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HOMOGENIZATION OF NEUMANN BOUNDARY DATA WITH FULLY NONLINEAR OPERATOR

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In this paper we study periodic homogenization problems for solutions of fully nonlinear PDEs in half-spaces with oscillatory Neumann boundary data. We show the existence and uniqueness of the homogenized Neumann data for a given half-space. Moreover, we show that there exists a continuous extension of the homogenized slope as the normal of the half-space varies over "irrational" directions.

1. Introduction

In this paper, we consider the averaging phenomena for solutions of uniformly elliptic nonlinear PDEs in half-spaces coupled with oscillatory Neumann boundary data. To be precise, let $\mathcal{M}^{n-1}$ be the normed space of symmetric $n \times n$ matrices and consider the function $F(M) : \mathcal{M}^{n-1} \to \mathbb{R}$, which satisfies:

(F1) F is uniformly elliptic, that is, there exist constants $0 < \lambda < \Lambda$ such that

$$\lambda \|N\| \leq F(M) - F(M+N) \leq \Lambda \|N\| \quad \text{for any} \quad N \geq 0;$$

(F2) (homogeneity) $F(tM) = tF(M)$ for any $M \in \mathcal{M}^{n-1}$ and $t > 0$. In particular, $F(0) = 0$.

(F3) $F(M)$ only depends on the eigenvalues of M.

The homogeneity condition (F2) can be relaxed (see condition (F4) of [Barles et al. 2008], for example). Typical examples of nonlinear operators that satisfy (F1)–(F3) are the Pucci extremal operators

$$\mathcal{P}^+(D^2u(x)) := \lambda \sum_{\mu_i<0} \mu_i + \Lambda \sum_{\mu_i \geq 0} \mu_i, \quad \mathcal{P}^-(D^2u(x)) := \Lambda \sum_{\mu_i<0} \mu_i + \lambda \sum_{\mu_i \geq 0} \mu_i,$$

where $\mu_1, \dots, \mu_n$ are eigenvalues of $D^2u(x)$.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ and suppose $g(x) : \mathbb{R}^n \to \mathbb{R}$ satisfies

(a) $g \in C^\beta(\mathbb{R}^n)$ for some $0 < \beta \leq 1$;

(b) $g(x + e_k) = g(x)$ for all $x \in \mathbb{R}^n$ and $k = 1, \dots, n$.

Next, for a given $p \in \mathbb{R}^n$, let $\Pi_\nu(p)$ be a strip domain in $\mathbb{R}^n$ with unit normal ν, that is,

$$\Pi_\nu(p) = \{x : -1 \leq (x-p) \cdot \nu \leq 0\}, \quad \text{where} \quad |\nu| = 1. \tag{1}$$

With F, g and Π_ν as given above, our goal is to describe the limiting behavior of u_ε as $\varepsilon \to 0$, where

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u_ε satisfies

$$(P_\varepsilon)\qquad \begin{cases} F(D^2u_\varepsilon)=0 & \text{in } \Pi_\nu(p),\\ \nu\cdot Du_\varepsilon = g(x/\varepsilon) & \text{on } \Gamma_0 := \{(x-p)\cdot\nu = 0\},\\ u=1 & \text{on } \Gamma_I := \{(x-p)\cdot\nu=-1\}.\end{cases}$$

The fixed boundary data on Γ_I is introduced to avoid discussion of the compatibility condition on g and to ensure the existence of u^ε.

Homogenization of elliptic, divergence-form equations with oscillatory coefficients and conormal boundary data is a classical subject. Let Ω be an open and bounded subset of $\mathbb{R}^n$. Consider $u^\varepsilon : \bar{\Omega}\to\mathbb{R}$ solving

$$\nabla\cdot\Big(A\Big(\frac{x}{\varepsilon}\Big)\nabla u_\varepsilon\Big)=0, \tag{2}$$

with the Neumann (conormal) condition

$$\nu\cdot\Big(A\Big(\frac{x}{\varepsilon}\Big)\nabla u\Big)(x) = g\Big(\frac{x}{\varepsilon}\Big),\quad x\in\partial\Omega. \tag{3}$$

The problem (2)–(3) has been widely studied, and by now has been well understood; see [Bensoussan et al. 1978] for an overview. We first consider the case when Ω is a half-space; thus, let

$$\Omega = \Sigma_\nu := \{x : (x-p)\cdot\nu \le 0\}.$$

We define the averaged Neumann data

$$\mu(\nu,\varepsilon) := \int_{(x-p)\cdot\nu=0,\,|x-p|\le 1} g\Big(\frac{x}{\varepsilon}\Big)\,dx. \tag{4}$$

Integrating by parts, one can show that u^ε locally uniformly converges to a continuous function $u^0 : \bar{\Omega}\to\mathbb{R}$ as $\varepsilon\to 0$ if and only if $\mu(\nu) := \lim_{\varepsilon\to 0}\mu(\nu,\varepsilon)$ exists, and that u^0 solves the averaged equation

$$(\bar{P}_{\text{div}})\qquad \begin{cases} -\nabla\cdot(A^0\nabla u^0)(x) = 0 & \text{for } x\in\Omega,\\ \nu\cdot(A^0\nabla u^0) = \mu(\nu) & \text{for } x\in\partial\Omega.\end{cases}$$

Therefore, different results hold depending on the choice of p and ν:

(a) If ν is a "rational" vector — one parallel to a vector in $\mathbb{Z}^n$ — then $\mu(\nu)$ exists if $p=0$, and

$$\mu(\nu) = \text{the average of } g(y) \text{ on the hyperplane } \{x\cdot\nu = 0\}.$$

(b) If ν is a rational vector and $p\ne 0$, then there may be no limit of $\mu(\nu,\varepsilon)$ and u^ε can have different subsequential limits.

(c) If ν is not a rational vector, then due to Weyl's equidistribution theorem (Lemma 2.5), $\mu(\nu,\varepsilon)$ converges to

$$\mu(\nu) = \langle g\rangle := \int_{[0,1]^n} g(y)\,dy,$$

independent of the choice of p. In particular, the homogenized slope $\mu(\nu)$ is discontinuous at every rational direction ν, but otherwise continuous.

From these results, the divergence form of the operator, and the fact that rational directions are of zero measure in $\mathcal{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$, the following results hold for the general domain Ω: if $\partial\Omega$ does not contain flat pieces whose normal vectors belong to $\mathbb{R}\mathbb{Z}^n$, then u^ε converges locally uniformly to the solution u^0 of $(\bar{P}_{\text{div}})$ with $\mu(\nu)$ replaced by $\langle g \rangle$. We refer to [Bensoussan et al. 1978] for detailed analysis. Note that u^0 is smooth up to the boundary, due to the fact that $\langle g \rangle$ is continuous (constant in particular).

For nonlinear or nondivergence operators, or for linear operators with oscillatory nonlinear boundary data, little is known for the homogenization of the oscillating Neumann boundary data. Most available results concern half-space domains going through the origin with its normal pointing to a rational direction. Tanaka [1984] considered some model problems in half-spaces whose boundary is parallel to the axes of the periodicity, by purely probabilistic methods. Arisawa [2003] studied special cases of problems in oscillatory domains near half-spaces going through the origin, using viscosity solutions as well as stochastic control theory. Generalizing her results, Barles, Da Lio and Souganidis [Barles et al. 2008] studied the problem for operators with oscillating coefficients, in half-space domains whose boundary is parallel to the axes of periodicity, with a series of assumptions which guarantee the existence of an approximate corrector.

In this paper, we extend the results above to the setting of general half-spaces Π_ν, defined in (1), where p is not necessarily zero and ν ranges over all directions in $\mathbb{R}^n$. In particular, we show the continuity properties of the homogenized slope $\mu(\nu)$ over the normal directions ν (see Theorem 1.2(ii)), with the hope that such results will lead to better understanding of homogenization phenomena in domains with general geometry (work in progress). Note that, as observed in the linear case, homogenized slopes may not exist if ν is parallel to a vector in $\mathbb{Z}^n$ and if $p \neq 0$, and therefore the best result we can hope for is the existence of the continuous function $\bar{\mu}(\nu) : S^{n-1} \to \mathbb{R}$ such that $\bar{\mu}(\nu) = \mu(\nu)$ for $\nu \in \mathcal{S}^{n-1} - \mathbb{R}\mathbb{Z}^n$. This is precisely what we will show.

Definition 1.1. A direction $\nu \in \mathcal{S}^{n-1}$ is called *rational* if $\nu \in \mathbb{R}\mathbb{Z}^n$, and *irrational* otherwise.

Theorem 1.2 (Main Theorem). *For a given $p \in \mathbb{R}^n$, let u_ε solve (P_ε).*

(i) *Let ν be an irrational direction. Then there is a unique constant $\mu(\nu) \in [\min g, \max g]$ such that u^ε locally uniformly converges to the solution of*

$$(\bar{P}) \qquad \begin{cases} F(D^2u) = 0 & \text{in } \Pi_\nu, \\ \nu \cdot Du = \mu(\nu) & \text{on } \Gamma_0, \\ u = 1 & \text{on } \Gamma_I. \end{cases}$$

(ii) $\mu(\nu) : (\mathcal{S}^{n-1} - \mathbb{R}\mathbb{Z}^n) \to \mathbb{R}$ *has a continuous extension* $\bar{\mu}(\nu) : \mathcal{S}^{n-1} \to \mathbb{R}$.

(iii) *For rational directions ν, if Γ_0 goes through the origin (that is if $p = 0$), then the statement in (i) holds for ν as well.*

(iv) *(Error estimate). Let ν be an irrational direction. Then for u^ε and u solving (P_ε) and $(\bar{P})$, we have the following estimate: for any $0 < \alpha < 1$, there exists a constant $C_\alpha > 0$ such that*

$$|u^\varepsilon - u| \leq C_\alpha \omega(\varepsilon)^\alpha \quad \text{in } \Pi_\nu. \tag{5}$$

Here $\omega(\varepsilon)$ depends on the "discrepancy" associated to ν as defined in (7).

