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CAUCHY PROBLEM FOR ULTRASOUND-MODULATED EIT

GUILLAUME BAL

Ultrasound modulation of electrical or optical properties of materials offers the possibility of devising hybrid imaging techniques that combine the high electrical or optical contrast observed in many settings of interest with the high resolution of ultrasound. Mathematically, these modalities require that we reconstruct a diffusion coefficient $\sigma(x)$ for $x \in X$, a bounded domain in $\mathbb{R}^n$, from knowledge of $\sigma(x)|\nabla u|^2(x)$ for $x \in X$, where u is the solution to the elliptic equation $-\nabla \cdot \sigma \nabla u = 0$ in X with $u = f$ on ∂X.

This inverse problem may be recast as a nonlinear equation, which formally takes the form of a 0-Laplacian. Whereas p-Laplacians with $p > 1$ are well-studied variational elliptic nonlinear equations, $p = 1$ is a limiting case with a convex but not strictly convex functional, and the case $p < 1$ admits a variational formulation with a functional that is not convex. In this paper, we augment the equation for the 0-Laplacian with Cauchy data at the domain's boundary, which results in a formally overdetermined, nonlinear hyperbolic equation.

This paper presents existence, uniqueness, and stability results for the Cauchy problem of the 0-Laplacian. In general, the diffusion coefficient $\sigma(x)$ can be stably reconstructed only on a subset of X described as the domain of influence of the space-like part of the boundary ∂X for an appropriate Lorentzian metric. Global reconstructions for specific geometries or based on the construction of appropriate complex geometric optics solutions are also analyzed.

1. Introduction

Electrical impedance tomography (EIT) and optical tomography (OT) are medical imaging modalities that take advantage of the high electrical and optical contrast exhibited by different tissues, and in particular, the high contrast often observed between healthy and unhealthy tissues. Electrical potentials and photon densities are modeled in such applications by a diffusion equation, which is known not to propagate singularities, and as a consequence, the reconstruction of the diffusion coefficient in such modalities often comes with poor resolution [Arridge and Schotland 2010; Bal 2009; Uhlmann 2009].

Ultrasound modulations have been proposed as a means to combine the high contrast of EIT and OT with the high resolution of ultrasonic waves propagating in an essentially homogeneous medium [Wang 2004]. In the setting of EIT, ultrasound-modulated electrical impedance tomography (UMEIT), also called acousto-electric tomography, has been proposed and analyzed in [Ammari et al. 2008; Bal et al. 2011a; Capdeboscq et al. 2009; Gebauer and Scherzer 2008; Kuchment and Kunyansky 2011; Zhang and Wang 2004]. In the setting of optical tomography, a similar model of ultrasound-modulated tomography (UMOT), also called acousto-optic tomography, has been derived in [Bal and Schotland 2010] in the

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so-called incoherent regime of wave propagation, while a large physical literature deals with the coherent regime [Atlan et al. 2005; Kempe et al. 1997; Wang 2004], whose mathematical structure is quite different. The 0-Laplacian model also finds applications in thermoacoustic tomography. For this and other hybrid imaging modalities, see, for example, [Bal 2013; Scherzer 2011].

Elliptic forward problem. In this paper, we aim to reconstruct an unknown coefficient $\sigma(x)$ from knowledge of a functional of the form $H(x) = \sigma(x)|\nabla u|^2(x)$, where $u(x)$ is the solution to the elliptic equation

$$-\nabla \cdot \sigma(x)\nabla u = 0 \quad \text{in } X, \qquad u = f \quad \text{on } \partial X. \tag{1}$$

Here, X is an open bounded domain in $\mathbb{R}^n$ with spatial dimension $n \geq 2$. We denote by ∂X the (sufficiently smooth) boundary of X and by $f(x)$ the Dirichlet boundary conditions prescribed in the physical experiments. Neumann or more general Robin boundary conditions could be analyzed similarly. We assume that the unknown diffusion coefficient σ is a real-valued, scalar function defined on X. It is bounded above and below by positive constants and assumed to be (sufficiently) smooth. The coefficient $\sigma(x)$ models the electrical conductivity in the setting of electrical impedance tomography and the diffusion coefficient of particles (photons) in the setting of optical tomography. Both EIT and OT are high-contrast modalities. We focus on the EIT setting here for concreteness, and refer to σ as the conductivity.

The derivation of such functionals as $H(x)$ from physical experiments, following similar derivations in [Bal et al. 2011a; Bal and Schotland 2010; Kuchment and Kunyansky 2011], is recalled in Section 2. For a derivation based on the focusing of acoustic pulses (in the time domain), we refer the reader to [Ammari et al. 2008]. This problem has been considered numerically in [Ammari et al. 2008; Gebauer and Scherzer 2008; Kuchment and Kunyansky 2011]. In those papers, it is shown numerically that UMEIT allows for high-resolution reconstructions, although typically more information than one measurement of the form $H(x) = \sigma(x)|\nabla u|^2(x)$ is required.

Following the methodology in [Capdeboscq et al. 2009], where the two-dimensional setting is analyzed, [Bal et al. 2011a] analyzes the reconstruction of σ in UMEIT from multiple measurements at least equal to the spatial dimension n. The stability estimates obtained in [Bal et al. 2011a] show that the reconstructions in UMEIT are indeed stable with respect to perturbations of the available measurements. Such results are confirmed by the theoretical investigations in a linearized setting and the numerical simulations proposed in [Kuchment and Kunyansky 2011]. In this paper, we consider the setting where a unique measurement $H(x) = \sigma(x)|\nabla u|^2(x)$ is available.

The inverse problem as a p-Laplacian. Following [Ammari et al. 2008; Bal and Schotland 2010; Gebauer and Scherzer 2008], we recast the inverse problem in UMEIT as a nonlinear partial differential equation; see (7) below. This equation is formally an extension to the case $p = 0$ of the p-Laplacian elliptic equations

$$-\nabla \cdot \frac{H(x)}{|\nabla u|^{2-p}}\nabla u = 0,$$

posed on a bounded, smooth, open domain $X \subset \mathbb{R}^n$, $n \geq 2$, with prescribed Dirichlet conditions, say. When $1 < p < \infty$, the above problem is known to admit a variational formulation with convex functional

$J[\nabla u] = \int_X H(x)|\nabla u|^p(x)\,dx$, which admits a unique minimizer (in an appropriate functional setting), this being a solution of the above associated Euler–Lagrange equation [Evans 1998].

The case $p = 1$ is a critical case, as the above functional remains convex but not strictly convex. Solutions are no longer unique in general. This problem has been extensively analyzed in the context of EIT perturbed by magnetic fields (CDII and MREIT) [Kwon et al. 2002; Nachman et al. 2007; 2009], where it is shown that slight modifications of the 1-Laplacian admit unique solutions in the setting of interest in MREIT. Of interest for this paper is the remark that the reconstruction when $p = 1$ exhibits some locality, in the sense that local perturbations of the source and boundary conditions of the 1-Laplacian do not influence the solution on the whole domain X. This behavior is characteristic of a transition from an elliptic equation when $p > 1$ to a hyperbolic equation when $p < 1$.

The inverse problem as a hyperbolic nonlinear equation. When $p < 1$, the above functional $J[\nabla u]$ is no longer convex. When $p = 0$, it should formally be replaced by $J[\nabla u] = \int_X H(x)\ln|\nabla u|(x)\,dx$, whose Euler–Lagrange equation is indeed (7) below. The resulting 0-Laplacian is not an elliptic problem. As we mentioned above, it should be interpreted as a hyperbolic equation, as the derivation of (8) below indicates.

Information then propagates in a local fashion, provided that compatible boundary conditions are imposed in order for the hyperbolic equation to be well-posed [Hörmander 1997; Taylor 1996]. We thus augment the nonlinear equation with Cauchy boundary measurements. As we shall see in the derivation of UMEIT in the next section, imposing such boundary conditions essentially amounts to assuming that $\sigma(x)$ is known at the domain's boundary. This results in an overdetermined problem in the same sense that a wave equation with Cauchy data at time $t = 0$ and at time $t = T > 0$ is overdetermined. Existence results are therefore only available in a local sense. We are primarily interested in showing a uniqueness (injectivity) result, which states that at most one coefficient σ is compatible with a given set of measurements, and a stability result, which characterizes how errors in measurements translate into errors in reconstructions. Redundant measurements clearly help in such analyses.

Space-like versus time-like boundary subsets. Once UMEIT is recast as a hyperbolic problem, we face several difficulties. The equation is hyperbolic in the sense that one of the spatial variables plays the usual role of "time" in a second-order wave equation. Such a "time" variable has an orientation that depends on position x in X and also on the solution of the hyperbolic equation itself, since the equation is nonlinear. Existence and uniqueness results for such equations need to be established, and we shall do so in Sections 3 and 4 below, adapting known results on linear and nonlinear hyperbolic equations that are summarized in [Hörmander 1997; Taylor 1996].

More damaging for the purpose of UMEIT and UMOT is the fact that hyperbolic equations propagate information in a stable fashion only when such information enters through a space-like surface, that is, a surface that is more orthogonal than it is tangent to the direction of "time". In two dimensions of space, the time-like and space-like variables can be interchanged so that when $n = 2$, unwanted singularities can propagate inside the domain only through points with "null-like" normal vector, and in most settings, such points have (surface Lebesgue) zero measure. In $n = 2$, it is therefore expected that spurious instabilities may propagate along a finite number of geodesics and that the reconstructions will be stable otherwise.

In dimensions $n \geq 3$, however, a large part of the boundary ∂X will in general be purely "time-like," so that the information available on such a part of the surface cannot be used to solve the inverse problem in a stable manner [Hörmander 1997]. Only on the domain of influence of the space-like part of the boundary do we expect to stably solve the nonlinear hyperbolic equation, and hence reconstruct the unknown conductivity $\sigma(x)$.

Special geometries and special boundary conditions. As we mentioned earlier, the partial reconstruction results described above can be improved in the setting of multiple measurements. Once several measurements, and hence several potential "time-like" directions are available, it becomes more likely that σ can be reconstructed on the whole domain X. In the setting of well-chosen multiple measurements, the theories developed in [Bal et al. 2011a; Capdeboscq et al. 2009] indeed show that σ can be uniquely and stably reconstructed on X.

An alternative solution is to devise geometries of X and of the boundary conditions that guarantee that the "time-like" part of the boundary ∂X is empty. Information can then be propagated uniquely and stably throughout the domain. In Section 4, we consider several such geometries. The first geometry consists of an annulus-shaped domain, to ensure that the two connected components of the boundary are level sets of the solution u. In such situations, the whole boundary ∂X turns out to be "space-like". Moreover, so long as u does not have any critical point, we can show that the reconstruction can be stably performed on the whole domain X.

Unfortunately, only in dimension $n = 2$ can we be sure that u does not have any critical point independent of the unknown conductivity σ. This is because critical points in a two-dimensional elliptic equation are necessarily isolated, as used in [Alessandrini 1986], for example, and our geometry simply prevents their existence. In three dimensions of space, however, critical points can arise. Such results are similar to those obtained in [Briane et al. 2004] in the context of homogenization theory, and are consistent with the analysis of critical points in elliptic equations, as in [Caffarelli and Friedman 1985; Hardt et al. 1999].

In dimension $n \geq 3$, we thus need to use another strategy to ensure that one vector field is always available for us to penetrate information inside the domain in a unique and stable manner. In this paper, such a result is obtained by means of boundary conditions f in (1) that are "close" to traces of appropriate complex geometric optics (CGO) solutions, which can be constructed provided that $\sigma(x)$ is sufficiently smooth. The CGO solutions are used to obtain required qualitative properties of the solutions to linear elliptic equations, as was done in the setting of other hybrid medical imaging modalities in, for example, [Bal and Ren 2011; Bal et al. 2011b; Bal and Uhlmann 2010; Triki 2010]; see also the review paper [Bal 2013].

The rest of the paper is structured as follows. Section 2 presents the derivation of the functional $H(x) = \sigma(x)|\nabla u|^2$ from ultrasound modulation of a domain of interest and the transformation of the inverse problem as a nonlinear hyperbolic equation. In Section 3, local results of uniqueness and stability are presented, adapting results on linear hyperbolic equations summarized in [Taylor 1996]. These results show that UMEIT and UMOT are indeed much more stable modalities than EIT and OT. The section concludes with a local reconstruction algorithm, which shows that the nonlinear equation admits a solution even if the available data are slightly perturbed by, say, noise. The existence result is obtained after an appropriate change of variables from the result for time-dependent second-order nonlinear hyperbolic equations in [Hörmander 1997]. Finally, in Section 4, we present global uniqueness and stability results for UMEIT for specific geometries or specific boundary conditions constructed by means of CGO solutions.

2. Derivation of a nonlinear equation

Ultrasound modulation. A methodology to combine high contrast with high resolution consists of perturbing the diffusion coefficient acoustically. Let an acoustic signal propagate throughout the domain. We assume here that the sound speed is constant and that the acoustic signal is a plane wave of the form $p\cos(k\cdot x+\varphi)$, where p is the amplitude of the acoustic signal, k its wavenumber, and φ an additional phase. The acoustic signal modifies the properties of the diffusion equation. We assume that such an effect is small but measurable and that the coefficient in (1) is modified as

$$\sigma_\varepsilon(x)=\sigma(x)(1+\varepsilon\cos(k\cdot x+\varphi)), \tag{2}$$

where $\varepsilon=p\Gamma$ is the product of the acoustic amplitude $p\in\mathbb{R}$ and a measure $\Gamma>0$ of the coupling between the acoustic signal and the modulations of the constitutive parameter in (1). For more information about similar derivations, we refer the reader to [Ammari et al. 2008; Bal and Schotland 2010; Kuchment and Kunyansky 2011].

Let u be a solution of (1) with fixed boundary condition f. When the acoustic field is turned on, the coefficients are modified as described in (2), and we denote by u_ε the corresponding solution. Note that $u_{-\varepsilon}$ is the solution obtained by changing the sign of p or, equivalently, by replacing φ by $\varphi+\pi$.

By the standard continuity of the solution to (1) with respect to changes in the coefficients and regular perturbation arguments, we find that $u_\varepsilon=u_0+\varepsilon u_1+O(\varepsilon^2)$. Let us multiply the equation for u_ε by $u_{-\varepsilon}$ and the equation for $u_{-\varepsilon}$ by u_ε, subtract the resulting equalities, and use standard integrations by parts. We obtain that

$$\int_X(\sigma_\varepsilon-\sigma_{-\varepsilon})\nabla u_\varepsilon\cdot\nabla u_{-\varepsilon}\,dx=\int_{\partial X}\sigma_{-\varepsilon}\frac{\partial u_{-\varepsilon}}{\partial\nu}u_\varepsilon-\sigma_\varepsilon\frac{\partial u_\varepsilon}{\partial\nu}u_{-\varepsilon}\,d\sigma. \tag{3}$$

Here, $\nu(x)$ is the outward unit normal to $X\subset\mathbb{R}^n$ at $x\in\partial X$, and as usual $\partial/\partial\nu:=\nu\cdot\nabla$. We assume that $\sigma_\varepsilon\partial_\nu u_\varepsilon$ is measured on ∂X, at least on the support of $u_\varepsilon=f$ for all values ε of interest. Note that the above equation still holds if the Dirichlet boundary conditions are replaced by Neumann (or more general Robin) boundary conditions. Let us define

$$J_\varepsilon:=\frac{1}{2}\int_{\partial X}\sigma_{-\varepsilon}\frac{\partial u_{-\varepsilon}}{\partial\nu}u_\varepsilon-\sigma_\varepsilon\frac{\partial u_\varepsilon}{\partial\nu}u_{-\varepsilon}d\sigma=\varepsilon J_1+O(\varepsilon^3). \tag{4}$$

The term of order $O(\varepsilon^2)$ vanishes by symmetry. We assume that the real-valued functions $J_1=J_1(k,\varphi)$ are known. This knowledge is based on the physical boundary measurement of the Cauchy data $(u_\varepsilon,\sigma_\varepsilon\partial_\nu u_\varepsilon)$ on ∂X.

Equating like powers of ε, we find at the leading order that

$$\int_X \big[\sigma(x)\nabla u_0 \cdot \nabla u_0(x)\big] \cos(k\cdot x+\varphi)\,dx = J_1(k,\varphi). \tag{5}$$

This may be acquired for all $k \in \mathbb{R}^n$ and $\varphi = 0, \pi/2$, and hence provides the Fourier transform of

$$H(x) = \sigma(x)|\nabla u_0|^2(x). \tag{6}$$

Upon taking the inverse Fourier transform of the measurements (5), we thus obtain the internal functional (6).

Nonlinear hyperbolic inverse problem. The forward problem consists of assuming σ and $f(x)$ known, solving (1) to get $u(x)$, and then constructing $H(x) = \sigma(x)|\nabla u|^2(x)$. The inverse problem consists of reconstructing σ and u from knowledge of $H(x)$ and $f(x)$.

As we shall see, the linearization of the latter inverse problem may involve an operator that is not injective, and so there is no guarantee that u and σ can be uniquely reconstructed; see Remark 3.4 below. In this paper, we instead assume that the Neumann data $\sigma\nu\cdot\nabla u$ and the conductivity $\sigma(x)$ on ∂X are also known. We saw that measurements of Neumann data were necessary in the construction of $H(x)$, and so our main new assumption is that $\sigma(x)$ is known on ∂X. This allows us to have access to $\nu\cdot\nabla u$ on ∂X. Note that for $x\in\partial X$, with the notation $\nabla_\tau u = u - \nu\cdot\nabla u\nu$, we find that $H(x) = \sigma|\nabla_\tau u|^2 + (1/\sigma)|\sigma\nu\cdot\nabla u|^2$, which provides a quadratic equation for σ when u and $\nu\cdot\nabla u$ are known at x.

Combining (1) and (6) with the above hypotheses, we can eliminate σ from the equations and obtain the following Cauchy problem for $u(x)$:

$$-\nabla\cdot\frac{H(x)}{|\nabla u|^2(x)}\nabla u = 0 \quad \text{in } X, \qquad u = f \quad\text{and}\quad \frac{\partial u}{\partial \nu} = j \quad \text{on } \partial X, \tag{7}$$

where (H, f, j) are now known while u is unknown. Thus the measurement operator maps (σ, u) to (H, f, j) constructed from a solution $u(x)$ of (1). Although this problem (7) may look elliptic at first, it is in fact hyperbolic as we already mentioned, and this is the reason why we augmented it with (redundant) Cauchy data. In the sequel, we also consider other redundant measurements given by the acquisition of $H(x) = \sigma(x)|\nabla u|^2(x)$ for solutions u corresponding to several boundary conditions $f(x)$. A general methodology to uniquely reconstruct $\sigma(x)$ from a sufficient number of redundant measurements has recently been analyzed in [Bal et al. 2011a; Capdeboscq et al. 2009].

The above equation may be transformed as

$$(I - 2\widehat{\nabla u}\otimes\widehat{\nabla u}) : \nabla^2 u + \nabla \ln H\cdot\nabla u = 0 \quad \text{in } X, \qquad u = f \quad\text{and}\quad \frac{\partial u}{\partial \nu} = j \quad \text{on } \partial X. \tag{8}$$

Here $\widehat{\nabla u} = \nabla u/|\nabla u|$. With

$$g^{ij} = g^{ij}(\nabla u) = -\delta^{ij} + 2(\widehat{\nabla u})_i(\widehat{\nabla u})_j \quad\text{and}\quad k^i = -(\nabla \ln H)_i, \tag{9}$$

(8) is recast as

$$g^{ij}(\nabla u)\partial^2_{ij}u + k^i\partial_i u = 0 \quad \text{in } X, \qquad u = f \quad\text{and}\quad \frac{\partial u}{\partial \nu} = j \quad \text{on } \partial X. \tag{10}$$

Note that g^{ij} is a definite matrix of signature $(1, n-1)$, so that (10) is a quasilinear strictly hyperbolic equation. The Cauchy data f and j then need to be provided on a space-like hypersurface in order for the hyperbolic problem to be well-posed [Hörmander 1983b]. This is the main difficulty when solving (7) with redundant Cauchy boundary conditions.

3. Local existence, uniqueness, and stability

Once we recast (7) as the nonlinear hyperbolic equation (10), we have a reasonable framework to perform local reconstructions. However, in general, we cannot hope to reconstruct $u(x)$, and hence $\sigma(x)$ on the whole domain X, at least not in a stable manner. The reason is that the direction of "time" in the second-order hyperbolic equation is $\widehat{\nabla u}(x)$. The normal $\nu(x)$ at the boundary ∂X separates the (good) part of ∂X that is "space-like" and the (bad) part of ∂X that is "time-like"; see definitions below. Cauchy data on space-like surfaces such as $t=0$ provide stable information to solve standard wave equations, where, as in general, it is known that arbitrary singularities can form in a wave equation from information on "time-like" surfaces such as $x=0$ or $y=0$ in a three-dimensional setting (where (t, x, y) are local coordinates of X) [Hörmander 1983b].

In the two-dimensional setting $n=2$, the numbers of space-like and time-like variables both equal 1 and "t" and "x" play a symmetric role. Nonetheless, if there exist points at the boundary of ∂X such that $\nu(x)$ is "light-like" (null), then singularities can form at such points and propagate inside the domain. As a consequence, even in two dimensions of space, instabilities are expected to occur in general.

We present local uniqueness and stability results for the reconstruction of u and σ in the next subsection. These results are based on the linear theory of hyperbolic equations with general Lorentzian metrics [Taylor 1996]. In Section 3B, we adapt results in [Hörmander 1997] to propose a local theory of reconstruction of $u(x)$, and hence $\sigma(x)$, by solving (10) with data (H, f, j) that are not necessarily in the range of the measurement operator $(u, \sigma) \mapsto (H, f, j)$, which to (u, σ) satisfying (1) associates the Cauchy data (f, j) and the internal functional H.

3A. *Uniqueness and stability.* Stability estimates may be obtained as follows. Let (u, σ) and $(\tilde{u}, \tilde{\sigma})$ be two solutions of (1) and the Cauchy problem (10) with measurements (H, f, j) and $(\tilde{H}, \tilde{f}, \tilde{j})$. Note that after solving (10), we then reconstruct the conductivities with

$$\sigma(x) = \frac{H}{|\nabla u|^2}(x), \qquad \tilde{\sigma}(x) = \frac{\tilde{H}}{|\nabla \tilde{u}|^2}(x). \tag{11}$$

The objective of stability estimates is to show that $(u-\tilde{u}, \sigma-\tilde{\sigma})$ are controlled by $(H-\tilde{H}, f-\tilde{f}, j-\tilde{j})$, that is, to show that small errors in measurements (that are in the range of the measurement operator) correspond to small errors in the coefficients that generated such measurements.

Some algebra shows that $v = \tilde{u} - u$ solves the linear equation

$$\nabla \cdot \left(\frac{H}{|\nabla \tilde{u}|^2} \left\{ I - \frac{\nabla u \otimes (\nabla u + \nabla \tilde{u})}{|\nabla u|^2} \right\} \nabla v + \frac{H - \tilde{H}}{|\nabla \tilde{u}|^2} \nabla \tilde{u} \right) = 0,$$

with Cauchy data $\tilde{f}-f$ and $\tilde{j}-j$, respectively. Changing the roles of u and $\tilde{u}$ and summing the two equalities, we get

$$\nabla\cdot\Big(\frac{H}{|\nabla\tilde{u}|^2|\nabla u|^2}\big\{(\nabla u+\nabla\tilde{u})\otimes(\nabla u+\nabla\tilde{u})-(|\nabla u|^2+|\nabla\tilde{u}|^2)I\big\}\nabla v+\delta H\Big(\frac{\nabla\tilde{u}}{|\nabla\tilde{u}|^2}+\frac{\nabla u}{|\nabla u|^2}\Big)\Big)=0.$$

The above operator is elliptic when $\nabla u\cdot\nabla\tilde{u}<0$ and is hyperbolic when $\nabla u\cdot\nabla\tilde{u}>0$. Note that $\nabla u\cdot\nabla\tilde{u}>0$ on ∂X when $j-\tilde{j}$ and $f-\tilde{f}$ are sufficiently small. We obtain a linear equation for v with a source term proportional to $\delta H=\tilde{H}-H$. For large amounts of noise, ∇u may significantly depart from $\nabla\tilde{u}$, in which case the above equation may lose its hyperbolic character. However, stability estimates are useful when δH is small, which should imply that u and $\tilde{u}$ are sufficiently close, in which case the above operator is hyperbolic. We assume here that the solutions u and $\tilde{u}$ are sufficiently close that the above equation is hyperbolic throughout the domain. We recast the above equation as the linear equation

$$\mathfrak{g}^{ij}(x)\partial^2_{ij}v+\mathfrak{k}^i\partial_i v+\partial_i(l^i\delta H)=0\quad\text{in } X,\qquad v=\tilde{f}-f,\quad \frac{\partial v}{\partial\nu}=\tilde{j}-j\quad\text{on }\partial X,\tag{12}$$

for appropriate coefficients $\mathfrak{g}^{ij}$, $\mathfrak{k}^i$ and l^i. Now $\mathfrak{g}^{ij}$ is strictly hyperbolic in X (of signature $(1,n-1)$) and is given explicitly by

$$\begin{aligned}\mathfrak{g}(x)&=\frac{H}{|\nabla\tilde{u}|^2|\nabla u|^2}\big\{(\nabla u+\nabla\tilde{u})\otimes(\nabla u+\nabla\tilde{u})-(|\nabla u|^2+|\nabla\tilde{u}|^2)I\big\}\\&=\alpha(x)\big(\boldsymbol{e}(x)\otimes\boldsymbol{e}(x)-\beta^2(x)(I-\boldsymbol{e}(x)\otimes\boldsymbol{e}(x))\big),\end{aligned}\tag{13}$$

where

$$\boldsymbol{e}(x)=\frac{\nabla u+\nabla\tilde{u}}{|\nabla u+\nabla\tilde{u}|}(x),\qquad \beta^2(x)=\frac{|\nabla u|^2+|\nabla\tilde{u}|^2}{|\nabla u+\nabla\tilde{u}|^2-(|\nabla u|^2+|\nabla\tilde{u}|^2)}(x),\tag{14}$$

and

$$\alpha(x)=\frac{H}{|\nabla\tilde{u}|^2|\nabla u|^2}\big(|\nabla u+\nabla\tilde{u}|^2-(|\nabla u|^2+|\nabla\tilde{u}|^2)\big)$$

is the appropriate (scalar) normalization constant. Here, $\boldsymbol{e}(x)$ is a normal vector that gives the direction of "time" and $\beta(x)$ should be seen as a speed of propagation (close to 1 when u and $\tilde{u}$ are close). When $\boldsymbol{e}$ is constant, then the above metric, up to normalization, corresponds to the operator $\partial_t^2-\beta^2(t,x')\Delta_{x'}$.

We also define the Lorentzian metric $\mathfrak{h}=\mathfrak{g}^{-1}$ so that $\mathfrak{h}_{ij}$ are the coordinates of the inverse of the matrix $\mathfrak{g}^{ij}$. We denote by $\langle\cdot,\cdot\rangle$ the bilinear product associated to $\mathfrak{h}$ so that $\langle u,v\rangle=\mathfrak{h}_{ij}u^iv^j$, where the two vectors u and v have coordinates u^i and v^i, respectively. We verify that

$$\mathfrak{h}(x)=\frac{1}{\alpha(x)}\Big(\boldsymbol{e}(x)\otimes\boldsymbol{e}(x)-\frac{1}{\beta^2(x)}(I-\boldsymbol{e}(x)\otimes\boldsymbol{e}(x))\Big).\tag{15}$$

The main difficulty in obtaining a solution v to (12) arises because $\nu(x)$ is not time-like for all points of ∂X. The space-like part Σ_g of ∂X is given by the points $x\in\partial X$ such that $\nu(x)$ is time-like, in the sense that $\mathfrak{h}(\nu(x),\nu(x))>0$, or equivalently,

$$|\nu(x)\cdot\boldsymbol{e}(x)|^2>\frac{1}{1+\beta^2(x)},\qquad x\in\partial X.\tag{16}$$

In (16), the dot product is with respect to the standard Euclidean metric and ν is a unit vector for the Euclidean metric, not for the metric $\mathfrak{h}$. The time-like part of ∂X is given by the points $x \in \partial X$ such that $\mathfrak{h}(\nu(x), \nu(x)) < 0$ (that is, $\nu(x)$ is a space-like vector), while the light-like (null) part of ∂X corresponds to x such that $\mathfrak{h}(\nu(x), \nu(x)) = 0$ (that is, $\nu(x)$ is a null vector).

When $j = \tilde{j}$ on ∂X so that $\nabla u(x) = \nabla \tilde{u}(x)$ and $\beta(x) = 1$ for $x \in \partial X$ (see also the proof of Theorem 3.1 below), then the above constraint becomes

$$|\nu(x) \cdot \widehat{\nabla u}(x)|^2 > \tfrac{1}{2}, \quad x \in \partial X. \tag{17}$$

In other words, when such a constraint is satisfied, the differential operator is strictly hyperbolic with respect to $\nu(x)$ on Σ_g. Once Σ_g is constructed, we need to define its domain of influence $X_g \subset X$, that is, the domain in which v can be calculated from knowledge of its Cauchy data on Σ_g. In order to do so, we apply the energy estimate method for hyperbolic equations described in [Taylor 1996, Section 2.8]. We need to introduce the notation used there; see Figure 1.

Let Σ_1 be an open connected component of Σ_g. We assume here that all coefficients and geometrical quantities are smooth. By assumption, Σ_1 is space-like, which means that the normal vector ν_1 is time-like and hence satisfies (16). Now let $\Sigma_2(s) \subset X$ be a family of (open) hypersurfaces that are also space-like with unit (with respect to the Euclidean metric) vector $\nu_2(x)$ that is thus time-like, that is, verifies (16). We assume that the boundary of $\Sigma_2(s)$ is a codimension-1 manifold of Σ_1. Let then

$$\mathbb{O}(s) = \bigcup_{0<\tau<s} \Sigma_2(\tau), \tag{18}$$

which we assume is an open subset of X. In other words, we look at domains of influence $\mathbb{O}(s)$ of Σ_1 that are foliated (swept out) by the space-like surfaces $\Sigma_2(\tau)$. Then we have the following result:

Theorem 3.1 (local uniqueness and stability). *Let u and $\tilde{u}$ be two solutions of (7) sufficiently close in $W^{1,\infty}(X)$ norm and such that $|\nabla u|$, $|\nabla \tilde{u}|$, H and $\tilde{H}$ are bounded above and below by positive constants.*

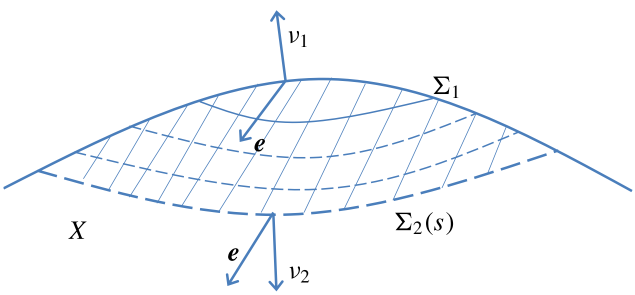

Figure 1. Construction of the domain of influence $\mathbb{O}$ (hatched area). The unit vectors $\boldsymbol{e}$ indicate the "time" direction of the Lorentzian metric $\mathfrak{h}$. The surface $\Sigma_2(s)$ has a normal vector $\nu_2(x)$ that forms a sufficiently small angle with $\boldsymbol{e}$ that $\Sigma_2(s)$ is a space-like surface, as is $\Sigma_1 \subset \Sigma_g$ with an angle such that $|\nu_1 \cdot \boldsymbol{e}|$ is also sufficiently close to 1.

This ensures that $\mathfrak{g}$ *constructed in* (13) *is strictly hyperbolic and that* $\alpha(x)$ *and* $\beta(x)$ *in* (14) *are bounded above and below by positive constants.*

Let Σ_1 *be an open connected component of* Σ_g, *the space-like component of* ∂X, *and let the domain of influence* $\mathbb{O} = \mathbb{O}(s)$ *for some* $s > 0$ *be constructed as above. Let us define the energy*

$$E(dv) = \langle dv, \nu_2\rangle^2 - \tfrac{1}{2}\langle dv, dv\rangle\langle \nu_2, \nu_2\rangle. \tag{19}$$

Here, dv *is the gradient of* v *in the metric* $\mathfrak{h}$, *and is thus given in coordinates by* $\mathfrak{g}^{ij}\partial_j v$. *Then we have the local stability result*

$$\int_{\mathbb{O}} E(dv)\, dx \le C\bigg(\int_{\Sigma_1} |f-\tilde f|^2 + |j-\tilde j|^2 d\sigma + \int_{\mathbb{O}} |\nabla \delta H|^2 dx\bigg), \tag{20}$$

where dx *and* $d\sigma$ *are the standard* (*Euclidean*) *volume and* (*hyper*)*surface measures on* $\mathbb{O}$ *and* Σ_1, *respectively.*

The above estimate is the natural estimate for the Lorentzian metric $\mathfrak{h}$. *For the Euclidean metric*, *the above estimate may be modified as follows. Let* $\nu_2(x)$ *be the unit* (*for the Euclidean metric*) *vector to* $x \in \Sigma_2(s)$, *and let us define* $c(x) := \nu_2(x)\cdot \boldsymbol{e}(x)$ *with* $\boldsymbol{e}(x)$ *as in* (14). *Let us define*

$$\theta := \min_{x\in\mathbb{O}} \Big[c^2(x) - \frac{1}{1+\beta^2(x)}\Big]. \tag{21}$$

We need $\theta > 0$ *for the metric* $\mathfrak{h}$ *to be hyperbolic with respect to* $\nu_2(x)$ *for all* $x \in \mathbb{O}$. *Then we have that*

$$\int_{\mathbb{O}} |v^2| + |\nabla v|^2 + (\sigma-\tilde\sigma)^2 dx \le \frac{C}{\theta^2}\bigg(\int_{\Sigma_1} |f-\tilde f|^2 + |j-\tilde j|^2 d\sigma + \int_{\mathbb{O}} |\nabla \delta H|^2 dx\bigg), \tag{22}$$

where σ *and* $\tilde\sigma$ *are the reconstructed conductivities given in* (11). *Provided that data are equal in the sense that* $f = \tilde f$, $j = \tilde j$, *and* $H = \tilde H$, *we obtain* $v = 0$ *and the uniqueness result* $u = \tilde u$ *and* $\sigma = \tilde\sigma$.

Proof. That $\mathfrak{h}$ is a hyperbolic metric is obtained, for instance, if u and $\tilde u$ are sufficiently close in the $W^{1,\infty}(X)$ norm and if $|\nabla u|$, $|\nabla \tilde u|$, H and $\tilde H$ are bounded above and below by positive constants. The derivation of (20) then follows from [Taylor 1996, Proposition 8.1] using the notation introduced earlier in this section. The volume and surface measures dx and $d\sigma$ are here the Euclidean measures and are of the same order as the volume and surface measures of the Lorentzian metric $\mathfrak{h}$. This can be seen in (15), since α and β are bounded above and below by positive constants.

Then (20) reflects the fact that the energy measured by the metric $\mathfrak{h}$ is controlled. However, this "energy" fails to remain definite for null-like vectors (vectors $\boldsymbol{v}$ such that $\mathfrak{h}(\boldsymbol{v},\boldsymbol{v}) = 0$), and as x approaches the boundary of the domain of influence of Σ_g, we expect the estimate to deteriorate.

Let $x \in \mathbb{O}$ be fixed and define $\nu = \nu_2(x)$ and $\boldsymbol{e} = \boldsymbol{e}(x)$. Let us decompose $\nu = c\boldsymbol{e} + s'\boldsymbol{e}^\perp$, where $c\boldsymbol{e}$ is the orthogonal projection of ν onto $\boldsymbol{e}$ and $s'\boldsymbol{e}^\perp := \nu - c\boldsymbol{e}$ the projection onto the orthogonal subspace of $\mathbb{R}^n$ with $\boldsymbol{e}^\perp$ a unit vector. For a vector $\boldsymbol{v} = v_1\boldsymbol{e} + v_2'\boldsymbol{e}^\perp + \boldsymbol{w}'$ (standing for dv) with $\boldsymbol{w}'$ orthogonal to $\boldsymbol{e}$ and $\boldsymbol{e}^\perp$ (and thus vanishing if $n = 2$), we need to estimate

$$E(\boldsymbol{v}) = h^2(\boldsymbol{v},\nu) - \tfrac{1}{2}h(\boldsymbol{v},\boldsymbol{v})h(\nu,\nu) = \frac{1}{\alpha^2}\big[(v_1c - v_2s)^2 - \tfrac{1}{2}\big(v_1^2 - (v_2^2 + |\boldsymbol{w}|^2)\big)(c^2 - s^2)\big],$$

where we have conveniently defined $v_2 = \beta^{-1} v_2'$, $\boldsymbol{w} = \beta^{-1}\boldsymbol{w}'$, and $s = \beta^{-1}s'$. After some straightforward algebra, we find that

$$E(\boldsymbol{v}) = \frac{1+\beta^2}{\alpha^2\beta^2}\theta|\boldsymbol{w}|^2 + \frac{1}{\alpha^2}\big(\tfrac{1}{2}(c^2+s^2)(v_1^2+v_2^2) - 2v_1v_2cs\big) \geq \frac{1+\beta^2}{\alpha^2\beta^2}\theta|\boldsymbol{w}|^2 + \frac{v_1^2+v_2^2}{2\alpha^2}(c-s)^2.$$

Since β is bounded above and below by positive constants, we need to bound $(c-s)$ from below, or equivalently, $(\beta c - s')^2$ from below. Some algebra shows that

$$\theta \leq c^2 - \frac{1}{1+\beta^2} = \frac{\beta c + s'}{1+\beta^2}(\beta c - s').$$

Since $\theta < 1$, this shows that

$$E(\boldsymbol{v}) \geq C\theta^2|\boldsymbol{v}|^2,$$

for a constant C that depends on the lower and upper bounds for β and α but not on the geometry of ν. Note that the behavior of the energy in θ^2 is sharp, as the bound is attained for $v_1 = v_2$ with $\boldsymbol{w} = 0$. This proves the error estimate for $\boldsymbol{v} = \nabla v$. Since v is controlled on Σ_1, we obtain control of v on $\mathbb{O}$ by the Poincaré inequality. Now $\sigma - \tilde{\sigma}$ is estimated by $H - \tilde{H}$ and by $\nabla u - \nabla \tilde{u} = \nabla v$, and hence the result.

In other words, the angle $\phi(x)$ between $\boldsymbol{e}(x)$ and $\nu_2(x)$ must be such that $\beta(x) - \tan\phi(x) \geq \theta^2$ in order to obtain a stable reconstruction. When δH is small, then $\nabla u - \tilde{\nabla} u$ is small, so that β is close to 1. As a consequence, we obtain that the constraint of hyperbolicity of $\mathfrak{h}$ is, to first order, $\tan\phi(x) < 1$, which is indeed the constraint (16) that holds when $\nabla u = \nabla\tilde{u}$ on ∂X.

For the uniqueness result, assume that u and $\tilde{u}$ are two solutions of (7). We define $\boldsymbol{e}(x) = \widehat{\nabla u}$ and $\beta^2 \equiv 1$. Then $v = 0$ on Σ_1 implies by the preceding results that $v = 0$ in a vicinity of Σ_1 in $\mathbb{O}$, so that $u = \tilde{u}$ in the vicinity of Σ_1. This shows that $u = \tilde{u}$ in $\mathbb{O}$, and hence in all the domain of dependence of Σ_g constructed as above. ☐

Remark 3.2. In two dimensions, we can interchange the roles of space-like and time-like variables, since both are one-dimensional, and find, at least for sufficiently simple geometries, that the complement of the domain of influence of Σ_g in X is the domain of influence of the complement of Σ_g in ∂X. We thus obtain stability of the reconstruction in all of X except in the vicinity of the geodesics for the metric $\mathfrak{g}$ that emanate from ∂X in a direction $\nu(x)$ that is null-like, that is, a vector such that $\mathfrak{h}(\nu(x), \nu(x)) = 0$, or equivalently such that $|\nu(x)\cdot\boldsymbol{e}(x)|^2 = \frac{1}{2}$.

In three (or higher) dimensions, however, no such interchange of the roles of time and space is possible. All we can hope for is a uniqueness and stability result in the domain of influence of Σ_g. The solution v and the conductivity σ are not stably reconstructed on the rest of the domain without additional information from, say, other boundary conditions $f(x)$. The case of redundant measurements of this type is considered in Section 4B below, and is analyzed in a different context in [Bal et al. 2011a; Capdeboscq et al. 2009].

Remark 3.3. Assuming that the errors on the Cauchy data f and j are negligible, we obtain the following stability estimate for the conductivity:

$$\|\sigma - \tilde{\sigma}\|_{L^2(\mathbb{O})} \leq \frac{C}{\theta}\|H - \tilde{H}\|_{H^1(X)}. \tag{23}$$

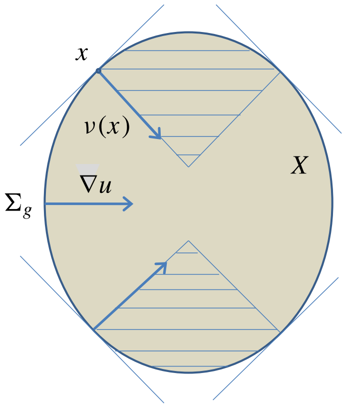

Figure 2. Geometry of the domain of influence in Euclidean geometry, with $\sigma \equiv 1$ and $u = x_1$, on a domain X given by an ovoid. In a three-dimensional geometry, we can regard the picture as a cross-section at $y = 0$ of a three-dimensional domain of revolution about the axis $\boldsymbol{e}_1 := \nabla u$. The vector $\nu(x)$ is a "null vector" making an angle of 45 degrees with ∇u. In two dimensions, Σ_g is the union of two connected components, whereas in three dimensions, Σ_g is composed of a unique connected component in ∂X. The hatched domain corresponds to $X \setminus X_g$, the part of the domain X that is not the domain of influence of Σ_g. In two dimensions, $\nabla u^{\perp} = \boldsymbol{e}_2$ may also play the role of "time", so that $X \setminus X_g$ is the domain of influence of $\Sigma \setminus \Sigma_g$. In three dimensions, the hatched region is not accessible with the techniques developed in this paper.

The measurements are of the form $H(x) = \sigma(x)|\nabla u|^2(x)$, which imposes reasonably restrictive assumptions on σ ensuring that ∇u is a solution in $H^2(\Omega)$. Under additional regularity assumptions on σ, for instance assuming that $H \in H^s(X)$ for $s \geq 2$, we find that

$$\|\sigma - \tilde{\sigma}\|_{L^2(\mathbb{O})} \leq \frac{C}{\theta} \|H - \tilde{H}\|_{L^2(X)}^{1-1/s} \|H + \tilde{H}\|_{H^s(X)}^{1/s}, \tag{24}$$

by standard interpolation. We thus obtain a standard Hölder estimate in the setting where the error in the measurements is quantified in the square integrable sense.

Remark 3.4. The linearization of (7) in the vicinity of $\sigma_0 = 1$ with only Dirichlet data is an ill-posed problem when X is a two-dimensional disc. Indeed, assume Dirichlet data of the form $f(x) = x_1$ in (1), so that the unperturbed solution is $u_0 = x_1$ in X. This shows that $\boldsymbol{e}(x) = \boldsymbol{e}_1$ in the definition (13), so that $\mathfrak{h} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2$ in (15). In other words, the linearized problem consists of solving

$$\frac{\partial^2 u}{\partial {x_1}^2} - \frac{\partial^2 u}{\partial {x_2}^2} = 0 \quad \text{in } X = \{x_1^2 + x_2^2 < 1\}, \qquad u = f \quad \text{on } \partial X.$$

The general solution to the above equation is of the form $F(x_1 - x_2) + G(x_1 + x_2)$, and there is an infinite number of linearly independent solutions to the above equation with $f = 0$. The linearization of the UMEIT problem without full Cauchy data and in this specific geometry provides an operator that is not injective.

3B. ***Reconstruction of the conductivity.*** The construction of the solution u, from which we deduce the reconstruction of $\sigma(x)$, requires that we solve the nonlinear equation (10). Let us assume that g^{ij} is given as in (9) and that the vector field h and the source terms f and j are smooth given functions. Then we can construct a unique solution to (10) locally in the vicinity of the part of ∂X that is space-like. In this section, we assume that the geometry and the coefficients of the wave equation are sufficiently smooth.

Let x^0 be a point in Σ_g, the space-like part of ∂X, so that $g(\nu(x^0), \nu(x^0)) \geq \eta > 0$. In the vicinity of x^0, which we now call 0, we parametrize ∂X by the variables $(y_1, \dots, y_{n-1})$ and denote by y_0 the signed distance to ∂X. In the vicinity of $x^0 = 0$, the map $y = F(x)$ is a diffeomorphism from a neighborhood U of $x = 0$ to the neighborhood $V = F(U)$ of $y = 0$. Moreover, locally, DF is close to the identity matrix (after an appropriate rotation of the domain if necessary) if U is sufficiently small. We denote by $J_F = \det(DF)$ the Jacobian of the transformation.

Let us come back to the equation

$$-\nabla\cdot\sigma(x)\nabla u = -\nabla\cdot\frac{H(x)}{|\nabla u|^2(x)}\nabla u = 0 \quad \text{in } X, \qquad u = f \quad \text{and} \quad \frac{\partial u}{\partial\nu} = j \quad \text{on } \partial X. \tag{25}$$

We define $v(y) = u(x)$, that is, $v = F_* u$, and then verify that $(\nabla u)(x) = DF^t \circ F^{-1}(y)\nabla v(y)$. In the y coordinates, we find that

$$-\nabla\cdot F_*\sigma\nabla v = 0, \qquad F(U),$$

where we have the standard expression in the y coordinates:

$$F_*\sigma(y) = \tilde{\sigma}(y) DF\, DF^t \circ F^{-1}(y), \qquad \tilde{\sigma} = J_F^{-1}\sigma\circ F^{-1}.$$

We may thus recast the above equation as the nonlinear equation

$$-\nabla\cdot\tilde{H}\frac{DF\,DF^t\circ F^{-1}}{|DF^t\nabla v|^2}\nabla v = 0, \qquad \tilde{H} = F_*(J_F^{-1}H) = J_F^{-1}H\circ F^{-1}. \tag{26}$$

Note that the boundary conditions are now posed on the surface $y_0 = 0$, where

$$v(0, y') = F_* f(0, y') \quad \text{and} \quad \partial_{y_0} v(0, y') = \alpha(y')F_* f(0, y') + \beta(y')F_* j(0, y'),$$

with α close to 0 and β close to 1 on $V = F(U)$. It remains to differentiate in (26) to obtain, after straightforward but tedious calculations, the expression

$$g_F^{ij}\partial^2_{ij}v + h^i_F\partial_i v = 0, \qquad F(U), \tag{27}$$

with the above "initial" conditions at $y_0 = 0$, where

$$\begin{aligned} g_F^{ij} &= -(DF\,DF^t)^i_k\delta^{jk} + 2(DF\,\widehat{DF^t\nabla v})^i(\widehat{DF^t\nabla v})^j, \\ h^i_F &= -(\nabla\ln\tilde{H}\cdot DF\,DF^t)^i - (\nabla\cdot DF\,DF^t)^i + 2(DF\,\widehat{DF^t\nabla v})^j(\widehat{DF^t\nabla v})\partial^k_j DF^i_k. \end{aligned} \tag{28}$$

When $F = I$, we recover (8). The nonlinear terms now involve functions of $\widehat{DF^t\nabla v}$.

Note that $g_F = DF\,g\,DF^t$ if we set $\widehat{DF^t\nabla v} := \widehat{\nabla u}$, and thus transforms as a tensor of type $(2, 0)$. As a consequence, the metric (a tensor of type $(0, 2)$) $g_F^{-1} = DF^{-1}g\,DF^{-t}$, since $g^{-1} = g$, as can be easily verified. Let $\nu_F = F_*\nu = DF\nu\circ F^{-1}$ be the push-forward of the normal vector seen as a vector field. At x^0, the change of variables is such that

$$g^{-1}_{F,F(x^0)}\Big(\frac{\partial}{\partial y_0},\frac{\partial}{\partial y_0}\Big) = g^{-1}_{F,F(x^0)}(\nu_F,\nu_F) = g^{-1}_{x^0}(\nu,\nu) = g_{x^0}(\nu,\nu)\geq \eta > 0.$$

This shows that g_F remains hyperbolic in the vicinity of $F(x^0)$ since $DFDF^t = I$ at $y=0$ by construction and DF is smooth. Moreover, the above is equivalent to

$$g_F^{ij}\partial^2_{ij} = \Box + \gamma^{ij}\partial^2_{ij},\qquad \Box = \frac{\partial^2}{\partial y^0\partial y^0} - \Delta_{y'},\qquad \Delta_{y'} = \sum_{j=1}^{n-1}\frac{\partial^2}{\partial y^j\partial y^j},$$

where $\sum_{i,j}|\gamma^{ij}|\leq (1-\eta)/2$. The above is nothing but the fact that g_F is hyperbolic in the vicinity of $y=0$. Note that $\gamma^{ij} = \gamma^{ij}(x,\widehat{DF^t\nabla v})$.

We thus have a nonlinear hyperbolic equation of the form

$$\begin{aligned}\big(\Box + \gamma^{ij}(x,\widehat{DF^t\nabla v})\partial_{ij} + h^i(x,\widehat{DF^t\nabla v})\partial_i\big)v = 0,&\qquad y_0>0,\ y'\in\mathbb{R}^{n-1},\\ v(0,y') = v_0(y'),\qquad \partial_{y_0}v(0,y') = j_0(y').&\end{aligned}\tag{29}$$

Since propagation in a wave equation is local, we can extend the boundary conditions for $y=(0,y')$ outside the domain $F(U)$ by $v_0=0$ and $\partial_{y_0}v=1$ and the functions γ^{ij} and h^i by 0 outside of $F(U)$. This allows us to obtain an equation posed on the half-space $y_0>0$.

The nonlinear functions $\gamma^{ij}(x,\widehat{DF^t\nabla v})$ and $h^i(x,\widehat{DF^t\nabla v})$ are smooth functions of ∇v except at the points where $\nabla v = 0$. However, we are interested in solutions such that ∇v does not reach 0, to preserve the hyperbolic structure of g^{ij}. Note that $|\nabla v|$ is bounded from below by a positive constant on $y_0=0$ by assumption. We obtain a bound on the uniform norm of the Hessian of v, which implies that at least for a sufficiently small interval $y_0\in(0,t_0)$, $|\nabla v|$ does not vanish and γ^{ij} and h^i can then be considered as smooth functions of x and ∇v.

Using [Hörmander 1997, Theorem 6.4.11 and remark following (6.4.24)], the above equation satisfies the hypotheses to obtain an a priori estimate for

$$M(y_0) = \sum_{|\alpha|\leq\kappa+2}\|\partial^\alpha u(y_0,\cdot)\|_{L^2(\mathbb{R}^{n-1})},$$

with κ the smallest integer strictly greater than $(n-1)/2$. By Sobolev embedding, this implies that the second derivatives of v are uniformly bounded so that for at least a small interval, $|\nabla v|$ is bounded away from 0.

Once v, and hence u, is reconstructed, at least in the vicinity of the part Σ_g of ∂X that is space-like for ∇u, we deduce that

$$\sigma(x) = \frac{H(x)}{|\nabla u|^2(x)}.$$

Note that ∇u cannot vanish, by construction, so that the above equality for $\sigma(x)$ is well-defined. We already know that a solution to the above nonlinear equation exists in the absence of noise, since we have constructed it by solving the original linear equation. In the presence of significant noise, the nonlinear equation may behave in a quite different manner than that for the exact solution. However, the above construction shows that the nonlinear equation can be solved locally if the measurement $H(x)$ is perturbed by a small amount of noise.

4. Global reconstructions of the diffusion coefficient

The picture in Figure 2 shows that in general we cannot hope to obtain a global reconstruction from a single measurement of $H(x)$ even augmented with full Cauchy data. Only Cauchy data on the space-like part of the boundary can be used to obtain stable reconstructions.

Global reconstructions have been obtained from redundant measurements of the form $H_{ij} = S_i \cdot S_j$, with $S_i = \sqrt{\sigma}\nabla u_i$ and u_i the solution of (1) with Dirichlet conditions $f = f_i$, in [Capdeboscq et al. 2009] in the two-dimensional setting and in [Bal et al. 2011a] in the two- and three-dimensional settings; see also [Kuchment and Kunyansky 2011].

This section analyzes geometries in which a unique measurement $H(x)$ or a small number of measurements of the form $H(x)$, augmented with Cauchy data (f, j), allow one to uniquely and stably reconstruct $\sigma(x)$ on the whole domain X. These reconstructions are obtained by (possibly) modifying the geometry of the problem so that the domain where $\sigma(x)$ is not known lies within the domain of dependence of Σ_g. We consider two scenarios. In the first scenario, considered in Section 4A, we slightly modify the problem to obtain a model with an internal source of radiation f. Such geometries are guaranteed to provide a unique global reconstruction in dimension $n = 2$, but not necessarily in higher spatial dimensions, where global reconstructions hold only for a certain class of coefficients $\sigma(x)$. In the second scenario, analyzed in Section 4B, we consider a setting where reconstructions are possible when the Lorentzian metric is the Euclidean (Lorentzian) metric, that is, $\alpha = \beta = 1$ in (15). We then show the existence of an open set of illuminations f for three different measurements of the form $H(x)$ such that the global result obtained for the Euclidean metric remains valid for arbitrary, sufficiently smooth coefficients $\sigma(x)$.

4A. ***Geometries with an internal source.*** From the geometric point of view, the Cauchy data are sufficient to allow for full reconstructions when $\Sigma_g = \partial X$, so that the whole boundary ∂X is space-like for the metric g, and X is the domain of dependence of Σ_g. This can happen, for instance, when ∂X is a level set of u and the normal derivative of u either points inwards or outwards at every point of ∂X. When X is a simply connected domain, the maximum principle prevents one from having such a geometry. However, when X is not simply connected, such a configuration can arise. We will show that such a configuration (with X the domain of dependence of Σ_g) is always possible in two dimensions of space. When $n \geq 3$, such configurations hold only for a restricted class of conductivities $\sigma(x)$ for which no critical points of $u(x)$ exist.

Let us consider the two-dimensional case $n = 2$. We assume that X is an open smooth domain diffeomorphic to an annulus and with boundary $\partial X = \partial X_0 \cup \partial X_1$; see Figure 3. We assume that $f = 0$ on the external boundary ∂X_0 and $f = 1$ on the internal boundary ∂X_1. The boundary of X is composed of

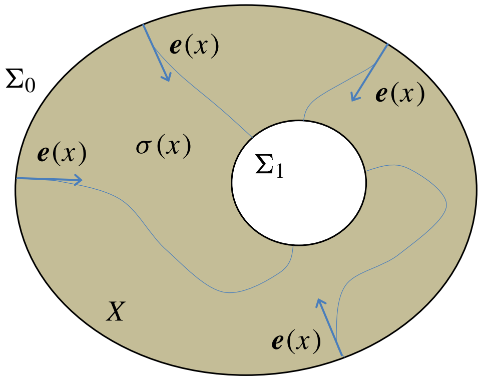

Figure 3. Geometry of an annulus in two space dimensions with boundaries Σ_0 and Σ_1, the level sets where $u = 0$ and $u = 1$, respectively. The four curves correspond to four integrals of the flow of the gradient vector field $\widehat{\nabla u}$.

two smooth connected components that are different level sets of the solution u to (1), which is uniquely defined in X.

In practice, such a domain X may be constructed as follows. As we do in the geometry depicted on page 769, we embed $\tilde{X}$, the domain where σ is unknown, into a larger domain X with, say, $\sigma(x) = \sigma_0$ on $X \setminus \tilde{X}$ and with a hole where we impose the aforementioned boundary conditions. Then we have the following result:

Proposition 4.1. *Let X be the geometry described above with $n = 2$ and $u(x)$ the solution to (1). We assume here that both the geometry and $\sigma(x)$ are sufficiently smooth. Then $|\nabla u|$ is bounded from above and below by positive constants. The level sets $\Sigma_c = \{x \in X,\ u(x) = c\}$ for $0 < c < 1$ are smooth curves that separate X into two disjoint subdomains.*

Proof. The proof of the first part is based on the fact that critical points of solutions to elliptic equations in two dimensions are isolated [Alessandrini 1986]. First of all, the Hopf lemma [Evans 1998] ensures that no critical point exists on the smooth closed curves Σ_0 and Σ_1. Let x_i be the finite number of points where $\nabla u(x_i) = 0$. At each x_i, the level set of u with value $0 < c_i = u(x_i) < 1$ is locally represented by n_i (n_i even) smooth simple arcs emanating from x_i that make an angle equal to $2\pi/n_i$ at x_i [Alessandrini 1986]. For instance, if only two simple arcs emanate from x_0, then these two arcs form a continuously differentiable curve in the vicinity of x_0. Between critical points, level sets of u are smooth by the inverse function theorem.

Let us assume that there is a point x_i with more than two simple arcs leaving x_i. Let γ_j, $1 \le j \le 4$, be such arcs. If γ_1 meets another critical point, we pick one of the possible other arcs emanating from this critical point to continue the curve γ_1. This is always possible, as critical points always have an even number of leaving simple arcs. The curve γ_1 cannot meet Σ_0 or Σ_1, and therefore must come back to the point x_i. Let us assume the existence of a closed subloop of γ_1 that does not self-intersect and does not wind around Σ_1 (that is, is homotopic to a point). In the interior of that closed subloop, u is then constant by the maximum principle and hence constant on X by the unique continuation theorem [Hörmander

1983a]. This is impossible, and therefore γ_1 must wind around Σ_1. Let us pick a subset of γ_1, which we still call γ_1, that winds around Σ_1 once. The loop meets one of the other γ_j to come back to x_i, which we call γ_2 if it is not γ_1. Now let us follow γ_3. Such a curve also has to come back to x_i. By the maximum principle and the unique continuation theorem, it cannot come back with a subloop homotopic to a point. So it must come back also winding around Σ_1. But γ_1 and γ_3 are then two different curves winding around Σ_1. This implies the existence of a connected (not necessarily simply connected) domain whose boundary is included in $\gamma_1 \cup \gamma_3$. Again, by the maximum principle and the unique continuation theorem, such a domain cannot exist. So any critical point cannot have more than two simple arcs of level curves of u leaving it.

So far, we have proved that any critical point x_i sees exactly two arcs leaving x_i at an angle equal to π, since by the maximum principle, critical points cannot be local minima or maxima. These two arcs again have to meet winding around Σ_1. This generates a single curve that we call γ_1, with no possible self-intersection. Moreover, since all angles at critical points are equal to π, the curve γ_1 is of class C^1 and piecewise of class C^2. Let X_c be the annulus with boundary equal to $\Sigma_1 \cup \gamma_1$. On X_c, u satisfies an elliptic equation with values $u = 1$ on Σ_1 and $0 < u = c_i < 1$ on γ_1. Since γ_1 is sufficiently smooth now (smooth on each arc with matching derivatives on each side of each critical point), it satisfies the interior sphere condition and we can apply the Hopf lemma [Gilbarg and Trudinger 1977, Lemma 3.4] to deduce that the normal derivative of u on γ_1 cannot vanish at x_i or anywhere along γ_1. There are therefore no critical points of u in $\bar{X}$. By continuity, this means that $|\nabla u|$ is uniformly bounded from below by a positive constant. Standard regularity results show that it is also bounded from above.

Now let $0 < c < 1$ and Σ_c be the level set where $u = c$. Such a level set separates X into two subdomains where $0 < u < c$ and $c < u < 1$, respectively, by the maximum principle. We therefore obtain a foliation of X into the union of the smooth curves Σ_c for $0 < c < 1$. Now let $x \in \Sigma_c$ and consider the flow of ∇u in both directions emanating from x. Then both curves are smooth and need to reach the boundary at a unique point. Since any point on Σ_0 is also mapped to a point on Σ_1 by the same flow, this shows that Σ_c is diffeomorphic to Σ_0 and Σ_1. □

The result extends to higher dimensions, provided that $|\nabla u|$ does not vanish, with exactly the same proof. Only the proof of the absence of critical points of u was purely two-dimensional. In the absence of critical points, we thus obtain that $\boldsymbol{e}(x) = \widehat{\nabla u} = \nu(x)$, so that $\nu(x)$ is clearly a time-like vector. Then the local results of Theorem 3.1 become global results, which yields the following proposition:

Proposition 4.2. *Let X be the geometry described above in dimension $n \geq 2$ and $u(x)$ the solution to (1). We assume here that both the geometry and $\sigma(x)$ are sufficiently smooth. We also assume that $|\nabla u|$ is bounded from above and below by positive constants. Then the nonlinear equation (10) admits a unique solution and the reconstruction of u and of σ is stable in X in the sense described in Theorem 3.1.*

Remark 4.3. The above geometry with a hole is not entirely necessary in practice. Formally, we can assume that the hole with boundary Σ_1 shrinks and converges to a point $x_0 \in \partial X$ at the boundary of the domain. Thus, the illumination f is an approximation of a delta function at x_0. The level sets of the solution are qualitatively similar to the level sets in the annulus. Away from x_0, the surface ∂X is a level set of the solution u, and hence the normal to the level set is a time-like vector for the Lorentzian metric with direction $\boldsymbol{e}(x) = \nu(x)$. Away from x_0, we can solve the wave equation inwards and obtain stable reconstructions in all of X but a small neighborhood of x_0. This construction should also provide stable reconstructions in arbitrary dimensions provided that u does not have any critical point.

In dimensions $n \geq 3$, however, we cannot guarantee that u does not have any critical point independent of the conductivity. If the conductivity is close to a constant where we know that no critical point exists, then by continuity of u with respect to small changes in $\sigma(x)$, u does not have any critical point and the above result applies. In the general case, however, we cannot guarantee that ∇u does not vanish, and in fact can produce a counterexample using the geometry introduced in [Briane et al. 2004] (see also [Melas 1993] for the existence of critical points of elliptic solutions):

Proposition 4.4. *There is an example of a smooth conductivity such that u admits critical points.*

Proof. Consider the geometry in three dimensions depicted in Figure 4. The domain X is a smooth, convex domain, invariant by rotation leaving $\boldsymbol{e}_z$ invariant and by symmetry $z \to -z$, and including two disjoint, interlocked tori T_1 and T_2. The first torus T_1 is centered at $c_1 = (0, 0, 1)$, with base circle

$$\left\{\boldsymbol{e}_z + 2\boldsymbol{e}_x + \alpha(\cos\phi\, \boldsymbol{e}_x + \sin\phi\, \boldsymbol{e}_y),\ 0 \leq \phi < 2\pi\right\}$$

rotating around c_1 in the plane $(\boldsymbol{e}_x, \boldsymbol{e}_z)$ (top torus in Figure 4) for $\alpha = \frac{1}{2}$, say. The second torus T_2 is centered at $c_2 = (0, 0, -1)$, with base circle

$$\left\{-\boldsymbol{e}_z + 2\boldsymbol{e}_y + \alpha(\cos\phi\, \boldsymbol{e}_x + \sin\phi\, \boldsymbol{e}_y),\ 0 \leq \phi < 2\pi\right\}$$

rotating around c_2 in the plane $(\boldsymbol{e}_y, \boldsymbol{e}_z)$ (bottom torus in Figure 4).

We consider the boundary condition $u = z$ on ∂X.

We assume that $\sigma(x) = 1 + \lambda\varphi(x)$ in (1), where $\varphi(x)$ is a smooth, nontrivial, nonnegative function with nonvanishing support inside each of the tori T_1 and T_2 that respects the invariance by rotation and the symmetries of the two tori. We normalize $\varphi(x)$ by 1 on the circles $\left\{\boldsymbol{e}_z + 2(\cos\phi\, \boldsymbol{e}_z + \sin\phi\, \boldsymbol{e}_y),\ 0 \leq \phi < 2\pi\right\}$

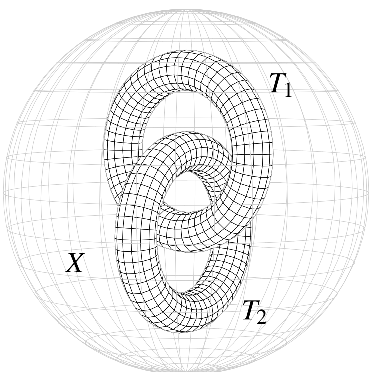

Figure 4. Geometry of a critical point: X is the ball of radius 4; the interlocked tori are the top torus T_1 and the bottom torus T_2.

and $\{-\boldsymbol{e}_z + 2(\cos\phi\, \boldsymbol{e}_z + \sin\phi\, \boldsymbol{e}_y),\ 0 \le \phi < 2\pi\}$ at the center of the volumes delimited by the two tori. When $\lambda = 0$, so that $\sigma(x) \equiv 1$, then $u = z$ is the solution of the problem (1). As λ, and hence σ inside the tori, converges to $+\infty$, the solution u is such that u converges to a constant $C_1 > 0$ on the support of φ inside T_1 and $C_2 < 0$ on the support of φ inside T_2. For λ sufficiently large, by continuity of the solution u with respect to σ, we obtain that $u(0, 0, 1) < 0$, since $(0, 0, 1)$ is inside T_2, and $u(0, 0, -1) > 0$, since $(0, 0, -1)$ is inside T_1. Since the geometry is invariant by symmetry $x \to -x$ and $y \to -y$, then so is the solution x, and hence $\partial_x u(0, 0, z) = \partial_y u(0, 0, z) = 0$ for all $(0, 0, z) \in X$. Now the function $z \to u(0, 0, z)$ goes from negative to positive to negative back to positive values as z increases, and so has at least two critical points. At these points, $\nabla u = 0$, and hence the possible presence of critical points in elliptic equations in dimensions three and higher. □

Note that the above symmetries are not necessary to obtain critical points, which appear generically in structures of the form of two interlocked rings with high conductivities, as indicated above. At an intuitive and informal level, small perturbations of the above geometry and the boundary conditions make it so that the level sets $\Sigma_c = \{u = c\}$ for c sufficiently large and c sufficiently small are simply connected codimension-1 manifolds with boundary on ∂X. When σ is sufficiently large, u converges to two different values c_1 and c_2 inside the two discs (say one positive in T_1 and one negative in T_2). Thus for σ sufficiently large, the level set $u = c_1$, assuming it does not have any critical point, is a smooth locally codimension-1 manifold, by the implicit function theorem, that can no longer be simply connected. Thus, as the level sets c decrease from high values to c_1, they go through a change of topology that can only occur at a critical point of u [Morse and Cairns 1969].

4B. ***Complex geometric optics solutions and global stability.*** Let us now consider a domain $\tilde{X}$, where $\sigma(x)$ is unknown and close to a constant σ_0. Let us assume that $\tilde{X}$ is embedded into a larger domain X and that we can assume that $\sigma(x)$ is known and also close to the constant σ_0. Then it is not difficult to construct X so that $\tilde{X}$ lies entirely within the domain of dependence of Σ_g; see, for instance, the geometry depicted in Figure 5.

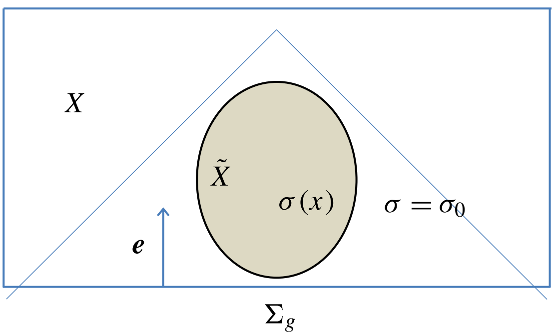

Figure 5. Geometry of a domain where the reconstruction of the unknown σ on $\tilde{X}$ is possible from a single measurement. The geometry of the Lorentzian metric is represented when $\sigma(x) = \sigma_0$. By continuity, the domain of influence of Σ_g includes $\tilde{X}$ for all smooth conductivities $\sigma(x)$ sufficiently close to σ_0.

For the rest of the section, we show that global reconstructions can be obtained for general sufficiently smooth metrics, provided that three well-chosen measurements are available. This result is independent of spatial dimension. The measurements are constructed by means of complex geometrical optics solutions.

Let k be a vector in $\mathbb{R}^n$ and $k^\perp$ be a vector orthogonal to k of the same length. Let $\rho = ik + k^\perp$ be a complex-valued vector so that $\rho \cdot \rho = 0$. Thus, $e^{\rho\cdot x}$ is harmonic and $\nabla e^{\rho\cdot x} = \rho e^{\rho\cdot x}$. The latter gradient has a privileged direction of propagation ρ, which is, however, complex-valued. Its real and imaginary parts are such that

$$e^{-k^\perp\cdot x}\Im\nabla e^{\rho\cdot x} = |k|\theta(x), \qquad e^{-k^\perp\cdot x}\Re\nabla e^{\rho\cdot x} = |k|\theta^\perp(x), \tag{30}$$

where $\theta(x) = \hat{k}\cos k\cdot x + \hat{k}^\perp \sin k\cdot x$ and $\theta^\perp(x) = -\hat{k}\sin k\cdot x + \hat{k}^\perp\cos k\cdot x$. As usual, $\hat{k} = k/|k|$.

Consider propagation with Cauchy data given on a hyperplane with normal vector $\hat{k} \in \mathbb{S}^{n-1}$. We want to make sure that we always have at our disposal a Lorentzian metric for which $\hat{k}$ is a time-like vector so that the available Cauchy data live on a space-like surface for that metric. For the rest of the section, we assume that $k = |k|\boldsymbol{e}_1$ and that $k^\perp = |k|\boldsymbol{e}_2$, so that

$$\theta(x) = \hat{k}\cos|k|x_1 + \hat{k}^\perp\sin|k|x_1 \quad \text{and} \quad \theta^\perp(x) = -\hat{k}\sin|k|x_1 + \hat{k}^\perp\cos|k|x_1. \tag{31}$$

For a vector field with unit vector $\theta(x)$, we associate the Lorentz metric with direction θ given by $\mathfrak{h}_\theta = 2\theta\otimes\theta - I$.

The Lorentzian metrics with directions $\theta(x)$ and $\theta^\perp(x)$ oscillate with x_1. A given vector $\hat{k}$ therefore cannot be time-like for all points x. However, we can always construct two different linear combinations of these two directions that form time-like vectors for a given range of $k\cdot x = |k|x_1$. Such combinations allow us to solve the wave equation forward and obtain unique and stable reconstructions on the whole domain X. The above construction with $e^{\rho\cdot x}$ harmonic can be applied when $\sigma(x) = \sigma_0$ a constant. It turns out that we can construct complex geometric optics solutions for arbitrary, sufficiently smooth conductivities $\sigma(x)$ and obtain global existence and uniqueness results in that setting. We state the following result.

Theorem 4.5. *Let σ be extended by $\sigma_0 = 1$ on $\mathbb{R}^n \setminus \tilde{X}$, where $\tilde{X}$ is the domain where σ is not known. We assume that σ is smooth on $\mathbb{R}^n$. Let $\sigma(x) - 1$ be supported without loss of generality on the cube $(0, 1)\times\left(-\frac{1}{2}, \frac{1}{2}\right)^{n-1}$. Define the domain $X = (0, 1)\times B_{n-1}(a)$, where $B_{n-1}(a)$ is the $(n-1)$-dimensional ball of radius a centered at 0 and where a is sufficiently large that the light cone for the Euclidean metric emerging from $B_{n-1}(a)$ strictly includes $\tilde{X}$. Then there is an open set of illuminations (f_1, f_2) such that if u_1 and u_2 are the corresponding solutions of (1), then the measurements*

$$H_{11}(x) = \sigma(x)|\nabla u_1|^2(x), \qquad H_{22}(x) = \sigma(x)|\nabla u_2|^2(x), \qquad H_{12}(x) = \sigma(x)|\nabla(u_1+u_2)|^2, \tag{32}$$

with the corresponding Cauchy data (f_1, j_1), (f_2, j_2) and (f_1+f_2, j_1+j_2) at $x_1 = 0$, uniquely determine $\sigma(x)$. Moreover, let $\tilde{H}_{ij}$ be measurements corresponding to $\tilde{\sigma}$ and $(\tilde{f}_1, \tilde{j}_1)$ and $(\tilde{f}_2, \tilde{j}_2)$, the corresponding Cauchy data at $x_1 = 0$. We assume that $\sigma(x) - 1$ and $\tilde{\sigma}(x) - 1$ (also supported in $(0, 1)\times\left(-\frac{1}{2}, \frac{1}{2}\right)^{n-1}$) are smooth and such that their norms in $H^{(n/2)+3+\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$ are bounded by M. Then for a

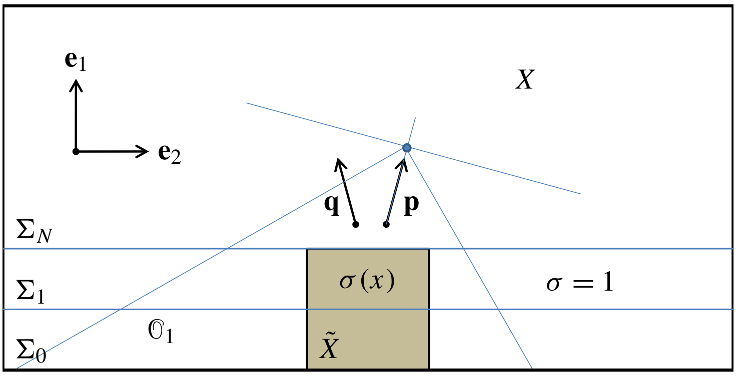

Figure 6. Extended geometry where complex geometric solutions are constructed.

constant C that depends on M, we have the global stability result

$$\|\sigma-\tilde\sigma\|_{L^2(\tilde X)} \le C\Big(\|d_C-\tilde d_C\|_{(L^2(B_{n-1}(a)))^4}+\sum_{(i,j)\in I}\|\nabla H_{ij}-\nabla\tilde H_{ij}\|_{L^2(X)}\Big). \tag{33}$$

Here, we have defined $I=\{(1,1),(1,2),(2,2)\}$ and $d_C=(f_1,j_1,f_2,j_2)$, with $\tilde d_C$ being defined similarly.

Proof. We recall that $k=|k|\boldsymbol{e}_1$ and $k^\perp=|k|\boldsymbol{e}_2$. The proof is performed iteratively on layers $t_{i-1}\le x_1\le t_i$, with $t_i=i/N$ for $0\le i\le N$ and $N=N(k)$ (to be determined) sufficiently large but finite for any given sufficiently smooth conductivity $\sigma(x)$. Here, $k=|k|\boldsymbol{e}_1$ is the vector in $\mathbb{R}^n$ used for the constructions of the CGO solutions. We define $y_i=(t_i,0,\dots,0)$ for $0\le i\le N$. Define two vectors close to $\boldsymbol{e}_1$ as

$$\boldsymbol{p}=w\boldsymbol{e}_1+\sqrt{1-w^2}\boldsymbol{e}_2,\qquad \boldsymbol{q}=w\boldsymbol{e}_1-\sqrt{1-w^2}\boldsymbol{e}_2,$$

with $w<1$ sufficiently close to 1 such that the light cones (for the Euclidean metric) emerging from $B_{n-1}(a)$ for the Lorentzian metric with main directions $\boldsymbol{p}$ and $\boldsymbol{q}$ still strictly include $\tilde X$; see Figure 6. All we need is that the radius a be chosen sufficiently large so that any Lorentzian metric with direction close to $\boldsymbol{e}_1$, $\boldsymbol{p}$ or $\boldsymbol{q}$, has a light cone emerging from $B_{n-1}(a)$ that includes $\tilde X$. This means that any time-like trajectory (geodesic) from a point in $\tilde X$ crosses $B_{n-1}(a)$ for all metrics with direction close to $\boldsymbol{e}_1$, $\boldsymbol{p}$ or $\boldsymbol{q}$. See Figure 6, where the light cone for $\boldsymbol{p}$ is shown to strictly include $\tilde X$.

Now consider the slab $t_0<x_1<t_1$. We prove a result on that slab and show that the Cauchy data at t_1 are controlled so that the same estimate may be used on $t_1<x_1<t_2$ and on all of $(0,1)$ by induction. Let α_1 and β_1 be the two angles in $(0,2\pi)$ such that

$$\cos\alpha_1\,\theta(y_0)+\sin\alpha_1\,\theta^\perp(y_0)=\boldsymbol{p},\qquad \cos\beta_1\,\theta(y_0)+\sin\beta_1\,\theta^\perp(y_0)=\boldsymbol{q},$$

where $\theta(x)\in\mathbb{S}^{n-1}$ is defined in (31).

The complex geometric optics solutions are constructed as follows. We define harmonic functions $v=\Im e^{\rho\cdot x}$ and $w=\Re e^{\rho\cdot x}$. Then we find that

$$\nabla v=e^{k^\perp\cdot x}|k|\theta(x),\qquad \nabla w=e^{k^\perp\cdot x}|k|\theta^\perp(x),$$

so that for the two harmonic functions $v_1 = \cos\alpha_1 v + \sin\alpha_1\, w$ and $w_1 = \cos\beta_1 v + \sin\beta_1\, w$, we have on the slab $0 < x_1 < t_1$ that

$$\widehat{\nabla v_1} = \cos\alpha_1\, \theta(x) + \sin\alpha_1\, \theta^\perp(x) = \boldsymbol{p} + O(|k|/N),$$

$$\widehat{\nabla w_1} = \cos\beta_1\, \theta(x) + \sin\beta_1\, \theta^\perp(x) = \boldsymbol{q} + O(|k|/N).$$

For $t_1 = 1/N$ such that $|k|t_1 = |k|/N$ is sufficiently small, $\widehat{\nabla v_1}$ and $\widehat{\nabla w_1}$, for all x such that $0 < x_1 < t_1$, are two vector fields such that the associated Lorentzian metrics $\mathfrak{h}_{\widehat{\nabla v_1}}$ and $\mathfrak{h}_{\widehat{\nabla w_1}}$ have $\boldsymbol{e}_1$ as a time-like vector.

Let us now assume that σ is arbitrary but smooth. The main idea of CGO solutions is that we can construct solutions for arbitrary σ that are close to the solutions corresponding to $\sigma = 1$ for $|k|$ sufficiently large. We construct CGO solutions u_ρ of (1) (and $\tilde{u}_\rho$ by replacing σ by $\tilde{\sigma}$) such that

$$u_\rho = \frac{1}{\sqrt{\sigma}} e^{\rho\cdot x}(1 + \psi_\rho),$$

with $|k|\psi_\rho$ bounded in the C^1 norm, since σ is sufficiently smooth by hypothesis. This result is proved in [Bal et al. 2011b] following earlier work in [Bal and Uhlmann 2010]. These solutions are constructed on $\mathbb{R}^n$ and then restricted to X; their boundary condition f_ρ is therefore specified by the construction. For such a solution, we find that

$$\nabla u_\rho = \frac{1}{\sqrt{\sigma}} e^{\rho\cdot x} |\rho| \big(\hat{\rho} + \phi_\rho\big),$$

where $|k||\phi_\rho|$ is also bounded in the uniform norm. This shows that

$$\widehat{\nabla \Im u_\rho}(x) = \theta(x) + \phi_{\rho,i}, \qquad \widehat{\nabla \Re u_\rho}(x) = \theta(x) + \phi_{\rho,r},$$

with $|k||\phi_{\rho,i}|$ and $|k||\phi_{\rho,r}|$ bounded in the uniform norm. As a consequence, we have constructed solutions of (1) with a gradient that is close to the prescribed $\theta(x)$ corresponding to harmonic functions. Construct now the two linear combinations

$$v_{1,\rho} = \cos\alpha_1\, v_\rho + \sin\alpha_1\, w_\rho, \quad w_{1,\rho} = \cos\beta_1\, v_\rho + \sin\beta_1\, w_\rho, \qquad \text{where } v_\rho := \Im u_\rho,\ w_\rho := \Re u_\rho. \tag{34}$$

Knowledge of the Cauchy data for $v_{1,\rho}$ and $w_{1,\rho}$ is inherited from that for v_ρ and w_ρ. Define $\tilde{v}_{1,\rho}$ and $\tilde{w}_{1,\rho}$ similarly with σ replaced by $\tilde{\sigma}$. We choose $|k|$ sufficiently large and then $t_1|k|$ sufficiently small so that ϕ_ρ is a negligible vector that does not perturb the Lorentzian metric much and so that

$$\widehat{\nabla v_{1,\rho}} = \boldsymbol{p} + O(|k|t_1) + O(M|k|^{-1}) \quad \text{and} \quad \widehat{\nabla w_{1,\rho}} = \boldsymbol{q} + O(|k|t_1) + O(M|k|^{-1}) \tag{35}$$

are directions of Lorentzian metrics for which (i) $\boldsymbol{e}_1$ is a time-like vector, and (ii) the light cone emerging from $B_{n-1}(a)$ includes $\tilde{X}$. Here, M is the uniform bound of σ in $H^{(n/2)+3+\varepsilon}(\mathbb{R}^n)$ [Bal et al. 2011b; Bal and Uhlmann 2010]. Note that this means that t_1 should be chosen on the order of $M|k|^{-2}$ once $|k|$ has been chosen, so that $M|k|^{-1}$ is sufficiently small.

The same properties hold for the vectors constructed by replacing σ by $\tilde{\sigma}$. Thus, the metric $\mathfrak{g}$ in (13) is given with α and β close to 1, and $\boldsymbol{e}(x)$ close to $\boldsymbol{p}$ for the function $v_{1,\rho}$ and close to $\boldsymbol{q}$ for the function $w_{1,\rho}$. Using Cauchy data on $\Sigma_0 := \{x_1 = 0\}$, we can then solve the linear equations on the slab $\mathbb{O}_1 := \{0 = t_0 < x_1 < t_1\}$ and get the solution at the surface $\Sigma_1 := \{x_1 = t_1\}$. For the solutions $v_{1,\rho}$ and $w_{1,\rho}$, we obtain as a slight modification of (20) the stability result [Taylor 1996]:

$$\int_{\Sigma_1} |f-\tilde{f}|^2 + |j-\tilde{j}|^2 d\sigma + \int_{\mathbb{O}_1} E(dv)\,dx \le C\biggl(\int_{\Sigma_0} |f-\tilde{f}|^2 + |j-\tilde{j}|^2 d\sigma + \int_{\mathbb{O}_1} |\nabla\delta H|^2\,dx\biggr). \quad (36)$$

The above measurements, $\delta H = H - \tilde{H}$, are those for the functions $(v_{1,\rho}, \tilde{v}_{1,\rho})$ and $(w_{1,\rho}, \tilde{w}_{1,\rho})$. Such measurements can be constructed from the three measurements for v_ρ, w_ρ and $v_\rho + w_\rho$. This is the place where we use the three measurements stated in the theorem: we need to ensure that $\sigma(x)|\mu\nabla v_\rho + \nu\nabla w_\rho|^2$ is available for any possible linear combination (μ, ν), since the values of α_1 and β_1 will vary (and will be called α_i and β_i) on each slab $t_i < x_i < t_{i+1}$. Since the measurements H for the 0-Laplacian problem are *quadratic* in the elliptic solution, three measurements are sufficient by polarization to allow us to construct $\sigma(x)|\nabla v_{1,\rho}|^2$ and $\sigma(x)|\nabla w_{1,\rho}|^2$.

On Σ_1, we have control on the Cauchy data of $v_{1,\rho}$ and $w_{1,\rho}$, and hence of $v_\rho = \Im u_\rho$ and $w_\rho = \Re u_\rho$ thanks to (36) and (34). Here, we need that $\boldsymbol{p}$ and $\boldsymbol{q}$ be not too close to one another (this is guaranteed by $w < 1$), so that the inversion of the 2×2 system is well-conditioned. On each slab, we define the angles α_i and β_i in order again to have Lorentzian metrics with directions close to $\boldsymbol{p}$ and $\boldsymbol{q}$. We then obtain a similar estimate to (36) and continue by induction until we reach the slab $\mathbb{O}_N := \{t_{N-1} < x_1 < t_N = 1\}$.

The stability results then apply to $\Im u_\rho$ and $\Re u_\rho$, and we thus obtain a global estimate for σ as in earlier sections. So far, the illuminations f prescribed on X to solve the elliptic problem are of a very specific type. In order for $\Im u_\rho$ and $\Re u_\rho$ to be the solutions to the elliptic problems on X, (f_1, f_2) needs to be the trace of $(\Im u_\rho, \Re u_\rho)$ on ∂X. It is for these illuminations that the three measurements $H_{ij}(x)$ for $(i, j) \in I$ generate Lorentzian metrics that satisfy the above sufficient properties. Since σ is not known, these traces are not known either.

However, any Lorentzian metric that is sufficiently close to the Lorentzian metrics constructed with the real and imaginary parts of u_ρ will inherit the same light cone properties and, in particular, the fact that $\boldsymbol{e}_1$ is a time-like vector for these new Lorentzian metrics throughout $X = (0, 1) \times B_{n-1}(a)$. Therefore, there is an open set of boundary conditions (f_1, f_2) close to $(\Im u_\rho|_{\partial X}, \Re u_\rho|_{\partial X})$ such that the conclusion (36) holds, as well as the same expressions on the other slabs $\mathbb{O}_i$. This concludes the proof of the result. □

Remark 4.6. The "three" measurements H_{ij} for $(i, j) \in I$ in (32) actually correspond to two physical measurements. Indeed, we can replace u_ε by $u_{1;\varepsilon}$ and $u_{-\varepsilon}$ by $u_{2;-\varepsilon}$ in (3) and obtain in the limit $\sigma\nabla u_1 \cdot \nabla u_2$, which, combined with H_{11} and H_{22}, yields H_{12} defined in (32). The experimental acquisition of H_{11} is in fact sufficient to also acquire $\sigma\nabla u_1 \cdot \nabla u_2$, as demonstrated in [Kuchment and Kunyansky 2011].

Remark 4.7. Theorem 4.5 is a uniqueness and stability result for arbitrary, sufficiently smooth conductivities. However, the boundary conditions f are quite specific, since they need to be sufficiently close to nonexplicit, σ-dependent traces of complex geometrical optics solutions. In some sense, the difficulty inherent to the spatially varying Lorentzian metric $\mathfrak{h}(x)$ in (15) has been shifted to the difficulty of constructing adapted boundary conditions (illuminations).

Note that the condition of flatness of the surfaces Σ_i in the above construction is not essential. Surfaces with a geometry such as that depicted in Figure 1 may also be considered. Such surfaces allow us to reduce the size of the domain X on which the conductivity $\sigma = 1$ needs to be extended. Unless the domain X has a specific geometry similar to that of the domain $\mathbb{O}$ between Σ_1 and Σ_2 in Figure 1, it seems necessary to augment the size of $\tilde{X}$ to that of X as described above to obtain a global uniqueness result.

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GUILLAUME BAL: gb2030@columbia.edu
Deparment of Applied Physics & Applied Mathematics, Columbia University, 500 W. 120th Street, S.W. Mudd Building Room 206, New York, NY 10027, United States

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SHARP WEIGHTED BOUNDS INVOLVING A_∞

TUOMAS HYTÖNEN AND CARLOS PÉREZ

We improve on several weighted inequalities of recent interest by replacing a part of the A_p bounds by weaker A_∞ estimates involving Wilson's A_∞ constant

$$[w]'_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q).$$

In particular, we show the following improvement of the first author's A_2 theorem for Calderón–Zygmund operators T:

$$\|T\|_{\mathscr{B}(L^2(w))} \le c_T\, [w]_{A_2}^{1/2} \big([w]'_{A_\infty} + [w^{-1}]'_{A_\infty}\big)^{1/2}.$$

Corresponding A_p type results are obtained from a new extrapolation theorem with appropriate mixed A_p-A_∞ bounds. This uses new two-weight estimates for the maximal function, which improve on Buckley's classical bound.

We also derive mixed A_1-A_∞ type results of Lerner, Ombrosi and Pérez (2009) of the form

$$\|T\|_{\mathscr{B}(L^p(w))} \le cpp'[w]_{A_1}^{1/p}([w]'_{A_\infty})^{1/p'}, \quad 1 < p < \infty,$$
$$\|Tf\|_{L^{1,\infty}(w)} \le c[w]_{A_1} \log(e + [w]'_{A_\infty}) \|f\|_{L^1(w)}.$$

An estimate dual to the last one is also found, as well as new bounds for commutators of singular integrals.

1. Introduction and statements of the main results

The weights w for which the usual operators T of classical analysis (like the Hardy–Littlewood maximal operator, the Hilbert transform, and general classes of Calderón–Zygmund operators) act boundedly on $L^p(w)$ were identified in works of Muckenhoupt [1972], Hunt, Muckenhoupt and Wheeden [Hunt et al. 1973], and Coifman and Fefferman [1974]. This class consists of the Muckenhoupt A_p weights, defined by the condition that (see [García-Cuerva and Rubio de Francia 1985])

$$[w]_{A_p} := \sup_Q \left(\fint_Q w\right)\left(\fint_Q w^{-1/(p-1)}\right)^{p-1} < \infty, \qquad p \in (1,\infty),$$

where the supremum is over all cubes in $\mathbb{R}^d$. Hence it is shown for any of these important operators T, whether it is linear or not, that

$$\|T\|_{\mathscr{B}(L^p(w))} := \sup_{f\neq 0} \frac{\|Tf\|_{L^p(w)}}{\|f\|_{L^p(w)}}$$

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is finite if and only if $[w]_{A_p} < \infty$.

It is a natural question to look for optimal quantitative bounds of $\|T\|_{\mathscr{B}(L^p(w))}$ in terms of $[w]_{A_p}$. The first author who studied that question was S. Buckley [1993], who proved

$$\|M\|_{\mathscr{B}(L^p(w))} \le c_{p,d}\,[w]_{A_p}^{1/(p-1)}, \quad 1 < p < \infty, \tag{1.1}$$

where M is the usual Hardy–Littlewood maximal function on $\mathbb{R}^d$. However, there has been a great impetus toward finding such precise dependence for more singular operators after the work of Astala, Iwaniec and Saksman [Astala et al. 2001], due to the connections with sharp regularity results for solutions to the Beltrami equation. The key fact was to prove that the operator norm of the Beurling–Ahlfors transform on $L^2(w)$ grows linearly in terms of the A_2 constant of w. This was proved by S. Petermichl and A. Volberg [2002] and by Petermichl [2007; 2008] for the Hilbert transform and the Riesz transforms. To be precise, in these papers it has been shown that if T is any of these operators, then

$$\|T\|_{\mathscr{B}(L^p(w))} \le c_{p,T}\,[w]_{A_p}^{\max\{1,1/(p-1)\}}. \tag{1.2}$$

The exponents are optimal in the sense that the exponent cannot be replaced by any smaller quantity. It was conjectured then that the same estimate holds for any Calderón–Zygmund operator T. This was proven first for special classes of integral transforms in [Cruz-Uribe et al. 2010; Lacey et al. 2010b], and eventually for general Calderón–Zygmund operators by the first author in [Hytönen 2012], using the main result from [Pérez et al. 2010], where it is shown that a weak type estimate is enough to prove the strong type. A direct proof of this result can be found in [Hytönen et al. 2010]. Other related works are [Cruz-Uribe et al. 2012; Hytönen et al. 2011; Lacey et al. 2010a; Lerner 2011; Vagharshakyan 2010].

The main purpose of this paper is to show that these results can be further improved. To do this, we recall the following definitions of the A_∞ constant of a weight w. First, there is the notion introduced by Hruščev [1984] (see also [García-Cuerva and Rubio de Francia 1985]),

$$[w]_{A_\infty} := \sup_Q \Big(\fint_Q w\Big) \exp\Big(\fint_Q \log w^{-1}\Big);$$

and second, there is the (as it turns out) smaller quantity that appeared with a different notation in the work of Wilson [1987; 1989; 2008] and was recently termed the "A_∞ constant" by Lerner [2011, Section 5.5]:

$$[w]'_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q).$$

Observe that

$$c_d[w]'_{A_\infty} \le [w]_{A_\infty} \le [w]_{A_p} \quad \text{for all } p \in [1, \infty),$$

where the second estimate is elementary, and the first will be checked in Proposition 2.2. While the constant $[w]_{A_\infty}$ is more widely used in the literature, and is also more flexible for our purposes, it is of interest to observe situations where the smaller constant $[w]'_{A_\infty}$ is sufficient for our estimates, thereby giving a sharper bound.

Now, if $\sigma = w^{-1/(p-1)}$ is the dual weight of w, we also have $[\sigma]_{A_\infty}^{p-1} \le [\sigma]_{A_{p'}}^{p-1} = [w]_{A_p}$. The point here is that these quantities can be much smaller for some classes of weights. Our results will be of the form

$$\|T\|_{\mathscr{B}(L^p(w))} \le c_{p,T} \sum [w]_{A_p}^{\alpha(p)} [w]_{A_\infty}^{\beta(p)} [\sigma]_{A_\infty}^{(p-1)\gamma(p)},$$

sometimes even with the smaller $[\ ]'_{A_\infty}$ constant instead of $[\ ]_{A_\infty}$, where the sum is over at most two triplets (α, β, γ), and the exponents satisfy $\alpha(p)+\beta(p)+\gamma(p) = \tau(p)$, where $\tau(p)$ is the exponent from the earlier sharp results. However, we will have $\alpha(p) < \tau(p)$, which shows that part of the necessary A_p control may in fact be replaced by weaker A_∞ control.

We now turn to a more detailed discussion of our results.

1A. ***The A_2 theory for Calderón–Zygmund operators.*** Our main result for Calderón–Zygmund operators is the following:

Theorem 1.3. *Let T be a Calderón–Zygmund operator and let $w \in A_2$ and $\sigma = w^{-1}$. Then there is a constant $c = c_{d,T}$ such that*

$$\|T\|_{\mathscr{B}(L^2(w))} \le c[w]_{A_2}^{1/2}\big([w]'_{A_\infty} + [w^{-1}]'_{A_\infty}\big)^{1/2} \le c[w]_{A_2}^{1/2}\big([w]_{A_\infty} + [w^{-1}]_{A_\infty}\big)^{1/2}. \tag{1.4}$$

We will prove this by following the approach from [Hytönen 2012; Hytönen et al. 2010] to the A_2 theorem $\|T\|_{\mathscr{B}(L^2(w))} \le c_T\, [w]_{A_2}$, and modifying the proof at some critical points. Indeed, the original argument uses the A_2 property basically *twice*, each time producing the factor $[w]_{A_2}^{1/2}$, and it suffices to observe that only the A_∞ property is actually needed in one of these estimates.

An interesting consequence of this theorem is the following: for any fixed Calderón–Zygmund operator T, we have

$$\inf_{w \in A_2} \frac{\|T\|_{\mathscr{B}(L^2(w))}}{[w]_{A_2}} = 0. \tag{1.5}$$

This follows once we describe, in Section 8, a family of weights $w \in A_2$ for which both $[w]'_{A_\infty}$ and $[\sigma]'_{A_\infty}$ (and even $[w]_{A_\infty}$ and $[\sigma]_{A_\infty}$) grow slower than $[w]_{A_2}$. In particular, the "reverse A_2 conjecture" $[w]_{A_2} \le c_T \|T\|_{\mathscr{B}(L^2(w))}$ is false.

