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RESTRICTION AND SPECTRAL MULTIPLIER THEOREMS ON ASYMPTOTICALLY CONIC MANIFOLDS

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The classical Stein–Tomas restriction theorem is equivalent to the fact that the spectral measure $dE(\lambda)$ of the square root of the Laplacian on $\mathbb{R}^n$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ for $1 \le p \le 2(n+1)/(n+3)$, where p' is the conjugate exponent to p, with operator norm scaling as $\lambda^{n(1/p-1/p')-1}$. We prove a geometric, or variable coefficient, generalization in which the Laplacian on $\mathbb{R}^n$ is replaced by the Laplacian, plus a suitable potential, on a nontrapping asymptotically conic manifold. It is closely related to Sogge's discrete L^2 restriction theorem, which is an $O(\lambda^{n(1/p-1/p')-1})$ estimate on the $L^p \to L^{p'}$ operator norm of the spectral projection for a spectral window of fixed length. From this, we deduce spectral multiplier estimates for these operators, including Bochner–Riesz summability results, which are sharp for p in the range above.

The paper divides naturally into two parts. In the first part, we show at an abstract level that restriction estimates imply spectral multiplier estimates, and are implied by certain pointwise bounds on the Schwartz kernel of λ-derivatives of the spectral measure. In the second part, we prove such pointwise estimates for the spectral measure of the square root of Laplace-type operators on asymptotically conic manifolds. These are valid for all $\lambda > 0$ if the asymptotically conic manifold is nontrapping, and for small λ in general. We also observe that Sogge's estimate on spectral projections is valid for any complete manifold with C^∞ bounded geometry, and in particular for asymptotically conic manifolds (trapping or not), while by contrast, the operator norm on $dE(\lambda)$ may blow up exponentially as $\lambda \to \infty$ when trapping is present.

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1. Introduction

The aim of this article is to prove some L^p multiplier properties for the Laplacian, and a Stein–Tomas-type restriction theorem for its spectral measure, on a class of Riemannian manifolds which include metric perturbations of Euclidean space. One of the first natural questions in harmonic analysis is to understand the L^p boundedness of Fourier multipliers M on $\mathbb{R}^n$, defined by

$$M(f)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix.\xi} m(\xi) \hat{f}(\xi)\, d\xi,$$

where m is a measurable function. Notice that for radial multipliers $m(\xi) = F(|\xi|)$, this amounts to study the L^p boundedness of $F(\sqrt{\Delta})$, where Δ is the nonnegative Laplacian. Of course, for $p = 2$, the necessary and sufficient condition on m for M to be bounded on L^2 is that $m \in L^\infty(\mathbb{R}^n)$, but the case $p \neq 2$ is much more difficult. The first results in this direction were given by Mikhlin [1965]: M acts boundedly on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ if

$$m \in C^\infty(\mathbb{R}^n \setminus \{0\}) \quad \text{and} \quad |\xi|^k |\nabla^k m(\xi)| \in L^\infty, \quad \forall k,\ 0 \le k \le \tfrac{1}{2}n + 1.$$

This was sharpened by Hörmander [1960; 1983, Theorem 7.9.5]: Let $\psi \in C_0^\infty(\frac{1}{2}, 2)$ be not identically zero, then M acts boundedly on $L^p(\mathbb{R}^n))$ for all $1 < p < \infty$ if

$$\sup_{t>0} \| m(t\,\cdot)\psi \|_{H^s(\mathbb{R}^n)} < \infty, \quad \tfrac{1}{2}n < s \in \mathbb{N}.$$

More generally, let $\boldsymbol{L}$ be a self-adjoint operator acting on L^2 of some measure space. Using the spectral theorem, "spectral multipliers" $F(\boldsymbol{L})$ can be defined for any bounded Borel function F, and they act continuously on L^2. A question which has attracted a lot of attention during the last thirty years is to find some necessary conditions on the function F to ensure that the operator $F(\boldsymbol{L})$ extends as a bounded operator for some range of L^p spaces for $p \neq 2$. Probably the most natural and concrete examples are functions of the Laplacian on complete Riemannian manifolds, or functions of Schrödinger operators with real potential $\Delta + V$, but these problems are also studied for abstract self-adjoint operators. Some particular families of functions F are also investigated in the theory of spectral multipliers: some of the most important examples include oscillatory integrals $e^{i(t\boldsymbol{L})^\alpha}(\mathrm{Id} + (t\boldsymbol{L})^\alpha)^{-\beta}$ and Bochner–Riesz means (2-18). The subject of Bochner–Riesz means and spectral multipliers is so broad that it is impossible to provide a comprehensive bibliography here, so we refer the reader to [Anker 1990; Christ and Sogge 1988; Clerc and Stein 1974; Cowling and Sikora 2001; Mauceri and Meda 1990; Müller and Stein 1994; Seeger and Sogge 1989; Sogge 1987; 1993; Taylor 1989; Thangavelu 1993], where further literature can be found.

The theory of Fourier multipliers and Bochner–Riesz analysis in this setting is related to the so-called *sphere restriction problem* for the Fourier transform: find the pairs (p, q) for which the *sphere restriction operator* $\mathrm{SR}(\lambda)$, defined by

$$\mathrm{SR}(\lambda) f(\omega) := \hat{f}(\lambda\omega), \quad \omega \in S^{n-1}, \lambda > 0,$$

acts boundedly from $L^p(\mathbb{R}^n)$ to $L^q(S^{n-1})$; see [Fefferman 1970; 1973]. Of course, the dependence in λ is trivial here since $\mathrm{SR}(\lambda)f = \lambda^{-n}\mathrm{SR}(1)(f(\lambda^{-1}\cdot))$, but this parameter λ will be important later on. There is a long list of results on this problem, but the first ones for general dimensions are due to Stein and Tomas. The theorem of Tomas [1975], improved by Stein [1993, Chapter IX, Section 2] for the endpoint $p = 2(n+1)/(n+3)$ is the following: SR(1) maps $L^p(\mathbb{R}^n)$ boundedly to $L^q(S^{n-1})$ if $p \le 2(n+1)/(n+3)$ and $q \le \frac{n-1}{n+1}\frac{p}{p-1}$ (notice that $q = 2$ when p reaches the endpoint). On the other hand, a necessary condition (based on the Knapp example) for boundedness is only given by $p < 2n/(n+1)$ and this leads to the conjecture that $p < 2n/(n+1)$ and $q \le \frac{n-1}{n+1}\frac{p}{p-1}$ is a necessary and sufficient condition. In fact, this has been shown by Zygmund [1974] in dimension 2, improving a result of Fefferman [1970] (by obtaining the endpoint estimate), but the conjecture is still open for $n > 2$. For more references and new results in this direction, we refer the interested reader to the survey by Tao [2003] on the subject.

Like the L^p multiplier problem, the sphere restriction problem has a corresponding natural generalization to certain types of manifolds (at least if we think of Fourier transform as a spectral diagonalization for the Laplacian), and in particular those which have similar structure at infinity as Euclidean space. On $\mathbb{R}^n$, the Schwartz kernel of the spectral measure $dE_{\sqrt{\Delta}}(\lambda)$ of $\sqrt{\Delta}$ is given by

$$dE_{\sqrt{\Delta}}(\lambda; z, z') = \frac{\lambda^{n-1}}{(2\pi)^n}\int_{S^{n-1}} e^{i(z-z').\lambda\omega}\,d\omega, \quad z, z' \in \mathbb{R}^n,$$

therefore $dE_{\sqrt{\Delta}}(\lambda) = (\lambda^{n-1})/((2\pi)^n)\mathrm{SR}(\lambda)^*\mathrm{SR}(\lambda)$ and the restriction theorem for $q = 2$ is equivalent to finding the largest $p < 2$ such that $dE_{\sqrt{\Delta}}$ maps L^p to $L^{p'}$. There is a natural class of Riemannian manifolds, called *scattering manifolds* or *asymptotically conic manifolds*, for which the spectral measure of the Laplacian admits an analogous factorization. Such manifolds, introduced by Melrose [1994], are by definition the interior M° of a compact manifold with boundary M, such that the metric g is smooth on M° and has the form

$$g = \frac{dx^2}{x^4} + \frac{h(x)}{x^2} \tag{1-1}$$

in a collar neighborhood near ∂M, where x is a smooth boundary defining function for M and $h(x)$ is a smooth one-parameter family of metrics on ∂M; the function $r := 1/x$ near $x = 0$ can be thought of as a radial coordinate near infinity and the given metric is asymptotic to the exact conic metric $((0,\infty)_r \times \partial M, dr^2 + r^2 h(0))$ as $r \to \infty$. Associated to the Laplacian on such a manifold is the family of Poisson operators $P(\lambda)$ defined for $\lambda > 0$. These form a sort of distorted Fourier transform for the Laplacian: they map $L^2(\partial M)$ into the null space of $\Delta_g - \lambda^2$ and satisfy $dE_{\sqrt{\Delta_g}}(\lambda) = (2\pi)^{-1}P(\lambda)P(\lambda)^*$ [Hassell and Vasy 1999]. Thus $(\lambda/2\pi)^{-(n-1)/2}P(\lambda)^*$ is an analogue of the restriction operator in this setting. The corresponding restriction problem is therefore to study the $L^p(M) \to L^q(\partial M)$ boundedness of $P(\lambda)^*$, and its norm in terms of the frequency λ (the dependence of $P(\lambda)$ in λ is no longer a scaling as it is for $\mathbb{R}^n$).

The aim of the present work is to address these multiplier and restriction problems in the geometric setting of asymptotically conic manifolds. In fact, we shall first show, in an abstract setting, that restriction-type estimates on the spectral measure of an operator imply spectral multiplier results for that operator. Then we will prove such restriction estimates for a class of operators which are 0-th order perturbations of the Laplacian on asymptotically conic manifolds. In particular, our results cover the following settings:

- Schrödinger operators, i.e., $\Delta + V$ on $\mathbb{R}^n$, where V is smooth and decaying sufficiently at infinity.
- The Laplacian with respect to metric perturbations of the flat metric on $\mathbb{R}^n$, again decaying sufficiently at infinity.
- The Laplacian on asymptotically conic manifolds.

Our first main result is that restriction estimates imply spectral multiplier estimates:

Theorem 1.1. *Let $\boldsymbol{L}$ be a nonnegative self-adjoint operator on $L^2(X, d\mu)$, where (X, d, μ) is a metric measure space such that the volume of balls satisfy the uniform bound $C_2 > \mu(B(x, \rho))/\rho^n > C_1$ for some $C_2 > C_1 > 0$. Suppose that the operator $\cos(t\sqrt{\boldsymbol{L}})$ satisfies finite speed propagation property* (2-2), *that the spectrum of $\boldsymbol{L}$ is absolutely continuous and that there exists $1 \le p < 2$ such that the spectral measure of $\boldsymbol{L}$ satisfies*

$$\|dE_{\sqrt{\boldsymbol{L}}}(\lambda)\|_{p\to p'} \le C\lambda^{n(1/p-1/p')-1}, \tag{1-2}$$

where p' is the exponent conjugate to p. Let $s > n(1/p - 1/2)$ be a Sobolev exponent. Then there exists C depending only on n, p, s, and the constant in (2-3) *such that, for every even $F \in H^s(\mathbb{R})$ supported in $[-1, 1]$, $F(\sqrt{\boldsymbol{L}})$ maps $L^p(X) \to L^p(X)$, and*

$$\sup_{\alpha>0} \|F(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p} \le C\|F\|_{H^s}. \tag{1-3}$$

Remark 1.2. As noted above, the hypothesis (1-2) is valid on the Euclidean space $\mathbb{R}^n$ and for exponents $1 \le p \le 2(n+1)/(n+3)$. In this case, the result is sharp in the sense that the hypothesis cannot be weakened to $F \in H^{s'}$ for any $s' < n(1/p - 1/2)$; see [Stein 1993, Section IX.2]. In fact, the proof shows that the theorem is true if we only assume $F \in B^{n(1/p-1/2)}_{1,2}$, which is slightly weaker, and gives an endpoint result. The result is sharp also in the sense that H^s cannot be replaced by the L^q Sobolev space W^s_q and $B^{n(1/p-1/2)}_{1,2}$ cannot be replaced by $B^{n(1/p-1/2)}_{1,q}$ for any $q < 2$; see Remark 2.11 below.

In the second part of the paper, we prove (1-2) for the spectral measure of the Laplacian Δ_g, plus a suitable potential, on asymptotically conic manifolds.

Theorem 1.3. *Let (M, g) be an asymptotically conic manifold of dimension $n \ge 3$, and let x be a smooth boundary defining function of ∂M. Let $\boldsymbol{H} := \Delta_g + V$ be a Schrödinger operator on M, with $V \in x^3C^\infty(M)$, and assume that $\boldsymbol{H}$ is a positive operator and that 0 is neither an eigenvalue nor a resonance. Then*:

(A) *For any $\lambda_0 > 0$ there exists a constant $C > 0$ such that the spectral measure $dE(\lambda)$ for $\sqrt{\boldsymbol{H}}$ satisfies*

$$\|dE_{\sqrt{\boldsymbol{H}}}(\lambda)\|_{L^p(M)\to L^{p'}(M)} \le C\lambda^{n(1/p-1/p')-1} \tag{1-4}$$

for $1 \le p \le 2(n+1)/(n+3)$ and $0 < \lambda \le \lambda_0$.

(B) *If (M, g) is nontrapping, then there exists $C > 0$ such that* (1-4) *holds for all $\lambda > 0$.*

(C) *If (M, g) is trapping and has asymptotically* ***Euclidean*** *ends, there exists $\chi \in C_0^\infty(M^\circ)$ and $C > 0$ such that*

$$\|(1-\chi)dE_{\sqrt{\boldsymbol{H}}}(\lambda)(1-\chi)\|_{L^p(M)\to L^{p'}(M)} \le C\lambda^{n(1/p-1/p')-1}, \quad \forall\lambda > 0, \tag{1-5}$$

for $1 < p \le 2(n+1)/(n+3)$. *However*, (1-4) *need not hold for all* $\lambda > 0$: *there exist* (*trapping*) *asymptotically Euclidean manifolds* (M, g), *sequences* $\lambda_n \to \infty$ *and* $C, c > 0$ *such that*

$$\|dE_{\sqrt{\Delta_g}}(\lambda_n)\|_{L^p(M)\to L^{p'}(M)} \ge Ce^{c\lambda_n}. \tag{1-6}$$

(D) *On the other hand*, *the Sogge-type spectral projection estimate*

$$\|\mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta}_g)\|_{L^p(M)\to L^{p'}(M)} \le C\lambda^{n(1/p-1/p')-1}, \quad \forall\lambda \ge 1, \tag{1-7}$$

holds for $1 \le p \le 2(n+1)/(n+3)$ *for all asymptotically conic manifolds*, *trapping or not*, *and indeed for the much larger class of complete manifolds with* C^∞ *bounded geometry.*

Remark 1.4. When the spectral measure estimate (1-4) holds, it trivially implies the Sogge-type spectral projection estimate (1-7), by integrating over a unit interval in λ. On the other hand, parts (C) and (D) of Theorem 1.3 show that the Sogge estimate holds in far greater generality than (1-4).

Remark 1.5. Probably the nontrapping condition is not necessary to obtain the estimate (1-4) for all $\lambda > 0$; it seems likely that asymptotically conic manifolds with a hyperbolic trapped set of sufficiently small dimension will also satisfy (1-4), by analogy with [Burq et al. 2010]. However, manifolds with elliptic trapping will typically have sequences of λ for which the norm on the left hand side of (1-4) grows superpolynomially; see Section 8C.

Remark 1.6. The spatially cut-off estimate (1-5) can be compared to the nontrapping L^2 estimate proved by Cardoso and Vodev [2002]

$$\|(1-\chi)(L-\lambda^2+i0)^{-1}(1-\chi)\|_{L^2_\alpha\to L^2_{-\alpha}} = O(\lambda^{-1}), \quad \forall\lambda > 1,\ \forall\alpha > \tfrac{1}{2},$$

where $L^2_\alpha := \langle r\rangle^{-\alpha}L^2(M)$. As a matter of fact, we use this estimate to prove (1-5).

Since $\boldsymbol{H}$ in Theorem 1.3 also satisfies the finite speed of propagation property (2-2), we deduce from the two theorems above

Corollary 1.7. *Let* $\boldsymbol{L} = \boldsymbol{H}$, *where* $\boldsymbol{H}$ *is as in Theorem 1.3*, *and assume that* (M, g) *in Theorem 1.3 is nontrapping. Then* $\boldsymbol{L}$ *satisfies* (1-3), *where* F *and* s *are as in Theorem 1.1 and* $p \in [1, 2(n+1)/(n+3)]$.

Remark 1.8. As far as we are aware, the restriction estimates for the spectral measure in Theorem 1.3 were previously known only for $\boldsymbol{H}$ being the Laplacian in the Euclidean space $\mathbb{R}^n$. As for the spectral multiplier result of Corollary 1.7, this was previously known for $s > n(1/p-1/2)+1/2$ [Duong et al. 2002]. Thus, for $p \in [1, 2(n+1)/(n+3)]$, we gain half a derivative over the best results previously known. The region in the $(1/p, s)$-plane in which we improve previous results is illustrated in Figure 1. The lower threshold of $n(1/p-1/2)$ for the Sobolev exponent s in Corollary 1.7 is known to be sharp in Euclidean space, and it is not hard to see that it is sharp for any asymptotically conic manifold.

Remark 1.9. There are not many examples of sharp spectral multiplier results in the literature. Those known to the authors are as follows. The sharp multiplier result in (1-3) for $p = 2(n+1)/(n+3)$ (the other p are obtained by interpolation) was proved for the Laplacian on any compact manifold by Seeger

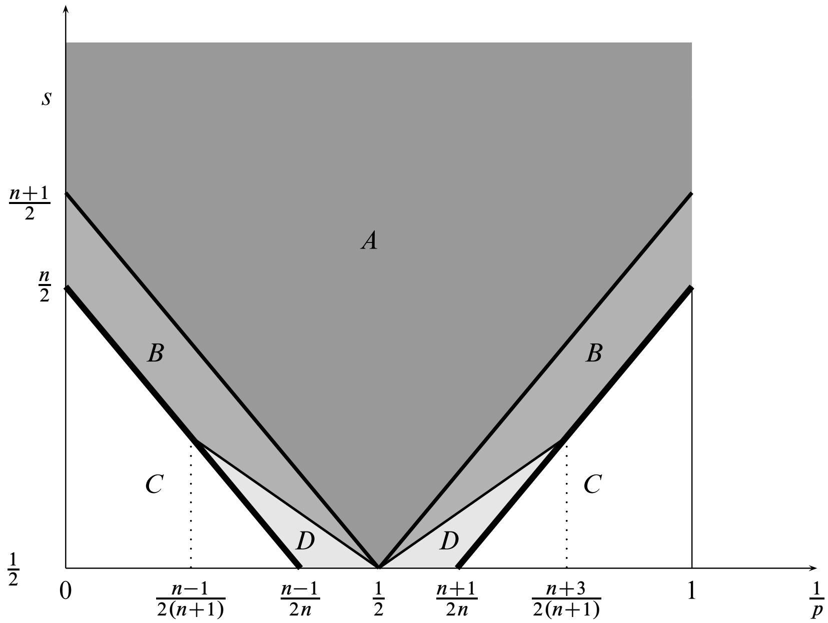

Figure 1. Map of where the statement of (1-3) has been established on nontrapping asymptotically conic manifolds, for different values of s and p. In region A this was previously known ([Duong et al. 2002]; see also Proposition 2.9). In the present paper we establish (1-3) also for region B (previously this was known only in the classical case of flat Euclidean space and the flat Laplacian). In region C it is known to be false, while region D is still unknown. For comparison with the Bochner–Riesz multiplier $F_\delta(\lambda) = (1-\lambda^2)_+^\delta$ observe that F_δ is in H^s for $s > \delta + 1/2$. For $F = F_\delta$, part of region D is known for flat Euclidean space [Lee 2004], and the celebrated Bochner–Riesz conjecture is that, for flat Euclidean space, (1-3) is true for $F = F_\delta$ in the whole of D.

and Sogge [1989]. In fact, they only needed the integrated estimate (1-7) to obtain the multiplier theorem in that setting. In the setting of the twisted Laplacian operator

$$\Delta_x + \Delta_y + \tfrac{1}{4}(\|x\|^2 + \|y\|^2) - i \sum_{j=1}^{n} (x_j \partial_{y_j} - y_j \partial_{x_j}),$$

the sharp multiplier result of (1-3) was proved by Stempak and Zienkiewicz [1998]. However, in this setting the required form of restriction estimates differs from both (1-4) and (1-7); see [Koch and Ricci 2007]. The last case of a sharp multiplier theorem known to us, although with a slightly different range of p, is for the harmonic oscillator; see [Karadzhov 1994; Koch and Tataru 2005; Thangavelu 1993].

Remark 1.10. A multiplier theorem of the type (1-3) does not hold for manifolds with exponential volume growth (like negatively curved complete manifolds); a *necessary* condition on the multiplier F in that case is typically a holomorphic extension of F into a strip. See for instance the work of Clerc and Stein [1974] or Anker [1990] for the case of noncompact symmetric spaces, or Taylor [1989] in the case of manifolds with bounded geometry, where sufficient conditions are also given.

Remark 1.11. Theorems 1.1 and 1.3 imply Bochner–Riesz summability for a range of exponents similar to those proved for the Euclidean Laplacian in [Stein 1993, page 390; Sogge 1993, Theorem 2.3.1] and for compact manifolds by Christ and Sogge [1988] and Sogge [1987]. See Corollary 2.10 below.

The heuristics one can extract from Theorem 1.3 and the last two remarks can be summarized as follows:

- The sharp restriction estimate on $dE(\lambda)$ at bounded and low frequencies λ only depends on the geometry near infinity.
- The high frequency restriction estimate on $dE(\lambda)$ also depends strongly on global dynamical properties (trapping/nontrapping).
- The integrated estimate (1-7) for all frequencies $\lambda > 1$ only depends on having uniform local geometry.

The proof of Theorem 1.1, given in Section 2, is based on a principle common to the proofs of most Fourier and spectral multiplier theorems. The rough idea is that one can control the L^p to L^p norm of operators with singular integral kernels by estimating the L^p to L^q norm of the operator for some $q > p$ (usually $q = 2$) and showing that a large part of the corresponding kernel is concentrated near the diagonal; see [Fefferman 1970; 1973; Seeger and Sogge 1989; Sogge 1987]. For calculations starting from $L^1 \to L^2$ estimates this principle can be equivalently stated in terms of weighted L^2 norms of the kernel; see [Cowling and Sikora 2001; Hörmander 1960; Mauceri and Meda 1990]. Our implementation of this principle in the proof of Theorem 1.1 is based on finite speed propagation of the wave equation, following [Cheeger et al. 1982; Cowling and Sikora 2001; Sikora 2004]. In the proof, we decompose the operator $F(\alpha\sqrt{L})$ as a sum over $\ell \in \mathbb{N}$ of multipliers $F_\ell(\alpha\sqrt{L})$ satisfying some finite speed propagation properties with F_ℓ Schwartz. The $L^p \to L^p$ norms for $F_\ell(\alpha\sqrt{L})$ are controlled by $C(\alpha 2^\ell)^{n(1/p-1/2)}$ times the $L^p \to L^2$ norms and then the TT^* argument reduces the problem to the bound of the $L^p \to L^{p'}$ norms of $|F_\ell|^2(\alpha\sqrt{L})$, which can be obtained using the restriction estimate of the spectral measure.

The proof of Theorem 1.3 proceeds in two steps. In the first step we suppose that we have an abstract operator $\boldsymbol{L}$ whose spectral measure can be factorized as $dE_{\sqrt{\boldsymbol{L}}}(\lambda) = (2\pi)^{-1}P(\lambda)P(\lambda)^*$ (see the discussion below (1-1)), where the initial space of $P(\lambda)$ is a Hilbert space. We then prove the following result in Section 3:

Proposition 1.12. *Let (X, d, μ) and $\boldsymbol{L}$ be as in Theorem 1.1, and assume $dE_{\sqrt{\boldsymbol{L}}}(\lambda) = (2\pi)^{-1}P(\lambda)P(\lambda)^*$ as described above. Also assume that for each λ we have an operator partition of unity on $L^2(X)$,*

$$\mathrm{Id} = \sum_{i=1}^{N(\lambda)} Q_i(\lambda), \tag{1-8}$$

where the Q_i are uniformly bounded as operators on $L^2(X)$ and $N(\lambda)$ is uniformly bounded. We assume that for $1 \le i \le N(\lambda)$, and some nonnegative function $w(z, z')$ on $X \times X$, the estimate

$$\bigl|\bigl(Q_i(\lambda)dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)Q_i(\lambda)\bigr)(z, z')\bigr| \le C\lambda^{n-1-j}\bigl(1 + \lambda w(z, z')\bigr)^{-(n-1)/2+j} \tag{1-9}$$

holds for $j=0$ and for $j=n/2-1$ and $j=n/2$ if n is even, or for $j=n/2-3/2$ and $j=n/2+1/2$ if n is odd. Here $dE^{(j)}_{\sqrt{L}}(\lambda)$ means $(d/d\lambda)^j dE_{\sqrt{L}}(\lambda)$, and C is independent of λ and i. Then the restriction estimates

$$\|dE_{\sqrt{L}}(\lambda)\|_{L^p(X)\to L^{p'}(X)} \le C'\lambda^{n(1/p-1/p')-1}, \quad 1\le p\le \frac{2(n+1)}{n+3}, \tag{1-10}$$

hold for all $\lambda>0$. Moreover, if the estimates above hold only for $0<\lambda\le\lambda_0$, then (1-10) *holds for $0<\lambda\le\lambda_0$.*

The key point here is that we only need to consider operators $Q_i(\lambda)dE^{(j)}_{\sqrt{L}}(\lambda)Q_k(\lambda)$ for $i=k$, which effectively means that we only need to analyze the kernel of $dE^{(j)}_{\sqrt{L}}(\lambda)$ close to the diagonal. The proof of this is based on the complex interpolation idea of Stein [1956] and appears in Section 3.

The second step is to prove estimates (1-9) in the case where $\boldsymbol{L}$ is the Laplacian or a Schrödinger operator on an asymptotically conic manifold:

Theorem 1.13. *Let (M,g) and $\boldsymbol{H}$ be as in Theorem 1.3. Then there exists an operator partition of unity,* (1-8), *where the Q_i are uniformly bounded as operators on $L^2(X)$ and $N(\lambda)$ is uniformly bounded, such that the estimates* (1-9) *hold for all integers $j\ge 0$ and for $0<\lambda\le\lambda_0$, where $w(z,z')$ is the Riemannian distance between points $z,z'\in M^\circ$. Moreover, if (M,g) is nontrapping, then estimates* (1-9) *hold for all $0<\lambda<\infty$.*

In the free Euclidean setting, this estimate is obvious (with the trivial partition of unity) by using the explicit formula of the spectral measure, but in our general setting it turns out to be quite involved and we really need to choose the partition of unity carefully. We use some results of [Hassell and Vasy 2001] on the resolvent of L on the spectrum, the high-energy (semiclassical) version of this [Hassell and Wunsch 2008] and the low energy estimates of our previous work [Guillarmou et al. 2012]. These three articles on which we build our estimates describe the Schwartz kernel of the spectral measure as a Legendrian distribution (a Fourier integral operator, in a sense) on a desingularized version of the compactification of the space $M\times M$, and this was done in a sort of uniform way with respect to the spectral parameter λ. The operators Q_i in the partition of unity will be pseudodifferential operators of a particular sort; see Section 6C for the estimate (1-9) for small λ, and Section 7D for the same estimate for large λ. By our discussion above, this establishes parts (A) and (B) of Theorem 1.3. Part (C) of Theorem 1.3 is proved in Section 8B and part (D) is proved in Section 8A.

Part I. Abstract self-adjoint operators

2. Restriction estimates imply spectral multiplier estimates

Let $\boldsymbol{L}$ be an abstract positive self-adjoint operator on $L^2(X)$, where X is a metric measure space with metric d and measure μ. We make the following assumptions about $\boldsymbol{L}$ and (X,d,μ):

- The space X is separable and has dimension n in the sense of the volume growth of balls: that is, there exist constants $0<c_1<c_2<\infty$ such that

$$c_1\rho^n \le \mu(B(x,\rho)) \le c_2\rho^n \tag{2-1}$$

for every $x \in X$ and $\rho > 0$;

- $\cos(t\sqrt{L})$ satisfies finite speed propagation in the sense that

$$\operatorname{supp} \cos(t\sqrt{L}) \subset \mathcal{D}_t := \{(z_1, z_2) \subset X \times X \mid d(z_1, z_2) \le |t|\}. \tag{2-2}$$

This statement says that $\langle f_1, \cos(t\sqrt{L}) f_2 \rangle = 0$ whenever supp $f_1 \in B(z_1, \rho_1)$, supp $f_2 \in B(z_2, \rho_2)$ and $|t| + \rho_1 + \rho_2 \le d(z_1, z_2)$.