Remark 1.3. Our method can be applied to the operators of the form $F(D^2u, x) = f(x)$, with F and f continuous in x, but we will restrict ourselves to the simple case discussed in (P_ε) for clarity of exposition. On the other hand, our proof for the continuity of $\mu(\nu)$ (Theorem 1.2(ii)) on page 965, cannot handle the case where the operator F depends on the oscillatory variable x/ε (see Remark 4.8).

2. Preliminary results

Let Ω be an open, bounded domain. Let Γ_I be a part of its boundary, and define $\Gamma_0 := \partial\Omega - \Gamma_I$. For a continuous function $f(x, \nu) : \mathbb{R}^n \times \mathcal{S}^{n-1} \to \mathbb{R}$, let us recall the definition of viscosity solutions for the following problem:

$$(P)_f \qquad \begin{cases} F(D^2u) = 0 & \text{in } \Omega, \\ \nu \cdot Du = f(x, \nu) & \text{on } \Gamma_0, \\ u = 1 & \text{on } \Gamma_I, \end{cases}$$

where $\nu = \nu_x$ denotes the outward normal at $x \in \partial\Omega$ with respect to Ω.

The following definition is equivalent to the ones given in [Crandall et al. 1992]:

Definition 2.1. (a) An upper semicontinuous function $u : \bar{\Omega} \to \mathbb{R}$ is a *viscosity subsolution* of $(P)_f$ if

(i) $u \le 1$ on Γ_I, and

(ii) for a given domain $\Sigma \subset \mathbb{R}^n$, u cannot cross from below any C^2 function ϕ in Σ which satisfies

$$\begin{cases} F(D^2\phi) > 0 & \text{in } \Omega \cap \Sigma, \\ \nu \cdot D\phi > f(x, \nu) & \text{on } \Gamma_0 \cap \Sigma, \\ \phi > u & \text{on } (\partial\Sigma \cup \Gamma_I) \cap \bar{\Sigma}. \end{cases}$$

(b) A lower semicontinuous function $u : \bar{\Omega} \to \mathbb{R}$ is a *viscosity supersolution* of $(P)_f$ if:

(i) $u \ge 1$ on Γ_I;

(ii) for a given domain $\Sigma \subset \mathbb{R}^n$, u cannot cross from above any C^2 function φ which satisfies

$$\begin{cases} F(D^2\phi) < 0 & \text{in } \Omega \cap \Sigma, \\ \nu \cdot D\phi < f(x, \nu) & \text{on } \Gamma_0 \cap \Sigma, \\ \phi < u & \text{on } (\partial\Sigma \cup \Gamma_I) \cap \bar{\Sigma}. \end{cases}$$

(c) u is a *viscosity solution* of $(P)_f$ if u is both a viscosity sub- and supersolution of $(P)_f$.

Existence and uniqueness of viscosity solutions of $(P)_f$ is based on the comparison principle we state below:

Theorem 2.2 [Ishii and Lions 1990, Section V]. *Suppose* Ω, Γ_I, Γ_0, F *and* ν *are as given above, and let* $f : \mathbb{R}^n \times \mathcal{S}^{n-1} \to \mathbb{R}$ *be continuous. Let u and v respectively be a viscosity sub- and supersolution of* $(P)_f$ *in a domain* $\Sigma \subset \mathbb{R}^n$. *If* $u \le v$ *on* $\partial\Sigma$, *then* $u \le v$ *in* Ω.

For details on the proof of this theorem as well as well-posedness of the problem $(P)_f$, we refer to [Crandall et al. 1992; Ishii 1991; Ishii and Lions 1990].

Next we state some regularity results that will be used in the paper.

Theorem 2.3 [Caffarelli and Cabré 1995, Chapter 8, modified for our setting]. *Let u be a viscosity solution of $F(D^2u) = 0$ in a domain Ω. For any $0 < \alpha < 1$ and for any compact subset Ω' of Ω, we have*

$$\|u\|_{C^\alpha(\Omega')} \le C d^{-\alpha} \|u\|_{L^\infty(\Omega)},$$

where $C > 0$ depends on n, λ, Λ and $d = d(\Omega', \partial\Omega)$.

Theorem 2.4 [Milakis and Silvestre 2006, Theorems 8.1 and 8.2]. *Let*

$$B_r^+ := \{|x| < r\} \cap \{x \cdot e_n \ge 0\} \quad \text{and} \quad \Gamma := \{x \cdot e_n = 0\} \cap B_1.$$

Let u be a viscosity solution of

$$\begin{cases} F(D^2u) = 0 & \text{in } B_1^+, \\ \nu \cdot Du = g & \text{in } \Gamma. \end{cases}$$

(a) *If g is bounded, then u is in $C^\alpha(\overline{B_{1/2}^+})$ for some $\alpha = \alpha(n, \lambda, \Lambda)$, and we have the estimate*

$$\|u\|_{C^\alpha(\overline{B_{1/2}^+})} \le C\big(\|u\|_{L^\infty(\overline{B_1^+})} + \max \|g\|\big).$$

(b) *Suppose $g \in C^\beta(\mathbb{R}^n)$, where $0 < \beta \le 1$. Then u is in $C^{1,\gamma}(\overline{B_{1/2}^+})$, where $\gamma = \min(\alpha_0, \beta)$ and $\alpha_0 = \alpha_0(n, \lambda, \Lambda)$. Moreover, we have the estimate*

$$\|u\|_{C^{1,\alpha}(\overline{B_{1/2}^+})} \le C\big(\|u\|_{L^\infty(\overline{B_1^+})} + \|g\|_{C^\beta}\big).$$

In (a) *and* (b), *the positive constant C depends only on n, λ, Λ and α.*

Let us next discuss the averaging property of the sequence $(nx)_n$ mod 1, where x is an irrational number, and its applications to dimensions greater than 1, which will prove useful in our analysis in Section 3. Since we obtain estimates on the convergence rate of solutions for (P_ε) in our result, we are particularly interested in the estimates on the rate of convergence of the sequence $(nx)_n$ to the uniform distribution (Definition 2.6). We begin by recalling the notion of equidistribution.

- A bounded sequence $(x_1, x_2, x_3 \dots)$ of real numbers is said to be *equidistributed* on an interval $[a, b]$ if for any $[c, d] \subset [a, b]$, we have

$$\lim_{n\to\infty} \frac{\big|\{x_1, \dots, x_n\} \cap [c, d]\big|}{n} = \frac{d - c}{b - a}.$$

  Here $|\{x_1, \dots, x_n\} \cap [c, d]|$ denotes the number of elements.

- The sequence $(x_1, x_2, x_3, \dots)$ is said to be equidistributed modulo 1 if $(x_1 - [x_1], x_2 - [x_2], \dots)$ is equidistributed in the interval $[0, 1]$.

Lemma 2.5 [Weyl 1910, Weyl's equidistribution theorem]. *If a is an irrational number, $(a, 2a, 3a, \dots)$ is equidistributed modulo* 1.

To discuss quantitative versions of Lemma 2.5, we introduce the notion of *discrepancy*.

Definition 2.6 [Kuipers and Niederreiter 1974]. Let (x_k), $k = 1, 2, \dots$ be a sequence in $\mathbb{R}$. For a subset $E \subset [0, 1]$, let $A(E; N)$ denote the number of points $\{x_n\}$, $1 \le n \le N$, that lie in E.

(a) The sequence (x_n), $n = 1, 2, \dots$ is said to be *uniformly distributed* mode 1 in $\mathbb{R}$ if

$$\lim_{N\to\infty} \frac{A(E; N)}{N} = \mu(E)$$

for all $E = [a, b)$. Here μ denotes the Lebesgue measure.

(b) For $x \in [0, 1]$, we define the discrepancy

$$D_N(x) := \sup_{E=[a,b)} \left| \frac{A(E; N)}{N} - \mu(E) \right|,$$

where $A(E; N)$ is defined with the sequence (kx), $k \in \mathbb{N}$, modulo 1.

It easily follows from Lemma 2.5 that the sequence $(x_k) = (kx)_{k\in\mathbb{N}}$ is uniformly distributed modulo 1 for any irrational number $x \in \mathbb{R}$. In particular, $D_N(x)$ converges to zero as $N \to \infty$.

Next, let $\mathcal{S}^{n-1} = \{\nu \in \mathbb{R}^n : |\nu| = 1\}$. For a direction $\nu = (\nu_1, \dots, \nu_n) \in \mathcal{S}^{n-1}$, let ν_i be the component with the biggest size, that is,

$$|\nu_i| = \max\{|\nu_j| : 1 \le j \le n\}.$$

(If there are multiple components, then we choose the one with largest index.)

Let H_ν be the hyperplane in $\mathbb{R}^n$ which passes through 0 and is normal to ν:

$$H_\nu = \{x \in \mathbb{R}^n : x \cdot \nu = 0\}.$$

Since $\nu_i \neq 0$, there exists $m(\nu)$ such that

$$(1, \dots, 1, m(\nu), 1, \dots, 1) \cdot \nu = 0, \tag{6}$$

where $m(\nu)$ is the i-th component of $(1, \dots, 1, m(\nu), 1, \dots, 1)$. Then we define

$$\omega_\nu(\varepsilon) := D_N(m(\nu)), \quad \text{where } N = \varepsilon^{-9/10}. \tag{7}$$

Note that, if $m(\nu)$ is irrational, then $\omega_\nu(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now we are ready to state our quantitative estimate on the averaging properties of the vector sequence $(n\nu)$ with an irrational direction ν, which will be used in the rest of the paper. Recall that for $\nu \in \mathcal{S}^{n-1}$, $\Pi_\nu(p) = \{x : -1 \le (x - p) \cdot \nu \le 0\}$. Write $\Gamma_0 = \{x : (x - p) \cdot \nu = 0\}$ and define

$$H_\nu = \{x : x \cdot \nu = 0\}.$$

Lemma 2.7. *For $\nu \in \mathbb{R}^n$ and $x_0 \in \Pi_\nu$, let $H(x_0) := H_\nu + x_0$. Let $0 < \varepsilon < \operatorname{dist}(x_0, \Gamma_0)$.*

(i) *Suppose that ν is a rational direction. Then for any $x \in H(x_0)$, there is $y \in H(x_0)$ such that*

$$|x - y| \le M_\nu \varepsilon, \quad y - x_0 \in \varepsilon\mathbb{Z}^n,$$

where $M_\nu > 0$ is a constant depending on ν.

