phi(a')^2$$

and on the other hand,

$$\phi(a')^2 - \phi(a)^2 = \int_a^{a'} \frac{\mathrm{d}}{\mathrm{d}x}\phi(x)^2\,\mathrm{d}x \leq \int_a^{a'} \Big\{\phi(x)^2 + \Big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big)^2\Big\}\,\mathrm{d}x.$$

Hence

$$\phi(a)^2 \leq \int_a^{a'} \Big\{\phi(x)^2 + \Big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big)^2\Big\}\,\mathrm{d}x + \int_a^{a+1}\phi(x)^2\,\mathrm{d}x \leq 2\int_a^{a+1} \Big\{\phi(x)^2 + \Big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big)^2\Big\}\,\mathrm{d}x. \qquad \square$$

Auxiliary currents. We have the same results for auxiliary currents of the form

$$J^{\mathrm{aux}}_\mu = \tfrac{1}{2}h(r)\partial_\mu(\phi^2). \tag{C-10}$$

Proposition C.8. *Let $h = \mathcal{O}(\frac{1}{r})$. Then there exists a constant $C(n,m)$ such that*

$$\int_{\partial^R\mathcal{D}_{\tau_1}^{\tau_2}\setminus\{r=R\}} {}^*J^{\mathrm{aux}} \leq C(n,m)\int_{\Sigma_{\tau_1}} \big(J^T(\phi), n\big), \tag{C-11}$$

and moreover, for a constant $C(n,m)$, we have the refinement

$$\int_{\partial^R\mathcal{D}_{\tau_1}^{\tau_2}\setminus\{r=R\}} {}^*J^{\mathrm{aux}} \leq C(n,m)\Big\{\int_{\Sigma_{\tau_1}^{\tau_2}} \big(J^T(\phi), n\big) + \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} r^{n-2}\phi^2\big|_{(\tau_1, R^*+\tau_2)}\Big\}. \tag{C-12}$$

Proof. Note that here, in comparison to the proof of Proposition C.1,

$$\Big|g\Big(J^{\mathrm{aux}}, \frac{\partial}{\partial v^*}\Big)\Big| \leq h^2\phi^2 + \Big(\frac{\partial\phi}{\partial v^*}\Big)^2. \qquad \square$$

Proposition C.9. *Let* $h = \mathcal{O}\big(\frac{1}{|r^*|}\big)$. *Then there exists a constant* $C(n, m)$ *such that*

$$\int_{\partial\mathcal{R}^{\infty}_{r_0,r_1}(t_0)} {}^*J^{\text{aux}} \leq C(n,m) \int_{\Sigma_{\tau_0}} \big(J^T(\phi), n\big), \tag{C-13}$$

where $\tau_0 = \frac{1}{2}(t_0 - r_1^*)$.

Remark C.10. In view of Proposition B.3, the function $h = \frac{1}{r}\big(1 - \frac{2m}{r^{n-2}}\big)$ satisfies the assumption of the proposition.

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CONDITIONAL GLOBAL REGULARITY OF SCHRÖDINGER MAPS: SUBTHRESHOLD DISPERSED ENERGY

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We consider the Schrödinger map initial value problem

$$\begin{cases} \partial_t \varphi = \varphi \times \Delta \varphi, \\ \varphi(x, 0) = \varphi_0(x), \end{cases}$$

with $\varphi_0 : \mathbb{R}^2 \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ a smooth H_Q^∞ map from the Euclidean space $\mathbb{R}^2$ to the sphere $\mathbb{S}^2$ with subthreshold ($< 4\pi$) energy. Assuming an a priori L^4 boundedness condition on the solution φ, we prove that the Schrödinger map system admits a unique global smooth solution $\varphi \in C(\mathbb{R} \to H_Q^\infty)$ provided that the initial data φ_0 is sufficiently energy-dispersed, i.e., sufficiently small in the critical Besov space $\dot{B}^1_{2,\infty}$. Also shown are global-in-time bounds on certain Sobolev norms of φ. Toward these ends we establish improved local smoothing and bilinear Strichartz estimates, adapting the Planchon–Vega approach to such estimates to the nonlinear setting of Schrödinger maps.

1. Introduction

We consider the Schrödinger map initial value problem

$$\begin{cases} \partial_t \varphi = \varphi \times \Delta \varphi, \\ \varphi(x, 0) = \varphi_0(x), \end{cases} \tag{1-1}$$

with $\varphi_0 : \mathbb{R}^d \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$.

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The system (1-1) enjoys conservation of energy,

$$E(\varphi(t)) := \frac{1}{2}\int_{\mathbb{R}^d} |\partial_x \varphi(t)|^2\, dx, \tag{1-2}$$

and mass,

$$M(\varphi(t)) := \int_{\mathbb{R}^d} |\varphi(t) - Q|^2\, dx,$$

where $Q \in \mathbb{S}^2$ is some fixed base point. When $d = 2$, both (1-1) and (1-2) are invariant with respect to the scaling

$$\varphi(x,t) \to \varphi(\lambda x, \lambda^2 t), \qquad \lambda > 0, \tag{1-3}$$

and in this case we call the equation (1-1) *energy-critical*. In this article we restrict ourselves to the energy-critical setting.

For the physical significance of (1-1), see [Chang et al. 2000; Nahmod et al. 2003; Papanicolaou and Tomaras 1991; Landau 1967]. The system also arises naturally from the (scalar-valued) free linear Schrödinger equation

$$(\partial_t + i\Delta)u = 0$$

by replacing the target manifold $\mathbb{C}$ with the sphere $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$, which then requires replacing Δu with $(u^*\nabla)_j \partial_j u = \Delta u - \perp(\Delta u)$ and i with the complex structure $u \times \cdot$. Here $\perp$ denotes orthogonal projection onto the normal bundle, which, for a given point (x,t), is spanned by $u(x,t)$. For more general analogues of (1-1), e.g., for Kähler targets other than $\mathbb{S}^2$, see [Ding and Wang 2001; McGahagan 2007; Nahmod et al. 2007]. See also [Kenig et al. 2000; Kenig and Nahmod 2005; Bejenaru et al. 2011b] for connections with other spin systems. The local theory for Schrödinger maps is developed in [Sulem et al. 1986; Chang et al. 2000; Ding and Wang 2001; McGahagan 2007]. For global results in the $d = 1$ setting, see [Chang et al. 2000; Rodnianski et al. 2009]. For $d \geq 3$, see [Bejenaru 2008a; 2008b; Bejenaru et al. 2007; 2011c; Ionescu and Kenig 2006; 2007b]. Concerning the related modified Schrödinger map system, see [Kato 2005; Kato and Koch 2007; Nahmod et al. 2007].

The small-energy (take $d = 2$) theory for (1-1) is now well-understood: building upon previous work (see below or [Bejenaru et al. 2011c, §1] for a brief history), global well-posedness and global-in-time bounds on certain Sobolev norms are shown in [Bejenaru et al. 2011c] given initial data with sufficiently small energy. The high-energy theory, however, is still very much in development. One of the main goals is to establish what is known as the *threshold conjecture*, which asserts that global well-posedness holds for (1-1) given initial data with energy below a certain energy threshold, and that finite-time blowup is possible for certain initial data with energy above this threshold. The threshold is directly tied to the nontrivial stationary solutions of (1-1), i.e., maps ϕ into $\mathbb{S}^2$ that satisfy

$$\phi \times \Delta\phi \equiv 0$$

and that do not send all of $\mathbb{R}^2$ to a single point of $\mathbb{S}^2$. Therefore we identify such stationary solutions with nontrivial harmonic maps $\mathbb{R}^2 \to \mathbb{S}^2$, which we refer to as *solitons* for (1-1). It turns out that there exist no nontrivial harmonic maps into the sphere $\mathbb{S}^2$ with energy less than 4π, and that the harmonic map given by the inverse of stereographic projection has energy precisely equal to $4\pi =: E_{\text{crit}}$. We therefore refer to the range of energies $[0, E_{\text{crit}})$ as *subthreshold*, and call E_{crit} the *critical* or *threshold energy*.

Recently, an analogous threshold conjecture was established for wave maps (see [Krieger et al. 2008; Rodnianski and Sterbenz 2010; Sterbenz and Tataru 2010a; 2010b] and, for hyperbolic space, [Krieger and Schlag 2012; Tao 2008a; 2008b; 2008c; 2009a; 2009b]). When $\mathcal{M}$ is a hyperbolic space, or, as in [Sterbenz and Tataru 2010a; 2010b], a generic compact manifold, we may define the associated energy threshold $E_{\text{crit}} = E_{\text{crit}}(\mathcal{M})$ as follows. Given a target manifold $\mathcal{M}$, consider the collection $\mathcal{S}$ of all nonconstant finite-energy harmonic maps $\phi : \mathbb{R}^2 \to \mathcal{M}$. If this set is empty, as is, for instance, the case when $\mathcal{M}$ is equal to a hyperbolic space $\mathbb{H}^m$, then we formally set $E_{\text{crit}} = +\infty$. If $\mathcal{S}$ is nonempty, then it turns out that the set $\{E(\phi) : \phi \in \mathcal{S}\}$ has a least element and that, moreover, this energy value is positive. In such case we call this least energy E_{crit}. The threshold E_{crit} depends upon geometric and topological properties of the target manifold $\mathcal{M}$; see [Lin and Wang 2008, Chapter 6] for further discussion. This definition yields $E_{\text{crit}} = 4\pi$ in the case of the sphere $\mathbb{S}^2$. For further discussion of the critical energy level in the wave maps setting, see [Sterbenz and Tataru 2010b; Tao 2008a].

We now summarize what is known for Schrödinger maps in $d = 2$. Asymptotic stability of harmonic maps of topological degree $|m| \geq 4$ under the Schrödinger flow is established in [Gustafson et al. 2008]. The result is extended to maps of degree $|m| \geq 3$ in [Gustafson et al. 2010]. A certain energy-class instability for degree-1 solitons of (1-1) is shown in [Bejenaru and Tataru 2010], where it is also shown that global solutions always exist for small localized equivariant perturbations of degree-1 solitons. Finite-time blowup for (1-1) is demonstrated in [Merle et al. 2011a; 2011b], using less-localized equivariant perturbations of degree-1 solitons, thus resolving the blowup assertion of the threshold conjecture. Blow-up dynamics for equivariant critical Schrödinger maps are studied in [Perelman 2012]. Global well-posedness given data with small critical Sobolev norm (in all dimensions $d \geq 2$) is shown in [Bejenaru et al. 2011c]. Recent work of the author [Smith 2012b] extends the result of Bejenaru et al. and the present conditional result to global regularity (in $d = 2$) assuming small critical Besov norm $\dot{B}^1_{2,\infty}$. In a different direction, [Dodson and Smith 2013] shows that the L^4 norm considered in this paper is in fact a controlling norm for critical Schrödinger maps. In the radial setting (which excludes harmonic maps), Gustafson and Koo [2011] established global well-posedness at any energy level. In the equivariant setting, Bejenaru et al. [2011a] established global existence and uniqueness as well as scattering given 1-equivariant data with energy less than 4π. They note that, although these results are stated only for data with energy less than 4π, their proofs remain valid for maps with energy slightly larger than 4π, suggesting that the "right" threshold conjecture for equivariant Schrödinger maps should be stated also in terms of homotopy class, leading to a threshold of 8π rather than 4π in the case where the target is $\mathbb{S}^2$. See the introduction of [Bejenaru et al. 2011a] for further discussion of this point. This global result has been extended to the $\mathbb{H}^2$ target in [Bejenaru et al. 2012], under the assumption that the initial data has finite energy.

The main purpose of this paper is to show that (1-1) admits a unique smooth global solution φ given smooth initial data φ_0 satisfying appropriate energy conditions and assuming a priori boundedness of a certain L^4 spacetime norm of the spatial gradient of the solution φ. In particular, we admit a restricted class of initial data with energy ranging over the entire subthreshold range.

In order to go beyond the small-energy results of [Bejenaru et al. 2011c], we introduce physical-space proofs of local smoothing and bilinear Strichartz estimates, in the spirit of [Planchon and Vega 2009; Planchon 2012, p. 1042-08; Tao 2010], that do not heavily depend upon perturbative methods. The local smoothing estimate that we establish is a nonlinear analogue of that shown in [Ionescu and Kenig 2006]. The bilinear Strichartz estimate is a nonlinear analogue of the improved bilinear Strichartz estimate of [Bourgain 1998]. These proofs more naturally account for magnetic nonlinearities, and we believe the technique developed here to be of independent interest and applicable to other settings. For local smoothing in the context of Schrödinger equations, see [Kenig et al. 1993; 1998; 2004; Ionescu and Kenig 2005; 2006; 2007b]. For other Strichartz and smoothing results for magnetic Schrödinger equations, see [Stefanov 2007; D'Ancona and Fanelli 2008; D'Ancona et al. 2010; Erdoğan et al. 2008; 2009; Fanelli and Vega 2009] and the references therein. We also use in a fundamental way the subthreshold *caloric gauge* of [Smith 2012a], which is an extension of a construction introduced in [Tao 2004].

To make these statements more precise, we now turn to some basic definitions and observations.

1A. ***Preliminaries.*** First we establish some basic notation. The boldfaced letters $\mathbb{Z}$ and $\mathbb{R}$ respectively denote the integers and real numbers. We use $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ to denote the nonnegative integers. Usual Lebesgue function spaces are denoted by L^p, and these sometimes include a subscript to indicate the variable or variables of integration. When function spaces are iterated, e.g., $L_t^\infty L_x^2$, the norms are applied starting with the rightmost one. When we use L^4 without subscripts, we mean $L^4_{t,x}$.

We use $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ to denote the standard 2-sphere embedded in 3-dimensional Euclidean space. The ambient space $\mathbb{R}^3$ carries the usual metric and $\mathbb{S}^2$ the inherited one. Throughout, $\mathbb{S}^1$ denotes the unit circle.

We use $\partial_x = (\partial_{x_1}, \partial_{x_2}) = (\partial_1, \partial_2)$ to denote the gradient operator, as throughout "∇" will stand for the Riemannian connection on $\mathbb{S}^2$. As usual, "Δ" denotes the (flat) spatial Laplacian.

The symbol $|\partial_x|^\sigma$ denotes the Fourier multiplier with symbol $|\xi|^\sigma$. We also use standard Littlewood–Paley Fourier multipliers P_k and $P_{\leq k}$, respectively denoting restrictions to frequencies $\sim 2^k$ and $\lesssim 2^k$; see Section 3 for details. We use $\hat{f}$ to denote the Fourier transform of a function f in the spatial variables.

We also employ without further comment (finite-dimensional) vector-valued analogues of the above.

We use $f \lesssim g$ to denote the estimate $|f| \leq C|g|$ for an absolute constant $C > 0$. As usual, the constant is allowed to change from line to line. To indicate dependence of the implicit constant upon parameters (which, for instance, can include functions), we use subscripts, e.g., $f \lesssim_k g$. As an equivalent alternative we write $f = O(g)$ (or, with subscripts, $f = O_k(g)$, for instance) to denote $|f| \leq C|g|$. If both $f \lesssim g$ and $g \lesssim f$, then we indicate this by writing $f \sim g$.

Now we introduce the notion of Sobolev spaces of functions mapping from Euclidean space into $\mathbb{S}^2$. The spaces are constructed with respect to a choice of base point $Q \in \mathbb{S}^2$, the purpose of which is to define a notion of decay: instead of decaying to zero at infinity, our Sobolev class functions decay to Q.

For $\sigma \in [0, \infty)$, let $H^\sigma = H^\sigma(\mathbb{R}^2)$ denote the usual Sobolev space of complex-valued functions on $\mathbb{R}^2$. For any $Q \in \mathbb{S}^2$, set

$$H_Q^\sigma := \{f : \mathbb{R}^2 \to \mathbb{R}^3 \text{ such that } |f(x)| \equiv 1 \text{ a.e. and } f - Q \in H^\sigma\}.$$

This is a metric space with induced distance $d_Q^\sigma(f,g)=\|f-g\|_{H^\sigma}$. For $f\in H_Q^\sigma$ we set $\|f\|_{H_Q^\sigma}=d_Q^\sigma(f,Q)$ for short. We also define the spaces

$$H^\infty := \bigcap_{\sigma\in\mathbb{Z}_+} H^\sigma \quad \text{and} \quad H_Q^\infty := \bigcap_{\sigma\in\mathbb{Z}_+} H_Q^\sigma.$$

For any time $T\in(0,\infty)$, these definitions may be extended to the spacetime slab $\mathbb{R}^2\times(-T,T)$ (or $\mathbb{R}^2\times[-T,T]$). For any $\sigma,\rho\in\mathbb{Z}_+$, let $H^{\sigma,\rho}(T)$ denote the Sobolev space of complex-valued functions on $\mathbb{R}^2\times(-T,T)$ with the norm

$$\|f\|_{H^{\sigma,\rho}(T)} := \sup_{t\in(-T,T)} \sum_{\rho'=0}^{\rho} \|\partial_t^{\rho'} f(\cdot,t)\|_{H^\sigma},$$

and for $Q\in\mathbb{S}^2$ endow

$$H_Q^{\sigma,\rho} := \{f:\mathbb{R}^2\times(-T,T)\to\mathbb{R}^3 \text{ such that } |f(x,t)|\equiv 1 \text{ a.e. and } f-Q\in H^{\sigma,\rho}(T)\}$$

with the metric induced by the $H^{\sigma,\rho}(T)$ norm. Also, define the spaces

$$H^{\infty,\infty}(T) = \bigcap_{\sigma,\rho\in\mathbb{Z}_+} H^{\sigma,\rho}(T) \quad \text{and} \quad H_Q^{\infty,\infty}(T) = \bigcap_{\sigma,\rho\in\mathbb{Z}_+} H_Q^{\sigma,\rho}(T).$$

For $f\in H^\infty$ and $\sigma\geq 0$ we define the homogeneous Sobolev norms as

$$\|f\|_{\dot H^\sigma} = \|\hat f(\xi)\cdot|\xi|^\sigma\|_{L^2}.$$

We mention two important conservation laws obeyed by solutions of the Schrödinger map system (1-1). In particular, if $\varphi\in C((T_1,T_2)\to H_Q^\infty)$ solves (1-1) on a time interval (T_1,T_2), then both

$$\int_{\mathbb{R}^2} |\varphi(t)-Q|^2\,dx \quad \text{and} \quad \int_{\mathbb{R}^2} |\partial_x\varphi(t)|^2\,dx$$

are conserved. Hence the Sobolev norms H_Q^0 and H_Q^1 are conserved, as well as the energy (1-2). Note also the time-reversibility obeyed by (1-1), which in particular permits the smooth extension to $(-T,T)$ of a smooth solution on $[0,T)$.

According to our conventions,

$$|\partial_x\varphi(t)|^2 := \sum_{m=1,2} |\partial_m\varphi(t)|^2.$$

We can now give a precise statement of a key known local result.

Theorem 1.1 (local existence and uniqueness). *If the initial data φ_0 is such that $\varphi_0\in H_Q^\infty$ for some $Q\in\mathbb{S}^2$, then there exists a time $T=T(\|\varphi_0\|_{H_Q^{25}})>0$ for which there exists a unique solution φ in $C([-T,T]\to H_Q^\infty)$ of the initial value problem (1-1).*

Proof. See [Sulem et al. 1986; Chang et al. 2000; Ding and Wang 2001; McGahagan 2007] and the references therein. □

1B. *Global theory.* Theorem 1.1 yields short-time existence and uniqueness as well as a blowup criterion; as such it is central to the continuity arguments used for global results. In the small-energy setting, global regularity (and more) was proved for (1-1) by Bejenaru, Ionescu, Kenig, and Tataru [Bejenaru et al. 2011c]. We now state a special case of their main result, omitting for the sake of brevity the consideration of higher spatial dimensions and continuity of the solution map.

Theorem 1.2 (global regularity). *Let $Q \in \mathbb{S}^2$. Then there exists an $\varepsilon_0 > 0$ such that, for any $\varphi_0 \in H_Q^\infty$ with $\|\partial_x \varphi_0\|_{L_x^2} \leq \varepsilon_0$, there is a unique solution $\varphi \in C(\mathbb{R} \to H_Q^\infty)$ of the initial value problem (1-1). Moreover, for any $T \in [0, \infty)$ and $\sigma \in \mathbb{Z}_+$,*

$$\sup_{t \in (-T,T)} \|\varphi(t)\|_{H_Q^\sigma} \lesssim_{\sigma, T, \|\varphi_0\|_{H_Q^\sigma}} 1.$$

Also, given any $\sigma_1 \in \mathbb{Z}_+$, there exists a positive $\varepsilon_1 = \varepsilon_1(\sigma_1) \leq \varepsilon_0$ such that the uniform bounds

$$\sup_{t \in \mathbb{R}} \|\varphi(t)\|_{H_Q^\sigma} \lesssim_\sigma \|\varphi_0\|_{H_Q^\sigma}$$

hold for all $1 \leq \sigma \leq \sigma_1$, provided $\|\partial_x \varphi_0\|_{L_x^2} \leq \varepsilon_1$.

A complete proof may be found in [Bejenaru et al. 2011c]. Among the key contributions of that work are the construction of the main function spaces and the completion of the linear estimate relating them, which includes an important maximal function estimate. A significant observation made in the same paper is that it is important that these spaces take into account a local smoothing effect; the authors crucially use this effect to help bring under control the worst term of the nonlinearity. Another novelty of [Bejenaru et al. 2011c] is its implementation of the caloric gauge, which was first introduced by Tao [2004] and subsequently recommended by him for use in studying Schrödinger maps [Tao 2006a]. As the caloric gauge is defined using harmonic map heat flow, it can be thought of as an intrinsic and nonlinear analogue of classical Littlewood–Paley theory. In [Bejenaru et al. 2011c], both the intrinsic caloric gauge and the extrinsic (and modern) Littlewood–Paley theory are used simultaneously.

Our main result extends Theorem 1.2.

Theorem 1.3. *Let $T > 0$ and $Q \in S^2$. Let $\varepsilon_0 > 0$ and let $\varphi \in H_Q^{\infty,\infty}(T)$ be a solution of the Schrödinger map system (1-1) whose initial data φ_0 has energy $E_0 := E(\varphi_0) < E_{\text{crit}}$ and satisfies the energy dispersion condition*

$$\sup_{k \in \mathbb{Z}} \|P_k \partial_x \varphi_0\|_{L_x^2} \leq \varepsilon_0. \tag{1-4}$$

Let $I \supset (-T, T)$ denote the maximal time interval for which there exists a smooth (necessarily unique) extension of φ satisfying (1-1). Suppose a priori that

$$\sum_{k \in \mathbb{Z}} \|P_k \partial_x \varphi\|_{L_{t,x}^4(I \times \mathbb{R}^2)}^2 \leq \varepsilon_0^2. \tag{1-5}$$

Then, for ε_0 sufficiently small,

$$\sup_{t \in (-T,T)} \|\varphi(t)\|_{H_Q^\sigma} \lesssim_{\sigma, T, \|\varphi_0\|_{H_Q^\sigma}} 1, \tag{1-6}$$

for all $\sigma \in \mathbb{Z}_+$. Additionally, $I = \mathbb{R}$, so that, in particular, φ admits a unique smooth global extension $\varphi \in C(\mathbb{R} \to H_Q^\infty)$. Moreover, for any $\sigma_1 \in \mathbb{Z}_+$, there exists a positive $\varepsilon_1 = \varepsilon_1(\sigma_1) \leq \varepsilon_0$ such that

$$\|\varphi\|_{L_t^\infty H_Q^\sigma(\mathbb{R}\times\mathbb{R}^2)} \lesssim_\sigma \|\varphi\|_{H_Q^\sigma(\mathbb{R}^2)} \tag{1-7}$$

holds for all $0 \leq \sigma \leq \sigma_1$ provided (1-4) and (1-5) hold with ε_1 in place of ε_0.

Note that the energy dispersion condition (1-4) holds automatically in the case of small energy. In such case, our proofs may be modified (essentially by collapsing to or reverting to the arguments of [Bejenaru et al. 2011c]) so that the a priori L^4 bound is not required. Such an L^4 bound, however, can then be seen to hold a posteriori.

Using time divisibility of the L^4 norm, we can replace (1-5) with

$$\sum_{k\in\mathbb{Z}} \|P_k \partial_x \varphi\|^2_{L^4_{t,x}(I\times\mathbb{R}^2)} \leq K$$

for any $K > 0$ provided we allow the threshold for ε_0 and the implicit constant in (1-7) to depend upon $K > 0$. We work with (1-5) as stated so as to avoid the additional technicalities that would arise otherwise.

We now turn to a very rough sketch of the proof of Theorem 1.3; for a detailed outline, see Section 4.

Basic setup and gauge selection. It suffices to prove homogeneous Sobolev variants of (1-6) and (1-7) over a suitable range. Thanks to mass and energy conservation, we need only consider $\sigma > 1$. For $\sigma \geq 1$, controlling $\|\varphi(t)\|_{\dot{H}^\sigma}$ is equivalent to controlling $\|\partial_x \varphi(t)\|_{\dot{H}^{\sigma-1}}$. We therefore consider the time evolution of $\partial_x\varphi$, which may be written entirely in terms of derivatives of the map φ. A more intrinsic way of expressing these equations is to select a *gauge* rather than an extrinsic embedding and coordinate system. We employ the caloric gauge, which is geometrically natural and is analytically well-suited for studying Schrödinger maps. See [Smith 2012a] for the complete details of the construction. It turns out that Sobolev bounds for the gauged derivative map imply corresponding Sobolev bounds for the ungauged derivative map. We schematically write the gauged equation as

$$(\partial_t - \Delta)\psi = \mathcal{N},$$

where ψ is $\partial_x\varphi$ placed in the caloric gauge and $\mathcal{N}$ is a nonlinearity constructed in part from ψ and $\partial_x\psi$.

Function spaces and their interrelation. To prove global results in the energy-critical setting, we of course must look for bounds other than energy estimates to control the solution. Local smoothing estimates and Strichartz estimates will be among the most important required. Our goal is to prove control over ψ within a suitable space through the use of a bootstrap argument. A standard setup requires a space, say G, for the functions ψ and a space, say N, for the nonlinearity $\mathcal{N}$. In fact, we work with stronger, frequency-localized spaces, G_k and N_k, to respectively hold $P_k\psi$ and $P_k\mathcal{N}$. We want them to be related at least by the linear estimate

$$\|P_k\psi\|_{G_k} \lesssim \|P_k\psi(t=0)\|_{L^2_x} + \|P_k\mathcal{N}\|_{N_k}.$$

The hope, then, is to control $\|P_k\mathcal{N}\|_{N_k}$ in terms of $\|P_k\psi(t=0)\|_{L^2_x}$ and $\varepsilon\|P_k\psi\|_{G_k}$ (with ε small), so that, by proving (under a bootstrap hypothesis) a statement such as

$$\|P_k\psi\|_{G_k} \lesssim \|P_k\psi(t=0)\|_{L^2_x} + \varepsilon\|P_k\psi\|_{G_k},$$

we may conclude

$$\|P_k\psi\|_{G_k} \lesssim \|P_k\psi(t=0)\|_{L^2_x}. \tag{1-8}$$

Once (1-8) is proved, showing (1-6) and (1-7) is reduced to the comparatively easy tasks of unwinding the gauging and frequency localization steps so as to conclude with a standard continuity argument.

Controlling the nonlinearity. In this context, the main contribution of this paper lies in showing that we may conclude (1-8) without assuming small energy. The most difficult-to-control terms in the nonlinearity $P_k\mathcal{N}$ are those involving a derivative landing on high-frequency pieces of the derivative fields; we represent them schematically as $A_{\mathrm{lo}}\partial\psi_{\mathrm{hi}}$. Local smoothing estimates controlling the linear evolution (introduced in [Ionescu and Kenig 2006; 2007b]) were successfully used in [Bejenaru et al. 2011c] to handle $A_{\mathrm{lo}}\partial_x\psi_{\mathrm{hi}}$. These are not strong enough to control $A_{\mathrm{lo}}\partial_x\psi_{\mathrm{hi}}$ in the subthreshold energy setting. We instead pursue a more covariant approach, working directly with a certain covariant frequency-localized Schrödinger equation (see Section 5). Our approach is also physical-space based, in the vein of [Planchon and Vega 2009; 2012; Tao 2010], and modular.

2. Gauge field equations

In Section 2A we pass to the derivative formulation of the Schrödinger map system (1-1). All of the main arguments of our subsequent analysis take place at this level. The derivative formulation is at once both overdetermined, reflecting geometric constraints, and underdetermined, exhibiting *gauge invariance*. Section 2B introduces the caloric gauge, which is the gauge we select and work with throughout. Both Tao [2006a] and Bejenaru et al. [2011c] give good explanations justifying the use of the caloric gauge in our setting as opposed to alternative gauges. The reader is referred to [Smith 2012a] for the requisite construction of the caloric gauge for maps with energy up to E_{crit}. Section 2C deals with frequency localizing components of the caloric gauge. Proofs are postponed to Section 6 so that we can more quickly turn our attention to the gauged Schrödinger map system.

2A. ***Derivative equations.*** We begin with some constructions that are valid for any smooth function $\phi : \mathbb{R}^2 \times (-T, T) \to \mathbb{S}^2$. For a more general and extensive introduction to the gauge formalism we now introduce, see [Tao 2004]. Space and time derivatives of ϕ are denoted by $\partial_\alpha\phi(x,t)$, where $\alpha = 1, 2, 3$ ranges over the spatial variables x_1, x_2 and time t with $\partial_3 = \partial_t$.

Select a (smooth) orthonormal frame $(v(x,t), w(x,t))$ for the bundle $T_{\phi(x,t)}\mathbb{S}^2$, that is, smooth functions $v, w : \mathbb{R}^2 \times (-T, T) \to T_{\phi(x,t)}\mathbb{S}^2$ such that at each point (x,t) in the domain the vectors $v(x,t), w(x,t)$ form an orthonormal basis for $T_{\phi(x,t)}\mathbb{S}^2$. As a matter of convention we assume that v and w are chosen so that $v \times w = \phi$.

With respect to this chosen frame we then introduce the derivative fields ψ_α, setting

$$\psi_\alpha := v \cdot \partial_\alpha \phi + i w \cdot \partial_\alpha \phi. \tag{2-1}$$

Then $\partial_\alpha \phi$ admits the representation

$$\partial_\alpha \phi = v \operatorname{Re} \psi_\alpha + w \operatorname{Im} \psi_\alpha \tag{2-2}$$

with respect to the frame (v, w). The derivative fields can be thought of as arising from the following process: First, rewrite the vector $\partial_\alpha \phi$ with respect to the orthonormal basis (v, w); then, identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$ according to $v \leftrightarrow 1$, $w \leftrightarrow i$. Note that this identification respects the complex structure of the target manifold.

Through this identification the Riemannian connection on $\mathbb{S}^2$ pulls back to a covariant derivative on $\mathbb{C}$, which we denote by

$$D_\alpha := \partial_\alpha + i A_\alpha.$$

The real-valued connection coefficients A_α are defined via

$$A_\alpha := w \cdot \partial_\alpha v, \tag{2-3}$$

so that in particular

$$\partial_\alpha v = -\phi \operatorname{Re} \psi_\alpha + w A_\alpha \quad \text{and} \quad \partial_\alpha w = -\phi \operatorname{Im} \psi_\alpha - v A_\alpha.$$

Due to the fact that the Riemannian connection on $\mathbb{S}^2$ is torsion-free, the derivative fields satisfy the relations

$$D_\beta \psi_\alpha = D_\alpha \psi_\beta. \tag{2-4}$$

or equivalently,

$$\partial_\beta A_\alpha - \partial_\alpha A_\beta = \operatorname{Im}(\psi_\beta \overline{\psi}_\alpha) =: q_{\beta\alpha}.$$

The curvature of the connection is therefore given by

$$[D_\beta, D_\alpha] := D_\beta D_\alpha - D_\alpha D_\beta = i q_{\beta\alpha}. \tag{2-5}$$

Assuming now that we are given a smooth solution φ of the Schrödinger map system (1-1), we derive the equations satisfied by the derivative fields ψ_α. The system (1-1) directly translates to

$$\psi_t = i D_l \psi_l \tag{2-6}$$

because

$$\varphi \times \Delta\varphi = J(\varphi)(\varphi^* \nabla)_j \partial_j \varphi,$$

where $J(\varphi)$ denotes the complex structure $\varphi\times$ and $(\varphi^*\nabla)_j$ the pullback of the Levi-Civita connection ∇ on the sphere.

Let us pause to note the following conventions regarding indices. Roman typeface letters are used to index spatial variables. Greek typeface letters are used to index the spatial variables along with time.

Repeated lettered indices within the same subscript or occurring in juxtaposed terms indicate an implicit summation over the appropriate set of indices.

Using (2-4) and (2-5) in (2-6) yields

$$D_t \psi_m = i D_l D_l \psi_m + q_{lm} \psi_l,$$

which is equivalent to the nonlinear Schrödinger equation

$$(i\partial_t + \Delta)\psi_m = \mathcal{N}_m, \tag{2-7}$$

where the nonlinearity $\mathcal{N}_m$ is defined by the formula

$$\mathcal{N}_m := -i A_l \partial_l \psi_m - i \partial_l (A_l \psi_m) + (A_t + A_x^2)\psi_m - i \psi_l \operatorname{Im}(\overline{\psi_l} \psi_m).$$

We split this nonlinearity as a sum $\mathcal{N}_m = B_m + V_m$, with B_m and V_m defined by

$$B_m := -i \partial_l (A_l \psi_m) - i A_l \partial_l \psi_m \tag{2-8}$$

and

$$V_m := (A_t + A_x^2)\psi_m - i \psi_l \operatorname{Im}(\overline{\psi_l} \psi_m), \tag{2-9}$$

thus separating the essentially semilinear magnetic potential terms and the essentially semilinear electric potential terms from each other.

We now state the gauge formulation of the differentiated Schrödinger map system:

$$\begin{cases} D_t \psi_m = i D_l D_l \psi_m + \operatorname{Im}(\psi_l \overline{\psi_m})\psi_l, \\ D_\alpha \psi_\beta = D_\beta \psi_\alpha, \\ \operatorname{Im}(\psi_\alpha \overline{\psi_\beta}) = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \end{cases} \tag{2-10}$$

A solution ψ_m to (2-10) cannot be determined uniquely without first choosing an orthonormal frame (v, w). Changing a given choice of orthonormal frame induces a gauge transformation and may be represented as

$$\psi_m \to e^{-i\theta} \psi_m \quad \text{and} \quad A_m \to A_m + \partial_m \theta$$

in terms of the gauge components. The system (2-10) is invariant with respect to such gauge transformations.

The advantage of working with this gauge formalism rather than the Schrödinger map system or the derivative equations directly is that a carefully selected choice of gauge tames the nonlinearity. In particular, when the caloric gauge is employed, the nonlinearity in (2-7) is nearly perturbative.

2B. *Introduction to the caloric gauge.* In this section we introduce the caloric gauge, which is the gauge we shall employ throughout the remainder of the paper. Gauges were first used to study (1-1) in the context of proving local wellposedness in [Chang et al. 2000]. We note here that the while the Coulomb gauge would seem an attractive choice, it turns out that this gauge is not well-suited to the study of Schrödinger maps in low dimension, as in low dimension parallel interactions of waves are more probable than in high dimension, resulting in unfavorable high × high → low cascades. See [Tao 2006a] and

[Bejenaru et al. 2011c] for further discussion and a comparison of the Coulomb and caloric gauges. Also see [Tao 2006b, Chapter 6] for a discussion of various gauges that have been used in the study of wave maps.

The caloric gauge was introduced by Tao [2004] in the setting of wave maps into hyperbolic space. In a series of unpublished papers [2008a; 2008b; 2008c; 2009a; 2009b], Tao used this gauge in establishing global regularity of wave maps into hyperbolic space. In his unpublished note [Tao 2006a], Tao also suggested the caloric gauge as a suitable gauge for the study of Schrödinger maps. The caloric gauge was first used in the Schrödinger maps problem by Bejenaru, Ionescu, Kenig, and Tataru [2011c] to establish global well-posedness in the setting of initial data with sufficiently small critical norm. We recommend [Tao 2004; 2006a; 2008b; Bejenaru et al. 2011c] for background on the caloric gauge and for helpful heuristics.

Theorem 2.1 (the caloric gauge). *Let $T \in (0, \infty)$, $Q \in S^2$, and let $\phi(x, t) \in H_Q^{\infty,\infty}(T)$ be such that $\sup_{t \in (-T,T)} E(\phi(t)) < E_{\text{crit}}$. There exists a unique smooth extension $\phi(s, x, t) \in C([0, \infty) \to H_Q^{\infty,\infty}(T))$ solving the covariant heat equation*

$$\partial_s \phi = \Delta \phi + \phi \cdot |\partial_x \phi|^2 \tag{2-11}$$

and with $\phi(0, x, t) = \phi(x, t)$. Moreover, for any given choice of a (constant) orthonormal basis (v_∞, w_∞) of $T_Q \mathbb{S}^2$, there exist smooth functions $v, w : [0, \infty) \times \mathbb{R}^2 \times (-T, T) \to S^2$ such that at each point (s, x, t), the set $\{v, w, \phi\}$ naturally forms an orthonormal basis for $\mathbb{R}^3$, the gauge condition

$$w \cdot \partial_s v \equiv 0, \tag{2-12}$$

is satisfied, and

$$|\partial_x^\rho f(s)| \lesssim_\rho \langle s \rangle^{-(|\rho|+1)/2} \tag{2-13}$$

for each $f \in \{\phi - Q, v - v_\infty, w - w_\infty\}$, multiindex ρ, and $s \geq 0$.

Proof. This is a special case of the more general result [Smith 2012a, Theorem 7.6]. Whereas in [Smith 2012a] everything is stated in terms of the category of Schwartz functions, in fact this requirement may be relaxed to $H_Q^{\infty,\infty}(T)$ without difficulty (at least in the case of compact target manifolds) since weighted decay in L^2-based Sobolev spaces is not used in any proofs. □

In our application in this paper, $E(\varphi(t))$ is conserved. Therefore, we set $E_0 := E(\varphi_0)$.

Having extended v, w along the heat flow, we may likewise extend A_x along the flow. We record here for reference a technical bound that proves useful; for the proof, see [Smith 2012a, §7.1].

Theorem 2.2. *Assume the conditions of Theorem 2.1 are in force. Then we have the bound*

$$\|A_x(s)\|_{L_x^2(\mathbb{R}^2)} \lesssim_{E_0} 1. \tag{2-14}$$

Corollary 2.3 (energy bounds for the frame). *Let φ be a Schrödinger map with energy $E_0 < E_{\text{crit}}$. Then*

$$\|\partial_x v\|_{L_t^\infty L_x^2} \lesssim_{E_0} 1. \tag{2-15}$$

Proof. Because $|v| \equiv 1$, we have $v \cdot \partial_m v \equiv 0$. Therefore, with respect to the orthonormal frame (v, w, φ), the vector $\partial_m v$ admits the representation

$$\partial_m v = A_m \cdot w - \operatorname{Re} \psi_m \cdot \varphi. \tag{2-16}$$

The bound (2-15) then follows from using $|w| \equiv 1 \equiv |\varphi|$, $\|\psi_m\|_{L_x^2} \equiv \|\partial_m \varphi\|_{L_x^2}$, energy conservation, and (2-14) all in (2-16). □

Adopting the convention $\partial_0 = \partial_s$, and now and hereafter allowing all Greek indices to range over heat time, spatial variables, and time, we define for all $(s, x, t) \in [0, \infty) \times \mathbb{R}^2 \times (-T, T)$ the various gauge components

$$\begin{aligned} \psi_\alpha &:= v \cdot \partial_\alpha \varphi + i w \cdot \partial_\alpha \varphi, \\ A_\alpha &:= w \cdot \partial_\alpha v, \\ D_\alpha &:= \partial_\alpha + A_\alpha, \\ q_{\alpha\beta} &:= \partial_\alpha A_\beta - \partial_\beta A_\alpha. \end{aligned}$$

For $\alpha = 0, 1, 2, 3$ we have

$$\partial_\alpha \varphi = v \operatorname{Re} \psi_\alpha + w \operatorname{Im} \psi_\alpha.$$

The parallel transport condition $w \cdot \partial_s v \equiv 0$ is equivalently expressed in terms of the connection coefficients as

$$A_s \equiv 0. \tag{2-17}$$

Expressed in terms of the gauge, the heat flow (2-11) lifts to

$$\psi_s = D_l \psi_l. \tag{2-18}$$

Using (2-4) and (2-5), we may rewrite the D_m covariant derivative of (2-18) as

$$\partial_s \psi_m = D_l D_l \psi_m + i \operatorname{Im}(\psi_m \overline{\psi_l}) \psi_l,$$

or equivalently

$$(\partial_s - \Delta)\psi_m = i A_l \partial_l \psi_m + i \partial_l (A_l \psi_m) - A_x^2 \psi_m + i \psi_l \operatorname{Im}(\overline{\psi_l} \psi_m). \tag{2-19}$$

More generally, taking the D_α covariant derivative, we obtain

$$(\partial_s - \Delta)\psi_\alpha = U_\alpha, \tag{2-20}$$

where we set

$$U_\alpha := i A_l \partial_l \psi_\alpha + i \partial_l (A_l \psi_\alpha) - A_x^2 \psi_\alpha + i \psi_l \operatorname{Im}(\overline{\psi_l} \psi_\alpha), \tag{2-21}$$

which admits the alternative representation

$$U_\alpha = 2i A_l \partial_l \psi_\alpha + i(\partial_l A_l)\psi_\alpha - A_x^2 \psi_\alpha + i \psi_l \operatorname{Im}(\overline{\psi_l} \psi_\alpha). \tag{2-22}$$

From (2-5) and (2-17) it follows that

$$\partial_s A_\alpha = \operatorname{Im}(\psi_s \overline{\psi_\alpha}).$$

Integrating back from $s = \infty$ (justified using (2-13)) yields

$$A_\alpha(s) = -\int_s^\infty \operatorname{Im}(\overline{\psi_\alpha}\psi_s)(s')\,ds'. \tag{2-23}$$

At $s = 0$, φ satisfies both (1-1) and (2-11), or equivalently, $\psi_t(s = 0) = i\psi_s(s = 0)$. While for $s > 0$ it continues to be the case that $\psi_s = D_l\psi_l$ by construction, we no longer necessarily have $\psi_t(s) = i D_l(s)\psi_l(s)$, i.e., $\varphi(s, x, t)$ is not necessarily a Schrödinger map at fixed $s > 0$. In the following lemma we derive an evolution equation for the commutator $\Psi = \psi_t - i\psi_s$.

Lemma 2.4 (flows do not commute). *Set* $\Psi := \psi_t - i\psi_s$. *Then*

$$\partial_s\Psi = D_l D_l\Psi + i\operatorname{Im}(\psi_t\overline{\psi_l})\psi_l - \operatorname{Im}(\psi_s\overline{\psi_l})\psi_l \tag{2-24}$$

$$= D_l D_l\Psi + i\operatorname{Im}(\Psi\overline{\psi_l})\psi_l + i\operatorname{Im}(i\psi_s\overline{\psi_l})\psi_l - \operatorname{Im}(\psi_s\overline{\psi_l})\psi_l. \tag{2-25}$$

Proof. We prove (2-24), since (2-25) is a trivial consequence of it.

Applying (2-19) and (2-20) to ψ_s and ψ_t and collapsing the covariant derivative terms yields

$$\partial_s\psi_t = D_l D_l\psi_t + i\operatorname{Im}(\psi_t\overline{\psi_l})\psi_l, \tag{2-26}$$

$$\partial_s\psi_s = D_l D_l\psi_s + i\operatorname{Im}(\psi_s\overline{\psi_l})\psi_l. \tag{2-27}$$

Multiply (2-27) by i to obtain the s-evolution of $i\psi_s$. Multiplication by i commutes with D_l, but fails to do so with $\operatorname{Im}(\,\cdot\,)$, and thus we obtain

$$\partial_s i\psi_m = D_l D_l i\psi_s - \operatorname{Im}(\psi_s\overline{\psi_l})\psi_l. \tag{2-28}$$

Together (2-26) and (2-28) imply (2-24). □

2C. *Frequency localization.* Frequency localization plays an indispensable role in our analysis. In this subsection we establish some basic concepts and then state some basic results for the caloric gauge.

Our notation for a standard Littlewood–Paley frequency localization of a function f to frequencies $\sim 2^k$ is $P_k f$ and to frequencies $\lesssim 2^k$ is $P_{\le k} f$. The particular localization chosen is of course immaterial to our analysis, but for definiteness is specified in the next section and chosen for convenience to coincide with that in [Bejenaru et al. 2011c].

We shall frequently make use of the following standard *Bernstein inequalities* for $\mathbb{R}^2$ with $\sigma \ge 0$ and $1 \le p \le q \le \infty$:

$$\|P_{\le k}|\partial_x|^\sigma f\|_{L^p_x(\mathbb{R}^2)} \lesssim_{p,\sigma} 2^{\sigma k}\|P_{\le k} f\|_{L^p_x(\mathbb{R}^2)}.$$

$$\|P_k|\partial_x|^{\pm\sigma} f\|_{L^p_x(\mathbb{R}^2)} \lesssim_{p,\sigma} 2^{\pm\sigma k}\|P_k f\|_{L^p_x(\mathbb{R}^2)}.$$

$$\|P_{\le k} f\|_{L^q_x(\mathbb{R}^2)} \lesssim_{p,q} 2^{2k(1/p-1/q)}\|P_{\le k} f\|_{L^p_x(\mathbb{R}^2)}.$$

$$\|P_k f\|_{L^q_x(\mathbb{R}^2)} \lesssim_{p,q} 2^{2k(1/p-1/q)}\|P_k f\|_{L^p_x(\mathbb{R}^2)}.$$

A particularly important notion for us is that of a frequency envelope, as it provides a way to rigorously manage the "frequency leakage" phenomenon and the frequency cascades produced by nonlinear interactions. We introduce a parameter δ in the definition; for the purposes of this paper $\delta = \frac{1}{40}$ suffices.

Definition 2.5 (frequency envelopes). A positive sequence $\{a_k\}_{k\in\mathbb{Z}}$ is a frequency envelope if it belongs to l^2 and is slowly varying:

$$a_k \le a_j 2^{\delta|k-j|}, \quad j,k\in\mathbb{Z}. \tag{2-29}$$

A frequency envelope $\{a_k\}_{k\in\mathbb{Z}}$ is ε-energy dispersed if it satisfies the additional condition

$$\sup_{k\in\mathbb{Z}} a_k \le \varepsilon.$$

Note in particular that frequency envelopes satisfy the summation rules

$$\sum_{k'\le k} 2^{pk'} a_{k'} \lesssim (p-\delta)^{-1} 2^{pk} a_k, \qquad p>\delta, \tag{2-30}$$

$$\sum_{k'\ge k} 2^{-pk'} a_{k'} \lesssim (p-\delta)^{-1} 2^{-pk} a_k, \quad p>\delta. \tag{2-31}$$

In practice we work with p bounded away from δ — for instance, $p>2\delta$ suffices — and iterate these inequalities only $O(1)$ times. Therefore, in applications we drop the factors $(p-\delta)^{-1}$ appearing in (2-30) and (2-31).

Finally, pick a positive integer σ_1 and hold it fixed throughout the remainder of this section. Results in this section hold for any such σ_1, though implicit constants are allowed to depend upon this choice.

Given initial data $\varphi_0 \in H_Q^\infty$, define for all $\sigma\ge 0$ and $k\in\mathbb{Z}$

$$c_k(\sigma) := \sup_{k'\in\mathbb{Z}} 2^{-\delta|k-k'|} 2^{\sigma k'} \|P_{k'}\partial_x \varphi_0\|_{L^2_x}. \tag{2-32}$$

Set $c_k := c_k(0)$ for short. For $\sigma\in[0,\sigma_1]$ we then have that

$$\|\partial_x\varphi_0\|^2_{\dot H^\sigma_x} \sim \sum_{k\in\mathbb{Z}} c_k^2(\sigma) \quad \text{and} \quad \|P_k\partial_x\varphi_0\|_{L^2_x} \le c_k(\sigma) 2^{-\sigma k}. \tag{2-33}$$

Similarly, for $\varphi\in H_Q^{\infty,\infty}(T)$, define for all $\sigma\ge 0$ and $k\in\mathbb{Z}$

$$\gamma_k(\sigma) := \sup_{k'\in\mathbb{Z}} 2^{-\delta|k-k'|} 2^{\sigma k'} \|P_{k'}\varphi\|_{L^\infty_t L^2_x}. \tag{2-34}$$

Set $\gamma_k := \gamma_k(1)$.

Theorem 2.6 (frequency-localized energy bounds for heat flow). *Let $f\in\{\varphi, v, w\}$. Then for $\sigma\in[1,\sigma_1]$ the bound*

$$\|P_k f(s)\|_{L^\infty_t L^2_x} \lesssim 2^{-\sigma k}\gamma_k(\sigma)(1+s2^{2k})^{-20} \tag{2-35}$$

holds and for any $\sigma,\rho\in\mathbb{Z}_+$ we have that

$$\sup_{k\in\mathbb{Z}} \sup_{s\in[0,\infty)} (1+s)^{\sigma/2} 2^{\sigma k} \|P_k \partial_t^\rho f(s)\|_{L^\infty_t L^2_x} < \infty. \tag{2-36}$$

Corollary 2.7 (frequency-localized energy bounds for the caloric gauge). *For $\sigma\in[0,\sigma_1-1]$, we have*

$$\|P_k\psi_x(s)\|_{L^\infty_t L^2_x} + \|P_k A_m(s)\|_{L^\infty_t L^2_x} \lesssim 2^k 2^{-\sigma k}\gamma_k(\sigma)(1+s2^{2k})^{-20}. \tag{2-37}$$

Moreover, for any $\sigma \in \mathbb{Z}_+$,

$$\sup_{k\in\mathbb{Z}} \sup_{s\in[0,\infty)} (1+s)^{\sigma/2} 2^{\sigma k} 2^{-k} \big(\| P_k(\partial_t^\rho \psi_x(s)) \|_{L_t^\infty L_x^2} + \| P_k(\partial_t^\rho A_x(s)) \|_{L_t^\infty L_x^2} \big) < \infty \tag{2-38}$$

and

$$\sup_{k\in\mathbb{Z}} \sup_{s\in[0,\infty)} (1+s)^{\sigma/2} 2^{\sigma k} \big(\| P_k(\partial_t^\rho \psi_t(s)) \|_{L_t^\infty L_x^2} + \| P_k(\partial_t^\rho A_t(s)) \|_{L_t^\infty L_x^2} \big) < \infty. \tag{2-39}$$

We prove Theorem 2.6 and its corollary in Section 6. Corollary 2.7 has an elementary consequence:

Corollary 2.8. *For* $\sigma \in [0, \sigma_1 - 1]$ *we have*

$$\| P_k \psi_x(0, \cdot, 0) \|_{L_x^2} \lesssim 2^{-\sigma k} c_k(\sigma). \tag{2-40}$$

3. Function spaces and basic estimates

3A. *Definitions.*

Definition 3.1 (Littlewood–Paley multipliers). Let $\eta_0 : \mathbb{R} \to [0, 1]$ be a smooth even function vanishing outside the interval $[-8/5, 8/5]$ and equal to 1 on $[-5/4, 5/4]$. For $j \in \mathbb{Z}$, set

$$\chi_j(\cdot) = \eta_0(\cdot/2^j) - \eta_0(\cdot/2^{j-1}), \quad \chi_{\le j}(\cdot) = \eta_0(\cdot/2^j).$$

Let P_k denote the operator on $L^\infty(\mathbb{R}^2)$ defined by the Fourier multiplier $\xi \to \chi_k(|\xi|)$. For any interval $I \subset \mathbb{R}$, define the Fourier multiplier

$$\chi_I = \sum_{j\in I\cap\mathbb{Z}} \chi_j$$

and let P_I denote its corresponding operator on $L^\infty(\mathbb{R}^2)$. We shall denote $P_{(-\infty,k]}$ by $P_{\le k}$ for short. For $\theta \in \mathbb{S}^1$ and $k \in \mathbb{Z}$, we define the operators $P_{k,\theta}$ by the Fourier multipliers $\xi \to \chi_k(\xi \cdot \theta)$.

Some frequency interactions in the nonlinearity of (2-7) can be controlled using the following lemma.

Lemma 3.2 (Strichartz estimate). *Let* $f \in L_x^2(\mathbb{R}^2)$ *and* $k \in \mathbb{Z}$. *Then the Strichartz estimate*

$$\| e^{it\Delta} f \|_{L_{t,x}^4} \lesssim \| f \|_{L_x^2}$$

holds, as does the maximal function bound

$$\| e^{it\Delta} P_k f \|_{L_x^4 L_t^\infty} \lesssim 2^{k/2} \| f \|_{L_x^2}.$$

The first bound is the original Strichartz estimate [1977] and the second follows from scaling. These will be augmented with certain lateral Strichartz estimates to be introduced shortly. Strichartz estimates alone are not sufficient for controlling the nonlinearity in (2-7). The additional control required comes from local smoothing and maximal function estimates. Certain local smoothing spaces localized to cubes were introduced in [Kenig et al. 1993] to study the local well-posedness of Schrödinger equations with general derivative nonlinearities. Stronger spaces were introduced in [Ionescu and Kenig 2007a] to prove a low-regularity global result. In the Schrödinger map setting, local smoothing spaces were first used in [Ionescu and Kenig 2006] and subsequently in [Ionescu and Kenig 2007b; Bejenaru et al. 2007;

Bejenaru 2008a]. The particular local smoothing/maximal function spaces we shall use were introduced in [Bejenaru et al. 2011c].

For a unit length $\theta \in \mathbb{S}^1$, we denote by H_θ its orthogonal complement in $\mathbb{R}^2$ with the induced measure. Define the lateral spaces $L_\theta^{p,q}$ as those consisting of all measurable f for which the norm

$$\|h\|_{L_\theta^{p,q}} = \left(\int_{\mathbb{R}} \left(\int_{H_\theta \times \mathbb{R}} |h(x_1\theta + x_2, t)|^q \, dx_2 \, dt \right)^{p/q} dx_1 \right)^{1/p},$$

is finite. We make the usual modifications when $p = \infty$ or $q = \infty$. The most important spaces for our analysis are the local smoothing space $L_\theta^{\infty,2}$ and the inhomogeneous local smoothing space $L_\theta^{1,2}$. To move between these spaces we use the maximal function space $L_\theta^{2,\infty}$.

Lemma 3.3 (local smoothing [Ionescu and Kenig 2006; 2007b]). *Let $f \in L_x^2(\mathbb{R}^2)$, $k \in \mathbb{Z}$, and $\theta \in \mathbb{S}^1$. Then*

$$\|e^{it\Delta} P_{k,\theta} f\|_{L_\theta^{\infty,2}} \lesssim 2^{-k/2} \|f\|_{L_x^2}.$$

For $f \in L_x^2(\mathbb{R}^d)$, the maximal function space bound

$$\|e^{it\Delta} P_k f\|_{L_\theta^{2,\infty}} \lesssim 2^{k(d-1)/2} \|f\|_{L_x^2}$$

holds for dimension $d \geq 3$.

In $d = 2$, the maximal function bound fails due to a logarithmic divergence. In order to overcome this, we exploit Galilean invariance as in [Bejenaru et al. 2011c] (the idea goes back to [Tataru 2001] in the setting of wave maps).

For $p, q \in [1, \infty]$, $\theta \in \mathbb{S}^1$, $\lambda \in \mathbb{R}$, define $L_{\theta,\lambda}^{p,q}$ using the norm

$$\|h\|_{L_{\theta,\lambda}^{p,q}} = \|T_{\lambda\theta}(h)\|_{L_\theta^{p,q}} = \left(\int_{\mathbb{R}} \left(\int_{H_\theta \times \mathbb{R}} |h((x_1 + t\lambda)\theta + x_2, t)|^q \, dx_2 \, dt \right)^{p/q} dx_1 \right)^{1/p},$$

where T_w denotes the Galilean transformation

$$T_w(f)(x,t) = e^{-ix\cdot w/2} e^{-it|w|^2/4} f(x + tw, t).$$

With $W \subset \mathbb{R}$ finite we define the spaces $L_{\theta,W}^{p,q}$ by

$$L_{\theta,W}^{p,q} = \sum_{\lambda \in W} L_{\theta,\lambda}^{p,q}, \quad \|f\|_{L_{\theta,W}^{p,q}} = \inf_{f = \sum_{\lambda \in W} f_\lambda} \sum_{\lambda \in W} \|f_\lambda\|_{L_{\theta,\lambda}^{p,q}}.$$

For $k \in \mathbb{Z}$, $\mathcal{K} \in \mathbb{Z}_+$, set

$$W_k := \{\lambda \in [-2^k, 2^k] : 2^{k+2\mathcal{K}} \lambda \in \mathbb{Z}\}.$$

In our application we shall work on a finite time interval $[-2^{2\mathcal{K}}, 2^{2\mathcal{K}}]$ in order to ensure that the W_k are finite. This still suffices for proving global results so long as our effective bounds are proved with constants independent of $T, \mathcal{K}$. As discussed in [Bejenaru et al. 2011c, §3], restricting T to a finite time interval avoids introducing additional technicalities.

Lemma 3.4 (local smoothing/maximal function estimates). *Let $f \in L^2_x(\mathbb{R}^2)$, $k \in \mathbb{Z}$, and $\theta \in \mathbb{S}^1$. Then*

$$\|e^{it\Delta} P_{k,\theta} f\|_{L^{\infty,2}_{\theta,\lambda}} \lesssim 2^{-k/2} \|f\|_{L^2_x}, \quad |\lambda| \le 2^{k-40},$$

and if $T \in (0, 2^{2\mathcal{K}}]$, then

$$\|1_{[-T,T]}(t) e^{it\Delta} P_k f\|_{L^{2,\infty}_{\theta, W_{k+40}}} \lesssim 2^{k/2} \|f\|_{L^2_x}.$$

Proof. The first bound follows from Lemma 3.3 via a Galilean boost. The second is more involved and is proven in [Bejenaru et al. 2011c, §7]. □

Lemma 3.5 (lateral Strichartz estimates). *Let $f \in L^2_x(\mathbb{R}^2)$, $k \in \mathbb{Z}$, and $\theta \in \mathbb{S}^1$. Let $2 < p \le \infty$, $2 \le q \le \infty$ and $1/p + 1/q = 1/2$. Then*

$$\|e^{it\Delta} P_{k,\theta} f\|_{L^{p,q}_\theta} \lesssim 2^{k(2/p-1/2)} \|f\|_{L^2_x}, \qquad p \ge q,$$
$$\|e^{it\Delta} P_k f\|_{L^{p,q}_\theta} \lesssim_p 2^{k(2/p-1/2)} \|f\|_{L^2_x}, \qquad p \le q.$$

Proof. Informally speaking, these bounds follow from interpolating between the L^4 Strichartz estimate and the local smoothing/maximal function estimates of Lemma 3.4. See [Bejenaru et al. 2011c, Lemma 7.1] for the rigorous argument. □

We now introduce the main function spaces. Let $T > 0$. For $k \in \mathbb{Z}$, let $I_k = \{\xi \in \mathbb{R}^2 : |\xi| \in [2^{k-1}, 2^{k+1}]\}$. Let

$$L^2_k(T) := \{f \in L^2(\mathbb{R}^2 \times [-T,T]) : \operatorname{supp} \hat{f}(\xi,t) \subset I_k \times [-T,T]\}.$$

For $f \in L^2(\mathbb{R}^2 \times [-T,T])$, let

$$\|f\|_{F^0_k(T)} := \|f\|_{L^\infty_t L^2_x} + \|f\|_{L^4_{t,x}} + 2^{-k/2}\|f\|_{L^4_x L^\infty_t} + 2^{-k/6} \sup_{\theta\in\mathbb{S}^1} \|f\|_{L^{3,6}_\theta} + + 2^{-k/2} \sup_{\theta\in\mathbb{S}^1} \|f\|_{L^{2,\infty}_{\theta,W_{k+40}}}.$$

We then define, similarly to what is done in [Bejenaru et al. 2011c], $F_k(T)$, $G_k(T)$, $N_k(T)$ as the normed spaces of functions in $L^2_k(T)$ for which the corresponding norms

$$\|f\|_{F_k(T)} := \inf_{J, m_1,\dots,m_J \in \mathbb{Z}_+} \inf_{f = f_{m_1} + \cdots + f_{m_J}} \sum_{j=1}^{J} 2^{m_j} \|f_{m_j}\|_{F^0_{k+m_j}},$$

$$\|f\|_{G_k(T)} := \|f\|_{F^0_k(T)} + 2^{k/6} \sup_{|j-k|\le 20} \sup_{\theta\in\mathbb{S}^1} \|P_{j,\theta} f\|_{L^{6,3}_\theta} + 2^{k/2} \sup_{|j-k|\le 20} \sup_{\theta\in\mathbb{S}^1} \sup_{|\lambda|<2^{k-40}} \|P_{j,\theta} f\|_{L^{\infty,2}_{\theta,\lambda}},$$

$$\|f\|_{N_k(T)} := \inf_{f = f_1+f_2+f_3+f_4+f_5+f_6} \|f_1\|_{L^{4/3}_{t,x}} + 2^{k/6}\|f_2\|_{L^{3/2,6/5}_{\hat{\theta}_1}} + 2^{k/6}\|f_3\|_{L^{3/2,6/5}_{\hat{\theta}_2}}$$
$$+ 2^{-k/6}\|f_4\|_{L^{6/5,3/2}_{\hat{\theta}_1}} + 2^{-k/6}\|f_5\|_{L^{6/5,3/2}_{\hat{\theta}_2}} + 2^{-k/2} \sup_{\theta\in\mathbb{S}^1} \|f_6\|_{L^{1,2}_{\theta,W_{k-40}}},$$

are finite, where $(\hat{\theta}_1, \hat{\theta}_2)$ denotes the canonical basis in $\mathbb{R}^2$.

There are a few minor differences between these spaces and those appearing in [Bejenaru et al. 2011c]. The space F^0_k now includes the lateral Strichartz space $L^{3,6}_\theta$, whereas in that reference, only G_k was endowed with this norm. The net effect on the space G_k is that it is left unchanged. The space F_k, however, now explicitly incorporates this particular lateral Strichartz structure. Note though, that for fixed $\theta \in \mathbb{S}^1$, we have by enough applications of Young's and Hölder's inequalities that

$$\begin{aligned}
2^{-k/6}\|f\|_{L^{3,6}_\theta} &= 2^{-k/6}\biggl(\int_{\mathbb{R}}\biggl(\int_{H_\theta\times\mathbb{R}}|f(x_1\theta+x_2,t)|^6\,dx_2\,dt\biggr)^{1/2}dx_1\biggr)^{1/3}\\
&\lesssim 2^{-k/6}\biggl(\int_{\mathbb{R}}\|f\|^2_{L^4_{\theta,t}}\|f\|_{L^\infty_{\theta,t}}\,dx_1\biggr)^{1/3}\\
&\lesssim 2^{-k/6}\biggl(\int_{\mathbb{R}}\|f\|^4_{L^4_{\theta,t}}\,dx_1\biggr)^{1/6}\biggl(\int_{\mathbb{R}}\|f\|^2_{L^\infty_{\theta,t}}dx_1\biggr)^{1/6}\\
&\lesssim \|f\|^{2/3}_{L^4}\cdot 2^{-k/6}\|f\|^{1/3}_{L^{2,\infty}_\theta} \lesssim \|f\|_{L^4}+2^{-k/2}\|f\|_{L^{2,\infty}_\theta}.
\end{aligned}$$

We also make one change to the N_k space: We explicitly incorporate $L^{6/5,3/2}_\theta$.

Incorporating these extra lateral Strichartz spaces affords us greater flexibility in certain estimates: We can avoid having to use local smoothing/maximal function spaces if we are willing to give up some decay. This tradeoff pays off in Section 5, where as a consequence we can prove a stronger local smoothing estimate for a certain magnetic nonlinear Schrödinger equation in the one regime where this improvement is absolutely essential.

Proposition 3.6 (main linear estimate). *Assume $\mathcal{K} \in \mathbb{Z}_+$, $T \in (0, 2^{2\mathcal{K}}]$ and $k \in \mathbb{Z}$. Then for each $u_0 \in L^2$ that is frequency-localized to I_k and for any $h \in N_k(T)$, the solution u of*

$$(i\partial_t + \Delta_x)u = h, \quad u(0) = u_0,$$

satisfies

$$\|u\|_{G_k(T)} \lesssim \|u(0)\|_{L^2_x} + \|h\|_{N_k(T)}.$$

Proof. See [Bejenaru et al. 2011c, Proposition 7.2] for details. Our changes to the spaces necessitate only minor changes in their proof, as we must incorporate $L^{6/5,3/2}_{\hat\theta_1}$ and $L^{6/5,3/2}_{\hat\theta_2}$ into the space $N^0_k(T)$. ☐

The spaces $G_k(T)$ are used to hold projections $P_k\psi_m$ of the derivative fields ψ_m satisfying (2-7). The main components of $G_k(T)$ are the local smoothing/maximal function spaces $L^{\infty,2}_{\theta,\lambda}$, $L^{2,\infty}_{\theta,W_{k+40}}$, and the lateral Strichartz spaces. The local smoothing and maximal function space components play an essential role in recovering the derivative loss that is due to the magnetic nonlinearity.

The spaces $N_k(T)$ hold frequency projections of the nonlinearities in (2-7). Here the main spaces are the inhomogeneous local smoothing spaces $L^{1,2}_{\theta,W_{k-40}}$ and the Strichartz spaces, both chosen to match those of $G_k(T)$.

The spaces $G_k(T)$ clearly embed in $F_k(T)$. Two key properties enjoyed only by the larger spaces $F_k(T)$ are

$$\|f\|_{F_k(T)} \approx \|f\|_{F_{k+1}(T)},$$

for $k \in \mathbb{Z}$ and $f \in F_k(T) \cap F_{k+1}(T)$, and

$$\|P_k(uv)\|_{F_k(T)} \lesssim \|u\|_{F_{k'}(T)}\|v\|_{L^\infty_{t,x}}$$

for $k, k' \in \mathbb{Z}$, $|k-k'| \le 20$, $u \in F_{k'}(T)$, $v \in L^\infty(\mathbb{R}^2 \times [-T, T])$. Both of these properties follow readily from the definitions.