- L satisfies restriction estimates, which come in a strong and a weak form. We say that L satisfies *L^p to $L^{p'}$ restriction estimates for all energies* if the spectral measure $dE_{\sqrt{L}}(\lambda)$ maps $L^p(X)$ to $L^{p'}(X)$ for some p satisfying $1 \le p < 2$ and all $\lambda > 0$, with an operator norm estimate

$$\| dE_{\sqrt{L}}(\lambda) \|_{L^p(X) \to L^{p'}(X)} \le C \lambda^{n(1/p - 1/p') - 1} \quad \text{for all } \lambda > 0. \tag{2-3}$$

We also consider a weaker form of these estimates: we say that L satisfies *low energy L^p to $L^{p'}$ restriction estimates* if $dE_{\sqrt{L}}(\lambda)$ maps $L^p(X)$ to $L^{p'}(X)$ for some p satisfying $1 \le p < 2$ and all $\lambda \in (0, \lambda_0]$, with an operator norm estimate

$$\| dE_{\sqrt{L}}(\lambda) \|_{L^p(X) \to L^{p'}(X)} \le C \lambda^{n(1/p - 1/p') - 1}, \quad 0 < \lambda \le \lambda_0, \tag{2-4}$$

for some C, together with weaker estimates for $\lambda \ge \lambda_0$,

$$\| E_{\sqrt{L}}[0, \lambda] \|_{L^p(X) \to L^{p'}(X)} \le C \lambda^{n(1/p - 1/p')}, \quad \lambda \ge \lambda_0, \tag{2-5}$$

with a uniform C. (Here $E_{\sqrt{L}}[0, \lambda]$ is the same as $\mathbb{1}_{[0,\lambda]}(\sqrt{L})$.)

Remark 2.1. The assumptions (with restriction estimates for all energies) are satisfied by taking $X = \mathbb{R}^n$ with the standard metric and measure, and L to be the (positive) Laplacian on $\mathbb{R}^n$ (with domain $H^2(\mathbb{R}^n)$). As we shall see, the assumptions are also satisfied for asymptotically conic manifolds, with the low energy restriction estimates holding unconditionally, and restriction estimates for all energies satisfied if the manifold is nontrapping.

Remark 2.2. Clearly, (2-5) follows from (2-3) by integrating over the interval $[0, \lambda]$. However, in Remark 8.8 we give an example where we have, by Proposition 8.1,

$$\| E_{\sqrt{L}}[\lambda, \lambda + 1] \|_{L^p(X) \to L^{p'}(X)} \le C \lambda^{n(1/p - 1/p') - 1}, \quad \lambda \ge \lambda_0,$$

(which implies (2-5)), but the pointwise estimate on the $L^p \to L^{p'}$ operator norm of $dE(\lambda)$ grows exponentially for a subsequence of λ tending to infinity.

Remark 2.3. Spectral projection estimate (2-5) is implied by a heat kernel bound

$$\| e^{-tL} \|_{L^p \to L^{p'}} \le C t^{-n(1/p - 1/p')/2}, \quad t \le \frac{1}{\lambda_0}. \tag{2-6}$$

This follows from short-time Gaussian bounds for the heat kernel, which hold for the Laplacian on any complete Riemannian manifold with bounded curvature and injectivity radius bounded below [Cheng et al. 1981, Theorem 4]. Estimate (2-6) implies, using T^*T, that $\|e^{-t\boldsymbol{L}}\|_{L^p\to L^2} \le Ct^{-n(1/p-1/p')/4}$. We then compute, using T^*T again,

$$E_{\sqrt{\boldsymbol{L}}}[0,\lambda] = E_{\sqrt{\boldsymbol{L}}}[0,\lambda]\, e^{\boldsymbol{L}/\lambda^2} e^{-\boldsymbol{L}/\lambda^2}$$
$$\implies \|E_{\sqrt{\boldsymbol{L}}}[0,\lambda]\|_{p\to p'} = \|E_{\sqrt{\boldsymbol{L}}}[0,\lambda]\|^2_{p\to 2} \le \big\|E_{\sqrt{\boldsymbol{L}}}[0,\lambda]e^{\boldsymbol{L}/\lambda^2}\big\|^2_{2\to 2}\cdot \big\|e^{-\boldsymbol{L}/\lambda^2}\big\|^2_{p\to 2}.$$

Conversely, (2-5) implies the heat kernel bound (2-6), which can be seen by writing $e^{-t\boldsymbol{L}}$ as in integral over the spectral measure, and then integrating by parts.

2A. ***The main result.*** The following theorem is the main result of this section.

Theorem 2.4. *Suppose that (X, d, μ) and $\boldsymbol{L}$ satisfy* (2-1) *and* (2-2), *and that $\boldsymbol{L}$ satisfies L^p to $L^{p'}$ restriction estimates for all energies*, (2-3), *for some p with $1 \le p < 2$. Let $s > n(1/p - 1/2)$ be a Sobolev exponent. Then there exists C depending only on n, p, s, and the constant in* (2-3) *such that, for every even $F \in H^s(\mathbb{R})$ supported in $[-1, 1]$, $F(\sqrt{\boldsymbol{L}})$ maps $L^p(X) \to L^p(X)$, and*

$$\sup_{\alpha>0} \|F(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p} \le C\|F\|_{H^s}. \tag{2-7}$$

If $\boldsymbol{L}$ only satisfies the weaker estimates (2-4), (2-5), *i.e., low energy L^p to $L^{p'}$ restriction estimates, then for all F as above, we have*

$$\sup_{\alpha\ge 4/\lambda_0} \|F(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p} \le C\|F\|_{H^s}, \tag{2-8}$$

where C depends on n, p, s, λ_0, and the constants in (2-4) *and* (2-5).

Remark 2.5. Notice that if $p > 2n/(n+1)$ then $s = 1/2$ satisfies $s > n(1/p - 1/2)$. However, $H^{1/2}$ functions need not be bounded, and such functions cannot be L^p multipliers even for $p = 2$, and a fortiori for $p \ne 2$. We deduce that, under the assumptions of Theorem 2.4, estimate (2-3), or even (2-4), is impossible for $p > 2n/(n+1)$.

In preparation for the proof of Theorem 2.4, we have (following [Cheeger et al. 1982]):

Lemma 2.6. *Assume that $\boldsymbol{L}$ satisfies* (2-2) *and that F is an even bounded Borel function with Fourier transform $\hat{F}$ satisfying* $\operatorname{supp}\widehat{F} \subset [-\rho, \rho]$. *Then*

$$\operatorname{supp} K_{F(\sqrt{\boldsymbol{L}})} \subset \mathcal{D}_\rho.$$

Proof. If F is an even function, then by the Fourier inversion formula,

$$F(\sqrt{\boldsymbol{L}}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{F}(t)\cos(t\sqrt{\boldsymbol{L}})\,dt.$$

But $\operatorname{supp}\widehat{F} \subset [-\rho, \rho]$ and Lemma 2.6 follows from (2-2). □

The next lemma is a crucial tool in using restriction type results, i.e., $L^p \to L^q$ continuity of spectral projectors, to obtain spectral multiplier type bounds, i.e., $L^p \to L^p$ estimates.

Lemma 2.7. *Suppose that (x, d, μ) satisfies (2-1) and S is a bounded linear operator from $L^p(X)$ to $L^q(X)$ such that*

$$\operatorname{supp} S \subset \mathcal{D}_\rho$$

for some $\rho > 0$. Then for any $1 \le p < q \le \infty$ there exists a constant $C = C_{p,q}$ such that

$$\|S\|_{p\to p} \le C\rho^{n(1/p-1/q)}\|S\|_{p\to q}.$$

Proof. We fix $\rho > 0$. Then we first choose a sequence $x_n \in M$ such that $d(x_i, x_j) > \rho/10$ for $i \ne j$ and $\sup_{x\in X}\inf_i d(x, x_i) \le \rho/10$. Such sequence exists because M is separable. Second, we define $\widetilde{B}_i$ by the formula

$$\widetilde{B}_i = \overline{B}\left(x_i, \tfrac{1}{10}\rho\right) - \left(\bigcup_{j<i} \overline{B}\left(x_j, \tfrac{1}{10}\rho\right)\right), \tag{2-9}$$

where $\overline{B}(x, \rho) = \{y \in M : d(z, z') \le \rho\}$. Third, we put $\chi_i = \chi_{\widetilde{B}_i}$, where $\chi_{\widetilde{B}_i}$ is the characteristic function of set $\widetilde{B}_i$. Fourth, we define the operator M_{χ_i} by the formula $M_{\chi_i} g = \chi_i g$.

Note that for $i \ne j$, $B(x_i, \frac{1}{20}\rho) \cap B(x_j, \frac{1}{20}\rho) = \varnothing$. Hence

$$K = \sup_i \#\{j;\ d(x_i, x_j) \le 2\rho\} \le \sup_x \frac{|\overline{B}(x, 2\rho)|}{|B(x, \frac{1}{20}\rho)|} < \frac{40^n c_2}{c_1} < \infty.$$

It is not difficult to see that if we set $I = \{i, j \mid d(x_i, x_j) < 2\rho\}$, then

$$\mathcal{D}_\rho \subset \bigcup_{i,j\in I} \widetilde{B}_i \times \widetilde{B}_j \subset \mathcal{D}_{4\rho}, \quad \text{so} \quad Sf = \sum_{i,j\in I} M_{\chi_i} S M_{\chi_j} f.$$

Hence, if we set $J_i = \{j \mid d(x_i, x_j) < 2\rho\}$ for a given i, then by the Hölder inequality

$$\begin{aligned}
\|Sf\|_p^p = \Big\| \sum_{i,j\in I} M_{\chi_i} S M_{\chi_j} f \Big\|_{L^p}^p &= \sum_i \Big\| \sum_{j\in J_i} M_{\chi_i} S M_{\chi_j} f \Big\|_p^p \\
&\le \sum_i |\widetilde{B}_i|^{p(1/p-1/q)} \Big\| \sum_{j\in J_i} M_{\chi_i} S M_{\chi_j} f \Big\|_q^p \\
&\le C\rho^{np(1/p-/q)} \sum_i \Big\| \sum_{j\in J_i} M_{\chi_i} S M_{\chi_j} f \Big\|_q^p \\
&\le CK^{p-1}\rho^{np(1/p-1/q)} \sum_i \sum_{j\in J_i} \big\| M_{\chi_i} S M_{\chi_j} f \big\|_q^p \\
&\le CK^p\rho^{np(1/p-1/q)} \sum_j \big\| S M_{\chi_j} f \big\|_q^p \\
&\le CK^p\rho^{np(1/p-1/q)} \|S\|_{p\to q}^p \sum_j \big\| M_{\chi_j} f \big\|_p^p \\
&= CK^p\rho^{np(1/p-1/q)} \|S\|_{p\to q}^p \|f\|_p^p.
\end{aligned}$$

This finishes the proof of Lemma 2.7. □

Proof of Theorem 2.4. We first assume that $\boldsymbol{L}$ satisfies L^p to $L^{p'}$ restriction estimates for all energies. We take $\eta \in C_c^\infty(-4,4)$ even and such that

$$\sum_{l\in\mathbb{Z}} \eta\Big(\frac{t}{2^l}\Big) = 1 \quad \text{for all } t \neq 0.$$

Then we set $\phi(t) = \sum_{l\leq 0} \eta(2^{-l}t)$,

$$F_0(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi(t)\widehat{F}(t)\cos(t\lambda)\,dt,$$

and

$$F_l(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \eta\Big(\frac{t}{2^l}\Big)\widehat{F}(t)\cos(t\lambda)\,dt. \tag{2-10}$$

Note that by virtue of the Fourier inversion formula,

$$F(\lambda) = \sum_{l\geq 0} F_l(\lambda),$$

and by Lemma 2.6,

$$\operatorname{supp} F_l(\alpha\sqrt{\boldsymbol{L}}) \subset \mathcal{D}_{2^{l+2}\alpha}.$$

Now by Lemma 2.7,

$$\|F(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p} \leq \sum_{l\geq 0}\|F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p} \leq C\sum_{l\geq 0}(2^l\alpha)^{n(1/p-1/2)}\|F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2}. \tag{2-11}$$

Unfortunately, F_l is no longer compactly supported. To remedy this we choose a function $\psi \in C_c^\infty(-4,4)$ such that $\psi(\lambda) = 1$ for $\lambda \in (-2,2)$ and note that

$$\|F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2} \leq \|(\psi F_l)(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2} + \|((1-\psi)F_l)(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2}.$$

To estimate the norm $\|\psi F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2}$ we use our restriction estimates (2-3). Using a T^*T argument and the fact that $\operatorname{supp}\psi \subset [-4,4]$, we note that

$$\begin{aligned}\|\psi F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2}^2 = \||\psi F_l|^2(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p'} &\leq \int_0^{4/\alpha} |\psi F_l(\alpha\lambda)|^2 \|dE_{\sqrt{\boldsymbol{L}}}(\lambda)\|_{p\to p'}\,d\lambda \\ &\leq \frac{C}{\alpha}\int_0^4 |\psi F_l(\lambda)|^2\|dE_{\sqrt{\boldsymbol{L}}}(\lambda/\alpha)\|_{p\to p'}\,d\lambda.\end{aligned} \tag{2-12}$$

It follows from the above calculation and (2-3) that

$$\alpha^{n(1/p-1/2)}\|\psi F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2} \leq C\|\psi F_l\|_2 \tag{2-13}$$

for all $\alpha > 0$. As a consequence, we obtain

$$\sum_{l\geq 0} 2^{ln(1/p-1/2)}\alpha^{n(1/p-1/2)}\|\psi F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2} \leq \sum_{l\geq 0} 2^{ln(1/p-1/2)}\|\psi F_l\|_2$$

for all $\alpha > 0$. Now let us recall that by the definition of a Besov space,

$$\sum_{l\geq 0} 2^{ln(1/p-1/2)}\|\psi F_l\|_2 \leq \sum_{l\geq 0} 2^{ln(1/p-1/2)}\|F_l\|_2 = \|F\|_{B^{n(1/p-1/2)}_{1,2}}.$$

See [Triebel 1992, Chapters I and II] for more details. We also recall that if $s > s'$ then $H^s \subset B^{s'}_{1,2}$ and $\|F\|_{B^{n(1/p-1/2)}_{1,2}} \leq C_s\|F\|_{H^s}$ for all $s > n(1/p-1/2)$ [ibid.]. Therefore, we have shown that

$$\sum_{l\geq 0} 2^{ln(1/p-1/2)}\alpha^{n(1/p-1/2)}\|\psi F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2} \leq C\|F\|_{H^s}. \tag{2-14}$$

Next we obtain bounds for the part of estimate (2-11) corresponding to the term $\|(1-\psi)F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2}$. This only requires the spectral projection estimates (2-5). We write

$$\begin{aligned}|(1-\psi)F_l|^2(\alpha\sqrt{\boldsymbol{L}}) &= \int_0^\infty |(1-\psi)(\alpha\lambda)F_l(\alpha\lambda)|^2 dE_{\sqrt{\boldsymbol{L}}}(\lambda)\\ &= -\int_0^\infty \Bigl(\frac{d}{d\lambda}|(1-\psi)(\alpha\lambda)F_l(\alpha\lambda)|^2\Bigr)E_{\sqrt{\boldsymbol{L}}}(\lambda)\,d\lambda\\ &= -\int_0^\infty \Bigl(\frac{d}{d\lambda}|(1-\psi)(\lambda)F_l(\lambda)|^2\Bigr)E_{\sqrt{\boldsymbol{L}}}(\lambda/\alpha)\,d\lambda.\end{aligned}$$

Hence, using (2-5),

$$\|(1-\psi)F_l(\alpha\sqrt{\boldsymbol{L}})\|^2_{p\to 2} \leq C\int_0^\infty \Bigl(\frac{d}{d\lambda}|(1-\psi)(\lambda)F_l(\lambda)|^2\Bigr)\Bigl(\frac{\lambda}{\alpha}\Bigr)^{n(1/p-1/p')}d\lambda. \tag{2-15}$$

We write

$$F_l(\lambda) = \frac{1}{2\pi}\int e^{it(\lambda-\lambda')}\eta\Bigl(\frac{t}{2^l}\Bigr)F(\lambda')\,d\lambda'\,dt,$$

use the identity

$$e^{it(\lambda-\lambda')} = i^{-N}(\lambda-\lambda')^{-N}(d/dt)^N e^{it(\lambda-\lambda')},$$

and integrate by parts N times. Note that if $\lambda \in \operatorname{supp} 1-\psi$ and $\lambda' \in \operatorname{supp} F$ then $\lambda \geq 2$ and $\lambda' \leq 1$, and hence $\lambda - \lambda' \geq \lambda/2$. It follows that

$$|((1-\psi)F_l)(\lambda)| \leq C\lambda^{-N}2^{-N(l-1)}\|F\|_2,$$

with C independent of N. Similarly,

$$\Bigl|\frac{d}{d\lambda}((1-\psi)F_l)(\lambda)\Bigr| \leq C\lambda^{-N}2^{-N(l-1)}2^l\|F\|_2.$$

Using this in (2-15) with N sufficiently large and $l \geq 2$, we obtain

$$(2^l\alpha)^{n(1/p-1/2)}\|((1-\psi)F_l)(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2} \leq C2^{-l}\|F\|_2.$$

Therefore, we have

$$\sum_l (2^l\alpha)^{n(1/p-1/2)}\|((1-\psi)F_l)(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2} \leq C\|F\|_2 \leq C\|F\|_{H^s}. \tag{2-16}$$

Equations (2-11), (2-14) and (2-16) prove (2-7).

The proof in the case that $\boldsymbol{L}$ satisfies low-energy restriction estimates (2-4) and (2-5) proceeds the same way, except that we require the condition $\alpha \le 4/\lambda_0$ at the step (2-12) in order that we can use the pointwise estimate (2-4) on the spectral measure in this integral. □

Remark 2.8. Note that if we only assume that (2-5) holds for all $\lambda > 0$ then we still have

$$\alpha^{n(1/p-1/2)} \|\psi F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2} \le \alpha^{n(1/p-1/2)} \left\|\psi F_l(\alpha\sqrt{\boldsymbol{L}}) e^{\alpha^2 \boldsymbol{L}}\right\|_{2\to 2} \cdot \left\|e^{-\alpha^2 \boldsymbol{L}}\right\|_{p\to 2} \le C\|\psi F_l\|_\infty,$$

Now the above estimate is just a version of (2-13) with norm $\|\psi F_l\|_2$ replaced by $\|\psi F_l\|_\infty$. Next if we replace the Besov space $B^{n(1/p-1/2)}_{1,2}$ by $B^{n(1/p-1/2)}_{1,\infty}$ then we can still follow the proof of Theorem 2.4. Recall also that if $s > s'$ then $W^s_\infty \subset B^{s'}_{1,\infty}$ and $\|F\|_{B^{n(1/p-1/2)}_{1,\infty}} \le C_s \|F\|_{W^s_\infty}$ for all $s > n(1/p-1/2)$, where $\|F\|_{W^s_\infty} = \|(I - d^2/dx^2)^{s/2} F\|_\infty$; see again [Triebel 1992]. This implies that (2-14) holds with the norm $\|F\|_{H^s}$ replaced by the norm $\|F\|_{W^s_\infty}$. As the rest of the proof of Theorem 2.4 does not require (2-3), the above argument proves the following proposition.

Proposition 2.9. *Suppose that* (X, d, μ) *and* $\boldsymbol{L}$ *satisfy* (2-1) *and* (2-2), *and that* $\boldsymbol{L}$ *satisfies* (2-5) *for all* $\lambda > 0$. *Let* $s > n|1/p - 1/2|$ *be a Sobolev exponent. Then there exists C depending only on* n, p, s, *and the constant in* (2-5) *such that, for every even* $F \in W^s_\infty(\mathbb{R})$ *supported in* $[-1, 1]$, $F(\sqrt{\boldsymbol{L}})$ *maps* $L^p(X) \to L^p(X)$, *and*

$$\sup_{\alpha>0} \|F(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p} \le C\|F\|_{W^s_\infty}. \tag{2-17}$$

Note also that if $s > s'$ then $\|F\|_{W^{s'}_\infty} \le C\|F\|_{H^{s+1/2}}$. That is, the multiplier result with exponent one-half bigger then the optimal exponent does not require (2-3) and holds just under assumption (2-5), which is equivalent with the standard heat kernel bounds (2-6) (for all t). For $p = 1$, Proposition 2.9 was proved in [Christ and Sogge 1988] and can be alternatively proved using Theorem 3.5 in the same paper and interpolation, see also [Duong et al. 2002, Theorem 3.1].

From this point of view, the key point about Theorem 2.4 is the gain of half a derivative over the more elementary (2-17).

2B. ***Bochner–Riesz summability.*** We use Theorem 2.4 to discuss boundedness of Bochner–Riesz means of the operator $\boldsymbol{L}$. Bochner–Riesz summability is technically speaking a slight weakening of Theorem 2.4 but is very close, and it allows us to compare our results with results described in [Stein 1993; Sogge 1993]. Let us recall that Bochner–Riesz means of order δ are defined by the formula

$$(1 - \boldsymbol{L}/\lambda^2)^\delta_+, \quad \lambda > 0. \tag{2-18}$$

For $\delta = 0$, this is the spectral projector $E_{\sqrt{\boldsymbol{L}}}([0, \lambda])$, while for $\delta > 0$ we think of (2-18) as a smoothed version of this spectral projector; the larger δ, the more smoothing. Bochner–Riesz summability describes the range of δ for which the above operators are bounded on L^p uniformly in λ.

Corollary 2.10. *Suppose that (X, d, μ) is as above, and that restriction estimates* (2-3) *for exponents* $1 \le p \le 2(n+1)/(n+3)$ *and finite speed propagation property* (2-2) *hold for operator* $\boldsymbol{L}$. *Then for all*

$$p \in \Big[1, \frac{2(n+1)}{n+3}\Big] \cup \Big[\frac{2(n+1)}{n-1}, \infty\Big] \quad \text{and} \quad \delta > n\Big|\frac{1}{p} - \frac{1}{2}\Big| - \frac{1}{2},$$

we have

$$\|(1 - \boldsymbol{L}/\lambda^2)_+^\delta\|_{p \to p} \le C \text{ for all } \lambda > 0. \tag{2-19}$$

For all $p \in \big(2(n+1)/(n+3), 2(n+1)/(n-1)\big)$ *these estimates hold if* $\delta > \frac{1}{2}(n-1)|1/p - 1/2|$.

Proof. Note that $(1-\lambda^2)_+^\delta \in H^s$ if and only if $\delta > s - 1/2$. Now for $p < 2(n+1)/(n+3)$ Corollary 2.10 follows from Theorem 2.4. For $2(n+1)/(n+3) < p < 2$ Corollary 2.10 follows from interpolating between (2-19) with $p = 2(n+1)/(n+3)$ and the trivial estimate for $p = 2$. For $p > 2$ the results follow by duality. □

Remark 2.11. We noted in the proof above that Corollary 2.10 follows from Theorem 2.4. In fact the Corollary 2.10 is slightly but essentially weaker than Theorem 2.4. Indeed Corollary 2.10 is equivalent to the version of Theorem 2.4 in which the H^s norm of a compactly supported function F is replaced by the L^1 norm of $F^s := F * \chi_+^{-s-1}$, where χ_+ is as in Section 3. To prove this we note that

$$F(\alpha\sqrt{\boldsymbol{L}}) = \int \chi_+^{\nu-1}(\lambda - \alpha\sqrt{\boldsymbol{L}}) F^\nu(\lambda)\, d\lambda, \qquad \nu \ge 0;$$

see (3-3) and (3-4). Hence if estimates (2-19) hold for some exponent δ then $\|F(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p} \le \|F^{\delta+1}\|_1$ and Bochner–Riesz summability of order δ implies Theorem 2.4 with the norm $\|F^{\delta+1}\|_1$. Note that if F, supported in $[-1, 1]$, is such that $F^{s+1/2}$ is in $L^1(\mathbb{R})$, then F is in $H^{s'}(\mathbb{R})$ for all $s' < s$ with an estimate $\|F\|_{H^{s'}} \le C\|F^{s+1/2}\|_{L^1}$. Hence, conversely, Theorem 2.4 with the stronger hypothesis $F^{s+1/2} \in L^1$ implies Bochner–Riesz summability of order δ for all $\delta > s - 1/2$.

2C. *Singular integrals.* Finally we will discuss a singular integral version of our spectral multiplier result. The following theorem is just reformulation of [Cowling and Sikora 2001, Theorem 3.5]. We write D_κ for the scaling operator $D_\kappa F(x) = F(\kappa x)$.

Theorem 2.12. *Suppose that operator* $\boldsymbol{L}$ *satisfies finite speed propagation property* (2-2), *that* $s > n/2$ *and that*

$$\|dE_{\sqrt{\boldsymbol{L}}}(\lambda)\|_{1\to\infty} \le \lambda^{n-1} \quad \text{for all } \lambda > 0. \tag{2-20}$$

Next let η be a smooth compactly supported nonzero function. Then for any Borel bounded function F such that $\sup_{\kappa>0} \|\eta\, D_\kappa F\|_{W_s^p} < \infty$ *the operator* $F(\sqrt{\boldsymbol{L}})$ *is of weak type* $(1, 1)$ *and is bounded on* $L^q(X)$ *for all* $1 < q < \infty$. *In addition*,

$$\|F(\sqrt{\boldsymbol{L}})\|_{L^1\to L^{1,\infty}} \le C_s\Big(\sup_{\kappa>0} \|\eta D_\kappa F\|_{W_s^p} + |F(0)|\Big). \tag{2-21}$$

Remark 2.13. It is a standard observation that up to equivalence the norm

$$\sup_{\kappa>0} \|\eta\, D_\kappa F\|_{W_s^p}$$

does not depend on the auxiliary function η as long as η is not identically equal zero.

Proof. Using T^*T trick we note that by (2-20) one has

$$\|F(\sqrt{L})\|_{1\to 2}^2 = \| \, |F|^2(\sqrt{L})\|_{1\to\infty} \le \int_0^\infty |F(\lambda)|^2 \|dE_{\sqrt{L}}(\lambda)\|_{1\to\infty} \, d\lambda \le C \int_0^\infty |F(\lambda)|^2 \lambda^{n-1} d\lambda.$$

Hence if $\operatorname{supp} F \subset [0, R)$ then

$$\|F(\sqrt{L})\|_{1\to 2}^2 \le C R^n \|D_R F\|_2^2,$$

that is, the estimates (3.22) of Theorem 3.5 of [Cowling and Sikora 2001] hold. Now Theorem 2.12 follows from the same Theorem 3.5. □

Remark 2.14. Theorem 2.12 is a singular integral version of Theorem 2.4 for $p = 1$. We expect that a similar extension to a singular integral version is possible for all p. That is if one assumes that $s > n|1/2 - 1/p|$ then one can prove weak-type (p, p) version of estimates (2-21). However the proof of such results seems to be more complex and not directly related to the rest of this paper, so we will not pursue this idea further here.

3. Kernel estimates imply restriction estimates

The goal of this section is to prove Proposition 1.12; that is, we show that restriction estimates (2-3) or (2-4) follow from certain pointwise estimates of λ-derivatives of the kernel of the spectral measure. We first prove a simplified version of Proposition 1.12 in which the partition of unity does not appear. We work in the same abstract setting as the previous section.

Proposition 3.1. *Let (X, d, μ) be a metric measure space and $\boldsymbol{L}$ an abstract positive self-adjoint operator on $L^2(X, \mu)$. Assume that the spectral measure $dE_{\sqrt{L}}(\lambda)$ for $\sqrt{\boldsymbol{L}}$ has a Schwartz kernel $dE_{\sqrt{L}}(\lambda)(z, z')$ that satisfies, for some nonnegative function w on $X \times X$ and some $n \ge 3$, the estimate*

$$\left| \left(\frac{d}{d\lambda}\right)^j dE_{\sqrt{L}}(\lambda)(z, z') \right| \le C\lambda^{n-1-j}(1 + \lambda w(z, z'))^{-(n-1)/2+j} \tag{3-1}$$

for $j = 0$ and for $j = n/2 - 1$ and $j = n/2$ if n is even, or for $j = n/2 - 3/2$ and $j = n/2 + 1/2$ if n is odd. Then (2-3) *holds for all p in the range* $[1, 2(n+1)/(n+3)]$. *Moreover, if the estimates above hold only for $0 < \lambda < \lambda_0$, then* (2-4) *hold for the same range of p.*

We prove this proposition via complex interpolation, embedding the derivatives of the spectral measure in an analytic family of operators, following the original (unpublished) proof of Stein in the classical case. To do this we use the distributions χ_+^a, defined by

$$\chi_+^a = x_+^a / \Gamma(a+1),$$

where Γ is the gamma function and

$$\begin{cases} x_+^a = x^a & \text{if } x \ge 0, \\ x_+^a = 0 & \text{if } x < 0. \end{cases}$$

The x_+^a are clearly distributions for $\operatorname{Re} a > -1$, and we have for $\operatorname{Re} a > 0$,

$$\frac{d}{dx} x_+^a = a x_+^{a-1} \quad \Longrightarrow \quad \frac{d}{dx} \chi_+^a = \chi_+^{a-1}, \tag{3-2}$$

which we use to extend the family of functions χ_+^a to a family of distributions on $\mathbb{R}$ defined for all $a \in \mathbb{C}$; see [Hörmander 1983] for details. Since $\chi_+^0(x) = H(x)$ is the Heaviside function, it follows that

$$\chi_+^{-k} = \delta_0^{(k-1)}, \quad k = 1, 2, \dots, \tag{3-3}$$

and therefore

$$\chi_+^0(\lambda - \sqrt{\boldsymbol{L}}) = E_{\sqrt{\boldsymbol{L}}}((0, \lambda]) \quad \text{and} \quad \chi_+^{-k}(\lambda - \sqrt{\boldsymbol{L}}) = \Big(\frac{d}{d\lambda}\Big)^{k-1} dE_{\sqrt{\boldsymbol{L}}}(\lambda), \quad k \geq 1.$$

A standard computation shows that for all $w, z \in \mathbb{C}$,

$$\chi_+^w * \chi_+^z = \chi_+^{w+z+1}, \tag{3-4}$$

where $\chi_+^w * \chi_+^z$ is the convolution of the distributions χ_+^w and χ_+^z see [Hörmander 1983, (3.4.10)]. We can use this relation to *define* the operators $\chi_+^z(\lambda - \sqrt{\boldsymbol{L}})$ for $\operatorname{Re} z < 0$, provided that the spectral measure of $\sqrt{\boldsymbol{L}}$ satisfies estimates of the type in Proposition 3.1:

Definition 3.2. Suppose that X, $\boldsymbol{L}$ and w are as in Proposition 3.1, and that $\boldsymbol{L}$ satisfies the kernel estimate

$$\left| \Big(\frac{d}{d\lambda}\Big)^k dE_{\sqrt{\boldsymbol{L}}}(\lambda)(z, z') \right| \leq C \lambda^l (1 + \lambda w(z, z'))^\beta \tag{3-5}$$

for some $k \geq 0$, $l \geq 0$ and β. Then, for $-(k+1) < \operatorname{Re} a < 0$ we define the operator $\chi_+^a(\lambda - \sqrt{\boldsymbol{L}})$ to be that operator with kernel

$$\chi_+^{k+a} * \chi_+^{-(k+1)}(\lambda - \sqrt{\boldsymbol{L}})(z, z') = (-1)^k \int_0^\lambda \frac{\sigma^{k+a}}{\Gamma(k+a+1)} \Big(\frac{d}{d\sigma}\Big)^k dE_{\sqrt{\boldsymbol{L}}}(\lambda - \sigma)(z, z')\, d\sigma. \tag{3-6}$$

Notice that the integral converges, since $\operatorname{Re}(k+a) > -1$ and $l \geq 0$ in (3-5). It is also independent of the choice of integer $k > -\operatorname{Re} a - 1$ (provided (3-5) holds), as we check by integrating by parts in σ in the integral above, and using (3-2). Note that the kernel $\chi_+^a(\lambda - \sqrt{\boldsymbol{L}})(z, z')$ is analytic in a, and as an integral operator maps $L^1_{\text{comp}}(X)$ to $L^\infty_{\text{loc}}(X)$. Therefore, for each fixed $\lambda > 0$, the family $\chi_+^a(\lambda - \sqrt{\boldsymbol{L}})$ is an analytic family of operators in the sense of Stein [1956] in the parameter a, for $\operatorname{Re} a > -k$.