(ii) *Suppose that ν is an irrational direction, and let $\omega_\nu : [0, 1) \to \mathbb{R}^+$ be defined as in* (7). *Then there exists a dimensional constant $M > 0$ such that the following is true*: *for any $x \in H(x_0)$, there is $y \in \mathbb{R}^n$ such that*

$$|x - y| \leq M\varepsilon^{1/10}, \quad y - x_0 \in \varepsilon\mathbb{Z}^n$$

and

$$\text{dist}(y, H(x_0)) < \varepsilon\omega_\nu(\varepsilon), \tag{8}$$

where ω_ν is as given in (7).

(iii) *If ν is an irrational direction, then for any $z \in \mathbb{R}^n$ and $\delta > 0$, there is $w \in H(x_0)$ such that*

$$|z - w| \leq \delta \textit{ mod } \varepsilon\mathbb{Z}^n.$$

Proof. The proof of (i) is immediate from the fact that for any rational direction ν, there exists an integer $M > 0$ depending on ν such that $M\nu \in \mathbb{Z}^n$.

Next, we prove (ii). Let ν be an irrational direction in $\mathbb{R}^n$. Without loss of generality, we may assume

$$|\nu_n| = \max\{|\nu_j| : 1 \leq j \leq n\}.$$

Let x be any point on $H(x_0)$: after a translation, we may assume that $x = 0$. Choose m such that

$$\varepsilon(1, 1, \ldots, 1, m) \in H(x_0).$$

Note that $M = |m| \leq n^2$. Also note that m is irrational since ν is an irrational direction. Since $H(x_0)$ contains $x = 0$, we have

$$k\varepsilon(1, 1, \ldots, 1, m) \in H(x_0) \text{ for any integer } k.$$

Consider the sequence (km), $k \in \mathbb{N}$. From the definition of $\omega_\nu(\varepsilon)$ and the discrepancy function $D_N(m)$, it follows that any interval $[a, b] \subset [0, 1]$ of length $\omega_\nu(\varepsilon)$ contains at least one point km (mod 1), for some $k \leq N = \varepsilon^{-9/10}$.

Hence for any $z = (0, 0, \ldots, 0, x_n) \in [0, \varepsilon]^n$, there exists

$$w = k\varepsilon(1, 1, \ldots, 1, m) \in H(x_0), \quad 0 \leq k \leq \varepsilon^{-9/10}$$

such that

$$|z - w| \leq \varepsilon\omega_\nu(\varepsilon) \mod \varepsilon\mathbb{Z}^n.$$

Similarly, for any $z \in [0, \varepsilon]^n$, there exists $w \in H(x_0) \cap \big(k\varepsilon(1, 1, \ldots, 1, m) + [0, \varepsilon]^n\big)$ such that

$$|z - w| \leq \varepsilon\omega_\nu(\varepsilon) \mod \varepsilon\mathbb{Z}^n, \quad 0 \leq k \leq \varepsilon^{-9/10}. \tag{9}$$

We continue with the proof of (ii). Recall that the coordinates are shifted so that $x = 0$. Thus it suffices to find $y \in \mathbb{R}^n$ such that

$$|x - y| = |y| \leq M\varepsilon^{1/10}, \quad |y - x_0| = 0 \mod \varepsilon\mathbb{Z}^n$$

and

$$\text{dist}(y, H(x_0)) < \varepsilon\omega_\nu(\varepsilon).$$

By (9), there exists $w \in H(x_0)$ such that

$$|x - w| = |w| \le Mk\varepsilon \le M\varepsilon^{1/10} \tag{10}$$

and

$$|x_0 - w| \le \varepsilon\omega_\nu(\varepsilon) \mod \varepsilon\mathbb{Z}^n. \tag{11}$$

Given w satisfying (11), we can take $y \in \mathbb{R}^n$ such that

$$|x_0 - y| = 0 \mod \varepsilon\mathbb{Z}^n, \quad |y - w| \le \varepsilon\omega_\nu(\varepsilon).$$

Then, by (10),

$$|y| \le |y - w| + |w| \le M\varepsilon^{1/10} + \varepsilon\omega_\nu(\varepsilon) \le M\varepsilon^{1/10}.$$

Also, since w is contained in $H(x_0)$, we have $\operatorname{dist}(y, H(x_0)) \le |y - w| \le \varepsilon\omega_\nu(\varepsilon)$, proving (ii).

Finally, (iii) is a direct consequence of (9). □

3. In the strip domain

Fix $p \in \mathbb{R}^n$ and $\nu \in \mathcal{S}^{n-1}$ such that $p \cdot \nu \ne 0$. Let

$$\Pi = \Pi_\nu = \{x \in \mathbb{R}^n : -1 \le (x - p) \cdot \nu \le 0\}.$$

We consider a bounded viscosity solution u_ε of

$$(P_\varepsilon) \qquad \begin{cases} F(D^2 u_\varepsilon) = 0 & \text{in } \Pi, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = g\Big(\dfrac{x}{\varepsilon}\Big) & \text{on } \Gamma_0 := \{x : (x - p) \cdot \nu = 0\}, \\ u_\varepsilon = 1 & \text{on } \Gamma_I := \{x : (x - p) \cdot \nu = -1\}. \end{cases}$$

Below we prove the existence and uniqueness of u_ε.

Lemma 3.1. *Let $f(x) : \mathbb{R}^n \to \mathbb{R}$ be continuous and bounded. Let Π be as given above and define $B_R(p) := \{|x - p| \le R\}$. Suppose w_1 and w_2 solve, in the viscosity sense,*

(a) *$F(D^2 w_1) = 0$ and $F(D^2 w_2) = 0$ in $\Sigma_R := \Pi \cap B_R(p)$;*

(b) *$\partial w_1/\partial \nu = f(x) = \partial w_2/\partial \nu$ on Γ_0;*

(c) *$w_1 = w_2$ on Γ_I;*

(d) *$w_1 = -M$, $w_2 = M$ on $\Pi \cap \partial B_R(p)$.*

Then, for $R > 2$ and $C = \dfrac{n\Lambda}{\lambda}$, we have

$$w_1 \le w_2 \le w_1 + \frac{3CM}{R^2} \quad \text{in } \Pi \cap B_1(p).$$

Proof. Without loss of generality, let us set $\nu = e_n$ and $p = 0$. The first inequality, $w_1 \le w_2$, directly follows from Theorem 2.2. To show the second inequality, consider $\tilde{\omega} := w_1 + M(h_1 + h_2)$, where

$$h_1 = \frac{1}{R^2}\big((x_1)^2 + \cdots + (x_n)^2\big) \quad \text{and} \quad h_2 = \frac{C}{R^2}\big(1 - (x_n)^2\big),$$

with $C = \dfrac{n\Lambda}{\lambda}$. We claim $w_2 \le \tilde{\omega}$. To see this, note that

$$F(D^2\tilde{\omega}) = F(D^2 w_1 + D^2 h_1 + D^2 h_2)$$
$$\ge F(D^2 w_1) + \frac{2}{R^2}(C\lambda - n\Lambda) \ge F(D^2 w_1) \text{ in } \Sigma_R.$$

On the boundary of Σ_R, $\tilde{\omega}$ satisfies

$$\partial_{x_n}\tilde{\omega} = \partial_{x_n}\omega_1 = \partial_{x_n}\omega_2 \quad \text{on } \Sigma_R \cap \{x_n = 0\}$$

and

$$w_2 \le \tilde{\omega} \quad \text{on } \Gamma_I \cap B_R(0) \text{ and on } \partial B_R(0) \cap \Pi.$$

It follows from Theorem 2.2 that $w_2 \le \tilde{\omega}$ in Σ_R, and we are done. □

Lemma 3.2. *There exists a unique bounded solution u of (P_ε).*

Proof. 1. Let Σ_R be as given in Lemma 3.1, and consider the viscosity solution $\omega_R(x)$ of (P_ε) in Σ_R with the lateral boundary data $M = 1$ on $\partial B_R(p) \cap \Pi$. The existence and uniqueness of the viscosity solution ω_R is shown, for example, in [Crandall et al. 1992; Ishii 1991; Ishii and Lions 1990].

By the maximum principle, $\omega_R \le 1 + \max(g)$ in Σ_R. Due to Theorem 2.4 and the Arzelà–Ascoli Theorem, ω_R locally uniformly converges to a continuous function $u_\varepsilon(x)$. Then by the stability property of viscosity solutions, it follows that $u_\varepsilon(x)$ is a viscosity solution of (P_ε).

2. To show uniqueness, suppose u_1 and u_2 are both viscosity solutions of (P_ε) with $|u_1|, |u_2| \le M$. Then Lemma 3.1 yields that, for any point $q \in \Gamma_0$ and any $R > 2$,

$$|u_1 - u_2| \le O\Big(\frac{1}{R^2}\Big) \quad \text{in } B_1(q) \cap \Pi.$$

Hence $u_1 = u_2$. □

The following is immediate from Theorem 2.2 and the construction of u_ε in the above lemma.