1B. ***The maximal function.*** We next discuss the sharp weighted bounds for the Hardy–Littlewood maximal function, which we first do in a two-weight setting. We need a new two-weight constant $B_p[w, \sigma]$ defined by the functional

$$B_p[w, \sigma] := \sup_Q \left(\fint_Q w\right)\left(\fint_Q \sigma\right)^p \exp\left(\fint_Q \log \sigma^{-1}\right), \tag{1.6}$$

which clearly satisfies

$$[w]_{A_p} \le B_p[w, \sigma] \le [w]_{A_p}[\sigma]_{A_\infty}.$$

Theorem 1.7. *Let M be the Hardy–Littlewood maximal operator and let $p \in (1, \infty)$. Then we have the estimates*

$$\|M(f\sigma)\|_{L^p(w)} \le C_d \cdot p' \cdot \big(B_p[w, \sigma]\big)^{1/p} \|f\|_{L^p(\sigma)} \tag{1.8}$$

and

$$\|M(f\sigma)\|_{L^p(w)} \le C_d \cdot p' \cdot \big([w]_{A_p}[\sigma]'_{A_\infty}\big)^{1/p} \|f\|_{L^p(\sigma)}. \tag{1.9}$$

We refer to Section 4 for the proof and for more information and background about this two-weight estimate for M. By a well-known change-of-weight argument, (1.9) implies:

Corollary 1.10. *For M and p as above, and $\sigma = w^{-1/(p-1)}$, we have*

$$\|M\|_{\mathcal{B}(L^p(w))} \le C_d \cdot p' \cdot \big([w]_{A_p}[\sigma]'_{A_\infty}\big)^{1/p}. \tag{1.11}$$

This improves on Buckley's theorem $\|M\|_{\mathcal{B}(L^p(w))} \le C_d \cdot p' \cdot [w]_{A_p}^{1/(p-1)}$. Corollary 1.10, at least for $p = 2$, was also independently discovered by A. Lerner and S. Ombrosi [2008].

1C. *The A_1 theory for Calderón–Zygmund operators.* It is an interesting fact that if we assume that the weight satisfies the stronger condition $w \in A_1$, then the estimate (1.2) can be considerably improved. Indeed, if T is any Calderón–Zygmund operator, then T is of course bounded on $L^p(w)$, because $A_1 \subset A_p$, but with a much better bound, namely

$$\|T\|_{\mathcal{B}(L^p(w))} \le cpp'[w]_{A_1}, \quad 1 < p < \infty. \tag{1.12}$$

Observe that the dependence on the A_1 constant is linear for any p, while in the A_p case it is highly nonlinear for $1 < p < 2$; see (1.2). The result is sharp both in terms of the dependence on $[w]_{A_1}$, and in terms of the dependence on p when taking $w = 1$ by the classical theory. This fact was used to get the following endpoint result:

$$\|Tf\|_{L^{1,\infty}(w)} \le c[w]_{A_1} \log(e + [w]_{A_1})\|f\|_{L^1(w)}. \tag{1.13}$$

See [Lerner et al. 2009a] and also [Lerner et al. 2008] for these results and for more information about the problem. It was conjectured in the first of these works that the growth of this bound would be linear; however, it was shown in [Nazarov et al. 2010] that the growth of the bound is worse than linear. It seems that most probably the $L \log L$ result (1.13) is the best possible.

On the other hand, in [Lerner et al. 2009b], a sort of "dual" estimate to the last bound was found, which is also of interest for related matters:

$$\left\|\frac{Tf}{w}\right\|_{L^{1,\infty}(w)} \le c[w]_{A_1} \log(e + [w]_{A_1}) \int_{\mathbb{R}^d} |f| \, dx.$$

In this paper, we improve these results following our new quantitative estimates, this time involving A_1 and A_∞ control. To be precise, we will prove the following new results:

Theorem 1.14. *Let T be a Calderón–Zygmund operator and let $1 < p < \infty$. Then*

$$\|T\|_{\mathcal{B}(L^p(w))} \le cpp'[w]_{A_1}^{1/p}([w]'_{A_\infty})^{1/p'},$$

where $c = c(d, T)$.

We will prove this by following the approach from [Lerner et al. 2008; 2009a] to (1.12), modifying the proof at several points. In analogy to (1.5), Theorem 1.14 disproves the "reverse A_1 conjecture"

$[w]_{A_1} \le c_T \|T\|_{\mathcal{B}(L^p(w))}$ for all $p \in (1, \infty)$: considering a family of weights $w \in A_1$ for which $[w]_{A_\infty}$ grows slower than $[w]_{A_1}$, for any fixed Calderón–Zygmund operator T, we have (see Section 8 for details)

$$\inf_{w \in A_1} \frac{\|T\|_{\mathcal{B}(L^p(w))}}{[w]_{A_1}} = 0, \qquad 1 < p < \infty.$$

Finally, we will also use the approach from [Lerner et al. 2009a; 2009b] to prove the following theorems, respectively.

Theorem 1.15. *Let T be a Calderón–Zygmund operator. Then*

$$\|Tf\|_{L^{1,\infty}(w)} \le c_{d,T}[w]_{A_1} \log(e + [w]'_{A_\infty})\|f\|_{L^1(w)}.$$

Theorem 1.16. *Let T be a Calderón–Zygmund operator. Then*

$$\left\|\frac{Tf}{w}\right\|_{L^{1,\infty}(w)} \le c_{d,T}[w]'_{A_\infty} \log(e + [w]_{A_1})\|f\|_{L^1(\mathbb{R}^d)}.$$

1D. *Commutators with BMO functions.* We further pursue the A_∞ point of view by proving a result in the spirit of Theorem 1.3 for commutators of linear operators T with BMO functions. These operators are defined formally by the expression

$$[b, T]f = bT(f) - T(bf).$$

More generally, we can consider the k-th order commutator defined by

$$T_b^k := [b, T_b^{k-1}].$$

When T is a singular integral operator, these operators were considered by Coifman, Rochberg and Weiss [Coifman et al. 1976], and since then many results have been obtained. We refer to [Chung et al. 2012] for more information about these operators; it is shown there that if T is a linear operator bounded on $L^2(w)$ for any $w \in A_2$ with bound

$$\|T\|_{\mathcal{B}(L^2(w))} \le \varphi([w]_{A_2}),$$

where φ is an increasing function $\varphi : [1, \infty) \to [0, \infty)$, then there is a dimensional constant c such that

$$\|[b, T]\|_{\mathcal{B}(L^2(w))} \le c\varphi(c[w]_{A_2})[w]_{A_2}\|b\|_{\mathrm{BMO}}.$$

In particular, if T is any Calderón–Zygmund operator, we can use the linear A_2 theorem for T to deduce

$$\|[b, T]\|_{\mathcal{B}(L^2(w))} \le c[w]_{A_2}^2\|b\|_{\mathrm{BMO}},$$

and the quadratic exponent cannot be improved.

An analogous result adapted to the A_∞ control reads as follows:

Theorem 1.17. *Let T be a linear operator bounded on $L^2(w)$ for any $w \in A_2$ and let $b \in$* BMO. *Suppose further that there is a function $\varphi : [1, \infty)^3 \to [0, \infty)$, increasing with respect to each component, such that*

$$\|T\|_{\mathcal{B}(L^2(w))} \le \varphi\big([w]_{A_2}, [w]'_{A_\infty}, [\sigma]'_{A_\infty}\big).$$

Then there is a dimensional constant c such that

$$\|[b,T]\|_{\mathscr{B}(L^2(w))} \le c\varphi\big(c[w]_{A_2}, c\,[w]'_{A_\infty}, c[\sigma]'_{A_\infty}\big)\big([w]'_{A_\infty}+[\sigma]'_{A_\infty}\big)\|b\|_{\mathrm{BMO}},$$

or more generally,

$$\|T_b^k\|_{\mathscr{B}(L^2(w))} \le c\varphi\big(c[w]_{A_2}, c[w]'_{A_\infty}, c[\sigma]'_{A_\infty}\big)\big([w]'_{A_\infty}+[\sigma]'_{A_\infty}\big)^k\|b\|^k_{\mathrm{BMO}}.$$

We can now apply Theorem 1.3.

Corollary 1.18. *Let T be any Calderón–Zygmund operator, and let $b \in$ BMO. Then*

$$\|[b,T]\|_{\mathscr{B}(L^2(w))} \le c[w]_{A_2}^{1/2}\big([w]'_{A_\infty}+[w^{-1}]'_{A_\infty}\big)^{3/2}\,\|b\|_{\mathrm{BMO}},$$

or more generally,

$$\|T_b^k\|_{\mathscr{B}(L^2(w))} \le c[w]_{A_2}^{1/2}\big([w]'_{A_\infty}+[\sigma]'_{A_\infty}\big)^{k+1/2}\|b\|^k_{\mathrm{BMO}}.$$

1E. ***An end-point estimate when $p=\infty$.*** We next discuss the limiting form of the estimate (1.2) as $p\to\infty$, that is, the sharp bounds for the norm of Calderón–Zygmund operators

$$T : L^\infty(w)\to \mathrm{BMO}(w), \qquad w\in A_\infty.$$

Qualitatively, this situation seems slightly uninteresting, as these end-point spaces simply reduce to their unweighted analogues: that $L^\infty(w)=L^\infty$ with equal norms is immediate from the fact that w and the Lebesgue measure share the same zero sets for $w\in A_\infty$. That the weighted norm

$$\|f\|_{\mathrm{BMO}(w)} := \sup_Q \inf_c \frac{1}{w(Q)}\int_Q |f-c|w<\infty$$

is equivalent to the usual $\|f\|_{\mathrm{BMO}}$ for $w\in A_\infty$ was proven by Muckenhoupt and Wheeden [1975, Theorem 5]. However, one may still investigate the quantitative bound of operators $T : L^\infty \to \mathrm{BMO} = \mathrm{BMO}(w)$, when the latter space is equipped with the norm $\|\ \|_{\mathrm{BMO}(w)}$. We start with:

Theorem 1.19. *For $w\in A_\infty$, we have a bounded embedding $I : \mathrm{BMO}\hookrightarrow \mathrm{BMO}(w)$ of norm at most $c[w]'_{A_\infty}$, where c is dimensional. This estimate is sharp in the following sense: if the norm of the embedding is bounded by $\phi([w]'_{A_\infty})$, or just by $\phi([w]_{A_\infty})$, for all $w\in A_\infty$, then $\phi(t)\ge ct$.*

The following corollary for Calderón–Zygmund operators can be seen as an easy endpoint estimate of the bound $\|T\|_{\mathscr{B}(L^p(w))}\le c_{p,T}\,[w]_{A_p}$ for $p\in[2,\infty)$.

Corollary 1.20. *Let T be any Calderón–Zygmund operator and let $w\in A_\infty$; then $T : L^\infty\to \mathrm{BMO}(w)$ with norm at most $c_T\,[w]'_{A_\infty}$. Furthermore, this estimate is sharp in terms of the dependence on $[w]'_{A_\infty}$ in the same way as Theorem 1.19.*

A related observation quantifying the known relation of A_∞ and BMO is as follows:

Proposition 1.21. *If $w\in A_\infty$, then $\log w\in$ BMO with*

$$\|\log w\|_{\mathrm{BMO}} \le \log(2e[w]_{A_\infty}).$$

1F. ***Extrapolation with A_∞ control.*** We recall the following quantitative version of Rubio de Francia's classical extrapolation theorem due to Dragičević, Grafakos, Pereyra, and Petermichl [Dragičević et al. 2005]: if an operator T satisfies

$$\|T\|_{\mathscr{B}(L^r(w))} \le \varphi([w]_{A_r})$$

for a fixed increasing function φ and for all $w \in A_r$, then it satisfies a similar estimate for all $p \in (1, \infty)$,

$$\|T\|_{\mathscr{B}(L^p(w))} \le 2\varphi\big(c_{p,r,d}[w]_{A_p}^{\max\{1,(r-1)/(p-1)\}}\big);$$

in particular, $\|T\|_{\mathscr{B}(L^r(w))} \lesssim [w]_{A_r}^{\tau(r)}$ implies that

$$\|T\|_{\mathscr{B}(L^p(w))} \lesssim [w]_{A_p}^{\tau(r)\max\{1,(r-1)/(p-1)\}}.$$

With our new quantitative estimates involving both A_2 and A_∞ control, it seems of interest to extrapolate such bounds as well. Hence we consider weighted estimates of the form

$$\|Tf\|_{L^r(w)} \le \varphi\big([w]_{A_r}, [w]_{A_\infty}, [w^{-1/(r-1)}]_{A_\infty}^{(r-1)}\big)\|f\|_{L^r(w)}, \tag{1.22}$$

where $\varphi : [1, \infty)^3 \to [0, \infty)$ is an increasing function with respect to each of the variables. An example is our bound for singular integrals 1.3, where

$$\varphi(x, y, z) = Cx^{1/2}(y + z)^{1/2}. \tag{1.23}$$

We now aim to extrapolate bounds like (1.22) from the given $r \in (1, \infty)$ to other exponents $p \in (1, \infty)$.

Theorem 1.24 (lower extrapolation). *Suppose that for some r and every $w \in A_r$, an operator T satisfies (1.22). Then for every $p \in (1, r)$, it satisfies*

$$\begin{aligned}\|Tf\|_{L^p(w)} &\le 2\varphi\Big(\big(2\|M\|_{\mathscr{B}(L^p(w))}\big)^{r-p}\big([w]_{A_r}, [w]_{A_\infty}, [w^{-1/(p-1)}]_{A_\infty}^{(p-1)}\big)\Big)\|f\|_{L^p(w)}\\ &\le 2\varphi\Big(\big(c_d([w]_{A_p}[w^{-1/(p-1)}]'_{A_\infty})^{1/p}\big)^{r-p}\big([w]_{A_p}, [w]_{A_\infty}, [w^{-1/(p-1)}]_{A_\infty}^{(p-1)}\big)\Big)\|f\|_{L^p(w)}.\end{aligned}$$

In typical applications, like (1.23), the function φ will have a homogeneity of the form $\varphi(\lambda x, \lambda y, \lambda z) = \lambda^s\varphi(x, y, z)$, and hence the common factor

$$\big(2\|M\|_{\mathscr{B}(L^p(w))}\big)^{r-p} \le \big(c_d([w]_{A_p}[w^{-1/(p-1)}]'_{A_\infty})^{1/p}\big)^{r-p}$$

may be extracted out of φ.

Observe that the condition (1.22) is of course implied by the stronger inequality

$$\|Tf\|_{L^r(w)} \le \varphi\big([w]_{A_r}, c_d^{-1}[w]'_{A_\infty}, (c_d^{-1}[w^{-1/(r-1)}]'_{A_\infty})^{(r-1)}\big)\|f\|_{L^r(w)};$$

however, even if we have this stronger inequality to start with (as is the case with the A_2 theorem for Calderón–Zygmund operators), we do not know how to exploit it to get a stronger conclusion than what we can derive from (1.22). A related difficulty will be pointed out in the proof. This is why we restrict to the assumption (1.22) only.

Theorem 1.25 (upper extrapolation). *Suppose that for some r and every $w \in A_r$, an operator T satisfies (1.22). Then for every $p \in (r, \infty)$, it satisfies*

$$\begin{aligned}\|Tf\|_{L^p(w)} &\le 2\varphi\Big(\big(2\|M\|_{\mathscr{B}(L^{p'}(w^{1-p'}))}\big)^{(p-r)/(p-1)} \\ &\qquad\qquad \times \big([w]_{A_p}^{(r-1)/(p-1)}, [w]_{A_\infty}^{(r-1)/(p-1)}, \big[w^{-1/(p-1)}\big]_{A_\infty}^{(r-1)}\big)\Big)\|f\|_{L^p(w)} \\ &\le 2\varphi\Big(\big(c_d[w]_{A_p}^{1/p}([w]'_{A_\infty})^{1/p'}\big)^{(p-r)/(p-1)} \\ &\qquad\qquad \times \big([w]_{A_p}^{(r-1)/(p-1)}, [w]_{A_\infty}^{(r-1)/(p-1)}, \big[w^{-1/(p-1)}\big]_{A_\infty}^{(r-1)}\big)\Big)\|f\|_{L^p(w)}.\end{aligned}$$

1G. ***The A_p theory for Calderón–Zygmund operators.*** As an application of the extrapolation theorems, we can deduce weighted L^p estimates for Calderón–Zygmund operators with mixed A_p and A_∞ control, akin to the A_2 bounds of Theorem 1.3. The same strategy has been earlier employed to prove the original A_p theorem (1.2) as a corollary of its A_2 version. However, in contrast to the "pure" A_p estimates, where the extrapolated result still exhibits the sharp dependence on the weight, it seems that the extrapolation of the mixed bounds is not equally efficient: the extrapolated bounds given below can be improved by methods directly adapted to L^p. Since the first public distribution of our present results, such further developments have been carried out in [Hytönen et al. 2011, Section 12; Lacey 2012; Hytönen and Lacey 2011]. Nevertheless, it seems worth recording the form of the A_p estimates directly delivered by the extrapolation method:

Corollary 1.26. *Let T be a Calderón–Zygmund operator and let $p \in (1, \infty)$. Then if $w \in A_p$ and $\sigma = w^{-1/(p-1)}$, we have*

$$\begin{aligned}\|T\|_{\mathscr{B}(L^p(w))} &\lesssim [w]_{A_p}^{2/p-1/2}\big([w]_{A_\infty}^{1/2} + [\sigma]_{A_\infty}^{(p-1)/2}\big)([\sigma]'_{A_\infty})^{2/p-1} \\ &\lesssim [w]_{A_p}^{2/p}([\sigma]'_{A_\infty})^{2/p-1} \quad \text{for } p \in (1, 2],\end{aligned} \tag{1.27}$$

and

$$\begin{aligned}\|T\|_{\mathscr{B}(L^p(w))} &\lesssim [w]_{A_p}^{2/p-1/[2(p-1)]}\big([w]_{A_\infty}^{1/[2(p-1)]} + [\sigma]_{A_\infty}^{1/2}\big)([w]'_{A_\infty})^{1-2/p} \\ &\lesssim [w]_{A_p}^{2/p}([w]'_{A_\infty})^{1-2/p} \quad \text{for } p \in [2, \infty).\end{aligned} \tag{1.28}$$

Here the simpler forms of the estimates in (1.27) and (1.28) are almost as good as the more complicated ones, since for many common weights, like power weights, we have $[w]_{A_\infty} + [\sigma]_{A_\infty}^{p-1} \eqsim [w]_{A_p}$; see Section 8.

It is immediate to check that Theorems 1.24 and 1.25, in combination with Theorem 1.3, give Corollary 1.26. Actually, the two statements (1.27) and (1.28) are equivalent to each other by using

$$\|T\|_{\mathscr{B}(L^p(w))} = \|T^*\|_{\mathscr{B}(L^{p'}(\sigma))}$$

and the fact that T^* is also a Calderón–Zygmund operator. Thanks to this equivalence, we would only need one of Theorems 1.24 and 1.25 to deduce this corollary. But for other classes of operators without a self-dual structure, it is useful to have both upper and lower extrapolation results available.

2. The two different A_∞ constants

Before pursuing further our analysis of inequalities with A_∞ control, we include this short section to compare the two A_∞ constants

$$[w]_{A_\infty} := \sup_Q \Big(\fint_Q w \Big) \exp\Big(\fint_Q \log w^{-1} \Big), \quad [w]'_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q).$$

We need the following auxiliary estimate, which is also used later in the paper:

Lemma 2.1. *The logarithmic maximal function*

$$M_0 f := \sup_Q \exp\Big(\fint_Q \log|f| \Big) \chi_Q$$

satisfies

$$\|M_0 f\|_{L^p} \le c_d^{1/p} \|f\|_{L^p}$$

for all $p \in (0, \infty)$. For the dyadic version, we can take $c_d = e$, independent of dimension d.

Proof. By Jensen's inequality and the basic properties of the logarithm, we have

$$M_0 f \le Mf, \qquad M_0 f = (M_0|f|^{1/q})^q \le (M|f|^{1/q})^q, \quad q \in (0, \infty),$$

where M is the Hardy–Littlewood maximal operator, or the dyadic maximal operator in the case of dyadic M_0. By the L^q boundedness of the usual maximal function for $q > 1$, we have

$$\int [M_0 f]^p \le \int \big[M|f|^{p/q}\big]^q \le (C_d \cdot q')^q \int (|f|^{p/q})^q = (C_d \cdot q')^q \int |f|^p.$$

In the nondyadic case, we simply take, say, $q = 2$, giving the claim with $c_d = (2C_d)^2$. In the dyadic case, we have $C_d = 1$, and we can take the limit $q \to \infty$, which gives

$$(q')^q = \Big(\frac{q}{q-1} \Big)^q = \Big(1 + \frac{1}{q-1} \Big)^q \to e,$$

and hence $\|M_0 f\|_{L^p}^p \le e \|f\|_{L^p}^p$. □

Proposition 2.2. *We have $[w]'_{A_\infty} \le c_d [w]_{A_\infty}$, where c_d is as in Lemma 2.1.*

Proof. For $x \in Q$, it is not difficult to see that for the computation of $M(w\chi_Q)(x)$, it suffices to take the supremum over cubes $R \ni x$ with $R \subseteq Q$:

$$M(w\chi_Q)(x) = \sup_{\substack{R \ni x \\ R \subseteq Q}} \fint_R w \quad \text{for all } x \in Q.$$

By the definition of $[w]_{A_\infty}$, we have

$$\fint_R w \le [w]_{A_\infty} \exp\Big(\fint_R \log w \Big),$$

and hence, taking the supremum over R,

$$M(w\chi_Q)(x) \le [w]_{A_\infty} M_0(w\chi_Q)(x) \quad \text{for all } x \in Q.$$

Integration over Q and application of Lemma 2.1 now give

$$\int_Q M(w\chi_Q) \le [w]_{A_\infty} \int M_0(w\chi_Q) \le [w]_{A_\infty} c_d \int w\chi_Q = c_d [w]_{A_\infty} w(Q);$$

thus $[w]'_{A_\infty} \le c_d [w]_{A_\infty}$. □

It is a well-known fact that any A_∞ weight satisfies a reverse Hölder inequality playing a central role in the area. In this paper, a sharp version of this property will also play a fundamental role. To be precise, if $w \in A_\infty$, we define

$$r(w) := 1 + \frac{1}{\tau_d\, [w]'_{A_\infty}},$$

where τ_d is a dimensional constant that we may take to be $\tau_d = 2^{11+d}$. Note that $r(w)' \approx [w]'_{A_\infty}$. The result we need is the following.

Theorem 2.3 (a new sharp reverse Hölder inequality). (a) *If $w \in A_\infty$, then*

$$\left(\fint_Q w^{r(w)} \right)^{1/r(w)} \le 2 \fint_Q w.$$

(b) *Furthermore, the result is optimal up to a dimensional factor: If a weight w satisfies the reverse Hölder inequality*

$$\left(\fint_Q w^r \right)^{1/r} \le K \fint_Q w,$$

then $[w]'_{A_\infty} \le c_d \cdot K \cdot r'$.

This result is new in the literature and has its own interest. In the classical situation, most of the available proofs do not give such explicit constants, which are important for us. Only under the stronger condition of A_1 was such a result found and used in a crucial way in [Lerner et al. 2009a]. Recently a very nice proof by A. de la Torre for the case $[w]_{A_\infty}$ was sent to us (personal communication, 2010). Another less precise proof, for the A_p case, $1 < p < \infty$, can be found in [Pérez 2013].

Part (b) follows from the boundedness of the maximal function in L^r with constant $c_d r'$:

$$\fint_Q M(\chi_Q w) \le \left(\fint_Q M(\chi_Q w)^r \right)^{1/r} \le c_d \cdot r' \left(\fint_Q w^r \right)^{1/r} \le c_d \cdot r' \cdot K \fint_Q w.$$

Remark 2.4. Results analogous to Proposition 2.2 and Theorem 2.3 have been independently obtained by O. Beznosova and A. Reznikov [2011]. Their formulation is slightly different, and involves yet another weight constant closely related to $[w]'_{A_\infty}$.

3. The A_2 theorem for Calderón–Zygmund operators

The purpose of this section is to prove Theorem 1.3, namely, the estimate

$$\|T\|_{\mathscr{B}(L^2(w))} \le c[w]_{A_2}^{1/2} \big([w]'_{A_\infty} + [\sigma]'_{A_\infty} \big)^{1/2},$$

where $c = c_{d,T}$ is a constant depending on the dimension and the operator T.

Here and throughout this section, $\sigma = w^{-1}$. This improves on the A_2 theorem [Hytönen 2012]:

$$\|T\|_{\mathscr{B}(L^2(w))} \le c\,[w]_{A_2},$$

and its proof follows the same outline, with the implementation of the A_∞ philosophy at key points.

3A. ***Reduction to a dyadic version.*** Fundamental to this proof strategy is the notion of *dyadic shifts*, which we recall. We work with a *general dyadic system* $\mathscr{D}$, this being a collection of axis-parallel cubes Q, whose sidelengths $\ell(Q)$ are of the form 2^k, $k \in \mathbb{Z}$, where moreover $Q \cap R \in \{Q, R, \varnothing\}$ for any two $Q, R \in \mathscr{D}$, and the cubes of a fixed sidelength 2^k form a partition of $\mathbb{R}^d$. Given such a dyadic system, a dyadic shift with parameters (m, n) is an operator of the form

$$Ш f = \sum_{K \in \mathscr{D}} A_K f, \qquad A_K f = \frac{1}{|K|} \sum_{\substack{I, J \in \mathscr{D};\, I, J \subseteq K \\ \ell(I) = 2^{-m}\ell(K) \\ \ell(J) = 2^{-n}\ell(K)}} \langle h_I^J, f\rangle k_J^I,$$

where h_I^J is a generalized Haar function on I (supported on I, constant on its dyadic subcubes, and normalized by $\|h_I^J\|_\infty \le 1$), and k_J^I on J. This implies that $|A_K f| \le \chi_K \cdot |K|^{-1} \cdot \int_K |f|$. For any subcollection $\mathscr{Q} \subset \mathscr{D}$, we write

$$Ш_{\mathscr{Q}} f := \sum_{K \in \mathscr{Q}} A_K f, \tag{3.1}$$

and we require that $\|Ш_{\mathscr{Q}} f\|_{L^2} \le \|f\|_{L^2}$ for all $\mathscr{Q} \subset \mathscr{D}$. This is automatic from straightforward orthogonality considerations in case we only have cancellative Haar functions with $\int h_I^J = \int k_J^I = 0$.

Dyadic shifts with parameters (0, 0) are well known in dyadic harmonic analysis under different names. Auscher et al. [2002] study such operators under the name *perfect dyadic* operators, which they decompose into a sum of a *Haar multiplier* (or *martingale transform*), a *paraproduct*, and a *dual paraproduct*. These three types of operators have of course been well known for a long time. The first dyadic shift (and this name) with parameters (0, 1) was introduced in [Petermichl 2000], and the definition in the above generality was given by Lacey, Petermichl and Reguera [Lacey et al. 2010b].

The importance of these dyadic shifts for the analysis of Calderón–Zygmund operators comes from the following:

Theorem 3.2 (dyadic representation theorem [Hytönen 2012, Theorem 4.2; Hytönen et al. 2010, Theorem 4.1]). *Let $T \in \mathscr{B}(L^2(\mathbb{R}^d))$ be a Calderón–Zygmund operator satisfying the standard estimates with the Hölder continuity exponent $\alpha \in (0, 1]$. Then T has the representation*

$$\langle g, Tf\rangle = c_{T,d}\,\mathbb{E}_{\mathscr{D}} \sum_{m,n=0}^{\infty} 2^{-(m+n)\alpha/2} \langle g, Ш_{\mathscr{D}}^{mn} f\rangle,$$

valid for all bounded and compactly supported functions f and g, where $Ш_{\mathscr{D}}^{mn}$ is a dyadic shift with parameters (m, n) related to the dyadic system $\mathscr{D}$, and $\mathbb{E}_{\mathscr{D}}$ is the expectation with respect to a probability measure on the space of all generalized dyadic systems; see [Hytönen 2012] for the details of the construction of this probability space.

This result was preceded by several versions restricted to special operators T: the Beurling–Ahlfors transform by Dragičević and Volberg [2003], the Hilbert transform by Petermichl [2000], the Riesz transforms by Petermichl, Treil and Volberg [Petermichl et al. 2002], and all one-dimensional convolution operators with an odd, smooth kernel by Vagharshakyan [2010]. An immediate consequence of the dyadic representation theorem is that Theorem 1.3 will be a consequence of the following dyadic version. (Similarly, the special cases of the representation theorem all played a role in proving the A_2 theorem for the mentioned particular operators.)

Theorem 3.3. *Let* Ш *be a dyadic shift with parameters* (m,n), *and* $r=\max\{m,n\}$. *For* $w\in A_2$ *and* $\sigma=w^{-1}$, *we have*

$$\|Ш f\|_{L^2(w)}\le C(r+1)^2[w]_{A_2}^{1/2}\big([w]'_{A_\infty}+[\sigma]'_{A_\infty}\big)^{1/2}\|f\|_{L^2(w)}.$$

The weighted norm of the shifts, in turn, is most conveniently deduced with the help of the following characterization of their boundedness in a two-weight situation:

Theorem 3.4 [Hytönen et al. 2010, Theorem 3.4]. *Let* Ш *be a dyadic shift with parameters* (m,n), *and let* $r=\max\{m,n\}$. *If for all* $Q\in\mathscr{D}$ *and some* B *there holds*

$$\bigg(\int_Q|Ш(\chi_Q\sigma)|^2w\bigg)^{1/2}\le B\sigma(Q)^{1/2},\qquad\bigg(\int_Q|Ш^*(\chi_Q w)|^2\sigma\bigg)^{1/2}\le Bw(Q)^{1/2},$$

then for a dimensional constant c, *we have*

$$\|Ш(f\sigma)\|_{L^2(w)}\le c\big((r+1)B+(r+1)^2(A_2[w,\sigma])^{1/2}\big)\|f\|_{L^2(\sigma)},$$

where $A_2[w,\sigma]$ *is defined by the functional*

$$A_2[w,\sigma]:=\sup_Q\bigg(\fint_Q w\bigg)\bigg(\fint_Q\sigma\bigg).$$

Since the last bound is equivalent to

$$\|Ш f\|_{L^2(w)}\le c\big((r+1)B+(r+1)^2[w]_{A_2}^{1/2}\big)\|f\|_{L^2(w)}$$

if $\sigma=w^{-1}$, and since $[w]_{A_\infty},[\sigma]_{A_\infty}\ge 1$, we are reduced to estimating the quantity B for $\sigma=w^{-1}$. Since Ш and Ш* are operators of the same form, and by the symmetry of w and σ, Theorem 3.4 shows that proving Theorem 3.3 amounts to showing that

$$\bigg(\int_Q|Ш(w\chi_Q)|^2\sigma\bigg)^{1/2}\le c\,(r+1)\big([w]_{A_2}[w]'_{A_\infty}w(Q)\big)^{1/2}.$$

We observe that

$$Ш(w\chi_Q)=\sum_{K\subseteq Q}A_K(w\chi_Q)+\sum_{K\supset Q}A_K(w\chi_Q),$$

and it suffices to consider the two parts separately. The big cubes are immediately handled by the maximal function estimate (see Corollary 1.10):

$$\int_Q \Big|\sum_{K\supset Q} A_K(w\chi_Q)\Big|^2 \sigma \le \int_Q \Big(\sum_{K\supset Q} \frac{w(Q)}{|K|}\chi_K\Big)^2 \sigma \lesssim \int_Q M_d(w\chi_Q)^2\sigma$$
$$\le [\sigma]_{A_2}[w]'_{A_\infty} w(Q) = [w]_{A_2}[w]'_{A_\infty} w(Q). \tag{3.5}$$

Hence, to prove Theorem 3.3, we are reduced to showing that

$$\Big(\int_Q \Big|\sum_{K\subseteq Q} A_K(w\chi_Q)\Big|^2 \sigma\Big)^{1/2} \le c(r+1)\big([w]_{A_2}[w]'_{A_\infty} w(Q)\big)^{1/2}. \tag{3.6}$$

This is the goal for the rest of this section.

3B. ***Proof of the key estimate* (3.6).** We follow the key steps from [Hytönen 2012; Hytönen et al. 2010; Lacey et al. 2010b]. The collection $\{K \in \mathscr{D} : K \subseteq Q\}$ is first split into $(r+1)$ subcollections according to the value of $\log_2 \ell(K) \mod (r+1)$; we henceforth work with one of these subcollections, which we denote by $\mathscr{K}$. This is the step which introduces the factor $(r+1)$, and we will estimate $Ш_{\mathscr{K}}(w\chi_Q)$ with a bound independent of r.

The collection $\mathscr{K}$ is further divided into the sets $\mathscr{K}^a$ of those cubes with

$$2^a < \frac{w(Q)}{|Q|}\frac{\sigma(Q)}{|Q|} \le 2^{a+1}, \tag{3.7}$$

where $a \le \log_2[w]_{A_2}$.

Among the cubes $K \in \mathscr{K}^a$, we choose the principal cubes $\mathscr{S}^a = \bigcup_{k=0}^\infty \mathscr{S}_k^a$ so that $\mathscr{S}_0^a$ consists of the maximal cubes in $\mathscr{K}^a$, and $\mathscr{S}_k^a$ the maximal cubes $S \in \mathscr{K}^a$ contained in some $S' \in \mathscr{S}_{k-1}^a$ with $\sigma(S)/|S| > 2\sigma(S')/|S'|$. Then

$$\mathscr{K}^a = \bigcup_{S\in\mathscr{S}^a} \mathscr{K}^a(S), \quad \mathscr{K}^a(S) := \big\{K \in \mathscr{K}^a \mid K \subseteq S,\ \text{there exists no } S' : K \subseteq S' \subset S\big\}.$$

It follows that, in the notation from (3.1),

$$Ш_{\mathscr{K}}(w\chi_Q) = \sum_{a\le\log_2[w]_{A_2}} \sum_{S\in\mathscr{S}^a} Ш_{\mathscr{K}^a(S)}(w\chi_Q). \tag{3.8}$$

To proceed, we recall the following distributional estimate:

Lemma 3.9 [Hytönen et al. 2010, (5.26)]. *With notation as above, we have*

$$\sigma\big(|Ш_{\mathscr{K}^a(S)}(w\chi_Q)| > t\langle w\rangle_S\big) \le Ce^{-ct}\sigma(S) \quad \textit{for all } S \in \mathscr{S}^a, \tag{3.10}$$

where the constants C and c are at worst dimensional.

This is a powerful estimate which readily leads to norm bounds for (3.8). The following computation, simplifying the corresponding ones from [Hytönen 2012; Hytönen et al. 2010; Lacey et al. 2010b], is borrowed from [Hytönen et al. 2011]: writing

$$E_j(S) := \big\{j \le \big|Ш_{\mathscr{K}^a(S)}(w\chi_Q)\big|/\langle w\rangle_S < j+1\big\} \subseteq S,$$

we have

$$\begin{aligned}
\Big\| \sum_{S\in\mathcal{S}^a} \text{Ш}_{\mathcal{K}^a(S)}(w\chi_Q) \Big\|_{L^2(\sigma)} &\leq \sum_{j=0}^{\infty} (j+1) \Big\| \sum_{S\in\mathcal{S}^a} \langle w\rangle_S \cdot \chi_{E_j(S)} \Big\|_{L^2(\sigma)} \\
&= \sum_{j=0}^{\infty} (j+1) \Big(\int \Big[\sum_{S\in\mathcal{S}^a} \langle w\rangle_S \cdot \chi_{E_j(S)}(x) \Big]^2 \sigma(x)\,\mathrm{d}x \Big)^{1/2} \\
&\overset{(*)}{\leq} C \sum_{j=0}^{\infty} (j+1) \Big(\int \sum_{S\in\mathcal{S}^a} \langle w\rangle_S^2 \cdot \chi_{E_j(S)}(x)\sigma(x)\,\mathrm{d}x \Big)^{1/2} \\
&= C \sum_{j=0}^{\infty} (j+1) \Big(\sum_{S\in\mathcal{S}^a} \langle w\rangle_S^2 \cdot \sigma(E_j(S)) \Big)^{1/2} \\
&\leq C \sum_{j=0}^{\infty} (j+1) \Big(\sum_{S\in\mathcal{S}^a} \langle w\rangle_S^2 \cdot C e^{-cj} \sigma(S) \Big)^{1/2} \quad \text{(by (3.10))} \\
&\leq C \sum_{j=0}^{\infty} e^{-cj} (j+1) \Big(2^a \sum_{S\in\mathcal{S}^a} w(S) \Big)^{1/2} \quad \text{(by (3.7) for } S\in\mathcal{S}^a\subset\mathcal{K}^a) \\
&\leq C \cdot 2^{a/2} \Big(\sum_{S\in\mathcal{S}^a} w(S) \Big)^{1/2}.
\end{aligned}$$

In $(*)$ we used the fact that at a fixed x, the numbers $\langle w\rangle_S$ for the principal cubes $S\supset E_j(S)\ni x$ increase at least geometrically, so their ℓ^1 and ℓ^2 norms are comparable.

We now come to the crucial point, where we can improve the earlier A_2 bounds to A_∞:

Lemma 3.11. *For the principal cubes as defined above, we have*

$$\sum_{S\in\mathcal{S}^a} w(S) \leq 2\cdot [w]'_{A_\infty} \cdot w(Q).$$

Proof. Let

$$E(S) := S \setminus \bigcup_{S'\subsetneq S} S'. \tag{3.12}$$

The union is the union of its maximal members S', which satisfy

$$|S'| = |S'|/w(S')\cdot w(S') \leq \tfrac{1}{2}|S|/w(S)\cdot w(S');$$

hence $\sum |S'| \leq \frac{1}{2}|S|$, and thus

$$|E(S)| \geq \tfrac{1}{2}|S|. \tag{3.13}$$

Therefore

$$\sum_{S\in\mathcal{S}^a} w(S) = \sum_{S\in\mathcal{S}^a} \frac{w(S)}{|S|}|S| \leq \sum_{S\in\mathcal{S}^a} \frac{w(S)}{|S|}\, 2|E(S)| \leq 2 \sum_{S\in\mathcal{S}^a} \int_{E(S)} M(w\chi_Q) = 2\int_Q M(w\chi_Q) \leq 2\,[w]'_{A_\infty} w(Q),$$

where the last step was the definition of $[w]'_{A_\infty}$. □

Substituting the estimates obtained back into (3.8), we conclude that

$$\|\text{Ш}_{\mathscr{K}}(w\chi_Q)\|_{L^2(\sigma)} \le \sum_{a\le \log_2[w]_{A_2}} \Big\| \sum_{S\in\mathscr{S}^a} \text{Ш}_{\mathscr{K}^a(S)}(w\chi_Q)\Big\|_{L^2(\sigma)} \le C \sum_{a\le \log_2[w]_{A_2}} 2^{a/2}\Big(\sum_{S\in\mathscr{S}^a} w(S)\Big)^{1/2}$$
$$\le C \sum_{a\le \log_2[w]_{A_2}} 2^{a/2}\big([w]'_{A_\infty}\cdot w(Q)\big)^{1/2} \le C[w]_{A_2}^{1/2}([w]'_{A_\infty})^{1/2} w(Q)^{1/2}.$$

Recalling the initial splitting of $\{K\in\mathscr{D} : K\subseteq Q\}$ into $r+1$ subcollections of the same form as $\mathscr{K}$, this concludes the proof of (3.6), and hence the proof of Theorem 3.3.

4. Two-weight theory for the maximal function

4A. ***Background.*** The two-weight problem was studied in the 1970s by Muckenhoupt and Wheeden and fully solved by E. Sawyer [1982]. The general question is to find a necessary and sufficient condition for a pair of unrelated weights w and σ for which the estimate

$$\|M(f\sigma)\|_{L^p(w)} \le B\|f\|_{L^p(\sigma)} \tag{4.1}$$

holds for a finite constant B. Then the main result of E. Sawyer shows that this is the case if and only if there exists a finite c such that

$$\int_Q M(\sigma\chi_Q)(y)^p w(y)\,dy \le c\,\sigma(Q)$$

for all cubes Q. Furthermore, it is shown in [Moen 2009] that if B denotes the best constant, then

$$B \approx \sup_Q \left(\frac{\int_Q M(\sigma\chi_Q)^p w\,dx}{\sigma(Q)}\right)^{1/p}$$

Since this condition is hard to verify in practice, the second author considered in [Pérez 1995] conditions closer in spirit to the classical two-weight A_p condition,

$$A_p[w,\sigma] := \sup_Q \left(\fint_Q w\right)\left(\fint_Q \sigma\right)^{p-1},$$

which reduces to $[w]_{A_p}$ if $\sigma = w^{-1/(p-1)}$. As a consequence of the main result in that work, if $\delta>0$ and

$$\sup_Q \left(\fint_Q w\right)\|\sigma\|^{p-1}_{L(\log L)^{p-1+\delta},Q} < \infty, \tag{4.2}$$

then the two-weight norm inequality (4.1) holds. Recent advances in collaboration with M. Mastyło [Mastyło and Pérez 2013] allow one to go beyond condition (4.2) and improve the main results from [Pérez 1995].

In this paper, we consider a different new quantity, namely

$$B_p[w,\sigma] := \sup_Q \left(\fint_Q w\right)\left(\fint_Q \sigma\right)^p \exp\left(\fint_Q \log\sigma^{-1}\right).$$

To understand this new quantity, we observe that it is simply the functional on Q defining the $A_p[w,\sigma]$ condition multiplied by $\fint_Q \sigma \exp\big(\fint_Q \log\sigma^{-1}\big) \geq 1$. Then it is immediate that

$$A_p[w,\sigma] \leq B_p[w,\sigma] \leq A_p[w,\sigma]A_\infty[\sigma],$$

the difference between the last two being that $A_p[w,\sigma]A_\infty[\sigma]$ involves two independent suprema, as opposed to just one in $B_p[u,v]$.

We will consider first the dyadic maximal operator M_d, for which we can prove a dimension-free bound. Let us also introduce the weighted dyadic maximal function

$$M_{d,\sigma} f := \sup_{Q\in\mathscr{D}} \frac{\chi_Q}{\sigma(Q)} \int_Q |f(y)|\sigma(y)\,\mathrm{d}y,$$

which controls $M_d(f\sigma)$ as follows:

Theorem 4.3. *Let $p\in(1,\infty)$; then*

$$\begin{aligned}\|M_d(f\sigma)\|_{L^p(w)} &\leq 4e\cdot\big(B_p[w,\sigma]\big)^{1/p}\|M_{d,\sigma}f\|_{L^p(\sigma)}\\ &\leq 4e\cdot p'\cdot\big(B_p[w,\sigma]\big)^{1/p}\|f\|_{L^p(\sigma)},\end{aligned}$$

and also

$$\begin{aligned}\|M_d(f\sigma)\|_{L^p(w)} &\leq 4e\cdot\big([w]_{A_p}[\sigma]'_{A_\infty}\big)^{1/p}\|M_{d,\sigma}f\|_{L^p(\sigma)}\\ &\leq 4e\cdot p'\cdot\big([w]_{A_p}[\sigma]'_{A_\infty}\big)^{1/p}\|f\|_{L^p(\sigma)}.\end{aligned}$$

The main estimate in both chains of inequalities is of course the first one, since the second is simply the universal estimate for the weighted dyadic maximal function on the weighted L^p space with the same weight:

$$\|M_{d,\sigma}\|_{\mathscr{B}(L^p(\sigma))} \leq p'.$$

Obviously, in this dyadic version, it suffices to have the supremum in the weight constants over dyadic cubes only, and to only use the dyadic square function in the definition of $[\sigma]'_{A_\infty}$. And specializing to the case $\sigma = w^{-1/(p-1)}$, by the standard dual weight trick, we also get the bounds

$$\|M_d f\|_{L^p(w)} \leq \begin{cases} 4e\cdot p'\cdot\big(B_p[w,w^{-1/(p-1)}]\big)^{1/p}\|f\|_{L^p(w)},\\ 4e\cdot p'\cdot\big([w]_{A_p}[w^{-1/(p-1)}]'_{A_\infty}\big)^{1/p}\|f\|_{L^p(w)}.\end{cases}$$

Let us also recall how such dyadic bounds yield corresponding results for the Hardy–Littlewood maximal operator by a standard argument.

Proof of Theorem 1.7. Consider the 2^d shifted dyadic systems

$$\mathscr{D}^\alpha := \big\{2^{-k}\big([0,1)^d + m + (-1)^k\alpha\big) : k\in\mathbb{Z}, m\in\mathbb{Z}^d\big\},\quad \alpha\in\big\{0,\tfrac{1}{3}\big\}^d.$$

One can check (perhaps best in dimension $d=1$ first) that any cube Q is contained in a shifted dyadic cube $Q^\alpha\in\mathscr{D}^\alpha$ with $\ell(Q^\alpha)\leq 6\ell(Q)$, for some α. Hence

$$\fint_Q |f| \leq 6^d \fint_{Q^\alpha} |f| \leq 6^d M_d^\alpha f,$$

and therefore

$$Mf \le 6^d \sum_{\alpha\in\{0,\frac{1}{3}\}^d} M_d^\alpha f.$$

Thus, the norm bound for M_d may be multiplied by 12^d to give a bound for M. □

Remark 4.4. A recent result in collaboration with A. Kairema [Hytönen and Kairema 2012] allows one to perform a similar trick with adjacent dyadic systems even in an abstract space of homogeneous type. Thus, Theorem 1.7 readily extends to this generality as well.

4B. ***Proof of Theorem 4.3.*** We start by observing that it suffices to have a uniform bound over all linearizations

$$\tilde{M}(f\sigma) = \sum_{Q\in\mathscr{D}} \chi_{E(Q)}\langle f\sigma\rangle_Q,$$

where the sets $E(Q)\subseteq Q$ are pairwise disjoint. Here we use the notation

$$\langle f\rangle_Q = \fint_Q f = \fint_Q f(x)\,\mathrm{d}x$$

and

$$\langle f\rangle_Q^\sigma = \frac{1}{\sigma(Q)}\int_Q f(x)\sigma(x)\,\mathrm{d}x,$$

where, as usual, $\sigma(E) = \int_Q \sigma(x)\,\mathrm{d}x$.

By this disjointness,

$$\|\tilde{M}(f\sigma)\|_{L^p(w)} = \Bigl(\sum_{Q\in\mathscr{D}} w(E(Q))\langle f\sigma\rangle_Q^p\Bigr)^{1/p} = \Bigl(\sum_{Q\in\mathscr{D}} w(E(Q))\Bigl(\frac{\sigma(Q)}{|Q|}\Bigr)^p(\langle f\rangle_Q^\sigma)^p\Bigr)^{1/p}.$$

Now recall:

Theorem 4.5 (dyadic Carleson embedding theorem). *Suppose that the nonnegative numbers a_Q satisfy*

$$\sum_{Q\subseteq R} a_Q \le A\sigma(R) \quad \textit{for all } R\in\mathscr{D}.$$

Then, for all $p\in[1,\infty)$ and $f\in L^p(\sigma)$,

$$\Bigl(\sum_{Q\in\mathscr{D}} a_Q(\langle f\rangle_Q^\sigma)^p\Bigr)^{1/p} \le A^{1/p}\|M_{d,\sigma}f\|_{L^p(\sigma)} \le A^{1/p}\cdot p'\cdot\|f\|_{L^p(\sigma)} \quad \textit{if } p>1.$$

Since this is a slightly nonstandard formulation, although immediate by inspection of the usual argument, we provide a proof for completeness:

Proof. We view the sum $\sum_Q a_Q(\langle f\rangle_Q)^p$ as an integral on a measure space $(\mathscr{D},\mu)$ built over the set of dyadic cubes $\mathscr{D}$, assigning to each $Q\in\mathscr{D}$ the measure a_Q. Thus

$$\sum_{Q\in\mathscr{D}} a_Q(\langle f\rangle_Q)^p = \int_0^\infty p\lambda^{p-1}\mu\bigl(\{Q\in\mathscr{D}:\langle f\rangle_Q>\lambda\}\bigr)\,\mathrm{d}\lambda =: \int_0^\infty p\lambda^{p-1}\mu(\mathscr{Q}_\lambda)\,\mathrm{d}\lambda.$$

Let $\mathscr{Q}_\lambda^*$ be the set of maximal dyadic cubes R with the property that $\langle f\rangle_R>\lambda$. The cubes $R\in\mathscr{Q}_\lambda^*$ are disjoint, and their union is equal to the set $\{M_{d,\sigma} f > \lambda\}$. Thus

$$\mu(\mathcal{Q}_\lambda) = \sum_{Q\in\mathcal{Q}_\lambda} a_Q \le \sum_{R\in\mathcal{Q}_\lambda^*} \sum_{Q\subseteq R} a_Q \le \sum_{R\in\mathcal{Q}_\lambda^*} A\sigma(R) = A\sigma(M_{d,\sigma} f > \lambda),$$

and hence

$$\sum_{Q\in\mathcal{D}} a_Q(\langle f\rangle_Q)^p \le A\int_0^\infty p\lambda^{p-1}\sigma(M_{d,\sigma} f > \lambda)\,\mathrm{d}\lambda = A\|M_{d,\sigma} f\|_{L^p(\sigma)}^p. \qquad \square$$

If we apply the Carleson embedding with $a_Q = w(E(Q))\bigl(\sigma(Q)/|Q|\bigr)^p$, we find that

$$\|\tilde{M}(f\sigma)\|_{L^p(w)} \le A^{1/p}\|M_{d,\sigma} f\|_{L^p(\sigma)}, \tag{4.6}$$

provided that

$$\sum_{Q\subseteq R} w(E(Q))\left(\frac{\sigma(Q)}{|Q|}\right)^p \le A\sigma(R) \quad \text{for all } R\in\mathcal{D}. \tag{4.7}$$

Note that on $E(Q)\subseteq Q\subseteq R$, we have $\sigma(Q)/|Q| \le M(\sigma\chi_R)$, and hence

$$\begin{aligned}\sum_{Q\subseteq R} w(E(Q))\left(\frac{\sigma(Q)}{|Q|}\right)^p &= \int \sum_{Q\subseteq R}\chi_{E(Q)}\left(\frac{\sigma(Q)}{|Q|}\right)^p w \\ &\le \int \sum_{Q\subseteq R}\chi_{E(Q)} M(\chi_R\sigma)^p w \le \int_R M(\chi_R\sigma)^p w.\end{aligned}$$

So if $\bigl\|\chi_R M(\chi_R\sigma)\bigr\|_{L^p(u)} \le A^{1/p}\sigma(R)^{1/p}$, then (4.7) holds, and hence by Carleson's embedding also (4.6), and therefore the original two-weight inequality

$$\|M(f\sigma)\|_{L^p(u)} \le A^{1/p}\,\|M_{d,\sigma} f\|_{L^p(\sigma)}.$$

Hence, we are reduced to proving that

$$\bigl\|\chi_R M(\chi_R\sigma)\bigr\|_{L^p(u)}^p \le A\sigma(R), \quad A = (4e)^{1/p}\cdot B_p[w,\sigma]. \tag{4.8}$$

(In fact, the argument up to this point was essentially reproving Sawyer's two-weight characterization for the maximal function, paying attention to the constants.)

To prove (4.8), we exploit another linearization of M involving the *principal cubes*, as in the proof of the A_2 theorem: let $\mathcal{S}_0 := \{R\}$ and recursively let

$$\mathcal{S}_k := \bigcup_{S\in\mathcal{S}_{k-1}} \bigl\{Q\subset S : \langle\sigma\rangle_Q > 2\langle\sigma\rangle_S,\ Q \text{ is a maximal such cube}\bigr\},$$

and then $\mathcal{S} := \bigcup_{k=0}^\infty \mathcal{S}_k$. The pairwise disjoint subsets $E(S)\subseteq S$, defined in (3.12), satisfy $|E(S)| \ge \frac{1}{2}|S|$ by (3.13), and they partition R.

If $x \in E(S)$ and $Q \ni x$, then $\langle\sigma\rangle_Q \le 2\langle\sigma\rangle_S$, and hence $\chi_R M(\chi_R\sigma) \le 2\langle\sigma\rangle_S$ on $\chi_{E(S)}$. So altogether,

$$\begin{aligned}
\|\chi_R M(\chi_R\sigma)\|_{L^p(w)}^p &\le 2^p \Big\| \sum_{S\in\mathscr{S}} \chi_{E(S)}\langle\sigma\rangle_S \Big\|_{L^p(w)}^p = 2^p \sum_{S\in\mathscr{S}} w(E(S))\Big(\frac{\sigma(S)}{|S|}\Big)^p \\
&\le 2^p \sum_{S\in\mathscr{S}} \frac{w(S)}{|S|}\Big(\frac{\sigma(S)}{|S|}\Big)^p |S| \\
&\le 2^{p+1} \sum_{S\in\mathscr{S}} B_p[w,\sigma] \exp\Big(\fint_S \log\sigma\Big)|E(S)| \\
&\le 2^{p+1} B_p[w,\sigma] \int_R \sum_{S\in\mathscr{S}} \exp\Big(\fint_S \log\sigma\Big)\chi_{E(S)} \\
&\le 2^{p+1} B_p[w,\sigma] \int_R \sup_{Q\in\mathscr{D}} \chi_Q \exp\Big(\fint_Q \log\sigma\chi_R\Big) \\
&= 2^{p+1} B_p[w,\sigma] \int_R M_0(\chi_R\sigma),
\end{aligned} \tag{4.9}$$

where M_0 is the (dyadic) logarithmic maximal function introduced in Lemma 2.1. By this lemma, we then have

$$\|\chi_R M(\chi_R\sigma)\|_{L^p(u)}^p \le 4^p B_p[w,\sigma]\cdot e\cdot\sigma(R),$$

which proves (4.8), and hence Theorem 4.3, upon taking the p-th root.

In order to prove the second version of Theorem 4.3, we only need to make a slight modification in the estimate (4.9). We then compute:

$$\begin{aligned}
\|\chi_R M(\chi_R\sigma)\|_{L^p(w)}^p &\le 2^p \sum_{S\in\mathscr{S}} \frac{w(S)}{|S|}\Big(\frac{\sigma(S)}{|S|}\Big)^p |S| \le 2^{p+1} \sum_{S\in\mathscr{S}} [w]_{A_p} \frac{\sigma(S)}{|S|}|E(S)| \\
&\le 2^{p+1}[w]_{A_p} \sum_{S\in\mathscr{S}} \int_{E(S)} M(\sigma\chi_Q) = 2^{p+1}[w]_{A_p} \int_Q M(\sigma\chi_Q) \\
&= 2^{p+1}[w]_{A_p}[\sigma]'_{A_\infty}\sigma(Q),
\end{aligned}$$

by a direct application of the definition of $[\sigma]'_{A_\infty}$ in the last step, and this completes the alternative argument.

4C. ***Another proof of Theorem 4.3.*** We finish this section by providing yet another proof variant for Theorem 4.3. This proof is more elementary, since it does not need the reduction to the testing condition (4.8), and it uses the more standard Calderón–Zygmund-type stopping cubes instead of the principal cubes. Its disadvantage is the fact the we cannot recover the dimension-independence by this argument. On the other hand, the proof may be extended to maximal functions defined in term of a general basis; see [García-Cuerva and Rubio de Francia 1985, Section IV.4].

A simpler proof of Theorem 4.3 with a dimension-dependent bound. Fix $a > 2^d$. For each integer k, let

$$\Omega_k = \{x \in \mathbb{R}^d : M_d(f\sigma)(x) > a^k\}.$$

By standard arguments, we consider the Calderón–Zygmund decomposition, and there is a family of maximal nonoverlapping dyadic cubes $\{Q_{k,j}\}$ for which $\Omega_k = \bigcup_j Q_{k,j}$ and

$$a^k < \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} |f(y)|\sigma(y)\,\mathrm{d}y \le 2^d a^k. \tag{4.10}$$

Now,

$$\begin{aligned}
\int_{\mathbb{R}^d} M_d(f\sigma)^p w\,\mathrm{d}x &= \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} M_d(f\sigma)^p w\,\mathrm{d}x \\
&\le a^p \sum_k a^{kp} w(\Omega_k) = a^p \sum_{k,j} a^{kp} w(Q_{k,j}) \\
&\le a^p \sum_{k,j} \Big(\frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} |f(y)|\sigma(y)\,\mathrm{d}y\Big)^p w(Q_{k,j}) \\
&= a^p \sum_{k,j} (\langle |f|\rangle^{\sigma}_{Q_{k,j}})^p \Big(\frac{\sigma(Q_{k,j})}{|Q_{k,j}|}\Big)^p w(Q_{k,j}) \\
&\le a^p B_p[w,\sigma] \sum_{k,j} (\langle |f|\rangle^{\sigma}_{Q_{k,j}})^p |Q_{k,j}| \exp\Big(\fint_{Q_{k,j}} \log \sigma(t)\,\mathrm{d}t\Big) \\
&= a^p B_p[w,\sigma] \sum_{Q\in\mathscr{D}} (\langle |f|\rangle^{\sigma}_{Q})^p a_Q,
\end{aligned}$$

where

$$a_Q = \begin{cases} |Q| \exp\big(\fint_Q \log\sigma\big) & \text{if } Q = Q_{k,j} \text{ for some } (k,j), \\ 0 & \text{else.} \end{cases}$$

By the dyadic Carleson embedding theorem, we can hence conclude that

$$\int_{\mathbb{R}^d} M^d(f\sigma)^p w\,\mathrm{d}x \le a^p B_p[w,\sigma] A \int_{\mathbb{R}^d} (M_{d,\sigma} f)^p \sigma\,\mathrm{d}x,$$

provided that we check the condition

$$\sum_{Q\subseteq R} a_Q = \sum_{k,j:Q_{k,j}\subseteq R} |Q_{k,j}| \exp\Big(\fint_{Q_{k,j}} \log\sigma\Big) \le A|R|. \tag{4.11}$$

To estimate the left side of (4.11), we first do the following: for each (k, j), we set $E_{k,j} = Q_{k,j} \setminus \Omega_{k+1}$. Observe that the sets of the family $E_{k,j}$ are pairwise disjoint. We claim that

$$|Q_{k,j}| < \frac{a}{a-2^d} |E_{k,j}| \tag{4.12}$$

for each k, j. Indeed, by (4.10) and Hölder's inequality,

$$|Q_{k,j} \cap \Omega_{k+1}| = \sum_{Q_{k+1,l}\subset Q_{k,j}} |Q_{k+1,l}| < \frac{1}{a^{k+1}} \sum_{Q_{k+1,l}\subset Q_{k,j}} \int_{Q_{k+1,l}} |f|\sigma \le \frac{1}{a^{k+1}} \int_{Q_{k,j}} |f|\sigma \le \frac{2^d}{a} |Q_{k,j}|,$$

which proves (4.12). With $\beta = a/(a-2^d)$, we can estimate the left side of (4.11) as follows:

$$\begin{aligned}\sum_{Q\subseteq R} a_Q &\le \beta \sum_{(k,j):Q_{k,j}\subseteq R} |E_{k,j}| \exp\Big(\fint_{Q_{k,j}} \log\sigma(t)\,\mathrm{d}t\Big)\\ &\le \beta \sum_{(k,j):Q_{k,j}\subseteq R} \int_{E_{k,j}} M_0(\sigma 1_R)(x)\,\mathrm{d}x\\ &\le \beta \int_R M_0(\sigma 1_R)(x)\,\mathrm{d}x \le \beta e\sigma(R),\end{aligned}$$

where we used the definition and the L^1 boundedness of the logarithmic dyadic maximal function. This proves (4.11) with $A = \beta e$, concluding the proof. □

5. Proof of the extrapolation theorems

We will prove in this section the upper and lower extrapolation theorems 1.24 and 1.25. Recall that the initial hypothesis is given by the expression

$$\|Tf\|_{L^r(w)} \le \varphi\big([w]_{A_r}, [w]_{A_\infty}, [w^{-1/(r-1)}]_{A_\infty}^{(r-1)}\big)\|f\|_{L^r(w)},$$

for some $r \in (1,\infty)$.

Proof of Theorem 1.24. Our argument is modeled after a simplified proof by Duoandikoetxea [2011] of the already cited result from [Dragičević et al. 2005] (see also [Cruz-Uribe et al. 2011]).

Fix some $p \in (1,r)$, $w \in A_p$, $f \in L^p(w)$ and $g := |f|/\|f\|_{L^p(w)}$. Let

$$Rg := \sum_{k=0}^{\infty} \frac{2^{-k}M^k g}{\|M\|^k_{\mathscr{B}(L^p(w))}},$$

so that

$$|g| \le Rg, \quad \|Rg\|_{L^p(w)} \le 2\|g\|_{L^p(w)} = 2, \quad [Rg]_{A_1} \le 2\|M\|_{L^p(w)}.$$

Then by Hölder's inequality,

$$\begin{aligned}\|Tf\|_{L^p(w)} &= \Big(\int |Tf|^p (Rg)^{-(r-p)p/r}(Rg)^{(r-p)p/r} w\Big)^{1/p}\\ &\le \Big(\int |Tf|^r (Rg)^{-(r-p)} w\Big)^{1/r}\Big(\int (Rg)^p w\Big)^{1/p-1/r}\\ &\le \|Tf\|_{L^r(W)}(2^p)^{1/p-1/r} \le 2\|Tf\|_{L^r(W)},\end{aligned}$$

where

$$W := (Rg)^{-(r-p)}w.$$

By assumption, we have

$$\|Tf\|_{L^r(W)} \le \varphi\big([W]_{A_r}, [W]_{A_\infty}, [W^{-1/(r-1)}]_{A_\infty}^{(r-1)}\big)\|f\|_{L^r(W)},$$

where

$$\|f\|_{L^r(W)} = \left(\int |f|^r (Rf)^{-(r-p)} w\right)^{1/r} \|f\|_{L^p(w)}^{(r-p)/r} \le \left(\int |f|^r |f|^{-(r-p)} w\right)^{1/r} \|f\|_{L^p(w)}^{(r-p)/r} = \|f\|_{L^p(w)},$$

so it remains to estimate the weight constants

$$[W]_{A_r}, \quad [W]_{A_\infty}, \quad [W^{-1/(r-1)}]_{A_\infty}.$$

Using $\sup_Q (Rg)^{-1} \le [Rg]_{A_1} \langle Rg\rangle_Q^{-1}$ or Hölder's or Jensen's inequality where appropriate, we compute

$$\begin{aligned}
\langle W\rangle_Q &= \big\langle (Rg)^{-(r-p)} w\big\rangle_Q \le [Rg]_{A_1}^{r-p} \langle Rg\rangle_Q^{-(r-p)} \langle w\rangle_Q,\\
\langle W^{-1/(r-1)}\rangle_Q^{r-1} &= \big\langle (Rg)^{(r-p)/(r-1)} w^{-1/(r-1)}\big\rangle_Q^{r-1} \le \langle Rg\rangle_Q^{r-p} \langle w^{-1/(p-1)}\rangle_Q^{p-1},\\
\exp\langle -\log W\rangle_Q &= \big(\exp\langle \log(Rg)\rangle_Q\big)^{r-p} \exp\langle -\log w\rangle_Q \le \langle Rg\rangle_Q^{r-p} \exp\langle -\log w\rangle_Q,
\end{aligned}$$

and

$$\begin{aligned}
\big(\exp\langle -\log W^{-1/(r-1)}\rangle_Q\big)^{r-1} &= \big(\exp\langle \log(Rg)^{-1}\rangle_Q\big)^{r-p} \big(\exp\langle -\log w^{-1/(r-1)}\rangle_Q\big)^{r-1}\\
&\le [Rg]_{A_1}^{r-p} \langle Rg\rangle^{-(r-p)} \big(\exp\langle -\log w^{-1/(p-1)}\rangle_Q\big)^{p-1}.
\end{aligned}$$

Multiplying the appropriate estimates and using the definition, we then have

$$[W]_{A_r} \le [Rg]_{A_1}^{r-p} [w]_{A_p}, \quad [W]_{A_\infty} \le [Rg]_{A_1}^{r-p} [w]_{A_\infty}, \quad \big[W^{-1/(r-1)}\big]_{A_\infty}^{r-1} \le [Rg]_{A_1}^{r-p} \big[w^{-1/(p-1)}\big]_{A_\infty}^{p-1}.$$

(We do not know whether it is possible to make similar estimates for $[W]'_{A_\infty}$ in terms of $[w]'_{A_\infty}$; this is the reason why we need to use the $[\ ]_{A_\infty}$ constants in this proof.)

Next, recall that

$$[Rg]_{A_1} \le 2\|M\|_{\mathcal{B}(L^p(w))} \le c_d \cdot p' \cdot [w]_{A_p}^{1/p} \big([w^{-1/(p-1)}]'_{A_\infty}\big)^{1/p}.$$

Thus we conclude the proof with

$$\begin{aligned}
\|Tf\|_{L^p(w)} &\le 2\|Tf\|_{L^r(W)} \le 2\varphi\big([W]_{A_r}, [W]_{A_\infty}, [W^{-1/(r-1)}]_{A_\infty}^{(r-1)}\big) \|f\|_{L^r(W)}\\
&\le 2\varphi\Big([Rg]_{A_1}^{r-p} \big([w]_{A_p}, [w]_{A_\infty}, [w^{-1/(p-1)}]_{A_\infty}^{(p-1)}\big)\Big) \|f\|_{L^p(w)}\\
&\le 2\varphi\Big(2^{r-p} \|M\|_{\mathcal{B}(L^p(w))}^{r-p} \big([w]_{A_p}, [w]_{A_\infty}, [w^{-1/(p-1)}]_{A_\infty}^{(p-1)}\big)\Big) \|f\|_{L^p(w)}. \qquad \square
\end{aligned}$$

Proof of Theorem 1.25. Again, our argument is inspired by Duoandikoetxea's simplification [2011] of the proof of a result in [Dragičević et al. 2005] (see also [Cruz-Uribe et al. 2011]).

Fix some $p \in (r, \infty)$, $w \in A_p$, $f \in L^p(w)$. By duality, we have

$$\|Tf\|_{L^p(w)} = \sup_{\substack{h \ge 0\\ \|h\|_{L^{p'}(w)} = 1}} \int |Tf| h w.$$

We fix one such h, and try to bound the expression on the right.

Observe that the pointwise multiplication operators

$$h \mapsto wh : L^{p'}(w) \to L^{p'}(w^{1-p'}), \quad g \mapsto \frac{1}{w} g : L^{p'}(w^{1-p'}) \to L^{p'}(w)$$

are isometric. Let R be as in the previous proof, except with p' and $\sigma = w^{1-p'}$ in place of p and w:

$$Rg := \sum_{k=0}^{\infty} \frac{2^{-k} M^k g}{\|M\|^k_{\mathcal{B}(L^{p'}(\sigma))}},$$

and $R'h := w^{-1} R(wh)$. Then

$$h \le R'h, \quad \|R'h\|_{L^{p'}(w)} \le 2\|h\|_{L^{p'}(w)} = 2, \quad [wR'h]_{A_1} \le 2\|M\|_{\mathcal{B}(L^{p'}(\sigma))}.$$

Then, by Hölder's inequality,

$$\begin{aligned}\int |Tf| hw \le \int |Tf|(R'h)w &= \int |Tf|(R'h)^{(p-r)/[r(p-1)]}(R'h)^{(r-1)p/[r(p-1)]}w \\ &\le \Big(\int |Tf|^r (R'h)^{(p-r)/(p-1)} w\Big)^{1/r} \Big(\int (R'h)^{p/(p-1)} w\Big)^{1/r'} \\ &\le \|Tf\|_{L^r(W)} 2^{p'/r'},\end{aligned}$$

where

$$W := (R'h)^{(p-r)/(p-1)} w.$$

By assumption,

$$\|Tf\|_{L^r(W)} \le \varphi\big([W]_{A_r}, [W]_{A_\infty}, [W^{-1/(r-1)}]^{(r-1)}_{A_\infty}\big) \|f\|_{L^r(W)}, \tag{5.1}$$

where, by Hölder's inequality with exponents p/r and $p/(p-r)$,

$$\begin{aligned}\|f\|_{L^r(W)} &= \Big(\int |f|^r w^{r/p} \cdot (R'h)^{(p-r)/(p-1)} w^{(p-r)/p}\Big)^{1/r} \\ &\le \Big(\int |f|^p w\Big)^{1/p} \Big(\int (R'h)^{p/(p-1)} w\Big)^{1/r-1/p} \le \|f\|_{L^p(w)} (2^{p'})^{1/r-1/p},\end{aligned}$$

so altogether, suppressing the arguments of φ from (5.1),

$$\begin{aligned}\int |Tf| hw \le \|Tf\|_{L^r(W)} 2^{p'/r'} &\le \varphi(\dots) \|f\|_{L^r(W)} 2^{p'/r'} \\ &\le \varphi(\dots)(2^{p'})^{1/r-1/p} \|f\|_{L^p(w)} 2^{p'/r'} = 2\varphi(\dots)\|f\|_{L^p(w)}.\end{aligned}$$

It remains to estimate

$$[W]_{A_r}, \quad [W]_{A_\infty}, \quad [W^{-1/(r-1)}]^{(r-1)}_{A_\infty}$$

for

$$W = (R'h)^{(p-r)/(p-1)} w = [(R'h)w]^{(p-r)/(p-1)} w^{(r-1)/(p-1)}.$$

We thus compute

$$\begin{aligned}
\langle W\rangle_Q &= \big\langle (R'h)^{(p-r)/(p-1)}w\big\rangle_Q \le \langle (R'h)w\rangle_Q^{(p-r)/(p-1)}\langle w\rangle_Q^{(r-1)/(p-1)},\\
\langle W^{-1/(r-1)}\rangle_Q^{r-1} &= \big\langle (wR'h)^{-(p-r)/[(p-1)(r-1)]}w^{-1/(p-1)}\big\rangle_Q^{r-1}\\
&\le [wR'h]_{A_1}^{(p-r)/(p-1)}\langle (R'h)w\rangle_Q^{-(p-r)/(p-1)}\langle w^{-1/(p-1)}\rangle_Q^{r-1},\\
\exp\big(-\langle \log W\rangle_Q\big) &= \big(\exp\langle \log (wR'h)^{-1}\rangle_Q\big)^{(p-r)/(r-1)}\big(\exp\langle -\log w\rangle_Q\big)^{(r-1)/(p-1)}\\
&\le [(R'h)w]_{A_1}^{(p-r)/(r-1)}\langle (R'h)w\rangle_Q^{-(p-r)/(r-1)}\big(\exp\langle -\log w\rangle_Q\big)^{(r-1)/(p-1)},
\end{aligned}$$

and

$$\begin{aligned}
\big(\exp(-\langle \log W^{-1/(r-1)}\rangle_Q)\big)^{r-1} &= \big(\exp(\langle \log (wR'h)\rangle_Q)\big)^{(p-r)/(p-1)}\big(\exp\langle -\log w^{-1/(p-1)}\rangle_Q\big)^{r-1}\\
&\le \langle (R'h)w\rangle_Q^{(p-r)/(r-1)}\big(\exp\langle -\log w^{-1/(p-1)}\rangle_Q\big)^{r-1}.
\end{aligned}$$

Multiplying the relevant quantities, it follows that

$$\begin{aligned}
[W]_{A_r} &\le [(R'h)w]_{A_1}^{(p-r)/(p-1)}[w]_{A_p}^{(r-1)/(p-1)},\\
[W]_{A_\infty} &\le [(R'h)w]_{A_1}^{(p-r)/(p-1)}[w]_{A_\infty}^{(r-1)/(p-1)},\\
\big[W^{-1/(r-1)}\big]_{A_\infty}^{r-1} &\le [(R'h)w]_{A_1}^{(p-r)/(p-1)}\big[w^{-1/(p-1)}\big]_{A_\infty}^{(r-1)}.
\end{aligned}$$

Also recall that

$$[(R'h)w]_{A_1} \le 2\|M\|_{\mathscr{B}(L^{p'}(w^{1-p'}))} \le c_d[w^{1-p'}]_{A_{p'}}^{1/p'}[w]_{A_\infty}^{1/p'} = c_d[w]_{A_p}^{1/p}[w]_{A_\infty}^{1/p'},$$

and thus we conclude with

$$\begin{aligned}
\|Tf\|_{L^p(w)} &\le \int |Tf|hw \le 2\varphi\big([W]_{A_r},[W]_{A_\infty},[W^{-1/(r-1)}]_{A_\infty}^{(r-1)}\big)\|f\|_{L^p(w)}\\
&\le 2\varphi\big([(R'h)w]_{A_1}^{(p-r)/(p-1)}\big([W]_{A_r},[W]_{A_\infty},[W^{-1/(r-1)}]_{A_\infty}^{(r-1)}\big)\big)\|f\|_{L^p(w)}\\
&\le 2\varphi\Big((2\|M\|_{\mathscr{B}(L^{p'}(w^{1-p'}))})^{(p-r)/(p-1)}\\
&\qquad\qquad \times\big([w]_{A_r}^{(r-1)/(p-1)},[w]_{A_\infty}^{(r-1)/(p-1)},[w^{-1/(p-1)}]_{A_\infty}^{(r-1)}\big)\Big)\|f\|_{L^p(w)}. \qquad\square
\end{aligned}$$

6. The A_1 theory, proof of Theorem 1.14 and its consequences

6A. *The main lemma.* The proofs of the theorems will be based on the following lemma.

Lemma 6.1. *Let T be any Calderón–Zygmund singular integral operator and let w be any weight. Also let $p,r\in(1,\infty)$. Then there is a constant $c=c_{d,T}$ such that*

$$\|Tf\|_{L^p(w)} \le cpp'(r')^{1/p'}\|f\|_{L^p(M_rw)},$$

where, as usual, we denote $M_rw = M(w^r)^{1/r}$.

This is a consequence of the estimate

$$\|Tf\|_{L^p(w)} \leq cpp'\Big(\frac{1}{r-1}\Big)^{1-1/pr}\|f\|_{L^p(M_r w)}$$

(which can be found in [Lerner et al. 2009a] when $r \in (1, 2]$), since

$$\Big(\frac{1}{r-1}\Big)^{1-1/pr} \leq (r')^{1-1/p+1/pr'} \leq 2(r')^{1/p'}$$

where we used $t^{1/t} \leq 2$, $t \geq 1$.

6B. ***Proof of the sharp reverse Hölder's inequality.***

Lemma 6.2. *For any cube Q and any measurable function w,*

$$\int_Q w\log\Big(e+\frac{w}{\langle w\rangle_Q}\Big)\,dx \leq 2^{d+1}\int_Q M(w\chi_Q)\,dx. \tag{6.3}$$

Hence, if $w \in A_\infty$,

$$\sup_Q \frac{1}{w(Q)}\int_Q w(y)\log\Big(e+\frac{w(y)}{\langle w\rangle_Q}\Big)\,dy \leq 2^{d+1}[w]'_{A_\infty}. \tag{6.4}$$

The essential idea of the proof can be traced back to the well-known $L\log L$ estimate for M in [Stein 1969]. However, these estimates are not homogeneous. A proof of this lemma within the context of spaces of homogeneous type can essentially be found in [Pérez and Wheeden 2001, Lemma 8.5] (see also [Wilson 2008, p. 17, inequality (2.15)] for a different proof).

Proof of Lemma 6.2. Fix a cube Q. By homogeneity, we assume that $\langle w\rangle_Q = 1$. The key estimate follows from the "reverse weak type (1, 1) estimate": if w is nonnegative and $t > \langle w\rangle_Q$,

$$\frac{1}{t}\int_{\{x\in Q:w(x)>t\}} w\,dx \leq 2^d\big|\{x \in Q : M(w\chi_Q)(x) > t\}\big|. \tag{6.5}$$

Now,

$$\frac{1}{|Q|}\int_Q w\log(e+w)\,dx = \frac{1}{|Q|}\int_0^\infty \frac{1}{e+t}\,w(\{x\in Q : w(x) > t\})\,dt = I + II.$$

Here

$$I := \frac{1}{|Q|}\int_0^1 \frac{1}{e+t}\,w(\{x \in Q : w(x) > t\})\,dt \leq 1 \leq \frac{1}{|Q|}\int_Q M(w\chi_Q)\,dx,$$

while for the complementary term II we use the estimate (6.5):

$$\begin{aligned} II &= \frac{1}{|Q|}\int_1^\infty \frac{1}{e+t} w(\{x \in Q : w(x) > t\})\,dt \\ &\leq \frac{2^d}{|Q|}\int_1^\infty \frac{t}{e+t}\big|\{x \in Q : M(w\chi_Q)(x) > t\}\big|\,dt \\ &\leq \frac{2^d}{|Q|}\int_0^\infty \big|\{x \in Q : M(w\chi_Q)(x) > t\}\big|\,dt \\ &= \frac{2^d}{|Q|}\int_Q M(w\chi_Q)(x)\,dx. \end{aligned}$$

This gives (6.3), and (6.4) follows from the definition of $[w]'_{A_\infty}$. □

The main use of the lemma is the following key observation.

Lemma 6.6 [Wilson 2008, p. 45]. *Let $S \subset Q$ and let $\lambda > 0$; then*

$$\frac{|S|}{|Q|} < e^{-\lambda} \quad \textit{implies} \quad \frac{w(S)}{w(Q)} < \frac{2^{d+2}[w]'_{A_\infty}}{\lambda} + e^{-\lambda/2}. \tag{6.7}$$

Proof. Indeed, if $E_\lambda = \{x \in Q : w(x) > e^\lambda \langle w\rangle_Q\}$, then $w(E_\lambda) \le (2^{d+1}/\lambda)[w]'_{A_\infty} w(Q)$ by (6.4). Therefore

$$\begin{aligned} w(S) \le w(S \cap E_{\lambda/2}) + w(S \setminus E_{\lambda/2}) &\le \frac{2^{d+2}[w]'_{A_\infty}}{\lambda} w(Q) + e^{\lambda/2}\langle w\rangle_Q |S| \\ &\le \frac{2^{d+2}[w]'_{A_\infty}}{\lambda} w(Q) + e^{\lambda/2} e^{-\lambda} w(Q) \qquad \text{by the hypothesis in (6.7)} \\ &= \frac{2^{d+2}[w]'_{A_\infty}}{\lambda} w(Q) + e^{-\lambda/2} w(Q), \end{aligned}$$

and this proves the claim (6.7). □

Proof of Theorem 2.3. Recall that we have to prove that

$$\left(\fint_Q w^{r(w)} \right)^{1/r(w)} \le 2 \fint_Q w,$$

where

$$r(w) := 1 + \frac{1}{\tau_d [w]'_{A_\infty}},$$

and where τ_d is a large dimensional constant.

Observe that by homogeneity, we can assume that $\fint_Q w = 1$. We use the dyadic maximal function on the dyadic subcubes of a given Q:

$$\begin{aligned} \int_Q w^{1+\varepsilon} &\le \int_Q M_d(w\chi_Q)^\varepsilon w = \int_0^\infty \varepsilon t^{\varepsilon-1} w\big(\{x \in Q : M_d(w\chi_Q) > t\}\big)\, \mathrm{d}t \\ &\le \int_0^1 \varepsilon t^{\varepsilon-1} w(Q)\, \mathrm{d}t + \varepsilon \int_1^\infty \varepsilon t^\varepsilon w\big(\{x \in Q : M_d(w\chi_Q) > t\}\big) \frac{\mathrm{d}t}{t} \\ &\le |Q| + \varepsilon \sum_{k\ge 0} \int_{a^k}^{a^{k+1}} t^\varepsilon w\big(\{x \in Q : M_d(w\chi_Q) > t\}\big) \frac{\mathrm{d}t}{t} \\ &\le |Q| + \varepsilon a^\varepsilon \sum_{k \ge 0} a^{k\varepsilon} \int_{a^k}^{a^{k+1}} w\big(\{x \in Q : M_d(w\chi_Q) > a^k\}\big) \frac{\mathrm{d}t}{t} \quad \text{for } a \gg 1, \\ &= |Q| + \varepsilon a^\varepsilon \log a \sum_{k\ge 0} a^{k\varepsilon}\, w(\Omega_k), \end{aligned}$$

where

$$\Omega_k = \{x \in Q : M_d(w\chi_Q)(x) > a^k\}.$$

Since $a^k \geq 1 = \fint_Q w$, we can consider the Calderón–Zygmund decomposition of w adapted to Q. There is a family of maximal nonoverlapping dyadic cubes $\{Q_{k,j}\}$ strictly contained in Q for which $\Omega_k = \bigcup_j Q_{k,j}$ and

$$a^k < \fint_{Q_{k,j}} w \leq 2^d a^k. \tag{6.8}$$

Now,

$$\sum_{k\geq 0} a^{k\varepsilon} w(\Omega_k) = \sum_{k,j} a^{k\varepsilon} w(Q_{k,j}) \leq \sum_{k,j} \left(\frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w(y)\,\mathrm{d}y \right)^{\varepsilon} w(Q_{k,j}).$$

We need to estimate $w(Q_{k,j})$, which we pursue similarly to Section 4C; see in particular (4.12). For each (k, j) we set $E_{k,j} = Q_{k,j} \setminus \Omega_{k+1}$. Observe that the sets of the family $E_{k,j}$ are pairwise disjoint. But exactly as in (4.12), we have that for $a > 2^d$ and for each k, j,

$$|Q_{k,j}| < \frac{a}{a-2^d} |E_{k,j}|. \tag{6.9}$$

We now apply (6.7) with $Q = Q_{k,j}$ and $S = Q_{k,j} \cap \Omega_{k+1}$. Choose λ such that $e^{-\lambda} = 2^d/a$, namely $\lambda = \log(a/2^d)$. Then applying (6.7), we have that

$$\frac{w(Q_{k,j} \cap \Omega_{k+1})}{w(Q_{k,j})} < \frac{2^{d+2}\,[w]'_{A_\infty}}{\log(a/2^d)} + \left(\frac{2^d}{a}\right)^{1/2}.$$

Since $a > 2^d$ is available, we choose $a = 2^d e^{L\,[w]'_{A_\infty}}$, with L a large dimensional constant to be chosen. If in particular $L \geq 2^{d+4}$, we have

$$\frac{w(Q_{k,j} \cap \Omega_{k+1})}{w(Q_{k,j})} < \frac{2^{d+2}}{L} + e^{-[w]'_{A_\infty} L/2} < \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

This yields that $w(Q_{k,j}) \leq 2w(E_{k,j})$, and we can continue with the sum estimate:

$$\begin{aligned} \sum_{k\geq 0} a^{k\varepsilon} w(\Omega_k) &\leq 2 \sum_{k,j} \left(\frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w(y)\,\mathrm{d}y \right)^{\varepsilon} w(E_{k,j}) \\ &\leq 2 \sum_{k,j} \int_{E_{k,j}} M_d(w\chi_Q)^{\varepsilon} w\,\mathrm{d}x \leq 2 \int_Q M_d(w\chi_Q)^{\varepsilon} w\,\mathrm{d}x. \end{aligned}$$

Combining estimates, we end up with

$$\fint_Q M_d(w\chi_Q)^{\varepsilon} w \leq 1 + 2\,\varepsilon a^{\varepsilon} \log a \fint_Q M_d(w\chi_Q)^{\varepsilon} w\,\mathrm{d}x,$$

for any $\varepsilon > 0$. Recall that $a = 2^d e^{L\,[w]'_{A_\infty}}$. Hence, if we choose

$$L = 2^{d+4}, \qquad \varepsilon = \frac{1}{2^7 L\,[w]'_{A_\infty}} = \frac{1}{2^{11+d}\,[w]'_{A_\infty}},$$

we can compute

$$2\varepsilon a^{\varepsilon} \log a < \tfrac{1}{2}, \qquad \fint_Q M_d(w\chi_Q)^{\varepsilon} w \leq 2,$$

concluding the proof of the theorem. □

6C. *Proof of Theorem 1.14: the strong case.* The proof is, as in [Lerner et al. 2009a], just an application of Lemma 6.1 with a specific parameter r coming from the sharp reverse Hölder inequality given by Theorem 2.3. Indeed, since $w \in A_1 \subset A_\infty$, and if we write

$$r(w) := 1 + \frac{1}{\tau_d\,[w]'_{A_\infty}},$$

we have

$$\left(\fint_Q w^{r(w)}\right)^{1/r(w)} \leq 2\fint_Q w. \tag{6.10}$$

Now, by Lemma 6.1 with $r = r(w)$, we have

$$\begin{aligned}\|Tf\|_{L^p(w)} &\leq cpp'(r')^{1/p'}\|f\|_{L^p(M_r w)} \leq cpp'([w]'_{A_\infty})^{1/p'}\|f\|_{L^p(2Mw)}\\ &\leq cpp'([w]'_{A_\infty})^{1/p'}[w]_{A_1}^{1/p}\|f\|_{L^p(w)},\end{aligned}$$

using the standard notation $M_r w = M(w^r)^{1/r}$. This concludes the proof of the theorem.

6D. *Proof of Theorem 1.15: the weak case.* We follow here the classical method of Calderón and Zygmund, with the modifications considered in [Pérez 1994]. Applying the Calderón–Zygmund decomposition to f at level λ, we get a family of pairwise disjoint cubes $\{Q_j\}$ such that

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j}|f| \leq 2^d\lambda.$$

Let $\Omega = \bigcup_j Q_j$ and $\widetilde{\Omega} = \bigcup_j 2Q_j$. The "good part" is defined by

$$g = \sum_j f_{Q_j}\chi_{Q_j}(x) + f(x)\chi_{\Omega^c}(x),$$

and the "bad part" b as

$$b = \sum_j b_j,$$

where

$$b_j(x) = (f(x) - f_{Q_j})\chi_{Q_j}(x).$$

Then $f = g + b$. We split the level set as

$$\begin{aligned} w\bigl\{x \in \mathbb{R}^d : |Tf(x)| > \lambda\bigr\} &\leq w(\widetilde{\Omega}) + w\Bigl\{x \in (\widetilde{\Omega})^c : |Tb(x)| > \frac{\lambda}{2}\Bigr\} + w\Bigl\{x \in (\widetilde{\Omega})^c : |Tg(x)| > \frac{\lambda}{2}\Bigr\}\\ &=: I + II + III.\end{aligned}$$

Exactly as in [Pérez 1994], the main term is III. We first deal with the easy terms I and II, which actually satisfy the better bound

$$I + II \lesssim \frac{1}{\lambda}[w]_{A_1}\|f\|_{L^1(w)}.$$

Indeed, the first term is essentially the level set of Mf:

$$I = w\{x \in \mathbb{R}^d : Mf(x) > c_d\,\lambda\},$$

and the result follows by the classical Fefferman–Stein inequality:

$$\|Mf\|_{L^{1,\infty}(w)} \le c_d \, \|f\|_{L^1(Mw)}.$$

For the second term, we use the following estimate: there is a dimensional constant c such that for any cube Q and any function b supported on Q such that $\int_Q b(x)\,\mathrm{d}x = 0$ and any weight w, we have

$$\int_{\mathbb{R}^n\setminus 2Q} |Tb(y)|w(y)\,\mathrm{d}y \le c_d \int_Q |b(y)|Mw(y)\,\mathrm{d}y. \tag{6.11}$$

This can be found in Lemma 3.3 of [García-Cuerva and Rubio de Francia 1985, p. 413]. Now, using this estimate with w replaced by $w\chi_{\mathbb{R}^n\setminus 2Q_j}$, we have

$$\begin{aligned} II &\le \frac{c}{\lambda}\int_{\mathbb{R}^n\setminus\tilde{\Omega}} |Tb(y)|w(y)\,\mathrm{d}y \\ &\le \frac{c}{\lambda}\sum_j \int_{\mathbb{R}^n\setminus 2Q_j} |Tb_j(y)|w(y)\,\mathrm{d}y \le \frac{c}{\lambda}\sum_j\int_{Q_j}|b_j(y)|M(w\chi_{\mathbb{R}^n\setminus 2Q_j})(y)\,\mathrm{d}y \\ &\le \frac{c}{\lambda}\int_{\mathbb{R}^d}|f(y)|Mw(y)\,\mathrm{d}y + \frac{c}{\lambda}\sum_j \frac{1}{|Q_j|}\int_{Q_j} M(w\chi_{\mathbb{R}^n\setminus 2Q_j})(x)\,\mathrm{d}x \int_{Q_j}|f(x)|\,\mathrm{d}x. \end{aligned}$$

To estimate the inner sum, we use that $M(\chi_{\mathbb{R}^n\setminus 2Q}\mu)$ is essentially constant on Q:

$$M(\chi_{\mathbb{R}^n\setminus 2Q}\mu)(y) \approx M(\chi_{\mathbb{R}^n\setminus 2Q}\mu)(z), \qquad y, z \in Q, \tag{6.12}$$

where the constants are dimensional. This fact that can be found in [ibid., p. 159]. Hence, the sum is controlled by

$$c_d \sum_j \inf_{x\in Q} M(w\chi_{\mathbb{R}^n\setminus 2Q_j})(x) \int_{Q_j}|f(x)|\,\mathrm{d}x \le c_d \int_{\mathbb{R}^n}|f(x)|Mw(x)\,\mathrm{d}x.$$

This gives the required estimate.

We now consider the singular term III, to which we apply the Chebyshev inequality and Lemma 6.1 with exponents $p, r \in (1,\infty)$ to be chosen soon:

$$\begin{aligned} III = w\Big\{x \in (\widetilde{\Omega})^c : |Tg(x)| > \frac{\lambda}{2}\Big\} &\le \frac{2^p}{\lambda^p}\|T(g)\|^p_{L^p(w\chi_{(\widetilde{\Omega})^c})} \\ &\le c(pp')^p(r')^{p/p'}\frac{1}{\lambda^p}\int_{\mathbb{R}^d}|g|^p M_r(w\chi_{(\widetilde{\Omega})^c})\,\mathrm{d}x = c(pp')^p(r')^{p-1}\frac{1}{\lambda}\int_{\mathbb{R}^d}|g|M_r(w\chi_{(\widetilde{\Omega})^c})\,\mathrm{d}x. \end{aligned}$$

Now, after using the definition of g, we apply the same argument as above, using (6.12) with M replaced by M_r. Then we have

$$\begin{aligned} \int_\Omega |g|M_r(w\chi_{(\widetilde{\Omega})^c})\,\mathrm{d}x &\le \sum_j \frac{1}{|Q_j|}\int_{Q_j}|f(x)|\,\mathrm{d}x \int_{Q_j} M_r(w\chi_{\mathbb{R}^n\setminus 2Q_j})(x)\,\mathrm{d}x \\ &\le c_d\sum_j \inf_{x\in Q} M_r(w\chi_{\mathbb{R}^n\setminus 2Q_j})(x)\int_{Q_j}|f(x)|\,\mathrm{d}x \le c_d\int_\Omega |f(x)|M_r w(x)\,\mathrm{d}x, \end{aligned}$$

and of course

$$\int_{\Omega^c} |g| M_r(w\chi_{(\tilde{\Omega})^c})\,\mathrm{d}x \le \int_{\Omega^c} |f| M_r w\,\mathrm{d}x.$$

Note that r is not chosen yet, and we conclude by choosing as above the exponent from Theorem 2.3,

$$r = r(w) := 1 + \frac{1}{\tau_d\,[w]'_{A_\infty}},$$

namely the sharp A_∞ reverse Hölder's exponent. We also choose

$$p = 1 + \frac{1}{\log(e+[w]'_{A_\infty})},$$

where $p < 2$ and $p' \approx \log(e+[w]'_{A_\infty})$. Then we continue with

$$\begin{aligned} w\Big\{x \in (\widetilde{\Omega})^c : |Tg(x)| > \frac{\lambda}{2}\Big\} &\le c\log(e+[w]'_{A_\infty})\frac{[w]'^{(p-1)}_{A_\infty}}{\lambda}\int_{\mathbb{R}^d} |f| 2Mw\,\mathrm{d}x. \\ &\le \frac{c[w]_{A_1}(e+\log[w]'_{A_\infty})}{\lambda}\int_{\mathbb{R}^d} |f| w\,\mathrm{d}x. \end{aligned}$$

This estimate combined with the previous ones for I and II completes the proof.

6E. ***Proof of Theorem 1.16: the dual weak case.*** We adapt here the method from [Lerner et al. 2009b], where a variant of the Calderón–Zygmund decomposition is used — namely, the Calderón–Zygmund cubes are replaced by Whitney cubes. Fix $\lambda > 0$, and set

$$\Omega_\lambda = \{x \in \mathbb{R}^d : M^c_w(f/w)(x) > \lambda\},$$

where M^c_w denotes the weighted centered maximal function. Let $\bigcup_j Q_j$ be the Whitney covering of Ω_λ, and set the Calderón–Zygmund decomposition $f = g + b$ with respect to these cubes: the "good part" is defined by

$$g = \sum_j f_{Q_j}\chi_{Q_j}(x) + f(x)\chi_{\Omega^c}(x),$$

and then the "bad part" b is given by

$$b = \sum_j b_j,$$

where

$$b_j(x) = (f(x) - \langle f\rangle_{Q_j})\chi_{Q_j}(x).$$

By the classical Besicovitch lemma, we have

$$w(\Omega_\lambda) \le \frac{c_n}{\lambda}\|f\|_{L^1(\mathbb{R}^d)}.$$

Hence, we have to estimate

$$w\Big\{x \notin \Omega_\lambda : \frac{|Tf(x)|}{w(x)} > \lambda\Big\} \le w\Big\{x \notin \Omega_\lambda : \frac{|Tb(x)|}{w(x)} > \frac{\lambda}{2}\Big\} + w\Big\{x \notin \Omega_\lambda : \frac{|Tg(x)|}{w(x)} > \frac{\lambda}{2}\Big\} =: I_1 + I_2.$$

By using (6.11) again, with $w=1$, we obtain

$$I_1 \le \frac{2}{\lambda}\int_{\mathbb{R}^d\setminus\Omega_\lambda}|Tb(x)|\,\mathrm{d}x \le \frac{c}{\lambda}\sum_j\int_{Q_j}|f-\langle f\rangle_{Q_j}|\,\mathrm{d}x \le \frac{c}{\lambda}\|f\|_{L^1(\mathbb{R}^d)},$$

where $c=c_{d,T}$.

To estimate I_2, we will use the dual version of Lemma 6.1, namely

$$\|Tf\|_{L^{p'}((M_rw)^{1-p'})} \le cpp'(r')^{1/p'}\|f\|_{L^{p'}((w)^{1-p'}))}. \tag{6.13}$$

As before, we use Theorem 2.3 with

$$r=r(w):=1+\frac{1}{\tau_d\,[w]'_{A_\infty}}$$

such that

$$\left(\fint_Q w^r\right)^{1/r} \le 2\fint_Q w.$$

Then $M_rw\le 2Mw\le 2[w]_{A_1}w$, where, as usual, $M_rw=M(w^r)^{1/r}$. Combining the Chebyshev inequality with (6.13) for a value of $p\in(1,\infty)$ to be chosen soon, we have

$$\begin{aligned}
I_2 &\le \frac{2^{p'}}{\lambda^{p'}}\int_{\mathbb{R}^d}|Tg|^{p'}w^{1-p'}\,\mathrm{d}x \le \frac{4^{p'}[w]_{A_1}^{p'-1}}{\lambda^{p'}}\int_{\mathbb{R}^d}|Tg|^{p'}M_rw^{1-p'}\,\mathrm{d}x\\
&\le (cpp')^{p'}r'\frac{[w]_{A_1}^{p'-1}}{\lambda^{p'}}\int_{\mathbb{R}^d}|g|^{p'}w^{1-p'}\,\mathrm{d}x\\
&\le (cp'p)^{p'}r'\frac{[w]_{A_1}^{p'-1}}{\lambda^{p'}}\left(\int_{\mathbb{R}^d\setminus\Omega_\lambda}|f|^{p'}w^{1-p'}\,\mathrm{d}x+\sum_j(\langle|f|\rangle_{Q_j})^{p'}\int_{Q_j}w^{1-p'}\,\mathrm{d}x\right).
\end{aligned}$$

We have that $|f|\le\lambda w$ almost everywhere in $\mathbb{R}^d\setminus\Omega_\lambda$, and hence

$$\int_{\mathbb{R}^d\setminus\Omega_\lambda}|f|^{p'}w^{1-p'}\,\mathrm{d}x \le \lambda^{p'-1}\,\|f\|_{L^1(\mathbb{R}^d)}.$$

Next, following again [Lerner et al. 2009b], by properties of the Whitney covering, it is easy to see that for any cube Q_j there exists a cube Q_j^* such that $Q_j\subset Q_j^*$, $|Q_j^*|\le c_n|Q_j|$, and the center of Q_j^* lies outside of Ω_λ. Therefore,

$$\begin{aligned}
(\langle|f|\rangle_{Q_j})^{p'-1}\int_{Q_j}w^{1-p'}\,\mathrm{d}x &\le [w]_{A_1}^{p'-1}(\langle|f|\rangle_{Q_j})^{p'-1}\int_{Q_j}(Mw)^{1-p'}\,\mathrm{d}x\\
&\le [w]_{A_1}^{p'-1}|Q_j|\left(\frac{c\langle|f|\rangle_{Q_j^*}}{\langle w\rangle_{Q_j^*}}\right)^{p'-1} \le (c\lambda[w]_{A_1})^{p'-1}|Q_j|,
\end{aligned}$$

which gives

$$\sum_j(\langle|f|\rangle_{Q_j})^{p'}\int_{Q_j}w^{1-p'}\,\mathrm{d}x \le (c\lambda[w]_{A_1})^{p'-1}\sum_j\langle|f|\rangle_{Q_j}|Q_j| \le (c\lambda[w]_{A_1})^{p'-1}\|f\|_{L^1(\mathbb{R}^d)}.$$

Combining the previous estimates and recalling that $r'\approx[w]'_{A_\infty}$, we obtain

$$I_2 \le c^{p'}[w]'_{A_\infty} p(p')^{p'} \frac{p^{p'-1}[w]_{A_1}^{2(p'-1)}}{\lambda} \|f\|_{L^1(\mathbb{R}^d)},$$

and choosing now p such that $p' = 1 + \dfrac{1}{\log(e+[w]_{A_1})} \le 2$, we get

$$I_2 \le \frac{c[w]'_{A_\infty} \log(e+[w]_{A_1})}{\lambda} \|f\|_{L^1(\mathbb{R}^d)}.$$

This, along with estimates for I_1 and for $w(\Omega_\lambda)$, completes the proof of Theorem 1.16.