In the proof of Proposition 3.1 we will need the following:

Lemma 3.3. *Suppose that $k \in \mathbb{N}$, that $-k < a < b < c$ and that $b = \theta a + (1-\theta)c$. Then there exists a constant C such that for any C^{k-1} function $f : \mathbb{R} \to \mathbb{C}$ with compact support, one has*

$$\|\chi_+^{b+is} * f\|_\infty \leq C(1+|s|)e^{\pi|s|/2} \|\chi_+^a * f\|_\infty^\theta \|\chi_+^c * f\|_\infty^{1-\theta}$$

for all $s \in \mathbb{R}$.

Remark 3.4. The convolution $\chi_+^a * f$, for $a > -k$ and $f \in C_c^{k-1}(\mathbb{R})$, may be defined to be $\chi_+^{a+k-1} * f^{(k-1)}$; this is independent of the choice of k.

Proof. Set, for $\zeta \in \mathbb{C}$,

$$I_\zeta f = \chi_+^\zeta * f$$

and consider the operator $I_{b+is}(\sigma I_c + I_a)^{-1}$, where the number $\sigma \in \mathbb{C}$ such that $|\sigma| = 1$ will be specified later. By (3-4)

$$I_{b+is}(\sigma I_c + I_a)^{-1} = I_{\beta+is}(\sigma I_{-1} + I_\alpha)^{-1} = I_{\beta+is}(\sigma I + I_\alpha)^{-1},$$

where $\beta = b - c - 1$ and $\alpha = a - c - 1$. Note that $\alpha < \beta < -1$. A standard calculation [Hörmander 1983, Example 7.1.17, page 167 and (3.2.9) page 72] shows that for $\operatorname{Re}\zeta \le -1$,

$$\widehat{\chi_+^\zeta}(\xi) = e^{-i\pi(\zeta+1)/2}(\xi - i0)^{-\zeta-1}.$$

It follows that $I_{\beta+is}(\sigma I + I_\alpha)^{-1} f = f * \eta_s$, where $\widehat{\eta_s}$ is the locally integrable function

$$\widehat{\eta_s}(\xi) = \frac{-ie^{-i\pi(\beta+is)/2}\xi_+^{-(\beta+is)-1} + ie^{i\pi(\beta+is)/2}\xi_-^{-(\beta+is)-1}}{\sigma - ie^{-i\pi\alpha/2}\xi_+^{-\alpha-1} + ie^{i\pi\alpha/2}\xi_-^{-\alpha-1}}.$$

Here $\xi_+ = \max(0, \xi)$ and $\xi_- = -\min(0, \xi)$. Note that if $|\sigma| = 1$ and $\sigma \notin \{ie^{-i\pi\alpha/2}, -ie^{-i\pi\alpha/2}\}$ then

$$\left|\frac{d}{d\xi}\widehat{\eta_s}(\xi)\right| \le C(1+|s|)e^{\pi|s|/2}\min\big(|\xi|^{-\beta-2}, |\xi|^{-\beta+\alpha-1}\big)$$

and $-\beta + \alpha - 1 < -1 < -\beta - 2$. It follows from these estimates that the function $\frac{d}{d\xi}\widehat{\eta_s}$ is in an $L^p(\mathbb{R})$ space for some $1 < p < 2$ and is also in some weighted space $L^1((1+|x|)^\epsilon dx, \mathbb{R})$. By the Sobolev embedding and Hausdorff–Young theorems, the function $x \to x\eta_s(x)$ is in $L^{p'}(\mathbb{R})$ for the conjugate exponent $p' < \infty$ and in $C^{\epsilon'}(\mathbb{R})$ for some $\epsilon' > 0$. Hence η_s is in L^1 and we have

$$\|\eta_s\|_1 \le C(1+|s|)e^{\pi|s|/2}.$$

Hence the operator $I_{b+is}(\sigma I_c + I_a)^{-1} = I_{\beta+is}(\sigma I + I_\alpha)^{-1}$ is bounded on $L^\infty(\mathbb{R})$ and

$$\|I_{b+is} f\|_\infty \le C(1+|s|)e^{\pi|s|/2}\|\sigma I_c f + I_a f\|_\infty \le C(1+|s|)e^{\pi|s|/2}(\|I_c f\|_\infty + \|I_a f\|_\infty).$$

Now if we set $D_\kappa f(x) = f(\kappa x)$ then for all $\zeta \in \mathbb{C}$,

$$I_\zeta D_\kappa f = \kappa^{-\zeta-1} D_\kappa I_\zeta f,$$

so

$$\kappa^{-b}\|I_{b+is} f\|_\infty = \kappa^{-b}\|D_\kappa I_{b+is} f\|_\infty = \kappa\|I_{b+is} D_\kappa f\|_\infty.$$

Hence

$$\begin{aligned}\kappa^{-b}\|I_{b+is} f\|_\infty = \kappa\|I_{b+is} D_\kappa f\|_\infty &\le C(1+|s|)e^{\pi|s|/2}\big(\kappa\|I_a(D_\kappa f)\|_\infty + \kappa\|I_c(D_\kappa f)\|_\infty\big)\\ &= C(1+|s|)e^{\pi|s|/2}\big(\kappa^{-a}\|I_a f\|_\infty + \kappa^{-c}\|I_c f\|_\infty\big).\end{aligned}$$

Putting $\kappa^{a-c} = \|I_a f\|_\infty \|I_c f\|_\infty^{-1}$ in this estimate yields Lemma 3.3. ☐

Proof of Proposition 3.1. To prove (2-3) in the range $1 \le p \le 2(n+1)/(n+3)$, it suffices by interpolation to establish the result for the endpoints $p = 1$ and $p = 2(n+1)/(n+3)$. The endpoint $p = 1$ is precisely (3-1) for $j = 0$, so it remains to obtain the endpoint $p = 2(n+1)/(n+3)$. This we will obtain through complex interpolation, applied to the analytic (in the parameter a) family $\chi_+^a(\lambda - \sqrt{\boldsymbol{L}})$ in the strip $-(n+1)/2 \le \operatorname{Re} a \le 0$.

On the line $\operatorname{Re} a = 0$, we have the estimate

$$\|\chi^{is}(\lambda - \sqrt{\boldsymbol{L}})\|_{L^2 \to L^2} \le \left|\frac{1}{\Gamma(1+is)}\right| = \sqrt{\frac{\sinh \pi s}{\pi s}} \le Ce^{\pi|s|/2}.$$

On the line $\operatorname{Re} a = -(n+1)/2$, we will prove an estimate of the form

$$\left\|\chi^{-(n+1)/2+is}(\lambda - \sqrt{\boldsymbol{L}})\right\|_{L^1 \to L^\infty} \le C(1+|s|)e^{\pi|s|/2}\lambda^{(n-1)/2} \qquad \text{for all } s \in \mathbb{R}. \tag{3-7}$$

Then, since we can write

$$dE_{\sqrt{\boldsymbol{L}}}(\lambda) = \chi_+^{-1}(\lambda - \sqrt{\boldsymbol{L}})$$

and

$$-1 = \frac{n-1}{n+1} \cdot 0 + \frac{2}{n+1} \cdot \left(-\frac{n+1}{2}\right) \quad \text{and} \quad \frac{n+3}{2(n+1)} = \frac{n-1}{n+1} \cdot \frac{1}{2} + \frac{2}{n+1} \cdot 1,$$

we obtain (2-3) at $p = 2(n+1)/(n+3)$ by complex interpolation.

It remains to prove (3-7). Let $\eta \in C_c^\infty(\mathbb{R})$ be a function such that $0 \le \eta(x) \le 1$ for all $x \in \mathbb{R}$ and $\eta(x) = 1$ for $|x| \le 2$ and $\eta(x) = 0$ for $|x| \ge 4$. Set

$$\begin{aligned} F_{z,z'}^{s,\Lambda}(\lambda) &= \chi_+^{-3/2-is} * \big(\eta(\,\cdot\,/\Lambda)\chi_+^{-k}(\,\cdot\, - \sqrt{\boldsymbol{L}})(z,z')\big)(\lambda), && n = 2k, \\ F_{z,z'}^{s,\Lambda}(\lambda) &= \chi_+^{-2-is} * \big(\eta(\,\cdot\,/\Lambda)\chi_+^{-k}(\,\cdot\, - \sqrt{\boldsymbol{L}})(z,z')\big)(\lambda), && n = 2k+1. \end{aligned}$$

Note that $\operatorname{supp}(\chi_+^z) \subset [0,\infty)$ for all z, and $\boldsymbol{L} \ge 0$. It follows that for $\lambda \le \Lambda$ and $n = 2k$,

$$F_{z,z'}^{s,\Lambda}(\lambda) = \chi_+^{-3/2-is} * \chi_+^{-k}(\lambda - \sqrt{\boldsymbol{L}})(z,z') = \chi_+^{-(n+1)/2-is}(\lambda - \sqrt{\boldsymbol{L}})(z,z')$$

and for $\lambda \le \Lambda$ and $n = 2k+1$,

$$F_{z,z'}^{s,\Lambda}(\lambda) = \chi_+^{-2-is} * \chi_+^{-k}(\lambda - \sqrt{\boldsymbol{L}})(z,z') = \chi_+^{-(n+1)/2-is}(\lambda - \sqrt{\boldsymbol{L}})(z,z'),$$

i.e., the cutoff function η has no effect for $\lambda \le \Lambda$. Hence

$$\left\|\chi_+^{-(n+1)/2-is}(\Lambda - \sqrt{\boldsymbol{L}})\right\|_{1\to\infty} \le \sup_{z,z'} |F_{z,z'}^{s,\Lambda}(\Lambda)|.$$

We consider first the odd-dimensional case $n = 2k+1$. By Lemma 3.3 and (3-3),

$$\begin{aligned}\big|F^{s,\Lambda}_{z,z'}(\Lambda)\big| &\le \big\|F^{s,\Lambda}_{z,z'}\big\|_\infty \\ &\le C(1+|s|)e^{\pi|s|/2}\sup_{\lambda>0}\big|\big(\chi_+^{-1}*(\eta(\cdot/\Lambda)\chi_+^{-k}(\cdot-\sqrt{\boldsymbol{L}})(z,z'))\big)(\lambda)\big|^{1/2} \\ &\qquad\qquad\times\sup_{\lambda>0}\big|\big(\chi_+^{-3}*(\eta(\cdot/\Lambda)\chi_+^{-k}(\cdot-\sqrt{\boldsymbol{L}})(z,z'))\big)(\lambda)\big|^{1/2} \\ &\le C(1+|s|)e^{\pi|s|/2}\sup_{\lambda>0}\big|\eta(\lambda/\Lambda)\chi_+^{-k}(\lambda-\sqrt{\boldsymbol{L}})(z,z')\big|^{1/2} \\ &\qquad\qquad\times\sup_{\lambda>0}\Big|\frac{d^2}{d\lambda^2}\eta(\lambda/\Lambda)\chi_+^{-k}(\lambda-\sqrt{\boldsymbol{L}})(z,z')\Big|^{1/2},\end{aligned}\tag{3-8}$$

where the presence of the η cutoff is now crucial. It follows from (3-1) with $j = n/2 - 3/2$ and $j = n/2+1/2$, i.e., $j = k-1$ and $j = k+1$, that

$$\sup_{\lambda>0}|\eta(\lambda/\Lambda)\chi_+^{-k}(\lambda-\sqrt{\boldsymbol{L}})(z,z')| \le C\Lambda^{k+1}(1+\Lambda w(z,z'))^{-1}.$$

(Here we used the fact that the function $\lambda^k(1+\lambda w)^\beta$ is an increasing function of λ provided $\lambda \ge 0$, $w \ge 0$, $k \ge 0$ and $k+\beta \ge 0$.) Similarly,

$$\begin{aligned}\sup_{\lambda>0}\Big|\frac{d^2}{d\lambda^2}\eta(\lambda/\Lambda)\chi_+^{-k}(\lambda-\sqrt{\boldsymbol{L}})(z,z')\Big| &\le \sup_{\lambda>0}\big|\eta(\lambda/\Lambda)\chi_+^{-k-2}(\lambda-\sqrt{\boldsymbol{L}})(z,z')\big| \\ &\qquad+\frac{1}{\Lambda}\sup_{\lambda>0}\big|\eta'(\lambda/\Lambda)\chi_+^{-k-1}(\lambda-\sqrt{\boldsymbol{L}})(z,z')\big| \\ &\qquad+\frac{1}{\Lambda^2}\sup_{\lambda>0}\big|\eta'(\lambda/\Lambda)\chi_+^{-k}(\lambda-\sqrt{\boldsymbol{L}})(z,z')\big| \\ &\le C\Lambda^{k-1}(1+\Lambda w(z,z')).\end{aligned}$$

Our estimate (3-7) for $n = 2k+1$ follows now from these two estimates and (3-8).

If $n = 2k$ is even, then by Lemma 3.3 and (3-3),

$$\begin{aligned}\big|F^{s,\Lambda}_{z,z'}(\Lambda)\big| &\le \big\|F^{s,\Lambda}_{z,z'}\big\|_\infty \\ &\le C(1+|s|)e^{\pi|s|/2}\sup_{\lambda>0}\big|\big(\chi_+^{-1}*(\eta(\cdot/\Lambda)\chi_+^{-k}(\cdot-\sqrt{\boldsymbol{L}})(z,z'))\big)(\lambda)\big|^{1/2} \\ &\qquad\qquad\times\sup_{\lambda>0}\big|\big(\chi_+^{-2}*(\eta(\cdot/\Lambda)\chi_+^{-k}(\cdot-\sqrt{\boldsymbol{L}})(z,z'))\big)(\lambda)\big|^{1/2} \\ &\le C(1+|s|)e^{\pi|s|/2}\sup_{\lambda>0}\big|\eta(\lambda/\Lambda)\chi_+^{-k}(\lambda-\sqrt{\boldsymbol{L}})(z,z')\big|^{1/2} \\ &\qquad\qquad\times\sup_{\lambda>0}\Big|\frac{d}{d\lambda}\eta(\lambda/\Lambda)\chi_+^{-k}(\lambda-\sqrt{\boldsymbol{L}})(z,z')\Big|^{1/2},\end{aligned}\tag{3-9}$$

and we follow the same argument as in the odd-dimensional case to establish (3-7) for $n = 2k$. □

In some situations, including the case of Laplace-type operators on asymptotically conic manifolds discussed later in this paper, we can express the spectral measure $dE(\lambda)$ in the form $P(\lambda)P(\lambda)^*$, where the initial space of $P(\lambda)$ is an auxiliary Hilbert space H. In this case, we can use a TT^* argument to show that the conclusions of Proposition 3.1 follow from localized estimates on $dE(\lambda)$, that is, on kernel estimates on $Q_i dE(\lambda) Q_i$, with respect to a operator partition of unity

$$\mathrm{Id} = \sum_{i=1}^{N(\lambda)} Q_i(\lambda), \quad 1 \le i \le N(\lambda).$$

Notice that we allow the partition of unity to depend on λ. However, we shall assume that $N(\lambda)$ is uniformly bounded in λ.

Remark 3.5. Here we assume that $Q_i(\lambda) dE^{(j)}_{\sqrt{\boldsymbol{L}}}(\lambda) Q_i(\lambda)$ can be defined somehow and has a Schwartz kernel; for example, we might know that there is some weight function ω on X such that $dE^{(j)}_{\sqrt{\boldsymbol{L}}}(\lambda)$ is a bounded map from $\omega^{j+1} L^2(X)$ to $\omega^{-j-1} L^2(X)$, and that $Q_i(\lambda)$ maps $\omega^a L^2(X)$ boundedly to itself for any a. This is the case in our application to asymptotically conic manifolds, with $\omega = x$ (where x is as in (1-1)).

Proof of Proposition 1.12. Observe that Proposition 1.12 reduces to Proposition 3.1 in the case that the partition of unity Q_i is trivial. We apply the argument in the proof of Proposition 3.1 to the operators $Q_i(\lambda) dE(\lambda) Q_i(\lambda)$, i.e., we replace $dE_{\sqrt{\boldsymbol{L}}}(\lambda)$ by $Q_i(\lambda) dE_{\sqrt{\boldsymbol{L}}}(\lambda) Q_i(\lambda)^*$ in (3-6). The conclusion is that

$$\| Q_i(\lambda) dE_{\sqrt{\boldsymbol{L}}}(\lambda) Q_i(\lambda)^* \|_{L^p(X) \to L^{p'}(X)} \le C \lambda^{n(1/p - 1/p') - 1} \quad \text{for all } \lambda > 0.$$

Using the fact that $dE_{\sqrt{\boldsymbol{L}}}(\lambda) = P(\lambda) P(\lambda)^*$ and the TT^* trick, we deduce that

$$\| Q_i(\lambda) P(\lambda) \|_{L^2(X) \to L^{p'}(X)} \le C \lambda^{n(1/2 - 1/p') - 1/2} \quad \text{for all } \lambda > 0.$$

Now we can sum over i, and find that

$$\| P(\lambda) \|_{L^2(X) \to L^{p'}(X)} \le C \lambda^{n(1/2 - 1/p') - 1/2} \quad \text{for all } \lambda > 0.$$

Finally, we use $dE_{\sqrt{\boldsymbol{L}}}(\lambda) = P(\lambda) P(\lambda)^*$ and the TT^* trick again to deduce that

$$\| dE_{\sqrt{\boldsymbol{L}}}(\lambda) \|_{L^p(X) \to L^{p'}(X)} \le C \lambda^{n(1/p - 1/p') - 1} \quad \text{for all } \lambda > 0,$$

yielding (2-3). Moreover, if the estimates hold only for $0 < \lambda \le \lambda_0$, then we obtain (2-4) instead. □

Remark 3.6. We acknowledge and thank Jared Wunsch for suggesting to us that the TT^* trick would be useful here.

Part II. Schrödinger operators on asymptotically conic manifolds

In this second part of the paper, we specialize to the case that (X, d, μ) is an asymptotically conic manifold (M°, g) with the Riemannian distance function d and Riemannian measure μ, and $\boldsymbol{L}$ is a Schrödinger operator $\boldsymbol{H}$ on $L^2(M^\circ, g)$, that is, an operator of the form $\boldsymbol{H} = \Delta_g + V$, where Δ_g is the positive Laplacian associated to g and $V \in C^\infty(M)$ is a potential function vanishing to third order at the boundary of the compactification M of M°. We assume that $\boldsymbol{H}$ has no L^2-eigenvalues (which implies that it is positive as an operator) and that zero is not a resonance.

The goal in this part of the paper is to show that $\boldsymbol{H}$ satisfies the low energy spectral measure estimates (2-4), and the full spectral measure estimates (2-3) provided that (M°, g) is nontrapping. To do this, we will establish the estimates (1-9) for a suitable partition of unity $Q_i(\lambda)$. In the case of low energy estimates, i.e., $\lambda \in (0, \lambda_0]$ for $\lambda_0 < \infty$, these Q_i will be pseudodifferential operators, lying in the calculus of operators introduced in [Guillarmou and Hassell 2008]. Thus our first task is to determine the nature of the operator $Q_i dE(\lambda) Q_i$ for such Q_i, which is the subject of Section 5. Before this, however, we recall some of the geometric preliminaries from [Guillarmou et al. 2012; Hassell and Wunsch 2008].

4. Geometric preliminaries

The Schwartz kernel of the spectral measure was constructed in [Guillarmou et al. 2012] for low energies and in [Hassell and Wunsch 2008] for high energies on a compactification of the space $[0, \lambda_0] \times (M^\circ)^2$, respectively $[0, h_0] \times (M^\circ)^2$, where we use $h = \lambda^{-1}$ in place of λ for high energies. We use the definitions and machinery from these papers extensively, and we do not review this material comprehensively here, since that would double the length of this paper. Nevertheless, we shall describe these compactifications, review some of their geometric properties, and define some coordinate systems that we shall use in the following sections.

Recall from the introduction that (M°, g) is asymptotically conic if M° is the interior of a compact manifold M with boundary, such that in a collar neighborhood of the boundary, the metric g takes the form $g = dx^2/x^4 + h(x)/x^2$, where x is a boundary defining function and $h(x)$ is a smooth family of metrics on the boundary ∂M. We use $y = (y_1, \dots, y_{n-1})$ for local coordinates on ∂M, so that (x, y) furnish local coordinates on M near ∂M. Away from ∂M, we use $z = (z_1, \dots, z_n)$ to denote local coordinates.

4A. ***The low energy space $M^2_{k,b}$.*** In [Guillarmou and Hassell 2008; Guillarmou et al. 2012], following unpublished work of Melrose and Sá Barreto, the low energy space $M^2_{k,b}$ is defined as follows: starting with $[0, \lambda_0] \times M^2$, we define submanifolds $C_3 := \{0\} \times \partial M \times \partial M$ and

$$C_{2,L} := \{0\} \times \partial M \times M, \quad C_{2,R} := \{0\} \times M \times \partial M, \quad C_{2,C} := [0, 1] \times \partial M \times \partial M.$$

The space $M^2_{k,b}$ is then defined as $[0, \lambda_0] \times M^2$ with the codimension 3 corner C_3 blown up, followed by the three codimension 2 corners $C_{2,*}$:

$$M^2_{k,b} := [[0, 1] \times M \times M; C_3, C_{2,R}, C_{2,L}, C_{2,C}].$$

The new boundary hypersurfaces created by these blowups are labeled bf_0, rb_0, lb_0 and bf, respectively, and the original boundary hypersurfaces $\{0\} \times M^2$, $[0, \lambda] \times M \times \partial M$ and $[0, \lambda] \times \partial M \times M$ are labeled zf, rb, lb, respectively. We remark that zf is canonically diffeomorphic to the b-double space

$$M^2_b = [M^2; \partial M \times \partial M].$$

Also, each section $M^2_{k,b} \cap \{\lambda = \lambda_*\}$, for fixed $0 < \lambda_* < \lambda_0$ is canonically diffeomorphic to M^2_b.

We define functions x and y on $M^2_{k,b}$ by lifting from the left copy of M (near ∂M), and x', y' by lifting from the right copy of M; similarly z, z' (away from ∂M). We also define $\rho = x/\lambda$, $\rho' = x'/\lambda$, and

$\sigma = \rho/\rho' = x/x'$. Near bf and away from rb, we use coordinates $y, y', \sigma, \rho', \lambda$, while near bf and away from lb, we use $y, y', \sigma^{-1}, \rho, \lambda$. We also use the notation $\rho_\bullet$, where $\bullet = \mathrm{bf}_0, \mathrm{lb}_0, \ldots$, to denote a generic boundary defining function for the boundary hypersurface $\bullet$.

This space has a compressed cotangent bundle ${}^{k,b}T^*M^2_{k,b}$, defined in [Guillarmou et al. 2012, Section 2]. A basis of sections of this space is given, in the region $\rho, \rho' \le C$ (which includes a neighborhood of bf), by

$$\frac{d\rho}{\rho^2}, \quad \frac{d\rho'}{\rho'^2}, \quad \frac{dy_i}{\rho}, \quad \frac{dy_i'}{\rho'}, \quad \frac{d\lambda}{\lambda}. \tag{4-1}$$

Therefore, any point in ${}^{k,b}T^*M^2_{k,b}$ lying over this region can be written as

$$\nu\frac{d\rho}{\rho^2} + \nu'\frac{d\rho'}{\rho'^2} + \mu_i\frac{dy_i}{\rho} + \mu_i'\frac{dy_i'}{\rho'} + T\frac{d\lambda}{\lambda}. \tag{4-2}$$

This defines local coordinates $(y, y', \sigma, \rho', \lambda, \mu, \mu', \nu, \nu', T)$ in ${}^{k,b}T^*M^2_{k,b}$, near bf and away from rb, where $(\mu, \mu', \nu, \nu', T)$ are linear coordinates on each fiber.

The compressed density bundle $\Omega_{k,b}(M^2_{k,b})$ is defined to be that line bundle whose smooth nonzero sections are given by the wedge product of a basis of sections for ${}^{k,b}T^*(M^2_{k,b})$. Using the coordinates above, we can write a smooth nonzero section $\boldsymbol{\omega}$ as

$$\boldsymbol{\omega} = \left|\frac{d\rho d\rho' dy dy' d\lambda}{\rho^{n+1}\rho'^{n+1}\lambda}\right| \sim \lambda^{2n}\left|\frac{dg\, dg'\, d\lambda}{\lambda}\right| \quad \text{in the region } \rho, \rho' \le C. \tag{4-3}$$

For $\rho, \rho' \ge C$, we can take $\boldsymbol{\omega} = (xx')^n |dg dg' d\lambda/\lambda|$. Here dg, respectively dg', denotes the Riemannian density with respect to g, lifted to $M^2_{k,b}$ by the left, respectively right, projection.

The boundary of ${}^{k,b}T^*M^2_{k,b}$ lying over boundary hypersurface $\bullet$ is denoted by ${}^{k,b}T^*_\bullet M^2_{k,b}$. The space ${}^{k,b}T^*_{\mathrm{lb}}M^2_{k,b}$ fibers over the space ${}^{\mathrm{sc}}T^*_{\partial M}M \times [0, \lambda]$ (the scattering cotangent bundle ${}^{\mathrm{sc}}T^*M$ over M is defined in [Melrose 1994; Hassell and Vasy 1999; 2001], and ${}^{\mathrm{sc}}T^*_{\partial M}M$ is that part of the bundle lying over ∂M). This fibration is given in local coordinates by

$$(y, y', \sigma, \lambda, \mu, \mu', \nu, \nu', T) \to (y, \mu, \nu, \lambda). \tag{4-4}$$

Similarly there is a natural fibration from ${}^{k,b}T^*_{\mathrm{rb}}M^2_{k,b}$ to ${}^{\mathrm{sc}}T^*_{\partial M}M \times [0, \lambda_0]$, which takes the form

$$(y, y', \sigma, \lambda, \mu, \mu', \nu, \nu', T) \to (y', \mu', \nu', \lambda). \tag{4-5}$$

We also note that there are natural maps π_L, π_R mapping ${}^{\mathrm{sc}}T^*_{\mathrm{bf}}M^2_b \times [0, \lambda_0]$ (see [Hassell and Vasy 1999; 2001]) to ${}^{\mathrm{sc}}T^*_{\partial M}M \times [0, \lambda_0]$ which are induced by the projections $T^*M^2 \to T^*M$ onto the left, respectively right, factor. In local coordinates, these are given by

$$\pi_L(y, y', \sigma, \mu, \mu', \nu, \nu', \lambda) = (y, \mu, \nu, \lambda), \quad \pi_R(y, y', \sigma, \mu, \mu', \nu, \nu', \lambda) = (y', \mu', \nu', \lambda). \tag{4-6}$$

We use these maps in Section 5.