In order to bound the nonlinearity of (2-7) in $N_k(T)$, it is important to gain regularity from the parabolic heat-time smoothing effect. The desired frequency-localized bounds do not (or at least not so readily) propagate in heat-time in the spaces $G_k(T)$, whereas these bounds do propagate with decay in the larger spaces $F_k(T)$. Note that since the $F_k(T)$ norm is translation invariant, we have

$$\|e^{s\Delta}h\|_{F_k(T)} \lesssim (1+s2^{2k})^{-20}\|h\|_{F_k(T)}, \quad s \ge 0,$$

for $h \in F_k(T)$. In certain bilinear estimates we do not need the full strength of the spaces $F_k(T)$ and instead can use the bound

$$\|f\|_{F_k(T)} \lesssim \|f\|_{L^2_x L^\infty_t} + \|f\|_{L^4_{t,x}}, \tag{3-1}$$

which follows from

$$\|f\|_{L^{2,\infty}_{\theta, W_{k+m_j}}} \le \|f\|_{L^{2,\infty}_\theta} \lesssim 2^{k/2}\|f\|_{L^2_x L^\infty_t}.$$

We introduce one more class of function spaces. These can be viewed as a refinement of the Strichartz part of $F_k(T)$. For $k \in \mathbb{Z}$ and $\omega \in [0, 1/2]$ we define $S^\omega_k(T)$ to be the normed space of functions belonging to $L^2_k(T)$ whose norm

$$\|f\|_{S^\omega_k(T)} = 2^{\omega k}\big(\|f\|_{L^\infty_t L^{2_\omega}_x} + \|f\|_{L^4_t L^{p_\omega}_x} + 2^{-k/2}\|f\|_{L^{p_\omega}_x L^\infty_t}\big) \tag{3-2}$$

is finite, where the exponents 2_ω and p_ω are determined by

$$\frac{1}{2_\omega} - \frac{1}{2} = \frac{1}{p_\omega} - \frac{1}{4} = \frac{\omega}{2}.$$

Note that $F_k(T) \hookrightarrow S^0_k(T)$ and that by Bernstein we have

$$\|f\|_{S^{\omega'}_k(T)} \lesssim \|f\|_{S^\omega_k(T)}, \quad \omega' \le \omega.$$

3B. *Bilinear estimates.*

Lemma 3.7 (bilinear estimates on $N_k(T)$). *For $k, k_1, k_3 \in \mathbb{Z}$, $h \in L^2_{t,x}$, $f \in F_{k_1}(T)$, and $g \in G_{k_3}(T)$, we have the following inequalities under the given restrictions on k_1, k_3:*

$$\|P_k(hf)\|_{N_k(T)} \lesssim \|h\|_{L^2_{t,x}}\|f\|_{F_{k_1}(T)} \qquad \text{if } |k_1-k| \le 80. \tag{3-3}$$

$$\|P_k(hf)\|_{N_k(T)} \lesssim 2^{-|k-k_1|/6}\|h\|_{L^2_{t,x}}\|f\|_{F_{k_1}(T)} \qquad \text{if } k_1 \le k-80. \tag{3-4}$$

$$\|P_k(hg)\|_{N_k(T)} \lesssim 2^{-|k-k_3|/6}\|h\|_{L^2_{t,x}}\|g\|_{G_{k_3}(T)} \qquad \text{if } k \le k_3-80. \tag{3-5}$$

Proof. Estimate (3-3) follows from Hölder's inequality and the definition of $F_k(T)$, $N_k(T)$:

$$\|Ff\|_{L^{4/3}} \le \|F\|_{L^2}\|f\|_{L^4}.$$

For (3-4) and (3-5), we use an angular partition of unity in frequency to write

$$f = f_1 + f_2, \quad \|f_1\|_{L^{3,6}_{\hat{\theta}_1}} + \|g_1\|_{L^{3,6}_{\hat{\theta}_2}} \lesssim 2^{k_1/6}\|f\|_{F_k(T)},$$
$$g = g_1 + g_2, \quad \|g_1\|_{L^{6,3}_{\hat{\theta}_1}} + \|g_1\|_{L^{6,3}_{\hat{\theta}_2}} \lesssim 2^{-k_1/6}\|g\|_{G_k(T)}.$$

Then

$$\begin{aligned}\|P_k(Ff)\|_{N_k(T)} &\lesssim 2^{-k/6}\big(\|Ff_1\|_{L^{6/5,3/2}_{\hat{\theta}_1}} + \|Ff_2\|_{L^{6/5,3/2}_{\hat{\theta}_2}}\big) \lesssim 2^{-k/6}\|F\|_{L^2}\big(\|f_1\|_{L^{3,6}_{\hat{\theta}_1}} + \|f_1\|_{L^{3,6}_{\hat{\theta}_2}}\big)\\ &\lesssim 2^{(k_1-k)/6}\|F\|_{L^2}\|f\|_{F_{k_1}(T)},\\ \|P_k(Fg)\|_{N_k(T)} &\lesssim 2^{k/6}\big(\|Fg_1\|_{L^{3/2,6/5}_{\hat{\theta}_1}} + \|Fg_2\|_{L^{3/2,6/5}_{\hat{\theta}_2}}\big) \lesssim 2^{k/6}\|F\|_{L^2}\big(\|g_1\|_{L^{6,3}_{\hat{\theta}_1}} + \|g_1\|_{L^{6,3}_{\hat{\theta}_2}}\big)\\ &\lesssim 2^{(k-k_1)/6}\|F\|_{L^2}\|g\|_{G_{k_3}(T)}.\end{aligned}$$

□

Lemma 3.8 (bilinear estimates on $L^2_{t,x}$). *For $k_1, k_2, k_3 \in \mathbb{Z}$, $f_1 \in F_{k_1}(T)$, $f_2 \in F_{k_2}(T)$, and $g \in G_{k_3}(T)$, we have*

$$\|f_1 \cdot f_2\|_{L^2_{t,x}} \lesssim \|f_1\|_{F_{k_1}(T)}\|f_2\|_{F_{k_2}(T)}, \tag{3-6}$$
$$\|f \cdot g\|_{L^2_{t,x}} \lesssim 2^{-|k_1-k_3|/6}\|f\|_{F_{k_1}(T)}\|g\|_{G_{k_3}(T)} \quad \text{for } k_1 \le k_3. \tag{3-7}$$

Proof. It suffices to show that

$$\|fg\|_{L^2} \lesssim \|f\|_{F^0_{k_1}(T)}\|g\|_{G_{k_2}(T)} \qquad \text{for } k_1 \ge k_2 - 100, \tag{3-8}$$
$$\|fg\|_{L^2} \lesssim 2^{(k_1-k_2)/6}\|f\|_{F^0_{k_1}(T)}\|g\|_{G_{k_2}(T)} \quad \text{for } k_1 < k_2 - 100. \tag{3-9}$$

Estimate (3-8) follows from estimating each factor in L^4. For (3-9), we first observe that, using a smooth partition of unity in frequency space, we may assume that $\hat{g}$ is supported in the set

$$\{\xi : |\xi| \in [2^{k_2-1}, 2^{k_2+1}] \text{ and } \xi \cdot \theta_0 \ge 2^{k_2-5}\}$$

for some direction $\theta_0 \in \mathbb{S}^1$. Then $\|fg\|_{L^2} \lesssim \|f\|_{L^{3,6}_{\theta_0}}\|g\|_{L^{6,3}_{\theta_0}} \lesssim 2^{(k_1-k_2)/6}\|f\|_{F^0_{k_1}(T)}\|g\|_{G_{k_2}(T)}$. □

We also have the following stronger estimates, which rely upon the local smoothing and maximal function spaces.

Lemma 3.9 (bilinear estimates using local smoothing/maximal function bounds). *For $k, k_1, k_2 \in \mathbb{Z}$, $h \in L^2_{t,x}$, $f \in F_{k_1}(T)$, $g \in G_{k_2}(T)$, we have, under the given restrictions on k_1, k_2:*

$$\|P_k(hf)\|_{N_k(T)} \lesssim 2^{-|k-k_1|/2}\|h\|_{L^2_{t,x}}\|f\|_{F_{k_1}(T)} \qquad \text{if } k_1 \le k - 80. \tag{3-10}$$
$$\|f \cdot g\|_{L^2_{t,x}} \lesssim 2^{-|k_1-k_2|/2}\|f\|_{F_{k_1}(T)}\|g\|_{G_{k_2}(T)} \quad \text{if } k_1 \le k_2. \tag{3-11}$$

Proof. Estimate (3-10) follows from the definitions since

$$\|P_k(hf)\|_{N_k(T)} \lesssim 2^{-k/2}\sup_{\theta\in\mathbb{S}^1}\|hf\|_{L^{1,2}_{\theta,W_{k-40}}} \lesssim 2^{-k/2}\sup_{\theta\in\mathbb{S}^1}\|f\|_{L^{2,\infty}_{\theta,W_{k_1+40}}}\|h\|_{L^2_{t,x}}.$$

The proof of (3-11) parallels that of (3-7) and is omitted (see [Bejenaru et al. 2011c, Lemma 6.5] for details). □

3C. *Trilinear estimates and summation.* We combine the bilinear estimates to establish some trilinear estimates. As we do not control local smoothing norms along the heat flow, we will oftentimes be able to put only one term in a G_k space. Nonetheless, such estimates still exhibit good off-diagonal decay.

Define the sets $Z_1(k)$, $Z_2(k)$, $Z_3(k) \subset \mathbb{Z}^3$ as follows:

$$\begin{aligned} Z_1(k) &:= \{(k_1,k_2,k_3) \in \mathbb{Z}^3 : k_1, k_2 \le k-40 \text{ and } |k_3 - k| \le 4\}. \\ Z_2(k) &:= \{(k_1,k_2,k_3) \in \mathbb{Z}^3 : k, k_3 \le k_1-40 \text{ and } |k_2 - k_1| \le 45\}. \\ Z_3(k) &:= \{(k_1,k_2,k_3) \in \mathbb{Z}^3 : |\max\{k,k_3\} - \max\{k_1,k_2\}| \le 40\}. \end{aligned} \tag{3-12}$$

In our main trilinear estimate, we avoid using local smoothing/maximal function spaces.

Lemma 3.10 (main trilinear estimate). *Let C_{k,k_1,k_2,k_3} denote the best constant C in the estimate*

$$\|P_k\left(P_{k_1} f_1 P_{k_2} f_2 P_{k_3} g\right)\|_{N_k(T)} \lesssim C \|P_{k_1} f_1\|_{F_{k_1}(T)} \|P_{k_2} f_2\|_{F_{k_2}(T)} \|P_{k_3} g\|_{G_{k_3}(T)}. \tag{3-13}$$

The best constant C_{k,k_1,k_2,k_3} satisfies the bounds

$$C_{k,k_1,k_2,k_3} \lesssim \begin{cases} 2^{-|(k_1+k_2)/6 - k/3|} & \text{if } (k_1,k_2,k_3) \in Z_1(k), \\ 2^{-|k-k_3|/6} & \text{if } (k_1,k_2,k_3) \in Z_2(k), \\ 2^{-|\Delta k|/6} & \text{if } (k_1,k_2,k_3) \in Z_3(k), \\ 0 & \text{if } (k_1,k_2,k_3) \in \mathbb{Z}^3 \setminus \{Z_1(k) \cup Z_2(k) \cup Z_3(k)\}, \end{cases}$$

where $\Delta k = \max\{k,k_1,k_2,k_3\} - \min\{k,k_1,k_2,k_3\} \ge 0$.

Proof. After placing the term $P_k(P_{k_1} f_1 P_{k_2} f_2 P_{k_3} g)$ in $L^{4/3}_{t,x}$ and then using Hölder's inequality to bound each factor in $L^4_{t,x}$, it follows from Bernstein that

$$C_{k,k_1,k_2,k_3} \lesssim 1, \tag{3-14}$$

and so, in particular, for any choice of integers k, k_1, k_2, k_3, such a constant C_{k,k_1,k_2,k_3} exists.

Frequencies not represented in one of $Z_1(k)$, $Z_2(k)$, $Z_3(k)$ cannot interact so as to yield a frequency in I_k. Over $Z_1(k)$, we apply (3-4) and (3-7).

On $Z_2(k)$ we apply (3-4) if $k > k_3$ and (3-5) if $k \le k_3$. We conclude with (3-6).

On $Z_3(k)$ we may assume without loss of generality that $k_1 \le k_2$. First suppose that $k_3 \le k$ and $|k-k_2| \le 40$. If $k_1 \le k_3$, then use (3-4), applying (3-6) to $P_{k_2} f_2 P_{k_3} g$. If $k_3 < k_1$, then use (3-6) on $P_{k_1} f_1 P_{k_2} f_2$ instead.

Now suppose that $k_3 > k$ and $|k_3 - k_2| \le 40$. If $k_1 \le k$, then use (3-3), applying (3-7) to $P_{k_1} f_1 P_{k_3} g$. If $k_{\min} = k$, then use (3-5) and (3-6). □

Corollary 3.11. *Let $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ be δ-frequency envelopes. Let C_{k,k_1,k_2,k_3} be as in Lemma 3.10. Then*

$$\sum_{(k_1,k_2,k_3) \in \mathbb{Z}^3 \setminus Z_2(k)} C_{k,k_1,k_2,k_3} a_{k_1} b_{k_2} c_{k_3} \lesssim a_k b_k c_k.$$

Proof. By Lemma 3.10, it suffices to restrict the sum to (k_1, k_2, k_3) lying in $Z_1(k) \cup Z_3(k)$. On $Z_1(k)$, the sum is bounded by

$$\begin{aligned}\sum_{(k_1,k_2,k_3)\in Z_1(k)} 2^{-|(k_1+k_2)/6-k/3|} a_{k_1} b_{k_2} c_{k_3} &\lesssim \sum_{k_1,k_2\le k-40} 2^{-|(k_1+k_2)/6-k/3|} 2^{\delta|2k-k_1-k_2|} a_k b_k c_k \\ &\lesssim a_k b_k c_k.\end{aligned}$$

On Z_3, we may assume without loss of generality that $k_2 \le k_1$. The sum is then controlled by

$$\begin{aligned}\sum_{(k_1,k_2,k_3)\in Z_3(k)} 2^{-|\Delta k|/6} a_{k_1} b_{k_2} c_{k_3} &\lesssim \sum_{\substack{k_2\le k\\ k_3\le k\\ |k_1-k|\le 40}} 2^{-|k-\min\{k_2,k_3\}|/6} a_{k_1} b_{k_2} c_{k_3} + \sum_{\substack{k_2\le k_1\\ k_1>k\\ |k_3-k_1|\le 40}} 2^{-|k_1-\min\{k_2,k\}|/6} a_{k_1} b_{k_2} c_{k_3} \\ &\lesssim \sum_{\substack{k_2\le k\\ k_3\le k}} 2^{-|k-\min\{k_2,k_3\}|/6} a_k b_{k_2} c_{k_3} + \sum_{\substack{k_2\le k_1\\ k_1>k}} 2^{-|k_1-\min\{k_2,k\}|/6} a_{k_1} b_{k_2} c_{k_1}.\end{aligned}$$

The first of these summands is controlled by

$$\begin{aligned}\sum_{k_3\le k_2\le k} 2^{-|k-k_3|/6} a_k b_{k_2} c_{k_3} + \sum_{k_2<k_3\le k} 2^{-|k-k_2|/6} a_k b_{k_2} c_{k_3} \\ \lesssim \sum_{k_3\le k_2\le k} 2^{-|k-k_3|/6} 2^{\delta|k-k_2|} a_k b_k c_{k_3} + \sum_{k_2<k_3\le k} 2^{-|k-k_2|/6} 2^{\delta|k-k_3|} a_k b_{k_2} c_k \\ \lesssim \sum_{k_3\le k} 2^{(\delta-1/6)|k-k_3|} a_k b_k c_{k_3} + \sum_{k_2<k} 2^{(\delta-1/6)|k-k_2|} a_k b_{k_2} c_k \\ \lesssim \sum_{k_3\le k} 2^{(2\delta-1/6)|k-k_3|} a_k b_k c_k + \sum_{k_2<k} 2^{(2\delta-1/6)|k-k_2|} a_k b_k c_k \\ \lesssim a_k b_k c_k.\end{aligned}$$

The second is controlled by

$$\begin{aligned}\sum_{k\le k_2\le k_1} 2^{-|k_1-k|/6} a_{k_1} b_{k_2} c_{k_1} + \sum_{k_2<k\le k_1} 2^{-|k_1-k_2|/6} a_{k_1} b_{k_2} c_{k_1} \\ \lesssim \sum_{k\le k_2\le k_1} 2^{-|k_1-k|/6} 2^{\delta|k_2-k|} a_{k_1} b_k c_{k_1} + \sum_{k_2<k\le k_1} 2^{|k_1-k_2|/6} 2^{\delta|k_2-k|} a_{k_1} b_k c_{k_1} \\ \lesssim \sum_{k\le k_1} 2^{(\delta-1/6)|k_1-k|} a_{k_1} b_k c_{k_1} + \sum_{k_2<k\le k_1} 2^{(\delta-1/6)|k_1-k_2|} a_{k_1} b_k c_{k_1} \\ \lesssim \sum_{k\le k_1} 2^{(3\delta-1/6)|k_1-k|} a_k b_k c_k + \sum_{k_2<k\le k_1} 2^{(3\delta-1/6)|k_1-k_2|} a_k b_k c_k \\ \lesssim a_k b_k c_k.\end{aligned}$$

□

Corollary 3.12. *Let $\{a_k\}$, $\{b_k\}$ be δ-frequency envelopes. Let C_{k,k_1,k_2,k_3} be as in Lemma 3.10. Then*

$$\sum_{(k_1,k_2,k_3)\in Z_2(k)\cup Z_3(k)} 2^{\max\{k,k_3\}-\max\{k_1,k_2\}} C_{k,k_1,k_2,k_3} a_{k_1} b_{k_2} c_{k_3} \lesssim a_k b_k c_k.$$

Proof. On $Z_3(k)$, $\max\{k_1, k_2\} \sim \max\{k, k_3\}$, and so the bound on $Z_3(k)$ follows from Corollary 3.11.

Note that $\max\{k_1, k_2\} > \max\{k, k_3\}$ on Z_2, where the sum is controlled by

$$\sum_{(k_1,k_2,k_3)\in Z_2(k)} 2^{\max\{k,k_3\}-\max\{k_1,k_2\}} 2^{-|k-k_3|/6} a_{k_1} b_{k_2} c_{k_3} \lesssim \sum_{k,k_3\le k_1-40} 2^{\max\{k,k_3\}-k_1} 2^{-|k-k_3|/6} a_{k_1} b_{k_1} c_{k_3},$$

Restricting the sum to $k_3 \le k$, we get

$$\sum_{k_3\le k\le k_1-40} 2^{-|k-k_1|} 2^{-|k-k_3|/6} a_{k_1} b_{k_1} c_{k_3} \lesssim a_k b_k c_k.$$

Over the complementary range $k \le k_3 \le k_1 - 40$, we have

$$\sum_{k\le k_3\le k_1-40} 2^{-|k_3-k_1|} 2^{-|k-k_3|/6} a_{k_1} b_{k_1} c_{k_3} \;\lesssim\; a_k b_k c_k \sum_{k\le k_3\le k_1-40} 2^{-|k_3-k_1|} 2^{-|k-k_3|/6} 2^{2\delta|k_1-k|} 2^{\delta|k-k-3|}.$$

Performing the change of variables $j := k_1 - k_3$, $l := k_3 - k$, we control the sum by

$$\sum_{j,l\ge 0} 2^{-j} 2^{-l/6} 2^{2\delta(j+l)} 2^{\delta l} \lesssim \sum_{j,l\ge 0} 2^{(2\delta-1)j} 2^{(3\delta-1/6)l} \lesssim 1. \qquad \square$$

Taking advantage of the local smoothing/maximal function spaces, we can obtain the following improvement.

Lemma 3.13 (main trilinear estimate improvement over Z_1). *The best constant C_{k,k_1,k_2,k_3} in (3-13) satisfies the improved estimate*

$$C_{k,k_1,k_2,k_3} \lesssim 2^{-|(k_1+k_2)/2-k|} \tag{3-15}$$

when $\{k_1, k_2, k_3\} \in Z_1(k)$.

4. Proof of Theorem 1.3

In this section we outline the proof of Theorem 1.3, taking as our starting point the local result stated in Theorem 1.1.

For technical reasons related to the function space definitions of the last section, it will be convenient to construct a solution φ on a time interval $(-2^{2\mathcal{K}}, 2^{2\mathcal{K}})$ for some given $\mathcal{K} \in \mathbb{Z}_+$ and proceed to prove bounds that are uniform in $\mathcal{K}$. We assume $1 \ll \mathcal{K} \in \mathbb{Z}_+$ is chosen and hereafter fixed. Invoking Theorem 1.1, we assume that we have a solution $\varphi \in C([-T, T] \to H_Q^\infty)$ of (1-1) on the time interval $[-T, T]$ for some $T \in (0, 2^{2\mathcal{K}})$. In order to extend φ to a solution on all of $(-2^{2\mathcal{K}}, 2^{2\mathcal{K}})$ with uniform bounds (uniform in $T, \mathcal{K}$), it suffices to prove uniform a priori estimates on

$$\sup_{t\in(-T,T)} \|\varphi(t)\|_{H_Q^\sigma}$$

for, say, σ in the interval $[1, \sigma_1]$, with $\sigma_1 \gg 1$ chosen sufficiently large ($\sigma_1 = 25$ will do).

The first step in our approach, carried out in Section 2, is to lift the Schrödinger map system (1-1) to the tangent bundle and view it with respect to the caloric gauge. Recall that the lift of (1-1) expressed in terms of the caloric gauge takes the form (2-7), or, equivalently,

$$(i\partial_t + \Delta)\psi_m = B_m + V_m, \tag{4-1}$$

with initial data $\psi_m(0)$. Here B_m and V_m respectively denote the magnetic and electric potentials (see (2-8) and (2-9) for definitions).

The goal then becomes proving a priori bounds on $\|\psi_m\|_{L_t^\infty H_x^\sigma}$. Herein lies the heart of the argument, and the purpose of this section is not only to give a high level description of the proof of Theorem 1.3, but also to outline the proof of the key a priori bounds. To establish these bounds, we in fact prove stronger frequency-localized estimates. The argument naturally splits into several components, and we consider each individually below.

Finally, to complete the proof of Theorem 1.3, we must transfer the a priori bounds on the derivative fields ψ_m back to bounds on the map φ, thereby allowing us to close a bootstrap argument. Once the derivative field bounds are established, this is, comparatively speaking, an easy task, and we take it up in the last subsection.

We return now to (4-1), projecting it to frequencies $\sim 2^k$ using the Littlewood–Paley multiplier P_k. Applying the linear estimate of Proposition 3.6 then yields

$$\|P_k\psi_m\|_{G_k(T)} \lesssim \|P_k\psi_m(0)\|_{L_x^2} + \|P_k V_m\|_{N_k(T)} + \|P_k B_m\|_{N_k(T)}. \tag{4-2}$$

In order to express control of the $G_k(T)$ norm of $P_k\psi_m$ in terms of the initial data, we introduce the following frequency envelopes. Let $\sigma_1 \in \mathbb{Z}_+$ be positive. For $\sigma \in [0, \sigma_1 - 1]$, set

$$b_k(\sigma) = \sup_{k' \in \mathbb{Z}} 2^{\sigma k'} 2^{-\delta|k-k'|} \|P_{k'}\psi_x\|_{G_k(T)}. \tag{4-3}$$

By (2-38), these envelopes are finite and in l^2. We abbreviate $b_k(0)$ by setting $b_k := b_k(0)$.

We now state the key result for solutions of the gauge field equation (4-1).

Theorem 4.1. *Assume $T \in (0, 2^{2\mathcal{K}})$ and $Q \in \mathbb{S}^2$. Choose $\sigma_1 \in \mathbb{Z}_+$ positive. Let $\varepsilon_1 > 0$ and let $\varphi \in H_Q^{\infty,\infty}(T)$ be a solution of the Schrödinger map system (1-1) whose initial data φ_0 has energy $E_0 := E(\varphi_0) < E_{\mathrm{crit}}$ and satisfies the energy dispersion condition*

$$\sup_{k \in \mathbb{Z}} c_k \le \varepsilon_1. \tag{4-4}$$

Assume moreover that

$$\sum_{k \in \mathbb{Z}} \|P_k\psi_x\|_{L_{t,x}^4(I \times \mathbb{R}^2)}^2 \le \varepsilon_1^2 \tag{4-5}$$

for any smooth extension φ on I, $[-T, T] \subset I \subset (-2^{2\mathcal{K}}, 2^{2\mathcal{K}})$. Suppose that the bootstrap hypothesis

$$b_k \le \varepsilon_1^{-1/10} c_k \tag{4-6}$$

is satisfied. Then, for ε_1 sufficiently small,

$$b_k(\sigma) \lesssim c_k(\sigma) \tag{4-7}$$

holds for all $\sigma \in [0, \sigma_1 - 1]$ and $k \in \mathbb{Z}$.

Proof. We use a continuity argument to prove Theorem 4.1. For $T' \in (0, T]$, let

$$\Psi(T') = \sup_{k \in \mathbb{Z}} c_k^{-1} \|P_k\psi_m(s = 0)\|_{G_k(T')}.$$

Then $\psi : (0, T] \to [0, \infty)$ is well-defined, increasing, continuous, and satisfies

$$\lim_{T' \to 0} \psi(T') \lesssim 1.$$

The critical implication to establish is

$$\Psi(T') \le \varepsilon_1^{-1/10} \implies \Psi(T') \lesssim 1,$$

which in particular follows from

$$b_k \lesssim c_k. \tag{4-8}$$

We also must similarly establish

$$b_k(\sigma) \lesssim c_k(\sigma) \tag{4-9}$$

for $\sigma \in (0, \sigma_1 - 1]$. The next several subsections describe the main steps of the proof of (4-8) and (4-9), to which the bulk of the remainder of this paper is dedicated. In Section 4E we complete the high level argument used to prove (4-8) and (4-9). □

Corollary 4.2. *Given the conditions of Theorem 4.1,*

$$\| P_k |\partial_x|^\sigma \partial_m \varphi \|_{L_t^\infty L_x^2((-T,T)\times\mathbb{R}^2)} \lesssim c_k(\sigma) \tag{4-10}$$

holds for all $\sigma \in [0, \sigma_1 - 1]$.

The proof we defer to Section 4F.

Together Theorem 1.1, Theorem 4.1, and Corollary 4.2 are almost enough to establish Theorem 1.3. The next lemma provides the final piece. We also defer its proof to Section 4F.

Lemma 4.3. *We have*

$$\sum_{k\in\mathbb{Z}} \| P_k \psi_x \|_{L^4_{t,x}}^2 \sim \sum_{k\in\mathbb{Z}} \| P_k \partial_x \varphi \|_{L^4_{t,x}}^2 .$$

Note that this lemma affords us a condition equivalent to (4-5) whose advantage lies in the fact that it is not expressed in terms of gauges.

Proof of Theorem 1.3. Fix $\sigma_1 \in \mathbb{Z}_+$ positive and let $\varepsilon_1 = \varepsilon_1(\sigma_1) \ge 0$. It suffices to prove (1-7) on the time interval $[-T, T]$ provided the estimate is uniform in T. In view of Theorem 1.1 and mass-conservation, proving

$$\| \partial_x \varphi \|_{L_t^\infty \dot{H}_Q^\sigma((-T,T)\times\mathbb{R}^2)} \lesssim_\sigma \| \partial_x \varphi \|_{\dot{H}_Q^\sigma(\mathbb{R}^2)} \tag{4-11}$$

for $\sigma \in [0, \sigma_1 - 1]$ with $\sigma_1 = 25$ is enough to establish (1-6).

By virtue of Lemma 4.3, the assumptions of Theorem 1.3 are equivalent to those of Theorem 4.1. Therefore we have access to Corollary 4.2, which states that (4-10) holds for $\sigma \in [0, \sigma_1 - 1]$. Using (2-33) and the Littlewood–Paley square function completes the proof of (4-11).

Global existence and (1-7) then follow via a standard bootstrap argument from Theorem 1.1 and from the fact that the constants in (4-11) are uniform in T. □

The remainder of this section is organized as follows. In Section 4A we state the key lemmas of parabolic type that are used to control the electric and magnetic nonlinearities. In Section 4B we state bounds that rely principally upon local smoothing, including a bilinear Strichartz estimate; they find application in controlling the worst magnetic nonlinearity terms.

In Section 4C we piece together the parabolic estimates to control the electric potential. In Section 4D we decompose the magnetic potential into two main pieces and demonstrate how to control one of these pieces.

In Section 4E we close the bootstrap argument proving Theorem 4.1. Here the remaining piece of the magnetic potential is addressed using a certain nonlinear version of a bilinear Strichartz estimate.

Finally, in Section 4F, we prove Corollary 4.2 and Lemma 4.3.

4A. ***Parabolic estimates.*** By "parabolic estimates" we mean those that principally rely upon the smoothing effect of the harmonic map heat flow. We include here only those that play a direct role in controlling the nonlinearity $\mathcal{N}$. These are proved in Section 7, where a host of auxiliary parabolic estimates are included as well. As the proofs rely upon a bootstrap argument that takes advantage of energy dispersion (4-4), these bounds rely upon this smallness constraint implicitly. On the other hand, L^4 smallness (4-5) is not used in the proofs of these bounds, but rather only in their application in this paper.

Lemma 4.4. *For $\sigma \in [0, \sigma_1 - 1]$, the derivative fields ψ_m satisfy*

$$\|P_k \psi_m(s)\|_{F_k(T)} \lesssim (1 + s2^{2k})^{-4} 2^{-\sigma k} b_k(\sigma) \tag{4-12}$$

for $s \geq 0$.

This estimate is used in Section 4D in controlling the magnetic nonlinearity, which schematically looks like $A\partial_x \psi$. To recover the loss of derivative, it is important to take advantage of parabolic smoothing by invoking representation (2-23) of A. Within the integral we schematically have $\psi(s) D_x \psi(s)$, and hence (4-12) allows us to take advantage of (3-3)–(3-7) in bounding this term. We prove (4-12) in Section 7A.

Lemma 4.5. *For $\sigma \in [0, \sigma_1 - 1]$, the derivative fields ψ_l and connection coefficients A_m satisfy*

$$\|P_k(A_m(s)\psi_l(s))\|_{F_k(T)} \lesssim (s2^{2k})^{-3/8}(1 + s2^{2k})^{-2} 2^{-(\sigma-1)k} b_k(\sigma). \tag{4-13}$$

Like the previous estimate, this estimate is also used in Section 4D in controlling the magnetic nonlinearity. Its proof is given in Section 7B. The need for this estimate arises from the need to control $D_x \psi$ appearing in representation (2-23) of A.

The next several estimates are used in Section 4C to control the electric potential. In particular, they provide a source of smallness crucial here for closing the bootstrap argument. They are proved in Section 7B.

Lemma 4.6. *For $\sigma \in [2\delta, \sigma_1 - 1]$, the connection coefficient A_x satisfies*

$$\|A_x^2\|_{L^2_{t,x}} \lesssim \sup_{j \in \mathbb{Z}} b_j^2 \cdot \sum_{k \in \mathbb{Z}} b_k^2, \tag{4-14}$$

$$\|P_k A_x^2(0)\|_{L^2_{t,x}} \lesssim 2^{-\sigma k} b_k(\sigma) \cdot \sup_j b_j \cdot \sum_{l \in \mathbb{Z}} b_l^2. \tag{4-15}$$

Lemma 4.7. *For $\sigma \in [2\delta, \sigma_1 - 1]$, the connection coefficient A_t satisfies*

$$\|A_t\|_{L^2_{t,x}} \lesssim \Big(1 + \sum_{j\in\mathbb{Z}} b_j^2\Big)^2 \sum_{k\in\mathbb{Z}} \|P_k\psi_x(0)\|^2_{L^4_{t,x}}, \tag{4-16}$$

$$\|P_k A_t\|_{L^2_{t,x}} \lesssim \Big(1 + \sum_p b_p^2\Big) \tilde{b}_k 2^{-\sigma k} b_k(\sigma). \tag{4-17}$$

In subsequent estimates the following shorthand will be useful:

$$\epsilon := \Big(1 + \sum_{j\in\mathbb{Z}} b_j^2\Big)^2 \sum_{l\in\mathbb{Z}} \|P_l\psi_x(0)\|^2_{L^4_{t,x}} + \Big(1 + \sum_l b_l^2\Big) \sup_{k\in\mathbb{Z}} b_k^2. \tag{4-18}$$

Under the assumptions of Theorem 4.1, ϵ is a very small quantity, being at least as good as $O(\varepsilon_1^{1/2})$.

4B. ***Smoothing and Strichartz.*** The key result of Section 5 is the following frequency-localized bilinear Strichartz estimate.

Theorem 4.8. *Suppose that ψ_m satisfies (2-7) on $[-T, T]$. Assume $\sigma \in [0, \sigma_1 - 1]$. Let the frequency envelopes b_j and c_j be defined as in (4-3) and (2-32). Let ϵ be given by (4-18). Suppose also that $2^{j-k} \ll 1$. Then*

$$2^{k-j}(1 + s2^{2j})^8 \|P_j\psi_l(s)\cdot P_k\psi_m(0)\|^2_{L^2_{t,x}} \lesssim 2^{-2\sigma k} c_j^2 c_k^2(\sigma) + \epsilon^2 b_j^2 b_k^2(\sigma). \tag{4-19}$$

In Section 5B we split the proof into two cases: $s = 0$ and $s > 0$, the more involved being the $s = 0$ case. In either case, if instead we only were to appeal to the local smoothing-based estimate (3-11) and the frequency envelope definition (4-3), then we would get the bound

$$2^{k-j}(1 + s2^{2j})^8 \|P_j\psi_l(s)\cdot P_k\psi_m(0)\|^2_{L^2_{t,x}} \lesssim b_j^2 b_k^2.$$

In practice this sort of bound must needs be summed over $j \ll k$. When initial energy is assumed to be small, as is done in [Bejenaru et al. 2011c], the sum $\sum_j b_j^2 \ll 1$ is small, and consequently the resulting term perturbative. In our subthreshold energy setting this is no longer the case, as in fact the sum may be large. What (4-19) reveals, though, is that any b_j contributions come with a power of ε. In view of additional work which we present in due course, this turns out to be sufficient for establishing that $b_k \lesssim c_k$.

An interesting related bound is the following local smoothing estimate, also proved in Section 5B. It arises as an easy corollary of our proof of Theorem 4.8.

Theorem 4.9. *Suppose that ψ_m satisfies (2-7) on $[-T, T]$. Assume $\sigma \in [0, \sigma_1 - 1]$. Let the frequency envelopes $b_j(\sigma)$ and $c_j(\sigma)$ be defined as in (4-3) and (2-32). Also, let ϵ be given by (4-18). Then*

$$2^k \sup_{|j-k|\le 20} \sup_{\theta\in\mathbb{S}^1} \|P_{j,\theta}P_k\psi_m\|^2_{L^{\infty,2}_\theta} \lesssim 2^{-2\sigma k} c_k^2(\sigma) + \epsilon 2^{-2\sigma k} b_k^2(\sigma) \tag{4-20}$$

holds for each $k \in \mathbb{Z}$.

We note that (4-20) likely extends to $L^{\infty,2}_{\theta,\lambda}$ for λ satisfying $|\lambda| < 2^{k-40}$, though we do not prove this. For comparison, note that from the definition of (4-3) we have

$$2^k \sup_{|j-k|\le 20} \sup_{\theta\in\mathbb{S}^1} \sup_{|\lambda|<2^{k-40}} \|P_{j,\theta} P_k \psi_m\|^2_{L^{\infty,2}_{\theta,\lambda}} \lesssim 2^{-2\sigma k} b_k^2(\sigma). \tag{4-21}$$

On the other hand, while the right-hand side of (4-20) may indeed be large, it so happens thanks to our hypotheses of energy dispersion and L^4 smallness that the $b_k(\sigma)$ term is perturbative. For our purposes, this is a substantial improvement over (4-21). However, it can be seen from the argument in Section 4E that even an extension of (4-20) to $L^{\infty,2}_{\theta,\lambda}$ spaces is not sufficient for proving $b_k(\sigma) \lesssim c_k(\sigma)$: it is important that we can replace two "b_j" terms with corresponding "c_j" terms as in (4-19).

4C. *Controlling the electric potential V.*

Lemma 4.10. *Suppose that $\sigma < \frac{1}{6} - 2\delta$. Then the electric potential term V_m satisfies the estimate*

$$\|P_k V_m\|_{N_k(T)} \lesssim \big(\|A_x^2\|_{L^2_{t,x}} + \|A_t\|_{L^2_{t,x}} + \|\psi_x^2\|_{L^2_{t,x}}\big) 2^{-\sigma k} b_k(\sigma). \tag{4-22}$$

Proof. Letting $f \in \{A_t, A_x^2, \psi_x^2\}$, we bound $P_k(f\psi_x)$ in $N_k(T)$. Begin with the following Littlewood–Paley decomposition of $P_k(f\psi_x)$:

$$P_k(f\psi_x) = P_k(P_{<k-80} f P_{k-5<\cdot<k+5}\psi_x) + \sum_{\substack{|k_1-k|\le 4\\ k_2\le k-80}} P_k(P_{k_1} f P_{k_2}\psi_x) + \sum_{\substack{|k_1-k_2|\le 90\\ k_1,k_2>k-80}} P_k(P_{k_1} f P_{k_2}\psi_x).$$

The first term is controlled using Hölder's inequality:

$$\begin{aligned}\|P_k(P_{<k-80} f P_{k-5<\cdot<k+5}\psi_x)\|_{N_k(T)} &\le \|P_k(P_{<k-80} f P_{k-5<\cdot<k+5}\psi_x)\|_{L^{4/3}_{t,x}}\\ &\le \|P_{<k-80} f\|_{L^2_{t,x}} \|P_{k-5<\cdot<k+5}\psi_x\|_{L^4_{t,x}}.\end{aligned}$$

To control the second term we apply (3-4):

$$\|P_k(P_{k_1} f P_{k_2}\psi_x)\|_{N_k(T)} \lesssim 2^{(k_2-k)/6} \|P_{k_1} f\|_{L^2_{t,x}} \|P_{k_2}\psi_x\|_{G_{k_2}(T)}.$$

Using (4-3), (2-30), and $\sigma < 1/6 - 2\sigma$, we conclude that

$$\Bigg\| \sum_{\substack{|k_1-k|\le 4\\ k_2<k-80}} P_k(P_{k_1} f P_{k_2}\psi_x) \Bigg\|_{N_k(T)} \lesssim 2^{-\sigma k} b_k(\sigma) \sum_{|k_1-k|\le 4} \|P_{k_1} f\|_{L^2_{t,x}}.$$

To control the high-high interaction, apply (3-5):

$$\|P_k(P_{k_1} f P_{k_2}\psi_x)\|_{N_k(T)} \lesssim 2^{(k-k_2)/6} \|P_{k_1} f\|_{L^2_{t,x}} \|P_{k_2}\psi_x\|_{G_{k_2}(T)}.$$

Therefore, by (4-3),

$$\sum_{\substack{|k_1-k_2|\le 90\\ k_1,k_2>k-80}} \|P_k(P_{k_1} f P_{k_2}\psi_x)\|_{N_k(T)} \lesssim \sum_{\substack{|k_1-k_2|\le 90\\ k_1,k_2>k-80}} 2^{(k-k_2)/6} \|P_{k_1} f\|_{L^2_{t,x}} 2^{-\sigma k_2} b_{k_2}(\sigma).$$

Using Cauchy–Schwarz and (2-31) yields

$$\sum_{\substack{|k_1-k_2|\le 90\\ k_1,k_2>k-80}} \|P_k(P_{k_1} f P_{k_2}\psi_x)\|_{N_k(T)} \lesssim 2^{-\sigma k} b_k(\sigma) \Big(\sum_{k_1\ge k-80} \|P_{k_1} f\|^2_{L^2_{t,x}} \Big)^{1/2},$$

and so, by switching the $L^2_{t,x}$ and l^2 norms, we get from the standard square function estimate that

$$\sum_{\substack{|k_1-k_2|\le 90\\ k_1,k_2>k-80}} \|P_k(P_{k_1} f P_{k_2}\psi_x)\|_{N_k(T)} \lesssim \|f\|_{L^2_{t,x}} 2^{-\sigma k} b_k(\sigma). \qquad \square$$

Corollary 4.11. *For $\sigma \in [0, \sigma_1 - 1]$ we have*

$$\|P_k V_m\|_{N_k(T)} \lesssim \epsilon 2^{-\sigma k} b_k(\sigma).$$

Proof. Given (4-22), this is a direct consequence of (4-14), (4-16), and the fact that

$$\|f\|^2_{L^4_{t,x}} \lesssim \sum_{k\in\mathbb{Z}} \|P_k f\|^2_{L^4_{t,x}}.$$

Therefore the result holds for $\sigma < 1/6 - 2\delta$.

To extend the proof to larger σ, we may mimic the proof of Lemma 4.10 by performing the same Littlewood–Paley decomposition and then, with regard to the first and third terms of the decomposition, proceeding as before in the proof of that lemma. The argument, however, must be modified in handling the term

$$\sum_{\substack{|k_1-k|\le 4\\ k_2\le k-80}} P_k(P_{k_1} f P_{k_2}\psi_x), \tag{4-23}$$

where $f \in \{A_t, A_x^2, \psi_x^2\}$. We take different approaches according to the choice of f.

When $f = A_x^2$, we apply (3-4) and invoke (4-15) to obtain

$$\begin{aligned}\Big\| \sum_{\substack{|k_1-k|\le 4\\ k_2<k-80}} P_k(P_{k_1} A_x^2 P_{k_2}\psi_x)\Big\|_{N_k(T)} &\lesssim \sum_{\substack{|k_1-k|\le 4\\ k_2<k-80}} 2^{(k_2-k)/6} \|P_{k_1} A_x^2\|_{L^2_{t,x}} \|P_{k_2}\psi_x\|_{G_{k_2}(T)} \\ &\lesssim \sum_{\substack{|k_1-k|\le 4\\ k_2<k-80}} 2^{(k_2-k)/6} 2^{-\sigma k_1} b_{k_1}(\sigma) b_{k_2} \cdot \sup_j b_j \cdot \sum_l b_l^2 \\ &\lesssim 2^{-\sigma k} b_k(\sigma)\cdot b_k \cdot \sup_j b_j \cdot \sum_j b_j^2,\end{aligned}$$

In the case where $f = A_t$, we apply (3-4) and use (4-17) to conclude that

$$\Big\| \sum_{\substack{|k_1-k|\le 4\\ k_2<k-80}} P_k(P_{k_1} A_t P_{k_2}\psi_x)\Big\|_{N_k(T)} \lesssim 2^{-\sigma k} b_k(\sigma) \tilde{b}_k b_k \Big(1 + \sum_p b_p^2\Big),$$

which suffices by Cauchy–Schwarz.

Finally we turn to $f = \psi_x^2$, which we further decompose as

$$f = 2 \sum_{\substack{|j_1-k|\le 4\\ j_2<k-80}} P_{j_1}\psi_x P_{j_2}\psi_x + \sum_{\substack{|j_1-j_2|\le 8\\ j_1,j_2\ge k-80}} P_{j_1}\psi_x P_{j_2}\psi_x.$$

To control the high-low term, we apply estimate (3-7) and get

$$\sum_{\substack{|j_1-k|\le 4\\ j_2<k-80}} \|P_{j_1}\psi_x P_{j_2}\psi_x\|_{L^2} \lesssim \sum_{\substack{|j_1-k|\le 4\\ j_2<k-80}} 2^{(j_2-j_1)/6} b_{j_2} 2^{-\sigma j_1} b_{j_1}(\sigma) \lesssim 2^{-\sigma k} b_k b_k(\sigma).$$

We turn to the high-high case. The full trilinear expression is given by

$$\sum_{\substack{|k_1-k|\le 4\\ k_2<k-80}} P_k\biggl(P_{k_1}\biggl(\sum_{\substack{|j_1-j_2|\le 8\\ j_1,j_2\ge k_1-80}} P_{j_1}\psi_x P_{j_2}\psi_x\biggr)\cdot P_{k_2}\psi_x\biggr).$$

We can drop the P_{k_1} factor because of the summation ranges, obtaining

$$\sum_{\substack{|k_1-k|\le 4\\ k_2<k-80}} \sum_{\substack{|j_1-j_2|\le 8\\ j_1,j_2\ge k_1-80}} P_k(P_{j_1}\psi_x P_{j_2}\psi_x \cdot P_{k_2}\psi_x).$$

We apply estimate (3-4) with $h = P_{j_2}\psi_x P_{k_2}\psi_x$ to get

$$\sum_{\substack{|k_1-k|\le 4\\ k_2<k-80}} \sum_{\substack{|j_1-j_2|\le 8\\ j_1,j_2\ge k_1-80}} \|P_k(P_{j_1}\psi_x P_{j_2}\psi_x \cdot P_{k_2}\psi_x)\|_{N_k(T)} \lesssim \sum_{\substack{|k_1-k|\le 4\\ k_2<k-80}} \sum_{\substack{|j_1-j_2|\le 8\\ j_1,j_2\ge k_1-80}} 2^{-|j_1-k|/6} \|P_{j_1}\psi_x\|_{G_{j_1}(T)} \|P_{j_2}\psi_x P_{k_2}\psi_x\|_{L^2}.$$

Next we use (3-7) to control the L^2 norm:

$$\sum_{\substack{|k_1-k|\le 4\\ k_2<k-80}} \sum_{\substack{|j_1-j_2|\le 8\\ j_1,j_2\ge k_1-80}} 2^{-|j_1-k|/6} \|P_{j_1}\psi_x\|_{G_{j_1}(T)} \|P_{j_2}\psi_x P_{k_2}\psi_x\|_{L^2} \lesssim \sum_{\substack{|k_1-k|\le 4\\ k_2<k-80}} \sum_{\substack{|j_1-j_2|\le 8\\ j_1,j_2\ge k_1-80}} 2^{-|j_1-k|/6} 2^{-|j_2-k_2|/6} 2^{-\sigma j_1} b_{j_1}(\sigma) b_{j_2} b_{k_2}.$$

In this sum we can replace the factor $2^{-|j_2-k_2|/6}$ by the larger factor $2^{-|k-k_2|/6}$, from which it is seen that the whole sum is controlled by

$$2^{-\sigma k} b_k(\sigma) b_k \sum_{k_2<k-80} 2^{-|k-k_2|/6} b_{k_2} \lesssim 2^{-\sigma k} b_k^2 b_k(\sigma).$$

□

4D. ***Decomposing the magnetic potential.*** We begin by introducing a paradifferential decomposition of the magnetic nonlinearity, splitting it into two pieces. This decomposition depends upon a frequency parameter $k \in \mathbb{Z}$, which we suppress in the notation; this same k will also be the output frequency whose behavior we are interested in controlling. The decomposition also depends upon the frequency gap parameter $\varpi \in \mathbb{Z}_+$. How ϖ is chosen and the exact role it plays are discussed in Section 5B. There it is shown that ϖ may be set equal to a sufficiently large universal constant (independent of ε, ε_1, k, etc.).

Define $A_{\mathrm{lo}\wedge\mathrm{lo}}$ as

$$A_{m,\mathrm{lo}\wedge\mathrm{lo}}(s) := -\sum_{k_1,k_2\leq k-\varpi} \int_s^\infty \mathrm{Im}(\overline{P_{k_1}\psi_m} P_{k_2}\psi_s)(s')\, ds'$$

and $A_{\mathrm{hi}\vee\mathrm{hi}}$ as

$$A_{m,\mathrm{hi}\vee\mathrm{hi}}(s) := -\sum_{\max\{k_1,k_2\}>k-\varpi} \int_s^\infty \mathrm{Im}(\overline{P_{k_1}\psi_m} P_{k_2}\psi_s)(s')\, ds',$$

so that $A_m = A_{m,\mathrm{lo}\wedge\mathrm{lo}} + A_{m,\mathrm{hi}\vee\mathrm{hi}}$. Similarly define $B_{\mathrm{lo}\wedge\mathrm{lo}}$ as

$$B_{m,\mathrm{lo}\wedge\mathrm{lo}} := -i \sum_{k_3} \left(\partial_l (A_{l,\mathrm{lo}\wedge\mathrm{lo}} P_{k_3}\psi_m) + A_{l,\mathrm{lo}\wedge\mathrm{lo}} \partial_l P_{k_3}\psi_m\right)$$

and $B_{\mathrm{hi}\vee\mathrm{hi}}$ as

$$B_{m,\mathrm{hi}\vee\mathrm{hi}} := -i \sum_{k_3} \left(\partial_l (A_{l,\mathrm{hi}\vee\mathrm{hi}} P_{k_3}\psi_m) + A_{l,\mathrm{hi}\vee\mathrm{hi}} \partial_l P_{k_3}\psi_m\right),$$

so that $B_m = B_{m,\mathrm{lo}\wedge\mathrm{lo}} + B_{m,\mathrm{hi}\vee\mathrm{hi}}$.

Our goal is to control $P_k B_m$ in $N_k(T)$. We consider first $P_k B_{m,\mathrm{hi}\vee\mathrm{hi}}$, performing a trilinear Littlewood–Paley decomposition. In order for frequencies k_1, k_2, k_3 to have an output in this expression at a frequency k, we must have $(k_1, k_2, k_3) \in Z_2(k) \cup Z_3(k) \cup Z_0(k)$, where

$$Z_0(k) := Z_1(k) \cap \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_1, k_2 > k - \varpi\} \tag{4-24}$$

and the other $Z_j(k)$ are defined in (3-12). We apply Lemma 3.10 to bound $P_k B_{m,\mathrm{hi}\vee\mathrm{hi}}$ in $N_k(T)$ by

$$\sum_{\substack{(k_1,k_2,k_3)\in\\ Z_2(k)\cup Z_3(k)\cup Z_0(k)}} \int_0^\infty 2^{\max\{k,k_3\}} C_{k,k_1,k_2,k_3} \|P_{k_1}\psi_x(s)\|_{F_{k_1}} \|P_{k_2}(D_l\psi_l(s))\|_{F_{k_2}} \|P_{k_3}\psi_m(0)\|_{G_{k_3}}\, ds,$$

which, thanks to (4-12) and (4-13), is controlled by

$$\sum_{\substack{(k_1,k_2,k_3)\in\\ Z_2(k)\cup Z_3(k)\cup Z_0(k)}} 2^{\max\{k,k_3\}} C_{k,k_1,k_2,k_3} b_{k_1} b_{k_2} b_{k_3} \int_0^\infty (1+s2^{2k_1})^{-4} 2^{k_2} (s2^{2k_2})^{-3/8} (1+s2^{2k_2})^{-2}\, ds.$$

As

$$\int_0^\infty (1+s2^{2k_1})^{-4} 2^{k_2} (s2^{2k_2})^{-3/8} (1+s2^{2k_2})^{-2}\, ds \lesssim 2^{-\max\{k_1,k_2\}}, \tag{4-25}$$

we reduce to

$$\sum_{\substack{(k_1,k_2,k_3) \text{ in}\\ Z_2(k)\cup Z_3(k)\cup Z_0(k)}} 2^{\max\{k,k_3\}-\max\{k_1,k_2\}} C_{k,k_1,k_2,k_3} b_{k_1} b_{k_2} b_{k_3}. \tag{4-26}$$

To estimate $P_k B_{m,\mathrm{hi}\vee\mathrm{hi}}$ on $Z_2 \cup Z_3$, we apply Corollary 3.12 and use the energy dispersion hypothesis. As for $Z_0(k)$, we note that its cardinality $|Z_0(k)|$ satisfies $|Z_0(k)| \lesssim \varpi$ independently of k. Hence for fixed ϖ summing over this set is harmless given sufficient energy dispersion. We obtain a bound of

$$\|P_k B_{m,\mathrm{hi}\vee\mathrm{hi}}\|_{N_k(T)} \lesssim b_k^2 b_k \lesssim \epsilon b_k. \tag{4-27}$$

Consider now the leading term $P_k B_{m,\mathrm{lo}\wedge\mathrm{lo}}$. Bounding this in N_k with any hope of summing requires the full strength of the decay that comes from the local smoothing/maximal function estimates. However, such bounds as are immediately at our disposal — (3-10) and (3-11) — do not bring $B_{m,\mathrm{lo}\wedge\mathrm{lo}}$ within the perturbative framework, instead yielding a bound of the form

$$\sum_{\substack{k_1,k_2 \le k-\varpi \\ |k_3-k|\le 4}} b_{k_1} b_{k_2} b_{k_3},$$

which is problematic since even $\sum_{j \ll k} c_j^2 \sim E_0^2 = O(1)$ for k large enough. This stands in sharp contrast with the small energy setting.

In the next section, however, we are able to capture enough improvement in such estimates so as to barely bring $B_{m,\mathrm{lo}\wedge\mathrm{lo}}$ back within reach of our bootstrap approach.

Finally, we need for $\sigma > 0$ an estimate analogous to (4-27). Returning to the proof of (4-26), we remark that any b_{k_j} may be replaced by $2^{-\sigma k_j} b_{k_j}$; in order to obtain an analogue of (4-27), we must make replacements judiciously so as to retain summability. In particular, for any (k_1, k_2, k_3) in $Z_2(k) \cup Z_3(k) \cup Z_0(k)$, we replace $b_{k_{\max}}$ with $2^{-\sigma k_{\max}} b_{k_{\max}}(\sigma)$ so that (4-26) becomes

$$\sum_{\substack{(k_1,k_2,k_3)\in \\ Z_2(k)\cup Z_3(k)\cup Z_0(k)}} 2^{\max\{k,k_3\}-\max\{k_1,k_2\}} C_{k,k_1,k_2,k_3} b_{k_{\min}} b_{k_{\mathrm{mid}}} 2^{-\sigma k_{\max}} b_{k_{\max}}(\sigma),$$

where $k_{\min}, k_{\mathrm{mid}}, k_{\max}$ denote, respectively, the min, mid, and max of $\{k_1, k_2, k_3\}$. We have $k_{\max} \gtrsim k$ over the set $Z_2(k) \cup Z_3(k) \cup Z_0(k)$ (see (3-12) and (4-24) for definitions), which guarantees summability due to straightforward modifications of Corollaries 3.11 and 3.12. Therefore

$$\| P_k B_{m,\mathrm{hi}\vee\mathrm{hi}} \|_{N_k(T)} \lesssim b_k^2 2^{-\sigma k} b_k(\sigma),$$

which, combined with (4-27) and the definition (4-18) of ϵ, implies this:

Corollary 4.12. *Assume $\sigma \in [0, \sigma_1 - 1]$. The term $B_{m,\mathrm{hi}\vee\mathrm{hi}}$ satisfies the estimate*

$$\| P_k B_{m,\mathrm{hi}\vee\mathrm{hi}} \|_{N_k(T)} \lesssim \epsilon 2^{-\sigma k} b_k(\sigma). \tag{4-28}$$

4E. *Closing the gauge field bootstrap.* We turn first to the completion of the proof of Theorem 4.1, as we now have in place all of the estimates that we need to prove (4-8).

Using the main linear estimate of Proposition 3.6 and the decomposition introduced in Section 4D, we obtain

$$\| P_k \psi_m \|_{G_k(T)} \lesssim \| P_k \psi_m(0) \|_{L_x^2} + \| P_k V_m \|_{N_k(T)} + \| P_k B_{m,\mathrm{hi}\vee\mathrm{hi}} \|_{N_k(T)} + \| P_k B_{m,\mathrm{lo}\wedge\mathrm{lo}} \|_{N_k(T)}. \tag{4-29}$$

In Sections 4C and 4D it is shown that $P_k V_m$ and $P_k B_{m,\mathrm{hi}\vee\mathrm{hi}}$ are perturbative in the sense that

$$\| P_k V_m \|_{N_k(T)} + \| P_k B_{m,\mathrm{hi}\vee\mathrm{hi}} \|_{N_k(T)} \lesssim \epsilon 2^{-\sigma k} b_k(\sigma),$$

To handle $P_k B_{m,\mathrm{lo}\wedge\mathrm{lo}}$, we first write

$$P_k B_{m,\mathrm{lo}\wedge\mathrm{lo}} = -i \partial_l (A_{l,\mathrm{lo}\wedge\mathrm{lo}} P_k \psi_m) + R,$$

where R is a perturbative remainder (thanks to a slight modification of technical Lemma 5.11). Therefore

$$\|P_k\psi_m\|_{G_k(T)} \lesssim 2^{-\sigma k}c_k(\sigma)+\epsilon 2^{-\sigma k}b_k(\sigma)+\|\partial_l(A_{l,\mathrm{lo}\wedge\mathrm{lo}}P_k\psi_m)\|_{N_k(T)}. \tag{4-30}$$

Thus it remains to control $-i\partial_l(A_{l,\mathrm{lo}\wedge\mathrm{lo}}P_k\psi_m)$, which we expand as

$$-iP_k\partial_l \sum_{\substack{k_1,k_2\le k-\varpi\\ |k_3-k|\le 4}} \int_0^\infty \mathrm{Im}(\overline{P_{k_1}\psi_l}P_{k_2}\psi_s)(s')P_{k_3}\psi_m(0)\,ds', \tag{4-31}$$

and whose $N_k(T)$ norm we denote by N_{lo}. In the $\sigma=0$ case the key is to apply Theorem 4.8 to $\overline{P_{k_1}\psi_l}(s')$ and $P_{k_3}\psi_m(0)$, after first placing all of (4-31) in $N_k(T)$ using (3-10). We obtain

$$\begin{aligned} N_{\mathrm{lo}} &\lesssim 2^k \sum_{\substack{k_1,k_2\le k-\varpi\\ |k_3-k|\le 4}} 2^{-|k-k_2|/2}2^{-|k_1-k_3|/2}2^{-\max\{k_1,k_2\}}b_{k_2}\left(c_{k_1}c_{k_3}+\epsilon^{1/2}b_{k_1}b_{k_3}\right)\\ &\lesssim 2^k \sum_{k_1,k_2\le k-\varpi} 2^{(k_1+k_2)/2-k}2^{-\max\{k_1,k_2\}}b_{k_2}(c_{k_1}c_k+\epsilon^{1/2}b_{k_1}b_k). \end{aligned}$$

Without loss of generality we restrict the sum to $k_1\le k_2$:

$$\sum_{k_1\le k_2\le k-\varpi} 2^{(k_1-k_2)/2}b_{k_2}(c_{k_1}c_k+\epsilon^{1/2}b_{k_1}b_k).$$

Using the frequency envelope property to sum off the diagonal, we reduce to

$$N_{\mathrm{lo}} \lesssim \sum_{j\le k-\varpi}(b_jc_jc_k+\epsilon^{1/2}b_j^2b_k).$$

Combining this with (4-30) and the fact that R is perturbative, we obtain

$$b_k \lesssim c_k+\epsilon b_k+\sum_{j\le k-\varpi}(b_jc_jc_k+\epsilon^{1/2}b_j^2b_k), \tag{4-32}$$

which, in view of our choice of ϵ, reduces to

$$b_k\lesssim c_k+c_k\sum_{j\le k-\varpi}b_jc_j.$$

Squaring and applying Cauchy–Schwarz yields

$$b_k^2\lesssim\left(1+\sum_{j\le k-\varpi}b_j^2\right)c_k^2. \tag{4-33}$$

Setting

$$B_k := 1+\sum_{j<k}b_j^2$$

in (4-33) leads to

$$B_{k+1}\le B_k(1+Cc_k^2)$$

with $C > 0$ independent of k. Therefore

$$B_{k+m} \le B_k \prod_{l=1}^{m}(1 + Cc_{k+l}^2) \le B_k \exp\Bigl(C \sum_{l=1}^{m} c_{k+l}^2\Bigr) \lesssim_{E_0} B_k.$$

Since $B_k \to 1$ as $k \to -\infty$, we conclude that

$$B_k \lesssim_{E_0} 1$$

uniformly in k, so that, in particular,

$$\sum_{j \in \mathbb{Z}} b_j^2 \lesssim 1, \tag{4-34}$$

which, joined with (4-33), implies (4-8).

The proof of (4-9) is almost an immediate consequence. Instead of (4-32), we obtain

$$b_k(\sigma) \lesssim c_k(\sigma) + \epsilon b_k(\sigma) + \sum_{j \le k-\varpi} (b_j c_j c_k(\sigma) + \epsilon^{1/2} b_j^2 b_k(\sigma)),$$

which suffices to prove (4-9) in view of (4-34).

4F. *De-gauging.* The previous subsections overcome the most significant obstacles encountered in proving conditional global regularity. All of the key estimates therein apply to the Schrödinger map system placed in the caloric gauge, and a bootstrap argument is in fact run and closed at that level. This final subsection justifies the whole approach, showing how to transfer these results obtained at the gauge level back to the underlying Schrödinger map itself.

Proof of (4-10). To gain control over the derivatives $\partial_m \varphi$ in $L_t^\infty L_x^2$, we utilize representation (2-2) and perform a Littlewood–Paley decomposition. We only indicate how to handle the term $v \cdot \operatorname{Re} \psi_m$, as the term $w \cdot \operatorname{Im} \psi_m$ may be handled similarly. Starting with

$$\begin{aligned} P_k(v \operatorname{Re} \psi_m) = \sum_{|k_2 - k| \le 4} & P_k(P_{\le k-5} v \cdot P_{k_2} \operatorname{Re} \psi_m) \\ & + \sum_{\substack{|k_1 - k| \le 4 \\ k_2 \le k-4}} P_k(P_{k_1} v \cdot P_{k_2} \operatorname{Re} \psi_m) + \sum_{\substack{|k_1 - k_2| \le 8 \\ k_1, k_2 \ge k-4}} P_k(P_{k_1} v \cdot P_{k_2} \operatorname{Re} \psi_m), \end{aligned} \tag{4-35}$$

we proceed to bound each term in $L_t^\infty L_x^2$.

In view of the fact that $|v| \equiv 1$, the low-high frequency interaction is controlled by

$$\begin{aligned} \sum_{|k_2 - k| \le 4} \| P_k(P_{\le k-5} v \cdot P_{k_2} \operatorname{Re} \psi_m) \|_{L_t^\infty L_x^2} &\lesssim \| P_{\le k-5} v \|_{L_{t,x}^\infty} \| P_k \psi_m \|_{L_t^\infty L_x^2} \\ &\lesssim \| P_k \psi_m \|_{L_t^\infty L_x^2} \lesssim c_k. \end{aligned} \tag{4-36}$$

To control the high-low frequency interaction, we use Hölder's inequality, Bernstein's inequality, (2-33) and Bernstein's inequality again, and finally the bound (2-15) along with the summation rule (2-30):

$$\begin{aligned}\sum_{\substack{|k_1-k|\le 4\\ k_2\le k-4}} \|P_k(P_{k_1}v\cdot P_{k_2}\operatorname{Re}\psi_m)\|_{L_t^\infty L_x^2} &\lesssim \sum_{\substack{|k_1-k|\le 4\\ k_2\le k-4}} \|P_{k_1}v\|_{L_t^\infty L_x^2}\|P_{k_2}\psi_m\|_{L_{t,x}^\infty}\\ &\lesssim \sum_{\substack{|k_1-k|\le 4\\ k_2\le k-4}} \|P_{k_1}v\|_{L_t^\infty L_x^2}\cdot 2^{k_2}\|P_{k_2}\psi_m\|_{L_t^\infty L_x^2}\\ &\lesssim \sum_{\substack{|k_1-k|\le 4\\ k_2\le k-4}} \|P_{k_1}\partial_x v\|_{L_t^\infty L_x^2}\cdot 2^{k_2-k}c_{k_2}\lesssim c_k. \end{aligned}\tag{4-37}$$

To control the high-high frequency interaction, we use Bernstein's inequality, Cauchy–Schwarz, Bernstein again, (2-15), and finally (2-31):

$$\begin{aligned}\sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} \|P_k(P_{k_1}v\cdot P_{k_2}\operatorname{Re}\psi_m)\|_{L_t^\infty L_x^2} &\lesssim \sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} 2^k\|P_{k_1}v\cdot P_{k_2}\operatorname{Re}\psi_m\|_{L_t^\infty L_x^1}\\ &\lesssim \sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} 2^k\|P_{k_1}v\|_{L_t^\infty L_x^2}\|P_{k_2}\psi_m\|_{L_t^\infty L_x^2}\\ &\lesssim \sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} 2^{k-k_1}\|P_{k_1}\partial_x v\|_{L_t^\infty L_x^2}\|P_{k_2}\psi_m\|_{L_t^\infty L_x^2}\\ &\lesssim \sum_{k_2\ge k-4} 2^{k-k_2}c_{k_2}\lesssim c_k. \end{aligned}\tag{4-38}$$

Combining (4-36), (4-37), and (4-38) and applying them in (4-35), we obtain

$$\|P_k(v\operatorname{Re}\psi_m)\|_{L_t^\infty L_x^2}\lesssim c_k.$$

As the above calculation holds with w in place of v, we conclude (recalling (2-2)) that

$$\|P_k\partial_x\varphi\|_{L_t^\infty L_x^2}\lesssim c_k.$$

Hence (4-10) holds for $\sigma=0$.