7. Commutators, proof of Theorem 1.17 and its consequences

For the proof, we need a sharp version of the John–Nirenberg theorem, which can be essentially found in [Journé 1983, pp. 31–32].

Lemma 7.1 (sharp John–Nirenberg theorem). *There are dimensional constants $0 \le \alpha_d < 1 < \beta_d$ such that*

$$\sup_Q \frac{1}{|Q|} \int_Q \exp\Big(\frac{\alpha_d}{\|b\|_{\mathrm{BMO}}} |b(y) - \langle b\rangle_Q|\Big)\, \mathrm{d}y \le \beta_d. \tag{7.2}$$

In fact, we can take $\alpha_d = 1/2^{d+2}$.

A key consequence of this lemma for the present purposes is that $e^{\operatorname{Re} z\, b} w$ inherits the good weight properties of w when the complex number z is small enough. More precisely, for the A_2 constant, we have:

Lemma 7.3. *There are dimensional constants ϵ_d and c_d such that*

$$[e^{\operatorname{Re} z\, b} w]_{A_2} \le c_d [w]_{A_2} \quad \textit{if } |z| \le \frac{\epsilon_d}{\|b\|_{\mathrm{BMO}}\big([w]'_{A_\infty} + [w^{-1}]_{A_\infty}\big)}.$$

Proof. From the reverse Hölder inequality with exponent $r = 1 + 1/(\tau_d\, [w]'_{A_\infty})$, and the John–Nirenberg inequality, we have for an arbitrary Q:

$$\begin{aligned} \fint_Q w e^{\operatorname{Re} z\, b} &\le \Big(\fint_Q w^r\Big)^{1/r} \Big(\fint_Q e^{r' \operatorname{Re} z\, (b - \langle b\rangle_Q)}\Big)^{1/r'} e^{\operatorname{Re} z \langle b\rangle_Q} \\ &\le \Big(2 \fint_Q w\Big) \cdot \beta_d \cdot e^{\operatorname{Re} z \langle b\rangle_Q}, \quad \text{if } |z| \le \frac{\epsilon_d}{\|b\|_{\mathrm{BMO}}[w]'_{A_\infty}}. \end{aligned}$$

By symmetry, we also have

$$\fint_Q w^{-1} e^{-\operatorname{Re} z\, b} \le 2\beta_d \Big(\fint_Q w^{-1}\Big) e^{-\operatorname{Re} z \langle b\rangle_Q} \quad \text{if } |z| \le \frac{\epsilon_d}{\|b\|_{\mathrm{BMO}}[w^{-1}]'_{A_\infty}}.$$

Multiplication of the two estimates gives

$$\Big(\fint_Q w e^{\operatorname{Re} z\, b}\Big)\Big(\fint_Q w^{-1} e^{-\operatorname{Re} z\, b}\Big) \le 4\beta_d^2 [w]_{A_2},$$

for all z as in the assertion, and completes the proof. $\square$

There is an analogous statement for the A_∞ constant $[\ ]'_{A_\infty}$. (A similar result for $[\ ]_{A_\infty}$ is also true, and easier, but we will have no need for it, and it is therefore left as an exercise for the reader.)

Lemma 7.4. *There are dimensional constants ϵ_d and c_d such that*

$$[e^{\operatorname{Re} z\, b} w]'_{A_\infty} \le c_d [w]'_{A_\infty} \quad \textit{if} \quad |z| \le \frac{\epsilon_d}{\|b\|_{BMO}[w]'_{A_\infty}}.$$

Proof. We know that w satisfies the reverse Hölder inequality $\left(\fint_Q w^{1+3\delta}\right)^{1/(1+3\delta)} \le 2 \fint_Q w$ with a constant $\delta = c_d/[w]'_{A_\infty} < 2^{-1}$, where c_d is a small dimensional constant. We will prove that $e^{\operatorname{Re} z\, b} w$ satisfies a reverse Hölder estimate

$$\left(\fint_Q (e^{\operatorname{Re} z\, b} w)^{1+\delta}\right)^{1/(1+\delta)} \le C_d \fint_Q e^{\operatorname{Re} z\, b} w, \tag{7.5}$$

for all z as in the assertion. By part (b) of Theorem 2.3, this shows that

$$[e^{\operatorname{Re} z\, b} w]'_{A_\infty} \le 2C_d/\delta \le c_d [w]'_{A_\infty},$$

To prove (7.5), we first have

$$\begin{aligned}\left(\fint_Q (e^{\operatorname{Re} z b} w)^{1+\delta}\right)^{1/(1+\delta)} &= e^{\operatorname{Re} z \langle b\rangle_Q} \left(\fint_Q (e^{\operatorname{Re} z (b-\langle b\rangle_Q)} w)^{1+\delta}\right)^{1/(1+\delta)} \\ &\le e^{\operatorname{Re} z \langle b\rangle_Q} \left(\fint_Q e^{\operatorname{Re} z (b-\langle b\rangle_Q)(1+\delta)^2/\delta}\right)^{\delta/(1+\delta)^2} \left(\fint_Q w^{(1+\delta)^2}\right)^{1/(1+\delta)^2},\end{aligned}$$

where we applied Hölder's inequality with exponents $(1+\delta)/\delta$ and $1+\delta$. Now

$$(1+\delta)^2 = 1 + 2\delta + \delta^2 \le 1 + 3\delta,$$

and hence the last factor is bounded by $2 \fint_Q w$. Moreover, by Lemma 7.1, we have

$$\fint_Q e^{\operatorname{Re} z (b-\langle b\rangle_Q)(1+\delta)^2/\delta} \le \beta_d \quad \text{if } |z| \le \frac{\alpha_d \delta}{4\|b\|_{\mathrm{BMO}}}.$$

So altogether,

$$\left(\fint_Q (e^{\operatorname{Re} z\, b} w)^{1+\delta}\right)^{1/(1+\delta)} \le e^{\operatorname{Re} z \langle b\rangle_Q} \cdot \beta_d \cdot 2 \fint_Q w, \tag{7.6}$$

and we concentrate on the last factor. We observe that

$$\left(\fint_Q w\right)^2 = \left(\fint_Q w^{(1+\delta)/2} w^{(1-\delta)/2}\right)^2 \le \left(\fint_Q w^{1+\delta}\right)\left(\fint_Q w^{1-\delta}\right) \le \left(2\fint_Q w\right)^{1+\delta}\left(\fint_Q w^{1-\delta}\right),$$

and hence

$$\begin{aligned}\fint_Q w \le 2^{(1+\delta)/(1-\delta)} \left(\fint_Q w^{1-\delta}\right)^{1/(1-\delta)} &\le 8\left(\fint_Q w^{1-\delta} e^{\operatorname{Re} z\, b(1-\delta)} e^{-\operatorname{Re} z\, b(1-\delta)}\right)^{1/(1-\delta)} \\ &\le 8\left(\fint_Q w e^{\operatorname{Re} z\, b}\right)\left(\fint_Q e^{-\operatorname{Re} z\, b(1-\delta)/\delta}\right)^{\delta/(1-\delta)},\end{aligned}$$

where we used Hölder's inequality with exponents $1/(1-\delta)$ and $1/\delta$.

Combining with (7.6), we have shown that

$$\begin{aligned}\Big(\fint_Q (e^{\operatorname{Re} z\, b} w)^{1+\delta}\Big)^{1/(1+\delta)} &\le e^{\operatorname{Re} z\langle b\rangle_Q}\cdot\beta_d\cdot 16\Big(\fint_Q w e^{\operatorname{Re} z\, b}\Big)\Big(\fint_Q e^{-\operatorname{Re} z\, b(1-\delta)/\delta}\Big)^{\delta/(1-\delta)}\\ &= 16\beta_d\cdot\Big(\fint_Q w e^{\operatorname{Re} z\, b}\Big)\Big(\fint_Q e^{-\operatorname{Re} z(b-\langle b\rangle_Q)(1-\delta)/\delta}\Big)^{\delta/(1-\delta)}\\ &\le 16\beta_d\cdot\Big(\fint_Q w e^{\operatorname{Re} z\, b}\Big)\cdot\beta_d,\end{aligned}$$

provided that $|z|\le\alpha_d\delta/\|b\|_{\mathrm{BMO}}$ in the last step. Altogether, we have proven (7.5) with $C_d=16\beta_d^2$, under the condition that $|z|\le\alpha_d\delta/(4\|b\|_{\mathrm{BMO}})$, and this completes the proof. □

Proof of Theorem 1.17. The proof is a revised version of that of [Chung et al. 2012], following the second proof in the classical L^p theorem for commutators that can be found in [Coifman et al. 1976]. Indeed, we begin by considering the "conjugate" of the operator given by

$$T_z(f)=e^{zb}T(e^{-zb}f),$$

where z is any complex number. Then a computation gives (for instance for "nice" functions)

$$[b,T](f)=\frac{\mathrm{d}}{\mathrm{d}z}T_z(f)|_{z=0}=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{T_z(f)}{z^2}\,\mathrm{d}z,\qquad \epsilon>0,$$

by the Cauchy integral theorem. Now, by Minkowski's inequality

$$\big\|[b,T](f)\big\|_{L^2(w)}\le\frac{1}{2\pi\epsilon^2}\int_{|z|=\epsilon}\|T_z(f)\|_{L^2(w)}|\mathrm{d}z|,\qquad\epsilon>0,\tag{7.7}$$

all we need to do is estimate $\|T_z(f)\|_{L^2(w)}=\big\|T(e^{-zb}f)\big\|_{L^2(e^{2\operatorname{Re} z\, b}w)}$, for $|z|=\epsilon$ with appropriate ϵ. By the main hypothesis of the theorem, we have

$$\big\|T(e^{-zb}f)\big\|_{L^2(w)}\le\varphi\Big(\big[e^{2\operatorname{Re} z\, b}w\big]_{A_2},\big[e^{2\operatorname{Re} z\, b}w\big]'_{A_\infty},\big[e^{2\operatorname{Re} z\, b}\sigma\big]'_{A_\infty}\Big)\|e^{-zb}f\|_{L^2(e^{2\operatorname{Re} z\, b}w)},$$

where $\|e^{-zb}f\|_{L^2(e^{2\operatorname{Re} z\, b}w)}=\|f\|_{L^2(w)}$.

By Lemmas 7.3 and 7.4 (the latter applied to both w and w^{-1}), we have

$$\big[we^{2\operatorname{Re} b\, z}\big]_{A_2}\le C_d[w]_{A_2},\quad \big[we^{2\operatorname{Re} b\, z}\big]'_{A_\infty}\le C_d[w]'_{A_\infty},\quad \big[w^{-1}e^{-2\operatorname{Re} b\, z}\big]'_{A_\infty}\le C_d[w^{-1}]'_{A_\infty},$$

provided that

$$|z|=\epsilon\le\frac{\epsilon_d}{\|b\|_{\mathrm{BMO}}\big([w]'_{A_\infty}+[w^{-1}]'_{A_\infty}\big)}.$$

Using this radius and the above estimates in (7.7), we obtain

$$\begin{aligned}\big\|[b,T](f)\big\|_{L^2(w)}&\le\frac{1}{2\pi\epsilon^2}\int_{|z|=\epsilon}\varphi\big(C_d[w]_{A_2},C_d[w]'_{A_\infty},C_d[w^{-1}]'_{A_\infty}\big)\|f\|_{L^2(w)}|\mathrm{d}z|\\ &\le C_d\|b\|_{\mathrm{BMO}}\big([w]'_{A_\infty}+[w^{-1}]'_{A_\infty}\big)\times\varphi\big(C_d[w]_{A_2},C_d[w]'_{A_\infty},C_d[w^{-1}]'_{A_\infty}\big)\|f\|_{L^2(w)}.\end{aligned}$$

This concludes the proof of the main part of the theorem. The estimate for T_b^k is deduced by iterating from the case $k = 1$. $\square$

8. Examples

We compare our new estimates with earlier quantitative results by means of some examples.

8A. ***Power weights and the maximal inequality.*** Let $d = 1$ and $p \in (1, \infty)$ be fixed; we do not pay attention to the dependence of multiplicative constants on p. For $w(x) = |x|^\alpha$ and $-1 < \alpha < p-1$, one easily checks that

$$[w]_{A_p} \eqsim \frac{1}{1+\alpha} \cdot \frac{1}{((p-1)-\alpha)^{p-1}}, \quad [w]_{A_\infty} \eqsim \frac{1}{1+\alpha}, \quad \big[w^{-1/(p-1)}\big]_{A_\infty} \eqsim \frac{1}{(p-1)-\alpha};$$

moreover, the functionals $[\ ]_{A_\infty}$ and $[\ ]'_{A_\infty}$ are comparable for these weights.

Letting $\alpha \to -1$ or $\alpha \to p-1$, this shows that we have power weights with $[w]_{A_p} = t \gg 1$ and either $[w]_{A_\infty} \eqsim t$ and $\big[w^{-1/(p-1)}\big]_{A_\infty} \eqsim 1$, or $[w]_{A_\infty} \eqsim 1$ and $\big[w^{-1/(p-1)}\big]_{A_\infty} \eqsim t^{1/(p-1)}$.

With $[w]_{A_p} \eqsim [w]_{A_\infty} \eqsim t \gg 1$ and $\big[w^{-1/(p-1)}\big]_{A_\infty} \eqsim 1$, our maximal estimate

$$\|M\|_{\mathscr{B}(L^p(w))} \lesssim \big([w]_{A_p}\big[w^{-1/(p-1)}\big]_{A_\infty}\big)^{1/p} \eqsim t^{1/p}$$

clearly improves on Buckley's bound

$$\|M\|_{\mathscr{B}(L^p(w))} \lesssim [w]_{A_p}^{1/(p-1)} \eqsim t^{1/(p-1)}.$$

Despite this improvement over earlier estimates, our bounds fail to provide a two-sided estimate for the norm of the maximal operator: A. Lerner and S. Ombrosi (personal communication, 2008) have constructed a family of weights which shows that

$$\inf_{w \in A_2} \frac{\|M\|_{\mathscr{B}(L^2(w))}}{\big([w]_{A_2}[w^{-1}]'_{A_\infty}\big)^{1/2}} = 0.$$

The weights of their example are products of power weights and the two-valued weights considered in the next subsection.

8B. ***Two-valued weights and Calderón–Zygmund operators.*** The estimates for the Muckenhoupt constants of power weights in the previous subsection show that

$$[w]_{A_2} \eqsim [w]_{A_\infty} + [w^{-1}]_{A_\infty} \eqsim [w]'_{A_\infty} + [w^{-1}]'_{A_\infty} \quad \text{for } w(x) = |x|^\alpha \text{ and } d = 1,$$

so the improvement of our bound

$$\|T\|_{\mathscr{B}(L^2(w))} \lesssim [w]_{A_2}^{1/2}\big([w]'_{A_\infty} + [\sigma]'_{A_\infty}\big)^{1/2}$$

over $\|T\|_{\mathscr{B}(L^2(w))} \lesssim [w]_{A_2}$ is invisible to such weights.

However, the difference can be observed with weights of the form $w = t \cdot \chi_E + \chi_{\mathbb{R}\setminus E}$, where $t > 0$ and $E \subset \mathbb{R}$ is a measurable set, so that both E and $\mathbb{R} \setminus E$ have positive Lebesgue measure. As I ranges over all intervals of $\mathbb{R}$, the ratio $|E \cap I|/|I|$ ranges (at least) over all values $\alpha \in (0, 1)$, and hence

$$[w]_{A_2} = \sup_{\alpha\in(0,1)} (\alpha t + 1-\alpha)(\alpha t^{-1} + 1 - \alpha) = \frac{(t+1)^2}{4t},$$

$$[w]_{A_\infty} = \sup_{\alpha\in(0,1)} f(\alpha), \quad \text{with } f(\alpha) := (\alpha t + 1 - \alpha)e^{-\alpha \log t}.$$

Now $f'(\alpha) = 0$ at the unique point $\hat{\alpha} = \dfrac{1}{\log t} - \dfrac{1}{t-1} \in (0,\,1)$, and so

$$[w]_{A_\infty} = f(\hat{\alpha}) = e^{-1}\frac{t-1}{\log t}\exp\frac{\log t}{t-1} \eqsim \begin{cases} \dfrac{t}{\log t} & \text{if } t \gg 1, \\ \dfrac{t^{-1}}{\log t^{-1}} & \text{if } 0 < t \ll 1. \end{cases}$$

Assume that $t \gg 1$, so $[w]_{A_\infty} \eqsim t/\log t$. Since σ is a weight of the same form with $t^{-1} \ll 1$ in place of t, we also have $[\sigma]_{A_\infty} \eqsim t/\log t$. Thus

$$[w]_{A_2} \eqsim t, \qquad \|T\|_{\mathscr{B}(L^2(w))} \lesssim [w]_{A_2}^{1/2}\big([w]_{A_\infty} + [\sigma]_{A_\infty}\big)^{1/2} \eqsim \frac{t}{\sqrt{\log t}}.$$

In particular, these estimates already show that

$$\inf_{w\in A_2} \frac{\|T\|_{\mathscr{B}(L^2(w))}}{[w]_{A_2}} = 0.$$

If we use the sharper version of our A_2 theorem with the weight constants $[\ ]'_{A_\infty}$ instead, we find that $\|T\|_{\mathscr{B}(L^2(w))}$ can actually grow much slower than $[w]_{A_2}$:

Lemma 8.1. *For $w = t\cdot\chi_E + \chi_{\mathbb{R}\setminus E}$ and $t \geq 3$, we have $[w]'_{A_\infty} \leq 4\log t$.*

With the earlier estimate for $[w]_{A_\infty}$, this shows that $[w]_{A_\infty}$ can be exponentially larger than $[w]'_{A_\infty}$. In fact, Lemma 3.11 of [Beznosova and Reznikov 2011] implies the even more surprising possibility that $[w]'_{A_\infty} \lesssim \log\log[w]_{A_\infty}$, which is also sharp, in that the converse always holds [ibid., Theorem 1.2]; however, the example there consists of the power weights $w(x) = |x|^t$ with $t \to \infty$, which fall outside A_2 as soon as $t \geq 1$, so they are not directly relevant for the present discussion of sharp A_2 bounds.

Proof. Note that

$$\begin{aligned}\chi_I M(w\chi_I) &= \chi_I \sup_{J\subseteq I} \chi_J \fint_J w = \chi_I \sup_{J\subseteq I}\chi_J \frac{1}{|J|}\big(|J\setminus E| + t|J\cap E|\big) \\ &= \chi_I \sup_{J\subseteq I}\chi_J\Big(1 + (t-1)\frac{|J\cap E|}{|J|}\Big) = \chi_I\big(1 + (t-1)M(\chi_{I\cap E})\big),\end{aligned}$$

and hence, with the abbreviations $\tau := t-1$ and $a := \frac{|I\cap E|}{|I|}$,

$$\begin{aligned}\int_I M(w\chi_I) &= |I| + \tau\int_I M(\chi_{I\cap E}) = |I| + \tau\int_0^1 \big|I\cap\{M(\chi_{I\cap E}) > \lambda\}\big|\,\mathrm{d}\lambda \\ &\leq |I| + \tau\Big(\int_0^a |I|\,\mathrm{d}\lambda + \int_a^1 \frac{2}{\lambda}|I\cap E|\,\mathrm{d}\lambda\Big) = |I| + \tau\Big(a|I| + 2|I\cap E|\log\frac{1}{a}\Big) \\ &= |I| + \tau|I\cap E|\Big(1 + 2\log\frac{|I|}{|I\cap E|}\Big),\end{aligned}$$

where the factor 2 is the weak-type (1, 1) norm of the maximal operator on the real line. Since $w(I) = |I| + \tau|I \cap E|$, we have

$$[w]'_{A_\infty} = \sup_I \frac{1}{w(I)} \int_I M(w\chi_I) \leq \sup_{\alpha\in(0,1)} \frac{1+\tau\alpha(1+2\log\alpha^{-1})}{1+\tau\alpha} = 1+2\sup_{\alpha\in(0,1)} \frac{\tau\alpha}{1+\tau\alpha}\log\frac{1}{\alpha}, \tag{8.2}$$

recalling that the ratio $|I\cap E|/|I|$ attains at least all values $\alpha \in (0, 1)$ as I ranges over all intervals.

If $\alpha \geq \tau^{-1}$, then $\log\alpha^{-1} \leq \log\tau$, while $\tau\alpha/(1+\tau\alpha) \leq 1$. If $\alpha \leq \tau^{-1}$, then

$$\tau\alpha\log\frac{1}{\alpha} = \tau\alpha\log\frac{1}{\tau\alpha} + \tau\alpha\log\tau \leq \frac{1}{e} + \log\tau,$$

as $x\log x^{-1} \leq e^{-1}$ and $x \leq 1$ for $x = \tau\alpha \in (0, 1)$. Altogether, recalling that $t = \tau+1 \geq 3$, we have

$$[w]'_{A_\infty} \leq 1+2\Big(\frac{1}{e}+\log\tau\Big) \leq \Big(1+\frac{2}{e}\Big)+2\log t \leq 4\log t. \qquad \square$$

Since $\sigma = w^{-1}$ is a weight of the same form, we find that for these particular weights,

$$[w]_{A_2} \eqsim t, \quad \|T\|_{\mathscr{B}(L^2(w))} \lesssim [w]_{A_2}^{1/2}\big([w]'_{A_\infty}+[\sigma]'_{A_\infty}\big)^{1/2} \lesssim (t\log t)^{1/2},$$

so indeed $\|T\|_{\mathscr{B}(L^2(w))}$ can grow much slower than $[w]_{A_2}$ for such particular families of weights. This example also motivates the use of the A_∞ constants $[w]'_{A_\infty}$, rather than $[w]_{A_\infty}$, whenever this is possible.

In a similar way, we can show that the main result from Theorem 1.14 strictly improves on the earlier estimate (1.12). Indeed, if we let w be the previous weight with $t \gg 1$ so that $[w]_{A_1} \eqsim t$ and $[w]'_{A_\infty} \eqsim \log t$, then

$$[w]_{A_1}^{1/p}[w]_{A_\infty}^{1/p'} \eqsim t^{1/p}(\log t)^{1/p'}.$$

As above, this family of weights shows that

$$\inf_{w\in A_1} \frac{\|T\|_{\mathscr{B}(L^p(w))}}{[w]_{A_1}} = 0, \quad 1 < p < \infty.$$

8C. *Two-valued weights and dyadic shifts.* Although it was not stated explicitly above, from the proof it is clear that our weighted bound for the dyadic shifts only depends on the dyadic Muckenhoupt constants, where the supremum is over dyadic cubes only, instead of all cubes. This makes a difference for the two-valued weights $w = t\cdot\chi_E + \chi_{\mathbb{R}\setminus E}$ considered above, when the set E is appropriately chosen. Indeed, with $E := \bigcup_{k\in\mathbb{Z}}[2k, 2k+1)$, one observes that the ratio $|E\cap I|/|I|$ only attains the values $0, \frac{1}{2}, 1$ as I ranges over the dyadic intervals. Consequently, the dyadic A_∞ constant has a different expression:

$$[w]^d_{A_\infty} = \max_{\alpha\in\{0,1/2,1\}} (\alpha t+1-\alpha)e^{-\alpha t} = \frac{t+1}{2\sqrt{t}} = ([w]^d_{A_2})^{1/2},$$

where $[w]^d_{A_2} = [w]_{A_2}$, as one easily observes. Repeating the proof of Lemma 8.1 in the dyadic case (recalling that the weak-type (1, 1) norm is $C_d = 1$ for the dyadic maximal operator), we get in place of (8.2) that

$$[w]'^{,d}_{A_\infty} \leq 1+\sup_{\alpha\in\{0,1/2,1\}} \frac{\tau\alpha}{1+\tau\alpha}\log\frac{1}{\alpha} = 1+\frac{\frac{1}{2}\tau}{1+\frac{1}{2}\tau}\log 2 \leq 1+\log 2.$$

So these constants are actually uniformly bounded over the choice of the parameter t.

By symmetry, we also have $[w^{-1}]^d_{A_\infty} = [w]^d_{A_\infty}$ and $[w^{-1}]'^{,d}_{A_\infty} = [w]'^{,d}_{A_\infty}$, and hence, for this particular E and $w = t\cdot\chi_E + \chi_{\mathbb{R}\setminus E}$,

$$\|\text{Ш}\|_{\mathscr{B}(L^2(w))} \lesssim (r+1)^2\big([w]^d_{A_2}\big)^{1/2}\big([w]'^{,d}_{A_\infty} + [w^{-1}]'^{,d}_{A_\infty}\big)^{1/2} \lesssim (r+1)^2\big([w]^d_{A_2}\big)^{1/2}.$$

Hruščev's A_∞ constants $[\ ]^d_{A_\infty}$ would have given the weaker bound $\|\text{Ш}\|_{\mathscr{B}(L^2(w))} \lesssim (r+1)^2\big([w]^d_{A_2}\big)^{3/4}$, instead.

8D. ***The extrapolated bounds for Calderón–Zygmund operators.*** It is interesting to compare our estimate (1.28), namely

$$\begin{aligned}\|T\|_{\mathscr{B}(L^p(w))} &\lesssim [w]_{A_p}^{2/p-1/[2(p-1)]}\big([w]_{A_\infty}^{1/[2(p-1)]} + [\sigma]_{A_\infty}^{1/2}\big)([w]'_{A_\infty})^{1-2/p} \\ &\lesssim [w]_{A_p}^{2/p}([w]'_{A_\infty})^{1-2/p},\end{aligned} \tag{8.3}$$

which is valid for any Calderón–Zygmund operator and for all $p \geq 2$, with an estimate implicitly contained in the proof of a related result by Lerner [2011, Theorem 1.2]. He considers maximal truncations T_* of convolution-type Calderón–Zygmund operators, and obtains the bound

$$\|T_*\|_{\mathscr{B}(L^p(w))} \lesssim [w]_{A_p}^{1/2}([w]'_{A_\infty})^{1/2} + \|M\|_{\mathscr{B}(L^p(w))} \lesssim [w]_{A_p}^{1/2}([w]'_{A_\infty})^{1/2}, \quad p \in [3,\infty), \tag{8.4}$$

where the second estimate is an application of Buckley's result (we do not even need our improvement at this point),

$$\|M\|_{\mathscr{B}(L^p(w))} \lesssim [w]_{A_p}^{1/(p-1)} \leq [w]_{A_p}^{1/2}, \quad p \in [3,\infty).$$

In (8.4), the factor $([w]'_{A_\infty})^{1/2}$ comes from an estimate of Wilson [1989] relating the weighted norms of the grand maximal function and a certain square function, while $[w]_{A_p}^{1/2}$ is Lerner's bound for the weighted norm of such square functions (whose exponent is optimal by [Cruz-Uribe et al. 2012]).

To simplify comparison, let us only consider the simpler form of our bound (8.3). Then the sum of the powers of $[w]_{A_p}$ and $[w]'_{A_\infty}$ in both (8.3) and (8.4) is $2/p + (1-2/p) = \frac{1}{2}+\frac{1}{2} = 1$, and the sharper bound is the one where the larger weight constant $[w]_{A_p}$ has the smaller power. We have $2/p \leq \frac{1}{2}$ if and only if $p \geq 4$, and hence Lerner's bound is sharper for $p \in [3,4)$ and ours for $p \in (4,\infty)$. This indicates that the present results are not the last word on joint A_p-A_∞-control, but there is place for further investigation (which indeed has already taken place since the first public distribution of this paper; see [Hytönen et al. 2011, Section 12; Lacey 2012; Hytönen and Lacey 2011]).

9. Proof of the end-point theory at $p = \infty$

The proof again relies on the sharp reverse Hölder inequality in Theorem 2.3: if $w \in A_\infty$ and if we let

$$r = r(w) := 1 + \frac{1}{c_d\,[w]'_{A_\infty}},$$

then

$$\left(\fint_Q w^r\,dx\right)^{1/r} \leq \frac{2}{|Q|}\int_Q w.$$

Proof of Theorem 1.19. For $c = \langle f\rangle_Q$,

$$\frac{1}{w(Q)}\int_Q |f-c|w = \frac{|Q|}{w(Q)}\fint_Q |f-c|w \le \frac{|Q|}{w(Q)}\Big(\fint_Q |f-c|^{r(w)'}\Big)^{1/r(w)'}\Big(\fint_Q w^{r(w)}\Big)^{1/r(w)}$$
$$\le \frac{|Q|}{w(Q)}\big(C_d r(w)'\|f\|_{\mathrm{BMO}}\big)\Big(2\fint_Q w\Big) = C_d r(w)'\|f\|_{\mathrm{BMO}} \le C_d[w]'_{A_\infty}\|f\|_{\mathrm{BMO}},$$

which shows that $\|f\|_{\mathrm{BMO}(w)} \le C_d[w]'_{A_\infty}\|f\|_{\mathrm{BMO}}$. Note that we used the sharp order of growth of the local L^p norms of BMO functions as $p\to\infty$, which follows easily from the exponential integrability.

To see the sharpness for $d=1$, consider $w(x) = |x|^{-1+\varepsilon}$, which has $[w]_{A_\infty} \eqsim [w]'_{A_\infty} \eqsim 1/\varepsilon$ and $f(x) = \log|x|$. We check that

$$\|f\|_{\mathrm{BMO}(w)} \ge \inf_a \frac{1}{w([0,1])}\int_0^1 \Big|\log\frac{1}{x} - a\Big| w(x)\,\mathrm{d}x \ge \frac{c}{\varepsilon} \ge c[w]_{A_\infty} \ge c[w]'_{A_\infty},$$

which proves the claim. It is immediate that $w([0,1]) = \displaystyle\int_0^1 x^{-1+\varepsilon}\,\mathrm{d}x = \frac{1}{\varepsilon}$. It remains to compute

$$\int_0^1 \Big|\log\frac{1}{x} - a\Big| x^{-1+\varepsilon}\,\mathrm{d}x = \int_0^\infty |t-a|e^{-\varepsilon t}\,\mathrm{d}t = \frac{1}{\varepsilon^2}\int_0^\infty |u-\varepsilon a|e^{-u}\,\mathrm{d}u.$$

It suffices to check that $\psi(\alpha) := \displaystyle\int_0^\infty |u-\alpha|e^{-u}\,\mathrm{d}u \ge c > 0$ for all $\alpha\in\mathbb{R}$. But this is an easy calculus exercise. $\square$

We now prove Corollary 1.20 on end-point estimates for Calderón–Zygmund operators.

Proof of Corollary 1.20. For the positive estimate, it suffices to factorize $T = I\circ T$, where $T : L^\infty \to \mathrm{BMO}$ and $I : \mathrm{BMO} \to \mathrm{BMO}(w)$ have norm bounds c_T and $c_d[w]'_{A_\infty}$, respectively. Concerning sharpness, note that the Hilbert transform of $\chi_{(-1,0)}$ is $\log(x+1) - \log x$ for $x>0$. Since $\log(x+1)$ is bounded on $[0,1]$, the computation proving the sharpness of the embedding $\mathrm{BMO} \hookrightarrow \mathrm{BMO}(w)$ also gives the lower bound

$$\big\|H\chi_{(-1,0)}\big\|_{\mathrm{BMO}(|x|^{-1+\varepsilon})} \ge \frac{c}{\varepsilon} = c[x^{-1+\varepsilon}]_{A_\infty}\|\chi_{(-1,0)}\|_{L^\infty} \ge c[x^{-1+\varepsilon}]'_{A_\infty}\|\chi_{(-1,0)}\|_{L^\infty}. \qquad \square$$

We conclude with the proof of Proposition 1.21 on the sharp relation of A_∞ and BMO. Note that here we use the larger constant $[w]_{A_\infty}$, not $[w]'_{A_\infty}$.

Proof of Proposition 1.21. Let Q be a cube. We estimate

$$\begin{aligned}\int_Q \big|\log w - \log c\big| &= \int_{Q\cap\{w\ge c\}} \log\frac{w}{c} + \int_{Q\cap\{w<c\}} \log\frac{c}{w} \\ &= \int_{Q\cap\{w\ge c\}} \log\frac{w}{c} + \Big(\int_Q - \int_{Q\cap\{w\ge c\}}\Big)\log\frac{c}{w} \\ &= 2\int_{Q\cap\{w\ge c\}} \log\frac{w}{c} + \int_Q \log c + \int_Q \log\frac{1}{w} \\ &\le 2\int_{Q\cap\{w\ge c\}} \frac{w}{c} + |Q|\log c + |Q|\log\Big([w]_{A_\infty}\Big/\fint_Q w\Big).\end{aligned}$$

Hence
$$\fint_Q \left|\log w - \log c\right| \le \frac{2}{c}\fint_Q w + \log c + \log[w]_{A_\infty} - \log\Big(\fint_Q w\Big).$$
Choosing $c = c_Q = 2\fint_Q w$, we get
$$\fint_Q \left|\log w - \log c_Q\right| \le 1 + \log 2 + \log\Big(\fint_Q w\Big) + \log[w]_{A_\infty} - \log\Big(\fint_Q w\Big) = \log(2e[w]_{A_\infty}),$$
and this proves that
$$\|\log w\|_{\mathrm{BMO}} \le \log(2e[w]_{A_\infty}). \qquad \square$$

Remark 9.1. In the last estimate, we cannot replace $[w]_{A_\infty}$ by $[w]'_{A_\infty}$. Indeed, for the two-valued weight $w = t\cdot 1_E + 1_{\mathbb{R}\setminus E}$, one readily checks that $\|\log w\|_{\mathrm{BMO}} \eqsim \log t$, whereas Lemma 8.1 shows that also $[w]'_{A_\infty} \lesssim \log t$. Thus $\|\log w\|_{\mathrm{BMO}} \le \log(c[w]'_{A_\infty})$ would lead to the obvious contradiction that $\log t \le c + \log\log t$.

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TUOMAS HYTÖNEN: tuomas.hytonen@helsinki.fi
Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68, FI-00014 Helsinki, Finland

CARLOS PÉREZ: carlosperez@us.es
Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad De Sevilla, 41080 Sevilla, Spain

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PERIODICITY OF THE SPECTRUM IN DIMENSION ONE

ALEX IOSEVICH AND MIHAL N. KOLOUNTZAKIS

A bounded measurable set Ω, of Lebesgue measure 1, in the real line is called *spectral* if there is a set Λ of real numbers ("frequencies") such that the exponential functions $e_\lambda(x) = \exp(2\pi i \lambda x)$, $\lambda \in \Lambda$, form a complete orthonormal system of $L^2(\Omega)$. Such a set Λ is called a *spectrum* of Ω. In this note we prove that any spectrum Λ of a bounded measurable set $\Omega \subseteq \mathbb{R}$ must be periodic.

1. Tilings, spectral sets and periodicity

Spectra of domains in Euclidean space and the Fuglede conjecture. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded measurable set and let us assume for simplicity that Ω has Lebesgue measure 1. The concept of a spectrum of Ω that we deal with in this paper may be interpreted as a way of using Fourier series for functions defined on Ω with nonstandard frequencies. It was introduced by Fuglede [1974] who was studying a problem of Segal on the extendability of the partial differential operators

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_d}$$

on $C_c(\Omega)$ to commuting operators on all of $L^2(\Omega)$.

Definition 1.1. A set $\Lambda \subseteq \mathbb{R}^d$ is called a *spectrum* of Ω (and Ω is said to be a *spectral set*) if the set of exponentials

$$E(\Lambda) = \{e_\lambda(x) = e^{2\pi i \lambda \cdot x} : \lambda \in \Lambda\}$$

is a complete orthonormal set in $L^2(\Omega)$.

(The inner product in $L^2(\Omega)$ is $\langle f, g\rangle = \int_\Omega f \overline{g}$.)

It is an easy result (see [Kolountzakis 2004], for instance) that the orthogonality of $E(\Lambda)$ is equivalent to the *packing condition*

$$\sum_{\lambda \in \Lambda} \left|\widehat{\chi_\Omega}\right|^2 (x - \lambda) \le 1, \quad \text{a.e. } (x), \tag{1}$$

as well as to the condition

$$\Lambda - \Lambda \subseteq \{0\} \cup \left\{\widehat{\chi_\Omega} = 0\right\}. \tag{2}$$

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The orthogonality and completeness of $E(\Lambda)$ is in turn equivalent to the *tiling condition*

$$\sum_{\lambda\in\Lambda} \left|\widehat{\chi_\Omega}\right|^2(x-\lambda) = 1, \quad \text{a.e. } (x). \tag{3}$$

These equivalent conditions follow from the identity

$$\langle e_\lambda, e_\mu\rangle = \int_\Omega e_\lambda \overline{e_\mu} = \widehat{\chi_\Omega}(\mu-\lambda) \tag{4}$$

and from the completeness of all the exponentials in $L^2(\Omega)$. Condition (1) roughly expresses the validity of Bessel's inequality for the system of exponentials $E(\Lambda)$, while condition (3) says that Bessel's inequality holds as an equality.

If Λ is a spectrum of Ω then so is any translate of Λ but there may be other spectra as well.

Example. If $Q_d = (-1/2, 1/2)^d$ is the cube of unit volume in $\mathbb{R}^d$ then $\mathbb{Z}^d$ is a spectrum of Q_d. Let us remark here that there are spectra of Q_d that are very different from translates of the lattice $\mathbb{Z}^d$ [Iosevich and Pedersen 1998; Lagarias et al. 2000; Kolountzakis 2000].

In the one-dimensional case, which will concern us in this paper, condition (2) implies that the set Λ has gaps bounded below by a positive number: the smallest positive zero of $\widehat{\chi_\Omega}$. (Note that since Ω is a bounded set, the function $\widehat{\chi_\Omega}(\xi)$ can be defined for all complex ξ and is an entire function. This guarantees that its zeros are a discrete set.)

The Fuglede or spectral set conjecture. Research on spectral sets has been driven for many years by a conjecture of Fuglede [1974] which stated that a set Ω is spectral if and only if it is a translational tile. A set Ω is a translational tile if we can translate copies of Ω around and fill space without overlaps. More precisely, there exists a set $S \subseteq \mathbb{R}^d$ such that

$$\sum_{s\in S} \chi_\Omega(x-s) = 1, \quad \text{a.e. } (x). \tag{5}$$

One can extend the definition of translational tiling to functions from sets.

Definition 1.2. We say that a nonnegative function $f:\mathbb{R}^d \to \mathbb{R}$ *tiles by translation with the set* $S \subseteq \mathbb{R}^d$ if

$$\sum_{s\in S} f(x-s) = \ell \quad \text{for almost every } x \in \mathbb{R}^d,$$

where ℓ is a constant (the *level* of the tiling).

Thus the question of spectrality for a set Ω is essentially a tiling question for the function $\left|\widehat{\chi_\Omega}\right|^2$ (the *power spectrum*). Taking into account the equivalent condition (3) one can now, more elegantly, restate the Fuglede conjecture as the equivalence

$$\chi_\Omega \text{ tiles } \mathbb{R}^d \text{ by translation at level } 1 \iff \left|\widehat{\chi_\Omega}\right|^2 \text{ tiles } \mathbb{R}^d \text{ by translation at level } 1. \tag{6}$$

In this form the conjectured equivalence is perhaps more justified. However this conjecture is now known to be false in both directions if $d \geq 3$ [Tao 2004; Matolcsi 2005; Kolountzakis and Matolcsi 2006a; 2006b;

Farkas et al. 2006; Farkas and Révész 2006], but remains open in dimensions 1 and 2 and it is not out of the question that the conjecture is true if one restricts the domain Ω to being convex. (It is known that the direction "tiling $\Rightarrow$ spectrality" is true in the case of convex domains; see [Kolountzakis 2004].) The equivalence (6) is also known, from the time of [Fuglede 1974], to be true if one adds the word *lattice* to both sides (that is, lattice tiles are the same as sets with a lattice spectrum).

Periodicity of spectra and tilings. The property of periodicity is very important for a tiling.

Definition 1.3. A set $S \subseteq \mathbb{R}^d$ is called (*fully*) *periodic* if there exists a lattice $L \subseteq \mathbb{R}^d$ (a discrete subgroup of $\mathbb{R}^d$ with d linearly independent generators: the period lattice) such that $S + t = S$ for all $t \in L$. We call a translation tiling *periodic* if the set of translations is periodic.

The so-called *periodic tiling conjecture* [Grünbaum and Shephard 1989; Lagarias and Wang 1997] should be mentioned at this point: if a set Ω tiles $\mathbb{R}^d$ by translations (at level 1) then it can also tile $\mathbb{R}^d$ by a periodic set of translations.

As an example of the importance of periodicity for a tiling we mention its connection to decidability [Robinson 1971], a question to which the study of tilings has provided several examples and problems. Although the general problem of tiling (not restricting the motions to be translations or allowing more than one tile) is undecidable, it is not hard to see that when the assumption of periodicity is added, the problem becomes decidable. Let us make this connection more clear by stating it in the discrete case:

> Assume that the *periodic tiling conjecture* is true. Then one can algorithmically decide if a given *finite* $\Omega \subseteq \mathbb{Z}^d$ admits tilings by translation or not.

Roughly, if one knows a priori that a set Ω admits periodic tilings, if it admits any, then the question "Does Ω admit a tiling?" can be answered algorithmically by simultaneously enumerating all possible counterexamples to tiling (if a tiling does not exist then the obstacle will show up at some finite stage) as well as all possible tilings of finite regions. If a tiling does not exist then the first enumeration will produce a counterexample. Otherwise, if a tiling exists then, by the periodic tiling conjecture, a periodic tiling exists and one of the finite regions that can be tiled with Ω will show this periodicity and can therefore be extended to all space. More details of this argument can be found in [Robinson 1971]. Both the periodic tiling conjecture and the question of decidability of tilings by translation are open for $d \geq 2$ (but see [Szegedy 1998; Wijshoff and van Leeuwen 1984] for some special cases). For $d = 1$ all translational tilings by finite subsets of $\mathbb{Z}$ are necessarily periodic [Newman 1977] and the problem is decidable. Another class of tilings where the periodic tiling conjecture holds is the case when Ω is assumed to be a convex polytope in $\mathbb{R}^d$, for any d [Venkov 1954; McMullen 1980].

In dimension $d = 1$ it is known [Leptin and Müller 1991; Lagarias and Wang 1996; Kolountzakis and Lagarias 1996] that all translational tilings by a bounded measurable set are necessarily periodic. More generally it is known that whenever $f \geq 0$ is an integrable function on the real line that tiles the real line by translation with a set of translates S, then S is of the form

$$S = \bigcup_{j=1}^{J} (\alpha_j \mathbb{Z} + \beta_j), \tag{7}$$

where the real numbers α_j are necessarily commensurable (and S is in that case periodic) if the tiling is indecomposable (cannot be made up by superimposing other tilings). But this result is not applicable to the periodicity of spectra, as the power-spectrum $\left|\widehat{\chi_\Omega}\right|^2$ is never of compact support when Ω is bounded (a qualitative expression of the *uncertainty principle*).

The question of periodicity of one-dimensional spectra was explicitly raised in [Łaba 2002]. It was recently proved (first in [Bose and Madan 2011] and then a simplified proof was given in [Kolountzakis 2012]) that if Ω is a finite union of intervals in the real line then any spectrum of Ω is periodic. See also [Lagarias and Wang 1997], where periodicity of spectra and of tilings plays an important role.

Theorem 1.4 [Bose and Madan 2011; Kolountzakis 2012]. *If $\Omega = \bigcup_{j=1}^n (a_j, b_j) \subseteq \mathbb{R}$ is a finite union of intervals of total length 1 and $\Lambda \subseteq \mathbb{R}$ is a spectrum of Ω, then there exists a positive integer T such that $\Lambda + T = \Lambda$.*

Our purpose in this note is to improve this result by removing the assumption that Ω is a finite union of intervals.

Theorem 1.5. *Suppose that Λ is a spectrum of $\Omega \subseteq \mathbb{R}$, where Ω is a bounded measurable set of measure 1. Then Λ is periodic and any period is a positive integer.*

The proof of Theorem 1.5 is given in Section 2.

Corollary 1.6. *If Ω, a bounded measurable set of measure 1, is spectral then Ω tiles the real line at some integer level T when translated at the locations $T^{-1}\mathbb{Z}$.*

Proof. Let Λ is a spectrum of Ω. By Theorem 1.5 we know that Λ is a periodic set and let T be one of its periods: $\Lambda + T = \Lambda$. Then we have $\Lambda = T\mathbb{Z} + \{\ell_1, \dots, \ell_T\}$ (the number of elements in each period must be T in order for Λ to have density 1, hence T is an integer), and, by (2), this implies that $\widehat{\chi_\Omega}(nT) = 0$ for all nonzero $n \in \mathbb{Z}$. Hence Ω tiles $\mathbb{R}$ when translated at $T^{-1}\mathbb{Z}$ (see [Kolountzakis 2004]) at level T. $\square$

Theorem 1.5 is not true in dimensions higher than 1. For instance, even when Ω is as simple as a cube, it may have spectra that are not periodic [Lagarias et al. 2000; Iosevich and Pedersen 1998; Kolountzakis 2000].

2. Proof of periodicity for spectra in dimension 1

The spectrum as a double sequence of symbols. Because of (2) we have that the gap between any two elements of Λ is bounded below by $\delta > 0$: the smallest positive zero of $\widehat{\chi_\Omega}$. Let us now observe that the gap between successive elements of Λ is also bounded above by a constant that depends only on Ω.

Lemma 2.1. *If $\Omega \subseteq \mathbb{R}$ is a bounded measurable set of measure 1 then there is a finite number $\Delta > 0$ such that if Λ is any spectrum of Ω then the gap between any two successive elements of Λ is at most Δ.*

Proof. Lemma 2.1 is essentially a special case of Lemma 2.3 of [Kolountzakis and Lagarias 1996]. In that lemma it is proved that if $0 \le f \in L^1(\mathbb{R})$ tiles the line with a set A,

$$\sum_{a \in A} f(x-a) = w \quad \text{for almost all } x \in \mathbb{R}, \text{ with } w > 0 \text{ a constant,}$$

then the set A has asymptotic density equal to $\rho = w/\int f$. This means that the ratio

$$|A \cap I|/|I|$$

tends to ρ as the length of the interval I tends to infinity. The convergence is *uniform* over the choice of the set A and the location of the interval I.[1] This uniformity of course implies that the maximum gap of A is bounded by a quantity that depends on f only.

Since $\sum_{\lambda\in\Lambda} |\chi_\Omega|^2(x-\lambda) = 1$ is a tiling and $0 \le |\chi_\Omega|^2 \in L^1(\mathbb{R})$ we deduce that Λ has gaps bounded above by a function of Ω alone. ☐

Let now

$$Z = \left\{\xi \in \mathbb{R} : \widehat{\chi_\Omega}(\xi) = 0\right\}$$

and define the finite set (as Z is discrete)

$$\Sigma = Z \cap (0, \Delta] = \{s_1, s_2, \dots, s_k\}, \tag{8}$$

where Δ is the quantity given by Lemma 2.1.

We now view the set Σ as a finite set of symbols (alphabet) and consider the set $\Sigma^{\mathbb{Z}}$ of all bidirectional sequences of elements of Σ equipped with the product topology. A sequence x^n of elements of $\Sigma^{\mathbb{Z}}$ converges to $x \in \Sigma^{\mathbb{Z}}$ if for all $k = 1, 2, \dots$ the double sequences x^n and x agree in the window $[-k, k]$ for large enough n. More precisely, for all $k = 1, 2, \dots$ there is n_0 such that for $n \ge n_0$ we have

$$x_j^n = x_j \quad \text{for } -k \le j \le k.$$

$\Sigma^{\mathbb{Z}}$ is a metrizable compact space so that each sequence $x^n \in \Sigma^{\mathbb{Z}}$ has a convergent subsequence. This is just another way of phrasing a diagonal argument that is somewhat more convenient to use. The proof below may of course be phrased avoiding topological notions altogether and replacing the convergence of each subsequence with a diagonal argument.

The space $\Sigma^{\mathbb{Z}}$ is the natural space in which to view a spectrum Λ of Ω, as the set Λ is locally of *finite complexity*: because of (2) the difference of any two successive elements of Λ can be only be an element of Σ. By demanding, as we may, that 0 is always in Λ we can therefore represent any set Λ with the sequence of its successive differences. More precisely, we map any set $\Lambda \subseteq \mathbb{R}$ whose successive differences are in Σ and which contains 0,

$$\Lambda = \{\dots < -\lambda_2 < -\lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots\},$$

to the element $(\Lambda_n) \in \Sigma^{\mathbb{Z}}$ given by

$$\Lambda_n = \lambda_{n+1} - \lambda_n \quad (n \in \mathbb{Z}).$$

This correspondence is a bijection and we will use one or the other form of the set Λ as it suits us.

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[1] Inequality (2.4) in [Kolountzakis and Lagarias 1996] speaks of $N_A(T) = |A \cap [-T, T]|$, but none of the other quantities that appear in it depend on A. This means that (2.4) holds even if we take $N_A(T)$ to be the number of elements of A in *any* interval of length $2T$. In fact, one can prove that $N_A(T)$ cannot be 0 if T is sufficiently large, depending on Ω, without taking the limit in (2.4) and without talking about asymptotic density.

Symbolic sequences determined by their values in a half-line. Suppose $X \subseteq \Sigma^{\mathbb{Z}}$. We say that X is determined by left half-lines if knowing an element of X to the left of any index n suffices to determine the element in the remaining positions to the right of n, i.e., if for any $x, y \in X$ and $n \in \mathbb{Z}$ we have

$$(x_i = y_i \text{ for } i \le n) \Longrightarrow (x_i = y_i \text{ for all } i \in \mathbb{Z}).$$

Determination of X by right half-lines is defined analogously.

We similarly say that X is determined by any window of size w (a positive integer) if for any $x \in X$ and any $n \in \mathbb{Z}$ knowing x_i for $i = n, n+1, \dots, n+w-1$ completely determines x.

Theorem 2.2. *Suppose $X \subseteq \Sigma^{\mathbb{Z}}$ is a closed, shift-invariant set that is determined by left half-lines and by right half-lines. Then there is a finite number w such that X is determined by windows of size w.*

Proof. It is enough to show that there is a finite window size w such that whenever two elements of X agree on a window of size w, then they necessarily agree at the first index to the right of that window. For in that case they necessarily agree at the entire right half-line to the right of the window and are by assumption equal elements of X.

Assume this is not true. Then there are elements x^n, y^n of X, $n = 1, 2, \dots$, which agree at some window of width n but disagree at the first location to the right of that window. Using the shift-invariance of X we may assume that

$$x^n_{-n} = y^n_{-n},\ x^n_{-n+1} = y^n_{-n+1},\ \dots,\ x^n_{-1} = y^n_{-1} \quad \text{and} \quad x^n_0 \ne y^n_0.$$

By the compactness of the space there are $x, y \in X$ and a subsequence (n_k) such that $x^{n_k} \to x$ and $y^{n_k} \to y$. By the meaning of convergence in the space $\Sigma^{\mathbb{Z}}$ we have that the sequences x and y agree for all negative indices and disagree at 0. This contradicts the assumption that X is determined by left half-lines. □

Theorem 2.3. *If $X \subseteq \Sigma^{\mathbb{Z}}$ is shift-invariant and is determined by windows of size w then all elements of X are periodic, and the period can be chosen to be at most $|\Sigma|^w$.*

Proof. Fix $x \in X$. Since there are at most $|\Sigma|^w$ different window-contents of length w, it follows that there are two indices $i, j \in \{0, 1, \dots, |\Sigma|^w\}$, $i < j$, such that

$$x_i = x_j,\ x_{i+1} = x_{j+1},\ \dots,\ x_{i+w-1} = x_{j+w-1}.$$

Writing Tx for the left shift of $x \in X$ (i.e., $(Tx)_n = x_{n+1}$) we have that x and $T^{j-i}x$ agree at the window $i, i+1, \dots, i+w-1$. By assumption then $x = T^{j-i}x$, which is another way of saying that the sequence x has period $j - i \le |\Sigma|^w$. □

Symbolic sequences with spectral gaps. Suppose $\Lambda \subseteq \mathbb{R}$ is a spectrum of the bounded set $\Omega \subseteq \mathbb{R}$ of measure 1. Write $\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$, where δ_λ is a unit point mass at point λ. It is well known (see [Kolountzakis 2004]) that the Fourier transform of the tempered distribution δ_Λ is supported by 0 plus the zeros of the function

$$\left(|\widehat{\chi_\Omega}|^2\right)^\wedge = \chi_\Omega * \chi_{-\Omega},$$

which is a continuous function with value 1 at the origin. Therefore there is an interval $(0,a)$, with $a=a(\Omega)>0$, such that δ_Λ has a spectral gap:

$$\operatorname{supp}\widehat{\delta_\Lambda}\cap(0,a)=\varnothing. \tag{9}$$

With $\Sigma=\Sigma(\Omega)$ defined by (8), let $X\subseteq\Sigma^{\mathbb{Z}}$ consist of all sequences which correspond to sets Λ with gaps from Σ such that (9) holds. The set X is obviously shift-invariant, as shifting a sequence in X corresponds to translation of the set Λ and translation will not affect the support of $\widehat{\delta_\Lambda}$.

Lemma 2.4. *The set X is closed in $\Sigma^{\mathbb{Z}}$.*

Proof. Suppose $\Lambda^n\in X$ and $\Lambda^n\to\Lambda\in\Sigma^{\mathbb{Z}}$ and that $\phi\in C^\infty(0,a)$. It is enough to show that $\widehat{\delta_\Lambda}(\phi)=0$, as this is what it means for $\widehat{\delta_\Lambda}$ to have no support in $(0,a)$ and therefore $\Lambda\in X$. By the definition of the Fourier transform,

$$\widehat{\delta_\Lambda}(\phi)=\delta_\Lambda(\widehat{\phi})=\sum_{\lambda\in\Lambda}\widehat{\phi}(\lambda)\overset{*}{=}\lim_{n\to\infty}\sum_{\lambda\in\Lambda^n}\widehat{\phi}(\lambda)=\lim_{n\to\infty}\delta_{\Lambda^n}(\widehat{\phi})=\lim_{n\to\infty}\widehat{\delta_{\Lambda^n}}(\phi)=0.$$

The justification for the starred equality above is very easy given the rapid decay of $\widehat{\phi}$, and the fact that all Λ^n have the same positive minimum gap. Indeed, these properties imply that for any $\epsilon>0$ we can find an $R>0$ such that

$$\Bigl|\sum_{\substack{\lambda\in\Lambda\\|\lambda|>R}}\widehat{\phi}(\lambda)\Bigr|<\epsilon\quad\text{for } L=\Lambda \text{ or } L=\Lambda^n,$$

and also an n_0 such that $\Lambda^n\cap[-R,R]=\Lambda\cap[-R,R]$ for $n\ge n_0$. It follows that for $n\ge n_0$ we have

$$\bigl|\widehat{\delta_\Lambda}(\phi)\bigr| = \bigl|\widehat{\delta_\Lambda}(\phi)-\widehat{\delta}_{\Lambda^n}(\phi)\bigr| = \Bigl|\sum_{\substack{\lambda\in\Lambda\\|\lambda|>R}}\widehat{\phi}(\lambda) - \sum_{\substack{\lambda\in\Lambda\\|\lambda|>R}}\widehat{\phi}(\lambda)\Bigr|\le 2\epsilon.$$

This implies that $\widehat{\delta_\Lambda}(\phi)=0$, as we had to show. □

Theorem 2.5. *The sequences in X are determined by both left half-lines and right half-lines.*

Proof. Suppose that X is not determined by left half-lines (the argument is similar for right half-lines). Then there are distinct $\Lambda^1,\Lambda^2\in X$ such that $\Lambda^1_i=\Lambda^2_i$ for all negative integers i. Both δ_{Λ^1} and δ_{Λ^2} have a spectral gap at $(0,a)$ and therefore so does their difference

$$\mu=\delta_{\Lambda^1}-\delta_{\Lambda^2}.$$

Notice that μ is supported in the half-line $[0,+\infty)$. Suppose $\psi\in C^\infty(-a/10,a/10)$. It follows from the rapid decay of $\widehat{\psi}$ that the measure

$$\nu=\widehat{\psi}\cdot\mu$$

is totally bounded and still has a spectral gap at the interval $(a/10,9a/10)$. But the measure ν is also supported in the half-line $[0,+\infty)$ and by the F. and M. Riesz theorem [Havin and Jöricke 1994] its

Fourier transform is mutually absolutely continuous with respect to the Lebesgue measure on the line.[2] But this is incompatible with the vanishing of $\widehat{\nu}$ in some interval. Therefore ν must be identically 0 and, since $\psi \in C^\infty(-a/10, a/10)$ is otherwise arbitrary, it follows that $\mu \equiv 0$, or $\Lambda^1 = \Lambda^2$, a contradiction. It follows that X is indeed determined by left half-lines. □

Conclusion of the argument. By Lemma 2.4 and Theorem 2.5 the set X defined above, right after (9), given Ω is a closed shift-invariant subset of $\Sigma^{\mathbb{Z}}$ and its elements are determined by half-lines. By Theorem 2.2 there exists a finite number w such that the elements of X are determined by their values at any window of width w. By Theorem 2.3 all elements of X are therefore periodic sequences. Since all spectra of Ω can also be viewed as elements of X, the periodicity of any spectrum of Ω follows from the periodicity of the sequence of its successive differences.

The fact that any period of Λ is a positive integer is a consequence of the fact that Λ has density 1: if T is a period of Λ this implies that there are exactly T elements of Λ in each interval $[x, x+T)$ hence T is an integer.

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ALEX IOSEVICH: iosevich@math.rochester.edu
Department of Mathematics, 915 Hylan Building, University of Rochester, Rochester, NY 14628, United States

MIHAL N. KOLOUNTZAKIS: kolount@math.uoc.gr
Department of Mathematics, University of Crete, Knossos Avenue, 71409 Iraklio, Greece

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A CODIMENSION-TWO STABLE MANIFOLD OF NEAR SOLITON EQUIVARIANT WAVE MAPS

IOAN BEJENARU, JOACHIM KRIEGER AND DANIEL TATARU

We consider finite-energy equivariant solutions for the wave map problem from $\mathbb{R}^{2+1}$ to $\mathbb{S}^2$ which are close to the soliton family. We prove asymptotic orbital stability for a codimension-two class of initial data which is small with respect to a stronger topology than the energy.

1. Introduction

We consider wave maps $U : \mathbb{R}^{2+1} \to \mathbb{S}^2$ which are equivariant with corotation index 1. In particular, they satisfy $U(t, \omega x) = \omega U(t, x)$ for $\omega \in SO(2, \mathbb{R})$, where the latter group acts in standard fashion on $\mathbb{R}^2$, and the action on $\mathbb{S}^2$ is induced from that on $\mathbb{R}^2$ via stereographic projection. Wave maps are characterized by being critical with respect to the functional

$$U \to \int_{\mathbb{R}^{2+1}} \langle \partial_\alpha U, \partial^\alpha U \rangle \, d\sigma, \quad \alpha = 0, 1, 2,$$

where Einstein's summation convention is in force, $\partial^\alpha = m^{\alpha\beta}\partial_\beta$, $m_{\alpha\beta} = (m^{\alpha\beta})^{-1}$ is the Minkowski metric on $\mathbb{R}^{2+1}$, and $d\sigma$ is the associated volume element. Also, $\langle \cdot , \cdot \rangle$ refers to the standard inner product on $\mathbb{R}^3$ if we use ambient coordinates to describe u, $\partial_\alpha u$, etc. Recall that the energy is preserved:

$$\mathscr{E}(u) = \frac{1}{2} \int_{\mathbb{R}^2} \langle DU(\cdot, t), DU(\cdot, t) \rangle \, dx = \text{const.}$$

The problem at hand is *energy critical*, meaning that the conserved energy is invariant under the natural re-scaling $U \to U(\lambda t, \lambda x)$.

We focus on a particular subset of equivariant maps characterized by the additional property that $U(t, r, \theta) = (u(t, r), \theta)$ in spherical coordinates, where, on the right-hand side, u stands for the longitudinal angle and θ stands for the latitudinal angle, while on the left-hand side, r, θ are the polar coordinates on $\mathbb{R}^2$. Now $u(t, r)$, a scalar function, satisfies the equation

$$-u_{tt} + u_{rr} + \frac{u_r}{r} = \frac{\sin(2u)}{2r^2}. \tag{1-1}$$

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Then the energy has the form

$$\mathcal{E}(u) = \pi \int_{\mathbb{R}^2} \left(|u_t|^2 + |u_r|^2 + \frac{\sin^2 u}{r^2} \right) r\,dr. \tag{1-2}$$

We shall be interested in corotational maps that are topologically nontrivial, that is, with

$$u(t, 0) = 0, \qquad u(t, \infty) = \pi.$$

A natural space adapted to the elliptic part of this energy is $\dot{H}^1_e$:

$$\|f\|^2_{\dot{H}^1_e} = \|\partial_r f\|^2_{L^2} + \left\| \frac{f}{r} \right\|^2_{L^2}.$$

This is the equivariant translation of the usual two-dimensional space $\dot{H}^1$. The size of the elliptic part of the energy of u in (1-2) and its $\dot{H}^1_e$ norm are comparable, provided that u is small pointwise. This is not true directly for u, but it is true after we subtract from u the "nearby" soliton that we describe below.

The solitons for (1-1) have the form

$$Q_\lambda(r) = Q(\lambda r), \qquad Q(r) = 2 \arctan r, \qquad \lambda \in \mathbb{R}_+ = (0, \infty),$$

and are global minimizers of the energy $\mathcal{E}$ within their homotopy class, $\mathcal{E}(Q_\lambda) = 4\pi$.

We consider solutions u which are close to the soliton in the sense that

$$\mathcal{E}(u) - \mathcal{E}(Q) \ll 1. \tag{1-3}$$

As it turns out, such solutions must stay close to the soliton family $\{Q_\lambda\}$, due to the bound

$$\inf_\lambda \|(u(t) - Q_\lambda)\|^2_{\dot{H}^1_e} + \|u_t(t)\|^2_{L^2} \sim \mathcal{E}(u) - \mathcal{E}(Q). \tag{1-4}$$

Indeed, this follows, for example, from [Cote 2005]. Thus at any given t, one can choose some $\lambda(t)$ such that

$$\|(u(t) - Q_\lambda)\|^2_{\dot{H}^1_e} + \|u_t(t)\|^2_{L^2} \sim \mathcal{E}(u) - \mathcal{E}(Q). \tag{1-5}$$

Such a parameter λ is uniquely determined up to an error of size $O((\mathcal{E}(u) - \mathcal{E}(Q))^1/2)$. One can, for instance, choose λ to be the minimizer in (1-4), though there are no obvious benefits to be derived from that. Another equivalent choice is more direct, namely by the relation

$$u(t, \lambda^{-1}(t)) = \frac{\pi}{2}, \tag{1-6}$$

and this still satisfies (1-5); see, for instance, [Bejenaru and Tataru 2014]. Since this problem is locally well-posed in the energy space, scaling considerations show that (for well-chosen $\lambda(t)$), we have

$$\left| \frac{d}{dt} \lambda(t) \right| \lesssim \lambda^{-2}, \tag{1-7}$$

so at least locally λ stays bounded. Then the main question to ask is as follows:

Open problem. *What is the behavior of the function $\lambda(t)$ for equivariant maps satisfying (1-3)?*

We can distinguish several interesting plausible scenarios:

- *Type 1:* $\lambda(t) \to \infty$ as $t \to t_0$ (finite time blow-up). By (1-7), this can only happen at rates $\lambda(t) \gtrsim |t-t_0|^{-1}$. The above extreme corresponds to self-similar concentration; this can also be thought of as a consequence of the finite speed of propagation. In effect, by the important work [Struwe 2003], it is known that such a blow-up can only occur with speed strictly faster than self-similar:
$$\lambda(t)|t-t_0| \to \infty.$$
- *Type 2:* $\lambda(t) \to \infty$ as $t \to \infty$ (infinite time focusing).
- *Type 3:* $\lambda(t) \to 0$ as $t \to \infty$ (infinite time relaxation). By (1-7), this can only happen at rates $\lambda(t) \gtrsim t^{-1}$, which corresponds to self-similar relaxation.
- *Type 4:* $\lambda(t)$ stays in a compact set globally in time. Then we have a global solution, and possibly a resolution into a soliton plus a dispersive part.

Blow-up solutions of Type 1 were constructed not long ago in two quite different papers, [Krieger et al. 2008] and [Rodnianski and Sterbenz 2010], and the result of the latter paper was significantly strengthened and generalized in [Raphaël and Rodnianski 2012]. The behavior of $\lambda(t)$ in [Krieger et al. 2008] as $t \to 0$ is given by
$$\lambda(t) = t^{-1-\nu}, \quad \nu \geq 1$$
(here the restriction $\nu \geq 1$ seems technical, and should really be $\nu > 0$), while that in [Raphaël and Rodnianski 2012] is
$$\lambda(t) \sim t^{-1} e^{c\sqrt{\log t}}.$$
The latter solutions were also proved to be stable with respect to a class of small smooth perturbations. It is not implausible that the set of all blow-up solutions is open in a suitable topology, although numerical evidence in [Bizoń et al. 2001] appears to suggest the existence of a codimension-one manifold of data leading to an unstable blow-up, which separates scattering solutions from a stable regime of finite time blow-up solutions.

Up to this point we are not aware of any examples of solutions of Type 2, 3 or 4 other than the Q_λ's in the wave maps context, although recent work [Gustafson et al. 2010] revealed unusual solutions of this type in the context of the Landau–Lifshitz equation. Earlier work [Krieger and Schlag 2007] showed the existence of Type 4 solutions for the critical focusing nonlinear wave equation on $\mathbb{R}^{3+1}$.

Understanding the general picture for data in the energy space seems out of reach for now. However, there is a simpler question one may ask, namely, what happens for data which is close to a soliton in a stronger topology which includes both extra regularity and extra decay at infinity. Neither the results of [Krieger et al. 2008] nor of [Raphaël and Rodnianski 2012] apply in this context. A good starting point for this investigation is the following:

Conjecture. *There exists a codimension-one set of (small) data leading to Type 4 solutions, which separates Type 1 and Type 3 solutions.*

One should take this only as a rough guide; some fine adjustments may be needed. Our main result is to construct a large class of Type 4 solutions:

Theorem 1.1. *There exists a codimension-two set of Type 4 equivariant wave maps satisfying (1-3).*

For a more precise formulation of the theorem, see page 834. Compared with the conjecture above, one can see that we are one dimension short. At this point it is not clear if this is a technical issue, or if something new happens. A plausible scenario might be that the missing dimension may include Type 2 solutions, as well as slowly relaxing Type 4 solutions.

One should also compare this result with the related problem for Schrödinger maps. Although the solitons are the same and the operator H arising below in the linearization is also the same for Schrödinger maps, in [Bejenaru and Tataru 2014] it is shown that the solitons are stable with respect to small localized perturbations. One way to explain this is that the linear growth in the resonant direction occurring in the H-wave equation has a stronger destabilizing effect than the corresponding lack of decay in the H-Schrödinger equation.

Notation. Here we introduce some notation which will be used throughout the paper. We slightly modify the use of $\langle \cdot \rangle$ in the following sense:

$$\langle x \rangle = \sqrt{4+x^2}, \quad x \in \mathbb{R}.$$

For a real number a, we define $a^+ = \max\{0, a\}$ and $a^- = \min\{0, a\}$.

We will use a dyadic partition of $\mathbb{R}_+$ into sets $\{A_m\}_{m\in\mathbb{Z}}$ given by

$$A_m = \{2^{m-1} < r < 2^{m+1}\}.$$

For given $M > 0$, we use smooth localization functions $\chi_{\lesssim M}, \chi_{\gtrsim M}$ forming a partition of unity for $\mathbb{R}_+$ and such that

$$|(r\partial_r)^\alpha \chi_{\lesssim M}| + |(r\partial_r)^\alpha \chi_{\gtrsim M}| \lesssim_\alpha 1.$$

2. The gauge derivative and linearizations

The linearized equation (1-1) around the soliton Q has the form

$$-v_{tt} - Hv = 0, \qquad H = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{\cos(2Q)}{r^2}. \tag{2-1}$$

The elliptic operator H admits the factorization

$$H = L^*L, \quad L = h_1 \partial_r h_1^{-1} = \partial_r + \frac{h_3}{r}, \quad L^* = -h_1^{-1}\partial_r h_1 - \frac{1}{r} = -\partial_r + \frac{h_3 - 1}{r}, \tag{2-2}$$

where[1] $h_1 = \sin Q = \dfrac{2r}{1+r^2}$, $h_3 = -\cos Q = \dfrac{r^2-1}{r^2+1}$. H is nonnegative and has a zero resonance

$$\phi_0 = h_1 = \frac{2r}{1+r^2}.$$

[1] Throughout this paper we use $\sin Q, \cos Q$ instead of h_1, h_3; however, the reader may need this correspondence in order to relate this work to [Bejenaru and Tataru 2014].

This resonance is the reason why (2-1) does not have good dispersive estimates. Since ϕ_0 fails to be an eigenvalue, we cannot project it away as is usually done in standard modulation theory. This suggests that working with the variable u and its equation (1-1) runs into problems due to the lack of good linear estimates needed to treat the nonlinearity. Therefore, instead of working with the solution u, we introduce a new variable

$$w = \partial_r u - \frac{1}{r}\sin u, \tag{2-3}$$

which has the nice property that

$$w = 0 \Longleftrightarrow u = Q_\lambda$$

for some $\lambda \in \mathbb{R}_+$. Indeed, by rearranging (1-2) and using $u(0) = 0$, $u(\infty) = \pi$, we obtain

$$\mathscr{E}(u) = \pi \int_0^\infty (|u_t|^2 + |w|^2) r\,dr + \pi \int_0^\infty 2\sin u \cdot \partial_r u\,dr = \pi \int_0^\infty (|u_t|^2 + |w|^2) r\,dr + 4\pi,$$

from which the above observation follows. This type of change of variables originates at least with the work [Gustafson et al. 2008]. If $\lambda(t)$ is chosen such that (1-5) holds, then using (1-3), a direct computation shows that

$$\|u - Q_\lambda\|_{\dot{H}^1_e} \approx \|w\|^2_{L^2}. \tag{2-4}$$

Then a direct computation shows that w solves

$$w_{tt} - \Delta w + \frac{2(1+\cos u)}{r^2} w = \frac{1}{r}\sin u(u_t^2 - w^2). \tag{2-5}$$

The function u appears in this equation, but it can be recovered from w by solving the ODE (2-3) with Q-like “data” at $r = \infty$.

We remark that the linearized form of (2-3) near Q is

$$z = \Big(\partial_r - \frac{1}{r}\cos Q\Big) v = Lv, \tag{2-6}$$

where L was introduced above in (2-2).

On the other hand, the linearized equation for w near Q has the form

$$z_{tt} - \Delta z + \frac{2(1+\cos Q)}{r^2} z = 0. \tag{2-7}$$

This wave equation is governed by the operator

$$\tilde{H} = -\Delta + \frac{2(1+\cos Q)}{r^2} = -\Delta + \frac{4}{r^2(1+r^2)} = LL^*.$$

This operator is better behaved than H; in particular, its zero mode ψ_0 grows logarithmically at infinity.

The plan is to treat (2-5) in a perturbative manner for the most part. To fix things, we will rewrite it in the form

$$(\partial_t^2 + \tilde{H}) w = \frac{2(\cos Q - \cos u)}{r^2} w + \frac{1}{r}\sin u(u_t^2 - w^2) := N(w, u) \tag{2-8}$$

and work with this from here on. Equation (2-8) for w is preferable due to the nice dispersive properties of its linear part. However, as u occurs in the w equation, one has to also keep track of it through the elliptic equation (2-3). In addition, u_t also appears in the above equation. This is related to w_t by differentiating (2-3):

$$w_t = \Big(\partial_r - \frac{1}{r}\cos u\Big)u_t. \tag{2-9}$$

In order to study this equation, we need to understand better the structure of its linear part, and, in particular, the spectral theory for the operator $\tilde{H}$. This is the subject of Section 3.

Setup of the problem. The starting point is to consider $\bar{w}$ to be an exact real solution to the linear homogeneous equation

$$(\partial_t^2 + \tilde{H})\bar{w} = 0, \quad w(0) = w_0, \quad w_t(0) = w_1, \tag{2-10}$$

where w_0 and w_1 are real Schwartz functions which are assumed to satisfy the nonresonance conditions

$$\langle w_0, \psi_0\rangle = 0, \quad \langle w_1, \psi_0\rangle = 0. \tag{2-11}$$

We denote by $\bar{u}$ the corresponding map, see (2-3) (this will be made precise in Proposition 5.2), obtained by solving the ODE

$$\partial_r \bar{u} - \frac{1}{r}\sin\bar{u} = \bar{w}, \quad \bar{u} \sim Q \text{ as } r \to \infty. \tag{2-12}$$

Now we seek a solution to the nonlinear equation u and its associated gauge derivative w close to $\bar{u}$, $\bar{w}$ respectively,

$$u = \bar{u} + \varepsilon, \quad w = \bar{w} + \gamma, \tag{2-13}$$

so that u and w match $\bar{u}$ and $\bar{w}$ asymptotically as $t \to \infty$.

By a slight abuse of notation, we use $\|\cdot\|_S$ to denote a norm obtained by adding sufficiently many seminorms of the Schwartz space S. We also use $\lesssim_S$ for inequalities where the implicit constant depends on $\|(w_0, w_1)\|_S$. Modulo defining the X and LX norms, we are now in a position to restate our main result in a more detailed fashion.

Theorem 2.1. *Let w_0, w_1 be Schwartz functions satisfying the nonresonance conditions (2-11). Let $\bar{u}$ and $\bar{w}$ be defined as above. Then there exist $T \lesssim_S 1$ and a unique wave map u in $[T,\infty)$ so that u and w match $\bar{u}$ and $\bar{w}$ as $t \to \infty$ in the following asymptotic fashion for $t \in [T,\infty)$:*

$$\|\gamma(t)\|_{LX} \lesssim_S t^{-3/2}, \quad \|\partial_t\gamma(t)\|_{LX} \lesssim_S t^{-5/2}, \quad \|\gamma(t)\|_{\dot{H}^1} \lesssim_S t^{-5/2}, \tag{2-14}$$

respectively

$$\|\varepsilon(t)\|_X \lesssim_S t^{-3/2}, \quad \|\partial_t\varepsilon(t)\|_{LX} \lesssim_S t^{-5/2}. \tag{2-15}$$

Furthermore, the map u and its corresponding gauge derivative w have a Lipschitz dependence on (w_0, w_1) with respect to the above norms.

One would expect the above result to be in terms of L^2 and $\dot{H}^1_e$ spaces. However, these spaces are very disconnected from the spectral structure of H and $\tilde{H}$, particularly at low frequencies, and this makes them unsuitable. The spaces $X \subset \dot{H}^1_e$ and $LX \subset L^2$ have been introduced in [Bejenaru and Tataru 2014] to address exactly this issue: they are low-frequency corrections of $\dot{H}^1_e$, respectively L^2. Their exact definition is provided in the next section.

In view of (2-8), the function γ solves

$$(\partial_t^2 + \tilde{H})\gamma = N(\bar{w} + \gamma, \bar{u} + \varepsilon) \tag{2-16}$$

with zero Cauchy data at infinity. By (2-3), (2-9), (2-13) and (2-12), the functions ε and ε_t are determined from the equations

$$\begin{aligned} \gamma &= \partial_r \varepsilon - \frac{\sin(\varepsilon + \bar{u}) - \sin \bar{u}}{r}, \\ \gamma_t &= \left(\partial_r - \frac{\cos(\varepsilon + \bar{u})}{r}\right)\varepsilon_t - \frac{\cos(\varepsilon + \bar{u}) - \cos \bar{u}}{r}\bar{u}_t. \end{aligned} \tag{2-17}$$

We proceed as follows. In the next section we recall from [Bejenaru and Tataru 2014] the spectral theory for H (which in fact originates in [Krieger et al. 2008]) and $\tilde{H}$ and the definitions and some properties of the spaces X and LX. Then, in Section 4, we provide linear estimates for the linear (inhomogeneous) wave equation corresponding to (2-10). In Section 5, we analyze the first approximations $\bar{w}$ and $\bar{u}$ using (2-12). Then, in Section 6, we continue with the study of the relation between ε and γ based on (2-17). All the analysis carried out in Sections 4–6 is done in the context of X and LX spaces. In the end, in Section 7, we study the solvability of Equation (2-16) using perturbative methods in LX based spaces.

3. The modified Fourier transform

In this section, we recall the spectral theory associated with the operators H, $\tilde{H}$. The spectral theory for H was developed in [Krieger et al. 2008], and the one for $\tilde{H}$ was derived from the one for H in [Bejenaru and Tataru 2014]. In this paper, we follow closely the exposition in [Bejenaru and Tataru 2014].

Generalized eigenfunctions. We consider H acting as an unbounded self-adjoint operator in $L^2(r\,dr)$. Then H is nonnegative, and its spectrum $[0, \infty)$ is absolutely continuous. H has a zero resonance, namely $\phi_0 = h_1$:

$$Hh_1 = 0.$$

For each $\xi > 0$, one can choose a normalized generalized eigenfunction ϕ_ξ,

$$H\phi_\xi = \xi^2 \phi_\xi.$$

These are unique up to a ξ dependent multiplicative factor, which is chosen as described below.

To these one associates a generalized Fourier transform $\mathcal{F}_H$ defined by

$$\mathcal{F}_H f(\xi) = \int_0^\infty \phi_\xi(r) f(r) r\,dr,$$

where the integral above is considered in the singular sense. This is an L^2 isometry, and we have the inversion formula

$$f(r) = \int_0^\infty \phi_\xi(r) \mathcal{F}_H f(\xi)\,d\xi.$$

The functions ϕ_ξ are smooth with respect to both r and ξ. To describe them, one considers two distinct regions, $r\xi \lesssim 1$ and $r\xi \gtrsim 1$.

In the first region, $r\xi \lesssim 1$, the functions ϕ_ξ admit a power series expansion of the form

$$\phi_\xi(r) = q(\xi)\Big(\phi_0 + \frac{1}{r}\sum_{j=1}^{\infty}(r\xi)^{2j}\phi_j(r^2)\Big), \quad r\xi \lesssim 1, \tag{3-1}$$

where $\phi_0 = h_1$ and the functions ϕ_j are analytic and satisfy

$$|(r\partial_r)^\alpha \phi_j| \lesssim_\alpha \frac{C^j}{(j-1)!}\log(1+r). \tag{3-2}$$

This bound is not spelled out in [Krieger et al. 2008], but it follows directly from the integral recurrence formula for the f_j given on p. 578 of that paper. The smooth positive weight q satisfies

$$q(\xi) \approx \begin{cases} \dfrac{1}{\xi^{1/2}|\log\xi|} & \text{if } \xi \ll 1, \\ \xi^{3/2} & \text{if } \xi \gg 1, \end{cases} \qquad |(\xi\partial_\xi)^\alpha q| \lesssim_\alpha q. \tag{3-3}$$

Defining the weight

$$m_k^1(r) = \begin{cases} \min\Big\{1, r2^k \dfrac{\log(1+r^2)}{\langle k\rangle}\Big\} & \text{if } k<0, \\ \min\{1, r^3 2^{3k}\} & \text{if } k \geq 0, \end{cases} \tag{3-4}$$

it follows that the nonresonant part of ϕ_ξ satisfies

$$\big|(\xi\partial_\xi)^\alpha (r\partial_r)^\beta \big(\phi_\xi(r) - q(\xi)\phi_0(r)\big)\big| \lesssim_{\alpha\beta} 2^{k/2} m_k^1(r), \quad \xi \approx 2^k,\ r\xi \lesssim 1. \tag{3-5}$$

In the other region, $r\xi \gtrsim 1$, we begin with the functions

$$\phi_\xi^+(r) = r^{-1/2}e^{ir\xi}\sigma(r\xi, r), \quad r\xi \gtrsim 1, \tag{3-6}$$

solving

$$H\phi_\xi^+ = \xi^2\phi_\xi^+,$$

where for σ, we have the asymptotic expansion

$$\sigma(q,r) \approx \sum_{j=0}^{\infty} q^{-j}\phi_j^+(r), \quad \phi_0^+ = 1, \quad \phi_1^+ = \frac{3i}{8} + O\Big(\frac{1}{1+r^2}\Big),$$

with $\sup_{r>0}|(r\partial_r)^k\phi_j^+| < \infty$ in the following sense:

$$\sup_{r>0}\Big|(r\partial r)^\alpha (q\partial_q)^\beta\Big(\sigma(q,r) - \sum_{j=0}^{j_0} q^{-j}\phi_j^+(r)\Big)\Big| \leq c_{\alpha,\beta,j_0} q^{-j_0-1}.$$

Then we have the representation

$$\phi_\xi(r) = a(\xi)\phi_\xi^+(r) + \overline{a(\xi)\phi_\xi^+(r)}, \tag{3-7}$$

where the complex-valued function a satisfies

$$|a(\xi)| = \sqrt{\frac{2}{\pi}}, \quad |(\xi\partial_\xi)^\alpha a(\xi)| \lesssim_\alpha 1. \tag{3-8}$$

The spectral theory for $\tilde{H}$ is derived from the spectral theory for H due to the conjugate representations

$$H = L^*L, \quad \tilde{H} = LL^*.$$

This allows us to define generalized eigenfunctions ψ_ξ for $\tilde{H}$ using the generalized eigenfunctions ϕ_ξ for H,

$$\psi_\xi = \xi^{-1}L\phi_\xi, \quad L^*\psi_\xi = \xi\phi_\xi. \tag{3-9}$$

It is easy to see that ψ_ξ are real and smooth, vanish at $r = 0$, and solve

$$\tilde{H}\psi_\xi = \xi^2\psi_\xi.$$

With respect to this frame, we can define the generalized Fourier transform adapted to $\tilde{H}$ by

$$\mathcal{F}_{\tilde{H}} f(\xi) = \int_0^\infty \psi_\xi(r) f(r) r\,dr,$$

where the integral above is considered in the singular sense. This is an L^2 isometry, and we have the inversion formula

$$f(r) = \int_0^\infty \psi_\xi(r)\mathcal{F}_{\tilde{H}} f(\xi)d\xi. \tag{3-10}$$

To see this, we compute, for a Schwartz function f,

$$\mathcal{F}_{\tilde{H}} Lf(\xi) = \int_0^\infty \psi_\xi(r)Lf(r)r\,dr = \int_0^\infty L^*\psi_\xi(r)f(r)r\,dr = \int_0^\infty \xi\phi_\xi(r)f(r)r\,dr = \xi\mathcal{F}_H f(\xi).$$

Hence

$$\|\mathcal{F}_{\tilde{H}} Lf\|_{L^2}^2 = \|\xi\mathcal{F}_H f(\xi)\|_{L^2}^2 = \langle Hf, f\rangle_{L^2(r\,dr)} = \|Lf\|_{L^2}^2,$$

which suffices, since Lf spans a dense subset of L^2.

The representation of ψ_ξ in the two regions $r\xi \lesssim 1$ and $r\xi \gtrsim 1$ is obtained from the similar representation of ϕ_ξ. In the first region, $r\xi \lesssim 1$, the functions ψ_ξ admit a power series expansion of the form

$$\psi_\xi = \xi q(\xi)\biggl(\psi_0(r) + \sum_{j\ge 1}(r\xi)^{2j}\psi_j(r^2)\biggr), \tag{3-11}$$

where

$$\psi_j(r) = (h_3 + 1 + 2j)\phi_{j+1}(r) + r\partial_r\phi_{j+1}(r).$$

From (3-2), it follows that

$$|(r\partial_r)^\alpha\psi_j| \lesssim_\alpha \frac{C^j}{(j-1)!}\log(1+r^2).$$

In addition, ψ_0 solves $L^*\psi_0 = \phi_0$, and therefore a direct computation shows that

$$\psi_0 = \frac{1}{2}\left(\frac{(1+r^2)\log(1+r^2)}{r^2} - 1\right).$$

In particular, defining the weights

$$m_k(r) = \begin{cases} \min\left\{1, \dfrac{\log(1+r^2)}{\langle k\rangle}\right\} & \text{if } k<0, \\ \min\{1, r^2 2^{2k}\} & \text{if } k \ge 0, \end{cases} \tag{3-12}$$

we have the pointwise bound for ψ_ξ

$$\left|(r\partial_r)^\alpha (\xi\partial_\xi)^\beta \psi_\xi(r)\right| \lesssim_{\alpha\beta} 2^{k/2} m_k(r), \quad \xi \approx 2^k,\ r\xi \lesssim 1. \tag{3-13}$$

On the other hand, in the regime $r\xi \gtrsim 1$, we define

$$\psi^+ = \xi^{-1} L\phi^+,$$

and we obtain the representation

$$\psi_\xi(r) = a(\xi)\psi_\xi^+(r) + \overline{a(\xi)\psi_\xi^+(r)}. \tag{3-14}$$

For ψ^+, we obtain the expression

$$\psi_\xi^+(r) = r^{-1/2} e^{ir\xi} \tilde\sigma(r\xi, r), \quad r\xi \gtrsim 1, \tag{3-15}$$

where $\tilde\sigma$ has the form

$$\tilde\sigma(q,r) = i\sigma(q,r) - \tfrac{1}{2} q^{-1}\sigma(q,r) + \frac{\partial}{\partial q}\sigma(q,r) + \xi^{-1} L\sigma(q,r),$$

and therefore it has exactly the same properties as σ. In particular, for fixed ξ, we obtain that

$$\tilde\sigma(r\xi, r) = i - \tfrac{7}{8} r^{-1}\xi^{-1} + O(r^{-2}). \tag{3-16}$$

We conclude our description of the generalized eigenfunctions and of the associated Fourier transforms with a bound on the $\tilde H$ Fourier transforms of Schwartz functions.

Lemma 3.1. *If f is a Schwartz function satisfying $\langle f, \psi_0\rangle = 0$, then*

$$\left|(\xi\partial_\xi)^\alpha \mathcal{F}_{\tilde H} f(\xi)\right| \lesssim_{\alpha,N} \begin{cases} \dfrac{\xi^{5/2}}{\langle \log\xi\rangle} & \text{if } \xi \lesssim 1, \\ \langle\xi\rangle^{-N} & \text{if } \xi \gtrsim 1. \end{cases} \tag{3-17}$$

Proof. We start from the definition of the modified Fourier transform and use that $\langle f, \psi_0\rangle = 0$:

$$\begin{aligned} |\mathcal{F}_{\tilde H} f(\xi)| &\lesssim \left(\left|\int_0^{\xi^{-1}} \psi_\xi(r) f(r) r\,dr\right| + \left|\int_{\xi^{-1}}^\infty \psi_\xi(r) f(r) r\,dr\right|\right) \\ &\lesssim \xi q(\xi)\left(\int_{\xi^{-1}}^\infty \left|\psi_0(r) f(r)\right| r\,dr + \int_0^{\xi^{-1}} \sum_{j\ge1} (r\xi)^{2j}\psi_j(r^2) f(r) r\,dr\right) + \int_{\xi^{-1}}^\infty |f(r)| r^{1/2}\,dr \\ &\lesssim \xi^3 q(\xi). \end{aligned}$$

A similar argument takes care of the case $\alpha > 0$. $\square$

The spaces X and LX. The operator L maps $\dot{H}^1_e$ into L^2. Conversely, one would like that, given some $f \in L^2$, we could solve $Lu = f$ and obtain a solution u which is in $\dot{H}^1_e$ and satisfies

$$\|u\|_{\dot{H}^1_e} \lesssim \|f\|_{L^2}.$$

However, this is not the case. The first observation is that the solution is only unique modulo a multiple of the resonance ϕ_0. Moreover, the inequality above is not expected to be true, even assuming that somehow we choose the "best" u from all candidates.

The spaces X and LX are in part introduced in order to remedy both the ambiguity in the inversion of L and the failing inequality.

Definition 3.2. (a) The space X is defined as the completion of the subspace of $L^2(r\,dr)$ for which the following norm is finite:

$$\|u\|_X = \Big(\sum_{k\geq 0} 2^{2k}\|P_k^H u\|_{L^2}^2\Big)^{1/2} + \sum_{k<0}\frac{1}{|k|}\|P_k^H u\|_{L^2},$$

where P_k^H is the Littlewood–Paley operator localizing at frequency $\xi \approx 2^k$ in the H calculus.

(b) LX is the space of functions of the form $f = Lu$ with $u \in X$, with norm $\|f\|_{LX} = \|u\|_X$. Expressed in the $\tilde{H}$ calculus, the LX norm is written as

$$\|f\|_{LX} = \Big(\sum_{k\geq 0}\|P_k^{\tilde{H}} f\|_{L^2}^2\Big)^{1/2} + \sum_{k<0}\frac{2^{-k}}{|k|}\|P_k^{\tilde{H}} f\|_{L^2}.$$

In this article we work with equivariant wave maps u for which $\|u - Q\|_X \ll 1$. This corresponds to functions w which satisfy $\|w\|_{LX} \ll 1$. The simplest properties of the space X are summarized as follows (see Proposition 4.2 in [Bejenaru and Tataru 2014]):

Proposition 3.3. *The following embeddings hold for the space X:*

$$H^1_e \subset X \subset \dot{H}^1_e. \tag{3-18}$$

In addition, for f in X, we have the bounds

$$\|\langle r\rangle^{1/2} f\|_{L^\infty} \lesssim \|f\|_X, \tag{3-19}$$

$$\Big\|\frac{f}{\log(1+r)}\Big\|_{L^2} \lesssim \|f\|_X, \tag{3-20}$$

$$\|\langle r\rangle^{1/2} f\|_{L^4} \lesssim \|f\|_X. \tag{3-21}$$

Now we turn our attention to the space LX. From [Bejenaru and Tataru 2014, Lemma 4.4 and Proposition 4.5], we have:

Lemma 3.4. *If $f \in L^2$ is localized at $\tilde{H}$-frequency 2^k, then*

$$|f(r)| \lesssim 2^k m_k(r)(1+2^k r)^{-1/2}\|f\|_{L^2}. \tag{3-22}$$

Proposition 3.5. *The following embeddings hold for LX:*

$$L^1 \cap L^2 \subset LX \subset L^2. \tag{3-23}$$

4. Linear estimates for the $\tilde{H}$ wave equation

In this section, we prove estimates for the linear equation

$$(\partial_t^2 + \tilde{H})\psi = f, \tag{4-1}$$

with zero Cauchy data at infinity. The solution is given by $\psi = Kf$, where

$$Kf(r,t) = -\mathcal{F}_{\tilde{H}}^{-1} \int_t^\infty \frac{\sin(t-s)\xi}{\xi} \mathcal{F}_{\tilde{H}} f(\xi, s)\, ds. \tag{4-2}$$

We also need its time derivative, which is given by

$$\partial_t Kf = -\mathcal{F}_{\tilde{H}}^{-1} \int_t^\infty \cos(t-s)\xi \cdot \mathcal{F}_{\tilde{H}} f(\xi, s)\, ds.$$

Finally, we need the following formula, which follows from (3-9):

$$L^* Kf = -\mathcal{F}_{H}^{-1} \int_t^\infty \sin(t-s)\xi \cdot \mathcal{F}_{\tilde{H}} f(\xi, s)\, ds.$$

The following result is a modification of the standard energy estimate for the wave equation:

Lemma 4.1. *Assume that $f(s) \in LX$. Then for every $\alpha > 0$, the solution of (4-1) with zero data at ∞ satisfies*

$$t^\alpha \|\psi(t)\|_{LX} + t^{\alpha+1}\big(\|\partial_t \psi(t)\|_{LX} + \|\psi(t)\|_{\dot{H}^1_e}\big) \lesssim \sup_s s^{\alpha+2} \|f(s)\|_{LX}. \tag{4-3}$$

Proof. The solution of (4-1) with zero data at ∞ is given by $\psi = Kf$. The estimate for the first term follows from the bound $\big|(\sin(t-s)\xi)/\xi\big| \lesssim |t-s|$ and the representation of the spaces LX on the Fourier side. The estimate for the second term is similar.