The space ${}^{k,b}T^*_{\mathrm{bf}}M^2_{k,b}$ is canonically diffeomorphic to ${}^{s\Phi}T^*_{\mathrm{bf}}M^2_b \times [0, \lambda_0]$, where ${}^{s\Phi}T^*_{\mathrm{bf}}M^2_b$ is the scattering-fibered cotangent bundle of M^2_b defined in [Hassell and Vasy 1999]. The space ${}^{s\Phi}T^*_{\mathrm{bf}}M^2_b$ has a natural contact structure, and Legendre submanifolds with respect to this structure play an important role in encoding the oscillations of the spectral measure at the boundary of $M^2_{k,b}$. In fact, three Legendre submanifolds of ${}^{s\Phi}T^*{}_{\mathrm{bf}}M^2_b$ arise in the identification of the spectral measure as a Legendre distribution (see [Guillarmou et al. 2012, Section 3]), which we now briefly describe. One is denoted ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$, which in coordinates used in (4-2) is given by

$$ {}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b = \{(y, y', \sigma, \mu, \mu', \nu, \nu') \mid y = y', \sigma = 1, \mu = -\mu', \nu = -\nu'\}; \tag{4-7} $$

it is a sort of conormal bundle to the boundary of the diagonal $\partial\mathrm{diag}_b$,

$$ \partial\mathrm{diag}_b = \{(y, y', \sigma) \mid y = y', \sigma = 1\}, \tag{4-8} $$

in M^2_b, and carries the "operator wavefront set" or "microlocal support" of scattering pseudodifferential operators. Another is the incoming/outgoing Legendrian submanifold $L^\sharp$, which in the coordinates used in (4-2) is given by

$$ L^\sharp = \{(y, y', \sigma, \mu, \mu', \nu, \nu') \mid \mu = \mu' = 0, \nu = \pm 1, \nu' = -\nu\}. \tag{4-9} $$

It has two components (corresponding to the sign of ν) and describes oscillations that are purely radial, that is, purely incoming or outgoing. The third and most interesting Legendre submanifold is the propagating Legendrian, denoted by L^{bf}. To describe it, let G denote the characteristic variety of $\boldsymbol{H} - \lambda^2$. Then L^{bf} is given by the flowout from ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b \cap G$ by the bicharacteristic flow of $\boldsymbol{H}$. It connects the incoming and outgoing components of $L^\sharp$ and has a conic singularity at each. As shown in [Hassell and Vasy 1999, Proposition 7.1], $(L^{\mathrm{bf}}, L^\sharp)$ is a Legendre conic pair, and has an associated class of polyhomogeneous-conormal Legendre distributions [Guillarmou et al. 2012, Section 3.2]

$$ I^{m,p;r_{\mathrm{lb}},r_{\mathrm{rb}};\mathcal{B}}(M^2_{k,b}, (L^{\mathrm{bf}}, L^{\sharp,\mathrm{bf}}); \Omega^{1/2}_{k,b}) \tag{4-10} $$

of order m at L^{bf} and p at $L^\sharp$, and with polyhomogeneous expansion with respect to the index family $\mathcal{B}$ at the boundary hypersurfaces at $\lambda = 0$. In terms of these space of half-densities we have:

Theorem 4.1 [Guillarmou et al. 2012, Theorem 3.10]. *The spectral measure* $dE_{\sqrt{\boldsymbol{H}}}(\lambda)$, *for* $0 < \lambda \le \lambda_0$, *is a conormal Legendre distribution in the space* (4-10) *tensored with* $|\lambda d\lambda|^{1/2}$ *(this makes it a full density, i.e., a measure, in* λ*), with* $m = -\frac{1}{2}$, $p = (n-2)/2$, $r_{\mathrm{lb}} = r_{\mathrm{rb}} = (n-1)/2$, *and where* $\mathcal{B}$ *is an index family with index sets at the faces* bf_0, lb_0, rb_0, zf *starting at order* -1, $n/2 - 1$, $n/2 - 1$, $n - 1$, *respectively.*

4B. ***The high energy space X.*** The high energy space $\boldsymbol{X}$ is defined by $\boldsymbol{X} = [0, h_0] \times M^2_b$. The boundary hypersurfaces $[0, h_0] \times M \times \partial M$, $[0, h_0] \times \partial M \times M$ and $\{0\} \times M^2_b$ are denoted by rb, lb and mf ("main face"), respectively, and the boundary hypersurface arising from $[0, h_0] \times \partial M \times \partial M$ is denoted by bf. Notice that this space fits together with the low energy space: in the range $\lambda \in (C^{-1}, C)$ (where $\lambda = 1/h$), the spaces both have the form $(C^{-1}, C) \times M^2_b$, and the labeling of boundary hypersurfaces is consistent. As before, we write $\sigma = x/x'$. We use the coordinates (y, y', σ, x', h) near bf and away from rb, and the coordinates $(y, y', \sigma^{-1}, x, h)$ near bf and away from lb. Away from bf, lb, rb we use the coordinates (z, z', h).

The compressed cotangent bundle ${}^{\mathrm{s}\Phi}T^*\boldsymbol{X}$ is described in [Hassell and Wunsch 2008]. A basis of sections of this bundle is given in the region $x, x' \le \epsilon$ by

$$\frac{dy_i}{xh}, \quad \frac{dy_i'}{x'h}, \quad d\Big(\frac{1}{xh}\Big), \quad d\Big(\frac{1}{x'h}\Big), \quad d\Big(\frac{1}{h}\Big).$$

In terms of this basis, any point in ${}^{\mathrm{s}\Phi}T^*\boldsymbol{X}$ lying over this region can be written as

$$\mu \cdot \frac{dy}{xh} + \mu' \cdot \frac{dy'}{x'h} + \nu d\Big(\frac{1}{xh}\Big) + \nu' d\Big(\frac{1}{x'h}\Big) + \tau d\Big(\frac{1}{h}\Big). \tag{4-11}$$

This defines local coordinates $(y, y', \sigma, x', h, \mu, \mu', \nu, \nu', \tau)$, where $(\mu, \mu', \nu, \nu', \tau)$ are local coordinates on each fiber. In the region $x, x' \ge \epsilon$, a basis of sections is

$$\frac{dz_i}{h}, \quad \frac{dz_i'}{h}, \quad d\Big(\frac{1}{h}\Big),$$

and in terms of this basis, any point in ${}^{\mathrm{s}\Phi}T^*\boldsymbol{X}$ lying over this region can be written as

$$\zeta \cdot \frac{dz}{h} + \zeta' \cdot \frac{dz'}{h} + \tau d\Big(\frac{1}{h}\Big). \tag{4-12}$$

This defines local coordinates $(z, z', h, \zeta, \zeta', \tau)$ on ${}^{\mathrm{s}\Phi}T^*\boldsymbol{X}$ over this region.

This compressed density bundle ${}^{\mathrm{s}\Phi}\Omega(\boldsymbol{X})$ is defined to be that line bundle whose smooth nonzero sections are given by a wedge product of a basis of sections for ${}^{\mathrm{s}\Phi}T^*\boldsymbol{X}$. We find that $|dg\, dg'\, dh/h^2| = |dg\, dg'\, d\lambda|$ is a smooth nonzero section of this bundle.

We also note that there are natural maps from ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}\boldsymbol{X} \to {}^{\mathrm{sc}}T^*M$, which (abusing notation) we will also denote by π_L, π_R, which are induced by the projections onto the left, respectively right, factor $T^*M^2 \to T^*M$. In local coordinates, these are given by

$$\pi_L(z, z', \zeta, \zeta', \tau) = (z, \zeta), \quad \pi_R(z, z', \zeta, \zeta', \tau) = (z', \zeta'), \tag{4-13}$$

away from the boundary hypersurface bf, or near bf by

$$\pi_L(x, y, x', y', \mu, \mu', \nu, \nu', \tau) = (x, y, \mu, \nu), \quad \pi_R(x, y, x', y', \mu, \mu', \nu, \nu', \tau) = (x', y', \mu', \nu'). \tag{4-14}$$

The space ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}\boldsymbol{X}$ has a natural contact structure, as described in [Hassell and Wunsch 2008]. Legendre submanifolds with respect to this contact structure are important in describing the singularities of the spectral measure at high energies. We need to define three Legendre submanifolds ${}^{\mathrm{s}\Phi}N^*\mathrm{diag}_b$ and L in order to describe the spectral measure at high energies as a Legendre distribution on $\boldsymbol{X}$ (see [ibid.]). The first of these, ${}^{\mathrm{s}\Phi}N^*\mathrm{diag}_b$, is associated to the diagonal submanifold $\mathrm{diag}_b \subset \{0\} \times M_b^2$, defined using the coordinates above by

$${}^{\mathrm{s}\Phi}N^*\mathrm{diag}_b = \{(z, z', h, \zeta, \zeta', \tau) \mid z = z', \zeta = -\zeta', h = 0, \tau = 0\} \tag{4-15}$$

away from bf, and

$${}^{\mathrm{s}\Phi}N^*\mathrm{diag}_b = \{(y, y', \sigma, x', h, \mu, \mu', \nu, \nu', \tau) \mid y = y', \sigma = 1, h = 0, \mu = -\mu', \nu = -\nu', \tau = 0\} \tag{4-16}$$

near bf. The second, $L^\sharp$, lives at ${}^{s\Phi}T^*_{\mathrm{bf}\cup\mathrm{mf}}\boldsymbol{X}$ and is defined in (4-9). The third, L, is obtained just as L^{bf} was obtained from ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$ in the previous subsection, namely as the flowout by the bicharacteristic flow of $\boldsymbol{H}$ starting from the intersection of ${}^{s\Phi}N^*\mathrm{diag}_b$ and the characteristic variety of $h^2\boldsymbol{H}-1$. Indeed, the submanifolds L^{bf} and ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$ are essentially the boundary hypersurfaces of L and ${}^{s\Phi}N^*\mathrm{diag}_b$ lying over $\mathrm{bf}\cap\mathrm{mf}$. Associated to $(L, L^\sharp)$ is a class of Legendre distributions [ibid., Section 6.5.2]

$$I^{m,p;r_{\mathrm{bf}},r_{\mathrm{lb}},r_{\mathrm{rb}}}(\boldsymbol{X}, (L, L^\#); {}^{s\Phi}\Omega^{1/2}). \tag{4-17}$$

In terms of this space of half-densities, we have:

Theorem 4.2 [Hassell and Wunsch 2008, Corollary 1.2]. *Suppose that (M, g) is nontrapping. Then the spectral measure $dE_{\sqrt{\boldsymbol{H}}}(\lambda)$ is a Legendre distribution on $\boldsymbol{X}$, lying in the space* (4-17) *tensored with $|d\lambda|^{1/2}$, with $m=\frac{1}{2}$, $p=(n-2)/2$, $r_{\mathrm{bf}}=-\frac{1}{2}$, $r_{\mathrm{lb}}=r_{\mathrm{rb}}=(n-1)/2$. Here we use the order conventions in Remark 4.3.*

Remark 4.3. We use different order conventions from [Hassell and Wunsch 2008], to agree with those used in [Guillarmou et al. 2012]. In terms of Equation (4.15) of [Hassell and Wunsch 2008], the order convention in the present paper corresponds to taking $N=2n$ (not $2n+1$ as in [ibid.]), that is, the total space dimension, but not including the λ dimension, and taking the fiber dimensions $f_{\mathrm{bf}}=0$ and $f_{\mathrm{lb}}=f_{\mathrm{rb}}=n$, again not including the λ dimension. This has the effect that the orders in the present paper are $\frac{1}{4}$ larger at $\mathrm{mf}=M_b^2\times\{h=0\}$, and $\frac{1}{4}$ smaller at bf, lb and rb, compared to [ibid.], and explains the discrepancies in the orders above compared to those given in Corollary 1.2 of [ibid.]. (An advantage of the ordering convention used here is that a semiclassical pseudodifferential operator of (semiclassical) order m, multiplied by $|dh/h^2|^{1/2}=|d\lambda|^{1/2}$ becomes a Legendre distribution of the same order m at the conormal bundle of the diagonal in mf.)

5. Microlocal support

Recall from the end of Section 1 our strategy for proving Theorem 1.3, involving estimates (1-9). The elements Q_i of our partition of unity will be chosen to be pseudodifferential operators lying in the calculus of operators introduced in [Guillarmou and Hassell 2008, Definition 2.7]. In view of Theorem 4.1, we need to understand what happens when a conormal Legendre distribution $F\in I^{m,r_{\mathrm{lb}},r_{\mathrm{rb}},\mathcal{B}}(M^2_{k,b}, \Lambda; \Omega^{1/2}_{k,b})$ is pre- and postmultiplied by such operators. We shall use the notation $\Psi_k^m(M, \Omega_{k,b}^{1/2})$ to denote what in [ibid.] was written $\Psi^{m,\mathcal{E}}(M, \tilde{\Omega}_b^{1/2})$, where the index family $\mathcal{E}$ assigns the C^∞ index family at sc, bf_0 and zf and the empty index family at all other boundary hypersurfaces. Such operators have kernels defined on the space $M^2_{k,\mathrm{sc}}$, defined in [ibid.], that are conormal of order m to the diagonal, uniformly to the boundary, smooth away from the diagonal, and rapidly vanishing at all boundary hypersurfaces not meeting the diagonal. As shown in [ibid., Proposition 2.10], $\Psi^0(M, \Omega^{1/2}_{k,b})$ is an algebra. It follows, using Hörmander's "square root trick" [1985, Section 18.1] that such kernels act as uniformly bounded (in λ) operators on $L^2(M)$.

In this section, we shall work exclusively on the low energy space $M^2_{k,b}$; the corresponding high energy estimates are given in Section 7A. We consider operators Q, Q' such that:

- Q, Q' are of order $-\infty$, i.e., $Q, Q' \in \Psi_k^{-\infty}(M, \Omega_{k,b}^{1/2})$, with compactly supported symbols. (5-1)
- Q, Q' have kernels supported close to the diagonal, inside the region $\{\sigma := x/x' \in [1/2, 2]\}$. (5-2)

With these assumptions, the kernels of Q, Q' are smooth (across the diagonal) on the space $M^2_{k,\mathrm{sc}}$. Viewed as distributions on $M^2_{k,b}$ (which has one fewer blowup than $M^2_{k,\mathrm{sc}}$) the kernels have a conic singularity at the boundary of the diagonal, $\partial\mathrm{diag}_b$. As shown in [Hassell and Vasy 2001, Section 5.1], this means that they are Legendre distributions in $I^{0,\infty,\infty;(0,0,\varnothing,\varnothing)}(M^2_{k,b}, {}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b; \Omega_{k,b}^{1/2})$, i.e., Legendre distributions of order 0 associated to ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$ (see (4-7)), with the C^∞ index set 0 at bf_0 and zf, and vanishing in a neighborhood of lb, rb, lb_0 and rb_0 (which is of course a trivial consequence of (5-2)).

Remark 5.1. The composition QF or FQ' is always well-defined when F is a Legendre distribution on $M^2_{k,b}$ and Q, Q' are as above, since F can be regarded as a map from $x^aL^2(M)$ to $x^{-a}L^2(M)$ for sufficiently large $a \in \mathbb{R}$, depending smoothly on $\lambda \in (0, \lambda_0)$, while pseudodifferential operators of order 0 are bounded on $x^aL^2(M)$ (uniformly in λ) for any a.

To state our results, we need to introduce some notation and define the notion of the microlocal support of F. Let $\Lambda \subset {}^{\mathrm{sc}}T^*_{\mathrm{bf}}M^2_b$ be the Legendre submanifold associated to F. We always assume that Λ is compact. Recall from [Hassell and Wunsch 2008, Section 4] that Λ determines two associated Legendre submanifolds Λ_{lb} and Λ_{rb} that are the bases of the fibrations on $\partial_{\mathrm{lb}}\Lambda$ and $\partial_{\mathrm{rb}}\Lambda$, respectively. These may be canonically identified with Legendre submanifolds of ${}^{\mathrm{sc}}T^*M$. We also define Λ' by negating the fiber coordinates corresponding to the right copy of M, i.e.,

$$q' = (y, y', x/x', \mu, \mu', \nu, \nu') \in \Lambda' \iff q = (y, y', x/x', \mu, -\mu', \nu, -\nu') \in \Lambda. \tag{5-3}$$

Similarly we define Λ'_{rb} by negating the fiber coordinates:

$$q' = (y', \mu', \nu') \in \Lambda'_{\mathrm{rb}} \iff q = (y', -\mu', -\nu') \in \Lambda_{\mathrm{rb}}.$$

We also define $\overline{\Lambda}'$, $\overline{\Lambda}_{\mathrm{lb}}$, $\overline{\Lambda}'_{\mathrm{rb}}$ by

$$\overline{\Lambda}' = \Lambda' \times [0, \lambda_0], \quad \overline{\Lambda}_{\mathrm{lb}} = \Lambda'_{\mathrm{lb}} \times [0, \lambda_0], \quad \overline{\Lambda}'_{\mathrm{rb}} = \Lambda'_{\mathrm{rb}} \times [0, \lambda_0]. \tag{5-4}$$

To define the microlocal support, $\mathrm{WF}'(F)$, of F we first recall from [Guillarmou et al. 2012] that $F \in I^{m,r_{\mathrm{lb}},r_{\mathrm{rb}},\mathcal{B}}(M^2_{k,b}, \Lambda; \Omega_{k,b}^{1/2})$ means F can be decomposed as $F = F_1 + F_2 + F_3 + F_4 + F_5 + F_6$, where

- F_1 is supported near bf and away from lb, rb;
- F_2 is supported near $\mathrm{bf} \cap \mathrm{lb}$;
- F_3 is supported near $\mathrm{bf} \cap \mathrm{rb}$;
- F_4 is supported near lb and away from bf;
- F_5 is supported near rb and away from bf;
- F_6 vanishes rapidly at bf, lb, rb and is polyhomogeneous on $M^2_{k,b}$ with index family $\mathcal{B}$;

and each F_i, $1 \le i \le 5$ has an oscillatory representation as follows:

- F_1 is a finite sum of terms of the form (up to rapidly vanishing terms which may be included in F_6)

$$\rho^{m-k/2+n/2} \int_{\mathbb{R}^k} e^{i\Phi(y,y',x/x',v)/\rho} a(\lambda, \rho, y, y', \sigma, v)\, dv\, \boldsymbol{\omega}, \tag{5-5}$$

where Φ locally parametrizes Λ, $\boldsymbol{\omega}$ is a nonzero section of the half-density bundle $\Omega_{k,b}^{1/2}$, compactly supported in v, and

$$a \text{ is polyhomogeneous conormal in } \lambda \text{ with index set } \mathcal{B}_{\mathrm{bf}_0} \text{ and smooth in all other variables.} \tag{5-6}$$

- F_2 is a finite sum of terms of the form (up to rapidly vanishing terms which may be included in F_6)

$$\sigma^{r_{\mathrm{lb}}-k/2} \rho'^{m-(k+k')/2+n/2} \int_{\mathbb{R}^{k+k'}} e^{i\Phi_1(y,v)/\rho} e^{i\Phi_2(y,y',\sigma,v,w)/\rho'} a(\lambda, \rho', y, y', \sigma, v, w)\, dv\, dw\, \boldsymbol{\omega}, \tag{5-7}$$

where $\Phi = \Phi_1 + \sigma\Phi_2$ locally parametrizes Λ (in particular, Φ_1 locally parametrizes Λ_{lb}), and a satisfies (5-6).

- F_3 is a finite sum of terms of the form (up to rapidly vanishing terms which may be included in F_6)

$$\rho^{m-(k+k')/2+n/2} \tilde{\sigma}^{r_{\mathrm{rb}}-k/2} \int_{\mathbb{R}^{k+k'}} e^{i\Phi'_1(y',v)/\rho'} e^{i\Phi'_2(y,y',\tilde{\sigma},v,w)/\rho} a(\lambda, \rho, y, y', \tilde{\sigma}, v, w)\, dv\, dw\, \boldsymbol{\omega}, \tag{5-8}$$

where $\tilde{\sigma} = \rho'/\rho = \sigma^{-1}$ and $\Phi = \Phi'_1 + \tilde{\sigma}\Phi'_2$ locally parametrizes Λ (in particular, Φ'_1 locally parametrizes Λ_{rb}), and a satisfies (5-6).

- F_4 is a finite sum of terms of the form

$$\rho^{r_{\mathrm{lb}}-k/2} \int_{\mathbb{R}^k} e^{i\Phi_1(y,v)/\rho} a(\lambda, \rho, y, z', v)\, dv\, \boldsymbol{\omega}, \tag{5-9}$$

where Φ parametrizes Λ_{lb} and a is polyhomogeneous at bf_0 and lb_0 with index sets $\mathcal{B}_{\mathrm{bf}_0}, \mathcal{B}_{\mathrm{lb}_0}$.

- F_5 is a finite sum of terms

$$(\rho')^{r_{\mathrm{rb}}-k/2} \int_{\mathbb{R}^k} e^{i\Phi'_1(y',v')/\rho'} a(\lambda, \rho', y', z, v)\, dv\, \boldsymbol{\omega}, \tag{5-10}$$

where Φ' parametrizes Λ_{rb} and a is polyhomogeneous at bf_0 and rb_0 with index sets $\mathcal{B}_{\mathrm{bf}_0}, \mathcal{B}_{\mathrm{rb}_0}$.

Then we define the microlocal support $\mathrm{WF}'(F)$ of F to be a closed subset of $\overline{\Lambda}' \cup \overline{\Lambda}_{\mathrm{lb}} \cup \overline{\Lambda}'_{\mathrm{rb}}$ as follows: We say that $(q', \lambda) \in \overline{\Lambda}'$ is not in $\mathrm{WF}'(F)$ if there is a neighborhood of $(q, \lambda) \in \Lambda \times [0, \lambda_0]$ in which F has order ∞. In terms of the oscillatory integral representation (5-5), say, the condition that F has order infinity at (q, λ) is equivalent to a vanishing rapidly in a neighborhood of the point $(\lambda, 0, y, y', \sigma, v)$ which corresponds under (5-3) to (q, λ) in the sense that $d_{y,y',\sigma,\rho}(\Phi(y, y', x/x', v)/\rho) = q$ and $d_v\Phi(y, y', x/x', v) = 0$ (by nondegeneracy there is only one v with this property). Similar considerations apply to (5-7) and (5-8). Likewise, we say that $(q, \lambda) \in \overline{\Lambda}_{\mathrm{lb}}$ is not in $\mathrm{WF}'(F)$ if there is a neighborhood of the *fiber* (see (4-4)) of $(q, \lambda) \in \Lambda_{\mathrm{lb}} \times [0, \lambda_0]$ in which F has order ∞, and $(q', \lambda) \in \overline{\Lambda}'_{\mathrm{rb}}$ is not in $\mathrm{WF}'(F)$ if there is a neighborhood of the *fiber* of $(q, \lambda) \in \Lambda_{\mathrm{rb}} \times [0, \lambda_0]$ in which F has order ∞. The fiber here is a copy of M. In terms of the oscillatory integral representation (5-7), the condition that F has order infinity in a neighborhood of the fiber of $(q, \lambda) = (y, \mu, \nu, \lambda) \in \overline{\Lambda}_{\mathrm{lb}}$ is equivalent to a vanishing rapidly in a neighborhood of the point $(\lambda, \rho', y, y', 0, v, w)$ for all (ρ', y', v, w) such that $d_{y,\rho}(\Phi_1/\rho) = q$ and $d_v \Phi_1 = 0$. Similarly, in (5-9) the condition is that a vanishes rapidly in a neighborhood of the point $(\lambda, 0, y, z', v)$ for all (z', v) such that $d_{y,\rho}(\Phi_1/\rho) = q$ and $d_v \Phi_1 = 0$.

These components of $\mathrm{WF}'(F)$ will be denoted by $\mathrm{WF}'_{\mathrm{bf}}(F)$, $\mathrm{WF}'_{\mathrm{lb}}(F)$ and $\mathrm{WF}'_{\mathrm{rb}}(F)$, respectively.

Note that if $F \in I^{m,r_{\mathrm{lb}},r_{\mathrm{rb}},\mathcal{B}}(\Lambda)$, then F is rapidly decreasing at bf, lb and rb if and only if $\mathrm{WF}'(F)$ is empty. Also note that if $\mathrm{WF}'_{\mathrm{lb}}(F)$ is empty, then $\partial_{\mathrm{lb}}\Lambda \times [0, \lambda_0]$ is disjoint from $\mathrm{WF}'_{\mathrm{bf}}(F)$, but the converse need not hold: if the kernel of F is supported away from bf then certainly $\mathrm{WF}'_{\mathrm{bf}}(F)$ will be empty, but $\mathrm{WF}'_{\mathrm{lb}}(F)$ need not be.

This definition makes sense also for pseudodifferential operators Q of order $-\infty$, with compact operator wavefront set. In the case of a pseudodifferential operator, the Legendre submanifold is ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$, defined in (4-7), and the components $\Lambda_{\mathrm{lb}} \cup \Lambda'_{\mathrm{rb}}$ are empty. Since ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$ is canonically diffeomorphic to ${}^{\mathrm{sc}}T^*{}_{\partial M}M$, we will always consider the microlocal support $\mathrm{WF}'(Q)$ of a pseudodifferential operator Q of differential order $-\infty$ to be a subset of ${}^{\mathrm{sc}}T^*{}_{\partial M}M \times [0, \lambda_0]$.

Lemma 5.2. *Assume that $F \in I^{m,r_{\mathrm{lb}},r_{\mathrm{rb}};\mathcal{B}}(M^2_{k,b}, \Lambda; \Omega^{1/2}_{k,b})$ is associated to a compact Legendre submanifold Λ and that $Q \in \Psi_k^{-\infty}(M; \Omega^{1/2}_{k,b})$ is of differential order $-\infty$, with compact operator wavefront set. Then QF is also a Legendre distribution in the space $I^{m,r_{\mathrm{lb}},r_{\mathrm{rb}};\mathcal{B}}(M^2_{k,b}, \Lambda; \Omega^{1/2}_{k,b})$ and we have*

$$\begin{aligned} \mathrm{WF}'_{\mathrm{lb}}(QF) &\subset \mathrm{WF}'(Q) \cap \mathrm{WF}'_{\mathrm{lb}}(F), \\ \mathrm{WF}'_{\mathrm{bf}}(QF) &\subset \pi_L^{-1}\,\mathrm{WF}'(Q) \cap \mathrm{WF}'_{\mathrm{bf}}(F), \\ \mathrm{WF}'_{\mathrm{rb}}(QF) &\subset \mathrm{WF}'_{\mathrm{rb}}(F), \end{aligned} \tag{5-11}$$

where π_L, π_R are as in (4-6). *Moreover, if Q is microlocally equal to the identity on $\pi_L(\mathrm{WF}'_{\mathrm{bf}}(F))$ and $\mathrm{WF}'_{\mathrm{lb}}(F)$, then $QF - F \in I^{\infty,\infty,r_{\mathrm{rb}};\mathcal{B}}(M^2_{k,b}, \Lambda; \Omega^{1/2}_{k,b})$, i.e., it vanishes to infinite order at* lb *and* bf.

There is of course a corresponding theorem for composition in the other order, which is obtained by taking the adjoint of the lemma above. Combining the two we obtain:

Corollary 5.3. *Suppose that F and Q, Q' are as above. Then*

$$\begin{aligned} \mathrm{WF}'_{\mathrm{lb}}(QFQ') &\subset \mathrm{WF}'(Q) \cap \mathrm{WF}'_{\mathrm{lb}}(F), \\ \mathrm{WF}'_{\mathrm{bf}}(QFQ') &\subset \pi_L^{-1}\,\mathrm{WF}'(Q) \cap \pi_R^{-1}\,\mathrm{WF}'(Q') \cap \mathrm{WF}'_{\mathrm{bf}}(F), \\ \mathrm{WF}'_{\mathrm{rb}}(QFQ') &\subset \mathrm{WF}'(Q') \cap \mathrm{WF}'_{\mathrm{rb}}(F). \end{aligned} \tag{5-12}$$

Proof of Lemma 5.2. We decompose as above $F = F_1 + F_2 + F_3 + F_4 + F_5 + F_6$, and consider each piece F_i separately.

• F_1 term. Using the notation in (5-5), the composition QF_1 takes the form

$$(2\pi)^{-n} \int_0^\infty \int e^{i((y-y'')\cdot\mu + (1-\rho/\rho'')\nu)/\rho} q(\lambda, \rho, y, \mu, \nu) \\ \times (\rho'')^{m-k/2+n/2} e^{i\Phi(y'',y',\rho'/\rho'',v)/\rho''} a(\lambda, \rho', y'', y', \rho'/\rho'', v)\, dv\, d\mu\, d\nu \, \frac{dy''\, d\rho''}{\rho''^{n+1}}\, \boldsymbol{\omega}. \tag{5-13}$$

Here the measure $\lambda^n dg''$, which arises from the combination of half-densities in Q and F, is equal to $dy'' d\rho''/\rho''^{n+1}$ times a smooth nonzero factor, which has been absorbed into the a term. Writing $\sigma'' = \rho/\rho''$, this can be expressed as

$$(2\pi)^{-n}\rho^{m-k/2-n+n/2}\int e^{i((y-y'')\cdot\mu+(1-\sigma'')\nu+\sigma''\Phi(y'',y',\sigma''/\sigma,v))/\rho} q(\lambda,\rho,y,\mu,\nu)(\sigma'')^{m-k/2+n/2-n-1} \times a(\lambda,\rho',y'',y',\sigma''\sigma^{-1},v)\,dv\,d\mu\,d\nu\,dy''\,d\sigma''\,\boldsymbol{\omega}.$$

For $\rho \geq \epsilon > 0$ the phase is not oscillating and this is polyhomogeneous conormal at bf_0 with the same index set $\mathcal{B}_{\mathrm{bf}_0}$ as for a. For ρ small, we perform stationary phase in the $(y'', \sigma'', \mu, \nu)$ variables. The phase has a nondegenerate stationary point where $y'' = y$, $\sigma'' = 1$, $\mu = d_y\Phi$, $\nu = \Phi + \sigma^{-1} d_\sigma \Phi$, and we obtain an asymptotic expansion as $\rho \to 0$ of the form

$$\rho^{m-k/2+n/2}\int_{\mathbb{R}^k} e^{i\Phi(y,y',\sigma,v)/\rho}\tilde{a}(\lambda,\rho,y,y',\sigma,v)\,dv\,\boldsymbol{\omega}, \tag{5-14}$$

where

$$\begin{aligned}\tilde{a}(\lambda,\rho,y,y',\sigma,v) &= \lambda^{-\frac{n}{2}}\sum_{j=0}^{M}\rho^j\left(\frac{(\partial_{y''}\cdot\partial_\mu+\partial_{\sigma''}\partial_\nu)^j}{i^j j!}q(\lambda,\rho,y,\mu,\nu)(\sigma'')^{m-\frac{k}{2}+\frac{n}{2}-n-1}a(\lambda,\rho',y'',y',\sigma''/\sigma,v)\right)g\Big|_{\substack{y=y'',\sigma''=1\\ \mu=d_y\Phi\\ \nu=\Phi+\sigma^{-1}d_\sigma\Phi}} \\ &\quad + O(\rho^{M+1}).\end{aligned} \tag{5-15}$$

In particular, this is a Legendre distribution associated to Λ of the same order, and with the same index family, as F. Moreover, we see from (5-14) and (5-15) that the microlocal support $\mathrm{WF}'_{\mathrm{bf}}(QF_1)$ is contained in $\mathrm{WF}'_{\mathrm{bf}}(F)$, as well as contained in $\pi_L^{-1}\,\mathrm{WF}'(Q)$.