Corollary 3.3. *Suppose u and v are bounded and continuous in $\bar{\Pi}_\nu(p)$, and solve*

a) $F(D^2u) \le 0 \le F(D^2v)$ *in* $\Pi_\nu(p)$;

b) $u \le v$ *on* Γ_I;

c) $\partial u/\partial\nu \le f(x) \le \partial v/\partial\nu$ *on* Γ_0;

where $f(x) : \mathbb{R}^n \to \mathbb{R}$ is continuous. Then $u \le v$ in $\Pi_\nu(p)$.

In the rest of this section, we will repeatedly use the fact that linear profiles as well as constants solve $F(D^2u) = 0$.

Lemma 3.4. *Let $\Pi_\nu(p)$ be as given in (P_ε) and let $0 < \varepsilon < 1$. Suppose that w_1 and w_2 are bounded and solve, in the viscosity sense,*

$$\begin{cases} F(D^2 w_i) = 0 & \text{in } \Pi_\nu(p), \\ |w_1 - w_2| \le \varepsilon & \text{on } \Gamma_I, \\ \dfrac{\partial w_1}{\partial\nu} - \dfrac{\partial w_2}{\partial\nu} = A & \text{on } \Gamma_0. \end{cases}$$

Then there exists a positive constant $C = C(A)$ such that

$$|w_1 - w_2| \geq C - \varepsilon \quad \text{in } \Pi_\nu(p) \cap B_{1/2}(p).$$

Proof. Let $\tilde{w} := w_2 + h$, where $h(x) = A(x-p)\cdot\nu + A - \varepsilon$. Then $\partial\tilde{\omega}/\partial\nu = \partial\omega_1/\partial\nu$ on Γ_0. Also, $\tilde{\omega} \leq w_1$ on Γ_I. Therefore, Corollary 3.3 yields that $w_2 + h \leq w_1$. Since $h \geq A/2 - \varepsilon$ in $B_{1/2}(p)$, we are done. □

Lemma 3.5. *Let $\tilde{\Pi} = \Pi + a\nu$ for some $0 \leq a \leq A\varepsilon$, where $0 < A < 1$. Suppose u_ε and $\tilde{u}_\varepsilon$ are bounded, and solve (P_ε) respectively in the domains Π and $\tilde{\Pi}$. Then we have*

$$|u_\varepsilon - \tilde{u}_\varepsilon| \leq C(A^\beta + \varepsilon^\alpha) \quad \text{in } \Pi \cap \tilde{\Pi},$$

where α is as given in Theorem 2.4 and β is the Hölder exponent of g.

Proof. 1. Let $v_\varepsilon(x) = \tilde{u}_\varepsilon(x + a\nu)$, so that v_ε and u_ε are defined in the same domain Π. Since $g(x) \in C^\beta(\mathbb{R}^n)$, $|\partial v_\varepsilon/\partial\nu - \partial u_\varepsilon/\partial\nu| \leq A^\beta$ on Γ_0.

2. On Γ_I, $u_\varepsilon = v_\varepsilon = 1$. Hence one can compare $u_\varepsilon \pm A^\beta(1 + (x-p)\cdot\nu)$ with v_ε and apply Theorem 2.2 to obtain

$$|u_\varepsilon - v_\varepsilon| \leq A^\beta \quad \text{in } \Pi.$$

Due to the Hölder continuity of u^ε given by Theorem 2.4, $|v_\varepsilon - \tilde{u}_\varepsilon| \leq C\varepsilon^\alpha$ in $\Pi \cap \tilde{\Pi}$. This finishes the proof. □

The next lemma follows from Theorem 2.4(b).

Lemma 3.6. *Let v_j be a bounded solution of (P_ε) with a constant Neumann condition $g(x) = \mu_j$. If $\mu_j \to \mu$, then v_j converges to v such that $\partial v/\partial\nu = \mu$ on Γ_0.*

4. Proof of the Main Theorem

We will prove first parts (i), (iii) and (iv) of Theorem 1.2; the proof of part (ii) starts on page 965.

Recall that

$$\Gamma_0 = \{x : (x-p)\cdot\nu = 0\}, \quad \Gamma_I = \{x : (x-p)\cdot\nu = -1\}.$$

Due to the uniform Hölder regularity of $\{u_\varepsilon\}$ (Theorem 2.4(a)), along subsequences $u_{\varepsilon_j} \to u$ in $\bar{\Pi}_\nu$. Note that there could be different limits along different subsequences (ε_j). Below, we will show that if ν is an irrational direction, all subsequential limits of (u_ε) coincide.

Suppose

$$0 \in \Pi_\nu = \{-1 < (x-p)\cdot\nu < 0\}.$$

Let us choose a convergent subsequence and rename it (u_j). For each j, there exists a constant μ_j and a function v_j in $\Pi_\nu(p)$ such that

$$(P_{\mu_j}) \qquad \begin{cases} F(D^2 v_j) = 0 & \text{in } \Pi_\nu(p), \\ \partial v_j/\partial\nu = \mu_j & \text{on } \Gamma_0, \\ v_j = u_j = 1 & \text{on } \Gamma_I, \\ v_j = u_j & \text{at } x = 0. \end{cases}$$

Lemma 4.1. *We have $\mu_j \to \mu$ for some μ as $j \to \infty$.* (The limit may depend on the subsequence chosen.)

Proof. Suppose not; then there is a constant $A > 0$ such that for any $N > 0$, $|\mu_m - \mu_n| \geq A$ for some $m, n > N$. Then, by Lemma 3.4,

$$|v_m(0) - v_n(0)| \geq C_A.$$

This contradicts the fact that $v_j(0) = u_j(0)$, since $u_j(0) \to u(0)$ as $j \to \infty$. □

The next lemma states that u_ε looks like a linear profile with respect to the direction ν as $\varepsilon \to 0$.

Lemma 4.2. *Away from the Neumann boundary Γ_0, u_ε is almost a constant on hyperplanes parallel to Γ_0. More precisely, let $x_0 \in \Pi_\nu(p)$ with $\mathrm{dist}(x_0, \Gamma_0) > \varepsilon^{1/20}$, and let $0 < \alpha < 1$. Then:*

(i) *If ν is a rational direction, there exists a constant $C > 0$ depending on ν, α and n, such that for any $x \in H(x_0) := \{(x - x_0) \cdot \nu = 0\}$,*

$$|u_\varepsilon(x) - u_\varepsilon(x_0)| \leq C\varepsilon^{\alpha/2}. \tag{12}$$

(ii) *If ν is any irrational direction, there exists a constant $C > 0$ depending on α and n, such that for any $x \in H(x_0)$,*

$$|u_\varepsilon(x) - u_\varepsilon(x_0)| \leq C\varepsilon^{\alpha/20} + C\omega_\nu(\varepsilon)^\beta, \tag{13}$$

where $\omega_\nu : [0, 1) \to [0, \infty)$ is a mode of continuity given as in (ii) of Lemma 2.7.

Proof. First, let ν be a rational direction. Lemma 2.7 implies that for any $x \in H(x_0)$, there is $y \in H(x_0)$ such that $|x - y| \leq M_\nu\varepsilon$ and $u_\varepsilon(y) = u_\varepsilon(x_0)$. Then by Theorem 2.3,

$$|u_\varepsilon(x_0) - u_\varepsilon(x)| \leq C\varepsilon^{-\alpha/20}(M_\nu\varepsilon)^\alpha \leq C\varepsilon^{\alpha/2}.$$

Next, we assume that ν is an irrational direction and $x \in H(x_0)$. By (ii) of Lemma 2.7, there exists $y \in \mathbb{R}^n$ such that $|x - y| \leq M\varepsilon^{1/10}$, $y - x_0 \in \varepsilon\mathbb{Z}^n$ and

$$\mathrm{dist}(y, H(x_0)) < \varepsilon\omega(\varepsilon). \tag{14}$$

Then we obtain

$$\begin{aligned} |u_\varepsilon(x_0) - u_\varepsilon(x)| &\leq |u_\varepsilon(x_0) - u_\varepsilon(y)| + |u_\varepsilon(y) - u_\varepsilon(x)| \\ &\leq C(\omega(\varepsilon)^\beta + \varepsilon^\alpha) + |u_\varepsilon(y) - u_\varepsilon(x)| \\ &\leq C\omega(\varepsilon)^\beta + C\varepsilon^{-\alpha/20}(M\varepsilon^{1/10})^\alpha \\ &\leq C\omega(\varepsilon)^\beta + C\varepsilon^{\alpha/20}, \end{aligned} \tag{15}$$

where the second inequality follows from Lemma 3.5 with (14), and the third inequality follows from Theorem 2.3. □

By Lemma 4.2 and by the comparison principle (Theorem 2.2), we obtain the following estimate: for $x \in \Pi$,

$$|u_\varepsilon(x) - v_\varepsilon(x)| \leq \Lambda(\varepsilon), \tag{16}$$

where

$$\Lambda(\varepsilon) = \begin{cases} C\varepsilon^{\alpha/2} & \text{if } \nu \text{ is a rational direction,} \\ C\varepsilon^{\alpha/20} + C\omega_\nu(\varepsilon)^\beta & \text{if } \nu \text{ is any irrational direction.} \end{cases}$$

Lemma 4.3. $\lim v_j = \lim u_j$, *and hence* $\partial u/\partial \nu = \mu$ *on* Γ_0.