The argument for the third term is more involved. We define g by

$$\mathcal{F}_{\tilde{H}} g(t,\xi) = -\int_t^\infty \sin((t-s)\xi) \mathcal{F}_{\tilde{H}} f(\xi, s)\, ds.$$

Then

$$\xi \mathcal{F}_{\tilde{H}} \psi(t,\xi) = \mathcal{F}_{\tilde{H}} g(t,\xi).$$

We estimate, as above,

$$\|g(t)\|_{LX} \lesssim \int_t^\infty \|f(s)\|_{LX}\, ds \lesssim t^{-\alpha-1} \sup_s s^{\alpha+2} \|f(s)\|_{LX}.$$

Hence it suffices to show that for ψ and g related as above, we have

$$\|\psi\|_{\dot{H}^1_e} \lesssim \|g\|_{LX}. \tag{4-4}$$

Here the time variable plays no role and is discarded. Recalling the form of L^* from (2-2), namely $L^* = -\partial_r + (h_3 - 1)/r$, it follows that

$$\|\psi\|_{\dot{H}^1_e} \lesssim \|L^*\psi\|_{L^2} + \left\|\frac{\psi}{r}\right\|_{L^2}.$$

For the first term, we use Plancherel to write

$$\|L^*\psi(t)\|_{L^2}^2 = \langle \psi(t), \tilde{H}\psi(t)\rangle = \|\xi \mathcal{F}_{\tilde{H}}\psi(\xi)\|_{L^2}^2 = \|g\|_{L^2} \lesssim \|g\|_{LX}^2.$$

For the second term, the L^2 bound for g no longer suffices, and we need to use the LX norm of g. We consider a Littlewood–Paley decomposition for both ψ and g, and denote their dyadic pieces by ψ_k, respectively g_k. Then

$$\|\psi_k\|_{L^2} \approx 2^{-k}\|g_k\|_{L^2}.$$

By using (3-13)–(3-14) and the Cauchy–Schwartz inequality, we obtain pointwise bounds for ψ_k, namely,

$$|\psi_k| \lesssim \frac{m_k(r)}{\langle 2^k r\rangle^{1/2}} 2^k \|\psi_k\|_{L^2} \lesssim \frac{m_k(r)}{\langle 2^k r\rangle^{1/2}} \|g_k\|_{L^2},$$

with m_k as in (3-12). For $k \geq 0$, the contributions are almost orthogonal, and we obtain

$$\left\|\frac{\psi_{\geq 0}}{r}\right\|_{L^2} \lesssim \|g_{\geq 0}\|_{L^2}.$$

However, if $k < 0$, then the weaker logarithmic decay for small r no longer suffices for such an argument. Instead, by direct computation, we obtain a weaker bound,

$$\left\|\frac{\psi_k}{r}\right\|_{L^2} \lesssim |k|^{1/2}\|g_k\|_{L^2} \lesssim |k|^{3/2} 2^k \|g\|_{LX}.$$

Then the k summation is easily accomplished. □

5. Analysis of the first approximations $\bar{w}$ and $\bar{u}$

Pointwise bounds for $\bar{w}$. We define f_0 and f_1 by $f_0 = \mathcal{F}_{\tilde{H}} w_0$ and $f_1 = \mathcal{F}_{\tilde{H}} w_1$. Then for $\bar{w}$, we have the representation

$$\bar{w}(t,r) = \int_0^\infty \psi_\xi(r)\Big(f_0(\xi)\cos(t\xi) + \frac{1}{\xi} f_1(\xi)\sin(t\xi)\Big)\,d\xi.$$

Since w_0, w_1 are Schwartz functions satisfying (2-11), from (3-17) we obtain

$$\left|(\xi\partial_\xi)^\alpha f_0(\xi)\right| + \left|(\xi\partial_\xi)^\alpha f_1(\xi)\right| \lesssim_{\alpha,N} \|(w_0,w_1)\|_S \begin{cases} \dfrac{\xi^{5/2}}{\langle \log \xi\rangle} & \text{if } \xi \lesssim 1, \\ \langle \xi\rangle^{-N} & \text{if } \xi \gtrsim 1. \end{cases} \tag{5-1}$$

Here, by a slight abuse of notation, we use $\|\,.\,\|_S$ to denote a finite collection of the S seminorms. This will allow us to obtain pointwise bounds for $\bar{w}$:

Lemma 5.1. *If w_0, w_1 are Schwartz functions satisfying the moment conditions (2-11), then $\bar{w}$ satisfies*

$$|\bar{w}(r,t)| \lesssim \frac{\log(1+r^2)}{\log\langle r+t\rangle} \frac{1}{\langle t+r\rangle^{1/2}\langle t-r\rangle^{5/2}\log\langle r-t\rangle} \|(w_0,w_1)\|_S. \tag{5-2}$$

Proof. We fix k and consider

$$\bar{w}_k(t,r) = \int_0^\infty \psi_\xi(r)\Big(f_0(\xi)\cos(t\xi) + \frac{1}{\xi} f_1(\xi)\sin(t\xi)\Big)\chi_k(\xi)\,d\xi.$$

For $\psi_\xi(r)$, we use the representation (3-11) in the region $\{r\xi \lesssim 1\}$, respectively (3-14) in the region $\{r\xi \gtrsim 1\}$. Then via a standard stationary phase argument, we obtain

$$|w_k(r,t)| \lesssim_N \frac{2^{k/2}\langle 2^k r\rangle^{-1/2} m_k(r)}{\langle 2^k|r-t|\rangle^N \langle k^-\rangle} 2^{5k/2} 2^{-Nk^+}.$$

The desired estimate (5-2) follows by summing these bounds with respect to k. □

Bounds for $\bar{u}$, $\bar{u}_t$. Next we consider $\bar{u}$, which is recovered from $\bar{w}$ via (2-12). This equation contains a nonlinear part coming from the sine function. Consequently, we split $\bar{u}$ into a linear and a nonlinear part:

$$\bar{u} = Q + \bar{u}^l + \bar{u}^{nl},$$

where $\bar{u}^l$ solves the linear part of (2-12),

$$L\bar{u}^l = \bar{w},$$

and $\bar{u}^{nl}$ solves

$$L\bar{u}^{nl} = N(\bar{u}^l, \bar{u}^{nl}), \tag{5-3}$$

where

$$N(u,v) = \frac{1}{r}\big[\sin Q\cdot\big(\cos(u+v)-1\big) + \cos Q\cdot\big(\sin(u+v)-(u+v)\big)\big].$$

Both of the above ODE's are taken with zero Cauchy data at infinity or, equivalently, can be interpreted via the diffeomorphism $L : X \to LX$. The linear part, $\bar{u}^l$, is recovered from the explicit formula

$$\bar{u}^l := L^{-1}\bar{w} = \int_0^\infty \xi^{-1}\phi_\xi(r)\Big(f_0(\xi)\cos(t\xi) + \frac{1}{\xi} f_1(\xi)\sin(t\xi)\Big)\,d\xi,$$

and will be split into a resonant and a nonresonant part: $\bar{u}^l = \bar{u}^{l,r} + \bar{u}^{l,nr}$.

For the nonlinear part, we use an iterative argument based on the fact that there is enough decay on the right-hand side that we can recover it via

$$\bar{u}^{nl} = h_1(r)\int_r^\infty \frac{N(\bar{u}^l, \bar{u}^{nl})}{h_1(s)}\,ds. \tag{5-4}$$

At this stage, we also want to keep track of the differences of solutions. For this, we denote by δw_0, δw_1, $\delta\bar{w}$, $\delta\bar{u}$ the corresponding differences.

Proposition 5.2. (a) *Assume that w_0, w_1 are Schwartz functions satisfying (2-11). Then*

$$\bar{u}^l = \bar{u}^{l,r} + \bar{u}^{l,nr}, \tag{5-5}$$

where $\bar{u}^{l,r}$ and $\bar{u}^{l,nr}$ satisfy the bounds

$$
\begin{aligned}
|\bar{u}^{l,r}| + r|\partial_r \bar{u}^{l,r}| + \langle r+t\rangle |\partial_t \bar{u}^{l,r}| &\lesssim \frac{h_1(r)}{\langle t+r\rangle \log^2\langle t+r\rangle} \|(w_0, w_1)\|_S, \\
|\bar{u}^{l,nr}| + \frac{r\langle r-t\rangle}{\langle t+r\rangle} |\partial_r \bar{u}^{l,nr}| + \langle r-t\rangle |\partial_t \bar{u}^{l,nr}| &\lesssim \frac{r}{r+\langle t\rangle} \frac{1}{\langle t+r\rangle^{1/2} \langle t-r\rangle^{3/2} \log\langle t-r\rangle} \|(w_0, w_1)\|_S.
\end{aligned}
\tag{5-6}
$$

In addition,

$$
\left| (\partial_r + \partial_t)\bar{u}^l + \frac{1}{2r}\bar{u}^l \right| \lesssim \frac{1}{t^{5/2}\langle r-t\rangle^{1/2} \log\langle t-r\rangle} \|(w_0, w_1)\|_S, \qquad r \sim t. \tag{5-7}
$$

(b) *For $t \gtrsim_S 1$, the nonlinear part $\bar{u}^{nl}$ satisfies the bounds*

$$
|\bar{u}^{nl}(r,t)| \lesssim_S \ h_1(r) t^{-1.5} \|(w_0, w_1)\|_S, \qquad \left| \partial_t \bar{u}^{nl} + \tfrac{1}{12} h_1 (\bar{u}^l)^3 \right| \lesssim_S \ h_1(r) t^{-2} \|(w_0, w_1)\|_S. \tag{5-8}
$$

(c) *The above estimates hold true for $\delta \bar{u}^{nl}$ and $\delta \partial_t \bar{u}_l$:*

$$
|\delta \bar{u}^{nl}(r,t)| \lesssim_S \ h_1(r) t^{-1.5} \|(\delta w_0, \delta w_1)\|_S, \qquad \left| \delta \partial_t \bar{u}^{nl} + \tfrac{1}{12} h_1 \delta (\bar{u}^l)^3 \right| \lesssim_S \ h_1(r) t^{-2} \|(\delta w_0, \delta w_1)\|_S. \tag{5-9}
$$

Remark 5.3. By finite speed of propagation arguments, it is not difficult to show that $\bar{u}^l$ decays rapidly outside the cone. However, for our purposes, the decay established in the above proposition suffices.

Remark 5.4. The bound (5-7) shows that a double cancellation occurs on the light cone, as opposed to the expected single cancellation. This is a consequence of the exact decay properties at infinity for the potential in $\tilde{H}$.

Remark 5.5. The second estimate in part (b) is the outcome of a more subtle nonlinear cancellation, rather then a brute force computation.

Proof. (a) We first split $\bar{u}^l$ into two parts,

$$
\bar{u}^l(r,t) = \sum_k \bar{u}^l_k(r,t) = \sum_{2^k \lesssim r^{-1}} \bar{u}^l_k(r,t) + \sum_{2^k \gtrsim r^{-1}} \bar{u}^l_k(r,t) := \bar{u}^l_{\text{low}}(r,t) + \bar{u}^l_{\text{hi}}(r,t),
$$

where

$$
\bar{u}^l_k = \int \xi^{-1} \phi_\xi(r) \chi_k(\xi) \left(\cos(t\xi) \cdot \widehat{f_0}(\xi) + \frac{\sin(t\xi)}{\xi} \widehat{f_1}(\xi) \right) d\xi.
$$

The functions $\widehat{f_0}(\xi)$ and $\widehat{f_1}(\xi)$ belong to the same class, and for large ξ they are smooth and rapidly decaying. Hence the first term in the above formula is better than the second, and will be neglected in the sequel. Then using the power series (3-1), we can write

$$
\bar{u}^l_k = \int \xi^{-2} q(\xi) \sin(t\xi) \left(\phi_0(r) + \frac{1}{r} \sum_{j \ge 1} (r\xi)^{2j} \phi_j(r^2) \right) \widehat{f_1}(\xi) \chi_k(\xi)\, d\xi, \qquad 2^k r \lesssim 1,
$$

which leads to a corresponding decomposition

$$
\bar{u}^l_{\text{low}} = \bar{u}^{l,0}_{\text{low}} + \sum_{j \ge 1} \bar{u}^{l,j}_{\text{low}}.
$$

Then we set

$$\bar{u}^{l,r} = \bar{u}^{l,0}_{\text{low}}, \quad \bar{u}^{l,nr} = \bar{u}^{l}_{\text{hi}} + \sum_{j\geq 1} \bar{u}^{l,j}_{\text{low}}, \tag{5-10}$$

and proceed to estimate all of the above components of $\bar{u}^l$.

The terms in $\bar{u}^l_{\text{hi}}$ are estimated by stationary phase using (5-1) and the ϕ_ξ representation in (3-7). This yields

$$|\bar{u}^l_k| \lesssim \frac{r^{-1/2}2^{3k/2}}{\langle 2^k|r-t|\rangle^N \langle k^-\rangle} 2^{-Nk^+}, \quad 2^k r \gtrsim 1, \tag{5-11}$$

which, after summation with respect to k, gives the bound

$$|\bar{u}^l_{\text{hi}}| \lesssim \sum_{2^k \gtrsim r^{-1}} |\bar{u}^l_k(r,t)| \lesssim \left(\frac{r}{\langle r+t\rangle}\right)^N \frac{1}{\langle r+t\rangle^{1/2}\langle r-t\rangle^{3/2}\log\langle r-t\rangle}.$$

The bounds for the time derivative are obtained from the explicit formula

$$\partial_t \bar{u}^l = \int_0^\infty \phi_\xi(r)\Big(-f_0(\xi)\sin(t\xi) + \frac{1}{\xi} f_1(\xi)\cos(t\xi)\Big)\, d\xi,$$

which shows that we produce an extra 2^k factor in (5-11). Similarly, an r derivative applied to ϕ_ξ yields an additional 2^k factor in the asymptotic expansion. Thus we obtain

$$|\partial_t \bar{u}^l_k| + |\partial_r \bar{u}^l_k| \lesssim \frac{r^{-1/2}2^{5k/2}}{\langle 2^k|r-t|\rangle^N \langle k^-\rangle} 2^{-Nk^+}, \quad 2^k r \gtrsim 1, \tag{5-12}$$

which leads to

$$|\partial_t \bar{u}^l_{hi}| + |\partial_r \bar{u}^l_{hi}| \lesssim \left(\frac{r}{\langle r+t\rangle}\right)^N \frac{1}{\langle r+t\rangle^{1/2}\langle r-t\rangle^{5/2}\log\langle r-t\rangle}.$$

We now consider the terms in $\bar{u}^{l,j}_{\text{low}}$. The main contribution comes from f_1, so we take $f_0 = 0$ for convenience. For $j = 0$, we have

$$\bar{u}^{l,0}_{\text{low}} = \phi_0(r)\sum_k \chi_{\lesssim 2^{-k}}(r)\int \xi^{-2} q(\xi)\sin(t\xi)\widehat{f_1}(\xi)\chi_k(\xi)\,d\xi := \phi_0(r)\sum_k \chi_{\lesssim 2^{-k}}(r) g^0_k(t) := \phi_0(r) g^0(r,t).$$

Using stationary phase and the properties of q, we have

$$|g^0_k(t)| + 2^{-k}|\partial_t g^0_k(t)| \lesssim \frac{2^k}{\langle k^-\rangle^2 \langle 2^k t\rangle^N} 2^{-Nk^+}.$$

By summing with respect to k, we obtain

$$|g^0(r,t)| + \langle t+r\rangle\big(|\partial_r g^0(r,t)| + |\partial_t g^0(r,t)|\big) \lesssim \frac{1}{\langle t+r\rangle \log^2\langle t+r\rangle}, \tag{5-13}$$

which yields the $\bar{u}^{l,r}$ bound in (5-6).

For $j \geq 1$, we have

$$u^{l,j}_{\mathrm{low}} = \sum_k \chi_{\{r \lesssim 2^{-k}\}} \frac{1}{r} \int \xi^{-2} q(\xi) \sin(t\xi) \sum_{j \geq 1} (r\xi)^{2j} \phi_j(r^2) \widehat{f_1}(\xi) \chi_k(\xi)\, d\xi$$
$$:= r^{2j-1}\phi_j(r^2) \sum_k \chi_{\lesssim 2^{-k}}(r) g^j_k(t) := r^{2j-1}\phi_j(r^2) g^j(r,t).$$

By stationary phase and the properties of q and $\widehat{f_1}$, we have

$$|g^j_k(r,t)| + 2^{-k}\big(|\partial_t g^j_k(r,t)| + |\partial_r g^j_k(r,t)|\big) \lesssim \frac{2^{(2j+1)k}}{\langle k^- \rangle^2 \langle 2^k t\rangle^N} 2^{-Nk^+}.$$

Summing up over k, we obtain

$$|g^j(r,t)| + \langle t+r\rangle\big(|\partial_r g^j(r,t)| + |\partial_t g^j(r,t)|\big) \lesssim \frac{1}{\langle t+r\rangle^{2j+1} \log^2\langle t+r\rangle}. \tag{5-14}$$

Hence, using the bound (3-2) for ϕ_j, we obtain a bound for $\bar{u}^{l,j}_{\mathrm{low}}$, namely

$$|\bar{u}^{l,j}_{\mathrm{low}}(r,t)| + |r\partial_r \bar{u}^{l,j}_{\mathrm{low}}(r,t)| + \langle t+r\rangle |\partial_t \bar{u}^{l,j}_{\mathrm{low}}(r,t)| \lesssim \frac{C^j}{j!} \frac{r^{2j-1}\log(1+r^2)}{\langle t+r\rangle^{2j+1}\log^2\langle t+r\rangle}. \tag{5-15}$$

Thus these contributions satisfy the bounds required of $\bar{u}^{l,nr}$.

We now turn our attention to the estimate (5-7), which applies in the region where $r \approx t$. By (5-6) (for $\bar{u}^l$) and (5-15), the contributions of the term $\bar{u}^l_{\mathrm{low}}$ are all below the required threshold, so it remains to consider $\bar{u}^l_{\mathrm{hi}}$. We have

$$\bar{u}^l_{\mathrm{hi}}(r,t) = \int_0^\infty \chi_{\gtrsim r^{-1}}(\xi)\xi^{-1}\phi_\xi(r)\Big(f_0(\xi)\cos(t\xi) + \frac{1}{\xi} f_1(\xi)\sin(t\xi)\Big)\, d\xi.$$

For ϕ_ξ, we use the representation (3-7), with ϕ^+_ξ as in (3-6),

$$\phi_\xi = r^{-1/2}\big(a(\xi)\sigma(r\xi,r)e^{ir\xi} + \bar{a}(\xi)\bar{\sigma}(r\xi,r)e^{-ir\xi}\big), \qquad r\xi \gtrsim 1.$$

We notice that the operator $\partial_r + \partial_t$ kills the resonant factors $e^{\pm i(r-t)\xi}$. Precisely, we have

$$\Big(\partial_r + \partial_t + \frac{1}{2r}\Big)\phi_\xi(r)\sin(t\xi) = 2r^{-1/2}\Re\big(e^{i\xi(r+t)}\xi a(\xi)\sigma(r\xi,r)\big) + 2r^{-1/2}\Re\big(e^{ir\xi}a(\xi)\partial_r\sigma(r\xi,r)\big)\sin(t\xi),$$

and a similar computation where $\sin(t\xi)$ is replaced by $\cos(t\xi)$. This leads to

$$\begin{aligned}\Big(\partial_r + \partial_t + \frac{1}{2r}\Big)\bar{u}^l_{\mathrm{hi}} \\ = \int_0^\infty \chi_{\gtrsim r^{-1}}(\xi) r^{-1/2}\Re\big(2\xi e^{i(r+t)\xi}a(\xi)\sigma(r\xi,r) + 2e^{ir\xi}a(\xi)\partial_r\sigma(r\xi,r)\cos(t\xi)\big)\frac{f_0(\xi)}{\xi}\, d\xi \\ + \int_0^\infty \chi_{\gtrsim r^{-1}}(\xi) r^{-1/2}\Re\big(2\xi e^{i(r+t)\xi}a(\xi)\sigma(r\xi,r) + 2e^{ir\xi}a(\xi)\partial_r\sigma(r\xi,r)\sin(t\xi)\big)\frac{f_1(\xi)}{\xi^2}\, d\xi.\end{aligned}$$

The two integrals above are treated as before, using stationary phase. The first term in each of the last integrals has a nonresonant phase; therefore each integration by parts gains a factor of $(\xi t)^{-1}$. Thus, taking (5-1) into account, their contributions can be estimated by

$$\int_0^\infty \chi_{\gtrsim t^{-1}}(\xi) t^{-1/2} \xi (t\xi)^{-N} \frac{\xi^{5/2}}{\xi^2 \log \xi} d\xi \approx \frac{1}{t^3 \log t}.$$

The second term contains the expression $\partial_r \sigma(r\xi, r)$, which (see the description of σ in Section 3) brings an additional factor of $r^{-1}(r\xi)^{-1} \approx t^{-2}\xi^{-1}$. The contribution of the part with phase $e^{i\xi(r+t)}$ is better than above, while the contribution of the part with phase $e^{i\xi(r-t)}$ is of the form

$$\int_0^\infty \chi_{\gtrsim t^{-1}}(\xi) a(\xi) t^{-1/2} t^{-1} (t\xi)^{-1} e^{i\xi(t-r)} \frac{\xi^{5/2}}{\xi^2 \log \xi} d\xi \approx \frac{1}{t^{5/2} \langle t-r \rangle^{1/2} \log \langle t-r \rangle},$$

as desired.

(b) We find u^{nl} from (5-4) using a fixed point argument in the Banach space Z^{nl} with norm

$$\|f\|_{Z^{nl}} = \|h_1^{-1} t^{1.5} f\|_{L^\infty}.$$

Denoting by Z^l the Banach space of functions of the form $\bar{u}^{l,r} + \bar{u}^{l,nr}$ with norm as in (5-5)–(5-6), we will show that the map

$$T : (u, v) \to L^{-1} N(u, v) = h_1(r) \int_r^\infty \frac{N(u, v)}{h_1(s)} ds$$

is locally Lipschitz from $Z^l \times Z^{nl}$ into Z^{nl}, and that in addition, the Lipschitz constant with respect to the second variable v can be made small if either both arguments are small or if u and v are in a bounded set B and the time t is large enough, depending on the size of B. This would imply the existence and uniqueness of $\bar{u}^{nl}$, as well as its Lipschitz dependence on $\bar{u}^l$ and, implicitly, on (w_0, w_1). Recall that

$$N(u, v) = \frac{1}{r}\big[\sin Q \cdot \big(\cos(u+v) - 1\big) + \cos Q \cdot \big(\sin(u+v) - (u+v)\big)\big]$$

Then

$$|N(u, v)| \lesssim \frac{1}{r^2+1}(|u|^2 + |v|^2) + \frac{1}{r}(|u|^3 + |v|^3),$$

$$|\nabla N(u, v)| \lesssim \frac{1}{r^2+1}(|u| + |v|) + \frac{1}{r}(|u|^2 + |v|^2).$$

Hence, it remains to show that

$$\int_0^\infty \frac{1}{r}(|u|^2 + |v|^2) + \frac{r^2+1}{r^2}(|u|^3 + |v|^3) dr \lesssim t^{-1.5}\big(\|u\|_{Z^l}^2 + \|v\|_{Z^{nl}}^2 + \|u\|_{Z^l}^3 + \|v\|_{Z^{nl}}^3\big).$$

For u, we have two components u^r and u^{nr}, and therefore we need to consider the six integrals

$$\int_0^\infty \frac{1}{r}|u^r|^2 dr \lesssim \int_0^\infty \frac{1}{r} \frac{h_1^2(r)}{(t \log^2 t)^2} dr \cdot \|u\|_{Z^l}^2 \approx \frac{1}{t^2 \log^4 t} \|u\|_{Z^l}^2,$$

$$\int_0^\infty \frac{1}{r}|u^{nr}|^2 dr \lesssim \int_0^\infty \frac{1}{r} \frac{r^2}{(t+r)^2 t \langle t-r \rangle^3 \log^2 \langle t-r \rangle} dr \cdot \|u\|_{Z^l}^2 \approx \frac{1}{t^2} \|u\|_{Z^l}^2,$$

$$\int_0^\infty \frac{1}{r}|v|^2 dr \lesssim \int_0^\infty \frac{1}{r} h_1^2(r) t^{-3} dr \cdot \|v\|_{Z^{nl}}^2 \approx \frac{1}{t^3} \|v\|_{Z^{nl}}^2,$$

$$\int_0^\infty \frac{r^2+1}{r^2}|u^r|^3 dr \lesssim \int_0^\infty \frac{r^2+1}{r^2}\frac{h_1^3(r)}{(t\log^2 t)^3}dr\cdot\|u\|_{Z^l}^3 \approx \frac{1}{t^3\log^6 t}\|u\|_{Z^l}^3,$$
$$\int_0^\infty \frac{r^2+1}{r^2}|u^{nr}|^3 dr \lesssim \int_0^\infty \frac{r^2+1}{r^2}\frac{r^3}{(t+r)^3 t^{3/2}\langle t-r\rangle^{9/2}\log^3\langle t-r\rangle}dr\cdot\|u\|_{Z^l}^3 \approx \frac{1}{t^{1.5}}\|u\|_{Z^l}^3,$$
$$\int_0^\infty \frac{r^2+1}{r^2}|v|^3 dr \lesssim \int_0^\infty \frac{r^2+1}{r^2}h_1^3(r)t^{-4.5}dr\cdot\|v\|_{Z^{nl}}^3 \approx \frac{1}{t^{4.5}}\|v\|_{Z^{nl}}^3.$$

We remark that the worst decay $t^{-1.5}$ comes from the penultimate integral above; all other terms are better. Furthermore, this term comes solely from the u dependence of $N(u,v)$. Thus, with our choice of norms, the Lipschitz constant for $L^{-1}N(u,v)$ with respect to u cannot be made small by taking t large; however, the Lipschitz constant with respect to v does have a negative power of t in it.

The argument for $\partial_t\bar{u}^{nl}$ is more involved. Differentiating (5-3), we obtain

$$\begin{aligned} L\left(\partial_t\bar{u}^{nl}+\frac{h_1}{12}(\bar{u}^l)^3\right) &= N_u(\bar{u}^l,\bar{u}^{nl})\partial_t\bar{u}^l + N_v(\bar{u}^l,\bar{u}^{nl})\partial_t\bar{u}^{nl} + \frac{h_1}{12}\partial_r(\bar{u}^l)^3 \\ &= N_v(\bar{u}^l,\bar{u}^{nl})\left(\partial_t\bar{u}^{nl}+\frac{h_1}{12}(\bar{u}^l)^3\right) + \left[N_u(\bar{u}^l,\bar{u}^{nl}) - \frac{h_1}{4}(\bar{u}^l)^2\right]\partial_t\bar{u}^l \\ &\quad - \tfrac{1}{12}N_v(\bar{u}^l,\bar{u}^{nl})h_1(\bar{u}^l)^3 + \frac{h_1}{12}(\partial_t+\partial_r)(\bar{u}^l)^3. \end{aligned} \tag{5-16}$$

We interpret this as a linear equation for $w = \partial_t\bar{u}^{nl} + (h_1/12)(\bar{u}^l)^3$, namely,

$$Lw = N_v(\bar{u}^l,\bar{u}^{nl})w + N_1(\bar{u}^l,\bar{u}^{nl}).$$

The approach is similar to what we have done before. We adjust the base space to

$$\|f\|_{\tilde{Z}^{nl}} = \|h_1^{-1}t^2 f\|_{L^\infty}$$

and prove that $w \to L^{-1}(N_v(\bar{u}^l,\bar{u}^{nl})w)$ is bounded from $\tilde{Z}^{nl}$ to $\tilde{Z}^{nl}$ with small norm, and also Lipschitz with respect to $(\bar{u}^l,\bar{u}^{nl}) \in Z^l\times Z^{nl}$ (but not necessarily with small Lipschitz constant), and also that $L^{-1}N_1$ is Lipschitz from $Z^l\times Z^{nl}$ to $\tilde{Z}^{nl}$ (no smallness needed).

The first bound above follows from the previous computation. The main cancellation occurs in the first term in N_1, where the $(\bar{u}^l)^2$ term disappears. Precisely, we have

$$N_u(u,v) - \tfrac{1}{4}h_1u^2 = -\frac{2}{1+r^2}\sin(u+v) - \frac{1-r^2}{r(1+r^2)}\bigl(1-\cos(u+v)\bigr) - \frac{r}{2(1+r^2)}u^2,$$

and therefore

$$\bigl|N_u(u,v) - \tfrac{1}{4}h_1u^2\bigr| \lesssim \frac{1}{1+r^2}(|u|+|v|) + \frac{1}{r}\bigl(|u|^3+|u||v|+|v|^2\bigr) + \frac{1}{r(1+r^2)}|u|^2.$$

For $\partial_t\bar{u}^l$, we use the same bounds as for $\bar{u}^l$. Then, compared with the previous computation, we need to reestimate the terms involving $|u|^3$, $|u||v|$ and $|u|^2$. The resonant part of u yields better bounds, so we only estimate terms involving u^{nr}:

$$\int_0^\infty \frac{r^2+1}{r^2}|u^{nr}|^4 dr \lesssim \|u\|_{Z^l}^4 \cdot \int_0^\infty \frac{r^2+1}{r^2}\,\frac{r^4}{(t+r)^4 t^2 \langle t-r\rangle^6 \log^4\langle t-r\rangle}\,dr \approx \frac{1}{t^2}\|u\|_{Z^l}^4,$$

$$\int_0^\infty \frac{r^2+1}{r^2}|u^{nr}|^2|v|\,dr \lesssim \|u\|_{Z^l}^2\|v\|_{Z^{nl}} \int_0^\infty \frac{r^2+1}{r^2}\,\frac{r^2}{(t+r)^2 t^{2.5}\langle t-r\rangle^3 \log^2\langle t-r\rangle}\,dr \approx \frac{1}{t^{2.5}}\|u\|_{Z^l}^2\|v\|_{Z^{nl}},$$

$$\int_0^\infty \frac{1}{r^2}|u^{nr}|^3 dr \lesssim \|u\|_{Z^l}^3 \cdot \int_0^\infty \frac{1}{r^2}\,\frac{r^3}{(t+r)^3 t^{1.5}\langle t-r\rangle^{4.5} \log^3\langle t-r\rangle}\,dr \approx \frac{1}{t^{3.5}}\|u\|_{Z^l}^4.$$

The third term on the right in (5-16) is better behaved than the second. Finally, for the last term in (5-16), we invoke (5-7) so that we use the same bounds for $(\partial_t+\partial_r)(\bar{u}^l)$ as for $r^{-1}\bar{u}^l$. Then the integral to estimate is

$$\int_0^\infty \frac{1}{r}|u|^3 dr \lesssim \frac{1}{t^{2.5}}\|u\|_{Z^l}^3.$$

(c) In the case of $\bar{u}^l$, this part follows from the linearity. In the case of $\bar{u}^{nl}$, the Lipschitz dependence on $\bar{u}^l$ has already been discussed above. An additional argument is required for $\delta\partial_t\bar{u}^{nl}$. However, nothing new happens there, and the details are left for the reader. □

6. The transition between γ and ε

In this section, we study the transition from γ to ε, which were both introduced in (2-13). This transition is described by (2-17), which we recall for convenience:

$$\gamma = \partial_r\varepsilon - \frac{\sin(\varepsilon+\bar{u}) - \sin\bar{u}}{r}.$$

The main result of this section is the following:

Proposition 6.1. (a) *Assume that $\gamma \in LX$ is small and $\bar{u}$, $\bar{w}$ are as in Proposition 5.2. Then for t large enough, there exists a unique solution $\varepsilon \in X$ of (2-17) which satisfies*

$$\|\varepsilon\|_X \lesssim_S \|\gamma\|_{LX}. \tag{6-1}$$

Furthermore, ε has a Lipschitz dependence on both γ and the linear data (w_0, w_1) for $\bar{w}$:

$$\|\delta\varepsilon\|_X \lesssim_S \|\delta\gamma\|_{LX} + \frac{1}{t\log^2 t}\|(\delta w_0, \delta w_1)\|_S\|\gamma\|_{LX}. \tag{6-2}$$

(b) *Also, if γ is a function of t, then*

$$\|\partial_t\varepsilon\|_X \lesssim_S \|\partial_t\gamma\|_{LX} + \frac{1}{t\log^2 t}\|\gamma\|_{LX}, \tag{6-3}$$

with the corresponding Lipschitz dependence

$$\|\delta\partial_t\varepsilon\|_X \lesssim_S \|\delta\partial_t\gamma\|_{LX} + \frac{1}{t\log^2 t}\|\delta\gamma\|_{LX} + \|(\delta w_0, \delta w_1)\|_S\Big(\|\partial_t\gamma\|_{LX} + \frac{1}{t\log^2 t}\|\gamma\|_{LX}\Big). \tag{6-4}$$

(c) *Assume in addition that $\gamma \in L^\infty$. Then*

$$|\varepsilon(r)| \lesssim_S r \log r \|\gamma\|_{LX \cap L^\infty}, \quad r \ll 1, \tag{6-5}$$

with a similar Lipschitz dependence.

Proof. (a) Equation (2-17) is rewritten as

$$L\varepsilon = \gamma + \frac{\sin(\varepsilon + \bar{u}) - \sin \bar{u} - \cos Q \cdot \varepsilon}{r} := \gamma + F(\varepsilon, \bar{u} - Q). \tag{6-6}$$

Hence, in order to prove both (6-1) and (6-2), it suffices to show that at fixed large enough time, the map F is Lipschitz:

$$F : X \times (Z^l + Z^{nl}) \to LX,$$

with a small Lipschitz constant in the second variable. For the X norm, we use the embeddings (3-18)–(3-21). For the LX norm, we use (3-23), which shows that it is enough to estimate $F(\bar{u}, \varepsilon)$ in $L^1 \cap L^2$. We expand F as follows:

$$\begin{aligned} F(\beta, v) &= \frac{\sin(\beta + Q + v) - \sin(Q + v) - \cos Q \cdot \beta}{r} \\ &= \frac{(\cos(Q + v) - \cos Q) \cdot \beta}{r} - \frac{\sin(Q + v) \cdot \beta^2}{2r} + \frac{O(\beta^3)}{r} \\ &= -\frac{\sin Q \cdot v\beta}{r} - \frac{\sin Q \cdot \beta^2}{2r} + \frac{O(v^2\beta)}{r} + \frac{O(\beta^3)}{r}. \end{aligned}$$

Hence

$$|F(\beta, v)| \lesssim \frac{|v||\beta|}{1 + r^2} + \frac{|\beta|^2}{1 + r^2} + \frac{|\beta|^3}{r} + \frac{|v|^2|\beta|}{r}. \tag{6-7}$$

By using (3-20), (3-18) and (5-6), we bound this first in L^2,

$$\begin{aligned} \|F(\beta, v)\|_{L^2} &\lesssim \left\| \frac{\beta}{\log(1 + r)} \right\|_{L^2} \left(\|\beta\|_{L^\infty} + \|\beta\|_{L^\infty}^2 + \left\| \frac{v}{1 + r} \right\|_{L^\infty} + \left\| \frac{v^2 \log(1 + r)}{r} \right\|_{L^\infty} \right) \\ &\lesssim \|\beta\|_X^2 + \|\beta\|_X^3 + \|\beta\|_X \left(\frac{1}{t \log^2 t} \|v\|_{Z^l + Z^{nl}} + \frac{\log t}{t^2} \|v\|_{Z^l + Z^{nl}}^2 \right), \end{aligned}$$

and then in L^1,

$$\begin{aligned} \|F(\beta, v)\|_{L^1} &\lesssim \left\| \frac{\beta}{\log(1 + r)} \right\|_{L^2}^2 (1 + \|\beta\|_{L^\infty}) \\ &\quad + \left\| \frac{\beta}{\log(1 + r)} \right\|_{L^2} \left(\left\| \frac{v \log(1 + r)}{1 + r^2} \right\|_{L^2} + \left\| \frac{v^2 \log(1 + r)}{r} \right\|_{L^2} \right) \\ &\lesssim \|\beta\|_X^2 + \|\beta\|_X^3 + \|\beta\|_X \left(\frac{1}{t \log^2 t} \|v\|_{Z^l + Z^{nl}} + \frac{\log t}{t^{3/2}} \|v\|_{Z^l + Z^{nl}}^2 \right). \end{aligned}$$

Hence we obtain

$$\|F(\beta, v)\|_{LX} \lesssim \|\beta\|_X^2 + \|\beta\|_X^3 + \|\beta\|_X \left(\frac{1}{t \log^2 t} \|v\|_{Z^l + Z^{nl}} + \frac{\log t}{t^{3/2}} \|v\|_{Z^l + Z^{nl}}^2 \right).$$

A similar analysis yields

$$\|\beta_1 F_\beta(\beta, v)\|_{LX} \lesssim \|\beta_1\|_X\left(\|\beta\|_X + \|\beta\|_X^2 + \frac{1}{t\log^2 t}\|v\|_{Z^l+Z^{nl}} + \frac{\log t}{t^{3/2}}\|v\|^2_{Z^l+Z^{nl}}\right),$$

$$\|v_1 F_v(\beta, v)\|_{LX} \lesssim \|v_1\|_{Z^l+Z^{nl}}\|\beta\|_X\left(\frac{1}{t\log^2 t} + \frac{\log t}{t^{3/2}}\|v\|_{Z^l+Z^{nl}}\right).$$

By the contraction principle, this proves both (6-1) and (6-2). The time decaying factors guarantee that for any size of $\bar{u} - Q$, the problem can be solved for large enough time.

(b) To prove (6-3), we differentiate with respect to t in (6-6):

$$L\partial_t\varepsilon = \partial_t\gamma + F_\varepsilon(\varepsilon, \bar{u})\partial_t\varepsilon + F_{\bar{u}}(\varepsilon, \bar{u})\partial_t\bar{u}.$$

Since $\partial_t\bar{u}$ satisfies the same pointwise bounds as $\bar{u}$, the last two estimates above show that the contraction principle still applies.

(c) Due to the embedding $X \subset \dot{H}^1_e \subset L^\infty$, we already have a small uniform bound for ε. We solve the ODE (6-6) in $[0, 1]$ with Cauchy data at $r = 1$. Making the bootstrap assumption

$$|\varepsilon| \le Mr\left|\log\frac{r}{2}\right|, \tag{6-8}$$

we rewrite (6-6) in the form

$$|L\varepsilon - \gamma| \le M^3 r^2\left|\log^3\frac{r}{2}\right| + C, \qquad C \approx_S \|\varepsilon\|_{L^\infty}.$$

Then solving the linear L evolution, we have

$$|\varepsilon| \lesssim r(|\gamma(1)| + M^3) + Cr\left|\log\frac{r}{2}\right| \lesssim_S M^3 r + r\left|\log\frac{r}{2}\right|\|\varepsilon\|_{L^\infty}.$$

If $\|\varepsilon\|_{L^\infty}$ is sufficiently small, then we can choose M small enough that the above bound is stronger than our bootstrap assumption (6-8). The proof of (6-5) is concluded. □

7. Perturbative analysis in the γ equation

Our main goal is to solve (2-16) for γ with zero Cauchy data at $t = \infty$. Using the backward linear parametrix K introduced in (4-2), Equation (2-16) is rewritten in the form

$$\gamma = KN(\bar{u} + \varepsilon, \bar{w} + \gamma), \tag{7-1}$$

where the auxiliary function ε and its time derivative ε_t are uniquely determined by γ and γ_t via Proposition 6.1.

Our strategy is to solve (7-1) using the contraction principle in the space $\boldsymbol{E}$ with norm

$$\|\gamma\|_{\boldsymbol{E}} = \sup_{t>t_0} t^{1.5}\|\gamma\|_{LX} + t^{2.5}\left(\|\partial_t\gamma\|_{LX} + \|\gamma\|_{\dot{H}^1_e}\right),$$

for a suitably chosen t_0. By Proposition 6.1, this yields control for ε in the space $\boldsymbol{G}$ with norm

$$\|\varepsilon\|_{\boldsymbol{G}} = \sup_{t>t_0} t^{1.5}\left(\|\varepsilon\|_X + \|r^{-1/2}\varepsilon\|_{L^\infty}\right) + t^{2.5}\|\partial_t\varepsilon\|_{LX}.$$

For the linear $\tilde{H}$ wave equation, we use the LX bounds in Lemma 4.1 with $\alpha = 1.5$. Thus we need to estimate the nonlinearity $N(\bar{u}+\varepsilon, \bar{w}+\gamma)$ in the space $\boldsymbol{N}$ with norm

$$\|N\|_{\boldsymbol{N}} = \sup_{t>t_0} t^{3.5}\|N(t)\|_{LX}.$$

Finally, all the implicit constants in our estimates depend on $\|(w_0, w_1)\|_S$ and need not be small. Thus we need a different source of smallness, which is an additional time decay factor, incorporated in the stronger norm $\boldsymbol{N}^\sharp$ defined by

$$\|N\|_{\boldsymbol{N}^\sharp} = \sup_{t>t_0} t^{3.5}(\log t)^2\|N(t)\|_{LX}.$$

With this notation, our main estimates for the nonlinearity $N(\bar{u}+\varepsilon, \bar{w}+\gamma)$ are as follows:

Proposition 7.1. *Assume that the Schwartz functions (w_0, w_1) satisfy the nonresonance conditions (2-11). Then*:

(a) *The map $(w_0, w_1) \to N(\bar{u}, \bar{w})$ is locally Lipschitz from S to $\boldsymbol{N}$.*

(b) *The map $(w_0, w_1, \gamma, \varepsilon) \to N(\bar{u}+\varepsilon, \bar{w}+\gamma) - N(\bar{u}, \bar{w})$ is locally Lipschitz from $S \times \boldsymbol{E} \times \boldsymbol{G}$ to $\boldsymbol{N}^\sharp$.*

In view of Lemma 4.1 and Proposition 6.1, the above result allows us to solve (7-1) for γ in the ball

$$B = \{\|\gamma - KN(\bar{u}, \bar{w})\|_{\boldsymbol{E}}\},$$

for $t > t_0$, via the contraction principle, provided that t_0 is chosen to be sufficiently large. This concludes the proof of Theorem 2.1.

We note that in terms of time decay we gain only logarithms, whereas the implicit constants in our estimates are all polynomial in $\|(w_0, w_1)\|_S$. This implies that for large Schwartz data (w_0, w_1) in the linear equation, our solutions are only defined for $t > T$, with T exponentially large.

Proof of Proposition 7.1. We recall that N is given by

$$N(w, u) = \frac{2(\cos Q - \cos u)}{r^2} w + \frac{1}{r}\sin u(u_t^2 - w^2).$$

We split the difference $N(\bar{w}+\gamma, \bar{u}+\varepsilon) - N(\bar{w}, \bar{u})$ as

$$N(w, u) - N(\bar{w}, \bar{u}) = N^l(\bar{w}, \bar{u}, \gamma, \varepsilon) + N^n(\bar{w}, \bar{u}, \gamma, \varepsilon).$$

The term N^l contains the linear contributions in ε, γ in the difference $N(w, u) - N(\bar{w}, \bar{u})$:

$$N^l = \frac{2(\cos Q - \cos \bar{u})}{r^2}\gamma + \frac{2\sin \bar{u}\cdot\varepsilon}{r^2}\bar{w} + \frac{\sin \bar{u}(2\bar{u}_t\varepsilon_t - 2\bar{w}\gamma) + \cos \bar{u}\cdot\varepsilon(\bar{u}_t^2 - \bar{w}^2)}{r}.$$

The remaining term N^n contains the genuinely nonlinear contributions in ε, γ in the difference $N(w, u) - N(\bar{w}, \bar{u})$:

$$N^n = \frac{2(\cos \bar{u} - \cos u - \sin \bar{u}\cdot\varepsilon)}{r^2}\bar{w} + \frac{2(\cos \bar{u} - \cos(\bar{u}+\varepsilon))}{r^2}\gamma + \frac{\sin \bar{u}(\varepsilon_t^2 - \gamma^2)}{r} + \frac{(\sin u - \sin \bar{u})(2\bar{u}_t\varepsilon_t - 2\bar{w}\gamma + \bar{u}_t^2 - \bar{w}^2)}{r} + \frac{(\sin u - \sin \bar{u} - \cos \bar{u}\cdot\varepsilon)(\bar{u}_t^2 - \bar{w}^2)}{r}.$$

We will consider separately the expressions $N(\bar{u}, \bar{w})$, N^l and N^n.

The term $N(\bar{w}, \bar{u})$. Our main goal here is to prove the estimate

$$\|N(\bar{w}, \bar{u})\|_{LX} \lesssim_S t^{-3.5}. \tag{7-2}$$

We also need to show that $N(\bar{w}, \bar{u})$ has a Lipschitz dependence on (w_0, w_1). However, as the leading order part of $N(\bar{w}, \bar{u})$ is multilinear, the proof of that follows the same lines as below and is omitted.

To establish (7-2), we split

$$N(\bar{w}, \bar{u}) = \chi_{r \ll t} N(\bar{w}, \bar{u}) + \chi_{r \gg t} N(\bar{w}, \bar{u}) + \chi_{r \approx t} N(\bar{w}, \bar{u}) = N_1 + N_2 + N_3.$$

For the first two terms, it suffices to use a direct estimate:

$$|N(\bar{w}, \bar{u})| \lesssim \frac{\sin Q}{r^2} |\bar{u} - Q||\bar{w}| + \frac{1}{r^2} |\bar{u} - Q|^2 |\bar{w}| + \frac{1}{r} (\sin Q + |\bar{u}|)(|\bar{u}_t|^2 + |\bar{w}|^2).$$

Using the bounds (5-6) and (5-8) for $\bar{u} - Q$, as well as the bound (5-2) for $\bar{w}$, this gives

$$|N_1(\bar{w}, \bar{u})| \lesssim_S \chi_{r \ll t} \frac{1}{\langle r \rangle^4 t^4},$$

where the leading contribution comes from $u^{l,r}$. This implies that

$$\|N_1\|_{L^1 \cap L^2} \lesssim_S t^{-4},$$

which suffices for (7-2) in view of the embedding (3-23). Similarly,

$$|N_2| \lesssim_S \chi_{r \gg t} \frac{1}{\langle r \rangle^8},$$

which also gives

$$\|N_2\|_{L^1 \cap L^2} \lesssim_S t^{-4}.$$

However, a similar direct computation for N_3 only gives

$$|N_3(\bar{w}, \bar{u})| \lesssim_S \chi_{r \sim t} \frac{1}{t^{2.5} \langle t - r \rangle^{5.5}},$$

which fails by two units,

$$\|N_3\|_{L^1 \cap L^2} \lesssim_S t^{-1.5}.$$

Hence, in order to conclude the proof of (7-2), we need to better exploit the structure of N and capture a double cancellation on the null cone. In the computations below (through the end of the subsection), we work in the regime $r \approx t$. We expand $N(\bar{w}, \bar{u})$ as

$$\begin{aligned} N(\bar{w}, \bar{u}) = {} & 2 \frac{\sin Q}{r^2} (\bar{u} - Q) \bar{w} + \frac{\cos Q}{r^2} (\bar{u} - Q)^2 \bar{w} + \frac{\sin Q}{r} (\bar{u}_t^2 - \bar{w}^2) + \frac{\cos Q}{r} (\bar{u}_t^2 - \bar{w}^2)(\bar{u} - Q) \\ & + \frac{\sin Q}{r^2} w O((\bar{u} - Q)^3) + \frac{\cos Q}{r^2} w O((\bar{u} - Q)^4) \\ & + \frac{\sin Q}{r} (\bar{u}_t^2 - \bar{w}^2) O((\bar{u} - Q)^2) + \frac{\cos Q}{r} (\bar{u}_t^2 - \bar{w}^2) O((\bar{u} - Q)^3). \end{aligned}$$

The terms on the second line are already acceptable; i.e., it can be estimated by $t^{-4.5}\langle t-r\rangle^{-3.5}$. For further progress, we observe that by (5-8) we have

$$\bar{u}^{nl} = O_S(t^{-2.5}), \quad \partial_t \bar{u}^{nl} = O_S\big(t^{-2.5}\langle t-r\rangle^{-0.5}\big),$$

and that by (5-7), we can write

$$\partial_t \bar{u} + \bar{w} = \partial_t \bar{u}^{nl} + \partial_t \bar{u}^l + \partial_r \bar{u}^l + \frac{\cos Q}{r}\bar{u}^l = O_S\big(t^{-1.5}\langle t-r\rangle^{-1.5}\big). \tag{7-3}$$

The first relation above allows us to dispense with $\bar{u}^{nl}$ everywhere and replace $\bar{u}-Q$ by $\bar{u}^l$, and the second allows us to estimate the third line in $N(\bar{w},\bar{u})$. We are left with

$$N(\bar{w},\bar{u}) = 2\frac{\sin Q}{r^2}\bar{u}^l\bar{w} + \frac{\cos Q}{r^2}(\bar{u}^l)^2\bar{w} + \frac{\sin Q}{r}\big((\bar{u}^l_t)^2 - \bar{w}^2\big) + \frac{\cos Q}{r}\big((\bar{u}^l_t)^2 - \bar{w}^2\big)\bar{u}^l + O_S\big(t^{-4.5}\langle t-r\rangle^{-3.5}\big).$$

To advance further, we substitute $\bar{w} = \partial_r \bar{u}^l - (\cos Q/r)\bar{u}^l$ everywhere. The $(\cos Q/r)\bar{u}^l$ is acceptable in the first two terms of N, that is, it gives contributions of $O_S(t^{-4.5}\langle t-r\rangle^{-3.5})$, and we discard it. For the last two terms, we use the better approximation from (5-7):

$$\bar{u}^l_t = -\partial_r \bar{u}^l - \frac{1}{2r}\bar{u}^l + O\big(t^{-2.5}\langle t-r\rangle^{-0.5}\big).$$

Then we can write

$$\begin{aligned}(\bar{u}^l_t)^2 - \bar{w}^2 &= \Big(\partial_r \bar{u}^l + \frac{1}{2r}\bar{u}^l\Big)^2 - \Big(\partial_r \bar{u}^l - \frac{\cos Q}{r}\bar{u}^l\Big)^2 + O_S\big(t^{-3}\langle t-r\rangle^{-3}\big)\\ &= -\frac{1}{r}\bar{u}^l\partial_r\bar{u}^l + O_S\big(t^{-3}\langle t-r\rangle^{-3}\big).\end{aligned}$$

It is also harmless to replace $\sin Q$ by r^{-1} and $\cos Q$ by -1 everywhere. Returning to N, we obtain

$$\begin{aligned}N(\bar{w},\bar{u}) &= \frac{2}{r^3}\bar{u}^l\partial_r\bar{u}^l - \frac{1}{r^2}(\bar{u}^l)^2\partial_r\bar{u}^l - \frac{1}{r^3}u^l\partial_r u^l + \frac{1}{r^2}(\bar{u}^l)^2\partial_r\bar{u}^l + O_S\big(\langle t\rangle^{-4.5}\langle t-r\rangle^{3.5}\big)\\ &= \frac{1}{2r^3}\partial_r(\bar{u}^l)^2 + O_S\big(t^{-4.5}\langle t-r\rangle^{-3.5}\big)\end{aligned}$$

in the region $r\approx t$, which we rewrite as

$$N_3 = Lg + \chi_{r\approx t}O_S\big(t^{-4.5}\langle t-r\rangle^{-3.5}\big), \quad g = \chi_{r\approx t}\frac{1}{2r^3}(\bar{u}^l)^2.$$

The last term can be directly estimated in $L^1\cap L^2$. For the leading term Lg, we estimate g in H^1_e and use the embedding (3-18). We have

$$|g| \lesssim_S \frac{1}{t^4\langle t-r\rangle^3}, \quad |\partial_r g| \lesssim_S \frac{1}{t^4\langle t-r\rangle^4},$$

and therefore

$$\|g\|_{H^1_e} \lesssim_S \frac{1}{t^{3.5}}.$$

This concludes the proof of (7-2).

The bound for N^l. Our goal here is to establish the bound

$$\|N^l(t)\|_{LX} \lesssim_S \frac{1}{t^{3.5}\log^2 t}\big(\|\gamma\|_{\boldsymbol{G}} + \|\varepsilon\|_{\boldsymbol{E}}\big). \tag{7-4}$$

The proof of the Lipschitz dependence on (w_0, w_1) is again similar and therefore omitted.

We recall that

$$N^l = \frac{2(\cos Q - \cos \bar{u})}{r^2}\gamma + \frac{2\sin\bar{u}\cdot\varepsilon}{r^2}\bar{w} + \frac{\sin\bar{u}(2\bar{u}_t\varepsilon_t - 2\bar{w}\gamma) + \cos\bar{u}\cdot\varepsilon(\bar{u}_t^2 - \bar{w}^2)}{r}.$$

The pointwise estimate

$$\left|\frac{2(\cos Q - \cos\bar{u})}{r}\right| \lesssim \frac{1}{r^2+1}|\bar{u} - Q| + \frac{1}{r}|\bar{u} - Q|^2,$$

combined with the pointwise bounds for $\bar{u}$ from (5-6), leads to

$$\left\|\frac{2(\cos Q - \cos\bar{u})}{r}\right\|_{L^\infty\cap L^2} \lesssim_S \frac{1}{t\log^2 t},$$

with the worst contribution arising from the resonant part of $\bar{u}$. From (3-23), it follows that

$$\left\|\frac{2(\cos Q - \cos\bar{u})}{r^2}\gamma\right\|_{LX} \lesssim \left\|\frac{2(\cos Q - \cos\bar{u})}{r}\right\|_{L^\infty\cap L^2}\cdot\left\|\frac{\gamma}{r}\right\|_{L^2} \lesssim_S \frac{1}{t^{3.5}\log^2 t}\|\gamma\|_{\boldsymbol{G}}.$$

Next, from (5-6) and (5-2), it follows that

$$\left\|\frac{\bar{u}\cdot\bar{w}}{r^2}\log(2+r)\right\|_{L^\infty\cap L^2} \lesssim_S \frac{\log t}{t^{2.5}},$$

which, combined with

$$\left\|\frac{\varepsilon}{\log(2+r)}\right\|_{L^2} \lesssim \|\varepsilon\|_X \lesssim t^{-1.5}\|\varepsilon\|_{\boldsymbol{E}}$$

(recall (3-20)), gives

$$\left\|\frac{2\sin\bar{u}\cdot\varepsilon}{r^2}\bar{w}\right\|_{LX} \lesssim_S \frac{\log t}{t^4}\|\varepsilon\|_{\boldsymbol{E}}.$$

Using (5-6), we obtain

$$\left\|\frac{\bar{u}\bar{u}_t}{r}\log(2+r)\right\|_{L^\infty\cap L^2} \lesssim_S \frac{\log t}{t^{1.5}},$$

and therefore, by invoking (3-23) and (3-20), it follows that

$$\left\|\frac{\sin(\bar{u})\cdot\bar{u}_t\varepsilon_t}{r}\right\|_{LX} \lesssim \left\|\frac{\bar{u}\bar{u}_t}{r}\log(2+r)\right\|_{L^\infty\cap L^2}\left\|\frac{\varepsilon_t}{\log(2+r)}\right\|_{L^2} \lesssim_S \frac{\log t}{t^4}\|\varepsilon\|_{\boldsymbol{E}}.$$

The following term in N^l requires some extra work. Using (5-6) and (5-2), we note that away from the cone, we have $|\sin(\bar{u})| \lesssim \sin Q$, and continue with

$$\left\|\chi_{r\not\approx t}\frac{\bar{w}\sin\bar{u}}{r}\right\|_{L^1\cap L^2} \lesssim_S t^{-2},$$

followed by

$$\left\| \chi_{r\not\approx t} \frac{\sin(\bar{u}) \cdot \bar{w}\gamma}{r} \right\|_{LX} \lesssim \left\| \chi_{r\not\approx t} \frac{\bar{u}\bar{w}}{r} \right\|_{L^1 \cap L^2} \|\gamma\|_{L^\infty} \lesssim_S t^{-4.5} \|\gamma\|_{\boldsymbol{G}}.$$

Near the cone, we write

$$\begin{aligned} \chi_{r\approx t} \frac{\bar{w} \sin \bar{u}}{r} &= \chi_{r\approx t} \left(\frac{2\bar{w}}{1+r^2} - \frac{\bar{w}(\bar{u}-Q)}{r} \cos Q + \frac{\bar{w}\, O((\bar{u}-Q)^2)}{1+r^2} + \frac{\bar{w}\, O((\bar{u}-Q)^3)}{r} \right) \\ &= \chi_{r\approx t} \frac{\bar{w}(\bar{u}-Q)}{r} + O_S\big(t^{-2.5} \langle t-r \rangle^{-2.5}\big) \\ &= L\big(\chi_{r\approx t} r^{-1} (\bar{u}^l)^2\big) + O_S\big(t^{-2.5} \langle t-r \rangle^{-2.5}\big). \end{aligned}$$

The output of the second term is estimated as above in $L^1 \cap L^2$, and yields a contribution of $t^{-4} \|\varepsilon\|_{\boldsymbol{E}}$ to the $\|N^l\|_{LX}$ bound. For the first term, we write its contribution to N^l in the form

$$L\big(\chi_{r\approx t} r^{-1} (\bar{u}^l)^2\big)\gamma = L\big(\chi_{r\approx t} r^{-1} (\bar{u}^l)^2 \gamma\big) + \chi_{r\approx t} r^{-1} (\bar{u}^l)^2 \partial_r \gamma.$$

Then, using (3-18) for the first term and (3-23) for the second term, we have

$$\begin{aligned} \big\| L\big(\chi_{r\approx t} r^{-1} (\bar{u}^l)^2\big)\gamma \big\|_{LX} &\lesssim \big\| \chi_{r\approx t} r^{-1} (\bar{u}^l)^2 \gamma \big\|_{H^1_e} + \big\| \chi_{r\approx t} r^{-1} (\bar{u}^l)^2 \partial_r \gamma \big\|_{L^1 \cap L^2} \\ &\lesssim \big\| \chi_{r\approx t} r^{-1} (\bar{u}^l)^2 \big\|_{H^1_e} \|\gamma\|_{\dot{H}^1_e} + \big\| \chi_{r\approx t} r^{-1} (\bar{u}^l)^2 \big\|_{L^2 \cap L^\infty} \|\partial_r \gamma\|_{L^2} \\ &\lesssim_S t^{-1.5} \|\gamma\|_{\dot{H}^1_e} \lesssim_S t^{-4} \|\gamma\|_{\boldsymbol{G}}. \end{aligned}$$

It remains to bound the last term in N^l. For this, we take advantage of the first-order cancellation on the cone in the expression $\bar{u}_t - \bar{w}$ (see (7-3)), which, combined with (5-6) and (5-2), gives

$$\left\| \frac{\cos \bar{u} (\bar{u}_t^2 - \bar{w}^2) \log(2+r)}{r} \right\|_{L^2 \cap L^\infty} \lesssim_S \frac{\log t}{t^{2.5}}.$$

This leads to

$$\left\| \varepsilon \frac{\cos \bar{u} (\bar{u}_t^2 - \bar{w}^2)}{r} \right\|_{L^1 \cap L^2} \lesssim_S \frac{\log t}{t^{2.5}} \left\| \frac{\varepsilon}{\log(2+r)} \right\|_{L^2} \lesssim_S \frac{\log t}{t^{2.5}} \|\varepsilon\|_X \lesssim_S \frac{\log t}{t^4} \|\varepsilon\|_{\boldsymbol{E}}.$$

This concludes the proof of the N^l bound (7-4).

The bound for N^n. Our goal here will be to prove the bound

$$\|N^n\|_{LX} \lesssim_S \frac{\log t}{t^4} (M^2 + M^3), \qquad M = \|\gamma\|_{\boldsymbol{G}} + \|\varepsilon\|_{\boldsymbol{E}}, \tag{7-5}$$

which is almost $t^{-.5}$ better than what we need. The corresponding Lipschitz dependence argument is similar and thus omitted. We recall the expression of N^n:

$$\begin{aligned} N^n = \frac{2(\cos \bar{u} - \cos u - \sin \bar{u} \cdot \varepsilon)}{r^2} \bar{w} + \frac{2(\cos \bar{u} - \cos(\bar{u}+\varepsilon))}{r^2} \gamma + \frac{\sin u (\varepsilon_t^2 - \gamma^2)}{r} \\ + \frac{(\sin u - \sin \bar{u})(2\bar{u}_t \varepsilon_t - 2\bar{w}\gamma)}{r} + \frac{(\sin u - \sin \bar{u} - \cos \bar{u} \cdot \varepsilon)(\bar{u}_t^2 - \bar{w}^2)}{r}. \end{aligned}$$

We successively consider the terms on the right. For the first one, we start with

$$\left|\frac{2(\cos\bar{u}-\cos u-\sin\bar{u}\cdot\varepsilon)}{r^2}\,\bar{w}\right| \lesssim \frac{\varepsilon^2|\bar{w}|}{r^2}.$$

Then, using (5-2) and (3-20), we obtain

$$\left\|\frac{\varepsilon^2\bar{w}}{r^2}\right\|_{L^1\cap L^2} \lesssim \left\|\frac{\varepsilon}{\log(2+r)}\right\|_{L^\infty\cap L^2}\left\|\frac{\varepsilon}{\log(2+r)}\right\|_{L^2}\left\|\frac{\bar{w}}{r^2}\log^2(2+r)\right\|_{L^\infty} \lesssim_S \frac{\log^2 t}{t^{5.5}}M^2.$$

The second term in N^n is estimated by

$$\left|\frac{\cos\bar{u}-\cos(\bar{u}+\varepsilon)}{r^2}\gamma\right| \lesssim \frac{|\sin\bar{u}\cdot\varepsilon\gamma|}{r^2}+\frac{|\varepsilon^2\gamma|}{r^2} \lesssim \frac{|\varepsilon\gamma|}{r\langle r\rangle^2}+\frac{|(\bar{u}-Q)\varepsilon\gamma|}{r^2}+\frac{|\varepsilon^2\gamma|}{r^2}.$$

The first two terms can be estimated in $L^1\cap L^2$ as before:

$$\left\|\frac{\varepsilon\gamma}{r\langle r\rangle^2}\right\|_{L^1\cap L^2} \lesssim \left\|\frac{\gamma}{r}\right\|_{L^2}\left\|\frac{\varepsilon}{\langle r\rangle^2}\right\|_{L^\infty\cap L^2} \lesssim_S t^{-4}M^2,$$

$$\left\|\frac{(\bar{u}-Q)\varepsilon\gamma}{r^2}\right\|_{L^1\cap L^2} \lesssim \left\|\frac{\gamma}{r}\right\|_{L^2}\left\|\frac{\bar{u}-Q}{r}\right\|_{L^2\cap L^\infty}\|\varepsilon\|_{L^\infty} \lesssim_S t^{-5}M^2.$$

For the last term, we first get the L^1 bound

$$\left\|\frac{\varepsilon^2\gamma}{r^2}\right\|_{L^1} \lesssim \|\varepsilon\|_{L^\infty}\left\|\frac{\varepsilon}{r}\right\|_{L^2}\left\|\frac{\gamma}{r}\right\|_{L^2} \lesssim \frac{1}{t^{5.5}}M^3.$$

However, getting the L^2 bound is more delicate:

$$\left\|\frac{\varepsilon^2\gamma}{r^2}\right\|_{L^2} \lesssim \left\|\frac{\varepsilon}{\sqrt{r}}\right\|_{L^\infty}^2\left\|\frac{\gamma}{r}\right\|_{L^2} \lesssim \frac{1}{t^{5.5}}M^3,$$

where the pointwise bound for $\varepsilon/\sqrt{r}$ near $r=0$ comes from (6-5).

The third term in N is estimated by using (5-6):

$$\left|\frac{\sin u(\varepsilon_t^2-\gamma^2)}{r}\right| \lesssim \frac{|\varepsilon_t|^2}{1+r}+\frac{|\gamma^2|}{1+r}.$$

We successively consider all terms:

$$\left\|\frac{|\varepsilon_t|^2}{1+r}\right\|_{L^1\cap L^2} \lesssim \left\|\frac{\varepsilon_t}{\log(2+r)}\right\|_{L^2\cap L^\infty}\left\|\frac{\varepsilon_t}{\log(2+r)}\right\|_{L^2} \lesssim \frac{1}{t^5}M^2,$$

$$\left\|\frac{|\gamma|^2}{1+r}\right\|_{L^1\cap L^2} \lesssim \|\gamma\|_{L^2\cap L^\infty}\left\|\frac{\gamma}{r}\right\|_{L^2} \lesssim \frac{1}{t^4}M^2.$$

Next we estimate the fourth term in N^n:

$$\left|\frac{(\sin u-\sin\bar{u})(2\bar{u}_t\varepsilon_t-2\bar{w}\gamma)}{r}\right| \lesssim \frac{|\varepsilon|\big(|\bar{u}_t\varepsilon_t|+|\bar{w}\gamma|\big)}{r}.$$

On behalf of (5-2), (5-6) and (3-20), we have

$$\left\| \frac{\varepsilon \bar{u}_t \varepsilon_t}{r} \right\|_{L^1 \cap L^2} \lesssim \|\varepsilon_t\|_{L^\infty} \left\| \frac{\varepsilon}{\log(2+r)} \right\|_{L^2} \left\| \frac{\bar{u}_t}{r} \log(2+r) \right\|_{L^\infty \cap L^2} \lesssim_S \frac{\log t}{t^4} M^2,$$

$$\left\| \frac{\varepsilon \bar{w} \gamma}{r} \right\|_{L^1 \cap L^2} \lesssim \|\varepsilon\|_{L^\infty} \|\bar{w}\|_{L^2 \cap L^\infty} \left\| \frac{\gamma}{r} \right\|_{L^2} \lesssim_S t^{-4} M^2.$$

Finally we consider the last term in N^n,

$$\left| \frac{(\sin u - \sin \bar{u} - \cos \bar{u} \cdot \varepsilon)(\bar{u}_t^2 - \bar{w}^2)}{r} \right| \lesssim \frac{\varepsilon^2 (\bar{u}_t^2 + \bar{w}^2)}{r},$$

which, by using (5-2), (5-6) and (3-20), we further bound as follows:

$$\left\| \frac{\varepsilon^2 (\bar{u}_t^2 + \bar{w}^2)}{r} \right\|_{L^1 \cap L^2} \lesssim \left\| \frac{\varepsilon}{\log(2+r)} \right\|_{L^2} \left\| \frac{\varepsilon}{\log(2+r)} \right\|_{L^2 \cap L^\infty} \left\| \frac{\bar{u}_t^2 + \bar{w}^2}{r} \log^2(2+r) \right\|_{L^\infty} \lesssim_S \frac{\log^2 t}{t^5} M^2. \qquad \square$$

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IOAN BEJENARU: ibejenar@ucsd.edu
Department of Mathematics, University of California at San Diego, 9500 Gilman Dr., La Jolla 92093-011, United States

JOACHIM KRIEGER: joachim.krieger@epfl.ch
Batiment des Mathématiques, École Polytechnique F'ed'erale de Lausanne, Station 8, CH-1015 Lausanne, France

DANIEL TATARU: tataru@math.berkeley.edu
Department of Mathematics, University of California at Berkeley, Evans Hall, Berkeley, CA 94720, United States

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DISCRETE FOURIER RESTRICTION ASSOCIATED WITH KDV EQUATIONS

YI HU AND XIAOCHUN LI

In this paper, we consider a discrete restriction associated with KdV equations. Some new Strichartz estimates are obtained. We also establish the local well-posedness for the periodic generalized Korteweg–de Vries equation with nonlinear term $F(u)\partial_x u$ provided $F \in C^5$ and the initial data $\phi \in H^s$ with $s > 1/2$.

1. Introduction

The discrete restriction problem associated with KdV equations is a problem asking the best constant $A_{p,N}$ satisfying

$$\sum_{n=-N}^{N} |\hat{f}(n, n^3)|^2 \le A_{p,N} \|f\|_{p'}^2, \tag{1-1}$$

where f is a periodic function on $\mathbb{T}^2$, $\hat{f}$ is the Fourier transform of f on $\mathbb{T}^2$, $p \ge 2$, and $p' = p/(p-1)$. It is natural to pose a conjecture asserting that for any $\varepsilon > 0$, $A_{p,N}$ satisfies

$$A_{p,N} \le \begin{cases} C_p N^{1-8/p+\varepsilon} & \text{for } p \ge 8, \\ C_p & \text{for } 2 \le p < 8. \end{cases} \tag{1-2}$$

It was proved by Bourgain that $A_{6,N} \le N^{\varepsilon}$. The desired upper bound for $A_{8,N}$ is not yet obtained; however, we are able to establish an affirmative answer for large p.

Theorem 1.1. *Let $A_{p,N}$ be defined as in (1-1). If $p \ge 14$, for any $\varepsilon > 0$, there exists a constant C_p independent of N such that*

$$A_{p,N} \le C_p N^{1-8/p+\varepsilon}. \tag{1-3}$$

The periodic Strichartz inequality associated to KdV equations is the inequality seeking the best constant $K_{p,N}$ satisfying

$$\left\| \sum_{n=-N}^{N} a_n e^{2\pi i t n^3 + 2\pi i x n} \right\|_{L^p_{x,t}(\mathbb{T}\times\mathbb{T})} \le K_{p,N} \left(\sum_{n=-N}^{N} |a_n|^2 \right)^{1/2}. \tag{1-4}$$

By duality, we immediately see that

$$K_{p,N} \sim \sqrt{A_{p,N}}.$$

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Hence, Theorem 1.1 is equivalent to Strichartz estimates,

$$K_{p,N} \leq C N^{1/2-4/p+\varepsilon}, \quad \text{for } p \geq 14. \tag{1-5}$$

It was observed by Bourgain that the periodic Strichartz inequalities (1-4) for $p = 4, 6$ are crucial for obtaining the local well-posedness of periodic KdV (mKdV or gKdV). The local (global) well-posedness of periodic KdV for $s \geq 0$ was first studied by Bourgain [1993b]. Via a bilinear estimate approach, Kenig, Ponce, and Vega [Kenig et al. 1996] established the local well-posedness of periodic KdV for $s > -1/2$. The sharp global well-posedness of the periodic KdV was proved by Colliander, Keel, Staffilani, Takaoka, and Tao [Colliander et al. 2003], by utilizing the I-method.

Inspired by Bourgain's work, we can obtain the following theorem on gKdV. Here the gKdV is the generalized Korteweg–de Vries (gKdV) equation

$$\begin{cases} u_t + u_{xxx} + u^k u_x = 0, \\ u(x,0) = \phi(x), \quad x \in \mathbb{T},\ t \in \mathbb{R}, \end{cases} \tag{1-6}$$

where $k \in \mathbb{N}$ and $k \geq 3$.

Theorem 1.2. *The Cauchy problem (1-6) is locally well-posed if the initial data $\phi \in H^s$ for $s > 1/2$.*

Theorem 1.2 is not new. It was proved by Colliander, Keel, Staffilani, Takaoka, and Tao [Colliander et al. 2004], but our method is different. The method used by those authors is based on a rescaling argument and the bilinear estimates proved by Kenig, Ponce and Vega [Kenig et al. 1996]. Our method is more straightforward and does not need the rescaling argument, the bilinear estimates, or the multilinear estimates in the earlier papers. This allows us to extend Theorem 1.2 to a very general setting. More precisely, consider the Cauchy problem for periodic generalized Korteweg–de Vries (gKdV) equation

$$\begin{cases} u_t + u_{xxx} + F(u) u_x = 0, \\ u(x,0) = \phi(x), \quad x \in \mathbb{T},\ t \in \mathbb{R}. \end{cases} \tag{1-7}$$

Here F is a suitable function. Then the following theorem can be established.

Theorem 1.3. *The Cauchy problem (1-7) is locally well-posed provided F is a C^5 function and the initial data $\phi \in H^s$ for $s > 1/2$.*

For sufficiently smooth F, say $F \in C^{15}$, the existence of a local solution of (1-7) for $s \geq 1$ and the global well-posedness of (1-7) for small data $\phi \in H^s$ with $s > 3/2$ were proved by Bourgain [1995]. The index $1/2$ is sharp because the ill-posedness of (1-6) for $s < 1/2$ is known; see [Colliander et al. 2004]. In order to make Theorem 1.3 well-posed for the initial data $\phi \in H^s$ with $s > 1/2$, the sharp regularity condition for F is perhaps C^4. But the method utilized in this paper, with a small modification, seems only to be able to reach an affirmative result for $F \in C^{(9/2)+}$ and $s > 1/2$. Moreover, the endpoint $s = 1/2$ case could possibly be done by combining the ideas from [Colliander et al. 2004] and this paper. We do not pursue this here.

2. Proof of Theorem 1.1

Proof. To prove Theorem 1.1, we need to introduce a level set. Since $\sqrt{A_{p,N}} \sim K_{p,N}$, it suffices to prove the Strichartz estimates (1-4). Let F_N be a periodic function on $\mathbb{T}^2$ given by

$$F_N(x,t) = \sum_{n=-N}^{N} a_n e^{2\pi i n x} e^{2\pi i n^3 t}, \tag{2-1}$$

where $\{a_n\}$ is a sequence with $\sum_n |a_n|^2 = 1$ and $(x,t) \in \mathbb{T}^2$. For any $\lambda > 0$, set a level set E_λ to be

$$E_\lambda = \{(x,t) \in \mathbb{T}^2 : |F_N(x,t)| > \lambda\}. \tag{2-2}$$

To obtain the desired estimate for the level set, let us first state a lemma on Weyl's sums.

Lemma 2.1. *Suppose that $t \in \mathbb{T}$ satisfies $|t - a/q| \le 1/q^2$, where a and q are relatively prime. Then if $q \ge N^2$,*

$$\left| \sum_{n=1}^{N} e^{2\pi i (t n^3 + b n^2 + c n)} \right| \le C N^{1/4+\varepsilon} q^{1/4}. \tag{2-3}$$

Here b and c are real numbers, and the constant C is independent of b, c, t, a, q, and N.

The proof of Lemma 2.1 relies on Weyl's squaring method. See [Hua 1965] or [Montgomery 1994] for details. We also need the following lemma.

Lemma 2.2 [Bourgain 1993a]. *For any integer $Q \ge 1$ and any integer $n \ne 0$, and any $\varepsilon > 0$,*

$$\sum_{Q \le q < 2Q} \left| \sum_{a \in \mathcal{P}_q} e^{2\pi i (a/q) n} \right| \le C_\varepsilon d(n, Q) Q^{1+\varepsilon}.$$

Here $\mathcal{P}_q$ is given by

$$\mathcal{P}_q = \{a \in \mathbb{N} : 1 \le a \le q \text{ and } (a,q) = 1\}, \tag{2-4}$$

and $d(n, Q)$ denotes the number of divisors of n less than Q and C_ε is a constant independent of Q, n.

Lemma 2.2 can be proved by observing that the arithmetic function defined by $f(q) = \sum_{a \in \mathcal{P}_q} e^{2\pi i (a/q) n}$ is multiplicative, and then utilizing the prime factorization for q to conclude the lemma.

Proposition 2.3. *Let K_N be a kernel defined by*

$$K_N(x,t) = \sum_{n=-N}^{N} e^{2\pi i t n^3 + 2\pi i x n}. \tag{2-5}$$

For any given positive number Q with $N^2 \le Q \le N^3$, the kernel K_N can be decomposed into $K_{1,Q} + K_{2,Q}$ such that

$$\|K_{1,Q}\|_\infty \le C_1 N^{1/4+\varepsilon} Q^{1/4}. \tag{2-6}$$

and

$$\|\widehat{K_{2,Q}}\|_\infty \le \frac{C_2 N^\varepsilon}{Q}. \tag{2-7}$$

Here the constants C_1, C_2 are independent of Q and N.

Proof. We can assume that Q is an integer, since otherwise we can take the integer part of Q. For a standard bump function φ supported on $[1/200, 1/100]$, we set

$$\Phi(t) = \sum_{Q \le q \le 5Q} \sum_{a \in \mathcal{P}_q} \varphi\left(\frac{t - a/q}{1/q^2}\right). \tag{2-8}$$

Clearly Φ is supported on $[0, 1]$. We can extend Φ to other intervals periodically to obtain a periodic function on $\mathbb{T}$. This periodic function, generated by Φ, will also be denoted by Φ. It is easy to see that

$$\widehat{\Phi}(0) = \sum_{q \sim Q} \sum_{a \in \mathcal{P}_q} \frac{\mathcal{F}_{\mathbb{R}}\varphi(0)}{q^2} = \sum_{q \sim Q} \frac{\phi(q)}{q^2} \mathcal{F}_{\mathbb{R}}\varphi(0) \tag{2-9}$$

is a constant independent of Q. Here ϕ is Euler's phi function, and $\mathcal{F}_{\mathbb{R}}$ denotes the Fourier transform of a function on $\mathbb{R}$. Also we have

$$\widehat{\Phi}(k) = \sum_{q \sim Q} \sum_{a \in \mathcal{P}_q} \frac{1}{q^2} e^{-2\pi i (a/q) k} \mathcal{F}_{\mathbb{R}}\varphi(k/q^2). \tag{2-10}$$

Applying Lemma 2.2 and the fact that $Q \le N^3$, we obtain

$$|\widehat{\Phi}(k)| \le \frac{N^{\varepsilon}}{Q}, \tag{2-11}$$

if $k \ne 0$.

We now define

$$K_{1,Q}(x, t) = \frac{1}{\widehat{\Phi}(0)} K_N(x, t) \Phi(t) \quad \text{and} \quad K_{2,Q} = K_N - K_{1,Q}.$$

Equation (2-6) follows from Lemma 2.1 since the intervals $J_{a/q} = \left[\frac{a}{q} + \frac{1}{100q^2}, \frac{a}{q} + \frac{1}{50q^2}\right]$ are pairwise disjoint for all $Q \le q \le 5Q$ and $a \in \mathcal{P}_q$.

We now prove (2-7). In fact, represent Φ as its Fourier series to get

$$K_{2,Q}(x, t) = -\frac{1}{\widehat{\Phi}(0)} \sum_{k \ne 0} \widehat{\Phi}(k) e^{2\pi i k t} K_N(x, t).$$

Thus its Fourier coefficient is

$$\widehat{K_{2,Q}}(n_1, n_2) = -\frac{1}{\widehat{\Phi}(0)} \sum_{k \ne 0} \widehat{\Phi}(k) \mathbf{1}_{\{n_2 = n_1^3 + k\}}(k).$$

Here $(n_1, n_2) \in \mathbb{Z}^2$ and $\mathbf{1}_A$ is the indicator function of a set A. This implies that $\widehat{K_{2,Q}}(n_1, n_2) = 0$ if $n_2 = n_1^3$, and if $n_2 \ne n_1^3$,

$$\widehat{K_{2,Q}}(n_1, n_2) = -\frac{1}{\widehat{\Phi}(0)} \widehat{\Phi}(n_2 - n_1^3).$$

Applying (2-11), we estimate $\widehat{K_{2,Q}}(n_1, n_2)$ by

$$|\widehat{K_{2,Q}}(n_1, n_2)| \le \frac{CN^{\varepsilon}}{Q},$$

since $N \le Q \le N^2$. Hence we obtain (2-7), completing the proof. □

Now we can state our theorem on the level set estimates.

Theorem 2.4. *For any positive numbers ε and $Q \ge N^2$, the level set defined as in (2-2) satisfies*

$$\lambda^2 |E_\lambda|^2 \le C_1 N^{1/4+\varepsilon} Q^{1/4} |E_\lambda|^2 + \frac{C_2 N^{\varepsilon}}{Q} |E_\lambda| \tag{2-12}$$

for all $\lambda > 0$. Here C_1 and C_2 are constants independent of N and Q.

Proof. Notice that if $Q \ge N^3$, (2-12) becomes trivial, since $E_\lambda = \varnothing$ if $\lambda \ge CN^{1/2}$. So we can assume that $N^2 \le Q \le N^3$. For the function F_N and the level set E_λ given in (2-1) and (2-2), respectively, we define f to be

$$f(x,t) = \frac{F_N(x,t)}{|F_N(x,t)|} \mathbf{1}_{E_\lambda}(x,t)\,.$$

Clearly

$$\lambda |E_\lambda| \le \int_{\mathbb{T}^2} \overline{F_N(x,t)}\, f(x,t)\, dx\, dt.$$

By the definition of F_N, we get

$$\lambda |E_\lambda| \le \sum_{n=-N}^{N} \overline{a_n}\, \hat{f}(n, n^3).$$

Utilizing the Cauchy–Schwarz inequality, we have

$$\lambda^2 |E_\lambda|^2 \le \sum_{n=-N}^{N} |\hat{f}(n, n^3)|^2.$$

The right hand side can be written as

$$\langle K_N * f, f \rangle. \tag{2-13}$$

For any Q with $N^2 \le Q \le N^3$, we employ Proposition 2.3 to decompose the kernel K_N. We then have

$$\lambda^2 |E_\lambda|^2 \le |\langle K_{1,Q} * f, f\rangle| + |\langle K_{2,Q} * f, f\rangle|. \tag{2-14}$$

From (2-6) and (2-7), we then obtain

$$\lambda^2 |E_\lambda|^2 \le C_1 N^{1/4+\varepsilon} Q^{1/4} \|f\|_1^2 + \frac{C_2 N^{\varepsilon}}{Q} \|f\|_2^2 \le C_1 N^{1/4+\varepsilon} Q^{1/4} |E_\lambda|^2 + \frac{C_2 N^{\varepsilon}}{Q} |E_\lambda|.$$ □

Corollary 2.5. *If $\lambda \ge 2C_1 N^{3/8+\varepsilon}$,*

$$|E_\lambda| \le \frac{CN^{1+\varepsilon}}{\lambda^{10}}. \tag{2-15}$$

Here C_1 is the constant C_1 in Theorem 2.4 and C is a constant independent of N and λ.

Proof. Since $\lambda \geq 2C_1 N^{3/8+\varepsilon}$, we simply take Q satisfying $2C_1 N^{1/4+\varepsilon} Q^{1/4} = \lambda^2$. Then Corollary 2.5 follows from Theorem 2.4. □

We are now ready to finish the proof of Theorem 1.1. In fact, let $p \geq 14$ and write $\|F\|_p^p$ as

$$p \int_0^{2C_1 N^{3/8+\varepsilon}} \lambda^{p-1} |E_\lambda| \, d\lambda + p \int_{2C_1 N^{3/8+\varepsilon}}^{2N^{1/2}} \lambda^{p-1} |E_\lambda| \, d\lambda. \tag{2-16}$$

Observe that $A_{6,N} \leq N^\varepsilon$ implies

$$|E_\lambda| \leq \frac{N^\varepsilon}{\lambda^6}. \tag{2-17}$$

Thus the first term in (2-16) is bounded by

$$C N^{3(p-6)/8+\varepsilon} \leq C N^{p/2-4+\varepsilon}, \tag{2-18}$$

since $p \geq 14$. From (2-15), the second term is majorized by

$$C N^{p/2-4+\varepsilon}. \tag{2-19}$$

Putting both estimates together, we complete the proof of Theorem 1.1. □

3. A Lower bound of $A_{p,N}$

In this section we show that $N^{1-8/p}$ is the best upper bound of $A_{p,N}$ if $p \geq 8$. Hence (1-3) can not be improved substantially, and it is sharp up to a factor of N^ε.

For $b \in \mathbb{N}$, let $J(N; b)$ be defined by

$$S(N; b) = \int_{\mathbb{T}^2} \left| \sum_{n=-N}^{N} e^{2\pi i t n^3 + 2\pi i x n} \right|^{2b} dx \, dt. \tag{3-1}$$

Proposition 3.1. *Let $S(N; b)$ be defined as in (3-1). Then*

$$S(N; b) \geq C(N^b + N^{2b-4}). \tag{3-2}$$

Here C is a constant independent of N.

Proof. Clearly $S(N; b)$ is equal to the number of solutions of

$$\begin{cases} n_1 + \cdots + n_b = m_1 + \cdots + m_b, \\ n_1^3 + \cdots + n_b^3 = m_1^3 + \cdots + m_b^3, \end{cases} \tag{3-3}$$

with $n_j, m_j \in \{-N, \ldots, N\}$ for all $j \in \{1, \ldots, b\}$. For each $(m_1, \ldots, m_b)$, we may obtain a solution of (3-3) by taking $(n_1, \ldots, n_b) = (m_1, \ldots, m_b)$. Thus

$$S(N; b) \geq N^b. \tag{3-4}$$

To derive a further lower bound for $S(N; b)$, we set Ω to be

$$\Omega = \left\{ (x, t) : |x| \leq \frac{1}{60N}, |t| \leq \frac{1}{60N^3} \right\}. \tag{3-5}$$

If $(x,t)\in\Omega$ and $|n|\le N$,

$$|tn^3+xn|\le\frac{1}{30}. \tag{3-6}$$

Hence, if $(x,t)\in\Omega$,

$$\left|\sum_{n=-N}^{N}e^{2\pi itn^3+2\pi ixn}\right|\ge\left|\operatorname{Re}\sum_{n=-N}^{N}e^{2\pi itn^3+2\pi ixn}\right|\ge\sum_{n=-N}^{N}\cos(2\pi(tn^3+xn))\ge CN. \tag{3-7}$$

Consequently, we have

$$S(N;b)\ge\int_\Omega\left|\sum_{n=-N}^{N}e^{2\pi itn^3+2\pi ixn}\right|^{2b}dx\,dt\ge CN^{2b}|\Omega|\ge CN^{2b-4}. \qquad \square$$

Proposition 3.2. *Let $p\ge 2$ be even. Then $A_{p,N}$ satisfies*

$$A_{p,N}\ge C(1+N^{1-8/p}). \tag{3-8}$$

Here C is a constant independent of N.

Proof. Let $p=2b$ since p is even. Setting $a_n=1$ for all n in the definition of $K_{p,N}$, we get

$$S(N;b)\le K_{p,N}^{p}(2N)^b. \tag{3-9}$$

By Proposition 3.1, we have

$$K_{p,N}\ge C(1+N^{1/2-4/p}). \tag{3-10}$$

Consequently, we conclude (3-8) since $A_{p,N}\sim K_{p,N}^2$. $\square$

4. An estimate of Hua

The following theorem was proved by Hua [1965] by an arithmetic argument. We provide a different proof.

Theorem 4.1. *Let $S(N;b)$ be defined as in (3-1). Then*

$$S(N;5)\le CN^{6+\varepsilon}. \tag{4-1}$$

By Proposition 3.1, we see that the estimate (4-1) is (almost) sharp. $S(N;4)\le N^{4+\varepsilon}$ is still open.

Proof of Theorem 4.1. Let G_λ be the level set given by

$$G_\lambda=\{(x,t)\in\mathbb{T}^2:|K_N(x,t)|\ge\lambda\}. \tag{4-2}$$

Here K_N is the function defined as in (2-5).

Letting $f=\mathbf{1}_{G_\lambda}K_N/|K_N|$, we have

$$\lambda|G_\lambda|\le\sum_{n=-N}^{N}\hat f(n,n^3)=\langle f_N,K_N\rangle, \tag{4-3}$$

where f_N is a rectangular Fourier partial sum defined by

$$f_N(x,t) = \sum_{\substack{|n_1|\le N\\ |n_2|\le N^3}} \hat{f}(n_1,n_2) e^{2\pi n_1 x} e^{2\pi i n_2 t}. \tag{4-4}$$

Employing Proposition 2.3 for K_N, we estimate the level set G_λ by

$$\lambda |G_\lambda| \le |\langle f_N, K_{1,Q}\rangle| + |\langle f_N, K_{2,Q}\rangle| \tag{4-5}$$

for any $Q \ge N^2$. From (2-6) and (2-7), $\lambda|G_\lambda|$ can be bounded further by

$$C\Biggl(N^{1/4+\varepsilon} Q^{1/4} \|f_N\|_1 + \sum_{\substack{|n_1|\le N\\ |n_2|\le N^3}} |\widehat{K_{2,Q}}(n_1,n_2)\hat{f}(n_1,n_2)| \Biggr). \tag{4-6}$$

Thus, from the fact that the L^1 norm of Dirichlet kernel D_N is comparable to $\log N$, (2-7), and the Cauchy–Schwarz inequality, we have

$$\lambda |G_\lambda| \le C N^{1/4+\varepsilon} Q^{1/4} |G_\lambda| + \frac{CN^{2+\varepsilon}}{Q} |G_\lambda|^{1/2}, \tag{4-7}$$

for all $Q \ge N^2$. For $\lambda \ge 2CN^{3/4+\varepsilon}$, take Q to be a number satisfying

$$2CN^{1/4+\varepsilon} Q^{1/4} = \lambda,$$

and obtain

$$|G_\lambda| \le \frac{CN^{6+\varepsilon}}{\lambda^{10}}. \tag{4-8}$$

Notice that

$$\|K_N\|_6 \le N^{1/2} K_{6,p} \le N^{1/2+\varepsilon}. \tag{4-9}$$

Hence, by (4-3), we majorize $|G_\lambda|$ by

$$|G_\lambda| \le \frac{CN^{3+\varepsilon}}{\lambda^6}. \tag{4-10}$$

We now estimate $S(N;5)$ by

$$S(N;5) \le C \int_{2CN^{3/4+\varepsilon}}^{2N} \lambda^{10-1} |G_\lambda|\, d\lambda + C \int_0^{2CN^{3/4+\varepsilon}} \lambda^{10-1} |G_\lambda|\, d\lambda. \tag{4-11}$$

From (4-8), the first term in the right hand side of (4-11) can be bounded by $CN^{6+\varepsilon}$. From (4-10), the second term is clearly bounded by $N^{6+\varepsilon}$. Putting both estimates together,

$$S(N;5) \le CN^{6+\varepsilon}, \tag{4-12}$$

as desired. $\square$

5. Estimates for the nonlinear term and Local well-posedness of (1-6)

For any integrable function u on $\mathbb{T}\times\mathbb{R}$, we define the space-time Fourier transform by

$$\hat{u}(n,\lambda)=\int_{\mathbb{R}}\int_{\mathbb{T}} u(x,t)e^{-inx}e^{-i\lambda t}\,dx\,dt \tag{5-1}$$

and set

$$\langle x\rangle := 1+|x|.$$

We now introduce the $X_{s,b}$ space, initially used by Bourgain.

Definition 5.1. Let I be a time interval in $\mathbb{R}$ and $s, b\in\mathbb{R}$. Let $X_{s,b}(I)$ be the space of functions u on $\mathbb{T}\times I$ that may be represented as

$$u(x,t)=\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\hat{u}(n,\lambda)e^{inx}e^{i\lambda t}d\lambda \quad \text{for } (x,t)\in\mathbb{T}\times I \tag{5-2}$$

with the space-time Fourier transform $\hat{u}$ satisfying

$$\|u\|_{X_{s,b}(I)}=\left(\sum_n\int\langle n\rangle^{2s}\langle\lambda-n^3\rangle^{2b}|\hat{u}(n,\lambda)|^2d\lambda\right)^{1/2}<\infty\,. \tag{5-3}$$

Here the norm should be understood as a restriction norm.

We take the time interval to be $[0,\delta]$ for a small positive number δ and abbreviate $\|u\|_{X_{s,b}(I)}$ as $\|u\|_{s,b}$ for any function u restricted to $\mathbb{T}\times[0,\delta]$. In this section, we always restrict the function u to $\mathbb{T}\times[0,\delta]$. Let w be the nonlinear function defined by

$$w=\left(u^k-\int u^k\,dx\right)u_x. \tag{5-4}$$

We also define

$$\|u\|_{Y_s}:=\|u\|_{s,1/2}+\left(\sum_n\langle n\rangle^{2s}\left(\int|\hat{u}(n,\lambda)|\,d\lambda\right)^2\right)^{1/2}. \tag{5-5}$$

We need the following estimate on the nonlinear function w, in order to establish a contraction on the space $\{u:\|u\|_{Y_s}\le M\}$ for some $M>0$.

Proposition 5.2. *For $s>1/2$, there exists $\theta>0$ such that, for the nonlinear function w given by (5-4),*

$$\|w\|_{s,-1/2}+\left(\sum_n\langle n\rangle^{2s}\left(\int\frac{|\widehat{w}(n,\lambda)|}{\langle\lambda-n^3\rangle}\,d\lambda\right)^2\right)^{1/2}\le C\delta^\theta\|u\|_{Y_s}^{k+1}. \tag{5-6}$$

Here C is a constant independent of δ and u.

The proof of Proposition 5.2 will appear in Section 6, and is based on the idea applied by Bourgain [1993b] while proving the special case $k=2$. In the proof, we write out the detailed treatment to some subcases, and omit the similar treatment of other subcases (but it is very easy to figure out). The main reason we include the proof of Proposition 5.2 in Section 6 is to provide the preparation so the readers can follow the (more technical) proof of the general case $F \in C^5$ more easily.

We now start to derive the local well-posedness of (1-6). For this purpose, we only need to consider the well-posedness of the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \left(u^k - \int_{\mathbb{T}} u^k dx\right) u_x = 0, \\ u(x,0) = \phi(x), \quad x \in \mathbb{T},\ t \in \mathbb{R}. \end{cases} \tag{5-7}$$

This is because if v is a solution of (5-7), the gauge transform

$$u(x,t) := v\left(x - \int_0^t \int_{\mathbb{T}} v^k(y,\tau)\, dy d\tau, t\right) \tag{5-8}$$

is a solution of (1-6) with the same initial value ϕ. Notice that this transform is invertible and preserves the initial data ϕ. The inverse transform is

$$v(x,t) := u\left(x + \int_0^t \int_{\mathbb{T}} u^k(y,\tau)\, dy d\tau, t\right). \tag{5-9}$$

It is easy to see that for any solution u of (1-6), this inverse transform of u defines a solution of (5-7). Hence, to establish the well-posedness of (1-6), it suffices to obtain the well-posedness of (5-7). This gauge transform was used in [Colliander et al. 2004].

By Duhamel's principle, the corresponding integral equation associated to (5-7) is

$$u(x,t) = e^{-t\partial_x^3}\phi(x) - \int_0^t e^{-(t-\tau)\partial_x^3} w(x,\tau)\, d\tau, \tag{5-10}$$

where w is defined as in (5-4).

Since we are only seeking the local well-posedness, we may use a bump function to truncate the time variable. Let ψ be a bump function supported in $[-2,2]$ with $\psi(t) = 1$, $|t| \le 1$, and let ψ_δ be

$$\psi_\delta(t) = \psi(t/\delta).$$

Then it suffices to find a local solution of

$$u(x,t) = \psi_\delta(t) e^{-t\partial_x^3}\phi(x) - \psi_\delta(t) \int_0^t e^{-(t-\tau)\partial_x^3} w(x,\tau)\, d\tau.$$

Let T be an operator given by

$$Tu(x,t) := \psi_\delta(t) e^{-t\partial_x^3}\phi(x) - \psi_\delta(t) \int_0^t e^{-(t-\tau)\partial_x^3} w(x,\tau)\, d\tau. \tag{5-11}$$

We denote the first term (the linear term) in (5-11) by $\mathscr{L}u$ and the second term (the nonlinear term) by $\mathscr{N}u$. Henceforth we represent Tu as $\mathscr{L}u + \mathscr{N}u$. The following two lemmas deal with $\mathscr{L}u$ and $\mathscr{N}u$ separately.

Lemma 5.3. *The linear term $\mathscr{L}$ satisfies*

$$\|\mathscr{L}u\|_{Y_s} \le C\|\phi\|_{H^s}. \tag{5-12}$$

Here C is a constant independent of δ.

Lemma 5.4. *The nonlinear term $\mathcal{N}$ satisfies*

$$\|\mathcal{N}u\|_{Y_s} \le C\biggl(\|w\|_{s,-1/2} + \biggl(\sum_n \langle n\rangle^{2s}\biggl(\int \frac{|\widehat{w}(n,\lambda)|}{\langle \lambda - n^3\rangle}\,d\lambda\biggr)^2\biggr)^{1/2}\biggr), \tag{5-13}$$

where C is a constant independent of δ.

Lemmas 5.3 and 5.4 are considered classical and their proofs can be found in many references, such as [Colliander et al. 2004].

Proposition 5.5. *Let $s > 1/2$ and T be the operator defined as in (5-11). Then there exists a positive number θ such that*

$$\|Tu\|_{Y_s} \le C(\|\phi\|_{H^s} + \delta^\theta \|u\|_{Y_s}^{k+1}). \tag{5-14}$$

Here C is a constant independent of δ.

Proof. Since $Tu = \mathcal{L}u + \mathcal{N}u$, Proposition 5.5 follows from Lemmas 5.3, 5.4, and Proposition 5.2. □

Proposition 5.5 yields that for δ sufficiently small, T maps a ball in Y_s into itself. Moreover, we write

$$\biggl(u^k - \int_{\mathbb{T}} u^k dx\biggr)u_x - \biggl(v^k - \int_{\mathbb{T}} v^k dx\biggr)v_x = \biggl(u^k - \int_{\mathbb{T}} u^k dx\biggr)(u-v)_x + \biggl((u^k - v^k) - \int_{\mathbb{T}} (u^k - v^k)\,dx\biggr)v_x$$

which equals

$$\biggl(u^k - \int_{\mathbb{T}} u^k dx\biggr)(u-v)_x + \sum_{j=0}^{k-1}\biggl((u-v)u^{k-1-j}v^j - \int_{\mathbb{T}} (u-v)u^{k-1-j}v^j\,dx\biggr)v_x. \tag{5-15}$$

For $k+1$ terms in (5-15), repeating similar argument as in the proof of Proposition 5.2, one obtains, for $s > 1/2$,

$$\|Tu - Tv\|_{Y_s} \le C\delta^\theta\biggl(\|u\|_{Y_s}^k + \sum_{j=1}^{k-1} \|u\|_{Y_s}^{k-1-j}\|v\|_{Y_s}^{j+1}\biggr)\|u-v\|_{Y_s}. \tag{5-16}$$

Hence, for $\delta > 0$ small enough, T is a contraction and the local well-posedness follows from Picard's fixed-point theorem.