If $q = 1 + O(\rho^\infty)$ on $\pi_L(\mathrm{WF}'_{\mathrm{bf}}(F))$, then in the sum over j in (5-15), only the $j = 0$ term is nonzero, because in all other terms, either $a = 0$ or $q = 1 + O(\rho^\infty)$ (implying that any derivative of q is $O(\rho^\infty)$) when evaluated at $y = y''$, $\sigma'' = 1$, $\mu = d_y\Phi$, $\nu = \Phi + \sigma d_\sigma \Phi$. Therefore, in this case, $QF_1 = F_1 \bmod O(\rho^\infty)$.

• F_2 term. In the notation (5-7), the composition QF_2 takes the form

$$(2\pi)^{-n}\int e^{i((y-y'')\cdot\mu+(1-\sigma'')\nu)/\rho}q(\lambda,\rho,y,\mu,\nu)\rho''^{r_{\mathrm{lb}}-k/2}\rho'^{m-r_{\mathrm{lb}}-k'/2+n/2}e^{i\Phi_1(y,v)/\rho''}e^{i\Phi_2(y'',y',\sigma''/\sigma,v,w)/\rho'} \times a(\lambda,\rho',y'',y',\sigma/\sigma'',v,w)\,dv\,dw\,d\mu\,d\nu\,\frac{dy''\,d\rho''}{\rho''^{n+1}}\,\boldsymbol{\omega}.$$

This can be written as

$$\begin{aligned}&(2\pi)^{-n}\rho^{r_{\mathrm{lb}}-k/2-n}\rho'^{m-r_{\mathrm{lb}}-k'/2+n/2} \\ &\quad\times\int e^{i((y-y'')\cdot\mu+(1-\sigma'')\nu+\sigma''\Phi_1(y'',v)+\sigma\Phi_2(y'',y',\sigma/\sigma'',v,w))/\rho} \\ &\qquad\times q(\lambda,\rho,y,\mu,\nu)(\sigma'')^{-r_{\mathrm{lb}}+k/2+n-1}a(\lambda,\rho',y'',y',\sigma/\sigma'',v,w)\,dv\,dw\,d\mu\,d\nu\,dy''\,d\sigma''\,\boldsymbol{\omega}.\end{aligned}$$

Now we perform stationary phase in the $(y'', \sigma'', \mu, \nu)$-variables. The phase has a nondegenerate stationary point where $y'' = y$, $\sigma'' = 1$, $\mu = d_y\Phi_1$, $\nu = \Phi_1 - d_\sigma\Phi$, and the rest of the argument to bound $\mathrm{WF}'_{\mathrm{bf}}(QF)$

is the same as for F_1. We also see from the stationary phase expansion that $\mathrm{WF}'_{\mathrm{lb}}(QF)$ is contained in both $\mathrm{WF}'(Q)$ and $\mathrm{WF}'_{\mathrm{lb}}(F)$.

• F_4 term. This works just as for the F_2 term.

• F_3 term. In the notation (5-8), the composition QF_3 takes the form

$$(2\pi)^{-n}\int e^{i((y-y'')\cdot\mu+(1-\sigma'')\nu)/\rho}q(\lambda,\rho,y,\mu,\nu)(\rho'')^{m-(k+k')/2+2n/4}(\tilde\sigma\sigma'')^{r_{\mathrm{rb}}-k/2} \times\int e^{i\Phi_1'(y',v)/\rho'}e^{i\Phi_2'(y',y'',\tilde\sigma\sigma'',v,w)/\rho''}a(\lambda,\rho'',y'',y',\tilde\sigma\sigma'',v,w)\,dv\,dw\,d\mu\,d\nu\,\frac{dy''\,d\rho''}{(\rho'')^{n+1}}\,\boldsymbol{\omega}.$$

This can be written as

$$(2\pi)^{-n}\int e^{i((y-y'')\cdot\mu+(1-\sigma'')\nu+\sigma''\Phi_2'(y',y'',\tilde\sigma\sigma'',v,w))/\rho}q(\lambda,\rho,y,\mu,\nu)(\rho/\sigma'')^{m-(k+k')/2} \times(\tilde\sigma\sigma'')^{r_{\mathrm{rb}}-k/2}e^{i\Phi_1'(y',v)/\rho'}a(\lambda,\rho/\sigma'',y'',y',\tilde\sigma\sigma'',v,w)\,dv\,dw\,d\mu\,d\nu\,\frac{dy''\,d\sigma''}{\sigma''}\,\boldsymbol{\omega}.$$

To investigate the behavior of this integral locally near a point $(x=0,\tilde\sigma=0,y,y')\in\mathrm{bf}\cap\mathrm{rb}$, we perform stationary phase in the (y'',σ'',μ,ν)-variables. The phase has a nondegenerate stationary point where $y''=y$, $\sigma''=1$, $\mu=d_y\Phi_2'$, $\nu=\Phi_2'+\tilde\sigma d_{\tilde\sigma}\Phi_2'$, and we get an asymptotic expansion as $\rho\to 0$ of the form

$$\rho^{m-(k+k')/2+2n/4}\tilde\sigma^{r_{\mathrm{rb}}-k/2}\int e^{i\Phi_1'(y',v)/\rho'}e^{i\Phi_2'(y,y',\tilde\sigma,v,w)/\rho}\tilde a(\lambda,\rho,y,y',\tilde\sigma,v,w)\,dv\,dw\,\boldsymbol{\omega},$$

where $\tilde a(\lambda,\rho,y,y',\tilde\sigma,v,w)$ is given by

$$\sum_{j=0}^{M}\rho^j\biggl(\frac{(-i(\partial_{y''}\cdot\partial_\mu+\partial_{\sigma''}\partial_\nu))^j}{j!}q(\lambda,\rho,y,\mu,\nu) \times(\sigma'')^{-m+r_{\mathrm{rb}}+k'/2}a(\lambda,\rho'',y'',y',\tilde\sigma\sigma'',v,w)\biggr)g\bigg|_{\substack{y=y'',\sigma''=1\\ \mu=d_y\Phi_1'\\ \nu=\Phi_2'+\tilde\sigma d_{\tilde\sigma}\Phi_2'}}+O(\rho^{M+1}). \tag{5-16}$$

This is a Legendre distribution associated to Λ of the same order as F, and with the same index family. Moreover, we see from the last two formulas that the microlocal support $\mathrm{WF}'_{\mathrm{bf}}(QF_3)$ is contained in $\mathrm{WF}'_{\mathrm{bf}}(F)$, as well as contained in $\pi_L^{-1}\,\mathrm{WF}'(Q)$. Finally, if $q=1+O(\rho^\infty)$ on $\pi_L(\mathrm{WF}'_{\mathrm{bf}}(F))$, then in the sum over j in (5-16), only the $j=0$ term is nonzero, because in all other terms, either $a=0$ or $q=1+O(\rho^\infty)$ (implying that any derivative of q is $O(\rho^\infty)$) when evaluated at $y=y''$, $\sigma''=1$, $\mu=d_y\Phi_2'$, $\nu=\Phi_2'+\sigma d_\sigma\Phi_2'$. Therefore, in this case, $QF_3=F_3 \bmod O(x^\infty)$.

• F_5 term. Writing F_5 in the form (5-10), we investigate QF_5 near a point (z,ρ',y'), where $z\in M^\circ$. In this case, we can find a neighborhood W of z with $\overline{W}\subset M^\circ$, and then the set

$$\{(z,z')\in\operatorname{supp}Q\mid z\in W\}$$

is contained in $W\times W'$ for some W' with $\overline{W'}\subset M^\circ$, since the support of Q is contained in the set where $\sigma\in[1/2,2]$. But in $W\times W'$, the kernel of Q is smooth since Q has differential order $-\infty$. Therefore, in this region the composition is given by an integral

$$\int Q(z, z'')(\rho')^{r_{\mathrm{rb}}-k/2} \int e^{i\Phi_1(y',v)/\rho'} a(\lambda, z'', y', \rho', v)\, dv\, dz''\, \boldsymbol{\omega},$$

with $Q(z, z'')$ smooth, and this has the form

$$(\rho')^{r_{\mathrm{rb}}-k/2} \int e^{i\Phi_1(y',v)/\rho'} \tilde{a}(\lambda, z, y', \rho', v)\, dv\, \boldsymbol{\omega}$$

for some $\tilde{a}$ depending polyhomogeneously on λ and smoothly in its other arguments. Moreover, if for a fixed (λ, y', v), a is $O((\rho')^\infty)$ in a neighborhood of $\{(\lambda, z, y', 0, v) \mid z \in M\}$, then the same is true of $\tilde{a}$. Therefore, $\mathrm{WF}'_{\mathrm{rb}}(QF_5)$ is contained in $\mathrm{WF}'_{\mathrm{rb}}(F_5)$ but is (in general) no smaller.

- Since $\mathrm{WF}'(F_6) = \mathrm{WF}'(QF_6) = \varnothing$, the F_6 term makes no contribution to the wavefront set.

This completes the proof. □

A similar result holds if F is associated to a Legendre conic pair rather than a single Legendre submanifold. However, rather than giving a full analogue of the result above, we give the following special cases which suffice for our needs.

Lemma 5.4. (i) *Suppose that* $F \in I^{m,p;r_{\mathrm{lb}},r_{\mathrm{rb}};\mathcal{B}}(M^2_{k,b}, (\Lambda, \Lambda^\sharp); \Omega^{1/2}_{k,b})$ *is a Legendre distribution on* $M^2_{k,b}$ *associated to a conic Legendrian pair* $(\Lambda, \Lambda^\sharp)$, *and suppose that* $Q \in \Psi_k^{-\infty}(M; \Omega^{1/2}_{k,b})$ *is a scattering pseudodifferential operator such that* Q *is microlocally equal to the identity operator near* $\pi_L(\Lambda \cup \Lambda^\sharp)$. *Then* $QF - F \in I^{\infty,\infty;\infty,r_{\mathrm{rb}};\mathcal{B}}(M^2_{k,b}, (\Lambda, \Lambda^\sharp); \Omega^{1/2}_{k,b})$, *so it vanishes to infinite order at* lb *and* bf. *Similarly, if* Q *is microlocally equal to the identity operator near* $\pi_R(\Lambda \cup \Lambda^\sharp)$, *then* $FQ - F \in I^{\infty,\infty;r_{\mathrm{lb}},\infty;\mathcal{B}}(M^2_{k,b}, (\Lambda, \Lambda^\sharp); \Omega^{1/2}_{k,b})$ *vanishes to infinite order at* bf *and* rb.

(ii) *Suppose that* F *is as above, and that* Q, Q' *are scattering pseudodifferential operators as above. If*

$$\pi_L^{-1}\,\mathrm{WF}'(Q) \cap \pi_R^{-1}\,\mathrm{WF}'(Q') \cap \Lambda^\sharp = \varnothing, \tag{5-17}$$

then $QFQ' \in I^{m,r_{\mathrm{lb}},r_{\mathrm{rb}};\mathcal{B}}(M^2_{k,b}, \Lambda; \Omega^{1/2}_{k,b})$; *in particular*, $\mathrm{WF}'_{\mathrm{bf}}(QFQ')$ *is disjoint from* $(\Lambda^\sharp)'$.

Proof. The proof of (i) is similar to the one above. To prove (ii), decompose $F = F_\Lambda + F_\sharp$, where $F_\Lambda \in I^{m,r}(M^2_{k,b}, \Lambda; \Omega^{1/2}_{k,b})$ is a Legendre distribution associated only to Λ and $F_\sharp$ is localized sufficiently close to $\Lambda^\sharp$. Here, sufficiently close means that when we write down $QF_\sharp Q'$ as a (sum of) integral(s), using a phase function that locally parametrizes of $(\Lambda, \Lambda^\sharp)$, then (5-17) implies that the total phase is nonstationary on the support of the integrand. The usual integration-by-parts argument then shows that this kernel is rapidly decreasing at bf, lb, rb and hence trivially satisfies the conclusion of the lemma. On the other hand, Lemma 5.2 applies to F_Λ and completes the proof. □

6. Low energy estimates on the spectral measure

6A. ***Pointwise bounds on Legendre distributions.*** Now we give a pointwise estimate on Legendre distributions of a particular type. We begin with a trivial estimate.

Proposition 6.1. *Let $\Lambda \subset {}^{sc}T^*_{\mathrm{bf}}(M_b^2)$ be a Legendre submanifold that projects diffeomorphically to* bf. *Suppose that $u \in I^{-n/2-\alpha,-\alpha,-\alpha;\mathcal{B}}(M^2_{k,b}, \Lambda; \Omega^{1/2}_{k,b})$. Let*

$$b = \min(\min \mathcal{B}_{\mathrm{bf}_0} + n,\ \min \mathcal{B}_{\mathrm{lb}_0} + n/2,\ \min \mathcal{B}_{\mathrm{rb}_0} + n/2,\ \min \mathcal{B}_{\mathrm{zf}}). \tag{6-1}$$

Then, as a multiple of the half-density $|dg\, dg'\, d\lambda/\lambda|^{1/2}$, we have a pointwise estimate

$$|u| \le C\lambda^b(\rho^{-1} + (\rho')^{-1})^\alpha.$$

This is trivial since in this case, u may be written as an oscillatory function with no integration, and the order of vanishing/growth at the boundary may be determined by inspection from (5-5)–(5-10). (The discrepancies of n and $n/2$ in (6-1) come about from comparing the nonvanishing half-density $\boldsymbol{\omega}$ on $M^2_{k,b}$ with the metric half-density $|dg\, dg'\, d\lambda/\lambda|^{1/2} = \rho_{\mathrm{lb}_0}^{-n/2}\rho_{\mathrm{rb}_0}^{-n/2}\rho_{\mathrm{bf}_0}^{-n}\,\boldsymbol{\omega}$.)

Now consider a situation in which the Legendre submanifold does not project diffeomorphically to bf. Let $\partial\mathrm{diag}_b$ denote the boundary of the diagonal in M_b^2, as in (4-8). Recall that we have coordinates (y, y', σ) on bf near $\partial\mathrm{diag}_b$. Let $w = (y - y', \sigma - 1)$, and let κ be the corresponding scattering coordinates dual to w. Then $\partial\mathrm{diag}_b$ is given by $\{w = 0\}$ as a submanifold of bf and the contact form on ${}^{sc}T^*_{\mathrm{bf}}M_b^2$ takes the form

$$d\nu - \mu \cdot dy - \kappa \cdot dw. \tag{6-2}$$

In these coordinates, the Legendre submanifold ${}^{sc}N^*\partial\mathrm{diag}_b$ is given by $\{w = 0, \mu = 0, \nu = 0\}$. Let Λ^{bf} be a Legendre submanifold contained in ${}^{sc}T^*_{\mathrm{bf}}M_b^2$, denote by π the natural projection from ${}^{sc}T^*_{\mathrm{bf}}M_b^2 \to \mathrm{bf}$, and for any $q \in \Lambda^{\mathrm{bf}}$ denote by $d\pi$ the induced map from $T_q\Lambda^{\mathrm{bf}} \to T_{\pi(q)}\mathrm{bf}$. We consider the following situation in which the rank of $d\pi$ is allowed to change.

Proposition 6.2. *Let Λ^{bf} be as above. Suppose that Λ^{bf} intersects ${}^{sc}N^*\partial\mathrm{diag}_b$ at $G^{\mathrm{bf}} = \Lambda^{\mathrm{bf}} \cap {}^{sc}N^*\partial\mathrm{diag}_b$ which is of codimension* 1 *in Λ^{bf}, and suppose that $\pi|_{G^{\mathrm{bf}}}$ is a fibration, with $(n-1)$-dimensional fibers, to $\partial\mathrm{diag}_b$. Assume further that $d\pi$ has full rank on $\Lambda^{\mathrm{bf}} \setminus G^{\mathrm{bf}}$, while*

$$\det d\pi \text{ vanishes to order exactly } n-1 \text{ at } G^{\mathrm{bf}}. \tag{6-3}$$

Suppose $u \in I^{-n/2-\alpha,-\alpha,-\alpha;\mathcal{B}}(M^2_{k,b}, \Lambda^{\mathrm{bf}}; \Omega^{1/2}_{k,b})$, and suppose that the (full) symbol of u vanishes to order $(n-1)/2 + \alpha$ on $G^{\mathrm{bf}} \times [0, \lambda_0]$, where $(n-1)/2 + \alpha \in \{0, 1, 2, \dots\}$. Then as a multiple of the scattering half-density $|dg\, dg'\, d\lambda/\lambda|^{1/2}$, we have a pointwise estimate

$$|u| \le C\lambda^b\Big(1 + \frac{|w|}{\rho}\Big)^\alpha \sim C\lambda^b(1 + \lambda d(z, z'))^\alpha, \tag{6-4}$$

with b as in (6-1). *Here $d(z, z')$ is the Riemannian distance between $z, z' \in M^\circ$.*

Remark 6.3. Notice that the condition on π at G^{bf} implies that $d\pi$ has corank at least $n-1$ on G^{bf}, hence that $\det d\pi$ must vanish to order at *least* $n-1$ there. Condition (6-3) is therefore that the order of vanishing at G^{bf} is the least possible, which is a nondegeneracy assumption concerning the manner in which the rank of the projection changes at G^{bf}. It implies, in particular, that Λ^{bf} intersects ${}^{sc}N^*\partial\mathrm{diag}_b$ cleanly.

Proof. Let q be an arbitrary point in G^{bf}. By rotating in the w variables, we can ensure that $d\kappa_1|_{G^{\mathrm{bf}}}$ vanishes at q (since $\kappa_1, \dots, \kappa_n$ are coordinates on the fibers of ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b \to \partial\mathrm{diag}_b$, and since $\pi|_{G^{\mathrm{bf}}} : G^{\mathrm{bf}} \to \partial\mathrm{diag}_b$ has $(n-1)$-dimensional fibers). We claim that $(y, w_1, \kappa_2, \dots, \kappa_n)$ furnish coordinates on Λ^{bf} locally near q. To see this, first note that $d\kappa_2|_{G^{\mathrm{bf}}}, \dots, d\kappa_n|_{G^{\mathrm{bf}}}$ are linearly independent at q, and furnish coordinates on the fibers of $G^{\mathrm{bf}} \to \partial\mathrm{diag}_b$. Next, since $\partial\mathrm{diag}_b$ is $(n-1)$-dimensional, G^{bf} is $2(n-1)$-dimensional, and the fibers of $G^{\mathrm{bf}} \to \partial\mathrm{diag}_b$ are $(n-1)$-dimensional, it follows that $G^{\mathrm{bf}} \to \partial\mathrm{diag}_b$ is a submersion. Since y_i are local coordinates on the base $\partial\mathrm{diag}_b$, we see that $(y, \kappa_2, \dots, \kappa_n)$ furnish coordinates on G^{bf} locally near q. Since $w_1 = 0$ on G^{bf}, to prove the claim it suffices to show that $dw_1|_{\Lambda^{\mathrm{bf}}} \neq 0$ at q.

To see this, we use (6-3) which implies that $d\pi$ has corank exactly $n-1$ at q, and hence there is a tangent vector $V \in T_q\Lambda^{\mathrm{bf}}$ such that $d\pi(V)$ is not tangent to $\partial\mathrm{diag}_b$. Therefore, it has a nonzero ∂_{w_j} component, which means that some dw_j does not vanish at q when restricted to Λ^{bf}. But since Λ^{bf} is Legendrian, the form (6-2) vanishes when restricted to Λ^{bf}, which implies that its differential $\omega \equiv d\mu \cdot dy + d\kappa \cdot dw$ also vanishes on Λ^{bf}. Hence $\omega(\partial_{\kappa_j}, V) = 0$ at q, $j \geq 2$, since ∂_{κ_j} and V are both tangent to Λ^{bf}. But this implies that $dw_j(V) = 0$ for $j \geq 2$, i.e., V has no ∂_{w_j} component for $j \geq 2$. It follows that $dw_1(V) \neq 0$, showing that $dw_1|_{\Lambda^{\mathrm{bf}}} \neq 0$ at q. It follows that $(y, w_1, \kappa_2, \dots, \kappa_n)$ indeed furnish coordinates on Λ^{bf} locally near q. We will use the notation $\overline{w} = (w_2, \dots, w_n)$ and $\bar{\kappa} = (\kappa_2, \dots, \kappa_n)$. Notice that $w_1|_{\Lambda^{\mathrm{bf}}}$ is a boundary defining function for G^{bf}, as a submanifold of Λ^{bf}, locally near q.

Now we write the other coordinates on Λ^{bf} as functions of $(y, w_1, \bar{\kappa})$ as follows:

$$\overline{w}_i = W_i(y, w_1, \bar{\kappa}), \quad \mu_i = M_i(y, w_1, \bar{\kappa}), \quad \kappa_1 = K(y, w_1, \bar{\kappa}), \quad \nu = N(y, w_1, \bar{\kappa}) \quad \text{on } \Lambda^{\mathrm{bf}}. \tag{6-5}$$

Notice that the vanishing of (6-2) on Λ^{bf} implies that

$$dN = \sum_{i=1}^{n-1} M_i dy_i + K dw_1 + \sum_{j=2}^{n} \kappa_j dW_j \quad \text{on } \Lambda^{\mathrm{bf}}. \tag{6-6}$$

By equating the coefficients of $d\bar{\kappa}$, dy and dw_1 on each side of (6-6), we obtain the identities

$$\begin{aligned} \sum_{j=2}^{n} v_j \frac{\partial W_j(y, w_1, v)}{\partial v_i} &= \frac{\partial N(y, w_1, v)}{\partial v_i}, \quad i = 2, \dots, n, \\ \sum_{j=2}^{n} v_j \frac{\partial W_j(y, w_1, v)}{\partial y_i} + M_i(y, w_1, v) &= \frac{\partial N(y, w_1, v)}{\partial y_i}, \quad i = 1, \dots, n-1, \\ \sum_{j=2}^{n} v_j \frac{\partial W_j(y, w_1, v)}{\partial w_1} + K(y, w_1, v) &= \frac{\partial N(y, w_1, v)}{\partial w_1}. \end{aligned} \tag{6-7}$$

We claim that the function

$$\Phi(y, w_1, \overline{w}, v) = \sum_{j=2}^{n} (\overline{w}_j - W_j(y, w_1, v)) v_j + N(y, w_1, v) \tag{6-8}$$

parametrizes Λ^{bf} locally near q. Notice that W, M and N are all $O(w_1)$ at q. Hence, $\Phi = \overline{w} \cdot v + O(w_1)$, so the $d_{v_j}\Phi = \overline{w}_j + O(w_1)$, where $2 \leq j \leq n$, have linearly independent differentials at the point

$\tilde{q} = (y(q), w = 0, \nu = 0, \mu = 0, \kappa_1 = 0, \bar{\kappa}(q))$ corresponding to q, i.e., Φ is a nondegenerate parametrization of Λ^{bf} near q. Next, using the first equation in (6-7) we find that

$$d_{v_j}\Phi = \overline{w}_j - W_j(y, w_1, v). \tag{6-9}$$

So $\overline{w} = W$ when $d_v\Phi = 0$. The Legendrian submanifold parametrized is then given by (using (6-7))

$$\left\{\left(y, w_1, W, -v\cdot\frac{\partial W}{\partial y} + \frac{\partial N}{\partial y}, -v\cdot\frac{\partial W}{\partial w_1} + \frac{\partial N}{\partial w_1}, v, N\right)\right\} = \{(y, w_1, W, M, K, v, N)\} = \Lambda^{\mathrm{bf}}. \tag{6-10}$$

Notice that the second derivative matrix $d^2_{vv}\Phi$ vanishes at $w_1 = 0$, so we can write $d^2_{vv}\Phi = w_1 A + O(w_1^2)$, where A is a smooth $(n-1)\times(n-1)$ matrix function of $(\bar{y}, v)$, where we write $\bar{y} = (y, w_1, \overline{w})$. We claim that A is invertible at (and therefore, near) $\tilde{q}$. To see this, we start from the fact that the map

$$\{(\bar{y}, v)\} \to \{(\bar{y}, d_{\bar{y}}\Phi, \Phi, d_v\Phi)\}$$

is locally a diffeomorphism onto its image. (This follows from the nondegeneracy condition on Φ, that the differentials $d(\partial\Phi/\partial v_j)$ are linearly independent.) Note that the determinant of the differential of the map

$$\{(\bar{y}, d_{\bar{y}}\Phi, \Phi, d_v\Phi)\} \to \{(\bar{y}, d_v\Phi)\}$$

is equal to the determinant of the differential of the map

$$\{(\bar{y}, d_{\bar{y}}\Phi, \Phi, d_v\Phi) \mid d_v\Phi = 0\} \to \bar{y},$$

and this map is $\pi|_{\Lambda^{\mathrm{bf}}}$ (in local coordinates). It follows that the order of vanishing of $\det d\pi$ at q is the same as the order of vanishing of the determinant of the differential of the map

$$\{(\bar{y}, v)\} \to \{(\bar{y}, d_v\Phi)\}$$

at $\tilde{q}$. But this determinant is simply $\det d^2_{vv}\Phi$. It follows from (6-3) that $\det d^2_{vv}\Phi$ vanishes to order exactly $n-1$ at $\tilde{q}$. But this implies that the matrix A is invertible at $\tilde{q}$, as claimed.

Now we write u as an oscillatory integral. It suffices to prove the proposition assuming that u has symbol supported close to q and that u itself is supported close to $\partial\mathrm{diag}_b$, since away from $\partial\mathrm{diag}_b$ the result follows from Proposition 6.1. It can then be written with respect to the phase function Φ: modulo a smooth term vanishing to order $O(\rho^\infty)$, u is a multiple of the scattering half-density $|dg\, dg'\, d\lambda/\lambda|^{1/2}$ given by

$$\rho^{-(n-1)/2-\alpha}\lambda^n \int e^{i\Phi(y,w,v)/\rho} a(\lambda, \rho, y, v)\, dv |dg\, dg'\, d\lambda/\lambda|^{1/2}. \tag{6-11}$$

Moreover, we may assume that a is a function only of λ, ρ, y, w_1 and v, polyhomogeneous conormal in λ with index set $\mathcal{B}_{\mathrm{bf}_0}$, smooth and compactly supported in the remaining variables, and vanishing to order $(n-1)/2+\alpha$ at $\rho = w_1 = 0$. It can therefore be written as

$$a = \sum_{j=0}^{(n-1)/2+\alpha-1} \rho^j w_1^{(n-1)/2+\alpha-j} a_j(\lambda, y, w_1, v) + \rho^{(n-1)/2+\alpha} b(\lambda, \rho, y, w_1, v), \tag{6-12}$$

with a_j and b polyhomogeneous in λ.

We begin with the easy case $|w_1| \le \rho$. In this case, a in (6-12) is uniformly bounded. We split into the regions where $|w_1| \ge c|w|$ for some $c > 0$, and $|w_1| \le c|w|$. The first region, where $|w_1| \ge c|w|$, is trivial since then $|w|/\rho$ is bounded, so all we are required to show is that the integral (6-11) is bounded by a multiple of λ^b, $b = \min \mathcal{B}_{\mathrm{bf}_0} + n$, which is clear since the integrand has this property pointwise. On the other hand, if $|w_1| \le c|w|$, then $|w_1| \le (n-1)c|w_j|$ for some $j \ge 2$. For suitably small c this means that $d_{v_j}\Phi \ne 0$ sufficiently close to $\tilde{q}$, as $d_{v_j}\Phi = w_j + O(w_1)$ using (6-8). Then, by integrating by parts N times with respect to v_j in (6-11), we can gain a factor of $C_N(1+|w|/\rho)^{-N}$ for any N, showing that a much stronger estimate than (6-4) holds.

From now on, then, we will assume that $|w_1| \ge \rho$. We begin by estimating the a_0 term. The case $|w_1| \le c|w|$ is treated just as above: by integrating by parts N times with respect to v_j in (6-11) we gain a factor $C_N(|w|/\rho)^N$. With $N = M + (n-1)/2 + \alpha$ the resulting integrand enjoys a pointwise estimate $\lambda^b(|w|/\rho)^{-M}$ for any desired M. So we assume in the rest of the proof that $|w_1| \ge c|w|$, and therefore we can replace the RHS $(1+|w|/\rho)^\alpha$ in (6-4) by the equivalent quantity $(|w_1|/\rho)^\alpha$.