Proof. Observe that v_j solves (P_{ε_j}) with $g = \mu_j$: note that v_j is then a linear profile, that is, $v_j(x) = \mu_j((x-p)\cdot\nu + 1) + 1$. Let x_0 be a point between Γ_0 and $H(0)$. Then by Lemma 4.2, applied to u_j and v_j,

$$\left|(u_j(x) - v_j(x)) - (u_j(x_0) - v_j(x_0))\right| \le \Lambda(\varepsilon_j), \tag{17}$$

for all $x \in H(x_0)$, if j is sufficiently large. Suppose now that

$$u_j(x_0) - v_j(x_0) > c > 0, \quad \text{for sufficiently large } j.$$

Then due to (17), $u_j - v_j \ge c/2$ on $H(x_0)$ if j is sufficiently large. Note that u_j can be constructed as the locally uniform limit of $u_{j,R}$, where $u_{j,R}$ solves

$$F(D^2 u_{j,R}) = 0 \quad \text{in } B_R(x_0) \cap \Pi, \qquad u_{j,R} = v_j \quad \text{on } \partial B_R(x_0) \cap \Pi,$$

with

$$u_{j,R} = 1 \quad \text{on } \Gamma_I, \qquad \frac{\partial}{\partial \nu} u_{j,R}(x) = g\Big(\frac{x}{\varepsilon_j}\Big) \quad \text{on } \Gamma_0.$$

Comparing $u_{j,R}$ and $v_j + c((x-x_0)\cdot\nu + 1)$ on the domain

$$B_R(x_0) \cap \big\{x : -1 \le (x-p)\cdot\nu \le (x-x_0)\cdot\nu\big\}$$

for sufficiently large R then yields that $u_{j,R}(0) \ge v_j(0) + c_0$ for all sufficiently large R, which would contradict the fact that $v_j(0) = u_j(0)$. Similarly, the case $\liminf_j (u_j(x_0) - v_j(x_0)) < 0$ can be excluded, and it follows that

$$|u_j(x_0) - v_j(x_0)| \to 0 \quad \text{as } j \to \infty.$$

Hence we get $v_j \to u$ in each compact subset of Π. By Lemmas 4.1 and 3.6, the limit $u = v$ of v_j satisfies $\partial u/\partial \nu = \mu$ on Γ_0. □

Lemma 4.4. *If* ν *is an irrational direction*, $\partial u/\partial \nu = \mu_\nu$ *for a constant* μ_ν *which depends on* ν, *not on the subsequence* ε_j.

Proof. 1. Let $0 < \eta < \varepsilon$ be sufficiently small. Let

$$w_\varepsilon(x) = \frac{u_\varepsilon(\varepsilon x)}{\varepsilon}, \quad w_\eta(x) = \frac{u_\eta(\eta x)}{\eta},$$

and denote by Γ_1 and Γ_2 the Neumann boundary of w_ε and w_η, respectively. By (iii) of Lemma 2.7, for the point $p \in \mathbb{R}^n$, there exist $q_1 \in \Gamma_1$ and $q_2 \in \Gamma_2$ such that

$$|p - q_1| \le \eta \mod \mathbb{Z}^n \quad \text{and} \quad |p - q_2| \le \eta \mod \mathbb{Z}^n.$$

Hence after translations by $p - q_1$ and $p - q_2$, we may suppose that $w_\varepsilon(x)$ and $w_\eta(x)$ are defined, respectively, on the extended strips

$$\Omega_\varepsilon := \Big\{x : -\frac{1}{\varepsilon} \le (x-p)\cdot\nu \le 0\Big\} \quad \text{and} \quad \Omega_\eta := \Big\{x : -\frac{1}{\eta} \le (x-p)\cdot\nu \le 0\Big\}.$$

Here, $w_\varepsilon = 1/\varepsilon$ on $\{(x-p)\cdot\nu = -1/\varepsilon\}$ and $w_\eta = 1/\eta$ on $\{(x-p)\cdot\nu = -1/\eta\}$. Moreover, on $\Gamma_0 := \{(x-p)\cdot\nu = 0\}$, we have

$$\frac{\partial w_\varepsilon}{\partial\nu} = g_1(x) := g(x-z_1) \quad \text{and} \quad \frac{\partial w_\eta}{\partial\nu} = g_2(x) := g(x-z_2),$$

where $|z_1|, |z_2| \le \eta$. Observe that since g has Hölder exponent $0<\beta\le 1$, we have $|g_1-g_2|\le\eta^\beta$.

Let v_ε be a solution of the problem (P_ε) with constant Neumann data $\partial v_\varepsilon/\partial\nu = \mu_\varepsilon$ on Γ_0 such that v_ε coincides with u_ε at $x=0$ and on Γ_I. By (16),

$$\Big|w_\varepsilon(x) - \frac{v_\varepsilon(\varepsilon x)}{\varepsilon}\Big| \le \frac{C\varepsilon^{\alpha/20} + C\omega(\varepsilon)^\beta}{\varepsilon}. \tag{18}$$

Note that v_ε is a linear profile: indeed,

$$\frac{v_\varepsilon(\varepsilon x)}{\varepsilon} = \mu_\varepsilon\Big((x-p)\cdot\nu + \frac{1}{\varepsilon}\Big) + \frac{1}{\varepsilon}.$$

From (18) and the comparison principle, it follows that, with $\Lambda(\varepsilon) = C\varepsilon^{\alpha/20} + C\omega(\varepsilon)^\beta$,

$$\big(\mu_\varepsilon - \Lambda(\varepsilon)\big)\Big((x-p)\cdot\nu + \frac{1}{\varepsilon}\Big) \le w_\varepsilon(x) - \frac{1}{\varepsilon} \le \big(\mu_\varepsilon + \Lambda(\varepsilon)\big)\Big((x-p)\cdot\nu + \frac{1}{\varepsilon}\Big), \tag{19}$$

2. (19) means that the slope of w_ε in the direction of ν (that is, $\nu\cdot Dw_\varepsilon$) is between $\mu_\varepsilon + \Lambda(\varepsilon)$ and $\mu_\varepsilon - \Lambda(\varepsilon)$ on $\{x : (x-p)\cdot\nu = -1/\varepsilon\}$. Now let us consider linear profiles

$$l_1(x) = a_1(x-p)\cdot\nu + b_1 \quad \text{and} \quad l_2(x) = a_2(x-p)\cdot\nu + b_2,$$

whose respective slopes are $a_1 = \mu_\varepsilon + \Lambda(\varepsilon)$ and $a_2 = \mu_\varepsilon - \Lambda(\varepsilon)$. Here b_1 and b_2 are chosen such that

$$l_1 = l_2 = \omega_\eta(x) \quad \text{on} \quad \Big\{x : (x-p)\cdot\nu = -\frac{1}{\eta}\Big\}.$$

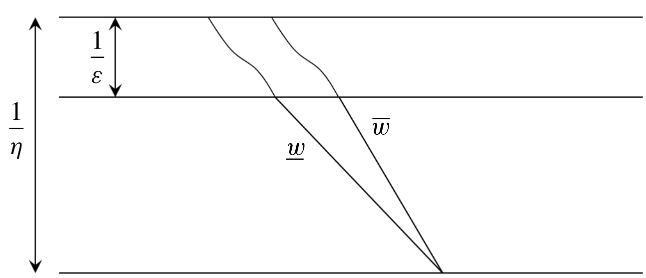
3. Now we define

$$\overline{w}(x) := \begin{cases} l_1(x) & \text{in } \big\{-1/\eta \le (x-p)\cdot\nu \le -1/\varepsilon\big\}, \\ w_\varepsilon(x) + c_1 & \text{in } \big\{-1/\varepsilon \le (x-p)\cdot\nu \le 0\big\} \end{cases}$$

and

$$\underline{w}(x) := \begin{cases} l_2(x) & \text{in } \big\{-1/\eta \le (x-p)\cdot\nu \le -1/\varepsilon\big\}, \\ w_\varepsilon(x) + c_2 & \text{in } \big\{-1/\varepsilon \le (x-p)\cdot\nu \le 0\big\}, \end{cases}$$

where c_1 and c_2 are constants satisfying $l_1 = w_\varepsilon + c_1$ and $l_2 = w_\varepsilon + c_2$ on $\{(x-p)\cdot\nu = -1/\varepsilon\}$. (See figure.)



Note that, due to (19), in $\{-1/\varepsilon \le (x-p)\cdot\nu \le 0\}$ we have

$$\overline{w}(x) = \min\big(l_1(x), w_\varepsilon(x)+c_1\big) \quad \text{and} \quad \underline{w}(x) = \max\big(l_2(x), w_\varepsilon(x)+c_2\big),$$

and thus it follows that $\overline{w}$ and $\underline{w}$ are respectively viscosity super- and subsolutions of (P).

4. Let us define

$$h_1(x) = \eta^\beta\Big((x-p)\cdot\nu + \frac{1}{\eta}\Big).$$

Then $w^+ := \overline{w} + h_1$ solves

$$\begin{cases} F(Dw^+) \ge 0 & \text{in } \Omega_\eta, \\ \partial w^+/\partial\nu = g(x) + \eta^\beta & \text{on } \Gamma_0, \end{cases}$$

and $w^- := \underline{w} - h_1$ solves

$$\begin{cases} F(Dw^-) \le 0 & \text{in } \Omega_\eta, \\ \partial w^-/\partial\nu = g(x) - \eta^\beta & \text{on } \Gamma_0. \end{cases}$$

Since $|g-\tilde{g}| \le \eta^\beta$ and $w^+ = w^- = w_\eta$ on $\{(x-p)\cdot\nu = -1/\eta\}$, it follows from the comparison principle for (P_ε) that

$$w^- \le w_\eta \le w^+ \quad \text{in } \Omega_\eta. \tag{20}$$

Hence we conclude

$$|\mu_\eta - \mu_\varepsilon| \le \Lambda(\varepsilon) + \eta^\beta, \tag{21}$$

where μ_η is the slope of v_η, and $\Lambda(\varepsilon) = C\varepsilon^{\alpha/20} + Cw(\varepsilon)^\beta \to 0$ as $\varepsilon \to 0$. □

The proof of the following lemma is immediate from Lemma 4.4 and (21) .