6. Proof of Proposition 5.2

Proof. From the definition of w in (5-4), we may write $\hat{w}(n,\lambda)$ as

$$\sum_{\substack{m+n_1+\cdots+n_k=n\\ n_1+\cdots+n_k\neq 0}} m \int \hat{u}(m, \lambda-\lambda_1-\cdots-\lambda_k)\hat{u}(n_1,\lambda_1)\cdots\hat{u}(n_k,\lambda_k)\,d\lambda_1\cdots d\lambda_k. \tag{6-1}$$

By duality, there exists a sequence $\{A_{n,\lambda}\}$ satisfying

$$\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} |A_{n,\lambda}|^2 d\lambda \le 1, \tag{6-2}$$

and $\|w\|_{s,-1/2}$ is bounded by

$$\sum_{\substack{m+n_1+\cdots+n_k=n\\ n_1+\cdots+n_k\neq 0}} \int \frac{\langle n\rangle^s |m|}{\langle \lambda-n^3\rangle^{1/2}} |\hat{u}(m,\lambda-\lambda_1-\cdots-\lambda_k)|\,|\hat{u}(n_1,\lambda_1)|\cdots|\hat{u}(n_k,\lambda_k)|\,|A_{n,\lambda}|\,d\lambda_1\cdots d\lambda_k\,d\lambda. \tag{6-3}$$

Since the $X_{s,b}$ is a restriction norm, we may assume that u is supported in $\mathbb{T}\times[0,\delta]$. However, the inverse space-time Fourier transform $|\hat{u}|^\vee$ in general may not be a function with compact support. The following standard trick allows us to assume $|\hat{u}|^\vee$ has a compact support too. In fact, let η be a bump function supported on $[-2\delta,2\delta]$ and with $\eta(t)=1$ in $|t|\le\delta$. Also $\hat{\eta}$ is positive. Then $u=u\eta$ and $\hat{u}=\hat{u}*\hat{\eta}$. Thus $|\hat{u}|\le|\hat{u}|*\hat{\eta}=(|\hat{u}|^\vee\eta)^\wedge$. Whenever we need to make $|\hat{u}|^\vee$ supported in a small time interval, we replace $|\hat{u}|$ by $(|\hat{u}|^\vee\eta)^\wedge$ since $|\hat{u}|^\vee\eta$ clearly is supported on $\mathbb{T}\times[-2\delta,2\delta]$. This will help us gain a positive power of δ in our estimates. Moreover, without loss of generality we can assume $|n_1|\ge|n_2|\ge\cdots\ge|n_k|$.

The trouble occurs mainly because of the factor $|m|$ resulting from $\partial_x u$. The idea (inspired by Bourgain [1993b]) is that either the factor $\langle\lambda-n^3\rangle^{-1/2}$ can be used to cancel $|m|$, or $|m|$ can be distributed to some of the $\hat{u}$. More precisely, we consider three cases:

$$|m| < 1000k^2|n_2|, \tag{6-4}$$

$$1000k^2|n_2| \le |m| \le 100k|n_1|, \tag{6-5}$$

$$|m| > 100k|n_1|. \tag{6-6}$$

Case 1: $|m|<1000k^2|n_2|$. This is the simplest case. In fact, in this case, it is easy to see that

$$\langle n\rangle^s|m| \le C\langle n_1\rangle^s\langle n_2\rangle^1/2\langle m\rangle^1/2. \tag{6-7}$$

Let

$$F_1(x,t)=\sum_n\int\frac{|A_{n,\lambda}|}{\langle\lambda-n^3\rangle^{1/2}}e^{i\lambda t}e^{inx}\,d\lambda; \tag{6-8}$$

$$G(x,t)=\sum_n\int\langle n\rangle^1/2|\hat{u}(n,\lambda)|e^{i\lambda t}e^{inx}\,d\lambda; \tag{6-9}$$

$$H(x,t)=\sum_n\int\langle n\rangle^s|\hat{u}(n,\lambda)|e^{i\lambda t}e^{inx}\,d\lambda; \tag{6-10}$$

$$U(x,t)=\sum_n\int|\hat{u}(n,\lambda)|e^{i\lambda t}e^{inx}\,d\lambda. \tag{6-11}$$

Using (6-7), we can estimate (6-3) by

$$C\sum_{m+n_1+\cdots+n_k=n}\int\widehat{F}(n,\lambda)\widehat{G}(m,\lambda-\lambda_1-\cdots-\lambda_k)\widehat{H}(n_1,\lambda_1)\widehat{G}(n_2,\lambda_2)\prod_{j=3}^{k}\widehat{U}(n_j,\lambda_j)\,d\lambda_1\cdots d\lambda_k\,d\lambda,$$

which clearly equals

$$C\left|\int_{\mathbb{T}\times\mathbb{R}}F_1(x,t)G(x,t)^2H(x,t)U(x,t)^{k-2}\,dx\,dt\right|. \tag{6-12}$$

Apply Hölder's inequality to majorize it by

$$C\|F_1\|_4\,\|G\|_{6+}^2\,\|H\|_4\,\|U\|_{6(k-2)-}^{k-2}.$$

Since U is supported on $\mathbb{T}\times[-2\delta,2\delta]$, one more use of Hölder inequality yields

$$\text{(6-3)}\le C\delta^{\theta}\|F_1\|_4\,\|G\|_{6+}^2\,\|H\|_4\,\|U\|_{6(k-2)}^{k-2}. \tag{6-13}$$

Let us recall some useful local embedding facts on $X_{s,b}$.

$$X_{0,1/3}\subseteq L^4_{x,t},\qquad X_{0+,1/2+}\subseteq L^6_{x,t}\qquad (t\text{ local}), \tag{6-14}$$

$$X_{\alpha,1/2}\subseteq L^q_{x,t},\qquad 0<\alpha<\tfrac{1}{2},\ 2\le q<\frac{6}{1-2\alpha}\qquad (t\text{ local}), \tag{6-15}$$

$$X_{1/2-\alpha,1/2-\alpha}\subseteq L^q_tL^r_x,\qquad 0<\alpha<\tfrac{1}{2},\ 2\le q,\ r<1/\alpha. \tag{6-16}$$

The two embedding results in (6-14) are consequences of the discrete restriction estimates on L^4 and L^6, respectively (see [Bourgain 1993b] for details). (6-15) and (6-16) follow by interpolation (see [Colliander et al. 2004] for details). (6-14) yields

$$\|F_1\|_4\le C\|F_1\|_{0,\frac{1}{3}}\le C\Bigl(\sum_n\int|A_{n,\lambda}|^2d\lambda\Bigr)^{1/2}\le C,$$

and

$$\|H\|_4\le C\|H\|_{0,1/3}\le C\|u\|_{s,1/2}\le C\|u\|_{Y_s}.$$

From (6-15) we have

$$\|G\|_{6+}\le C\|G\|_{0+,1/2}\le C\|u\|_{s,1/2}\le C\|u\|_{Y_s}.$$

Using (6-16), we get

$$\|U\|_{6(k-2)}\le C\|U\|_{1/2-,1/2-}\le C\|u\|_{s,1/2}\le C\|u\|_{Y_s}.$$

Hence, for Case 1, we have

$$\text{(6-3)}\le C\delta^{\theta}\|u\|_{Y_s}^{k+1}. \tag{6-17}$$

Case 2: $1000k^2|n_2|\le|m|\le 100k|n_1|$. In this case, we further consider two subcases:

$$|m+n_1|\le 1000k^2|n_2|, \tag{6-18}$$

$$|m+n_1|> 1000k^2|n_2|. \tag{6-19}$$

If $|m+n_1|\le 1000k^2|n_2|$, we use the triangle inequality to get

$$|n|=|m+n_1+n_2+\cdots+n_k|\le C|n_2|. \tag{6-20}$$

Hence we have

$$\langle n\rangle^s|m|\le C\langle n_2\rangle^s\langle m\rangle^{1/2}\langle n_1\rangle^{1/2}. \tag{6-21}$$

Thus this subcase can be treated exactly the same as Case 1. We omit the details.

In the second subcase, $|m+n_1| > 1000k^2|n_2|$, the crucial arithmetic observation is

$$n^3-(m^3+n_1^3+\cdots+n_k^3)=3(m+n_1)(m+a)(n_1+a)+a^3-(n_2^3+\cdots+n_k^3), \tag{6-22}$$

where $a=n_2+\cdots+n_k$. This observation can be easily verified since $n=m+n_1+\cdots+n_k$. From (6-5) and (6-19), we get

$$|n^3-(m^3+n_1^3+\cdots+n_k^3)| \geq Ck^2\langle n_2\rangle |m|\,|n_1| \geq Ck|m|^2. \tag{6-23}$$

This implies that at least one of following statements holds:

$$|\lambda-n^3| \geq C|m|^2, \tag{6-24}$$

$$|(\lambda-\lambda_1-\cdots-\lambda_k)-m^3| \geq C|m|^2, \tag{6-25}$$

$$\text{there exists an } i\in\{1,\ldots,k\} \text{ such that } |\lambda_i-n_i^3| \geq C|m|^2. \tag{6-26}$$

For (6-24), (6-3) can be bounded by

$$\sum_{m+n_1+\cdots+n_k=n}\int \langle n_1\rangle^s |\hat{u}(m,\lambda-\lambda_1-\cdots-\lambda_k)|\,|\hat{u}(n_1,\lambda_1)|\cdots|\hat{u}(n_k,\lambda_k)|\,|A_{n,\lambda}|\,d\lambda_1\cdots d\lambda_k\,d\lambda. \tag{6-27}$$

Let F_2 be defined by

$$F_2(x,t)=\sum_n\int |A_{n,\lambda}|e^{i\lambda t}e^{inx}\,d\lambda. \tag{6-28}$$

Then we represent (6-27) as

$$\sum_{m+n_1+\cdots+n_k=n}\int \widehat{F_2}(n,\lambda)\widehat{U}(m,\lambda-\lambda_1-\cdots-\lambda_k)\widehat{H}(n_1,\lambda_1)\prod_{j=2}^{k}\widehat{U}(n_j,\lambda_j)\,d\lambda_1\cdots d\lambda_k\,d\lambda. \tag{6-29}$$

Here H and U are the functions defined in (6-10) and (6-11). Clearly (6-29) equals

$$\int_{\mathbb{T}\times\mathbb{R}} F_2(x,t)H(x,t)U(x,t)^k\,dx\,dt. \tag{6-30}$$

Utilizing Hölder's inequality, we estimate it further by

$$\|F_2\|_2\,\|H\|_4\,\|U\|_{4k}^k \leq C\delta^\theta\|u\|_{Y_s}^{k+1}. \tag{6-31}$$

This yields the desired estimate for subcase (6-24).

One can similarly complete the proofs of subcases (6-25) and (6-26), and hence the proof of Case 2.

Case 3: $|m| > 100k|n_1|$. The arithmetic observation (6-22) again plays an important role. In this case, let us further consider two subcases:

$$|m|^2 \leq 1000k^2|n_2|^2|n_3|, \tag{6-32}$$

$$|m|^2 > 1000k^2|n_2|^2|n_3|. \tag{6-33}$$

For the first subcase, we observe that, from (6-32),

$$|m|^2 \le C|n_1||n_2||n_3|,$$

since $|n_2| \le |n_1|$. Hence we have

$$|m| = |m|^{1/2}|m|^{2/3} \le C|m|^{1/3}|n_1|^{1/3}|n_2|^{1/3}|n_3|^{1/3}. \tag{6-34}$$

This immediately implies

$$\langle n\rangle^s|m| \le C|m|^{s+1} \le \langle m\rangle^{(s+1)/3}\langle n_1\rangle^{(s+1)/3}\langle n_2\rangle^{(s+1)/3}\langle n_3\rangle^{(s+1)/3}. \tag{6-35}$$

Note that $(s+1)/3 < s$ for $s > 1/2$. By distributing the four factors to the corresponding functions, one can mimic the proof of Case 1 to finish subcase (6-32).

We now turn to the contribution of (6-33). Clearly we have

$$|(n_2+\cdots+n_k)^3 - (n_2^3+\cdots+n_k^3)| \le 10k|n_2|^2|n_3|, \tag{6-36}$$

since $|n_2| \ge |n_3| \ge \cdots \ge |n_k|$. From the crucial arithmetic observation (6-22), (6-36), and (6-33), we have

$$|n^3 - (m^3 + n_1^3 + \cdots + n_k^3)| \ge Ck|m|^2. \tag{6-37}$$

This is the same as (6-23). Hence we again reduce the problems to (6-24), (6-25), and (6-26), which were all done in Case 2. Therefore Case 3 is finished.

Putting all the cases together, we obtain

$$\|w\|_{s,-1/2} \le C\delta^\theta\|u\|_{Y_s}^{k+1}. \tag{6-38}$$

Finally we need to estimate

$$\left(\sum_n \langle n\rangle^{2s}\left(\int \frac{|\widehat{w}(n,\lambda)|}{\langle\lambda-n^3\rangle}\,d\lambda\right)^2\right)^{1/2}. \tag{6-39}$$

Let $\{A_n\}$ be a sequence with

$$\left(\sum_n |A_n|^2\right)^{1/2} \le 1.$$

By duality, it suffices to estimate

$$\sum_{\substack{m+n_1+\cdots+n_k=n\\ n_1+\cdots+n_k\neq 0}} \int \frac{\langle n\rangle^s|m|}{\langle\lambda-n^3\rangle}|\hat{u}(m,\lambda-\lambda_1-\cdots-\lambda_k)||\hat{u}(n_1,\lambda_1)|\cdots|\hat{u}(n_k,\lambda_k)||A_n|\,d\lambda_1\cdots d\lambda_k\,d\lambda. \tag{6-40}$$

By the same idea and similar techniques, one can bound (6-40) by mimicking the treatment of (6-3) and get

$$\left(\sum_n \langle n\rangle^{2s}\left(\int \frac{|\widehat{w}(n,\lambda)|}{\langle\lambda-n^3\rangle}\,d\lambda\right)^2\right)^{1/2} \le C\delta^\theta\|u\|_{Y_s}^{k+1}. \tag{6-41}$$

We complete the proof of Proposition 5.2 by combining (6-38) and (6-41). □

7. Proof of Theorem 1.3

The argument is similar to that in Section 5. By using a gauge transform as in (5-8) with v^k replaced by $F(v)$, the well-posedness of (1-7) is equivalent to the well-posedness of the following equation:

$$\begin{cases} u_t + u_{xxx} + (F(u) - \int_{\mathbb{T}} F(u)\,dx)u_x = 0, \\ u(x,0) = \phi(x), \quad x \in \mathbb{T},\ t \in \mathbb{R}. \end{cases} \tag{7-1}$$

Now the nonlinear function w is defined by

$$w = \partial_x u\Big(F(u) - \int_{\mathbb{T}} F(u)\,dx\Big). \tag{7-2}$$

Let T_F be an operator given by

$$T_F u(x,t) := \psi_\delta(t) e^{-t\partial_x^3}\phi(x) - \psi_\delta(t)\int_0^t e^{-(t-\tau)\partial_x^3} w(x,\tau)\,d\tau. \tag{7-3}$$

As in Section 5, the local well-posedness is a consequence of the following proposition.

Proposition 7.1. *Let $s > 1/2$. There exists $\theta > 0$ such that, for the nonlinear function w given by (7-2) and any u satisfying $\|u\|_{Y_s} \le C_0\|\phi\|_{H^s}$,*

$$\|w\|_{s,-1/2} + \bigg(\sum_n \langle n\rangle^{2s}\Big(\int \frac{|\widehat{w}(n,\lambda)|}{\langle \lambda - n^3\rangle}\,d\lambda\Big)^2\bigg)^{1/2} \le C(\|\phi\|_{H^s}, F)\delta^\theta \|u\|_{Y_s}^4, \tag{7-4}$$

provided $F \in C^5$. Here C_0 is a suitably large constant, and $C(\|\phi\|_{H^s}, F)$ is a constant independent of δ and u, but which may depend on $\|\phi\|_{H^s}$ and F.

The constant $C(\|\phi\|_{H^s}, F)$ will be specified in the proof of Proposition 7.1, which we postpone to Section 8. We now return to the proof of Theorem 1.3. Proposition 7.1 implies that for δ sufficiently small, T maps a ball

$$\{u \in Y_s : \|u\|_{Y_s} \le C_0\|\phi\|_{H^s}\}$$

into itself. Moreover, using Lemma 5.4 and repeating similar argument as in the proof of Proposition 7.1, one obtains, for $s > 1/2$ and $F \in C^5$,

$$\|T_F u - T_F v\|_{Y_s} \le \delta^\theta C(\|\phi\|_{H^s}, F)\|u - v\|_{Y_s} \tag{7-5}$$

for all u, v in the ball $\{u \in Y_s : \|u\|_{Y_s} \le C_0\|\phi\|_{H^s}\}$. Therefore, for $\delta > 0$ small enough, T_F is a contraction on the ball and the local well-posedness again follows from Picard's fixed-point theorem. This completes the proof of Theorem 1.3.

8. Proof of Proposition 7.1

First we introduce a decomposition of $F(u)$ which was used by Bourgain. Let K be a dyadic number, and define a Fourier multiplier operator P_K by setting

$$P_K u(x,t) = \int \psi_K(y) u(x-y,t)\,dy. \tag{8-1}$$

Here the Fourier transform of ψ_K is a standard bump function supported on $[-2K, 2K]$ and $\widehat{\psi_K}(x) = 1$ for $x \in [-K, K]$. Let u_K denote the Littlewood–Paley Fourier multiplier, that is,

$$u_K = P_K u - P_{K/2} u. \tag{8-2}$$

Then we may decompose $F(u)$ by

$$F(u) = \sum_K (F(P_K u) - F(P_{K/2} u)) = \sum_K F_1(P_K u, P_{K/2} u) u_K + R_1,$$

where R_1 is a function independent of the space variable x. Repeating this procedure for F_1, we obtain

$$\begin{aligned} F(u) &= \sum_{K_1 \ge K_2} F_2(P_{2K_2} u, \dots, P_{K_2/4} u) u_{K_1} u_{K_2} + \sum_{K_1} R_2 u_{K_1} + R_1 \\ &= \sum_{K_1 \ge K_2 \ge K_3} F_3(P_{4K_3} u, \dots, P_{K_3/8} u) u_{K_1} u_{K_2} u_{K_3} + \sum_{K_1 \ge K_2} R_3 u_{K_1} u_{K_2} + \sum_{K_1} R_2 u_{K_1} + R_1 \end{aligned}$$

where R_1, R_2, R_3 are functions independent of the space variable. Set

$$G_{K_3}(x, t) = F_3(P_{4K_3} u, \dots, P_{K_3/8} u). \tag{8-3}$$

Hence we represent w, defined in (7-2), as

$$\begin{aligned} w = &\sum_{K_0, K_1 \ge K_2 \ge K_3} \partial_x u_{K_0} \Big(u_{K_1} u_{K_2} u_{K_3} G_{K_3} - \int_{\mathbb{T}} u_{K_1} u_{K_2} u_{K_3} G_{K_3}\, dx \Big) \\ &+ \sum_{K_0, K_1 \ge K_2} \partial_x u_{K_0} \Big(u_{K_1} u_{K_2} - \int_{\mathbb{T}} u_{K_1} u_{K_2}\, dx \Big) R_3 + \sum_{K_0, K_1} \partial_x u_{K_0} \Big(u_{K_1} - \int_{\mathbb{T}} u_{K_1}\, dx \Big) R_2. \end{aligned}$$

The main contribution of w is from the first term. The remaining terms can be handled by the method presented in Section 6, because R_2, R_3 are functions independent of the space variable x (actually they only depend on the conserved quantity $\int_{\mathbb{T}} u\, dx$). Hence in what follows we only focus on estimating the first term — the most difficult one. Denote the first term by w_1:

$$w_1 = \sum_{K_0, K_1 \ge K_2 \ge K_3} \partial_x u_{K_0} \Big(u_{K_1} u_{K_2} u_{K_3} G_{K_3} - \int_{\mathbb{T}} u_{K_1} u_{K_2} u_{K_3} G_{K_3}\, dx \Big). \tag{8-4}$$

We should prove

$$\|w_1\|_{s,-1/2} + \bigg(\sum_n \langle n \rangle^{2s} \Big(\int \frac{|\hat{w}_1(n, \lambda)|}{\langle \lambda - n^3 \rangle}\, d\lambda \Big)^2 \bigg)^{1/2} \le C \delta^\theta C(\|\phi\|_{H^s}, F) \|u\|_{Y_s}^4. \tag{8-5}$$

In order to specify the constant $C(\|\phi\|_{H^s}, F)$, we define $\mathfrak{M}$ by setting

$$\mathfrak{M} = \sup\big\{ |D^\alpha F_3(u_1, \dots, u_6)| : u_j \text{ satisfies } \|u_j\|_{Y_s} \le C_0 \|\phi\|_{H^s} \text{ for all } j = 1, \dots, 6; \alpha \big\}. \tag{8-6}$$

Here $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_6}^{\alpha_6}$ and α is taken over all tuples $(\alpha_1, \dots, \alpha_6) \in (\mathbb{N} \cup \{0\})^6$ with $\sum |\alpha_j| \le 2$. $\mathfrak{M}$ is a real number. This is because, for $s > 1/2$, $\|u\|_{Y_s} \le C_0 \|\phi\|_{H^s}$ yields that u is bounded by $C \|\phi\|_{H^s}$, and the previous claim follows from $F_3 \in C^2$.

In order to bound $\|w_1\|_{s,-1/2}$, by duality, it suffices to bound

$$\sum_{\substack{K_0,K_1\geq K_2\geq K_3\\ n_0+n_1+n_2+n_3+m=n\\ n_1+n_2+n_3+m\neq 0}} \int \frac{A_{n,\lambda}\langle n\rangle^s n_0}{\langle \lambda-n^3\rangle^{1/2}}\hat{u}_{K_0}(n_0,\lambda-\lambda_1-\lambda_2-\lambda_3-\mu) \cdot \prod_{j=1}^{3}\hat{u}_{K_j}(n_j,\lambda_j)\widehat{G_{K_3}}(m,\mu)\,d\lambda_1\cdots d\lambda_4 d\lambda d\mu, \tag{8-7}$$

where $A_{n,\lambda}$ satisfies

$$\sum_n \int |A_{n,\lambda}|^2 d\lambda = 1.$$

The trouble maker is G_{K_3} since there is no way to find a suitable upper bound for its $X_{s,b}$ norm. Because of this, the method in Section 6 is no longer valid, and we have to treat m and μ differently from n and λ, respectively. A delicate analysis must be done to overcome the difficulty caused by G_{K_3}. For simplicity, we assume that $\delta=1$. One can modify the argument to gain a decay of δ^θ by using the technical treatment from Section 6.

For a dyadic number M, define the Littlewood–Paley Fourier multiplier by

$$g_{K_3,M} = P_M G_{K_3} - P_{M/2}G_{K_3} = (G_{K_3})_M. \tag{8-8}$$

Let v be defined by

$$v(x,t) = \sum_n \int \frac{A_{n,\lambda}}{\langle\lambda-n^3\rangle^{1/2}} e^{i\lambda t}e^{inx}\,d\lambda. \tag{8-9}$$

To estimate (8-7), it suffices to estimate

$$\sum_{\substack{K,K_0,K_1\geq K_2\geq K_3,M\\ n_0+n_1+n_2+n_3+m=n\\ n_1+n_2+n_3+m\neq 0}} \int \widehat{\langle\partial_x\rangle^s v_K}(n,\lambda)\widehat{\partial_x u_{K_0}}(n_0,\lambda-\lambda_1-\lambda_2-\lambda_3-\mu) \cdot \prod_{j=1}^{3}\hat{u}_{K_j}(n_j,\lambda_j)\widehat{g_{K_3,M}}(m,\mu)\,d\lambda_1\cdots d\lambda_4 d\lambda d\mu. \tag{8-10}$$

Here K is a dyadic number.

As we did in Section 6, we consider three cases:

$$K_0 < 2^{100}K_2; \tag{8-11}$$
$$2^{100}K_2 \leq K_0 \leq 2^{10}K_1; \tag{8-12}$$
$$K_0 > 2^{10}K_1. \tag{8-13}$$

The rest of the paper is devoted to a proof of these three cases. In what follows, we will only provide the details for the estimates of $\|w_1\|_{s,-1/2}$ with $1/2<s<1$ (the case $s\geq 1$ is easier). For the desired estimate of

$$\biggl(\sum_n \langle n\rangle^{2s}\biggl(\int \frac{|\hat{w}_1(n,\lambda)|}{\langle\lambda-n^3\rangle}\,d\lambda\biggr)^2\biggr)^{1/2},$$

simply replace v by

$$v_1(x,t)=\sum_n \int \frac{C_{n,\lambda}A_n}{\langle \lambda-n^3\rangle} e^{i\lambda t}e^{inx}\,d\lambda, \tag{8-14}$$

and then the desired estimate follows similarly. Here $C_{n,\lambda}\in\mathbb{C}$ satisfies $\sup_\lambda |C_n,\lambda|\le 1$ and $\{A_n\}$ satisfies $\sum_n |A_n|^2\le 1$.

9. Proof of case (8-11)

In this case, we should consider further two subcases:

$$M\le 2^{10}K_1, \tag{9-1}$$

$$M> 2^{10}K_1. \tag{9-2}$$

For the contribution of (9-1), noticing that $K\le CK_1$ in this subcase, we estimate (8-10) by

$$\sum_{K_1\ge K_2\ge K_3}\int_{\mathbb{T}\times\mathbb{R}}\biggl|\Bigl(\sum_{K\le CK_1}\partial_x^s v_K\Bigr)\Bigl(\sum_{K_0\le CK_2}\partial_x u_{K_0}\Bigr)u_{K_1}u_{K_2}u_{K_3}(P_{2^{10}K_1}G_{K_3})\biggr|\,dx\,dt, \tag{9-3}$$

which is bounded by

$$\sum_{K_3}\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\int_{\mathbb{T}\times\mathbb{R}}\sum_{K_1}\sum_{K\le CK_1}K^s v_K^*|u_{K_1}|\sum_{K_2}\sum_{K_0\le CK_2}K_0u_{K_0}^*|u_{K_2}|\,dx\,dt, \tag{9-4}$$

where f^* stands for the Hardy–Littlewood maximal function of f. By the Schur's test, (9-4) can be estimated by

$$\sum_{K_3}K_3^{-(2s-1)/2}\|u\|_{Y_s}\mathfrak{M}\int\Bigl(\sum_K|v_K^*|^2\Bigr)^{1/2}\Bigl(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\Bigr)^{1/2} \cdot\Bigl(\sum_{K_0}K_0|u_{K_0}^*|^2\Bigr)^{1/2}\Bigl(\sum_{K_2}K_2|u_{K_2}|^2\Bigr)^{1/2}dx\,dt. \tag{9-5}$$

Since $s>1/2$, we obtain, by a use of Hölder's inequality, that (9-4) is majorized by

$$C\mathfrak{M}\|u\|_{Y_s}\biggl\|\Bigl(\sum_K|v_K^*|^2\Bigr)^{1/2}\biggr\|_4\biggl\|\Bigl(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\Bigr)^{1/2}\biggr\|_4\biggl\|\Bigl(\sum_{K_0}K_0|u_{K_0}^*|^2\Bigr)^{1/2}\biggr\|_4\biggl\|\Bigl(\sum_{K_2}K_2|u_{K_2}|^2\Bigr)^{1/2}\biggr\|_4. \tag{9-6}$$

Observe that

$$\biggl\|\Bigl(\sum_K|v_K^*|^2\Bigr)^{1/2}\biggr\|_4\le\biggl\|\Bigl(\sum_K|v_K|^2\Bigr)^{1/2}\biggr\|_4\le C\|v\|_4\le C\|v\|_{0,1/3}\le C. \tag{9-7}$$

Here the first inequality is obtained by using Fefferman and Stein's vector-valued inequality on the maximal function, and the second is a consequence of the classical Littlewood–Paley theorem. Similarly,

$$\biggl\|\Bigl(\sum_{K_0}K_0|u_{K_0}^*|^2\Bigr)^{1/2}\biggr\|_4\le\biggl\|\Bigl(\sum_{K_0}K_0|u_{K_0}|^2\Bigr)^{1/2}\biggr\|_4\le C\|\partial_x^{1/2}u\|_4\le C\|u\|_{1/2,1/3}\le C\|u\|_{Y_s} \tag{9-8}$$

and

$$\left\| \left(\sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{1/2} \right\|_4 \le C \|\partial_x^s u\|_4 \le C \|u\|_{s,1/3} \le C \|u\|_{Y_s}. \tag{9-9}$$

Hence, from (9-7), (9-8) and (9-9), we have

$$\text{(8-10)} \le C \mathfrak{M} \|u\|_{Y_s}^4. \tag{9-10}$$

For the contribution of (9-2), since in this subcase $K \le CM$, we estimate (8-10) by

$$\sum_{K_1} \|u_{K_1}\|_\infty \int_{\mathbb{T}\times\mathbb{R}} \sum_{K_3 \le K_1} |u_{K_3}| \sum_M \sum_{K \le CM} K^s v_K^* |g_{K_3,M}| \sum_{K_2} \sum_{K_0 \le CK_2} K_0 u_{K_0}^* |u_{K_2}| \, dx \, dt, \tag{9-11}$$

which is bounded by

$$\sum_{K_1} K_1^{-(2s-1)/2} \|u\|_{Y_s} \int_{\mathbb{T}\times\mathbb{R}} \sum_{K_3 \le K_1} |u_{K_3}| \left(\sum_K |v_K^*|^2 \right)^{1/2} \left(\sum_M M^{2s} |g_{K_3,M}|^2 \right)^{1/2} \cdot \left(\sum_{K_0} K_0 |u_{K_0}^*|^2 \right)^{1/2} \left(\sum_{K_2} K_2 |u_{K_2}|^2 \right)^{1/2} dx \, dt. \tag{9-12}$$

By a use of the Cauchy–Schwarz inequality, (9-12) is estimated by

$$\sum_{K_1} K_1^{-(2s-1)/2} \|u\|_{Y_s} \int_{\mathbb{T}\times\mathbb{R}} \left(\sum_K |v_K^*|^2 \right)^{1/2} \left(\sum_{K_0} K_0 |u_{K_0}^*|^2 \right)^{1/2} \left(\sum_{K_2} K_2 |u_{K_2}|^2 \right)^{1/2} \cdot \left(\sum_{K_3} K_3^{2s} |u_{K_3}|^2 \right)^{1/2} \left(\sum_{K_3 \le K_1} \sum_M \frac{M^{2s}}{K_3^{2s}} |g_{K_3,M}|^2 \right)^{1/2} dx \, dt. \tag{9-13}$$

Using Hölder's inequality, we then bound it further by

$$\sum_{K_1} K_1^{-(2s-1)/2} \|u\|_{Y_s} \left\| \left(\sum_K |v_K^*|^2 \right)^{1/2} \right\|_4 \left\| \left(\sum_{K_2} K_0 |u_{K_0}^*|^2 \right)^{1/2} \right\|_6 \left\| \left(\sum_{K_2} K_2 |u_{K_2}|^2 \right)^{1/2} \right\|_6 \cdot \left\| \left(\sum_{K_3} K_3^{2s} |u_{K_3}|^2 \right)^{1/2} \right\|_4 \left\| \left(\sum_{K_3 \le K_1} \sum_M \frac{M^{2s}}{K_3^{2s}} |g_{K_3,M}|^2 \right)^{1/2} \right\|_6, \tag{9-14}$$

which is majorized by

$$\sum_{K_1} K_1^{-(2s-1)/2} \|u\|_{Y_s}^4 \sum_{K_3 \le K_1} K_3^{-s} \left\| \left(\sum_M M^{2s} |g_{K_3,M}|^2 \right)^{1/2} \right\|_6 \le \sum_{K_1} K_1^{-(2s-1)/2} \|u\|_{Y_s}^4 \sum_{K_3 \le K_1} K_3^{-s} \|\partial_x^s G_{K_3}\|_\infty.$$

From the definition of G_{K_3}, we have

$$\partial_x G_{K_3}(x,t) = \mathrm{O}(\mathfrak{M} K_3) \|u\|_{Y_s} = \mathrm{O}(\mathfrak{M} K_3) \|\phi\|_{H^s}. \tag{9-15}$$

Hence, for $s < 1$,

$$\|\partial_x^s G_{K_3}\|_\infty \le C \mathfrak{M} K_3^s \|\phi\|_{H^s}. \tag{9-16}$$

Since $s > 1/2$, we then have

$$(9\text{-}14) \leq C\mathfrak{M}\|\phi\|_{H^s} \sum_{K_1} K_1^{-(2s-1)/2+\varepsilon} \|u\|_{Y_s}^4 \leq C\mathfrak{M}\|\phi\|_{H^s} \|u\|_{Y_s}^4. \tag{9-17}$$

This completes our discussion of Case (8-11).

10. Proof of case (8-12)

In this case, it suffices to consider the following subcases:

$$K \leq 2^{10} K_2, \tag{10-1}$$

$$K \leq 2^{10} M, \tag{10-2}$$

$$K > 2^9(K_2 + M) \quad \text{and} \quad K_3 \geq K_0^{1/2}, \tag{10-3}$$

$$K > 2^9(K_2 + M), \quad K_3 \leq K_0^{1/2}, \quad \text{and} \quad M \geq 2^{-10} K_0^{2/3}, \tag{10-4}$$

$$K > 2^9(K_2 + M), \quad K_3 \leq K_0^{1/2}, \quad \text{and} \quad M < 2^{-10} K_0^{2/3}. \tag{10-5}$$

The first two cases can be handled in exactly the same way as cases (9-1) and (9-2).

For case (10-3), observe that (8-12) and (10-3) imply

$$K \leq C K_1 \tag{10-6}$$

and

$$K_0^{1/2} \leq K_2^{1/2} K_3^{1/2}. \tag{10-7}$$

Hence (8-10) is bounded by

$$\int \sum_{K_1} \sum_{K \leq C K_1} K^s v_K^* |u_{K_1}| \sum_{\substack{K_0 \geq K_2 \geq K_3 \\ K_0 \leq K_3^2}} K_0 u_{K_0}^* |u_{K_2}| \, |u_{K_3}| \, \|G_{K_3}\|_\infty \, dx\, dt. \tag{10-8}$$

Applying Hölder's inequality, we estimate (10-8) by

$$C\mathfrak{M} \int \Big(\sum_K |v_K^*|^2\Big)^{1/2} \Big(\sum_{K_1} K_1^{2s} |u_{K_1}|^2\Big)^{1/2} \prod_{j=0,2,3} \Big(\sum_{K_j} K_j^{1+\varepsilon} |u_{K_j}|^2\Big)^{1/2} dx\, dt. \tag{10-9}$$

One more use of Hölder's inequality yields that (10-8) is bounded by

$$C\mathfrak{M} \Big\| \Big(\sum_K |v_K|^2\Big)^{1/2} \Big\|_4 \Big\| \Big(\sum_{K_1} K_1^{2s} |u_{K_1}|^2\Big)^{1/2} \Big\|_4 \prod_{j=0,2,3} \Big\| \Big(\sum_{K_j} K_j^{1+\varepsilon} |u_{K_j}|^2\Big)^{1/2} \Big\|_6.$$

Hence we obtain

$$(10\text{-}8) \leq C\mathfrak{M} \|u\|_{Y_s}^4. \tag{10-10}$$

This finishes the proof of (10-3).

For case (10-4), we estimate (8-10) by

$$\sum_{K_2,K_3}\int\sum_{K_1}\sum_{K\le CK_1}K^s v_K^*|u_{K_1}|\sum_{K_0}K_0|u_{K_0}^*||u_{K_2}||u_{K_3}|\sum_{M\ge CK_0^{2/3}}|g_{K_3,M}|\,dx\,dt, \tag{10-11}$$

which is dominated by

$$C\sum_{K_2,K_3}\int\Big(\sum_K|v_K^*|^2\Big)^{1/2}\Big(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\Big)^{1/2}|u_{K_2}||u_{K_3}| \cdot\Big(\sum_{K_0}K_0|u_{K_0}^*|^2\Big)^{1/2}\Big(\sum_M M^{3/2}|g_{K_3,M}|^2\Big)^{1/2}dx\,dt. \tag{10-12}$$

By Hölder's inequality with L^4 norms for the first two functions in the integrand, L^{6+} norms for the next three functions, and an L^p norm (very large p) for the last one, (10-12) is dominated by

$$C\|u\|_{Y_s}\sum_{K_2,K_3}\|u_{K_2}\|_{6+}\|u_{K_3}\|_{6+}\Big\|\Big(\sum_{K_0}K_0|u_{K_0}^*|^2\Big)^{1/2}\Big\|_{6+}\|\partial_x^{3/4}G_{K_3}\|_\infty. \tag{10-13}$$

Applying (9-16), we estimate (10-12) by

$$C\mathfrak{M}\|\phi\|_{H_s}\|u\|_{Y_s}^2\prod_{j=2}^3\sum_{K_j}K_j^{3/8}\|u_{K_j}\|_{6+}\le C\mathfrak{M}\|\phi\|_{H_s}\|u\|_{Y_s}^2\prod_{j=2}^3\sum_{K_j}K_j^{3/8}\|u_{K_j}\|_{0+,1/2}\le C\mathfrak{M}\|\phi\|_{H_s}\|u\|_{Y_s}^4,$$

as desired. This completes the discussion of (10-4).

We now turn to case (10-5). In this case, we have

$$|n_0+n_1|+2K_2+M\ge|n|\ge K/2\ge 2^8(K_2+M), \tag{10-14}$$

which implies

$$|n_0+n_1|\ge 2^5(K_2+M). \tag{10-15}$$

Notice that

$$\begin{aligned}(n_0+n_1+n_2+n_3+m)^3-n_0^3-n_1^3-n_2^3-n_3^3-m^3\\ =3(n_0+n_1)(n_0+n_2+n_3+m)(n_1+n_2+n_3+m)+(n_2+n_3+m)^3-n_2^3-n_3^3-m^3.\end{aligned} \tag{10-16}$$

From (10-15), (10-16), and (10-5), we obtain

$$|n^3-n_0^3-n_1^3-n_2^3-n_3^3-m^3|\ge C(K_2+M)K_0K_1\ge CK_0K_1\ge CK_0^2. \tag{10-17}$$

Hence one of the following four statements must be true:

$$|\lambda-n^3|\ge K_0^2, \tag{10-18}$$

$$|(\lambda-\lambda_1-\lambda_2-\lambda_3-\mu)-n_0^3|\ge K_0^2, \tag{10-19}$$

$$\text{there exists an } i\in\{1,2,3\} \text{ such that } |\lambda_i-n_i^3|\ge K_0^2, \tag{10-20}$$

$$|\mu|\ge K_0^2. \tag{10-21}$$

For case (10-18), we set

$$\tilde{v}(x,t) = (\hat{v}\mathbf{1}_{|\lambda-n^3|\geq K_0^2})^{\vee}(x,t). \tag{10-22}$$

We then estimate (8-10) by

$$\sum_{K_2,K_3} \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \int |\partial_x u_{K_0}| \sum_{K_1} \sum_{K\leq CK_1} K^s \tilde{v}_K^* |u_{K_1}|\, dx\, dt. \tag{10-23}$$

This is clearly bounded by

$$C\mathfrak{M}\|u\|_{Y_s}^2 \sum_{K_0} \int K_0 |u_{K_0}^*| \Big(\sum_K |\tilde{v}_K^*|^2\Big)^{1/2} \Big(\sum_{K_1} K_1^{2s} |u_{K_1}|^2\Big)^{1/2} dx\, dt. \tag{10-24}$$

Using the Cauchy–Schwarz inequality, we bound (10-24) by

$$C\mathfrak{M}\|u\|_{Y_s}^2 \int \Big(\sum_{K_0} K_0^{\varepsilon} |u_{K_0}^*|^2\Big)^{1/2} \Big(\sum_{K_0} K_0^{2-\varepsilon} \sum_K |\tilde{v}_K^*|^2\Big)^{1/2} \Big(\sum_{K_1} K_1^{2s} |u_{K_1}|^2\Big)^{1/2} dx\, dt. \tag{10-25}$$

By Hölder's inequality, (10-25) is majorized by

$$C\mathfrak{M}\|u\|_{Y_s}^2 \bigg\| \Big(\sum_{K_0} K_0^{\varepsilon} |u_{K_0}^*|^2\Big)^{1/2} \bigg\|_4 \bigg\| \Big(\sum_{K_0} K_0^{2-\varepsilon} \sum_K |\tilde{v}_K^*|^2\Big)^{1/2} \bigg\|_2 \bigg\| \Big(\sum_{K_1} K_1^{2s} |u_{K_1}|^2\Big)^{1/2} \bigg\|_4,$$

which is controlled by

$$C\mathfrak{M}\|u\|_{Y_s}^3 \|\partial_x^{\varepsilon} u\|_4 \Big(\sum_{K_0} K_0^{2-\varepsilon} \|\tilde{v}\|_2^2\Big)^{1/2} \leq C\mathfrak{M}\|u\|_{Y_s}^3 \|\partial_x^{\varepsilon} u\|_4 \sum_{K_0} K_0^{-\varepsilon/2} \leq C\mathfrak{M}\|u\|_{Y_s}^4. \tag{10-26}$$

This finishes the proof of case (10-18).

For case (10-19), let $\tilde{u}$ be defined by

$$\tilde{u} = (\hat{u}\mathbf{1}_{|\lambda-n^3|\geq K_0^2})^{\vee}. \tag{10-27}$$

Then (8-10) can be estimated by

$$\sum_{K_2,K_3} \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \int |\partial_x \tilde{u}_{K_0}| \sum_{K_1} \sum_{K\leq CK_1} K^s v_K^* |u_{K_1}|\, dx\, dt. \tag{10-28}$$

By Schur's test and Hölder's inequality, we control (10-28) by

$$\sum_{K_2,K_3} \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \|\partial_x \tilde{u}_{K_0}\|_2 \bigg\| \Big(\sum_K |v_K|^2\Big)^{1/2} \bigg\|_4 \bigg\| \Big(\sum_{K_1} K_1^{2s} |u_{K_1}|^2\Big)^{1/2} \bigg\|_4, \tag{10-29}$$

which is bounded by

$$C\mathfrak{M}\|u\|_{Y_s}^3 \sum_{K_0} \|u_{K_0}\|_{0,1/2} \leq C\mathfrak{M}\|u\|_{Y_s}^4. \tag{10-30}$$

This completes the proof of case (10-19).

For case (10-20), if $j=1$, we dominate (8-10) by

$$\sum_{K_2,K_3}\|u_{K_2}\|_\infty\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\sum_{K_0}\int|\partial_x u_{K_0}|\sum_{K_1}\sum_{K\le CK_1}K^s v_K^*|\tilde u_{K_1}|\,dx\,dt. \tag{10-31}$$

As we did in case (10-19), we bound (10-31) by

$$C\mathfrak{M}\|u\|_{Y_s}^2\sum_{K_0}\|\partial_x u_{K_0}\|_4\,\|v\|_4\,\Bigg\|\bigg(\sum_{K_1}K_1^{2s}|\tilde u_{K_1}|^2\bigg)^{1/2}\Bigg\|_2. \tag{10-32}$$

This can be further controlled by

$$C\mathfrak{M}\|u\|_{Y_s}^3\sum_{K_0}\frac{1}{K_0}\|\partial_x u_{K_0}\|_4\,\|v\|_4\le\mathfrak{M}\|u\|_{Y_s}^3\sum_{K_0}\frac{1}{K_0}\|u_{K_0}\|_{1,1/3}\le C\mathfrak{M}\|u\|_{Y_s}^4, \tag{10-33}$$

as desired.

We now consider $j=2$ or $j=3$. Without loss of generality, assume $j=2$. In this case, we estimate (8-10) by

$$\sum_{K_3}\|u_{K_3}\|\,\|G_{K_3}\|_\infty\sum_{K_0}\int|\partial_x u_{K_0}|\sum_{K_1}\sum_{K\le CK_1}K^s v_K^*|u_{K_1}|\sum_{K_2\le CK_0}|\tilde u_{K_2}|\,dx\,dt, \tag{10-34}$$

which is bounded by

$$C\mathfrak{M}\|u\|_{Y_s}\sum_{K_0}\|\partial_x u_{K_0}\|_\infty\sum_{K_2\le K_0}\|\tilde u_{K_2}\|_2\,\|v\|_4\,\Bigg\|\bigg(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\bigg)^{1/2}\Bigg\|_4.$$

Notice that

$$\begin{aligned}\sum_{K_0}\|\partial_x u_{K_0}\|_\infty\sum_{K_2\le K_0}\|\tilde u_{K_2}\|_2&\le C\sum_{K_0}\frac{1}{K_0}\|\partial_x u_{K_0}\|_\infty\|u\|_{Y_s}\\&\le C\sum_n\int|\hat u(n,\lambda)|\,d\lambda\|u\|_{Y_s}\le C\|u\|_{Y_s}^2.\end{aligned}$$

Hence (10-34) is dominated by

$$C\mathfrak{M}\|u\|_{Y_s}^4. \tag{10-35}$$

This completes case (10-20).

We now turn to the most difficult case, (10-21) in case (8-12). We should decompose G_{K_3}, with respect to the t-variable, into Littlewood–Paley multipliers in the same spirit as before. More precisely, for any dyadic number L, let Q_L be

$$Q_L u(x,t)=\int\psi_L(\tau)u(x,t-\tau)\,d\tau. \tag{10-36}$$

Here the Fourier transform of ψ_L is a bump function supported on $[-2L,2L]$ and $\widehat{\psi_L}(x)=1$ if $x\in[-L,L]$. Let

$$\Pi_L u=Q_L u-Q_{L/2}u. \tag{10-37}$$

Then $\Pi_L u$ gives a Littlewood–Paley multiplier with respect to the time variable t. Using this multiplier, we represent

$$u_K = \sum_L u_{K,L}. \tag{10-38}$$

Here $u_{K,L} = \Pi_L(u_K)$. We decompose G_{K_3} as

$$\begin{aligned} G_{K_3} &= C + \sum_L (F_3(Q_L P_{4K_3}u, \dots, Q_L P_{K_3/8}u) - F_3(Q_{L/2} P_{4K_3}u, \dots, Q_{L/2} P_{K_3/8}u)) \\ &= C + \sum_{\substack{j=4,2,1,1/2,1/4,1/8 \\ L}} H_{K_3,L} u_{jK_3,L}, \end{aligned} \tag{10-39}$$

where $H_{K_3,L}$ is given by

$$H_{K_3,L} = F_4\big(Q_{\ell L} P_{4K_3}u, \dots, Q_{\ell L} P_{K_3/8}u;\ \ell = 1, \tfrac{1}{2}\big). \tag{10-40}$$

Let $\mathfrak{M}_1$ be defined by

$$\mathfrak{M}_1 = \sup\{|D^\alpha F_4(u_1, \dots, u_{12})| : u_j \text{ satisfies } \|u_j\|_{Y_s} \le C_0 \|\phi\|_{H^s} \text{ for all } j = 1, \dots, 12;\ \alpha\}. \tag{10-41}$$

Here $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_{12}}^{\alpha_{12}}$ and α is taken over all tuples $(\alpha_1, \dots, \alpha_{12}) \in (\mathbb{N} \cup \{0\})^{12}$ with $\sum |\alpha_j| \le 1$. $\mathfrak{M}_1$ is a real number because $F_4 \in C^1$.

In order to finish the proof, we need to consider a further three subcases:

$$L \le 2^{10} K_3^3, \tag{10-42}$$

$$2^{10} K_3^3 < L \le 2^{-5} K_0^2, \tag{10-43}$$

$$L > 2^{-5} K_0^2. \tag{10-44}$$

For the contribution of (10-42), we set

$$h_{K_0,jK_3,L} = (\widehat{H_{K_3,L} u_{jK_3,L}} \mathbf{1}_{|\mu| \ge K_0^2})^\vee. \tag{10-45}$$

Here $j = 4, 2, 1, 1/2, 1/4, 1/8$. From the definition of $H_{K_3,L}$, we get

$$\|h_{K_0,jK_3,L}\|_4 \le C \mathfrak{M}_1 \|\phi\|_{H^s} \frac{L}{K_0^2} \|u_{jK_3,L}\|_4. \tag{10-46}$$

Then (8-10) is bounded by

$$\sum_{K_2} \|u_{K_2}\|_\infty \sum_{K_0} \int K_0 u_{K_0}^* \sum_{K_3 \le C K_0^{1/2}} \|u_{K_3}\|_\infty \sum_{L \le C K_3^3} |h_{K_0,jK_3,L}| \sum_{K_1} \sum_{K \le C K_1} K^s v_K^* |u_{K_1}|\, dx\, dt, \tag{10-47}$$

which is majorized by

$$\begin{aligned} &\sum_{K_2} \|u_{K_2}\|_\infty \sum_{K_0} K_0 \sum_{K_3 \le C K_0^{1/2}} \|u_{K_3}\|_\infty \int u_{K_0}^* \\ &\qquad \cdot \sum_{L \le C K_3^3} |h_{K_0,jK_3,L}| \Big(\sum_K |v_K^*|^2\Big)^{1/2} \Big(\sum_{K_1} K_1^{2s} |u_{K_1}|^2\Big)^{1/2} dx\, dt. \end{aligned} \tag{10-48}$$

Using Hölder's inequality with L^4 norms for the four functions in the integrand, we estimate (10-48) as follows:

$$C\mathfrak{M}_1\|\phi\|_{H^s}\|u\|_{Y_s}^2\sum_{K_0}K_0\|u_{K_0}\|_4\sum_{K_3\le K_0^{1/2}}\|u_{K_3}\|_\infty\sum_{L\le CK_3^3}\frac{L}{K_0^2}\|u_{jK_3,L}\|_4$$

$$\le C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^3\sum_{K_0}K_0^{1/2}\|u_{K_0}\|_{0,1/3}$$

$$\le C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^4. \tag{10-49}$$

This finishes case (10-42).

For the contribution of (10-43), we bound (8-10) by

$$\sum_{K_2}\|u_{K_2}\|_\infty\sum_{K_3}\|u_{K_3}\|_\infty\int\sum_{K_0}|\partial_x u_{K_0}|\sum_{2^{10}K_3^3<L\le 2^{-10}K_0^2}|h_{K_0,jK_3,L}|\sum_{K_1}\sum_{K\le CK_1}K^s v_K^*|u_{K_1}|\,dx\,dt, \tag{10-50}$$

which is dominated by

$$C\|u\|_{Y_s}\sum_{K_3}\|u_{K_3}\|_\infty\sum_{\substack{\Delta\le 2^{-10}\\ \Delta\,\text{dyadic}}}\int\sum_{K_0}|\partial_x u_{K_0}|\sum_{\substack{2^{10}K_3^3<L\\ (\Delta/2)K_0^2<L\le\Delta K_0^2}}|h_{K_0,jK_3,L}|$$

$$\cdot\Big(\sum_K|v_K^*|^2\Big)^{1/2}\Big(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\Big)^{1/2}dx\,dt, \tag{10-51}$$

By the Cauchy–Schwarz inequality, we further estimate (10-51) by

$$C\|u\|_{Y_s}\sum_{K_3}\|u_{K_3}\|_\infty\sum_{\substack{\Delta\le 2^{-10}\\ \Delta\,\text{dyadic}}}\Delta^{-1/2}\int\sum_{K_0}\frac{|\partial_x u_{K_0}|}{K_0}$$

$$\cdot\Big(\sum_{\substack{2^{10}K_3^3<L\\ (\Delta/2)K_0^2<L\le\Delta K_0^2}}L|h_{K_0,jK_3,L}|^2\Big)^{1/2}\Big(\sum_K|v_K^*|^2\Big)^{1/2}\Big(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\Big)^{1/2}dx\,dt. \tag{10-52}$$

Applying Hölder's inequality with an L^∞ norm for the first function in the integrand, an L^2 norm for the second, and L^4 norms for the last two functions, we then majorize (10-52) by

$$C\|u\|_{Y_s}^2\sum_{K_3}\|u_{K_3}\|_\infty\sum_{\substack{\Delta\le 2^{-10}\\ \Delta\,\text{dyadic}}}\Delta^{-1/2}\sum_{K_0}\frac{\|\partial_x u_{K_0}\|_\infty}{K_0}\Big\|\Big(\sum_{\substack{2^{10}K_3^3<L\\ (\Delta/2)K_0^2<L\le\Delta K_0^2}}L|h_{K_0,jK_3,L}|^2\Big)^{1/2}\Big\|_2. \tag{10-53}$$

Notice that if $L\sim\Delta K_0^2$,

$$\|h_{K_0,jK_3,L}\|_2\le C\mathfrak{M}_1\|\phi\|_{H^s}\Delta\|u_{jK_3,L}\|_2. \tag{10-54}$$

Thus we have

$$\begin{aligned}\Bigg\|\bigg(\sum_{\substack{2^{10}K_3^3<L\\(\Delta/2)K_0^2<L\le\Delta K_0^2}} L|h_{K_0,jK_3,L}|^2\bigg)^{1/2}\Bigg\|_2 &\le C\mathfrak{M}_1\|\phi\|_{H^s}\Delta\bigg(\sum_{\substack{2^{10}K_3^3<L\\(\Delta/2)K_0^2<L\le\Delta K_0^2}} L\|u_{jK_3,L}\|_2^2\bigg)^{1/2}\\ &\le C\mathfrak{M}_1\|\phi\|_{H^s}\Delta\|u_{jK_3}\|_{0,1/2}\\ &\le C\mathfrak{M}_1\|\phi\|_{H^s}^2\Delta. \end{aligned}\tag{10-55}$$

From (10-55), (10-53) is bounded by

$$C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^2\sum_{K_3}\|u_{K_3}\|_\infty\sum_{\substack{\Delta\le 2^{-10}\\ \Delta\,\text{dyadic}}}\Delta^{1/2}\sum_{K_0}\frac{\|\partial_x u_{K_0}\|_\infty}{K_0},\tag{10-56}$$

which is clearly majorized by

$$C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^4.\tag{10-57}$$

This finishes case (10-43).

For the contribution of (10-44), we estimate (8-10) by

$$\sum_{K_2}\|u_{K_2}\|_\infty\sum_{K_3}\|u_{K_3}\|_\infty\int\sum_{K_0}|\partial_x u_{K_0}|\sum_{L>2^{-5}K_0^2}|h_{K_0,jK_3,L}|\sum_{K_1}\sum_{K\le CK_1}K^s v_K^*|u_{K_1}|\,dx\,dt,\tag{10-58}$$

which is bounded by

$$\begin{aligned}\sum_{K_2}\|u_{K_2}\|_\infty\sum_{K_3}\|u_{K_3}\|_\infty\int\bigg(\sum_{K_0}\frac{|\partial_x u_{K_0}|^2}{K_0^2}\bigg)^{1/2}&\\ \cdot\bigg(\sum_{L>2^{-5}K_0^2}L|h_{K_0,jK_3,L}|^2\bigg)^{1/2}\bigg(\sum_K|v_K^*|^2\bigg)^{1/2}\bigg(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\bigg)^{1/2}&dx\,dt.\end{aligned}\tag{10-59}$$

Applying Hölder's inequality, we further have

$$\begin{aligned}(10\text{-}59)&\le C\mathfrak{M}_1\|u\|_{Y_s}^2\sum_{K_3}\|u_{K_3}\|_\infty\sum_{K_0}\frac{\|\partial_x u_{K_0}\|_\infty}{K_0}\bigg(\sum_{L>2^{-5}K_0^2}L\|u_{jK_3,L}\|_2^2\bigg)^{1/2}\\ &\le C\mathfrak{M}_1\|u\|_{Y_s}^2\sum_{K_3}\|u_{K_3}\|_\infty\sum_{K_0}\frac{\|\partial_x u_{K_0}\|_\infty}{K_0}\,\|u_{jK_3}\|_{0,1/2}.\end{aligned}\tag{10-60}$$

This is clearly majorized by

$$C\mathfrak{M}_1\|\phi\|_{H_s}\|u\|_{Y_s}^4.\tag{10-61}$$

Hence we complete case (10-44).

11. Proof of case (8-13)

In this case, it suffices to consider the following subcases:

$$M \geq 2^{-10} K_0^{2/3}, \tag{11-1}$$

$$M < 2^{-10} K_0^{2/3} \quad \text{and} \quad K_2^2 K_3 \geq 2^{-10} K_0^2, \tag{11-2}$$

$$M < 2^{-10} K_0^{2/3} \quad \text{and} \quad K_2^2 M \geq 2^{-10} K_0^2, \tag{11-3}$$

$$M < 2^{-10} K_0^{2/3}, \quad K_2^2 K_3 < 2^{-10} K_0^2 \quad \text{and} \quad K_2^2 M < 2^{-10} K_0^2. \tag{11-4}$$

For case (11-1), notice that we have

$$K \leq C M^{3/2}. \tag{11-5}$$

Hence we estimate (8-10) by

$$\int \sum_{K_1 \geq K_2 \geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \sum_M \sum_{K \leq CM^{3/2}} K^s v_K^* \sum_{K_0 \leq CM^{3/2}} K_0 u_{K_0}^* |g_{K_3,M}| \, dx \, dt, \tag{11-6}$$

which is bounded by

$$\int \sum_{K_1 \geq K_2 \geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \sum_M M^{(3/2)(1-s)} |g_{K_3,M}| \sum_{K \leq CM^{3/2}} K^s v_K^* \biggl(\sum_{K_0} K_0^{2s} |u_{K_0}^*|^2 \biggr)^{1/2} dx \, dt, \tag{11-7}$$

since $1/2 < s < 1$. Applying Schur's test, we estimate (11-7) by

$$\int \sum_{K_1 \geq K_2 \geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \biggl(\sum_M M^3 |g_{K_3,M}|^2 \biggr)^{1/2} \biggl(\sum_K |v_K^*|^2 \biggr)^{1/2} \biggl(\sum_{K_0} K_0^{2s} |u_{K_0}^*|^2 \biggr)^{1/2} dx \, dt. \tag{11-8}$$

By Hölder's inequality and $s > 1/2$, (11-8) is majorized by

$$\begin{aligned} C \sum_{K_1 \geq K_2 \geq K_3} \|\partial_x^{3/2} G_{K_3}\|_\infty \biggl(\prod_{j=1}^3 \|u_{K_j}\|_{6+} \biggr) \biggl\| \biggl(\sum_K |v_K|^2 \biggr)^{1/2} \biggr\|_4 \biggl\| \biggl(\sum_{K_0} K_0^{2s} |u_{K_0}^*|^2 \biggr)^{1/2} \biggr\|_4 \\ \leq C\mathfrak{M}(\|\phi\|_{H^s} + \|\phi\|_{H^s}^2) \|u\|_{Y_s} \sum_{K_1 \geq K_3 \geq K_3} K_3^{3/2} \prod_{j=1}^3 \|u_{K_j}\|_{6+} \\ \leq C\mathfrak{M}(\|\phi\|_{H^s} + \|\phi\|_{H^s}^2) \|u\|_{Y_s} \prod_{j=1}^3 \sum_{K_j} K_j^{1/2} \|u_{K_j}\|_{0+,1/2} \\ \leq C\mathfrak{M}(\|\phi\|_{H^s} + \|\phi\|_{H^s}^2) \|u\|_{Y_s}^4. \end{aligned} \tag{11-9}$$

This finishes case (11-1).

For case (11-2), observe that, in this case,

$$K_0 \leq C K_1^{1/2} K_2^{1/2} K_3^{1/2}. \tag{11-10}$$

We estimate (8-10) by

$$\int \sum_{K_1\geq K_2\geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \sum_{K\leq CK_0} K^s v_K^* \sum_{K_0\leq C(K_1K_2K_3)^{1/2}} K_0 u_{K_0}^* \|G_{K_3}\|_\infty dx\, dt, \tag{11-11}$$

which is bounded by

$$C\mathfrak{M} \int \Big(\sum_K |v_K^*|^2\Big)^{1/2} \Big(\sum_{K_0} K_0^{2s} |u_{K_0}^*|^2\Big)^{1/2} \prod_{j=1}^3 \sum_{K_j} K_j^{1/2} |u_{K_j}|\, dx\, dt. \tag{11-12}$$

Using Hölder's inequality with L^4 norms for the first two functions and L^6 norms for the last three functions in the integrand, we obtain

$$C\mathfrak{M}\|u\|_{Y_s} \prod_{j=1}^3 \Big\| \sum_{K_j} K_j^{1/2} |u_{K_j}| \Big\|_6 \leq C\mathfrak{M}\|u\|_{Y_s}^4. \tag{11-13}$$

This completes case (11-2).

For case (11-3) we have

$$K_0 \leq C K_1^{1/2} K_2^{1/2} M^{1/2}. \tag{11-14}$$

Hence we dominate (8-10) by

$$\int \sum_{K_1\geq K_2\geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \sum_M |g_{K_3,M}| \sum_{K\leq CK_0} K^s v_K^* \sum_{K_0\leq C(K_1K_2M)^{1/2}} K_0 u_{K_0}^* dx\, dt, \tag{11-15}$$

which is bounded by

$$C \sum_{K_3} \int \Big(\sum_K |v_K^*|^2\Big)^{1/2} \Big(\sum_{K_0} K_0^{2s} |u_{K_0}^*|^2\Big)^{1/2} |u_{K_3}| \cdot \Big(\sum_M M |g_{K_3,M}|^2\Big)^{1/2} \prod_{j=1}^2 \sum_{K_j} K_j^{1/2} |u_{K_j}|\, dx\, dt. \tag{11-16}$$

Using Hölder's inequality with L^4 norms for the first two functions, L^6 norms for the third, an L^p norm with p very large for the fourth, and L^{6+} for the last two functions in the integrand, we obtain

$$C\|u\|_{Y_s} \prod_{j=1}^2 \Big\| \sum_{K_j} K_j^{1/2} |u_{K_j}| \Big\|_{6+} \sum_{K_3} \|u_{K_3}\|_6 \, \|\partial_x^{1/2} G_{K_3}\|_\infty. \tag{11-17}$$

Clearly (11-17) is dominated by

$$C\mathfrak{M}\|\phi\|_{H^s} \|u\|_{Y_s}^3 \sum_{K_3} K_3^{1/2} \|u_{K_3}\|_6 \leq C\mathfrak{M}\|\phi\|_{H^s}\|u\|_{Y_s}^4. \tag{11-18}$$

Hence case (11-3) is done.

For case (11-4) we observe that

$$M^2 K_2 \leq 2^{-10} K_0^2. \tag{11-19}$$

In fact, if (11-19) does not hold, then, from (11-4),

$$M^2K_2 > 2^{-10}K_0^2 > K_2^2 M.$$

Thus $M > K_2$, which immediately yields

$$M^3 > M^2K_2 > 2^{-10}K_0^2,$$

contradicting $M < 2^{-10}K_0^{2/3}$. Hence (11-19) must be true. From (11-19), $K_2^2K_3 + K_2^2M < 2^{-9}K_0^2$, we get

$$|(n_2+n_3+m)^3 - n_2^3 - n_3^3 - m^3| \le 2^{-5}K_0^2. \tag{11-20}$$

Since $n_1+n_2+n_3+m \ne 0$, from (8-13), (11-4), and (11-20), the crucial arithmetic observation (10-16) yields

$$|n^3 - n_0^3 - n_1^3 - n_2^3 - n_3^3 - m^3| \ge 2K_0^2. \tag{11-21}$$

Hence one of the following statements must be true:

$$|\lambda - n^3| \ge K_0^2, \tag{11-22}$$

$$|(\lambda-\lambda_1-\lambda_2-\lambda_3-\mu) - n_0^3| \ge K_0^2, \tag{11-23}$$

$$\text{there exists an } i \in \{1,2,3\} \text{ such that } |\lambda_i - n_i^3| \ge K_0^2, \tag{11-24}$$

$$|\mu| \ge K_0^2. \tag{11-25}$$

For case (11-22), we estimate (8-10) by

$$\sum_{K_1,K_2,K_3} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \int K_0 |u^*_{K_0}| \Big| \sum_{K \le CK_0} \partial_x^s \tilde{v}_K \Big| \, dx\, dt. \tag{11-26}$$

Then the Cauchy–Schwarz inequality yields

$$\begin{aligned} C\mathfrak{M}\|u\|_{Y_s}^3 &\Big\| \Big(\sum_{K_0} K_0^{2-2s} \Big| \sum_{K\le CK_0} \partial_x^s \tilde{v}_K \Big|^2 \Big)^{1/2} \Big\|_2 \Big\| \Big(\sum_{K_0} K_0^{2s} |u^*_{K_0}|^2 \Big)^{1/2} \Big\|_2 \\ &\le C\mathfrak{M}\|u\|_{Y_s}^4 \Big(\sum_{K_0} K_0^{2-2s} \sum_{K\le CK_0} \big\| \partial_x^s \tilde{v}_K \big\|_2^2 \Big)^{1/2} \le C\mathfrak{M}\|u\|_{Y_s}^4. \end{aligned} \tag{11-27}$$

This finishes the proof of case (11-22).

For case (11-23), (8-10) can be estimated by

$$\sum_{K_1,K_2,K_3} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \int K_0 |\tilde{u}^*_{K_0}| \sum_{K \le CK_0} K^s v^*_K \, dx\, dt. \tag{11-28}$$

By Schur's test and Hölder's inequality, we control (11-28) by

$$C\mathfrak{M}\|u\|_{Y_s}^3 \Big\| \Big(\sum_K |v^*_K|^2 \Big)^{1/2} \Big\|_2 \Big\| \Big(\sum_{K_0} K_0^{2s+2} |\tilde{u}_{K_0}|^2 \Big)^{1/2} \Big\|_2, \tag{11-29}$$

which is clearly bounded by

$$C\mathfrak{M}\|u\|_{Y_s}^3\Big(\sum_{K_0} K_0^{2s}\|u_{K_0}\|_{0,1/2}^2\Big)^{1/2} \le C\mathfrak{M}\|u\|_{Y_s}^4. \tag{11-30}$$

This completes the proof of case (11-23).

For case (11-24), without loss of generality, assume $j=1$. We then dominate (8-10) by

$$\sum_{K_2,K_3}\|u_{K_2}\|_\infty\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\sum_{K_1}\sum_{K_0}\int K_0|u_{K_0}^*||\tilde{u}_{K_1}|\sum_{K\le CK_0}K^s v_K^*\,dx\,dt. \tag{11-31}$$

By Hölder's inequality, we bound (11-31) by

$$\begin{aligned}\sum_{K_2,K_3}&\|u_{K_2}\|_\infty\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\sum_{K_1}\sum_{K_0}\sum_{K\le CK_0}K^sK_0\|u_{K_0}\|_4\,\|\tilde{u}_{K_1}\|_2\,\|v_K\|_4\\ &\le\sum_{K_2,K_3}\|u_{K_2}\|_\infty\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\sum_{K_1}\|u_{K_1}\|_{0,1/2}\sum_{K_0}\sum_{K\le CK_0}K^s\|u_{K_0}\|_4\|v_K\|_4. \end{aligned}\tag{11-32}$$

By Schur's test, we dominate (11-32) by

$$\begin{aligned}C\mathfrak{M}\|u\|_{Y_s}^2&\sum_{K_1}\|u_{K_1}\|_{0,1/2}\Big(\sum_{K_0}K_0^{2s}\|u_{K_0}\|_4^2\Big)^{1/2}\Big(\sum_K\|v_K\|_4^2\Big)^{1/2}\\ &\le C\mathfrak{M}\|u\|_{Y_s}^3\Big(\sum_{K_0}K_0^{2s}\|u_{K_0}\|_{0,1/3}^2\Big)^{1/2}\Big(\sum_K\|v_K\|_{0,1/3}^2\Big)^{1/2}\\ &\le C\mathfrak{M}\|u\|_{Y_s}^4. \end{aligned}\tag{11-33}$$

Hence case (11-24) is done.

In order to finish the proof, as is done in (10-36), we need to consider three further subcases:

$$L\le 2^{10}K_3^3, \tag{11-34}$$

$$2^{10}K_3^3<L\le 2^{-5}K_0^2, \tag{11-35}$$

$$L>2^{-5}K_0^2. \tag{11-36}$$

For the contribution of (11-34), notice that

$$\|h_{K_0,jK_3,L}\|_6\le C\mathfrak{M}_1\|\phi\|_{H^s}\frac{L}{K_0^2}\|u_{jK_3,L}\|_6. \tag{11-37}$$

Here $h_{K_0,jK_3,L}$ is defined as in (10-45). In this case we also have $K_3\le K_0^{2/3}$, from

$$K_2^2K_3\le 2^{-10}K_0^2.$$

Then (8-10) is bounded by

$$\int\sum_{K_0}K_0u_{K_0}^*\sum_{K\le CK_0}K^sv_K^*\sum_{\substack{K_1\ge K_2\ge K_3\\ K_3\le K_0^{2/3}}}|u_{K_1}||u_{K_2}||u_{K_3}|\sum_{L\le CK_3^3}|h_{K_0,jK_3,L}|\,dx\,dt. \tag{11-38}$$

Write (11-38) as

$$\sum_{\substack{\Delta \text{dyadic}\\ \Delta\le 1}} \int \sum_{K_0} K_0 u^*_{K_0} \sum_{K\le CK_0} K^s v^*_K \sum_{\substack{K_1\ge K_2\ge K_3\\ \Delta K_0^{2/3}/2<K_3\le \Delta K_0^{2/3}}} |u_{K_1}||u_{K_2}||u_{K_3}| \sum_{L\le CK_3^3} |h_{K_0,jK_3,L}|\,dx\,dt. \tag{11-39}$$

Observe that if $\Delta K_0^{2/3}/2 < K_3 \le \Delta K_0^{2/3}$, we have

$$K_0 \le \Delta^{-3/2} K_1^{1/2} K_2^{1/2} K_3^{1/2}. \tag{11-40}$$

Hence

$$\begin{aligned} C\|u\|_{Y_s} \sum_{K_0}\sum_{K\le K_0} K^s \sum_{K_1,K_2} K_1^{1/2}K_2^{1/2} \sum_{\Delta\le 1}\Delta^{-3/2} \sum_{K_3\sim \Delta K_0^{2/3}} K_3^{1/2} \\ \cdot \int u^*_{K_0} v^*_K |u_{K_1}||u_{K_2}| \sum_{L\le CK_3^3} |h_{K_0,jK_3,L}|\,dx\,dt. \end{aligned} \tag{11-41}$$

Applying Hölder's inequality with L^4 norms for first two functions and L^6 for the last three, and then using (11-37), we get

$$\begin{aligned} C\mathfrak{M}_1\|\phi\|_{H^s}\|u\|_{Y_s} \sum_{K_0}\sum_{K\le K_0} K^s \sum_{K_1,K_2} K_1^{1/2}K_2^{1/2} \sum_{\Delta\le 1}\Delta^{-3/2} \\ \cdot \sum_{K_3\sim \Delta K_0^{2/3}} K_3^{1/2} \|u_{K_0}\|_4\,\|v^*_K\|_4\,\|u_{K_1}\|_6\,\|u_{K_2}\|_6 \sum_{L\le CK_3^3} \frac{L}{K_0^2}\|u_{jK_3,L}\|_6, \end{aligned} \tag{11-42}$$

which is bounded by

$$\begin{aligned} &C\mathfrak{M}_1\|\phi\|_{H^s}\|u\|_{Y_s} \sum_{K_0}\sum_{K\le K_0} K^s \sum_{\Delta\le 1}\Delta^{-3/2} \sum_{L\le C\Delta^3K_0^2} \frac{L}{K_0^2}\|u_{K_0}\|_4\,\|v^*_K\|_4 \\ &\quad\cdot \sum_{K_1} K_1^{1/2}\|u_{K_1}\|_{0+,1/2} \sum_{K_2} K_2^{1/2}\|u_{K_2}\|_{0+,1/2} \sum_{K_3} K_3^{1/2}\|u_{jK_3,L}\|_{0+,1/2} \\ &\qquad\le C\mathfrak{M}_1\|\phi\|^2_{H^s}\|u\|^3_{Y_s} \sum_{\Delta\le 1}\Delta^{3/2}\sum_{K_0}\sum_{K\le CK_0} K^s\|u_{K_0}\|_4\,\|v_K\|_4 \\ &\qquad\le C\mathfrak{M}_1\|\phi\|^2_{H^s}\|u\|^3_{Y_s} \Big(\sum_{K_0} K_0^{2s}\|u_{K_0}\|^2_{0,1/3}\Big)^{1/2}\Big(\sum_K \|v_K\|^2_{0,1/3}\Big)^{1/2} \\ &\qquad\le C\mathfrak{M}_1\|\phi\|^2_{H^s}\|u\|^4_{Y_s}. \end{aligned} \tag{11-43}$$

This completes case (11-34).

For the contribution of (11-35), we bound (8-10) by

$$\sum_{K_1}\|u_{K_1}\|_\infty \sum_{K_2}\|u_{K_2}\|_\infty \sum_{K_3}\|u_{K_3}\|_\infty \int \sum_{K_0}\sum_{K\le CK_0} K^s v^*_K K_0 u^*_{K_0} \sum_{2^{10}K_3^3<L\le 2^{-5}K_0^2} |h_{K_0,jK_3,L}|\,dx\,dt, \tag{11-44}$$

which is dominated by

$$C\|u\|_{Y_s}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\substack{\Delta\le 2^{-5}\\ \Delta\text{dyadic}}} \sum_{K_0} \sum_{K\le CK_0} K^s \int K_0 u_{K_0}^* v_K^* \sum_{\substack{2^{10}K_3^3<L\\ (\Delta/2)K_0^2<L\le\Delta K_0^2}} |h_{K_0,jK_3,L}|\, dx\, dt. \tag{11-45}$$

Using the Cauchy–Schwarz inequality, we further estimate (11-45) by

$$C\|u\|_{Y_s}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\substack{\Delta\le 2^{-5}\\ \Delta\text{dyadic}}} \Delta^{-\frac{1}{2}} \sum_{K_0} \sum_{K\le CK_0} K^s \int u_{K_0}^* v_K^* \Bigg(\sum_{\substack{2^{10}K_3^3<L\\ (\Delta/2)K_0^2<L\le\Delta K_0^2}} L|h_{K_0,jK_3,L}|^2\Bigg)^{1/2} dx\, dt. \tag{11-46}$$

Employing Hölder's inequality with L^4 norms for the first two functions and an L^2 for the last one, we bound (11-46) by

$$C\|u\|_{Y_s}^2 \cdot \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\substack{\Delta\le 2^{-10}\\ \Delta\text{dyadic}}} \Delta^{-1/2} \sum_{K_0} \sum_{K\le CK_0} K^s \|u_{K_0}\|_4\, \|v_K\|_4 \Bigg\| \Bigg(\sum_{\substack{2^{10}K_3^3<L\\ (\Delta/2)K_0^2<L\le\Delta K_0^2}} L|h_{K_0,jK_3,L}|^2\Bigg)^{1/2} \Bigg\|_2. \tag{11-47}$$

From (10-55), (11-47) is majorized by

$$\begin{aligned} C\mathfrak{M}_1 \|\phi\|_{H^s}^2 \|u\|_{Y_s}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\substack{\Delta\le 2^{-10}\\ \Delta\text{dyadic}}} \Delta^{\frac{1}{2}} \sum_{K_0} \sum_{K\le CK_0} K^s \|u_{K_0}\|_4\, \|v_K\|_4 \\ \le C\mathfrak{M}_1 \|\phi\|_{H^s}^2 \|u\|_{Y_s}^3 \Bigg(\sum_{K_0} K_0^{2s} \|u_{K_0}\|_{0,\frac{1}{3}}^2 \Bigg)^{1/2} \Bigg(\sum_{K} \|v_K\|_{0,\frac{1}{3}}^2 \Bigg)^{1/2} \\ \le C\mathfrak{M}_1 \|\phi\|_{H^s}^2 \|u\|_{Y_s}^4. \end{aligned} \tag{11-48}$$

This finishes the proof for case (11-35).

For the contribution of (11-36), we estimate (8-10) by

$$\sum_{K_1,K_2} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \sum_{K_3} \|u_{K_3}\|_\infty \int \sum_{K_0} K_0 u_{K_0}^* \sum_{L>2^{-5}K_0^2} |h_{K_0,jK_3,L}| \sum_{K\le CK_0} K^s v_K^*\, dx\, dt. \tag{11-49}$$

By the Cauchy–Schwarz inequality, (11-49) is bounded by

$$\sum_{K_1,K_2} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \sum_{K_3} \|u_{K_3}\|_\infty \sum_{K_0} \sum_{K\le CK_0} K^s \int v_K^* u_{K_0}^* \Bigg(\sum_{L>2^{-10}K_0^2} L|h_{K_0,jK_3,L}|^2\Bigg)^{1/2} dx\, dt. \tag{11-50}$$

Employing Hölder's inequality with L^4 norms for the first two functions and an L^2 for the last one, we dominate (11-50) by

$$C\mathfrak{M}_1\|u\|_{Y_s}^2 \sum_{K_3}\|u_{K_3}\|_\infty \sum_{K_0}\sum_{K\le CK_0} K^s\|u_{K_0}\|_4\,\|v_K\|_4 \left\|\Bigl(\sum_{L>2^{-5}K_0^2} L|u_{jK_3,L}|^2\Bigr)^{1/2}\right\|_2$$
$$\le C\mathfrak{M}_1\|u\|_{Y_s}^2 \sum_{K_3}\|u_{K_3}\|_\infty \sum_{K_0}\sum_{K\le CK_0} K^s\|u_{K_0}\|_{0,1/3}\|v_K\|_{0,1/3}\|u\|_{0,1/2}$$
$$\le C\mathfrak{M}_1\|\phi\|_{H^s}\|u\|_{Y_s}^4. \tag{11-51}$$

Hence we complete case (11-36).

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YI HU: yihu1@illinois.edu
Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801, United States
Current address: *Department of Mathematical Sciences, Georgia Southern University, 65 Georgia Ave. Room 3008, Statesboro, GA 30458, United States*

XIAOCHUN LI: xcli@math.uiuc.edu
Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801, United States

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RESTRICTION AND SPECTRAL MULTIPLIER THEOREMS ON ASYMPTOTICALLY CONIC MANIFOLDS

COLIN GUILLARMOU, ANDREW HASSELL AND ADAM SIKORA

The classical Stein–Tomas restriction theorem is equivalent to the fact that the spectral measure $dE(\lambda)$ of the square root of the Laplacian on $\mathbb{R}^n$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ for $1 \le p \le 2(n+1)/(n+3)$, where p' is the conjugate exponent to p, with operator norm scaling as $\lambda^{n(1/p-1/p')-1}$. We prove a geometric, or variable coefficient, generalization in which the Laplacian on $\mathbb{R}^n$ is replaced by the Laplacian, plus a suitable potential, on a nontrapping asymptotically conic manifold. It is closely related to Sogge's discrete L^2 restriction theorem, which is an $O(\lambda^{n(1/p-1/p')-1})$ estimate on the $L^p \to L^{p'}$ operator norm of the spectral projection for a spectral window of fixed length. From this, we deduce spectral multiplier estimates for these operators, including Bochner–Riesz summability results, which are sharp for p in the range above.

The paper divides naturally into two parts. In the first part, we show at an abstract level that restriction estimates imply spectral multiplier estimates, and are implied by certain pointwise bounds on the Schwartz kernel of λ-derivatives of the spectral measure. In the second part, we prove such pointwise estimates for the spectral measure of the square root of Laplace-type operators on asymptotically conic manifolds. These are valid for all $\lambda > 0$ if the asymptotically conic manifold is nontrapping, and for small λ in general. We also observe that Sogge's estimate on spectral projections is valid for any complete manifold with C^∞ bounded geometry, and in particular for asymptotically conic manifolds (trapping or not), while by contrast, the operator norm on $dE(\lambda)$ may blow up exponentially as $\lambda \to \infty$ when trapping is present.

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Keywords: restriction estimates, spectral multipliers, Bochner–Riesz summability, asymptotically conic manifolds .

1. Introduction

The aim of this article is to prove some L^p multiplier properties for the Laplacian, and a Stein–Tomas-type restriction theorem for its spectral measure, on a class of Riemannian manifolds which include metric perturbations of Euclidean space. One of the first natural questions in harmonic analysis is to understand the L^p boundedness of Fourier multipliers M on $\mathbb{R}^n$, defined by

$$M(f)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix.\xi} m(\xi) \hat{f}(\xi)\, d\xi,$$

where m is a measurable function. Notice that for radial multipliers $m(\xi) = F(|\xi|)$, this amounts to study the L^p boundedness of $F(\sqrt{\Delta})$, where Δ is the nonnegative Laplacian. Of course, for $p = 2$, the necessary and sufficient condition on m for M to be bounded on L^2 is that $m \in L^\infty(\mathbb{R}^n)$, but the case $p \neq 2$ is much more difficult. The first results in this direction were given by Mikhlin [1965]: M acts boundedly on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ if

$$m \in C^\infty(\mathbb{R}^n \setminus \{0\}) \quad \text{and} \quad |\xi|^k |\nabla^k m(\xi)| \in L^\infty, \quad \forall k,\ 0 \le k \le \tfrac{1}{2}n + 1.$$

This was sharpened by Hörmander [1960; 1983, Theorem 7.9.5]: Let $\psi \in C_0^\infty(\frac{1}{2}, 2)$ be not identically zero, then M acts boundedly on $L^p(\mathbb{R}^n))$ for all $1 < p < \infty$ if

$$\sup_{t>0} \|m(t\,\cdot\,)\psi\|_{H^s(\mathbb{R}^n)} < \infty, \quad \tfrac{1}{2}n < s \in \mathbb{N}.$$

More generally, let $\boldsymbol{L}$ be a self-adjoint operator acting on L^2 of some measure space. Using the spectral theorem, "spectral multipliers" $F(\boldsymbol{L})$ can be defined for any bounded Borel function F, and they act continuously on L^2. A question which has attracted a lot of attention during the last thirty years is to find some necessary conditions on the function F to ensure that the operator $F(\boldsymbol{L})$ extends as a bounded operator for some range of L^p spaces for $p \neq 2$. Probably the most natural and concrete examples are functions of the Laplacian on complete Riemannian manifolds, or functions of Schrödinger operators with real potential $\Delta + V$, but these problems are also studied for abstract self-adjoint operators. Some particular families of functions F are also investigated in the theory of spectral multipliers: some of the most important examples include oscillatory integrals $e^{i(t\boldsymbol{L})^\alpha}(\mathrm{Id} + (t\boldsymbol{L})^\alpha)^{-\beta}$ and Bochner–Riesz means (2-18). The subject of Bochner–Riesz means and spectral multipliers is so broad that it is impossible to provide a comprehensive bibliography here, so we refer the reader to [Anker 1990; Christ and Sogge 1988; Clerc and Stein 1974; Cowling and Sikora 2001; Mauceri and Meda 1990; Müller and Stein 1994; Seeger and Sogge 1989; Sogge 1987; 1993; Taylor 1989; Thangavelu 1993], where further literature can be found.

The theory of Fourier multipliers and Bochner–Riesz analysis in this setting is related to the so-called *sphere restriction problem* for the Fourier transform: find the pairs (p, q) for which the *sphere restriction operator* $\mathrm{SR}(\lambda)$, defined by

$$\mathrm{SR}(\lambda) f(\omega) := \hat{f}(\lambda\omega), \quad \omega \in S^{n-1}, \lambda > 0,$$

acts boundedly from $L^p(\mathbb{R}^n)$ to $L^q(S^{n-1})$; see [Fefferman 1970; 1973]. Of course, the dependence in λ is trivial here since $\mathrm{SR}(\lambda)f = \lambda^{-n}\mathrm{SR}(1)(f(\lambda^{-1}\cdot))$, but this parameter λ will be important later on. There is a long list of results on this problem, but the first ones for general dimensions are due to Stein and Tomas. The theorem of Tomas [1975], improved by Stein [1993, Chapter IX, Section 2] for the endpoint $p = 2(n+1)/(n+3)$ is the following: SR(1) maps $L^p(\mathbb{R}^n)$ boundedly to $L^q(S^{n-1})$ if $p \le 2(n+1)/(n+3)$ and $q \le \frac{n-1}{n+1}\frac{p}{p-1}$ (notice that $q = 2$ when p reaches the endpoint). On the other hand, a necessary condition (based on the Knapp example) for boundedness is only given by $p < 2n/(n+1)$ and this leads to the conjecture that $p < 2n/(n+1)$ and $q \le \frac{n-1}{n+1}\frac{p}{p-1}$ is a necessary and sufficient condition. In fact, this has been shown by Zygmund [1974] in dimension 2, improving a result of Fefferman [1970] (by obtaining the endpoint estimate), but the conjecture is still open for $n > 2$. For more references and new results in this direction, we refer the interested reader to the survey by Tao [2003] on the subject.

Like the L^p multiplier problem, the sphere restriction problem has a corresponding natural generalization to certain types of manifolds (at least if we think of Fourier transform as a spectral diagonalization for the Laplacian), and in particular those which have similar structure at infinity as Euclidean space. On $\mathbb{R}^n$, the Schwartz kernel of the spectral measure $dE_{\sqrt{\Delta}}(\lambda)$ of $\sqrt{\Delta}$ is given by

$$dE_{\sqrt{\Delta}}(\lambda; z, z') = \frac{\lambda^{n-1}}{(2\pi)^n}\int_{S^{n-1}} e^{i(z-z').\lambda\omega}\,d\omega, \qquad z, z' \in \mathbb{R}^n,$$

therefore $dE_{\sqrt{\Delta}}(\lambda) = (\lambda^{n-1})/((2\pi)^n)\mathrm{SR}(\lambda)^*\mathrm{SR}(\lambda)$ and the restriction theorem for $q = 2$ is equivalent to finding the largest $p < 2$ such that $dE_{\sqrt{\Delta}}$ maps L^p to $L^{p'}$. There is a natural class of Riemannian manifolds, called *scattering manifolds* or *asymptotically conic manifolds*, for which the spectral measure of the Laplacian admits an analogous factorization. Such manifolds, introduced by Melrose [1994], are by definition the interior M° of a compact manifold with boundary M, such that the metric g is smooth on M° and has the form

$$g = \frac{dx^2}{x^4} + \frac{h(x)}{x^2} \tag{1-1}$$

in a collar neighborhood near ∂M, where x is a smooth boundary defining function for M and $h(x)$ is a smooth one-parameter family of metrics on ∂M; the function $r := 1/x$ near $x = 0$ can be thought of as a radial coordinate near infinity and the given metric is asymptotic to the exact conic metric $((0,\infty)_r \times \partial M, dr^2 + r^2 h(0))$ as $r \to \infty$. Associated to the Laplacian on such a manifold is the family of Poisson operators $P(\lambda)$ defined for $\lambda > 0$. These form a sort of distorted Fourier transform for the Laplacian: they map $L^2(\partial M)$ into the null space of $\Delta_g - \lambda^2$ and satisfy $dE_{\sqrt{\Delta_g}}(\lambda) = (2\pi)^{-1}P(\lambda)P(\lambda)^*$ [Hassell and Vasy 1999]. Thus $(\lambda/2\pi)^{-(n-1)/2}P(\lambda)^*$ is an analogue of the restriction operator in this setting. The corresponding restriction problem is therefore to study the $L^p(M) \to L^q(\partial M)$ boundedness of $P(\lambda)^*$, and its norm in terms of the frequency λ (the dependence of $P(\lambda)$ in λ is no longer a scaling as it is for $\mathbb{R}^n$).

The aim of the present work is to address these multiplier and restriction problems in the geometric setting of asymptotically conic manifolds. In fact, we shall first show, in an abstract setting, that restriction-type estimates on the spectral measure of an operator imply spectral multiplier results for that operator. Then we will prove such restriction estimates for a class of operators which are 0-th order perturbations of the Laplacian on asymptotically conic manifolds. In particular, our results cover the following settings:

- Schrödinger operators, i.e., $\Delta + V$ on $\mathbb{R}^n$, where V is smooth and decaying sufficiently at infinity.
- The Laplacian with respect to metric perturbations of the flat metric on $\mathbb{R}^n$, again decaying sufficiently at infinity.
- The Laplacian on asymptotically conic manifolds.

Our first main result is that restriction estimates imply spectral multiplier estimates:

Theorem 1.1. *Let $\boldsymbol{L}$ be a nonnegative self-adjoint operator on $L^2(X, d\mu)$, where (X, d, μ) is a metric measure space such that the volume of balls satisfy the uniform bound $C_2 > \mu(B(x, \rho))/\rho^n > C_1$ for some $C_2 > C_1 > 0$. Suppose that the operator $\cos(t\sqrt{\boldsymbol{L}})$ satisfies finite speed propagation property (2-2), that the spectrum of $\boldsymbol{L}$ is absolutely continuous and that there exists $1 \le p < 2$ such that the spectral measure of $\boldsymbol{L}$ satisfies*

$$\|dE_{\sqrt{\boldsymbol{L}}}(\lambda)\|_{p\to p'} \le C\lambda^{n(1/p-1/p')-1}, \tag{1-2}$$

where p' is the exponent conjugate to p. Let $s > n(1/p - 1/2)$ be a Sobolev exponent. Then there exists C depending only on n, p, s, and the constant in (2-3) such that, for every even $F \in H^s(\mathbb{R})$ supported in $[-1, 1]$, $F(\sqrt{\boldsymbol{L}})$ maps $L^p(X) \to L^p(X)$, and

$$\sup_{\alpha>0} \|F(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p} \le C\|F\|_{H^s}. \tag{1-3}$$

Remark 1.2. As noted above, the hypothesis (1-2) is valid on the Euclidean space $\mathbb{R}^n$ and for exponents $1 \le p \le 2(n+1)/(n+3)$. In this case, the result is sharp in the sense that the hypothesis cannot be weakened to $F \in H^{s'}$ for any $s' < n(1/p - 1/2)$; see [Stein 1993, Section IX.2]. In fact, the proof shows that the theorem is true if we only assume $F \in B^{n(1/p-1/2)}_{1,2}$, which is slightly weaker, and gives an endpoint result. The result is sharp also in the sense that H^s cannot be replaced by the L^q Sobolev space W^s_q and $B^{n(1/p-1/2)}_{1,2}$ cannot be replaced by $B^{n(1/p-1/2)}_{1,q}$ for any $q < 2$; see Remark 2.11 below.

In the second part of the paper, we prove (1-2) for the spectral measure of the Laplacian Δ_g, plus a suitable potential, on asymptotically conic manifolds.

Theorem 1.3. *Let (M, g) be an asymptotically conic manifold of dimension $n \ge 3$, and let x be a smooth boundary defining function of ∂M. Let $\boldsymbol{H} := \Delta_g + V$ be a Schrödinger operator on M, with $V \in x^3C^\infty(M)$, and assume that $\boldsymbol{H}$ is a positive operator and that 0 is neither an eigenvalue nor a resonance. Then*:

(A) *For any $\lambda_0 > 0$ there exists a constant $C > 0$ such that the spectral measure $dE(\lambda)$ for $\sqrt{\boldsymbol{H}}$ satisfies*

$$\|dE_{\sqrt{\boldsymbol{H}}}(\lambda)\|_{L^p(M)\to L^{p'}(M)} \le C\lambda^{n(1/p-1/p')-1} \tag{1-4}$$

for $1 \le p \le 2(n+1)/(n+3)$ and $0 < \lambda \le \lambda_0$.

(B) *If (M, g) is nontrapping, then there exists $C > 0$ such that (1-4) holds for all $\lambda > 0$.*

(C) *If (M, g) is trapping and has asymptotically **Euclidean** ends, there exists $\chi \in C_0^\infty(M^\circ)$ and $C > 0$ such that*

$$\|(1-\chi)dE_{\sqrt{\boldsymbol{H}}}(\lambda)(1-\chi)\|_{L^p(M)\to L^{p'}(M)} \le C\lambda^{n(1/p-1/p')-1}, \quad \forall\lambda > 0, \tag{1-5}$$

for $1 < p \le 2(n+1)/(n+3)$. *However*, (1-4) *need not hold for all* $\lambda > 0$: *there exist* (*trapping*) *asymptotically Euclidean manifolds* (M, g), *sequences* $\lambda_n \to \infty$ *and* $C, c > 0$ *such that*

$$\|dE_{\sqrt{\Delta_g}}(\lambda_n)\|_{L^p(M)\to L^{p'}(M)} \ge Ce^{c\lambda_n}. \tag{1-6}$$

(D) *On the other hand*, *the Sogge-type spectral projection estimate*

$$\|\mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta_g})\|_{L^p(M)\to L^{p'}(M)} \le C\lambda^{n(1/p-1/p')-1}, \quad \forall \lambda \ge 1, \tag{1-7}$$

holds for $1 \le p \le 2(n+1)/(n+3)$ *for all asymptotically conic manifolds*, *trapping or not*, *and indeed for the much larger class of complete manifolds with* C^∞ *bounded geometry.*

Remark 1.4. When the spectral measure estimate (1-4) holds, it trivially implies the Sogge-type spectral projection estimate (1-7), by integrating over a unit interval in λ. On the other hand, parts (C) and (D) of Theorem 1.3 show that the Sogge estimate holds in far greater generality than (1-4).

Remark 1.5. Probably the nontrapping condition is not necessary to obtain the estimate (1-4) for all $\lambda > 0$; it seems likely that asymptotically conic manifolds with a hyperbolic trapped set of sufficiently small dimension will also satisfy (1-4), by analogy with [Burq et al. 2010]. However, manifolds with elliptic trapping will typically have sequences of λ for which the norm on the left hand side of (1-4) grows superpolynomially; see Section 8C.

Remark 1.6. The spatially cut-off estimate (1-5) can be compared to the nontrapping L^2 estimate proved by Cardoso and Vodev [2002]

$$\|(1-\chi)(L-\lambda^2+i0)^{-1}(1-\chi)\|_{L^2_\alpha\to L^2_{-\alpha}} = O(\lambda^{-1}), \quad \forall \lambda > 1, \ \forall \alpha > \tfrac{1}{2},$$

where $L^2_\alpha := \langle r\rangle^{-\alpha}L^2(M)$. As a matter of fact, we use this estimate to prove (1-5).

Since $\boldsymbol{H}$ in Theorem 1.3 also satisfies the finite speed of propagation property (2-2), we deduce from the two theorems above

Corollary 1.7. *Let* $\boldsymbol{L} = \boldsymbol{H}$, *where* $\boldsymbol{H}$ *is as in Theorem 1.3*, *and assume that* (M, g) *in Theorem 1.3 is nontrapping. Then* $\boldsymbol{L}$ *satisfies* (1-3), *where* F *and* s *are as in Theorem 1.1 and* $p \in [1, 2(n+1)/(n+3)]$.

Remark 1.8. As far as we are aware, the restriction estimates for the spectral measure in Theorem 1.3 were previously known only for $\boldsymbol{H}$ being the Laplacian in the Euclidean space $\mathbb{R}^n$. As for the spectral multiplier result of Corollary 1.7, this was previously known for $s > n(1/p-1/2)+1/2$ [Duong et al. 2002]. Thus, for $p \in [1, 2(n+1)/(n+3)]$, we gain half a derivative over the best results previously known. The region in the $(1/p, s)$-plane in which we improve previous results is illustrated in Figure 1. The lower threshold of $n(1/p-1/2)$ for the Sobolev exponent s in Corollary 1.7 is known to be sharp in Euclidean space, and it is not hard to see that it is sharp for any asymptotically conic manifold.