For fixed $w_1 \ne 0$, let us change variable from $v_1, \dots, v_{n-1}$ to $\theta_1, \dots, \theta_{n-1}$, where

$$\theta_i = w_1^{-1/2} d_{v_i}\Phi. \tag{6-13}$$

Then

$$\frac{\partial \theta_i}{\partial v_j} = w_1^{-1/2} d^2_{v_i v_j}\Phi = w_1^{1/2} A_{ij}, \tag{6-14}$$

where A_{ij} is nonsingular as we have noted above. Therefore,

$$\frac{\partial \Phi}{\partial \theta} = \Big(\frac{\partial \theta}{\partial v}\Big)^{-1} \frac{\partial \Phi}{\partial v} = A^{-1}\theta. \tag{6-15}$$

This shows that the θ coordinates are suitable coordinates in which to perform stationary phase computations. We proceed with a standard argument, which can be found in Sogge's book [1993], for example. We use the identity

$$e^{i\Phi/\rho} = \Big(\frac{\rho}{w_1^{1/2} i\theta_j}\frac{\partial}{\partial v_j}\Big) e^{i\Phi/\rho},$$

which can be written as

$$e^{i\Phi/\rho} = \Big(\sum_k \frac{\rho}{i\theta_j} A_{jk} \frac{\partial}{\partial \theta_k}\Big) e^{i\Phi/\rho}. \tag{6-16}$$

We also need the following observation: by applying (6-14) repeatedly, we obtain

$$\left|\frac{\partial^{|\alpha|} A}{\partial^\alpha \theta}\right| \le C|w_1|^{-|\alpha|/2} \le C\rho^{-|\alpha|/2}. \tag{6-17}$$

In the θ coordinates, we are trying to prove the estimate

$$\left|\rho^{-(n-1)/2-\alpha} \int_{\mathbb{R}^{n-1}} w_1^\alpha e^{i\Phi(y,w,\theta)/\rho} \tilde{a}_0(\lambda, \rho, y, w_1, \theta)\, d\theta\right| \le C\Big(\frac{w_1}{\rho}\Big)^\alpha \lambda^b. \tag{6-18}$$

Here the $w_1^{(n-1)/2}$ factor was absorbed as a Jacobian factor, and $\tilde{a}_0$ is again smooth. Clearly this is equivalent to a uniform bound on

$$\left|\rho^{-(n-1)/2}\lambda^{-b}\int_{\mathbb{R}^{n-1}} e^{i\Phi(y,w,\theta)/\rho}\tilde{a}_0(\lambda,\rho,y,w_1,\theta)\,d\theta\right|. \tag{6-19}$$

We introduce a partition of unity in (ρ,θ)-space, $1=\chi_0+\sum_{j=1}^{n-1}\chi_j$, where χ_0 is a compactly supported function of $\theta/\sqrt{\rho}$, and χ_j is supported where $|\theta|\geq\sqrt{\rho}$, and where $\theta_j\geq|\theta|/(n-1)$. We can do this with derivatives estimated by

$$|\nabla_\theta^{(k)}\chi_k|\leq C\rho^{-k/2}. \tag{6-20}$$

The integral with χ_0 inserted is trivial to estimate since it occurs on a set of measure $\rho^{(n-1)/2}$. With χ_j inserted, we use the identity (6-16) M times, for M a sufficiently large integer. Thus we consider

$$\rho^{-(n-1)/2}\int\chi_j\biggl(\sum_k\frac{\rho}{i\theta_j}A_{jk}(y,\theta)\frac{\partial}{\partial\theta_k}\biggr)^M e^{i\Phi(y,w,\theta)/\rho}\tilde{a}_0(\lambda,\rho,y,w_1,\theta)\,d\theta$$

and integrate by parts M times. The result can be estimated by

$$C\rho^{-(n-1)/2+M}\sum_{k=0}^{M}\rho^{-(M-k)/2}\int_{|\theta|\geq\sqrt{\rho}}1_{\operatorname{supp}\chi_j}\theta_j^{-M-k}\,d\theta, \tag{6-21}$$

where $M-k$ derivatives fall on the χ_j or A_{jk} terms (via (6-17) and (6-20)), and at most k fall on a θ_j^{-p} term. Note that on the support of χ_j, we can estimate $\theta_j^{-1}\leq c|\theta|^{-1}$. The θ integral is absolutely convergent for $M>n-1$, and

$$\int_{|\theta|\geq\sqrt{\rho}}|\theta|^{-M-k}\,d\theta=C_k\rho^{-(M+k)/2+(n-1)/2}$$

since $\dim\theta=n-1$. Substitution of this into (6-21) gives a uniform bound since $\tilde{a}$ is polyhomogeneous in λ with index set $\mathcal{B}_{\mathrm{bf}_0}+n$. Moreover, since Φ and $\tilde{a}$ are smooth in w_1, the bound is uniform as $w_1\to 0$.

To treat the terms a_i for $i>0$ and b in (6-12), we perform the same manipulations as above, and we end up with a uniform bound times $C\rho^i w_1^{-i}$, which is bounded for $\rho\leq w_1$. This completes the proof. □

6B. ***Geometry of L^{bf}.*** We collect here some facts concerning the geometry of the Legendre submanifold L^{bf} (see Section 4A). We begin by defining

$$\begin{aligned}G^{\mathrm{bf}}&=\{(y,y',\sigma,\mu,\mu',\nu,\nu')\in{}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b\mid \nu^2+h^{ij}\mu_i\mu_j=1\}\\&=\{(y,y,1,\mu,-\mu,\nu,-\nu)\mid\nu^2+h^{ij}\mu_i\mu_j=1\}.\end{aligned}$$

Clearly, G^{bf} is an S^{n-1}-bundle over $\partial\mathrm{diag}_b$.

Lemma 6.4. *The Legendre submanifold ${}^{\mathrm{sc}}N^*\partial diag_b$ intersects L^{bf} cleanly at G^{bf}, and the projection $\pi: L^{\mathrm{bf}}\to\mathrm{bf}$ satisfies* (6-3).

Proof. According to [Hassell and Vasy 2001], the Legendre submanifold L^{bf} is given by the flowout from G^{bf} by the vector field

$$V_l = -\nu\left(\sigma\frac{\partial}{\partial\sigma} + \mu\frac{\partial}{\partial\mu}\right) + h\frac{\partial}{\partial\nu} + \frac{\partial h}{\partial\mu_i}\frac{\partial}{\partial y_i} - \frac{\partial h}{\partial y_i}\frac{\partial}{\partial\mu_i}, \quad h = \sum_{i,j} h^{ij}(y)\mu_i\mu_j \tag{6-22}$$

(see [Guillarmou et al. 2012, Section 3.1]). Observe that at least one of the coefficients of ∂_σ or ∂_ν is nonvanishing, so either $\dot\sigma \neq 0$ or $\dot\nu + \dot\nu' \neq 0$ under the flowout by V_l. Since $\sigma = 1$ and $\nu + \nu' = 0$ at ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$, we see that V_l is everywhere transverse to ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$, so G^{bf} has codimension 1 in L^{bf}, and intersects L^{bf} cleanly.

It remains to show that the projection π from L^{bf} to bf satisfies (6-3). First we choose coordinates on L^{bf}. Near a point on L^{bf} at which $|\mu|_h^2 := h^{ij}\mu_i\mu_j < 1$, and therefore $\nu \neq 0$, we can choose coordinates (μ, y', ϵ), where ϵ is the flowout time from G^{bf} along the vector field V_l. Coordinates on the base are (y, y', σ). With the dot indicating derivative along the flow of V_l, i.e., $d/d\epsilon$, we have

$$\dot\sigma = -\nu \quad \text{and} \quad \dot y^i = 2h^{ij}\mu_j \quad \text{on } G^{\mathrm{bf}}.$$

It follows that

$$\begin{aligned} \sigma &= 1 - \nu\epsilon + O(\epsilon^2), \\ y^i &= (y')^i + 2h^{ij}\mu_j\epsilon + O(\epsilon^2), \end{aligned}$$

and we see that near G^{bf},

$$\frac{\partial\sigma}{\partial\epsilon} \neq 0, \quad \frac{\partial y^i}{\partial\mu_j} = \epsilon h^{ij} + O(\epsilon^2),$$

which, using the positive-definiteness of h^{ij}, shows that $\det d\pi$, where π is the map

$$L^{\mathrm{bf}} \ni (\mu, y', \epsilon) \mapsto (y(\mu, y', \epsilon), y', \sigma(\mu, y', \epsilon)),$$

vanishes to order exactly $n-1$ as $\epsilon \to 0$.

On the other hand, near a point on L^{bf} at which $|\mu| = 1$, we can choose a coordinate μ_i which is nonzero. Without loss of generality we suppose that $i = 1$. Then write $\overline{y} = (y_2, \dots, y_{n-1})$ and $\overline{\mu} = (\mu_2, \dots, \mu_{n-1})$. We can take $(\nu, \overline{\mu}, y', \epsilon)$ as coordinates on L^{bf}. Calculating as above, we find that

$$\begin{aligned} y^1 &= y_1' + 2h^{1j}\mu_j\epsilon + O(\epsilon^2), \\ y^i &= (y')^i + 2h^{ij}\mu_j\epsilon + O(\epsilon^2), \quad i \geq 2, \\ \sigma &= 1 - \nu\epsilon + O(\epsilon^2), \end{aligned}$$

which shows that

$$\frac{\partial y_1}{\partial\epsilon} > 0, \quad \frac{\partial\overline{y}^i}{\partial\overline{\mu}_j} = \epsilon h^{ij} + O(\epsilon^2), \quad \frac{\partial\sigma}{\partial\nu} = -\epsilon + O(\epsilon^2).$$

Again we find that $\det d\pi$, where π is the map

$$L^{\mathrm{bf}} \ni (\nu, \overline{\mu}, y', \epsilon) \mapsto (y(\nu, \overline{\mu}, y', \epsilon), y', \sigma(\nu, \overline{\mu}, y', \epsilon)),$$

vanishes to order exactly $n-1$ as $\epsilon \to 0$. □

Lemma 6.5. *There exists* $\delta > 0$ *such that, if*

$$q = (y, y', \sigma, \mu, \mu', \nu, \nu') \in L^{\mathrm{bf}} \quad \textit{and} \quad |\nu + \nu'| < \delta,$$

then either $q \in G^{\mathrm{bf}}$, *or* $d\pi : T_q L^{\mathrm{bf}} \to T_{\pi(q)}\mathrm{bf}$ *is invertible, and hence* $\pi : L \to \mathrm{bf}$ *is a diffeomorphism locally near* q.

Proof. We use the explicit description of L^{bf} given in [Hassell and Vasy 2001, Section 4]:

$$L^{\mathrm{bf}} = \left\{ \begin{pmatrix} y, y', \sigma, \\ \nu, \nu', \mu, \mu' \end{pmatrix} \;\middle|\; \begin{array}{l} \exists (y_0, \hat{\mu}_0) \in S^*(\partial M),\ s, s' \in (0, \pi),\ \text{such that} \\ \qquad \sigma = \sin s / \sin s',\ \nu = -\cos s,\ \nu' = \cos s', \\ (y, \mu) = \sin s \exp(s H_{\frac{1}{2}h})(y_0, \hat{\mu}_0), \\ (y', \mu') = -\sin s' \exp(s' H_{\frac{1}{2}h})(y_0, \hat{\mu}_0), \end{array} \right\} \cup T_+ \cup T_- \cup F_+ \cup F_-, \tag{6-23}$$

where

$$T_\pm = \{(y, y, \sigma, \pm 1, \mp 1, 0, 0) \mid \sigma > 0,\ y \in \partial M\},$$
$$F_\pm = \{(y, y', \sigma, \pm 1, \pm 1, 0, 0) \mid \sigma > 0,\ \exists \text{ geodesic of length } \pi \text{ connecting } y, y'\}.$$

We see that $\nu = -\nu'$ on L^{bf} only on $G^{\mathrm{bf}} \cup T_+ \cup T_-$. A compactness argument shows that for any neighborhood U of $G^{\mathrm{bf}} \cup T_+ \cup T_-$, the set

$$\{(y, y', \sigma, \mu, \mu', \nu, \nu') \in L^{\mathrm{bf}} \mid |\nu + \nu'| < \delta\}$$

is contained in U if δ is sufficiently small. So it is enough to show that L^{bf} projects diffeomorphically to bf in some neighborhood of $G^{\mathrm{bf}} \cup T_+ \cup T_-$, except at G^{bf} itself. Lemma 6.4 shows that $L^{\mathrm{bf}} \subset {}^{\mathrm{sc}}T^*{}_{\mathrm{bf}} M_b^2$ projects diffeomorphically to the base bf in a sufficiently small deleted neighborhood of G^{bf}. Now consider a neighborhood of $T_+ \cap \{\sigma \le 1 - \epsilon\}$ for some small ϵ. As shown in [Hassell and Vasy 2001], near this set, (y', μ', σ) are smooth coordinates. Also, we have from (6-23) that

$$(y, \mu) = \sigma \exp\left(\frac{s' - s}{\sin s'} H_{\frac{1}{2}h} \right)(y', \mu').$$

Using the expression (6-22) for the Hamilton vector field, we find that, near T_+,

$$y^i = y'^i + \frac{s' - s}{\sin s'} h^{ij} \mu'_j + O(|\mu'|^2) = (1 - \sigma) h^{ij} \mu'_j + O\big((\sin s)^2 + (\sin s')^2 + |\mu'|^2\big),$$

which shows that at T_+, where $\sin s = \sin s' = \mu' = 0$, we have

$$\left. \frac{\partial y^i}{\partial \mu'_j} \right|_{y', \sigma} = (1 - \sigma) h^{ij}.$$

Since (y', μ', σ) furnish smooth coordinates near T_+, this equation and the positive-definiteness of h^{ij} show that also (y, y', σ) furnish smooth coordinates in a neighborhood of T_+ when $\sigma < 1 - \epsilon$. (Of course, we know from Lemma 6.4 that this cannot hold uniformly up to $\sigma = 1$). A similar argument holds for $\sigma > 1 + \epsilon$ and for T_-. □

Remark 6.6. These lemmas will be applied to distributions of the form

$$Q(\lambda) dE_{\sqrt{L}}(\lambda) Q(\lambda), \tag{6-24}$$

where Q is a pseudodifferential operator with small microsupport. Notice that by taking the microsupport sufficiently small, we can localize the microsupport of (6-24) to points $(y, y', \sigma, \mu, \mu', \nu, \nu')$ such that y is close to y', μ is close to μ' and ν is close to ν'. However, we cannot localize so that σ is close to 1, simply because if $x, x' \in (0, \epsilon)$, then $\sigma = x/x'$ can take any value in $(0, \infty)$. Therefore, it is important to understand the properties of π on L near the whole of the sets $T_\pm$, not just close to ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$.

6C. *Proof of Theorem 1.3, part (A).* By Proposition 1.12, to prove part (A) of Theorem 1.3 it is sufficient to prove Theorem 1.13 for $\boldsymbol{L} = \boldsymbol{H}$ and for $\lambda \le \lambda_0$, that is, to prove the estimates

$$\big|(Q_i(\lambda) dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) Q_i(\lambda))(z, z')\big| \le C\lambda^{n-1-j}(1 + \lambda d(z, z'))^{-(n-1)/2+j}, \quad j \ge 0. \tag{6-25}$$

Our starting point is Theorem 4.1. As an immediate consequence of this theorem, the j-th λ-derivative $dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)$ is a Legendre distribution in the space

$$I^{m-j, p-j; r_{\mathrm{lb}}-j, r_{\mathrm{rb}}-j; \mathcal{B}^{(j)}}(M^2_{k,b}, (L^{\mathrm{bf}}, L^{\sharp,\mathrm{bf}}); \Omega^{1/2}_{k,b}),$$

where $\mathcal{B}^{(j)}$ is an index family with index sets at the faces $\mathrm{bf}_0, \mathrm{lb}_0, \mathrm{rb}_0, \mathrm{zf}$ starting at order $-1-j$, $n/2-1-j$, $n/2-1-j$, $n-1-j$ respectively.

Next we choose a partition of unity. We choose Q_0 to be multiplication by the function $1 - \chi(\rho)$, where $\chi(\rho) = 1$ for $\rho \le \epsilon$ and $\chi(\rho) = 0$ for $\rho \ge 2\epsilon$, for some sufficiently small ϵ. Then $Q_0 dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) Q_0$ is polyhomogeneous on $M^2_{k,b}$, with index sets as above at $\mathrm{bf}_0, \mathrm{lb}_0, \mathrm{rb}_0, \mathrm{zf}$ and supported away from the remaining boundary hypersurfaces. Now recall that $|dg\, dg'\, d\lambda/\lambda|^{1/2}$ is equal to $\rho_{\mathrm{bf}_0}^{-n} \rho_{\mathrm{lb}_0}^{-n/2} \rho_{\mathrm{rb}_0}^{-n/2}$ multiplied with a smooth nonvanishing section of the half-density bundle $\Omega^{1/2}_{k,b}$. It is then immediate that $Q_0 dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) Q_0$ is bounded, as a multiple of $|dg\, dg'\, d\lambda/\lambda|^{1/2}$ by λ^{n-1-j}, which yields (6-25) for $i = 0$ since in this region we have $\lambda d(z, z') \le C$.

Next, we choose Q_1' such that $\mathrm{Id} - Q_1'$ is microlocally equal to the identity for $|\mu|_h^2 + \nu^2 \le \frac{3}{2}$, and microsupported in $|\mu|_h^2 + \nu^2 \le 2$. Let $Q_1 = \chi(\rho) Q_1'$. Then, we claim that $Q_1 dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) Q_1$ has empty wavefront set, and is therefore polyhomogeneous with index sets at the faces $\mathrm{bf}_0, \mathrm{lb}_0, \mathrm{rb}_0, \mathrm{zf}$ starting at order -1, $n/2-1$, $n/2-1$, $n-1$ respectively. To see this, we write

$$\begin{aligned} &Q_1 dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) Q_1 \\ &\quad = dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) - (\mathrm{Id} - Q_1) dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) - dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)(\mathrm{Id} - Q_1) + (\mathrm{Id} - Q_1) dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)(\mathrm{Id} - Q_1). \end{aligned} \tag{6-26}$$

Since $\mathrm{Id} - Q_1$ is microlocally equal to the identity on $\pi_L(\mathrm{WF}'_{\mathrm{bf}}\, dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda))$ and on $\mathrm{WF}'_{\mathrm{lb}}(dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda))$, Lemma 5.2 shows that the sum of the first two terms on the right hand side above vanishes to infinite order at lb and bf, and similarly the sum of the third and fourth terms vanishes to infinite order at lb and bf. Now consider the multiplication of $\mathrm{Id} - Q_1$ on the right, and group together the first and third terms, and the second and fourth terms on the right-hand side. We see, using the adjoint of Lemma 5.2 (since $\mathrm{Id} - Q_1$ is also microlocally equal to the identity on $\mathrm{WF}'_{\mathrm{rb}}(dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda))$), that the sum of the first and third terms vanishes to infinite order at rb, and similarly the sum of the second and fourth terms vanishes at rb. Hence $Q_1 dE^{(j)}_{\sqrt{H}}(\lambda) Q_1$ vanishes to all orders at bf, lb, rb and has empty wavefront set as claimed. This piece therefore also satisfies (6-25).

We now further decompose $\mathrm{Id} - Q_0 - Q_1 = \chi(\mathrm{Id} - Q_1')$, which has compact microsupport, into a sum of terms. Choosing δ as in Lemma 6.5, we partition the interval $[-2, 2]$ into $N-1$ intervals B_i each of length $\delta/2$, and choose a decomposition $\mathrm{Id} - Q_1 = \sum_{i=2}^N Q_i$, where Q_i, and hence also Q_i^*, is microsupported in the set $\{|\mu|_h^2 + \nu^2 \le 2, \nu \in 2B_i\}$ (where $2B_i$ is the interval with the same center as B_i and twice the length). It follows that if $q' = (y, y', \sigma, \mu, \mu', \nu, \nu') \in (L^{\mathrm{bf}})'$ is such that $\pi_L(q') \in \mathrm{WF}'(Q_i)$ and $\pi_R(q') \in \mathrm{WF}'(Q_i^*)$, then $|\nu - \nu'| \le \delta$. Together with Lemma 5.4, this means that $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$ is associated only to the Legendrian L^{bf} and not to $L^{\sharp,\mathrm{bf}}$, since on $(L^{\sharp,\mathrm{bf}})'$ we have $|\nu - \nu'| = 2 > \delta$.

Next, by Lemma 6.5, if $q' = (y, y', \sigma, \mu, \mu', \nu, \nu') \in (L^{\mathrm{bf}})'$ is such that $\pi_L(q')$ is in $\mathrm{WF}'(Q_i)$ and $\pi_R(q')$ is in $\mathrm{WF}'(Q_i^*)$, then due to our choice of δ, either $q \in G^{\mathrm{bf}}$, or locally near q, L^{bf} projects diffeomorphically to bf. Therefore, the microsupport of $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$, $i \ge 2$, is a subset of $(L^{\mathrm{bf}})'$ which satisfies the conditions of either Proposition 6.1 or Proposition 6.2.

In the case of Proposition 6.1, we have $b = n-1-j$, $\alpha = -(n-1)/2 + j$ and estimate (6-25) follows directly. Next consider the case of Proposition 6.2. In this case, we have to determine the order of vanishing of the symbol of $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$ at G^{bf}. Locally near $q \in G^{\mathrm{bf}} \cap L^{\mathrm{bf}}$, L^{bf} can be parametrized by a phase function Φ that *vanishes* at G^{bf} when $d_v \Phi = 0$; see (6-8). The kernel $Q_i dE_{\sqrt{H}}(\lambda) Q_i^*$ is a Legendrian of order $-1/2$. Each time we apply a λ derivative to $dE_{\sqrt{H}}(\lambda)$, it hits either the phase function or the symbol. If it hits the phase, then the order of the Legendrian is reduced by 1, but it brings down a factor of Φ that vanishes at $G^{\mathrm{bf}} \times [0, \lambda_0]$. If it hits the symbol, then the order of the Legendrian is not reduced. Therefore, as a Legendrian of order $-1/2 - j$, the full symbol of $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$ vanishes to order j at $G^{\mathrm{bf}} \times [0, \lambda_0]$. Therefore, we can apply Proposition 6.2 with $b = n-1-j$ and $\alpha = -(n-1)/2 + j$, and we deduce (6-25) in this case. This concludes the proof of (6-25) and hence establishes Theorem 1.13 for low energies $\lambda \le \lambda_0$.

7. High energy estimates (in the nontrapping case)

In the previous section we proved estimates on the spectral measure $dE_{\sqrt{H}}(\lambda)$ for $\lambda \in (0, \lambda_0]$. We now prove high energy estimates, i.e., estimates for $\lambda \in [\lambda_0, \infty)$. For convenience, we introduce the semiclassical parameter $h = \lambda^{-1}$, so that we are interested in estimates for $h \in (0, h_0]$, where $h_0 = \lambda_0^{-1}$. To do this, we use the description of the high-energy asymptotics of the spectral measure from [Hassell and Wunsch 2008]. The structure of the argument will be the same as in the previous section, and our main task is to adapt each of the intermediate results — Lemmas 5.2 and 5.4, Propositions 6.1 and 6.2, Lemma 6.4 and Lemma 6.5 — to the high-energy setting. *Throughout this section we assume that the manifold (M, g) is nontrapping.*

7A. ***Microlocal support.*** We begin by defining, by analogy with the discussion in Section 5, the notion of microlocal support of a Legendre distribution on $\boldsymbol{X}$.

Let $\Lambda \subset {}^{sc}T^*{}_{mf}X$ be the Legendre submanifold associated to F. We assume that Λ is compact. Recall from [Hassell and Wunsch 2008, Section 3] that Λ determines associated Legendre submanifolds Λ_{bf}, Λ_{lb} and Λ_{rb} which are the bases of the fibrations on $\partial_{bf}\Lambda$, $\partial_{lb}\Lambda$ and $\partial_{rb}\Lambda$, respectively. The Legendre submanifold Λ_{bf} can be canonically identified with a Legendre submanifold of ${}^{sc}T^*{}_{bf}M^2_b$, while $\partial_{lb}\Lambda$ and $\partial_{rb}\Lambda$ may be canonically identified with Legendre submanifolds of ${}^{sc}T^*{}_{\partial M}M$. We define Λ' by negating the fiber coordinates corresponding to the right copy of M, i.e.,

$$q' = (z, z', \zeta, \zeta') \in \Lambda' \iff q = (z, z', \zeta, -\zeta') \in \Lambda.$$

Similarly we define Λ'_{bf} and Λ'_{rb} as in the previous section.

Then we define the microlocal support $\mathrm{WF}'(F)$ of $F \in I^m(\Lambda)$ to be a closed subset of

$$\Lambda' \cup (\Lambda'_{bf} \times [0, h_0]) \cup (\Lambda_{lb} \times [0, h_0]) \cup (\Lambda'_{rb} \times [0, h_0])$$

in the same way as before: we say that $q' \in \Lambda'$ is not in $\mathrm{WF}'(F)$ if there is a neighborhood of $q \in \Lambda$ in which F has order $-\infty$, in the sense of Section 5. That is, in a local oscillatory representation for F of the form (for simplicity, where q lies over the interior of M^2_b),

$$h^{m-k/2-n} \int_{\mathbb{R}^k} e^{i\psi(z,v)/h} a(z, v, h)\, dv |dg dg' dh/h^2|^{1/2},$$

where $q = (z_*, d_z\psi(z_*, v_*))$ and $d_v\psi(z_*, v_*) = 0$ (these conditions determining (z_*, v_*) locally uniquely provided that ψ is a nondegenerate parametrization of Λ), the condition that F has order $-\infty$ in a neighborhood of q is equivalent to a being $O(h^\infty)$ in a neighborhood of the point $(z_*, v_*, 0)$. Similarly, $q' \in \Lambda'_{bf} \times [0, h_0]$ is not in $\mathrm{WF}'(F)$ if there is a neighborhood of $q \in \Lambda_{bf} \times [0, h_0]$ in which F has order $-\infty$.

Similarly, $(\tilde{q}, h) \in \Lambda_{lb} \times [0, h_0]$ is not in $\mathrm{WF}'(F)$ if F can be written modulo $(hxx')^\infty C^\infty(M^2_b)$ using local oscillatory integral representations with symbols that vanish in a neighborhood of the *fiber* in their domain corresponding to $(\tilde{q}, h)$, and $(\tilde{q}', h) \in \Lambda'_{rb} \times [0, h_0]$ is not in $\mathrm{WF}'(F)$ if F can be written modulo $(hxx')^\infty C^\infty(M^2_b)$ using local oscillatory integral representations with symbols that vanish in a neighborhood of the fiber in their domain corresponding to $(\tilde{q}, h)$. These components of $\mathrm{WF}'(F)$ will be denoted $\mathrm{WF}'_{mf}(F)$, $\mathrm{WF}'_{lb}(F)$, $\mathrm{WF}'_{bf}(F)$ and $\mathrm{WF}'_{rb}(F)$, respectively.

If $F \in I^m(\Lambda)$, then $F \in (hxx')^\infty C^\infty(M^2)$ if and only if $\mathrm{WF}'(F)$ is empty. Also note that if $\mathrm{WF}'_*(F)$ is empty, then $\partial_*\Lambda'$ is disjoint from $\mathrm{WF}'_{mf}(F)$, but the converse need not hold: if the kernel of F is supported away from mf then certainly $\mathrm{WF}'_{mf}(F)$ will be empty, but $\mathrm{WF}'_*(F)$ need not be.

Particular examples of Legendre distributions on X are the kernels of semiclassical scattering pseudodifferential operators Q of differential order $-\infty$ with compact operator wavefront set. In the case of such a pseudodifferential[1] operator, the Legendre submanifold Λ is a compact subset of ${}^{s\Phi}N^*\mathrm{diag}_b$, defined in (4-15), and the components $\Lambda_{lb} \cup \Lambda'_{rb}$ are empty. Thus in this case we may (and will) identify the microlocal support $\mathrm{WF}'_{mf}(Q)$ with a compact subset of ${}^{sc}T^*M$, and $\mathrm{WF}'_{bf}(Q)$ may be identified with a compact subset of ${}^{sc}T^*{}_{\partial M}M \times [0, h_0)$.

[1]Throughout this section we deal with semiclassical scattering pseudodifferential operators. The words "semiclassical scattering" will usually be omitted.

In the next lemma, π_L and π_R denote the maps defined in either (4-6) or (4-14), as the case may be.

Lemma 7.1. *Suppose that F is a Legendre distribution on $\boldsymbol{X}$ and Q is a semiclassical scattering pseudodifferential operator. Assume that $F \in I^{m;r_{\mathrm{bf}},r_{\mathrm{lb}},r_{\mathrm{rb}}}(\boldsymbol{X}, \Lambda; {}^{\mathrm{s}\Phi}\Omega^{1/2})$ is associated to a compact Legendre submanifold Λ and that Q is of differential order $-\infty$ and semiclassical order 0, with compact operator wavefront set. Then QF is also a Legendre distribution in $I^{m;r_{\mathrm{bf}},r_{\mathrm{lb}},r_{\mathrm{rb}}}(\boldsymbol{X}, \Lambda; {}^{\mathrm{s}\Phi}\Omega^{1/2})$ and we have*

$$\begin{aligned} \mathrm{WF}'_{\mathrm{mf}}(QF) &\subset \pi_L^{-1}\,\mathrm{WF}'_{\mathrm{mf}}(Q) \cap \mathrm{WF}'_{\mathrm{mf}}(F),\\ \mathrm{WF}'_{\mathrm{bf}}(QF) &\subset \pi_L^{-1}\,\mathrm{WF}'_{\mathrm{bf}}(Q) \cap \mathrm{WF}'_{\mathrm{bf}}(F),\\ \mathrm{WF}'_{\mathrm{lb}}(QF) &\subset \mathrm{WF}'_{\mathrm{bf}}(Q) \cap \mathrm{WF}'_{\mathrm{lb}}(F),\\ \mathrm{WF}'_{\mathrm{rb}}(QF) &\subset \mathrm{WF}'_{\mathrm{rb}}(F). \end{aligned} \tag{7-1}$$

Moreover, if Q is microlocally equal to the identity on $\pi_L(\mathrm{WF}'_{\mathrm{mf}}(F))$, $\pi_L(\mathrm{WF}'_{\mathrm{bf}}(F))$ and $\mathrm{WF}'_{\mathrm{lb}}(F)$, then $QF - F \in I^{\infty,\infty,\infty,r_{\mathrm{rb}}}(\boldsymbol{X}, \Lambda; {}^{\mathrm{s}\Phi}\Omega^{1/2})$, i.e., it vanishes to infinite order at mf, lb *and* bf.