Lemma 4.5 (error estimate: Theorem 1.2(iv)). *For any irrational direction ν, there is a unique homogenized slope $\mu(\nu) \in \mathbb{R}$ and $\varepsilon_0 = \varepsilon_0(\nu) > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the following holds: for any $0 < \alpha < 1$, there exists a constant $C = C(\alpha, n, \lambda, \Lambda)$ such that*

$$\big|u_\varepsilon(x) - \big(1 + \mu(\nu)((x-p)\cdot\nu + 1)\big)\big| \le \Lambda(\varepsilon) := C\varepsilon^{\alpha/20} + C\omega_\nu(\varepsilon)^\beta \quad \text{in } \Pi_\nu(p), \tag{22}$$

where $\omega_\nu(\varepsilon)$ is as given in (7).

Lemma 4.6. *Let ν be a rational direction. If the Neumann boundary Γ_0 passes through $p = 0$, then there is a unique homogenized slope $\mu(\nu)$ for which the result of Lemma 4.5 holds with $\Lambda(\varepsilon) = C\varepsilon^{\alpha/2}$.*

Proof. The proof is parallel to that of Lemma 4.4. Let w_ε and w_η be as given in the proof of Lemma 4.4. Note that since Ω_ε and Ω_η have their Neumann boundaries passing through the origin, $\partial w_\varepsilon/\partial\nu = g(x) = \partial w_\eta/\partial\nu$ without translation of the x variable, and thus we do not need to use the properties of hyperplanes with an irrational normal (Lemma 2.7(b)) to estimate the error between the shifted Neumann boundary data. □

Remark 4.7. As mentioned in the introduction, if ν is a rational direction with $p \ne 0$, the values of $g(\cdot/\varepsilon)$ on $\partial\Omega_\varepsilon$ and $\partial\Omega_\eta$ may be very different under any translation, and thus the proof of Lemma 4.4 fails. In this case, u_ε may converge to solutions of different Neumann boundary data, depending on the subsequence.

Proof of Theorem 1.2(ii). Recall that we must show that *the homogenized limit $\mu(\nu)$, defined in Lemma 4.5 for irrational directions in $\mathcal{S}^{n-1}$, has a continuous extension $\bar{\mu}(\nu) : \mathcal{S}^{n-1} \to \mathbb{R}$.*

Fix a unit vector $\nu \in \mathcal{S}^{n-1}$. Then we will show that there exists a positive constant $C > 0$ depending on ν such that the following holds: given $\delta > 0$, there exists $\varepsilon > 0$ such that for any two irrational directions $\nu_1, \nu_2 \in \mathcal{S}^{n-1}$,

$$|\mu(\nu_1) - \mu(\nu_2)| < C\delta^{1/2} \quad \text{whenever } 0 < |\nu_1 - \nu|,\ |\nu_2 - \nu| < \varepsilon. \tag{23}$$

1. To simplify the proof, we first present the case $n = 2$. For simplicity of notation, we may assume that $|\nu \cdot e_1| \le |\nu \cdot e_2|$ and $p = 0$. First we introduce several notations. Again for notational simplicity and clarity in the proof, we assume that $\nu = e_2$: we will explain in the paragraph below how to modify the notations and the proof for $\nu \ne e_2$. Let us define

$$\Omega_0 := \Pi_\nu(0) = \{(x, y) \in \mathbb{R}^2 : -1 \le y \le 0\},$$

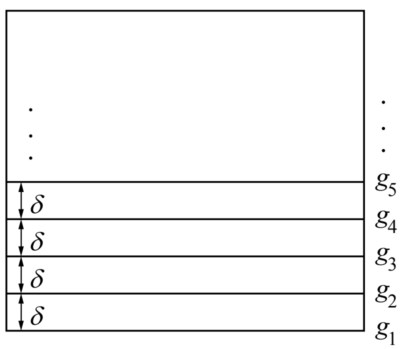
and for $i = 1, 2$,

$$\Omega_i := \Pi_{\nu_i}(0) = \{(x, y) \in \mathbb{R}^2 : -1 \le (x, y) \cdot \nu_i \le 0\}.$$

Let us also define the family of functions

$$g_i(x_1, x_2) = g_i(x_1) = g(x_1, \delta(i-1)),$$

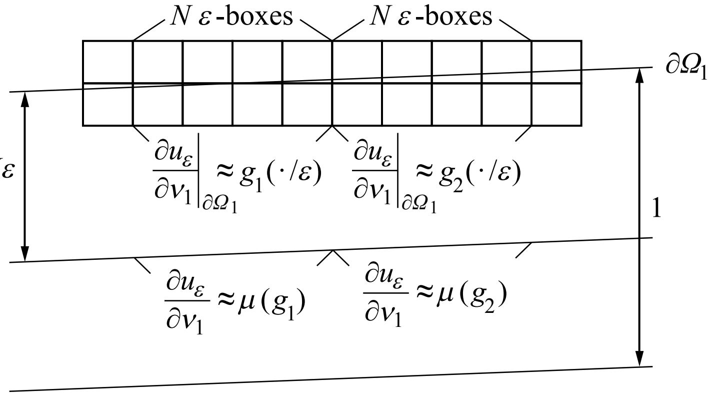
where $i = 1, \dots, m := [1/\delta] + 1$ (see figure).



If ν is a rational direction different from e_2, take the smallest $K_\nu \in \mathbb{N}$ such that $K_\nu \nu = 0 \bmod \mathbb{N}^2$. Then g can be considered as a K_ν-periodic function with the new direction of axis of ν. If ν is an irrational direction, take the smallest $K_\nu \in \mathbb{N}$ such that $|K_\nu \nu| \le \delta \bmod \mathbb{N}^2$. Then g is almost K_ν-periodic up to the order of δ with the new axis of ν. We point out that it does not make any difference in the proof if we replace the periodicity of g by the fact that g is almost periodic up to the order δ.

Before moving on to the next step, we briefly discuss the heuristics in the proof.

Proof by heuristics. Since the domains Ω_1 and Ω_2 point toward different directions ν_1 and ν_2, we cannot directly compare their boundary data, even if $\partial\Omega_1$ and $\partial\Omega_2$ cover most of the unit cell in $\mathbb{R}^n/\mathbb{Z}^n$. To overcome this difficulty, we perform a two-scale homogenization.



First we consider the functions g_i $(i = 1, \dots, m)$ whose profiles cover most values of g in $\mathbb{R}^2$ up to the order of δ^β, where β is the Hölder exponent of g. Note that most values of g in $\mathbb{R}^2$ are taken on $\partial\Omega_1$ and on $\partial\Omega_2$, since ν_1 and ν_2 are both irrational directions. On the other hand, since ν_1 and ν_2 are very close to ν, which may be a rational direction, the averaging behavior of a solution u_ε in Ω_1 (or Ω_2) would occur only if ε gets very small.

If $|\nu_1 - \nu| = |\nu_1 - e_2|$ is chosen much smaller than δ, we can say that the Neumann data $g_1(\cdot/\varepsilon)$ is (almost) repeated $N := [\delta/|\nu_1 - \nu|]$ times on $\partial\Omega_1$ with period ε, up to the error $O(\delta^\beta)$. (See figure at the top of the page.) Similarly, on the next piece of the boundary, $g_2(\cdot/\varepsilon)$ is (almost) repeated N times, and then $g_3(\cdot/\varepsilon)$ is repeated N times: this pattern will repeat with g_k ($k \in \mathbb{N}$ mod m).

If N is sufficiently large, that is, if $|\nu_1 - \nu|$ is sufficiently small compared to δ, the solution u_ε in Ω_1 will exhibit averaging behavior, $N\varepsilon$-away from $\partial\Omega_1$. More precisely, on the $N\varepsilon$-sized segments of hyperplane H located $N\varepsilon$-away from $\partial\Omega_1$, u_ε would be homogenized by repeating the profiles of g_i (for some fixed i) with an error of $O(\delta^\beta)$. This is the first homogenization of u_ε near the boundary of Ω_1: we denote by $\mu(g_i)$ the corresponding values of the homogenized slopes of u_ε on H.

Now a unit distance away from $\partial\Omega_1$, we obtain the second homogenization of u_ε, whose slope is determined by $\mu(g_i)$, $i = 1, \dots, m$. Note that this estimate does not depend on the direction ν_1, but on the quantity $|\nu_1 - \nu|$. Hence, applying the same argument for ν_2, we conclude that $|\mu(\nu_1) - \mu(\nu_2)|$ is small. Note that $\mu(\nu_1)$ and $\mu(\nu_2)$ are uniquely determined because ν_1 and ν_2 are irrational directions (Lemma 4.6).[1]

A rigorous proof of the above observation is rather lengthy: the main difficulty lies in the fact that to perform the first homogenization $N\varepsilon$-away from the boundary, one requires the solution u_ε to be sufficiently flat in tangential directions to ν, which we do not know a priori. We will go around this difficulty by constructing sub- and supersolutions by patching up solutions from the near-boundary region and from the region away from the boundary. The proof is given in steps 2–8 below.

[1]By (F3), we may assume that the arrangement of $g_1, \dots, g_m$ is the same for the directions ν_1 and ν_2, after appropriate rotation and reflection (note that $(F3)$ implies rotation and reflection invariance of the operator F).