Remark 1.9. There are not many examples of sharp spectral multiplier results in the literature. Those known to the authors are as follows. The sharp multiplier result in (1-3) for $p = 2(n+1)/(n+3)$ (the other p are obtained by interpolation) was proved for the Laplacian on any compact manifold by Seeger

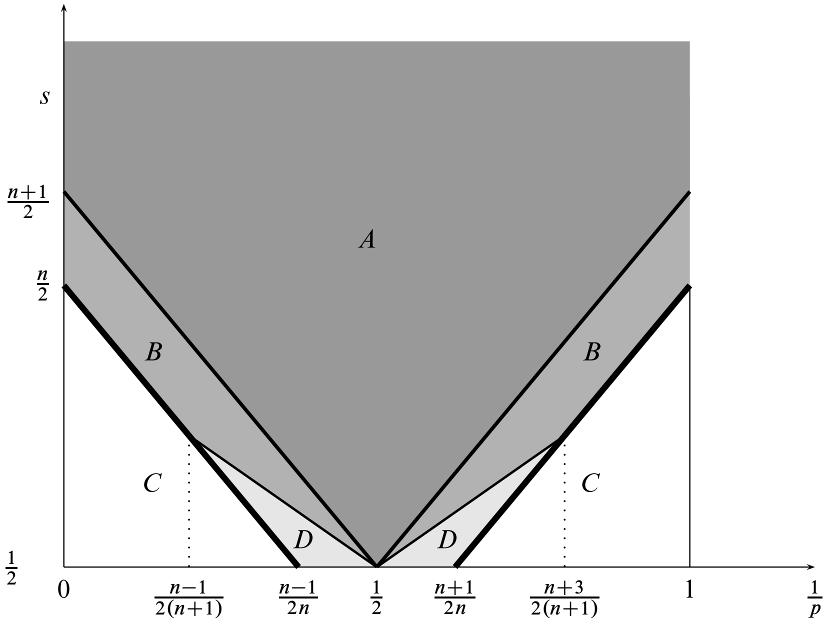

Figure 1. Map of where the statement of (1-3) has been established on nontrapping asymptotically conic manifolds, for different values of s and p. In region A this was previously known ([Duong et al. 2002]; see also Proposition 2.9). In the present paper we establish (1-3) also for region B (previously this was known only in the classical case of flat Euclidean space and the flat Laplacian). In region C it is known to be false, while region D is still unknown. For comparison with the Bochner–Riesz multiplier $F_\delta(\lambda) = (1-\lambda^2)_+^\delta$ observe that F_δ is in H^s for $s > \delta + 1/2$. For $F = F_\delta$, part of region D is known for flat Euclidean space [Lee 2004], and the celebrated Bochner–Riesz conjecture is that, for flat Euclidean space, (1-3) is true for $F = F_\delta$ in the whole of D.

and Sogge [1989]. In fact, they only needed the integrated estimate (1-7) to obtain the multiplier theorem in that setting. In the setting of the twisted Laplacian operator

$$\Delta_x + \Delta_y + \tfrac{1}{4}(\|x\|^2 + \|y\|^2) - i \sum_{j=1}^{n} (x_j \partial_{y_j} - y_j \partial_{x_j}),$$

the sharp multiplier result of (1-3) was proved by Stempak and Zienkiewicz [1998]. However, in this setting the required form of restriction estimates differs from both (1-4) and (1-7); see [Koch and Ricci 2007]. The last case of a sharp multiplier theorem known to us, although with a slightly different range of p, is for the harmonic oscillator; see [Karadzhov 1994; Koch and Tataru 2005; Thangavelu 1993].

Remark 1.10. A multiplier theorem of the type (1-3) does not hold for manifolds with exponential volume growth (like negatively curved complete manifolds); a *necessary* condition on the multiplier F in that case is typically a holomorphic extension of F into a strip. See for instance the work of Clerc and Stein [1974] or Anker [1990] for the case of noncompact symmetric spaces, or Taylor [1989] in the case of manifolds with bounded geometry, where sufficient conditions are also given.

Remark 1.11. Theorems 1.1 and 1.3 imply Bochner–Riesz summability for a range of exponents similar to those proved for the Euclidean Laplacian in [Stein 1993, page 390; Sogge 1993, Theorem 2.3.1] and for compact manifolds by Christ and Sogge [1988] and Sogge [1987]. See Corollary 2.10 below.

The heuristics one can extract from Theorem 1.3 and the last two remarks can be summarized as follows:

- The sharp restriction estimate on $dE(\lambda)$ at bounded and low frequencies λ only depends on the geometry near infinity.
- The high frequency restriction estimate on $dE(\lambda)$ also depends strongly on global dynamical properties (trapping/nontrapping).
- The integrated estimate (1-7) for all frequencies $\lambda > 1$ only depends on having uniform local geometry.

The proof of Theorem 1.1, given in Section 2, is based on a principle common to the proofs of most Fourier and spectral multiplier theorems. The rough idea is that one can control the L^p to L^p norm of operators with singular integral kernels by estimating the L^p to L^q norm of the operator for some $q > p$ (usually $q = 2$) and showing that a large part of the corresponding kernel is concentrated near the diagonal; see [Fefferman 1970; 1973; Seeger and Sogge 1989; Sogge 1987]. For calculations starting from $L^1 \to L^2$ estimates this principle can be equivalently stated in terms of weighted L^2 norms of the kernel; see [Cowling and Sikora 2001; Hörmander 1960; Mauceri and Meda 1990]. Our implementation of this principle in the proof of Theorem 1.1 is based on finite speed propagation of the wave equation, following [Cheeger et al. 1982; Cowling and Sikora 2001; Sikora 2004]. In the proof, we decompose the operator $F(\alpha\sqrt{L})$ as a sum over $\ell \in \mathbb{N}$ of multipliers $F_\ell(\alpha\sqrt{L})$ satisfying some finite speed propagation properties with F_ℓ Schwartz. The $L^p \to L^p$ norms for $F_\ell(\alpha\sqrt{L})$ are controlled by $C(\alpha 2^\ell)^{n(1/p-1/2)}$ times the $L^p \to L^2$ norms and then the TT^* argument reduces the problem to the bound of the $L^p \to L^{p'}$ norms of $|F_\ell|^2(\alpha\sqrt{L})$, which can be obtained using the restriction estimate of the spectral measure.

The proof of Theorem 1.3 proceeds in two steps. In the first step we suppose that we have an abstract operator $\boldsymbol{L}$ whose spectral measure can be factorized as $dE_{\sqrt{\boldsymbol{L}}}(\lambda) = (2\pi)^{-1}P(\lambda)P(\lambda)^*$ (see the discussion below (1-1)), where the initial space of $P(\lambda)$ is a Hilbert space. We then prove the following result in Section 3:

Proposition 1.12. *Let (X, d, μ) and $\boldsymbol{L}$ be as in Theorem 1.1, and assume $dE_{\sqrt{\boldsymbol{L}}}(\lambda) = (2\pi)^{-1}P(\lambda)P(\lambda)^*$ as described above. Also assume that for each λ we have an operator partition of unity on $L^2(X)$,*

$$\mathrm{Id} = \sum_{i=1}^{N(\lambda)} Q_i(\lambda), \tag{1-8}$$

where the Q_i are uniformly bounded as operators on $L^2(X)$ and $N(\lambda)$ is uniformly bounded. We assume that for $1 \le i \le N(\lambda)$, and some nonnegative function $w(z, z')$ on $X \times X$, the estimate

$$\bigl|\bigl(Q_i(\lambda)dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)Q_i(\lambda)\bigr)(z, z')\bigr| \le C\lambda^{n-1-j}\bigl(1 + \lambda w(z, z')\bigr)^{-(n-1)/2+j} \tag{1-9}$$

holds for $j = 0$ *and for* $j = n/2 - 1$ *and* $j = n/2$ *if* n *is even, or for* $j = n/2 - 3/2$ *and* $j = n/2 + 1/2$ *if* n *is odd. Here* $dE^{(j)}_{\sqrt{L}}(\lambda)$ *means* $(d/d\lambda)^j dE_{\sqrt{L}}(\lambda)$, *and* C *is independent of* λ *and* i. *Then the restriction estimates*

$$\left\| dE_{\sqrt{L}}(\lambda) \right\|_{L^p(X) \to L^{p'}(X)} \le C' \lambda^{n(1/p - 1/p') - 1}, \quad 1 \le p \le \frac{2(n+1)}{n+3}, \tag{1-10}$$

hold for all $\lambda > 0$. *Moreover, if the estimates above hold only for* $0 < \lambda \le \lambda_0$, *then* (1-10) *holds for* $0 < \lambda \le \lambda_0$.

The key point here is that we only need to consider operators $Q_i(\lambda) dE^{(j)}_{\sqrt{L}}(\lambda) Q_k(\lambda)$ for $i = k$, which effectively means that we only need to analyze the kernel of $dE^{(j)}_{\sqrt{L}}(\lambda)$ close to the diagonal. The proof of this is based on the complex interpolation idea of Stein [1956] and appears in Section 3.

The second step is to prove estimates (1-9) in the case where $\boldsymbol{L}$ is the Laplacian or a Schrödinger operator on an asymptotically conic manifold:

Theorem 1.13. *Let* (M, g) *and* $\boldsymbol{H}$ *be as in Theorem 1.3. Then there exists an operator partition of unity,* (1-8), *where the* Q_i *are uniformly bounded as operators on* $L^2(X)$ *and* $N(\lambda)$ *is uniformly bounded, such that the estimates* (1-9) *hold for all integers* $j \ge 0$ *and for* $0 < \lambda \le \lambda_0$, *where* $w(z, z')$ *is the Riemannian distance between points* $z, z' \in M^\circ$. *Moreover, if* (M, g) *is nontrapping, then estimates* (1-9) *hold for all* $0 < \lambda < \infty$.

In the free Euclidean setting, this estimate is obvious (with the trivial partition of unity) by using the explicit formula of the spectral measure, but in our general setting it turns out to be quite involved and we really need to choose the partition of unity carefully. We use some results of [Hassell and Vasy 2001] on the resolvent of L on the spectrum, the high-energy (semiclassical) version of this [Hassell and Wunsch 2008] and the low energy estimates of our previous work [Guillarmou et al. 2012]. These three articles on which we build our estimates describe the Schwartz kernel of the spectral measure as a Legendrian distribution (a Fourier integral operator, in a sense) on a desingularized version of the compactification of the space $M \times M$, and this was done in a sort of uniform way with respect to the spectral parameter λ. The operators Q_i in the partition of unity will be pseudodifferential operators of a particular sort; see Section 6C for the estimate (1-9) for small λ, and Section 7D for the same estimate for large λ. By our discussion above, this establishes parts (A) and (B) of Theorem 1.3. Part (C) of Theorem 1.3 is proved in Section 8B and part (D) is proved in Section 8A.

Part I. Abstract self-adjoint operators

2. Restriction estimates imply spectral multiplier estimates

Let $\boldsymbol{L}$ be an abstract positive self-adjoint operator on $L^2(X)$, where X is a metric measure space with metric d and measure μ. We make the following assumptions about $\boldsymbol{L}$ and (X, d, μ):

- The space X is separable and has dimension n in the sense of the volume growth of balls: that is, there exist constants $0 < c_1 < c_2 < \infty$ such that

$$c_1 \rho^n \le \mu(B(x, \rho)) \le c_2 \rho^n \tag{2-1}$$

for every $x \in X$ and $\rho > 0$;

- $\cos(t\sqrt{L})$ satisfies finite speed propagation in the sense that

$$\operatorname{supp}\cos(t\sqrt{L}) \subset \mathcal{D}_t := \{(z_1, z_2) \subset X \times X \mid d(z_1, z_2) \le |t|\}. \tag{2-2}$$

This statement says that $\langle f_1, \cos(t\sqrt{L}) f_2\rangle = 0$ whenever $\operatorname{supp} f_1 \in B(z_1, \rho_1)$, $\operatorname{supp} f_2 \in B(z_2, \rho_2)$ and $|t| + \rho_1 + \rho_2 \le d(z_1, z_2)$.

- L satisfies restriction estimates, which come in a strong and a weak form. We say that L satisfies *L^p to $L^{p'}$ restriction estimates for all energies* if the spectral measure $dE_{\sqrt{L}}(\lambda)$ maps $L^p(X)$ to $L^{p'}(X)$ for some p satisfying $1 \le p < 2$ and all $\lambda > 0$, with an operator norm estimate

$$\|dE_{\sqrt{L}}(\lambda)\|_{L^p(X)\to L^{p'}(X)} \le C\lambda^{n(1/p-1/p')-1} \quad \text{for all } \lambda > 0. \tag{2-3}$$

We also consider a weaker form of these estimates: we say that L satisfies *low energy L^p to $L^{p'}$ restriction estimates* if $dE_{\sqrt{L}}(\lambda)$ maps $L^p(X)$ to $L^{p'}(X)$ for some p satisfying $1 \le p < 2$ and all $\lambda \in (0, \lambda_0]$, with an operator norm estimate

$$\|dE_{\sqrt{L}}(\lambda)\|_{L^p(X)\to L^{p'}(X)} \le C\lambda^{n(1/p-1/p')-1}, \quad 0 < \lambda \le \lambda_0, \tag{2-4}$$

for some C, together with weaker estimates for $\lambda \ge \lambda_0$,

$$\|E_{\sqrt{L}}[0, \lambda]\|_{L^p(X)\to L^{p'}(X)} \le C\lambda^{n(1/p-1/p')}, \quad \lambda \ge \lambda_0, \tag{2-5}$$

with a uniform C. (Here $E_{\sqrt{L}}[0, \lambda]$ is the same as $\mathbb{1}_{[0,\lambda]}(\sqrt{L})$.)

Remark 2.1. The assumptions (with restriction estimates for all energies) are satisfied by taking $X = \mathbb{R}^n$ with the standard metric and measure, and L to be the (positive) Laplacian on $\mathbb{R}^n$ (with domain $H^2(\mathbb{R}^n)$). As we shall see, the assumptions are also satisfied for asymptotically conic manifolds, with the low energy restriction estimates holding unconditionally, and restriction estimates for all energies satisfied if the manifold is nontrapping.

Remark 2.2. Clearly, (2-5) follows from (2-3) by integrating over the interval $[0, \lambda]$. However, in Remark 8.8 we give an example where we have, by Proposition 8.1,

$$\|E_{\sqrt{L}}[\lambda, \lambda+1]\|_{L^p(X)\to L^{p'}(X)} \le C\lambda^{n(1/p-1/p')-1}, \quad \lambda \ge \lambda_0,$$

(which implies (2-5)), but the pointwise estimate on the $L^p \to L^{p'}$ operator norm of $dE(\lambda)$ grows exponentially for a subsequence of λ tending to infinity.

Remark 2.3. Spectral projection estimate (2-5) is implied by a heat kernel bound

$$\|e^{-tL}\|_{L^p\to L^{p'}} \le Ct^{-n(1/p-1/p')/2}, \quad t \le \frac{1}{\lambda_0}. \tag{2-6}$$

This follows from short-time Gaussian bounds for the heat kernel, which hold for the Laplacian on any complete Riemannian manifold with bounded curvature and injectivity radius bounded below [Cheng et al. 1981, Theorem 4]. Estimate (2-6) implies, using T^*T, that $\|e^{-t\boldsymbol{L}}\|_{L^p\to L^2} \le Ct^{-n(1/p-1/p')/4}$. We then compute, using T^*T again,

$$E_{\sqrt{\boldsymbol{L}}}[0,\lambda] = E_{\sqrt{\boldsymbol{L}}}[0,\lambda]\, e^{\boldsymbol{L}/\lambda^2} e^{-\boldsymbol{L}/\lambda^2}$$
$$\implies \|E_{\sqrt{\boldsymbol{L}}}[0,\lambda]\|_{p\to p'} = \|E_{\sqrt{\boldsymbol{L}}}[0,\lambda]\|^2_{p\to 2} \le \big\|E_{\sqrt{\boldsymbol{L}}}[0,\lambda]e^{\boldsymbol{L}/\lambda^2}\big\|^2_{2\to 2}\cdot\big\|e^{-\boldsymbol{L}/\lambda^2}\big\|^2_{p\to 2}.$$

Conversely, (2-5) implies the heat kernel bound (2-6), which can be seen by writing $e^{-t\boldsymbol{L}}$ as in integral over the spectral measure, and then integrating by parts.

2A. ***The main result.*** The following theorem is the main result of this section.

Theorem 2.4. *Suppose that* (X, d, μ) *and* $\boldsymbol{L}$ *satisfy* (2-1) *and* (2-2), *and that* $\boldsymbol{L}$ *satisfies* L^p *to* $L^{p'}$ *restriction estimates for all energies*, (2-3), *for some* p *with* $1 \le p < 2$. *Let* $s > n(1/p - 1/2)$ *be a Sobolev exponent. Then there exists* C *depending only on* n, p, s, *and the constant in* (2-3) *such that, for every even* $F \in H^s(\mathbb{R})$ *supported in* $[-1, 1]$, $F(\sqrt{\boldsymbol{L}})$ *maps* $L^p(X) \to L^p(X)$, *and*

$$\sup_{\alpha>0} \|F(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p} \le C\|F\|_{H^s}. \tag{2-7}$$

If $\boldsymbol{L}$ *only satisfies the weaker estimates* (2-4), (2-5), *i.e., low energy* L^p *to* $L^{p'}$ *restriction estimates, then for all* F *as above, we have*

$$\sup_{\alpha\ge 4/\lambda_0} \|F(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p} \le C\|F\|_{H^s}, \tag{2-8}$$

where C *depends on* n, p, s, λ_0, *and the constants in* (2-4) *and* (2-5).

Remark 2.5. Notice that if $p > 2n/(n+1)$ then $s = 1/2$ satisfies $s > n(1/p - 1/2)$. However, $H^{1/2}$ functions need not be bounded, and such functions cannot be L^p multipliers even for $p = 2$, and a fortiori for $p \ne 2$. We deduce that, under the assumptions of Theorem 2.4, estimate (2-3), or even (2-4), is impossible for $p > 2n/(n+1)$.

In preparation for the proof of Theorem 2.4, we have (following [Cheeger et al. 1982]):

Lemma 2.6. *Assume that* $\boldsymbol{L}$ *satisfies* (2-2) *and that* F *is an even bounded Borel function with Fourier transform* $\hat{F}$ *satisfying* $\operatorname{supp}\widehat{F} \subset [-\rho, \rho]$. *Then*

$$\operatorname{supp} K_{F(\sqrt{\boldsymbol{L}})} \subset \mathfrak{D}_\rho.$$

Proof. If F is an even function, then by the Fourier inversion formula,

$$F(\sqrt{\boldsymbol{L}}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{F}(t)\cos(t\sqrt{\boldsymbol{L}})\,dt.$$

But $\operatorname{supp}\widehat{F} \subset [-\rho, \rho]$ and Lemma 2.6 follows from (2-2). □

The next lemma is a crucial tool in using restriction type results, i.e., $L^p \to L^q$ continuity of spectral projectors, to obtain spectral multiplier type bounds, i.e., $L^p \to L^p$ estimates.

Lemma 2.7. *Suppose that (x, d, μ) satisfies (2-1) and S is a bounded linear operator from $L^p(X)$ to $L^q(X)$ such that*

$$\operatorname{supp} S \subset \mathfrak{D}_\rho$$

for some $\rho > 0$. Then for any $1 \le p < q \le \infty$ there exists a constant $C = C_{p,q}$ such that

$$\|S\|_{p\to p} \le C\rho^{n(1/p-1/q)}\|S\|_{p\to q}.$$

Proof. We fix $\rho > 0$. Then we first choose a sequence $x_n \in M$ such that $d(x_i, x_j) > \rho/10$ for $i \ne j$ and $\sup_{x\in X} \inf_i d(x, x_i) \le \rho/10$. Such sequence exists because M is separable. Second, we define $\widetilde{B}_i$ by the formula

$$\widetilde{B}_i = \overline{B}\left(x_i, \tfrac{1}{10}\rho\right) - \left(\bigcup_{j<i} \overline{B}\left(x_j, \tfrac{1}{10}\rho\right)\right), \tag{2-9}$$

where $\overline{B}(x, \rho) = \{y \in M : d(z, z') \le \rho\}$. Third, we put $\chi_i = \chi_{\widetilde{B}_i}$, where $\chi_{\widetilde{B}_i}$ is the characteristic function of set $\widetilde{B}_i$. Fourth, we define the operator M_{χ_i} by the formula $M_{\chi_i} g = \chi_i g$.

Note that for $i \ne j$, $B(x_i, \frac{1}{20}\rho) \cap B(x_j, \frac{1}{20}\rho) = \varnothing$. Hence

$$K = \sup_i \#\{j;\ d(x_i, x_j) \le 2\rho\} \le \sup_x \frac{|\overline{B}(x, 2\rho)|}{|B(x, \frac{1}{20}\rho)|} < \frac{40^n c_2}{c_1} < \infty.$$

It is not difficult to see that if we set $I = \{i, j \mid d(x_i, x_j) < 2\rho\}$, then

$$\mathfrak{D}_\rho \subset \bigcup_{i,j\in I} \widetilde{B}_i \times \widetilde{B}_j \subset \mathfrak{D}_{4\rho}, \quad \text{so} \quad Sf = \sum_{i,j\in I} M_{\chi_i} S M_{\chi_j} f.$$

Hence, if we set $J_i = \{j \mid d(x_i, x_j) < 2\rho\}$ for a given i, then by the Hölder inequality

$$\begin{aligned}
\|Sf\|_p^p = \Big\| \sum_{i,j\in I} M_{\chi_i} S M_{\chi_j} f \Big\|_{L^p}^p &= \sum_i \Big\| \sum_{j\in J_i} M_{\chi_i} S M_{\chi_j} f \Big\|_p^p \\
&\le \sum_i |\widetilde{B}_i|^{p(1/p-1/q)} \Big\| \sum_{j\in J_i} M_{\chi_i} S M_{\chi_j} f \Big\|_q^p \\
&\le C\rho^{np(1/p-/q)} \sum_i \Big\| \sum_{j\in J_i} M_{\chi_i} S M_{\chi_j} f \Big\|_q^p \\
&\le C K^{p-1} \rho^{np(1/p-1/q)} \sum_i \sum_{j\in J_i} \| M_{\chi_i} S M_{\chi_j} f \|_q^p \\
&\le C K^p \rho^{np(1/p-1/q)} \sum_j \| S M_{\chi_j} f \|_q^p \\
&\le C K^p \rho^{np(1/p-1/q)} \|S\|_{p\to q}^p \sum_j \| M_{\chi_j} f \|_p^p \\
&= C K^p \rho^{np(1/p-1/q)} \|S\|_{p\to q}^p \|f\|_p^p.
\end{aligned}$$

This finishes the proof of Lemma 2.7. ∎

Proof of Theorem 2.4. We first assume that $\boldsymbol{L}$ satisfies L^p to $L^{p'}$ restriction estimates for all energies. We take $\eta \in C_c^\infty(-4,4)$ even and such that

$$\sum_{l\in\mathbb{Z}} \eta\Big(\frac{t}{2^l}\Big) = 1 \quad \text{for all } t \neq 0.$$

Then we set $\phi(t) = \sum_{l\le 0} \eta(2^{-l}t)$,

$$F_0(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi(t)\widehat{F}(t)\cos(t\lambda)\,dt,$$

and

$$F_l(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \eta\Big(\frac{t}{2^l}\Big)\widehat{F}(t)\cos(t\lambda)\,dt. \tag{2-10}$$

Note that by virtue of the Fourier inversion formula,

$$F(\lambda) = \sum_{l\ge 0} F_l(\lambda),$$

and by Lemma 2.6,

$$\operatorname{supp} F_l(\alpha\sqrt{\boldsymbol{L}}) \subset \mathcal{D}_{2^{l+2}\alpha}.$$

Now by Lemma 2.7,

$$\|F(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p} \le \sum_{l\ge 0}\|F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p} \le C\sum_{l\ge 0}(2^l\alpha)^{n(1/p-1/2)}\|F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2}. \tag{2-11}$$

Unfortunately, F_l is no longer compactly supported. To remedy this we choose a function $\psi \in C_c^\infty(-4,4)$ such that $\psi(\lambda) = 1$ for $\lambda \in (-2,2)$ and note that

$$\|F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2} \le \|(\psi F_l)(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2} + \|((1-\psi)F_l)(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2}.$$

To estimate the norm $\|\psi F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2}$ we use our restriction estimates (2-3). Using a T^*T argument and the fact that $\operatorname{supp}\psi \subset [-4,4]$, we note that

$$\begin{aligned}\|\psi F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2}^2 = \||\psi F_l|^2(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p'} &\le \int_0^{4/\alpha} |\psi F_l(\alpha\lambda)|^2 \|dE_{\sqrt{\boldsymbol{L}}}(\lambda)\|_{p\to p'}\,d\lambda \\ &\le \frac{C}{\alpha}\int_0^4 |\psi F_l(\lambda)|^2\|dE_{\sqrt{\boldsymbol{L}}}(\lambda/\alpha)\|_{p\to p'}\,d\lambda. \end{aligned} \tag{2-12}$$

It follows from the above calculation and (2-3) that

$$\alpha^{n(1/p-1/2)}\|\psi F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2} \le C\|\psi F_l\|_2 \tag{2-13}$$

for all $\alpha > 0$. As a consequence, we obtain

$$\sum_{l\ge 0} 2^{ln(1/p-1/2)}\alpha^{n(1/p-1/2)}\|\psi F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2} \le \sum_{l\ge 0} 2^{ln(1/p-1/2)}\|\psi F_l\|_2$$

for all $\alpha > 0$. Now let us recall that by the definition of a Besov space,

$$\sum_{l\geq 0} 2^{ln(1/p-1/2)}\|\psi F_l\|_2 \leq \sum_{l\geq 0} 2^{ln(1/p-1/2)}\|F_l\|_2 = \|F\|_{B^{n(1/p-1/2)}_{1,2}}.$$

See [Triebel 1992, Chapters I and II] for more details. We also recall that if $s > s'$ then $H^s \subset B^{s'}_{1,2}$ and $\|F\|_{B^{n(1/p-1/2)}_{1,2}} \leq C_s\|F\|_{H^s}$ for all $s > n(1/p-1/2)$ [ibid.]. Therefore, we have shown that

$$\sum_{l\geq 0} 2^{ln(1/p-1/2)}\alpha^{n(1/p-1/2)}\|\psi F_l(\alpha\sqrt{L})\|_{p\to 2} \leq C\|F\|_{H^s}. \tag{2-14}$$

Next we obtain bounds for the part of estimate (2-11) corresponding to the term $\|(1-\psi)F_l(\alpha\sqrt{L})\|_{p\to 2}$. This only requires the spectral projection estimates (2-5). We write

$$\begin{aligned}
|(1-\psi)F_l|^2(\alpha\sqrt{L}) &= \int_0^\infty |(1-\psi)(\alpha\lambda)F_l(\alpha\lambda)|^2 dE_{\sqrt{L}}(\lambda)\\
&= -\int_0^\infty \Big(\frac{d}{d\lambda}|(1-\psi)(\alpha\lambda)F_l(\alpha\lambda)|^2\Big)E_{\sqrt{L}}(\lambda)\,d\lambda\\
&= -\int_0^\infty \Big(\frac{d}{d\lambda}|(1-\psi)(\lambda)F_l(\lambda)|^2\Big)E_{\sqrt{L}}(\lambda/\alpha)\,d\lambda.
\end{aligned}$$

Hence, using (2-5),

$$\|(1-\psi)F_l(\alpha\sqrt{L})\|^2_{p\to 2} \leq C\int_0^\infty \Big(\frac{d}{d\lambda}|(1-\psi)(\lambda)F_l(\lambda)|^2\Big)\Big(\frac{\lambda}{\alpha}\Big)^{n(1/p-1/p')}d\lambda. \tag{2-15}$$

We write

$$F_l(\lambda) = \frac{1}{2\pi}\int e^{it(\lambda-\lambda')}\eta\Big(\frac{t}{2^l}\Big)F(\lambda')\,d\lambda'\,dt,$$

use the identity

$$e^{it(\lambda-\lambda')} = i^{-N}(\lambda-\lambda')^{-N}(d/dt)^N e^{it(\lambda-\lambda')},$$

and integrate by parts N times. Note that if $\lambda \in \operatorname{supp} 1-\psi$ and $\lambda' \in \operatorname{supp} F$ then $\lambda \geq 2$ and $\lambda' \leq 1$, and hence $\lambda - \lambda' \geq \lambda/2$. It follows that

$$|((1-\psi)F_l)(\lambda)| \leq C\lambda^{-N}2^{-N(l-1)}\|F\|_2,$$

with C independent of N. Similarly,

$$\Big|\frac{d}{d\lambda}((1-\psi)F_l)(\lambda)\Big| \leq C\lambda^{-N}2^{-N(l-1)}2^l\|F\|_2.$$

Using this in (2-15) with N sufficiently large and $l \geq 2$, we obtain

$$(2^l\alpha)^{n(1/p-1/2)}\|((1-\psi)F_l)(\alpha\sqrt{L})\|_{p\to 2} \leq C2^{-l}\|F\|_2.$$

Therefore, we have

$$\sum_l (2^l\alpha)^{n(1/p-1/2)}\|((1-\psi)F_l)(\alpha\sqrt{L})\|_{p\to 2} \leq C\|F\|_2 \leq C\|F\|_{H^s}. \tag{2-16}$$

Equations (2-11), (2-14) and (2-16) prove (2-7).

The proof in the case that $\boldsymbol{L}$ satisfies low-energy restriction estimates (2-4) and (2-5) proceeds the same way, except that we require the condition $\alpha \le 4/\lambda_0$ at the step (2-12) in order that we can use the pointwise estimate (2-4) on the spectral measure in this integral. □

Remark 2.8. Note that if we only assume that (2-5) holds for all $\lambda > 0$ then we still have

$$\alpha^{n(1/p-1/2)} \|\psi F_l(\alpha\sqrt{\boldsymbol{L}})\|_{p\to 2} \le \alpha^{n(1/p-1/2)} \big\|\psi F_l(\alpha\sqrt{\boldsymbol{L}}) e^{\alpha^2 \boldsymbol{L}}\big\|_{2\to 2} \cdot \big\|e^{-\alpha^2\boldsymbol{L}}\big\|_{p\to 2}$$
$$\le C\|\psi F_l\|_\infty,$$

Now the above estimate is just a version of (2-13) with norm $\|\psi F_l\|_2$ replaced by $\|\psi F_l\|_\infty$. Next if we replace the Besov space $B^{n(1/p-1/2)}_{1,2}$ by $B^{n(1/p-1/2)}_{1,\infty}$ then we can still follow the proof of Theorem 2.4. Recall also that if $s > s'$ then $W^s_\infty \subset B^{s'}_{1,\infty}$ and $\|F\|_{B^{n(1/p-1/2)}_{1,\infty}} \le C_s \|F\|_{W^s_\infty}$ for all $s > n(1/p - 1/2)$, where $\|F\|_{W^s_\infty} = \|(I - d^2/dx^2)^{s/2} F\|_\infty$; see again [Triebel 1992]. This implies that (2-14) holds with the norm $\|F\|_{H^s}$ replaced by the norm $\|F\|_{W^s_\infty}$. As the rest of the proof of Theorem 2.4 does not require (2-3), the above argument proves the following proposition.

Proposition 2.9. *Suppose that (X, d, μ) and $\boldsymbol{L}$ satisfy (2-1) and (2-2), and that $\boldsymbol{L}$ satisfies (2-5) for all $\lambda > 0$. Let $s > n|1/p - 1/2|$ be a Sobolev exponent. Then there exists C depending only on n, p, s, and the constant in (2-5) such that, for every even $F \in W^s_\infty(\mathbb{R})$ supported in $[-1, 1]$, $F(\sqrt{\boldsymbol{L}})$ maps $L^p(X) \to L^p(X)$, and*

$$\sup_{\alpha>0} \|F(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p} \le C\|F\|_{W^s_\infty}. \tag{2-17}$$

Note also that if $s > s'$ then $\|F\|_{W^{s'}_\infty} \le C\|F\|_{H^{s+1/2}}$. That is, the multiplier result with exponent one-half bigger then the optimal exponent does not require (2-3) and holds just under assumption (2-5), which is equivalent with the standard heat kernel bounds (2-6) (for all t). For $p = 1$, Proposition 2.9 was proved in [Christ and Sogge 1988] and can be alternatively proved using Theorem 3.5 in the same paper and interpolation, see also [Duong et al. 2002, Theorem 3.1].

From this point of view, the key point about Theorem 2.4 is the gain of half a derivative over the more elementary (2-17).

2B. *Bochner–Riesz summability.* We use Theorem 2.4 to discuss boundedness of Bochner–Riesz means of the operator $\boldsymbol{L}$. Bochner–Riesz summability is technically speaking a slight weakening of Theorem 2.4 but is very close, and it allows us to compare our results with results described in [Stein 1993; Sogge 1993]. Let us recall that Bochner–Riesz means of order δ are defined by the formula

$$(1 - \boldsymbol{L}/\lambda^2)^\delta_+, \quad \lambda > 0. \tag{2-18}$$

For $\delta = 0$, this is the spectral projector $E_{\sqrt{\boldsymbol{L}}}([0, \lambda])$, while for $\delta > 0$ we think of (2-18) as a smoothed version of this spectral projector; the larger δ, the more smoothing. Bochner–Riesz summability describes the range of δ for which the above operators are bounded on L^p uniformly in λ.

Corollary 2.10. *Suppose that (X, d, μ) is as above, and that restriction estimates (2-3) for exponents $1 \le p \le 2(n+1)/(n+3)$ and finite speed propagation property (2-2) hold for operator $\boldsymbol{L}$. Then for all*

$$p \in \Big[1, \frac{2(n+1)}{n+3}\Big] \cup \Big[\frac{2(n+1)}{n-1}, \infty\Big] \quad \textit{and} \quad \delta > n\Big|\frac{1}{p} - \frac{1}{2}\Big| - \frac{1}{2},$$

we have

$$\|(1 - \boldsymbol{L}/\lambda^2)_+^\delta\|_{p\to p} \le C \textit{ for all } \lambda > 0. \tag{2-19}$$

For all $p \in \big(2(n+1)/(n+3), 2(n+1)/(n-1)\big)$ these estimates hold if $\delta > \frac{1}{2}(n-1)|1/p - 1/2|$.

Proof. Note that $(1-\lambda^2)_+^\delta \in H^s$ if and only if $\delta > s - 1/2$. Now for $p < 2(n+1)/(n+3)$ Corollary 2.10 follows from Theorem 2.4. For $2(n+1)/(n+3) < p < 2$ Corollary 2.10 follows from interpolating between (2-19) with $p = 2(n+1)/(n+3)$ and the trivial estimate for $p = 2$. For $p > 2$ the results follow by duality. □

Remark 2.11. We noted in the proof above that Corollary 2.10 follows from Theorem 2.4. In fact the Corollary 2.10 is slightly but essentially weaker than Theorem 2.4. Indeed Corollary 2.10 is equivalent to the version of Theorem 2.4 in which the H^s norm of a compactly supported function F is replaced by the L^1 norm of $F^s := F * \chi_+^{-s-1}$, where χ_+ is as in Section 3. To prove this we note that

$$F(\alpha\sqrt{\boldsymbol{L}}) = \int \chi_+^{\nu-1}(\lambda - \alpha\sqrt{\boldsymbol{L}})F^\nu(\lambda)\, d\lambda, \qquad \nu \ge 0;$$

see (3-3) and (3-4). Hence if estimates (2-19) hold for some exponent δ then $\|F(\alpha\sqrt{\boldsymbol{L}})\|_{p\to p} \le \|F^{\delta+1}\|_1$ and Bochner–Riesz summability of order δ implies Theorem 2.4 with the norm $\|F^{\delta+1}\|_1$. Note that if F, supported in $[-1, 1]$, is such that $F^{s+1/2}$ is in $L^1(\mathbb{R})$, then F is in $H^{s'}(\mathbb{R})$ for all $s' < s$ with an estimate $\|F\|_{H^{s'}} \le C\|F^{s+1/2}\|_{L^1}$. Hence, conversely, Theorem 2.4 with the stronger hypothesis $F^{s+1/2} \in L^1$ implies Bochner–Riesz summability of order δ for all $\delta > s - 1/2$.

2C. *Singular integrals.* Finally we will discuss a singular integral version of our spectral multiplier result. The following theorem is just reformulation of [Cowling and Sikora 2001, Theorem 3.5]. We write D_κ for the scaling operator $D_\kappa F(x) = F(\kappa x)$.

Theorem 2.12. *Suppose that operator $\boldsymbol{L}$ satisfies finite speed propagation property (2-2), that $s > n/2$ and that*

$$\|dE_{\sqrt{\boldsymbol{L}}}(\lambda)\|_{1\to\infty} \le \lambda^{n-1} \quad \textit{for all } \lambda > 0. \tag{2-20}$$

Next let η be a smooth compactly supported nonzero function. Then for any Borel bounded function F such that $\sup_{\kappa>0} \|\eta\, D_\kappa F\|_{W_s^p} < \infty$ the operator $F(\sqrt{\boldsymbol{L}})$ is of weak type $(1, 1)$ and is bounded on $L^q(X)$ for all $1 < q < \infty$. In addition,

$$\|F(\sqrt{\boldsymbol{L}})\|_{L^1\to L^{1,\infty}} \le C_s\Big(\sup_{\kappa>0} \|\eta D_\kappa F\|_{W_s^p} + |F(0)|\Big). \tag{2-21}$$

Remark 2.13. It is a standard observation that up to equivalence the norm

$$\sup_{\kappa>0} \|\eta\, D_\kappa F\|_{W_s^p}$$

does not depend on the auxiliary function η as long as η is not identically equal zero.

Proof. Using T^*T trick we note that by (2-20) one has

$$\|F(\sqrt{L})\|_{1\to 2}^2 = \| |F|^2(\sqrt{L})\|_{1\to\infty} \le \int_0^\infty |F(\lambda)|^2 \|dE_{\sqrt{L}}(\lambda)\|_{1\to\infty}\, d\lambda \le C\int_0^\infty |F(\lambda)|^2\lambda^{n-1} d\lambda.$$

Hence if $\operatorname{supp} F \subset [0, R)$ then

$$\|F(\sqrt{L})\|_{1\to 2}^2 \le CR^n \|D_R F\|_2^2,$$

that is, the estimates (3.22) of Theorem 3.5 of [Cowling and Sikora 2001] hold. Now Theorem 2.12 follows from the same Theorem 3.5. □

Remark 2.14. Theorem 2.12 is a singular integral version of Theorem 2.4 for $p = 1$. We expect that a similar extension to a singular integral version is possible for all p. That is if one assumes that $s > n|1/2 - 1/p|$ then one can prove weak-type (p, p) version of estimates (2-21). However the proof of such results seems to be more complex and not directly related to the rest of this paper, so we will not pursue this idea further here.

3. Kernel estimates imply restriction estimates

The goal of this section is to prove Proposition 1.12; that is, we show that restriction estimates (2-3) or (2-4) follow from certain pointwise estimates of λ-derivatives of the kernel of the spectral measure. We first prove a simplified version of Proposition 1.12 in which the partition of unity does not appear. We work in the same abstract setting as the previous section.

Proposition 3.1. *Let (X, d, μ) be a metric measure space and L an abstract positive self-adjoint operator on $L^2(X, \mu)$. Assume that the spectral measure $dE_{\sqrt{L}}(\lambda)$ for $\sqrt{L}$ has a Schwartz kernel $dE_{\sqrt{L}}(\lambda)(z, z')$ that satisfies, for some nonnegative function w on $X \times X$ and some $n \ge 3$, the estimate*

$$\left|\left(\frac{d}{d\lambda}\right)^j dE_{\sqrt{L}}(\lambda)(z, z')\right| \le C\lambda^{n-1-j}(1 + \lambda w(z, z'))^{-(n-1)/2+j} \tag{3-1}$$

for $j = 0$ and for $j = n/2 - 1$ and $j = n/2$ if n is even, or for $j = n/2 - 3/2$ and $j = n/2 + 1/2$ if n is odd. Then (2-3) holds for all p in the range $[1, 2(n+1)/(n+3)]$. Moreover, if the estimates above hold only for $0 < \lambda < \lambda_0$, then (2-4) hold for the same range of p.

We prove this proposition via complex interpolation, embedding the derivatives of the spectral measure in an analytic family of operators, following the original (unpublished) proof of Stein in the classical case. To do this we use the distributions χ_+^a, defined by

$$\chi_+^a = x_+^a/\Gamma(a+1),$$

where Γ is the gamma function and

$$\begin{cases} x_+^a = x^a & \text{if } x \ge 0, \\ x_+^a = 0 & \text{if } x < 0. \end{cases}$$

The x_+^a are clearly distributions for $\operatorname{Re} a > -1$, and we have for $\operatorname{Re} a > 0$,

$$\frac{d}{dx} x_+^a = a x_+^{a-1} \quad \Longrightarrow \quad \frac{d}{dx} \chi_+^a = \chi_+^{a-1}, \tag{3-2}$$

which we use to extend the family of functions χ_+^a to a family of distributions on $\mathbb{R}$ defined for all $a \in \mathbb{C}$; see [Hörmander 1983] for details. Since $\chi_+^0(x) = H(x)$ is the Heaviside function, it follows that

$$\chi_+^{-k} = \delta_0^{(k-1)}, \quad k = 1, 2, \dots, \tag{3-3}$$

and therefore

$$\chi_+^0(\lambda - \sqrt{\boldsymbol{L}}) = E_{\sqrt{\boldsymbol{L}}}((0, \lambda]) \quad \text{and} \quad \chi_+^{-k}(\lambda - \sqrt{\boldsymbol{L}}) = \Big(\frac{d}{d\lambda}\Big)^{k-1} dE_{\sqrt{\boldsymbol{L}}}(\lambda), \quad k \geq 1.$$

A standard computation shows that for all $w, z \in \mathbb{C}$,

$$\chi_+^w * \chi_+^z = \chi_+^{w+z+1}, \tag{3-4}$$

where $\chi_+^w * \chi_+^z$ is the convolution of the distributions χ_+^w and χ_+^z see [Hörmander 1983, (3.4.10)]. We can use this relation to *define* the operators $\chi_+^z(\lambda - \sqrt{\boldsymbol{L}})$ for $\operatorname{Re} z < 0$, provided that the spectral measure of $\sqrt{\boldsymbol{L}}$ satisfies estimates of the type in Proposition 3.1:

Definition 3.2. Suppose that X, $\boldsymbol{L}$ and w are as in Proposition 3.1, and that $\boldsymbol{L}$ satisfies the kernel estimate

$$\Big| \Big(\frac{d}{d\lambda}\Big)^k dE_{\sqrt{\boldsymbol{L}}}(\lambda)(z, z') \Big| \leq C \lambda^l (1 + \lambda w(z, z'))^\beta \tag{3-5}$$

for some $k \geq 0$, $l \geq 0$ and β. Then, for $-(k+1) < \operatorname{Re} a < 0$ we define the operator $\chi_+^a(\lambda - \sqrt{\boldsymbol{L}})$ to be that operator with kernel

$$\chi_+^{k+a} * \chi_+^{-(k+1)}(\lambda - \sqrt{\boldsymbol{L}})(z, z') = (-1)^k \int_0^\lambda \frac{\sigma^{k+a}}{\Gamma(k+a+1)} \Big(\frac{d}{d\sigma}\Big)^k dE_{\sqrt{\boldsymbol{L}}}(\lambda - \sigma)(z, z')\, d\sigma. \tag{3-6}$$

Notice that the integral converges, since $\operatorname{Re}(k+a) > -1$ and $l \geq 0$ in (3-5). It is also independent of the choice of integer $k > -\operatorname{Re} a - 1$ (provided (3-5) holds), as we check by integrating by parts in σ in the integral above, and using (3-2). Note that the kernel $\chi_+^a(\lambda - \sqrt{\boldsymbol{L}})(z, z')$ is analytic in a, and as an integral operator maps $L^1_{\mathrm{comp}}(X)$ to $L^\infty_{\mathrm{loc}}(X)$. Therefore, for each fixed $\lambda > 0$, the family $\chi_+^a(\lambda - \sqrt{\boldsymbol{L}})$ is an analytic family of operators in the sense of Stein [1956] in the parameter a, for $\operatorname{Re} a > -k$.

In the proof of Proposition 3.1 we will need the following:

Lemma 3.3. *Suppose that $k \in \mathbb{N}$, that $-k < a < b < c$ and that $b = \theta a + (1 - \theta) c$. Then there exists a constant C such that for any C^{k-1} function $f : \mathbb{R} \to \mathbb{C}$ with compact support, one has*

$$\|\chi_+^{b+is} * f\|_\infty \leq C(1 + |s|) e^{\pi |s|/2} \|\chi_+^a * f\|_\infty^\theta \|\chi_+^c * f\|_\infty^{1-\theta}$$

for all $s \in \mathbb{R}$.

Remark 3.4. The convolution $\chi_+^a * f$, for $a > -k$ and $f \in C_c^{k-1}(\mathbb{R})$, may be defined to be $\chi_+^{a+k-1} * f^{(k-1)}$; this is independent of the choice of k.

Proof. Set, for $\zeta \in \mathbb{C}$,

$$I_\zeta f = \chi_+^\zeta * f$$

and consider the operator $I_{b+is}(\sigma I_c + I_a)^{-1}$, where the number $\sigma \in \mathbb{C}$ such that $|\sigma| = 1$ will be specified later. By (3-4)

$$I_{b+is}(\sigma I_c + I_a)^{-1} = I_{\beta+is}(\sigma I_{-1} + I_\alpha)^{-1} = I_{\beta+is}(\sigma I + I_\alpha)^{-1},$$

where $\beta = b - c - 1$ and $\alpha = a - c - 1$. Note that $\alpha < \beta < -1$. A standard calculation [Hörmander 1983, Example 7.1.17, page 167 and (3.2.9) page 72] shows that for $\operatorname{Re}\zeta \le -1$,

$$\widehat{\chi_+^\zeta}(\xi) = e^{-i\pi(\zeta+1)/2}(\xi - i0)^{-\zeta-1}.$$

It follows that $I_{\beta+is}(\sigma I + I_\alpha)^{-1} f = f * \eta_s$, where $\widehat{\eta_s}$ is the locally integrable function

$$\widehat{\eta}_s(\xi) = \frac{-ie^{-i\pi(\beta+is)/2}\xi_+^{-(\beta+is)-1} + ie^{i\pi(\beta+is)/2}\xi_-^{-(\beta+is)-1}}{\sigma - ie^{-i\pi\alpha/2}\xi_+^{-\alpha-1} + ie^{i\pi\alpha/2}\xi_-^{-\alpha-1}}.$$

Here $\xi_+ = \max(0, \xi)$ and $\xi_- = -\min(0, \xi)$. Note that if $|\sigma| = 1$ and $\sigma \notin \{ie^{-i\pi\alpha/2}, -ie^{-i\pi\alpha/2}\}$ then

$$\left|\frac{d}{d\xi}\widehat{\eta}_s(\xi)\right| \le C(1+|s|)e^{\pi|s|/2}\min\left(|\xi|^{-\beta-2}, |\xi|^{-\beta+\alpha-1}\right)$$

and $-\beta + \alpha - 1 < -1 < -\beta - 2$. It follows from these estimates that the function $\frac{d}{d\xi}\widehat{\eta}_s$ is in an $L^p(\mathbb{R})$ space for some $1 < p < 2$ and is also in some weighted space $L^1((1+|x|)^\epsilon dx, \mathbb{R})$. By the Sobolev embedding and Hausdorff–Young theorems, the function $x \to x\eta_s(x)$ is in $L^{p'}(\mathbb{R})$ for the conjugate exponent $p' < \infty$ and in $C^{\epsilon'}(\mathbb{R})$ for some $\epsilon' > 0$. Hence η_s is in L^1 and we have

$$\|\eta_s\|_1 \le C(1+|s|)e^{\pi|s|/2}.$$

Hence the operator $I_{b+is}(\sigma I_c + I_a)^{-1} = I_{\beta+is}(\sigma I + I_\alpha)^{-1}$ is bounded on $L^\infty(\mathbb{R})$ and

$$\|I_{b+is} f\|_\infty \le C(1+|s|)e^{\pi|s|/2}\|\sigma I_c f + I_a f\|_\infty \le C(1+|s|)e^{\pi|s|/2}(\|I_c f\|_\infty + \|I_a f\|_\infty).$$

Now if we set $D_\kappa f(x) = f(\kappa x)$ then for all $\zeta \in \mathbb{C}$,

$$I_\zeta D_\kappa f = \kappa^{-\zeta-1} D_\kappa I_\zeta f,$$

so

$$\kappa^{-b}\|I_{b+is} f\|_\infty = \kappa^{-b}\|D_\kappa I_{b+is} f\|_\infty = \kappa\|I_{b+is} D_\kappa f\|_\infty.$$

Hence

$$\begin{aligned}\kappa^{-b}\|I_{b+is} f\|_\infty = \kappa\|I_{b+is} D_\kappa f\|_\infty &\le C(1+|s|)e^{\pi|s|/2}\big(\kappa\|I_a(D_\kappa f)\|_\infty + \kappa\|I_c(D_\kappa f)\|_\infty\big)\\ &= C(1+|s|)e^{\pi|s|/2}\big(\kappa^{-a}\|I_a f\|_\infty + \kappa^{-c}\|I_c f\|_\infty\big).\end{aligned}$$

Putting $\kappa^{a-c} = \|I_a f\|_\infty \|I_c f\|_\infty^{-1}$ in this estimate yields Lemma 3.3. □

Proof of Proposition 3.1. To prove (2-3) in the range $1 \le p \le 2(n+1)/(n+3)$, it suffices by interpolation to establish the result for the endpoints $p = 1$ and $p = 2(n+1)/(n+3)$. The endpoint $p = 1$ is precisely (3-1) for $j = 0$, so it remains to obtain the endpoint $p = 2(n+1)/(n+3)$. This we will obtain through complex interpolation, applied to the analytic (in the parameter a) family $\chi_+^a(\lambda - \sqrt{\boldsymbol{L}})$ in the strip $-(n+1)/2 \le \operatorname{Re} a \le 0$.

On the line $\operatorname{Re} a = 0$, we have the estimate

$$\|\chi^{is}(\lambda - \sqrt{\boldsymbol{L}})\|_{L^2 \to L^2} \le \left|\frac{1}{\Gamma(1+is)}\right| = \sqrt{\frac{\sinh \pi s}{\pi s}} \le Ce^{\pi|s|/2}.$$

On the line $\operatorname{Re} a = -(n+1)/2$, we will prove an estimate of the form

$$\left\|\chi^{-(n+1)/2+is}(\lambda - \sqrt{\boldsymbol{L}})\right\|_{L^1 \to L^\infty} \le C(1+|s|)e^{\pi|s|/2}\lambda^{(n-1)/2} \qquad \text{for all } s \in \mathbb{R}. \tag{3-7}$$

Then, since we can write

$$dE_{\sqrt{\boldsymbol{L}}}(\lambda) = \chi_+^{-1}(\lambda - \sqrt{\boldsymbol{L}})$$

and

$$-1 = \frac{n-1}{n+1}\cdot 0 + \frac{2}{n+1}\cdot\left(-\frac{n+1}{2}\right) \quad \text{and} \quad \frac{n+3}{2(n+1)} = \frac{n-1}{n+1}\cdot\frac{1}{2} + \frac{2}{n+1}\cdot 1,$$

we obtain (2-3) at $p = 2(n+1)/(n+3)$ by complex interpolation.

It remains to prove (3-7). Let $\eta \in C_c^\infty(\mathbb{R})$ be a function such that $0 \le \eta(x) \le 1$ for all $x \in \mathbb{R}$ and $\eta(x) = 1$ for $|x| \le 2$ and $\eta(x) = 0$ for $|x| \ge 4$. Set

$$\begin{aligned} F_{z,z'}^{s,\Lambda}(\lambda) &= \chi_+^{-3/2-is} * \big(\eta(\,\cdot\,/\Lambda)\chi_+^{-k}(\,\cdot\, - \sqrt{\boldsymbol{L}})(z,z')\big)(\lambda), \qquad && n = 2k, \\ F_{z,z'}^{s,\Lambda}(\lambda) &= \chi_+^{-2-is} * \big(\eta(\,\cdot\,/\Lambda)\chi_+^{-k}(\,\cdot\, - \sqrt{\boldsymbol{L}})(z,z')\big)(\lambda), && n = 2k+1. \end{aligned}$$

Note that $\operatorname{supp}(\chi_+^z) \subset [0,\infty)$ for all z, and $\boldsymbol{L} \ge 0$. It follows that for $\lambda \le \Lambda$ and $n = 2k$,

$$F_{z,z'}^{s,\Lambda}(\lambda) = \chi_+^{-3/2-is} * \chi_+^{-k}(\lambda - \sqrt{\boldsymbol{L}})(z,z') = \chi_+^{-(n+1)/2-is}(\lambda - \sqrt{\boldsymbol{L}})(z,z')$$

and for $\lambda \le \Lambda$ and $n = 2k+1$,

$$F_{z,z'}^{s,\Lambda}(\lambda) = \chi_+^{-2-is} * \chi_+^{-k}(\lambda - \sqrt{\boldsymbol{L}})(z,z') = \chi_+^{-(n+1)/2-is}(\lambda - \sqrt{\boldsymbol{L}})(z,z'),$$

i.e., the cutoff function η has no effect for $\lambda \le \Lambda$. Hence

$$\left\|\chi_+^{-(n+1)/2-is}(\Lambda - \sqrt{\boldsymbol{L}})\right\|_{1\to\infty} \le \sup_{z,z'} |F_{z,z'}^{s,\Lambda}(\Lambda)|.$$

We consider first the odd-dimensional case $n = 2k + 1$. By Lemma 3.3 and (3-3),

$$\begin{aligned} \left|F^{s,\Lambda}_{z,z'}(\Lambda)\right| &\le \left\|F^{s,\Lambda}_{z,z'}\right\|_\infty \\ &\le C(1+|s|)e^{\pi|s|/2} \sup_{\lambda>0}\left|\left(\chi_+^{-1} * (\eta(\cdot/\Lambda)\chi_+^{-k}(\cdot - \sqrt{\boldsymbol{L}})(z,z'))\right)(\lambda)\right|^{1/2} \\ &\qquad\qquad \times \sup_{\lambda>0}\left|\left(\chi_+^{-3} * (\eta(\cdot/\Lambda)\chi_+^{-k}(\cdot - \sqrt{\boldsymbol{L}})(z,z'))\right)(\lambda)\right|^{1/2} \\ &\le C(1+|s|)e^{\pi|s|/2} \sup_{\lambda>0}\left|\eta(\lambda/\Lambda)\chi_+^{-k}(\lambda - \sqrt{\boldsymbol{L}})(z,z')\right|^{1/2} \\ &\qquad\qquad \times \sup_{\lambda>0}\left|\frac{d^2}{d\lambda^2}\eta(\lambda/\Lambda)\chi_+^{-k}(\lambda - \sqrt{\boldsymbol{L}})(z,z')\right|^{1/2}, \end{aligned} \tag{3-8}$$

where the presence of the η cutoff is now crucial. It follows from (3-1) with $j = n/2 - 3/2$ and $j = n/2 + 1/2$, i.e., $j = k - 1$ and $j = k + 1$, that

$$\sup_{\lambda>0}|\eta(\lambda/\Lambda)\chi_+^{-k}(\lambda - \sqrt{\boldsymbol{L}})(z,z')| \le C\Lambda^{k+1}(1 + \Lambda w(z,z'))^{-1}.$$

(Here we used the fact that the function $\lambda^k(1+\lambda w)^\beta$ is an increasing function of λ provided $\lambda \ge 0$, $w \ge 0$, $k \ge 0$ and $k + \beta \ge 0$.) Similarly,

$$\begin{aligned} \sup_{\lambda>0}\left|\frac{d^2}{d\lambda^2}\eta(\lambda/\Lambda)\chi_+^{-k}(\lambda - \sqrt{\boldsymbol{L}})(z,z')\right| &\le \sup_{\lambda>0}\left|\eta(\lambda/\Lambda)\chi_+^{-k-2}(\lambda - \sqrt{\boldsymbol{L}})(z,z')\right| \\ &\qquad + \frac{1}{\Lambda}\sup_{\lambda>0}\left|\eta'(\lambda/\Lambda)\chi_+^{-k-1}(\lambda - \sqrt{\boldsymbol{L}})(z,z')\right| \\ &\qquad + \frac{1}{\Lambda^2}\sup_{\lambda>0}\left|\eta'(\lambda/\Lambda)\chi_+^{-k}(\lambda - \sqrt{\boldsymbol{L}})(z,z')\right| \\ &\le C\Lambda^{k-1}(1 + \Lambda w(z,z')). \end{aligned}$$

Our estimate (3-7) for $n = 2k + 1$ follows now from these two estimates and (3-8).

If $n = 2k$ is even, then by Lemma 3.3 and (3-3),

$$\begin{aligned} \left|F^{s,\Lambda}_{z,z'}(\Lambda)\right| &\le \left\|F^{s,\Lambda}_{z,z'}\right\|_\infty \\ &\le C(1+|s|)e^{\pi|s|/2} \sup_{\lambda>0}\left|\left(\chi_+^{-1} * (\eta(\cdot/\Lambda)\chi_+^{-k}(\cdot - \sqrt{\boldsymbol{L}})(z,z'))\right)(\lambda)\right|^{1/2} \\ &\qquad\qquad \times \sup_{\lambda>0}\left|\left(\chi_+^{-2} * (\eta(\cdot/\Lambda)\chi_+^{-k}(\cdot - \sqrt{\boldsymbol{L}})(z,z'))\right)(\lambda)\right|^{1/2} \\ &\le C(1+|s|)e^{\pi|s|/2} \sup_{\lambda>0}\left|\eta(\lambda/\Lambda)\chi_+^{-k}(\lambda - \sqrt{\boldsymbol{L}})(z,z')\right|^{1/2} \\ &\qquad\qquad \times \sup_{\lambda>0}\left|\frac{d}{d\lambda}\eta(\lambda/\Lambda)\chi_+^{-k}(\lambda - \sqrt{\boldsymbol{L}})(z,z')\right|^{1/2}, \end{aligned} \tag{3-9}$$

and we follow the same argument as in the odd-dimensional case to establish (3-7) for $n = 2k$. □

In some situations, including the case of Laplace-type operators on asymptotically conic manifolds discussed later in this paper, we can express the spectral measure $dE(\lambda)$ in the form $P(\lambda)P(\lambda)^*$, where the initial space of $P(\lambda)$ is an auxiliary Hilbert space H. In this case, we can use a TT^* argument to show that the conclusions of Proposition 3.1 follow from localized estimates on $dE(\lambda)$, that is, on kernel estimates on $Q_i dE(\lambda) Q_i$, with respect to a operator partition of unity

$$\mathrm{Id} = \sum_{i=1}^{N(\lambda)} Q_i(\lambda), \quad 1 \le i \le N(\lambda).$$

Notice that we allow the partition of unity to depend on λ. However, we shall assume that $N(\lambda)$ is uniformly bounded in λ.

Remark 3.5. Here we assume that $Q_i(\lambda) dE^{(j)}_{\sqrt{\boldsymbol{L}}}(\lambda) Q_i(\lambda)$ can be defined somehow and has a Schwartz kernel; for example, we might know that there is some weight function ω on X such that $dE^{(j)}_{\sqrt{\boldsymbol{L}}}(\lambda)$ is a bounded map from $\omega^{j+1} L^2(X)$ to $\omega^{-j-1} L^2(X)$, and that $Q_i(\lambda)$ maps $\omega^a L^2(X)$ boundedly to itself for any a. This is the case in our application to asymptotically conic manifolds, with $\omega = x$ (where x is as in (1-1)).

Proof of Proposition 1.12. Observe that Proposition 1.12 reduces to Proposition 3.1 in the case that the partition of unity Q_i is trivial. We apply the argument in the proof of Proposition 3.1 to the operators $Q_i(\lambda) dE(\lambda) Q_i(\lambda)$, i.e., we replace $dE_{\sqrt{\boldsymbol{L}}}(\lambda)$ by $Q_i(\lambda) dE_{\sqrt{\boldsymbol{L}}}(\lambda) Q_i(\lambda)^*$ in (3-6). The conclusion is that

$$\| Q_i(\lambda) dE_{\sqrt{\boldsymbol{L}}}(\lambda) Q_i(\lambda)^* \|_{L^p(X) \to L^{p'}(X)} \le C \lambda^{n(1/p - 1/p') - 1} \quad \text{for all } \lambda > 0.$$

Using the fact that $dE_{\sqrt{\boldsymbol{L}}}(\lambda) = P(\lambda) P(\lambda)^*$ and the TT^* trick, we deduce that

$$\| Q_i(\lambda) P(\lambda) \|_{L^2(X) \to L^{p'}(X)} \le C \lambda^{n(1/2 - 1/p') - 1/2} \quad \text{for all } \lambda > 0.$$

Now we can sum over i, and find that

$$\| P(\lambda) \|_{L^2(X) \to L^{p'}(X)} \le C \lambda^{n(1/2 - 1/p') - 1/2} \quad \text{for all } \lambda > 0.$$

Finally, we use $dE_{\sqrt{\boldsymbol{L}}}(\lambda) = P(\lambda) P(\lambda)^*$ and the TT^* trick again to deduce that

$$\| dE_{\sqrt{\boldsymbol{L}}}(\lambda) \|_{L^p(X) \to L^{p'}(X)} \le C \lambda^{n(1/p - 1/p') - 1} \quad \text{for all } \lambda > 0,$$

yielding (2-3). Moreover, if the estimates hold only for $0 < \lambda \le \lambda_0$, then we obtain (2-4) instead. □

Remark 3.6. We acknowledge and thank Jared Wunsch for suggesting to us that the TT^* trick would be useful here.

Part II. Schrödinger operators on asymptotically conic manifolds

In this second part of the paper, we specialize to the case that (X, d, μ) is an asymptotically conic manifold (M°, g) with the Riemannian distance function d and Riemannian measure μ, and $\boldsymbol{L}$ is a Schrödinger operator $\boldsymbol{H}$ on $L^2(M^\circ, g)$, that is, an operator of the form $\boldsymbol{H} = \Delta_g + V$, where Δ_g is the positive Laplacian associated to g and $V \in C^\infty(M)$ is a potential function vanishing to third order at the boundary of the compactification M of M°. We assume that $\boldsymbol{H}$ has no L^2-eigenvalues (which implies that it is positive as an operator) and that zero is not a resonance.

The goal in this part of the paper is to show that $\boldsymbol{H}$ satisfies the low energy spectral measure estimates (2-4), and the full spectral measure estimates (2-3) provided that (M°, g) is nontrapping. To do this, we will establish the estimates (1-9) for a suitable partition of unity $Q_i(\lambda)$. In the case of low energy estimates, i.e., $\lambda \in (0, \lambda_0]$ for $\lambda_0 < \infty$, these Q_i will be pseudodifferential operators, lying in the calculus of operators introduced in [Guillarmou and Hassell 2008]. Thus our first task is to determine the nature of the operator $Q_i dE(\lambda) Q_i$ for such Q_i, which is the subject of Section 5. Before this, however, we recall some of the geometric preliminaries from [Guillarmou et al. 2012; Hassell and Wunsch 2008].

4. Geometric preliminaries

The Schwartz kernel of the spectral measure was constructed in [Guillarmou et al. 2012] for low energies and in [Hassell and Wunsch 2008] for high energies on a compactification of the space $[0, \lambda_0] \times (M^\circ)^2$, respectively $[0, h_0] \times (M^\circ)^2$, where we use $h = \lambda^{-1}$ in place of λ for high energies. We use the definitions and machinery from these papers extensively, and we do not review this material comprehensively here, since that would double the length of this paper. Nevertheless, we shall describe these compactifications, review some of their geometric properties, and define some coordinate systems that we shall use in the following sections.

Recall from the introduction that (M°, g) is asymptotically conic if M° is the interior of a compact manifold M with boundary, such that in a collar neighborhood of the boundary, the metric g takes the form $g = dx^2/x^4 + h(x)/x^2$, where x is a boundary defining function and $h(x)$ is a smooth family of metrics on the boundary ∂M. We use $y = (y_1, \dots, y_{n-1})$ for local coordinates on ∂M, so that (x, y) furnish local coordinates on M near ∂M. Away from ∂M, we use $z = (z_1, \dots, z_n)$ to denote local coordinates.

4A. ***The low energy space $M^2_{k,b}$.*** In [Guillarmou and Hassell 2008; Guillarmou et al. 2012], following unpublished work of Melrose and Sá Barreto, the low energy space $M^2_{k,b}$ is defined as follows: starting with $[0, \lambda_0] \times M^2$, we define submanifolds $C_3 := \{0\} \times \partial M \times \partial M$ and

$$C_{2,L} := \{0\} \times \partial M \times M, \quad C_{2,R} := \{0\} \times M \times \partial M, \quad C_{2,C} := [0, 1] \times \partial M \times \partial M.$$

The space $M^2_{k,b}$ is then defined as $[0, \lambda_0] \times M^2$ with the codimension 3 corner C_3 blown up, followed by the three codimension 2 corners $C_{2,*}$:

$$M^2_{k,b} := [[0, 1] \times M \times M; C_3, C_{2,R}, C_{2,L}, C_{2,C}].$$

The new boundary hypersurfaces created by these blowups are labeled bf_0, rb_0, lb_0 and bf, respectively, and the original boundary hypersurfaces $\{0\} \times M^2$, $[0, \lambda] \times M \times \partial M$ and $[0, \lambda] \times \partial M \times M$ are labeled zf, rb, lb, respectively. We remark that zf is canonically diffeomorphic to the b-double space

$$M^2_b = [M^2; \partial M \times \partial M].$$

Also, each section $M^2_{k,b} \cap \{\lambda = \lambda_*\}$, for fixed $0 < \lambda_* < \lambda_0$ is canonically diffeomorphic to M^2_b.

We define functions x and y on $M^2_{k,b}$ by lifting from the left copy of M (near ∂M), and x', y' by lifting from the right copy of M; similarly z, z' (away from ∂M). We also define $\rho = x/\lambda$, $\rho' = x'/\lambda$, and

$\sigma = \rho/\rho' = x/x'$. Near bf and away from rb, we use coordinates $y, y', \sigma, \rho', \lambda$, while near bf and away from lb, we use $y, y', \sigma^{-1}, \rho, \lambda$. We also use the notation $\rho_\bullet$, where $\bullet = \mathrm{bf}_0, \mathrm{lb}_0, \ldots$, to denote a generic boundary defining function for the boundary hypersurface $\bullet$.

This space has a compressed cotangent bundle ${}^{k,b}T^*M^2_{k,b}$, defined in [Guillarmou et al. 2012, Section 2]. A basis of sections of this space is given, in the region $\rho, \rho' \le C$ (which includes a neighborhood of bf), by

$$\frac{d\rho}{\rho^2}, \quad \frac{d\rho'}{\rho'^2}, \quad \frac{dy_i}{\rho}, \quad \frac{dy_i'}{\rho'}, \quad \frac{d\lambda}{\lambda}. \tag{4-1}$$

Therefore, any point in ${}^{k,b}T^*M^2_{k,b}$ lying over this region can be written as

$$\nu\frac{d\rho}{\rho^2} + \nu'\frac{d\rho'}{\rho'^2} + \mu_i\frac{dy_i}{\rho} + \mu_i'\frac{dy_i'}{\rho'} + T\frac{d\lambda}{\lambda}. \tag{4-2}$$

This defines local coordinates $(y, y', \sigma, \rho', \lambda, \mu, \mu', \nu, \nu', T)$ in ${}^{k,b}T^*M^2_{k,b}$, near bf and away from rb, where $(\mu, \mu', \nu, \nu', T)$ are linear coordinates on each fiber.

The compressed density bundle $\Omega_{k,b}(M^2_{k,b})$ is defined to be that line bundle whose smooth nonzero sections are given by the wedge product of a basis of sections for ${}^{k,b}T^*(M^2_{k,b})$. Using the coordinates above, we can write a smooth nonzero section $\boldsymbol{\omega}$ as

$$\boldsymbol{\omega} = \left|\frac{d\rho d\rho' dy dy' d\lambda}{\rho^{n+1}\rho'^{n+1}\lambda}\right| \sim \lambda^{2n}\left|\frac{dg\, dg'\, d\lambda}{\lambda}\right| \quad \text{in the region } \rho, \rho' \le C. \tag{4-3}$$

For $\rho, \rho' \ge C$, we can take $\boldsymbol{\omega} = (xx')^n |dg dg' d\lambda/\lambda|$. Here dg, respectively dg', denotes the Riemannian density with respect to g, lifted to $M^2_{k,b}$ by the left, respectively right, projection.

The boundary of ${}^{k,b}T^*M^2_{k,b}$ lying over boundary hypersurface $\bullet$ is denoted by ${}^{k,b}T^*_\bullet M^2_{k,b}$. The space ${}^{k,b}T^*_{\mathrm{lb}}M^2_{k,b}$ fibers over the space ${}^{\mathrm{sc}}T^*{}_{\partial M}M \times [0, \lambda]$ (the scattering cotangent bundle ${}^{\mathrm{sc}}T^*M$ over M is defined in [Melrose 1994; Hassell and Vasy 1999; 2001], and ${}^{\mathrm{sc}}T^*{}_{\partial M}M$ is that part of the bundle lying over ∂M). This fibration is given in local coordinates by

$$(y, y', \sigma, \lambda, \mu, \mu', \nu, \nu', T) \to (y, \mu, \nu, \lambda). \tag{4-4}$$

Similarly there is a natural fibration from ${}^{k,b}T^*_{\mathrm{rb}}M^2_{k,b}$ to ${}^{\mathrm{sc}}T^*{}_{\partial M}M \times [0, \lambda_0]$, which takes the form

$$(y, y', \sigma, \lambda, \mu, \mu', \nu, \nu', T) \to (y', \mu', \nu', \lambda). \tag{4-5}$$

We also note that there are natural maps π_L, π_R mapping ${}^{\mathrm{sc}}T^*{}_{\mathrm{bf}}M^2_b \times [0, \lambda_0]$ (see [Hassell and Vasy 1999; 2001]) to ${}^{\mathrm{sc}}T^*{}_{\partial M}M \times [0, \lambda_0]$ which are induced by the projections $T^*M^2 \to T^*M$ onto the left, respectively right, factor. In local coordinates, these are given by

$$\pi_L(y, y', \sigma, \mu, \mu', \nu, \nu', \lambda) = (y, \mu, \nu, \lambda), \quad \pi_R(y, y', \sigma, \mu, \mu', \nu, \nu', \lambda) = (y', \mu', \nu', \lambda). \tag{4-6}$$

We use these maps in Section 5.

The space ${}^{k,b}T^*_{\mathrm{bf}}M^2_{k,b}$ is canonically diffeomorphic to ${}^{\mathrm{s}\Phi}T^*{}_{\mathrm{bf}}M^2_b \times [0, \lambda_0]$, where ${}^{\mathrm{s}\Phi}T^*{}_{\mathrm{bf}}M^2_b$ is the scattering-fibered cotangent bundle of M^2_b defined in [Hassell and Vasy 1999]. The space ${}^{\mathrm{s}\Phi}T^*{}_{\mathrm{bf}}M^2_b$ has a natural contact structure, and Legendre submanifolds with respect to this structure play an important role in encoding the oscillations of the spectral measure at the boundary of $M^2_{k,b}$. In fact, three Legendre submanifolds of ${}^{s\Phi}T^*{}_{\mathrm{bf}}M^2_b$ arise in the identification of the spectral measure as a Legendre distribution (see [Guillarmou et al. 2012, Section 3]), which we now briefly describe. One is denoted ${}^{sc}N^*\partial\mathrm{diag}_b$, which in coordinates used in (4-2) is given by

$$ {}^{sc}N^*\partial\mathrm{diag}_b = \{(y, y', \sigma, \mu, \mu', \nu, \nu') \mid y = y', \sigma = 1, \mu = -\mu', \nu = -\nu'\}; \tag{4-7}$$

it is a sort of conormal bundle to the boundary of the diagonal $\partial\mathrm{diag}_b$,

$$\partial\mathrm{diag}_b = \{(y, y', \sigma) \mid y = y', \sigma = 1\}, \tag{4-8}$$

in M^2_b, and carries the "operator wavefront set" or "microlocal support" of scattering pseudodifferential operators. Another is the incoming/outgoing Legendrian submanifold $L^\sharp$, which in the coordinates used in (4-2) is given by

$$L^\sharp = \{(y, y', \sigma, \mu, \mu', \nu, \nu') \mid \mu = \mu' = 0, \nu = \pm 1, \nu' = -\nu\}. \tag{4-9}$$

It has two components (corresponding to the sign of ν) and describes oscillations that are purely radial, that is, purely incoming or outgoing. The third and most interesting Legendre submanifold is the propagating Legendrian, denoted by L^{bf}. To describe it, let G denote the characteristic variety of $\boldsymbol{H} - \lambda^2$. Then L^{bf} is given by the flowout from ${}^{sc}N^*\partial\mathrm{diag}_b \cap G$ by the bicharacteristic flow of $\boldsymbol{H}$. It connects the incoming and outgoing components of $L^\sharp$ and has a conic singularity at each. As shown in [Hassell and Vasy 1999, Proposition 7.1], $(L^{\mathrm{bf}}, L^\sharp)$ is a Legendre conic pair, and has an associated class of polyhomogeneous-conormal Legendre distributions [Guillarmou et al. 2012, Section 3.2]

$$I^{m,p;r_{\mathrm{lb}},r_{\mathrm{rb}};\mathcal{B}}(M^2_{k,b}, (L^{\mathrm{bf}}, L^{\sharp,\mathrm{bf}}); \Omega^{1/2}_{k,b}) \tag{4-10}$$

of order m at L^{bf} and p at $L^\sharp$, and with polyhomogeneous expansion with respect to the index family $\mathcal{B}$ at the boundary hypersurfaces at $\lambda = 0$. In terms of these space of half-densities we have:

Theorem 4.1 [Guillarmou et al. 2012, Theorem 3.10]. *The spectral measure $dE_{\sqrt{\boldsymbol{H}}}(\lambda)$, for $0 < \lambda \le \lambda_0$, is a conormal Legendre distribution in the space (4-10) tensored with $|\lambda d\lambda|^{1/2}$ (this makes it a full density, i.e., a measure, in λ), with $m = -\frac{1}{2}$, $p = (n-2)/2$, $r_{\mathrm{lb}} = r_{\mathrm{rb}} = (n-1)/2$, and where $\mathcal{B}$ is an index family with index sets at the faces* bf_0, lb_0, rb_0, zf *starting at order* -1, $n/2-1$, $n/2-1$, $n-1$, *respectively.*

4B. *The high energy space X.* The high energy space X is defined by $X = [0, h_0] \times M^2_b$. The boundary hypersurfaces $[0, h_0] \times M \times \partial M$, $[0, h_0] \times \partial M \times M$ and $\{0\} \times M^2_b$ are denoted by rb, lb and mf ("main face"), respectively, and the boundary hypersurface arising from $[0, h_0] \times \partial M \times \partial M$ is denoted by bf. Notice that this space fits together with the low energy space: in the range $\lambda \in (C^{-1}, C)$ (where $\lambda = 1/h$), the spaces both have the form $(C^{-1}, C) \times M^2_b$, and the labeling of boundary hypersurfaces is consistent. As before, we write $\sigma = x/x'$. We use the coordinates (y, y', σ, x', h) near bf and away from rb, and the coordinates $(y, y', \sigma^{-1}, x, h)$ near bf and away from lb. Away from bf, lb, rb we use the coordinates (z, z', h).

The compressed cotangent bundle ${}^{s\Phi}T^*X$ is described in [Hassell and Wunsch 2008]. A basis of sections of this bundle is given in the region $x, x' \leq \epsilon$ by

$$\frac{dy_i}{xh}, \quad \frac{dy_i'}{x'h}, \quad d\Big(\frac{1}{xh}\Big), \quad d\Big(\frac{1}{x'h}\Big), \quad d\Big(\frac{1}{h}\Big).$$

In terms of this basis, any point in ${}^{s\Phi}T^*X$ lying over this region can be written as

$$\mu \cdot \frac{dy}{xh} + \mu' \cdot \frac{dy'}{x'h} + \nu d\Big(\frac{1}{xh}\Big) + \nu' d\Big(\frac{1}{x'h}\Big) + \tau d\Big(\frac{1}{h}\Big). \tag{4-11}$$

This defines local coordinates $(y, y', \sigma, x', h, \mu, \mu', \nu, \nu', \tau)$, where $(\mu, \mu', \nu, \nu', \tau)$ are local coordinates on each fiber. In the region $x, x' \geq \epsilon$, a basis of sections is

$$\frac{dz_i}{h}, \quad \frac{dz_i'}{h}, \quad d\Big(\frac{1}{h}\Big),$$

and in terms of this basis, any point in ${}^{s\Phi}T^*X$ lying over this region can be written as

$$\zeta \cdot \frac{dz}{h} + \zeta' \cdot \frac{dz'}{h} + \tau d\Big(\frac{1}{h}\Big). \tag{4-12}$$

This defines local coordinates $(z, z', h, \zeta, \zeta', \tau)$ on ${}^{s\Phi}T^*X$ over this region.

This compressed density bundle ${}^{s\Phi}\Omega(X)$ is defined to be that line bundle whose smooth nonzero sections are given by a wedge product of a basis of sections for ${}^{s\Phi}T^*X$. We find that $|dg\, dg'\, dh/h^2| = |dg\, dg'\, d\lambda|$ is a smooth nonzero section of this bundle.

We also note that there are natural maps from ${}^{s\Phi}T^*_{\mathrm{mf}}X \to {}^{sc}T^*M$, which (abusing notation) we will also denote by π_L, π_R, which are induced by the projections onto the left, respectively right, factor $T^*M^2 \to T^*M$. In local coordinates, these are given by

$$\pi_L(z, z', \zeta, \zeta', \tau) = (z, \zeta), \quad \pi_R(z, z', \zeta, \zeta', \tau) = (z', \zeta'), \tag{4-13}$$

away from the boundary hypersurface bf, or near bf by

$$\pi_L(x, y, x', y', \mu, \mu', \nu, \nu', \tau) = (x, y, \mu, \nu), \quad \pi_R(x, y, x', y', \mu, \mu', \nu, \nu', \tau) = (x', y', \mu', \nu'). \tag{4-14}$$

The space ${}^{s\Phi}T^*_{\mathrm{mf}}X$ has a natural contact structure, as described in [Hassell and Wunsch 2008]. Legendre submanifolds with respect to this contact structure are important in describing the singularities of the spectral measure at high energies. We need to define three Legendre submanifolds ${}^{s\Phi}N^*\mathrm{diag}_b$ and L in order to describe the spectral measure at high energies as a Legendre distribution on X (see [ibid.]). The first of these, ${}^{s\Phi}N^*\mathrm{diag}_b$, is associated to the diagonal submanifold $\mathrm{diag}_b \subset \{0\} \times M_b^2$, defined using the coordinates above by

$${}^{s\Phi}N^*\mathrm{diag}_b = \{(z, z', h, \zeta, \zeta', \tau) \mid z = z', \zeta = -\zeta', h = 0, \tau = 0\} \tag{4-15}$$

away from bf, and

$${}^{s\Phi}N^*\mathrm{diag}_b = \{(y, y', \sigma, x', h, \mu, \mu', \nu, \nu', \tau) \mid y = y', \sigma = 1, h = 0, \mu = -\mu', \nu = -\nu', \tau = 0\} \tag{4-16}$$

near bf. The second, $L^\sharp$, lives at ${}^{s\Phi}T^*_{\mathrm{bf}\cup\mathrm{mf}}\boldsymbol{X}$ and is defined in (4-9). The third, L, is obtained just as L^{bf} was obtained from ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$ in the previous subsection, namely as the flowout by the bicharacteristic flow of $\boldsymbol{H}$ starting from the intersection of ${}^{s\Phi}N^*\mathrm{diag}_b$ and the characteristic variety of $h^2\boldsymbol{H}-1$. Indeed, the submanifolds L^{bf} and ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$ are essentially the boundary hypersurfaces of L and ${}^{s\Phi}N^*\mathrm{diag}_b$ lying over $\mathrm{bf}\cap\mathrm{mf}$. Associated to $(L, L^\sharp)$ is a class of Legendre distributions [ibid., Section 6.5.2]

$$I^{m,p;r_{\mathrm{bf}},r_{\mathrm{lb}},r_{\mathrm{rb}}}(\boldsymbol{X}, (L, L^{\#}); {}^{s\Phi}\Omega^{1/2}). \tag{4-17}$$

In terms of this space of half-densities, we have:

Theorem 4.2 [Hassell and Wunsch 2008, Corollary 1.2]. *Suppose that (M, g) is nontrapping. Then the spectral measure $dE_{\sqrt{\boldsymbol{H}}}(\lambda)$ is a Legendre distribution on $\boldsymbol{X}$, lying in the space (4-17) tensored with $|d\lambda|^{1/2}$, with $m=\frac{1}{2}$, $p=(n-2)/2$, $r_{\mathrm{bf}}=-\frac{1}{2}$, $r_{\mathrm{lb}}=r_{\mathrm{rb}}=(n-1)/2$. Here we use the order conventions in Remark 4.3.*

Remark 4.3. We use different order conventions from [Hassell and Wunsch 2008], to agree with those used in [Guillarmou et al. 2012]. In terms of Equation (4.15) of [Hassell and Wunsch 2008], the order convention in the present paper corresponds to taking $N=2n$ (not $2n+1$ as in [ibid.]), that is, the total space dimension, but not including the λ dimension, and taking the fiber dimensions $f_{\mathrm{bf}}=0$ and $f_{\mathrm{lb}}=f_{\mathrm{rb}}=n$, again not including the λ dimension. This has the effect that the orders in the present paper are $\frac{1}{4}$ larger at $\mathrm{mf}=M_b^2\times\{h=0\}$, and $\frac{1}{4}$ smaller at bf, lb and rb, compared to [ibid.], and explains the discrepancies in the orders above compared to those given in Corollary 1.2 of [ibid.]. (An advantage of the ordering convention used here is that a semiclassical pseudodifferential operator of (semiclassical) order m, multiplied by $|dh/h^2|^{1/2}=|d\lambda|^{1/2}$ becomes a Legendre distribution of the same order m at the conormal bundle of the diagonal in mf.)

5. Microlocal support

Recall from the end of Section 1 our strategy for proving Theorem 1.3, involving estimates (1-9). The elements Q_i of our partition of unity will be chosen to be pseudodifferential operators lying in the calculus of operators introduced in [Guillarmou and Hassell 2008, Definition 2.7]. In view of Theorem 4.1, we need to understand what happens when a conormal Legendre distribution $F\in I^{m,r_{\mathrm{lb}},r_{\mathrm{rb}},\mathcal{B}}(M^2_{k,b}, \Lambda; \Omega^{1/2}_{k,b})$ is pre- and postmultiplied by such operators. We shall use the notation $\Psi^m_k(M, \Omega^{1/2}_{k,b})$ to denote what in [ibid.] was written $\Psi^{m,\mathcal{E}}(M, \tilde{\Omega}^{1/2}_b)$, where the index family $\mathcal{E}$ assigns the C^∞ index family at sc, bf_0 and zf and the empty index family at all other boundary hypersurfaces. Such operators have kernels defined on the space $M^2_{k,\mathrm{sc}}$, defined in [ibid.], that are conormal of order m to the diagonal, uniformly to the boundary, smooth away from the diagonal, and rapidly vanishing at all boundary hypersurfaces not meeting the diagonal. As shown in [ibid., Proposition 2.10], $\Psi^0(M, \Omega^{1/2}_{k,b})$ is an algebra. It follows, using Hörmander's "square root trick" [1985, Section 18.1] that such kernels act as uniformly bounded (in λ) operators on $L^2(M)$.

In this section, we shall work exclusively on the low energy space $M^2_{k,b}$; the corresponding high energy estimates are given in Section 7A. We consider operators Q, Q' such that:

- Q, Q' are of order $-\infty$, i.e., $Q, Q' \in \Psi_k^{-\infty}(M, \Omega_{k,b}^{1/2})$, with compactly supported symbols. (5-1)
- Q, Q' have kernels supported close to the diagonal, inside the region $\{\sigma := x/x' \in [1/2, 2]\}$. (5-2)

With these assumptions, the kernels of Q, Q' are smooth (across the diagonal) on the space $M_{k,\mathrm{sc}}^2$. Viewed as distributions on $M_{k,b}^2$ (which has one fewer blowup than $M_{k,\mathrm{sc}}^2$) the kernels have a conic singularity at the boundary of the diagonal, $\partial\mathrm{diag}_b$. As shown in [Hassell and Vasy 2001, Section 5.1], this means that they are Legendre distributions in $I^{0,\infty,\infty;(0,0,\varnothing,\varnothing)}(M_{k,b}^2, {}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b; \Omega_{k,b}^{1/2})$, i.e., Legendre distributions of order 0 associated to ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$ (see (4-7)), with the C^∞ index set 0 at bf_0 and zf, and vanishing in a neighborhood of lb, rb, lb_0 and rb_0 (which is of course a trivial consequence of (5-2)).

Remark 5.1. The composition QF or FQ' is always well-defined when F is a Legendre distribution on $M_{k,b}^2$ and Q, Q' are as above, since F can be regarded as a map from $x^a L^2(M)$ to $x^{-a}L^2(M)$ for sufficiently large $a \in \mathbb{R}$, depending smoothly on $\lambda \in (0, \lambda_0)$, while pseudodifferential operators of order 0 are bounded on $x^a L^2(M)$ (uniformly in λ) for any a.

To state our results, we need to introduce some notation and define the notion of the microlocal support of F. Let $\Lambda \subset {}^{\mathrm{sc}}T^*_{\mathrm{bf}}M_b^2$ be the Legendre submanifold associated to F. We always assume that Λ is compact. Recall from [Hassell and Wunsch 2008, Section 4] that Λ determines two associated Legendre submanifolds Λ_{lb} and Λ_{rb} that are the bases of the fibrations on $\partial_{\mathrm{lb}}\Lambda$ and $\partial_{\mathrm{rb}}\Lambda$, respectively. These may be canonically identified with Legendre submanifolds of ${}^{\mathrm{sc}}T^*M$. We also define Λ' by negating the fiber coordinates corresponding to the right copy of M, i.e.,

$$q' = (y, y', x/x', \mu, \mu', \nu, \nu') \in \Lambda' \iff q = (y, y', x/x', \mu, -\mu', \nu, -\nu') \in \Lambda. \tag{5-3}$$

Similarly we define Λ'_{rb} by negating the fiber coordinates:

$$q' = (y', \mu', \nu') \in \Lambda'_{\mathrm{rb}} \iff q = (y', -\mu', -\nu') \in \Lambda_{\mathrm{rb}}.$$

We also define $\overline{\Lambda}'$, $\overline{\Lambda}_{\mathrm{lb}}$, $\overline{\Lambda}'_{\mathrm{rb}}$ by

$$\overline{\Lambda}' = \Lambda' \times [0, \lambda_0], \quad \overline{\Lambda}_{\mathrm{lb}} = \Lambda'_{\mathrm{lb}} \times [0, \lambda_0], \quad \overline{\Lambda}'_{\mathrm{rb}} = \Lambda'_{\mathrm{rb}} \times [0, \lambda_0]. \tag{5-4}$$

To define the microlocal support, $\mathrm{WF}'(F)$, of F we first recall from [Guillarmou et al. 2012] that $F \in I^{m,r_{\mathrm{lb}},r_{\mathrm{rb}},\mathcal{B}}(M_{k,b}^2, \Lambda; \Omega_{k,b}^{1/2})$ means F can be decomposed as $F = F_1 + F_2 + F_3 + F_4 + F_5 + F_6$, where

- F_1 is supported near bf and away from lb, rb;
- F_2 is supported near $\mathrm{bf} \cap \mathrm{lb}$;
- F_3 is supported near $\mathrm{bf} \cap \mathrm{rb}$;
- F_4 is supported near lb and away from bf;
- F_5 is supported near rb and away from bf;
- F_6 vanishes rapidly at bf, lb, rb and is polyhomogeneous on $M_{k,b}^2$ with index family $\mathcal{B}$;

and each F_i, $1 \le i \le 5$ has an oscillatory representation as follows:

- F_1 is a finite sum of terms of the form (up to rapidly vanishing terms which may be included in F_6)

$$\rho^{m-k/2+n/2}\int_{\mathbb{R}^k} e^{i\Phi(y,y',x/x',v)/\rho} a(\lambda,\rho,y,y',\sigma,v)\,dv\,\boldsymbol{\omega}, \tag{5-5}$$

where Φ locally parametrizes Λ, $\boldsymbol{\omega}$ is a nonzero section of the half-density bundle $\Omega^{1/2}_{k,b}$, compactly supported in v, and

$$a \text{ is polyhomogeneous conormal in } \lambda \text{ with index set } \mathcal{B}_{\mathrm{bf}_0} \text{ and smooth in all other variables.} \tag{5-6}$$

- F_2 is a finite sum of terms of the form (up to rapidly vanishing terms which may be included in F_6)

$$\sigma^{r_{\mathrm{lb}}-k/2}\rho'^{m-(k+k')/2+n/2}\int_{\mathbb{R}^{k+k'}} e^{i\Phi_1(y,v)/\rho}e^{i\Phi_2(y,y',\sigma,v,w)/\rho'} a(\lambda,\rho',y,y',\sigma,v,w)\,dv\,dw\,\boldsymbol{\omega}, \tag{5-7}$$

where $\Phi=\Phi_1+\sigma\Phi_2$ locally parametrizes Λ (in particular, Φ_1 locally parametrizes Λ_{lb}), and a satisfies (5-6).

- F_3 is a finite sum of terms of the form (up to rapidly vanishing terms which may be included in F_6)

$$\rho^{m-(k+k')/2+n/2}\tilde{\sigma}^{r_{\mathrm{rb}}-k/2}\int_{\mathbb{R}^{k+k'}} e^{i\Phi'_1(y',v)/\rho'}e^{i\Phi'_2(y,y',\tilde{\sigma},v,w)/\rho} a(\lambda,\rho,y,y',\tilde{\sigma},v,w)\,dv\,dw\,\boldsymbol{\omega}, \tag{5-8}$$

where $\tilde{\sigma}=\rho'/\rho=\sigma^{-1}$ and $\Phi=\Phi'_1+\tilde{\sigma}\Phi'_2$ locally parametrizes Λ (in particular, Φ'_1 locally parametrizes Λ_{rb}), and a satisfies (5-6).

- F_4 is a finite sum of terms of the form

$$\rho^{r_{\mathrm{lb}}-k/2}\int_{\mathbb{R}^k} e^{i\Phi_1(y,v)/\rho} a(\lambda,\rho,y,z',v)\,dv\,\boldsymbol{\omega}, \tag{5-9}$$

where Φ parametrizes Λ_{lb} and a is polyhomogeneous at bf_0 and lb_0 with index sets $\mathcal{B}_{\mathrm{bf}_0}$, $\mathcal{B}_{\mathrm{lb}_0}$.

- F_5 is a finite sum of terms

$$(\rho')^{r_{\mathrm{rb}}-k/2}\int_{\mathbb{R}^k} e^{i\Phi'_1(y',v')/\rho'} a(\lambda,\rho',y',z,v)\,dv\,\boldsymbol{\omega}, \tag{5-10}$$

where Φ' parametrizes Λ_{rb} and a is polyhomogeneous at bf_0 and rb_0 with index sets $\mathcal{B}_{\mathrm{bf}_0}$, $\mathcal{B}_{\mathrm{rb}_0}$.

Then we define the microlocal support $\mathrm{WF}'(F)$ of F to be a closed subset of $\overline{\Lambda}'\cup\overline{\Lambda}_{\mathrm{lb}}\cup\overline{\Lambda}'_{\mathrm{rb}}$ as follows: We say that $(q',\lambda)\in\overline{\Lambda}'$ is not in $\mathrm{WF}'(F)$ if there is a neighborhood of $(q,\lambda)\in\Lambda\times[0,\lambda_0]$ in which F has order ∞. In terms of the oscillatory integral representation (5-5), say, the condition that F has order infinity at (q,λ) is equivalent to a vanishing rapidly in a neighborhood of the point $(\lambda,0,y,y',\sigma,v)$ which corresponds under (5-3) to (q,λ) in the sense that $d_{y,y',\sigma,\rho}(\Phi(y,y',x/x',v)/\rho)=q$ and $d_v\Phi(y,y',x/x',v)=0$ (by nondegeneracy there is only one v with this property). Similar considerations apply to (5-7) and (5-8). Likewise, we say that $(q,\lambda)\in\overline{\Lambda}_{\mathrm{lb}}$ is not in $\mathrm{WF}'(F)$ if there is a neighborhood of the *fiber* (see (4-4)) of $(q,\lambda)\in\Lambda_{\mathrm{lb}}\times[0,\lambda_0]$ in which F has order ∞, and $(q',\lambda)\in\overline{\Lambda}'_{\mathrm{rb}}$ is not in $\mathrm{WF}'(F)$ if there is a neighborhood of the *fiber* of $(q,\lambda)\in\Lambda_{\mathrm{rb}}\times[0,\lambda_0]$ in which F has order ∞. The fiber here is a copy of M. In terms of the oscillatory integral representation (5-7), the condition that F has order infinity in a neighborhood of the fiber of $(q,\lambda)=(y,\mu,\nu,\lambda)\in\overline{\Lambda}_{\mathrm{lb}}$ is equivalent to a vanishing rapidly in a neighborhood of the point $(\lambda, \rho', y, y', 0, v, w)$ for all (ρ', y', v, w) such that $d_{y,\rho}(\Phi_1/\rho) = q$ and $d_v \Phi_1 = 0$. Similarly, in (5-9) the condition is that a vanishes rapidly in a neighborhood of the point $(\lambda, 0, y, z', v)$ for all (z', v) such that $d_{y,\rho}(\Phi_1/\rho) = q$ and $d_v \Phi_1 = 0$.

These components of $\mathrm{WF}'(F)$ will be denoted by $\mathrm{WF}'_{\mathrm{bf}}(F)$, $\mathrm{WF}'_{\mathrm{lb}}(F)$ and $\mathrm{WF}'_{\mathrm{rb}}(F)$, respectively.

Note that if $F \in I^{m,r_{\mathrm{lb}},r_{\mathrm{rb}},\mathcal{B}}(\Lambda)$, then F is rapidly decreasing at bf, lb and rb if and only if $\mathrm{WF}'(F)$ is empty. Also note that if $\mathrm{WF}'_{\mathrm{lb}}(F)$ is empty, then $\partial_{\mathrm{lb}}\Lambda \times [0, \lambda_0]$ is disjoint from $\mathrm{WF}'_{\mathrm{bf}}(F)$, but the converse need not hold: if the kernel of F is supported away from bf then certainly $\mathrm{WF}'_{\mathrm{bf}}(F)$ will be empty, but $\mathrm{WF}'_{\mathrm{lb}}(F)$ need not be.

This definition makes sense also for pseudodifferential operators Q of order $-\infty$, with compact operator wavefront set. In the case of a pseudodifferential operator, the Legendre submanifold is ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$, defined in (4-7), and the components $\Lambda_{\mathrm{lb}} \cup \Lambda'_{\mathrm{rb}}$ are empty. Since ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$ is canonically diffeomorphic to ${}^{\mathrm{sc}}T^*{}_{\partial M}M$, we will always consider the microlocal support $\mathrm{WF}'(Q)$ of a pseudodifferential operator Q of differential order $-\infty$ to be a subset of ${}^{\mathrm{sc}}T^*{}_{\partial M}M \times [0, \lambda_0]$.

Lemma 5.2. *Assume that $F \in I^{m,r_{\mathrm{lb}},r_{\mathrm{rb}};\mathcal{B}}(M^2_{k,b}, \Lambda; \Omega^{1/2}_{k,b})$ is associated to a compact Legendre submanifold Λ and that $Q \in \Psi_k^{-\infty}(M; \Omega^{1/2}_{k,b})$ is of differential order $-\infty$, with compact operator wavefront set. Then QF is also a Legendre distribution in the space $I^{m,r_{\mathrm{lb}},r_{\mathrm{rb}};\mathcal{B}}(M^2_{k,b}, \Lambda; \Omega^{1/2}_{k,b})$ and we have*

$$\begin{aligned} &\mathrm{WF}'_{\mathrm{lb}}(QF) \subset \mathrm{WF}'(Q) \cap \mathrm{WF}'_{\mathrm{lb}}(F), \\ &\mathrm{WF}'_{\mathrm{bf}}(QF) \subset \pi_L^{-1}\,\mathrm{WF}'(Q) \cap \mathrm{WF}'_{\mathrm{bf}}(F), \\ &\mathrm{WF}'_{\mathrm{rb}}(QF) \subset \mathrm{WF}'_{\mathrm{rb}}(F), \end{aligned} \tag{5-11}$$

where π_L, π_R are as in (4-6). Moreover, if Q is microlocally equal to the identity on $\pi_L(\mathrm{WF}'_{\mathrm{bf}}(F))$ and $\mathrm{WF}'_{\mathrm{lb}}(F)$, then $QF - F \in I^{\infty,\infty,r_{\mathrm{rb}};\mathcal{B}}(M^2_{k,b}, \Lambda; \Omega^{1/2}_{k,b})$, i.e., it vanishes to infinite order at lb *and* bf.