Now we turn to the case $\sigma\in[0,\sigma_1-1]$. Using Bernstein's inequality in (4-36) and (4-38), we obtain

$$\sum_{|k_2-k|\le 4} \|P_k(P_{\le k-5}v\cdot P_{k_2}\operatorname{Re}\psi_m)\|_{L_t^\infty L_x^2}\lesssim 2^{-\sigma k}c_k(\sigma),\tag{4-39}$$

$$\sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} \|P_k(P_{k_1}v\cdot P_{k_2}\operatorname{Re}\psi_m)\|_{L_t^\infty L_x^2}\lesssim 2^{-\sigma k}c_k(\sigma),\tag{4-40}$$

as well as analogous estimates with w in place of v. Such a direct argument, however, does not yield the analogue of (4-37). We circumvent this obstruction as follows. Let $\mathscr{C}\in(0,\infty)$ be the best constant for which

$$\|P_k\partial_x\varphi\|_{L_t^\infty L_x^2}\le \mathscr{C}2^{-\sigma k}c_k(\sigma)\tag{4-41}$$

holds for $\sigma \in [0, \sigma_1 - 1]$. Such a constant exists by smoothness and the fact that the $c_k(\sigma)$ are frequency envelopes. In view of definition (2-34) and estimate (2-35), we similarly have

$$\| P_k \partial_x v(0) \|_{L_t^\infty L_x^2} \lesssim \mathscr{C} 2^{-\sigma k} c_k(\sigma). \tag{4-42}$$

Using (4-42) in (4-37), we obtain

$$\sum_{\substack{|k_1 - k| \le 4 \\ k_2 \le k-4}} \| P_k (P_{k_1} v \cdot P_{k_2} \operatorname{Re} \psi_m) \|_{L_t^\infty L_x^2} \lesssim \mathscr{C} 2^{-\sigma k} c_k c_k(\sigma). \tag{4-43}$$

From the representations (2-2) and (4-35), and from the estimates (4-39), (4-40), and (4-43), along with the analogous estimates for w, it follows that

$$\| P_k \partial_x \varphi \|_{L_t^\infty L_x^2} \lesssim (1 + c_k \mathscr{C}) 2^{-\sigma k} c_k(\sigma).$$

In view of energy dispersion ($c_k \le \varepsilon$) and the optimality of $\mathscr{C}$ in (4-41), we conclude that $\mathscr{C} \lesssim 1 + \varepsilon \mathscr{C}$, so that $\mathscr{C} \lesssim 1$. Therefore

$$\| P_k \partial_x^\sigma \partial_m \varphi \|_{L_t^\infty L_x^2} \sim 2^{\sigma k} \| P_k \partial_m \varphi \|_{L_t^\infty L_x^2} \lesssim c_k(\sigma),$$

which completes the proof of (4-10). □

It will be convenient in certain arguments to use the weaker frequency envelope defined by

$$\tilde{b}_k = \sup_{k' \in \mathbb{Z}} 2^{-\delta |k - k'|} \| P_{k'} \psi_x \|_{L_{t,x}^4}. \tag{4-44}$$

Proof of Lemma 4.3. Let us first establish

$$\sum_{k \in \mathbb{Z}} \| P_k \psi_x \|_{L_{t,x}^4}^2 \lesssim \sum_{k \in \mathbb{Z}} \| P_k \partial_x \varphi \|_{L_{t,x}^4}^2.$$

We use (2-1), i.e., $\psi_m = v \cdot \partial_m \varphi + i w \cdot \partial_m \varphi$, but for the sake of exposition only treat $v \cdot \partial_m \varphi$. We start with the Littlewood–Paley decomposition

$$P_k \psi_m(0) = \sum_{|k_2 - k| \le 4} P_k (P_{\le k-5} v \cdot P_{k_2} \partial_m \varphi) + \sum_{\substack{|k_1 - k| \le 4 \\ k_2 \le k-4}} P_k (P_{k_1} v \cdot P_{k_2} \partial_m \varphi) + \sum_{\substack{|k_1 - k_2| \le 8 \\ k_1, k_2 \ge k-4}} P_k (P_{k_1} v \cdot P_{k_2} \partial_m \varphi).$$

In view of $|v| \equiv 1$, the $L_{t,x}^4$ norm of the low-high interaction is controlled by $\tilde{b}_k$ (see (4-44)). To control the high-low interaction, we use Hölder's and Bernstein's inequalities along with (2-15):

$$\begin{aligned}
\sum_{\substack{|k_1 - k| \le 4 \\ k_2 \le k-4}} \| P_k (P_{k_1} v \cdot P_{k_2} \partial_m \varphi) \|_{L_{t,x}^4} &\lesssim \sum_{\substack{|k_1 - k| \le 4 \\ k_2 \le k-4}} \| P_{k_1} v \|_{L_t^\infty L_x^4} \cdot \| P_{k_2} \partial_m \varphi \|_{L_t^4 L_x^\infty} \\
&\lesssim \sum_{\substack{|k_1 - k| \le 4 \\ k_2 \le k-4}} 2^{k_1/2} \| P_{k_1} v \|_{L_t^\infty L_x^2} 2^{k_2/2} \| P_{k_2} \partial_m \varphi \|_{L_{t,x}^4} \\
&\lesssim \sum_{\substack{|k_1 - k| \le 4 \\ k_2 \le k-4}} 2^{k_1} \| P_{k_1} v \|_{L_t^\infty L_x^2} \tilde{b}_k \lesssim \tilde{b}_k.
\end{aligned}$$

To control the high-high interaction, we use Bernstein, Hölder, Bernstein again, and (2-15):

$$\begin{aligned}\sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} \|P_k(P_{k_1}v\cdot P_{k_2}\partial_m\varphi)\|_{L^4_{t,x}} &\lesssim \sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} 2^{k/2}\|P_{k_1}v\cdot P_{k_2}\partial_m\varphi\|_{L^4_tL^2_x}\\ &\lesssim \sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} 2^{k/2}\|P_{k_1}v\|_{L^\infty_tL^4_x}\|P_{k_2}\partial_m\varphi\|_{L^4_{t,x}}\\ &\lesssim \sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} 2^{(k+k_1)/2}\|P_{k_1}v\|_{L^\infty_tL^2_x}\|P_k\partial_m\varphi\|_{L^4_{t,x}}\\ &\lesssim \sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} 2^{(k-k_1)/2}\|P_{k_1}\partial_xv\|_{L^\infty_tL^2_x}\|P_{k_2}\partial_m\varphi\|_{L^4_{t,x}} \lesssim \sum_{k_2\ge k-4} 2^{(k-k_2)/4}\tilde b_{k_2}\lesssim \tilde b_k.\end{aligned}$$

Therefore

$$\|P_k\psi_m(0)\|_{L^4_{t,x}}\lesssim \tilde b_k$$

and

$$\sum_{k\in\mathbb{Z}}\|P_k\psi_m(0)\|^2_{L^4_{t,x}}\lesssim \sum_{k\in\mathbb{Z}}\tilde b_k^2\sim\sum_{k\in\mathbb{Z}}\|P_k\partial_m\varphi(0)\|^2_{L^4_{t,x}}.$$

By using (2-2), creating an L^4 frequency envelope for $P_k\partial_m\varphi(0)$, and reversing the roles of ψ_α and $\partial_\alpha\varphi$ in the preceding argument, we conclude the reverse inequality

$$\sum_{k\in\mathbb{Z}}\|P_k\partial_m\varphi(0)\|^2_{L^4_{t,x}}\lesssim\sum_{k\in\mathbb{Z}}\|P_k\psi_m(0)\|^2_{L^4_{t,x}}.\qquad\square$$

5. Local smoothing and bilinear Strichartz

The main goal of this section is to establish the improved bilinear Strichartz estimate of Theorem 4.8. As a by-product we also obtain the frequency-localized local smoothing estimate of Theorem 4.9.

Our approach is to first establish abstract local smoothing and bilinear Strichartz estimates for solutions to certain magnetic nonlinear Schrödinger equations. These are in the spirit of [Planchon and Vega 2009; 2012; Tao 2010]. We shall then apply these to Schrödinger maps, in particular to the paralinearized derivative field equations written with respect to the caloric gauge.

We introduce some notation. Let

$$I_k(\mathbb{R}^d)=\{\xi\in\mathbb{R}^d:|\xi|\in[-2^{k-1},2^{k+1}]\}\quad\text{and}\quad I_{(-\infty,k]}:=\bigcup_{j\le k}I_j.$$

For a d-vector-valued function $B=(B_l)$ on $\mathbb{R}^d$ with real entries, define the magnetic Laplacian Δ_B, acting on complex-valued functions f, via

$$\Delta_Bf:=(\partial_x+iB)((\partial_x+iB)f)=\Delta f+i(\partial_lB_l)f+2iB_l\partial_lf-B_l^2f.\tag{5-1}$$

For a unit vector $\boldsymbol{e}\in\mathbb{S}^{d-1}$, denote by $\{x\cdot\boldsymbol{e}=0\}$ the orthogonal complement in $\mathbb{R}^d$ of the span of $\boldsymbol{e}$, equipped with the induced measure. Given $\boldsymbol{e}$, we can construct a positively oriented orthonormal basis

$\boldsymbol{e}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_{d-1}$ of $\mathbb{R}^d$ so that $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{d-1}$ form an orthonormal basis for $\{x \cdot \boldsymbol{e} = 0\}$. For complex-valued functions f on $\mathbb{R}^d$, define $E_{\boldsymbol{e}}(f) : \mathbb{R} \to \mathbb{R}$ as

$$E_{\boldsymbol{e}}(f)(x_0) := \int_{x \cdot \boldsymbol{e}=0} |f|^2 \, dx' = \int_{\mathbb{R}^{d-1}} |f(x_0 \boldsymbol{e} + x_j \boldsymbol{e}_j)|^2 \, dx', \tag{5-2}$$

where the implicit sum runs over $1, 2, \dots, d-1$, and dx' is the standard $(d-1)$-dimensional Lebesgue measure. We also adopt the following notation for this section: for z, ζ complex,

$$z \wedge \zeta := z\bar{\zeta} - \bar{z}\zeta = 2i \,\mathrm{Im}(z\bar{\zeta}).$$

5A. ***Key lemmas.***

Lemma 5.1 (abstract almost-conservation of energy). *Let $d \geq 1$ and $\boldsymbol{e} \in \mathbb{S}^{d-1}$. Let v be a $C_t^\infty(H_x^\infty)$ function on $\mathbb{R}^d \times [0, T]$ solving*

$$(i\partial_t + \Delta_{\mathcal{A}})v = \Lambda_v \tag{5-3}$$

with initial data v_0. Take $\mathcal{A}_l$ to be real-valued, smooth, and bounded, with $\Delta_{\mathcal{A}}$ defined via (5-1). Then

$$\|v\|^2_{L_t^\infty L_x^2} \leq \|v_0\|^2_{L_x^2} + \left| \int_0^T \int_{\mathbb{R}^d} v \wedge \Lambda_v \, dx dt \right|. \tag{5-4}$$

Proof. We begin with

$$\frac{1}{2}\partial_t \int |v|^2 \, dx = \int \mathrm{Im}(\bar{v}\partial_t v) \, dx,$$

which may equivalently be written as

$$i\partial_t \int |v|^2 \, dx = -\int v \wedge i\partial_t v \, dx.$$

Substituting from (5-3) yields

$$i\partial_t \int |v|^2 \, dx = \int v \wedge (\Delta_{\mathcal{A}} v - \Lambda_v) \, dx.$$

Expanding $\Delta_{\mathcal{A}}$ using (5-1) and using the straightforward relations

$$\partial_l(v \wedge i\mathcal{A}_l v) = v \wedge i(\partial_l \mathcal{A}_l)v + v \wedge 2i\mathcal{A}_l \partial_l v \quad \text{and} \quad \partial_l(v \wedge \partial_l v) = v \wedge \Delta v,$$

we get

$$i\partial_t \int |v|^2 \, dx = \int \partial_l(v \wedge \partial_l v) \, dx + \int \partial_l(v \wedge i\mathcal{A}_l v) \, dx - \int v \wedge \mathcal{A}_l^2 v \, dx - \int v \wedge \Lambda_v \, dx.$$

The first two terms on the right-hand side vanish upon integration in x; the third is equal to zero because $\mathcal{A}_l^2$ is real. Integrating in time and taking absolute values therefore yields

$$\left| \int_{\mathbb{R}^d} |v(T')|^2 - |v_0|^2 \, dx \right| = \left| \int_0^{T'} \int_{\mathbb{R}^d} v \wedge \Lambda_v \, dx dt \right|$$

for any time $T' \in (0, T]$. □

Lemma 5.2 (local smoothing preparation). *Let $d \geq 1$ and $\boldsymbol{e} \in \mathbb{S}^{d-1}$. Let $j, k \in \mathbb{Z}$ and $j = k + O(1)$. Let $\varepsilon_m > 0$ be a small positive number such that $\varepsilon_m 2^{O(1)} \ll 1$. Let v be a $C_t^\infty(H_x^\infty)$ function on $\mathbb{R}^d \times [0, T]$ solving*

$$(i\partial_t + \Delta_{\mathscr{A}})v = \Lambda_v, \tag{5-5}$$

where $\mathscr{A}_l$ is real-valued, smooth, and satisfies the estimate

$$\|\mathscr{A}\|_{L^\infty_{t,x}} \leq \varepsilon_m 2^k. \tag{5-6}$$

The solution v is assumed to have (spatial) frequency support in I_k, with the additional constraint that $\boldsymbol{e} \cdot \xi \in [2^{j-1}, 2^{j+1}]$ for all ξ in the support of $\hat{v}$. Then

$$2^j \int_0^T E_{\boldsymbol{e}}(v)\, dt \lesssim \|v\|^2_{L_t^\infty L_x^2} + \left| \int_0^T \int_{x \cdot \boldsymbol{e} \geq 0} v \wedge \Lambda_v \, dx dt \right| + 2^j \int_0^T E_{\boldsymbol{e}}(v + i 2^{-j} \partial_{\boldsymbol{e}} v)\, dt. \tag{5-7}$$

Proof. We begin by introducing

$$M_{\boldsymbol{e}}(t) := \int_{x \cdot \boldsymbol{e} \geq 0} |v(x,t)|^2 \, dx.$$

Then

$$0 \leq M_{\boldsymbol{e}}(t) \leq \|v(t)\|^2_{L_x^2(\mathbb{R}^d)} \leq \|v\|^2_{L_t^\infty L_x^2([-T,T]\times\mathbb{R}^d)}. \tag{5-8}$$

Differentiating in time yields

$$i\dot{M}_{\boldsymbol{e}}(t) = \int_{x \cdot \boldsymbol{e} \geq 0} v \wedge (i\partial_t v)\, dx = \int_{x \cdot \boldsymbol{e} \geq 0} v \wedge (\Delta_{\mathscr{A}} v - \Lambda_v)\, dx,$$

which may be rewritten as

$$i\dot{M}_{\boldsymbol{e}}(t) = \int_{x \cdot \boldsymbol{e} \geq 0} \partial_l (v \wedge (\partial_l + i\mathscr{A}_l)v)\, dx - \int_{x \cdot \boldsymbol{e} \geq 0} v \wedge \Lambda_v \, dx. \tag{5-9}$$

By integrating by parts,

$$\int_{x \cdot \boldsymbol{e} \geq 0} \partial_l (v \wedge (\partial_l + i\mathscr{A}_l)v)\, dx = -\int_{x \cdot \boldsymbol{e} = 0} v \wedge (\partial_{\boldsymbol{e}} v + i\boldsymbol{e} \cdot \mathscr{A} v)\, dx',$$

and therefore (5-9) may be rewritten as

$$-\int_{x \cdot \boldsymbol{e} = 0} v \wedge (\partial_{\boldsymbol{e}} v + i\boldsymbol{e} \cdot \mathscr{A} v)\, dx' = i\dot{M}_{\boldsymbol{e}}(t) + \int_{x \cdot \boldsymbol{e} \geq 0} v \wedge \Lambda_v \, dx. \tag{5-10}$$

On the one hand, we have the heuristic that $\partial_{\boldsymbol{e}} v \approx i 2^j v$ since v has localized frequency support. On the other hand, since $\mathscr{A}$ is real-valued, we have

$$\int_0^T \int_{x \cdot \boldsymbol{e} = 0} v \wedge i\boldsymbol{e} \cdot \mathscr{A} v \, dx' dt = 2 \int_0^T \int_{x \cdot \boldsymbol{e} = 0} \boldsymbol{e} \cdot \mathscr{A} |v|^2 \, dx' dt \tag{5-11}$$

and hence by assumption (5-6) also

$$\int_0^T \int_{x \cdot \boldsymbol{e} = 0} |\mathscr{A}|\, |v|^2 \, dx' dt \leq \varepsilon_m 2^k \int_0^T \int_{x \cdot \boldsymbol{e} = 0} |v|^2 \, dx' dt. \tag{5-12}$$

Together these facts motivate rewriting $v \wedge \partial_{\boldsymbol{e}} v$ as

$$v \wedge \partial_{\boldsymbol{e}} v = 2 \cdot i2^j |v|^2 + v \wedge (\partial_{\boldsymbol{e}} v - i2^j v). \tag{5-13}$$

Using (5-11), (5-13), and the bounds (5-12) and (5-8) in (5-10), we obtain by time-integration that

$$(1-\varepsilon_m 2^{k-j})2^j \int_0^T E_{\boldsymbol{e}}(v)\,dt \le \|v\|^2_{L_t^\infty L_x^2} + \left| \int_0^T \int_{x\cdot \boldsymbol{e}\ge 0} v \wedge \Lambda_v \,dx dt \right| + 2\cdot 2^j \int_0^T \int_{x\cdot \boldsymbol{e}=0} |v+i2^{-j}\partial_{\boldsymbol{e}} v||v|\,dx' dt.$$

Applying Cauchy–Schwarz to the last term yields

$$2^j \int_0^T \int_{x\cdot \boldsymbol{e}=0} |v+i2^{-j}\partial_{\boldsymbol{e}} v||v|\,dx'dt \le 8\cdot 2^j \int_0^T E_{\boldsymbol{e}}(v+i2^{-j}\partial_{\boldsymbol{e}} v)\,dt + \frac{1}{8}\cdot 2^j \int_0^T E_{\boldsymbol{e}}(v)\,dt.$$

Therefore (5-7). □

We now describe the constraints on the nonlinearity that we shall require in the abstract setting

Definition 5.3. Let $\mathcal{P}$ be a fixed finite subset of $\{1 < p < \infty\}$. A bilinear form $B(\cdot\,,\cdot)$ is said to be *adapted* to $\mathcal{P}$ provided it measures its arguments in Strichartz-type spaces, the estimate

$$\left| \int_0^T \int_{\mathbb{R}^d} f \wedge g \,dx dt \right| \lesssim B(f, g)$$

holds for all complex-valued functions f, g on $\mathbb{R}^d \times [0, T]$, Bernstein's inequalities hold in both arguments of B, and these arguments are measured in L_x^p only for $p \in \mathcal{P}$. Given $B(\cdot\,,\cdot)$ and $\boldsymbol{e} \in \mathbb{S}^{d-1}$, we define $B_{\boldsymbol{e}}(\cdot\,,\cdot)$ via

$$B_{\boldsymbol{e}}(f, g) := B(f, \chi_{\{x\cdot \boldsymbol{e}\ge 0\}} g).$$

Definition 5.4. Let $\boldsymbol{e} \in \mathbb{S}^{d-1}$ and let $\mathcal{A}_l$ be real-valued and smooth. Let v be a $C_t^\infty(H_x^\infty)$ function on $\mathbb{R}^d \times [0, T]$ solving

$$(i\partial_t + \Delta_{\mathcal{A}})v = \Lambda_v.$$

Assume v is (spatially) frequency-localized to I_k with the additional constraint that $\boldsymbol{e}\cdot\xi \in [2^{j-1}, 2^{j+1}]$ for all ξ in the support of $\hat{v}$. Define a sequence of functions $\{v^{(m)}\}_{m=1}^\infty$ by setting $v^{(1)} = v$ and

$$v^{(m+1)} := v^{(m)} + i2^{-j}\partial_{\boldsymbol{e}} v^{(m)}.$$

By (5-1) and the Leibniz rule,

$$(i\partial_t + \Delta_{\mathcal{A}})v^{(m)} = \Lambda_{v^{(m)}},$$

where

$$\Lambda_{v^{(m)}} := (1 + i2^{-j}\partial_{\boldsymbol{e}})\Lambda_{v^{(m-1)}} + i2^{-j}(i\partial_{\boldsymbol{e}}\partial_l A_l - \partial_{\boldsymbol{e}} A_l^2)v^{(m-1)} - 2^{-j+1}(\partial_{\boldsymbol{e}} A_l)\partial_l v^{(m-1)}.$$

The sequence $\{v^{(m)}\}_{m=1}^\infty$ is called the *derived sequence* corresponding to v.

Suppose we are given a form B adapted to $\mathcal{P}$. The derived sequence is said to be *controlled* with respect to $B_{\boldsymbol{e}}$ provided that $B_{\boldsymbol{e}}(v^{(m)}, \Lambda_{v^{(m)}}) < \infty$ for each $m \ge 1$.

We remark that if the derived sequence $\{v^{(m)}\}_{m=1}^{\infty}$ of v is controlled, then for all $l \geq 1$, the derived sequences $\{v^{(m)}\}_{m=l}^{\infty}$ are also controlled.

Theorem 5.5 (abstract local smoothing). *Let $d \geq 1$ and $\boldsymbol{e} \in \mathbb{S}^{d-1}$. Let $j, k \in \mathbb{Z}$ and $j = k + O(1)$. Let $\varepsilon_m > 0$ be a small positive number such that $\varepsilon_m 2^{O(1)} \ll 1$. Let $\eta > 0$. Let $\mathcal{P}$ be a fixed finite subset of $(1, \infty)$ with $2 \in \mathcal{P}$, and let B be a form adapted to $\mathcal{P}$. Let v be a $C_t^{\infty}(H_x^{\infty})$ function on $\mathbb{R}^d \times [0, T]$ solving*

$$(i\partial_t + \Delta_{\mathcal{A}})v = \Lambda_v, \tag{5-14}$$

where $\mathcal{A}_l$ is real-valued, smooth, has spatial Fourier support in $I_{(-\infty,k]}$, and satisfies the estimate

$$\|\mathcal{A}\|_{L^{\infty}_{t,x}} \leq \varepsilon_m 2^k. \tag{5-15}$$

The solution v is assumed to have (spatial) frequency support in I_k. We take Λ_v to be frequency-localized to $I_{(-\infty,k]}$. Assume moreover that

$$\boldsymbol{e} \cdot \xi \in [(1-\eta)2^j, (1+\eta)2^j] \tag{5-16}$$

for all ξ in the support of $\hat{v}$.

If the derived sequence of v is controlled with respect to $B_{\boldsymbol{e}}$, then there exists $\eta^* > 0$ such that, for all $0 \leq \eta < \eta^*$, the local smoothing estimate

$$2^j \int_0^T E_{\boldsymbol{e}}(v)\, dt \lesssim \|v\|^2_{L_t^{\infty} L_x^2} + B_{\boldsymbol{e}}(v, \Lambda_v) \tag{5-17}$$

holds uniformly in T and $j = k + O(1)$.

Proof. The foundation for proving (5-17) is (5-7), which for an adapted form $B_{\boldsymbol{e}}$ implies

$$2^j \int_0^T E_{\boldsymbol{e}}(v)\, dt \lesssim \|v\|^2_{L_t^{\infty} L_x^2} + B_{\boldsymbol{e}}(v, \Lambda_v) + 2^j \int_0^T E_{\boldsymbol{e}}(v + i2^{-j}\partial_{\boldsymbol{e}} v)\, dt. \tag{5-18}$$

Therefore our goal is control the last term in (5-18). This we do using a bootstrap argument that hinges upon the fact that $\tilde{v} := v + i2^{-j}\partial_{\boldsymbol{e}} v$ is the second term in the derived sequence of v, and that being "controlled" is an inherited property (in the sense of the comments following Definition 5.4).

By Bernstein's and Hölder's inequalities, we have

$$2^j \int_0^T E_{\boldsymbol{e}}(v)\, dt \lesssim 2^{2j} T \|v\|^2_{L_t^{\infty} L_x^2}.$$

for any v. For fixed $T > 0$ and $k \in \mathbb{Z}$, let $K_{T,k} \geq 1$ be the best constant for which the inequality

$$2^j \int_0^T E_{\boldsymbol{e}}(v)\, dt \leq K_{T,k}\big(\|v\|^2_{L_x^2} + B_{\boldsymbol{e}}(v, \Lambda_v)\big) \tag{5-19}$$

holds for all controlled sequences. Applying (5-19) to $\tilde{v}$ results in

$$2^j \int_0^T E_{\boldsymbol{e}}(\tilde{v})\, dt \leq K_{T,k}\big(\|\tilde{v}\|^2_{L_x^2} + B_{\boldsymbol{e}}(\tilde{v}, \Lambda_{\tilde{v}})\big), \tag{5-20}$$

and thus we seek to control norms of $\tilde{v}$ in terms of those of v.

Let $\widetilde{P}_k$, $\widetilde{P}_{j,\boldsymbol{e}}$ denote slight fattenings of the Fourier multipliers P_k, $P_{j,\boldsymbol{e}}$. On the one hand, Plancherel implies

$$\|(1+i2^{-j}\partial_{\boldsymbol{e}})\widetilde{P}_{j,\boldsymbol{e}}\widetilde{P}_k\|_{L_x^2\to L_x^2}\lesssim \eta. \tag{5-21}$$

On the other hand, Bernstein's inequalities imply

$$\|(1+i2^{-j}\partial_{\boldsymbol{e}})\widetilde{P}_{j,\boldsymbol{e}}\widetilde{P}_k\|_{L_x^p\to L_x^p}\lesssim 1,\quad 1\le p\le\infty.$$

Therefore it follows from Riesz–Thorin interpolation that

$$\|(1+i2^{-j}\partial_{\boldsymbol{e}})\widetilde{P}_{j,\boldsymbol{e}}\widetilde{P}_k\|_{L_x^p\to L_x^p}\lesssim \begin{cases}\eta^{2/p} & 2\le p<\infty,\\ \eta^{2-2/p} & 1<p\le 2.\end{cases}$$

Restricting to $p\in\mathcal{P}$, we conclude that there exists a $q>0$ such that

$$\|(1+i2^{-j}\partial_{\boldsymbol{e}})\widetilde{P}_{j,\boldsymbol{e}}\widetilde{P}_k\|_{L_x^p\to L_x^p}\lesssim \eta^q \tag{5-22}$$

for all $p\in\mathcal{P}$ and all η small enough.

Applying (5-22) and Bernstein to $\tilde{v}$ yields

$$\|\tilde{v}\|_{L_x^2}\lesssim \eta^q\|v\|_{L_x^2},\qquad B_{\boldsymbol{e}}(\tilde{v},\Lambda_{\tilde{v}})\lesssim \eta^q B_{\boldsymbol{e}}(v,\Lambda_v),$$

which, combined with (5-20) and (5-18), leads to

$$2^j\int_0^T E_{\boldsymbol{e}}(v)\,dt\lesssim (1+\eta^q K_{T,k})\big(\|v\|^2_{L_t^\infty L_x^2}+B_{\boldsymbol{e}}(v,\Lambda_v)\big).$$

As $K_{T,k}$ is the best constant for which (5-19) holds, it follows that

$$K_{T,k}\lesssim 1+\eta^q K_{T,k}$$

and hence that $K_{T,k}\lesssim 1$ for η small enough. □

Corollary 5.6. *Given the assumptions of Theorem 5.5, we have*

$$2^j\int_0^T E_{\boldsymbol{e}}(v)\,dt\lesssim \|v_0\|^2_{L_x^2}+B(v,\Lambda_v)+B_{\boldsymbol{e}}(v,\Lambda_v).$$

Proof. This is an immediate consequence of Theorem 5.5 and Lemma 5.1. □

Corollary 5.7 (abstract bilinear Strichartz). *Let $d\ge 1$ and $\boldsymbol{e}\in\mathbb{S}^{d-1}$. Set $\tilde{\boldsymbol{e}}=(-\boldsymbol{e},\boldsymbol{e})/\sqrt{2}$. Let $j,k\in\mathbb{Z}$ and $j=k+O(1)$. Let $\varepsilon_m>0$ be a small positive number such that $\varepsilon_m 2^{O(1)}\ll 1$. Let $\eta>0$. Let $\mathcal{P}$ be a fixed finite subset of $(1,\infty)$ with $2\in\mathcal{P}$, and let $B_{\tilde{\boldsymbol{e}}}$ be a form that is adapted to $\mathcal{P}$.*

Let $w(x,y)$ be a $C_t^\infty(H^\infty_{x,y})$ function on $\mathbb{R}^{2d}\times[0,T]$, equal to w_0 at $t=0$ and solving

$$(i\partial_t+\Delta_{\mathcal{A}})w=\Lambda_w,$$

where $\mathcal{A}_{k'}$ is real-valued, smooth, has spatial Fourier support in $I_{(-\infty,k]}$, and satisfies the estimate

$$\|\mathcal{A}\|_{L^\infty_{t,x,y}}\le \varepsilon_m 2^k.$$

Assume w has (spatial) frequency support in I_k and that

$$\tilde{\boldsymbol{e}}\cdot\xi\in[(1-\eta)2^j,(1+\eta)2^j]$$

for all ξ *in the support of* $\widehat{w}$*. Take* Λ_w *to be frequency-localized to* $I_{(-\infty,k]}$*.*

Suppose that $w(x,y)$ *admits a decomposition* $w(x,y)=u(x)v(y)$*, where* u *has frequency support in* I_l*,* $l\ll k$*. Use* u_0*,* v_0 *to denote* $u(t=0)$*,* $v(t=0)$*. If the derived sequence of* w *is controlled with respect to* $B_{\tilde{e}}$*, then*

$$\|uv\|^2_{L^2_{t,x}}\lesssim 2^{l(d-1)}2^{-j}\big(\|u_0\|^2_{L^2_x}\|v_0\|^2_{L^2_x}+B(w,\Lambda_w)+B_{\tilde{e}}(w,\Lambda_w)\big) \tag{5-23}$$

uniformly in T *and* $j=k+O(1)$ *provided* η *is small enough.*

Proof. Taking into account that

$$\|w_0\|_{L^2_{x,y}}=\|u_0\|_{L^2_x}\|v_0\|_{L^2_x},$$

we apply Corollary 5.6 to w at $(x,y)=0$ and get

$$2^j\int_0^T E_{\tilde{e}}(w)\,dt\lesssim\|u_0\|^2_{L^2_x}\|v_0\|^2_{L^2_x}+B(w,\Lambda_w)+B_{\tilde{e}}(w,\Lambda_w). \tag{5-24}$$

We complete $(-\boldsymbol{e},\boldsymbol{e})/\sqrt{2}$ to a basis as follows:

$$(-\boldsymbol{e},\boldsymbol{e})/\sqrt{2},\ (0,\boldsymbol{e}_1),\ \dots,\ (0,\boldsymbol{e}_{d-1}),\ (\boldsymbol{e},\boldsymbol{e})/\sqrt{2},\ (\boldsymbol{e}_1,0),\ \dots,\ (\boldsymbol{e}_{d-1},0).$$

On the one hand, $E_{\tilde{e}}(w)(0)$ is by definition (see (5-2)) equal to

$$\int_{\mathbb{R}}\int_{\mathbb{R}^{2d-2}}|u(0\cdot\boldsymbol{e}+r\boldsymbol{e}+x_j\boldsymbol{e}_j,t)v(0\cdot\boldsymbol{e}+r\boldsymbol{e}+y_j\boldsymbol{e}_j,t)|^2\,dx'dy'dr.$$

We rewrite it as

$$\int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}|v(r\boldsymbol{e}+y_j\boldsymbol{e}_j,t)|^2dy'\int_{\mathbb{R}^{d-1}}|u(r\boldsymbol{e}+x_j\boldsymbol{e}_j,t)|^2\,dx'dr. \tag{5-25}$$

On the other hand,

$$\|uv\|^2_{L^2_y}=\int_{\mathbb{R}^d}|u(y,t)|^2|v(y,t)|^2dy=\int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}|u(r\boldsymbol{e}+y_j\boldsymbol{e}_j)|^2|v(r\boldsymbol{e}+y_j\boldsymbol{e}_j)|^2dy'dr,$$

and by applying Bernstein to u in the y' variables, we obtain

$$\|uv\|^2_{L^2_y}\lesssim 2^{l(d-1)}\int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}|v(r\boldsymbol{e}+y_j\boldsymbol{e}_j)|^2dy'\int_{\mathbb{R}^{d-1}}|u(r\boldsymbol{e}+x_j\boldsymbol{e}_j)|^2\,dx'dr. \tag{5-26}$$

Together (5-26), (5-25), and (5-24) imply (5-23). □

5B. ***Applying the abstract lemmas.*** We would like to apply the abstract estimates just developed to the evolution equation (2-7). We work in the caloric gauge and adopt the magnetic potential decomposition introduced in Section 4D. Throughout we take ϵ as defined in (4-18).

Our starting point is the equation

$$(i\partial_t+\Delta)\psi_m=B_{m,\mathrm{lo}\wedge\mathrm{lo}}+B_{m,\mathrm{hi}\vee\mathrm{hi}}+V_m. \tag{5-27}$$

Applying Fourier multipliers P_k, $P_{j,\theta}P_k$, or variants thereof, we easily obtain corresponding evolution equations for $P_k\psi_m$, $P_{j,\theta}P_k$, etc. In rewriting a projection P of (5-27) in the form (5-3), evidently $\Delta_{\mathcal{A}}\psi_m$ should somehow come from $\Delta P\psi_m-PB_{m,\mathrm{lo}\wedge\mathrm{lo}}$, whereas $PB_{m,\mathrm{hi}\vee\mathrm{hi}}+PV_m$ ought to constitute the leading part of the nonlinearity Λ. Fourier multipliers P, however, do not commute with the connection coefficients A, and therefore in order to use the abstract machinery we must first track and control certain commutators. Toward this end we adopt some notation from [Tao 2001].

Following [Tao 2001, §1], we use $L_O(f_1, \dots, f_m)(s, x, t)$ to denote any multilinear expression of the form

$$L_O(f_1, \dots, f_m)(s, x, t) := \int K(y_1, \dots, y_{M(c)}) f_1(s, x - y_1, t) \dots f_m(s, x - y_{M(c)}, t)\, dy_1 \dots dy_{M(c)},$$

where the kernel K is a measure with bounded mass (and K may change from line to line). Moreover, the kernel of L_O does not depend upon the index α. Also, we extend this notation to vector or matrices by making K into an appropriate tensor. The expression $L_O(f_1, \dots, f_m)$ may be thought of as a variant of $O(f_1, \dots, f_m)$. It obeys two key properties. The first is a simple consequence of Minkowski's inequality; see, for example, [Tao 2001, Lemma 1].

Lemma 5.8. *Let $X_1, \dots, X_m$, X be spatially translation-invariant Banach spaces such that the product estimate*

$$\|f_1 \cdots f_m\|_X \le C_0 \|f_1\|_{X_1} \cdots \|f_m\|_{X_m}$$

holds for all scalar-valued $f_i \in X_i$ and for some constant $C_0 > 0$. Then

$$\|L_O(f_1, \dots, f_m)\|_X \lesssim (Cd)^{Cm} C_0 \|f_1\|_{X_1} \cdots \|f_m\|_{X_m}$$

holds for all $f_i \in X_i$ that are scalars, d-dimensional vectors, or $d \times d$ matrices.

The next lemma is an adaptation of Lemma 2 in [Tao 2001].

Lemma 5.9 (Leibniz rule). *Let P'_k be a C^∞ Fourier multiplier whose frequency support lies in some compact subset of $I_k(\mathbb{R}^d)$. The commutator identity*

$$P'_k(fg) = f P'_k g + L_O(\partial_x f, 2^{-k} g)$$

holds.

Proof. Rescale so that $k = 0$ and let $m(\xi)$ denote the symbol of P'_0 so that

$$\widehat{P'_0 h}(\xi) := m(\xi)\hat{h}(\xi).$$

By the fundamental theorem of calculus, we have

$$\begin{aligned}\left(P'_0(fg) - f P'_0 g\right)(s, x, t) &= \int_{\mathbb{R}^d} \check{m}(y)(f(s, x - y, t) - f(s, x, t)) g(s, x - y, t)\, dy \\ &= -\int_0^1 \int_{\mathbb{R}^d} \check{m}(y) y \cdot \partial_x f(s, x - ry, t) g(s, x - y, t)\, dy dr.\end{aligned}$$

The conclusion follows from the rapid decay of $\hat{m}$. □

We are interested in controlling $P_{\theta,j} P_k \psi_m$ in $L^{\infty,2}_\theta$ over all $\theta \in \mathbb{S}^1$ and $|j-k| \leq 20$. In the abstract framework, however, we assumed a much tighter localization than $P_{\theta,j}$ provides. Therefore we decompose $P_{\theta,j}$ as a sum

$$P = \sum_{l=1,\dots,O((\eta^*)^{-1})} P_{\theta,j,l}, \tag{5-28}$$

and it suffices by the triangle inequality to bound $P_{\theta,j,l} P_k \psi_m$. We note that this does not affect perturbative estimates since η^* is universal and in particular does not depend upon ε_1, ε.

For notational convenience set $P := P_{\theta,j,l} P_k$. Applying P to (5-27) yields

$$(i\partial_t + \Delta) P\psi_m = P\left(B_{m,\mathrm{lo}\wedge\mathrm{lo}} + B_{m,\mathrm{hi}\vee\mathrm{hi}} + V_m\right).$$

Now

$$P B_{m,\mathrm{lo}\wedge\mathrm{lo}} = -iP \sum_{|k_3-k|\leq 4} (\partial_l(A_{l,\mathrm{lo}\wedge\mathrm{lo}} P_{k_3}\psi_m) + A_{l,\mathrm{lo}\wedge\mathrm{lo}}\partial_l P_{k_3}\psi_m),$$

as P localizes to a region of the annulus I_k. Applying Lemma 5.9, we obtain

$$P B_{m,\mathrm{lo}\wedge\mathrm{lo}} = -i(\partial_l(A_{l,\mathrm{lo}\wedge\mathrm{lo}} P\psi_m) - iA_{l,\mathrm{lo}\wedge\mathrm{lo}}\partial_l P\psi_m) + R,$$

where

$$R := \sum_{|k_3-k|\leq 4} \left(L_O(\partial_x\partial_l A_{l,\mathrm{lo}\wedge\mathrm{lo}}, 2^{-k} P_{k_3}\psi_m) + L_O(\partial_x A_{l,\mathrm{lo}\wedge\mathrm{lo}}, 2^{-k} P_{k_3}\partial_l\psi_m)\right). \tag{5-29}$$

Set

$$\mathcal{A}_m := A_{m,\mathrm{lo}\wedge\mathrm{lo}}.$$

Then

$$(i\partial_t + \Delta_{\mathcal{A}}) P\psi_m = P(B_{m,\mathrm{hi}\vee\mathrm{hi}} + V_m) + \mathcal{A}_x^2 P\psi_m + R. \tag{5-30}$$

It is this equation that we shall show fits within the abstract local smoothing framework.

First we check that Lemmas 5.1 and 5.2 apply. The main condition to check is (5-6). Key are the bound (2-14) and Bernstein, which together with the fact that $\mathcal{A}$ is frequency-localized to $I_{(-\infty,k]}$ provide the estimate

$$\|\mathcal{A}\|_{L^\infty_{t,x}} \lesssim 2^k.$$

To achieve the ε_m gain, we adjust ϖ, which forces a gap between I_k and the frequency support of $\mathcal{A}$, i.e., we localize $\mathcal{A}$ to $I_{(-\infty,k-\varpi]}$ instead. Thus it suffices to set $\varpi \in \mathbb{Z}_+$ equal to a sufficiently large universal constant.

There is more to check in showing that (5-30) falls within the purview of Theorem 5.5. Already we have $d = 2$, $\boldsymbol{e} = \theta$, $\varepsilon_m \sim 2^{-\varpi}$, $\mathcal{A}_m := A_{m,\mathrm{lo}\wedge\mathrm{lo}}$, $v = P_{\theta,j,l} P_k \psi_m$, and $\Lambda_v = P(B_{m,\mathrm{hi}\vee\mathrm{hi}} + V_m) + \mathcal{A}_x^2 P\psi_m + R$.

Next we choose $\mathcal{P}$ based upon the norms used in N_k, with the exception of the local smoothing/maximal function estimates. To be precise, define the new norms $\widetilde{N}_k$ via

$$\|f\|_{\widetilde{N}_k(T)} := \inf_{f=f_1+f_2+f_3+f_4+f_5} \|f_1\|_{L^{4/3}_{t,x}} + 2^{k/6}\|f_2\|_{L^{3/2,6/5}_{\hat{\theta}_1}} + 2^{k/6}\|f_3\|_{L^{3/2,6/5}_{\hat{\theta}_2}} + 2^{-k/6}\|f_4\|_{L^{6/5,3/2}_{\hat{\theta}_1}} + 2^{-k/6}\|f_5\|_{L^{6/5,3/2}_{\hat{\theta}_2}}$$

and similarly $\widetilde{G}_k$ via

$$\|f\|_{\widetilde{G}_k(T)} := \|f\|_{L^\infty_t L^2_x} + \|f\|_{L^4_{t,x}} + 2^{-k/2}\|f\|_{L^4_x L^\infty_t} + 2^{-k/6}\sup_{\theta\in\mathbb{S}^1}\|f\|_{L^{3,6}_\theta} + 2^{k/6}\sup_{|j-k|\le 20}\sup_{\theta\in\mathbb{S}^1}\|P_{j,\theta}f\|_{L^{6,3}_\theta}.$$

Set $\mathcal{P} = \{2, 3, 3/2, 4, 4/3, 6, 5/6\}$. We define the form $B(\cdot,\cdot)$ via

$$B(f, g) := \|f\|_{\widetilde{G}_k(T)}\|g\|_{\widetilde{N}_k(T)} \tag{5-31}$$

and B_θ by

$$B_\theta(f, g) := B(f, \chi_{\{x\cdot\theta\ge 0\}}g) \tag{5-32}$$

as in Definition 5.3. That B_θ is adapted to $\mathcal{P}$ is a direct consequence of the definition.

Proposition 5.10. *Let $\eta > 0$ be a parameter to be specified later. Let also $d = 2$, $\boldsymbol{e} = \theta$, $\varepsilon_m \sim 2^{-\varpi}$, $\mathcal{A}_m := A_{m,\mathrm{lo}\wedge\mathrm{lo}}$, $v = P^{(\eta)}_{\theta,j,l}P_k\psi_m$, $\Lambda_v = P(B_{m,\mathrm{hi}\vee\mathrm{hi}}+V_m)+\mathcal{A}^2_x P\psi_m+R$, and $\mathcal{P} = \{2, 3, 3/2, 4, 4/3, 6, 5/6\}$. Let B, B_θ be given by (5-31) and (5-32) respectively. Then the conditions of Theorem 5.5 are satisfied and the derived sequence of v is controlled with respect to B_θ so that conclusion (5-17) holds for $v = P^{(\eta)}_{\theta,j,l}P_k\psi_m$ given η sufficiently small.*

Proof. The only claim of Proposition 5.10 that remains to be verified is that the derived sequence of $v = P_{\theta,j,l}P_k\psi_m$ is controlled with respect to B_θ. In particular, we need to show that for each $q \ge 1$ we have

$$B_\theta(v^{(q)}, \Lambda_{v^{(q)}}) < \infty,$$

where $v^{(1)} := P_{\theta,j,l}P_k\psi_m$,

$$v^{(q+1)} := v^{(q)} + i2^{-j}\partial_\theta v^{(q)},$$

and

$$\Lambda_{v^{(q+1)}} := (1 + i2^{-j}\partial_\theta)\Lambda_{v^{(q)}} + i2^{-j}(i\partial_\theta\partial_l\mathcal{A}_l - \partial_\theta\mathcal{A}^2_l)v^{(q)} - 2^{-j+1}(\partial_\theta\mathcal{A}_l)\partial_l v^{(q)}.$$

We first prove the following lemma.

Lemma 5.11. *Let $\sigma \in [0, \sigma_1 - 1]$. The right-hand side of (5-30) satisfies*

$$\|P(B_{m,\mathrm{hi}\vee\mathrm{hi}} + V_m) + \mathcal{A}^2_x P\psi_m + R\|_{\widetilde{N}_k(T)} \lesssim \epsilon 2^{-\sigma k}b_k(\sigma).$$

Proof. We will repeatedly use implicitly the fact that the multiplier $P_{\theta,j,l}$ is bounded on L^p, $1 \le p \le \infty$, so that in particular P obeys estimates that are at least as good as those obeyed by P_k.

From Corollaries 4.11 and 4.12 of Sections 4C and 4D it follows that $P_k(B_{m,\mathrm{hi}\vee\mathrm{hi}} + V_m)$ is perturbative and bounded in $\widetilde{N}_k(T)$ by $\epsilon 2^{-\sigma k}b_k(\sigma)$. The $\widetilde{N}_k(T)$ estimates on PV_m immediately imply the boundedness of $\mathcal{A}^2_x P\psi_m$.

To estimate R, we apply Lemma 3.10 to bound $PB_{m,\mathrm{lo}\wedge\mathrm{lo}}$ by

$$\sum_{(k_1,k_2,k_3)\in Z_1(k)} \int_0^\infty 2^{\max\{k_1,k_2\}} 2^{k_3-k} C_{k,k_1,k_2,k_3} \|P_{k_1}\psi_x(s)\|_{F_{k_1}} \|P_{k_2}(D_l\psi_l(s))\|_{F_{k_2}} \|P_{k_3}\psi_m(0)\|_{G_{k_3}}\, ds,$$

which, in view of (4-12), (4-13), and (4-25), is controlled by

$$\sum_{(k_1,k_2,k_3)\in Z_1(k)} C_{k,k_1,k_2,k_3} b_{k_1} b_{k_2} 2^{-\sigma k_3} b_{k_3}(\sigma).$$

Summation is achieved thanks to Corollary 3.11. □

We return to the proof of the proposition, and in particular to showing that $B_\theta(v,\Lambda_v)<\infty$. With the important observation that the spatial multiplier $\chi_{x\cdot\theta\geq 0}$ is bounded on the spaces $\widetilde{N}_k(T)$, we may apply Lemma 5.11 to control $\chi_{x\cdot\theta\geq 0}\Lambda_v$ in $\widetilde{N}_k$. Since by assumption $P\psi_m$ is bounded in $\widetilde{G}_k(T)$ (even in $G_k(T)$), we conclude that $B_\theta(v,\Lambda_v)<\infty$.

Next we need to show $B_\theta(v^q,\Lambda_{v^q})<\infty$ for $q>1$. By Bernstein,

$$\|v^{(q)}\|_{\widetilde{G}_k(T)} \lesssim \|v^{(q-1)}\|_{\widetilde{G}_k(T)}.$$

Similarly,

$$\|(1+i2^{-j})\partial_\theta \Lambda_{v^{(q)}}\|_{\widetilde{N}_k(T)} \lesssim \|\Lambda_{v^{(q-1)}}\|_{\widetilde{N}_k(T)}.$$

Thus it remains to control $i2^{-j}(i\partial_\theta\partial_l\mathcal{A}_l-\partial_\theta\mathcal{A}_l^2)v^{(q)}$ and $2^{-j+1}(\partial_\theta\mathcal{A}_l)\partial_l v^{(q)}$ in $\widetilde{N}_k$ for each $q>1$. Both are consequences of arguments in Lemma 5.11: Boundedness of $2^{-j}(\partial_\theta\partial_l\mathcal{A}_l)v^{(q)}$ and $2^{-j+1}(\partial_\theta\mathcal{A}_l)\partial_l v^{(q)}$ follows directly from the argument used to control R and from Bernstein's inequality, whereas boundedness of $2^{-j}(\partial_\theta\mathcal{A}_l^2)v^{(q)}$ is a consequence of Bernstein and the estimates on $\mathcal{A}_x^2 P\psi_m$ from Section 4C. □

Combining Lemma 5.11 and Proposition 5.10, we conclude that Corollary 5.6 applies to $v=P\psi_m$, with right-hand side bounded by $2^{-2\sigma k}c_k(\sigma)^2+\epsilon 2^{-2\sigma k}b_k(\sigma)^2$. In view of the decomposition (5-28), we conclude this:

Corollary 5.12. *Assume $\sigma\in[0,\sigma_1-1]$. The function $P_k\psi_m$ satisfies*

$$\sup_{|j-k|\leq 20}\sup_{\theta\in\mathbb{S}^1}\|P_{j,\theta}P_k\psi_m\|_{L_\theta^{\infty,2}} \lesssim 2^{-k/2}(2^{-\sigma k}c_k(\sigma)+\epsilon^{1/2}2^{-\sigma k}b_k(\sigma)).$$

This proves Theorem 4.9.

Our next objective is to apply Corollary 5.7 to the case where w splits as a product $u(x)v(y)$ where u, v are appropriate frequency localizations of ψ_m or $\overline{\psi_m}$. First we must find function spaces suitable for defining an adapted form. We start with $(i\partial_t+\Delta_{\mathcal{A}})w=\Lambda_w$ and observe how it behaves with respect to separation of variables. If $w(x,y)=u(x)v(y)$, then the left-hand side may be rewritten as $u\cdot(i\partial_t+\Delta_{\mathcal{A}_y})v+v\cdot(i\partial_t+\Delta_{\mathcal{A}_x})u$. Let $\Lambda_u:=(i\partial_t+\Delta_{\mathcal{A}_x})u$ and $\Lambda_v:=(i\partial_t+\Delta_{\mathcal{A}_y})v$. Then

$$(i\partial_t+\Delta_{\mathcal{A}})(uv)=u\Lambda_v+v\Lambda_u.$$

We control

$$\int_0^T\int_{\mathbb{R}^2\times\mathbb{R}^2} u(x)v(y)(\Lambda_u(x)v(y)+u(x)\Lambda_v(y))\,dxdydt$$

as follows: in the case of the first term $u(x)v(y)\Lambda_u(x)v(y)$ we place each $v(y)$ in $L_t^\infty L_y^2$; we bound $u(x)\Lambda_u(x)$ by placing $u(x)$ in G_j and $\Lambda_u(x)$ in $\widetilde{N}_j$. To control $u(x)v(y)u(x)\Lambda_v(y)$, we simply reverse the roles of u and v (and of x and y). This leads us to the spaces $\overline{N}_{k,l}$ defined by

$$\|f\|_{\overline{N}_{k,l}(T)} := \inf\Big\{ \|g_{2j-1}\|_{\widetilde{N}_l(T)} \|h_{2j-1}\|_{L_t^\infty L_y^2} + \|g_{2j}\|_{L_t^\infty L_x^2} \|h_{2j}\|_{\widetilde{N}_k(T)} : J \in \mathbb{Z}_+ \text{ and } f(x,y) = \sum_{j=1}^{2J} \big(g_{2j-1}(x)h_{2j-1}(y) + g_{2j}(x)h_{2j}(y)\big) \Big\}, \tag{5-33}$$

and the spaces $\overline{G}_{k,l}$ defined via

$$\|f\|_{\overline{G}_{k,l}(T)} := \big\| \|f(x,y)\|_{\widetilde{G}_k(T)(y)} \big\|_{\widetilde{G}_l(T)(x)}. \tag{5-34}$$

We use these spaces to define the form $\overline{B}(\cdot,\cdot)$ by

$$\overline{B}(f,g) := \|f\|_{\overline{G}_{k,l}(T)} \|g\|_{\overline{N}_{k,l}(T)}, \tag{5-35}$$

and the form $\overline{B}_\Theta$ by

$$\overline{B}_\Theta(f,g) := \overline{B}(f, \chi_{\{(x,y)\cdot\Theta\ge 0\}} g), \tag{5-36}$$

where $\Theta := (-\theta, \theta)$.

Proposition 5.13. *Let $\eta > 0$ be a small parameter and $\varpi \in \mathbb{Z}_+$ a large parameter, both to be specified later. Let $j, k, l \in \mathbb{Z}$, $j = k + O(1)$, $l \ll k$. Let $d = 2$, $\boldsymbol{e} = \theta$, $\varepsilon_m \sim 2^{-\varpi}$, $\mathcal{A}_x := A_{m,\mathrm{lo}\wedge\mathrm{lo}}$, $v = P_{\theta,j,l}^{(\eta)} P_k \psi_m$, $\Lambda_v = P(B_{m,\mathrm{hi}\vee\mathrm{hi}} + V_m) + \mathcal{A}_x^2 P\psi_m + R$, and $\mathcal{P} = \{2, 3, 3/2, 4, 4/3, 6, 5/6\}$. Here R is given by (5-29). Also, let $u = \overline{P_l\psi_p}$, $p \in \{1,2\}$ and $\Lambda_u = P_l(B_{p,\mathrm{hi}\vee\mathrm{hi}} + V_p) + \mathcal{A}_x^2 P_l\psi_p + R'$, where R' is given by (5-29), but defined in terms of derivative field ψ_l and frequency l rather than ψ_m and k.*

Let $w(x,y) := u(x)v(y)$, $\mathcal{A} := (\mathcal{A}_x, \mathcal{A}_y)$, $\Lambda_w := \Lambda_u v + u\Lambda_v$. Then, for ϖ sufficiently large and η sufficiently small, the conditions of Corollary 5.7 are satisfied and (5-23) applies to $u(x)v(x)$.

Proof. The frequency support conditions on $\mathcal{A}$ and Λ_w are easily verified. That the L^∞ bound on $\mathcal{A}$ holds follows from (2-14) and Bernstein provided ϖ is large enough (see the discussion preceding Proposition 5.10). In order to guarantee the frequency support conditions on w, it is necessary to make the gap $l \ll k$ sufficiently large with respect to η.

That $\overline{B}_\Theta$ is adapted to $\mathcal{P}$ is a straightforward consequence of its definition. To see that the derived sequence of w is controllable, we look to the proof of Proposition 5.10 and the definitions of the $\overline{N}_{k,l}$, $\overline{G}_{k,l}$ spaces. □

In a spirit similar to that of the proof of Corollary 5.12, we may combine Lemma 5.11 and the proof of Proposition 5.10 to control $B(w, \Lambda_w) + B_\Theta(w, \Lambda_w)$; in fact, in measuring Λ_w in the $\overline{N}_{k,l}$ spaces, it suffices to take $J = 1$ (see (5-33)). Then we obtain $B(w, \Lambda_w) + B_\Theta(w, \Lambda_w) \lesssim \epsilon b_j 2^{-\sigma k} b_k(\sigma)$. Using decomposition (5-28) and the triangle inequality to bound $P_k\psi_m$ in terms of the bounds on $P_{\theta,j,l}^{(\eta)} P_k\psi_m$, we obtain the bilinear Strichartz analogue of Corollary 5.12. In our application, however, the lower-frequency term will not simply be $\overline{P_j\psi_l}$, but rather its heat flow evolution $\overline{P_j\psi_l(s)}$.

Corollary 5.14 (improved bilinear Strichartz). *Let $j, k \in \mathbb{Z}$, $j \ll k$, and let*

$$u \in \{P_j\psi_l, \overline{P_j\psi_l} : j \le k - \varpi, l \in \{1, 2\}\}.$$

Then for $s \ge 0$, $\sigma \in [0, \sigma_1 - 1]$,

$$\|u(s)P_k\psi_m(0)\|_{L^2_{t,x}} \lesssim 2^{(j-k)/2}(1+s2^{2j})^{-4}2^{-\sigma k}\left(c_j c_k(\sigma) + \epsilon b_j b_k(\sigma)\right). \tag{5-37}$$

Proof. It only remains to prove (5-37) when $s > 0$. Let $v := P_k\psi_m$. Using the Duhamel formula, we write

$$u(s)v = \left(e^{s\Delta}u(0)\right)v(0) + \int_0^s e^{(s-s')\Delta}U(s')\,ds' \cdot v(0), \tag{5-38}$$

where U is defined by (2-21) in terms of u.

To control the nonlinear term $\int_0^s e^{(s-s')\Delta}U(s')\,ds' \cdot v(0)$ in L^2, we apply local smoothing estimate (3-11), which places the nonlinear evolution in $F_j(T)$ and $v(0)$ in $G_k(T)$. Using Lemma 7.11 to bound the $F_j(T)$ norm, we conclude that

$$\left\|\int_0^s e^{(s-s')\Delta}\widetilde{U}(s')\,ds' \cdot v(0)\right\|_{L^2_{t,x}} \lesssim \epsilon 2^{(j-k)/2}(1+s2^{2j})^{-4}2^{-\sigma k}b_j b_k(\sigma). \tag{5-39}$$

It remains to show that

$$\|(e^{s\Delta}u)v\|_{L^2_{t,x}} \lesssim (1+s2^{2j})^{-4}2^{(j-k)/2}2^{-\sigma k}(c_j c_k(\sigma) + \epsilon b_j b_k(\sigma)), \tag{5-40}$$

which is not a direct consequence of the time $s = 0$ bound. Let $\mathcal{T}_a$ denote the spatial translation operator that acts on functions $f(x,t)$ according to $\mathcal{T}_a f(x,t) := f(x-a,t)$. If

$$\|(\mathcal{T}_{x_1}u)(\mathcal{T}_{x_2}v)\|_{L^2_{t,x}} \lesssim 2^{(j-k)/2}2^{-\sigma k}(c_j c_k(\sigma) + \epsilon b_j b_k(\sigma)) \tag{5-41}$$

can be shown to hold for all $x_1, x_2 \in \mathbb{R}^2$, then (5-40) follows from Minkowski's and Young's inequalities.

Consider, then, a solution w to

$$(i\partial_t + \Delta_{\mathcal{A}}(x,t))w(x,t) = \Lambda_w(x,t)$$

satisfying the conditions of Theorem 5.5. The translate $\mathcal{T}_{x_0}w(x,t)$ then satisfies

$$(i\partial_t + \Delta_{\mathcal{T}_{x_0}(\mathcal{A})(x,t)})(\mathcal{T}_{x_0}w)(x,t) = (\mathcal{T}_{x_0}\Lambda_w)(x,t).$$

The operator $\mathcal{T}_{x_0}$ clearly does not affect $L^\infty_{t,x}$ bounds or frequency support conditions. The only possible obstruction to concluding (5-17) is this: whereas the derived sequence of w is controlled with respect to B_e, in the abstract setting it may no longer be the case that the derived sequence of $\mathcal{T}_{x_0}w$ is controlled. This is due to the presence of the spatial multiplier in the definition of B_e. Fortunately, as already alluded to in the proof of Proposition 5.10, in our applications we do enjoy uniform boundedness with respect to *any* spatial multipliers appearing in the second argument of an adapted form B_e. Therefore Proposition 5.13 holds for spatial translates of frequency projections of ψ_m, from which we conclude (5-41). □

This establishes Theorem 4.8.

6. The caloric gauge

In Section 6A we briefly recall from [Smith 2012a] the construction of the caloric gauge and some useful quantitative estimates. In Section 6B we prove the frequency-localized estimates stated in Section 2C.

6A. ***Construction and basic results.*** In brief, the basic caloric gauge construction goes as follows. Starting with H^∞_Q-class data $\varphi_0 : \mathbb{R}^2 \to \mathbb{S}^2$ with energy $E(\varphi_0) < E_{\text{crit}}$, evolve φ_0 in s via the heat flow equation (2-11). At $s = \infty$ the map trivializes. Place an arbitrary orthonormal frame $e(\infty)$ on $T_{\varphi(s=\infty)}\mathbb{S}^2$. Evolving this frame backward in time via parallel transport in the s direction yields a caloric gauge on $\varphi^* T_{\varphi(s=\infty)}\mathbb{S}^2$.

For energies $E(\varphi_0)$ sufficiently small, global existence and decay bounds may be proven directly using Duhamel's formula. In order to extend these results to all energies less than E_{crit}, we employ in [Smith 2012a] a concentration compactness argument that exploits the symmetries of (2-11) via concentration compactness.

In [Smith 2012a] the following energy densities play an important role in the quantitative arguments.

Definition 6.1. For each positive integer k, define the energy densities $\boldsymbol{e}_k$ of a heat flow φ by

$$\begin{aligned} \boldsymbol{e}_k &:= |(\varphi^*\nabla)_x^{k-1}\partial_x\varphi|^2 \\ &:= \langle (\varphi^*\nabla)_{j_1}\dots(\varphi^*\nabla)_{j_{k-1}}\partial_{j_k}\varphi, (\varphi^*\nabla)_{j_1}\dots(\varphi^*\nabla)_{j_{k-1}}\partial_{j_k}\varphi\rangle, \end{aligned} \tag{6-1}$$

where $j_1, \dots, j_k$ are summed over 1, 2 and ∇ denotes the Riemannian connection on the sphere, i.e., for vector fields X, Y on the sphere $\nabla_X Y$ denotes the orthogonal projection of $\partial_X Y$ onto the sphere.

Theorem 6.2 [Smith 2012a]. *For any initial data $\varphi_0 \in H^\infty_Q$ with $E(\varphi_0) < E_{\text{crit}}$ there exists a unique global smooth heat flow φ with initial data φ_0. Moreover, φ satisfies the estimates*

$$\int_0^\infty \int_{\mathbb{R}^2} s^{k-1}\boldsymbol{e}_{k+1}(s,x)\,dx\,ds \lesssim_{E_0,k} 1, \tag{6-2}$$

$$\sup_{0<s<\infty} s^{k-1}\int_{\mathbb{R}^2}\boldsymbol{e}_k(s,x)\,dx \lesssim_{E_0,k} 1,$$

$$\sup_{\substack{0<s<\infty\\ x\in\mathbb{R}^2}} s^k\boldsymbol{e}_k(s,x) \lesssim_{E_0,k} 1,$$

$$\int_0^\infty s^{k-1}\sup_{x\in\mathbb{R}^2}\boldsymbol{e}_k(s,x)\,ds \lesssim_{E_0,k} 1, \tag{6-3}$$

for each $k \geq 1$, as well as the estimate

$$\int_0^\infty\int_{\mathbb{R}^2}\boldsymbol{e}_1^2(s,x)\,dx\,ds \lesssim_{E_0} 1. \tag{6-4}$$

We employ (6-2), (6-3), and (6-4) below.

6B. ***Frequency-localized caloric gauge estimates.*** The key estimate to establish is (2-35) for φ; most of the remaining estimates will be derived as corollaries of it. Our strategy is to exploit energy dispersion so that we can apply the Duhamel formula to a frequency localization of the heat flow equation (2-11), which for convenience we rewrite as

$$\partial_s \varphi = \Delta\varphi + \varphi \boldsymbol{e}_1. \tag{6-5}$$

Proof of (2-35) *for* φ. Let $\sigma_1 \in \mathbb{Z}_+$ be positive and let $S' \geq S \gg 0$. Let $\mathcal{K} \in \mathbb{Z}_+$, $T \in (0, 2^{2\mathcal{K}}]$ be fixed. Define for each $t \in (-T, T)$ the quantity

$$\mathscr{C}(S,t) := \sup_{\sigma\in[2\delta,\sigma_1]} \sup_{s\in[0,S]} \sup_{k\in\mathbb{Z}} (1+s2^{2k})^{\sigma_1} 2^{\sigma k} \gamma_k(\sigma)^{-1} \| P_k \varphi(s,\cdot,t)\|_{L_x^2(\mathbb{R}^2)}. \tag{6-6}$$

For fixed t the function $\mathscr{C}(S,t) : [0, S'] \to (0,\infty)$ is well-defined, continuous, and nondecreasing. Moreover, in view of the definition (2-34) of $\gamma_k(\sigma)$, it follows that $\lim_{S\to 0} \mathscr{C}(S,t) \lesssim 1$. A simple consequence of (6-6) is

$$\| P_k \varphi(s,\cdot,t)\|_{L_x^2(\mathbb{R}^2)} \leq \mathscr{C}(S,t)(1+s2^{2k})^{-\sigma_1} 2^{-\sigma k} \gamma_k(\sigma) \tag{6-7}$$

for $0 \leq s \leq S \leq S'$.

Our goal is to show $\mathscr{C}(S,t) \lesssim 1$ uniformly in S and t and our strategy is to apply Duhamel's formula to (6-5) and run a bootstrap argument. Beginning with the decomposition

$$P_k(\varphi\boldsymbol{e}_1) = \sum_{|k_2-k|\leq 4} P_k(P_{\leq k-5}\varphi \cdot P_{k_2}\boldsymbol{e}_1) + \sum_{|k_1-k|\leq 4} P_k(P_{k_1}\varphi \cdot P_{\leq k-5}\boldsymbol{e}_1) + \sum_{\substack{k_1,k_2\geq k-4\\ |k_1-k_2|\leq 8}} P_k(P_{k_1}\varphi\cdot P_{k_2}\boldsymbol{e}_1),$$

we proceed to place in L_x^2 each of the three terms on the right-hand side; we then integrate in s and consider separately the low-high, high-low, and high-high frequency interactions.