2. Given $\delta > 0$, let us choose irrational unit vectors $\nu_1, \nu_2 \in \mathbb{R}^2$ such that

$$0 < \bar{\varepsilon}_0^{1/1000} \le \varepsilon_0^{1/1000} \le \delta,$$

where $\varepsilon_0 = |\nu_1 - e_2|$ and $\bar{\varepsilon}_0 = |\nu_2 - e_2|$. Let $\varepsilon = \varepsilon_0^{21/20}$ and $\bar{\varepsilon} = \bar{\varepsilon}_0^{21/20}$. Let us also define

$$N = \left[\frac{\delta}{|\nu_1 - e_2|}\right] = \left[\frac{\delta}{\varepsilon_0}\right]. \tag{24}$$

Then $N\varepsilon = \delta\varepsilon_0^{1/20} := \delta_0$. Note that

$$\delta_0 \ge \varepsilon^{1/20} \quad \text{and} \quad \delta_0 \ge \delta^{100}.$$

With the above definition of ε and N, consider the strip regions $I_0 = [-N\varepsilon, 0] \times \mathbb{R}$, $I_1 = [0, N\varepsilon] \times \mathbb{R}$, $I_{-1} = [-2N\varepsilon, -N\varepsilon] \times \mathbb{R}$, $I_2 = [N\varepsilon, 2N\varepsilon] \times \mathbb{R}, \dots$, that is,

$$I_k = [(k-1)N\varepsilon, kN\varepsilon] \times \mathbb{R} \quad \text{for } k \in \mathbb{Z}.$$

Let $\tilde{k} \in [1, m]$ denote k in modulo m, where $m = [1/\delta] + 1$. Note that, since $N|\nu_1 - e_2| = \delta$, $g_{\tilde{k}}(\cdot/\varepsilon)$ is (almost) repeated N times on $I_k \cap \partial\Omega_1$. This fact and the Hölder continuity of g yield that

$$\left| g\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - g_{\tilde{k}}\left(\frac{x}{\varepsilon}\right) \right| < C\delta^\beta \quad \text{on } \partial\Omega_1 \cap I_k, \quad \text{for } k \in \mathbb{Z}. \tag{25}$$

3. Let w_ε solve $(P) : F(D^2 w_\varepsilon) = 0$ in Ω_0, with

$$\begin{cases} \dfrac{\partial w_\varepsilon}{\partial \nu}(x, 0) = g_{\tilde{k}}\left(\dfrac{x}{\varepsilon}\right) & \text{for } (x, 0) \in I_k, \\ w_\varepsilon = 1 & \text{on } \{y = -1\}. \end{cases}$$

Next let u_ε solve (P) in Ω_1, with

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial \nu_1}(x, 0) = g\left(\dfrac{x}{\varepsilon}, \dfrac{y}{\varepsilon}\right) & \text{on } \{(x, y) \cdot \nu_1 = 0\}, \\ u_\varepsilon = 1 & \text{on } \{(x, y) \cdot \nu_1 = -1\}. \end{cases}$$

Let $\mu(w_\varepsilon)$ ($\mu(u_\varepsilon)$) be chosen as the slope μ_j in the linearized problem (P_{μ_j}) in Section 4, where u_j is replaced by w_ε (u_ε) and the reference point $x = 0$ is replaced by $x = -e_2/2 = (0, -\frac{1}{2})$. (Recall that we assumed $0 \in \partial\Omega_1$, and $(0, -\frac{1}{2}) \in \Omega_i$ for $i = 1, 2$.) Then $\mu(w_\varepsilon)$ and $\mu(u_\varepsilon)$ denote the slopes of a linear approximation of w_ε and u_ε. From (25) it follows that

$$|\mu(w_\varepsilon) - \mu(u_\varepsilon)| < C\delta^\beta. \tag{26}$$

We point out that $\mu(w_\varepsilon)$ and $\mu(u_\varepsilon)$ respectively converge to a unique limit as $\varepsilon \to 0$, since ν_1 is irrational.

4. We begin by introducing $\mu_{1/N}(g_k)$, which denotes the average slope of a solution with Neumann data $g_k(x/\varepsilon)$, δ_0-away from the Neumann boundary $\{y = 0\}$. (Here note that $\delta_0 = N\varepsilon$.)

Let us define

$$H := \partial\Omega_0 - N\varepsilon e_2 = \{(x, y) : y = -\delta_0\}.$$

Let $\eta = 1/N$ and let $w_{\eta,1}$ solve

$$\begin{cases} F(D^2 w_{\eta,1}) = 0 & \text{in } \{-\delta_0 \le y \le 0\}, \\ w_{\eta,1} = w_\varepsilon(0, -\delta_0) & \text{on } H = \{y = -\delta_0\}, \\ \dfrac{\partial w_{\eta,1}}{\partial y}(x, 0) = g_1\Big(\dfrac{x}{\varepsilon}, 0\Big) & \text{on } \partial\Omega_0 = \{y = 0\}, \end{cases}$$

where $g_1(x, 0) = g_1(x + k, 0)$ for $k \in \mathbb{Z}$. Let $\mu_{1/N}(g_1)$ be the slope of the linear approximation of $w_{\eta,1}$, defined as follows: choose a linear solution $v_{\eta,1}(\cdot)$ such that

$$\begin{cases} F(D^2 v_{\eta,1}) = 0 & \text{in } \{-\delta_0 \le y \le 0\}, \\ v_{\eta,1} = w_{\eta,1}(0, -\delta_0) & \text{on } H = \{y = -\delta_0\}, \\ v_{\eta,1}\Big(0, -\dfrac{\delta_0}{2}\Big) = w_{\eta,1}\Big(0, -\dfrac{\delta_0}{2}\Big), & \\ \dfrac{\partial v_{\eta,1}}{\partial y}(x, 0) = \mu_{1/N}(g_1) & \text{on } \partial\Omega_0 = \{y = 0\}. \end{cases}$$

Since $g_1(x/\varepsilon, 0)$ is periodic on $\{y = 0\}$ with period ε and $\delta_0 = N\varepsilon$, we can apply Lemma 4.2(i), using the fact that $\delta_0 \ge \varepsilon^{1/20}$, to conclude that

$$\Big| w_{\eta,1}(x, y) - \Big(w_{\eta,1}\Big(0, -\frac{\delta_0}{2}\Big) + \mu_{1/N}(g_1)\Big(y + un\frac{\delta_0}{2}\Big)\Big)\Big| \le C\delta_0^{1+\beta} \tag{27}$$

on $\{y = -\delta_0/2\} \cap I_1$. Similarly, one can define $w_{\eta,k}$ and $v_{\eta,k}$ for $k \in \mathbb{Z}$ to conclude that

$$\Big| w_{\eta,k}(x, y) - \Big(w_{\eta,k}\Big((k-1)\delta_0, -\frac{\delta_0}{2}\Big) + \mu_{1/N}(g_{\tilde{k}})\Big(y + \frac{\delta_0}{2}\Big)\Big)\Big| \le C\delta_0^{1+\beta} \tag{28}$$

on $\{y = -\delta_0/2\} \cap I_k$.

5. We will now construct barriers which bound w_ε from above and below, by pasting together the near-boundary and the rest of the region together as follows. First we construct a supersolution of (P_ε). Let ρ_ε solve the Neumann boundary problem away from the boundary $\{y = 0\}$:

$$\begin{cases} F(D^2 \rho_\varepsilon) = 0 & \text{in } \{-1 \le y \le -\delta_0\}, \\ \dfrac{\partial \rho_\varepsilon}{\partial y} = \Lambda(x) & \text{on } H = \{y = -\delta_0\}, \\ \rho_\varepsilon = 1 & \text{on } \{y = -1\}. \end{cases}$$

Here $\Lambda(x)$ is a Hölder continuous function obtained by approximating $\mu_{1/N}(g_k) + 2\delta_0^{\alpha_0}$ in each $N\varepsilon$-strip, where the constant $0 < \alpha_0 < 1$ will be decided below. Here the Hölder continuity of $\Lambda(x)$ is obtained by the fact that g_k and g_j differ from each other by $((k - j)\delta)^\beta$ and they are apart by $(k - j)N\varepsilon \ge (k - j)\delta^{100}$.

Then Theorem 2.4(b) yields that $\rho_\varepsilon \in C^{1,\gamma}$ up to H, where γ depends on β and n. Therefore there exists a constant $0 < \alpha_0 < 1$ such that the following holds: in each $\delta_0^{1-\alpha_0}$-neighborhood of a point $(x_0, -\delta_0) \in H$, we have

$$\big|\rho_\varepsilon(x, -\delta_0) - \rho_\varepsilon(x_0, -\delta_0) - \alpha(x_0)(x - x_0)\big| \le \delta_0^{1+\alpha_0}, \tag{29}$$

where $\alpha(x_0)$ is the tangential derivative of ρ_ε at $(x_0, -\delta_0)$.

6. Next we construct the near-boundary barrier:

$$\begin{cases} F(D^2 f_\varepsilon) = 0 & \text{in } \{-\delta_0 \le y \le 0\}, \\ f_\varepsilon = \rho_\varepsilon & \text{on } H = \{y = -\delta_0\}, \\ \dfrac{\partial f_\varepsilon}{\partial y} = g_{\tilde{k}}\Big(\dfrac{x}{\varepsilon}\Big) & \text{on } \{y = 0\} \cap I_k. \end{cases}$$

Let us now estimate the slope of f_ε on H. Let us choose a constant μ_ε and the corresponding linear profile ϕ_ε such that

$$\begin{cases} F(D^2 \phi_\varepsilon) = 0 & \text{in } \{-\delta_0 \le y \le 0\}, \\ \phi_\varepsilon(x, -\delta) = f_\varepsilon(0, -\delta_0) & \text{on } H, \\ \phi_\varepsilon\Big(0, -\dfrac{\delta}{2}\Big) = f_\varepsilon\Big(0, -\dfrac{\delta_0}{2}\Big), & \\ \dfrac{\partial \phi_\varepsilon}{\partial y} = \mu_\varepsilon & \text{on } \partial\Omega_0 = \{y = 0\}. \end{cases}$$

Equation (29) and the comparison principle (Theorem 2.2), as well as the localization argument as in the proof of Lemma 3.1 applied to the rescaled function

$$(\delta_0)^{-1} f_\varepsilon\bigg(\frac{(x - x_0)}{\delta_0} + x_0, \frac{y}{\delta_0}\bigg) - \alpha(x_0)(x - x_0)$$

in the region $\{-1 \le y \le 0\} \cap \{|x| \le \delta_0^{-\alpha_0}\}$, yields that

$$|\phi_\varepsilon - f_\varepsilon| \le C\delta_0^{1+\alpha_0} \quad \text{in } \{-\delta_0 \le y \le 0\} \cap \{|x| \le \delta_0^{1-\alpha_0}\}. \tag{30}$$