We omit the proof, as it is essentially identical to that of Lemma 5.2. There is of course a corresponding theorem for composition in the other order, which is obtained by taking the adjoint of the lemma above. Combining the two we obtain:

Corollary 7.2. *Suppose that F and Q, Q' are as above. Then*

$$\begin{aligned} \mathrm{WF}'_{\mathrm{mf}}(QFQ') &\subset \pi_L^{-1}\,\mathrm{WF}'_{\mathrm{mf}}(Q) \cap \pi_R^{-1}\,\mathrm{WF}'_{\mathrm{mf}}(Q') \cap \mathrm{WF}'_{\mathrm{mf}}(F),\\ \mathrm{WF}'_{\mathrm{bf}}(QFQ') &\subset \pi_L^{-1}\,\mathrm{WF}'_{\mathrm{bf}}(Q) \cap \pi_R^{-1}\,\mathrm{WF}'_{\mathrm{bf}}(Q') \cap \mathrm{WF}'_{\mathrm{bf}}(F),\\ \mathrm{WF}'_{\mathrm{lb}}(QFQ') &\subset \mathrm{WF}'_{\mathrm{bf}}(Q) \cap \mathrm{WF}'_{\mathrm{lb}}(F),\\ \mathrm{WF}'_{\mathrm{rb}}(QFQ') &\subset \mathrm{WF}'_{\mathrm{bf}}(Q') \cap \mathrm{WF}'_{\mathrm{rb}}(F). \end{aligned} \tag{7-2}$$

A similar result holds if F is associated to a Legendre conic pair rather than a single Legendre submanifold.

Lemma 7.3. (i) *Suppose that $F \in I^{m,p;r_{\mathrm{bf}},r_{\mathrm{lb}},r_{\mathrm{rb}}}(\boldsymbol{X}, (\Lambda, \Lambda^\sharp); {}^{\mathrm{s}\Phi}\Omega^{1/2})$ is a Legendre distribution on $\boldsymbol{X}$ associated to a conic Legendrian pair $(\Lambda, \Lambda^\sharp)$, and suppose that Q is a pseudodifferential operator such that Q is microlocally equal to the identity operator near $\pi_L(\Lambda \cup \Lambda^\sharp)$. Then $QF - F \in I^{\infty,\infty;\infty,\infty,r_{\mathrm{rb}}}(\boldsymbol{X}, (\Lambda, \Lambda^\sharp), {}^{\mathrm{s}\Phi}\Omega^{1/2})$, so it vanishes to infinite order at* mf, lb *and* bf. *If Q' is microlocally equal to the identity operator near $\pi_R(\Lambda \cup \Lambda^\sharp)$, then $FQ' - F \in I^{\infty,\infty;\infty,r_{\mathrm{lb}},\infty}(\boldsymbol{X}, (\Lambda, \Lambda^\sharp), {}^{\mathrm{s}\Phi}\Omega^{1/2})$ vanishes to infinite order at* mf, bf *and* rb.

(ii) *Suppose that F is as above, a Legendre distribution on M_b^2 associated to a conic Legendrian pair $(\Lambda, \Lambda^\sharp)$ of order $(m, p; r_{\mathrm{bf}}, r_{\mathrm{lb}}, r_{\mathrm{rb}})$, and suppose that Q, Q' are pseudodifferential operators. If*

$$\pi_L^{-1}\,\mathrm{WF}'_{\mathrm{bf}}(Q) \cap \pi_R^{-1}\,\mathrm{WF}'_{\mathrm{bf}}(Q') \cap \Lambda^\sharp = \varnothing, \tag{7-3}$$

then $QFQ' \in I^{m;r_{\mathrm{bf}},r_{\mathrm{lb}},r_{\mathrm{rb}}}(M_b^2, \Lambda; {}^{\mathrm{s}\Phi}\Omega^{1/2})$; in particular, $\mathrm{WF}'_{\mathrm{bf}}(QFQ')$ is disjoint from $(\Lambda^\sharp)'$.

We omit the proof, which is a straightforward modification of the arguments in Section 5.

7B. ***Pointwise estimates on Legendre distributions.*** Now we give a pointwise estimate on Legendre distributions of a particular type. First we begin with the trivial case.

Proposition 7.4. *Let* $\Lambda \subset {}^{sc}T^*{}_{mf}(X)$ *be a Legendre distribution that projects diffeomorphically to* mf. *Suppose that* $u \in I^{m,r_{bf},r_{lb},r_{rb}}(X, \Lambda; {}^{s\Phi}\Omega^{1/2})$ *with*

$$m = n/2 - l, \quad r_{bf} = -n/2 - \alpha, \quad r_{lb} = r_{rb} = -\alpha.$$

Then, as a multiple of the half-density $|dg\, dg'\, d\lambda|^{1/2}$, *we have a pointwise estimate*

$$|u| \le C\lambda^l (x^{-1} + (x')^{-1})^\alpha.$$

Generalizing Proposition 6.2 to the case of $X = M_b^2 \times [0, h_0]$ is straightforward.

Proposition 7.5. *Let* Λ *be a Legendrian submanifold of* ${}^{s\Phi}T^*_{mf}X$. *Assume that* Λ *intersects* ${}^{s\Phi}N^*diag_b$, *defined in* (4-15), *at* $G = \Lambda \cap {}^{s\Phi}N^*diag_b$ *which is codimension* 1 *in* Λ *and transversal to the boundary at* bf, *and that* $d\pi$ *has full rank on* $\Lambda \setminus G$, *while* $\pi|_G$ *is a fibration* $G \to diag_b$ *with* $(n-1)$*-dimensional fibers, with condition* (6-3) *holding at* G.

Assume that $u \in I^{m,r_{bf},r_{lb},r_{rb}}(X, \Lambda; {}^{s\Phi}\Omega^{1/2})$, *with* $m, r_{bf}, r_{lb}, r_{rb}$ *as in Proposition 7.4 and that the full symbol of* u *vanishes to order* $(n-1)/2 + \alpha$ *both at* $G \subset \Lambda$ *and at* $\partial_{bf}G \times [0, h_0] \subset \partial_{bf}\Lambda \times [0, h_0]$. *Then, as a multiple of the half-density* $|dg\, dg'\, d\lambda|^{1/2}$, *we have a pointwise estimate*

$$|u| \le C\lambda^{l-\alpha}(1 + \lambda d(z, z'))^\alpha. \tag{7-4}$$

Proof. First consider u on a neighborhood of X disjoint from diag_b. In that case, the result follows from Proposition 7.4.

Next consider u near diag_b, but away from bf. Then if u is microlocally trivial at ${}^{s\Phi}N^*\mathrm{diag}_b$, the result follows from Proposition 7.4. If not, then the geometry is the same as that considered in Proposition 6.2 (with ρ replaced by h; also note that the estimate in Proposition 6.2 is respect to the half-density $\lambda^n|dg\, dg'\, d\lambda|^{1/2}$), and the result follows from that proposition.

So we are reduced to the case where we are microlocally close to $\Lambda \cap \partial_{bf}{}^{s\Phi}N^*\mathrm{diag}_b = \partial_{bf}G$. Let $q \in \partial_{bf}G$. In a neighborhood of $\partial_{bf}\mathrm{diag}_b$, we have coordinates (x, y, w), where $w = (y - y', \sigma - 1)$ as before. In terms of these we can write points in ${}^{s\Phi}T^*{}_{mf}X$ in the form

$$\kappa \cdot \frac{dw}{xh} + \mu \cdot \frac{dy}{xh} + \tau \cdot \frac{dx}{xh} + \nu d\Big(\frac{1}{xh}\Big),$$

and this defines local coordinates $(x, y, w; \tau, \mu, \kappa, \nu)$ on ${}^{s\Phi}T^*{}_{mf}X$. Then, contracting the symplectic form with $xh^2\partial_h$ and restricting to ${}^{s\Phi}T^*_{mf}X$ gives the contact form on ${}^{s\Phi}T^*_{mf}X$, which in these coordinates takes the form

$$d\nu - \tau dx - \mu \cdot dy - \kappa \cdot dw. \tag{7-5}$$

Using the transversality of Λ to ${}^{s\Phi}T^*_{bf\cap mf}X$ we see, as in the proof of Proposition 6.2 that $(x, y, w_1, \bar\kappa)$ form coordinates on Λ. Then as in the proof of Proposition 6.2, we can write the remaining coordinates as functions of $(x, y, w_1, \bar{\kappa})$ on Λ:

$$\overline{w}_i = W_i(x, y, w_1, \bar{\kappa}), \quad \mu_i = M_i(x, y, w_1, \bar{\kappa}), \quad i = 2, \dots, n,$$
$$\kappa_1 = K(x, y, w_1, \bar{\kappa}), \quad \nu = N(x, y, w_1, \bar{\kappa}), \quad \tau = T(x, y, w_1, \bar{\kappa}).$$

In the same way as before, we find that

$$\widetilde{\Phi}(x, y, w, v) = \sum_{j=2}^{n} (\overline{w}_j - W_j(x, y, w_1, v))v_j + N(x, y, w, v), \quad v = (v_2, \dots, v_n),$$

parametrizes Λ locally, and has the properties that $\widetilde{\Phi} = O(w_1)$ when $d_v\widetilde{\Phi} = 0$, and $\widetilde{\Phi} = \Phi + O(x)$, where Φ is precisely as in the proof of Proposition 6.2. We can then follow the proof given there, where (6-11) is replaced by

$$x^{-(n-1)/2-\alpha}\lambda^{(n-1)/2+k} \int e^{i\widetilde{\Phi}(x,y,w,v)/xh}\tilde{a}(x, y, w_1, v, h)\, dv, \tag{7-6}$$

in which the function $\tilde{a}$ vanishes to order $(n-1)/2+\alpha$ at $x = 0$ and at $w_1 = 0$. In effect we have replaced the large parameter $1/x$ in the phase of (6-11) by $1/xh$, while x plays the role of a smooth parameter.

The rest of the argument is parallel to the proof of Proposition 6.2. We deal with the cases $|w_1| \le xh$ and $|w_1| \le c|w|$ exactly as in the previous proof. Assuming then that $|w_1| \ge xh$ and $|w_1| \sim |w|$, we make the change of variables (6-13). By continuity, the matrix A in (6-15) remains nonsingular, and (6-17) remains valid, for small x. Hence, we can integrate by parts using the identity

$$e^{i\widetilde{\Phi}/x} = \left(\sum_k \frac{xh}{i\theta_j} A_{jk} \frac{\partial}{\partial\theta_k}\right) e^{i\widetilde{\Phi}/x},$$

analogous to (6-16).

In the θ coordinates, we are trying to prove the estimate

$$\left| x^{-(n-1)/2-\alpha} h^{-(n-1)/2-l} \int_{\mathbb{R}^{n-1}} w_1^{\alpha} e^{i\widetilde{\Phi}(x,y,w,\theta)/xh} \tilde{a}_0(x, y, w_1, \theta)\, d\theta \right| \le Ch^{-l}\left(\frac{w_1}{x}\right)^{\alpha},$$

since when $|w| \ge xh$,

$$\frac{|w|}{xh} \sim \lambda d(z, z') \sim 1 + \lambda d(z, z').$$

As before, the $w_1^{(n-1)/2}$ factor was absorbed as a Jacobian factor, and $\tilde{a}$ is again smooth. This estimate is equivalent to a uniform bound on

$$\left| (xh)^{-(n-1)/2} \int_{\mathbb{R}^{n-1}} e^{i\widetilde{\Phi}(x,y,w,\theta)/x} \tilde{a}_0(x, y, w_1, \theta)\, d\theta \right|. \tag{7-7}$$

We introduce a modified partition of unity in (x, θ)-space, $1 = \chi_0 + \sum_{j=1}^{n-1} \chi_j$, where χ_0 is a compactly supported function of $\theta/\sqrt{xh}$, and χ_j is supported where $|\theta| \ge \sqrt{xh}$, and where $\theta_j \ge |\theta|/(n-1)$, with derivatives estimated by

$$\left|\nabla_\theta^{(k)} \chi_k\right| \le C(xh)^{-k/2}. \tag{7-8}$$

Then the rest of the argument proceeds just as before, leading to (7-7). □

7C. ***Geometry of the Legendre submanifold L.*** We prove results analogous to Lemmas 6.4 and 6.5. First, we define

$$G = \{q \in {}^{s\Phi}N^*\mathrm{diag}_b \mid \sigma(h^2\Delta_g)(q) = 1\},$$

where σ is the semiclassical principal symbol. This is an S^{n-1}-bundle over diag_b.

Lemma 7.6. *The Legendre submanifold L introduced in Section 4B intersects ${}^{s\Phi}N^*\mathrm{diag}_b$ cleanly at G, and the projection $\pi : L \to \mathrm{mf}$ satisfies* (6-3).

Proof. This is proved just as for Lemma 6.4. As shown in [Hassell and Wunsch 2008], L can be obtained as the flowout from G by a vector field V_l, which is obtained from the Hamilton vector field of $\Delta_g - \lambda^2$ by dividing by boundary defining function factors (see [ibid., Section 11]), so that it becomes smooth up to the boundary of ${}^{s\Phi}T^*X$. This vector field takes the form (6-22) up to $O(x)$ near bf, and repeating the argument below (6-22) with x as a smooth parameter establishes the lemma in a neighborhood of $\partial_{\mathrm{bf}}G$, i.e., for $x + x' \le \epsilon$ for some small $\epsilon > 0$.

Away from bf, we can use coordinates (z, z') on mf, and writing points in ${}^{s\Phi}T^*_{\mathrm{mf}}X$ in the form

$$\zeta \cdot \frac{dz}{h} + \zeta' \cdot \frac{dz'}{h} + \tau d\Big(\frac{1}{h}\Big)$$

defines fiber coordinates (ζ, ζ', τ) on ${}^{s\Phi}T^*_{\mathrm{mf}}X$. In terms of these coordinates, we have

$$V_l = g^{ij}(z)\zeta_i \frac{\partial}{\partial z^j} - \frac{1}{2}\frac{\partial g^{ij}(z)}{\partial z_k}\zeta_i\zeta_j \frac{\partial}{\partial \zeta_k} + g^{ij}(z)\zeta_i\zeta_j \frac{\partial}{\partial \tau}. \tag{7-9}$$

We recognize the equations for (z, ζ) as equations for geodesic flow. Moreover, letting $|\zeta|_g = g^{ij}(z)\zeta_i\zeta_j$, we find that $(|\zeta|_g^2)^{\cdot} = 0$ and $|\zeta|_g = 1$ on G, hence $|\zeta|_g = 1$ on L; similarly $|\zeta'|_g = 1$ on L. Finally, $\dot\tau = 1$ and $\tau = 0$ on G. It follows that near a point on G where (say) $\zeta_1 \neq 0$, we can use coordinates $(\bar\zeta, z', \tau)$ as coordinates on L, where $\bar\zeta = (\zeta_2, \dots, \zeta_n)$, $\bar z = (z_2, \dots, z_n)$. We then find, from (7-9), that

$$\begin{aligned} z^1 &= (z')^1 + g^{ij}\zeta_j\tau + O(\tau^2), \\ z^i &= (z')^i + g^{ij}\zeta_j\tau + O(\tau^2), \quad i \ge 2, \end{aligned}$$

and we see that near G,

$$\frac{\partial z^1}{\partial \tau} \neq 0, \quad \frac{\partial \bar z^i}{\partial \bar\zeta_j} = \tau g^{ij},$$

which shows that $\det d\pi$, where π is the map

$$L \ni (\bar\zeta, z', \tau) \mapsto (z^1(\bar\zeta, z', \tau), \bar\zeta(\bar\zeta, z', \tau), z'),$$

vanishes to order exactly $n - 1$ at G. □

Lemma 7.7. (i) *There exists $0 < \delta < 1$ and $\epsilon > 0$ such that the Legendre submanifold $L \subset {}^{s\Phi}T^*{}_{\mathrm{mf}}X$ projects diffeomorphically to the base* mf *locally near all points $(x, y, x', y', \mu, \mu', \nu, \nu', \tau) \in L \setminus G$ such that $x + x' < 2\epsilon$ and $|\nu + \nu'| < \delta$.*

(ii) *For any $\epsilon > 0$ there exists $\iota > 0$ such that L projects diffeomorphically to the base near all points $(z, z', \zeta, \zeta', \tau) \in L \setminus G$ such that $x + x' > \epsilon$ and $|\tau| < \iota$.*

Proof. (i) A topological argument shows that for sufficiently small ϵ, depending on δ, the subset of L where $x + x' < 2\epsilon$ and $|\nu + \nu'| < \delta$ is contained in a small neighborhood of the set $G \cup T_+ \cup T_-$, where $T_\pm \subset \partial_{\mathrm{bf}} L = L^{\mathrm{bf}}$ are as in (6-23). Lemma 7.6 shows that L projects diffeomorphically to mf in a deleted neighborhood of G. Near the sets $T_\pm$, we use Lemma 6.5 and the fact, proved in [Hassell and Wunsch 2008], that L is transverse to the boundary at bf to show that $(y, y', \sigma, \rho_{\mathrm{bf}})$ form coordinates locally near $T_\pm$ away from G. Here ρ_{bf} is a boundary defining function for bf and can be taken to be x for $\sigma > 1$ or x' for $\sigma < 1$. Therefore, L projects diffeomorphically to mf locally near $T_\pm$ and away from G.

(ii) The calculation above shows that if τ is small, then $d(z, z')$ is small and $|\zeta + \zeta'|$ is small, i.e., $(z, \zeta, z', \zeta', \tau)$ is close to G. So by taking ι sufficiently small, we restrict attention to a small neighborhood of $G \cap \{x + x' \geq \epsilon\}$. The result then follows directly from Lemma 7.6. □

Remark 7.8. In fact, we can take ι to be the injectivity radius of M.

Let M' be the compact subset of M° given by $\{x \geq \epsilon\}$, where ϵ is as in Lemma 7.7, and let ι be the injectivity radius of M. For any $z_0 \in M'$, let z denote the Riemannian normal coordinates centered at z_0, and ζ the corresponding dual coordinates. Define the quantity

$$\eta = \inf_{z_0 \in M'} \min\{|z - z'| + |\zeta - \zeta'| : |z - z_0| \leq \iota/4, |z' - z_0| \leq \iota/4, \gamma(0) = (z, \zeta),\ \gamma(t) = (z', \zeta'),\ t \geq \iota\},$$

where the minimum is taken over all geodesics $\gamma : \mathbb{R} \to M^\circ$ that are arc-length parametrized.

Lemma 7.9. *The quantity η is strictly positive.*

Proof. We use the nontrapping assumption; then there is no geodesic γ with $\gamma(0) = (z, \zeta) = \gamma(t)$, if $t > \iota$. Therefore, by compactness, the minimum for a fixed z_0 in the expression above is strictly positive. This minimum varies continuously with z_0 and therefore the inf over all z_0 in the compact set M' is also strictly positive. □

7D. ***Proof of Theorem 1.3, part (B).*** We now assemble our results to prove (1-9) for $\lambda \geq \lambda_0$, i.e., $h \leq h_0$, which by Proposition 1.12 and Section 6C is sufficient to prove part (B) of Theorem 1.3.

We now choose a partition of unity consisting of pseudodifferential operators. This is done similarly to the previous section. In particular, we will choose Q_1 to have microsupport disjoint from the characteristic variety of $h^2 \boldsymbol{H} - 1$, while the others will have compact microsupport, that is, they will be pseudodifferential operators of differential order $-\infty$. In detail, we choose Q_1 such that $\mathrm{Id} - Q_1$ is microlocally equal to the identity where $\sigma(h^2 \Delta_g) \leq 3/2$, and microsupported where $\sigma(h^2 \Delta_g) \leq 2$ (here σ denotes the semiclassical principal symbol). Then, we claim that $dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)$ is in $(hxx')^\infty C^\infty(M^2)$. To see this, we write

$$Q_1 dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) Q_1 = dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) - (\mathrm{Id} - Q_1) dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) - dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)(\mathrm{Id} - Q_1) + (\mathrm{Id} - Q_1) dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)(\mathrm{Id} - Q_1)$$

and use Theorem 4.2 and the microlocal support estimates as in the discussion below (6-26) to show that $\mathrm{WF}'(dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda))$ is empty. This piece therefore is in $(hxx')^\infty C^\infty(M^2)$, and trivially satisfies (6-25).

We now further decompose $\mathrm{Id} - Q_1$, which has compact microsupport, into a sum of terms. We first choose a function $m \in C^\infty(M_b^2)$ that is equal to 1 in a neighborhood of ∂M_b^2 and supported where $x + x' < 2\epsilon$, where ϵ is as in Lemma 7.7. Choosing δ as in Lemma 7.7, we divide up the interval

$[-2, 2]$ into $N-1$ intervals B_i each of width $\le \delta/4$, and choose a decomposition $(\mathrm{Id} - Q_1)m = \sum_{i=2}^{N} Q_i$, where the operators Q_i, and hence also Q_i^*, are supported on the set $x + x' < 2\epsilon$ and microsupported in the set $\{\sigma(h^2\Delta_g) \le 2, \nu \in 2B_i\}$. It follows that if $q' = (x, y, x', y', \mu, \mu', \nu, \nu', \tau) \in L'$ is such that $\pi_L(q') \in \mathrm{WF}'_{\mathrm{mf}}(Q_i)$ and $\pi_R(q') \in \mathrm{WF}'_{\mathrm{mf}}(Q_i^*)$, then $|\nu - \nu'| \le \delta/2$. Together with Theorem 4.2 and Lemma 7.3, this means that $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$ is a Legendrian distribution associated only to L and not to $L^\sharp$, since on $(L^\sharp)'$ we have $|\nu - \nu'| = 2 > \delta/2$. Then Lemma 7.6 guarantees that on the microsupport of $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$, the projection π to mf is either a diffeomorphism or satisfies the conditions of Proposition 7.5.

We finally decompose $(\mathrm{Id} - Q_1)(1 - m)$ as $\sum_{i=N+1}^{N+N'} Q_i$, where Q_i is microsupported in a sufficiently small set so that $\mathrm{WF}_{\mathrm{mf}}(Q_i)$ is a subset of

$$\{(z, \zeta) \mid |z - z_0| + |\zeta - \zeta_0| < \eta/2\} \tag{7-10}$$

for some $z_0 \in M' = \{x \ge \epsilon\} \subset M^\circ$ and some ζ_0 (where we use Riemannian normal coordinates as in Lemma 7.9). By construction, then, if $q' = (z, z', \zeta, \zeta', \tau) \in \mathrm{WF}'_{\mathrm{mf}}(Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*)$, then we must have $|z - z'| + |\zeta - \zeta'| < \eta$ from (7-10), and also $\gamma(0) = (z, \zeta)$, $\gamma(t) = (z', \zeta')$ for some geodesic γ. From Lemma 7.9 we conclude $t < \iota$, thus γ is the short geodesic between z and z'. Consequently, $\tau < \iota$ and by Lemma 7.7 either L locally projects diffeomorphically to mf, or $q' \in {}^{\mathrm{sc}}N^*\mathrm{diag}_b$.

We next consider the symbol of $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$. As in the previous section, this symbol vanishes to order j both at $G \subset \mathrm{mf}$ and at $\partial G \times [0, h_0] \subset \mathrm{bf}$, due to the vanishing of the phase function $\widetilde{\Phi}$ at G when $d_v \widetilde{\Phi} = 0$. Therefore, in all cases, $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$ satisfies the conditions of Proposition 7.5 with $l = j$, and the required estimate (6-25) follows from this proposition. This completes the proof of (1-4) for $\lambda_0 \le \lambda < \infty$.

8. Trapping results

8A. ***Spectral projection estimates.*** In this section we study the Laplacian on a manifold N with C^∞ bounded geometry, in the sense that the local injectivity radius $\iota(z)$, $z \in N$ has a positive lower bound, say ϵ; the metric g_{ij}, expressed in normal coordinates in the ball of radius $\epsilon/2$ around any point z is uniformly bounded in $C^\infty(B(0, \epsilon/2))$, as z ranges over N; and the inverse metric g^{ij} is uniformly bounded in supremum norm. (In fact, we only need g_{ij} to be bounded in C^k for some k depending on dimension n, but k tends to infinity as $n \to \infty$.) This implies that the distance function $d(q, q')$ satisfies the $n \times n$ Carleson–Sjölin condition (see [Sogge 1993, Section 2.2]) uniformly over all $z \in N$ and $q, q' \in B(z, \epsilon/2)$ with $d(q, q') \ge \epsilon/4$.

Then the following Sogge-type restriction theorem holds:

Proposition 8.1. *Let N be a complete Riemannian manifold of dimension n with C^∞ bounded geometry. Then the Laplacian Δ_N on N satisfies for $\lambda \ge 1$*

$$\big\| \mathbb{1}_{[\lambda, \lambda+1]}(\sqrt{\Delta_N}) \big\|_{L^p(N) \to L^{p'}(N)} \le C \lambda^{n(1/p - 1/p') - 1}, \quad 1 \le p \le \frac{2(n+1)}{n+3}. \tag{8-1}$$

This is quite likely well-known to experts, but to our knowledge such a result has not appeared in the literature, so we sketch a proof.

Proof. It is enough to prove (8-1) for the endpoints $p = 1$ and $p = 2(n+1)/(n+3)$, and use interpolation. We adapt Sogge's argument. Let ϵ be as above. We then choose an nonzero Schwartz function χ such that its Fourier transform $\widehat{\chi}$ is nonnegative and supported in $[\epsilon/4, \epsilon/2]$. It follows that $\chi(0) > 0$, and by taking ϵ sufficiently small, we can arrange that $\operatorname{Re} \chi \geq c > 0$ on $[0, 1]$.

Now let $\chi_\lambda^{\mathrm{ev}}(\sigma) = \chi(\sigma - \lambda) + \chi(-\sigma - \lambda)$. This is an even function, and since χ is rapidly decreasing, for sufficiently large λ we have

$$\operatorname{Re} \chi_\lambda^{\mathrm{ev}} \geq \tfrac{1}{2} c \quad \text{on } [\lambda, \lambda + 1].$$

That is,

$$(\operatorname{Re} \chi_\lambda^{\mathrm{ev}})^2 - \tfrac{1}{8} c^2 = F_\lambda, \quad \text{where } F_\lambda \geq 0 \text{ on } [\lambda, \lambda+1].$$

Then for $f \in L^p$,

$$\begin{aligned}
\tfrac{1}{8} c^2 \big\| \mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta_N}) f \big\|_{L^2}^2 &= \big\langle \mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta_N}) f, \big(\operatorname{Re} \chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})\big)^2 - F_\lambda(\sqrt{\Delta_N}) f \big\rangle \\
&= \big\langle \mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta_N}) \operatorname{Re} \chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N}) f, \operatorname{Re} \chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N}) f \big\rangle \\
&\qquad - \big\langle F_\lambda(\sqrt{\Delta_N}) \mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta_N}) f, \mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta_N}) f \big\rangle \\
&\leq \big\| \operatorname{Re} \chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N}) f \big\|_{L^2}^2 \\
&\leq \big\| \chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N}) f \big\|_{L^2}^2.
\end{aligned}$$

So it is enough to estimate the operator norm of the operator $\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})$ from L^p to L^2. To do this we express $\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})$ in terms of the half-wave group $e^{it\sqrt{\Delta_N}}$:

$$\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N}) = \frac{1}{\pi} \int e^{it\sqrt{\Delta_N}} \widehat{\chi_\lambda^{\mathrm{ev}}}(t) \, dt. \tag{8-2}$$

Since $\widehat{\chi_\lambda^{\mathrm{ev}}} = e^{-it\lambda} \widehat{\chi}(t) + e^{it\lambda} \widehat{\chi}(-t)$ is even in t, we can write this as

$$\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N}) = \frac{1}{\pi} \int \cos t\sqrt{\Delta_N} \big(e^{-it\lambda} \widehat{\chi}(t) + e^{it\lambda} \widehat{\chi}(-t) \big) \, dt. \tag{8-3}$$

Using the fact that the kernel of $\cos t\sqrt{\Delta_N}$ is supported in $\mathcal{D}_t$ for any complete Riemannian manifold, we see that $\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})$ is supported in $\mathcal{D}_{\epsilon/2}$. The estimate (8-1) for $p = 1$ then follows from [Sogge 1993, Lemma 4.2.4], or alternatively from the kernel bound $C\lambda^{(n-1)/2}$ that follows from the description of $\cos t\sqrt{\Delta_N}$ as a Fourier integral operator of order 0 associated to the conormal bundle of $\{d(x, y) = t\}$. For the other endpoint $p = 2(n+1)/(n+3)$, the argument in [Sogge 1993, Section 5.1] shows that $\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})$ maps any $f \in L^p(N)$ and supported in a ball of radius $\epsilon/2$ to $L^2(N)$ with a bound

$$\| \chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N}) f \|_2 \leq C \lambda^{n(1/p - 1/2) - 1/2} \| f \|_p,$$

where C is uniform over N due to the bounded geometry. We then choose a sequence of balls $B(x_i, \epsilon/2)$ that cover N, such that $B(x_i, \epsilon)$ have uniformly bounded overlap, i.e., such that $\sum_i \mathbb{1}_{B(x_i,\epsilon)}$ is uniformly bounded. Then for any $f \in L^p(N)$, and using the continuous embedding from $l^p \to l^2$ for $1 \leq p < 2$,

$$\begin{aligned}\|\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})f\|_2^2 \le \sum_i \|\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})f\|^2_{L^2(B(x_i,\epsilon/2))} &\le C\lambda^{2n(1/p-1/2)-1}\sum_i \|f\|^2_{L^p(B(x_i,\epsilon))}\\ &\le C\lambda^{2n(1/p-1/2)-1}\Big(\sum_i \|f\|^p_{L^p(B(x_i,\epsilon))}\Big)^{2/p}\\ &\le C\lambda^{2n(1/p-1/2)-1}\|f\|^2_{L^p}, \qquad (8\text{-}4)\end{aligned}$$

showing that $\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})$, and hence also $\mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta_N})$, maps from $L^p(N)$ to $L^2(N)$ with a bound $C\lambda^{n(1/p-1/2)-1/2}$. Using the T^*T trick we obtain (8-1). □

8B. ***Spatially localized results for trapping manifolds.*** Let us assume now that M° is asymptotically *Euclidean* and has several ends $\mathscr{E}_1,\dots,\mathscr{E}_k$. By an *end* here we mean a connected component $\mathscr{E}_i$ of $\{x<2\epsilon\}$, where x is a boundary defining function and $\epsilon>0$ is a small fixed number, so that $\mathscr{E}_i$ is diffeomorphic to $(r_i,\infty)\times S^{n-1}$ with a metric of the form $dr^2+r^2h(y,dy,1/r)$, with h smooth, and such that the projection of the trapped set to M° is disjoint from $\mathscr{E}_i$.