Low-high interaction. By Duhamel and the triangle inequality it suffices to bound

$$\mathrm{LH}(s,t) := \int_0^s e^{-(s-s')2^{2k-2}} \sum_{|k_2-k|\leq 4} \| P_k(P_{\leq k-5}\varphi(s',\cdot,t)\cdot P_{k_2}\boldsymbol{e}_1(s',\cdot,t))\|_{L_x^2}\, ds'. \tag{6-8}$$

By Hölder's inequality, $|\varphi| \equiv 1$, and L^p-boundedness of the Littlewood–Paley multipliers,

$$\begin{aligned}\mathrm{LH}(s,t) &\lesssim \int_0^s e^{-(s-s')2^{2k-2}} \sum_{|k_2-k|\leq 4} \|P_{\leq k-5}\varphi\|_{L_x^\infty} \|P_{k_2}\boldsymbol{e}_1\|_{L_x^2}\, ds' \\ &\lesssim \int_0^s e^{-(s-s')2^{2k-2}} \sum_{|k_2-k|\leq 4} \|P_{k_2}\boldsymbol{e}_1(s',\cdot,t)\|_{L_x^2}\, ds'.\end{aligned}$$

To control the sum we further decompose $P_l\boldsymbol{e}_1 = P_l(\partial_x\varphi\cdot\partial_x\varphi)$ into low-high and high-high frequency interactions:

$$P_l\boldsymbol{e}_1 = 2\sum_{|l_1-l|\leq 4} P_l(P_{\leq l-5}\partial_x\varphi \cdot P_{l_1}\partial_x\varphi) + \sum_{\substack{l_1,l_2\geq l-4\\ |l_1-l_2|\leq 8}} P_l(P_{l_1}\partial_x\varphi\cdot P_{l_2}\partial_x\varphi). \tag{6-9}$$

Low-high interaction (i). We first attend to the low-high subcase. For convenience set Ξ_{lh} equal to the first term of the right-hand side of (6-9), i.e.,

$$\Xi_{lh}(s,x,t) := \sum_{|l_1-l|\leq 4} P_l(P_{\leq l-5}\partial_x\varphi(s,x,t)\cdot P_{l_1}\partial_x\varphi(s,x,t)).$$

By the triangle inequality, Hölder's inequality, Bernstein's inequality, the definition (6-1) for $\boldsymbol{e}_1(s,\cdot,t)$, and (6-7), it follows that

$$\begin{aligned}\|\Xi_{lh}(s,\cdot,t)\|_{L^2_x} &\lesssim \sum_{|l_1-l|\le 4}\|P_l(P_{\le l-5}\partial_x\varphi\cdot P_{l_1}\partial_x\varphi)\|_{L^2_x} \lesssim \sum_{|l_1-l|\le 4}\|P_{\le l-5}\partial_x\varphi\|_{L^\infty_x}\|P_{l_1}\partial_x\varphi\|_{L^2_x}\\ &\lesssim \sum_{|l_1-l|\le 4}\|P_{\le l-5}\partial_x\varphi\|_{L^\infty_x}2^{l_1}\|P_{l_1}\varphi\|_{L^2_x} \lesssim \|\sqrt{\boldsymbol{e}_1}\|_{L^\infty_x}2^l\sum_{|l_1-l|\le 4}\|P_{l_1}\varphi\|_{L^2_x}\\ &\lesssim \|\sqrt{\boldsymbol{e}_1}(s,\cdot,t)\|_{L^\infty_x}2^l 2^{-\sigma l}\gamma_l(\sigma)\mathscr{C}(S,t)(1+s2^{2l})^{-\sigma_1}.\end{aligned}$$

As we apply this inequality in the case where $l=k_2$, $|k_2-k|\le 4$, we have

$$\int_0^s e^{-(s-s')2^{2k-2}}\|\Xi_{lh}(s',\cdot,t)\|_{L^2_x}ds' \lesssim 2^k2^{-\sigma k}\gamma_k(\sigma)\mathscr{C}(S,t)\int_0^s e^{-(s-s')2^{2k-2}}\|\sqrt{\boldsymbol{e}_1}(s',\cdot,t)\|_{L^\infty_x}(1+s'2^{2k})^{-\sigma_1}ds'. \tag{6-10}$$

Apply Cauchy–Schwarz. Clearly

$$\left(\int_0^s\|\sqrt{\boldsymbol{e}_1}(s',\cdot,t)\|^2_{L^\infty_x}\,ds'\right)^{1/2}\le\|\boldsymbol{e}_1(\cdot,\cdot,t)\|^{1/2}_{L^1_sL^\infty_x}. \tag{6-11}$$

We postpone applying (6-3) with $k=1$ to (6-11). As for the other factor, we have

$$\left(\int_0^s e^{-(s-s')2^{2k-1}}(1+s'2^{2k})^{-2\sigma_1}ds'\right)^{1/2}\lesssim\left(s(1+s2^{2k-1})^{-2\sigma_1}(1+s2^{2k})^{-1}\right)^{1/2} \tag{6-12}$$

since

$$\int_0^s e^{-(s-s')\lambda}(1+s'\lambda')^{-\alpha}ds'\lesssim s(1+\lambda s)^{-\alpha}(1+\lambda' s)^{-1}$$

for $s\ge 0$, $0\le\lambda\le\lambda'$, and $\alpha>1$. Hence, applying Cauchy–Schwarz to (6-10) and using (6-11) and (6-12), we get

$$\int_0^s e^{-(s-s')2^{2k-2}}\|\Xi_{lh}(s',\cdot,t)\|_{L^2_x}\,ds' \lesssim 2^{-\sigma k}\gamma_k(\sigma)\mathscr{C}(S,t)2^k s^{1/2}(1+s2^{2k-1})^{-\sigma_1}(1+s2^{2k})^{-1/2}\|\boldsymbol{e}_1(t)\|^{1/2}_{L^1_sL^\infty_x([0,s]\times\mathbb{R}^2)}.$$

Discarding $s^{1/2}2^k(1+s2^{2k})^{1/2}\le 1$, we conclude that

$$\int_0^s e^{-(s-s')2^{2k-2}}\|\Xi_{lh}(s',\cdot,t)\|_{L^2_x}\,ds'\lesssim 2^{-\sigma k}\gamma_k(\sigma)\mathscr{C}(S,t)(1+s2^{2k-1})^{-\sigma_1}\|\boldsymbol{e}_1(t)\|^{1/2}_{L^1_sL^\infty_x([0,s]\times\mathbb{R}^2)}. \tag{6-13}$$

Low-high interaction (ii). We now move on to the high-high interaction subcase, setting Ξ_{hh} equal to the second term of the right-hand side of (6-9):

$$\Xi_{hh}(s,x,t):=\sum_{\substack{l_1,l_2\ge l-4\\|l_1-l_2|\le 8}}P_l(P_{l_1}\partial_x\varphi(s,x,t)\cdot P_{l_2}\partial_x\varphi(s,x,t)).$$

By the triangle inequality, Bernstein, and Cauchy–Schwarz,

$$\|\Xi_{hh}\|_{L^2_x} \lesssim \sum_{\substack{l_1,l_2\geq l-4\\ |l_1-l_2|\leq 8}} \|P_l(P_{l_1}\partial_x\varphi\cdot P_{l_2}\partial_x\varphi)\|_{L^2_x} \lesssim \sum_{\substack{l_1,l_2\geq l-4\\ |l_1-l_2|\leq 8}} 2^l\|P_{l_1}\partial_x\varphi\cdot P_{l_2}\partial_x\varphi\|_{L^1_x}$$
$$\lesssim \sum_{\substack{l_1,l_2\geq l-4\\ |l_1-l_2|\leq 8}} 2^l\|P_{l_1}\partial_x\varphi\|_{L^2_x}\|P_{l_2}\partial_x\varphi\|_{L^2_x}.$$

At this stage we apply Bernstein twice, exploiting $|l_1-l_2|\leq 8$, and get

$$\|P_{l_1}\partial_x\varphi\|_{L^2_x}\|P_{l_2}\partial_x\varphi\|_{L^2_x} \lesssim 2^{l_2}\|P_{l_1}\partial_x\varphi\|_{L^2_x}\|P_{l_2}\varphi\|_{L^2_x} \lesssim \|P_{l_1}|\partial_x|^2\varphi\|_{L^2_x}\|P_{l_2}\varphi\|_{L^2_x}.$$

So

$$\|\Xi_{hh}\|_{L^2_x} \lesssim 2^l \sum_{\substack{l_1,l_2\geq l-4\\ |l_1-l_2|\leq 8}} \|P_{l_1}|\partial_x|^2\varphi\|_{L^2_x}\|P_{l_2}\varphi\|_{L^2_x}.$$

Applying Cauchy–Schwarz yields

$$\|\Xi_{hh}\|_{L^2_x} \lesssim 2^l\biggl(\sum_{l_1\geq l-4}\|P_{l_1}|\partial_x|^2\varphi\|^2_{L^2_x}\biggr)^{1/2}\biggl(\sum_{l_2\geq l-4}\|P_{l_2}\varphi\|^2_{L^2_x}\biggr)^{1/2} \lesssim \||\partial_x|^2\varphi\|_{L^2_x}2^l\biggl(\sum_{l_2\geq l-4}\|P_{l_2}\varphi\|^2_{L^2_x}\biggr)^{1/2}. \tag{6-14}$$

As φ takes values in $\mathbb{S}^2$, which has constant curvature, we readily estimate ordinary derivatives by covariant ones:

$$|\partial_x^2\varphi| \lesssim \sqrt{\boldsymbol{e}_2}+\boldsymbol{e}_1. \tag{6-15}$$

Applying (6-15) in (6-14) and using (6-7), we arrive at

$$\|\Xi_{hh}(s,\cdot,t)\|_{L^2_x} \lesssim \|\sqrt{\boldsymbol{e}_2}+\boldsymbol{e}_1\|_{L^2_x}2^l\biggl(\sum_{l_2\geq l-4}\|P_{l_2}\varphi\|^2_{L^2_x}\biggr)^{1/2}$$
$$\lesssim \|(\sqrt{\boldsymbol{e}_2}+\boldsymbol{e}_1)(s,\cdot,t)\|_{L^2_x}2^l\mathscr{C}(S,t)\biggl(\sum_{l_2\geq l-4}(1+s2^{2l_2})^{-2\sigma_1}2^{-2\sigma l_2}\gamma^2_{l_2}(\sigma)\biggr)^{1/2}$$
$$\lesssim \|(\sqrt{\boldsymbol{e}_2}+\boldsymbol{e}_1)(s,\cdot,t)\|_{L^2_x}2^l\mathscr{C}(S,t)(1+s2^{2l})^{-\sigma_1}\biggl(\sum_{l_2\geq l-4}2^{-2\sigma l_2}\gamma^2_{l_2}(\sigma)\biggr)^{1/2}. \tag{6-16}$$

As $\sigma>\delta$ is bounded away from δ uniformly, we may apply summation rule (2-31) in (6-16). Recalling $l=k_2$ where $|k_2-k|\leq 4$, we conclude that

$$\|\Xi_{hh}(s,\cdot,t)\|_{L^2_x} \lesssim \|(\sqrt{\boldsymbol{e}_2}+\boldsymbol{e}_1)(s,\cdot,t)\|_{L^2_x}2^k2^{-\sigma k}\gamma_k(\sigma)\mathscr{C}(S,t)(1+s2^{2k})^{-\sigma_1}.$$

Integrating in s yields

$$\int_0^s e^{-(s-s')2^{2k-2}}\|\Xi_{hh}(s',\cdot,t)\|_{L^2_x}\,ds'$$
$$\lesssim 2^k2^{-\sigma k}\gamma_k(\sigma)\mathscr{C}(S,t)\int_0^s e^{-(s-s')2^{2k-2}}\bigl\|(\sqrt{\boldsymbol{e}_2}+\boldsymbol{e}_1)(s',\cdot,t)\bigr\|_{L^2_x}(1+s'2^{2k})^{-\sigma_1}\,ds'. \tag{6-17}$$

We use the triangle inequality to write $\|\sqrt{\boldsymbol{e}_2}+\boldsymbol{e}_1\|_{L^2_x} \le \|\sqrt{\boldsymbol{e}_2}\|_{L^2_x} + \|\boldsymbol{e}_1\|_{L^2_x}$ and split the integral in (6-17) into two pieces. By Cauchy–Schwarz and (6-12),

$$\begin{aligned}\int_0^s e^{-(s-s')2^{2k-2}} \|\boldsymbol{e}_1(s',\cdot,t)\|_{L^2_x}(1+s'2^{2k})^{-\sigma_1}\,ds' \\ &\le \left(\int_0^s \|\boldsymbol{e}_1(s',\cdot,t)\|^2_{L^2_x}\,ds'\right)^{1/2}\left(\int_0^s e^{-(s-s')2^{2k-1}}(1+s'2^{2k})^{-2\sigma_1}\,ds'\right)^{1/2} \\ &\lesssim \|\boldsymbol{e}_1(t)\|_{L^2_{s,x}}\big(s(1+s2^{2k-1})^{-2\sigma_1}(1+s2^{2k})^{-1}\big)^{1/2}. \end{aligned} \tag{6-18}$$

To the remaining integral we also apply Cauchy–Schwarz and (6-12):

$$\begin{aligned}\int_0^s e^{-(s-s')2^{2k-2}} \|\sqrt{\boldsymbol{e}_2}(s',\cdot,t)\|_{L^2_x}(1+s'2^{2k})^{-\sigma_1}\,ds' \\ &\le \left(\int_0^s \|\boldsymbol{e}_2(s',\cdot,t)\|_{L^1_x}\,ds'\right)^{1/2}\left(\int_0^s e^{-(s-s')2^{2k-1}}(1+s'2^{2k})^{-2\sigma_1}\,ds'\right)^{1/2} \\ &\lesssim \|\boldsymbol{e}_2(t)\|^{1/2}_{L^1_{s,x}}\big(s(1+s2^{2k-1})^{-2\sigma_1}(1+s2^{2k})^{-1}\big)^{1/2}. \end{aligned} \tag{6-19}$$

Hence, using Cauchy–Schwarz, (6-18), and (6-19) in (6-17), we conclude that

$$\int_0^s e^{-(s-s')2^{2k-2}} \|\Xi_{hh}(s',\cdot,t)\|_{L^2_x}\,ds' \lesssim 2^{-\sigma k}\gamma_k(\sigma)\mathscr{C}(S,t)(1+s2^{2k-1})^{-\sigma_1}\big(\|\boldsymbol{e}_1(t)\|_{L^2_{s,x}}+\|\boldsymbol{e}_2(t)\|^{1/2}_{L^1_{s,x}}\big). \tag{6-20}$$

Low-high interaction: conclusion. Combining (6-13) and (6-20), we conclude in view of (6-8) and the decomposition (6-9) that

$$\mathrm{LH}(s,t) \lesssim 2^{-\sigma k}\gamma_k(\sigma)\mathscr{C}(S,t)(1+s2^{2k-1})^{-\sigma_1}\big(\|\boldsymbol{e}_1(t)\|^{1/2}_{L^1_s L^\infty_x}+\|\boldsymbol{e}_1(t)\|_{L^2_{s,x}}+\|\boldsymbol{e}_2(t)\|^{1/2}_{L^1_{s,x}}\big). \tag{6-21}$$

High-low interaction. We now go on to bound the high-low interaction. By Duhamel and the triangle inequality it suffices to bound

$$\mathrm{HL}(s,t) := \int_0^s e^{-(s-s')2^{2k-2}} \sum_{|k_1-k|\le 4} \|P_k(P_{k_1}\varphi(s',\cdot,t)\cdot P_{\le k-5}\boldsymbol{e}_1(s',\cdot,t))\|_{L^2_x}\,ds'.$$

By Hölder's inequality, (6-7), and Bernstein's inequality, we have

$$\begin{aligned}\sum_{|k_1-k|\le 4} \|P_k(P_{k_1}\varphi(s,\cdot,t)\cdot P_{\le k-5}\boldsymbol{e}_1(s,\cdot,t))\|_{L^2_x} \\ &\lesssim \sum_{|k_1-k|\le 4}\|P_{k_1}\varphi\|_{L^2_x}\|P_{\le k-5}\boldsymbol{e}_1\|_{L^\infty_x} \\ &\lesssim \|P_{\le k-5}\boldsymbol{e}_1(s,\cdot,t)\|_{L^\infty_x}\sum_{|k_1-k|\le 4}(1+s'2^{2k_1})^{-\sigma_1}2^{-\sigma k_1}\gamma_{k_1}(\sigma)\mathscr{C}(S,t) \\ &\lesssim 2^k\|P_{\le k-5}\boldsymbol{e}_1(s,\cdot,t)\|_{L^2_x}2^{-\sigma k}\gamma_k(\sigma)\mathscr{C}(S,t)(1+s'2^{2k})^{-\sigma_1}.\end{aligned}$$

Hence

$$\mathrm{HL}(s,t) \lesssim 2^k 2^{-\sigma k}\gamma_k(\sigma)\mathscr{C}(S,t)\int_0^s e^{-(s-s')2^{2k-2}}(1+s'2^{2k})^{-\sigma_1}\|\boldsymbol{e}_1(s',\cdot,t)\|_{L^2_x}\,ds'.$$

Bounding the integral as in (6-18), we obtain

$$\mathrm{HL}(s,t) \lesssim 2^{-\sigma k}\gamma_k(\sigma)\mathscr{C}(S,t)(1+s2^{2k-1})^{-\sigma_1}\|\boldsymbol{e}_1(t)\|_{L^2_{s,x}}. \tag{6-22}$$

High-high interaction. We conclude with the high-high interaction. Set

$$\mathrm{HH}(s,x,t) := \int_0^s e^{-(s-s')2^{2k-2}} \sum_{\substack{k_1,k_2\geq k-4\\ |k_1-k_2|\leq 8}} \|P_k(P_{k_1}\varphi(s,x,t)\cdot P_{k_2}\boldsymbol{e}_1(s,x,t))\|_{L^2_x}\,ds'.$$

By Bernstein, Cauchy–Schwarz, and (6-7),

$$\begin{aligned}\sum_{\substack{k_1,k_2\geq k-4\\ |k_1-k_2|\leq 8}} \|P_k(P_{k_1}\varphi\cdot P_{k_2}\boldsymbol{e}_1)\|_{L^2_x} &\lesssim \sum_{\substack{k_1,k_2\geq k-4\\ |k_1-k_2|\leq 8}} 2^k\|P_{k_1}\varphi\|_{L^2_x}\|P_{k_2}\boldsymbol{e}_1\|_{L^2_x}\\ &\lesssim 2^k\Bigl(\sum_{k_1\geq k-4}\|P_{k_1}\varphi\|^2_{L^2_x}\Bigr)^{1/2}\Bigl(\sum_{k_2\geq k-4}\|P_{k_2}\boldsymbol{e}_1\|^2_{L^2_x}\Bigr)^{1/2}\\ &\lesssim 2^k\Bigl(\sum_{k_1\geq k-4}(1+s'2^{2k_1})^{-2\sigma_1}2^{-2\sigma k_1}\gamma_{k_1}(\sigma)^2\mathscr{C}(S,t)^2\Bigr)^{1/2}\|\boldsymbol{e}_1(s,\cdot,t)\|_{L^2_x}\\ &= \|\boldsymbol{e}_1(s,\cdot,t)\|_{L^2_x}2^k\mathscr{C}(S,t)\Bigl(\sum_{k_1\geq k-4}(1+s'2^{2k_1})^{-2\sigma_1}2^{-2\sigma k_1}\gamma_{k_1}(\sigma)^2\Bigr)^{1/2}.\end{aligned}$$

We handle the sum as in (6-16), taking advantage of the frequency envelope summation rule (2-31), and conclude that

$$\mathrm{HH}(s,t) \lesssim 2^{-\sigma k}\gamma_k(\sigma)\mathscr{C}(S,t)(1+s2^{2k-1})^{-\sigma_1}\|\boldsymbol{e}_1(t)\|_{L^2_{s,x}}. \tag{6-23}$$

Wrapping up. For the linear term $e^{s\Delta}P_k\varphi$ we have

$$\|e^{s\Delta}P_k\varphi_0\|_{L^2_x} \leq e^{-s2^{2k-2}}\|P_k\varphi_0\|_{L^2_x} \leq e^{-s2^{2k-2}}2^{-\sigma k}\gamma_k(\sigma). \tag{6-24}$$

Using (6-21)–(6-24) in Duhamel's formula applied to the covariant heat equation (6-5), we have that for any $s\in[0,S]$, $t\in(-T,T)$,

$$\begin{aligned}2^{\sigma k}\|P_k\varphi(s,\cdot,t)\|_{L^2_x}(1+s2^{2k})^{\sigma_1} &\lesssim \gamma_k(\sigma)+\mathrm{LL}(s,t)+\mathrm{LH}(s,t)+\mathrm{HH}(s,t)\\ &\lesssim \gamma_k(\sigma)+\gamma_k(\sigma)\mathscr{C}(S,t)\bigl(\|\boldsymbol{e}_1(t)\|^{1/2}_{L^1_sL^\infty_x}+\|\boldsymbol{e}_2(t)\|^{1/2}_{L^1_{s,x}}+\|\boldsymbol{e}_1(t)\|_{L^2_{s,x}}\bigr).\end{aligned}$$

In view of (6-3) with $k=1$, (6-2) with $k=1$, and (6-4), we may split up the s-time interval $[0,\infty)$ into $O_{E_0}(1)$ intervals I_ρ on which

$$\|\boldsymbol{e}_1(t)\|^{1/2}_{L^1_sL^\infty_x(I_\rho\times\mathbb{R}^2)},\quad \|\boldsymbol{e}_2(t)\|^{1/2}_{L^1_sL^1_x(I_\rho\times\mathbb{R}^2)},\quad\text{and}\quad \|\boldsymbol{e}_1(t)\|_{L^2_sL^2_x(I_\rho\times\mathbb{R}^2)}$$

are all simultaneously small uniformly in t. By iterating a bootstrap argument $O_{E_0}(1)$ times beginning with interval I_1, we conclude that $\mathscr{C}(s,t)\lesssim 1$ for all $s>0$, uniformly in t. Therefore

$$\|P_k\varphi(s)\|_{L^\infty_tL^2_x} \lesssim (1+s2^{2k})^{-\sigma_1}2^{-\sigma k}\gamma_k(\sigma) \tag{6-25}$$

for $s\in[0,\infty)$ and $\sigma\geq 2\delta$. □

Remark 6.3. Having proven the quantitative bounds (2-35) for φ, one may establish as a corollary the qualitative bounds (2-36) for φ by using an inductive argument as in the proof of [Bejenaru et al. 2011c, Lemma 8.3]. We omit the proof, noting in particular that the argument deriving (2-36) from (2-35) does not require a small-energy hypothesis.

Proof of (2-35) *for* v, w. We begin by introducing the matrix-valued function

$$R(s,x,t) := \partial_s\varphi(s,x,t)\cdot\varphi(s,x,t)^\dagger - \varphi(s,x,t)\cdot\partial_s\varphi(s,x,t)^\dagger, \tag{6-26}$$

where here φ is thought of as a column vector. The dagger "†" denotes transpose. Using the heat flow equation (2-11) in (6-26), we rewrite R as

$$R = \Delta\varphi\cdot\varphi^\dagger - \varphi\cdot\Delta\varphi^\dagger \tag{6-27}$$

$$= \partial_m(\partial_m\varphi\cdot\varphi^\dagger - \varphi\cdot\partial_m\varphi^\dagger) \tag{6-28}$$

and proceed to bound its Littlewood–Paley projections $P_k R$ in L_x^2. Noting that by Bernstein we have

$$\|P_k(\partial_m(\partial_m\varphi\cdot\varphi^\dagger))\|_{L_x^2} \sim 2^k\|P_k(\partial_m\varphi\cdot\varphi^\dagger)\|_{L_x^2}, \tag{6-29}$$

we further decompose the nonlinearity $P_k(\partial_m\varphi\cdot\varphi^\dagger)$ as

$$P_k(\partial_m\varphi\cdot\varphi^\dagger) = \sum_{|k_2-k|\le 4} P_{\le k-4}\partial_m\varphi\cdot P_{k_2}\varphi^\dagger + \sum_{|k_1-k|\le 4} P_{k_1}\partial_m\varphi\cdot P_{\le k-4}\varphi^\dagger + \sum_{\substack{k_1,k_2\ge k-4\\ |k_1-k_2|\le 8}} P_k(P_{k_1}\partial_m\varphi\cdot P_{k_2}\varphi^\dagger). \tag{6-30}$$

By Hölder's and Bernstein's inequalities, and by $|\varphi|\equiv 1$ and (6-25) with Bernstein,

$$\sum_{|k_2-k|\le 4}\|P_{\le k-4}\partial_m\varphi\cdot P_{k_2}\varphi\|_{L_x^2} \lesssim \sum_{|k_2-k|\le 4} 2^k\|P_{\le k-4}\varphi\|_{L_x^\infty}\|P_{k_2}\varphi\|_{L_x^2}$$
$$\lesssim 2^k(1+s2^{2k})^{-\sigma_1}2^{-\sigma k}\gamma_k(\sigma). \tag{6-31}$$

Similarly,

$$\sum_{|k_1-k|\le 4}\|P_{k_1}\partial_m\varphi\cdot P_{\le k-4}\varphi\|_{L_x^2} \lesssim \sum_{|k_1-k|\le 4}\|P_{k_1}\partial_m\varphi\|_{L_x^2}\|P_{\le k-4}\varphi\|_{L_x^\infty}$$
$$\lesssim 2^k(1+s2^{2k})^{-\sigma_1}2^{-\sigma k}\gamma_k(\sigma). \tag{6-32}$$

Finally, by Bernstein and Cauchy–Schwarz, energy decay, (6-25), and frequency envelope summation rule (2-31), we get

$$\sum_{\substack{k_1,k_2\ge k-4\\ |k_1-k_2|\le 8}}\|P_k(P_{k_1}\partial_m\varphi\cdot P_{k_2}\varphi)\|_{L_x^2} \lesssim \sum_{\substack{k_1,k_2\ge k-4\\ |k_1-k_2|\le 8}} 2^k\|P_{k_1}\partial_m\varphi\|_{L_x^2}\|P_{k_2}\varphi\|_{L_x^2}$$
$$\lesssim 2^k\sum_{k_2\ge k-4}\|P_{k_2}\varphi\|_{L_x^2}$$
$$\lesssim 2^k\sum_{k_1\ge k-4}(1+s2^{2k_1})^{-\sigma_1}2^{-\sigma k_1}\gamma_{k_1}(\sigma)$$
$$\lesssim 2^k(1+s2^{2k})^{-\sigma_1}2^{-\sigma k}\gamma_k(\sigma). \tag{6-33}$$

Using the decomposition (6-30) and combining the cases (6-31), (6-32), and (6-33) to control (6-29), we conclude from the representation (6-28) of R that for fixed $t \in (-T, T)$,

$$2^{\sigma k} \| P_k R(s, \cdot, t) \|_{L_x^2} \lesssim 2^{2k} (1 + s 2^{2k})^{-\sigma_1} \gamma_k(\sigma).$$

As this estimate is uniform in T, it follows that

$$2^{\sigma k} \| P_k R(s) \|_{L_t^\infty L_x^2} \lesssim 2^{2k} (1 + s 2^{2k})^{-\sigma_1} \gamma_k(\sigma). \tag{6-34}$$

By arguing as in [Bejenaru et al. 2011c, Lemma 8.4], one may obtain the qualitative estimate

$$\sup_{s \geq 0} \big((1+s)^{(\sigma+2)/2} \| \partial_x^\sigma \partial_t^\rho R(s) \|_{L_t L_x^2} (1+s)^{(\sigma+3)/2} \| \partial_x^\sigma \partial_t^\rho R(s) \|_{L_{t,x}^\infty} \big) < \infty. \tag{6-35}$$

From the Duhamel representation of φ and the explicit formula for the heat kernel, one can easily show the qualitative bound[1]

$$\int_0^\infty \| R(s, \cdot, t) \|_{L_x^\infty} \, ds \lesssim_\varphi 1$$

as in [Smith 2012a, §7]. Hence we may define v as the unique solution of the ODE

$$\partial_s v = R(s) \cdot v \quad \text{and} \quad v(\infty) = Q', \tag{6-36}$$

where $Q' \in \mathbb{S}^2$ is chosen so that $Q \cdot Q' = 0$. This indeed coincides with the definition given in [Smith 2012a], since (6-36) is nothing other than the parallel transport condition $(\varphi^* \nabla)_s v = 0$ written explicitly in the setting $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$. Smoothness and basic convergence properties follow as in [Smith 2012a], to which we refer the reader for the precise results and proofs. Our goal here is to exploit (6-36) and (6-34) to prove (2-35) for v.

Using $\int_0^\infty \| \partial_x^\sigma \partial_t^\rho R(s) \|_{L_{t,x}^\infty} \, ds < \infty$ from (6-35), we conclude that

$$\sup_{s \geq 0} (1+s)^{(\sigma+1)/2} \| \partial_x^\sigma \partial_t^\rho (v(s) - Q') \|_{L_{t,x}^\infty} < \infty \tag{6-37}$$

for $\sigma, \rho \in \mathbb{Z}_+$. Integrating (6-36) in s from infinity, we get

$$v(s) - Q' + \int_s^\infty R(s') \cdot Q' \, ds' = -\int_s^\infty R(s') \cdot (v(s') - Q') \, ds', \tag{6-38}$$

which, combined with estimates (6-35) and (6-37), implies

$$\sup_{s \geq 0} \sup_{k \in \mathbb{Z}} (1+s)^{\sigma/2} 2^{\sigma k} \| P_k \partial_t^\rho v(s) \|_{L_t^\infty L_x^2} < \infty, \tag{6-39}$$

i.e., (2-36) for v. Projecting (6-36) to frequencies $\sim 2^k$ and integrating in s, we obtain

$$P_k(v(s)) = -\int_s^\infty P_k(R(s') \cdot v(s')) \, ds'. \tag{6-40}$$

[1] We may alternatively invoke (6-35) as in [Bejenaru et al. 2011c].

Set

$$\mathscr{C}_1(S,t) := \sup_{\sigma\in[2\delta,\sigma_1]} \sup_{s\in[S,\infty)} \sup_{k\in\mathbb{Z}} \gamma_k(\sigma)^{-1}(1+s2^{2k})^{\sigma_1-1}2^{\sigma k}\|P_k v(s,\cdot,t)\|_{L^2_x}.$$

That $\mathscr{C}_1(S,t)<\infty$ follows from (6-39) and $\sup_{k\in\mathbb{Z}}\gamma_k(\sigma)^{-1}2^{-\delta|k|}<\infty$. Consequently, for $s\in[S,\infty)$,

$$\|P_k v(s,\cdot,t)\|_{L^2_x} \lesssim \mathscr{C}_1(S,t)(1+s2^{2k})^{-\sigma_1+1}2^{-\sigma k}\gamma_k(\sigma). \tag{6-41}$$

We perform the Littlewood–Paley decomposition

$$P_k(R(s)v(s)) = \sum_{|k_2-k|\le 4} P_k(P_{\le k-4}R(s)P_{k_2}v(s)) + \sum_{|k_1-k|\le 4} P_k(P_{k_1}R(s)P_{\le k-4}v(s)) + \sum_{k_2\ge k-4} P_k(P_{\ge k-4}R(s)P_{k_2}v(s)) \tag{6-42}$$

and proceed to consider individually the various frequency interactions. By Hölder's inequality, Bernstein's inequality, and (6-41),

$$\begin{aligned}\sum_{|k_2-k|\le 4}\|P_k(P_{\le k-4}R(s)P_{k_2}v(s))\|_{L^2_x} &\lesssim \sum_{|k_2-k|\le 4}\|P_{\le k-4}R(s)\|_{L^2_x}\|P_{k_2}v(s)\|_{L^\infty_x}\\ &\lesssim \|R(s)\|_{L^2_x}\sum_{|k_2-k|\le 4}2^{k_2}\|P_{k_2}v(s)\|_{L^2_x}\\ &\lesssim \|R(s)\|_{L^2_x}2^k2^{-\sigma k}\gamma_k(\sigma)(1+s2^{2k})^{-\sigma_1+1}\mathscr{C}_1(S,t).\end{aligned} \tag{6-43}$$

By Hölder's inequality, $|v|\equiv 1$, and (6-34),

$$\begin{aligned}\sum_{|k_1-k|\le 4}\|P_k(P_{k_1}R(s)P_{\le k-4}v(s))\|_{L^2_x} &\lesssim \|P_{\le k-4}v(s)\|_{L^\infty_x}\sum_{|k_1-k|\le 4}\|P_{k_1}R(s)\|_{L^2_x}\\ &\lesssim 2^{2k}(1+s2^{2k})^{-\sigma_1}2^{-\sigma k}\gamma_k(\sigma).\end{aligned} \tag{6-44}$$

From Bernstein's inequality, Cauchy–Schwarz, (6-41), and $\sigma>2\delta$ with (2-31), it follows that

$$\begin{aligned}\sum_{k_2\ge k-4}\|P_k(P_{\ge k-4}R(s)P_{k_2}v(s))\|_{L^2_x} &\lesssim \sum_{k_2\ge k-4}2^k\|P_{\ge k-4}R(s)P_{k_2}v(s)\|_{L^1_x}\\ &\lesssim \|R(s)\|_{L^2_x}\,2^k\sum_{k_2\ge k-4}\|P_{k_2}v(s)\|_{L^2_x}\\ &\lesssim \|R(s)\|_{L^2_x}\,2^k\sum_{k_2\ge k-4}2^{-\sigma k_2}\gamma_{k_2}(\sigma)(1+s2^{2k_2})^{-\sigma_1+1}\mathscr{C}_1(S,t)\\ &\lesssim \|R(s)\|_{L^2_x}\,2^k2^{-\sigma k}\gamma_k(\sigma)(1+s2^{2k})^{-\sigma_1+1}\mathscr{C}_1(S,t).\end{aligned} \tag{6-45}$$

Using the decomposition (6-42) in (6-40) and combining the estimates (6-43), (6-44), and (6-45) gives

$$\begin{aligned}2^{\sigma k}\|P_k v(s)\|_{L^2_x} &\le \int_s^\infty 2^{\sigma k}\|P_k(R(s')v(s'))\|_{L^2_x}\,ds'\\ &\lesssim \gamma_k(\sigma)\int_s^\infty 2^{2k}(1+s'2^{2k})^{-\sigma_1}\,ds' + \mathscr{C}_1(s,t)\gamma_k(\sigma)\int_s^\infty \|R(s')\|_{L^2_x}2^k(1+s'2^{2k})^{-\sigma_1+1}ds'.\end{aligned}$$

Applying Cauchy–Schwarz in s, we obtain

$$2^{\sigma k}\|P_k v(s)\|_{L^2_x} \lesssim \gamma_k(\sigma)\int_s^\infty 2^{2k}(1+s'2^{2k})^{-\sigma_1}\,ds' + \mathscr{C}_1(s,t)\gamma_k(\sigma)\left(\int_s^\infty \|R(s')\|_{L^2_x}^2\,ds'\right)^{1/2}\left(\int_s^\infty 2^{2k}(1+s'2^{2k})^{-2\sigma_1+2}\,ds'\right)^{1/2}$$
$$\lesssim \gamma_k(\sigma)+\mathscr{C}_1(s,t)\gamma_k(\sigma)\left(\int_s^\infty \|R(s')\|_{L^2_x}^2\,ds'\right)^{1/2}. \tag{6-46}$$

As noted in (6-15), we have $|\Delta\varphi| \le \sqrt{\boldsymbol{e}_2}+\boldsymbol{e}_1$, so it follows from the representation (6-27) of R that

$$|R(s,x,t)| \le |\boldsymbol{e}_1(s,x,t)|+|\sqrt{\boldsymbol{e}_2}(s,x,t)|. \tag{6-47}$$

As (6-47) implies

$$\int_0^\infty \|R(s)\|_{L^2_x}^2\,ds \lesssim \|\boldsymbol{e}_2\|_{L^1_{s,x}}+\|\boldsymbol{e}_1\|_{L^2_{s,x}}^2,$$

we therefore, in view of (6-2) with $k=1$ and (6-4), may choose S large so that the integral of the R term in (6-46) is small, say $\le \varepsilon$. Then

$$\mathscr{C}_1(S,t) \lesssim 1+\varepsilon\mathscr{C}_1(S,t),$$

so that $\mathscr{C}_1(S) \lesssim 1$ for such S. In fact, together (6-2) and (6-4) imply that we may divide the time interval $[0,\infty)$ into $O_{E_0}(1)$ subintervals I_ρ so that on each such subinterval

$$\int_{I_\rho} \|R(s)\|_{L^2_x}^2\,ds \le \varepsilon^2.$$

Hence by a simple iterative bootstrap argument we conclude that

$$\mathscr{C}_1(0,t) \lesssim 1. \tag{6-48}$$

As (6-48) is uniform in t, we have

$$\|P_k v(s,\cdot,t)\|_{L^2_x} \lesssim (1+s2^{2k})^{-\sigma_1+1}2^{-\sigma k}\gamma_k(\sigma). \tag{6-49}$$

By repeating the argument above with w in place of v (and appropriately modifying the boundary condition at ∞ in (6-36)), we get

$$\|P_k w(s,\cdot,t)\|_{L^2_x} \lesssim (1+s2^{2k})^{-\sigma_1+1}2^{-\sigma k}\gamma_k(\sigma) \tag{6-50}$$

and $\sup_{s\ge 0}\sup_{k\in\mathbb{Z}}(1+s)^{\sigma/2}2^{\sigma k}\|P_k\partial_t^\rho w(s)\|_{L^\infty_t L^2_x}<\infty$, and so (2-35) and (2-36) follow for w. □

Proof of (2-37). Recall that

$$\psi_m = v\cdot\partial_m\varphi + iw\cdot\partial_m\varphi = -\partial_m v\cdot\varphi - i\partial_m w\cdot\varphi. \tag{6-51}$$

Our first aim is to control $\|P_k\psi_x\|_{L^\infty_t L^2_x}$. We start with a Littlewood–Paley decomposition of $\partial_m v\cdot\varphi$:

$$P_k(\partial_m v\cdot\varphi) = \sum_{|k_2-k|\le 4} P_k(P_{\le k-5}\partial_m v\cdot P_{k_2}\varphi) + \sum_{|k_1-k|\le 4} P_k(P_{k_1}\partial_m v\cdot P_{\le k-5}\varphi) + \sum_{\substack{k_1,k_2\ge k-4\\ |k_1-k_2|\le 8}} P_k(P_{k_1}\partial_m v\cdot P_{k_2}\varphi). \tag{6-52}$$

To control the low-high frequency term we apply Hölder's inequality, energy decay, and (6-25) with Bernstein's inequality:

$$\sum_{|k_2-k|\le 4} \|P_k(P_{\le k-5}\partial_m v\cdot P_{k_2}\varphi)\|_{L^2_x} \lesssim \sum_{|k_2-k|\le 4} \|P_{\le k-5}\partial_m v\|_{L^2_x}\|P_{k_2}\varphi\|_{L^\infty_x} \lesssim (1+s2^{2k})^{-\sigma_1}2^k 2^{-\sigma k}\gamma_k(\sigma). \tag{6-53}$$

We control the high-low frequency term by using Hölder's inequality, $|\varphi|\equiv 1$, and (6-49):

$$\sum_{|k_1-k|\le 4} \|P_k(P_{k_1}\partial_m v\cdot P_{\le k-5}\varphi)\|_{L^2_x} \lesssim \sum_{|k_1-k|\le 4} \|P_{k_1}\partial_m v\|_{L^2_x}\|P_{\le k-5}\varphi\|_{L^\infty_x} \lesssim (1+s2^{2k})^{-\sigma_1}2^k 2^{-\sigma k}\gamma_k(\sigma). \tag{6-54}$$

To control the high-high frequency term, we use Bernstein's inequality and Cauchy–Schwarz, energy conservation and (6-25), and (2-31):

$$\begin{aligned}\sum_{\substack{k_1,k_2\ge k-4\\ |k_1-k_2|\le 8}} \|P_k(P_{k_1}\partial_m v\cdot P_{k_2}\varphi)\|_{L^2_x} &\lesssim \sum_{\substack{k_1,k_2\ge k-4\\ |k_1-k_2|\le 8}} 2^k\|P_{k_1}\partial_m v\|_{L^2_x}\|P_{k_2}\varphi\|_{L^2_x}\\ &\lesssim 2^k\sum_{k_2\ge k-4}(1+s2^{2k_2})^{-\sigma_1}2^{-\sigma k_2}\gamma_{k_2}(\sigma)\\ &\lesssim (1+s2^{2k})^{-\sigma_1}2^k 2^{-\sigma k}\gamma_k(\sigma).\end{aligned} \tag{6-55}$$

We conclude using (6-53), (6-54), and (6-55) in representation (6-52) that

$$\|P_k(\partial_m v\cdot\varphi)\|_{L^2_x} \lesssim (1+s2^{2k})^{-\sigma_1}2^k 2^{-\sigma k}\gamma_k(\sigma). \tag{6-56}$$

By repeating the argument with w in place of v, it follows that (6-56) also holds with w in place of v. Therefore, referring back to (6-51), we conclude that

$$\|P_k\psi_m\|_{L^2_x} \lesssim (1+s2^{2k})^{-\sigma_1}2^k 2^{-\sigma k}\gamma_k(\sigma).$$

As this bound is uniform in t, (2-37) holds for ψ_m.

Recalling that

$$A_m = \partial_m v\cdot w,$$

and repeating the argument with w in place of φ and (6-50) in place of (6-25), we conclude that

$$\|P_k A_x(s)\|_{L^\infty_t L^2_x} \lesssim (1+s2^{2k})^{-\sigma_1+1}2^k 2^{-\sigma k}\gamma_k(\sigma). \qquad \square$$

7. Proofs of parabolic estimates

The purpose of this section is to prove the parabolic heat-time estimates stated in Section 4A. Many of these estimates have counterparts in [Bejenaru et al. 2011c]. Nevertheless, our proofs are more involved since we only require energy dispersion, which is weaker than the small-energy assumption made in [Bejenaru et al. 2011c]. Some of the L^p estimates in Section 7B are new.

Throughout we assume ε_1 energy dispersion on the initial data as stated in (4-4) and we assume that the bootstrap hypothesis (4-6) holds. Let $\sigma_1 \in \mathbb{Z}_+$ be positive and fixed. We work exclusively with $\sigma \in [0, \sigma_1 - 1]$, even if this is not always explicitly stated. Set $\varepsilon = \varepsilon_1^{7/5}$ for short.

In this section we extensively use the spaces defined via (3-2). They provide a crucial gain in high-high frequency interactions, which is captured in Lemmas 7.2 and 7.14.

Lemma 7.1. *Let $f \in L^2_{k_1}(T)$, where $|k_1 - k| \le 20$, let $0 \le \omega' \le 1/2$, and let $h \in L^2_k(T)$. Then*

$$\|P_k(fg)\|_{F_k(T)} \lesssim \|f\|_{F_{k_1}(T)} \|g\|_{L^\infty_{t,x}},$$
$$\|P_k(fg)\|_{S^{\omega'}_k(T)} \lesssim \|f\|_{F_{k_1}(T)} 2^{k\omega'} \|g\|_{L^{2/\omega'}_x L^\infty_t},$$
$$\|h\|_{L^\infty_{t,x}} + 2^{k\omega'} \|h\|_{L^{2/\omega'}_x L^\infty_t} \lesssim 2^k \|h\|_{F_k(T)}.$$

Moreover, for f_{k_1}, g_{k_2} belonging to $L^2_{k_1}(T)$, $L^2_{k_2}(T)$ respectively, and with $|k_1 - k_2| \le 8$, we have

$$\|P_k(f_{k_1} g_{k_2})\|_{F_k(T) \cap S^{1/2}_k(T)} \lesssim 2^k 2^{(k_2-k)(1-\omega)} \|f_{k_1}\|_{S^\omega_{k_1}(T)} \|g_{k_2}\|_{S^0_{k_2}(T)}.$$

Proof. For the proofs, see [Bejenaru et al. 2011c, §3]. □

Lemma 7.2. *Assume that $T \in (0, 2^{2\mathcal{K}}]$, $f, g \in H^{\infty,\infty}(T)$, $P_k f \in F_k(T) \cap S^\omega_k(T)$, $P_k g \in F_k(T)$ for some $\omega \in [0, 1/2]$ and all $k \in \mathbb{Z}$, and*

$$\alpha_k = \sum_{|j-k| \le 20} \|P_j f\|_{F_j(T) \cap S^\omega_j(T)}, \quad \beta_k = \sum_{|j-k| \le 20} \|P_j g\|_{F_j(T)}.$$

Then, for any $k \in \mathbb{Z}$,

$$\|P_k(fg)\|_{F_k(T) \cap S^{1/2}_k(T)} \lesssim \sum_{j \le k} 2^j (\beta_k \alpha_j + \alpha_k \beta_j) + 2^k \sum_{j \ge k} 2^{(j-k)(1-\omega)} \alpha_j \beta_j.$$

Proof. For the proof, see [Bejenaru et al. 2011c, §5]. □

7A. *Derivative field control.* The main purpose of this subsection is to establish the estimate (4-12), which states that

$$\|P_k \psi_m(s)\|_{F_k(T)} \lesssim (1 + s 2^{2k})^{-4} 2^{-\sigma k} b_k(\sigma).$$

In the course of the proof we shall also establish auxiliary estimates useful elsewhere. Estimate (4-12) plays a key role in controlling the nonlinear paradifferential flow, allowing us to gain regularity by integrating in heat time. The proof uses a bootstrap argument and exploits the Duhamel formula.

Recall that the fields ψ_α, A_α, $\alpha = 1, 2, 3$, ($\psi_3 \equiv \psi_t$, $A_3 \equiv A_t$) satisfy (2-20), which states that

$$(\partial_s - \Delta)\psi_\alpha = U_\alpha.$$

We use representation (2-22) of the heat nonlinearity:

$$U_\alpha := 2i A_l \partial_l \psi_\alpha + i(\partial_l A_l)\psi_\alpha - A_x^2 \psi_\alpha + i \operatorname{Im}(\psi_\alpha \overline{\psi_l})\psi_l.$$

Hence ψ_α admits the representation

$$\psi_\alpha(s) = e^{s\Delta}\psi_\alpha(s_0) + \int_{s_0}^{s} e^{(s-s')\Delta} U_\alpha(s')\,ds' \tag{7-1}$$

for any $s \geq s_0 \geq 0$.

For each $k \in \mathbb{Z}$, set

$$a(k) := \sup_{s\in[0,\infty)} (1+s2^{2k})^4 \sum_{m=1,2} \|P_k\psi_m(s)\|_{F_k(T)},$$

and for $\sigma \in [0, \sigma_1 - 1]$ introduce the frequency envelopes

$$a_k(\sigma) = \sup_{j\in\mathbb{Z}} 2^{-\delta|k-j|} 2^{\sigma j} a(j). \tag{7-2}$$

The frequency envelopes $a_k(\sigma)$ are finite and in l^2 by (2-38) and (3-1).

Our goal is to show $a_k(\sigma) \lesssim b_k(\sigma)$, which in particular implies (4-12).

Lemma 7.3. *Suppose that ψ_x satisfies the bootstrap condition*

$$\|P_k\psi_x(s)\|_{F_k(T)\cap S_k^{1/2}(T)} \leq \varepsilon_p^{-1/2} b_k (1+s2^{2k})^{-4}. \tag{7-3}$$

Then (4-12) holds.

We can take $\varepsilon_p = \varepsilon_1^{1/10}$, for instance. As in [Bejenaru et al. 2011c], this result may be strengthened:

Corollary 7.4. *The estimate (4-12) holds even when the bootstrap hypothesis (7-3) is dropped.*

Proof. Directly apply the argument of [Bejenaru et al. 2011c, Corollary 4.4], which we omit. □

The sequence of lemmas we prove in order to establish Lemma 7.3 culminates in Lemma 7.11, which controls the nonlinear term of the Duhamel formula (7-1) by $2^{-\sigma k} a_k(\sigma)$ along with suitable decay and an epsilon-gain arising from energy dispersion. Its immediate predecessor, Lemma 7.10, controls $P_k U_m$ in $F_k(T)$.

Referring back to (2-22) and seeing as how U_m contains the term $2iA_l\partial_l\psi_m$, we see that in order to apply the parabolic estimates of Lemma 7.1 toward controlling $P_k U_m$, it is necessary that we first control $P_k A_m$ in $F_k(T)$ in terms of the frequency envelopes $\{a_l(\sigma)\}$, and it is to this that we now turn.

For $k, k_0 \in \mathbb{Z}$ and $s \in [2^{2k_0-1}, 2^{2k_0+1})$, set

$$b_{k,s}(\sigma) = \begin{cases} \sum_{j=k}^{-k_0} a_j a_j(\sigma) & \text{if } k+k_0 \leq 0, \\ 2^{k+k_0} a_{-k_0} a_k(\sigma) & \text{if } k+k_0 \geq 0. \end{cases}$$

Let $\mathscr{C}$ be the smallest number in $[1,\infty)$ such that

$$\|P_k A_m(s)\|_{F_k(T)\cap S_k^{1/2}(T)} \leq \mathscr{C}(1+s2^{2k})^{-4} 2^{-\sigma k} b_{k,s}(\sigma) \tag{7-4}$$

for all $s \in [0,\infty)$, $k \in \mathbb{Z}$, $m = 1, 2$, and $\sigma \in [0, \sigma_1 - 1]$. While this constant is indeed finite, it is not a priori controlled by energy. To show that $\mathscr{C}$ is indeed controlled by energy, we use the integral representation

$$A_m(s) = -\sum_{l=1,2} \int_s^\infty \operatorname{Im}(\overline{\psi_m}(\partial_l\psi_l + iA_l\psi_l))(r)\,dr \tag{7-5}$$

and seek to control the Littlewood–Paley projection of the integrand in $F_k(T) \cap S_k^{1/2}(T)$. We treat differently the two types of terms in (7-5) that need to be controlled. In Lemma 7.5 we bound terms of the sort $P_k(\psi_x \overline{\psi_x})$ and $P_k(\psi_x \partial_x \overline{\psi_x})$ in $F_k(T) \cap S_k^{1/2}(T)$. In Lemma 7.6 we combine the estimate on $P_k(\psi_x \overline{\psi_x})$ with (7-4) to obtain control on $P_k(\psi_x \overline{\psi_x} A_x)$, gaining an epsilon from energy dispersion. Using (7-5) and exploiting the epsilon gain from energy dispersion will lead us to the conclusion of Lemma 7.7: $\mathscr{C} \lesssim 1$.

We use the following bracket notation in the sequel:

$$\langle f \rangle := (1 + f^2)^{1/2}.$$

Lemma 7.5. *For any* $f, g \in \{\psi_m, \overline{\psi_m} : m = 1, 2\}$, $r \in [2^{2j-2}, 2^{2j+2}]$, $j \in \mathbb{Z}$, $i = 1, 2$, *and* $\sigma \in [0, \sigma_1 - 1]$, *we have the bounds*

$$\| P_k(f(r)g(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} 2^{-j} a_{-j} a_{\max(k,-j)}(\sigma) \tag{7-6}$$

and

$$\| P_k(f(r)\partial_i g(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} 2^{-j} a_{-j} (2^k a_k(\sigma) + 2^{-j} a_{-j}(\sigma)). \tag{7-7}$$

Proof. By Lemma 7.2 with $\omega = 0$ we have

$$\| P_k(fg) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \sum_{l \le k} 2^l \alpha_k \beta_l + \sum_{l \ge k} 2^l \alpha_l \beta_l, \tag{7-8}$$

where, due to the definition (7-2), α_k and β_k satisfy

$$\alpha_k \lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} a_k(\sigma), \quad \beta_k \lesssim \langle 2^{j+k} \rangle^{-8} a_k. \tag{7-9}$$

Turning to the high-low frequency interaction first, we have using (7-9) and the frequency envelope property (2-29) that

$$\sum_{l \le k} 2^l \alpha_k \beta_l \lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} 2^{-j} a_{-j} \sum_{l \le k} \langle 2^{j+l} \rangle^{-8} 2^{j+l} 2^{\delta |j+l|} a_k(\sigma). \tag{7-10}$$

Thus it remains to show that

$$\sum_{l \le k} \langle 2^{j+l} \rangle^{-8} 2^{j+l} 2^{\delta |j+l|} a_k(\sigma) \lesssim a_{\max(k,-j)}(\sigma), \tag{7-11}$$

which follows from pulling out a factor of $a_k(\sigma)$ or $a_{-j}(\sigma)$, according to whether $k + j \ge 0$ or $k + j < 0$, and then summing the remaining geometric series. In case $k + j < 0$ we pull out a factor of $a_{-j}(\sigma)$ via (2-29).

Turning to the high-high frequency interaction term, we have

$$\sum_{l \ge k} 2^l \alpha_l \beta_l \lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} 2^{-j} a_{-j} \sum_{l \ge k} \langle 2^{j+l} \rangle^{-8} 2^{j+l} 2^{\delta |j+l|} a_l(\sigma), \tag{7-12}$$

and so it remains to show that

$$\sum_{l \ge k} \langle 2^{j+l} \rangle^{-8} 2^{j+l} 2^{\delta |j+l|} a_l(\sigma) \lesssim a_{\max(k,-j)}(\sigma). \tag{7-13}$$

When $k+j\geq 0$, we have, using (2-31),

$$\sum_{l\geq k}\langle 2^{j+l}\rangle^{-8}2^{j+l}2^{\delta|j+l|}a_l(\sigma)\lesssim a_k(\sigma)\sum_{l\geq k}2^{(2\delta-1)(j+l)}\lesssim a_k(\sigma).$$

If $k+j\leq 0$, we control the sum with (2-30) if $l+j<0$ and with (2-31) if $l+j\geq 0$. Hence (7-13) holds.

Together (7-8)–(7-13) imply (7-6).

To establish (7-7) we follow a similar strategy. By Lemma 7.2 with $\omega=0$ we have

$$\|P_k(f\partial_i g)\|_{F_k(T)\cap S_k^{1/2}(T)}\lesssim\sum_{l\leq k}2^l\alpha_l\beta_k+\sum_{l\geq k}2^l\alpha_k\beta_l+\sum_{l\geq k}2^l\alpha_l\beta_l,\tag{7-14}$$

where for any $\sigma\in[0,\sigma_1-1]$ we have

$$\alpha_k\lesssim\langle 2^{j+k}\rangle^{-8}2^{-\sigma k}a_k(\sigma)\quad\text{and}\quad\beta_k\lesssim\langle 2^{j+k}\rangle^{-8}2^k2^{-\sigma k}a_k(\sigma).\tag{7-15}$$

Beginning with the low-high frequency interaction, we have

$$\sum_{l\leq k}2^l\alpha_l\beta_k\lesssim\langle 2^{j+k}\rangle^{-8}2^{-\sigma k}2^k a_k(\sigma)\sum_{l\leq k}\langle 2^{j+l}\rangle^{-8}2^l a_l,\tag{7-16}$$

and so it remains to show that

$$\sum_{l\leq k}\langle 2^{j+l}\rangle^{-8}2^l a_l\lesssim 2^{-j}a_{-j}.\tag{7-17}$$

If $k+j\leq 0$, then (7-17) holds due to (2-30). If $k+j\geq 0$, then we apply (2-30) and (2-31) according to whether $l+j\leq 0$ or $l+j>0$.

Turning now to the high-low frequency interaction, we have

$$\sum_{l\leq k}2^l\alpha_k\beta_l\lesssim\langle 2^{j+k}\rangle^{-8}2^{-\sigma k}2^{-j}a_{-j}2^k a_k(\sigma)\sum_{l\leq k}\langle 2^{j+l}\rangle^{-8}2^{l-k}2^{l+j}2^{\delta|l+j|}.\tag{7-18}$$

We need only check that

$$\sum_{l\leq k}\langle 2^{j+l}\rangle^{-8}2^{l-k}2^{l+j}2^{\delta|l+j|}\lesssim 1,$$

which can be seen to hold by breaking into cases $k+j\leq 0$ and $k+j\geq 0$.

We conclude with the high-high frequency interaction:

$$\begin{aligned}\sum_{l\geq k}2^l\alpha_l\beta_l&\lesssim\langle 2^{j+k}\rangle^{-8}2^{-\sigma k}\sum_{l\geq k}\langle 2^{j+l}\rangle^{-8}2^{2l}a_l(\sigma)a_l\\&\lesssim\langle 2^{j+k}\rangle^{-8}2^{-\sigma k}2^{-2j}a_ja_j(\sigma)\sum_{l\geq k}\langle 2^{j+l}\rangle^{-8}2^{2l+2j}2^{2\delta|l+j|}.\end{aligned}\tag{7-19}$$

Here

$$\sum_{l\geq k}\langle 2^{j+l}\rangle^{-8}2^{2l+2j}2^{2\delta|l+j|}\lesssim 1,\tag{7-20}$$

which is seen to hold by considering separately the cases $k+j\geq 0$, $k+j<0$.

Combining (7-16)–(7-20), we conclude (7-7). □

Lemma 7.6. *Let*

$$f(r) \in \{\overline{\psi_m}(r)\psi_l(r) : m, l = 1, 2\}, \quad g(r) \in \{A_m(r) : m = 1, 2\},$$

and $r \in [2^{2j-2}, 2^{2j+2}]$. *Then*

$$\|P_k(fg)(r)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \begin{cases} \varepsilon\mathscr{C}2^{-\sigma k}2^{-2j}a_{-j}a_{-j}(\sigma) & k+j \le 0, \\ \varepsilon\mathscr{C}\langle 2^{j+k}\rangle^{-8}2^{-\sigma k}2^{-2j}b_{k,r}(\sigma) & k+j \ge 0. \end{cases}$$

Proof. We apply Lemma 7.2. By (7-6) and (7-4) we have

$$\alpha_k(r) \lesssim 2^{-\sigma k}\langle 2^{j+k}\rangle^{-8}2^{-j}a_{-j}a_{\max(k,-j)}(\sigma), \tag{7-21}$$

and

$$\beta_k(r) \lesssim \mathscr{C}2^{-\sigma k}\langle 2^{j+k}\rangle^{-8}b_{k,r}(\sigma), \tag{7-22}$$

for any $\sigma \in [0, \sigma_1 - 1]$.

We consider six cases, treating separately the low-high, high-low, and high-high frequency interactions, which we further divide according to whether $k + j \ge 0$ or $k + j \le 0$.

Low-high frequency interaction with $k + j \ge 0$. Using (7-21) and (7-22), we have

$$\sum_{l\le k} 2^l\alpha_l\beta_k \lesssim \mathscr{C}\langle 2^{j+k}\rangle^{-8}2^{-\sigma k}2^{-2j}b_{k,r}(\sigma)\sum_{l\le k}2^l2^{2j}\alpha_l, \tag{7-23}$$

and so it remains to verify that

$$\sum_{l\le k}2^l2^{2j}\alpha_l \lesssim \varepsilon. \tag{7-24}$$

Taking $\sigma = 0$ in the bounds (7-21) for α_l and using (2-29), (2-31) yields

$$\begin{aligned}\sum_{l\le k}2^l2^{2j}\alpha_l &\lesssim \sum_{l\le k}\langle 2^{j+l}\rangle^{-8}2^l2^{2j}2^{-j}a_{-j}a_{\max(l,-j)} \\ &= \sum_{l\le -j}2^{l+j}a_{-j}^2 + \sum_{-j<l\le k}\langle 2^{j+l}\rangle^{-8}2^{l+j}a_{-j}a_l \lesssim a_{-j}^2 + a_{-j}^2\sum_{-j<l\le k}\langle 2^{j+l}\rangle^{-8}2^{(1+\delta)(l+j)} \lesssim \varepsilon,\end{aligned}$$

which proves (7-23).

High-low frequency interaction with $k + j \ge 0$. Taking $\sigma = 0$ in the bounds for $b_{l,r}$, we have

$$\sum_{l\le k}2^l\alpha_k\beta_l \lesssim \mathscr{C}\langle 2^{j+k}\rangle^{-8}2^{-\sigma k}2^{-2j}b_{k,r}(\sigma)\sum_{l\le k}\langle 2^{j+l}\rangle^{-8}2^{l-k}b_{l,r}, \tag{7-25}$$

and so it remains to show that

$$\sum_{l\le k}\langle 2^{j+l}\rangle^{-8}2^{l-k}b_{l,r} \lesssim \varepsilon. \tag{7-26}$$

We split the sum as follows:

$$\sum_{l\le k}\langle 2^{j+l}\rangle^{-8}2^{l-k}b_{l,r} = \sum_{l\le -j}\langle 2^{j+l}\rangle^{-8}2^{l-k}\sum_{q=l}^{-j}a_q^2 + \sum_{-j<l\le k}\langle 2^{j+l}\rangle^{-8}2^{l-k}2^{l+j}a_{-j}a_l.$$

The first summand is controlled by

$$\sum_{l\le -j}\langle 2^{j+l}\rangle^{-8}2^{l-k}\sum_{q=l}^{-j}a_q^2\lesssim a_{-j}^2\sum_{l\le -j}2^{l-k}\sum_{q=l}^{-j}2^{-2\delta(j+q)}\lesssim a_{-j}^2\lesssim\varepsilon.$$

The second summand by may be handled similarly, thus proving (7-26).

High-high frequency interaction with $k+j\ge 0$. Taking $\sigma=0$ in the bound (7-22) for β_l, we have

$$\begin{aligned}\sum_{l\ge k}2^l\alpha_l\beta_l &\lesssim\langle 2^{j+l}\rangle^{-8}2^k\sum_{l\ge k}2^{l-k}2^{-\sigma l}2^{-j}a_{-j}a_l(\sigma)\mathscr{C}2^{l+j}a_{-j}a_l\\ &\lesssim\mathscr{C}\langle 2^{j+k}\rangle^{-8}2^{-\sigma k}2^{-2j}b_{k,r}(\sigma)\sum_{l\ge k}\langle 2^{j+l}\rangle^{-8}2^{l-k}2^{\delta(l-k)}2^{l+j}a_{-j}a_l,\end{aligned}\tag{7-27}$$

and so it remains to show that

$$\sum_{l\ge k}\langle 2^{j+l}\rangle^{-8}2^{l-k}2^{\delta(l-k)}2^{l+j}a_{-j}a_l\lesssim\varepsilon,\tag{7-28}$$

which follows, for instance, from pulling out a_{-j}^2 via (2-29) and summing.

In view of (7-23)–(7-28), it follows from Lemma 7.2, with $\omega=0$ that

$$\|P_k(fg)(r)\|_{F_k(T)\cap S_k^{1/2}(T)}\lesssim\varepsilon\mathscr{C}\langle 2^{j+k}\rangle^{-8}2^{-\sigma k}2^{-2j}b_{k,r}(\sigma)\quad\text{for } k+j\ge 0\tag{7-29}$$

as required.

Low-high frequency interaction with $k+j\le 0$. In this case it follows from (7-22) that

$$\beta_k\lesssim\mathscr{C}2^{-\sigma k}\sum_{p=k}^{-j}a_pa_p(\sigma),$$

so that

$$\begin{aligned}\sum_{l\le k}2^l\alpha_l\beta_k &\lesssim\mathscr{C}2^{-\sigma k}2^{-j}a_{-j}a_{-j}\sum_{p=k}^{-j}a_pa_p(\sigma)\sum_{l\le k}\langle 2^{j+l}\rangle^{-8}2^l\\ &\lesssim\mathscr{C}2^{-\sigma k}2^{-2j}a_{-j}a_{-j}(\sigma)\cdot a_{-j}\sum_{p=k}^{-j}a_p2^{-\delta(j+p)}\sum_{l\le k}2^{l+j}.\end{aligned}\tag{7-30}$$

It remains to show that

$$a_{-j}\sum_{p=k}^{-j}a_p2^{-\delta(j+p)}\sum_{l\le k}2^{l+j}\lesssim\varepsilon,$$

which follows from pulling out a_p as an a_{-j} via (2-29) and summing.

High-low frequency interaction with $k+j\le 0$. In this case

$$\sum_{l\le k}2^l\alpha_k\beta_l\lesssim\mathscr{C}2^{-2j}a_{-j}a_{-j}(\sigma)\sum_{l\le k}2^{l+j}\sum_{p=l}^{-j}a_p^2,\tag{7-31}$$

and so we need to show that

$$\sum_{l\le k} 2^{l+j} \sum_{p=l}^{-j} a_p^2 \lesssim \varepsilon,$$

which follows by pulling out a_{-j}^2 and summing.

High-high frequency interaction with $k+j\le 0$. As a first step we write

$$2^k \sum_{l\ge k} 2^{(l-k)/2}\alpha_l\beta_l = 2^k \sum_{k\le l<-j} 2^{(l-k)/2}\alpha_l\beta_l + 2^k \sum_{l\ge -j} 2^{(l-k)/2}\alpha_l\beta_l. \tag{7-32}$$

The first summand is controlled by

$$2^k \sum_{k\le l<-j} 2^{(l-k)/2}\alpha_l\beta_l \lesssim \mathscr{C} 2^{-\sigma k}2^{-2j}a_{-j}a_{-j}(\sigma) \sum_{k\le l<-j} 2^{(l-k)/2}2^{k+j}2^{-\sigma(l-k)} \sum_{p=l}^{-j} a_p^2. \tag{7-33}$$

We have

$$\sum_{k\le l<-j} 2^{(l-k)/2}2^{k+j}2^{-\sigma(l-k)} \sum_{p=l}^{-j} a_p^2 \lesssim a_{-j}^2 2^{(k+j)/2} \sum_{k\le l<-j} 2^{-2\delta(j+l)} \lesssim \varepsilon,$$

which establishes the desired control on the first summand.

The second summand is controlled by

$$\begin{aligned} 2^k \sum_{l\ge -j} 2^{(l-k)/2}\alpha_l\beta_l &\lesssim 2^k \sum_{l\ge -j} 2^{(l-k)/2}\langle 2^{j+l}\rangle^{-8}2^{-\sigma l}2^{-j}a_{-j}a_l(\sigma)\mathscr{C}\langle 2^{j+l}\rangle^{-8}2^{l+j}a_{-j}a_l \\ &\lesssim \mathscr{C}2^{-\sigma k}2^{-2j}a_{-j}a_{-j}(\sigma) \sum_{l\ge -j} 2^{(l-k)/2}2^{k+j}2^{(1+\delta)(l+j)}a_{-j}a_l, \end{aligned} \tag{7-34}$$

and so it remains to show that

$$\sum_{l\ge -j} 2^{(l-k)/2}2^{k+j}2^{(1+\delta)(l+j)}a_{-j}a_l \lesssim \varepsilon, \tag{7-35}$$

which follows from pulling out a_{-j}^2 and summing.

Combining (7-30)–(7-35), we conclude from applying Lemma 7.2 with $\omega = 1/2$ that

$$\|P_k(fg)(r)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \varepsilon\mathscr{C}2^{-\sigma k}2^{-2j}a_{-j}a_{-j}(\sigma) \quad \text{for } k+j\le 0,$$

which, combined with (7-29) completes the proof of the lemma. □

Lemma 7.7. *For any $k\in\mathbb{Z}$ and $s\in[0,\infty)$ we have*

$$\|P_kA_m(s)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim (1+s2^{2k})^{-4}2^{-\sigma k}b_{k,s}(\sigma).$$

Proof. From the representation (7-5) for A_m it follows that

$$\begin{aligned} \|P_kA_m(s)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \int_s^\infty \|P_k(\overline{\psi_m}(r)\partial_l\psi_l(r))\|_{F_k(T)\cap S_k^{1/2}(T)}\,dr \\ + \int_s^\infty \|P_k(\overline{\psi_m}(r)\psi_lA_l(r))\|_{F_k(T)\cap S_k^{1/2}(T)}\,dr. \end{aligned} \tag{7-36}$$

Taking $k_0 \in \mathbb{Z}$ so that $s \in [2^{2k_0-1}, 2^{2k_0+1})$ and using (7-7), we see that the first term is dominated by

$$\sum_{j \geq k_0} \int_{2^{2j-1}}^{2^{2j+1}} \| P_k(\overline{\psi}_m(r) \partial_l \psi_l(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} dr \lesssim 2^{-\sigma k} \sum_{j \geq k_0} \langle 2^{j+k} \rangle^{-8} (2^{j+k} a_{-j} a_k(\sigma) + a_{-j} a_{-j}(\sigma)). \quad (7\text{-}37)$$

We claim that

$$\sum_{j \geq k_0} \langle 2^{j+k} \rangle^{-8} (2^{j+k} a_{-j} a_k(\sigma) + a_{-j} a_{-j}(\sigma)) \lesssim (1 + s2^{2k})^{-4} b_{k,s}(\sigma). \quad (7\text{-}38)$$

When $k + k_0 \geq 0$, it follows from (2-29) that the left-hand side of (7-38) is bounded by

$$2^{k_0+k} a_{-k_0} a_k(\sigma) \sum_{j \geq k_0} \langle 2^{j+k} \rangle^{-8} \left(2^{j-k_0} 2^{\delta(j-k_0)} + 2^{-k_0-k} 2^{\delta(j-k_0)} 2^{\delta(k+j)}\right)$$
$$\lesssim b_{k,s}(\sigma) \sum_{j \geq k_0} \langle 2^{j+k} \rangle^{-8} (2^{(1+\delta)(j-k_0)} + 2^{(\delta-1)(k_0+k)} 2^{2\delta(j-k_0)}), \quad (7\text{-}39)$$

and so it suffices to show that

$$\sum_{j \geq k_0} \langle 2^{j+k} \rangle^{-8} 2^{2(j-k_0)} \lesssim \langle 2^{j+k_0} \rangle^{-8}, \quad (7\text{-}40)$$

which follows from series comparison, for instance.

Together (7-40) and (7-39), show that (7-38) holds for $k + k_0 \geq 0$.

If, on the other hand, $k + k_0 \leq 0$, then we split the sum in (7-38) according to whether $j + k \leq 0$ or $j + k > 0$. In the first case,

$$\sum_{k_0 \leq j \leq -k} \langle 2^{j+k} \rangle^{-8} (2^{j+k} a_{-j} a_k(\sigma) + a_{-j} a_{-j}(\sigma)) \lesssim \langle 2^{k_0+k} \rangle^{-8} b_{k,s}(\sigma) + \sum_{k_0 \leq j \leq -k} \langle 2^{j+k} \rangle^{-8} 2^{j+k} a_{-j} a_k(\sigma). \quad (7\text{-}41)$$

Then

$$\sum_{k_0 \leq j \leq -k} \langle 2^{j+k} \rangle^{-8} 2^{j+k} a_{-j} a_k(\sigma) \lesssim \sum_{k_0 \leq j \leq -k} \langle 2^{j+k} \rangle^{-8} 2^{j+k} a_{-j} a_{-j}(\sigma) 2^{-\delta(j+k)} \sim (1 + s2^{2k})^{-4} b_{k,s}(\sigma). \quad (7\text{-}42)$$

When $j + k > 0$ we have

$$\sum_{j > -k} \langle 2^{j+k} \rangle^{-8} (2^{j+k} a_{-j} a_k(\sigma) + a_{-j} a_{-j}(\sigma)) \lesssim a_k a_k(\sigma) \sum_{j > -k} \langle 2^{j+k} \rangle^{-8} (2^{j+k} 2^{\delta(j+k)} + 2^{2\delta(j+k)})$$
$$\lesssim b_{k,s}(\sigma). \quad (7\text{-}43)$$

Therefore (7-41) and (7-42) imply (7-38) holds when $k + k_0 \leq 0$ and $j + k \leq 0$ and (7-43) implies it holds when both $k + k_0 \leq 0$ and $j + k > 0$.