There is of course a corresponding theorem for composition in the other order, which is obtained by taking the adjoint of the lemma above. Combining the two we obtain:

Corollary 5.3. *Suppose that F and Q, Q' are as above. Then*

$$\begin{aligned} &\mathrm{WF}'_{\mathrm{lb}}(QFQ') \subset \mathrm{WF}'(Q) \cap \mathrm{WF}'_{\mathrm{lb}}(F), \\ &\mathrm{WF}'_{\mathrm{bf}}(QFQ') \subset \pi_L^{-1}\,\mathrm{WF}'(Q) \cap \pi_R^{-1}\,\mathrm{WF}'(Q') \cap \mathrm{WF}'_{\mathrm{bf}}(F), \\ &\mathrm{WF}'_{\mathrm{rb}}(QFQ') \subset \mathrm{WF}'(Q') \cap \mathrm{WF}'_{\mathrm{rb}}(F). \end{aligned} \tag{5-12}$$

Proof of Lemma 5.2. We decompose as above $F = F_1 + F_2 + F_3 + F_4 + F_5 + F_6$, and consider each piece F_i separately.

- F_1 term. Using the notation in (5-5), the composition QF_1 takes the form

$$(2\pi)^{-n} \int_0^\infty \int e^{i((y-y'')\cdot\mu + (1-\rho/\rho'')\nu)/\rho} q(\lambda, \rho, y, \mu, \nu) \\ \times (\rho'')^{m-k/2+n/2} e^{i\Phi(y'',y',\rho'/\rho'',v)/\rho''} a(\lambda, \rho', y'', y', \rho'/\rho'', v)\, dv\, d\mu\, d\nu \,\frac{dy''\, d\rho''}{\rho''^{n+1}}\, \boldsymbol{\omega}. \tag{5-13}$$

Here the measure $\lambda^n dg''$, which arises from the combination of half-densities in Q and F, is equal to $dy''d\rho''/\rho''^{n+1}$ times a smooth nonzero factor, which has been absorbed into the a term. Writing $\sigma'' = \rho/\rho''$, this can be expressed as

$$(2\pi)^{-n}\rho^{m-k/2-n+n/2}\int e^{i((y-y'')\cdot\mu+(1-\sigma'')\nu+\sigma''\Phi(y'',y',\sigma''/\sigma,v))/\rho} q(\lambda,\rho,y,\mu,\nu)(\sigma'')^{m-k/2+n/2-n-1} \times a(\lambda,\rho',y'',y',\sigma''\sigma^{-1},v)\,dv\,d\mu\,d\nu\,dy''\,d\sigma''\,\boldsymbol{\omega}.$$

For $\rho \ge \epsilon > 0$ the phase is not oscillating and this is polyhomogeneous conormal at bf_0 with the same index set $\mathcal{B}_{\mathrm{bf}_0}$ as for a. For ρ small, we perform stationary phase in the (y'',σ'',μ,ν) variables. The phase has a nondegenerate stationary point where $y''=y$, $\sigma''=1$, $\mu=d_y\Phi$, $\nu=\Phi+\sigma^{-1}d_\sigma\Phi$, and we obtain an asymptotic expansion as $\rho\to 0$ of the form

$$\rho^{m-k/2+n/2}\int_{\mathbb{R}^k} e^{i\Phi(y,y',\sigma,v)/\rho}\tilde{a}(\lambda,\rho,y,y',\sigma,v)\,dv\,\boldsymbol{\omega}, \tag{5-14}$$

where

$$\begin{aligned}\tilde{a}(\lambda,\rho,y,y',\sigma,v) &= \lambda^{-\frac{n}{2}}\sum_{j=0}^{M}\rho^j\left(\frac{(\partial_{y''}\cdot\partial_\mu+\partial_{\sigma''}\partial_\nu)^j}{i^j j!}q(\lambda,\rho,y,\mu,\nu)(\sigma'')^{m-\frac{k}{2}+\frac{n}{2}-n-1}a(\lambda,\rho',y'',y',\sigma''/\sigma,v)\right)g\Bigg|_{\substack{y=y'',\sigma''=1\\ \mu=d_y\Phi\\ \nu=\Phi+\sigma^{-1}d_\sigma\Phi}} \\ &\quad + O(\rho^{M+1}).\end{aligned} \tag{5-15}$$

In particular, this is a Legendre distribution associated to Λ of the same order, and with the same index family, as F. Moreover, we see from (5-14) and (5-15) that the microlocal support $\mathrm{WF}'_{\mathrm{bf}}(QF_1)$ is contained in $\mathrm{WF}'_{\mathrm{bf}}(F)$, as well as contained in $\pi_L^{-1}\,\mathrm{WF}'(Q)$.

If $q = 1 + O(\rho^\infty)$ on $\pi_L(\mathrm{WF}'_{\mathrm{bf}}(F))$, then in the sum over j in (5-15), only the $j=0$ term is nonzero, because in all other terms, either $a=0$ or $q=1+O(\rho^\infty)$ (implying that any derivative of q is $O(\rho^\infty)$) when evaluated at $y=y''$, $\sigma''=1$, $\mu=d_y\Phi$, $\nu=\Phi+\sigma d_\sigma\Phi$. Therefore, in this case, $QF_1 = F_1 \bmod O(\rho^\infty)$.

- F_2 term. In the notation (5-7), the composition QF_2 takes the form

$$(2\pi)^{-n}\int e^{i((y-y'')\cdot\mu+(1-\sigma'')\nu)/\rho}q(\lambda,\rho,y,\mu,\nu)\rho''^{r_{\mathrm{lb}}-k/2}\rho'^{m-r_{\mathrm{lb}}-k'/2+n/2}e^{i\Phi_1(y,v)/\rho''}e^{i\Phi_2(y'',y',\sigma''/\sigma,v,w)/\rho'} \times a(\lambda,\rho',y'',y',\sigma/\sigma'',v,w)\,dv\,dw\,d\mu\,d\nu\,\frac{dy''\,d\rho''}{\rho''^{n+1}}\,\boldsymbol{\omega}.$$

This can be written as

$$\begin{aligned}&(2\pi)^{-n}\rho^{r_{\mathrm{lb}}-k/2-n}\rho'^{m-r_{\mathrm{lb}}-k'/2+n/2}\\ &\quad\times\int e^{i((y-y'')\cdot\mu+(1-\sigma'')\nu+\sigma''\Phi_1(y'',v)+\sigma\Phi_2(y'',y',\sigma/\sigma'',v,w))/\rho}\\ &\qquad\times q(\lambda,\rho,y,\mu,\nu)(\sigma'')^{-r_{\mathrm{lb}}+k/2+n-1}a(\lambda,\rho',y'',y',\sigma/\sigma'',v,w)\,dv\,dw\,d\mu\,d\nu\,dy''\,d\sigma''\,\boldsymbol{\omega}.\end{aligned}$$

Now we perform stationary phase in the (y'',σ'',μ,ν)-variables. The phase has a nondegenerate stationary point where $y''=y$, $\sigma''=1$, $\mu=d_y\Phi_1$, $\nu=\Phi_1-d_\sigma\Phi$, and the rest of the argument to bound $\mathrm{WF}'_{\mathrm{bf}}(QF)$

is the same as for F_1. We also see from the stationary phase expansion that $\mathrm{WF}'_{\mathrm{lb}}(QF)$ is contained in both $\mathrm{WF}'(Q)$ and $\mathrm{WF}'_{\mathrm{lb}}(F)$.

- F_4 term. This works just as for the F_2 term.
- F_3 term. In the notation (5-8), the composition QF_3 takes the form

$$(2\pi)^{-n}\int e^{i((y-y'')\cdot\mu+(1-\sigma'')\nu)/\rho}q(\lambda,\rho,y,\mu,\nu)(\rho'')^{m-(k+k')/2+2n/4}(\tilde\sigma\sigma'')^{r_{\mathrm{rb}}-k/2} \\ \times\int e^{i\Phi_1'(y',v)/\rho'}e^{i\Phi_2'(y',y'',\tilde\sigma\sigma'',v,w)/\rho''}a(\lambda,\rho'',y'',y',\tilde\sigma\sigma'',v,w)\,dv\,dw\,d\mu\,d\nu\,\frac{dy''\,d\rho''}{(\rho'')^{n+1}}\,\boldsymbol{\omega}.$$

This can be written as

$$(2\pi)^{-n}\int e^{i((y-y'')\cdot\mu+(1-\sigma'')\nu+\sigma''\Phi_2'(y',y'',\tilde\sigma\sigma'',v,w))/\rho}q(\lambda,\rho,y,\mu,\nu)(\rho/\sigma'')^{m-(k+k')/2} \\ \times(\tilde\sigma\sigma'')^{r_{\mathrm{rb}}-k/2}e^{i\Phi_1'(y',v)/\rho'}a(\lambda,\rho/\sigma'',y'',y',\tilde\sigma\sigma'',v,w)\,dv\,dw\,d\mu\,d\nu\,\frac{dy''\,d\sigma''}{\sigma''}\,\boldsymbol{\omega}.$$

To investigate the behavior of this integral locally near a point $(x=0,\tilde\sigma=0,y,y')\in\mathrm{bf}\cap\mathrm{rb}$, we perform stationary phase in the (y'',σ'',μ,ν)-variables. The phase has a nondegenerate stationary point where $y''=y$, $\sigma''=1$, $\mu=d_y\Phi_2'$, $\nu=\Phi_2'+\tilde\sigma d_{\tilde\sigma}\Phi_2'$, and we get an asymptotic expansion as $\rho\to 0$ of the form

$$\rho^{m-(k+k')/2+2n/4}\tilde\sigma^{r_{\mathrm{rb}}-k/2}\int e^{i\Phi_1'(y',v)/\rho'}e^{i\Phi_2'(y,y',\tilde\sigma,v,w)/\rho}\tilde a(\lambda,\rho,y,y',\tilde\sigma,v,w)\,dv\,dw\,\boldsymbol{\omega},$$

where $\tilde a(\lambda,\rho,y,y',\tilde\sigma,v,w)$ is given by

$$\sum_{j=0}^{M}\rho^j\biggl(\frac{(-i(\partial_{y''}\cdot\partial_\mu+\partial_{\sigma''}\partial_\nu))^j}{j!}q(\lambda,\rho,y,\mu,\nu) \\ \times(\sigma'')^{-m+r_{\mathrm{rb}}+k'/2}a(\lambda,\rho'',y'',y',\tilde\sigma\sigma'',v,w)\biggr)g\bigg|_{\substack{y=y'',\sigma''=1\\ \mu=d_y\Phi_1'\\ \nu=\Phi_2'+\tilde\sigma d_{\tilde\sigma}\Phi_2'}}+O(\rho^{M+1}).\quad(5\text{-}16)$$

This is a Legendre distribution associated to Λ of the same order as F, and with the same index family. Moreover, we see from the last two formulas that the microlocal support $\mathrm{WF}'_{\mathrm{bf}}(QF_3)$ is contained in $\mathrm{WF}'_{\mathrm{bf}}(F)$, as well as contained in $\pi_L^{-1}\,\mathrm{WF}'(Q)$. Finally, if $q=1+O(\rho^\infty)$ on $\pi_L(\mathrm{WF}'_{\mathrm{bf}}(F))$, then in the sum over j in (5-16), only the $j=0$ term is nonzero, because in all other terms, either $a=0$ or $q=1+O(\rho^\infty)$ (implying that any derivative of q is $O(\rho^\infty)$) when evaluated at $y=y''$, $\sigma''=1$, $\mu=d_y\Phi_2'$, $\nu=\Phi_2'+\sigma d_\sigma\Phi_2'$. Therefore, in this case, $QF_3=F_3 \bmod O(x^\infty)$.

- F_5 term. Writing F_5 in the form (5-10), we investigate QF_5 near a point (z,ρ',y'), where $z\in M^\circ$. In this case, we can find a neighborhood W of z with $\overline{W}\subset M^\circ$, and then the set

$$\{(z,z')\in\operatorname{supp}Q\mid z\in W\}$$

is contained in $W\times W'$ for some W' with $\overline{W}'\subset M^\circ$, since the support of Q is contained in the set where $\sigma\in[1/2,2]$. But in $W\times W'$, the kernel of Q is smooth since Q has differential order $-\infty$. Therefore, in this region the composition is given by an integral

$$\int Q(z,z'')(\rho')^{r_{\mathrm{rb}}-k/2}\int e^{i\Phi_1(y',v)/\rho'}a(\lambda,z'',y',\rho',v)\,dv\,dz''\,\boldsymbol{\omega},$$

with $Q(z,z'')$ smooth, and this has the form

$$(\rho')^{r_{\mathrm{rb}}-k/2}\int e^{i\Phi_1(y',v)/\rho'}\tilde{a}(\lambda,z,y',\rho',v)\,dv\,\boldsymbol{\omega}$$

for some $\tilde{a}$ depending polyhomogeneously on λ and smoothly in its other arguments. Moreover, if for a fixed (λ, y', v), a is $O((\rho')^\infty)$ in a neighborhood of $\{(\lambda, z, y', 0, v) \mid z \in M\}$, then the same is true of $\tilde{a}$. Therefore, $\mathrm{WF}'_{\mathrm{rb}}(QF_5)$ is contained in $\mathrm{WF}'_{\mathrm{rb}}(F_5)$ but is (in general) no smaller.

- Since $\mathrm{WF}'(F_6) = \mathrm{WF}'(QF_6) = \varnothing$, the F_6 term makes no contribution to the wavefront set.

This completes the proof. □

A similar result holds if F is associated to a Legendre conic pair rather than a single Legendre submanifold. However, rather than giving a full analogue of the result above, we give the following special cases which suffice for our needs.

Lemma 5.4. (i) *Suppose that* $F \in I^{m,p;r_{\mathrm{lb}},r_{\mathrm{rb}};\mathcal{B}}(M^2_{k,b}, (\Lambda, \Lambda^\sharp); \Omega^{1/2}_{k,b})$ *is a Legendre distribution on* $M^2_{k,b}$ *associated to a conic Legendrian pair* $(\Lambda, \Lambda^\sharp)$, *and suppose that* $Q \in \Psi_k^{-\infty}(M; \Omega^{1/2}_{k,b})$ *is a scattering pseudodifferential operator such that* Q *is microlocally equal to the identity operator near* $\pi_L(\Lambda \cup \Lambda^\sharp)$. *Then* $QF - F \in I^{\infty,\infty;\infty,r_{\mathrm{rb}};\mathcal{B}}(M^2_{k,b}, (\Lambda, \Lambda^\sharp); \Omega^{1/2}_{k,b})$, *so it vanishes to infinite order at* lb *and* bf. *Similarly, if* Q *is microlocally equal to the identity operator near* $\pi_R(\Lambda \cup \Lambda^\sharp)$, *then* $FQ - F \in I^{\infty,\infty;r_{\mathrm{lb}},\infty;\mathcal{B}}(M^2_{k,b}, (\Lambda, \Lambda^\sharp); \Omega^{1/2}_{k,b})$ *vanishes to infinite order at* bf *and* rb.

(ii) *Suppose that* F *is as above, and that* Q, Q' *are scattering pseudodifferential operators as above. If*

$$\pi_L^{-1}\,\mathrm{WF}'(Q) \cap \pi_R^{-1}\,\mathrm{WF}'(Q') \cap \Lambda^\sharp = \varnothing, \tag{5-17}$$

then $QFQ' \in I^{m,r_{\mathrm{lb}},r_{\mathrm{rb}};\mathcal{B}}(M^2_{k,b}, \Lambda; \Omega^{1/2}_{k,b})$; *in particular,* $\mathrm{WF}'_{\mathrm{bf}}(QFQ')$ *is disjoint from* $(\Lambda^\sharp)'$.

Proof. The proof of (i) is similar to the one above. To prove (ii), decompose $F = F_\Lambda + F_\sharp$, where $F_\Lambda \in I^{m,r}(M^2_{k,b}, \Lambda; \Omega^{1/2}_{k,b})$ is a Legendre distribution associated only to Λ and $F_\sharp$ is localized sufficiently close to $\Lambda^\sharp$. Here, sufficiently close means that when we write down $QF_\sharp Q'$ as a (sum of) integral(s), using a phase function that locally parametrizes of $(\Lambda, \Lambda^\sharp)$, then (5-17) implies that the total phase is nonstationary on the support of the integrand. The usual integration-by-parts argument then shows that this kernel is rapidly decreasing at bf, lb, rb and hence trivially satisfies the conclusion of the lemma. On the other hand, Lemma 5.2 applies to F_Λ and completes the proof. □

6. Low energy estimates on the spectral measure

6A. ***Pointwise bounds on Legendre distributions.*** Now we give a pointwise estimate on Legendre distributions of a particular type. We begin with a trivial estimate.

Proposition 6.1. *Let $\Lambda \subset {}^{\mathrm{sc}}T^*_{\mathrm{bf}}(M_b^2)$ be a Legendre submanifold that projects diffeomorphically to* bf. *Suppose that $u \in I^{-n/2-\alpha,-\alpha,-\alpha;\mathcal{B}}(M^2_{k,b}, \Lambda; \Omega^{1/2}_{k,b})$. Let*

$$b = \min(\min \mathcal{B}_{\mathrm{bf}_0} + n,\ \min \mathcal{B}_{\mathrm{lb}_0} + n/2,\ \min \mathcal{B}_{\mathrm{rb}_0} + n/2,\ \min \mathcal{B}_{\mathrm{zf}}). \tag{6-1}$$

Then, as a multiple of the half-density $|dg\, dg'\, d\lambda/\lambda|^{1/2}$, we have a pointwise estimate

$$|u| \le C\lambda^b(\rho^{-1} + (\rho')^{-1})^\alpha.$$

This is trivial since in this case, u may be written as an oscillatory function with no integration, and the order of vanishing/growth at the boundary may be determined by inspection from (5-5)–(5-10). (The discrepancies of n and $n/2$ in (6-1) come about from comparing the nonvanishing half-density $\boldsymbol{\omega}$ on $M^2_{k,b}$ with the metric half-density $|dg\, dg'\, d\lambda/\lambda|^{1/2} = \rho_{\mathrm{lb}_0}^{-n/2} \rho_{\mathrm{rb}_0}^{-n/2} \rho_{\mathrm{bf}_0}^{-n}\, \boldsymbol{\omega}$.)

Now consider a situation in which the Legendre submanifold does not project diffeomorphically to bf. Let $\partial\mathrm{diag}_b$ denote the boundary of the diagonal in M_b^2, as in (4-8). Recall that we have coordinates (y, y', σ) on bf near $\partial\mathrm{diag}_b$. Let $w = (y - y', \sigma - 1)$, and let κ be the corresponding scattering coordinates dual to w. Then $\partial\mathrm{diag}_b$ is given by $\{w = 0\}$ as a submanifold of bf and the contact form on ${}^{\mathrm{sc}}T^*_{\mathrm{bf}} M_b^2$ takes the form

$$d\nu - \mu \cdot dy - \kappa \cdot dw. \tag{6-2}$$

In these coordinates, the Legendre submanifold ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$ is given by $\{w = 0, \mu = 0, \nu = 0\}$. Let Λ^{bf} be a Legendre submanifold contained in ${}^{\mathrm{sc}}T^*_{\mathrm{bf}} M_b^2$, denote by π the natural projection from ${}^{\mathrm{sc}}T^*_{\mathrm{bf}} M_b^2 \to \mathrm{bf}$, and for any $q \in \Lambda^{\mathrm{bf}}$ denote by $d\pi$ the induced map from $T_q\Lambda^{\mathrm{bf}} \to T_{\pi(q)}\mathrm{bf}$. We consider the following situation in which the rank of $d\pi$ is allowed to change.

Proposition 6.2. *Let Λ^{bf} be as above. Suppose that Λ^{bf} intersects ${}^{\mathrm{sc}}N^*\partial diag_b$ at $G^{\mathrm{bf}} = \Lambda^{\mathrm{bf}} \cap {}^{\mathrm{sc}}N^*\partial diag_b$ which is of codimension* 1 *in Λ^{bf}, and suppose that $\pi|_{G^{\mathrm{bf}}}$ is a fibration, with $(n-1)$-dimensional fibers, to $\partial diag_b$. Assume further that $d\pi$ has full rank on $\Lambda^{\mathrm{bf}} \setminus G^{\mathrm{bf}}$, while*

$$\det d\pi \text{ vanishes to order exactly } n-1 \text{ at } G^{\mathrm{bf}}. \tag{6-3}$$

Suppose $u \in I^{-n/2-\alpha,-\alpha,-\alpha;\mathcal{B}}(M^2_{k,b}, \Lambda^{\mathrm{bf}}; \Omega^{1/2}_{k,b})$, and suppose that the (full) symbol of u vanishes to order $(n-1)/2 + \alpha$ on $G^{\mathrm{bf}} \times [0, \lambda_0]$, where $(n-1)/2 + \alpha \in \{0, 1, 2, \dots\}$. Then as a multiple of the scattering half-density $|dg\, dg'\, d\lambda/\lambda|^{1/2}$, we have a pointwise estimate

$$|u| \le C\lambda^b\Big(1 + \frac{|w|}{\rho}\Big)^\alpha \sim C\lambda^b(1 + \lambda d(z, z'))^\alpha, \tag{6-4}$$

with b as in (6-1). Here $d(z, z')$ is the Riemannian distance between $z, z' \in M^\circ$.

Remark 6.3. Notice that the condition on π at G^{bf} implies that $d\pi$ has corank at least $n-1$ on G^{bf}, hence that $\det d\pi$ must vanish to order at *least* $n-1$ there. Condition (6-3) is therefore that the order of vanishing at G^{bf} is the least possible, which is a nondegeneracy assumption concerning the manner in which the rank of the projection changes at G^{bf}. It implies, in particular, that Λ^{bf} intersects ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$ cleanly.

Proof. Let q be an arbitrary point in G^{bf}. By rotating in the w variables, we can ensure that $d\kappa_1|_{G^{\mathrm{bf}}}$ vanishes at q (since $\kappa_1, \dots, \kappa_n$ are coordinates on the fibers of ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b \to \partial\mathrm{diag}_b$, and since $\pi|_{G^{\mathrm{bf}}} : G^{\mathrm{bf}} \to \partial\mathrm{diag}_b$ has $(n-1)$-dimensional fibers). We claim that $(y, w_1, \kappa_2, \dots, \kappa_n)$ furnish coordinates on Λ^{bf} locally near q. To see this, first note that $d\kappa_2|_{G^{\mathrm{bf}}}, \dots, d\kappa_n|_{G^{\mathrm{bf}}}$ are linearly independent at q, and furnish coordinates on the fibers of $G^{\mathrm{bf}} \to \partial\mathrm{diag}_b$. Next, since $\partial\mathrm{diag}_b$ is $(n-1)$-dimensional, G^{bf} is $2(n-1)$-dimensional, and the fibers of $G^{\mathrm{bf}} \to \partial\mathrm{diag}_b$ are $(n-1)$-dimensional, it follows that $G^{\mathrm{bf}} \to \partial\mathrm{diag}_b$ is a submersion. Since y_i are local coordinates on the base $\partial\mathrm{diag}_b$, we see that $(y, \kappa_2, \dots, \kappa_n)$ furnish coordinates on G^{bf} locally near q. Since $w_1 = 0$ on G^{bf}, to prove the claim it suffices to show that $dw_1|_{\Lambda^{\mathrm{bf}}} \neq 0$ at q.

To see this, we use (6-3) which implies that $d\pi$ has corank exactly $n-1$ at q, and hence there is a tangent vector $V \in T_q\Lambda^{\mathrm{bf}}$ such that $d\pi(V)$ is not tangent to $\partial\mathrm{diag}_b$. Therefore, it has a nonzero ∂_{w_j} component, which means that some dw_j does not vanish at q when restricted to Λ^{bf}. But since Λ^{bf} is Legendrian, the form (6-2) vanishes when restricted to Λ^{bf}, which implies that its differential $\omega \equiv d\mu \cdot dy + d\kappa \cdot dw$ also vanishes on Λ^{bf}. Hence $\omega(\partial_{\kappa_j}, V) = 0$ at q, $j \geq 2$, since ∂_{κ_j} and V are both tangent to Λ^{bf}. But this implies that $dw_j(V) = 0$ for $j \geq 2$, i.e., V has no ∂_{w_j} component for $j \geq 2$. It follows that $dw_1(V) \neq 0$, showing that $dw_1|_{\Lambda^{\mathrm{bf}}} \neq 0$ at q. It follows that $(y, w_1, \kappa_2, \dots, \kappa_n)$ indeed furnish coordinates on Λ^{bf} locally near q. We will use the notation $\overline{w} = (w_2, \dots, w_n)$ and $\bar\kappa = (\kappa_2, \dots, \kappa_n)$. Notice that $w_1|_{\Lambda^{\mathrm{bf}}}$ is a boundary defining function for G^{bf}, as a submanifold of Λ^{bf}, locally near q.

Now we write the other coordinates on Λ^{bf} as functions of $(y, w_1, \bar\kappa)$ as follows:

$$\overline{w}_i = W_i(y, w_1, \bar\kappa), \quad \mu_i = M_i(y, w_1, \bar\kappa), \quad \kappa_1 = K(y, w_1, \bar\kappa), \quad \nu = N(y, w_1, \bar\kappa) \quad \text{on } \Lambda^{\mathrm{bf}}. \tag{6-5}$$

Notice that the vanishing of (6-2) on Λ^{bf} implies that

$$dN = \sum_{i=1}^{n-1} M_i dy_i + K dw_1 + \sum_{j=2}^{n} \kappa_j dW_j \quad \text{on } \Lambda^{\mathrm{bf}}. \tag{6-6}$$

By equating the coefficients of $d\bar\kappa$, dy and dw_1 on each side of (6-6), we obtain the identities

$$\begin{aligned}
\sum_{j=2}^{n} v_j \frac{\partial W_j(y, w_1, v)}{\partial v_i} &= \frac{\partial N(y, w_1, v)}{\partial v_i}, \quad i = 2, \dots, n, \\
\sum_{j=2}^{n} v_j \frac{\partial W_j(y, w_1, v)}{\partial y_i} + M_i(y, w_1, v) &= \frac{\partial N(y, w_1, v)}{\partial y_i}, \quad i = 1, \dots, n-1, \\
\sum_{j=2}^{n} v_j \frac{\partial W_j(y, w_1, v)}{\partial w_1} + K(y, w_1, v) &= \frac{\partial N(y, w_1, v)}{\partial w_1}.
\end{aligned} \tag{6-7}$$

We claim that the function

$$\Phi(y, w_1, \overline{w}, v) = \sum_{j=2}^{n} (\overline{w}_j - W_j(y, w_1, v))v_j + N(y, w_1, v) \tag{6-8}$$

parametrizes Λ^{bf} locally near q. Notice that W, M and N are all $O(w_1)$ at q. Hence, $\Phi = \overline{w} \cdot v + O(w_1)$, so the $d_{v_j}\Phi = \overline{w}_j + O(w_1)$, where $2 \leq j \leq n$, have linearly independent differentials at the point

$\tilde{q}=(y(q),\, w=0,\, \nu=0,\, \mu=0,\, \kappa_1=0,\, \bar{\kappa}(q))$ corresponding to q, i.e., Φ is a nondegenerate parametrization of Λ^{bf} near q. Next, using the first equation in (6-7) we find that

$$d_{v_j}\Phi = \overline{w}_j - W_j(y, w_1, v). \tag{6-9}$$

So $\overline{w} = W$ when $d_v\Phi = 0$. The Legendrian submanifold parametrized is then given by (using (6-7))

$$\left\{\left(y, w_1, W, -v\cdot\frac{\partial W}{\partial y}+\frac{\partial N}{\partial y}, -v\cdot\frac{\partial W}{\partial w_1}+\frac{\partial N}{\partial w_1}, v, N\right)\right\} = \{(y, w_1, W, M, K, v, N)\} = \Lambda^{\mathrm{bf}}. \tag{6-10}$$

Notice that the second derivative matrix $d^2_{vv}\Phi$ vanishes at $w_1=0$, so we can write $d^2_{vv}\Phi = w_1 A + O(w_1^2)$, where A is a smooth $(n-1)\times(n-1)$ matrix function of $(\bar{y}, v)$, where we write $\bar{y} = (y, w_1, \overline{w})$. We claim that A is invertible at (and therefore, near) $\tilde{q}$. To see this, we start from the fact that the map

$$\{(\bar{y}, v)\} \to \{(\bar{y}, d_{\bar{y}}\Phi, \Phi, d_v\Phi)\}$$

is locally a diffeomorphism onto its image. (This follows from the nondegeneracy condition on Φ, that the differentials $d(\partial\Phi/\partial v_j)$ are linearly independent.) Note that the determinant of the differential of the map

$$\{(\bar{y}, d_{\bar{y}}\Phi, \Phi, d_v\Phi)\} \to \{(\bar{y}, d_v\Phi)\}$$

is equal to the determinant of the differential of the map

$$\{(\bar{y}, d_{\bar{y}}\Phi, \Phi, d_v\Phi) \mid d_v\Phi = 0\} \to \bar{y},$$

and this map is $\pi|_{\Lambda^{\mathrm{bf}}}$ (in local coordinates). It follows that the order of vanishing of $\det d\pi$ at q is the same as the order of vanishing of the determinant of the differential of the map

$$\{(\bar{y}, v)\} \to \{(\bar{y}, d_v\Phi)\}$$

at $\tilde{q}$. But this determinant is simply $\det d^2_{vv}\Phi$. It follows from (6-3) that $\det d^2_{vv}\Phi$ vanishes to order exactly $n-1$ at $\tilde{q}$. But this implies that the matrix A is invertible at $\tilde{q}$, as claimed.

Now we write u as an oscillatory integral. It suffices to prove the proposition assuming that u has symbol supported close to q and that u itself is supported close to $\partial\mathrm{diag}_b$, since away from $\partial\mathrm{diag}_b$ the result follows from Proposition 6.1. It can then be written with respect to the phase function Φ: modulo a smooth term vanishing to order $O(\rho^\infty)$, u is a multiple of the scattering half-density $|dg\, dg'\, d\lambda/\lambda|^{1/2}$ given by

$$\rho^{-(n-1)/2-\alpha}\lambda^n \int e^{i\Phi(y,w,v)/\rho} a(\lambda, \rho, y, v)\, dv |dg\, dg'\, d\lambda/\lambda|^{1/2}. \tag{6-11}$$

Moreover, we may assume that a is a function only of λ, ρ, y, w_1 and v, polyhomogeneous conormal in λ with index set $\mathcal{B}_{\mathrm{bf}_0}$, smooth and compactly supported in the remaining variables, and vanishing to order $(n-1)/2+\alpha$ at $\rho = w_1 = 0$. It can therefore be written as

$$a \;=\; \sum_{j=0}^{(n-1)/2+\alpha-1} \rho^j w_1^{(n-1)/2+\alpha-j} a_j(\lambda, y, w_1, v) + \rho^{(n-1)/2+\alpha} b(\lambda, \rho, y, w_1, v), \tag{6-12}$$

with a_j and b polyhomogeneous in λ.

We begin with the easy case $|w_1| \le \rho$. In this case, a in (6-12) is uniformly bounded. We split into the regions where $|w_1| \ge c|w|$ for some $c > 0$, and $|w_1| \le c|w|$. The first region, where $|w_1| \ge c|w|$, is trivial since then $|w|/\rho$ is bounded, so all we are required to show is that the integral (6-11) is bounded by a multiple of λ^b, $b = \min \mathcal{B}_{\mathrm{bf}_0} + n$, which is clear since the integrand has this property pointwise. On the other hand, if $|w_1| \le c|w|$, then $|w_1| \le (n-1)c|w_j|$ for some $j \ge 2$. For suitably small c this means that $d_{v_j}\Phi \ne 0$ sufficiently close to $\tilde{q}$, as $d_{v_j}\Phi = w_j + O(w_1)$ using (6-8). Then, by integrating by parts N times with respect to v_j in (6-11), we can gain a factor of $C_N(1+|w|/\rho)^{-N}$ for any N, showing that a much stronger estimate than (6-4) holds.

From now on, then, we will assume that $|w_1| \ge \rho$. We begin by estimating the a_0 term. The case $|w_1| \le c|w|$ is treated just as above: by integrating by parts N times with respect to v_j in (6-11) we gain a factor $C_N(|w|/\rho)^N$. With $N = M + (n-1)/2 + \alpha$ the resulting integrand enjoys a pointwise estimate $\lambda^b(|w|/\rho)^{-M}$ for any desired M. So we assume in the rest of the proof that $|w_1| \ge c|w|$, and therefore we can replace the RHS $(1+|w|/\rho)^\alpha$ in (6-4) by the equivalent quantity $(|w_1|/\rho)^\alpha$.

For fixed $w_1 \ne 0$, let us change variable from $v_1, \dots, v_{n-1}$ to $\theta_1, \dots, \theta_{n-1}$, where

$$\theta_i = w_1^{-1/2} d_{v_i}\Phi. \tag{6-13}$$

Then

$$\frac{\partial \theta_i}{\partial v_j} = w_1^{-1/2} d^2_{v_i v_j}\Phi = w_1^{1/2} A_{ij}, \tag{6-14}$$

where A_{ij} is nonsingular as we have noted above. Therefore,

$$\frac{\partial \Phi}{\partial \theta} = \left(\frac{\partial \theta}{\partial v}\right)^{-1} \frac{\partial \Phi}{\partial v} = A^{-1}\theta. \tag{6-15}$$

This shows that the θ coordinates are suitable coordinates in which to perform stationary phase computations. We proceed with a standard argument, which can be found in Sogge's book [1993], for example. We use the identity

$$e^{i\Phi/\rho} = \left(\frac{\rho}{w_1^{1/2} i\theta_j} \frac{\partial}{\partial v_j}\right) e^{i\Phi/\rho},$$

which can be written as

$$e^{i\Phi/\rho} = \left(\sum_k \frac{\rho}{i\theta_j} A_{jk} \frac{\partial}{\partial \theta_k}\right) e^{i\Phi/\rho}. \tag{6-16}$$

We also need the following observation: by applying (6-14) repeatedly, we obtain

$$\left| \frac{\partial^{|\alpha|} A}{\partial^\alpha \theta} \right| \le C|w_1|^{-|\alpha|/2} \le C\rho^{-|\alpha|/2}. \tag{6-17}$$

In the θ coordinates, we are trying to prove the estimate

$$\left| \rho^{-(n-1)/2-\alpha} \int_{\mathbb{R}^{n-1}} w_1^\alpha e^{i\Phi(y,w,\theta)/\rho} \tilde{a}_0(\lambda, \rho, y, w_1, \theta)\, d\theta \right| \le C\left(\frac{w_1}{\rho}\right)^\alpha \lambda^b. \tag{6-18}$$

Here the $w_1^{(n-1)/2}$ factor was absorbed as a Jacobian factor, and $\tilde{a}_0$ is again smooth. Clearly this is equivalent to a uniform bound on

$$\left| \rho^{-(n-1)/2} \lambda^{-b} \int_{\mathbb{R}^{n-1}} e^{i\Phi(y,w,\theta)/\rho} \tilde{a}_0(\lambda, \rho, y, w_1, \theta)\, d\theta \right|. \tag{6-19}$$

We introduce a partition of unity in (ρ, θ)-space, $1 = \chi_0 + \sum_{j=1}^{n-1} \chi_j$, where χ_0 is a compactly supported function of $\theta/\sqrt{\rho}$, and χ_j is supported where $|\theta| \geq \sqrt{\rho}$, and where $\theta_j \geq |\theta|/(n-1)$. We can do this with derivatives estimated by

$$|\nabla_\theta^{(k)} \chi_k| \leq C\rho^{-k/2}. \tag{6-20}$$

The integral with χ_0 inserted is trivial to estimate since it occurs on a set of measure $\rho^{(n-1)/2}$. With χ_j inserted, we use the identity (6-16) M times, for M a sufficiently large integer. Thus we consider

$$\rho^{-(n-1)/2} \int \chi_j \left(\sum_k \frac{\rho}{i\theta_j} A_{jk}(y, \theta) \frac{\partial}{\partial \theta_k} \right)^M e^{i\Phi(y,w,\theta)/\rho} \tilde{a}_0(\lambda, \rho, y, w_1, \theta)\, d\theta$$

and integrate by parts M times. The result can be estimated by

$$C\rho^{-(n-1)/2+M} \sum_{k=0}^{M} \rho^{-(M-k)/2} \int_{|\theta| \geq \sqrt{\rho}} 1_{\operatorname{supp} \chi_j} \theta_j^{-M-k}\, d\theta, \tag{6-21}$$

where $M-k$ derivatives fall on the χ_j or A_{jk} terms (via (6-17) and (6-20)), and at most k fall on a θ_j^{-p} term. Note that on the support of χ_j, we can estimate $\theta_j^{-1} \leq c|\theta|^{-1}$. The θ integral is absolutely convergent for $M > n-1$, and

$$\int_{|\theta| \geq \sqrt{\rho}} |\theta|^{-M-k}\, d\theta = C_k \rho^{-(M+k)/2+(n-1)/2}$$

since $\dim \theta = n-1$. Substitution of this into (6-21) gives a uniform bound since $\tilde{a}$ is polyhomogeneous in λ with index set $\mathcal{B}_{\mathrm{bf}_0} + n$. Moreover, since Φ and $\tilde{a}$ are smooth in w_1, the bound is uniform as $w_1 \to 0$.

To treat the terms a_i for $i > 0$ and b in (6-12), we perform the same manipulations as above, and we end up with a uniform bound times $C\rho^i w_1^{-i}$, which is bounded for $\rho \leq w_1$. This completes the proof. □

6B. *Geometry of L^{bf}*. We collect here some facts concerning the geometry of the Legendre submanifold L^{bf} (see Section 4A). We begin by defining

$$\begin{aligned} G^{\mathrm{bf}} &= \{(y, y', \sigma, \mu, \mu', \nu, \nu') \in {}^{\mathrm{sc}}N^* \partial \mathrm{diag}_b \mid \nu^2 + h^{ij} \mu_i \mu_j = 1\} \\ &= \{(y, y, 1, \mu, -\mu, \nu, -\nu) \mid \nu^2 + h^{ij} \mu_i \mu_j = 1\}. \end{aligned}$$

Clearly, G^{bf} is an S^{n-1}-bundle over $\partial \mathrm{diag}_b$.

Lemma 6.4. *The Legendre submanifold ${}^{\mathrm{sc}}N^* \partial \mathrm{diag}_b$ intersects L^{bf} cleanly at G^{bf}, and the projection $\pi : L^{\mathrm{bf}} \to \mathrm{bf}$ satisfies (6-3).*

Proof. According to [Hassell and Vasy 2001], the Legendre submanifold L^{bf} is given by the flowout from G^{bf} by the vector field

$$V_l = -\nu\left(\sigma\frac{\partial}{\partial\sigma} + \mu\frac{\partial}{\partial\mu}\right) + h\frac{\partial}{\partial\nu} + \frac{\partial h}{\partial\mu_i}\frac{\partial}{\partial y_i} - \frac{\partial h}{\partial y_i}\frac{\partial}{\partial\mu_i}, \quad h = \sum_{i,j} h^{ij}(y)\mu_i\mu_j \tag{6-22}$$

(see [Guillarmou et al. 2012, Section 3.1]). Observe that at least one of the coefficients of ∂_σ or ∂_ν is nonvanishing, so either $\dot\sigma \neq 0$ or $\dot\nu + \dot\nu' \neq 0$ under the flowout by V_l. Since $\sigma = 1$ and $\nu + \nu' = 0$ at ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$, we see that V_l is everywhere transverse to ${}^{\mathrm{sc}}N^*\partial\mathrm{diag}_b$, so G^{bf} has codimension 1 in L^{bf}, and intersects L^{bf} cleanly.

It remains to show that the projection π from L^{bf} to bf satisfies (6-3). First we choose coordinates on L^{bf}. Near a point on L^{bf} at which $|\mu|_h^2 := h^{ij}\mu_i\mu_j < 1$, and therefore $\nu \neq 0$, we can choose coordinates (μ, y', ϵ), where ϵ is the flowout time from G^{bf} along the vector field V_l. Coordinates on the base are (y, y', σ). With the dot indicating derivative along the flow of V_l, i.e., $d/d\epsilon$, we have

$$\dot\sigma = -\nu \quad\text{and}\quad \dot y^i = 2h^{ij}\mu_j \quad \text{on } G^{\mathrm{bf}}.$$

It follows that

$$\begin{aligned} \sigma &= 1 - \nu\epsilon + O(\epsilon^2), \\ y^i &= (y')^i + 2h^{ij}\mu_j\epsilon + O(\epsilon^2), \end{aligned}$$

and we see that near G^{bf},

$$\frac{\partial\sigma}{\partial\epsilon} \neq 0, \quad \frac{\partial y^i}{\partial\mu_j} = \epsilon h^{ij} + O(\epsilon^2),$$

which, using the positive-definiteness of h^{ij}, shows that $\det d\pi$, where π is the map

$$L^{\mathrm{bf}} \ni (\mu, y', \epsilon) \mapsto (y(\mu, y', \epsilon), y', \sigma(\mu, y', \epsilon)),$$

vanishes to order exactly $n-1$ as $\epsilon \to 0$.

On the other hand, near a point on L^{bf} at which $|\mu| = 1$, we can choose a coordinate μ_i which is nonzero. Without loss of generality we suppose that $i = 1$. Then write $\bar y = (y_2, \dots, y_{n-1})$ and $\bar\mu = (\mu_2, \dots, \mu_{n-1})$. We can take $(\nu, \bar\mu, y', \epsilon)$ as coordinates on L^{bf}. Calculating as above, we find that

$$\begin{aligned} y^1 &= y_1' + 2h^{1j}\mu_j\epsilon + O(\epsilon^2), \\ y^i &= (y')^i + 2h^{ij}\mu_j\epsilon + O(\epsilon^2), \quad i \geq 2, \\ \sigma &= 1 - \nu\epsilon + O(\epsilon^2), \end{aligned}$$

which shows that

$$\frac{\partial y_1}{\partial\epsilon} > 0, \quad \frac{\partial\bar y^i}{\partial\bar\mu_j} = \epsilon h^{ij} + O(\epsilon^2), \quad \frac{\partial\sigma}{\partial\nu} = -\epsilon + O(\epsilon^2).$$

Again we find that $\det d\pi$, where π is the map

$$L^{\mathrm{bf}} \ni (\nu, \bar\mu, y', \epsilon) \mapsto (y(\nu, \bar\mu, y', \epsilon), y', \sigma(\nu, \bar\mu, y', \epsilon)),$$

vanishes to order exactly $n-1$ as $\epsilon \to 0$. $\square$

Lemma 6.5. *There exists $\delta > 0$ such that, if*

$$q = (y, y', \sigma, \mu, \mu', \nu, \nu') \in L^{\mathrm{bf}} \quad \textit{and} \quad |\nu + \nu'| < \delta,$$

then either $q \in G^{\mathrm{bf}}$, or $d\pi : T_q L^{\mathrm{bf}} \to T_{\pi(q)}\mathrm{bf}$ is invertible, and hence $\pi : L \to \mathrm{bf}$ is a diffeomorphism locally near q.

Proof. We use the explicit description of L^{bf} given in [Hassell and Vasy 2001, Section 4]:

$$L^{\mathrm{bf}} = \left\{ \begin{pmatrix} y, y', \sigma, \\ \nu, \nu', \mu, \mu' \end{pmatrix} \middle| \begin{array}{c} \exists (y_0, \hat{\mu}_0) \in S^*(\partial M),\ s, s' \in (0, \pi),\ \text{such that} \\ \sigma = \sin s / \sin s',\ \nu = -\cos s,\ \nu' = \cos s', \\ (y, \mu) = \sin s \exp(s H_{\frac{1}{2}h})(y_0, \hat{\mu}_0), \\ (y', \mu') = -\sin s' \exp(s' H_{\frac{1}{2}h})(y_0, \hat{\mu}_0), \end{array} \right\} \cup T_+ \cup T_- \cup F_+ \cup F_-, \tag{6-23}$$

where

$$T_\pm = \{(y, y, \sigma, \pm 1, \mp 1, 0, 0) \mid \sigma > 0,\ y \in \partial M\},$$
$$F_\pm = \{(y, y', \sigma, \pm 1, \pm 1, 0, 0) \mid \sigma > 0,\ \exists \text{ geodesic of length } \pi \text{ connecting } y, y'\}.$$

We see that $\nu = -\nu'$ on L^{bf} only on $G^{\mathrm{bf}} \cup T_+ \cup T_-$. A compactness argument shows that for any neighborhood U of $G^{\mathrm{bf}} \cup T_+ \cup T_-$, the set

$$\{(y, y', \sigma, \mu, \mu', \nu, \nu') \in L^{\mathrm{bf}} \mid |\nu + \nu'| < \delta\}$$

is contained in U if δ is sufficiently small. So it is enough to show that L^{bf} projects diffeomorphically to bf in some neighborhood of $G^{\mathrm{bf}} \cup T_+ \cup T_-$, except at G^{bf} itself. Lemma 6.4 shows that $L^{\mathrm{bf}} \subset {}^{\mathrm{sc}}T^*{}_{\mathrm{bf}} M_b^2$ projects diffeomorphically to the base bf in a sufficiently small deleted neighborhood of G^{bf}. Now consider a neighborhood of $T_+ \cap \{\sigma \le 1 - \epsilon\}$ for some small ϵ. As shown in [Hassell and Vasy 2001], near this set, (y', μ', σ) are smooth coordinates. Also, we have from (6-23) that

$$(y, \mu) = \sigma \exp\left(\frac{s' - s}{\sin s'} H_{\frac{1}{2}h}\right)(y', \mu').$$

Using the expression (6-22) for the Hamilton vector field, we find that, near T_+,

$$y^i = y'^i + \frac{s' - s}{\sin s'} h^{ij} \mu'_j + O(|\mu'|^2) = (1 - \sigma) h^{ij} \mu'_j + O\big((\sin s)^2 + (\sin s')^2 + |\mu'|^2\big),$$

which shows that at T_+, where $\sin s = \sin s' = \mu' = 0$, we have

$$\left. \frac{\partial y^i}{\partial \mu'_j} \right|_{y', \sigma} = (1 - \sigma) h^{ij}.$$

Since (y', μ', σ) furnish smooth coordinates near T_+, this equation and the positive-definiteness of h^{ij} show that also (y, y', σ) furnish smooth coordinates in a neighborhood of T_+ when $\sigma < 1 - \epsilon$. (Of course, we know from Lemma 6.4 that this cannot hold uniformly up to $\sigma = 1$). A similar argument holds for $\sigma > 1 + \epsilon$ and for T_-. $\square$

Remark 6.6. These lemmas will be applied to distributions of the form

$$Q(\lambda)dE_{\sqrt{L}}(\lambda)Q(\lambda), \tag{6-24}$$

where Q is a pseudodifferential operator with small microsupport. Notice that by taking the microsupport sufficiently small, we can localize the microsupport of (6-24) to points $(y, y', \sigma, \mu, \mu', \nu, \nu')$ such that y is close to y', μ is close to μ' and ν is close to ν'. However, we cannot localize so that σ is close to 1, simply because if $x, x' \in (0, \epsilon)$, then $\sigma = x/x'$ can take any value in $(0, \infty)$. Therefore, it is important to understand the properties of π on L near the whole of the sets $T_\pm$, not just close to ${}^{sc}N^*\partial\mathrm{diag}_b$.

6C. ***Proof of Theorem 1.3, part (A).*** By Proposition 1.12, to prove part (A) of Theorem 1.3 it is sufficient to prove Theorem 1.13 for $\boldsymbol{L} = \boldsymbol{H}$ and for $\lambda \le \lambda_0$, that is, to prove the estimates

$$\bigl|(Q_i(\lambda)dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)Q_i(\lambda))(z, z')\bigr| \le C\lambda^{n-1-j}(1 + \lambda d(z, z'))^{-(n-1)/2+j}, \quad j \ge 0. \tag{6-25}$$

Our starting point is Theorem 4.1. As an immediate consequence of this theorem, the j-th λ-derivative $dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)$ is a Legendre distribution in the space

$$I^{m-j,p-j;r_{\mathrm{lb}}-j,r_{\mathrm{rb}}-j;\mathcal{B}^{(j)}}(M^2_{k,b}, (L^{\mathrm{bf}}, L^{\sharp,\mathrm{bf}}); \Omega^{1/2}_{k,b}),$$

where $\mathcal{B}^{(j)}$ is an index family with index sets at the faces $\mathrm{bf}_0, \mathrm{lb}_0, \mathrm{rb}_0, \mathrm{zf}$ starting at order $-1-j$, $n/2-1-j$, $n/2-1-j$, $n-1-j$ respectively.

Next we choose a partition of unity. We choose Q_0 to be multiplication by the function $1-\chi(\rho)$, where $\chi(\rho) = 1$ for $\rho \le \epsilon$ and $\chi(\rho) = 0$ for $\rho \ge 2\epsilon$, for some sufficiently small ϵ. Then $Q_0 dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)Q_0$ is polyhomogeneous on $M^2_{k,b}$, with index sets as above at $\mathrm{bf}_0, \mathrm{lb}_0, \mathrm{rb}_0, \mathrm{zf}$ and supported away from the remaining boundary hypersurfaces. Now recall that $|dg\, dg'\, d\lambda/\lambda|^{1/2}$ is equal to $\rho_{\mathrm{bf}_0}^{-n}\rho_{\mathrm{lb}_0}^{-n/2}\rho_{\mathrm{rb}_0}^{-n/2}$ multiplied with a smooth nonvanishing section of the half-density bundle $\Omega^{1/2}_{k,b}$. It is then immediate that $Q_0 dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)Q_0$ is bounded, as a multiple of $|dg\, dg'\, d\lambda/\lambda|^{1/2}$ by λ^{n-1-j}, which yields (6-25) for $i = 0$ since in this region we have $\lambda d(z, z') \le C$.

Next, we choose Q_1' such that $\mathrm{Id} - Q_1'$ is microlocally equal to the identity for $|\mu|_h^2 + \nu^2 \le \frac{3}{2}$, and microsupported in $|\mu|_h^2 + \nu^2 \le 2$. Let $Q_1 = \chi(\rho)Q_1'$. Then, we claim that $Q_1 dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)Q_1$ has empty wavefront set, and is therefore polyhomogeneous with index sets at the faces $\mathrm{bf}_0, \mathrm{lb}_0, \mathrm{rb}_0, \mathrm{zf}$ starting at order -1, $n/2-1$, $n/2-1$, $n-1$ respectively. To see this, we write

$$\begin{aligned} &Q_1 dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)Q_1 \\ &\quad= dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) - (\mathrm{Id} - Q_1)dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) - dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)(\mathrm{Id} - Q_1) + (\mathrm{Id} - Q_1)dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)(\mathrm{Id} - Q_1). \end{aligned} \tag{6-26}$$

Since $\mathrm{Id} - Q_1$ is microlocally equal to the identity on $\pi_L(\mathrm{WF}'_{\mathrm{bf}}\, dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda))$ and on $\mathrm{WF}'_{\mathrm{lb}}(dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda))$, Lemma 5.2 shows that the sum of the first two terms on the right hand side above vanishes to infinite order at lb and bf, and similarly the sum of the third and fourth terms vanishes to infinite order at lb and bf. Now consider the multiplication of $\mathrm{Id} - Q_1$ on the right, and group together the first and third terms, and the second and fourth terms on the right-hand side. We see, using the adjoint of Lemma 5.2 (since $\mathrm{Id} - Q_1$ is also microlocally equal to the identity on $\mathrm{WF}'_{\mathrm{rb}}(dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda))$), that the sum of the first and third terms vanishes to infinite order at rb, and similarly the sum of the second and fourth terms vanishes at rb. Hence $Q_1 dE^{(j)}_{\sqrt{H}}(\lambda) Q_1$ vanishes to all orders at bf, lb, rb and has empty wavefront set as claimed. This piece therefore also satisfies (6-25).

We now further decompose $\mathrm{Id} - Q_0 - Q_1 = \chi(\mathrm{Id} - Q_1')$, which has compact microsupport, into a sum of terms. Choosing δ as in Lemma 6.5, we partition the interval $[-2, 2]$ into $N-1$ intervals B_i each of length $\delta/2$, and choose a decomposition $\mathrm{Id} - Q_1 = \sum_{i=2}^N Q_i$, where Q_i, and hence also Q_i^*, is microsupported in the set $\{|\mu|_h^2 + \nu^2 \le 2, \nu \in 2B_i\}$ (where $2B_i$ is the interval with the same center as B_i and twice the length). It follows that if $q' = (y, y', \sigma, \mu, \mu', \nu, \nu') \in (L^{\mathrm{bf}})'$ is such that $\pi_L(q') \in \mathrm{WF}'(Q_i)$ and $\pi_R(q') \in \mathrm{WF}'(Q_i^*)$, then $|\nu - \nu'| \le \delta$. Together with Lemma 5.4, this means that $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$ is associated only to the Legendrian L^{bf} and not to $L^{\sharp,\mathrm{bf}}$, since on $(L^{\sharp,\mathrm{bf}})'$ we have $|\nu - \nu'| = 2 > \delta$.

Next, by Lemma 6.5, if $q' = (y, y', \sigma, \mu, \mu', \nu, \nu') \in (L^{\mathrm{bf}})'$ is such that $\pi_L(q')$ is in $\mathrm{WF}'(Q_i)$ and $\pi_R(q')$ is in $\mathrm{WF}'(Q_i^*)$, then due to our choice of δ, either $q \in G^{\mathrm{bf}}$, or locally near q, L^{bf} projects diffeomorphically to bf. Therefore, the microsupport of $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$, $i \ge 2$, is a subset of $(L^{\mathrm{bf}})'$ which satisfies the conditions of either Proposition 6.1 or Proposition 6.2.

In the case of Proposition 6.1, we have $b = n - 1 - j$, $\alpha = -(n-1)/2 + j$ and estimate (6-25) follows directly. Next consider the case of Proposition 6.2. In this case, we have to determine the order of vanishing of the symbol of $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$ at G^{bf}. Locally near $q \in G^{\mathrm{bf}} \cap L^{\mathrm{bf}}$, L^{bf} can be parametrized by a phase function Φ that *vanishes* at G^{bf} when $d_v \Phi = 0$; see (6-8). The kernel $Q_i dE_{\sqrt{H}}(\lambda) Q_i^*$ is a Legendrian of order $-1/2$. Each time we apply a λ derivative to $dE_{\sqrt{H}}(\lambda)$, it hits either the phase function or the symbol. If it hits the phase, then the order of the Legendrian is reduced by 1, but it brings down a factor of Φ that vanishes at $G^{\mathrm{bf}} \times [0, \lambda_0]$. If it hits the symbol, then the order of the Legendrian is not reduced. Therefore, as a Legendrian of order $-1/2 - j$, the full symbol of $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$ vanishes to order j at $G^{\mathrm{bf}} \times [0, \lambda_0]$. Therefore, we can apply Proposition 6.2 with $b = n - 1 - j$ and $\alpha = -(n-1)/2 + j$, and we deduce (6-25) in this case. This concludes the proof of (6-25) and hence establishes Theorem 1.13 for low energies $\lambda \le \lambda_0$.

7. High energy estimates (in the nontrapping case)

In the previous section we proved estimates on the spectral measure $dE_{\sqrt{H}}(\lambda)$ for $\lambda \in (0, \lambda_0]$. We now prove high energy estimates, i.e., estimates for $\lambda \in [\lambda_0, \infty)$. For convenience, we introduce the semiclassical parameter $h = \lambda^{-1}$, so that we are interested in estimates for $h \in (0, h_0]$, where $h_0 = \lambda_0^{-1}$. To do this, we use the description of the high-energy asymptotics of the spectral measure from [Hassell and Wunsch 2008]. The structure of the argument will be the same as in the previous section, and our main task is to adapt each of the intermediate results — Lemmas 5.2 and 5.4, Propositions 6.1 and 6.2, Lemma 6.4 and Lemma 6.5 — to the high-energy setting. *Throughout this section we assume that the manifold (M, g) is nontrapping.*

7A. *Microlocal support.* We begin by defining, by analogy with the discussion in Section 5, the notion of microlocal support of a Legendre distribution on X.

Let $\Lambda \subset {}^{sc}T^*{}_{mf}X$ be the Legendre submanifold associated to F. We assume that Λ is compact. Recall from [Hassell and Wunsch 2008, Section 3] that Λ determines associated Legendre submanifolds Λ_{bf}, Λ_{lb} and Λ_{rb} which are the bases of the fibrations on $\partial_{bf}\Lambda$, $\partial_{lb}\Lambda$ and $\partial_{rb}\Lambda$, respectively. The Legendre submanifold Λ_{bf} can be canonically identified with a Legendre submanifold of ${}^{sc}T^*{}_{bf}M_b^2$, while $\partial_{lb}\Lambda$ and $\partial_{rb}\Lambda$ may be canonically identified with Legendre submanifolds of ${}^{sc}T^*{}_{\partial M}M$. We define Λ' by negating the fiber coordinates corresponding to the right copy of M, i.e.,

$$q' = (z, z', \zeta, \zeta') \in \Lambda' \quad \Longleftrightarrow \quad q = (z, z', \zeta, -\zeta') \in \Lambda.$$

Similarly we define Λ'_{bf} and Λ'_{rb} as in the previous section.

Then we define the microlocal support $\mathrm{WF}'(F)$ of $F \in I^m(\Lambda)$ to be a closed subset of

$$\Lambda' \cup (\Lambda'_{bf} \times [0, h_0]) \cup (\Lambda_{lb} \times [0, h_0]) \cup (\Lambda'_{rb} \times [0, h_0])$$

in the same way as before: we say that $q' \in \Lambda'$ is not in $\mathrm{WF}'(F)$ if there is a neighborhood of $q \in \Lambda$ in which F has order $-\infty$, in the sense of Section 5. That is, in a local oscillatory representation for F of the form (for simplicity, where q lies over the interior of M_b^2),

$$h^{m-k/2-n} \int_{\mathbb{R}^k} e^{i\psi(z,v)/h} a(z, v, h)\, dv |dg dg' dh/h^2|^{1/2},$$

where $q = (z_*, d_z\psi(z_*, v_*))$ and $d_v\psi(z_*, v_*) = 0$ (these conditions determining (z_*, v_*) locally uniquely provided that ψ is a nondegenerate parametrization of Λ), the condition that F has order $-\infty$ in a neighborhood of q is equivalent to a being $O(h^\infty)$ in a neighborhood of the point $(z_*, v_*, 0)$. Similarly, $q' \in \Lambda'_{bf} \times [0, h_0]$ is not in $\mathrm{WF}'(F)$ if there is a neighborhood of $q \in \Lambda_{bf} \times [0, h_0]$ in which F has order $-\infty$.

Similarly, $(\tilde{q}, h) \in \Lambda_{lb} \times [0, h_0]$ is not in $\mathrm{WF}'(F)$ if F can be written modulo $(hxx')^\infty C^\infty(M_b^2)$ using local oscillatory integral representations with symbols that vanish in a neighborhood of the *fiber* in their domain corresponding to $(\tilde{q}, h)$, and $(\tilde{q}', h) \in \Lambda'_{rb} \times [0, h_0]$ is not in $\mathrm{WF}'(F)$ if F can be written modulo $(hxx')^\infty C^\infty(M_b^2)$ using local oscillatory integral representations with symbols that vanish in a neighborhood of the fiber in their domain corresponding to $(\tilde{q}, h)$. These components of $\mathrm{WF}'(F)$ will be denoted $\mathrm{WF}'_{mf}(F)$, $\mathrm{WF}'_{lb}(F)$, $\mathrm{WF}'_{bf}(F)$ and $\mathrm{WF}'_{rb}(F)$, respectively.

If $F \in I^m(\Lambda)$, then $F \in (hxx')^\infty C^\infty(M^2)$ if and only if $\mathrm{WF}'(F)$ is empty. Also note that if $\mathrm{WF}'_*(F)$ is empty, then $\partial_*\Lambda'$ is disjoint from $\mathrm{WF}'_{mf}(F)$, but the converse need not hold: if the kernel of F is supported away from mf then certainly $\mathrm{WF}'_{mf}(F)$ will be empty, but $\mathrm{WF}'_*(F)$ need not be.

Particular examples of Legendre distributions on X are the kernels of semiclassical scattering pseudodifferential operators Q of differential order $-\infty$ with compact operator wavefront set. In the case of such a pseudodifferential[1] operator, the Legendre submanifold Λ is a compact subset of ${}^{s\Phi}N^*\mathrm{diag}_b$, defined in (4-15), and the components $\Lambda_{lb} \cup \Lambda'_{rb}$ are empty. Thus in this case we may (and will) identify the microlocal support $\mathrm{WF}'_{mf}(Q)$ with a compact subset of ${}^{sc}T^*M$, and $\mathrm{WF}'_{bf}(Q)$ may be identified with a compact subset of ${}^{sc}T^*{}_{\partial M}M \times [0, h_0)$.

[1] Throughout this section we deal with semiclassical scattering pseudodifferential operators. The words "semiclassical scattering" will usually be omitted.

In the next lemma, π_L and π_R denote the maps defined in either (4-6) or (4-14), as the case may be.

Lemma 7.1. *Suppose that F is a Legendre distribution on $\boldsymbol{X}$ and Q is a semiclassical scattering pseudodifferential operator. Assume that $F \in I^{m;r_{\mathrm{bf}},r_{\mathrm{lb}},r_{\mathrm{rb}}}(\boldsymbol{X}, \Lambda; {}^{s\Phi}\Omega^{1/2})$ is associated to a compact Legendre submanifold Λ and that Q is of differential order $-\infty$ and semiclassical order 0, with compact operator wavefront set. Then QF is also a Legendre distribution in $I^{m;r_{\mathrm{bf}},r_{\mathrm{lb}},r_{\mathrm{rb}}}(\boldsymbol{X}, \Lambda; {}^{s\Phi}\Omega^{1/2})$ and we have*

$$\begin{aligned} \mathrm{WF}'_{\mathrm{mf}}(QF) &\subset \pi_L^{-1}\,\mathrm{WF}'_{\mathrm{mf}}(Q) \cap \mathrm{WF}'_{\mathrm{mf}}(F), \\ \mathrm{WF}'_{\mathrm{bf}}(QF) &\subset \pi_L^{-1}\,\mathrm{WF}'_{\mathrm{bf}}(Q) \cap \mathrm{WF}'_{\mathrm{bf}}(F), \\ \mathrm{WF}'_{\mathrm{lb}}(QF) &\subset \mathrm{WF}'_{\mathrm{bf}}(Q) \cap \mathrm{WF}'_{\mathrm{lb}}(F), \\ \mathrm{WF}'_{\mathrm{rb}}(QF) &\subset \mathrm{WF}'_{\mathrm{rb}}(F). \end{aligned} \tag{7-1}$$

Moreover, if Q is microlocally equal to the identity on $\pi_L(\mathrm{WF}'_{\mathrm{mf}}(F))$, $\pi_L(\mathrm{WF}'_{\mathrm{bf}}(F))$ and $\mathrm{WF}'_{\mathrm{lb}}(F)$, then $QF - F \in I^{\infty,\infty,\infty,r_{\mathrm{rb}}}(\boldsymbol{X}, \Lambda; {}^{s\Phi}\Omega^{1/2})$, i.e., it vanishes to infinite order at mf, lb *and* bf.

We omit the proof, as it is essentially identical to that of Lemma 5.2. There is of course a corresponding theorem for composition in the other order, which is obtained by taking the adjoint of the lemma above. Combining the two we obtain:

Corollary 7.2. *Suppose that F and Q, Q' are as above. Then*

$$\begin{aligned} \mathrm{WF}'_{\mathrm{mf}}(QFQ') &\subset \pi_L^{-1}\,\mathrm{WF}'_{\mathrm{mf}}(Q) \cap \pi_R^{-1}\,\mathrm{WF}'_{\mathrm{mf}}(Q') \cap \mathrm{WF}'_{\mathrm{mf}}(F), \\ \mathrm{WF}'_{\mathrm{bf}}(QFQ') &\subset \pi_L^{-1}\,\mathrm{WF}'_{\mathrm{bf}}(Q) \cap \pi_R^{-1}\,\mathrm{WF}'_{\mathrm{bf}}(Q') \cap \mathrm{WF}'_{\mathrm{bf}}(F), \\ \mathrm{WF}'_{\mathrm{lb}}(QFQ') &\subset \mathrm{WF}'_{\mathrm{bf}}(Q) \cap \mathrm{WF}'_{\mathrm{lb}}(F), \\ \mathrm{WF}'_{\mathrm{rb}}(QFQ') &\subset \mathrm{WF}'_{\mathrm{bf}}(Q') \cap \mathrm{WF}'_{\mathrm{rb}}(F). \end{aligned} \tag{7-2}$$

A similar result holds if F is associated to a Legendre conic pair rather than a single Legendre submanifold.

Lemma 7.3. (i) *Suppose that $F \in I^{m,p;r_{\mathrm{bf}},r_{\mathrm{lb}},r_{\mathrm{rb}}}(\boldsymbol{X}, (\Lambda, \Lambda^\sharp); {}^{s\Phi}\Omega^{1/2})$ is a Legendre distribution on $\boldsymbol{X}$ associated to a conic Legendrian pair $(\Lambda, \Lambda^\sharp)$, and suppose that Q is a pseudodifferential operator such that Q is microlocally equal to the identity operator near $\pi_L(\Lambda \cup \Lambda^\sharp)$. Then $QF - F \in I^{\infty,\infty;\infty,\infty,r_{\mathrm{rb}}}(\boldsymbol{X}, (\Lambda, \Lambda^\sharp), {}^{s\Phi}\Omega^{1/2})$, so it vanishes to infinite order at* mf, lb *and* bf. *If Q' is microlocally equal to the identity operator near $\pi_R(\Lambda \cup \Lambda^\sharp)$, then $FQ' - F \in I^{\infty,\infty;\infty,r_{\mathrm{lb}},\infty}(\boldsymbol{X}, (\Lambda, \Lambda^\sharp), {}^{s\Phi}\Omega^{1/2})$ vanishes to infinite order at* mf, bf *and* rb.

(ii) *Suppose that F is as above, a Legendre distribution on M_b^2 associated to a conic Legendrian pair $(\Lambda, \Lambda^\sharp)$ of order $(m, p; r_{\mathrm{bf}}, r_{\mathrm{lb}}, r_{\mathrm{rb}})$, and suppose that Q, Q' are pseudodifferential operators. If*

$$\pi_L^{-1}\,\mathrm{WF}'_{\mathrm{bf}}(Q) \cap \pi_R^{-1}\,\mathrm{WF}'_{\mathrm{bf}}(Q') \cap \Lambda^\sharp = \varnothing, \tag{7-3}$$

then $QFQ' \in I^{m;r_{\mathrm{bf}},r_{\mathrm{lb}},r_{\mathrm{rb}}}(M_b^2, \Lambda; {}^{s\Phi}\Omega^{1/2})$; in particular, $\mathrm{WF}'_{\mathrm{bf}}(QFQ')$ is disjoint from $(\Lambda^\sharp)'$.

We omit the proof, which is a straightforward modification of the arguments in Section 5.

7B. *Pointwise estimates on Legendre distributions.* Now we give a pointwise estimate on Legendre distributions of a particular type. First we begin with the trivial case.

Proposition 7.4. *Let $\Lambda \subset {}^{\mathrm{sc}}T^*{}_{\mathrm{mf}}(X)$ be a Legendre distribution that projects diffeomorphically to* mf. *Suppose that $u \in I^{m,r_{\mathrm{bf}},r_{\mathrm{lb}},r_{\mathrm{rb}}}(X, \Lambda; {}^{\mathrm{s}\Phi}\Omega^{1/2})$ with*

$$m = n/2 - l, \quad r_{\mathrm{bf}} = -n/2 - \alpha, \quad r_{\mathrm{lb}} = r_{\mathrm{rb}} = -\alpha.$$

Then, as a multiple of the half-density $|dg\, dg'\, d\lambda|^{1/2}$, we have a pointwise estimate

$$|u| \le C\lambda^l (x^{-1} + (x')^{-1})^\alpha.$$

Generalizing Proposition 6.2 to the case of $X = M_b^2 \times [0, h_0]$ is straightforward.

Proposition 7.5. *Let Λ be a Legendrian submanifold of ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}X$. Assume that Λ intersects ${}^{\mathrm{s}\Phi}N^*\mathrm{diag}_b$, defined in (4-15), at $G = \Lambda \cap {}^{\mathrm{s}\Phi}N^*\mathrm{diag}_b$ which is codimension 1 in Λ and transversal to the boundary at* bf, *and that $d\pi$ has full rank on $\Lambda \setminus G$, while $\pi|_G$ is a fibration $G \to \mathrm{diag}_b$ with $(n-1)$-dimensional fibers, with condition (6-3) holding at G.*

Assume that $u \in I^{m,r_{\mathrm{bf}},r_{\mathrm{lb}},r_{\mathrm{rb}}}(X, \Lambda; {}^{\mathrm{s}\Phi}\Omega^{1/2})$, with $m, r_{\mathrm{bf}}, r_{\mathrm{lb}}, r_{\mathrm{rb}}$ as in Proposition 7.4 and that the full symbol of u vanishes to order $(n-1)/2 + \alpha$ both at $G \subset \Lambda$ and at $\partial_{\mathrm{bf}} G \times [0, h_0] \subset \partial_{\mathrm{bf}}\Lambda \times [0, h_0]$. Then, as a multiple of the half-density $|dg\, dg'\, d\lambda|^{1/2}$, we have a pointwise estimate

$$|u| \le C\lambda^{l-\alpha}(1 + \lambda d(z, z'))^\alpha. \tag{7-4}$$

Proof. First consider u on a neighborhood of X disjoint from diag_b. In that case, the result follows from Proposition 7.4.

Next consider u near diag_b, but away from bf. Then if u is microlocally trivial at ${}^{\mathrm{s}\Phi}N^*\mathrm{diag}_b$, the result follows from Proposition 7.4. If not, then the geometry is the same as that considered in Proposition 6.2 (with ρ replaced by h; also note that the estimate in Proposition 6.2 is respect to the half-density $\lambda^n |dg\, dg'\, d\lambda|^{1/2}$), and the result follows from that proposition.

So we are reduced to the case where we are microlocally close to $\Lambda \cap \partial_{\mathrm{bf}}{}^{\mathrm{s}\Phi}N^*\mathrm{diag}_b = \partial_{\mathrm{bf}} G$. Let $q \in \partial_{\mathrm{bf}} G$. In a neighborhood of $\partial_{\mathrm{bf}}\mathrm{diag}_b$, we have coordinates (x, y, w), where $w = (y - y', \sigma - 1)$ as before. In terms of these we can write points in ${}^{\mathrm{s}\Phi}T^*{}_{\mathrm{mf}}X$ in the form

$$\kappa \cdot \frac{dw}{xh} + \mu \cdot \frac{dy}{xh} + \tau \cdot \frac{dx}{xh} + \nu d\Big(\frac{1}{xh}\Big),$$

and this defines local coordinates $(x, y, w; \tau, \mu, \kappa, \nu)$ on ${}^{\mathrm{s}\Phi}T^*{}_{\mathrm{mf}}X$. Then, contracting the symplectic form with $xh^2\partial_h$ and restricting to ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}X$ gives the contact form on ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}X$, which in these coordinates takes the form

$$d\nu - \tau dx - \mu \cdot dy - \kappa \cdot dw. \tag{7-5}$$

Using the transversality of Λ to ${}^{\mathrm{s}\Phi}T^*_{\mathrm{bf}\cap\mathrm{mf}}X$ we see, as in the proof of Proposition 6.2 that $(x, y, w_1, \bar\kappa)$ form coordinates on Λ. Then as in the proof of Proposition 6.2, we can write the remaining coordinates as functions of $(x, y, w_1, \bar{\kappa})$ on Λ:

$$\overline{w}_i = W_i(x, y, w_1, \bar{\kappa}), \quad \mu_i = M_i(x, y, w_1, \bar{\kappa}), \quad i = 2, \dots, n,$$
$$\kappa_1 = K(x, y, w_1, \bar{\kappa}), \quad v = N(x, y, w_1, \bar{\kappa}), \quad \tau = T(x, y, w_1, \bar{\kappa}).$$

In the same way as before, we find that

$$\widetilde{\Phi}(x, y, w, v) = \sum_{j=2}^{n} (\overline{w}_j - W_j(x, y, w_1, v))v_j + N(x, y, w, v), \quad v = (v_2, \dots, v_n),$$

parametrizes Λ locally, and has the properties that $\widetilde{\Phi} = O(w_1)$ when $d_v \widetilde{\Phi} = 0$, and $\widetilde{\Phi} = \Phi + O(x)$, where Φ is precisely as in the proof of Proposition 6.2. We can then follow the proof given there, where (6-11) is replaced by

$$x^{-(n-1)/2-\alpha} \lambda^{(n-1)/2+k} \int e^{i\widetilde{\Phi}(x,y,w,v)/xh} \tilde{a}(x, y, w_1, v, h)\, dv, \tag{7-6}$$

in which the function $\tilde{a}$ vanishes to order $(n-1)/2+\alpha$ at $x = 0$ and at $w_1 = 0$. In effect we have replaced the large parameter $1/x$ in the phase of (6-11) by $1/xh$, while x plays the role of a smooth parameter.

The rest of the argument is parallel to the proof of Proposition 6.2. We deal with the cases $|w_1| \le xh$ and $|w_1| \le c|w|$ exactly as in the previous proof. Assuming then that $|w_1| \ge xh$ and $|w_1| \sim |w|$, we make the change of variables (6-13). By continuity, the matrix A in (6-15) remains nonsingular, and (6-17) remains valid, for small x. Hence, we can integrate by parts using the identity

$$e^{i\widetilde{\Phi}/x} = \left(\sum_k \frac{xh}{i\theta_j} A_{jk} \frac{\partial}{\partial \theta_k} \right) e^{i\widetilde{\Phi}/x},$$

analogous to (6-16).

In the θ coordinates, we are trying to prove the estimate

$$\left| x^{-(n-1)/2-\alpha} h^{-(n-1)/2-l} \int_{\mathbb{R}^{n-1}} w_1^{\alpha} e^{i\widetilde{\Phi}(x,y,w,\theta)/xh} \tilde{a}_0(x, y, w_1, \theta)\, d\theta \right| \le Ch^{-l} \left(\frac{w_1}{x} \right)^{\alpha},$$

since when $|w| \ge xh$,

$$\frac{|w|}{xh} \sim \lambda d(z, z') \sim 1 + \lambda d(z, z').$$

As before, the $w_1^{(n-1)/2}$ factor was absorbed as a Jacobian factor, and $\tilde{a}$ is again smooth. This estimate is equivalent to a uniform bound on

$$\left| (xh)^{-(n-1)/2} \int_{\mathbb{R}^{n-1}} e^{i\widetilde{\Phi}(x,y,w,\theta)/x} \tilde{a}_0(x, y, w_1, \theta)\, d\theta \right|. \tag{7-7}$$

We introduce a modified partition of unity in (x, θ)-space, $1 = \chi_0 + \sum_{j=1}^{n-1} \chi_j$, where χ_0 is a compactly supported function of $\theta/\sqrt{xh}$, and χ_j is supported where $|\theta| \ge \sqrt{xh}$, and where $\theta_j \ge |\theta|/(n-1)$, with derivatives estimated by

$$\left| \nabla_\theta^{(k)} \chi_k \right| \le C(xh)^{-k/2}. \tag{7-8}$$

Then the rest of the argument proceeds just as before, leading to (7-7). □

7C. *Geometry of the Legendre submanifold L.* We prove results analogous to Lemmas 6.4 and 6.5. First, we define

$$G = \{q \in {}^{s\Phi}N^*\mathrm{diag}_b \mid \sigma(h^2\Delta_g)(q) = 1\},$$

where σ is the semiclassical principal symbol. This is an S^{n-1}-bundle over diag_b.

Lemma 7.6. *The Legendre submanifold L introduced in Section 4B intersects ${}^{s\Phi}N^*\mathrm{diag}_b$ cleanly at G, and the projection $\pi : L \to \mathrm{mf}$ satisfies (6-3).*

Proof. This is proved just as for Lemma 6.4. As shown in [Hassell and Wunsch 2008], L can be obtained as the flowout from G by a vector field V_l, which is obtained from the Hamilton vector field of $\Delta_g - \lambda^2$ by dividing by boundary defining function factors (see [ibid., Section 11]), so that it becomes smooth up to the boundary of ${}^{s\Phi}T^*X$. This vector field takes the form (6-22) up to $O(x)$ near bf, and repeating the argument below (6-22) with x as a smooth parameter establishes the lemma in a neighborhood of $\partial_{\mathrm{bf}}G$, i.e., for $x + x' \le \epsilon$ for some small $\epsilon > 0$.

Away from bf, we can use coordinates (z, z') on mf, and writing points in ${}^{s\Phi}T^*_{\mathrm{mf}}X$ in the form

$$\zeta \cdot \frac{dz}{h} + \zeta' \cdot \frac{dz'}{h} + \tau d\Big(\frac{1}{h}\Big)$$

defines fiber coordinates (ζ, ζ', τ) on ${}^{s\Phi}T^*_{\mathrm{mf}}X$. In terms of these coordinates, we have

$$V_l = g^{ij}(z)\zeta_i \frac{\partial}{\partial z^j} - \frac{1}{2}\frac{\partial g^{ij}(z)}{\partial z_k}\zeta_i\zeta_j \frac{\partial}{\partial \zeta_k} + g^{ij}(z)\zeta_i\zeta_j \frac{\partial}{\partial \tau}. \tag{7-9}$$

We recognize the equations for (z, ζ) as equations for geodesic flow. Moreover, letting $|\zeta|_g = g^{ij}(z)\zeta_i\zeta_j$, we find that $(|\zeta|_g^2)^{\cdot} = 0$ and $|\zeta|_g = 1$ on G, hence $|\zeta|_g = 1$ on L; similarly $|\zeta'|_g = 1$ on L. Finally, $\dot\tau = 1$ and $\tau = 0$ on G. It follows that near a point on G where (say) $\zeta_1 \neq 0$, we can use coordinates $(\bar\zeta, z', \tau)$ as coordinates on L, where $\bar\zeta = (\zeta_2, \dots, \zeta_n)$, $\bar z = (z_2, \dots, z_n)$. We then find, from (7-9), that

$$\begin{aligned} z^1 &= (z')^1 + g^{ij}\zeta_j\tau + O(\tau^2), \\ z^i &= (z')^i + g^{ij}\zeta_j\tau + O(\tau^2), \quad i \ge 2, \end{aligned}$$

and we see that near G,

$$\frac{\partial z^1}{\partial \tau} \neq 0, \qquad \frac{\partial \bar z^i}{\partial \bar\zeta_j} = \tau g^{ij},$$

which shows that $\det d\pi$, where π is the map

$$L \ni (\bar\zeta, z', \tau) \mapsto (z^1(\bar\zeta, z', \tau), \bar\zeta(\bar\zeta, z', \tau), z'),$$

vanishes to order exactly $n - 1$ at G. □

Lemma 7.7. (i) *There exists $0 < \delta < 1$ and $\epsilon > 0$ such that the Legendre submanifold $L \subset {}^{s\Phi}T^*{}_{\mathrm{mf}}X$ projects diffeomorphically to the base* mf *locally near all points $(x, y, x', y', \mu, \mu', \nu, \nu', \tau) \in L \setminus G$ such that $x + x' < 2\epsilon$ and $|\nu + \nu'| < \delta$.*

(ii) *For any $\epsilon > 0$ there exists $\iota > 0$ such that L projects diffeomorphically to the base near all points $(z, z', \zeta, \zeta', \tau) \in L \setminus G$ such that $x + x' > \epsilon$ and $|\tau| < \iota$.*

Proof. (i) A topological argument shows that for sufficiently small ϵ, depending on δ, the subset of L where $x + x' < 2\epsilon$ and $|\nu + \nu'| < \delta$ is contained in a small neighborhood of the set $G \cup T_+ \cup T_-$, where $T_\pm \subset \partial_{\mathrm{bf}} L = L^{\mathrm{bf}}$ are as in (6-23). Lemma 7.6 shows that L projects diffeomorphically to mf in a deleted neighborhood of G. Near the sets $T_\pm$, we use Lemma 6.5 and the fact, proved in [Hassell and Wunsch 2008], that L is transverse to the boundary at bf to show that $(y, y', \sigma, \rho_{\mathrm{bf}})$ form coordinates locally near $T_\pm$ away from G. Here ρ_{bf} is a boundary defining function for bf and can be taken to be x for $\sigma > 1$ or x' for $\sigma < 1$. Therefore, L projects diffeomorphically to mf locally near $T_\pm$ and away from G.

(ii) The calculation above shows that if τ is small, then $d(z, z')$ is small and $|\zeta + \zeta'|$ is small, i.e., $(z, \zeta, z', \zeta', \tau)$ is close to G. So by taking ι sufficiently small, we restrict attention to a small neighborhood of $G \cap \{x + x' \geq \epsilon\}$. The result then follows directly from Lemma 7.6. ☐

Remark 7.8. In fact, we can take ι to be the injectivity radius of M.

Let M' be the compact subset of M° given by $\{x \geq \epsilon\}$, where ϵ is as in Lemma 7.7, and let ι be the injectivity radius of M. For any $z_0 \in M'$, let z denote the Riemannian normal coordinates centered at z_0, and ζ the corresponding dual coordinates. Define the quantity

$$\eta = \inf_{z_0 \in M'} \min\{|z - z'| + |\zeta - \zeta'| : |z - z_0| \leq \iota/4, |z' - z_0| \leq \iota/4, \gamma(0) = (z, \zeta),\ \gamma(t) = (z', \zeta'),\ t \geq \iota\},$$

where the minimum is taken over all geodesics $\gamma : \mathbb{R} \to M^\circ$ that are arc-length parametrized.

Lemma 7.9. *The quantity η is strictly positive.*

Proof. We use the nontrapping assumption; then there is no geodesic γ with $\gamma(0) = (z, \zeta) = \gamma(t)$, if $t > \iota$. Therefore, by compactness, the minimum for a fixed z_0 in the expression above is strictly positive. This minimum varies continuously with z_0 and therefore the inf over all z_0 in the compact set M' is also strictly positive. ☐

7D. *Proof of Theorem 1.3, part (B).* We now assemble our results to prove (1-9) for $\lambda \geq \lambda_0$, i.e., $h \leq h_0$, which by Proposition 1.12 and Section 6C is sufficient to prove part (B) of Theorem 1.3.

We now choose a partition of unity consisting of pseudodifferential operators. This is done similarly to the previous section. In particular, we will choose Q_1 to have microsupport disjoint from the characteristic variety of $h^2 \boldsymbol{H} - 1$, while the others will have compact microsupport, that is, they will be pseudodifferential operators of differential order $-\infty$. In detail, we choose Q_1 such that $\mathrm{Id} - Q_1$ is microlocally equal to the identity where $\sigma(h^2 \Delta_g) \leq 3/2$, and microsupported where $\sigma(h^2 \Delta_g) \leq 2$ (here σ denotes the semiclassical principal symbol). Then, we claim that $dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)$ is in $(hxx')^\infty C^\infty(M^2)$. To see this, we write

$$Q_1 dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) Q_1 = dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) - (\mathrm{Id} - Q_1) dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda) - dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)(\mathrm{Id} - Q_1) + (\mathrm{Id} - Q_1) dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda)(\mathrm{Id} - Q_1)$$

and use Theorem 4.2 and the microlocal support estimates as in the discussion below (6-26) to show that $\mathrm{WF}'(dE^{(j)}_{\sqrt{\boldsymbol{H}}}(\lambda))$ is empty. This piece therefore is in $(hxx')^\infty C^\infty(M^2)$, and trivially satisfies (6-25).

We now further decompose $\mathrm{Id} - Q_1$, which has compact microsupport, into a sum of terms. We first choose a function $m \in C^\infty(M_b^2)$ that is equal to 1 in a neighborhood of ∂M_b^2 and supported where $x + x' < 2\epsilon$, where ϵ is as in Lemma 7.7. Choosing δ as in Lemma 7.7, we divide up the interval

$[-2, 2]$ into $N-1$ intervals B_i each of width $\le \delta/4$, and choose a decomposition $(\mathrm{Id}-Q_1)m = \sum_{i=2}^{N} Q_i$, where the operators Q_i, and hence also Q_i^*, are supported on the set $x + x' < 2\epsilon$ and microsupported in the set $\{\sigma(h^2\Delta_g) \le 2, \nu \in 2B_i\}$. It follows that if $q' = (x, y, x', y', \mu, \mu', \nu, \nu', \tau) \in L'$ is such that $\pi_L(q') \in \mathrm{WF}'_{\mathrm{mf}}(Q_i)$ and $\pi_R(q') \in \mathrm{WF}'_{\mathrm{mf}}(Q_i^*)$, then $|\nu - \nu'| \le \delta/2$. Together with Theorem 4.2 and Lemma 7.3, this means that $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$ is a Legendrian distribution associated only to L and not to $L^\sharp$, since on $(L^\sharp)'$ we have $|\nu - \nu'| = 2 > \delta/2$. Then Lemma 7.6 guarantees that on the microsupport of $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$, the projection π to mf is either a diffeomorphism or satisfies the conditions of Proposition 7.5.

We finally decompose $(\mathrm{Id}-Q_1)(1-m)$ as $\sum_{i=N+1}^{N+N'} Q_i$, where Q_i is microsupported in a sufficiently small set so that $\mathrm{WF}_{\mathrm{mf}}(Q_i)$ is a subset of

$$\{(z, \zeta) \mid |z - z_0| + |\zeta - \zeta_0| < \eta/2\} \tag{7-10}$$

for some $z_0 \in M' = \{x \ge \epsilon\} \subset M^\circ$ and some ζ_0 (where we use Riemannian normal coordinates as in Lemma 7.9). By construction, then, if $q' = (z, z', \zeta, \zeta', \tau) \in \mathrm{WF}'_{\mathrm{mf}}(Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*)$, then we must have $|z - z'| + |\zeta - \zeta'| < \eta$ from (7-10), and also $\gamma(0) = (z, \zeta)$, $\gamma(t) = (z', \zeta')$ for some geodesic γ. From Lemma 7.9 we conclude $t < \iota$, thus γ is the short geodesic between z and z'. Consequently, $\tau < \iota$ and by Lemma 7.7 either L locally projects diffeomorphically to mf, or $q' \in {}^{\mathrm{sc}}N^*\mathrm{diag}_b$.

We next consider the symbol of $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$. As in the previous section, this symbol vanishes to order j both at $G \subset \mathrm{mf}$ and at $\partial G \times [0, h_0] \subset \mathrm{bf}$, due to the vanishing of the phase function $\widetilde{\Phi}$ at G when $d_v\widetilde{\Phi} = 0$. Therefore, in all cases, $Q_i dE^{(j)}_{\sqrt{H}}(\lambda) Q_i^*$ satisfies the conditions of Proposition 7.5 with $l = j$, and the required estimate (6-25) follows from this proposition. This completes the proof of (1-4) for $\lambda_0 \le \lambda < \infty$.

8. Trapping results

8A. ***Spectral projection estimates.*** In this section we study the Laplacian on a manifold N with C^∞ bounded geometry, in the sense that the local injectivity radius $\iota(z)$, $z \in N$ has a positive lower bound, say ϵ; the metric g_{ij}, expressed in normal coordinates in the ball of radius $\epsilon/2$ around any point z is uniformly bounded in $C^\infty(B(0, \epsilon/2))$, as z ranges over N; and the inverse metric g^{ij} is uniformly bounded in supremum norm. (In fact, we only need g_{ij} to be bounded in C^k for some k depending on dimension n, but k tends to infinity as $n \to \infty$.) This implies that the distance function $d(q, q')$ satisfies the $n \times n$ Carleson–Sjölin condition (see [Sogge 1993, Section 2.2]) uniformly over all $z \in N$ and $q, q' \in B(z, \epsilon/2)$ with $d(q, q') \ge \epsilon/4$.

Then the following Sogge-type restriction theorem holds:

Proposition 8.1. *Let N be a complete Riemannian manifold of dimension n with C^∞ bounded geometry. Then the Laplacian Δ_N on N satisfies for $\lambda \ge 1$*

$$\big\| \mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta_N}) \big\|_{L^p(N) \to L^{p'}(N)} \le C\lambda^{n(1/p-1/p')-1}, \quad 1 \le p \le \frac{2(n+1)}{n+3}. \tag{8-1}$$

This is quite likely well-known to experts, but to our knowledge such a result has not appeared in the literature, so we sketch a proof.

Proof. It is enough to prove (8-1) for the endpoints $p=1$ and $p=2(n+1)/(n+3)$, and use interpolation. We adapt Sogge's argument. Let ϵ be as above. We then choose an nonzero Schwartz function χ such that its Fourier transform $\widehat{\chi}$ is nonnegative and supported in $[\epsilon/4, \epsilon/2]$. It follows that $\chi(0) > 0$, and by taking ϵ sufficiently small, we can arrange that $\operatorname{Re}\chi \geq c > 0$ on $[0, 1]$.

Now let $\chi_\lambda^{\mathrm{ev}}(\sigma) = \chi(\sigma-\lambda) + \chi(-\sigma-\lambda)$. This is an even function, and since χ is rapidly decreasing, for sufficiently large λ we have

$$\operatorname{Re}\chi_\lambda^{\mathrm{ev}} \geq \tfrac{1}{2}c \quad \text{on } [\lambda, \lambda+1].$$

That is,

$$(\operatorname{Re}\chi_\lambda^{\mathrm{ev}})^2 - \tfrac{1}{8}c^2 = F_\lambda, \quad \text{where } F_\lambda \geq 0 \text{ on } [\lambda, \lambda+1].$$

Then for $f \in L^p$,

$$\begin{aligned}
\tfrac{1}{8}c^2 \big\| \mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta_N}) f \big\|_{L^2}^2 &= \big\langle \mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta_N}) f, \big(\operatorname{Re}\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})\big)^2 - F_\lambda(\sqrt{\Delta_N}) f \big\rangle \\
&= \big\langle \mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta_N}) \operatorname{Re}\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N}) f, \operatorname{Re}\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N}) f \big\rangle \\
&\qquad - \big\langle F_\lambda(\sqrt{\Delta_N}) \mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta_N}) f, \mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta_N}) f \big\rangle \\
&\leq \big\| \operatorname{Re}\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N}) f \big\|_{L^2}^2 \\
&\leq \big\| \chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N}) f \big\|_{L^2}^2.
\end{aligned}$$

So it is enough to estimate the operator norm of the operator $\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})$ from L^p to L^2. To do this we express $\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})$ in terms of the half-wave group $e^{it\sqrt{\Delta_N}}$:

$$\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N}) = \frac{1}{\pi} \int e^{it\sqrt{\Delta_N}} \,\widehat{\chi_\lambda^{\mathrm{ev}}}(t)\, dt. \tag{8-2}$$

Since $\widehat{\chi_\lambda^{\mathrm{ev}}} = e^{-it\lambda}\widehat{\chi}(t) + e^{it\lambda}\widehat{\chi}(-t)$ is even in t, we can write this as

$$\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N}) = \frac{1}{\pi} \int \cos t\sqrt{\Delta_N} \big(e^{-it\lambda}\widehat{\chi}(t) + e^{it\lambda}\widehat{\chi}(-t)\big)\, dt. \tag{8-3}$$

Using the fact that the kernel of $\cos t\sqrt{\Delta_N}$ is supported in $\mathscr{D}_t$ for any complete Riemannian manifold, we see that $\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})$ is supported in $\mathscr{D}_{\epsilon/2}$. The estimate (8-1) for $p=1$ then follows from [Sogge 1993, Lemma 4.2.4], or alternatively from the kernel bound $C\lambda^{(n-1)/2}$ that follows from the description of $\cos t\sqrt{\Delta_N}$ as a Fourier integral operator of order 0 associated to the conormal bundle of $\{d(x,y)=t\}$. For the other endpoint $p = 2(n+1)/(n+3)$, the argument in [Sogge 1993, Section 5.1] shows that $\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})$ maps any $f \in L^p(N)$ and supported in a ball of radius $\epsilon/2$ to $L^2(N)$ with a bound

$$\|\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N}) f\|_2 \leq C\lambda^{n(1/p-1/2)-1/2} \|f\|_p,$$

where C is uniform over N due to the bounded geometry. We then choose a sequence of balls $B(x_i, \epsilon/2)$ that cover N, such that $B(x_i,\epsilon)$ have uniformly bounded overlap, i.e., such that $\sum_i \mathbb{1}_{B(x_i,\epsilon)}$ is uniformly bounded. Then for any $f \in L^p(N)$, and using the continuous embedding from $l^p \to l^2$ for $1 \leq p < 2$,

$$
\begin{aligned}
\|\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})f\|_2^2 \le \sum_i \|\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})f\|_{L^2(B(x_i,\epsilon/2))}^2 &\le C\lambda^{2n(1/p-1/2)-1}\sum_i \|f\|_{L^p(B(x_i,\epsilon))}^2 \\
&\le C\lambda^{2n(1/p-1/2)-1}\Big(\sum_i \|f\|_{L^p(B(x_i,\epsilon))}^p\Big)^{2/p} \\
&\le C\lambda^{2n(1/p-1/2)-1}\|f\|_{L^p}^2, \qquad (8\text{-}4)
\end{aligned}
$$

showing that $\chi_\lambda^{\mathrm{ev}}(\sqrt{\Delta_N})$, and hence also $\mathbb{1}_{[\lambda,\lambda+1]}(\sqrt{\Delta_N})$, maps from $L^p(N)$ to $L^2(N)$ with a bound $C\lambda^{n(1/p-1/2)-1/2}$. Using the T^*T trick we obtain (8-1). □

8B. ***Spatially localized results for trapping manifolds.*** Let us assume now that M° is asymptotically *Euclidean* and has several ends $\mathscr{E}_1,\dots,\mathscr{E}_k$. By an *end* here we mean a connected component $\mathscr{E}_i$ of $\{x<2\epsilon\}$, where x is a boundary defining function and $\epsilon>0$ is a small fixed number, so that $\mathscr{E}_i$ is diffeomorphic to $(r_i,\infty)\times S^{n-1}$ with a metric of the form $dr^2+r^2h(y,dy,1/r)$, with h smooth, and such that the projection of the trapped set to M° is disjoint from $\mathscr{E}_i$.