Putting the estimates (28) and (30) together, it follows that for any $(x_0, -\delta_0) \in H$, we have

$$\Big| f_\varepsilon(x, y) - \Big(\alpha(x_0)(x - x_0) + \mu_{1/N}(g_k)\Big(y + \frac{\delta_0}{2}\Big)\Big)\Big| \le \delta_0^{1+\alpha_0} \quad \text{on } \Big\{y = -\frac{\delta_0}{2}\Big\} \cap \big\{|x - x_0| \le \delta_0^{1-\alpha_0}\big\},$$

for appropriate k in each δ-strip. Using the above inequality, (29), and the $C^{1,\gamma}$ regularity of f_ε up to its Dirichlet boundary, we obtain that

$$\frac{\partial f_\varepsilon}{\partial y} \le \Lambda(x),$$

which then makes the following function a supersolution of (P_ε):

$$\underline{\rho}_\varepsilon := \begin{cases} \rho_\varepsilon & \text{in } \{-1 \le y \le -\delta_0\}, \\ f_\varepsilon & \text{in } \{-\delta_0 \le y \le 0\}. \end{cases}$$

Similarly, one can construct a subsolution $\bar{\rho}_\varepsilon$ of (P_ε) by replacing $\Lambda(x)$ given in the construction of ρ_ε by $\tilde{\Lambda}(x) := \Lambda(x) - 4\delta_0^{\alpha_0}$, such that

$$\bar{\rho}_\varepsilon \le w_\varepsilon \le \underline{\rho}_\varepsilon. \tag{31}$$

7. Parallel arguments as in steps 2–6 apply to the other direction, ν_2: if we define $\bar{\varepsilon}_0$, M and $\bar{H}$ by

$$|\nu_2 - e_2| = \bar{\varepsilon}_0 < \varepsilon_0, \quad M = \Big[\frac{\delta}{\bar{\varepsilon}_0}\Big], \quad \bar{\varepsilon} = \bar{\varepsilon}_0^{21/20} \quad \text{and} \quad \bar{H} = \{y = -M\bar{\varepsilon}\},$$

then we can construct barriers $\bar{\rho}_{\bar{\varepsilon}}$ and $\underline{\rho}_{\bar{\varepsilon}}$ such that

$$\bar{\rho}_{\bar{\varepsilon}} \leq w_{\bar{\varepsilon}}(x) \leq \underline{\rho}_{\bar{\varepsilon}}, \tag{32}$$

with their corresponding Neumann boundary conditions on H:

$$\frac{\partial}{\partial y}\bar{\rho}_{\bar{\varepsilon}}, \quad \frac{\partial}{\partial y}\underline{\rho}_{\bar{\varepsilon}} = \mu_{1/M}(g_{\tilde{k}}) + O(\bar{\delta}_0^{\alpha_0}) \quad \text{and} \quad \bar{H} \cap I_k, \tag{33}$$

where their respective derivative is taken as a limit from the region $\{-1 \leq y < -\bar{\delta}_0\}$.

8. Now we proceed to estimate the averaging behavior of u^ε away from the Neumann boundary. By (21) of Lemmas 4.4 and 4.6,

$$\left|\mu_{1/N}(g_{\tilde{k}}) - \mu_{1/M}(g_{\tilde{k}})\right| < \Lambda\Big(\frac{1}{N}\Big) + \Big(\frac{1}{M}\Big)^\beta, \tag{34}$$

where $\Lambda\Big(\dfrac{1}{N}\Big) = CN^{-\alpha/2}$. Let us write $\mu_{1/N}(g_{\tilde{k}}) = \mu_{\tilde{k},N}$, and let h and $\bar{h}$ respectively solve

$$\begin{cases} F(D^2 h) = 0 & \text{in } \{-1 \leq y \leq -N\varepsilon\}, \\ h = 1 & \text{on } \{y = -1\}, \\ \dfrac{\partial h}{\partial \nu} = \mu_{\tilde{k},N} & \text{on } H \cap I_k, \end{cases}$$

and

$$\begin{cases} F(D^2 \bar{h}) = 0 & \text{in } \{-1 \leq y \leq -M\bar{\varepsilon}\}, \\ \bar{h} = 1 & \text{on } \{y = -1\}, \\ \dfrac{\partial \bar{h}}{\partial \nu} = \mu_{\tilde{k},M} & \text{on } \bar{H} \cap I_k. \end{cases}$$

Let $\mu(h)$ and $\mu(\bar{h})$ be the respective slope of linear approximation for h and $\bar{h}$.

Then it follows from (34) that if $\delta_0 \sim N\varepsilon$ and $\bar{\delta}_0 \sim M\bar{\varepsilon}$ are sufficiently small,

$$|\mu(h) - \mu(\bar{h})| < C\Big(m\Big(\frac{1}{N}\Big) + \Big(\frac{1}{M}\Big)^\beta\Big). \tag{35}$$

Lastly, observe that by (31) and (32), there exists $0 < \gamma < 1$ such that

$$|\mu(w_\varepsilon) - \mu(h)| < C\delta^\gamma \quad \text{and} \quad |\mu(w_{\bar{\varepsilon}}) - \mu(\bar{h})| < C\delta^\gamma.$$

The above inequalities and (35) yield

$$|\mu(w_\varepsilon) - \mu(w_{\bar{\varepsilon}})| < C\Big(\delta^\gamma + m\Big(\frac{1}{N}\Big) + \Big(\frac{1}{M}\Big)^\beta\Big).$$

Then we conclude from (26) that

$$|\mu(u_\varepsilon) - \mu(u_{\bar{\varepsilon}})| < C\Big(\delta^\gamma + m\Big(\frac{1}{N}\Big) + \Big(\frac{1}{M}\Big)^\beta\Big). \tag{36}$$

9. Lastly, we estimate the rate of convergence of $\mu(u_\varepsilon)$ to $\mu(v_1)$ as $\varepsilon \to 0$. The claim is that

$$|\mu(\nu_1)-\mu(u_\varepsilon)|\le C\big(\varepsilon_0^\beta+\varepsilon_0^{21\alpha/200}+\varepsilon_0^{1/20}\big).$$

We will argue similarly as in the proof of Lemma 4.2(ii). Let us define v^ε, the linear approximation of u_ε, as in (P_{μ_j}) of page 960, where the reference function u_j is replaced by u_ε.

Recall that $\Omega_1=\{y:-1\le y\cdot\nu_1\le 0\}$. We define

$$\tilde{\Omega}_1:=\Omega_1\cap\{y:y\cdot\nu_1\le -N\varepsilon\delta^{-1}\nu_1\}$$

and $L:=\partial\Omega_1-N\varepsilon\delta^{-1}\nu_1$. For any given $x_0\in L$ and for any $x\in L$, there exists $y\in\mathbb{R}^2$ such that $|x-y|\le N\varepsilon m$, $x_0-y=0 \bmod \varepsilon\mathbb{Z}^2$, and

$$\operatorname{dist}(y,L)\le\varepsilon|\nu_1-e_2|=\varepsilon\varepsilon_0.$$

(Recall that $m=\left[\frac{1}{\delta}\right]+1$.) Then by arguing as in (15), for $x\in L$,

$$|u_\varepsilon(x_0)-u_\varepsilon(x)|\le C\varepsilon_0^\beta+C(N\varepsilon\delta^{-1})^\alpha(N\varepsilon m)^\alpha\le C(\varepsilon_0^\beta+\varepsilon^{\alpha/10}).$$

Hence, due to the comparison principle (Theorem 2.2) applied to u_ε and v_ε in the domain $\tilde{\Omega}_1$, we obtain

$$|u_\varepsilon-v_\varepsilon|\le C(\varepsilon_0^\beta+\varepsilon^{\alpha/10}+N\varepsilon\delta^{-1})=C(\varepsilon_0^\beta+\varepsilon_0^{21\alpha/200}+\varepsilon_0^{1/20}). \tag{37}$$

Following the proof of (21) using (37) instead of (13), we conclude

$$|\mu(u_\varepsilon)-\mu(\nu_1)|\le C(\varepsilon_0^\beta+\varepsilon_0^{21\alpha/200}+\varepsilon_0^{1/20})\le\delta.$$

Parallel arguments apply to ν_2. Combining the above inequality with (36),

$$|\mu(\nu_1)-\mu(\nu_2)|\le C\Big(\delta^\gamma+m\Big(\frac{1}{N}\Big)+\Big(\frac{1}{M}\Big)^\beta\Big).$$

Since N and M grow to infinity as ε and $\bar{\varepsilon}$ go to zero, the above inequality proves the lemma.

10. For the general dimensions $n>2$, let us define

$$g_i(x_1,\dots,x_{n-1},x_n)=g_i(x_1,\dots,x_{n-1})=g(x_1,\dots,x_{n-1},\delta(i-1))$$

for $i=0,1,\dots,m:=[\delta^{-1}]$. Let us also define

$$I_{k_1,k_2,\dots,k_{n-1}}:=\big[(k_1-1)N\varepsilon,k_1N\varepsilon\big]\times\cdots\times\big[(k_{n-1}-1)N\varepsilon,k_{n-1}N\varepsilon\big]\times\mathbb{R}.$$

Then parallel arguments as in steps 1–9 would apply to yield the proposition in $\mathbb{R}^n$. □

Remark 4.8. The proof breaks down for $F=F(D^2u,x/\varepsilon)$, since the idea of perturbing the problem by tilting the Neumann boundary and its boundary data, that is, the approximation of u_η by w_η in step 3, does not apply if the inside operator also depends on x/ε.

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