Having shown (7-38), we combine it with (7-37), concluding that

$$\int_s^\infty \| P_k(\overline{\psi}_m(r) \partial_l \psi_l(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} dr \lesssim (1 + s2^{2k})^{-4} 2^{-\sigma k} b_{k,s}(\sigma). \quad (7\text{-}44)$$

We move on to control the second term in (7-36). By Lemma 7.6 and (7-38), this term is bounded by

$$\sum_{j\geq k_0}\int_{2^{2j-1}}^{2^{2j+1}}\|P_k(\overline{\psi_x}(r)\psi_x(r)A_x(r))\|_{F_k(T)\cap S_k^{1/2}(T)}\,dr$$
$$\lesssim \mathscr{C}2^{-\sigma k}\varepsilon\sum_{j\geq k_0}\langle 2^{j+k}\rangle^{-8}(\mathbf{1}_-(k+j)a_{-j}a_{-j}(\sigma)+\mathbf{1}_+(k+j)b_{k,2^{2j}}(\sigma))$$
$$\lesssim \mathscr{C}2^{-\sigma k}\varepsilon\langle 2^{k_0+k}\rangle^{-8}b_{k,2^{2k_0}}(\sigma). \tag{7-45}$$

Together (7-36), (7-44), and (7-45) imply that

$$\|P_kA_m(s)\|_{F_k(T)\cap S_k^{1/2}(T)}\lesssim 2^{-\sigma k}(1+s2^{2k})^{-4}b_{k,s}(\sigma)(1+\mathscr{C}\varepsilon),$$

from which it follows that $\mathscr{C}\lesssim 1+\mathscr{C}\varepsilon$ and hence $\mathscr{C}\lesssim 1$, proving the lemma. □

Lemma 7.8. *We have*

$$\|P_kA_l^2(r)\|_{F_k(T)\cap S_k^{1/2}(T)}\lesssim\begin{cases}\varepsilon 2^{-\sigma k}2^{-j}a_{-j}a_{-j}(\sigma) & \text{if } k+j\leq 0,\\ \varepsilon 2^{-\sigma k}2^{-j}b_{k,2^{2j}}(\sigma) & \text{if } k+j\geq 0.\end{cases}$$

Proof. We apply Lemma 7.2 with $f=g=A_l$ and $\omega=0$ so that

$$\|P_k(A_l^2(r))\|_{F_k(T)\cap S_k^{1/2}(T)}\lesssim\sum_{l\leq k}2^l\alpha_k\beta_l+\sum_{l\geq k}2^l\alpha_l\beta_l,$$

where

$$\alpha_k\lesssim 2^{-\sigma k}\langle 2^{j+k}\rangle^{-8}b_{k,s}(\sigma),\quad \beta_k\lesssim\langle 2^{j+k}\rangle^{-8}b_{k,s}.$$

Case $k+j\leq 0$. We first consider the case $k+j\leq 0$ and proceed to control the high-low frequency interaction. We have

$$\sum_{l\leq k}2^l\alpha_k\beta_l\lesssim 2^{-\sigma k}\sum_{l\leq k}2^lb_{k,2^{2j}}(\sigma)b_{l,2^{2j}}\lesssim 2^{-\sigma k}\sum_{p=k}^{-j}a_pa_p(\sigma)2^l\sum_{l\leq k}\sum_{q=l}^{-j}a_q^2$$
$$\lesssim 2^{-\sigma k}a_{-j}a_{-j}(\sigma)\sum_{p=k}^{-j}2^{-2\delta(j+p)}\sum_{l\leq k}2^la_{-j}^2\sum_{q=l}^{-j}2^{-2\delta(j+q)}. \tag{7-46}$$

It remains to show that

$$\sum_{p=k}^{-j}2^{-2\delta(j+p)}\sum_{l\leq k}2^la_{-j}^2\sum_{q=l}^{-j}2^{-2\delta(j+q)}\lesssim\varepsilon, \tag{7-47}$$

which follows from bounding a_{-j}^2 by ε and summing. To control the high-high interaction term we first split the sum as

$$\sum_{l\geq k}2^l\alpha_l\beta_l\lesssim\sum_{k\leq l<-j}2^l\alpha_l\beta_l+\sum_{l\geq -j}2^l\alpha_l\beta_l. \tag{7-48}$$

The first summand is controlled by

$$\sum_{k\le l<-j} 2^l \alpha_l \beta_l \lesssim 2^{-\sigma k} \sum_{k\le l<-j} 2^l b_{l,2^{2j}}(\sigma) b_{l,2^{2j}} \lesssim 2^{-\sigma k} 2^{-j} \sum_{k\le l<-j} 2^{j+l} \sum_{p=l}^{-j} a_p a_p(\sigma) \sum_{q=l}^{-j} a_q^2.$$

Pulling out $a_{-j}^3 a_{-j}(\sigma)$ and summing implies

$$\sum_{k\le l<-j} 2^l \alpha_l \beta_l \lesssim \varepsilon 2^{-\sigma k} 2^{-j} a_{-j} a_{-j}(\sigma). \tag{7-49}$$

The second summand is controlled by

$$\begin{aligned} \sum_{l\ge -j} 2^l \alpha_l \beta_l &\lesssim 2^{-\sigma k} \sum_{l\ge -j} 2^l \langle 2^{j+l}\rangle^{-8} b_{l,2^{2j}}(\sigma) b_{l,2^{2j}} \\ &\lesssim 2^{-\sigma k} \sum_{l\ge -j} 2^l \langle 2^{j+l}\rangle^{-8} 2^{2(l+j)} a_{-j}^2 a_l a_l(\sigma) \lesssim \varepsilon 2^{-\sigma k} 2^{-j} a_{-j} a_{-j}(\sigma). \end{aligned} \tag{7-50}$$

Combining (7-46)–(7-50), we conclude that

$$\| P_k A_l^2(r) \|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \varepsilon 2^{-\sigma k} 2^{-j} a_{-j} a_{-j}(\sigma) \quad \text{for } k+j \le 0. \tag{7-51}$$

Case $k+j\ge 0$. We now consider the case $k+j\ge 0$ and turn to the high-low frequency interaction, splitting it into two pieces:

$$\sum_{l\le k} 2^l \alpha_k \beta_l \le \sum_{l\le -j} 2^l \alpha_k \beta_l + \sum_{-j<l\le k} 2^l \alpha_k \beta_l. \tag{7-52}$$

The first summand is controlled by

$$\sum_{l\le -j} 2^l \alpha_k \beta_l \lesssim 2^{-\sigma k} 2^{-j} b_{k,2^{2j}}(\sigma) \sum_{l\le -j} 2^{l+j} \langle 2^{j+k}\rangle^{-8} b_{l,2^{2j}}, \tag{7-53}$$

and so we need to show that

$$\sum_{l\le -j} 2^{l+j} \langle 2^{j+k}\rangle^{-8} b_{l,2^{2j}} \lesssim \varepsilon, \tag{7-54}$$

which follows from

$$\sum_{l\le -j} 2^{l+j} b_{l,2^{2j}} \lesssim \sum_{l\le -j} 2^{l+j} \sum_{p=l}^{-j} a_p^2 \lesssim a_{-j}^2 \sum_{l\le -j} 2^{(1-2\delta)(l+j)} \lesssim \varepsilon.$$

The second summand in (7-52) is controlled by

$$\sum_{-j<l\le k} 2^l \alpha_k \beta_l \lesssim 2^{-\sigma k} 2^{-j} b_{k,2^{2j}}(\sigma) \sum_{j<l\le k} \langle 2^{j+l}\rangle^{-8} 2^{l+j} \langle 2^{j+k}\rangle^{-8} 2^{l+j} a_{-j} a_l, \tag{7-55}$$

where we note that

$$\sum_{-j<l\le k} \langle 2^{j+l}\rangle^{-8} 2^{2l+2j} a_{-j} a_l \lesssim a_{-j}^2 \sum_{-j<l\le k} \langle 2^{j+l}\rangle^{-8} 2^{(2+\delta)(l+j)} \lesssim \varepsilon. \tag{7-56}$$

We now turn to the high-high frequency interaction. We have

$$\begin{aligned}\sum_{l\geq k} 2^l \alpha_l \beta_l &\lesssim \sum_{l\geq k} 2^l 2^{-\sigma l} \langle 2^{j+l}\rangle^{-8} 2^{2(l+j)} a_{-j}^2 a_l a_l(\sigma)\\ &\lesssim 2^{-\sigma k} 2^{-j} 2^{k+j} a_{-j} \sum_{l\geq k} 2^{l-k} 2^{-\sigma(l-k)} \langle 2^{j+l}\rangle^{-8} 2^{2(l+j)} a_{-j} a_l a_l(\sigma)\\ &\lesssim 2^{-\sigma k} 2^{-j} b_{k,2^{2j}}(\sigma) \sum_{l\geq k} \langle 2^{j+l}\rangle^{-8} 2^{(1+\delta)(l-k)} 2^{2(l+j)} a_{-j} a_l. \end{aligned} \tag{7-57}$$

It remains to show that

$$\sum_{l\geq k} \langle 2^{j+l}\rangle^{-8} 2^{(1+\delta)(l-k)} 2^{2(l+j)} a_{-j} a_l \lesssim \varepsilon, \tag{7-58}$$

which follows from bounding $a_{-j} a_l$ by ε and summing.

Together (7-52)–(7-58) imply that

$$\| P_k A_l^2(r)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \varepsilon 2^{-\sigma k} 2^{-j} b_{k,2^{2j}}(\sigma) \quad \text{for } k+j\geq 0,$$

which combined with (7-51) implies the lemma. □

Set

$$c_{k,j}(\sigma) = \begin{cases} 2^{-j} a_{-j} a_{-j}(\sigma) & \text{if } k+j\leq 0,\\ 2^{2k+j} a_{-j} a_k(\sigma) & \text{if } k+j\geq 0.\end{cases} \tag{7-59}$$

Lemma 7.9. *Let* $r\in[2^{2j-2}, 2^{2j+2}]$ *and let*

$$F \in \{A_l^2,\ \partial_l A_l,\ fg : l=1,2;\ f,g\in\{\psi_m, \overline{\psi_m} : m=1,2\}\}.$$

Then

$$\| P_k F(r)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} c_{k,j}(\sigma). \tag{7-60}$$

Proof. If $F = A_l^2$, then (7-60) is an immediate consequence of Lemma 7.8 when $k+j\leq 0$. If $k+j\geq 0$, then Lemma 7.8 implies

$$\| P_k A_l^2(r)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \varepsilon 2^{-\sigma k} 2^{-j} 2^{k+j} a_{-j} a_{-j}(\sigma),$$

and multiplying the right-hand side by 2^{k+j} yields the desired estimate.

Consider now the case where $F = \partial_l A_l$. By Lemma 7.7, we have

$$\| P_k(\partial_l A_l)(r)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim 2^k \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} b_{k,2^{2j}}(\sigma). \tag{7-61}$$

When $k+j\geq 0$, we rewrite (7-61) as

$$\| P_k(\partial_l A_l)(r)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} 2^k 2^{k+j} a_{-j} a_k(\sigma),$$

which is the desired bound (7-60). If $k+j \le 0$, then (7-61) becomes

$$\|P_k(\partial_l A_l)(r)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} 2^k \sum_{p=k}^{-j} a_p a_p(\sigma)$$
$$\lesssim \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} 2^{-j} a_{-j} a_{-j}(\sigma) = \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} c_{k,j}(\sigma).$$

If $F = fg$, fg as in the statement of the lemma, then (7-60) follows directly from (7-6) when $k+j \le 0$. If $k+j \ge 0$, then to get (7-60) we multiply the right-hand side of (7-6) by 2^{2j+2k}. □

Set

$$d_{k,j} := \varepsilon \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} 2^{2k} (a_k(\sigma) + 2^{-3(k+j)/2} a_{-j}(\sigma)). \tag{7-62}$$

Lemma 7.10. *We have*

$$\|P_k U_m(r)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \varepsilon \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} 2^{2k} (a_k(\sigma) + 2^{-3(k+j)/2} a_{-j}(\sigma)) =: d_{k,j}.$$

Proof. Using now (2-21) instead of (2-22), i.e., taking now

$$U_\alpha = i A_l \partial_l \psi_\alpha + i \partial_l (A_l \psi_\alpha) - A_x^2 \psi_\alpha + i \,\mathrm{Im}(\psi_\alpha \psi_l) \psi_l,$$

we have that it suffices to prove that

$$\|P_k(F(r) f(r))\|_{F_k(T)\cap S_k^{1/2}(T)} + 2^k \|P_k(A_l(r) f(r))\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim d_{k,j},$$

where

$$F \in \{A_l^2, \partial_l A_l, gh : l = 1, 2;\ f, h \in \{\psi_m, \overline{\psi_m} : m = 1, 2\}\}$$

and $f \in \{\psi_m, \overline{\psi_m} : m = 1, 2\}$. We consider the terms $P_k(Ff)$, and $P_k(Af)$ separately.

Controlling $P_k(Ff)$. We apply Lemma 7.2 to $P_k(Ff)$, handling the different frequency interactions separately and according to cases. We record a consequence of (7-60):

$$\alpha_k \lesssim \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} c_{k,j}(\sigma),$$

Let us begin by assuming $k+j \le 0$. For the low-high frequency interaction, we have

$$\sum_{l\le k} 2^l \alpha_l \beta_k \lesssim 2^{-\sigma k} a_k(\sigma) \sum_{l\le k} 2^l c_{l,j} \lesssim 2^{-\sigma k} a_k(\sigma) \sum_{l\le k} 2^{l-j} a_{-j}^2 \lesssim \varepsilon 2^{-\sigma k} 2^{k-j} 2^{-\delta(k+j)} a_{-j}(\sigma). \tag{7-63}$$

In a similar manner we control the high-low frequency interaction by

$$\sum_{l\le k} 2^l \alpha_k \beta_l \lesssim 2^{-\sigma k} c_{k,j}(\sigma) \sum_{l\le k} 2^l a_l \lesssim 2^{-\sigma k} 2^{-j} a_{-j} a_{-j}(\sigma) \sum_{l\le k} 2^l a_l \lesssim \varepsilon 2^{-\sigma k} 2^{k-j} a_{-j}(\sigma). \tag{7-64}$$

The high-high frequency interaction we split into two sums:

$$2^k \sum_{l\ge k} 2^{(l-k)/2} \alpha_l \beta_l \lesssim 2^k \sum_{k\le l<-j} 2^{(l-k)/2} \alpha_l \beta_l + 2^k \sum_{l\ge -j} 2^{(l-k)/2} \alpha_l \beta_l. \tag{7-65}$$

We control the first summand using the definition (7-59) of $c_{k,j}(\sigma)$, the frequency envelope properties (2-29), (2-30), and energy dispersion:

$$\begin{aligned}2^k \sum_{k\le l<-j} 2^{(l-k)/2}\alpha_l\beta_l &\lesssim 2^k \sum_{k\le l<-j} 2^{(l-k)/2}2^{-\sigma l}c_{l,j}(\sigma)a_l \\ &\lesssim 2^{-\sigma k}2^{k-j}a_{-j}(\sigma)a_{-j}\sum_{k\le l<-j}2^{(l-k)/2}a_l \\ &\lesssim 2^{-\sigma k}2^{k-j}2^{-(k+j)/2}a_{-j}(\sigma)a_{-j}\sum_{k\le l<-j}2^{(l+j)/2}a_l \\ &\lesssim \varepsilon 2^{-\sigma k}2^{k-j}2^{-(k+j)/2}a_{-j}(\sigma). \end{aligned} \tag{7-66}$$

In like manner we control the second summand:

$$\begin{aligned}2^k \sum_{l\ge -j} 2^{(l-k)/2}\alpha_l\beta_l &\lesssim 2^k \sum_{l\ge -j} \langle 2^{j+l}\rangle^{-8}2^{(l-k)/2}2^{-\sigma l}c_{l,j}(\sigma)a_l \\ &\lesssim 2^k \sum_{l\ge -j} \langle 2^{j+l}\rangle^{-8}2^{(l-k)/2}2^{-\sigma l}2^{2l+j}a_{-j}a_l(\sigma)a_l \\ &\lesssim \varepsilon 2^{-\sigma k}2^{k-j}2^{-(k+j)/2}a_{-j}(\sigma). \end{aligned} \tag{7-67}$$

Combining (7-63)–(7-67), we conclude that

$$\|P_k(F(r)f(r))\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \varepsilon 2^{-\sigma k}2^{k-j}2^{-(k+j)/2}a_{-j}(\sigma), \quad k+j\le 0. \tag{7-68}$$

We now turn to the case $k+j\ge 0$. In the low-high frequency interaction case, we have

$$\begin{aligned}\sum_{l\le k} 2^l\alpha_l\beta_k &\lesssim \langle 2^{j+k}\rangle^{-8}2^{-\sigma k}a_k(\sigma)\sum_{l\le k}\langle 2^{j+l}\rangle^{-8}2^l c_{l,j} \\ &\lesssim \langle 2^{j+k}\rangle^{-8}2^{-\sigma k}2^{2k}a_k(\sigma)\biggl(\sum_{l\le -j}2^{l-2k}2^{-j}a_{-j}^2+\sum_{-j<l\le k}\langle 2^{j+l}\rangle^{-8}2^{l-2k}2^{l+j}a_{-j}a_l\biggr). \end{aligned} \tag{7-69}$$

To estimate the first term we use

$$a_{-j}^2\sum_{l\le -j}2^{l-k}2^{-j-k} \lesssim \varepsilon 2^{-(j+k)}\cdot 2^{-(j+k)} \le \varepsilon, \tag{7-70}$$

and for the second

$$\begin{aligned}a_{-j}\sum_{-j<l\le k}\langle 2^{j+l}\rangle^{-8}2^{3l+j-2k}a_l &= a_{-j}\sum_{-j<l\le k}\langle 2^{j+l}\rangle^{-8}2^{l+j}2^{2l-2k}a_l \\ &\lesssim a_{-j}a_k\sum_{-j<l\le k}\langle 2^{j+l}\rangle^{-8}2^{l+j}2^{(2-\delta)(l-k)} \lesssim \varepsilon. \end{aligned} \tag{7-71}$$

In the high-low frequency interaction case, we have

$$\begin{aligned}\sum_{l\le k} 2^l \alpha_k \beta_l &\lesssim \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} c_{k,j}(\sigma) \sum_{l\le k} \langle 2^{j+l}\rangle^{-8} 2^l a_l \\ &\lesssim \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} 2^{2k+j} a_{-j} a_k(\sigma) \sum_{l\le k} \langle 2^{j+l}\rangle^{-8} 2^l a_l \\ &\lesssim \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} 2^{2k} a_k(\sigma) a_{-j}^2. \qquad (7\text{-}72)\end{aligned}$$

In the high-high frequency interaction case we have

$$\begin{aligned}\sum_{l\ge k} 2^l \alpha_l \beta_l &\lesssim \sum_{l\ge k} \langle 2^{j+l}\rangle^{-8} 2^l 2^{-\sigma l} a_l(\sigma) c_{l,j} \\ &\lesssim \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} \sum_{l\ge k} \langle 2^{j+l}\rangle^{-8} 2^l a_l(\sigma) 2^{2l+j} a_{-j} a_l \\ &\lesssim \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} 2^{2k} a_k(\sigma) a_{-j}^2. \qquad (7\text{-}73)\end{aligned}$$

From (7-69)–(7-73) we conclude that

$$\|P_k(F(r) f(r))\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \varepsilon \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} 2^{2k} a_k(\sigma), \quad k+j\ge 0. \qquad (7\text{-}74)$$

Controlling $2^k P_k(Af)$. We now apply Lemma 7.2 to $P_k(A_l f)$. Note that

$$\alpha_k \lesssim \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} b_{k,r}(\sigma)$$

because of Lemma 7.7, and that

$$\beta_k \lesssim \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} a_k(\sigma).$$

We begin by assuming $k+j\le 0$. The low-high frequency interaction is controlled by

$$\begin{aligned}\sum_{l\le k} 2^l \alpha_l \beta_k &\lesssim 2^{-\sigma k} a_k(\sigma) \sum_{l\le k} 2^l \sum_{p=l}^{-j} a_l^2 \\ &\lesssim 2^{-\sigma k} 2^{-\delta(k+j)} a_{-j}^2 a_{-j}(\sigma) \sum_{l\le k} 2^l \sum_{p=l}^{-j} 2^{-2(j+p)}.\end{aligned}$$

Summing yields

$$2^k \sum_{l\le k} 2^l \alpha_l \beta_k \lesssim 2^{2k} 2^{-\sigma k} 2^{-(k+j)/2} a_{-j}^2 a_{-j}(\sigma). \qquad (7\text{-}75)$$

Control over the high-low frequency interaction follows from

$$\begin{aligned}\sum_{l\le k} 2^l \alpha_k \beta_l &\lesssim 2^{-\sigma k} \sum_{p=k}^{-j} a_p a_p(\sigma) \sum_{l\le k} 2^l a_l \\ &\lesssim 2^k 2^{-\sigma k} 2^{-2\delta(k+j)} a_{-j} a_k a_{-j}(\sigma). \qquad (7\text{-}76)\end{aligned}$$

We now turn to the high-high frequency interaction. We begin by splitting the sum:

$$2^k \sum_{l \geq k} 2^{(l-k)/2} \alpha_l \beta_l \lesssim 2^k \sum_{k \leq l < -j} 2^{(l-k)/2} \alpha_l \beta_l + 2^k \sum_{l \geq -j} 2^{(l-k)/2} \alpha_l \beta_l. \tag{7-77}$$

Then

$$\begin{aligned} 2^k \sum_{k \leq l < -j} 2^{(l-k)/2} \alpha_l \beta_l &\lesssim 2^k 2^{-\sigma k} a_{-j}(\sigma) \sum_{k \leq l < -j} 2^{(l-k)/2} 2^{-\delta(j+l)} \sum_{p=l}^{-j} a_p^2 \\ &\lesssim 2^k 2^{-\sigma k} 2^{-(k+j)/2} a_{-j}^2 a_{-j}(\sigma). \end{aligned} \tag{7-78}$$

As for the second summand, we have

$$\begin{aligned} 2^k \sum_{l \geq -j} 2^{(l-k)/2} \alpha_l \beta_l &\lesssim 2^k \sum_{l \geq -j} \langle 2^{j+l} \rangle^{-8} 2^{(l-k)/2} 2^{l+j} a_{-j} a_l(\sigma) 2^{-\sigma l} a_l \\ &\lesssim 2^k 2^{-\sigma k} 2^{-(k+j)/2} a_{-j}^2 a_{-j}(\sigma). \end{aligned} \tag{7-79}$$

Combining (7-75)– (7-79) yields

$$2^k \| P_k(A_l(r) f(r)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \varepsilon 2^{2k} 2^{-\sigma k} 2^{-(k+j)/2} a_{-j}(\sigma), \quad k+j \leq 0. \tag{7-80}$$

Now let us assume that $k+j \geq 0$. The low-high frequency interaction we first split into two pieces:

$$\sum_{l \leq k} 2^l \alpha_l \beta_k \lesssim \sum_{l \leq -j} 2^l \alpha_l \beta_k + \sum_{-j < l \leq k} 2^l \alpha_l \beta_k. \tag{7-81}$$

For the first term, we have

$$\begin{aligned} \sum_{l \leq -j} 2^l \alpha_l \beta_k &\lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} a_k(\sigma) \sum_{l \leq -j} \sum_{p=l}^{-j} a_p^2 \\ &\lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} a_{-j}^2 a_k(\sigma) \sum_{l \leq -j} 2^l \sum_{p=l}^{-j} 2^{-2\delta(j+p)}. \end{aligned} \tag{7-82}$$

Then

$$\sum_{l \leq -j} 2^l \sum_{p=l}^{-j} 2^{-2\delta(j+p)} \lesssim \sum_{l \leq -j} 2^l 2^{-2\delta(j+l)} \lesssim 2^{-j} \leq 2^k. \tag{7-83}$$

As for the second summand,

$$\begin{aligned} \sum_{-j < l \leq k} 2^l \alpha_l \beta_k &\lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} a_k(\sigma) \sum_{-j < l \leq k} \langle 2^{j+l} \rangle^{-8} 2^l 2^{l+j} a_{-j} a_l \\ &\lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} 2^k a_{-j}^2 a_k(\sigma). \end{aligned} \tag{7-84}$$

The high-low frequency interaction is controlled by

$$\begin{aligned} \sum_{l \leq k} 2^l \alpha_k \beta_l &\lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} 2^{k+j} a_{-j} a_k(\sigma) \sum_{l \leq k} \langle 2^{j+l} \rangle^{-8} 2^l a_l \\ &\lesssim \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} 2^k a_{-j}^2 a_k(\sigma). \end{aligned} \tag{7-85}$$

Finally, the high-high frequency interaction is controlled by

$$\sum_{l\geq k} 2^l \alpha_l \beta_l \lesssim \sum_{l\geq k} \langle 2^{j+l}\rangle^{-8} 2^l 2^{l+j} a_{-j} a_l 2^{-\sigma l} a_l(\sigma) \lesssim \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} 2^k a_{-j} a_k a_k(\sigma). \tag{7-86}$$

Thus, in view of (7-81)–(7-86), we have shown that

$$2^k \| P_k(A_l(r) f(r))\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \varepsilon \langle 2^{j+k}\rangle^{-8} 2^{-\sigma k} 2^{2k} a_k(\sigma), \quad k+j\geq 0. \tag{7-87}$$

Combining (7-68), (7-74), (7-80), and (7-87) proves the lemma. □

Lemma 7.11. *We have*

$$\left\| \int_0^s e^{(s-s')\Delta} P_k U_m(s')\, ds' \right\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \varepsilon (1+s2^{2k})^{-4} 2^{-\sigma k} a_k(\sigma).$$

Proof. Let $k_0\in\mathbb{Z}$ be such that $s\in[2^{2k_0-1}, 2^{2k_0+1})$. If $k+k_0\leq 0$, then it follows from Lemma 7.10 that

$$\begin{aligned}\left\| \int_0^s e^{(s-r)\Delta} P_k U_m(r)\, dr \right\|_{F_k(T)\cap S_k^{1/2}(T)} &\lesssim \sum_{j\leq k_0} \int_{2^{2j-1}}^{2^{2j+1}} \|P_k U_m(r)\|_{F_k(T)\cap S_k^{1/2}(T)}\, dr \\ &\lesssim \sum_{j\leq k_0} 2^{2j} \varepsilon 2^{-\sigma k} 2^{2k} (a_k(\sigma) + 2^{-3(k+j)/2} a_{-j}(\sigma)) \\ &\lesssim \varepsilon 2^{-\sigma k} a_k(\sigma) \sum_{j\leq k_0} 2^{2k+2j} (1+2^{-3(k+j)/2} 2^{-\delta(k+j)}) \\ &\lesssim \varepsilon 2^{-\sigma k} a_k(\sigma).\end{aligned}$$

On the other hand, if $k+k_0>0$, then

$$\begin{aligned}\left\| \int_0^s e^{(s-r)\Delta} P_k U_m(r)\, dr \right\|_{F_k(T)\cap S_k^{1/2}(T)} &\lesssim \int_0^{s/2} \|e^{(s-r)\Delta} P_k U_m(r)\|_{F_k(T)\cap S_k^{1/2}(T)}\, dr \\ &\qquad + \int_{s/2}^s \|e^{(s-r)\Delta} P_k U_m(r)\|_{F_k(T)\cap S_k^{1/2}(T)}\, dr \\ &\lesssim \sum_{j\leq k_0} 2^{-20(k+k_0)} 2^{2j} d_{k,j} + 2^{2k_0} d_{k,k_0} \\ &\lesssim 2^{-20(k_0+k)} \sum_{j\leq k_0} 2^{2j} d_{k,j} + 2^{-2k} d_{k,k_0}.\end{aligned} \tag{7-88}$$

By Lemma 7.10 and the fact that $k+k_0>0$, we have

$$2^{-2k} d_{k,k_0} \lesssim \varepsilon \langle 2^{k_0+k}\rangle^{-8} 2^{-\sigma k} a_k(\sigma)$$

and

$$
\begin{aligned}
2^{-20(k_0+k)} \sum_{j \le k_0} 2^{2j} d_{k,j} &\lesssim 2^{-20(k_0+k)} \sum_{j \le k_0} \varepsilon \langle 2^{j+k} \rangle^{-8} 2^{-\sigma k} 2^{2k} \left(2^{2j} a_k(\sigma) + 2^{j/2} 2^{-3k/2} a_{-j}(\sigma) \right) \\
&\lesssim \varepsilon 2^{-\sigma k} a_k(\sigma) 2^{-20(k_0+k)} \sum_{j \le k_0} \langle 2^{j+k} \rangle^{-8} \left(2^{2j+2k} + 2^{(j+k)/2} 2^{\delta |j+k|} \right) \\
&\lesssim \varepsilon \langle 2^{k_0+k} \rangle^{-8} 2^{-\sigma k} a_k(\sigma),
\end{aligned}
$$

which, combined with (7-88), completes the proof of the lemma. □

Lemma 7.12. *The following bound from* (4-12) *holds*:

$$\| P_k \psi_m(s) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim (1 + s 2^{2k})^{-4} 2^{-\sigma k} b_k(\sigma).$$

Proof. In view of (7-1), we have

$$P_k \psi_m(s) = e^{s\Delta} P_k \psi_m(0) + \int_0^s e^{(s-r)\Delta} P_k U_m(r)\, dr.$$

Then it follows from Lemma 7.11 that

$$\| P_k \psi_m(s) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim 2^{-\sigma k} (1 + s 2^{2k})^{-4} (b_k(\sigma) + \varepsilon a_k(\sigma)), \quad 0 \le \sigma \le \sigma_1 - 1.$$

Therefore $a_k(\sigma) \lesssim b_k(\sigma) + \varepsilon a_k(\sigma)$ and hence

$$a_k(\sigma) \lesssim b_k(\sigma), \tag{7-89}$$

as required. □

7B. *Connection coefficient control.* The main results of this subsection are the $L^2_{t,x}$ bounds (4-14) and (4-16), respectively proven in Corollary 7.19 and Lemma 7.21, and the frequency-localized $L^2_{t,x}$ bounds (4-15) and (4-17), respectively proven in Corollaries 7.20 and 7.22.

Lemma 7.13. *Let* $s \in [2^{2j-2}, 2^{2j+2}]$. *Then*

$$\| P_k (A_l(s) \psi_m(s)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \varepsilon (1 + s 2^{2k})^{-3} (s 2^{2k})^{-3/8} 2^k 2^{-\sigma k} b_k(\sigma).$$

Proof. Using (7-80) and (2-29), we have

$$2^k \| P_k (A_l(s) \psi_m(s)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \varepsilon 2^{2k} 2^{-\sigma k} 2^{-(1/2+\delta)(k+j)} a_k(\sigma). \tag{7-90}$$

Combining (7-90), (7-87), and (7-89) then yields

$$\| P_k (A_l(s) \psi_m(s)) \|_{F_k(T) \cap S_k^{1/2}(T)} \lesssim \begin{cases} \varepsilon (s 2^{2k})^{-3/8} 2^k 2^{-\sigma k} b_k(\sigma) & \text{if } k + j \le 0, \\ \varepsilon (1 + s 2^{2k})^{-4} 2^k 2^{-\sigma k} b_k(\sigma) & \text{if } k + j \ge 0, \end{cases}$$

which proves the lemma. □

Lemma 7.14 [Bejenaru et al. 2011c, §5]. *Assume that* $T \in (0, 2^{2\mathcal{K}}]$, $f, g \in H^{\infty,\infty}(T)$, $P_k f \in S_k^\omega(T)$, *and* $P_k g \in L^4_{t,x}$ *for some* $\omega \in [0, 1/2]$ *and all* $k \in \mathbb{Z}$. *Set*

$$\mu_k := \sum_{|j-k|\le 20} \|P_j f\|_{S^\omega_{k'}(T)}, \qquad \nu_k := \sum_{|j-k|\le 20} \|P_j g\|_{L^4_{t,x}}.$$

Then, for any $k \in \mathbb{Z}$,

$$\|P_k(fg)\|_{L^4_{t,x}} \lesssim \sum_{j\le k} 2^j \mu_j \nu_k + \sum_{j\le k} 2^{(k+j)/2} \mu_k \nu_j + 2^k \sum_{j\ge k} 2^{-\omega(j-k)} \mu_j \nu_j.$$

Lemma 7.15. *We have*

$$\|P_k \psi_s(0)\|_{L^4_{t,x}} + \|P_k \psi_t(0)\|_{L^4_{t,x}} \lesssim 2^k \tilde{b}_k \Big(1 + \sum_j b_j^2\Big).$$

Proof. We only treat $\psi_t(0)$ since $\psi_s(0)$ and $\psi_t(0)$ differ only by a factor of i. As $\psi_t(0) = i D_l(0)\psi_l(0)$, we have

$$\psi_t(0) = i\partial_l \psi_l(0) - A_l(0)\psi_l(0).$$

Clearly

$$\|P_k \partial_l \psi_l(0)\|_{L^4_{t,x}} \lesssim 2^k \|P_k \psi_x(0)\|_{L^4_{t,x}} \lesssim 2^k \tilde{b}_k.$$

For the remaining term, we apply Lemma 7.14, bounding $P_j A_l(0)$ in $S_j^{1/2}$ by $\sum_p b_p^2$, which follows from Lemma 7.7. We get

$$\|P_k(A_l(0)\psi_l(0))\|_{L^4_{t,x}} \lesssim \sum_{j\le k} 2^j \Big(\sum_p b_p^2\Big)\tilde{b}_k + \sum_{j\le k} 2^{(k+j)/2}\Big(\sum_p b_p^2\Big)\tilde{b}_j + 2^k \sum_{j\ge k} 2^{-(j-k)/2}\Big(\sum_p b_p^2\Big)\tilde{b}_j.$$

Therefore

$$\|P_k(A_l \psi_l(0))\|_{L^4_{t,x}} \lesssim 2^k \tilde{b}_k \Big(\sum_j b_j^2\Big). \qquad \square$$

Corollary 7.16. *We have*

$$\|P_k \psi_s(0)\|_{L^4_{t,x}} + \|P_k \psi_t(0)\|_{L^4_{t,x}} \lesssim 2^k 2^{-\sigma k} b_k(\sigma)\Big(1 + \sum_j b_j^2\Big).$$

Proof. Without loss of generality, we prove the bound only for ψ_t. We have

$$\|P_k \partial_l \psi_l(0)\|_{L^4_{t,x}} \lesssim 2^k \|P_k \psi_x(0)\|_{L^4_{t,x}} \lesssim 2^k 2^{-\sigma k} b_k(\sigma).$$

It remains to control $P_k(A_l(0)\psi_l(0))$ in $L^4_{t,x}$. The obstruction to applying Lemma 7.14 as we did in Lemma 7.15 is the high-low interaction, for which summation can be achieved only for small σ. If we restrict the range of σ to $\sigma < 1/2 - 2\delta$, then we ensure the constant remains bounded and can apply Lemma 7.14 as in Lemma 7.15.

For $\sigma \ge 1/2 - 2\delta$, we can still apply the bounds of Lemma 7.14 to the low-high and high-high interactions. For the remaining high-low interaction, we bound $A_l(0)$ in $L^4_{t,x}$ and $\psi_l(0)$ in $L^\infty_{t,x}$. In particular, we have, thanks to (7-95) and Bernstein, that

$$\begin{aligned}\sum_{\substack{|j_1-k|\le 4\\ j_2\le k+4}} \|P_k(P_{j_1}A_l(0)P_{j_2}\psi_l(0))\|_{L^4_{t,x}} &\lesssim \sum_{\substack{|j_1-k|\le 4\\ j_2\le k+4}} \|P_{j_1}A_l(0)\|_{L^4_{t,x}}\|P_{j_2}\psi_l(0)\|_{L^\infty_{t,x}}\\ &\lesssim \sum_{\substack{|j_1-k|\le 4\\ j_2\le k+4}} 2^{-\sigma j_1}b_{j_1}b_{j_1}(\sigma)2^{j_2}\|P_{j_2}\psi_l(0)\|_{L^\infty_t L^2_x}\\ &\lesssim \sum_{j_2\le k+4} 2^{-\sigma k}b_k b_k(\sigma)2^{j_2}b_{j_2}\\ &\lesssim 2^{-\sigma k}b_k^2 b_k(\sigma)\sum_{j_2\le k+4} 2^k 2^{(j_2-k)+(k-j_2)\delta} \lesssim 2^{-\sigma k}2^k b_k^2 b_k(\sigma). \qquad \square\end{aligned}$$

Lemma 7.17. *We have*

$$\|P_k\psi_s(s)\|_{L^4_{t,x}} + \|P_k\psi_t(s)\|_{L^4_{t,x}} \lesssim (1+s2^{2k})^{-2}2^k\tilde{b}_k\Big(1+\sum_j b_j^2\Big).$$

Proof. We treat only $\psi_t(s)$ since the proof for $\psi_s(s)$ is analogous. From (7-1) we have

$$\psi_t(s) = e^{s\Delta}\psi_t(0) + \int_0^s e^{(s-r)\Delta}U_t(r)\,dr.$$

We claim that

$$\Big\|\int_0^s e^{(s-r)\Delta}P_kU_t(r)\,dr\Big\|_{L^4_{t,x}} \lesssim \varepsilon(1+s2^{2k})^{-2}2^k\tilde{b}_k\Big(1+\sum_j b_j^2\Big), \tag{7-91}$$

which combined with Lemma 7.15 and a standard iteration argument proves the lemma.

As in the proof of Lemma 7.11, we take

$$F \in \{A_l^2,\ \partial_l A_l,\ fg : l=1,2;\ f,g\in\{\psi_m,\overline{\psi_m} : m=1,2\}\}.$$

By (7-60) and (7-89) we have

$$\|P_kF(r)\|_{S_k^{1/2}(T)} \lesssim \varepsilon^{1/2}(1+s2^{2k})^{-2}(s2^{2k})^{-5/8}2^k b_k. \tag{7-92}$$

Moreover, by Lemma 7.7,

$$\|P_kA_l(r)\|_{S_k^{1/2}(T)} \lesssim \varepsilon^{1/2}(1+s2^{2k})^{-3}(s2^{2k})^{-1/8}b_k. \tag{7-93}$$

Applying Lemma 7.14 with $\omega = 1/2$ yields

$$\|P_k(F(r)\psi_t(r))\|_{L^4_{t,x}} + 2^k\|P_k(A_l(r)\psi_t(r))\|_{L^4_{t,x}} \lesssim \varepsilon(1+s2^{2k})^{-2}(s2^{2k})^{-7/8}2^k\tilde{b}_k\Big(1+\sum_j b_j^2\Big). \tag{7-94}$$

Integrating with respect to s yields

$$\int_0^s (1+(s-r)2^{2k})^{-N}(1+r2^{2k})^{-2}(r2^{2k})^{-7/8}\,dr \lesssim 2^{-2k}(1+s2^{2k})^{-2},$$

which, together with (7-94), implies (7-91). $\square$

Lemma 7.18. *We have*

$$\|P_k A_m(0)\|_{L^4_{t,x}} \lesssim 2^{-\sigma k} b_k b_k(\sigma). \tag{7-95}$$

Proof. We have

$$\|P_k \psi_m(s)\|_{S^0_k} \lesssim (1+s2^{2k})^{-4} 2^{-\sigma k} b_k(\sigma)$$

and

$$\|P_k(D_l \psi_l)(s)\|_{L^4_{t,x}} \lesssim (1+s2^{2k})^{-3}(s2^{2k})^{-3/8} 2^k 2^{-\sigma k} b_k(\sigma).$$

Applying Lemma 7.14 with $\omega = 0$, we get

$$\begin{aligned}\|P_k A_m(0)\|_{L^4_{t,x}} &\lesssim \sum_{l=1,2} \int_0^\infty \|P_k(\overline{\psi_m(s)} D_l \psi_l(s))\|_{L^4_{t,x}}\, ds \\ &\lesssim 2^{-\sigma k} \sum_{j \le k} b_j b_k(\sigma) 2^{j+k} \int_0^\infty (1+s2^{2k})^{-3}(s2^{2k})^{-3/8}\, ds \\ &\qquad + 2^{-\sigma k} \sum_{j\le k} b_k(\sigma) b_j 2^{(k+j)/2} 2^j \int_0^\infty (1+s2^{2k})^{-4}(s2^{2j})^{-3/8}\, ds \\ &\qquad + \sum_{j \ge k} 2^{-\sigma j} b_j(\sigma) b_j 2^{k-j} 2^{2j} \int_0^\infty (1+s2^{2j})^{-7}(s2^{2j})^{-3/8}\, ds.\end{aligned}$$

Call the integrals I_1, I_2, and I_3, respectively. Clearly I_1 and I_3 satisfy $I_1 \lesssim 2^{-2k}$ and $I_3 \lesssim 2^{-2j}$. By Cauchy–Schwarz, I_2 satisfies

$$I_2 \lesssim \left(\int_0^\infty (1+s2^{2k})^{-8}(1+s2^{2j})^4\, ds\right)^{1/2} \left(\int_0^\infty (1+s2^{2j})^{-4}(s2^{2j})^{-3/8}\, ds\right)^{1/2} \lesssim 2^{-j-k}.$$

Therefore

$$\|P_k A_m(0)\|_{L^4_{t,x}} \lesssim 2^{-\sigma k} b_k(\sigma) \sum_{j\le k} \bigl(b_j 2^{j-k} + b_j 2^{(j-k)/2}\bigr) + 2^{-\sigma k} \sum_{j \ge k} b_j(\sigma) b_j 2^{k-j} \lesssim 2^{-\sigma k} b_k b_k(\sigma). \quad \square$$

Corollary 7.19. *We have*

$$\|A_x^2(0)\|_{L^2_{t,x}} \lesssim \sup_{j\in\mathbb{Z}} b_j^2 \cdot \sum_{k\in\mathbb{Z}} b_k^2.$$

Proof.

$$\|A_x^2(0)\|_{L^2_{t,x}} \lesssim \|A_x(0)\|^2_{L^4_{t,x}} \lesssim \sum_{k\in\mathbb{Z}} \|P_k A_x(0)\|^2_{L^4_{t,x}} \lesssim \sup_{j\in\mathbb{Z}} b_j^2 \cdot \sum_{k\in\mathbb{Z}} b_k^2. \quad \square$$

Corollary 7.20. *Let $\sigma \ge 2\delta$. Then*

$$\|P_k A_x^2(0)\|_{L^2_{t,x}} \lesssim 2^{-\sigma k} b_k(\sigma) \cdot \sup_j b_j \cdot \sum_{l\in\mathbb{Z}} b_l^2.$$

Proof. We perform a Littlewood–Paley decomposition and invoke Corollary 7.19.

Consider first the high-low interactions:

$$\sum_{\substack{|j_2-k|\le 4\\ j_1\le k-5}} \|P_k(P_{j_1}A_x P_{j_2}A_x)\|_{L^2} \lesssim \sum_{\substack{|j_2-k|\le 4\\ j_1\le k-5}} \|P_{j_1}A_x\|_{L^4}\|P_{j_2}A_x\|_{L^4} \lesssim 2^{-\sigma k} b_k b_k(\sigma) \sum_{j_1\le k-5} b_{j_1}^2.$$

Next consider the high-high interactions:

$$\sum_{\substack{j_1,j_2\geq k-4\\|j_1-j_2|\leq 8}} \|P_k(P_{j_1}A_x P_{j_2}A_x)\|_{L^2} \lesssim \sum_{\substack{j_1,j_2\geq k-4\\|j_1-j_2|\leq 8}} \|P_{j_1}A_x\|_{L^4}\|P_{j_2}A_x\|_{L^4} \lesssim \sum_{j\geq k-4} 2^{-\sigma j} b_j(\sigma) b_j^3.$$

Using the frequency envelope property, we bound this last sum by

$$\sum_{j\geq k-4} 2^{-\sigma j} b_j(\sigma) b_j^3 \lesssim 2^{-\sigma k} b_k(\sigma) \sum_{j\geq k-4} 2^{-\sigma(j-k)} 2^{\delta(j-k)} b_j^3 \lesssim 2^{-\sigma k} b_k(\sigma) \sup_{j\geq k-4} b_j \cdot \sum_{j\geq k-4} b_j^2.$$

It is in controlling this last sum that we use $\sigma > \delta+$. □

Lemma 7.21. *We have*

$$\|A_t(0)\|_{L^2_{t,x}} \lesssim \Big(1+\sum_j b_j^2\Big)^2 \sum_k \|P_k\psi_x(0)\|^2_{L^4_{t,x}}.$$

Proof. We begin with

$$\|A_t(0)\|_{L^2_{t,x}} \lesssim \int_0^\infty \|(\overline{\psi_t}\cdot D_l\psi_l)(s)\|_{L^2_{t,x}}\, ds. \tag{7-96}$$

If we define

$$\mu_k(s) := \sup_{k'\in\mathbb{Z}} 2^{-\delta|k-k'|}\|P_k\psi_t(s)\|_{L^4_{t,x}} \quad\text{and}\quad \nu_k(s) := \sup_{k'\in\mathbb{Z}} 2^{-\delta|k-k'|}\|P_k(D_l\psi_l)(s)\|_{L^4_{t,x}}, \tag{7-97}$$

then

$$\|(\overline{\psi_t}\cdot D_l\psi_l)(s)\|_{L^2_{t,x}} \lesssim \sum_k \mu_k(s)\sum_{j\leq k}\nu_j(s) + \sum_k \nu_k(s)\sum_{j\leq k}\mu_j(s). \tag{7-98}$$

From Lemmas 7.15, 7.12, and 7.13, it follows that

$$\mu_k(s),\, \nu_k(s) \lesssim (1+s2^{2k})^{-2} 2^k \tilde{b}_k \Big(1+\sum_p b_p^2\Big). \tag{7-99}$$

Combining (7-96), (7-98), and (7-99), we have

$$\begin{aligned}
\|A_t(0)\|_{L^2_{t,x}} &\lesssim \sum_k \mu_k(s)\sum_{j\leq k}\nu_j(s)\\
&\lesssim \Big(1+\sum_p b_p^2\Big)^2 \sum_k 2^k\tilde{b}_k \sum_{j\leq k} 2^j\tilde{b}_j \int_0^\infty (1+s2^{2j})^{-2}(1+s2^{2k})^{-2}\, ds\\
&\lesssim \Big(1+\sum_p b_p^2\Big)^2 \sum_k 2^k\tilde{b}_k \sum_{j\leq k} 2^j\tilde{b}_j \int_0^\infty (1+s2^{2k})^{-2}\, ds\\
&\lesssim \Big(1+\sum_p b_p^2\Big)^2 \sum_k 2^{2k}\tilde{b}_k^2 \int_0^\infty (1+s2^{2k})^{-2}\, ds\\
&\lesssim \Big(1+\sum_p b_p^2\Big)^2 \sum_k \tilde{b}_k^2.
\end{aligned}$$

□

As a corollary of the proof, we also obtain this:

Corollary 7.22. *Let $\sigma \geq 2\delta$. Then*

$$\|P_k A_t\|_{L^2_{t,x}} \lesssim \Big(1+\sum_p b_p^2\Big)\tilde{b}_k 2^{-\sigma k} b_k(\sigma).$$

Proof. We start by modifying the proof of Lemma 7.21, taking μ_k and ν_k as in (7-97). Then

$$\begin{aligned}\|P_k A_t\|_{L^2} &\lesssim \int_0^\infty \|P_k(\overline{\psi_t}\cdot D_l\psi_l)(s)\|_{L^2_{t,x}}\,ds\\ &\lesssim \int_0^\infty \Big(\mu_k(s)\sum_{j\leq k}\nu_j(s) + \nu_k\sum_{j\leq k}\mu_j(s) + \sum_{j\geq k}\mu_j(s)\nu_j(s)\Big)ds.\end{aligned}$$

Combining Lemmas 7.12 and 7.13 gives a bound on ν_k of

$$\|\nu_k(s)\|_{L^4} \lesssim (1+s2^{2k})^{-3}(s2^{2k})^{-3/8}2^k 2^{-\sigma k} b_k(\sigma), \tag{7-100}$$

which leads to

$$\int_0^\infty \nu_k \sum_{j\leq k}\mu_j(s)\,ds \lesssim \Big(1+\sum_p b_p^2\Big)\tilde{b}_k 2^{-\sigma k} b_k(\sigma).$$

Also, by using (7-99) for μ_k and (7-100) for ν_k yields

$$\begin{aligned}\int_0^\infty \sum_{j\geq k}\mu_j(s)\nu_j(s)\,ds &\lesssim \Big(1+\sum_p b_p^2\Big)\sum_{j\geq k} 2^{2j}2^{-\sigma j} b_j(\sigma)\tilde{b}_j \int_0^\infty (1+s2^{2j})^{-3}(s2^{2j})^{-3/8}\,ds\\ &\lesssim \Big(1+\sum_p b_p^2\Big)\sum_{j\geq k} 2^{-\sigma j} b_j(\sigma)\tilde{b}_j\\ &\lesssim \Big(1+\sum_p b_p^2\Big)2^{-\sigma k} b_k(\sigma)\sum_{j\geq k} 2^{(\delta-\sigma)(j-k)}\tilde{b}_j \lesssim \Big(1+\sum_p b_p^2\Big)2^{-\sigma k}\tilde{b}_k b_k(\sigma).\end{aligned}$$

Here we have used $\sigma \geq 2\delta$. It remains to consider

$$\int_0^\infty \mu_k(s)\sum_{j\leq k}\nu_j(s)ds.$$

Suppose that

$$\mu_k(s) \lesssim (1+s2^{2k})^{-2}2^k 2^{-\sigma k} b_k(\sigma)\Big(1+\sum_p b_p^2\Big). \tag{7-101}$$

Then

$$\begin{aligned}\int_0^\infty \mu_k(s)\sum_{j\leq k}\nu_j(s)\,ds &\lesssim \Big(1+\sum_p b_p^2\Big)^2 2^{-\sigma k} b_k(\sigma)2^k \sum_{j\leq k}\int_0^\infty (1+s2^{2k})^{-2}(1+s2^{2j})^{-2}2^j\tilde{b}_j\,ds\\ &\lesssim \Big(1+\sum_p b_p^2\Big)^2 2^{-\sigma k} b_k(\sigma)2^k \sum_{j\leq k} 2^j\tilde{b}_j \int_0^\infty (1+s2^{2k})^{-2}\,ds\\ &\lesssim \Big(1+\sum_p b_p^2\Big)^2 2^{-\sigma k} b_k(\sigma)2^{2k}\tilde{b}_k\cdot 2^{-2k} = \Big(1+\sum_p b_p^2\Big)^2 2^{-\sigma k} b_k(\sigma)\tilde{b}_k.\end{aligned}$$

Hence it remains to establish (7-101).

By Corollary 7.16, (7-101) holds when $s = 0$. To extend this estimate to $s > 0$, we proceed as in the proof of Lemma 7.17, replacing bounds (7-92) and (7-93) with their $\sigma > 0$ analogues as needed; that these analogues hold follows from the bounds referenced in establishing (7-92) and (7-93). To obtain the analogue of (7-94), we apply Lemma 7.14, choosing to use $\sigma > 0$ bounds only over the high frequency ranges. □

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BILINEAR DISPERSIVE ESTIMATES VIA SPACE-TIME RESONANCES I: THE ONE-DIMENSIONAL CASE

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We prove new bilinear dispersive estimates. They are obtained and described via a bilinear time-frequency analysis following the space-time resonances method, introduced by Masmoudi, Shatah, and the second author. They allow us to understand the large time behavior of solutions of quadratic dispersive equations.

1. Introduction

Linear dispersive and Strichartz estimates. A linear, hyperbolic equation is called dispersive if the group velocity of a wave packet depends on its frequency. In order to remain concise, we discuss in this section only the Schrödinger equation

$$\begin{cases} \partial_t u - i\Delta u = 0, \\ u_{|t=0} = f, \end{cases}$$

whose solution we denote $u(t) = e^{it\Delta} f$. This is the prototype of a dispersive equation. A first way to quantify dispersion is provided by the "dispersive estimates", which, in the case of the linear Schrödinger equation, read

$$\|e^{it\Delta} f\|_{L^p(\mathbb{R}^d)} \lesssim t^{d/p-d/2} \|f\|_{L^{p'}(\mathbb{R}^d)} \quad \text{if } 2 \le p \le \infty.$$

Another way of quantifying dispersion is provided by Strichartz estimates, which first appeared in [Strichartz 1977] (and were later extended by Ginibre and Velo [1992], with the endpoints due to Keel and Tao [1998]). They read

$$\|e^{it\Delta} f\|_{L^p L^q(\mathbb{R}^+\times\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}$$

for every admissible exponents (p, q), which means $2 \le p, q \le \infty$, $(p, q, d) \neq (2, \infty, 2)$ and

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

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Let us just point out the situation if the Euclidean space $\mathbb{R}^d$ is replaced by a compact Riemannian manifold. In that case, any constant function is a solution of the free Schrödinger equation and therefore the dispersive estimate fails for large t. It also fails locally in time. Then Strichartz estimates may only hold with a finite time scale and a loss of derivatives (the data f is controlled in a positive order Sobolev space), which were obtained for the torus by Bourgain [1993b; 1993a] and then extended to general manifolds by Burq, Gérard, and Tzvetkov [Burq et al. 2004].

Bilinear Strichartz estimates. Recently bilinear (and more generally multilinear) analogs of such inequalities have appeared. They correspond to controlling the size of the (pointwise) product of two linear solutions, for instance

$$\|vw\|_{L^pL^q(\mathbb{R}^+\times\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)} \quad \text{with} \begin{cases} i\partial_t v + \Delta v = 0, & v(t=0)=f, \\ i\partial_t w + \Delta w = 0, & w(t=0)=g, \end{cases} \tag{1-1}$$

or the solution to the inhomogeneous linear problem, the right hand side being given by the product of two linear solutions:

$$\|u\|_{L^pL^q(\mathbb{R}^+\times\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)} \quad \text{with} \begin{cases} i\partial_t v + \Delta v = 0, & v(t=0)=f, \\ i\partial_t w + \Delta w = 0, & w(t=0)=g, \\ i\partial_t u + \Delta u = vw, & u(t=0)=0. \end{cases} \tag{1-2}$$

A first line of research, where $p=q=2$, is related to the use of $X^{s,b}$ spaces in order to solve nonlinear dispersive equations; see, in particular, [Bourgain 1993b] and [Tao 2001]. If the Euclidean space is replaced by a manifold, we refer to [Burq et al. 2005] and [Hani 2010]. The case of the wave equation is treated by Klainerman, Machedon, Bourgain, and Tataru [Klainerman and Machedon 1996], and Foschi and Klainerman [2000]. In all these works, f and g are chosen with vastly different frequency supports, and the focus is on understanding the effect on the implicit constant.

Another line of research considers the case where p and q are not 2: see [Wolff 2001] for the case of the wave equation and [Tao 2003] for the Schrödinger equation. The problem then becomes related to deep harmonic analysis questions (the restriction conjecture), and the optimal estimates are not known in high dimension.

In this article our goal is different from the two directions mentioned: we aim at finding a decay rate in time (rather than integrability properties), and at understanding the effect of localized data.

The set up. From now on, the dimension d of the ambient space is set equal to 1. Let a, b, c be smooth real-valued functions on $\mathbb{R}$, and fix a smooth, compactly supported bilinear symbol m on the frequency plane $\mathbb{R}^2$. We denote by T_m the associated pseudoproduct operator. (a precise definition of T_m is given in Section 1; T_m can be thought of as a generalized product operator, and our setting of course includes classical products between functions that are compactly supported in Fourier space.) Consider then the equation

$$\begin{cases} i\partial_t u + a(D)u = T_m(v,w), \\ i\partial_t v + b(D)v = 0, \\ i\partial_t w + c(D)w = 0, \end{cases} \quad \text{with} \quad \begin{cases} u(t=0)=0, \\ v(t=0)=f, \\ w(t=0)=g. \end{cases} \tag{1-3}$$

The unknown functions are complex-valued, and this system is set in the whole space: f and g map $\mathbb{R}$ to $\mathbb{C}$, whereas u, v, and w map $\mathbb{R}^2$ to $\mathbb{C}$. The above system is meant to help understand the nonlinear interaction of free waves, which is of course the first step towards understanding a nonlinear problem.

Most of the time, but not always, we assume

$$\text{The second derivatives } a'', b'', c'' \text{ are bounded away from zero.} \tag{H}$$

Under this hypothesis, it is well known that the groups $e^{ita(D)}$, $e^{itb(D)}$, $e^{itc(D)}$ satisfy the following estimates (we denote by $S(t)$ any of these groups):

- Dispersive estimates: $\|S(t)f\|_{L^{p'}} \lesssim |t|^{1/2-1/p}\|f\|_{L^p}$ for $p \in [1, 2]$.
- Strichartz estimates: $\|S(t)f\|_{L^p_t L^q} \lesssim \|f\|_{L^2}$ if $2/p + 1/q = \frac{1}{2}$ and $2 \le p, q \le \infty$.

The question we want to answer is this: *Given f and g in L^2 (or weighted L^2 spaces), how does u grow or decay in L^p spaces*, $2 \le p \le \infty$?

The answer of course depends on a, b, c, and the crucial notion is that of space-time resonance.

Space-time resonances. Using Duhamel's formula, we see that $u(t, \cdot)$ is given by the bilinear operator T_t defined by

$$T_t(f, g)(x) = \int_0^t \iint e^{ix(\xi+\eta)} e^{ita(\xi)} e^{is\phi(\xi,\eta)} m(\xi, \eta) \hat{f}(\eta)\hat{g}(\xi - \eta)\, d\xi\, d\eta\, ds,$$

or, more concisely,

$$T_t(f, g) \stackrel{\text{def}}{=} -ie^{ita(D)} \int_0^t T_{me^{is\phi}}(f, g)\, ds,$$

where

$$\phi(\xi, \eta) \stackrel{\text{def}}{=} -a(\xi + \eta) + b(\xi) + c(\eta).$$

Thus the goal of this article is to understand the behavior for large time $t \gg 1$ and some exponent $q \in [2, \infty]$ of

$$\|T_t(f, g)\|_{L^q}, \quad f, g \in L^2.$$

We sometimes find it convenient to write $u(t)$ as

$$u(t) = \mathcal{F}^{-1} \int_0^t \int_{\mathbb{R}} e^{ita(\xi)} e^{is\Phi(\xi,\eta)} \mu(\xi, \eta) \hat{f}(\xi - \eta)\hat{g}(\eta)\, d\eta\, ds,$$

where

$$\Phi(\xi, \eta) \stackrel{\text{def}}{=} -a(\xi) + b(\xi - \eta) + c(\eta) = \phi(\xi - \eta, \eta) \quad \text{and} \quad \mu(\xi, \eta) \stackrel{\text{def}}{=} m(\xi - \eta, \eta).$$

Viewing this double integral as a stationary phase problem, it becomes clear that the sets where the phase is stationary in s or η,

$$\Gamma \stackrel{\text{def}}{=} \{(\xi, \eta) \text{ such that } \Phi(\xi, \eta) = 0\} \quad \text{and} \quad \Delta \stackrel{\text{def}}{=} \{(\xi, \eta) \text{ such that } \partial_\eta \Phi(\xi, \eta) = 0\},$$

play a crucial role. Even more important is their intersection $\Gamma \cap \Delta$.

The sets Γ and Δ are, respectively, the sets of time and space resonances; their intersection is the set of space-time resonant sets. A general presentation, stressing their relevance to PDE problems, can be found in [Germain 2010b]; for applications see [Germain et al. 2009; 2012a; 2012b; Germain 2010a; Germain and Masmoudi 2011].

In order to answer the question from the previous page, one has to distinguish between various possible geometries of Γ and Δ (which can be reduced to a discrete set, or curves, with vanishing curvature or not, etc....), possible orders of vanishing of Φ and $\partial_\eta \Phi$ on Γ and Δ, respectively, and different types of intersections of Γ and Δ (at a point or on a dimension 1 set, transverse or not, etc....). Considering all the possible configurations would be a daunting task. We therefore focus on a few relevant and "generic" examples.

- We study the influence of time resonances alone, ignoring space resonances: in other words, we study various configurations for Γ, without making any assumptions on Δ. This essentially amounts to considering the worst possible case as far as Δ is concerned.
- Similarly, we study the influence of space resonances alone, ignoring about time resonances.
- When putting space and time resonances together, we assume a "generic" configuration: Γ and Δ are smooth curves, and they intersect transversally at a point. Aside from being generic, this configuration is of key importance for many nonlinear PDE; this is explained in the next subsection.

Space-time resonant set reduced to a point. As was just mentioned, the case where Γ and Δ are curves which intersect transversally at a point will be examined carefully in this article. It is of course the generic situation, but it also occurs in a number of important models from physics; we give a few examples here. We restrict the discussion to one-dimensional models.

For simple equations of the form $i\partial_t u + \tau(D)u = Q(u, \bar{u})$, where u is scalar-valued, Q quadratic (that is, we retain only the quadratic part of the nonlinearity), and $\tau(\xi) = |\xi|^\alpha$ is homogeneous, the space-time resonant set of the various possible interactions between u and $\bar{u}$ is never reduced to a point. This is the case for standard equations such as NLS, KdV, and wave equations.

However, if τ is no longer supposed to be homogeneous, the space-time resonant set might be reduced to a point. In particular, this is the case for the water wave equation (ideal fluid with a free surface) in the following setting: close to the equilibrium given by a flat surface and zero velocity, including the effects of gravity g and capillarity c, with a constant depth d (perhaps infinite). The dispersion relation for the linearized problem is then

$$\tau(\xi) = \tanh(d|\xi|)\sqrt{g|\xi| + c|\xi|^3}.$$

For more complex models, u is vector-valued, and the system accounts for the interaction of waves with different dispersion relations. It is then often the case that the space-time resonance set is reduced to a point. We mention in particular the following.

- The Euler–Maxwell system, describing the interaction of a charged fluid with an electromagnetic field (see [Germain and Masmoudi 2011] for a mathematical treatment of this equation dealing with space-time resonances). Many other models of plasma physics could also be mentioned here.

- Systems where wave and (generalized) Schrödinger equations are coupled: for instance, the Davey–Stewartson, Ishimori, Maxwell–Schrödinger, and Zakharov systems.

A sample of our results. Among the many results proved in this paper, we record in Theorem 1.1 a few that are illustrative and interesting. We need a definition: a curve in (ξ, η) is *characteristic* if it has tangents parallel to one or more of the directions $\xi = 0$, $\eta = 0$, or $\xi + \eta = 0$, and *noncharacteristic* otherwise.

Theorem 1.1. *Recall that m is smooth and compactly supported. Assume that (H) holds.*

(i) *If Γ is a noncharacteristic curve along which Φ vanishes at order* 1,

$$\|u(t)\|_{L^q} \lesssim \langle \log t \rangle \|f\|_{L^{2,s}} \|g\|_{L^{2,s}} \quad \textit{for } s > \tfrac{1}{4}.$$

(ii) *If $\Delta = \varnothing$, then, for any $\delta > 0$,*

$$\|u(t)\|_{L^q} \lesssim t^{1/q-1/2+\delta} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}} \quad \textit{for } s > 1 - \tfrac{1}{q}.$$

Furthermore, this rate of decay is optimal.

(iii) *If Γ and Δ intersect transversely at a single point in the support of m, then, for any $\delta > 0$*

$$\|u(t)\|_{L^q} \lesssim t^{-(1/4-1/(2q))+\delta} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}} \quad \textit{for } s > 1.$$

Furthermore, this rate of decay is optimal (up to the loss δ as small as we want).

Organization of the article. In Section 2 we derive asymptotic equivalents for u when f and g smooth and localized. Three cases are considered: $\Gamma = \varnothing$, $\Delta = \varnothing$, and Γ and Δ are curves intersecting transversally at a point (in particular we prove the second part of Theorem 1.1). In Section 3, relying only on time resonances, we establish estimates for u when f and g belong to L^2. In Section 4, we establish estimates for u when f and g belong to weighted L^2 spaces. In particular we consider the case when the space-time resonant set is reduced to a point, and thereby prove the first part of Theorem 1.1. In Appendix A, we detail some results on boundedness of multilinear operators. Finally, in Appendix B, one-dimensional oscillatory integrals are studied.

Notations. We adopt the following notations.

- $A \lesssim B$ if $A \le CB$ for some implicit constant C. The value of C may change from line to line.
- $A \sim B$ means that both $A \lesssim B$ and $B \lesssim A$.
- If f is a function over $\mathbb{R}^d$, its Fourier transform, denoted $\hat{f}$, or $\mathscr{F}(f)$, is given by

$$\hat{f}(\xi) = \mathscr{F} f(\xi) = \frac{1}{(2\pi)^{d/2}} \int e^{-ix\xi} f(x)\, dx, \quad \text{thus } f(x) = \frac{1}{(2\pi)^{d/2}} \int e^{ix\xi} \hat{f}(\xi)\, d\xi.$$

  (In the text, we systematically drop constants such as $1/(2\pi)^{d/2}$ since they are not relevant.)
- The Fourier multiplier with symbol $m(\xi)$ is defined by

$$m(D) f = \mathscr{F}^{-1} [m \mathscr{F} f].$$

- The bilinear Fourier multiplier with symbol m is given by

$$T_m(f,g)(x) \overset{\text{def}}{=} \int_{\mathbb{R}^2} e^{ix(\xi+\eta)} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)\,d\xi d\eta = \mathcal{F}^{-1}\int m(\xi-\eta,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta.$$

- The Japanese bracket $\langle\cdot\rangle$ stands for $\langle x\rangle = \sqrt{1+x^2}$.
- The weighted Fourier space $L^{p,s}$ is given by the norm $\|f\|_{L^{p,s}} = \|\langle x\rangle^s f\|_{L^p}$.
- If E is a set in $\mathbb{R}^d$, E_ϵ is the set of points of $\mathbb{R}^d$ that are within ϵ of a point of E.

2. Asymptotic equivalents

Preliminary discussion. Our aim in this section is to obtain asymptotic equivalents, as $t \to \infty$, for the solution u of (1-3), under the simplifying assumption that f and g are very smooth and localized. Hypotheses on a, b, c are needed, and the variety of possible situations is huge; we try to focus on the most representative, or generic situations. First, we assume in this whole section that (H) holds: this gives decay for the linear waves. For bilinear estimates, everything hinges on the vanishing properties of Φ and $\partial_\eta\Phi$, where

$$\Phi(\xi,\eta) = -a(\xi) + b(\xi-\eta) + c(\eta).$$

We distinguish three situations: Φ does not vanish (Theorem 2.2), Φ_η does not vanish (Theorem 2.4), $\{\Phi = 0\}$ and $\{\Phi_\eta = 0\}$ are curves intersecting transversally (Theorem 2.5). Additional assumptions will be specified as needed.

Asymptotics for the linear Cauchy problem. They are obtained easily by stationary phase; see for instance [Stein 1993].

Lemma 2.1. *Assume that $F \in \mathcal{S}$ is such that $\widehat{F}$ is compactly supported; and suppose that a'' does not vanish on* Supp F. *Then*

$$e^{ita(D)}F(x) = e^{it[a(\xi_0)+X\xi_0]} e^{i(\pi/4)\sigma} \frac{1}{\sqrt{|a''(\xi_0)|}} \frac{1}{\sqrt{t}} \widehat{F}(\xi_0) + O\Big(\frac{1}{t}\Big),$$

where

$$X \overset{\text{def}}{=} \frac{x}{t}, \quad a'(\xi_0) + X \overset{def}{=} 0, \quad \sigma \overset{\text{def}}{=} \operatorname{sign} a''(\xi_0).$$

The point of view of stationary phase. The solution of (1-3) is

$$u(t,x) = -\frac{i}{\sqrt{2\pi}} \int_0^t \iint e^{ix\xi} e^{i[(t-s)a(\xi)+sb(\xi-\eta)+sc(\eta)]} \mu(\xi,\eta) \hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta\,d\xi\,ds.$$

Recalling that $X \overset{\text{def}}{=} x/t$ and $\mu(\xi,\eta) \overset{\text{def}}{=} m(\xi-\eta,\eta)$, this is equal to

$$u(t,x) = -\frac{i}{\sqrt{2\pi}} t \int_0^1 \iint e^{it[(1-\sigma)a(\xi)+\sigma b(\xi-\eta)+\sigma c(\eta)+X\xi]} \mu(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta\,d\xi\,d\sigma. \tag{2-0}$$

This is now a (nonstandard) stationary phase problem, with phase

$$\psi(\xi,\eta,\sigma) \overset{\text{def}}{=} (1-\sigma)a(\xi) + \sigma b(\xi-\eta) + \sigma c(\eta) + X\xi = a(\xi) + \sigma\Phi(\xi,\eta) + X\xi.$$

The phase of the gradient is

$$\nabla_{\xi,\eta,\sigma}\psi = \begin{pmatrix} a'+\sigma\Phi_\xi+X \\ \sigma\Phi_\eta \\ \Phi \end{pmatrix},$$

which vanishes if either

$$\begin{cases} \sigma=0, \\ \Phi=0, \\ a'+X=0, \end{cases} \quad \text{or} \quad \begin{cases} \Phi=0, \\ \Phi_\eta=0, \\ a'+\sigma\Phi_\xi+X=0. \end{cases} \tag{2-1}$$

The Hessian of ψ is given by

$$\operatorname{Hess}_{\xi,\eta,\sigma}\psi = \begin{pmatrix} a''+\sigma\Phi_{\xi\xi} & \sigma\Phi_{\xi\eta} & \Phi_\xi \\ \sigma\Phi_{\xi\eta} & \sigma\Phi_{\eta\eta} & \Phi_\eta \\ \Phi_\xi & \Phi_\eta & 0 \end{pmatrix}.$$

On stationary points of the first type in (2-1), the Hessian is degenerate if and only if (ξ,η) belongs to the space-time resonant set. On stationary points of the second type in (2-1), the Hessian is generically nondegenerate.