Proposition 8.2. *Assume M° is asymptotically Euclidean, possibly with several ends. Let $\chi\in C^\infty(M)$ be supported in $\{x<\epsilon\}$ and let $\boldsymbol{H}$ be as in Theorem 1.3. Then one has*

$$\|\chi\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\chi\|_{L^p\to L^{p'}} \le C\lambda^{n(1/p-1/p')-1} \quad \textit{for } 1<p\le \frac{2(n+1)}{n+3}. \tag{8-5}$$

Proof. As in [Hassell and Vasy 1999], we can write $dE_{\sqrt{\boldsymbol{H}}}(\lambda)=(2\pi)^{-1}P(\lambda)P(\lambda)^*$, where $P(\lambda)$ is the Poisson operator associated to $\boldsymbol{H}$. Hence one needs to get $L^p(M)\to L^2(\partial M)$ bounds for $P(\lambda)^*\chi$. The Schwartz kernel of $P(\lambda)^*$ is given by

$$P^*(\lambda;y,z')=[x^{-(n-1)/2}e^{i\lambda/x}R(\lambda;x,y;z')]|_{x=0}. \tag{8-6}$$

Let $\chi_1,\chi_2,\chi_3\in C^\infty(M)$ be supported in $\{x<2\epsilon\}$ and equal to 1 in $\{x<\epsilon\}$, and $\chi_i\chi_j=\chi_j$ if $j<i$. Let (M_i,g_i) be a nontrapping asymptotically Euclidean manifold with one unique end isometric to $\mathscr{E}_i$. The existence of such a manifold can be easily proved if one takes ϵ small enough. There is a natural identification $\iota_j : M_j\cap\{x<2\epsilon\}\to M\cap\{x<2\epsilon\}$, and so functions supported in $\{x<2\epsilon\}$ can be considered as functions on M or $\bigcup_j M_j$. To simplify notations, we shall implicitly use this identification in what follows, instead of writing ι_j^*, ι_{j*}. Let $\boldsymbol{H}_j=\Delta_{M_j}+V_j$, where V_j is equal to V in the identified region, such that $\boldsymbol{H}_j$ satisfies the conditions of Theorem 1.3 (which can always be achieved by making V_j sufficiently positive in a compact set away from the identified region). For $\lambda\in\{z\in\mathbb{C};\operatorname{Im}\lambda>0\}$, we define the resolvent $R_j(\lambda):=(\boldsymbol{H}_j-\lambda^2)^{-1}$, and by [Hassell and Vasy 2001] the Schwartz kernel of this operator extends continuously to $\lambda\in\mathbb{R}$ as a Legendre distribution. For $\lambda>0$ it corresponds to the outgoing resolvent while for $\lambda<0$ it is the incoming resolvent. For what follows, we consider $\operatorname{Re}\lambda>0$ to deal with the outgoing case. We have the following identities for $\operatorname{Im}\lambda>0$:

$$(\boldsymbol{H}_j-\lambda^2)\sum_j \chi_2R_j(\lambda)\chi_1=\chi_1+\sum_j[\boldsymbol{H}_j,\chi_2]R_j(\lambda)\chi_1,$$

$$\sum_j \chi_2R_j(\lambda)\chi_3(\boldsymbol{H}_j-\lambda^2)=\chi_2+\sum_j \chi_2R_j(\lambda)[\chi_3,\boldsymbol{H}_j],$$

which can be also written as

$$\sum_j \chi_2 R_j(\lambda)\chi_1 = R(\lambda)\chi_1 + \sum_j R(\lambda)[\boldsymbol{H}_j, \chi_2]R_j(\lambda)\chi_1,$$
$$\sum_j \chi_2 R_j(\lambda)\chi_3 = \chi_2 R(\lambda) + \sum_j \chi_2 R_j(\lambda)[\chi_3, \boldsymbol{H}_j]R(\lambda).$$

Multiplying the second identity by χ_1 on the right and combining with the first one, we deduce that

$$\chi_2 R(\lambda)\chi_1 = \sum_j \chi_2 R_j(\lambda)\chi_1 + \sum_{i,j} \chi_2 R_i(\lambda)[\chi_3, \boldsymbol{H}]R(\lambda)[\boldsymbol{H}, \chi_2]R_j(\lambda)\chi_1. \tag{8-7}$$

Since $R_j(\lambda)$, $R(\lambda)$ extend to $\lambda \in \mathbb{R}$ as operators mapping $C_0^\infty(M)$ to $C^\infty(M)$, (8-7) also extends to $\lambda \in \mathbb{R}$ as a map from $C_0^\infty(M)$ to C^∞ (since $[\boldsymbol{H}, \chi_i]$ is a compactly supported differential operator). Now to obtain the Poisson operator $P(\lambda)^*$, we use (8-6) and deduce from (8-7) that

$$P(\lambda)^*\chi_1 = \sum_j P_j(\lambda)^*\chi_1 + \sum_{i,j} P_i^*(\lambda)[\chi_3, \boldsymbol{H}]R(\lambda)[\boldsymbol{H}, \chi_2]R_j(\lambda)\chi_1, \tag{8-8}$$

where $P_j(\lambda)^*$ is the adjoint of the Poisson operator for $\boldsymbol{H}_j$ on (M_j, g_j) (mapping to ∂M by the natural identification of ∂M_i with ∂M). Since $\nabla\chi_2$ and $\nabla\chi_3$ are compactly supported, we can choose $\eta \in C_0^\infty(M^\circ)$, supported in $\{x < 2\epsilon\}$, such that $\eta = 1$ on $\operatorname{supp}\nabla\chi_2 \cup \operatorname{supp}\nabla\chi_3$, and write (8-8) in the form

$$P(\lambda)^*\chi_1 = \sum_j P_j(\lambda)^*\chi_1 + \sum_{i,j} P_i^*(\lambda)\eta[\chi_3, \boldsymbol{H}]\eta R(\lambda)\eta[\boldsymbol{H}, \chi_2]\eta R_j(\lambda)\chi_1. \tag{8-9}$$

In [Cardoso and Vodev 2002, Equation (1.5)],[2] Cardoso and Vodev prove the following L^2 estimate: If $\eta \in C_0^\infty(M)$ (respectively $\eta_j \in C_0^\infty(M_j)$) is supported in $\{x < 2\epsilon\}$, then for ϵ small enough, there is $C > 0$ such that, for all $\lambda > 1$,

$$\begin{aligned} &\|\eta R(\lambda)\eta\|_{L^2\to L^2} \le C\lambda^{-1} \qquad (\text{respectively } \|\eta_j R_j(\lambda)\eta_j\|_{L^2\to L^2} \le C\lambda^{-1}),\\ &\|\eta R(\lambda)\eta\|_{H^{-1}\to H^1} \le C\lambda \qquad (\text{respectively } \|\eta_j R_j(\lambda)\eta_j\|_{H^{-1}\to H^1} \le C\lambda). \end{aligned} \tag{8-10}$$

Since the spectral measure $dE_j(\lambda)$ for $\sqrt{\boldsymbol{H}_j}$ on (M_j, g_j) satisfies

$$dE_j(\lambda) = \frac{\lambda}{\pi i}(R_j(\lambda) - R_j(-\lambda)) = \frac{1}{2\pi} P_j(\lambda)P_j(\lambda)^*,$$

we deduce by the TT^* argument and (8-10) that

$$\|\eta_j P_j(\lambda)\|_{L^2(\partial M_j)\to L^2(M_j)} \le C \tag{8-11}$$

[2]In [Cardoso and Vodev 2002, Theorem 1.1], for $\lambda \in \mathbb{R}^*$ and $|\lambda| \gg 1$, only the $\|\eta R(\lambda)\eta\|_{L^2\to L^2} = O(|\lambda|^{-1})$ norm appears but it is a direct consequence of [ibid., Equation (4.9)] that $\|\eta R(\lambda)\eta\|_{L^2\to H^1} = O(1)$ if η has support far enough in the end. (Note that the H^1 space in [ibid.] involves a semiclassical scaling, unlike our standard H^1 space.) Then combining with $\Delta\eta R(\lambda)\eta = \eta^2 + ([\Delta, \eta] + \lambda^2\eta)R(\lambda)\eta$, we get $\|\eta R(\lambda)\eta\|_{L^2\to H^2} = O(|\lambda|)$ for all $\lambda \in \mathbb{R}^*$ and taking adjoints give $\|\eta R(\lambda)\eta\|_{H^{-2}\to L^2}$, which by interpolating show that the $H^{-1} \to H^1$ norm is $O(|\lambda|)$.

if η_j is as above. Now since M_j is nontrapping, we also know from Theorem 1.3 and the TT^* argument that for $p \in [1, 2(n+1)/(n+3)]$ we have

$$\|P_j(\lambda)^* \chi_1\|_{L^p(M_j)\to L^2(\partial M_j)} \le C\lambda^{n(1/p-1/2)-1/2}. \tag{8-12}$$

We now use the following:

Lemma 8.3. *Assume that M_j is asymptotically Euclidean and nontrapping. Let $\chi \in C^\infty(M_j)$ be equal to 1 in $\{x < \epsilon\}$ and supported in $\{x < 2\epsilon\}$ and let $\eta \in C_0^\infty(M_j)$ be supported in $\{x < 2\epsilon\}$ such that*

$$\inf\{x \mid \exists\,(x, y) \in \operatorname{supp} \eta\} \ge \gamma \sup\{x \mid \exists\,(x, y) \in \operatorname{supp} \chi\} \tag{8-13}$$

for some $\gamma > 1$; in particular, the distance between the support of η and χ is positive. Then the following estimate holds for $1 < p \le 2(n+1)/(n+3)$ and $\lambda \ge 1$:

$$\|\eta R_j(\lambda)\chi\|_{L^p(M_j)\to L^2(M_j)} \le \frac{C}{\lambda}\|\eta\, dE_j(\lambda)\chi\|_{L^p(M_j)\to L^2(M_j)} + O(\lambda^{-\infty}).$$

Assuming for a moment the validity of Lemma 8.3, we complete the proof of Proposition 8.2. Since $\eta dE_j(\lambda)\chi = \eta P_j(\lambda)P_j(\lambda)^*\chi$, we deduce from Lemma 8.3 and equations (8-11) and (8-12) that

$$\|\eta R_j(\lambda)\chi\|_{L^p(M_j)\to L^2(M_j)} \le C\lambda^{n(1/p-1/2)-1/2-1}, \quad \lambda \ge 1. \tag{8-14}$$

Now we can analyze the boundedness of the right-hand term of (8-9) as follows: $\eta R_j(\lambda)\chi$ maps $L^p(M_j) \to L^2(M_j)$ with norm $C\lambda^{n(1/p-1/2)-1/2-1}$ by (8-14); $[\boldsymbol{H}, \chi_2]$ maps $L^2(M_j)$ to $H^{-1}(M)$ with norm independent of λ; $\eta R(\lambda)\eta$ maps $H^{-1}(M)$ to $H^1(M)$ with norm $C\lambda$ by (8-10); $[\chi_3, \boldsymbol{H}]$ maps $H^1(M_j)$ to $L^2(M)$ with norm independent of λ; and $P_i^*(\lambda)\eta$ maps $L^2(M)$ to $L^2(M)$ with uniformly bounded norm by (8-12). This concludes the proof of Proposition 8.2. □

Proof of Lemma 8.3. Recall that $R_j(\pm\lambda)$ is the sum of a pseudodifferential operator and of Legendre distributions associated to the Legendre submanifolds $({}^{s\Phi}N^*\mathrm{diag}_b, L_\pm)$ and to $(L_\pm, L_\pm^\sharp)$. Since the distance between the supports of η and χ is positive, we see that $\eta R_j(\pm\lambda)\chi$ are, like $dE_j(\lambda)$, both Legendre distributions (conic pairs) associated to $(L, L^\sharp)$ with disjoint microlocal support; indeed, the nontrapping assumption implies that L_+ and L_- intersect only at G, which is contained in ${}^{s\Phi}N^*\mathrm{diag}_b$, while $L_+^\sharp$ and $L_-^\sharp$ are disjoint. We claim that we can choose a microlocal partition of unity,

$$\sum_{i=1}^N Q_i = \mathrm{Id},$$

where the Q_i are semiclassical scattering pseudodifferential operators, such that for each pair (i, k), either $Q_i\eta R_j(\lambda)\chi Q_k$ or $Q_i\eta R_j(-\lambda)\chi Q_k$ is microlocally trivial. This does not quite follow from the disjointness of the microlocal supports of $\eta R_j(\pm\lambda)\chi$; we must also check that at $T_\pm$, there are no points $(y, y', \sigma, \mu, \mu', \nu, \nu'), (y, y', \sigma^*, \mu, \mu', \nu, \nu') \in {}^{s\Phi}T^*_{\mathrm{bf}}\boldsymbol{X}$, differing only in the σ coordinate, such that the first point is in $\mathrm{WF}'_{\mathrm{bf}}(\eta R_j(\lambda)\chi)$ and the second point is in $\mathrm{WF}'_{\mathrm{bf}}(\eta R_j(-\lambda)\chi)$ (see Remark 6.6). This follows from (6-23); in fact, the coordinates (ν, ν') determine σ except on the sets $T_\pm$. However, on $T_\pm$, we find that $(y, y', \sigma, \mu = 0, \mu' = 0, \nu = \pm 1, \nu' = \mp 1)$ is in L_+ if and only if $\sigma \le 1$ and $\nu = 1$, or $\sigma \ge 1$ and $\nu = -1$, while it is in L_- if and only if $\sigma \le 1$ and $\nu = -1$, or $\sigma \ge 1$ and $\nu = 1$. But condition (8-13) implies that $\sigma \geq \gamma > 1$ on the support of the kernel of $\eta R_j(\pm\lambda)\chi$, so we see that indeed it is not possible to have $(y, y', \sigma, \mu, \mu', \nu, \nu') \in \mathrm{WF}'_{\mathrm{bf}}(\eta R_j(\lambda)\chi)$ and $(y, y', \sigma^*, \mu, \mu', \nu, \nu') \in \mathrm{WF}'_{\mathrm{bf}}(\eta R_j(-\lambda)\chi)$.

Now let $\mathcal{N}$ be the set of pairs (i, k), with $1 \leq i, k \leq N$, such that $Q_i \eta R_j(\lambda)\chi Q_k$ is not microlocally trivial. This means that if $(i, k) \in \mathcal{N}$, then $Q_i \eta R_j(-\lambda)\chi Q_k$ *is* microlocally trivial. Let us also observe that as the Q_i are uniformly bounded as operators $L^2 \to L^2$, and as they are Calderón–Zygmund operators in a uniform sense as $h \to 0$, then they are uniformly bounded as operators $L^p \to L^p$ for $1 < p < \infty$. Therefore we can compute that

$$\begin{aligned}
\|\eta R_j(\lambda)\chi\|_{L^p(M_j)\to L^2(M_j)} &\leq \sum_{i,k=1}^{N} \|Q_i \eta R_j(\lambda)\chi Q_k\|_{L^p(M_j)\to L^2(M_j)} \\
&= \sum_{(i,k)\in\mathcal{N}} \|Q_i \eta R_j(\lambda)\chi Q_k\|_{L^p(M_j)\to L^2(M_j)} + O(\lambda^{-\infty}) \\
&= \sum_{(i,k)\in\mathcal{N}} \|Q_i \eta (R_j(\lambda) - R_j(-\lambda))\chi Q_k\|_{L^p(M_j)\to L^2(M_j)} + O(\lambda^{-\infty}) \\
&= \frac{1}{2\pi\lambda} \sum_{(i,k)\in\mathcal{N}} \|Q_i \eta\, dE_j(\lambda)\chi Q_k\|_{L^p(M_j)\to L^2(M_j)} + O(\lambda^{-\infty}) \\
&\leq \frac{CN^2}{\lambda} \|\eta\, dE_j(\lambda)\chi\|_{L^p(M_j)\to L^2(M_j)} + O(\lambda^{-\infty}), \qquad (8\text{-}15)
\end{aligned}$$

proving the lemma. □

Remark 8.4. Observe that we missed the endpoint $p = 1$ due to our use of Calderón–Zygmund theory. In the case that M is exactly Euclidean for $x < 2\epsilon$ we can take M_j to be flat Euclidean space and then it is straightforward to check that $\eta R_j(\lambda)\chi$ is bounded $L^1(M_j) \to L^2(M_j)$ with norm $O(\lambda^{(n-3)/2})$, which gives us Proposition 8.2 for $p = 1$ in this case.

In [Seeger and Sogge 1989], spectral multiplier estimates are proved for compact manifolds for the same exponents as in Theorem 1.1. This was done using Sogge's discrete L^2 restriction theorem, i.e., Proposition 8.1. One may suspect that, since spectral multiplier estimates can be proved in the compact case, and since we have localized restriction estimates outside the trapped sets, that one should be able to prove spectral multiplier estimates on asymptotically conic manifolds unconditionally, i.e., without any nontrapping assumption. We have not been able to prove this, however, but have the following localized results:

Proposition 8.5. *Let M° be a manifold with Euclidean ends, and let $p \in [1, 2(n+1)/(n+3)]$. Let $\boldsymbol{H}$ be as in Theorem 1.3, let χ be a cutoff function as in Proposition 8.2, let F be a multiplier satisfying the assumption of Theorem 1.1, i.e., $F \in H^s$ for some $s > \max\left(n\left(\frac{1}{p} - \frac{1}{2}\right), \frac{1}{2}\right)$. Then we have*

$$\sup_{\alpha>0} \|F(\alpha\sqrt{\boldsymbol{H}})\chi\|_{p\to p} \leq C\|F\|_{H^s}.$$

This is proved by following the proof of Theorem 1.1, using (8-5) in place of (2-3).

Proposition 8.6. *Let $\omega \in C_c^\infty(M^\circ)$ be compactly supported and let $\boldsymbol{H}$ and F be as above. Then the following estimate holds*:

$$\sup_{\alpha>0} \|\omega F(\alpha\sqrt{\boldsymbol{H}})\|_{L^p\to L^p} \le \|F\|_{H^s}.$$

This is proved by following the method of [Seeger and Sogge 1989], using the compact support of ω to obtain the embedding from L^2 to L^p as in [ibid., Equation (3.11)].

8C. ***Examples with elliptic trapping.*** Here we show that the restriction estimate at high frequency generically fails for asymptotically conic manifolds with elliptic closed geodesics. Indeed, it has been proved by Babich and Lazutkin [1968] and Ralston [1977] that if there exists a closed geodesic γ in M such that the eigenvalues of the linearized Poincaré map of γ are of modulus 1 and are not roots of unity, then there exists a sequence of quasimodes $u_j \in C_0^\infty(K)$ with K a fixed compact set containing the geodesic, a sequence of positive real numbers $\lambda_j \to \infty$ such that for all $N > 0$ there is $C_N > 0$ such that

$$\|u_j\|_{L^2} = 1, \quad \|(\Delta_g - \lambda_j^2)u_j\|_{L^2} \le C_N \lambda_j^{-N}. \tag{8-16}$$

Proposition 8.7. *Assume that (M, g) is an asymptotically conic manifold with an elliptic closed geodesic such that the eigenvalues of the linearized Poincaré map of γ are of modulus 1 and are not roots of unity. Then for all $p \in [1, 2)$ and $M \ge 0$ the spectral measure $dE_{\sqrt{\Delta_g}}(\lambda)$ does* ***not*** *satisfy the restriction estimate*

$$\exists C > 0,\ \exists \lambda_0 > 0,\ \forall \lambda \ge \lambda_0, \quad \|dE_{\sqrt{\Delta_g}}(\lambda)\|_{L^p\to L^{p'}} \le C\lambda^M.$$

Proof. Let u_j be the quasimodes above. Then the inequality

$$\|(\Delta_g - \lambda_j^2)u_j\|_{L^2} \le C_N \lambda_j^{-N}$$

implies that

$$\left\| \mathbb{1}_{\mathbb{R}\setminus[\lambda_j^2 - 2C_N\lambda_j^{-N}, \lambda_j^2 + 2C_N\lambda_j^{-N}]}(\Delta_g)u_j \right\|_{L^2} \le \frac{1}{2}$$

since $\|(\Delta_g - \lambda_j^2)v\| \ge c\|v\|$ if v is in the range of the spectral projector $\mathbb{1}_{\mathbb{R}\setminus[\lambda_j^2 - c, \lambda_j^2 + c]}(\Delta_g)$. Therefore

$$\left\| \mathbb{1}_{[\lambda_j^2 - 2C_N\lambda_j^{-N}, \lambda_j^2 + 2C_N\lambda_j^{-N}]}(\Delta_g)u_j \right\|_{L^2} \ge \frac{\sqrt{3}}{2}, \tag{8-17}$$

and using the fact that $\mathbb{1}_{[\lambda_j^2 - 2C_N\lambda_j^{-N}, \lambda_j^2 + 2C_N\lambda_j^{-N}]}(\Delta_g)$ is a projection,

$$\left\langle u_j, \mathbb{1}_{[\lambda_j^2 - 2C_N\lambda_j^{-N}, \lambda_j^2 + 2C_N\lambda_j^{-N}]}(\Delta_g)u_j \right\rangle \ge \frac{3}{4}. \tag{8-18}$$

This implies that for large enough λ we have

$$\left\langle u_j, \mathbb{1}_{[\lambda_j - 2C_N\lambda_j^{-N-1}, \lambda_j + 2C_N\lambda_j^{-N-1}]}(\sqrt{\Delta_g})u_j \right\rangle \ge \frac{3}{4}. \tag{8-19}$$

Now assume that there exists C such that $\|dE_{\sqrt{\Delta_g}}(\lambda)\|_{L^p\to L^{p'}} \le C\lambda^M$. Then using the continuous embeddings from $L^2(K) \to L^p(K)$ and $L^{p'}(K)$ to $L^2(K)$, we see that there is $C' > 0$ such that

$$\langle u_j, dE_{\sqrt{\Delta_g}}(\lambda)u_j \rangle \le C'\lambda^M \|u_j\|_{L^2} \le 2C'\lambda^M.$$

By integrating this on the interval $[\lambda_j - 2C_N\lambda_j^{-N-1}, \lambda_j + 2C_N\lambda_j^{-N-1}]$, we contradict (8-19) if $N+1$ is chosen larger than M and j is large enough. □

Remark 8.8. In fact, one can construct examples where the spectral measure blows up exponentially with respect to the frequency λ. Consider a Riemannian manifold (M, g) which is a connected sum of flat $\mathbb{R}^n$ and a sphere S^n, so that it contains an open set S isometric to part of a round sphere S^n, namely

$$S = \{x = (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} ; |x| = 1, x_1^2 + x_2^2 > \tfrac{1}{4}\}.$$

Consider the functions $u_N(x) := (x_1 + ix_2)^N$ (as functions on $\mathbb{R}^{n+1}$). These restrict to eigenfunctions on S^n with corresponding eigenvalue $N(N+n-1)$ and with norm $\|u_N\|_{L^2} \sim cN^{-1/4}$ for some $c > 0$ as $N \to \infty$. Let $\chi \in C_0^\infty(S)$ be equal to 1 on $S \cap \{x_1^2 + x_2^2 \geq 1/2\}$ and extend it by 0 on $M \setminus S$. The modified function $v_N = \chi u_N / \|\chi u_N\|_{L^2}$ satisfies

$$(\Delta_g - N(N+n-1))v_N = [\Delta_g, \chi]u_N / \|\chi u_N\|_{L^2}.$$

But since $|x_1 + ix_2| < 1/2$ on the support of $[\Delta_g, \chi]$ and since $\|\chi u_N\| > CN^{-1/4}$ for some $C > 0$ when N is large, we deduce that $(\Delta_g - N(N+n-1))v_N = O_{L^2}(e^{-\alpha N})$ for some $\alpha > 0$. Applying the argument of Proposition 8.7, we deduce that there exist $C > 0$, $\beta > 0$ and a sequence $\lambda_N \sim \sqrt{N(N+n-1)}$ such that $\|dE(\lambda_N)\|_{L^p \to L^{p'}} \geq Ce^{\beta\lambda_N}$.

9. Conclusion: application and open problems

The restriction theorem can be applied to prove Sobolev estimates. Recall that the Hardy–Littlewood–Sobolev theorem tells us the inverse of the Laplacian, i.e., the resolvent at zero energy, on $\mathbb{R}^n$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ when $n \geq 3$ and $p = 2n/(n+2)$; this holds true on any asymptotically conic manifold. Since the resolvent looks like the spectral measure microlocally away from the diagonal, and since this value of p is in the range $[1, 2(n+1)/(n+3)]$ in which the spectral measure is bounded $L^p \to L^{p'}$ by Theorem 1.3, this suggests that the resolvent kernel $(\Delta - (\lambda \pm i0)^2)^{-1}$ on an asymptotically conic manifold should be bounded from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ when $p = 2n/(n+2)$. This result has been recently proved in [Guillarmou and Hassell 2012] and if in addition the metric is nontrapping, we have the following uniform Sobolev estimate: For $p = 2n/(n+2)$, $p' = 2n/(n-2)$ there exists $C > 0$ independent of $\lambda \in \mathbb{C}$ such that

$$\forall u \in W^{2,p}(M), \quad \|(\Delta - \lambda^2)u\|_{L^p} \geq C\|u\|_{L^{p'}}.$$

This was proved by Kenig–Ruiz–Sogge [1987] for constant coefficient operators on $\mathbb{R}^n$. The boundedness of the resolvent for $p \in [2n/(n+2), 2(n+1)/(n+3)]$ is also satisfied for $\lambda \neq 0$ but the constant is $O(|\lambda|^{n(1/p-1/p')-2)})$.

We mention several ways in which the investigations of this paper could be extended.

Theorem 1.3 is only stated for dimensions $n \geq 3$. This is because the proof relies on the analysis of [Guillarmou and Hassell 2008; Guillarmou et al. 2012], which is only done for $n \geq 3$. It would be interesting to treat also the case $n = 2$. The main difficulty in doing this is to write down a suitable inverse for the model operator at the zf face in the construction of [Guillarmou and Hassell 2008, Section 3], which is not invertible as an operator on $L^2(M)$ in two dimensions as it is in all higher dimensions.

One could also extend Theorem 1.3 by allowing potential functions which are $O(x^2)$ instead of only $O(x^3)$ at infinity, i.e., inverse-square decay near infinity. This should be relatively straightforward, because all the analysis has been done in the two papers cited above. For potentials of the form $V = V_0 x^2$, with V_0 strictly negative at ∂M, this would have the effect of changing the "numerology", i.e., the range of p and the power of λ in (1-4), for example. Here we preferred not to treat this case, in order not to complicate the statement of Theorem 1.3, but rather to keep the numerology as it is in the familiar setting of the classical Stein–Tomas theorem, and in Sogge's discrete L^2 restriction theorem.

Another way to extend Theorem 1.3 would be to allow operators $\boldsymbol{H}$ with eigenvalues. In this case, we would consider the positive part $\mathbb{1}_{(0,\infty)}(\boldsymbol{H})$ of the operator $\boldsymbol{H}$. We expect such a generalization to be straightforward, as the analysis has been carried out in [Guillarmou and Hassell 2008; Guillarmou et al. 2012], with the only complication being that $\mathbb{1}_{(0,\infty)}(\boldsymbol{H})$ does not satisfy the finite speed propagation property (2-2).

We close by posing, as open problems, some possible generalizations that seem to be a little less straightforward:

- Prove (or disprove) the restriction theorem for high energies in the presence of trapping, in the case that the trapped set is hyperbolic and the topological pressure assumption of [Nonnenmacher and Zworski 2009] and [Burq et al. 2010] is satisfied.
- Prove (or disprove) the spectral multiplier result for high energies in the trapping case, i.e., Propositions 8.5 and 8.6 without the cutoff functions.

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