The main difficulty in the analysis is handling the stationary points on the boundary of the integration domain, namely those for which $\sigma=0$ or 1; this is even more complicated when they are degenerate.

Theorem 2.2 (absence of time resonances). *Assume that $\Phi(\xi,\eta)$ does not vanish on* $\operatorname{Supp} m$ (*that is*, $\Gamma=\varnothing$), *and that f and g belong to $\mathcal{S}$. Then, as $t\to\infty$,*

$$u(t)=e^{ita(D)}F+O\Big(\frac{1}{t}\Big).$$

with

$$F=T_{m/\phi}(f,g).$$

Remark 2.3. The asymptotic behavior of $e^{ita(D)}F$ is given by Lemma 2.1.

Proof. The proof is very easy: u is given by

$$u(t)=-ie^{ita(D)}\int_0^t T_{me^{is\phi}}(f,g)\,ds,$$

or

$$u(t)=-T_{m/\phi}(e^{itb(D)}f,e^{itc(D)}g)+e^{ita(D)}T_{m/\phi}(f,g).$$

The theorem follows since the first term above is $O(1/t)$, by the linear decay estimates. □

Theorem 2.4 (absence of space resonances). *Assume that ψ_η does not vanish on* $\operatorname{Supp} m$ (*that is*, $\Delta=\varnothing$), *that $\psi_{\xi\xi}(\xi,\eta,\sigma)$ does not vanish on* $\operatorname{Supp} m\times[0,1]$, *and that f, g belong to $\mathcal{S}$. Fix $M>0$ and $N\in\mathbb{N}$. Then, as $t\to\infty$,*

$$u(t)=e^{ita(D)}F+O\Big(\frac{1}{M^N\sqrt{t}}\Big)$$

where

$$F=-i\int_0^M e^{i(t-s)a(D)}T_m(e^{isb(D)}f,e^{isc(D)}g)\,ds.$$

(*In other words*, $e^{ita(D)}F$ *is the solution of*

$$i\partial_t u + a(D)u = \begin{cases} T_m(v,w) & \text{if } 0<t<M,\\ 0 & \text{if } t>M, \end{cases} \qquad i\partial_t v + b(D)v = 0, \qquad i\partial_t w + c(D)w = 0,$$

with the data $u(t=0)=0$, $v(t=0)=f$, *and* $w(t=0)=g$.)

Proof. Starting from the stationary phase formulation (see page 692), it suffices to show that

$$\int_{M/t}^{1} \iint e^{it\psi(\xi,\eta,\sigma)} \mu(\xi,\eta) \hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta\,d\xi\,d\sigma \tag{2-2}$$

is $O(1/(M^N t^{3/2}))$.

First apply the stationary phase lemma in ξ in the above. The vanishing set of ψ_ξ depends on X. If X is such that ψ_ξ does not vanish, (2-2) is $O(1/t^N)$ for any N and we are done. Otherwise, ψ_ξ vanishes for some ξ, which we denote ξ_0, and which is a function of X, η, and σ. We can assume without loss of generality that ξ_0 is unique. Since $\psi_{\xi\xi}$ does not vanish by assumption, the stationary phase lemma gives

$$(2\text{-}2) = \int_{M/t}^{1} \int e^{it\psi(\xi_0,\eta,\sigma)} \left(\frac{\alpha(\xi,\eta,\sigma)}{\sqrt{t}} + \frac{\beta(\xi,\eta,\sigma)}{t} + \frac{\gamma(\xi,\eta,\sigma)}{t\sqrt{t}} + O\left(\frac{1}{t^2}\right) \right) d\eta\,d\sigma,$$

where α, β, and γ are smooth functions which we do not specify. The fourth summand in (2-2) is already small enough. We will now show how to deal with the first one, and this will conclude the proof since the second and third ones are easier (better decay). Thus we now want to show that

$$\int_{M/t}^{1} \int e^{it\psi(\xi_0,\eta,\sigma)} \frac{\alpha(\xi,\eta,\sigma)}{\sqrt{t}}\,d\eta\,d\sigma \tag{2-3}$$

is $O(1/(M^N t))$. In order to take advantage of oscillations in η, observe that

$$\partial_\eta[\psi(\xi_0(\eta,\sigma,X),\eta,\sigma)] = \partial_\eta \xi_0 [\partial_\xi \psi](\xi_0,\eta,\sigma) + [\partial_\eta \psi](\xi_0,\eta,\sigma) = [\partial_\eta \psi](\xi_0,\eta,\sigma) = \sigma[\partial_\eta \Phi](\xi_0,\eta).$$

By hypothesis, $\partial_\eta \Phi$ does not vanish, therefore

$$|\partial_\eta[\psi(\xi_0,\eta,\sigma)]| \gtrsim \sigma.$$

Integrating by parts $N+1$ times with the help of the identity

$$\frac{1}{t\partial_\eta[\psi(\xi_0,\eta,\sigma)]} \partial_\eta e^{it\psi(\xi_0,\eta,\sigma)} = ie^{it\psi(\xi_0,\eta,\sigma)},$$

we obtain

$$|(2\text{-}3)| \lesssim \int_{M/t}^{1} \frac{1}{(\sigma t)^{N+1}\sqrt{t}}\,d\sigma \lesssim \frac{1}{M^N t^{3/2}},$$

which concludes the proof. ☐

Theorem 2.5 (space-time resonance set reduced to a point). *Assume that f, g belong to $\mathcal{S}$, that there exists a unique (ξ_0, η_0) such that*

$$\Phi(\xi_0, \eta_0) = \Phi_\eta(\xi_0, \eta_0) = 0,$$

and that the following technical, generic hypotheses are satisfied:

- (*we are under the standing assumption* (H), *but only the fact that a'' is nonvanishing is used here*;)
- $\Phi_\xi(\xi_0, \eta_0) \neq 0$;
- $\Phi_{\eta\eta}(\xi_0, \eta_0) \neq 0$;

and that Supp m *is contained in a small enough neighborhood of* (ξ_0, η_0).

Recall that $X = x/t$, and set

$$\Sigma(X) \overset{\text{def}}{=} -\frac{1}{\Phi_\xi(\xi_0, \eta_0)}(a'(\xi_0) + X).$$

Let $\epsilon > 0$ be small enough. Assume without loss of generality that $\Phi_\xi(\xi_0, \eta_0) > 0$. Then:

- *If $X < -\Phi_\xi(\xi_0, \eta_0) - a'(\xi_0) - \epsilon$,*
$$u(t) = O\Big(\frac{1}{t^N}\Big)$$
for any N.
- *If $-\Phi_\xi(\xi_0, \eta_0) - a'(\xi_0) - \epsilon < X < -\Phi_\xi(\xi_0, \eta_0) - a'(\xi_0) + \epsilon$,*
$$u(t) = \frac{1}{\sqrt{t}} A_2(\Sigma)\mathcal{G}_1(\sqrt{t}[\Sigma - 1]) + O\Big(\frac{1}{t}\Big)$$
for a smooth function A_2.
- *If $-\Phi_\xi(\xi_0, \eta_0) - a'(\xi_0) + \epsilon < X < -a'(\xi_0) - \epsilon$,*
$$u(t, x) = \frac{1}{\sqrt{t}} \frac{A_1}{\sqrt{\Sigma(X)}} e^{it\psi(\xi_0,\eta_0,\Sigma)} + O\Big(\frac{1}{t}\Big)$$
for a constant A_1.
- *If $-a'(\xi_0) - \epsilon < X < -a'(\xi_0) + \epsilon$,*
$$u(t) = A_0(\Sigma)\frac{1}{t^{1/4}}\mathcal{G}_2(\sqrt{t}\Sigma) + \begin{cases} O(t^{-3/4}) & \text{if } |\sqrt{t}\Sigma| < 1, \\ O(|\log t|/\sqrt{t}) & \text{if } |\sqrt{t}\Sigma| > 1. \end{cases}$$
for a smooth function A_0;
- *if $-a'(\xi_0) + \epsilon < X$,*
$$u(t) = O\Big(\frac{1}{t^N}\Big)$$
for any N.

Remark 2.6. (1) Theorem 2.5 provides an efficient equivalent of $u(t)$ for large t in all the zones of the space-time plane (x,t), except where Σ is small, but larger than $1/|\log t|^2$ (because then $|\log t|/\sqrt{t} > (1/t^{1/4})|\mathscr{G}_2(\sqrt{t}\Sigma)|$). Dealing with this region would require fairly technical developments, from which we refrain.

(2) If Φ vanishes at order 1 on Γ and Δ, the conditions $\Phi_\xi(\xi_0,\eta_0) \neq 0$ and $\Phi_{\eta\eta}(\xi_0,\eta_0) \neq 0$ are equivalent to Γ and Δ intersecting transversally at (ξ_0,η_0). Indeed, a tangent vector to Γ (respectively, Δ) at (ξ_0,η_0) is given by

$$\begin{pmatrix} \partial_\eta \Phi(\xi_0,\eta_0) \\ -\partial_\xi \Phi(\xi_0,\eta_0) \end{pmatrix} = \begin{pmatrix} 0 \\ -\partial_\xi \Phi(\xi_0,\eta_0) \end{pmatrix} \qquad \left(\text{respectively } \begin{pmatrix} \partial_\eta^2 \Phi(\xi_0,\eta_0) \\ -\partial_\eta\partial_\xi \Phi(\xi_0,\eta_0) \end{pmatrix} \right).$$

These two vectors are not collinear if $\partial_\xi \Phi(\xi_0,\eta_0)\partial_\eta^2\Phi(\xi_0,\eta_0) \neq 0$.

(3) The hypothesis that $\operatorname{Supp} m$ is restricted in a small enough neighborhood is not restrictive: away from (ξ_0,η_0), either Φ or Φ_η is nonzero, so either Theorem 2.2 or Theorem 2.4 applies.

The proof distinguishes three regions: σ away from 0 and 1, σ close to 0, and σ close to 1. Starting from Equation (2-0), we split the time integral as follows:

$$\begin{aligned} u(t,x) &= -\frac{i}{\sqrt{2\pi}} t \int_0^1 \iint e^{it\psi(\xi,\eta,\sigma)} \mu(\xi,\eta) \hat{f}(\xi-\eta)\hat{g}(\eta)\, d\xi\, d\eta\, d\sigma \\ &= -\frac{i}{\sqrt{2\pi}} t \int_0^t \iint \big(\chi_I(\sigma)+\chi_{II}(\sigma)+\chi_{III}(\sigma)\big) \dots d\xi\, d\eta\, d\sigma \overset{\text{def}}{=} I + II + III. \end{aligned} \tag{2-4}$$

Here χ_I, χ_{II}, and χ_{III} are three smooth, positive functions, adding up to 1 for each σ and such that

$$\chi_{II}(\sigma) = \begin{cases} 0 & \text{if } \sigma < \delta, \\ 1 & \text{if } \sigma > 2\delta, \end{cases} \qquad \chi_I(\sigma) = \begin{cases} 0 & \text{if } \sigma < \delta \text{ or } \sigma > 1-\delta, \\ 1 & \text{if } 2\delta < \sigma < 1-2\delta, \end{cases} \qquad \chi_{III}(\sigma) = \begin{cases} 0 & \text{if } \sigma < 1-2\delta, \\ 1 & \text{if } \sigma > 1-\delta. \end{cases}$$

Here $\delta > 0$ is a sufficiently small number.

The contribution of σ away from 0 *and* 1. This is the simplest case since it can be settled by resorting to elementary stationary phase considerations. Our aim is to estimate

$$I = -\frac{i}{\sqrt{2\pi}} t \int_0^1 \iint \chi_I(\sigma) e^{it\psi(\xi,\eta,\sigma)} \mu(\xi,\eta) \hat{f}(\xi-\eta)\hat{g}(\eta)\, d\xi\, d\eta\, d\sigma.$$

The phase $\psi(\xi,\eta,\sigma)$ is also a function of X, but from now on we consider X to be fixed.

Since σ does not vanish on $\operatorname{Supp}\chi_I$, the gradient

$$\nabla_{\xi,\eta,\sigma}\psi = \begin{pmatrix} a' + \sigma\Phi_\xi + X \\ \sigma\Phi_\eta \\ \Phi \end{pmatrix}$$

vanishes if

$$\Phi(\xi,\eta) = \Phi_\eta(\xi,\eta) = 0 \quad \text{and} \quad a'(\xi) + \sigma\Phi_\xi(\xi,\eta) + X = 0.$$

The first two conditions impose $(\xi,\eta)=(\xi_0,\eta_0)$ whereas the third one gives

$$\sigma=\Sigma(X)\overset{\text{def}}{=}-\frac{1}{\Phi_\xi(\xi_0,\eta_0)}(a'(\xi_0)+X).$$

(This makes sense under the assumption that $\Phi_\xi(\xi_0,\eta_0)\neq 0$.) We assume that X is such that σ given by the above line lies in $\operatorname{Supp} m$; if this is not the case, the contribution of I is negligible. The Hessian at (Σ,ξ_0,η_0) is

$$\operatorname{Hess}_{\xi,\eta,\sigma}\psi(\xi_0,\eta_0,\Sigma)=\begin{pmatrix} a''+\Sigma\Phi_{\xi\xi}(\xi_0,\eta_0) & \Sigma\Phi_{\xi\eta}(\xi_0,\eta_0) & \Phi_\xi(\xi_0,\eta_0)\\ \Sigma\Phi_{\xi\eta}(\xi_0,\eta_0) & \Sigma\Phi_{\eta\eta}(\xi_0,\eta_0) & 0\\ \Phi_\xi(\xi_0,\eta_0) & 0 & 0\end{pmatrix}$$

with determinant

$$\det\operatorname{Hess}_{\xi,\eta,\sigma}\psi(\xi_0,\eta_0,\Sigma)=-\Sigma\Phi_\xi(\xi_0,\eta_0)^2\Phi_{\eta\eta}(\xi_0,\eta_0).$$

Let us assume that $\Phi_{\eta\eta}(\xi_0,\eta_0)$ is not zero, which is generically satisfied. The stationary phase principle then gives [Stein 1993]

$$u(t,x)=\frac{1}{\sqrt{t}}\frac{\chi_I(\Sigma(X))}{\sqrt{\Sigma(X)}}e^{it\psi(\xi_0,\eta_0,\Sigma)}A_1+O\Big(\frac{1}{t}\Big)$$

with

$$A_1\overset{\text{def}}{=}\frac{(2\pi)^{3/2}e^{i(\pi/4)S}}{|\Phi_\xi(\xi_0,\eta_0)|\sqrt{|\Phi_{\eta\eta}(\xi_0,\eta_0)|}}\mu(\xi_0,\eta_0)\hat f(\xi_0-\eta_0)\hat g(\eta_0)$$

where S is the signature of $\operatorname{Hess}_{\xi,\eta,\sigma}\psi(\xi_0,\eta_0,\Sigma)$.

The contribution of σ *close to* 0.

Step 1: splitting between small and large times. Our aim is to estimate

$$II=-\frac{i}{\sqrt{2\pi}}t\int_0^1\iint\chi_{II}(\sigma)e^{it\psi(\xi,\eta,\sigma)}\mu(\xi,\eta)\hat f(\xi-\eta)\hat g(\eta)\,d\xi\,d\eta\,d\sigma,$$

which we split into

$$II=-\frac{i}{\sqrt{2\pi}}t\bigg[\int_0^{1/t}+\int_{1/t}^1\dots d\sigma\bigg]\overset{\text{def}}{=}II_1+II_2.$$

Rescaling II_1, we see that it can be written

$$u(t)=e^{ita(D)}F\quad\text{where } F=-\frac{i}{\sqrt{2\pi}}\int_0^1\int e^{i(s-t)a(D)}T_m(e^{-isb(D)}f,e^{-isc(D)}g)\,ds,$$

so that it reduces to a linear solution for t sufficiently large. We now focus on II_2.

Step 2: stationary phase in ξ. We want to apply the stationary phase lemma in the variable ξ. Observe that

$$\partial_\xi\psi(\xi,\eta,\sigma)=a'(\xi)+\sigma\Phi_\xi+X.$$

Thus for η, σ, and X fixed, $\partial_\xi\psi(\xi,\eta,\sigma)=0$ may or may not have a solution in $\operatorname{Supp} m$. If not, the contribution is negligible, so let us assume that this equation has a solution $\xi = \Xi(X, \eta, \sigma)$. Next,

$$\partial_\xi^2 \psi(\xi, \eta, \sigma) = a''(\xi) + \sigma \Phi_{\xi\xi}.$$

Since we are assuming that a'' does not vanish, taking δ small enough, we can ensure that $\partial_\xi^2 \psi(\xi, \eta, \sigma)$ does not vanish. Applying the stationary phase lemma then gives

$$II_2 = t \int_{1/t}^{1} \int G(\Xi, \eta) e^{it\psi(\Xi,\eta,\sigma)} \left(\frac{\sqrt{2\pi}\, e^{iS_0\pi/4}}{\sqrt{\psi_{\xi\xi}(\Xi, \eta, \sigma)}\sqrt{t}} + \frac{\alpha(\eta, \sigma)}{t} + \frac{\beta(\eta, \sigma)}{t\sqrt{t}} + O\Big(\frac{1}{t^2}\Big) \right) d\eta\, d\sigma, \quad (2\text{-}5)$$

where $S_0 = \operatorname{sign}(\psi_{\xi\xi}(\Xi, \eta, \sigma))$, α and β are smooth functions, and for simplicity we denoted

$$G(\xi, \eta, \sigma) = -\frac{i}{\sqrt{2\pi}} \chi_{II}(\sigma) \mu(\xi, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta).$$

The last term in (2-5), containing $O(1/t^2)$, contributes $O(1/t^2)$ to u; thus we can discard it and focus on

$$t \int_{1/t}^{1} \int G(\Xi, \eta) e^{it\psi(\Xi,\eta,\sigma)} \left(\frac{\sqrt{2\pi}\, e^{iS_0\frac{\pi}{4}}}{\sqrt{\psi_{\xi\xi}(\Xi, \eta, \sigma)}\sqrt{t}} + \frac{\alpha(\xi, \eta, \sigma)}{t} + \frac{\beta(\xi, \eta, \sigma)}{t\sqrt{t}} \right) d\eta\, d\sigma. \quad (2\text{-}6)$$

Step 3: stationary phase in η. Observe that

$$\partial_\eta[\psi(\Xi(\eta, \sigma), \eta, \sigma)] = \partial_\eta \Xi[\partial_\xi \psi](\Xi, \eta, \sigma) + [\partial_\eta \psi](\Xi, \eta, \sigma) = [\partial_\eta \psi](\Xi, \eta, \sigma) = \sigma[\partial_\eta \Phi](\Xi, \eta).$$

Just as for the stationary phase in ξ, we denote by $\eta = H(\sigma, X)$ the solution of $[\partial_\eta \Phi](\Xi, \eta) = 0$ (if no solution exist, the contribution is negligible). Next, set

$$\partial_\eta^2[\psi(\Xi(\eta, \sigma), \eta, \sigma)] = \sigma \partial_\eta \Xi[\partial_\xi \partial_\eta \Phi](\Xi, \eta) + \sigma[\partial_\eta^2 \Phi](\Xi, \eta) \overset{\text{def}}{=} \sigma Z(\eta, \sigma).$$

We need to assume that

$$Z(\eta, \sigma) \neq 0$$

if $(\sigma, \Xi, \eta) \in \operatorname{Supp} m\chi_{II}$. Since the support of m, as well as δ, is assumed to be small enough, it suffices that $Z(\eta_0, 0) \neq 0$; but a simple computation reveals that $Z(\eta_0, 0) = \phi_{\eta\eta}(\eta_0, \xi_0)$, which is nonzero by hypothesis. The stationary phase lemma in η applied to Theorem 2.5 then gives

$$\begin{aligned}(2\text{-}6) = t \int_{1/t}^{1} G(\Xi, H) e^{it\psi(\Xi,H,\sigma)} &\frac{\sqrt{2\pi}\, e^{iS_1\pi/4}}{\sqrt{t}\sqrt{\sigma} Z(H, \sigma)} \\ &\times \left(\frac{\sqrt{2\pi}\, e^{iS_0\pi/4}}{\sqrt{\psi_{\xi\xi}(\Xi, H, \sigma)}\sqrt{t}} + \frac{\alpha(H, \sigma)}{t} + \frac{\beta(H, \sigma)}{t\sqrt{t}} + O\Big(\frac{1}{t\sqrt{t\sigma}}\Big) \right) d\sigma, \quad (2\text{-}7)\end{aligned}$$

where $S_1 = \operatorname{sign}(Z(H, \sigma))$, $\tilde{\alpha}$ and $\tilde{\beta}$ are smooth functions. The last summand in (2-7) contributes

$$O\Big(\frac{t}{t\sqrt{t}} \int_{1/t}^{1} \frac{d\sigma}{\sigma}\Big) = O\Big(\frac{\log t}{\sqrt{t}}\Big).$$

We discard this and focus on

$$t\int_{1/t}^{1} G(\Xi,H)e^{it\psi(\Xi,H,\sigma)}\frac{\sqrt{2\pi}\,e^{iS_1\pi/4}}{\sqrt{t}\sqrt{\sigma}Z(H,\sigma)}\left(\frac{\sqrt{2\pi}\,e^{iS_0\pi/4}}{\sqrt{\psi_{\xi\xi}(H,\sigma)}\sqrt{t}}+\frac{\alpha(H,\sigma)}{t}+\frac{\beta(\xi,H,\sigma)}{t\sqrt{t}}\right)d\sigma. \tag{2-8}$$

Step 4: stationary phase in σ. In this final step, we are not going to apply the standard stationary phase lemma, but rather its variant given in Proposition B.2. Differentiating in σ, the phase in (2-7) gives

$$\partial_\sigma\big[\psi\big(\Xi(H(\eta,\sigma),\sigma),H(\sigma),\sigma\big)\big]=[\partial_\sigma\psi](\Xi,H,\sigma)=\Phi(\Xi(H,\sigma),H(\sigma)),$$

since $\partial_\xi\psi=\partial_\eta\psi=0$ at the point (Ξ,H,σ). Thus $\partial_\sigma\phi=0$ if $\Phi(\Xi,H)=0$. On the other hand, since $\partial_\eta\Phi(\Xi,H)=0$ by definition of H,

$$\Phi(\Xi(H,\sigma),H(\sigma))=0 \quad\text{if and only if}\quad H(\sigma)=\eta_0 \quad\text{and}\quad \Xi(\eta_0,\sigma)=\xi_0.$$

But by definition of Ξ this implies

$$\sigma=\Sigma(X)=-\frac{X+a'(\xi_0)}{\psi_\xi(\xi_0,\eta_0)}.$$

In order to apply Proposition B.2, we need to check that

$$\partial_\sigma^2\big[\psi\big(\Xi(H(\eta,\sigma),\sigma),H(\sigma),\sigma\big)\big](\Sigma)=\partial_\sigma[\Phi(\Xi(H,\sigma),H(\sigma))](\Sigma)\neq 0.$$

Since δ is chosen small enough, it suffices to check that it holds for $\Sigma=0$ (that is, when X is such that $\Sigma(X)=0$). This follows from the following computation:

$$\begin{aligned}\partial_\sigma[\Phi(\Xi(H,\Sigma),H(\Sigma))](0)&=\partial_\xi\Phi(\xi_0,\eta_0)(\partial_\sigma\Xi(\eta_0,0)+\partial_\sigma H(0)\partial_\eta\Xi(\eta_0,0))+\partial_\eta\Phi(\xi_0,\eta_0)\partial_\sigma H(0)\\&=\partial_\xi\Phi(\xi_0,\eta_0)\partial_\sigma\Xi(\eta_0,0)=-\frac{\phi_\xi(\xi_0,\eta_0)^2}{a''(\xi_0)}\neq 0,\end{aligned}$$

where we used that $\partial_\eta\Phi(\xi_0,\eta_0)=\partial_\eta\Xi(\eta_0,0)=0$ and $\partial_\sigma\Xi(\eta_0,0)=-\phi_\xi(\xi_0,\eta_0)/a''(\xi_0)$. We now write

$$\text{(2-8)}=t\int_{1/t}^{1}\dots\,d\sigma=t\int_0^1-t\int_0^{1/t}\dots\,d\sigma.$$

The second summand, $t\int_0^{1/t}\dots\,d\sigma$, is directly estimated to be $O(1/\sqrt{t})$. As for the first summand, $t\int_0^1\dots\,d\sigma$, apply Proposition B.2(iv) to obtain

$$\text{(2-8)}=A_0(\Sigma)\mathscr{G}_2(\sqrt{t}\Sigma)+\begin{cases}O(t^{-3/4}) & \text{if } |\sqrt{t}\Sigma|<1,\\ O(\sqrt{|\Sigma|/t}) & \text{if } |\sqrt{t}\Sigma|>1,\end{cases}$$

where A_0 is a smooth function which we do not detail here.

The contribution of σ close to 1. In order to estimate

$$III=-\frac{i}{\sqrt{2\pi}}t\int_0^1\iint\chi_{III}(\sigma)e^{it\psi(\xi,\eta,\sigma)}\mu(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta\,d\xi\,d\sigma.$$

an approach similar to the one used for II can be followed, the details being simpler: first apply the stationary phase Lemma in the (ξ,η) variables, then Proposition B.2(i). We do not give details here.

Conclusion.

Space-time localization of the waves. As a conclusion of the asymptotic analysis of waves which has just been carried out, it is interesting to compare the space-time localizations of the emerging wave u, the solution of (1-3), in the three situations we examined. To simplify, suppose that f and g are localized in space close to 0, and in frequency close to $\tilde{\xi} - \tilde{\eta}$ and $\tilde{\eta}$, respectively. Then

- in the absence of space-time resonances, u will be localized where $X \sim -a'(\tilde{\xi})$, where it will have size $\sim 1/\sqrt{t}$;
- if the space-time resonant set is reduced to a point, then, under the assumptions of Theorem 2.5, u will have size $\sim 1/t^{1/4}$ if $-\Phi_\xi(\eta_0, \xi_0) - a'(\xi_0) < X < -a'(\xi_0)$, and size $\sim 1/\sqrt{t}$ if $X \sim -a'(\xi_0)$.

Lower bound. The asymptotic equivalents which have been computed also provide lower bounds for L^p norms of u. In the absence of space time resonances, we do not learn anything, since the equivalent for u is similar to a linear solution. However, in the case when Theorem 2.5 applies (that is, when Δ and Γ intersect transversally at a point), for t large we get

$$\|u(t)\|_{L^q} \gtrsim \begin{cases} \log t & \text{for } q = 2, \\ t^{1/(2q)-1/4} & \text{for } 2 < q \le \infty, \end{cases} \tag{2-9}$$

which corresponds to the lower bound states in Theorem 1.1.

3. Nonlocalized data

In this section, the data are only supposed to belong to L^2, as opposed to in Section 4, where the data will belong to weighted L^2 spaces.

Main results.

Theorem 3.1. *Assume that m is smooth and compactly supported and a, b, c are real-valued. In the various possible situations that follow, for $q \in [2, \infty]$, the solution u of (1-3) satisfies*

$$\|u(t)\|_{L^q} \lesssim \alpha(t) \|f\|_{L^2} \|g\|_{L^2}$$

with $\alpha(t)$ as follows:

$$\alpha(t) = \begin{cases} t & \text{in general}, \\ 1 & \text{if } \Gamma \text{ is empty}, \\ t^{1/2+1/(2q)} & \text{if } \Gamma \text{ is a point where } \phi \text{ vanishes at order two}, \\ t^{1/q} & \text{if } 2 \le q < \infty \text{ and } \Gamma \text{ is a noncharacteristic curve where } \phi \text{ vanishes at order one}, \\ \langle \log t \rangle & \text{if } q = \infty \text{ and } \Gamma \text{ is a noncharacteristic curve where } \phi \text{ vanishes at order one}, \\ t^{1/4+1/(2q)} & \text{when } \Gamma \text{ is a curve with nonvanishing curvature where } \phi \text{ vanishes at order one}, \\ t^{1/2} & \text{if } \Gamma \text{ is a general curve where } \phi \text{ vanishes at order one}. \end{cases}$$

In the two first situations above, the bound can be improved if the unitary groups $e^{ita(D)}$, $e^{itb(D)}$, and $e^{itc(D)}$ give decay.

In this setting, using precisely the structure of the product of two linear solutions (which cannot be described by using only the set $\phi^{-1}(\{0\})$ as previously), we get the following improvement.

Theorem 3.2. *Assume that m is smooth and compactly supported, and that (H) holds. For all $q \in [2, \infty]$, the solution u of (1-3) satisfies*

$$\|u(t)\|_{L^q} \lesssim t^{1/2+1/(2q)} \|f\|_{L^2} \|g\|_{L^2}.$$

If, moreover, we assume that $\Gamma = \varnothing$, then, for $p, q \in [2, \infty)$ with $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$, we get

$$\|u(t)\|_{L^p_t L^q} \lesssim \|f\|_{L^2} \|g\|_{L^2}. \tag{3-1}$$

Remark 3.3. The last statement of the previous theorem gives decay in an integrated form (u belonging to some $L^p L^q$), as opposed to the pointwise in time rate of decay obtained earlier; of course, this has to do with the use of Strichartz estimates. Heuristically, (3-1) can be understood as giving the rate of decay $\|u(t)\|_{L^q} \lesssim t^{1/q-1/2} \|f\|_{L^2} \|g\|_{L^2}$.

Then, if the smooth symbol m does not have bounded support, we have the following result.

Corollary 3.4. *We want to track the dependence of the bounds in the above theorem on the size of the support of m. So assume that m is bounded by 1 along with sufficiently many of its derivatives, and that it is supported on $B(0, R)$. Then all the previous boundedness results hold with an extra factor R.*

Proof. In Theorems 3.1 or 3.2, we have obtained boundedness from $L^2 \times L^2$ to $\mathscr{B}$ (where $\mathscr{B} = L^q$ or $L^p_T L^q$) of the operator $T_t = T_\sigma$ with the symbol

$$\sigma(\xi, \eta) \stackrel{\text{def}}{=} e^{ita(\xi+\eta)} \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi, \eta)} m(\xi, \eta),$$

when m has a bounded support. So now, considering a smooth symbol m supported on $B(0, R)$, we split it (using a smooth partition of the unity) as

$$m = \sum_{k,l} m_{k,l}$$

with $m_{k,l}$ smooth symbols supported on $[k-1, k+1] \times [l-1, l+1]$. Applying the previous results (invariant by modulation), we get

$$\|T_\sigma\|_{\mathscr{B}} \le \sum_{k,l} \|T_{\sigma_{k,l}}\|_{\mathscr{B}} \le c_{\mathscr{B}} \sum_{k,l} \|\pi_k f\|_{L^2} \|\pi_l g\|_{L^2},$$

where $c_{\mathscr{B}}$ is the constant previously obtained for compactly supported symbols and $\pi_k f$ is a smooth truncation of f for frequencies around $[k-1, k+1]$. Using orthogonality, it follows that

$$\|T_\sigma\|_{\mathscr{B}} \le c_{\mathscr{B}} \Big(\sum_{k,l} 1 \Big)^{1/2} \|f\|_{L^2} \|g\|_{L^2},$$

which gives the desired results, since $k, l \in \{-R-1, \dots, R+1\}$. ☐

Proof of Theorem 3.1: the general case. Using no properties on Γ or a, b, c, we can get the following general bound.

Lemma 3.5. *Assume that m is compactly supported. For all $q \in [2, \infty]$, the solution u to (1-3) satisfies*

$$\|u(t)\|_{L^q} \lesssim t \|f\|_{L^2} \|g\|_{L^2}.$$

Proof. The solution $u(t)$ is given by

$$u(t) = T_t(f, g)(x) = \int_{\mathbb{R}^2} e^{ix(\xi+\eta)} e^{ita(\xi+\eta)} \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi, \eta)} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) \, d\xi d\eta = T_\sigma(f, g)(x),$$

with symbol

$$\sigma(\xi, \eta) \overset{\text{def}}{=} e^{ita(\xi+\eta)} \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi, \eta)} m(\xi, \eta).$$

In this general setting, we only know that σ is bounded by t and compactly supported. Lemma A.1 implies that

$$\|T_t(f, g)\|_{L^q} \lesssim t \|f\|_{L^2} \|g\|_{L^2}. \qquad \square$$

The next several results improve on this bound under two different kinds of assumptions:

- using geometric properties of the resonance set Γ, or
- assuming linear Strichartz inequalities for the unitary groups $e^{ita(D)}$, $e^{itb(D)}$, and $e^{itc(D)}$, and using the structure of the product of two linear solutions.

Proof of Theorems 3.1 and 3.2: the case without resonances. We assume here that the phase function ϕ does not vanish.

Proposition 3.6. *Assume that $\Gamma = \varnothing$ and that m is compactly supported. For $q \in [2, \infty]$, the solution u of (1-3) satisfies*

$$\|u\|_{L^q} \lesssim \|f\|_{L^2} \|g\|_{L^2}. \tag{3-2}$$

Proof. The solution $u(t)$ is given by

$$T_t(f, g)(x) = \int_{\mathbb{R}^2} e^{ix(\xi+\eta)} e^{ita(\xi+\eta)} \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi, \eta)} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta = T_\sigma(f, g)(x),$$

with symbol

$$\sigma(\xi, \eta) \overset{\text{def}}{=} e^{ita(\xi+\eta)} \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi, \eta)} m(\xi, \eta).$$

Since a is real-valued and ϕ is nonvanishing, σ is bounded by a constant and compactly supported. Lemma A.1 yields

$$\|T_t(f, g)\|_{L^q} \lesssim \|f\|_{L^2} \|g\|_{L^2}. \qquad \square$$

Let us now deal with the improved bounds of Theorem 3.2 (using dispersive and Strichartz estimates on the linear evolution groups).

Proof of Theorem 3.2. Let us check the first claim. For every $s \in (0, t)$, we use the dispersive inequality (1-3) and the $L^2 \times L^2 \to L^1$ boundedness of T_m to get

$$\|e^{ita(D)} T_{me^{is\phi}}(f, g)\|_{L^\infty} = \|e^{i(t-s)a(D)} T_m(e^{isb(D)} f, e^{isc(D)} g)\|_{L^\infty} \lesssim \frac{1}{\sqrt{t-s}} \|f\|_{L^2} \|g\|_{L^2}.$$

Integrating for $s \in (0, t)$, it follows that

$$\|T_t(f, g)\|_{L^\infty} \lesssim t^{1/2} \|f\|_{L^2} \|g\|_{L^2}.$$

Similarly, using the $L^2 \times L^\infty \to L^2$ boundedness of T_m, we have for all $s > 0$

$$\|e^{ita(D)} T_{me^{is\phi}}(f, g)\|_{L^2} = \|T_m(e^{isb(D)} f, e^{isc(D)} g)\|_{L^2} \lesssim \|f\|_{L^2} \|e^{isc(D)} g\|_{L^\infty},$$

which yields (using the Strichartz inequality)

$$\|T_t(f, g)\|_{L^2} \lesssim t^{3/4} \|f\|_{L^2} \|g\|_{L^2}.$$

The proof is concluded by interpolating between L^2 and L^∞.

Next, assume that $\Gamma = \varnothing$, which means that ϕ is nonvanishing on the support of m. Computing the integration over $s \in [0, t]$, we can split

$$i T_t(f, g)(x) = I_t(f, g) - II_t(f, g),$$

with

$$I_t(f, g)(x) \overset{\text{def}}{=} \int_{\mathbb{R}^2} e^{ix(\xi+\eta)} e^{it(b(\xi)+c(\eta))} \frac{1}{\phi(\xi, \eta)} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta$$

and

$$II_t(f, g)(x) \overset{\text{def}}{=} \int_{\mathbb{R}^2} e^{ix(\xi+\eta)} e^{ita(\xi+\eta)} \frac{1}{\phi(\xi, \eta)} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta.$$

In other words,

$$I_t(f, g) = T_{m/\phi}(e^{itb(D)} f, e^{itc(D)} g) \quad \text{and} \quad II_t = e^{ita(D)} T_{m/\phi}(f, g).$$

Since ϕ is assumed to be smooth and nonvanishing, m/ϕ is also smooth and compactly supported so that the bilinear operator $T_{m/\phi}$ is bounded from $L^P \times L^Q$ into L^R as soon as $1/P + 1/Q \geq 1/R$.

Choose now p and q as in the statement of the theorem. Using the dispersive estimates and Bernstein's inequality (indeed, since m has a compact support, it is possible to assume that $\hat{f}$ and $\hat{g}$ are compactly supported) gives

$$\|I_t(f, g)\|_{L^p L^q} \lesssim \|T_{m/\phi}(f, g)\|_{L^1} \lesssim \|f\|_{L^2} \|g\|_{L^2}.$$

Therefore, $e^{itb(D)} f$ enjoys the usual Strichartz estimates, as well as, by Bernstein's inequality, the bounds $\|e^{itb(D)} f\|_{L^{2q}} \lesssim \|f\|_{L^q}$; the case of g is similar. This gives

$$\|II_t(f, g)\|_{L^p L^q} \lesssim \|e^{itb(D)} f\|_{L^{2p} L^q} \|e^{itc(D)} g\|_{L^{2p} L^q} \lesssim \|f\|_{L^2} \|g\|_{L^2}. \qquad \square$$

The case with resonance at only one point.

Proposition 3.7. *Assume that ϕ only vanishes at the point (ξ_0, η_0). Assume further that $\nabla\phi$ also vanishes at (ξ_0, η_0), but that* Hess ϕ *has a definite sign at that point. If $q \in [2, \infty]$, the solution u of (1-3) satisfies*

$$\|u(t)\|_{L^q} \lesssim t^{1/2+1/(2q)} \|f\|_{L^2} \|g\|_{L^2}.$$

Proof. Assume for simplicity that ϕ vanishes at order 2 at 0. Take a smooth, compactly supported function χ, equal to 1 on $B(0, 1)$, and set $\psi = \chi(\cdot/2) - \chi$, so that

$$1 = \chi + \sum_{j \geq 1} \psi(2^{-j} \cdot).$$

Then decompose the symbol as

$$e^{ita(\xi)} m \frac{e^{it\phi} - 1}{\phi} = \left(\chi(\sqrt{t}(\xi, \eta)) + \sum_{j \geq 1} \psi(2^{-j} \sqrt{t}(\xi, \eta)) \right) e^{ita(\xi)} m \frac{e^{it\phi} - 1}{\phi} \overset{\text{def}}{=} m_0(\xi, \eta) + \sum_{j \geq 1} m_j(\xi, \eta).$$

Obviously,

$$T_t = T_{m_0} + \sum_{j \geq 1} T_{m_j},$$

so it suffices to bound the summands above. The symbol m_j ($j \geq 0$) is supported on a ball of radius $\sim 2^j/\sqrt{t}$, and bounded by $2^{-2j} t$. It follows by Lemma A.1 that

$$\|T_{m_0}\|_{L^2 \times L^2 \to L^q} \lesssim t^{1/2+1/2q} \quad \text{and} \quad \|T_{m_j}\|_{L^2 \times L^2 \to L^q} \lesssim t^{1/2+1/(2q)} 2^{j(-1-1/q)}.$$

Therefore,

$$\|T_t\|_{L^2 \times L^2 \to L^q} \lesssim t^{1/2+1/(2q)} \left(1 + \sum_{j \geq 1} 2^{j(-1-1/q)} \right) \lesssim t^{1/2+1/(2q)},$$

which is the desired result. □

The case of resonances along a curve.

Proposition 3.8. *Assume that Γ is a smooth curve, where ϕ vanishes at order* 1. *If $q \in [2, \infty]$, the solution u of (1-3) satisfies the following.*

- *If Γ is noncharacteristic,*

$$\|u(t)\|_{L^q} \lesssim \|f\|_{L^2} \|g\|_{L^2} \begin{cases} t^{1/q} & \text{if } 2 \leq q < \infty, \\ \langle \log t \rangle & \text{if } q = \infty. \end{cases}$$

- *If Γ has nonvanishing curvature,*

$$\|u(t)\|_{L^q} \lesssim t^{1/4+1/(2q)} \|f\|_{L^2} \|g\|_{L^2}.$$

- *Else,*

$$\|u(t)\|_{L^q} \lesssim t^{1/2} \|f\|_{L^2} \|g\|_{L^2}.$$

As explained in Remark A.4, the estimate for a noncharacteristic curve Γ still holds if the only characteristic points are characteristic along the variable $\xi + \eta$.

Proof. We only treat the case where Γ is noncharacteristic and $2 \le q < \infty$; the other cases can be obtained by a similar argument. Similarly to Proposition 3.7, consider a smooth, compactly supported function χ, equal to 1 on $[0, 1]$, and set $\psi = \chi(\cdot/2) - \chi$, so that

$$1 = \chi + \sum_{j \ge 1} \psi(2^{-j} \cdot).$$

We denote the distance function by $d_\Gamma(\xi, \eta) = d((\xi, \eta), \Gamma)$; since Γ is supposed to be a smooth curve and $\nabla\phi$ is nonvanishing near Γ, it follows that

$$d_\Gamma(\xi, \eta) \simeq |\phi(\xi, \eta)|.$$

Then decompose the symbol as

$$e^{ita(\xi)} m(\xi, \eta) \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi, \eta)} = \Big(\chi\,(t\phi(\xi, \eta)) + \sum_{j \ge 1} \psi(2^{-j} t\phi(\xi, \eta))\Big) e^{ita(\xi)} m(\xi, \eta) \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi, \eta)}$$
$$\stackrel{\text{def}}{=} m_0(\xi, \eta) + \sum_{j \ge 1} m_j(\xi, \eta).$$

Obviously,

$$T_t = T_{m_0} + \sum_{j \ge 1} T_{m_j},$$

so it suffices to bound the summands above. The symbol m_0 $(j \ge 0)$ is supported on a neighborhood $\Gamma_{2^j/t}$ and bounded by $t2^{-j}$, up to a numerical constant. If Γ is noncharacteristic, it follows by Lemma A.3 that $\|T_{m_0}\|_{L^2 \times L^2 \to L^q} \lesssim t^{1/q}$ and $\|T_{m_j}\|_{L^2 \times L^2 \to L^q} \lesssim t^{1/q} 2^{-j/q}$. Therefore,

$$\|T_t\|_{L^2 \times L^2 \to L^q} \lesssim t^{1/q} \Big(1 + \sum_{j \ge 1} 2^{-j/q}\Big) \lesssim t^{1/q},$$

which is the desired result. □

4. Localized data

We will now assume that the data belongs to a weighted Sobolev space, and study the decay of the solution of (1-3).

The role of time resonances.

Proposition 4.1. *Recall that m is smooth and compactly supported. Assume that ϕ only vanishes at (ξ_0, η_0), that $\nabla\phi$ also vanishes at that point, and that* Hess ϕ *at that point has a definite sign. If $q \in [2, \infty]$, the solution u of (1-3) satisfies the following.*

- *If $0 \le s < 1/2$, $\|u(t)\|_{L^q} \lesssim t^{1/2 + 1/(2q) - s} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.*
- *If $s > 1/2$ and $q < \infty$, $\|u(t)\|_{L^q} \lesssim t^{-1/(2q)} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.*
- *If $s > 1/2$ and $q = \infty$, $\|u(t)\|_{L^\infty} \lesssim \langle \log t \rangle \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.*

Proof. As in the proof of Proposition 3.7, we decompose the symbol, giving the decomposition

$$u(t) = T_{m_0}(f, g) + \sum_{j \geq 1} T_{m_j}(f, g).$$

Again the symbol m_j is supported on a ball of radius $2^j t^{-1/2}$ and is bounded by $2^{-2j} t$. We conclude with Lemma A.1. □

Theorem 4.2. *Assume that ϕ vanishes at first order along a noncharacteristic curve Γ. Then for $2 \leq q < \infty$ and $s \geq 0$, the solution u of (1-3) satisfies the following estimates*:

- *If* $0 \leq s < 1/4$, $\|u(t)\|_{L^q} \lesssim t^{(1-4s)/q} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.
- *If* $s > 1/4$, $\|u(t)\|_{L^q} \lesssim \langle \log t \rangle \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.

If $q = \infty$, the solution u of (1-3) satisfies

$$\|u(t)\|_{L^\infty} \lesssim \langle \log t \rangle \|f\|_{L^2} \|g\|_{L^2}.$$

Remark 4.3. The L^∞ estimate of Proposition 3.8 does not improve if the data belong to weighted L^2 spaces. Also, notice that the L^2 estimate for $s > \frac{1}{4}$ is already as good as allowed by the lower bound Equation (2-9): any further assumption on space resonances will not improve the estimate.

Proof of Theorem 4.2. Just as in the proof of Proposition 3.8, split the symbol as

$$e^{ita(\xi)} m(\xi, \eta) \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi, \eta)} = m_0(\xi, \eta) + \sum_{j \geq 1} m_j(\xi, \eta).$$

Obviously,

$$T_t = T_{m_0} + \sum_{j \geq 1} T_{m_j},$$

so it suffices to bound the summands above. The symbol m_j is supported on a neighborhood $\Gamma_{2^j/t}$ and bounded by $t 2^{-2j}$, up to a constant. Since Γ is noncharacteristic, it follows by Lemma A.5 that for $s < \frac{1}{4}$

$$\begin{aligned} \|T_{m_j}\|_{L^{2,s} \times L^{2,s} \to L^q} &\lesssim (t^{-1} 2^j)^{1-1/q+4s/q} (t 2^{-2j}), \\ \|T_{m_0}\|_{L^{2,s} \times L^{2,s} \to L^q} &\lesssim t^{-1+1/q-4s/q} t, \end{aligned}$$

with corresponding estimates if $s > \frac{1}{4}$. The proof of the proposition is concluded by summing the above bounds for the elementary operators T_{m_j}. □

Following the same reasoning and estimates as in [Bernicot and Germain 2012], it is possible to get similar results for a curve admitting characteristic points.

Theorem 4.4. *Assume that ϕ vanishes at first order along a curve Γ with nonvanishing curvature. Then, for $2 \leq q < \infty$, $s \geq 0$, and $\delta > 0$, the solution u of (1-3) satisfies the following estimates*:

- *If* $0 \leq s \leq 1/2$, $\|u(t)\|_{L^q} \lesssim t^{(1-2s)/q} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.
- *If* $s \geq 1/2$, $\|u(t)\|_{L^q} \lesssim \langle t \rangle^\delta \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.

If $q=\infty$, the solution u of (1-3) satisfies

$$\|u(t)\|_{L^\infty} \lesssim \langle \log t\rangle \|f\|_{L^2}\|g\|_{L^2}.$$

Proof. Use Lemma A.6 instead of Lemma A.5 and follow the proof of Theorem 4.2. □

The role of space resonances.

Theorem 4.5. *Assume that (H) holds and $\Delta=\varnothing$, or in other words that $(\partial_\xi-\partial_\eta)\phi$ never vanishes. Then the solution u of (1-3) satisfies the following bounds for any $\delta>0$.*

- *If $0\le s<1/q$, $\|u(t)\|_{L^q} \lesssim t^{1/(2q)+1/2-(3/2)s+\delta}\|f\|_{L^{2,s}}\|g\|_{L^{2,s}}$.*
- *If $1/q<s<1-1/q$, $\|u(t)\|_{L^q} \lesssim t^{1/2-s+\delta}\|f\|_{L^{2,s}}\|g\|_{L^{2,s}}$.*
- *If $s>1-1/q$, $\|u(t)\|_{L^q} \lesssim t^{1/q-1/2+\delta}\|f\|_{L^{2,s}}\|g\|_{L^{2,s}}$.*

Proof. The proof proceeds by interpolating between the following L^2 and the L^∞ estimates. Indeed if $s<1/q$, then, for $\theta=2/q$, we have $L^q:=(L^2,L^\infty)_\theta$ and $L^{2,s}=(L^{2,qs/2},L^{2,0})_\theta$ with $qs/2\le 1/2$. We conclude similarly for the two other cases.

Recall that

$$\Phi(\xi,\eta) \stackrel{\text{def}}{=} \phi(\xi-\eta,\eta),$$

so that the hypothesis on ϕ translates into $\partial_\eta\Phi\neq 0$, and, in Fourier space, u reads

$$\hat{u}(t,\xi)=e^{ita(\xi)}\int_0^t\int e^{i\tau\Phi(\xi,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)m(\xi-\eta,\eta)\,d\eta\,d\tau.$$

The L^2 estimate. We want to prove that, for every exponent $\delta>0$ (as small as we want),

$$\|u(t)\|_{L^2} \lesssim \begin{cases} t^{3/4-(3/2)s+\delta}\|f\|_{L^{2,s}}\|g\|_{L^{2,s}} & \text{if } 0\le s\le \frac{1}{2},\\ \|f\|_{L^{2,s}}\|g\|_{L^{2,s}} & \text{if } s>\frac{1}{2}.\end{cases} \tag{4-1}$$

The result for $s=0$ is given by Theorem 3.2. So let us study the case $s=\frac{1}{2}$ so that (4-1) will follow by interpolation.

We first observe that the embedding $L^{2,1/2+\delta}\subset L^1$ and the dispersive estimates $L^1\to L^\infty$ give

$$\begin{aligned}\left\|\int e^{i\tau\phi(\xi-\eta,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)m(\xi-\eta,\eta)\,d\eta\right\|_{L^2} &\lesssim \|T_m(e^{i\tau b(D)}f,e^{i\tau c(D)}g)\|_{L^2}\\ &\lesssim \|e^{i\tau b(D)}f\|_{L^\infty}\|g\|_{L^2} \lesssim \tau^{-1/2}\|f\|_{L^{2,1/2+\delta}}\|g\|_{L^2}.\end{aligned}$$

Moreover, integrating by parts in η via the identity $is\partial_\eta\Phi e^{is\Phi}=\partial_\eta e^{is\Phi}$ gives

$$\begin{aligned}&\left\|\int e^{i\tau\Phi(\xi,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)m(\xi-\eta,\eta)\,d\eta\right\|_{L^2}\\ &\qquad\lesssim \tau^{-1}\left\|\int e^{i\tau\Phi(\xi,\eta)}\partial_\eta[\partial_\eta\Phi(\xi,\eta)^{-1}\hat{f}(\xi-\eta)\hat{g}(\eta)m(\xi-\eta,\eta)]\,d\eta\right\|_{L^2}\\ &\qquad\lesssim \tau^{-3/2}[\|f\|_{L^{2,1/2+\delta}}\|g\|_{L^{2,1}}+\|f\|_{L^{2,1}}\|g\|_{L^{2,1/2+\delta}}],\end{aligned} \tag{4-2}$$

where we repeat the same arguments as previously.

So let us fix τ and consider the bilinear operator

$$U \stackrel{\text{def}}{=} (f, g) \to \int e^{i\tau\Phi(\xi,\eta)} \hat{f}(\xi-\eta)\hat{g}(\eta)m(\xi-\eta,\eta)\, d\eta. \tag{4-3}$$

We have obtained that

$$\|U\|_{L^{2,1/2+\delta}\times L^2\to L^2} + \|U\|_{L^2\times L^{2,1/2+\delta}\to L^2} \lesssim \tau^{-1/2} \tag{4-4}$$

and

$$\|U(f,g)\|_{L^2} \lesssim \tau^{-3/2}[\|f\|_{L^{2,1/2+\delta}}\|g\|_{L^{2,1}} + \|f\|_{L^{2,1}}\|g\|_{L^{2,1/2+\delta}}]. \tag{4-5}$$

We now explain how we can interpolate between these two estimates to obtain

$$\|U(f,g)\|_{L^2} \lesssim \tau^{-1+\delta}\|f\|_{L^{2,1/2}}\|g\|_{L^{2,1/2}}, \tag{4-6}$$

for any $\delta > 0$. We first consider the collection of dyadic intervals

$$I_0 \stackrel{\text{def}}{=} [-1,1] \qquad I_n \stackrel{\text{def}}{=} [-2^n, 2^{n-1}] \cup [2^{n-1}, 2^n] \quad \text{for } n \geq 1.$$

On each set I_n, the weight $\langle x\rangle$ is equivalent to 2^n, so for $n \leq m$, two integers, we know from (4-4) that

$$\|U\|_{L^2(I_n)\times L^2(I_m)\to L^2} \lesssim \tau^{-1/2}2^{n(1/2+\delta)}$$

and from (4-5) that

$$\|U\|_{L^2(I_n)\times L^2(I_m)\to L^2} \lesssim \tau^{-3/2}[2^{n(1/2+\delta)}2^m + 2^n 2^{m(1/2+\delta)}] \lesssim \tau^{-3/2}2^{n(1/2+\delta)}2^m.$$

Consequently, taking the geometric average with $\delta' > 2\delta$, we get

$$\|U\|_{L^2(I_n)\times L^2(I_m)\to L^2} \lesssim \tau^{-1+\delta'}2^{n(1/2+\delta)}2^{m(1/2-\delta')} \lesssim \tau^{-1+\delta}2^{(n+m)(1/2-\delta)}.$$

So we have

$$\begin{aligned} \|U(f,g)\|_{L^2} &\lesssim \tau^{-1+\delta'} \sum_{n,m\geq 0} 2^{(n+m)(1/2-\delta)}\|f\|_{L^2(I_n)}\|g\|_{L^2(I_m)} \\ &\lesssim \tau^{-1+\delta'}\Big(\sum_{n,m\geq 0} 2^{-(n+m)\delta}\Big)\|f\|_{L^{2,1/2}}\|g\|_{L^{2,1/2}} \\ &\lesssim \tau^{-1+\delta'}\|f\|_{L^{2,1/2}}\|g\|_{L^{2,1/2}}. \end{aligned}$$

Since δ, δ' can be chosen as small as we want with $\delta' > 2\delta > 0$, δ' can be chosen arbitrarily small, which concludes the proof of (4-6).

Finally from (4-6), we obtain (4-1) for $s = \frac{1}{2}$ by integrating in time for $\tau \in (0,t)$.

The L^∞ estimate. We want to prove that

$$\|u(t)\|_{L^\infty} \lesssim \begin{cases} t^{1/2-s+\delta}\|f\|_{L^{2,s}}\|g\|_{L^{2,s}} & \text{if } 0 \leq s \leq 1, \\ t^{-1/2}\|f\|_{L^{2,s}}\|g\|_{L^{2,s}} & \text{if } s > 1. \end{cases} \tag{4-7}$$

The case $s=0$ was stated in Theorem 3.2. Recall that, writing

$$u(t) \overset{\text{def}}{=} \int_0^t F(t,s)\,ds,$$

the $L^1 \to L^\infty$ dispersive estimate gives

$$\|F(t,s)\|_{L^\infty} \lesssim \frac{1}{\sqrt{t-s}} \|f\|_{L^2} \|g\|_{L^2}. \tag{4-8}$$

Next, integrating by parts via the formula $is\partial_\eta \Phi e^{is\Phi} = \partial_\eta e^{is\Phi}$ gives

$$\begin{aligned}\int e^{ita(\xi)} e^{is\Phi(\xi,\eta)} \hat{f}(\xi-\eta)\hat{g}(\eta) m(\xi-\eta,\eta)\,d\eta \\ = \int e^{ita(\xi)} e^{is\Phi(\xi,\eta)} \frac{1}{is\partial_\eta \Phi(\xi,\eta)} \partial_\eta \hat{f}(\xi-\eta)\hat{g}(\eta) m(\xi-\eta,\eta)\,d\eta \\ + \int e^{ita(\xi)} e^{is\Phi(\xi,\eta)} \frac{1}{is\partial_\eta \Phi(\xi,\eta)} \hat{f}(\xi-\eta)\partial_\eta \hat{g}(\eta) m(\xi-\eta,\eta)\,d\eta \\ + \int e^{ita(\xi)} e^{is\Phi(\xi,\eta)} \partial_\eta \left(\frac{m(\xi-\eta,\eta)}{is\partial_\eta \Phi(\xi,\eta)} \right) \hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta,\end{aligned}$$

which becomes, in physical space,

$$F(t,s) = I + II + III, \tag{4-9}$$

with

$$\begin{aligned} I &\overset{\text{def}}{=} \frac{1}{s} e^{i(t-s)a(D)} T_{\frac{m}{i\partial_\eta \Phi}} (e^{itb(D)} xf, e^{itc(D)} g), \\ II &\overset{\text{def}}{=} \frac{1}{s} e^{i(t-s)a(D)} T_{\frac{m}{i\partial_\eta \Phi}} (e^{itb(D)} f, e^{itc(D)} xg), \\ III &\overset{\text{def}}{=} \frac{1}{s} e^{i(t-s)a(D)} T_{\partial_\eta \frac{m}{i\partial_\eta \Phi}} (e^{itb(D)} f, e^{itc(D)} g). \end{aligned}$$

Using the $L^1 \to L^\infty$ dispersive estimate,

$$\|I\|_{L^\infty} \lesssim \frac{1}{s\sqrt{t-s}} \left\| T_{\frac{m}{i\partial_\eta \Phi}} (e^{itb(D)} xf,\ e^{itc(D)} g) \right\|_{L^1} \lesssim \frac{1}{s\sqrt{t-s}} \|xf\|_{L^2} \|g\|_{L^2} \lesssim \frac{1}{s\sqrt{t-s}} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}.$$

Similar estimates for II and III give

$$\|F(t,s)\|_{L^\infty} \lesssim \frac{1}{s\sqrt{t-s}} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}. \tag{4-10}$$

Repeating the argument, but integrating by parts twice via the identity $\dfrac{1}{is\partial_\eta \Phi} \partial_\eta e^{is\Phi} = e^{is\Phi}$, yields

$$\|F(t,s)\|_{L^\infty} \lesssim \frac{1}{s^2\sqrt{t-s}} \|f\|_{L^{2,2}} \|g\|_{L^{2,2}}. \tag{4-11}$$

Finally, interpolating between (4-8), (4-10), and (4-11) gives

$$\|F(t,s)\|_{L^\infty} \lesssim \frac{1}{s^\sigma \sqrt{t-s}} \|f\|_{L^{2,\sigma}} \|g\|_{L^{2,\sigma}} \quad \text{for } 0 \le \sigma \le 2.$$

Integrating this inequality in s (recall that $u(t) = \int_0^t F(t,s)\,ds$) gives the desired estimate. □

The role of space-time resonances. We want to consider here the case of a point which would be resonant both in space and in time; we need to combine the two approaches previously presented.

Theorem 4.6. *Assume as usual that m is smooth and compactly supported and that (H) holds. Assume further that the point*

$$p_0 \stackrel{\text{def}}{=} (\xi_0, \eta_0)$$

is the only point in the support of m that is resonant in space and time — in other words, the only point such that $\phi(p_0) = (\partial_\xi - \partial_\eta)\phi(p_0) = 0$. Moreover, assume that ϕ and $(\partial_\xi - \partial_\eta)\phi$ vanish at order one on their zero sets, and that the two smooth curves $\{\phi = 0\}$ and $\{(\partial_\xi - \partial_\eta)\phi = 0\}$ are non tangentially intersecting at p_0 with $\partial_\xi \phi(p_0) \neq 0$. Then the solution u of (1-3) satisfies the following bounds for $q \in [2, \infty]$ and every $\delta > 0$.

- *If $s \in \left[0, \frac{1}{2}\right]$,*
$$\|u(t)\|_{L^q} \lesssim t^{1/q - s(1/4+3/(2q))+\delta} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}.$$
- *If $s \in \left(\frac{1}{2}, 1\right]$,*
$$\|u(t)\|_{L^q} \lesssim t^{-s(1/4-1/(2q))+\delta} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}.$$

Remark 4.7.

- For $q = \infty$, the estimates follow from the ones with $q < \infty$ with the Bessel inequality (since δ can be as small as we want).
- The assumptions of the theorem imply that, if ϕ and $\nabla_\eta \phi$ vanish at order 1 on Γ and Δ, respectively, then, at the intersection point of Γ and Δ, Γ is characteristic along $\xi + \eta$. Fortunately, this turns out not be a problem in the estimates.
- The technical assumption $\partial_\xi \phi(p_0) \neq 0$ is exactly the same as that of Theorem 2.5: $\Phi_\xi(\xi_0, \eta_0) \neq 0$.
- In the previous results, for $s = 1$, we get that, for large $t \gg 1$,
$$\|u(t)\|_{L^q} \lesssim t^{1/(2q) - 1/4 + \delta} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}},$$
for every $\delta > 0$. This estimate is optimal (up to δ which can be chosen as small as we want) due to the lower bound in (2-9).

Proof. The L^2 inequalities ($q = 2$) have already been proved in Theorems 4.2 and 4.4. Indeed, from Theorem 4.4 we know that $u(t)$ can be estimated in L^2 with a bound $t^{(1-s)/2+\delta}$ if $s \leq \frac{1}{2}$ and t^δ for every $\delta > 0$ if $s \geq \frac{1}{2}$. Moreover, Theorem 4.2 yields that, for every $\delta > 0$,

$$\|u(t)\|_{L^\infty} \lesssim t^\delta \|f\|_{L^2} \|g\|_{L^2}.$$

So it suffices to check the only remaining extremal point, $q = \infty$ with $s = 1$. We now aim at proving that

$$\|u(t)\|_{BMO} \lesssim t^{-1/4+\delta} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}, \tag{4-12}$$

which implies the desired result by interpolation.

To prove (4-12), the main idea is to combine the two previous situations, so let us consider small parameters $\epsilon_1, \epsilon_2 \in (t^{-1/2}, 1)$ and a smooth partition of the unity with respect to the domains

$$\Omega_1 \stackrel{\text{def}}{=} \{(\xi,\eta),\ |\phi(\xi,\eta)| > \epsilon_1 + \tfrac{1}{2}|(\partial_\xi - \partial_\eta)\phi(\xi,\eta)|\},$$
$$\Omega_2 \stackrel{\text{def}}{=} \{(\xi,\eta),\ |(\partial_\xi - \partial_\eta)\phi(\xi,\eta)| > \epsilon_2 + \tfrac{1}{2}|\phi(\xi,\eta)|\},$$
$$\Omega_3 \stackrel{\text{def}}{=} \{(\xi,\eta),\ |\phi(\xi,\eta)| < 2\epsilon_1 \quad \text{and} \quad |(\partial_\xi - \partial_\eta)\phi(\xi,\eta)| < 2\epsilon_2\}.$$

More precisely, Ω_1 can be thought of as a truncated "cone" around the curve $|(\partial_\xi - \partial_\eta)\phi| = 0$ and of top p_0. Ω_2 can be thought of similarly, but around the other curve. This decomposition, from the smooth symbol m, gives rise to three symbols m_i, and we have

$$u(t) = u_1(t) + u_2(t) + u_3(t)$$

with

$$\hat{u_i}(t,\xi) := e^{ita(\xi)} \int_0^t \int e^{is\phi(\xi-\eta,\eta)} m_i(\xi-\eta,\eta) \hat{f}(\xi-\eta)\hat{g}(\eta)\, d\eta\, ds.$$

Step 1: estimate of u_1 in BMO with $s = 1$. We perform the same decomposition as was used in the proof of Theorem 3.2, so

$$u_1(t) = I_t(f,g) - II_t(f,g),$$

with

$$I_t(f,g) = T_{m_1/\phi}(e^{itb(D)} f, e^{itc(D)} g) \quad \text{and} \quad II_t = e^{ita(D)} T_{m_1/\phi}(f,g).$$

The symbol m_1 is of Coifman–Meyer type [Coifman and Meyer 1978] (up to a translation from p_0 to 0) and ϕ is smooth and lower-bounded by ϵ_1 so $T_{m_1/\phi}$ is bounded from $L^\infty \times L^\infty$ to a modulated BMO space [Meyer and Coifman 1991] with norm $\lesssim \epsilon_1^{-1}$. Using the dispersive inequalities for the linear evolution groups,

$$\|I_t(f,g)\|_{BMO} \lesssim \epsilon_1^{-1} \|e^{itb(D)} f\|_{L^\infty} \|e^{itc(D)} g\|_{L^\infty} \lesssim \epsilon_1^{-1} t^{-1} \|f\|_{L^1} \|g\|_{L^1} \lesssim \epsilon_1^{-1} t^{-1} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}},$$

where we used $L^{2,1} \subset L^1$. Then we decompose the symbol m_1 around p_0 for scales 2^j from ϵ_1 to 1 as follows (here the scale means the distance in the frequency plane to the point p_0, which in Ω_1 is equivalent to $|\phi|$):

$$m_1 = \sum_{\epsilon_1 \le 2^j \lesssim 1} m_1 \chi(2^{-j}\phi),$$

where χ is a compactly supported and smooth function. The symbol $m_1\chi(2^{-j}\phi)/\phi$ is of Coifman–Meyer type (up to a translation) with a bound 2^{-j} so the operator $T_{m_1\chi(2^{-j}\phi)/\phi}$ is bounded from $L^2 \times L^2$ to L^1 with a bound 2^{-j}. Since when we evaluate $T_{m_1\chi(2^{-j}\phi)/\phi}(f,g)$, the functions f and g may be assumed supported in frequency on an interval of length 2^j, we deduce from Lemma A.2 that

$$\begin{aligned}\|II_t(f,g)\|_{L^\infty} &\lesssim t^{-1/2} \|T_{m_1/\phi}(f,g)\|_{L^1} \\ &\lesssim t^{-1/2} \Big(\sum_{\epsilon_1 \le 2^j \lesssim 1} 2^j 2^{-j} \Big) \|f\|_{L^{2,1}} \|g\|_{L^{2,1}} \lesssim t^{-1/2} |\log \epsilon_1| \, \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}.\end{aligned}$$

So, since $\epsilon_1 \in [t^{-1/2}, 1]$, for every $\delta > 0$, we obtain

$$\|u_1(t)\|_{BMO} \lesssim t^{-1/2}\epsilon_1^{-\delta}\|f\|_{L^{2,1}}\|g\|_{L^{2,1}}. \tag{4-13}$$

Step 2: estimate of u_2 in L^∞ with $s = 1$. For u_2, we follow the proof of Theorem 4.5, with the symbol m_2 supported on a cone with $|(\xi, \eta) - p_0| \geq \epsilon_2$. In our current situation, the symbol m_2 satisfies the Hörmander regularity condition (which means $|\partial^\alpha m_2(\xi, \eta)| \lesssim |(\xi, \eta) - p_0|^{-|\alpha|}$) and is supported on Ω_2, which can be considered as a cone of top p_0. So Ω_2 can be split into different parts at distance 2^j from p_0 for $\epsilon_2 \leq 2^j \lesssim 1$:

$$m_2 = \sum_{\epsilon_2 \leq 2^j \lesssim 1} m_2\chi(2^{-j}(\,\cdot - p_0)),$$

where χ is a smooth and compactly supported function. For each of these pieces, $\chi(2^{-j}(\,\cdot - p_0))$ restricts frequencies to a ball of radius $\sim 2^j$, so it is possible to add projections π_j on f and g, where π_j projects on intervals of length $\sim 2^j$ which we do not specify.

These considerations lead to the following modification of (4-10):

$$\|F(t, s)\|_{L^\infty} \lesssim \frac{1}{s\sqrt{t-s}} \sum_{\epsilon_2 \leq 2^j \lesssim 1} \big(I_j + II_j + III_j\big), \tag{4-14}$$

where

$$I_j \overset{\text{def}}{=} \Big\| T_{\frac{m_2\chi(2^{-j}(\cdot - p_0))}{\partial_\eta \Phi}} \Big\|_{L^2\times L^2 \to L^1} \|xf\|_{L^2}\|\pi_j g\|_{L^2},$$

$$II_j \overset{\text{def}}{=} \Big\| T_{\frac{m_2\chi(2^{-j}(\cdot - p_0))}{\partial_\eta \Phi}} \Big\|_{L^2\times L^2 \to L^1} \|\pi_j f\|_{L^2}\|xg\|_{L^2},$$

$$III_j \overset{\text{def}}{=} \Big\| T_{\partial_\eta \frac{m_2\chi(2^{-j}(\cdot - p_0))}{\partial_\eta \Phi}} \Big\|_{L^2\times L^2 \to L^1} \|\pi_j f\|_{L^2}\|\pi_j g\|_{L^2}.$$

To bound I_j, observe that $2^j \dfrac{m_2\chi(2^{-j}(\,\cdot - p_0))}{\partial_\eta \Phi}$ is a Coifman–Meyer symbol; thus

$$\Big\| T_{\frac{m_2\chi(2^{-j}(\cdot - p_0))}{\partial_\eta \Phi}} \Big\|_{L^2\times L^2 \to L^1} \lesssim 2^j.$$

Furthermore, by Lemma A.2, $\|\pi_j g\|_{L^2} \lesssim 2^{j/2}\|g\|_{L^{2,1}}$. Therefore,

$$I_j \lesssim 2^{-j}\|f\|_{L^{2,1}}\|\pi_j g\|_{L^2} \lesssim 2^{-j/2}\|f\|_{L^{2,1}}\|\pi_j g\|_{L^{2,1}}.$$

Similarly,

$$II_j \lesssim 2^{-j}\|f\|_{L^2}\|g\|_{L^{2,1}} \lesssim 2^{-j/2}\|f\|_{L^{2,1}}\|g\|_{L^{2,1}}.$$

Finally, $2^{2j}\partial_\eta \dfrac{m_2\chi(2^{-j}(\,\cdot - p_0))}{\partial_\eta \Phi}$ is also a Coifman–Meyer symbol. Applying this and Lemma A.2 gives

$$III_j \lesssim 2^{-j}\|f\|_{L^{2,1}}\|g\|_{L^{2,1}}.$$

It follows that

$$\|F(t,s)\|_{L^\infty} \lesssim \Big(\sum_{\epsilon_2 \le 2^j \lesssim 1} 2^{-j/2} + 2^{-j}\Big) \frac{1}{s\sqrt{t-s}} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}} \lesssim \epsilon_2^{-1} \frac{1}{s\sqrt{t-s}} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}},$$

which means that in (4-10) we get a new extra factor ϵ_2^{-1}. Finally, applying similar arguments as for Theorem 4.5, we conclude that, for any $\delta > 0$, we have

$$\|u_2(t)\|_{L^\infty} \lesssim \epsilon_2^{-1-\delta} t^{-1/2+\delta} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}. \tag{4-15}$$

Step 3: Estimate of u_3 in L^∞ with $s = 1$. For u_3, we know that the symbol m_3 is supported on a ball of radius $\epsilon := \max\{\epsilon_1, \epsilon_2\}$ around the space-time resonant point p_0.

We follow similar arguments as for Proposition 3.8, so we split the ball $B(p_0, \epsilon)$ into "strips" with scale ϕ from 0 to ϵ:

$$m_3 = \sum_{0 < 2^j \lesssim \epsilon} m_3 \chi(2^{-j}\phi),$$

which implies

$$u_3(t) = \sum_{0 < 2^k \lesssim \epsilon} T_{m_3^j}(f, g)$$

where $T_{m_3^j}$ is the bilinear Fourier multiplier associated to the symbol

$$m_3^j(\xi,\eta) = e^{ita(\xi+\eta)} m_3(\xi,\eta) \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi,\eta)} \chi(2^{-j}\phi(\xi,\eta)).$$

For each scale 2^j, the symbol m_3^j is bounded by $\max\{t, 2^{-j}\}$, so Lemmas A.3 and A.2 with Remark A.4 imply (the functions f, g may be supposed to be frequentially supported on an interval of length ϵ)

$$\|T_{m_3^j}(f,g)\|_{L^\infty} \lesssim \max\{t, 2^{-j}\} 2^j \|f\|_{L^2} \|g\|_{L^2} \lesssim \max\{t, 2^{-j}\} 2^j \epsilon \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}.$$

By summing all these inequalities over the scale 2^j, we get

$$\begin{aligned} \|u_3(t)\|_{L^\infty} &\lesssim \Big(t \sum_{2^j \le t^{-1}} 2^j + \sum_{t^{-1} \le 2^j \le \epsilon} 1\Big) \epsilon \|f\|_{L^{2,1}} \|g\|_{L^{2,1}} \\ &\lesssim \langle \log(\epsilon t)\rangle \epsilon \|f\|_{L^{2,1}} \|g\|_{L^{2,1}} \lesssim (\epsilon t)^\delta \epsilon \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}, \end{aligned} \tag{4-16}$$

for every $\delta > 0$, since $\epsilon t > 1$.

Step 4: End of the proof. Optimizing over ϵ_1 and ϵ_2 leads to

$$\epsilon_1 = \epsilon_2 = \epsilon_t := t^{-1/4+\delta}.$$

As required, we have $\epsilon_t \in [t^{-1/2}, 1]$. So by summing (4-13) and (4-16) with the estimate for u_2, we now have, for every small enough $\delta > 0$,

$$\|u(t)\|_{BMO} \lesssim [t^{-1/2}\epsilon_t^{-\delta} + (\epsilon t)^\delta \epsilon_t] \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}.$$

Since $\epsilon_t \geq t^{-1/2}$, the main term in the previous inequality is the second one, so we deduce for every $\delta > 0$

$$\|u(t)\|_{BMO} \lesssim t^{-1/4+\delta},$$

which concludes the proof of (4-12). □

Appendix A: Multilinear estimates

Lemma A.1. *Suppose that the symbol $\sigma(\xi, \eta)$ is bounded (that is, $\|\sigma\|_{L^\infty} \lesssim 1$) and supported on a ball of radius ϵ, say $B(0, \epsilon)$. For $q \in [2, \infty]$ and $s < \frac{1}{2}$,*

$$\|T_\sigma(f, g)\|_{L^q} \lesssim \epsilon^{1-1/q+2s} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$$

and

$$\|T_\sigma(f, g)\|_{L^q} \lesssim \epsilon^{2-1/q} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$$

if $s > \frac{1}{2}$.

Proof. Consider the first claim in the case $s = 0$. The lemma is obtained by interpolating between the endpoints $q = 2$ and $q = \infty$. If $q = 2$, it follows from an application of the Plancherel equality and the Cauchy–Schwarz inequality that

$$\begin{aligned} \|T_\sigma(f, g)\|_{L^2}^2 &= \int \left| \int \sigma(\xi - \eta, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta)\, d\eta \right|^2 d\xi \\ &\leq \int \left(\int |\sigma(\xi - \eta, \eta)|^2\, d\eta \right) \left(\int |\hat{f}(\xi - \eta) \hat{g}(\eta)|^2 d\eta \right) d\xi \\ &\lesssim \epsilon \|f\|_{L^2}^2 \|g\|_{L^2}^2. \end{aligned} \tag{A-1}$$

If $q = \infty$, use Cauchy–Schwarz again to get

$$\begin{aligned} \|T_\sigma(f, g)\|_{L^\infty} &= \left\| \iint e^{ix(\xi+\eta)} \sigma(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta)\, d\eta\, d\xi \right\|_{L^\infty} \lesssim \iint_{B(0,\epsilon)} |\hat{f}(\xi) \hat{g}(\eta)|\, d\eta\, d\xi \\ &\lesssim \epsilon \left(\iint |\hat{f}(\xi) \hat{g}(\eta)|^2\, d\eta\, d\xi \right)^{1/2} \lesssim \epsilon \|f\|_{L^2} \|g\|_{L^2}. \end{aligned} \tag{A-2}$$

Then, for $s > 0$, we use that the symbol is supported on a ball of radius ϵ, so f (respectively g) can be replaced with $\pi_I(f)$ (respectively $\pi_J(g)$), corresponding to the frequency-truncation of f on an interval I of length 2ϵ. We conclude by applying the previous reasoning with $\pi_I(f)$ and $\pi_J(g)$ and Lemma A.2. □

Lemma A.2. *Assume that I is an interval and consider π_I the Fourier multiplier, given by a smooth function supported on $2I$ and equal to 1 on I. For $q \in [2, \infty]$ and $s < \frac{1}{2}$,*

$$\|\pi_I(f)\|_{L^2} \lesssim |I|^s \|f\|_{L^{s,2}}.$$

Proof. The proof relies on the Sobolev embedding as follows:

$$\|\pi_I(f)\|_{L^2} \lesssim |I|^{1/2-1/\sigma} \|\hat{f}\|_{L^\sigma(2I)} \lesssim |I|^s \|\hat{f}\|_{L^\sigma} \lesssim |I|^s \|\hat{f}\|_{W^{s,2}} \lesssim |I|^s \|f\|_{L^{s,2}},$$

where the exponent σ is given by $1/\sigma = \frac{1}{2} - s$. □

Lemma A.3. *Consider a smooth curve* Γ *and a bounded symbol* σ ($\|\sigma\|_\infty \lesssim 1$) *supported on* $\Gamma_\epsilon \cap B(0, M)$, *for a positive constant* M. *Suppose* $q \in [2, \infty]$.

- *If the curve* Γ *is noncharacteristic*,

$$\|T_\sigma(f, g)\|_{L^q} \lesssim \epsilon^{1-1/q} \|f\|_{L^2} \|g\|_{L^2}.$$

- *If the curve* Γ *has nonvanishing curvature*,

$$\|T_\sigma(f, g)\|_{L^q} \lesssim \epsilon^{3/4-1/(2q)} \|f\|_{L^2} \|g\|_{L^2}.$$

- *Otherwise*,

$$\|T_\sigma(f, g)\|_{L^q} \lesssim \epsilon^{1/2} \|f\|_{L^2} \|g\|_{L^2}.$$

Proof. As for Lemma A.1, by interpolation it suffices to study the two extremal situations, $q = 2$ and $q = \infty$. First, for $q = 2$, employ the same reasoning as in Lemma A.1 (relying on the Plancherel equality). Since the support Γ_ϵ now has a measure bounded by ϵ (up to a constant), we get

$$\|T_\sigma(f, g)\|_{L^2} \lesssim \epsilon^{1/2} \|f\|_{L^2} \|g\|_{L^2}. \tag{A-3}$$

Let us point out that this estimate is the easiest situation (when the three exponents are equal to 2) described by Theorem 1.5 of [Bernicot and Germain 2012]. Moreover this estimate does not depend on geometric properties of the curve Γ.

Let us now study the case where $q = \infty$. If the curve Γ is noncharacteristic, then Proposition 6.2 of [Bernicot and Germain 2012] implies that

$$\|T_\sigma(f, g)\|_{L^\infty} \lesssim \epsilon \|f\|_{L^2} \|g\|_{L^2},$$

which, by interpolating with (A-3), proves the desired result. If the curve Γ has a nonvanishing curvature, the proposition just cited yields

$$\|T_\sigma(f, g)\|_{L^\infty} \lesssim \epsilon^{3/4} \|f\|_{L^2} \|g\|_{L^2},$$

and we similarly conclude by interpolation. □

Remark A.4. The estimate for a noncharacteristic curve Γ still holds if the curve admits some points that are characteristic only along the variable $\xi + \eta$, which means when the tangential vector of the curve at this point is parallel to $(-1, 1)$. Indeed the proof of Proposition 6.2 of [Bernicot and Germain 2012] only requires appropriate decompositions in the variables ξ and η for f and g and do not use specific properties on the third frequency variable $\xi + \eta$.

Lemma A.5. *Assume that* Γ *is a noncharacteristic curve. Consider a bounded symbol* σ ($\|\sigma\|_\infty \lesssim 1$) *supported on* $\Gamma_\epsilon \cap B(0, M)$, *for a positive constant* M.

- *If* $0 \le s < 1/4$, $\|T_\sigma(f, g)\|_{L^q} \lesssim \epsilon^{1-1/q+4s/q} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.
- *If* $s > 1/4$, $\|T_\sigma(f, g)\|_{L^q} \lesssim \epsilon \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.

Proof. We follow the same steps as for the previous lemma. The $L^2 \times L^2$ to L^∞ estimate cannot be improved by replacing L^2 by $L^{2,s}$, so we simply focus on $L^2 \times L^2$ to L^2 estimates. Since the curve is assumed to be noncharacteristic, it follows that

$$|\langle T_\sigma(f,g),h\rangle| \lesssim \sum_i \epsilon^{1/2} \|\hat{f}\|_{L^2(I_i^1)} \|\hat{g}\|_{L^2(I_i^2)} \|\hat{h}\|_{L^2(I_i^3)}, \tag{A-4}$$

where the $(I_i^k)_i$ are collections of almost disjoint intervals of length ϵ for $k = 1, 2, 3$. As a consequence, from the Cauchy–Schwartz inequality it turns out

$$\|T_\sigma(f,g)\|_{L^2} \lesssim \epsilon^{1/2} (\sup_i \|\hat{f}\|_{L^2(I_i^1)}) \|g\|_{L^2}.$$

Using Sobolev embedding on the whole space $\mathbb{R}$, we get

$$\|\hat{f}\|_{L^2(I_i^1)} \lesssim \epsilon^{1/2-1/\sigma} \|\hat{f}\|_{L^\sigma(I_i^1)} \lesssim \epsilon^{2s} \|\hat{f}\|_{L^\sigma} \lesssim \epsilon^{2s} \|\hat{f}\|_{W^{2s,2}}$$

with the exponent σ given by $1/\sigma = \frac{1}{2} - 2s$ (we recall that $s \le \frac{1}{4}$). So finally we get

$$\|T_\sigma(f,g)\|_{L^2} \lesssim \epsilon^{1/2+2s} \|f\|_{L^{2,2s}} \|g\|_{L^2}.$$

By symmetry and then interpolation, we deduce

$$\|T_\sigma(f,g)\|_{L^2} \lesssim \epsilon^{1/2+2s} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}. \qquad \square$$

Lemma A.6. *Assume that Γ has a nonvanishing curvature. Consider a bounded symbol σ ($\|\sigma\|_\infty \lesssim 1$) supported on $\Gamma_\epsilon \cap B(0, M)$, for a positive constant M. If $0 \le s \le \frac{1}{2}$, then, for every $\delta > 0$,*

$$\|T_\sigma(f,g)\|_{L^2} \lesssim \begin{cases} \epsilon^{1/2+s+\delta} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}, & \text{if } s < 1/2, \\ \epsilon \sqrt{|\log \epsilon|}\, \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}, & \text{if } s > 1/2. \end{cases}$$

Proof. The case $s = 0$ is included in Lemma A.3, so by interpolation (with $L^{2,s} \subset L^1$ for $s > \frac{1}{2}$) it suffices to check that

$$\|T_\sigma(f,g)\|_{L^2} \lesssim \epsilon \sqrt{|\log \epsilon|}\, \|f\|_{L^1} \|g\|_{L^1}.$$

This estimate was already proved in [Bernicot and Germain 2012, Proposition 5.1]. For readability we quickly sketch the proof here. Assume that $(0, 0) \in \Gamma$ and let us work around this point. Then note that, for every L^2-function h,

$$\begin{aligned} \big|\langle T_\sigma(f,g),h\rangle\big| &= \left| \int \sigma(\xi,\eta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi+\eta)\, d\xi\, d\eta \right| \lesssim \|\hat{f}\|_{L^\infty} \|\hat{g}\|_{L^\infty} \int \left| \sigma\Big(\frac{u+v}{2}, \frac{u-v}{2}\Big) \right| |\hat{h}(u)|\, du\, dv \\ &\lesssim \|f\|_{L^1} \|g\|_{L^1} \int |\hat{h}(u)| \frac{\epsilon}{\sqrt{1+|u|}} \lesssim \|f\|_{L^1} \|g\|_{L^1} \|h\|_{L^2} \epsilon \sqrt{|\log \epsilon|}, \end{aligned}$$

where we have used (because of the nonvanishing curvature) that, uniformly with respect to λ_0,

$$\big|\{\xi-\eta,\ (\xi,\eta) \in \Gamma_\epsilon, \xi+\eta = \lambda_0\}\big| \lesssim \frac{\epsilon}{\sqrt{\epsilon+|\lambda_0|}}. \qquad \square$$

Appendix B: One-dimensional oscillatory integrals

Before stating the main proposition, we need to define the functions[1]

$$\mathcal{G}_1(x) \overset{\text{def}}{=} \int_x^\infty e^{i\sigma^2}\,d\sigma \quad\text{and}\quad \mathcal{G}_2(x) \overset{\text{def}}{=} \int_x^\infty e^{i\sigma^2}\frac{d\sigma}{\sqrt{\sigma-x}}.$$

Their qualitative behavior is given by the following lemma.

Lemma B.1. (i) *$\mathcal{G}_1$ is a smooth function such that*

$$\begin{cases} \mathcal{G}_1(x) = -e^{ix^2}/(2ix) + O(1/x^2) & \text{as } x\to\infty,\\ \mathcal{G}_1(x) = C_0 + O(1/x) & \text{as } x\to-\infty,\end{cases}$$

where C_0 is the constant $C_0=\int_{-\infty}^{\infty} e^{i\sigma^2}\,d\sigma$.

(ii) *$\mathcal{G}_2$ is a smooth function such that*

$$\mathcal{G}_2(x) = \begin{cases} C_+e^{ix^2}\sqrt{2/x} + O(1/|x|^{5/6}) & \text{as } x\to\infty,\\ C_-e^{ix^2}\sqrt{2/|x|} + \sqrt{\pi}e^{i\pi/4}e^{-ix^2}(1/\sqrt{|x|}) + O(1/|x|^{5/7}) & \text{as } x\to-\infty,\end{cases}$$

where $C_\pm = \int_0^\infty e^{\pm i\sigma^2}\,d\sigma$.

We now state the main result. Recall that $C_0=\int_{-\infty}^{+\infty} e^{i\sigma^2}\,d\sigma$.

Proposition B.2. *Let χ be a smooth, compactly supported function, and let ζ be a smooth function.*

(i) *If $\zeta''\ge c>0$ and $\zeta'(\sigma_0)=0$,*

$$\int_0^\infty e^{it\zeta(\sigma)}\chi(\sigma)\,d\sigma = \chi(\sigma_0)\sqrt{\frac{2}{\zeta''(\sigma_0)}}\frac{1}{\sqrt{t}}\mathcal{G}_1(\sqrt{t}\sigma_0) + O_c\Big(\frac{1}{t}\Big).$$

(ii) *If $|\zeta'|\ge c>0$ does not vanish,*

$$\int_0^\infty e^{it\zeta(\sigma)}\chi(\sigma)\frac{d\sigma}{\sqrt{\sigma}} = \frac{\chi(0)}{\sqrt{\zeta'(0)}}e^{it\zeta(0)}\frac{C_0}{\sqrt{t}} + O_c\Big(\frac{1}{t}\Big).$$

(iii) *If $|\zeta''|\ge c>0$, $\zeta'(\sigma_0)=0$ with $\sigma_0\ge c$,*

$$\int_0^\infty e^{it\zeta(\sigma)}\chi(\sigma)\frac{d\sigma}{\sqrt{\sigma}} = \frac{\chi(0)}{\sqrt{\zeta'(0)}}e^{it\zeta(0)}\frac{C_0}{\sqrt{t}} + \sqrt{2\pi}\,e^{it\zeta(\sigma_0)}e^{i\,\mathrm{sign}(\zeta''(\sigma_0))\frac{\pi}{4}}\frac{\chi(\sigma_0)}{\sqrt{\sigma_0\zeta''(\sigma_0)}}\frac{1}{\sqrt{t}} + O_c\Big(\frac{1}{t}\Big).$$

(iv) *If $\zeta''\ge c>0$ and $\zeta'(\sigma_0)=0$,*

$$\int_0^\infty e^{it\zeta(\sigma)}\chi(\sigma)\frac{d\sigma}{\sqrt{\sigma}} = C(\chi,\zeta)\mathcal{G}_2(\sqrt{t}\sigma_0) + \begin{cases} O_c(t^{-3/4}) & \text{if } |\sqrt{t}\sigma_0|<1,\\ O_c(\sqrt{\sigma_0/t}) & \text{if } |\sqrt{t}\sigma_0|<1,\end{cases}$$

where $C(\chi,\zeta)$ is a function of χ and ζ (and hence also of σ_0) which we do not make explicit here.

[1] The function $\mathcal{G}_1$ can obviously be obtained from the Fresnel integrals $S(x)=\int_0^x \sin t^2\,dt$ and $C(x)=\int_0^x\cos t^2\,dt$. In particular, the constants C_0 and $C_\pm$ appearing below can be computed via Fresnel integrals to yield $C_0=(1+i)\sqrt{\pi/2}$ and $C_\pm=((1\pm i)/2)\sqrt{\pi/2}$. See [Abramowitz and Stegun 1964].

Remark B.3. Statements (ii) and (iii) on the one hand, and (iv) on the other, are complementary: (ii) and (iii) apply when ζ' vanishes away from zero, or not at all, whereas (iv) is meaningful if the point of vanishing of ζ' approaches zero.

Proof of Lemma B.1. Assertion (i) is proved by a simple integration by parts, so we skip it and focus on (ii). After the change of variable of integration to $\tau = \sqrt{\sigma - x}$, $\mathscr{G}_2$ becomes

$$\mathscr{G}_2(x) = 2e^{ix^2} \int_0^\infty e^{i\tau^2(\tau^2+2x)} d\tau \stackrel{\text{def}}{=} 2e^{ix^2} g(x).$$

The case $x \to \infty$. Split

$$g(x) = \int_0^R + \int_R^\infty \dots d\tau \stackrel{\text{def}}{=} I + II.$$

Start with

$$I = \int_0^R e^{i2x\tau^2} d\tau + \int_0^R [e^{i\tau^2(\tau^2+2x)} - e^{i2x\tau^2}] d\tau \stackrel{\text{def}}{=} I_1 + I_2.$$

The term I_1 can be written

$$I_1 = \int_0^\infty e^{i2x\tau^2} d\tau - \int_R^\infty e^{i2x\tau^2} d\tau = \frac{1}{\sqrt{2x}} \int_0^\infty e^{i\sigma^2} d\sigma + O\left(\frac{1}{xR}\right),$$

where the inequality $\int_R^\infty e^{i2x\tau^2} d\tau = O(1/(\alpha R))$ follows by integration by parts.

As for I_2, estimate it brutally by

$$|I_2| \lesssim \int_0^R \tau^4 d\tau = O(R^5).$$

Finally, an integration by parts gives

$$II = \int_0^\infty e^{i\tau^2(\tau^2+2\tau x)} d\tau \lesssim \frac{1}{R^2 x}.$$

Gathering the above gives

$$g(x) = \sqrt{\frac{1}{2x}} \int_0^\infty e^{i\sigma^2} d\sigma + O(R^5) + O\left(\frac{1}{xR^2}\right);$$

finally, optimizing over R gives

$$g(x) = \sqrt{\frac{1}{2x}} \int_0^\infty e^{i\sigma^2} d\sigma + O\left(\frac{1}{x^{5/7}}\right),$$

which is the desired result.

The case $x \to -\infty$. Split

$$g(x) = \int_0^{\sqrt{-x/2}} + \int_{\sqrt{-x/2}}^\infty \dots d\tau \stackrel{\text{def}}{=} III + IV.$$

Start with III. Similarly to g in the case $x \to \infty$, we use the split

$$III = \int_0^R + \int_R^{\sqrt{-x/2}} \dots d\tau = III_1 + III_2,$$

and estimate

$$III_1 = \frac{1}{\sqrt{2x}} \int_0^\infty e^{i\sigma^2}\, d\sigma + O\Big(R^5 + \frac{1}{|x|R}\Big) \quad \text{and} \quad III_2 = O\Big(\frac{1}{R^2 x}\Big).$$

Optimizing over R gives

$$III = \frac{1}{\sqrt{2x}} \int_0^\infty e^{i\sigma^2}\, d\sigma + O\Big(\frac{1}{|x|^{5/7}}\Big).$$

Turning now to IV, observe that the change of variable $\rho = -\tau^2/x$ gives

$$IV = \sqrt{-x} \int_{1/2}^\infty e^{ix^2\rho(\rho-2)} \frac{d\rho}{2\sqrt{\rho}} = \frac{\sqrt{\pi}}{2} e^{i\pi/4} e^{-ix^2} \frac{1}{\sqrt{|x|}} + O\Big(\frac{1}{|x|}\Big),$$

where the last equality follows by the stationary phase lemma. Putting together our estimates on III and IV gives the desired result. □

An intermediate result. The following proposition essentially corresponds to Proposition B.2, where ζ is replaced by either σ or $\sigma-\epsilon$ (in which case $\sigma_0 = \epsilon$).

Proposition B.4. *Let χ be a smooth function.*

(i) $\displaystyle\int_\epsilon^\infty e^{it\sigma^2} \chi(\sigma)\, d\sigma = \frac{\chi(0)}{\sqrt{t}} \mathscr{G}_1(\sqrt{t}\epsilon) + O\Big(\frac{1}{t}\Big).$

(ii) $\displaystyle\int_0^\infty e^{it\sigma} \chi(\sigma) \frac{d\sigma}{\sqrt{\sigma}} = \frac{C_0}{\sqrt{t}} \chi(0) + O\Big(\frac{1}{t}\Big)$ *(recall that $C_0 = \int_{-\infty}^{+\infty} e^{i\sigma^2}\, d\sigma$).*

(iii) $\displaystyle\int_\epsilon^\infty e^{it\sigma^2} \frac{1}{\sqrt{\sigma-\epsilon}} \chi(\sigma)\, d\sigma = \frac{\chi(0)}{t^{1/4}} \mathscr{G}_2(\sqrt{t}\epsilon) + \begin{cases} O(t^{-3/4}) & \text{if } |\sqrt{t}\epsilon| < 1 \\ O(\sqrt{\epsilon/t}) & \text{if } |\sqrt{t}\epsilon| > 1 \end{cases}$

Proof. We prove only (iii), since (i) and (ii) are simpler and can be proved using a similar procedure.
<u>First reduction for (iii).</u> The change of variable $\tau = \sqrt{t}\sigma$ gives

$$\int_\epsilon^\infty e^{it\sigma^2} \frac{1}{\sqrt{\sigma-\epsilon}} \chi(\sigma)\, d\sigma = t^{-1/4} \int_{\sqrt{t}\epsilon}^\infty e^{i\tau^2} \frac{1}{\sqrt{\tau - \sqrt{t}\epsilon}} \chi\Big(\frac{\tau}{\sqrt{t}}\Big)\, d\tau.$$

Thus the proposition is proved if we show that

$$\int_{\sqrt{t}\epsilon}^\infty e^{i\tau^2} \frac{1}{\sqrt{\tau - \sqrt{t}\epsilon}} \Big[\chi\Big(\frac{\tau}{\sqrt{t}}\Big) - \chi(0)\Big]\, d\tau = \begin{cases} O(t^{-1/2}) & \text{if } |\sqrt{t}\epsilon| < 1, \\ O(\sqrt{\epsilon} t^{-1/4}) & \text{if } |\sqrt{t}\epsilon| > 1. \end{cases} \tag{B-1}$$

Define β a smooth, compactly supported function, equal to 1 on the support of χ. We can write

$$\text{(B-1)} = \chi(0) \int_{\sqrt{t}\epsilon}^\infty e^{i\tau^2} \Big[\beta\Big(\frac{\tau}{\sqrt{t}}\Big) - 1\Big] \frac{d\tau}{\sqrt{\tau - \sqrt{t}\epsilon}} + \int_{\sqrt{t}\epsilon}^\infty e^{i\tau^2} \beta\Big(\frac{\tau}{\sqrt{t}}\Big) \Big[\chi\Big(\frac{\tau}{\sqrt{t}}\Big) - \chi(0)\Big] \frac{d\tau}{\sqrt{\tau - \sqrt{t}\epsilon}}.$$

Since the first summand is easier to deal with, we focus on the second. Setting

$$Z(y) \stackrel{\text{def}}{=} \beta(y)[\chi(y) - \chi(0)],$$

reduces the question to proving that

$$\int_{\sqrt{t}\epsilon}^{\infty} e^{i\tau^2} Z\left(\frac{\tau}{\sqrt{t}}\right) \frac{d\tau}{\sqrt{\tau-\sqrt{t}\epsilon}} \lesssim \begin{cases} t^{-1/2} & \text{if } |\sqrt{t}\epsilon|<1, \\ \sqrt{\epsilon}t^{-1/4} & \text{if } |\sqrt{t}\epsilon|>1, \end{cases} \tag{B-2}$$

where Z is a smooth function vanishing at 0.

Proof of (B-2). Split the left-hand side of (B-2) as

$$\int_{\sqrt{t}\epsilon}^{\sqrt{t}\epsilon+R} + \int_{\sqrt{t}\epsilon+R}^{\infty} \dots d\tau \overset{\text{def}}{=} I + II.$$

The term I is estimated directly, giving

$$I \leq \int_{\sqrt{t}\epsilon}^{\sqrt{t}\epsilon+R} \left| Z\left(\frac{\tau}{\sqrt{t}}\right)\right| \frac{d\tau}{\sqrt{\tau-\sqrt{t}\epsilon}} \lesssim \begin{cases} \epsilon\sqrt{R} & \text{if } R<\sqrt{t}|\epsilon|, \\ t^{-1/2}R^{3/2} & \text{if } R>\sqrt{t}|\epsilon|. \end{cases}$$

The term II is submitted first to an integration by parts using the identity $\frac{1}{2\tau}\partial_\tau e^{i\tau^2} = e^{i\tau^2}$:

$$\begin{aligned} II &= \int_{\sqrt{t}\epsilon+R}^{\infty} \frac{1}{2\tau}\partial_\tau e^{i\tau^2} Z\left(\frac{\tau}{\sqrt{t}}\right)\frac{d\tau}{\sqrt{\tau-\sqrt{t}\epsilon}} = \frac{1}{\sqrt{t}}\int_{\sqrt{t}\epsilon+R}^{\infty} \tfrac{1}{2}\partial_\tau e^{i\tau^2}\widetilde{Z}\left(\frac{\tau}{\sqrt{t}}\right)\frac{d\tau}{\sqrt{\tau-\sqrt{t}\epsilon}} \\ &= -\frac{1}{2\sqrt{t}\sqrt{R}} e^{i(\sqrt{t}\epsilon+R)^2}\widetilde{Z}\left(\frac{\sqrt{t}\epsilon+R}{\sqrt{t}}\right) - \frac{1}{2t}\int_{\sqrt{t}\epsilon+R}^{\infty} e^{i\tau^2}\widetilde{Z}'\left(\frac{\tau}{\sqrt{t}}\right)\frac{d\tau}{\sqrt{\tau-\sqrt{t}\epsilon}} \\ &\qquad + \frac{1}{4\sqrt{t}}\int_{\sqrt{t}\epsilon+R}^{\infty} e^{i\tau^2}\widetilde{Z}\left(\frac{\tau}{\sqrt{t}}\right)\frac{d\tau}{(\tau-\sqrt{t}\epsilon)^{3/2}}, \end{aligned}$$

where we set $\widetilde{Z}(y) \overset{\text{def}}{=} Z(y)/y$. The term II is then estimated directly:

$$\begin{aligned} II &\lesssim \frac{1}{2\sqrt{t}\sqrt{R}}\left|\widetilde{Z}\left(\frac{\sqrt{t}\epsilon+R}{\sqrt{t}}\right)\right| + \frac{1}{t}\int_{\sqrt{t}\epsilon+R}^{\infty}\left|\widetilde{Z}'\left(\frac{\tau}{\sqrt{t}}\right)\right|\frac{d\tau}{\sqrt{\tau-\sqrt{t}\epsilon}} + \frac{1}{\sqrt{t}}\int_{\sqrt{t}\epsilon+R}^{\infty}\left|\widetilde{Z}\left(\frac{\tau}{\sqrt{t}}\right)\right|\frac{d\tau}{(\tau-\sqrt{t}\epsilon)^{3/2}} \\ &\lesssim t^{-1/2}R^{-1/2}. \end{aligned}$$

Summing up, we have

$$I + II \lesssim \begin{cases} \epsilon\sqrt{R} + t^{-1/2}R^{-1/2} & \text{if } R<\sqrt{t}|\epsilon|, \\ t^{-1/2}R^{3/2} + t^{-1/2}R^{-1/2} & \text{if } R>\sqrt{t}|\epsilon|. \end{cases}$$

Optimizing over R (distinguishing between the cases $\sqrt{t}|\epsilon|>1$ and $\sqrt{t}|\epsilon|<1$) gives (B-2). □

Proof of Proposition B.2. We only prove (iv); the proofs of (i) and (ii) closely follow that of (iv), and (iii) simply requires an additional application of the stationary phase lemma. The idea is simply to perform a change of variable which reduces matters to Proposition B.4. We want to estimate

$$\int_0^{\infty} e^{it\zeta(\sigma)}\chi(\sigma)\frac{d\sigma}{\sqrt{\sigma}} \tag{B-3}$$

where $\zeta'' \geq c > 0$ and $\zeta'(\sigma_0) = 0$. Now set

$$y = \Phi(\sigma) \overset{\text{def}}{=} \operatorname{sign}(\sigma-\sigma_0)\sqrt{\zeta(\sigma)}.$$

Notice that Φ is smooth, and that

$$\Phi^{-1}(0)=\sigma_0, \quad \Phi'(\sigma_0)=\sqrt{\frac{\zeta''(\sigma_0)}{2}}, \quad (\Phi^{-1})'(0)=\sqrt{\frac{2}{\zeta''(\sigma_0)}}.$$

Furthermore,

$$(\Phi^{-1})'(\Phi(0))=\operatorname{sign}(\Phi(0)-\sigma_0)\frac{2\sqrt{\zeta}(\Phi(0))}{\zeta'(\Phi(0))}\stackrel{\text{def}}{=}C(\zeta)^2,$$

which implies that $\sqrt{\Phi^{-1}(y)}$ can be written

$$\sqrt{\Phi^{-1}(y)}=C(\zeta)\sqrt{y}\gamma(y)$$

for some smooth, positive function γ. Performing the change of variable $y=\Phi(\sigma)$ gives

$$\text{(B-3)}=\int_{\Phi(0)}^{\infty}e^{ity^2}\chi\circ\Phi^{-1}(y)(\Phi^{-1})'(y)C(\zeta)^{-1}\frac{dy}{\sqrt{y}\sqrt{\gamma(y)}}.$$

Applying Proposition B.4 gives the desired result:

$$\text{(B-3)}=\chi(\sigma_0)\sqrt{\frac{2}{\zeta''(\sigma_0)}}\frac{1}{C(\zeta)}\frac{1}{\gamma(0)}\frac{1}{t^{1/4}}\mathscr{G}_2(\sqrt{t}\epsilon)+\begin{cases}O(t^{-3/4}) & \text{if } |\sqrt{t}\epsilon|<1,\\ O(\sqrt{\epsilon/t}) & \text{if } |\sqrt{t}\epsilon|>1.\end{cases} \qquad \square$$

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SMOOTHING AND GLOBAL ATTRACTORS FOR THE ZAKHAROV SYSTEM ON THE TORUS

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We consider the Zakharov system with periodic boundary conditions in dimension one. In the first part of the paper, it is shown that for fixed initial data in a Sobolev space, the difference of the nonlinear and the linear evolution is in a smoother space for all times the solution exists. The smoothing index depends on a parameter distinguishing the resonant and nonresonant cases. As a corollary, we obtain polynomial-in-time bounds for the Sobolev norms with regularity above the energy level. In the second part of the paper, we consider the forced and damped Zakharov system and obtain analogous smoothing estimates. As a corollary we prove the existence and smoothness of global attractors in the energy space.

1. Introduction

We study the system of nonlinear partial differential equations, introduced in [Zakharov 1972]. It describes the propagation of Langmuir waves in an ionized plasma. The system with periodic boundary conditions consists of a complex field u (Schrödinger part) and a real field n (wave part) satisfying the equation

$$\begin{cases} iu_t + \alpha u_{xx} = nu, \quad x \in \mathbb{T}, \quad t \in [-T, T], \\ n_{tt} - n_{xx} = (|u|^2)_{xx}, \\ u(x,0) = u_0(x) \in H^{s_0}(\mathbb{T}), \\ n(x,0) = n_0(x) \in H^{s_1}(\mathbb{T}), \quad n_t(x,0) = n_1(x) \in H^{s_1-1}(\mathbb{T}), \end{cases} \tag{1}$$

where $\alpha > 0$ and T is the time of existence of the solutions. The function $u(x,t)$ denotes the slowly varying envelope of the electric field with a prescribed frequency and the function $n(x,t)$ denotes the deviation of the ion density from the equilibrium. Here α is the dispersion coefficient. In the literature (see, e.g., [Takaoka 1999]) it is standard to include the speed of an ion acoustic wave in a plasma as a coefficient β^{-2} in front of n_{tt} where $\beta > 0$. One can scale away this parameter using time and amplitude coefficients of the form $t \to \beta t$, $u \to \sqrt{\beta} u$, and $n \to \beta n$ and reduce the system to (1). Smooth solutions of the Zakharov system obey the conservation laws

$$\|u(t)\|_{L^2(\mathbb{T})} = \|u_0\|_{L^2(\mathbb{T})}$$

and

$$E(u,n,\nu)(t) = \alpha \int_{\mathbb{T}} |\partial_x u|^2\,dx + \frac{1}{2}\int_{\mathbb{T}} n^2\,dx + \frac{1}{2}\int_{\mathbb{T}} \nu^2\,dx + \int_{\mathbb{T}} n|u|^2\,dx = E(u_0,n_0,n_1)$$

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where ν is such that $n_t = \nu_x$ and $\nu_t = (n + |u|^2)_x$. These conservation laws identify $H^1 \times L^2 \times H^{-1}$ as the energy space for the system.

For $\alpha = 1$, Bourgain [1994] proved that the problem is locally well-posed in the energy space using the restricted norm method (see, e.g., [Bourgain 1993]). The solutions are well-posed in the sense of the following definition

Definition 1.1. Let X, Y, Z be Banach spaces. We say that the system of equations (1) is locally well-posed in $H^{s_0}(\mathbb{T}) \times H^{s_1}(\mathbb{T}) \times H^{s_1-1}(\mathbb{T})$, if for given initial data

$$(u_0, n_0, n_1) \in H^{s_0}(\mathbb{T}) \times H^{s_1}(\mathbb{T}) \times H^{s_1-1}(\mathbb{T}),$$

there exists $T = T(\|u_0\|_{H^{s_0}}, \|n_0\|_{H^{s_1}}, \|n_1\|_{H^{s_1-1}}) > 0$ and a unique solution

$$(u, n, n_t) \in \big(X \cap C_t^0 H_x^{s_0}([-T,T] \times \mathbb{T}),\ Y \cap C_t^0 H_x^{s_1}([-T,T] \times \mathbb{T}),\ Z \cap C_t^0 H_x^{s_1-1}([-T,T] \times \mathbb{T})\big).$$

We also demand that there is continuity with respect to the initial data in the appropriate topology. If T can be taken to be arbitrarily large then we say that the problem is globally well-posed.

Thus, the energy solutions exist for all times due to the a priori bounds on the local theory norms. We should note that although the quantity $\int_{\mathbb{T}} n|u|^2\,dx$ has no definite sign it can be controlled using Sobolev inequalities by the H^1 norm of u and the L^2 norm of n. This gives the a priori bound (see [Pecher 2001])

$$\|u(t)\|_{H^1} + \|n(t)\|_{L^2} + \|n_t(t)\|_{H^{-1}} \lesssim \|u(0)\|_{H^1} + \|n(0)\|_{L^2} + \|n_t(0)\|_{H^{-1}}, \quad t \in \mathbb{R} \tag{2}$$

Takaoka [1999] extended the local-in-time theory of Bourgain and proved that when $\frac{1}{\alpha} \in \mathbb{N}$ we have local well-posedness in $H^{s_0} \times H^{s_1} \times H^{s_1-1}$ for $s_1 \geq 0$ and $\max(s_1, \frac{s_1}{2} + \frac{1}{2}) \leq s_0 \leq s_1 + 1$. In the case that $\frac{1}{\alpha} \notin \mathbb{N}$ one has local well-posedness for $s_1 \geq -\frac{1}{2}$, $\max(s_1, \frac{s_1}{2} + \frac{1}{4}) \leq s_0 \leq s_1 + 1$. A recent result [Kishimoto 2011] establishes well-posedness in the case of the higher dimensional torus.

The corresponding Cauchy problem on $\mathbb{R}^d$ has a long history. In this case it is somehow easier to establish the well-posedness of the system due to the dispersive effects of the solution waves. We cite the following papers [Added and Added 1984; 1988, Bejenaru and Herr 2011; Bejenaru et al. 2009; Bourgain and Colliander 1996; Colliander et al. 2008; Ginibre et al. 1997; Kenig et al. 1995; Sulem and Sulem 1979] as a historical summary of the results. It is expected that (see, e.g., [Kishimoto 2011]) the optimal regularity range for local well-posedness is on the line $s_1 = s_0 - \frac{1}{2}$ because the two equations in the Zakharov system equally share the loss of derivative. The Zakharov system is not scale invariant but it can be reduced to a simplified system like in [Ginibre et al. 1997], and one can then define a critical regularity. This is given by the pair $(s_0, s_1) = (\frac{d-3}{2}, \frac{d-4}{2})$, which is also on the line. In dimensions 1 and 2, the lowest regularity for the system to have local solutions has been found to be $(s_0, s_1) = (0, -\frac{1}{2})$ [Ginibre et al. 1997]. It is harder to establish the global ! solutions at this level since there is no conservation law controlling the wave part. This has been done only in one dimension [Colliander et al. 2008].

In the first part of this paper we study the dynamics of the solutions in the periodic case in more detail.[1] We prove that the difference between the nonlinear and the linear evolution for both the Schrödinger

[1] We restrict ourselves to the one-dimensional periodic case because the resonance structure is simpler. The corresponding problem in higher dimensions, $\mathbb{T}^d$ or $\mathbb{R}^d$, appears to be much harder.

and the wave part is in a smoother space than the corresponding initial data, see Theorems 2.3 and 2.4 below. This smoothing property is not apparent if one views the nonlinear evolution as a perturbation of the linear flow and apply standard Picard iteration techniques to absorb the nonlinear terms. The result will follow from a combination of the method of normal forms (through differentiation by parts) inspired by the result in [Babin et al. 2011], and the restricted norm method of Bourgain [1993]. Here the method is applied to a dispersive system of equations where the resonances are harder to control and the coupling nonlinear terms introduce additional difficulties in estimating the first order corrections. As a corollary, in the case $\alpha > 0$, we obtain polynomial-in-time bounds for Sobolev norms above the energy level $(s_0, s_1) = (1, 0)$ by a bootstrapping argument utilizing the a priori bounds and the smoothing estimates, see Corollary 2.5 below. We have applied this method in [Erdoğan and Tzirakis 2012] to obtain similar results for the periodic KdV with a smooth space-time potential. We note that the resonance structure in one-dimensional is easier to handle.

In the second part we study the existence of a global attractor (see the next section for a definition of global attractors and the statement of our result) for the dissipative Zakharov system in the energy space. Our motivation comes from the smoothing estimates that we obtained in the first part of the paper and our work in [Erdoğan and Tzirakis 2011] (also see [Goubet and Molinet 2009] in which the existence of global attractors was obtained as a corollary of a Kato type smoothing estimate). More precisely we consider

$$\begin{cases} iu_t + \alpha u_{xx} + i\gamma u = nu + f, & x \in \mathbb{T}, \quad t \in [-T, T], \\ n_{tt} - n_{xx} + \nu n_t = (|u|^2)_{xx} + g, \\ u(x,0) = u_0(x) \in H^1(\mathbb{T}), \\ n(x,0) = n_0(x) \in L^2(\mathbb{T}), \quad n_t(x,0) = n_1(x) \in H^{-1}(\mathbb{T}), \quad f \in H^1(\mathbb{T}), \quad g \in L^2(\mathbb{T}) \end{cases} \tag{3}$$

where $f,\ g$ are time-independent, g is mean-zero, $\int_{\mathbb{T}} g(x)dx = 0$, and the damping coefficients $\nu,\ \gamma > 0$. For simplicity we set $\gamma = \nu$, and $g = 0$. Our calculations apply equally well to the full system and all proofs go through with minor modifications (in particular, one does not need any other a priori estimates).

The problem with Dirichlet boundary conditions has been considered in [Flahaut 1991; Goubet and Moise 1998] in more regular spaces than the energy space. The regularity of the attractor in Gevrey spaces with periodic boundary problem was considered in [Shcherbina 2003].

Notation. To avoid the use of multiple constants, we write $A \lesssim B$ to denote that there is an absolute constant C such that $A \le CB$. We also write $A \sim B$ to denote both $A \lesssim B$ and $B \lesssim A$. We also define $\langle \cdot \rangle = 1 + |\cdot|$.

We define the Fourier sequence of a 2π-periodic L^2 function u as

$$u_k = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx} dx, \quad k \in \mathbb{Z}.$$

With this normalization we have

$$u(x) = \sum_k e^{ikx} u_k \quad \text{and} \quad (uv)_k = u_k * v_k = \sum_{m+n=k} u_n v_m.$$

As usual, for $s<0$, H^s is the completion of L^2 under the norm

$$\|u\|_{H^s} = \|\hat{u}(k)\langle k\rangle^s\|_{\ell^2}.$$

Note that for a mean-zero L^2 function u, $\|u\|_{H^s} \sim \|\hat{u}(k)|k|^s\|_{\ell^2}$. For a sequence u_k, with $u_0=0$, we will use $\|u\|_{H^s}$ notation to denote $\|u_k|k|^s\|_{\ell^2}$. We also define $\dot{H}^s = \{u \in H^s : u \text{ is mean-zero}\}$. For $s=0$ we write $\dot{H}^0 = \dot{L}^2$.

The following function will appear many times in the proofs below.

$$\phi_\beta(k) := \sum_{|n|\le|k|} \frac{1}{|n|^\beta} \sim \begin{cases} 1 & \text{if } \beta>1, \\ \log(1+\langle k\rangle) & \text{if } \beta=1, \\ \langle k\rangle^{1-\beta} & \text{if } \beta<1. \end{cases}$$

2. Statement of results

Smoothing estimates for the Zakharov system. First note that if n_0 and n_1 are mean-zero then n, n_t remain mean-zero during the evolution since by integrating the wave part of the system (1) we obtain $\partial_t^2 \int_{\mathbb{T}} n(x,t)dx = 0$. We will work with this mean-zero assumption in this paper. This is no loss of generality since if $\int_T n_0(x)dx = A$ and $\int_T n_1(x)dx = B$, then one can consider the new variables $n \to n - A - Bt$ and $u \to e^{i(Bt^2/2+At)}u$, and obtain the same system with mean-zero data.

By considering the operator $d = (-\partial_{xx})^{1/2}$, and writing $n_\pm = n \pm i d^{-1} n_t$, the system (1) can be rewritten as

$$\begin{cases} iu_t + \alpha u_{xx} = \frac{1}{2}(n_+ + n_-)u, \quad x \in \mathbb{T}, \quad t \in [-T,T], \\ (i\partial_t \mp d)n_\pm = \pm d(|u|^2), \\ u(x,0) = u_0(x) \in H^{s_0}(\mathbb{T}), \quad n_\pm(x,0) = n_0(x) \pm i d^{-1} n_1(x) \in H^{s_1}(\mathbb{T}). \end{cases} \tag{4}$$

Note that $d^{-1}n_1(x)$ is well-defined because of the mean-zero assumption, and that $n_+ = \overline{n_-}$.

The local well posedness of the system was established in the framework of $X^{s,b}$ spaces introduced by Bourgain [1993]. Let

$$\|u\|_{X^{s,b}} = \big\|\langle k\rangle^s \langle \tau - \alpha k^2\rangle^b \hat{u}(k,\tau)\big\|_{\ell^2_k L^2_\tau},$$

$$\|n\|_{Y^{s,b}_\pm} = \big\|\langle k\rangle^s \langle \tau \mp |k|\rangle^b \hat{n}(k,\tau)\big\|_{\ell^2_k L^2_\tau}.$$

Here $\pm$ corresponds to the norm of $n_\pm$ in the system (4). As usual we also define the restricted norm

$$\|u\|_{X^{s,b}_T} = \inf_{\substack{\tilde{u}=u \\ t\in[-T,T]}} \|\tilde{u}\|_{X^{s,b}}.$$

The norms $Y^{s,b}_{\pm,T}$ are defined accordingly. We also abbreviate $n_\pm(x,0) = n_{\pm,0}$.

Definition 2.1. We say (s_0,s_1) is α-admissible if $s_1 \ge -\frac{1}{2}$ and $\max(s_1, \frac{s_1}{2}+\frac{1}{4}) \le s_0 \le s_1+1$ for $\frac{1}{\alpha} \notin \mathbb{N}$, or if $s_1 \ge 0$ and $\max(s_1, \frac{s_1}{2}+\frac{1}{2}) \le s_0 \le s_1+1$ for $\frac{1}{\alpha} \in \mathbb{N}$.

Takaoka's theorem on local well-posedness can be stated as follows:

Theorem 2.2 [Takaoka 1999]. *Suppose $\alpha \neq 0$ and (s_0, s_1) is α-admissible. Then given initial data $(u_0, n_{+,0}, n_{-,0}) \in H^{s_0} \times H^{s_1} \times H^{s_1}$ there exists*

$$T \gtrsim \left(\|u_0\|_{H^{s_0}} + \|n_{+,0}\|_{H^{s_1}} + \|n_{-,0}\|_{H^{s_1}}\right)^{-\frac{1}{12}+},$$

and a unique solution $(u, n_+, n_-) \in C\left([-T, T] : H^{s_0} \times H^{s_1} \times H^{s_1}\right)$. Moreover, we have

$$\|u\|_{X_T^{s,\frac{1}{2}}} + \|n_{+,0}\|_{Y_{+,T}^{s_1,\frac{1}{2}}} + \|n_{-,0}\|_{Y_{-,T}^{s_1,\frac{1}{2}}} \leq 2\left(\|u_0\|_{H^{s_0}} + \|n_{+,0}\|_{H^{s_1}} + \|n_{-,0}\|_{H^{s_1}}\right).$$

Now, we can state our results on the smoothing estimates:

Theorem 2.3. *Suppose $\frac{1}{\alpha} \notin \mathbb{N}$, and (s_0, s_1) is α-admissible. Consider the solution of (4) with initial data $(u_0, n_{+,0}, n_{-,0}) \in H^{s_0} \times H^{s_1} \times H^{s_1}$. Assume that we have a growth bound*

$$\|u(t)\|_{H^{s_0}} + \|n_+(t)\|_{H^{s_1}} + \|n_-(t)\|_{H^{s_1}} \leq C\left(\|u_0\|_{H^{s_0}} + \|n_{+,0}\|_{H^{s_1}} + \|n_{-,0}\|_{H^{s_1}}\right)(1+|t|)^{\eta(s_0,s_1)}.$$

Then, for any $a_0 \leq \min(1, 2s_0, 1+2s_1)$ (the inequality has to be strict if $s_0 - s_1 = 1$) and for any $a_1 \leq \min(1, 2s_0, 2s_0 - s_1)$, we have

$$u(t) - e^{i\alpha t\partial_x^2} u_0 \in C_t^0 H_x^{s_0+a_0}(\mathbb{R} \times \mathbb{T}), \tag{5}$$

$$n_\pm(t) - e^{\mp itd} n_{\pm,0} \in C_t^0 H_x^{s_1+a_1}(\mathbb{R} \times \mathbb{T}). \tag{6}$$

Moreover, for $\beta > 1 + 15\eta(s_0, s_1)$, we have

$$\|u(t) - e^{i\alpha t\partial_x^2} u_0\|_{H^{s_0+a_0}} + \|n_\pm(t) - e^{\mp itd} n_{\pm,0}\|_{H^{s_1+a_1}} \leq C(1+|t|)^\beta, \tag{7}$$

where $C = C\left(s_0, s_1, a_0, a_1, \|u_0\|_{H^{s_0}}, \|n_{+,0}\|_{H^{s_1}}, \|n_{-,0}\|_{H^{s_1}}\right)$.

Theorem 2.4. *Suppose $\frac{1}{\alpha} \in \mathbb{N}$, and (s_0, s_1) is α-admissible. Assume that we have a growth bound*

$$\|u(t)\|_{H^{s_0}} + \|n_+(t)\|_{H^{s_1}} + \|n_-(t)\|_{H^{s_1}} \leq C\left(\|u_0\|_{H^{s_0}} + \|n_{+,0}\|_{H^{s_1}} + \|n_{-,0}\|_{H^{s_1}}\right)(1+|t|)^{\alpha(s_0,s_1)}.$$

Then, for any $a_0 \leq \min(1, s_1)$ (the inequality has to be strict if $s_0 - s_1 = 1$ and $s_1 \geq 1$) and for any $a_1 \leq \min(1, 2s_0 - s_1 - 1)$, we have (5), (6) and (7).

The growth bound assumption in the theorems above follows from (2) in the case $s_0 = 1$ and $s_1 = 0$. This is used in the corollary below together with a bootstrapping argument to obtain norm growth bounds in all regularity levels above energy. Although the actual growth bounds can be calculated explicitly we won't do so here since we don't believe that the rates are optimal.

Corollary 2.5. *For any $\alpha > 0$, and for any α-admissible (s_0, s_1) with $s_0 \geq 1$, $s_1 \geq 0$, the global solution of (4) with $H^{s_0} \times H^{s_1} \times H^{s_1}$ data satisfies the growth bound*

$$\|u(t)\|_{H^{s_0}} + \|n_+(t)\|_{H^{s_1}} + \|n_-(t)\|_{H^{s_1}} \leq C_1(1+|t|)^{C_2},$$

where C_1 depends on s_0, s_1, and $\|u_0\|_{H^{s_0}} + \|n_{+,0}\|_{H^{s_1}} + \|n_{-,0}\|_{H^{s_1}}$, and C_2 depends on s_0, s_1.

Proof. We drop the $\pm$ signs and work with u and n. First note that because of the energy conservation, $\|u\|_{H^1}$ and $\|n\|_{L^2}$ are bounded for all times. Assume that the claim holds for regularity levels (s_0, s_1). Let (a_0, a_1) be given by Theorem 2.3 or Theorem 2.4. Note that for initial data in $H^{s_0+a_0} \times H^{s_1+a_1}$, applying the theorem with (s_0, s_1) and (a_0, a_1), we have

$$\|u(t) - e^{i\alpha t\partial_x^2} u_0\|_{H^{s_0+a_0}} + \|n_\pm(t) - e^{\mp itd} n_{\pm,0}\|_{H^{s_1+a_1}} \le C(1+|t|)^\beta.$$

Therefore, since the linear groups are unitary, we have

$$\|u(t)\|_{H^{s_0+a_0}} + \|n(t)\|_{H^{s_1+a_1}} \le C(1+|t|)^\beta + \|u_0\|_{H^{s_0+a_0}} + \|n_0\|_{H^{s_1+a_0}}.$$

The statement follows by induction on the regularity.

We note that in the case $\frac{1}{\alpha} \in \mathbb{N}$, $s_0 = 1$, $s_1 = 0$, we have $a_0 = 0$. However, since $a_1 \in [0, 1]$, we obtain the statement for α-admissible $(1, s_1)$, $0 \le s_1 \le 1$. From then on we can take both $a_0 > 0$ and $a_1 > 0$. □

Existence of a global attractor for the dissipative Zakharov system. The problem of global attractors for nonlinear PDEs is concerned with the description of the nonlinear dynamics for a given problem as $t \to \infty$. In particular assuming that one has a well-posed problem for all times we can define the semigroup operator $U(t) : u_0 \in H \to u(t) \in H$ where H is the phase space. We want to describe the long time asymptotics of the solution by an invariant set $X \subset H$ (a global attractor) to which the orbit converges as $t \to \infty$:

$$U(t)X = X, \quad t \in \mathbb{R}_+, \qquad d(u(t), X) \to 0.$$

For dissipative systems there are many results (see, e.g., [Temam 1997]) establishing the existence of a compact set that satisfies the above properties. Dissipativity is characterized by the existence of a bounded absorbing set into which all solutions enter eventually. The candidate for the attractor set is the omega limit set of an absorbing set, B, defined by

$$\omega(B) = \bigcap_{s\ge 0} \overline{\bigcup_{t\ge s} U(t)B},$$

where the closure is taken on H. To state our result we need some definitions from [Temam 1997] (also see [Erdoğan and Tzirakis 2011; Flahaut 1991; Goubet and Moise 1998] for more discussion).

Definition 2.6. We say that a compact subset $\mathcal{A}$ of H is a global attractor for the semigroup $\{U(t)\}_{t\ge 0}$ if $\mathcal{A}$ is invariant under the flow and if for every $u_0 \in H$, $d(U(t)u_0, \mathcal{A}) \to 0$ as $t \to \infty$.

The distance is understood to be the distance of a point to the set $d(x, Y) = \inf_{y\in Y} d(x, y)$.

To state a general theorem for the existence of a global attractor we need one more definition:

Definition 2.7. We say a bounded subset $\mathcal{B}_0$ of H is absorbing if for any bounded $\mathcal{B} \subset H$ there exists $T = T(\mathcal{B})$ such that for all $t \ge T$, $U(t)\mathcal{B} \subset \mathcal{B}_0$.

It is not hard to see that the existence of a global attractor $\mathcal{A}$ for a semigroup $U(t)$ implies the existence of an absorbing set. For the converse we cite the following theorem from [Temam 1997] which gives a general criterion for the existence of a global attractor.

Theorem A. *We assume that H is a metric space and that the operator $U(t)$ is a continuous semigroup from H to itself for all $t \geq 0$. We also assume that there exists an absorbing set $\mathcal{B}_0$. If the semigroup $\{U(t)\}_{t\geq 0}$ is asymptotically compact, i.e., for every bounded sequence x_k in H and every sequence $t_k \to \infty$, $\{U(t_k)x_k\}_k$ is relatively compact in H, then $\omega(\mathcal{B}_0)$ is a global attractor.*

Using Theorem A and a smoothing estimate as above, we will prove the following

Theorem 2.8. *Fix $\alpha > 0$. Consider the dissipative Zakharov system (3) on $\mathbb{T} \times [0, \infty)$ with $u_0 \in H^1$ and with mean-zero $n_0 \in L^2$, $n_1 \in H^{-1}$. Then the equation possesses a global attractor in $H^1 \times \dot{L}^2 \times \dot{H}^{-1}$. Moreover, for any $a \in (0, 1)$, the global attractor is a compact subset of $H^{1+a} \times H^a \times H^{-1+a}$, and it is bounded in $H^{1+a} \times H^a \times H^{-1+a}$ by a constant depending only on a, α, γ, and $\|f\|_{H^1}$.*

To prove Theorem 2.8 in the case $\frac{1}{\alpha} \notin \mathbb{N}$ we will demonstrate that the solution decomposes into two parts; a linear one which decays to zero as time goes to infinity and a nonlinear one which always belongs to a smoother space. As a corollary we prove that all solutions are attracted by a ball in $H^{1+a} \times H^a \times H^{-1+a}$, $a \in (0, 1)$, whose radius depends only on a, the H^1 norm of the forcing term and the damping parameter. This implies the existence of a smooth global attractor and provides quantitative information on the size of the attractor set in $H^{1+a} \times H^a \times H^{-1+a}$. In addition it implies that higher order Sobolev norms are bounded for all positive times; see [Erdoğan and Tzirakis 2011]. In the case $\frac{1}{\alpha} \in \mathbb{N}$ the proof is slightly different because of a resonant term.

We close this section with a discussion of the well-posedness of (3) in $H^1 \times L^2 \times H^{-1}$. We first rewrite the system (when $\gamma = \nu, g = 0$) by passing to $n_\pm$ variables as above:

$$\begin{cases} (i\partial_t + \alpha\partial_x^2 + i\gamma)u = \frac{1}{2}(n_+ + n_-)u + f, \quad x \in \mathbb{T}, \quad t \in [-T, T], \\ (i\partial_t \mp d + i\gamma)n_\pm = \pm d(|u|^2), \\ u(x, 0) = u_0(x) \in H^1(\mathbb{T}), \quad n_\pm(x, 0) = n_{\pm,0}(x) = n_0(x) \pm i d^{-1} n_1(x) \in L^2(\mathbb{T}). \end{cases} \tag{8}$$

Theorem 2.9. *Given initial data $(u_0, n_{+,0}, n_{-,0}) \in H^1 \times L^2 \times L^2$ there exists*

$$T = T\big(\|u_0\|_{H^1}, \|n_{+,0}\|_{L^2}, \|n_{-,0}\|_{L^2}, \|f\|_{H^1}, \gamma\big),$$

and a unique solution $(u, n_+, n_-) \in C\big([-T, T] : H^1 \times L^2 \times L^2\big)$ of (8). Moreover, we have

$$\|u\|_{X_T^{1,\frac{1}{2}}} + \|n_{+,0}\|_{Y_{+,T}^{0,\frac{1}{2}}} + \|n_{-,0}\|_{Y_{-,T}^{0,\frac{1}{2}}} \leq 2\big(\|u_0\|_{H^1} + \|n_{+,0}\|_{L^2} + \|n_{-,0}\|_{L^2}\big).$$

This theorem follows by using the a priori estimates of Takaoka [1999]. In the case of forced and damped KdV, this was done in [Erdoğan and Tzirakis 2011, Theorem 2.1, Lemma 2.2]. We should note that the spaces where the contraction argument is done are independent of γ. One can possibly use dissipative variants of Bourgain spaces in the spirit of [Molinet and Ribaud 2002] but we don't need to do so here.

The global well-posedness follows from the following a priori estimate for the system (8) which was obtained in [Flahaut 1991] (recall that $n_\pm = n \pm i d^{-1} n_t$):

$$\|u\|_{H^1} + \|n_+\|_{L^2} + \|n_-\|_{L^2} \leq C_1 + C_2 e^{-C_3 t}, \quad t > 0, \tag{9}$$

where $C_1 = C_1(\alpha, \gamma, \|f\|_{H^1})$, $C_2 = C_2(\alpha, \gamma, \|f\|_{H^1}, \|u_0\|_{H^1}, \|n_{\pm,0}\|_{L^2})$, and $C_3 = C_3(\alpha, \gamma)$. In fact this was proved in [Flahaut 1991] for Dirichlet boundary conditions. In the case of periodic boundary conditions, the proof remains valid. Note that (9) also implies the existence of an absorbing set $\mathcal{B}_0$ in $H^1 \times L^2 \times L^2$ of radius $C_1(\alpha, \gamma, \|f\|_{H^1})$.

3. Proofs of 2.3 and 2.4

In this section we drop the $\pm$ signs and work with one n. We also set $Y = Y_+$.

$$\begin{cases} iu_t + \alpha u_{xx} = nu, \quad x \in \mathbb{T}, \quad t \in [-T, T], \\ (i\partial_t - d)n = d(|u|^2), \\ u(x,0) = u_0(x) \in H^{s_0}(\mathbb{T}), \quad n(x,0) = n_0(x) + id^{-1}n_1(x) \in H^{s_1}(\mathbb{T}). \end{cases} \tag{10}$$

Remark 3.1. We note that since $n_+ = \overline{n_-}$ all of our claims about (10) is also valid for (4). The difference in the proof will arise in the differentiation by parts process and the $X^{s,b}$ estimates. Because of (15), in formulas (16) and (17) there will additional sums in which every term, in the phase and in the multiplier with an $|\cdot|$ sign, will have a $\pm$ sign in front. This change won't alter the proofs for the $X^{s,b}$ estimates, in fact, all the cases we considered will work exactly the same way. Also it won't change the structure of the resonant sets in the case $\frac{1}{\alpha} \in \mathbb{N}$.

We will prove Theorem 2.4 only for $\alpha = 1$. Therefore, below we either have $\frac{1}{\alpha} \notin \mathbb{N}$ or $\alpha = 1$. The case $\alpha \neq 1$, $\frac{1}{\alpha} \in \mathbb{N}$ can be handled by only cosmetic changes in the proof. Writing

$$u(x,t) = \sum_k u_k(t)e^{ikx}, \quad n(x,t) = \sum_{j \neq 0} n_j(t)e^{ijx},$$

we obtain the following system for the Fourier coefficients:

$$\begin{cases} i\partial_t u_k - \alpha k^2 u_k = \sum_{\substack{k_1+k_2=k \\ k_1 \neq 0}} n_{k_1} u_{k_2}, \\ i\partial_t n_j - |j| n_j = |j| \sum_{j_1+j_2=j} u_{j_1} \overline{u_{-j_2}}, \quad j \neq 0 \\ u_k(0) = (u_0)_k, \quad n_j(0) = (n_0)_j + i|j|^{-1}(n_1)_j, \quad j \neq 0. \end{cases} \tag{11}$$

We start with the following proposition, which follows from differentiation by parts.

Proposition 3.2. *The system (11) can be written in the following form:*

$$i\partial_t\big[e^{it\alpha k^2}u_k + e^{it\alpha k^2}B_1(n,u)_k\big] = e^{it\alpha k^2}\big[\rho_1(k) + R_1(u)(\hat{k},t) + R_2(u,n)(\hat{k},t)\big], \tag{12}$$

$$i\partial_t\big[e^{it|j|}n_j + e^{it|j|}B_2(u)_j\big] = e^{it|j|}\big[\rho_2(j) + R_3(u,n)(\hat{j},t) + R_4(u,n)(\hat{j},t)\big], \tag{13}$$

where

$$B_1(n,u)_k = \sum^{*}_{\substack{k_1+k_2=k \\ k_1 \neq 0}} \frac{n_{k_1}u_{k_2}}{\alpha k^2 - \alpha k_2^2 - |k_1|}, \quad B_2(u)_j = |j| \sum^{*}_{j_1+j_2=j} \frac{u_{j_1}\overline{u_{-j_2}}}{|j| - \alpha j_1^2 + \alpha j_2^2},$$

$$R_1(u)(\hat{k},t) = \sum_{k_1,k_2}^{*} \frac{|k_1+k_2| u_{k_1} \overline{u_{-k_2}} u_{k-k_1-k_2}}{\alpha k^2 - \alpha(k-k_1-k_2)^2 - |k_1+k_2|},$$

$$R_2(u,n)(\hat{k},t) = \sum_{k_1,k_2\neq 0}^{*} \frac{n_{k_1} n_{k_2} u_{k-k_1-k_2}}{\alpha k^2 - \alpha(k-k_1)^2 - |k_1|},$$

$$R_3(u,n)(\hat{j},t) = |j| \sum_{j_1\neq 0, j_2}^{*} \frac{n_{j_1} u_{j_2} \overline{u_{j_1+j_2-j}}}{|j| - \alpha(j_1+j_2)^2 + \alpha(j-j_1-j_2)^2},$$

$$R_4(u,n)(\hat{j},t) = |j| \sum_{j_1\neq 0, j_2}^{*} \frac{\overline{n_{-j_1}} u_{j_2} \overline{u_{j_1+j_2-j}}}{|j| - \alpha j_2^2 + \alpha(j-j_2)^2}.$$

Here, $\overset{*}{\sum}$ means that the sum is over all nonresonant terms, i.e., over all indices for which the denominator is not zero. Moreover, the resonant terms ρ_1 and ρ_2 are zero if $\frac{1}{\alpha} \notin \mathbb{N}$. For $\alpha = 1$,

$$\rho_1(k) = n_{2k-\operatorname{sgn}(k)} u_{\operatorname{sgn}(k)-k}, \qquad k \neq 0,$$
$$\rho_2(j) = |j| u_{\frac{1}{2}(j+\operatorname{sgn} j)} \overline{u_{\frac{1}{2}(j-\operatorname{sgn} j)}}, \quad j \text{ odd}.$$

Proof of Proposition 3.2. Changing the variables $m_j = n_j e^{i|j|t}$ and $v_k = u_k e^{i\alpha k^2 t}$ in (11), we obtain

$$\begin{cases} i\partial_t v_k = \sum_{\substack{k_1+k_2=k \\ k_1\neq 0}} e^{it(\alpha k^2 - \alpha k_2^2 - |k_1|)} m_{k_1} v_{k_2}, \\ i\partial_t m_j = |j| \sum_{j_1+j_2=j} e^{it(|j| - \alpha j_1^2 + \alpha j_2^2)} v_{j_1} \overline{v_{-j_2}}, \quad j \neq 0, \\ v_k(0) = (u_0)_k, \quad m_j(0) = (n_0)_j + i|j|^{-1}(n_1)_j, \quad j \neq 0. \end{cases} \tag{14}$$

It is easy to check that if we define m_j^+ and m_j^- accordingly, then

$$\partial_t m_j^- = \overline{\partial_t m_{-j}^+}. \tag{15}$$

Note that the exponents do not vanish if $1/\alpha$ is not an integer. On the other hand if $\alpha = 1$, then the resonant set is

$$(k_1, k_2) = \big(2k - \operatorname{sgn}(k), \operatorname{sgn}(k) - k\big), \quad k \neq 0.$$
$$(j_1, j_2) = \Big(\frac{j+\operatorname{sgn}(j)}{2}, \frac{j-\operatorname{sgn}(j)}{2}\Big), \quad j \text{ odd}.$$

The contribution of the corresponding terms give ρ_1 and ρ_2 in the case $\alpha = 1$. Below, we assume that $\frac{1}{\alpha} \notin \mathbb{N}$.

Differentiating by parts in the v equation we obtain

$$i\partial_t v_k = \sum_{\substack{k_1+k_2=k\\k_1\neq 0}} e^{it(\alpha k^2-\alpha k_2^2-|k_1|)} m_{k_1} v_{k_2}$$
$$= \sum_{\substack{k_1+k_2=k\\k_1\neq 0}} \frac{\partial_t\big(e^{it(\alpha k^2-\alpha k_2^2-|k_1|)} m_{k_1} v_{k_2}\big)}{i(\alpha k^2-\alpha k_2^2-|k_1|)} + i \sum_{\substack{k_1+k_2=k\\k_1\neq 0}} \frac{e^{it(\alpha k^2-\alpha k_2^2-|k_1|)}\partial_t\big(m_{k_1} v_{k_2}\big)}{\alpha k^2-\alpha k_2^2-|k_1|}.$$

The second sum can be rewritten using the equation as follows:

$$\sum_{\substack{k_1+k_2+k_3=k\\k_1+k_2\neq 0}} \frac{e^{it\alpha(k^2-k_1^2+k_2^2-k_3^2)}|k_1+k_2| v_{k_1}\overline{v_{-k_2}} v_{k_3}}{\alpha k^2-\alpha k_3^2-|k_1+k_2|} + \sum_{\substack{k_1+k_2+k_3=k\\k_1+k_2\neq 0}} \frac{e^{it(\alpha k^2-\alpha k_3^2-|k_1|-|k_2|)} m_{k_1} m_{k_2} v_{k_3}}{\alpha k^2-\alpha(k_2+k_3)^2-|k_1|}. \quad (16)$$

Now, we differentiate by parts in the m equation:

$$i\partial_t m_j = |j| \sum_{j_1+j_2=j} e^{it(|j|-\alpha j_1^2+\alpha j_2^2)} v_{j_1}\overline{v_{-j_2}}$$
$$= |j| \sum_{j_1+j_2=j} \frac{\partial_t\big(e^{it(|j|-\alpha j_1^2+\alpha j_2^2)} v_{j_1}\overline{v_{-j_2}}\big)}{i(|j|-\alpha j_1^2+\alpha j_2^2)} + i|j| \sum_{j_1+j_2=j} \frac{e^{it(|j|-\alpha j_1^2+\alpha j_2^2)}\partial_t\big(v_{j_1}\overline{v_{-j_2}}\big)}{|j|-\alpha j_1^2+\alpha j_2^2}.$$

The second sum can be rewritten using the equation as follows:

$$|j| \sum_{\substack{j_1+j_2+j_3=k\\j_1\neq 0}} \frac{e^{it(|j|+\alpha j_3^2-\alpha j_2^2-|j_1|)} m_{j_1} v_{j_2}\overline{v_{-j_3}}}{|j|-\alpha(j_1+j_2)^2+\alpha j_3^2} + |j| \sum_{\substack{j_1+j_2+j_3=k\\j_2\neq 0}} \frac{e^{it(|j|-\alpha j_1^2+\alpha j_3^2+|j_2|)} v_{j_1}\overline{m_{-j_2}}\,\overline{v_{-j_3}}}{|j|-\alpha j_1^2+\alpha(j_2+j_3)^2}. \quad (17)$$

The statement follows by going back to the variables u and n. □

Integrating (12) and (13) from 0 to t, we obtain

$$u_k(t) - e^{-it\alpha k^2} u_k(0) = e^{-it\alpha k^2} B_1(n,u)_k(0) - B_1(n,u)_k(t) - i\int_0^t e^{-i\alpha k^2(t-s)}\big[\rho_1(k) + R_1(u)(\hat{k},s) + R_2(u,n)(\hat{k},s)\big]ds. \quad (18)$$

$$n_j(t) - e^{-it|j|} n_j(0) = e^{-it|j|} B_2(u)_j(0) - B_2(u)_j(t) - i\int_0^t e^{-i|j|(t-s)}\big[\rho_2(j) + R_3(u,n)(\hat{j},s) + R_4(u,n)(\hat{j},s)\big]ds. \quad (19)$$

Below we obtain a priori estimates for ρ_1, ρ_2, B_1, and B_2. Before that we state a technical lemma that will be used many times in the proofs.

Lemma 3.3. (a) *If* $\kappa \geq \lambda \geq 0$ *and* $\kappa + \lambda > 1$, *then*

$$\sum_n \frac{1}{\langle n-k_1\rangle^\kappa \langle n-k_2\rangle^\lambda} \lesssim \langle k_1-k_2\rangle^{-\lambda}\phi_\kappa(k_1-k_2).$$

(b) *For* $\kappa \in (0,1]$, *we have*

$$\int_{\mathbb{R}} \frac{d\tau}{\langle \tau+\rho_1\rangle^\kappa \langle \tau+\rho_2\rangle} \lesssim \frac{1}{\langle \rho_1-\rho_2\rangle^{\kappa-}}.$$

(c) *If* $\kappa > 1/2$, *then*

$$\sum_n \frac{1}{\langle n^2+c_1 n+c_2\rangle^\kappa} \lesssim 1,$$

where the implicit constant is independent of c_1 *and* c_2.

We will prove this lemma in the Appendix.

Lemma 3.4. *Under the conditions of Theorem 2.3 and Theorem 2.4, for each* t, *we have*

$$\begin{aligned}
\|\rho_1(t)\|_{H^s} &\lesssim \|n(t)\|_{H^{s_1}}\|u(t)\|_{H^{s_0}} && \text{if } s \leq s_0+s_1,\\
\|\rho_2(t)\|_{H^s} &\lesssim \|u(t)\|^2_{H^{s_0}} && \text{if } s \leq 2s_0-1,\\
\|B_1(n,u)(t)\|_{H^s} &\lesssim \|n(t)\|_{H^{s_1}}\|u(t)\|_{H^{s_0}} && \text{if } s \leq 1+s_0+\min(s_1,0),\\
\|B_2(u)(t)\|_{H^s} &\lesssim \|u(t)\|^2_{H^{s_0}} && \text{if } s \leq \min(2s_0, 1+s_0).
\end{aligned}$$

Proof. The proof for ρ_1 and ρ_2 is immediate from their definition.

To estimate B_1, first note that

$$\left|\alpha k^2 - \alpha k_2^2 - |k_1|\right| = |\alpha|\,|k_1|\left|2k-k_1-\frac{1}{\alpha}\operatorname{sgn}(k_1)\right| \sim \langle k_1\rangle\langle 2k-k_1\rangle.$$

The last equality is immediate in the case $\frac{1}{\alpha} \notin \mathbb{N}$, when $\alpha = 1$, it follows from the nonresonant condition. Therefore we have

$$|B_1(n,u)_k| \lesssim \sum_{k_1\neq 0} \frac{|n_{k_1}|\,|u_{k-k_1}|}{\langle k_1\rangle\langle 2k-k_1\rangle}.$$

We estimate the H^s norm as follows:

$$\|B_1\|^2_{H^s} \lesssim \left\| \sum_{k_1\neq 0} \langle k_1\rangle^{2s_1}|n_{k_1}|^2\langle k-k_1\rangle^{2s_0}|u_{k-k_1}|^2 \right\|_{\ell^1_k} \left\| \sum_{k_1} \frac{\langle k\rangle^{2s}}{\langle k_1\rangle^{2+2s_1}\langle k-k_1\rangle^{2s_0}\langle 2k-k_1\rangle^2} \right\|_{\ell^\infty_k}$$

The first sum is bounded by $\|n\|^2_{H^{s_1}}\|u\|^2_{H^{s_0}}$ since it is a convolution of two ℓ^1 sequences. To estimate the second sum we distinguish the cases $|k_1| < |k|/2$, $|k_1| > 4|k|$, and $|k_1| \sim |k|$. In the first case, we bound the sum by

$$\sum_{k_1} \frac{\langle k\rangle^{2s-2-2s_0}}{\langle k_1\rangle^{2+2s_1}} \lesssim \langle k\rangle^{2s-2-2s_0},$$

since $2+2s_1>1$. In the second case, we bound the sum by

$$\sum_{|k_1|>4|k|}\frac{\langle k\rangle^{2s}}{\langle k_1\rangle^{4+2s_1+2s_0}}\lesssim\langle k\rangle^{2s-3-2s_1-2s_0}\le\langle k\rangle^{2s-2-2s_0}.$$

In the final case, we have

$$\sum_{|k_1|\sim|k|}\frac{\langle k\rangle^{2s-2-2s_1}}{\langle k-k_1\rangle^{2s_0}\langle 2k-k_1\rangle^{2}}\lesssim\langle k\rangle^{2s-2-2s_1-2\min(s_0,1)}.$$

In the last inequality we used part (a) of Lemma 3.3.

Combining these cases we see that $B_1\in H^s$ for $s\le 1+\min(s_0,s_1+\min(s_0,1))$. In particular, $B_1\in H^s$ if $s\le 1+s_0+\min(s_1,0)$ which can be seen by distinguishing the cases $s_0\ge 1$ and $s_0<1$ and using the condition $1+s_1\ge s_0$.

Similarly, we estimate

$$|B_2(u)_j|\lesssim\sum_{j_1}\frac{|u_{j_1}|\,|u_{j_1-j}|}{\langle j-2j_1\rangle}.$$

As in the case of B_1, we see that $B_2\in H^s$ if

$$\sup_j\sum_{j_1}\frac{\langle j\rangle^{2s}}{\langle j-2j_1\rangle^{2}\langle j_1\rangle^{2s_0}\langle j-j_1\rangle^{2s_0}}<\infty.$$

We distinguish the cases $|j_1|<|j|/4$, $|j_1|>2|j|$, and $|j_1|\sim|j|$. In the first case, we bound the sum by

$$\sum_{|j_1|<|j|/4}\frac{\langle j\rangle^{2s-2-2s_0}}{\langle j_1\rangle^{2s_0}}\lesssim\langle j\rangle^{2s-2-2s_0}\phi_{2s_0}(j).$$

In the second case, we bound the sum by

$$\sum_{|j_1|>2|j|}\frac{\langle j\rangle^{2s}}{\langle j_1\rangle^{2+4s_0}}\lesssim\langle j\rangle^{2s-1-4s_0}.$$

In the final case, we have

$$\sum_{|j_1|\sim|j|}\frac{\langle j\rangle^{2s-2s_0}}{\langle j-2j_1\rangle^{2}\langle j-j_1\rangle^{2s_0}}\lesssim\langle j\rangle^{2s-2s_0-2\min(s_0,1)}.$$

Combining this cases, we see that B_2 is in H^s if $s\le\min(2s_0,1+s_0)$. □

Using the estimates in Lemma 3.4 in the equations (18) and (19) after writing the equations in the x variable, we obtain

$$\begin{aligned}\|u(t)-e^{it\alpha\partial_x^2}u_0\|_{H^{s_0+a_0}}&\lesssim\|n_0\|_{H^{s_1}}\|u_0\|_{H^{s_0}}+\|n(t)\|_{H^{s_1}}\|u(t)\|_{H^{s_0}}\\&+\int_0^t\|n(s)\|_{H^{s_1}}\|u(s)\|_{H^{s_0}}ds+\Big\|\int_0^t e^{i\alpha(t-s)\partial_x^2}\big[R_1(u)(s)+R_2(u,n)(s)\big]ds\Big\|_{H^{s_0+a_0}},\end{aligned}\tag{20}$$

$$\|n(t)-e^{-itd}n_0\|_{H^{s_1+a_1}} \lesssim \|u_0\|_{H^{s_0}}^2 + \|u(t)\|_{H^{s_0}}^2 + \int_0^t \|u(s)\|_{H^{s_0}}^2 ds + \left\| \int_0^t e^{-id(t-s)}\big[R_3(u,n)(s)+R_4(u,n)(s)\big]ds \right\|_{H^{s_1+a_1}}, \quad (21)$$

where

$$R_\ell(s) = \sum_k R_\ell(\hat{k},s)e^{ikx}, \quad \ell = 1,2,3,4.$$

Above, the smoothing indexes a_0 and a_1 depend on α as stated in Theorem 2.3 and Theorem 2.4. The dependence arises only from the contribution of the resonant terms ρ_1 and ρ_2.

Note that, with δ as in Theorem 2.2,

$$\left\| \int_0^t e^{i\alpha(t-s)\partial_x^2}\big[R_1(u)(s)+R_2(u,n)(s)\big]ds \right\|_{L^\infty_{t\in[-\delta,\delta]}H_x^{s_0+a_0}}$$
$$\lesssim \left\| \psi_\delta(t) \int_0^t e^{i\alpha(t-s)\partial_x^2}\big[R_1(u)(s)+R_2(u,n)(s)\big]ds \right\|_{X^{s_0+a_0,b}} \lesssim \|R_1(u)+R_2(u,n)\|_{X_\delta^{s_0+a_0,b-1}}, \quad (22)$$

for $b > 1/2$. Here we used the imbedding $X^{s_0+a_0,b} \subset L_t^\infty H_x^{s_0+a_0}$. Similarly,

$$\left\| \int_0^t e^{-id(t-s)}\big[R_3(u,n)(s)+R_4(u,n)(s)\big]ds \right\|_{L^\infty_{t\in[-\delta,\delta]}H_x^{s_1+a_1}} \lesssim \|R_3(u,n)+R_4(u,n)\|_{X_\delta^{s_1+a_1,b-1}}. \quad (23)$$

Remark 3.5. We note that the inequalities (22) and (23) remain valid in the case the linear group is modified with a damping term; see Lemma 3.3 from [Erdoğan and Tzirakis 2011]. It is important to note that we don't need to alter the definition of the $X^{s,b}$ norm.

Proposition 3.6. *Given $s_1 > -\frac{1}{2}$, $\max(s_1, \frac{s_1}{2}+\frac{1}{4}) \le s_0 \le s_1+1$, and $\frac{1}{2} < b < \min(\frac{3}{4}, \frac{s_0+1}{2})$, we have*

$$\|R_1(u)\|_{X^{s,b-1}} \lesssim \|u\|^3_{X^{s_0,\frac{1}{2}}}, \quad \textit{provided } s \le s_0 + \min(1, 2s_0).$$

We also have

$$\|R_2(u,n)\|_{X^{s,b-1}} \lesssim \|n\|^2_{Y^{s_1,\frac{1}{2}}} \|u\|_{X^{s_0,\frac{1}{2}}},$$

provided $s \le \min(s_0+1+2s_1, s_0+1, 3+2s_1-2b, 3+s_1-2b)$.

Proposition 3.7. *Given $s_1 > -\frac{1}{2}$, $\max(s_1, \frac{s_1}{2}+\frac{1}{4}) \le s_0 \le s_1+1$, and $\frac{1}{2} < b < \frac{3}{4} + \min(0, \frac{s_0+s_1}{2})$, we have*

$$\|R_3(u,n)\|_{X^{s,b-1}} + \|R_4(u,n)\|_{X^{s,b-1}} \lesssim \|n\|_{Y^{s_1,\frac{1}{2}}} \|u\|^2_{X^{s_0,\frac{1}{2}}},$$

provided $s \le s_1 + \min(1, 2s_0, 2s_0 - s_1)$.

We will prove these propositions later on. Using (22), (23) and the propositions above (with $b-1/2$ sufficiently small depending on a_0, a_1, s_0, s_1) in (20) and (21), we see that for $t \in [-\delta, \delta]$, we have

$$\|u(t)-e^{it\alpha\partial_x^2}u_0\|_{H^{s_0+a_0}}+\|n(t)-e^{-itd}n_0\|_{H^{s_1+a_1}} \lesssim \big[\|n_0\|_{H^{s_1}}+\|u_0\|_{H^{s_0}}\big]^2 \\ +\big[\|n(t)\|_{H^{s_1}}+\|u(t)\|_{H^{s_0}}\big]^2+\int_0^t \big[\|n(s)\|_{H^{s_1}}+\|u(s)\|_{H^{s_0}}\big]^2 ds+\big[\|n\|_{Y^{s_1,\frac12}}+\|u\|_{X^{s_0,\frac12}}\big]^3.$$

In the rest of the proof the implicit constants depend on $\|n_0\|_{H^{s_1}}, \|u_0\|_{H^{s_0}}$. Fix T large. For $t \le T$, we have the bound (with $\gamma = \gamma(s_0, s_1)$)

$$\|u(t)\|_{H^{s_0}}+\|n(t)\|_{H^{s_1}} \lesssim (1+|t|)^\gamma \lesssim T^\gamma.$$

Thus, with $\delta \sim T^{-12\gamma-}$, we have

$$\|u(j\delta)-e^{i\delta\alpha\partial_x^2}u((j-1)\delta)\|_{H^{s_0+a_0}}+\|n(j\delta)-e^{-i\delta d}n((j-1)\delta)\|_{H^{s_1+a_1}} \lesssim T^{3\gamma},$$

for any $j \le T/\delta \sim T^{1+12\gamma+}$. Here we used the local theory bound

$$\|u\|_{X^{s_0,1/2}_{[(j-1)\delta,\, j\delta]}} \lesssim \|u((j-1)\delta)\|_{H^{s_0}} \lesssim T^\gamma,$$

and similarly for n. Using this we obtain (with $J = T/\delta \sim T^{1+12\gamma+}$)

$$\begin{aligned}\|u(J\delta)-e^{i\alpha J\delta\partial_x^2}u(0)\|_{H^{s_0+a_0}} &\le \sum_{j=1}^J \|e^{i(J-j)\delta\alpha\partial_x^2}u(j\delta)-e^{i(J-j+1)\delta\alpha\partial_x^2}u((j-1)\delta)\|_{H^{s_0+a_0}}\\ &= \sum_{j=1}^J \|u(j\delta)-e^{i\delta\alpha\partial_x^2}u((j-1)\delta)\|_{H^{s_0+a_0}}\\ &\lesssim JT^{3\gamma} \sim T^{1+15\gamma+}.\end{aligned}$$

The analogous bound follows similarly for the wave part n.

The continuity in $H^{s_0+a_0} \times H^{s_1+a_1}$ follows from dominated convergence theorem, the continuity of u and n in H^{s_0}, H^{s_1}, respectively, and from the embedding $X^{s,b} \subset C_t^0 H_x^s$ (for $b > 1/2$). For details, see [Erdoğan and Tzirakis 2012; Ginibre et al. 1997].

4. Proof of Proposition 3.6

First note that the denominator in the definition of R_1 satisfies

$$\begin{aligned}\big|\alpha k^2-\alpha(k-k_1-k_2)^2-|k_1+k_2|\big| &= |\alpha|\,|k_1+k_2|\,\Big|2k-k-k_1-\frac{1}{\alpha}\operatorname{sgn}(k_1+k_2)\Big|\\ &\sim \langle k_1+k_2\rangle\langle 2k-k_1-k_2\rangle.\end{aligned} \tag{24}$$

The last equality holds trivially if $1/\alpha$ is not an integer. In the case that $1/\alpha$ is an integer it holds since the sum is over the nonresonant terms. Similarly, we shall see that the denominators of R_2, R_3, R_4 are respectively comparable to

$$\langle k_1\rangle\langle 2k-k_1\rangle,\quad \langle j\rangle\langle j-2j_1-2j_2\rangle,\quad \langle j\rangle\langle j-2j_2\rangle, \tag{25}$$

We start with the proof for R_2. We have

$$\|R_2(u,n)\|^2_{X^{s,b-1}} = \Big\| \int_{\tau_1,\tau_2} \sum^*_{k_1,k_2\neq 0} \frac{\langle k\rangle^s \hat{n}(k_1,\tau_1)\hat{n}(k_2,\tau_2)\hat{u}(k-k_1-k_2,\tau-\tau_1-\tau_2)}{(\alpha k^2-\alpha(k-k_1)^2-|k_1|)\langle \tau-k^2\rangle^{1-b}} \Big\|^2_{\ell^2_k L^2_\tau}.$$

Let

$$f(k,\tau)=|\hat{n}(k,\tau)|\langle k\rangle^{s_1}\langle \tau-|k|\rangle^{\frac12},\quad g(k,\tau)=|\hat{u}(k,\tau)|\langle k\rangle^{s_0}\langle \tau-\alpha k^2\rangle^{\frac12}.$$

It suffices to prove that

$$\Big\| \int_{\tau_1,\tau_2} \sum^*_{k_1,k_2\neq 0} M(k_1,k_2,k,\tau_1,\tau_2,\tau) f(k_1,\tau_1) f(k_2,\tau_2) g(k-k_1-k_2,\tau-\tau_1-\tau_2)\Big\|^2_{\ell^2_k L^2_\tau} \lesssim \|f\|_2^4\|g\|_2^2,$$

where

$$M(k_1,k_2,k,\tau_1,\tau_2,\tau) = \frac{\langle k\rangle^s\langle k_1\rangle^{-s_1}\langle k_2\rangle^{-s_1}\langle k-k_1-k_2\rangle^{-s_0}}{(\alpha k^2-\alpha(k-k_1)^2-|k_1|)\langle \tau-\alpha k^2\rangle^{1-b}\langle \tau_1-|k_1|\rangle^{\frac12}\langle \tau_2-|k_2|\rangle^{\frac12}\langle \tau-\tau_1-\tau_2-\alpha(k-k_1-k_2)^2\rangle^{\frac12}}.$$

By Cauchy–Schwarz in the variables τ_1,τ_2,k_1,k_2, we estimate the norm above by

$$\sup_{k,\tau}\Big(\int_{\tau_1,\tau_2}\sum^*_{k_1,k_2\neq 0} M^2(k_1,k_2,k,\tau_1,\tau_2,\tau)\Big) \times \Big\|\int_{\tau_1,\tau_2}\sum_{k_1,k_2\neq 0} f^2(k_1,\tau_1) f^2(k_2,\tau_2) g^2(k-k_1-k_2,\tau-\tau_1-\tau_2)\Big\|_{\ell^1_k L^1_\tau}.$$

Note that the norm above is equal to $\|f^2 * f^2 * g^2\|_{\ell^1_k L^1_\tau}$, which can be estimated by $\|f\|_2^4\|g\|_2^2$ by Young's inequality. Therefore, it suffices to prove that the supremum above is finite.

Using part (b) of Lemma 3.3 in τ_1 and τ_2 integrals, we obtain

$$\sup_{k,\tau}\int_{\tau_1,\tau_2}\sum^*_{k_1,k_2\neq 0} M^2 \lesssim \sup_{k,\tau}\sum^*_{k_1,k_2\neq 0}\frac{\langle k\rangle^{2s}\langle k_1\rangle^{-2s_1}\langle k_2\rangle^{-2s_1}\langle k-k_1-k_2\rangle^{-2s_0}}{(\alpha k^2-\alpha(k-k_1)^2-|k_1|)^2\langle \tau-\alpha k^2\rangle^{2-2b}\langle \tau-|k_1|-|k_2|-\alpha(k-k_1-k_2)^2\rangle^{1-}}$$
$$\lesssim \sup_k \sum_{k_1,k_2\neq 0}\frac{\langle k\rangle^{2s}\langle k_1\rangle^{-2s_1}\langle k_2\rangle^{-2s_1}\langle k-k_1-k_2\rangle^{-2s_0}}{\langle k_1\rangle^2\langle 2k-k_1\rangle^2\langle \alpha k^2-|k_1|-|k_2|-\alpha(k-k_1-k_2)^2\rangle^{2-2b}}.$$

The last line follows by (25) and by the simple fact

$$\langle \tau - n\rangle\langle \tau - m\rangle \gtrsim \langle n-m\rangle. \tag{26}$$

Setting $k_2 = l+k-k_1$, we rewrite the sum as

$$\sup_k \sum_{k_1\geq 0,n} \frac{\langle k\rangle^{2s}\langle l+k-k_1\rangle^{-2s_1}}{\langle k_1\rangle^{2+2s_1}\langle 2k-k_1\rangle^2\langle l\rangle^{2s_0}\langle \alpha(l^2-k^2)+k_1+|k_1-l-k|\rangle^{2-2b}}.$$

Here, without loss of generality (since $(k_1,k_2,k)\to(-k_1,-k_2,-k)$ is a symmetry for the sum), we only considered the case $k_1\geq 0$.

Case (i): $-1/2 < s_1 < 0$, $0 < \frac{s_1}{2}+\frac{1}{4} \leq s_0 \leq s_1+1$. We write the sum as

$$\sum_{\substack{|l|\sim|k|\\ k_1\geq 0}} + \sum_{\substack{|l|\ll|k|\\ 0\leq k_1\leq|l+k|}} + \sum_{\substack{|l|\ll|k|\\ k_1\geq|l+k|}} + \sum_{\substack{|l|\gg|k|\\ k_1\geq|l+k|}} + \sum_{\substack{|l|\gg|k|\\ 0\leq k_1\leq|l+k|}} =: S_1+S_2+S_3+S_4+S_5.$$

In the sum S_1, we have

$$\langle l\rangle\sim\langle k\rangle, \quad \langle l+k-k_1\rangle \lesssim \langle k_1\rangle + \langle 2k-k_1\rangle.$$

Using this, we have

$$S_1 \lesssim \sum_{k_1\geq 0,l} \frac{\langle k\rangle^{2s-2s_0}\big(\langle k_1\rangle^{-2s_1}+\langle 2k-k_1\rangle^{-2s_1}\big)}{\langle k_1\rangle^{2+2s_1}\langle 2k-k_1\rangle^2\langle \alpha(l^2-k^2)+k_1+|k_1-l-k|\rangle^{2-2b}}.$$

Summing in l using part (c) of Lemma 3.3 and then summing in k_1 using part (a) of Lemma 3.3, we obtain

$$S_1 \lesssim \langle k\rangle^{2s-2s_0-2-4s_1} + \langle k\rangle^{2s-2s_0-2-2s_1} \lesssim \langle k\rangle^{2s-2s_0-2-4s_1}.$$

Note that S_1 is bounded in k for $s\leq s_0+1+2s_1$.

In the case of S_2, we have

$$|l\pm k|\sim|k|, \quad |2k-k_1|\sim|k|, \quad |l+k-k_1|\lesssim|k|.$$

Also note that (since we can assume that $|k|\gg 1$)

$$\big|\alpha(l^2-k^2)+k_1+|k_1-l-k|\big| = \alpha(k^2-l^2)+O(|k|)\sim k^2.$$

Using these, and then summing in k_1, we have

$$S_2 \lesssim \sum_{\substack{|l|\ll|k|\\ 0\leq k_1\leq|l+k|}} \frac{\langle k\rangle^{2s-6+4b-2s_1}}{\langle k_1\rangle^{2+2s_1}\langle l\rangle^{2s_0}} \lesssim \langle k\rangle^{2s-6-2s_1+4b}\phi_{2s_0}(k)$$

Note that S_2 is bounded in k if $s<\min(s_0+\frac{5}{2}+s_1-2b, 3+s_1-2b)$, and in particular, if $s\leq\min(s_0+1+2s_1, 3+2s_1-2b)$.

In the case of S_3, we have $k_1 \geq |l+k| \gtrsim |k|$. Using this we estimate

$$S_3 \lesssim \sum_{\substack{|l|\ll|k| \\ k_1\geq|l+k|}} \frac{\langle k\rangle^{2s-2-4s_1}}{\langle 2k-k_1\rangle^2 \langle l\rangle^{2s_0} \langle \alpha(l^2-k^2)+2k_1-l-k\rangle^{2-2b}}$$
$$\lesssim \sum_{|l|\ll|k|} \frac{\langle k\rangle^{2s-2-4s_1}}{\langle l\rangle^{2s_0} \langle \alpha(l^2-k^2)+3k-l\rangle^{2-2b}}.$$

The second inequality follows from part (a) of Lemma 3.3. Note that

$$\langle \alpha(l^2-k^2)+3k-l\rangle \sim k^2,$$

since $|l| \ll |k|$. Using this and then summing in l, we have

$$S_3 \lesssim \langle k\rangle^{2s-6-4s_1+4b} \phi_{2s_0}(k).$$

Note that this is also bounded in k if $s \leq \min(s_0+1+2s_1, 3+2s_1-2b)$.

In the case of S_4, we have $k_1 \gg |k|$. Therefore

$$S_4 \lesssim \sum_{|l|,k_1\gg|k|} \frac{\langle k\rangle^{2s-2s_0}}{\langle k_1\rangle^{4+4s_1} \langle \alpha(l^2-k^2)+2k_1-l-k\rangle^{2-2b}} \lesssim \sum_{k_1\gg|k|} \frac{\langle k\rangle^{2s-2s_0}}{\langle k_1\rangle^{4+4s_1}} \lesssim \langle k\rangle^{2s-2s_0-3-4s_1}.$$

We used part (c) of Lemma 3.3 in the second inequality.

In the case of S_5, we have $|l+k-k_1| \lesssim |l|$ and

$$\big|\alpha(l^2-k^2)+k_1+|k_1-l-k|\big| = \alpha(k^2-l^2)+O(|l|) \sim l^2.$$

Thus, we estimate using part (a) of Lemma 3.3

$$S_5 \lesssim \sum_{|l|\gg|k|,k_1} \frac{\langle k\rangle^{2s}}{\langle k_1\rangle^{2+2s_1} \langle 2k-k_1\rangle^2 \langle l\rangle^{2s_0+2s_1+4-4b}} \lesssim \langle k\rangle^{2s-2s_0-5-4s_1+4b}.$$

Note that to sum in l we need $2s_0+2s_1+4-4b > 1$, which holds under the conditions of the proposition.

Case (ii): $0 \leq s_1$, $\max(s_1, \frac{s_1}{2}+\frac{1}{4}) \leq s_0 \leq s_1+1$. We write the sum as

$$\sum_{k_1\geq 0,\, |l|\gtrsim|k|} + \sum_{|l|\ll|k|,\, 0\leq k_1\ll k^2} + \sum_{|l|\ll|k|,\, k_1\gtrsim k^2} =: S_1+S_2+S_3.$$

In the case of S_1 we have

$$S_1 \lesssim \sum_{k_1\geq 0,\, |l|\gtrsim|k|} \frac{\langle k\rangle^{2s-2s_0}}{\langle k_1\rangle^{2+2s_1} \langle 2k-k_1\rangle^2 \langle \alpha(l^2-k^2)+k_1+|k_1-l-k|\rangle^{2-2b}} \lesssim \langle k\rangle^{2s-2s_0-2}.$$

We obtained the second inequality by first summing in l using part (c) of Lemma 3.3, and then in k_1 using part (a) of the Lemma. Thus S_1 is bounded in k if $s \leq s_0+1$.

In the case of S_2, we have

$$\langle \alpha(l^2-k^2)+k_1+|k_1-l-k|\rangle \gtrsim k^2, \text{ and } \langle k_1\rangle\langle l+k-k_1\rangle \gtrsim \langle l+k\rangle \gtrsim \langle k\rangle.$$

Therefore,

$$S_2 \lesssim \langle k\rangle^{2s-4+4b-2s_1} \sum_{|l|\ll|k|,\, 0\le k_1\ll k^2} \frac{1}{\langle k_1\rangle^2\langle 2k-k_1\rangle^2\langle l\rangle^{2s_0}} \lesssim \langle k\rangle^{2s-6+4b-2s_1}\phi_{2s_0}(k).$$

Note that S_2 is bounded in k if $s\le \min(s_0+1, s_1+3-2b)$.

Finally we estimate S_3 as follows

$$\begin{aligned} S_3 &\lesssim \sum_{|l|\ll|k|,\, k_1\gtrsim k^2} \frac{\langle k\rangle^{2s}}{\langle k_1\rangle^{4+4s_1}\langle \alpha(l^2-k^2)+k_1+|k_1-l-k|\rangle^{2-2b}} \\ &\lesssim \langle k\rangle^{2s-6-8s_1}\sum_l \frac{1}{\langle \alpha(l^2-k^2)+k_1+|k_1-l-k|\rangle^{2-2b}} \lesssim \langle k\rangle^{2s-6-8s_1}. \end{aligned}$$

In the last inequality we used part (c) of Lemma 3.3. Note that this term is bounded in k if $s\le s_0+1$.

We now consider R_1. By using Cauchy–Schwarz, the convolution structure, and then integrating in τ_1,τ_2 as in the previous case, it suffices to prove that

$$\sup_k \sum_{k_1,k_2}^{*} \frac{\langle k\rangle^{2s}\langle k_1\rangle^{-2s_0}\langle k_2\rangle^{-2s_0}\langle k-k_1-k_2\rangle^{-2s_0}|k_1+k_2|^2}{(\alpha k^2-\alpha(k-k_1-k_2)^2-|k_1+k_2|)^2\langle k^2-k_1^2+k_2^2-(k-k_1-k_2)^2\rangle^{2-2b}} < \infty.$$

Recalling (24), and using

$$\langle k^2-k_1^2+k_2^2-(k-k_1-k_2)^2\rangle \sim \langle (k_1+k_2)(k-k_1)\rangle,$$

it suffices to prove that

$$\sup_k \sum_{k_1,k_2}^{*} \frac{\langle k\rangle^{2s}\langle k_1\rangle^{-2s_0}\langle k_2\rangle^{-2s_0}\langle k-k_1-k_2\rangle^{-2s_0}}{\langle 2k-k_1-k_2\rangle^2\langle (k_1+k_2)(k-k_1)\rangle^{2-2b}} < \infty.$$

Note that the contribution of the case $k_1=k$ is

$$\lesssim \sum_{k_2} \frac{\langle k\rangle^{2s-2s_0}}{\langle k-k_2\rangle^2\langle k_2\rangle^{4s_0}} \lesssim \langle k\rangle^{2s-2s_0-\min(2,4s_0)},$$

so it satisfies the claim. For $k_1\neq k$ (since we also have $k_1+k_2\neq 0$ by nonresonant condition), we have $\langle (k_1+k_2)(k-k_1)\rangle \sim \langle k_1+k_2\rangle\langle k-k_1\rangle$. Also letting $l=k_1+k_2$ it suffices to consider the sum

$$\sum_{k_1,l} \frac{\langle k\rangle^{2s}}{\langle 2k-l\rangle^2\langle k-l\rangle^{2s_0}\langle l\rangle^{2-2b}\langle l-k_1\rangle^{2s_0}\langle k_1\rangle^{2s_0}\langle k-k_1\rangle^{2-2b}} = \sum_{\substack{k_1\\ |l-2k|>|k|/2}} + \sum_{\substack{k_1\\ |l-2k|\le|k|/2}} =: S_1+S_2.$$

We have

$$S_1 \lesssim \langle k\rangle^{2s-2} \sum_{l,k_1} \frac{1}{\langle k-l\rangle^{2s_0}\langle l\rangle^{2-2b}\langle l-k_1\rangle^{2s_0}\langle k_1\rangle^{2s_0}\langle k-k_1\rangle^{2-2b}}.$$

Using $\max(\langle k-l\rangle^{2s_0}, \langle l-k_1\rangle^{2s_0}) \gtrsim \langle k-k_1\rangle^{2s_0}$ and part (a) of Lemma 3.3 (recall that $2s_0+2-2b>1$), we have

$$S_1 \lesssim \langle k\rangle^{2s-2} \sum_{l,k_1} \frac{1}{\langle l\rangle^{2-2b}\min\big(\langle k-l\rangle^{2s_0}, \langle l-k_1\rangle^{2s_0}\big)\langle k_1\rangle^{2s_0}\langle k-k_1\rangle^{2s_0+2-2b}}$$

$$\lesssim \langle k\rangle^{2s-2} \sum_{k_1} \frac{1}{\langle k_1\rangle^{2s_0}\langle k-k_1\rangle^{2s_0+2-2b}} \lesssim \langle k\rangle^{2s-2-2s_0}.$$

In the case of S_2 we have $\langle l\rangle, \langle k-l\rangle \gtrsim \langle k\rangle$, and hence

$$S_2 \lesssim \langle k\rangle^{2s-2s_0-2+2b} \sum_{\substack{k_1 \\ |l-2k|\le|k|/2}} \frac{1}{\langle 2k-l\rangle^{2}\langle l-k_1\rangle^{2s_0}\langle k_1\rangle^{2s_0}\langle k-k_1\rangle^{2-2b}}.$$

Note that $\max(\langle l-k_1\rangle^{2s_0}, \langle k_1\rangle^{2s_0}) \gtrsim \langle l\rangle^{2s_0} \ge \langle k\rangle^{2s_0}$. Thus,

$$S_2 \lesssim \langle k\rangle^{2s-4s_0-2+2b} \sum_{\substack{k_1 \\ |l-2k|\le|k|/2}} \frac{1}{\langle 2k-l\rangle^{2}\min(\langle l-k_1\rangle^{2s_0}, \langle k_1\rangle^{2s_0})\langle k-k_1\rangle^{2-2b}}.$$

Using part (a) of Lemma 3.3 (noting that $|l-k|\gtrsim|k|$ and that $\langle k\rangle^{-\lambda}\phi_\kappa(k)=\langle k\rangle^{-\lambda}\phi_\kappa(k)$ if $0<\kappa,\lambda<1$), we obtain

$$S_2 \lesssim \langle k\rangle^{2s-4s_0-2+2b} \sum_{l} \frac{1}{\langle 2k-l\rangle^{2}} \langle k\rangle^{-2+2b}\phi_{2s_0}(k) \lesssim \langle k\rangle^{2s-4s_0-4+4b}\phi_{2s_0}(k).$$

Note that S_2 is bounded in k if $s\le s_0+\min(1,2s_0)$.

5. Proof of Proposition 3.7

We first consider R_3. By using Cauchy–Schwarz, the convolution structure, and then integrating in τ_1, τ_2 as in the proof of the previous proposition, it suffices to prove that

$$\sup_{j} \sum_{j_1\neq 0, j_2}^{*} \frac{\langle j\rangle^{2s}|j|^2\langle j_1\rangle^{-2s_1}\langle j_2\rangle^{-2s_0}\langle j-j_1-j_2\rangle^{-2s_0}}{\big||j|-\alpha(j_1+j_2)^2+\alpha(j-j_1-j_2)^2\big|^2\langle |j|-|j_1|+\alpha(j-j_1-j_2)^2-\alpha j_2^2\rangle^{2-2b}} < \infty.$$

Recalling (25), it suffices to prove that

$$\sum_{j_1\neq 0, j_2} \frac{\langle j\rangle^{2s}\langle j_1\rangle^{-2s_1}\langle j_2\rangle^{-2s_0}\langle j-j_1-j_2\rangle^{-2s_0}}{\langle j-2j_1-2j_2\rangle^{2}\langle |j|-|j_1|+\alpha(j-j_1-j_2)^2-\alpha j_2^2\rangle^{2-2b}}$$

is bounded in j. Letting $l=j-j_1-j_2$ and $m=j_2$, we rewrite the sum as

$$\sum_{m,l}\frac{\langle j\rangle^{2s}\langle j-l-m\rangle^{-2s_1}}{\langle 2l-j\rangle^2\langle m\rangle^{2s_0}\langle l\rangle^{2s_0}\langle \alpha l^2-\alpha m^2+|j|-|j-l-m|+\rangle^{2-2b}}. \tag{27}$$

We note that a similar argument gives us the following sum for R_4:

$$\sum_{m,l}\frac{\langle j\rangle^{2s}\langle j-l-m\rangle^{-2s_1}}{\langle 2l-j\rangle^2\langle m\rangle^{2s_0}\langle l\rangle^{2s_0}\langle \alpha l^2-\alpha m^2-|j|-|j-l-m|+\rangle^{2-2b}}. \tag{28}$$

We note that, by symmetry, if we can prove that

$$\sum_{m,l}\frac{\langle j\rangle^{2s}\langle j-l-m\rangle^{-2s_1}}{\langle 2l-j\rangle^2\langle m\rangle^{2s_0}\langle l\rangle^{2s_0}\langle \alpha l^2-\alpha m^2+j-|j-l-m|+\rangle^{2-2b}} \tag{29}$$

is bounded in $j\neq 0$, then the boundedness of (27) and (28) follow.

Case (i): $-\frac{1}{2}<s_1<0$. We rewrite (27) as

$$\sum_{|l|\sim|m|\lesssim|j|}+\sum_{|l|\sim|m|\gg|j|}+\sum_{\substack{|l|\ll|m|\\|j|\geq|m+l|}}+\sum_{\substack{|l|\gg|m|\\|j|\geq|m+l|}}+\sum_{\substack{|l|\ll|m|\\|j|\leq|m+l|}}+\sum_{\substack{|l|\gg|m|\\|j|\leq|m+l|}}=:S_1+S_2+S_3+S_4+S_5+S_6.$$

For S_1 we have

$$S_1\lesssim\sum_{|l|\sim|m|\lesssim|j|}\frac{\langle j\rangle^{2s-2s_1}}{\langle 2l-j\rangle^2\langle l\rangle^{4s_0}\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle^{2-2b}}\lesssim\langle j\rangle^{2s-2s_1-\min(2,4s_0)}.$$

In the second inequality we first summed in m using part (c) of Lemma 3.3, and then in n using part (a) of the lemma.

For S_2 we have

$$S_2\lesssim\sum_{|l|\sim|m|\gg|j|}\frac{\langle j\rangle^{2s}}{\langle l\rangle^{2+4s_0+2s_1}\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle^{2-2b}}\lesssim\langle j\rangle^{2s-2s_1-4s_0-1}.$$

Again, we first summed in m using part (c) of Lemma 3.3.

In the case of S_3 we have $|l|\ll|m|\lesssim|j|$, and hence

$$\begin{aligned}S_3&\lesssim\sum_{|l|\ll|m|\lesssim|j|}\frac{\langle j\rangle^{2s-2s_1-2}}{\langle l\rangle^{4s_0}\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle^{2-2b}}\\&\lesssim\sum_{|l|\lesssim|j|}\frac{\langle j\rangle^{2s-2s_1-2}}{\langle l\rangle^{4s_0}}\lesssim\langle j\rangle^{2s-2s_1-2}\phi_{4s_0}(j)\lesssim\langle j\rangle^{2s-2s_1-\min(2,4s_0)}.\end{aligned}$$

In the case of S_4 we have

$$\langle 2l-j\rangle+\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle\gtrsim l^2.$$

Since $\langle 2l-j\rangle \gtrsim l^2$ implies that $\langle 2l-j\rangle \gtrsim \langle j\rangle$, we have

$$\frac{1}{\langle 2l-j\rangle^2\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle^{2-2b}} \lesssim \frac{1}{\langle j\rangle^2\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle^{2-2b}}+\frac{1}{\langle 2l-j\rangle^2\langle l\rangle^{4-4b}}.$$

Therefore we estimate

$$S_4 \lesssim \sum_{|m|\ll|l|\lesssim|j|}\frac{\langle j\rangle^{2s-2s_1-2}}{\langle m\rangle^{4s_0}\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle^{2-2b}}+\sum_{|m|\ll|l|\lesssim|j|}\frac{\langle j\rangle^{2s-2s_1}}{\langle 2l-j\rangle^2\langle l\rangle^{2s_0+4-4b}\langle m\rangle^{2s_0}}.$$

The first sum can be estimated as in S_3 switching the roles of l and m. To estimate the second, we first sum in l using part (a) of Lemma 3.3, and then in m to obtain

$$\lesssim \langle j\rangle^{2s-2s_1-\min(2,2s_0+4-4b)}\phi_{2s_0}(j) \lesssim \langle j\rangle^{2s-2s_1-\min(2,4s_0)}.$$

In the case of S_5, we have

$$\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle \sim \langle m\rangle^2, \quad |m|\gtrsim |j|.$$

Therefore, noting that $2s_0+2s_1+4-4b>1$, we have

$$S_5 \lesssim \sum_{|l|\ll|m|}\frac{\langle j\rangle^{2s}}{\langle 2l-j\rangle^2\langle l\rangle^{2s_0}\langle m\rangle^{2s_0+2s_1+4-4b}} \lesssim \sum_{l}\frac{\langle j\rangle^{2s}}{\langle 2l-j\rangle^2\langle l\rangle^{4s_0+2s_1+3-4b}}$$
$$\lesssim \langle j\rangle^{2s-\min(2,4s_0+2s_1+3-4b)}.$$

In the case of S_6, we have

$$\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle \sim \langle l\rangle^2, \quad |l|\gtrsim |j|. \tag{30}$$

Therefore,

$$S_6 \lesssim \sum_{|m|\ll|l|\gtrsim|j|}\frac{\langle j\rangle^{2s}}{\langle 2l-j\rangle^2\langle l\rangle^{2s_0+2s_1+4-4b}\langle m\rangle^{2s_0}} \lesssim \sum_{|l|\gtrsim|j|}\frac{\langle j\rangle^{2s}\phi_{2s_0}(l)}{\langle 2l-j\rangle^2\langle l\rangle^{2s_0+2s_1+4-4b}}$$
$$\lesssim \langle j\rangle^{2s-2s_0-2s_1-4+4b}\phi_{2s_0}(j).$$

In the last inequality we used $|l|\gtrsim|j|$ and then summed in l.

Case (ii): $s_1\ge 0$. We rewrite (27) as

$$\sum_{|l|\lesssim|m|}+\sum_{|m|\ll|l|\ll|j|}+\sum_{|m|\ll|l|\gtrsim|j|}=: S_1+S_2+S_3.$$

In the case of S_1, we have $|j|\le|j-l-m|+|m+l|\lesssim|j-l-m|+|m|$, and hence

$$\langle j-l-m\rangle\langle m\rangle \gtrsim \langle j\rangle.$$

Using this and noting that $s_0 \ge s_1$, we have

$$S_1 \lesssim \sum_{|l|\lesssim|m|} \frac{\langle j\rangle^{2s-2s_1}}{\langle 2l-j\rangle^2\langle l\rangle^{4s_0-2s_1}\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle^{2-2b}} \lesssim \langle j\rangle^{2s-2s_1-\min(2,4s_0-2s_1)}.$$

In the last inequality we summed in m using part (c) of Lemma 3.3 and then in l using part (a) of the lemma.

In the case of S_2 we have

$$S_2 \lesssim \sum_{|m|\ll|l|\ll|j|} \frac{\langle j\rangle^{2s-2-2s_1}}{\langle m\rangle^{4s_0}\langle j-|j-l-m|+\alpha l^2-\alpha m^2\rangle^{2-2b}} \lesssim \langle j\rangle^{2s-2-2s_1}\phi_{4s_0}(j).$$

Note that in the case of S_3 we have (30). Therefore

$$S_3 \lesssim \sum_{|m|\ll|l|\gtrsim|j|} \frac{\langle j\rangle^{2s}}{\langle 2l-j\rangle^2\langle l\rangle^{2s_0+4-4b}\langle m\rangle^{2s_0}\langle j-l-m\rangle^{2s_1}}.$$

If $s_0+s_1>1/2$, we sum in m and then in n using part (a) of Lemma 3.3 to obtain

$$S_3 \lesssim \sum_{|l|\gtrsim|j|} \frac{\langle j\rangle^{2s-2s_0-4+4b}}{\langle 2l-j\rangle^2\langle j-l\rangle^{2s_1+\min(0,2s_0-1)-}} \lesssim \langle j\rangle^{2s-2s_0-4+4b-\min(2,2s_1,2s_1+2s_0-1)+}.$$

If $s_0+s_1\in(0,1/2]$, we have

$$S_3 \lesssim \sum_{|l|\gtrsim|j|} \frac{\langle j\rangle^{2s}\langle l\rangle^{1-2s_0-2s_1+}}{\langle 2l-j\rangle^2\langle l\rangle^{2s_0+4-4b}} \lesssim \langle j\rangle^{2s-4s_0-2s_1-3+4b+}.$$

Note that each term above is bounded in j if $s\le s_1+\min(1,2s_0-s_1)$.

6. Existence of global attractor

In this section we prove Theorem 2.8. As in the previous sections we drop the $\pm$ signs and work with the system

$$\begin{cases} (i\partial_t+\alpha\partial_x^2+i\gamma)u = nu+f, \quad x\in\mathbb{T}, \quad t\in[-T,T],\\ (i\partial_t-d+i\gamma)n = d(|u|^2),\\ u(x,0)=u_0(x)\in H^1(\mathbb{T}), \quad n(x,0)=n_0(x)\in \dot{L}^2(\mathbb{T}). \end{cases} \tag{31}$$

We start with a smoothing estimate for (31) that implies the existence of a global attractor:

Theorem 6.1. *Consider the solution of (31) with initial data $(u_0,n_0)\in H^1\times\dot{L}^2$. Then, for $\frac{1}{\alpha}\notin\mathbb{N}$, and for any $a<1$, we have*

$$u(t)-e^{i\alpha t\partial_x^2-\gamma t}u_0\in C_t^0H_x^{1+a}([0,\infty)\times\mathbb{T}) \quad \text{and} \quad n(t)-e^{-itd-\gamma t}n_0\in C_t^0H_x^a([0,\infty)\times\mathbb{T}). \tag{32}$$

Moreover,

$$\|u(t)-e^{i\alpha t\partial_x^2-\gamma t}u_0\|_{H^{1+a}}+\|n(t)-e^{-itd-\gamma t}n_0\|_{H^a}\le C\big(a,\alpha,\gamma,\|f\|_{H_1},\|u_0\|_{H^1},\|n_0\|_{L^2}\big). \tag{33}$$

In the case $\alpha = 1$ *we have, for any* $a < 1$,

$$\left\| u(t) - e^{it\partial_x^2 - \gamma t} u_0 + i \int_0^t e^{(i\partial_x^2 - \gamma)(t-t')} \rho_1 dt' \right\|_{H^{1+a}} + \left\| n(t) - e^{-itd - \gamma t} n_0 \right\|_{H^a} \leq C\left(a, \gamma, \|f\|_{H_1}, \|u_0\|_{H^1}, \|n_0\|_{L^2}\right), \quad (34)$$

where ρ_1 *is as in Proposition 3.2. The analogous continuity statements as in (32) are also valid.*

Proof. Writing

$$u(x,t) = \sum_k u_k(t) e^{ikx}, \quad n(x,t) = \sum_{j \neq 0} n_j(t) e^{ijx}, \quad f(x) = \sum_k f_k e^{ikx}$$

we obtain the following system for the Fourier coefficients:

$$\begin{cases} i\partial_t u_k + (i\gamma - \alpha k^2) u_k = \sum_{\substack{k_1 + k_2 = k \\ k_1 \neq 0}} n_{k_1} u_{k_2} + f_k, \\ i\partial_t n_j + (i\gamma - |j|) n_j = |j| \sum_{j_1 + j_2 = j} u_{j_1} \overline{u_{-j_2}}. \end{cases} \quad (35)$$

We have the following proposition, which follows from differentiation by parts as in Proposition 3.2 by using the change of variables $m_j = n_j e^{i|j|t + \gamma t}$, and $v_k = u_k e^{i\alpha k^2 t + \gamma t}$.

Proposition 6.2. *The system (35) can be written in the form*

$$i\partial_t\left[e^{it\alpha k^2 + \gamma t} u_k\right] + ie^{-\gamma t} \partial_t\left[e^{it\alpha k^2 + 2\gamma t} B_1(n,u)_k\right] = e^{it\alpha k^2 + \gamma t}\left[\rho_1(k) + f_k + B_1(n,f) + R_1(u)(\hat{k},t) + R_2(u,n)(\hat{k},t)\right], \quad (36)$$

$$i\partial_t\left[e^{it|j| + \gamma t} n_j\right] + ie^{-\gamma t} \partial_t\left[e^{it|j| + 2\gamma t} B_2(u)_j\right] = e^{it|j| + \gamma t}\left[\rho_2(j) + B_2(f,u) + B_2(u,f) + R_3(u,n)(\hat{j},t) + R_4(u,n)(\hat{j},t)\right]. \quad (37)$$

where $B_i, \rho_i, i = 1, 2,$ *and* R_j, $j = 1, 2, 3, 4$ *are as in Proposition 3.2.*

Integrating (36) from 0 to t, we obtain

$$\begin{aligned} u_k(t) - e^{-it\alpha k^2 - \gamma t} u_k(0) &= -B_1(n,u)_k + e^{-it\alpha k^2 - \gamma t} B_1(n_0, u_0)_k \\ &+ \int_0^t e^{-(i\alpha k^2 + \gamma)(t-t')} \Big[-\gamma B_1(n,u)_k - i\rho_1(k) - if_k - iB_1(n,f)_k \Big] dt' \\ &- i \int_0^t e^{-(i\alpha k^2 + \gamma)(t-t')} \left[R_1(u)(\hat{k},t') + R_2(u,n)(\hat{k},t')\right] dt'. \end{aligned}$$

First note that (identifying the function with its Fourier sequence) we have

$$\left\| \int_0^t e^{-(i\alpha k^2 + \gamma)(t-t')} f_k dt' \right\|_{H^{1+a}} = \left\| \frac{f_k}{i\alpha k^2 + \gamma} (1 - e^{-it\alpha k^2 - \gamma t}) \right\|_{H^{1+a}} \lesssim \|f_k\|_{H^{a-1}}. \quad (38)$$

In the case $\frac{1}{\alpha} \notin \mathbb{N}$, using (38), the estimates in Lemma 3.4 and Proposition 3.6 (see Remark 3.5) as above, and also using the growth bound in (9), we obtain for any $a<1$

$$\|u(t)-e^{i\alpha\partial_x^2t-\gamma t}u_0\|_{H^{1+a}}\lesssim\|f\|_{H^{a-1}}+\big[\|f\|_{H^1}+\|n(0)\|_{L^2}+\|u(0)\|_{H^1}\big]^2+\big[\|u\|_{X_\delta^{1,\frac12}}+\|n\|_{Y_\delta^{1,\frac12}}\big]^3.$$

Using the local theory (Theorem 2.9) bound for $X_\delta^{1,\frac12}$, $Y_\delta^{1,\frac12}$ norms for a $\delta=\delta(\|n_0\|_{L^2},\|u_0\|_{H^1},\|f\|_{H^1})$, we obtain for $t<\delta$

$$\|u(t)-e^{i\alpha\partial_x^2t-\gamma t}u_0\|_{H^{1+a}}\lesssim C(a,\gamma,\|f\|_{H^1},\|n_0\|_{L^2}+\|u_0\|_{H^1}).$$

In the rest of the proof the implicit constants depend on $a,\gamma,\|f\|_{H^1},\|n_0\|_{L^2}+\|u_0\|_{H^1}$. Fix t large, and δ as above. We have

$$\big\|u(j\delta)-e^{i\alpha\partial_x^2\delta-\gamma\delta}u((j-1)\delta)\big\|_{H^{1+a}}\lesssim 1,$$

for any $j\le t/\delta$. Using this we obtain (with $J=t/\delta$)

$$\begin{aligned}\|u(J\delta)-e^{J\delta(i\alpha\partial_x^2-\gamma)}u(0)\|_{H^{1+a}}&\le\sum_{j=1}^J\big\|e^{(J-j)\delta(i\alpha\partial_x^2-\gamma)}u(j\delta)-e^{(J-j+1)\delta(i\alpha\partial_x^2-\gamma)}u((j-1)\delta)\big\|_{H^{1+a}}\\&=\sum_{j=1}^J e^{-(J-j)\delta\gamma}\|u(j\delta)-e^{\delta(i\alpha\partial_x^2-\gamma)}u((j-1)\delta)\|_{H^{1+a}}\\&\lesssim\sum_{j=1}^J e^{-(J-j)\delta\gamma}\lesssim\frac{1}{1-e^{-\delta\gamma}}.\end{aligned}$$

In the case $\alpha=1$, we have to separate the resonant term in this argument. We have the following inequality for $t<\delta$

$$\Big\|u(t)-e^{i\alpha\partial_x^2t-\gamma t}u_0+i\int_0^t e^{(i\alpha\partial_x^2-\gamma)(t-t')}\rho_1\,dt'\Big\|_{H^{1+a}}\lesssim C(a,\gamma,\|f\|_{H^1},\|n_0\|_{L^2}+\|u_0\|_{H^1}).$$

Accordingly we have

$$\begin{aligned}&\Big\|u(J\delta)-e^{J\delta(i\alpha\partial_x^2-\gamma)}u(0)+\int_0^{J\delta}e^{(i\alpha\partial_x^2-\gamma)(J\delta-t')}\rho_1\,dt'\Big\|_{H^{1+a}}\\&\quad\le\sum_{j=1}^J\Big\|e^{(J-j)\delta(i\alpha\partial_x^2-\gamma)}\Big(u(j\delta)-e^{\delta(i\alpha\partial_x^2-\gamma)}u((j-1)\delta)+i\int_{(j-1)\delta}^{j\delta}e^{(i\alpha\partial_x^2-\gamma)(j\delta-t')}\rho_1\,dt'\Big)\Big\|_{H^{1+a}}\\&\quad=\sum_{j=1}^J e^{-(J-j)\delta\gamma}\Big\|u(j\delta)-e^{\delta(i\alpha\partial_x^2-\gamma)}u((j-1)\delta)+i\int_{(j-1)\delta}^{j\delta}e^{(i\alpha\partial_x^2-\gamma)(j\delta-t')}\rho_1\,dt'\Big\|_{H^{1+a}}\\&\quad\lesssim\sum_{j=1}^J e^{-(J-j)\delta\gamma}\lesssim\frac{1}{1-e^{-\delta\gamma}}.\end{aligned}$$

The corresponding inequalities for the wave part follow similarly. The only difference is that we don't need to separate the resonant term, since $\rho_2\in H^1$ by Lemma 3.4.

This completes the proof of the global bound stated in Theorem 6.1. Finally the continuity in $H^1 \times \dot{L}^2$ follows as in [Erdoğan and Tzirakis 2012]. We omit the details. □

Proof of Theorem 2.8. We follow the strategy we outlined in [Erdoğan and Tzirakis 2011]. We start with the case $\frac{1}{\alpha} \notin \mathbb{N}$. First of all note that the existence of an absorbing set, $\mathcal{B}_0 \subset H^1 \times \dot{L}^2$, is immediate from (9). Second, we need to verify the asymptotic compactness of the propagator U_t. It suffices to prove that for any sequence $t_r \to \infty$ and for any sequence $(u_{0,r}, n_{0,r})$ in $\mathcal{B}_0$, the sequence $U_{t_r}(u_{0,r}, n_{0,r})$ has a convergent subsequence in $H^1 \times \dot{L}^2$.

To see this note that by Theorem 6.1, (if $(u_0, n_0) \in \mathcal{B}_0$)

$$U_t\big(u_0, n_0\big) = \big(e^{i\alpha t\partial_x^2 - \gamma t}u_0, e^{-itd-\gamma t}n_0\big) + N_t\big(u_0, n_0\big)$$

where $N_t\big(u_0, n_0\big)$ is in a ball in $H^{1+a} \times H^a$ with radius depending on $a \in (0,1), \alpha, \gamma$, and $\|f\|_{H^1}$. By Rellich's theorem, $\{N_t\big(u_0, n_0\big) : t > 0, (u_0, n_0) \in \mathcal{B}_0\}$ is precompact in $H^1 \times \dot{L}^2$. Since

$$\big\|\big(e^{i\alpha t\partial_x^2 - \gamma t}u_0, e^{-itd-\gamma t}n_0\big)\big\|_{H^1 \times \dot{L}^2} \lesssim e^{-\gamma t} \to 0, \qquad \text{as } t \to \infty,$$

uniformly on $\mathcal{B}_0$, we conclude that $\{U_{t_r}\big(u_{0,r}, n_{0,r}\big) : r \in \mathbb{N}\}$ is precompact in $H^1 \times \dot{L}^2$. Thus, U_t is asymptotically compact. This and Theorem A imply the existence of a global attractor $\mathcal{A} \subset H^1 \times \dot{L}^2$.

We now prove that the attractor set $\mathcal{A}$ is a compact subset of $H^{1+a} \times H^a$ for any $a \in (0,1)$. By Rellich's theorem, it suffices to prove that for any $a \in (0,1)$, there exists a closed ball $B_a \subset H^{1+a} \times H^a$ of radius $C(a, \alpha, \gamma, \|f\|_{H^1})$ such that $\mathcal{A} \subset B_a$. By definition

$$\mathcal{A} = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} U_t \mathcal{B}_0} =: \bigcap_{\tau \geq 0} V_\tau.$$

By Theorem 6.1 and the discussion above, V_τ is contained in a δ_τ neighborhood, N_τ, of a ball B_a in $H^1 \times \dot{L}^2$ whose radius depends only on $a, \alpha, \gamma, \|f\|_{H^1}$, and where $\delta_\tau \to 0$ as τ tends to infinity. Since B_a is a compact subset of $H^1 \times \dot{L}^2$, we have

$$\mathcal{A} = \bigcap_{\tau \geq 0} V_\tau \subset \bigcap_{\tau > 0} N_\tau = B_a.$$

Now consider the case $\frac{1}{\alpha} \in \mathbb{N}$. For simplicity, we take $\alpha = 1$. We have to be slightly more careful in this case because of the contribution of the resonant term, ρ_1, which is does not belong to H^{1+a} for any $a > 0$. Recall that, by Theorem 6.1, for $(u_0, n_0) \in \mathcal{B}_0$

$$U_t\big(u_0, n_0\big) = \big(e^{i\alpha t\partial_x^2 - \gamma t}u_0, e^{-itd-\gamma t}n_0\big) + N_t\big(u_0, n_0\big) + i\bigg(\int_0^t e^{(i\partial_x^2 - \gamma)(t-t')}\rho_1 dt', 0\bigg), \tag{39}$$

where $N_t\big(u_0, n_0\big)$ is in a ball in $H^{1+a} \times H^a$ with radius depending on $a \in (0,1), \gamma$, and $\|f\|_{H^1}$. Recall from Proposition 3.2, that the Fourier coefficients of ρ_1 are

$$(\rho_1)_k = \rho_1(n, u)_k = n_{2k-\operatorname{sgn}(k)}u_{\operatorname{sgn}(k)-k}, \quad k \neq 0.$$

In light of the proof of the case $\frac{1}{\alpha} \notin \mathbb{N}$ above, it suffices to consider the contribution of the resonant term under the assumption that $(u_0, n_0) \in \mathcal{B}_0$. Using (39), we write

$$\rho_1\big(n(t'), u(t')\big) = \rho_1\big(e^{-it'd-\gamma t'} n_0, u(t')\big) + \rho_1\big(N_{t'}(n_0), u(t')\big). \tag{40}$$

Now note that, by Lemma 3.4, we have

$$\big\|\rho_1(n,u)\big\|_{H^{1+a}} \lesssim \|n\|_{H^a}\|u\|_{H^1}.$$

Using this with $a = 0$, we see that the contribution of the first summand in (40) to the resonant term in (39) satisfies

$$\begin{aligned}\bigg\| \int_0^t e^{(i\partial_x^2-\gamma)(t-t')} \rho_1\big(e^{-it'd-\gamma t'} n_0, u(t')\big) dt' \bigg\|_{H^1} &\lesssim \int_0^t e^{-\gamma(t-t')} \|e^{-it'd-\gamma t'} n_0\|_{L^2} \|u(t')\|_{H^1} dt' \\ &\le t e^{-\gamma t} C(a, \gamma, \|f\|_{H^1}),\end{aligned}$$

which goes to zero uniformly in $\mathcal{B}_0$. Similarly, the contribution of the second summand in (40) to the resonant term in (39) satisfies

$$\begin{aligned}\bigg\| \int_0^t e^{(i\partial_x^2-\gamma)(t-t')} \rho_1\big(N_{t'}(n_0), u(t')\big) dt' \bigg\|_{H^{1+a}} &\lesssim \int_0^t e^{-\gamma(t-t')} \|N_{t'}(n_0)\|_{H^a} \|u(t')\|_{H^1} dt' \\ &\le C(a, \gamma, \|f\|_{H^1}).\end{aligned}$$

The rest of the proof is same as the case $\frac{1}{\alpha} \notin \mathbb{N}$. □

Appendix

We prove Lemma 3.3. Note that, with $m = k_2 - k_1$, we can rewrite the sum in part (a) as

$$\sum_n \frac{1}{\langle n\rangle^{\kappa} \langle n-m\rangle^{\lambda}}.$$

For $|n| < |m|/2$, we estimate the sum by

$$\sum_{|n|<|m|/2} \frac{1}{\langle n\rangle^{\kappa} \langle m\rangle^{\lambda}} \le \langle m\rangle^{-\lambda} \phi_\kappa(m).$$

For $|n| > 2|m|$, we estimate by

$$\sum_{|n|>2|m|} \frac{1}{\langle n\rangle^{\kappa+\lambda}} \lesssim \langle m\rangle^{1-\kappa-\lambda} \lesssim \langle m\rangle^{-\lambda} \phi_\kappa(m).$$

Finally for $|n| \sim |m|$, we estimate by

$$\sum_{|n|\sim|m|} \frac{1}{\langle m\rangle^{\kappa} \langle n-m\rangle^{\lambda}} \lesssim \langle m\rangle^{-\kappa} \phi_\lambda(m) \lesssim \langle m\rangle^{-\lambda} \phi_\kappa(m).$$

The last inequality follows from the definition of ϕ_κ and the hypothesis $\kappa \ge \lambda$.

Part (b) follows from part (a). To obtain part (c), write

$$|n^2 + c_1 n + c_2| = |(n + z_1)(n + z_2)| \geq |n + x_1|\,|n + x_2|$$

where x_i is the real part of z_i. The contribution of the terms $|n + x_1| < 1$ or $|n + x_2| < 1$ is $\lesssim 1$. Therefore, we estimate the sum in part (c) by

$$\lesssim 1 + \sum_n \frac{1}{\langle n + x_1 \rangle^{\kappa} \langle n + x_2 \rangle^{\kappa}} \lesssim 1$$

by part (a).

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