Proposition 8.2. *Assume M° is asymptotically Euclidean, possibly with several ends. Let $\chi\in C^\infty(M)$ be supported in $\{x<\epsilon\}$ and let $\boldsymbol{H}$ be as in Theorem 1.3. Then one has*

$$
\|\chi\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\chi\|_{L^p\to L^{p'}} \le C\lambda^{n(1/p-1/p')-1} \quad \textit{for } 1<p\le \frac{2(n+1)}{n+3}. \tag{8-5}
$$

Proof. As in [Hassell and Vasy 1999], we can write $dE_{\sqrt{\boldsymbol{H}}}(\lambda)=(2\pi)^{-1}P(\lambda)P(\lambda)^*$, where $P(\lambda)$ is the Poisson operator associated to $\boldsymbol{H}$. Hence one needs to get $L^p(M)\to L^2(\partial M)$ bounds for $P(\lambda)^*\chi$. The Schwartz kernel of $P(\lambda)^*$ is given by

$$
P^*(\lambda;y,z') = [x^{-(n-1)/2}e^{i\lambda/x}R(\lambda;x,y;z')]|_{x=0}. \tag{8-6}
$$

Let $\chi_1,\chi_2,\chi_3\in C^\infty(M)$ be supported in $\{x<2\epsilon\}$ and equal to 1 in $\{x<\epsilon\}$, and $\chi_i\chi_j=\chi_j$ if $j<i$. Let (M_i,g_i) be a nontrapping asymptotically Euclidean manifold with one unique end isometric to $\mathscr{E}_i$. The existence of such a manifold can be easily proved if one takes ϵ small enough. There is a natural identification $\iota_j: M_j\cap\{x<2\epsilon\}\to M\cap\{x<2\epsilon\}$, and so functions supported in $\{x<2\epsilon\}$ can be considered as functions on M or $\bigcup_j M_j$. To simplify notations, we shall implicitly use this identification in what follows, instead of writing ι_j^*, ι_{j*}. Let $\boldsymbol{H}_j=\Delta_{M_j}+V_j$, where V_j is equal to V in the identified region, such that $\boldsymbol{H}_j$ satisfies the conditions of Theorem 1.3 (which can always be achieved by making V_j sufficiently positive in a compact set away from the identified region). For $\lambda\in\{z\in\mathbb{C};\operatorname{Im}\lambda>0\}$, we define the resolvent $R_j(\lambda):=(\boldsymbol{H}_j-\lambda^2)^{-1}$, and by [Hassell and Vasy 2001] the Schwartz kernel of this operator extends continuously to $\lambda\in\mathbb{R}$ as a Legendre distribution. For $\lambda>0$ it corresponds to the outgoing resolvent while for $\lambda<0$ it is the incoming resolvent. For what follows, we consider $\operatorname{Re}\lambda>0$ to deal with the outgoing case. We have the following identities for $\operatorname{Im}\lambda>0$:

$$
\begin{gathered}
(\boldsymbol{H}_j-\lambda^2)\sum_j \chi_2R_j(\lambda)\chi_1 = \chi_1+\sum_j[\boldsymbol{H}_j,\chi_2]R_j(\lambda)\chi_1, \\
\sum_j \chi_2R_j(\lambda)\chi_3(\boldsymbol{H}_j-\lambda^2) = \chi_2+\sum_j \chi_2R_j(\lambda)[\chi_3,\boldsymbol{H}_j],
\end{gathered}
$$

which can be also written as

$$\sum_j \chi_2 R_j(\lambda)\chi_1 = R(\lambda)\chi_1 + \sum_j R(\lambda)[\boldsymbol{H}_j, \chi_2]R_j(\lambda)\chi_1,$$
$$\sum_j \chi_2 R_j(\lambda)\chi_3 = \chi_2 R(\lambda) + \sum_j \chi_2 R_j(\lambda)[\chi_3, \boldsymbol{H}_j]R(\lambda).$$

Multiplying the second identity by χ_1 on the right and combining with the first one, we deduce that

$$\chi_2 R(\lambda)\chi_1 = \sum_j \chi_2 R_j(\lambda)\chi_1 + \sum_{i,j} \chi_2 R_i(\lambda)[\chi_3, \boldsymbol{H}]R(\lambda)[\boldsymbol{H}, \chi_2]R_j(\lambda)\chi_1. \tag{8-7}$$

Since $R_j(\lambda)$, $R(\lambda)$ extend to $\lambda \in \mathbb{R}$ as operators mapping $C_0^\infty(M)$ to $C^\infty(M)$, (8-7) also extends to $\lambda \in \mathbb{R}$ as a map from $C_0^\infty(M)$ to C^∞ (since $[\boldsymbol{H}, \chi_i]$ is a compactly supported differential operator). Now to obtain the Poisson operator $P(\lambda)^*$, we use (8-6) and deduce from (8-7) that

$$P(\lambda)^*\chi_1 = \sum_j P_j(\lambda)^*\chi_1 + \sum_{i,j} P_i^*(\lambda)[\chi_3, \boldsymbol{H}]R(\lambda)[\boldsymbol{H}, \chi_2]R_j(\lambda)\chi_1, \tag{8-8}$$

where $P_j(\lambda)^*$ is the adjoint of the Poisson operator for $\boldsymbol{H}_j$ on (M_j, g_j) (mapping to ∂M by the natural identification of ∂M_i with ∂M). Since $\nabla\chi_2$ and $\nabla\chi_3$ are compactly supported, we can choose $\eta \in C_0^\infty(M^\circ)$, supported in $\{x < 2\epsilon\}$, such that $\eta = 1$ on $\operatorname{supp}\nabla\chi_2 \cup \operatorname{supp}\nabla\chi_3$, and write (8-8) in the form

$$P(\lambda)^*\chi_1 = \sum_j P_j(\lambda)^*\chi_1 + \sum_{i,j} P_i^*(\lambda)\eta[\chi_3, \boldsymbol{H}]\eta R(\lambda)\eta[\boldsymbol{H}, \chi_2]\eta R_j(\lambda)\chi_1. \tag{8-9}$$

In [Cardoso and Vodev 2002, Equation (1.5)],[2] Cardoso and Vodev prove the following L^2 estimate: If $\eta \in C_0^\infty(M)$ (respectively $\eta_j \in C_0^\infty(M_j)$) is supported in $\{x < 2\epsilon\}$, then for ϵ small enough, there is $C > 0$ such that, for all $\lambda > 1$,

$$\begin{aligned} &\|\eta R(\lambda)\eta\|_{L^2\to L^2} \le C\lambda^{-1} && (\text{respectively } \|\eta_j R_j(\lambda)\eta_j\|_{L^2\to L^2} \le C\lambda^{-1}),\\ &\|\eta R(\lambda)\eta\|_{H^{-1}\to H^1} \le C\lambda && (\text{respectively } \|\eta_j R_j(\lambda)\eta_j\|_{H^{-1}\to H^1} \le C\lambda). \end{aligned} \tag{8-10}$$

Since the spectral measure $dE_j(\lambda)$ for $\sqrt{\boldsymbol{H}_j}$ on (M_j, g_j) satisfies

$$dE_j(\lambda) = \frac{\lambda}{\pi i}(R_j(\lambda) - R_j(-\lambda)) = \frac{1}{2\pi}P_j(\lambda)P_j(\lambda)^*,$$

we deduce by the TT^* argument and (8-10) that

$$\|\eta_j P_j(\lambda)\|_{L^2(\partial M_j)\to L^2(M_j)} \le C \tag{8-11}$$

[2] In [Cardoso and Vodev 2002, Theorem 1.1], for $\lambda \in \mathbb{R}^*$ and $|\lambda| \gg 1$, only the $\|\eta R(\lambda)\eta\|_{L^2\to L^2} = O(|\lambda|^{-1})$ norm appears but it is a direct consequence of [ibid., Equation (4.9)] that $\|\eta R(\lambda)\eta\|_{L^2\to H^1} = O(1)$ if η has support far enough in the end. (Note that the H^1 space in [ibid.] involves a semiclassical scaling, unlike our standard H^1 space.) Then combining with $\Delta\eta R(\lambda)\eta = \eta^2 + ([\Delta, \eta] + \lambda^2\eta)R(\lambda)\eta$, we get $\|\eta R(\lambda)\eta\|_{L^2\to H^2} = O(|\lambda|)$ for all $\lambda \in \mathbb{R}^*$ and taking adjoints give $\|\eta R(\lambda)\eta\|_{H^{-2}\to L^2}$, which by interpolating show that the $H^{-1} \to H^1$ norm is $O(|\lambda|)$.

if η_j is as above. Now since M_j is nontrapping, we also know from Theorem 1.3 and the TT^* argument that for $p \in [1, 2(n+1)/(n+3)]$ we have

$$\|P_j(\lambda)^* \chi_1\|_{L^p(M_j) \to L^2(\partial M_j)} \le C\lambda^{n(1/p-1/2)-1/2}. \tag{8-12}$$

We now use the following:

Lemma 8.3. *Assume that M_j is asymptotically Euclidean and nontrapping. Let $\chi \in C^\infty(M_j)$ be equal to 1 in $\{x < \epsilon\}$ and supported in $\{x < 2\epsilon\}$ and let $\eta \in C_0^\infty(M_j)$ be supported in $\{x < 2\epsilon\}$ such that*

$$\inf\{x \mid \exists\,(x, y) \in \operatorname{supp} \eta\} \ge \gamma \sup\{x \mid \exists\,(x, y) \in \operatorname{supp} \chi\} \tag{8-13}$$

for some $\gamma > 1$; in particular, the distance between the support of η and χ is positive. Then the following estimate holds for $1 < p \le 2(n+1)/(n+3)$ and $\lambda \ge 1$:

$$\|\eta R_j(\lambda)\chi\|_{L^p(M_j)\to L^2(M_j)} \le \frac{C}{\lambda}\|\eta\, dE_j(\lambda)\chi\|_{L^p(M_j)\to L^2(M_j)} + O(\lambda^{-\infty}).$$

Assuming for a moment the validity of Lemma 8.3, we complete the proof of Proposition 8.2. Since $\eta dE_j(\lambda)\chi = \eta P_j(\lambda)P_j(\lambda)^*\chi$, we deduce from Lemma 8.3 and equations (8-11) and (8-12) that

$$\|\eta R_j(\lambda)\chi\|_{L^p(M_j)\to L^2(M_j)} \le C\lambda^{n(1/p-1/2)-1/2-1}, \quad \lambda \ge 1. \tag{8-14}$$

Now we can analyze the boundedness of the right-hand term of (8-9) as follows: $\eta R_j(\lambda)\chi$ maps $L^p(M_j) \to L^2(M_j)$ with norm $C\lambda^{n(1/p-1/2)-1/2-1}$ by (8-14); $[\boldsymbol{H}, \chi_2]$ maps $L^2(M_j)$ to $H^{-1}(M)$ with norm independent of λ; $\eta R(\lambda)\eta$ maps $H^{-1}(M)$ to $H^1(M)$ with norm $C\lambda$ by (8-10); $[\chi_3, \boldsymbol{H}]$ maps $H^1(M_j)$ to $L^2(M)$ with norm independent of λ; and $P_i^*(\lambda)\eta$ maps $L^2(M)$ to $L^2(M)$ with uniformly bounded norm by (8-12). This concludes the proof of Proposition 8.2. □

Proof of Lemma 8.3. Recall that $R_j(\pm\lambda)$ is the sum of a pseudodifferential operator and of Legendre distributions associated to the Legendre submanifolds $({}^{s\Phi}N^*\mathrm{diag}_b, L_\pm)$ and to $(L_\pm, L_\pm^\sharp)$. Since the distance between the supports of η and χ is positive, we see that $\eta R_j(\pm\lambda)\chi$ are, like $dE_j(\lambda)$, both Legendre distributions (conic pairs) associated to $(L, L^\sharp)$ with disjoint microlocal support; indeed, the nontrapping assumption implies that L_+ and L_- intersect only at G, which is contained in ${}^{s\Phi}N^*\mathrm{diag}_b$, while $L_+^\sharp$ and $L_-^\sharp$ are disjoint. We claim that we can choose a microlocal partition of unity,

$$\sum_{i=1}^N Q_i = \mathrm{Id},$$

where the Q_i are semiclassical scattering pseudodifferential operators, such that for each pair (i, k), either $Q_i\eta R_j(\lambda)\chi Q_k$ or $Q_i\eta R_j(-\lambda)\chi Q_k$ is microlocally trivial. This does not quite follow from the disjointness of the microlocal supports of $\eta R_j(\pm\lambda)\chi$; we must also check that at $T_\pm$, there are no points $(y, y', \sigma, \mu, \mu', \nu, \nu'), (y, y', \sigma^*, \mu, \mu', \nu, \nu') \in {}^{s\Phi}T^*{}_{\mathrm{bf}}\boldsymbol{X}$, differing only in the σ coordinate, such that the first point is in $\mathrm{WF}'_{\mathrm{bf}}(\eta R_j(\lambda)\chi)$ and the second point is in $\mathrm{WF}'_{\mathrm{bf}}(\eta R_j(-\lambda)\chi)$ (see Remark 6.6). This follows from (6-23); in fact, the coordinates (ν, ν') determine σ except on the sets $T_\pm$. However, on $T_\pm$, we find that $(y, y', \sigma, \mu = 0, \mu' = 0, \nu = \pm 1, \nu' = \mp 1)$ is in L_+ if and only if $\sigma \le 1$ and $\nu = 1$, or $\sigma \ge 1$ and $\nu = -1$, while it is in L_- if and only if $\sigma \le 1$ and $\nu = -1$, or $\sigma \ge 1$ and $\nu = 1$. But condition (8-13) implies that $\sigma \geq \gamma > 1$ on the support of the kernel of $\eta R_j(\pm\lambda)\chi$, so we see that indeed it is not possible to have $(y, y', \sigma, \mu, \mu', \nu, \nu') \in \mathrm{WF}'_{\mathrm{bf}}(\eta R_j(\lambda)\chi)$ and $(y, y', \sigma^*, \mu, \mu', \nu, \nu') \in \mathrm{WF}'_{\mathrm{bf}}(\eta R_j(-\lambda)\chi)$.

Now let $\mathcal{N}$ be the set of pairs (i, k), with $1 \leq i, k \leq N$, such that $Q_i \eta R_j(\lambda)\chi Q_k$ is not microlocally trivial. This means that if $(i, k) \in \mathcal{N}$, then $Q_i \eta R_j(-\lambda)\chi Q_k$ *is* microlocally trivial. Let us also observe that as the Q_i are uniformly bounded as operators $L^2 \to L^2$, and as they are Calderón–Zygmund operators in a uniform sense as $h \to 0$, then they are uniformly bounded as operators $L^p \to L^p$ for $1 < p < \infty$. Therefore we can compute that

$$\begin{aligned}
\|\eta R_j(\lambda)\chi\|_{L^p(M_j)\to L^2(M_j)} &\leq \sum_{i,k=1}^{N} \|Q_i \eta R_j(\lambda)\chi Q_k\|_{L^p(M_j)\to L^2(M_j)} \\
&= \sum_{(i,k)\in\mathcal{N}} \|Q_i \eta R_j(\lambda)\chi Q_k\|_{L^p(M_j)\to L^2(M_j)} + O(\lambda^{-\infty}) \\
&= \sum_{(i,k)\in\mathcal{N}} \|Q_i \eta (R_j(\lambda) - R_j(-\lambda))\chi Q_k\|_{L^p(M_j)\to L^2(M_j)} + O(\lambda^{-\infty}) \\
&= \frac{1}{2\pi\lambda} \sum_{(i,k)\in\mathcal{N}} \|Q_i \eta \, dE_j(\lambda)\chi Q_k\|_{L^p(M_j)\to L^2(M_j)} + O(\lambda^{-\infty}) \\
&\leq \frac{CN^2}{\lambda} \|\eta \, dE_j(\lambda)\chi\|_{L^p(M_j)\to L^2(M_j)} + O(\lambda^{-\infty}),
\end{aligned} \tag{8-15}$$

proving the lemma. $\square$

Remark 8.4. Observe that we missed the endpoint $p = 1$ due to our use of Calderón–Zygmund theory. In the case that M is exactly Euclidean for $x < 2\epsilon$ we can take M_j to be flat Euclidean space and then it is straightforward to check that $\eta R_j(\lambda)\chi$ is bounded $L^1(M_j) \to L^2(M_j)$ with norm $O(\lambda^{(n-3)/2})$, which gives us Proposition 8.2 for $p = 1$ in this case.

In [Seeger and Sogge 1989], spectral multiplier estimates are proved for compact manifolds for the same exponents as in Theorem 1.1. This was done using Sogge's discrete L^2 restriction theorem, i.e., Proposition 8.1. One may suspect that, since spectral multiplier estimates can be proved in the compact case, and since we have localized restriction estimates outside the trapped sets, that one should be able to prove spectral multiplier estimates on asymptotically conic manifolds unconditionally, i.e., without any nontrapping assumption. We have not been able to prove this, however, but have the following localized results:

Proposition 8.5. *Let M° be a manifold with Euclidean ends, and let $p \in [1, 2(n+1)/(n+3)]$. Let $\boldsymbol{H}$ be as in Theorem 1.3, let χ be a cutoff function as in Proposition 8.2, let F be a multiplier satisfying the assumption of Theorem 1.1, i.e., $F \in H^s$ for some $s > \max\big(n\big(\frac{1}{p} - \frac{1}{2}\big), \frac{1}{2}\big)$. Then we have*

$$\sup_{\alpha>0} \|F(\alpha\sqrt{\boldsymbol{H}})\chi\|_{p\to p} \leq C\|F\|_{H^s}.$$

This is proved by following the proof of Theorem 1.1, using (8-5) in place of (2-3).

Proposition 8.6. *Let $\omega \in C_c^\infty(M^\circ)$ be compactly supported and let $\boldsymbol{H}$ and F be as above. Then the following estimate holds*:

$$\sup_{\alpha>0} \|\omega F(\alpha\sqrt{\boldsymbol{H}})\|_{L^p\to L^p} \le \|F\|_{H^s}.$$

This is proved by following the method of [Seeger and Sogge 1989], using the compact support of ω to obtain the embedding from L^2 to L^p as in [ibid., Equation (3.11)].

8C. ***Examples with elliptic trapping.*** Here we show that the restriction estimate at high frequency generically fails for asymptotically conic manifolds with elliptic closed geodesics. Indeed, it has been proved by Babich and Lazutkin [1968] and Ralston [1977] that if there exists a closed geodesic γ in M such that the eigenvalues of the linearized Poincaré map of γ are of modulus 1 and are not roots of unity, then there exists a sequence of quasimodes $u_j \in C_0^\infty(K)$ with K a fixed compact set containing the geodesic, a sequence of positive real numbers $\lambda_j \to \infty$ such that for all $N > 0$ there is $C_N > 0$ such that

$$\|u_j\|_{L^2} = 1, \quad \|(\Delta_g - \lambda_j^2)u_j\|_{L^2} \le C_N\lambda_j^{-N}. \tag{8-16}$$

Proposition 8.7. *Assume that (M, g) is an asymptotically conic manifold with an elliptic closed geodesic such that the eigenvalues of the linearized Poincaré map of γ are of modulus 1 and are not roots of unity. Then for all $p \in [1, 2)$ and $M \ge 0$ the spectral measure $dE_{\sqrt{\Delta_g}}(\lambda)$ does **not** satisfy the restriction estimate*

$$\exists C > 0,\ \exists\lambda_0 > 0,\ \forall\lambda \ge \lambda_0, \quad \|dE_{\sqrt{\Delta_g}}(\lambda)\|_{L^p\to L^{p'}} \le C\lambda^M.$$

Proof. Let u_j be the quasimodes above. Then the inequality

$$\|(\Delta_g - \lambda_j^2)u_j\|_{L^2} \le C_N\lambda_j^{-N}$$

implies that

$$\left\|\mathbb{1}_{\mathbb{R}\setminus[\lambda_j^2-2C_N\lambda_j^{-N},\lambda_j^2+2C_N\lambda_j^{-N}]}(\Delta_g)u_j\right\|_{L^2} \le \frac{1}{2}$$

since $\|(\Delta_g - \lambda_j^2)v\| \ge c\|v\|$ if v is in the range of the spectral projector $\mathbb{1}_{\mathbb{R}\setminus[\lambda_j^2-c,\lambda_j^2+c]}(\Delta_g)$. Therefore

$$\left\|\mathbb{1}_{[\lambda_j^2-2C_N\lambda_j^{-N},\lambda_j^2+2C_N\lambda_j^{-N}]}(\Delta_g)u_j\right\|_{L^2} \ge \frac{\sqrt{3}}{2}, \tag{8-17}$$

and using the fact that $\mathbb{1}_{[\lambda_j^2-2C_N\lambda_j^{-N},\lambda_j^2+2C_N\lambda_j^{-N}]}(\Delta_g)$ is a projection,

$$\left\langle u_j, \mathbb{1}_{[\lambda_j^2-2C_N\lambda_j^{-N},\lambda_j^2+2C_N\lambda_j^{-N}]}(\Delta_g)u_j\right\rangle \ge \frac{3}{4}. \tag{8-18}$$

This implies that for large enough λ we have

$$\left\langle u_j, \mathbb{1}_{[\lambda_j-2C_N\lambda_j^{-N-1},\lambda_j+2C_N\lambda_j^{-N-1}]}(\sqrt{\Delta}_g)u_j\right\rangle \ge \frac{3}{4}. \tag{8-19}$$

Now assume that there exists C such that $\|dE_{\sqrt{\Delta}_g}(\lambda)\|_{L^p\to L^{p'}} \le C\lambda^M$. Then using the continuous embeddings from $L^2(K) \to L^p(K)$ and $L^{p'}(K)$ to $L^2(K)$, we see that there is $C' > 0$ such that

$$\langle u_j, dE_{\sqrt{\Delta}_g}(\lambda)u_j\rangle \le C'\lambda^M\|u_j\|_{L^2} \le 2C'\lambda^M.$$

By integrating this on the interval $[\lambda_j - 2C_N\lambda_j^{-N-1}, \lambda_j + 2C_N\lambda_j^{-N-1}]$, we contradict (8-19) if $N+1$ is chosen larger than M and j is large enough. □

Remark 8.8. In fact, one can construct examples where the spectral measure blows up exponentially with respect to the frequency λ. Consider a Riemannian manifold (M, g) which is a connected sum of flat $\mathbb{R}^n$ and a sphere S^n, so that it contains an open set S isometric to part of a round sphere S^n, namely

$$S = \{x = (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1}\,;\, |x| = 1, x_1^2 + x_2^2 > \tfrac{1}{4}\}.$$

Consider the functions $u_N(x) := (x_1 + ix_2)^N$ (as functions on $\mathbb{R}^{n+1}$). These restrict to eigenfunctions on S^n with corresponding eigenvalue $N(N+n-1)$ and with norm $\|u_N\|_{L^2} \sim cN^{-1/4}$ for some $c > 0$ as $N \to \infty$. Let $\chi \in C_0^\infty(S)$ be equal to 1 on $S \cap \{x_1^2 + x_2^2 \ge 1/2\}$ and extend it by 0 on $M \setminus S$. The modified function $v_N = \chi u_N / \|\chi u_N\|_{L^2}$ satisfies

$$(\Delta_g - N(N+n-1))v_N = [\Delta_g, \chi]u_N / \|\chi u_N\|_{L^2}.$$

But since $|x_1 + ix_2| < 1/2$ on the support of $[\Delta_g, \chi]$ and since $\|\chi u_N\| > CN^{-1/4}$ for some $C > 0$ when N is large, we deduce that $(\Delta_g - N(N+n-1))v_N = O_{L^2}(e^{-\alpha N})$ for some $\alpha > 0$. Applying the argument of Proposition 8.7, we deduce that there exist $C > 0$, $\beta > 0$ and a sequence $\lambda_N \sim \sqrt{N(N+n-1)}$ such that $\|dE(\lambda_N)\|_{L^p \to L^{p'}} \ge Ce^{\beta\lambda_N}$.

9. Conclusion: application and open problems

The restriction theorem can be applied to prove Sobolev estimates. Recall that the Hardy–Littlewood–Sobolev theorem tells us the inverse of the Laplacian, i.e., the resolvent at zero energy, on $\mathbb{R}^n$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ when $n \ge 3$ and $p = 2n/(n+2)$; this holds true on any asymptotically conic manifold. Since the resolvent looks like the spectral measure microlocally away from the diagonal, and since this value of p is in the range $[1, 2(n+1)/(n+3)]$ in which the spectral measure is bounded $L^p \to L^{p'}$ by Theorem 1.3, this suggests that the resolvent kernel $(\Delta - (\lambda \pm i0)^2)^{-1}$ on an asymptotically conic manifold should be bounded from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ when $p = 2n/(n+2)$. This result has been recently proved in [Guillarmou and Hassell 2012] and if in addition the metric is nontrapping, we have the following uniform Sobolev estimate: For $p = 2n/(n+2)$, $p' = 2n/(n-2)$ there exists $C > 0$ independent of $\lambda \in \mathbb{C}$ such that

$$\forall u \in W^{2,p}(M), \quad \|(\Delta - \lambda^2)u\|_{L^p} \ge C\|u\|_{L^{p'}}.$$

This was proved by Kenig–Ruiz–Sogge [1987] for constant coefficient operators on $\mathbb{R}^n$. The boundedness of the resolvent for $p \in [2n/(n+2), 2(n+1)/(n+3)]$ is also satisfied for $\lambda \ne 0$ but the constant is $O(|\lambda|^{n(1/p - 1/p') - 2})$.

We mention several ways in which the investigations of this paper could be extended.

Theorem 1.3 is only stated for dimensions $n \ge 3$. This is because the proof relies on the analysis of [Guillarmou and Hassell 2008; Guillarmou et al. 2012], which is only done for $n \ge 3$. It would be interesting to treat also the case $n = 2$. The main difficulty in doing this is to write down a suitable inverse for the model operator at the zf face in the construction of [Guillarmou and Hassell 2008, Section 3], which is not invertible as an operator on $L^2(M)$ in two dimensions as it is in all higher dimensions.

One could also extend Theorem 1.3 by allowing potential functions which are $O(x^2)$ instead of only $O(x^3)$ at infinity, i.e., inverse-square decay near infinity. This should be relatively straightforward, because all the analysis has been done in the two papers cited above. For potentials of the form $V = V_0 x^2$, with V_0 strictly negative at ∂M, this would have the effect of changing the "numerology", i.e., the range of p and the power of λ in (1-4), for example. Here we preferred not to treat this case, in order not to complicate the statement of Theorem 1.3, but rather to keep the numerology as it is in the familiar setting of the classical Stein–Tomas theorem, and in Sogge's discrete L^2 restriction theorem.

Another way to extend Theorem 1.3 would be to allow operators $\boldsymbol{H}$ with eigenvalues. In this case, we would consider the positive part $\mathbb{1}_{(0,\infty)}(\boldsymbol{H})$ of the operator $\boldsymbol{H}$. We expect such a generalization to be straightforward, as the analysis has been carried out in [Guillarmou and Hassell 2008; Guillarmou et al. 2012], with the only complication being that $\mathbb{1}_{(0,\infty)}(\boldsymbol{H})$ does not satisfy the finite speed propagation property (2-2).

We close by posing, as open problems, some possible generalizations that seem to be a little less straightforward:

- Prove (or disprove) the restriction theorem for high energies in the presence of trapping, in the case that the trapped set is hyperbolic and the topological pressure assumption of [Nonnenmacher and Zworski 2009] and [Burq et al. 2010] is satisfied.
- Prove (or disprove) the spectral multiplier result for high energies in the trapping case, i.e., Propositions 8.5 and 8.6 without the cutoff functions.

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COLIN GUILLARMOU: cguillar@dma.ens.fr
DMA, U.M.R. 8553 CNRS, École Normale Supérieure, 45 rue d'Ulm, 75005 Paris CEDEX 05, France

ANDREW HASSELL: Andrew.Hassell@anu.edu.au
Department of Mathematics, Australian National University, Canberra ACT 0200, Australia

ADAM SIKORA: sikora@mq.edu.au
Department of Mathematics, Macquarie University, Sydney NSW 2109, Australia

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HOMOGENIZATION OF NEUMANN BOUNDARY DATA WITH FULLY NONLINEAR OPERATOR

SUNHI CHOI, INWON C. KIM AND KI-AHM LEE

In this paper we study periodic homogenization problems for solutions of fully nonlinear PDEs in half-spaces with oscillatory Neumann boundary data. We show the existence and uniqueness of the homogenized Neumann data for a given half-space. Moreover, we show that there exists a continuous extension of the homogenized slope as the normal of the half-space varies over "irrational" directions.

1. Introduction

In this paper, we consider the averaging phenomena for solutions of uniformly elliptic nonlinear PDEs in half-spaces coupled with oscillatory Neumann boundary data. To be precise, let $\mathcal{M}^{n-1}$ be the normed space of symmetric $n \times n$ matrices and consider the function $F(M) : \mathcal{M}^{n-1} \to \mathbb{R}$, which satisfies:

(F1) F is uniformly elliptic, that is, there exist constants $0 < \lambda < \Lambda$ such that

$$\lambda \|N\| \leq F(M) - F(M+N) \leq \Lambda \|N\| \quad \text{for any} \quad N \geq 0;$$

(F2) (homogeneity) $F(tM) = tF(M)$ for any $M \in \mathcal{M}^{n-1}$ and $t > 0$. In particular, $F(0) = 0$.

(F3) $F(M)$ only depends on the eigenvalues of M.

The homogeneity condition (F2) can be relaxed (see condition (F4) of [Barles et al. 2008], for example). Typical examples of nonlinear operators that satisfy (F1)–(F3) are the Pucci extremal operators

$$\mathcal{P}^+(D^2u(x)) := \lambda \sum_{\mu_i<0} \mu_i + \Lambda \sum_{\mu_i \geq 0} \mu_i, \quad \mathcal{P}^-(D^2u(x)) := \Lambda \sum_{\mu_i<0} \mu_i + \lambda \sum_{\mu_i \geq 0} \mu_i,$$

where $\mu_1, \dots, \mu_n$ are eigenvalues of $D^2u(x)$.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ and suppose $g(x) : \mathbb{R}^n \to \mathbb{R}$ satisfies

(a) $g \in C^\beta(\mathbb{R}^n)$ for some $0 < \beta \leq 1$;

(b) $g(x + e_k) = g(x)$ for all $x \in \mathbb{R}^n$ and $k = 1, \dots, n$.

Next, for a given $p \in \mathbb{R}^n$, let $\Pi_\nu(p)$ be a strip domain in $\mathbb{R}^n$ with unit normal ν, that is,

$$\Pi_\nu(p) = \{x : -1 \leq (x-p) \cdot \nu \leq 0\}, \quad \text{where} \quad |\nu| = 1. \tag{1}$$

With F, g and Π_ν as given above, our goal is to describe the limiting behavior of u_ε as $\varepsilon \to 0$, where

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u_ε satisfies

$$(P_\varepsilon) \qquad \begin{cases} F(D^2u_\varepsilon) = 0 & \text{in } \Pi_\nu(p), \\ \nu \cdot Du_\varepsilon = g(x/\varepsilon) & \text{on } \Gamma_0 := \{(x-p)\cdot\nu = 0\}, \\ u = 1 & \text{on } \Gamma_I := \{(x-p)\cdot\nu = -1\}. \end{cases}$$

The fixed boundary data on Γ_I is introduced to avoid discussion of the compatibility condition on g and to ensure the existence of u^ε.

Homogenization of elliptic, divergence-form equations with oscillatory coefficients and conormal boundary data is a classical subject. Let Ω be an open and bounded subset of $\mathbb{R}^n$. Consider $u^\varepsilon : \bar{\Omega} \to \mathbb{R}$ solving

$$\nabla \cdot \Big(A\Big(\frac{x}{\varepsilon}\Big) \nabla u_\varepsilon \Big) = 0, \tag{2}$$

with the Neumann (conormal) condition

$$\nu \cdot \Big(A\Big(\frac{x}{\varepsilon}\Big) \nabla u \Big)(x) = g\Big(\frac{x}{\varepsilon}\Big), \qquad x \in \partial\Omega. \tag{3}$$

The problem (2)–(3) has been widely studied, and by now has been well understood; see [Bensoussan et al. 1978] for an overview. We first consider the case when Ω is a half-space; thus, let

$$\Omega = \Sigma_\nu := \{x : (x-p)\cdot\nu \le 0\}.$$

We define the averaged Neumann data

$$\mu(\nu, \varepsilon) := \int_{(x-p)\cdot\nu=0, |x-p|\le 1} g\Big(\frac{x}{\varepsilon}\Big)\, dx. \tag{4}$$

Integrating by parts, one can show that u^ε locally uniformly converges to a continuous function $u^0 : \bar{\Omega} \to \mathbb{R}$ as $\varepsilon \to 0$ if and only if $\mu(\nu) := \lim_{\varepsilon\to 0} \mu(\nu, \varepsilon)$ exists, and that u^0 solves the averaged equation

$$(\bar{P}_{\text{div}}) \qquad \begin{cases} -\nabla \cdot (A^0 \nabla u^0)(x) = 0 & \text{for } x \in \Omega, \\ \nu \cdot (A^0 \nabla u^0) = \mu(\nu) & \text{for } x \in \partial\Omega. \end{cases}$$

Therefore, different results hold depending on the choice of p and ν:

(a) If ν is a "rational" vector — one parallel to a vector in $\mathbb{Z}^n$ — then $\mu(\nu)$ exists if $p = 0$, and

$$\mu(\nu) = \text{the average of } g(y) \text{ on the hyperplane } \{x \cdot \nu = 0\}.$$

(b) If ν is a rational vector and $p \ne 0$, then there may be no limit of $\mu(\nu, \varepsilon)$ and u^ε can have different subsequential limits.

(c) If ν is not a rational vector, then due to Weyl's equidistribution theorem (Lemma 2.5), $\mu(\nu, \varepsilon)$ converges to

$$\mu(\nu) = \langle g \rangle := \int_{[0,1]^n} g(y)\, dy,$$

independent of the choice of p. In particular, the homogenized slope $\mu(\nu)$ is discontinuous at every rational direction ν, but otherwise continuous.

From these results, the divergence form of the operator, and the fact that rational directions are of zero measure in $\mathcal{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$, the following results hold for the general domain Ω: if $\partial\Omega$ does not contain flat pieces whose normal vectors belong to $\mathbb{R}\mathbb{Z}^n$, then u^ε converges locally uniformly to the solution u^0 of $(\bar{P}_{\text{div}})$ with $\mu(\nu)$ replaced by $\langle g \rangle$. We refer to [Bensoussan et al. 1978] for detailed analysis. Note that u^0 is smooth up to the boundary, due to the fact that $\langle g \rangle$ is continuous (constant in particular).

For nonlinear or nondivergence operators, or for linear operators with oscillatory nonlinear boundary data, little is known for the homogenization of the oscillating Neumann boundary data. Most available results concern half-space domains going through the origin with its normal pointing to a rational direction. Tanaka [1984] considered some model problems in half-spaces whose boundary is parallel to the axes of the periodicity, by purely probabilistic methods. Arisawa [2003] studied special cases of problems in oscillatory domains near half-spaces going through the origin, using viscosity solutions as well as stochastic control theory. Generalizing her results, Barles, Da Lio and Souganidis [Barles et al. 2008] studied the problem for operators with oscillating coefficients, in half-space domains whose boundary is parallel to the axes of periodicity, with a series of assumptions which guarantee the existence of an approximate corrector.

In this paper, we extend the results above to the setting of general half-spaces Π_ν, defined in (1), where p is not necessarily zero and ν ranges over all directions in $\mathbb{R}^n$. In particular, we show the continuity properties of the homogenized slope $\mu(\nu)$ over the normal directions ν (see Theorem 1.2(ii)), with the hope that such results will lead to better understanding of homogenization phenomena in domains with general geometry (work in progress). Note that, as observed in the linear case, homogenized slopes may not exist if ν is parallel to a vector in $\mathbb{Z}^n$ and if $p \neq 0$, and therefore the best result we can hope for is the existence of the continuous function $\bar{\mu}(\nu) : S^{n-1} \to \mathbb{R}$ such that $\bar{\mu}(\nu) = \mu(\nu)$ for $\nu \in \mathcal{S}^{n-1} - \mathbb{R}\mathbb{Z}^n$. This is precisely what we will show.

Definition 1.1. A direction $\nu \in \mathcal{S}^{n-1}$ is called *rational* if $\nu \in \mathbb{R}\mathbb{Z}^n$, and *irrational* otherwise.

Theorem 1.2 (Main Theorem). *For a given $p \in \mathbb{R}^n$, let u_ε solve (P_ε).*

(i) *Let ν be an irrational direction. Then there is a unique constant $\mu(\nu) \in [\min g, \max g]$ such that u^ε locally uniformly converges to the solution of*

$$(\bar{P}) \qquad \begin{cases} F(D^2u) = 0 & \textit{in } \Pi_\nu, \\ \nu \cdot Du = \mu(\nu) & \textit{on } \Gamma_0, \\ u = 1 & \textit{on } \Gamma_I. \end{cases}$$

(ii) *$\mu(\nu) : (\mathcal{S}^{n-1} - \mathbb{R}\mathbb{Z}^n) \to \mathbb{R}$ has a continuous extension $\bar{\mu}(\nu) : \mathcal{S}^{n-1} \to \mathbb{R}$.*

(iii) *For rational directions ν, if Γ_0 goes through the origin (that is if $p = 0$), then the statement in (i) holds for ν as well.*

(iv) *(Error estimate). Let ν be an irrational direction. Then for u^ε and u solving (P_ε) and $(\bar{P})$, we have the following estimate: for any $0 < \alpha < 1$, there exists a constant $C_\alpha > 0$ such that*

$$|u^\varepsilon - u| \le C_\alpha \omega(\varepsilon)^\alpha \quad \textit{in } \Pi_\nu. \tag{5}$$

Here $\omega(\varepsilon)$ depends on the "discrepancy" associated to ν as defined in (7).

Remark 1.3. Our method can be applied to the operators of the form $F(D^2u, x) = f(x)$, with F and f continuous in x, but we will restrict ourselves to the simple case discussed in (P_ε) for clarity of exposition. On the other hand, our proof for the continuity of $\mu(\nu)$ (Theorem 1.2(ii)) on page 965, cannot handle the case where the operator F depends on the oscillatory variable x/ε (see Remark 4.8).

2. Preliminary results

Let Ω be an open, bounded domain. Let Γ_I be a part of its boundary, and define $\Gamma_0 := \partial\Omega - \Gamma_I$. For a continuous function $f(x, \nu) : \mathbb{R}^n \times \mathcal{S}^{n-1} \to \mathbb{R}$, let us recall the definition of viscosity solutions for the following problem:

$$(P)_f \qquad \begin{cases} F(D^2u) = 0 & \text{in } \Omega, \\ \nu \cdot Du = f(x, \nu) & \text{on } \Gamma_0, \\ u = 1 & \text{on } \Gamma_I, \end{cases}$$

where $\nu = \nu_x$ denotes the outward normal at $x \in \partial\Omega$ with respect to Ω.

The following definition is equivalent to the ones given in [Crandall et al. 1992]:

Definition 2.1. (a) An upper semicontinuous function $u : \bar{\Omega} \to \mathbb{R}$ is a *viscosity subsolution* of $(P)_f$ if

(i) $u \leq 1$ on Γ_I, and

(ii) for a given domain $\Sigma \subset \mathbb{R}^n$, u cannot cross from below any C^2 function ϕ in Σ which satisfies

$$\begin{cases} F(D^2\phi) > 0 & \text{in } \Omega \cap \Sigma, \\ \nu \cdot D\phi > f(x, \nu) & \text{on } \Gamma_0 \cap \Sigma, \\ \phi > u & \text{on } (\partial\Sigma \cup \Gamma_I) \cap \bar{\Sigma}. \end{cases}$$

(b) A lower semicontinuous function $u : \bar{\Omega} \to \mathbb{R}$ is a *viscosity supersolution* of $(P)_f$ if:

(i) $u \geq 1$ on Γ_I;

(ii) for a given domain $\Sigma \subset \mathbb{R}^n$, u cannot cross from above any C^2 function φ which satisfies

$$\begin{cases} F(D^2\phi) < 0 & \text{in } \Omega \cap \Sigma, \\ \nu \cdot D\phi < f(x, \nu) & \text{on } \Gamma_0 \cap \Sigma, \\ \phi < u & \text{on } (\partial\Sigma \cup \Gamma_I) \cap \bar{\Sigma}. \end{cases}$$

(c) u is a *viscosity solution* of $(P)_f$ if u is both a viscosity sub- and supersolution of $(P)_f$.

Existence and uniqueness of viscosity solutions of $(P)_f$ is based on the comparison principle we state below:

Theorem 2.2 [Ishii and Lions 1990, Section V]. *Suppose Ω, Γ_I, Γ_0, F and ν are as given above, and let $f : \mathbb{R}^n \times \mathcal{S}^{n-1} \to \mathbb{R}$ be continuous. Let u and v respectively be a viscosity sub- and supersolution of $(P)_f$ in a domain $\Sigma \subset \mathbb{R}^n$. If $u \leq v$ on $\partial\Sigma$, then $u \leq v$ in Ω.*

For details on the proof of this theorem as well as well-posedness of the problem $(P)_f$, we refer to [Crandall et al. 1992; Ishii 1991; Ishii and Lions 1990].

Next we state some regularity results that will be used in the paper.

Theorem 2.3 [Caffarelli and Cabré 1995, Chapter 8, modified for our setting]. *Let u be a viscosity solution of $F(D^2u)=0$ in a domain Ω. For any $0<\alpha<1$ and for any compact subset Ω' of Ω, we have*

$$\|u\|_{C^\alpha(\Omega')} \le Cd^{-\alpha}\|u\|_{L^\infty(\Omega)},$$

where $C>0$ depends on n, λ, Λ and $d=d(\Omega',\partial\Omega)$.

Theorem 2.4 [Milakis and Silvestre 2006, Theorems 8.1 and 8.2]. *Let*

$$B_r^+ := \{|x|<r\}\cap\{x\cdot e_n \ge 0\} \quad \textit{and} \quad \Gamma := \{x\cdot e_n = 0\}\cap B_1.$$

Let u be a viscosity solution of

$$\begin{cases} F(D^2u)=0 & \text{in } B_1^+, \\ \nu\cdot Du = g & \text{in } \Gamma. \end{cases}$$

(a) *If g is bounded, then u is in $C^\alpha(\overline{B_{1/2}^+})$ for some $\alpha=\alpha(n,\lambda,\Lambda)$, and we have the estimate*

$$\|u\|_{C^\alpha(\overline{B_{1/2}^+})} \le C\big(\|u\|_{L^\infty(\overline{B_1^+})} + \max\|g\|\big).$$

(b) *Suppose $g\in C^\beta(\mathbb{R}^n)$, where $0<\beta\le 1$. Then u is in $C^{1,\gamma}(\overline{B_{1/2}^+})$, where $\gamma=\min(\alpha_0,\beta)$ and $\alpha_0=\alpha_0(n,\lambda,\Lambda)$. Moreover, we have the estimate*

$$\|u\|_{C^{1,\alpha}(\overline{B_{1/2}^+})} \le C\big(\|u\|_{L^\infty(\overline{B_1^+})} + \|g\|_{C^\beta}\big).$$

In (a) *and* (b), *the positive constant C depends only on n, λ, Λ and α.*

Let us next discuss the averaging property of the sequence $(nx)_n$ mod 1, where x is an irrational number, and its applications to dimensions greater than 1, which will prove useful in our analysis in Section 3. Since we obtain estimates on the convergence rate of solutions for (P_ε) in our result, we are particularly interested in the estimates on the rate of convergence of the sequence $(nx)_n$ to the uniform distribution (Definition 2.6). We begin by recalling the notion of equidistribution.

- A bounded sequence $(x_1,x_2,x_3\dots)$ of real numbers is said to be *equidistributed* on an interval $[a,b]$ if for any $[c,d]\subset[a,b]$, we have

  $$\lim_{n\to\infty}\frac{\big|\{x_1,\dots,x_n\}\cap[c,d]\big|}{n} = \frac{d-c}{b-a}.$$

  Here $|\{x_1,\dots,x_n\}\cap[c,d]|$ denotes the number of elements.
- The sequence $(x_1,x_2,x_3,\dots)$ is said to be equidistributed modulo 1 if $(x_1-[x_1],x_2-[x_2],\dots)$ is equidistributed in the interval $[0,1]$.

Lemma 2.5 [Weyl 1910, Weyl's equidistribution theorem]. *If a is an irrational number, $(a,2a,3a,\dots)$ is equidistributed modulo* 1.

To discuss quantitative versions of Lemma 2.5, we introduce the notion of *discrepancy*.

Definition 2.6 [Kuipers and Niederreiter 1974]. Let (x_k), $k=1,2,\dots$ be a sequence in $\mathbb{R}$. For a subset $E\subset[0,1]$, let $A(E;N)$ denote the number of points $\{x_n\}$, $1\le n\le N$, that lie in E.

(a) The sequence (x_n), $n = 1, 2, \dots$ is said to be *uniformly distributed* mode 1 in $\mathbb{R}$ if

$$\lim_{N\to\infty} \frac{A(E;N)}{N} = \mu(E)$$

for all $E = [a, b)$. Here μ denotes the Lebesgue measure.

(b) For $x \in [0, 1]$, we define the discrepancy

$$D_N(x) := \sup_{E=[a,b)} \left| \frac{A(E;N)}{N} - \mu(E) \right|,$$

where $A(E; N)$ is defined with the sequence (kx), $k \in \mathbb{N}$, modulo 1.

It easily follows from Lemma 2.5 that the sequence $(x_k) = (kx)_{k\in\mathbb{N}}$ is uniformly distributed modulo 1 for any irrational number $x \in \mathbb{R}$. In particular, $D_N(x)$ converges to zero as $N \to \infty$.

Next, let $\mathcal{S}^{n-1} = \{\nu \in \mathbb{R}^n : |\nu| = 1\}$. For a direction $\nu = (\nu_1, \dots, \nu_n) \in \mathcal{S}^{n-1}$, let ν_i be the component with the biggest size, that is,

$$|\nu_i| = \max\{|\nu_j| : 1 \le j \le n\}.$$

(If there are multiple components, then we choose the one with largest index.)

Let H_ν be the hyperplane in $\mathbb{R}^n$ which passes through 0 and is normal to ν:

$$H_\nu = \{x \in \mathbb{R}^n : x \cdot \nu = 0\}.$$

Since $\nu_i \neq 0$, there exists $m(\nu)$ such that

$$(1, \dots, 1, m(\nu), 1, \dots, 1) \cdot \nu = 0, \tag{6}$$

where $m(\nu)$ is the i-th component of $(1, \dots, 1, m(\nu), 1, \dots, 1)$. Then we define

$$\omega_\nu(\varepsilon) := D_N(m(\nu)), \quad \text{where } N = \varepsilon^{-9/10}. \tag{7}$$

Note that, if $m(\nu)$ is irrational, then $\omega_\nu(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now we are ready to state our quantitative estimate on the averaging properties of the vector sequence $(n\nu)$ with an irrational direction ν, which will be used in the rest of the paper. Recall that for $\nu \in \mathcal{S}^{n-1}$, $\Pi_\nu(p) = \{x : -1 \le (x - p) \cdot \nu \le 0\}$. Write $\Gamma_0 = \{x : (x - p) \cdot \nu = 0\}$ and define

$$H_\nu = \{x : x \cdot \nu = 0\}.$$

Lemma 2.7. *For $\nu \in \mathbb{R}^n$ and $x_0 \in \Pi_\nu$, let $H(x_0) := H_\nu + x_0$. Let $0 < \varepsilon < \operatorname{dist}(x_0, \Gamma_0)$.*

(i) *Suppose that ν is a rational direction. Then for any $x \in H(x_0)$, there is $y \in H(x_0)$ such that*

$$|x - y| \le M_\nu \varepsilon, \quad y - x_0 \in \varepsilon\mathbb{Z}^n,$$

where $M_\nu > 0$ is a constant depending on ν.

(ii) *Suppose that ν is an irrational direction, and let $\omega_\nu : [0,1) \to \mathbb{R}^+$ be defined as in (7). Then there exists a dimensional constant $M > 0$ such that the following is true: for any $x \in H(x_0)$, there is $y \in \mathbb{R}^n$ such that*

$$|x-y| \leq M\varepsilon^{1/10}, \quad y - x_0 \in \varepsilon\mathbb{Z}^n$$

and

$$\operatorname{dist}(y, H(x_0)) < \varepsilon\omega_\nu(\varepsilon), \tag{8}$$

where ω_ν is as given in (7).

(iii) *If ν is an irrational direction, then for any $z \in \mathbb{R}^n$ and $\delta > 0$, there is $w \in H(x_0)$ such that*

$$|z-w| \leq \delta \ mod\ \varepsilon\mathbb{Z}^n.$$

Proof. The proof of (i) is immediate from the fact that for any rational direction ν, there exists an integer $M > 0$ depending on ν such that $M\nu \in \mathbb{Z}^n$.

Next, we prove (ii). Let ν be an irrational direction in $\mathbb{R}^n$. Without loss of generality, we may assume

$$|\nu_n| = \max\{|\nu_j| : 1 \leq j \leq n\}.$$

Let x be any point on $H(x_0)$: after a translation, we may assume that $x = 0$. Choose m such that

$$\varepsilon(1, 1, \ldots, 1, m) \in H(x_0).$$

Note that $M = |m| \leq n^2$. Also note that m is irrational since ν is an irrational direction. Since $H(x_0)$ contains $x = 0$, we have

$$k\varepsilon(1, 1, \ldots, 1, m) \in H(x_0) \text{ for any integer } k.$$

Consider the sequence (km), $k \in \mathbb{N}$. From the definition of $\omega_\nu(\varepsilon)$ and the discrepancy function $D_N(m)$, it follows that any interval $[a, b] \subset [0, 1]$ of length $\omega_\nu(\varepsilon)$ contains at least one point km (mod 1), for some $k \leq N = \varepsilon^{-9/10}$.

Hence for any $z = (0, 0, \ldots, 0, x_n) \in [0, \varepsilon]^n$, there exists

$$w = k\varepsilon(1, 1, \ldots, 1, m) \in H(x_0), \quad 0 \leq k \leq \varepsilon^{-9/10}$$

such that

$$|z - w| \leq \varepsilon\omega_\nu(\varepsilon) \mod \varepsilon\mathbb{Z}^n.$$

Similarly, for any $z \in [0, \varepsilon]^n$, there exists $w \in H(x_0) \cap \big(k\varepsilon(1, 1, \ldots, 1, m) + [0, \varepsilon]^n\big)$ such that

$$|z - w| \leq \varepsilon\omega_\nu(\varepsilon) \mod \varepsilon\mathbb{Z}^n, \quad 0 \leq k \leq \varepsilon^{-9/10}. \tag{9}$$

We continue with the proof of (ii). Recall that the coordinates are shifted so that $x = 0$. Thus it suffices to find $y \in \mathbb{R}^n$ such that

$$|x - y| = |y| \leq M\varepsilon^{1/10}, \quad |y - x_0| = 0 \mod \varepsilon\mathbb{Z}^n$$

and

$$\operatorname{dist}(y, H(x_0)) < \varepsilon\omega_\nu(\varepsilon).$$

By (9), there exists $w \in H(x_0)$ such that

$$|x - w| = |w| \le Mk\varepsilon \le M\varepsilon^{1/10} \tag{10}$$

and

$$|x_0 - w| \le \varepsilon\omega_\nu(\varepsilon) \mod \varepsilon\mathbb{Z}^n. \tag{11}$$

Given w satisfying (11), we can take $y \in \mathbb{R}^n$ such that

$$|x_0 - y| = 0 \mod \varepsilon\mathbb{Z}^n, \quad |y - w| \le \varepsilon\omega_\nu(\varepsilon).$$

Then, by (10),

$$|y| \le |y - w| + |w| \le M\varepsilon^{1/10} + \varepsilon\omega_\nu(\varepsilon) \le M\varepsilon^{1/10}.$$

Also, since w is contained in $H(x_0)$, we have $\operatorname{dist}(y, H(x_0)) \le |y - w| \le \varepsilon\omega_\nu(\varepsilon)$, proving (ii).

Finally, (iii) is a direct consequence of (9). □

3. In the strip domain

Fix $p \in \mathbb{R}^n$ and $\nu \in \mathcal{S}^{n-1}$ such that $p \cdot \nu \ne 0$. Let

$$\Pi = \Pi_\nu = \{x \in \mathbb{R}^n : -1 \le (x - p) \cdot \nu \le 0\}.$$

We consider a bounded viscosity solution u_ε of

$$(P_\varepsilon) \qquad \begin{cases} F(D^2 u_\varepsilon) = 0 & \text{in } \Pi, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = g\left(\dfrac{x}{\varepsilon}\right) & \text{on } \Gamma_0 := \{x : (x - p) \cdot \nu = 0\}, \\ u_\varepsilon = 1 & \text{on } \Gamma_I := \{x : (x - p) \cdot \nu = -1\}. \end{cases}$$

Below we prove the existence and uniqueness of u_ε.

Lemma 3.1. *Let $f(x) : \mathbb{R}^n \to \mathbb{R}$ be continuous and bounded. Let Π be as given above and define $B_R(p) := \{|x - p| \le R\}$. Suppose w_1 and w_2 solve, in the viscosity sense,*

(a) *$F(D^2 w_1) = 0$ and $F(D^2 w_2) = 0$ in $\Sigma_R := \Pi \cap B_R(p)$;*

(b) *$\partial w_1/\partial \nu = f(x) = \partial w_2/\partial \nu$ on Γ_0;*

(c) *$w_1 = w_2$ on Γ_I;*

(d) *$w_1 = -M$, $w_2 = M$ on $\Pi \cap \partial B_R(p)$.*

Then, for $R > 2$ and $C = \dfrac{n\Lambda}{\lambda}$, we have

$$w_1 \le w_2 \le w_1 + \frac{3CM}{R^2} \quad \text{in } \Pi \cap B_1(p).$$

Proof. Without loss of generality, let us set $\nu = e_n$ and $p = 0$. The first inequality, $w_1 \le w_2$, directly follows from Theorem 2.2. To show the second inequality, consider $\tilde{\omega} := w_1 + M(h_1 + h_2)$, where

$$h_1 = \frac{1}{R^2}\big((x_1)^2 + \cdots + (x_n)^2\big) \quad \text{and} \quad h_2 = \frac{C}{R^2}\big(1 - (x_n)^2\big),$$

with $C = \frac{n\Lambda}{\lambda}$. We claim $w_2 \le \tilde{\omega}$. To see this, note that

$$F(D^2\tilde{\omega}) = F(D^2 w_1 + D^2 h_1 + D^2 h_2)$$
$$\ge F(D^2 w_1) + \frac{2}{R^2}(C\lambda - n\Lambda) \ge F(D^2 w_1) \text{ in } \Sigma_R.$$

On the boundary of Σ_R, $\tilde{\omega}$ satisfies

$$\partial_{x_n}\tilde{\omega} = \partial_{x_n}\omega_1 = \partial_{x_n}\omega_2 \quad \text{on } \Sigma_R \cap \{x_n = 0\}$$

and

$$w_2 \le \tilde{\omega} \quad \text{on } \Gamma_I \cap B_R(0) \text{ and on } \partial B_R(0) \cap \Pi.$$

It follows from Theorem 2.2 that $w_2 \le \tilde{\omega}$ in Σ_R, and we are done. □

Lemma 3.2. *There exists a unique bounded solution u of (P_ε).*

Proof. 1. Let Σ_R be as given in Lemma 3.1, and consider the viscosity solution $\omega_R(x)$ of (P_ε) in Σ_R with the lateral boundary data $M = 1$ on $\partial B_R(p) \cap \Pi$. The existence and uniqueness of the viscosity solution ω_R is shown, for example, in [Crandall et al. 1992; Ishii 1991; Ishii and Lions 1990].

By the maximum principle, $\omega_R \le 1 + \max(g)$ in Σ_R. Due to Theorem 2.4 and the Arzelà–Ascoli Theorem, ω_R locally uniformly converges to a continuous function $u_\varepsilon(x)$. Then by the stability property of viscosity solutions, it follows that $u_\varepsilon(x)$ is a viscosity solution of (P_ε).

2. To show uniqueness, suppose u_1 and u_2 are both viscosity solutions of (P_ε) with $|u_1|, |u_2| \le M$. Then Lemma 3.1 yields that, for any point $q \in \Gamma_0$ and any $R > 2$,

$$|u_1 - u_2| \le O\left(\frac{1}{R^2}\right) \quad \text{in } B_1(q) \cap \Pi.$$

Hence $u_1 = u_2$. □

The following is immediate from Theorem 2.2 and the construction of u_ε in the above lemma.

Corollary 3.3. *Suppose u and v are bounded and continuous in $\bar{\Pi}_\nu(p)$, and solve*

a) $F(D^2u) \le 0 \le F(D^2v)$ *in* $\Pi_\nu(p)$;

b) $u \le v$ *on* Γ_I;

c) $\partial u/\partial\nu \le f(x) \le \partial v/\partial\nu$ *on* Γ_0;

where $f(x) : \mathbb{R}^n \to \mathbb{R}$ is continuous. Then $u \le v$ in $\Pi_\nu(p)$.

In the rest of this section, we will repeatedly use the fact that linear profiles as well as constants solve $F(D^2u) = 0$.

Lemma 3.4. *Let $\Pi_\nu(p)$ be as given in (P_ε) and let $0 < \varepsilon < 1$. Suppose that w_1 and w_2 are bounded and solve, in the viscosity sense,*

$$\begin{cases} F(D^2 w_i) = 0 & \text{in } \Pi_\nu(p), \\ |w_1 - w_2| \le \varepsilon & \text{on } \Gamma_I, \\ \dfrac{\partial w_1}{\partial\nu} - \dfrac{\partial w_2}{\partial\nu} = A & \text{on } \Gamma_0. \end{cases}$$

Then there exists a positive constant $C = C(A)$ such that

$$|w_1 - w_2| \geq C - \varepsilon \quad \text{in } \Pi_\nu(p) \cap B_{1/2}(p).$$

Proof. Let $\tilde{w} := w_2 + h$, where $h(x) = A(x-p)\cdot\nu + A - \varepsilon$. Then $\partial\tilde{\omega}/\partial\nu = \partial\omega_1/\partial\nu$ on Γ_0. Also, $\tilde{\omega} \leq w_1$ on Γ_I. Therefore, Corollary 3.3 yields that $w_2 + h \leq w_1$. Since $h \geq A/2 - \varepsilon$ in $B_{1/2}(p)$, we are done. □

Lemma 3.5. *Let $\tilde{\Pi} = \Pi + a\nu$ for some $0 \leq a \leq A\varepsilon$, where $0 < A < 1$. Suppose u_ε and $\tilde{u}_\varepsilon$ are bounded, and solve (P_ε) respectively in the domains Π and $\tilde{\Pi}$. Then we have*

$$|u_\varepsilon - \tilde{u}_\varepsilon| \leq C(A^\beta + \varepsilon^\alpha) \quad \text{in } \Pi \cap \tilde{\Pi},$$

where α is as given in Theorem 2.4 and β is the Hölder exponent of g.

Proof. 1. Let $v_\varepsilon(x) = \tilde{u}_\varepsilon(x + a\nu)$, so that v_ε and u_ε are defined in the same domain Π. Since $g(x) \in C^\beta(\mathbb{R}^n)$, $|\partial v_\varepsilon/\partial\nu - \partial u_\varepsilon/\partial\nu| \leq A^\beta$ on Γ_0.

2. On Γ_I, $u_\varepsilon = v_\varepsilon = 1$. Hence one can compare $u_\varepsilon \pm A^\beta(1 + (x-p)\cdot\nu)$ with v_ε and apply Theorem 2.2 to obtain

$$|u_\varepsilon - v_\varepsilon| \leq A^\beta \quad \text{in } \Pi.$$

Due to the Hölder continuity of u^ε given by Theorem 2.4, $|v_\varepsilon - \tilde{u}_\varepsilon| \leq C\varepsilon^\alpha$ in $\Pi \cap \tilde{\Pi}$. This finishes the proof. □

The next lemma follows from Theorem 2.4(b).

Lemma 3.6. *Let v_j be a bounded solution of (P_ε) with a constant Neumann condition $g(x) = \mu_j$. If $\mu_j \to \mu$, then v_j converges to v such that $\partial v/\partial\nu = \mu$ on Γ_0.*

4. Proof of the Main Theorem

We will prove first parts (i), (iii) and (iv) of Theorem 1.2; the proof of part (ii) starts on page 965.

Recall that

$$\Gamma_0 = \{x : (x-p)\cdot\nu = 0\}, \quad \Gamma_I = \{x : (x-p)\cdot\nu = -1\}.$$

Due to the uniform Hölder regularity of $\{u_\varepsilon\}$ (Theorem 2.4(a)), along subsequences $u_{\varepsilon_j} \to u$ in $\bar{\Pi}_\nu$. Note that there could be different limits along different subsequences (ε_j). Below, we will show that if ν is an irrational direction, all subsequential limits of (u_ε) coincide.

Suppose

$$0 \in \Pi_\nu = \{-1 < (x-p)\cdot\nu < 0\}.$$

Let us choose a convergent subsequence and rename it (u_j). For each j, there exists a constant μ_j and a function v_j in $\Pi_\nu(p)$ such that

$$(P_{\mu_j}) \qquad \begin{cases} F(D^2 v_j) = 0 & \text{in } \Pi_\nu(p), \\ \partial v_j/\partial\nu = \mu_j & \text{on } \Gamma_0, \\ v_j = u_j = 1 & \text{on } \Gamma_I, \\ v_j = u_j & \text{at } x = 0. \end{cases}$$

Lemma 4.1. *We have $\mu_j \to \mu$ for some μ as $j \to \infty$.* (The limit may depend on the subsequence chosen.)

Proof. Suppose not; then there is a constant $A > 0$ such that for any $N > 0$, $|\mu_m - \mu_n| \geq A$ for some $m, n > N$. Then, by Lemma 3.4,

$$|v_m(0) - v_n(0)| \geq C_A.$$

This contradicts the fact that $v_j(0) = u_j(0)$, since $u_j(0) \to u(0)$ as $j \to \infty$. □

The next lemma states that u_ε looks like a linear profile with respect to the direction ν as $\varepsilon \to 0$.

Lemma 4.2. *Away from the Neumann boundary Γ_0, u_ε is almost a constant on hyperplanes parallel to Γ_0. More precisely, let $x_0 \in \Pi_\nu(p)$ with $\operatorname{dist}(x_0, \Gamma_0) > \varepsilon^{1/20}$, and let $0 < \alpha < 1$. Then:*

(i) *If ν is a rational direction, there exists a constant $C > 0$ depending on ν, α and n, such that for any $x \in H(x_0) := \{(x - x_0) \cdot \nu = 0\}$,*

$$|u_\varepsilon(x) - u_\varepsilon(x_0)| \leq C\varepsilon^{\alpha/2}. \tag{12}$$

(ii) *If ν is any irrational direction, there exists a constant $C > 0$ depending on α and n, such that for any $x \in H(x_0)$,*

$$|u_\varepsilon(x) - u_\varepsilon(x_0)| \leq C\varepsilon^{\alpha/20} + C\omega_\nu(\varepsilon)^\beta, \tag{13}$$

where $\omega_\nu : [0, 1) \to [0, \infty)$ is a mode of continuity given as in (ii) of Lemma 2.7.

Proof. First, let ν be a rational direction. Lemma 2.7 implies that for any $x \in H(x_0)$, there is $y \in H(x_0)$ such that $|x - y| \leq M_\nu \varepsilon$ and $u_\varepsilon(y) = u_\varepsilon(x_0)$. Then by Theorem 2.3,

$$|u_\varepsilon(x_0) - u_\varepsilon(x)| \leq C\varepsilon^{-\alpha/20}(M_\nu \varepsilon)^\alpha \leq C\varepsilon^{\alpha/2}.$$

Next, we assume that ν is an irrational direction and $x \in H(x_0)$. By (ii) of Lemma 2.7, there exists $y \in \mathbb{R}^n$ such that $|x - y| \leq M\varepsilon^{1/10}$, $y - x_0 \in \varepsilon\mathbb{Z}^n$ and

$$\operatorname{dist}(y, H(x_0)) < \varepsilon\omega(\varepsilon). \tag{14}$$

Then we obtain

$$\begin{aligned} |u_\varepsilon(x_0) - u_\varepsilon(x)| &\leq |u_\varepsilon(x_0) - u_\varepsilon(y)| + |u_\varepsilon(y) - u_\varepsilon(x)| \\ &\leq C(\omega(\varepsilon)^\beta + \varepsilon^\alpha) + |u_\varepsilon(y) - u_\varepsilon(x)| \\ &\leq C\omega(\varepsilon)^\beta + C\varepsilon^{-\alpha/20}(M\varepsilon^{1/10})^\alpha \\ &\leq C\omega(\varepsilon)^\beta + C\varepsilon^{\alpha/20}, \end{aligned} \tag{15}$$

where the second inequality follows from Lemma 3.5 with (14), and the third inequality follows from Theorem 2.3. □

By Lemma 4.2 and by the comparison principle (Theorem 2.2), we obtain the following estimate: for $x \in \Pi$,

$$|u_\varepsilon(x) - v_\varepsilon(x)| \leq \Lambda(\varepsilon), \tag{16}$$

where

$$\Lambda(\varepsilon) = \begin{cases} C\varepsilon^{\alpha/2} & \text{if } \nu \text{ is a rational direction,} \\ C\varepsilon^{\alpha/20} + C\omega_\nu(\varepsilon)^\beta & \text{if } \nu \text{ is any irrational direction.} \end{cases}$$

Lemma 4.3. $\lim v_j = \lim u_j$, *and hence* $\partial u/\partial \nu = \mu$ *on* Γ_0.

Proof. Observe that v_j solves (P_{ε_j}) with $g = \mu_j$: note that v_j is then a linear profile, that is, $v_j(x) = \mu_j((x-p)\cdot\nu + 1) + 1$. Let x_0 be a point between Γ_0 and $H(0)$. Then by Lemma 4.2, applied to u_j and v_j,

$$\left|(u_j(x) - v_j(x)) - (u_j(x_0) - v_j(x_0))\right| \le \Lambda(\varepsilon_j), \tag{17}$$

for all $x \in H(x_0)$, if j is sufficiently large. Suppose now that

$$u_j(x_0) - v_j(x_0) > c > 0, \quad \text{for sufficiently large } j.$$

Then due to (17), $u_j - v_j \ge c/2$ on $H(x_0)$ if j is sufficiently large. Note that u_j can be constructed as the locally uniform limit of $u_{j,R}$, where $u_{j,R}$ solves

$$F(D^2 u_{j,R}) = 0 \quad \text{in } B_R(x_0)\cap\Pi, \qquad u_{j,R} = v_j \quad \text{on } \partial B_R(x_0)\cap\Pi,$$

with

$$u_{j,R} = 1 \quad \text{on } \Gamma_I, \qquad \frac{\partial}{\partial\nu} u_{j,R}(x) = g\left(\frac{x}{\varepsilon_j}\right) \quad \text{on } \Gamma_0.$$

Comparing $u_{j,R}$ and $v_j + c((x-x_0)\cdot\nu + 1)$ on the domain

$$B_R(x_0)\cap\left\{x : -1 \le (x-p)\cdot\nu \le (x-x_0)\cdot\nu\right\}$$

for sufficiently large R then yields that $u_{j,R}(0) \ge v_j(0) + c_0$ for all sufficiently large R, which would contradict the fact that $v_j(0) = u_j(0)$. Similarly, the case $\liminf_j (u_j(x_0) - v_j(x_0)) < 0$ can be excluded, and it follows that

$$|u_j(x_0) - v_j(x_0)| \to 0 \quad \text{as } j \to \infty.$$

Hence we get $v_j \to u$ in each compact subset of Π. By Lemmas 4.1 and 3.6, the limit $u = v$ of v_j satisfies $\partial u/\partial\nu = \mu$ on Γ_0. □

Lemma 4.4. *If* ν *is an irrational direction,* $\partial u/\partial\nu = \mu_\nu$ *for a constant* μ_ν *which depends on* ν, *not on the subsequence* ε_j.

Proof. 1. Let $0 < \eta < \varepsilon$ be sufficiently small. Let

$$w_\varepsilon(x) = \frac{u_\varepsilon(\varepsilon x)}{\varepsilon}, \quad w_\eta(x) = \frac{u_\eta(\eta x)}{\eta},$$

and denote by Γ_1 and Γ_2 the Neumann boundary of w_ε and w_η, respectively. By (iii) of Lemma 2.7, for the point $p \in \mathbb{R}^n$, there exist $q_1 \in \Gamma_1$ and $q_2 \in \Gamma_2$ such that

$$|p - q_1| \le \eta \mod \mathbb{Z}^n \quad \text{and} \quad |p - q_2| \le \eta \mod \mathbb{Z}^n.$$

Hence after translations by $p - q_1$ and $p - q_2$, we may suppose that $w_\varepsilon(x)$ and $w_\eta(x)$ are defined, respectively, on the extended strips

$$\Omega_\varepsilon := \left\{x : -\frac{1}{\varepsilon} \le (x-p)\cdot\nu \le 0\right\} \quad \text{and} \quad \Omega_\eta := \left\{x : -\frac{1}{\eta} \le (x-p)\cdot\nu \le 0\right\}.$$

Here, $w_\varepsilon = 1/\varepsilon$ on $\{(x-p)\cdot\nu = -1/\varepsilon\}$ and $w_\eta = 1/\eta$ on $\{(x-p)\cdot\nu = -1/\eta\}$. Moreover, on $\Gamma_0 := \{(x-p)\cdot\nu = 0\}$, we have

$$\frac{\partial w_\varepsilon}{\partial\nu} = g_1(x) := g(x-z_1) \quad\text{and}\quad \frac{\partial w_\eta}{\partial\nu} = g_2(x) := g(x-z_2),$$

where $|z_1|, |z_2| \le \eta$. Observe that since g has Hölder exponent $0<\beta\le 1$, we have $|g_1-g_2|\le\eta^\beta$.

Let v_ε be a solution of the problem (P_ε) with constant Neumann data $\partial v_\varepsilon/\partial\nu = \mu_\varepsilon$ on Γ_0 such that v_ε coincides with u_ε at $x=0$ and on Γ_I. By (16),

$$\left|w_\varepsilon(x) - \frac{v_\varepsilon(\varepsilon x)}{\varepsilon}\right| \le \frac{C\varepsilon^{\alpha/20} + C\omega(\varepsilon)^\beta}{\varepsilon}. \tag{18}$$

Note that v_ε is a linear profile: indeed,

$$\frac{v_\varepsilon(\varepsilon x)}{\varepsilon} = \mu_\varepsilon\Big((x-p)\cdot\nu + \frac{1}{\varepsilon}\Big) + \frac{1}{\varepsilon}.$$

From (18) and the comparison principle, it follows that, with $\Lambda(\varepsilon) = C\varepsilon^{\alpha/20} + C\omega(\varepsilon)^\beta$,

$$\big(\mu_\varepsilon - \Lambda(\varepsilon)\big)\Big((x-p)\cdot\nu + \frac{1}{\varepsilon}\Big) \le w_\varepsilon(x) - \frac{1}{\varepsilon} \le \big(\mu_\varepsilon + \Lambda(\varepsilon)\big)\Big((x-p)\cdot\nu + \frac{1}{\varepsilon}\Big), \tag{19}$$

2. (19) means that the slope of w_ε in the direction of ν (that is, $\nu\cdot Dw_\varepsilon$) is between $\mu_\varepsilon + \Lambda(\varepsilon)$ and $\mu_\varepsilon - \Lambda(\varepsilon)$ on $\{x : (x-p)\cdot\nu = -1/\varepsilon\}$. Now let us consider linear profiles

$$l_1(x) = a_1(x-p)\cdot\nu + b_1 \quad\text{and}\quad l_2(x) = a_2(x-p)\cdot\nu + b_2,$$

whose respective slopes are $a_1 = \mu_\varepsilon + \Lambda(\varepsilon)$ and $a_2 = \mu_\varepsilon - \Lambda(\varepsilon)$. Here b_1 and b_2 are chosen such that

$$l_1 = l_2 = \omega_\eta(x) \quad\text{on}\quad \Big\{x : (x-p)\cdot\nu = -\frac{1}{\eta}\Big\}.$$

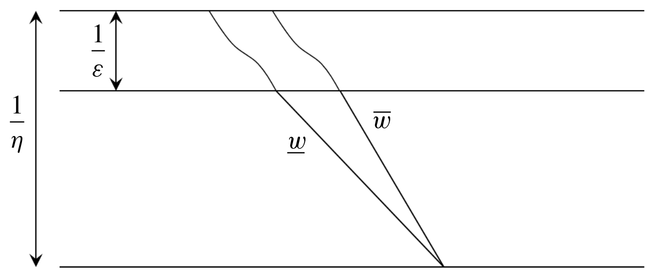
3. Now we define

$$\overline{w}(x) := \begin{cases} l_1(x) & \text{in } \big\{-1/\eta \le (x-p)\cdot\nu \le -1/\varepsilon\big\}, \\ w_\varepsilon(x) + c_1 & \text{in } \big\{-1/\varepsilon \le (x-p)\cdot\nu \le 0\big\} \end{cases}$$

and

$$\underline{w}(x) := \begin{cases} l_2(x) & \text{in } \big\{-1/\eta \le (x-p)\cdot\nu \le -1/\varepsilon\big\}, \\ w_\varepsilon(x) + c_2 & \text{in } \big\{-1/\varepsilon \le (x-p)\cdot\nu \le 0\big\}, \end{cases}$$

where c_1 and c_2 are constants satisfying $l_1 = w_\varepsilon + c_1$ and $l_2 = w_\varepsilon + c_2$ on $\{(x-p)\cdot\nu = -1/\varepsilon\}$. (See figure.)



Note that, due to (19), in $\{-1/\varepsilon \le (x-p)\cdot\nu \le 0\}$ we have

$$\overline{w}(x) = \min\big(l_1(x), w_\varepsilon(x)+c_1\big) \quad \text{and} \quad \underline{w}(x) = \max\big(l_2(x), w_\varepsilon(x)+c_2\big),$$

and thus it follows that $\overline{w}$ and $\underline{w}$ are respectively viscosity super- and subsolutions of (P).

4. Let us define

$$h_1(x) = \eta^\beta\Big((x-p)\cdot\nu + \frac{1}{\eta}\Big).$$

Then $w^+ := \overline{w} + h_1$ solves

$$\begin{cases} F(Dw^+) \ge 0 & \text{in } \Omega_\eta, \\ \partial w^+/\partial\nu = g(x) + \eta^\beta & \text{on } \Gamma_0, \end{cases}$$

and $w^- := \underline{w} - h_1$ solves

$$\begin{cases} F(Dw^-) \le 0 & \text{in } \Omega_\eta, \\ \partial w^-/\partial\nu = g(x) - \eta^\beta & \text{on } \Gamma_0. \end{cases}$$

Since $|g-\tilde{g}| \le \eta^\beta$ and $w^+ = w^- = w_\eta$ on $\{(x-p)\cdot\nu = -1/\eta\}$, it follows from the comparison principle for (P_ε) that

$$w^- \le w_\eta \le w^+ \quad \text{in } \Omega_\eta. \tag{20}$$

Hence we conclude

$$|\mu_\eta - \mu_\varepsilon| \le \Lambda(\varepsilon) + \eta^\beta, \tag{21}$$

where μ_η is the slope of v_η, and $\Lambda(\varepsilon) = C\varepsilon^{\alpha/20} + Cw(\varepsilon)^\beta \to 0$ as $\varepsilon \to 0$. □

The proof of the following lemma is immediate from Lemma 4.4 and (21) .

Lemma 4.5 (error estimate: Theorem 1.2(iv)). *For any irrational direction ν, there is a unique homogenized slope $\mu(\nu) \in \mathbb{R}$ and $\varepsilon_0 = \varepsilon_0(\nu) > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the following holds: for any $0 < \alpha < 1$, there exists a constant $C = C(\alpha, n, \lambda, \Lambda)$ such that*

$$\big|u_\varepsilon(x) - \big(1+\mu(\nu)((x-p)\cdot\nu+1)\big)\big| \le \Lambda(\varepsilon) := C\varepsilon^{\alpha/20} + C\omega_\nu(\varepsilon)^\beta \quad \text{in } \Pi_\nu(p), \tag{22}$$

where $\omega_\nu(\varepsilon)$ is as given in (7).

Lemma 4.6. *Let ν be a rational direction. If the Neumann boundary Γ_0 passes through $p = 0$, then there is a unique homogenized slope $\mu(\nu)$ for which the result of Lemma 4.5 holds with $\Lambda(\varepsilon) = C\varepsilon^{\alpha/2}$.*

Proof. The proof is parallel to that of Lemma 4.4. Let w_ε and w_η be as given in the proof of Lemma 4.4. Note that since Ω_ε and Ω_η have their Neumann boundaries passing through the origin, $\partial w_\varepsilon/\partial\nu = g(x) = \partial w_\eta/\partial\nu$ without translation of the x variable, and thus we do not need to use the properties of hyperplanes with an irrational normal (Lemma 2.7(b)) to estimate the error between the shifted Neumann boundary data. □

Remark 4.7. As mentioned in the introduction, if ν is a rational direction with $p \ne 0$, the values of $g(\cdot/\varepsilon)$ on $\partial\Omega_\varepsilon$ and $\partial\Omega_\eta$ may be very different under any translation, and thus the proof of Lemma 4.4 fails. In this case, u_ε may converge to solutions of different Neumann boundary data, depending on the subsequence.

Proof of Theorem 1.2(ii). Recall that we must show that *the homogenized limit $\mu(\nu)$, defined in Lemma 4.5 for irrational directions in $\mathcal{S}^{n-1}$, has a continuous extension $\bar{\mu}(\nu) : \mathcal{S}^{n-1} \to \mathbb{R}$.*

Fix a unit vector $\nu \in \mathcal{S}^{n-1}$. Then we will show that there exists a positive constant $C > 0$ depending on ν such that the following holds: given $\delta > 0$, there exists $\varepsilon > 0$ such that for any two irrational directions $\nu_1, \nu_2 \in \mathcal{S}^{n-1}$,

$$|\mu(\nu_1) - \mu(\nu_2)| < C\delta^{1/2} \quad \text{whenever } 0 < |\nu_1 - \nu|,\ |\nu_2 - \nu| < \varepsilon. \tag{23}$$

1. To simplify the proof, we first present the case $n = 2$. For simplicity of notation, we may assume that $|\nu \cdot e_1| \le |\nu \cdot e_2|$ and $p = 0$. First we introduce several notations. Again for notational simplicity and clarity in the proof, we assume that $\nu = e_2$: we will explain in the paragraph below how to modify the notations and the proof for $\nu \neq e_2$. Let us define

$$\Omega_0 := \Pi_\nu(0) = \{(x, y) \in \mathbb{R}^2 : -1 \le y \le 0\},$$

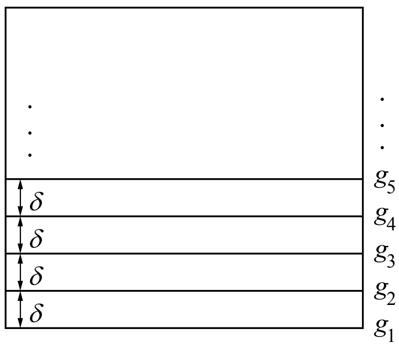
and for $i = 1, 2$,

$$\Omega_i := \Pi_{\nu_i}(0) = \{(x, y) \in \mathbb{R}^2 : -1 \le (x, y) \cdot \nu_i \le 0\}.$$

Let us also define the family of functions

$$g_i(x_1, x_2) = g_i(x_1) = g(x_1, \delta(i-1)),$$

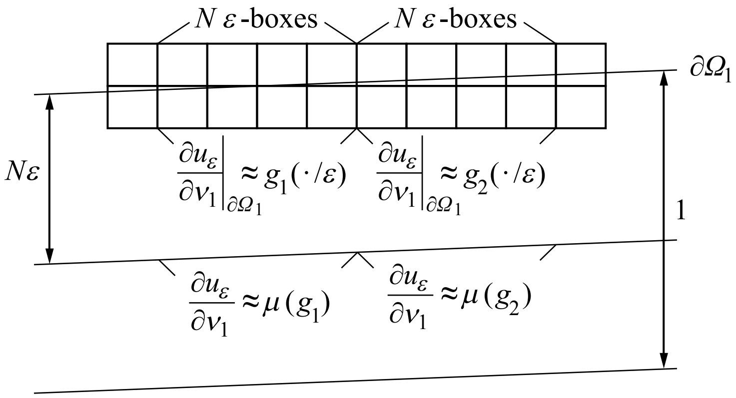
where $i = 1, \dots, m := [1/\delta] + 1$ (see figure).



If ν is a rational direction different from e_2, take the smallest $K_\nu \in \mathbb{N}$ such that $K_\nu \nu = 0 \bmod \mathbb{N}^2$. Then g can be considered as a K_ν-periodic function with the new direction of axis of ν. If ν is an irrational direction, take the smallest $K_\nu \in \mathbb{N}$ such that $|K_\nu \nu| \le \delta \bmod \mathbb{N}^2$. Then g is almost K_ν-periodic up to the order of δ with the new axis of ν. We point out that it does not make any difference in the proof if we replace the periodicity of g by the fact that g is almost periodic up to the order δ.

Before moving on to the next step, we briefly discuss the heuristics in the proof.

Proof by heuristics. Since the domains Ω_1 and Ω_2 point toward different directions ν_1 and ν_2, we cannot directly compare their boundary data, even if $\partial\Omega_1$ and $\partial\Omega_2$ cover most of the unit cell in $\mathbb{R}^n/\mathbb{Z}^n$. To overcome this difficulty, we perform a two-scale homogenization.



First we consider the functions g_i $(i = 1, \dots, m)$ whose profiles cover most values of g in $\mathbb{R}^2$ up to the order of δ^β, where β is the Hölder exponent of g. Note that most values of g in $\mathbb{R}^2$ are taken on $\partial\Omega_1$ and on $\partial\Omega_2$, since ν_1 and ν_2 are both irrational directions. On the other hand, since ν_1 and ν_2 are very close to ν, which may be a rational direction, the averaging behavior of a solution u_ε in Ω_1 (or Ω_2) would occur only if ε gets very small.

If $|\nu_1 - \nu| = |\nu_1 - e_2|$ is chosen much smaller than δ, we can say that the Neumann data $g_1(\cdot/\varepsilon)$ is (almost) repeated $N := [\delta/|\nu_1 - \nu|]$ times on $\partial\Omega_1$ with period ε, up to the error $O(\delta^\beta)$. (See figure at the top of the page.) Similarly, on the next piece of the boundary, $g_2(\cdot/\varepsilon)$ is (almost) repeated N times, and then $g_3(\cdot/\varepsilon)$ is repeated N times: this pattern will repeat with g_k $(k \in \mathbb{N} \bmod m)$.

If N is sufficiently large, that is, if $|\nu_1 - \nu|$ is sufficiently small compared to δ, the solution u_ε in Ω_1 will exhibit averaging behavior, $N\varepsilon$-away from $\partial\Omega_1$. More precisely, on the $N\varepsilon$-sized segments of hyperplane H located $N\varepsilon$-away from $\partial\Omega_1$, u_ε would be homogenized by repeating the profiles of g_i (for some fixed i) with an error of $O(\delta^\beta)$. This is the first homogenization of u_ε near the boundary of Ω_1: we denote by $\mu(g_i)$ the corresponding values of the homogenized slopes of u_ε on H.

Now a unit distance away from $\partial\Omega_1$, we obtain the second homogenization of u_ε, whose slope is determined by $\mu(g_i)$, $i = 1, \dots, m$. Note that this estimate does not depend on the direction ν_1, but on the quantity $|\nu_1 - \nu|$. Hence, applying the same argument for ν_2, we conclude that $|\mu(\nu_1) - \mu(\nu_2)|$ is small. Note that $\mu(\nu_1)$ and $\mu(\nu_2)$ are uniquely determined because ν_1 and ν_2 are irrational directions (Lemma 4.6).[1]

A rigorous proof of the above observation is rather lengthy: the main difficulty lies in the fact that to perform the first homogenization $N\varepsilon$-away from the boundary, one requires the solution u_ε to be sufficiently flat in tangential directions to ν, which we do not know a priori. We will go around this difficulty by constructing sub- and supersolutions by patching up solutions from the near-boundary region and from the region away from the boundary. The proof is given in steps 2–8 below.

[1] By (F3), we may assume that the arrangement of $g_1, \dots, g_m$ is the same for the directions ν_1 and ν_2, after appropriate rotation and reflection (note that $(F3)$ implies rotation and reflection invariance of the operator F).

2. Given $\delta > 0$, let us choose irrational unit vectors $\nu_1, \nu_2 \in \mathbb{R}^2$ such that

$$0 < \bar{\varepsilon}_0^{1/1000} \le \varepsilon_0^{1/1000} \le \delta,$$

where $\varepsilon_0 = |\nu_1 - e_2|$ and $\bar{\varepsilon}_0 = |\nu_2 - e_2|$. Let $\varepsilon = \varepsilon_0^{21/20}$ and $\bar{\varepsilon} = \bar{\varepsilon}_0^{21/20}$. Let us also define

$$N = \left[\frac{\delta}{|\nu_1 - e_2|}\right] = \left[\frac{\delta}{\varepsilon_0}\right]. \tag{24}$$

Then $N\varepsilon = \delta\varepsilon_0^{1/20} := \delta_0$. Note that

$$\delta_0 \ge \varepsilon^{1/20} \quad \text{and} \quad \delta_0 \ge \delta^{100}.$$

With the above definition of ε and N, consider the strip regions $I_0 = [-N\varepsilon, 0] \times \mathbb{R}$, $I_1 = [0, N\varepsilon] \times \mathbb{R}$, $I_{-1} = [-2N\varepsilon, -N\varepsilon] \times \mathbb{R}$, $I_2 = [N\varepsilon, 2N\varepsilon] \times \mathbb{R}$, ..., that is,

$$I_k = [(k-1)N\varepsilon, kN\varepsilon] \times \mathbb{R} \quad \text{for } k \in \mathbb{Z}.$$

Let $\tilde{k} \in [1, m]$ denote k in modulo m, where $m = [1/\delta] + 1$. Note that, since $N|\nu_1 - e_2| = \delta$, $g_{\tilde{k}}(\,\cdot\,/\varepsilon)$ is (almost) repeated N times on $I_k \cap \partial\Omega_1$. This fact and the Hölder continuity of g yield that

$$\left|g\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - g_{\tilde{k}}\left(\frac{x}{\varepsilon}\right)\right| < C\delta^\beta \quad \text{on } \partial\Omega_1 \cap I_k, \quad \text{for } k \in \mathbb{Z}. \tag{25}$$

3. Let w_ε solve $(P) : F(D^2 w_\varepsilon) = 0$ in Ω_0, with

$$\begin{cases} \dfrac{\partial w_\varepsilon}{\partial \nu}(x, 0) = g_{\tilde{k}}\left(\dfrac{x}{\varepsilon}\right) & \text{for } (x, 0) \in I_k, \\ w_\varepsilon = 1 & \text{on } \{y = -1\}. \end{cases}$$

Next let u_ε solve (P) in Ω_1, with

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial \nu_1}(x, 0) = g\left(\dfrac{x}{\varepsilon}, \dfrac{y}{\varepsilon}\right) & \text{on } \{(x, y) \cdot \nu_1 = 0\}, \\ u_\varepsilon = 1 & \text{on } \{(x, y) \cdot \nu_1 = -1\}. \end{cases}$$

Let $\mu(w_\varepsilon)$ ($\mu(u_\varepsilon)$) be chosen as the slope μ_j in the linearized problem (P_{μ_j}) in Section 4, where u_j is replaced by w_ε (u_ε) and the reference point $x = 0$ is replaced by $x = -e_2/2 = (0, -\frac{1}{2})$. (Recall that we assumed $0 \in \partial\Omega_1$, and $(0, -\frac{1}{2}) \in \Omega_i$ for $i = 1, 2$.) Then $\mu(w_\varepsilon)$ and $\mu(u_\varepsilon)$ denote the slopes of a linear approximation of w_ε and u_ε. From (25) it follows that

$$|\mu(w_\varepsilon) - \mu(u_\varepsilon)| < C\delta^\beta. \tag{26}$$

We point out that $\mu(w_\varepsilon)$ and $\mu(u_\varepsilon)$ respectively converge to a unique limit as $\varepsilon \to 0$, since ν_1 is irrational.

4. We begin by introducing $\mu_{1/N}(g_k)$, which denotes the average slope of a solution with Neumann data $g_k(x/\varepsilon)$, δ_0-away from the Neumann boundary $\{y = 0\}$. (Here note that $\delta_0 = N\varepsilon$.)

Let us define

$$H := \partial\Omega_0 - N\varepsilon e_2 = \{(x, y) : y = -\delta_0\}.$$

Let $\eta = 1/N$ and let $w_{\eta,1}$ solve

$$\begin{cases} F(D^2 w_{\eta,1}) = 0 & \text{in } \{-\delta_0 \le y \le 0\}, \\ w_{\eta,1} = w_\varepsilon(0, -\delta_0) & \text{on } H = \{y = -\delta_0\}, \\ \dfrac{\partial w_{\eta,1}}{\partial y}(x, 0) = g_1\Big(\dfrac{x}{\varepsilon}, 0\Big) & \text{on } \partial\Omega_0 = \{y = 0\}, \end{cases}$$

where $g_1(x, 0) = g_1(x+k, 0)$ for $k \in \mathbb{Z}$. Let $\mu_{1/N}(g_1)$ be the slope of the linear approximation of $w_{\eta,1}$, defined as follows: choose a linear solution $v_{\eta,1}(\cdot)$ such that

$$\begin{cases} F(D^2 v_{\eta,1}) = 0 & \text{in } \{-\delta_0 \le y \le 0\}, \\ v_{\eta,1} = w_{\eta,1}(0, -\delta_0) & \text{on } H = \{y = -\delta_0\}, \\ v_{\eta,1}\Big(0, -\dfrac{\delta_0}{2}\Big) = w_{\eta,1}\Big(0, -\dfrac{\delta_0}{2}\Big), & \\ \dfrac{\partial v_{\eta,1}}{\partial y}(x, 0) = \mu_{1/N}(g_1) & \text{on } \partial\Omega_0 = \{y = 0\}. \end{cases}$$

Since $g_1(x/\varepsilon, 0)$ is periodic on $\{y = 0\}$ with period ε and $\delta_0 = N\varepsilon$, we can apply Lemma 4.2(i), using the fact that $\delta_0 \ge \varepsilon^{1/20}$, to conclude that

$$\Big| w_{\eta,1}(x, y) - \Big(w_{\eta,1}\Big(0, -\frac{\delta_0}{2}\Big) + \mu_{1/N}(g_1)\Big(y + un\frac{\delta_0}{2}\Big)\Big)\Big| \le C\delta_0^{1+\beta} \tag{27}$$

on $\{y = -\delta_0/2\} \cap I_1$. Similarly, one can define $w_{\eta,k}$ and $v_{\eta,k}$ for $k \in \mathbb{Z}$ to conclude that

$$\Big| w_{\eta,k}(x, y) - \Big(w_{\eta,k}\Big((k-1)\delta_0, -\frac{\delta_0}{2}\Big) + \mu_{1/N}(g_{\tilde{k}})\Big(y + \frac{\delta_0}{2}\Big)\Big)\Big| \le C\delta_0^{1+\beta} \tag{28}$$

on $\{y = -\delta_0/2\} \cap I_k$.

5. We will now construct barriers which bound w_ε from above and below, by pasting together the near-boundary and the rest of the region together as follows. First we construct a supersolution of (P_ε). Let ρ_ε solve the Neumann boundary problem away from the boundary $\{y = 0\}$:

$$\begin{cases} F(D^2 \rho_\varepsilon) = 0 & \text{in } \{-1 \le y \le -\delta_0\}, \\ \dfrac{\partial \rho_\varepsilon}{\partial y} = \Lambda(x) & \text{on } H = \{y = -\delta_0\}, \\ \rho_\varepsilon = 1 & \text{on } \{y = -1\}. \end{cases}$$

Here $\Lambda(x)$ is a Hölder continuous function obtained by approximating $\mu_{1/N}(g_k) + 2\delta_0^{\alpha_0}$ in each $N\varepsilon$-strip, where the constant $0 < \alpha_0 < 1$ will be decided below. Here the Hölder continuity of $\Lambda(x)$ is obtained by the fact that g_k and g_j differ from each other by $((k-j)\delta)^\beta$ and they are apart by $(k-j)N\varepsilon \ge (k-j)\delta^{100}$.

Then Theorem 2.4(b) yields that $\rho_\varepsilon \in C^{1,\gamma}$ up to H, where γ depends on β and n. Therefore there exists a constant $0 < \alpha_0 < 1$ such that the following holds: in each $\delta_0^{1-\alpha_0}$-neighborhood of a point $(x_0, -\delta_0) \in H$, we have

$$\big|\rho_\varepsilon(x, -\delta_0) - \rho_\varepsilon(x_0, -\delta_0) - \alpha(x_0)(x - x_0)\big| \le \delta_0^{1+\alpha_0}, \tag{29}$$

where $\alpha(x_0)$ is the tangential derivative of ρ_ε at $(x_0, -\delta_0)$.

6. Next we construct the near-boundary barrier:

$$\begin{cases} F(D^2 f_\varepsilon) = 0 & \text{in } \{-\delta_0 \le y \le 0\}, \\ f_\varepsilon = \rho_\varepsilon & \text{on } H = \{y = -\delta_0\}, \\ \dfrac{\partial f_\varepsilon}{\partial y} = g_{\tilde{k}}\Big(\dfrac{x}{\varepsilon}\Big) & \text{on } \{y = 0\} \cap I_k. \end{cases}$$

Let us now estimate the slope of f_ε on H. Let us choose a constant μ_ε and the corresponding linear profile ϕ_ε such that

$$\begin{cases} F(D^2 \phi_\varepsilon) = 0 & \text{in } \{-\delta_0 \le y \le 0\}, \\ \phi_\varepsilon(x, -\delta) = f_\varepsilon(0, -\delta_0) & \text{on } H, \\ \phi_\varepsilon\Big(0, -\dfrac{\delta}{2}\Big) = f_\varepsilon\Big(0, -\dfrac{\delta_0}{2}\Big), & \\ \dfrac{\partial \phi_\varepsilon}{\partial y} = \mu_\varepsilon & \text{on } \partial\Omega_0 = \{y = 0\}. \end{cases}$$

Equation (29) and the comparison principle (Theorem 2.2), as well as the localization argument as in the proof of Lemma 3.1 applied to the rescaled function

$$(\delta_0)^{-1} f_\varepsilon\bigg(\frac{(x - x_0)}{\delta_0} + x_0, \frac{y}{\delta_0}\bigg) - \alpha(x_0)(x - x_0)$$

in the region $\{-1 \le y \le 0\} \cap \{|x| \le \delta_0^{-\alpha_0}\}$, yields that

$$|\phi_\varepsilon - f_\varepsilon| \le C\delta_0^{1+\alpha_0} \quad \text{in } \{-\delta_0 \le y \le 0\} \cap \{|x| \le \delta_0^{1-\alpha_0}\}. \tag{30}$$

Putting the estimates (28) and (30) together, it follows that for any $(x_0, -\delta_0) \in H$, we have

$$\bigg| f_\varepsilon(x, y) - \Big(\alpha(x_0)(x - x_0) + \mu_{1/N}(g_k)\Big(y + \frac{\delta_0}{2}\Big)\Big)\bigg| \le \delta_0^{1+\alpha_0} \quad \text{on } \Big\{y = -\frac{\delta_0}{2}\Big\} \cap \big\{|x - x_0| \le \delta_0^{1-\alpha_0}\big\},$$

for appropriate k in each δ-strip. Using the above inequality, (29), and the $C^{1,\gamma}$ regularity of f_ε up to its Dirichlet boundary, we obtain that

$$\frac{\partial f_\varepsilon}{\partial y} \le \Lambda(x),$$

which then makes the following function a supersolution of (P_ε):

$$\underline{\rho}_\varepsilon := \begin{cases} \rho_\varepsilon & \text{in } \{-1 \le y \le -\delta_0\}, \\ f_\varepsilon & \text{in } \{-\delta_0 \le y \le 0\}. \end{cases}$$

Similarly, one can construct a subsolution $\bar{\rho}_\varepsilon$ of (P_ε) by replacing $\Lambda(x)$ given in the construction of ρ_ε by $\tilde{\Lambda}(x) := \Lambda(x) - 4\delta_0^{\alpha_0}$, such that

$$\bar{\rho}_\varepsilon \le w_\varepsilon \le \underline{\rho}_\varepsilon. \tag{31}$$

7. Parallel arguments as in steps 2–6 apply to the other direction, ν_2: if we define $\bar{\varepsilon}_0$, M and $\bar{H}$ by

$$|\nu_2 - e_2| = \bar{\varepsilon}_0 < \varepsilon_0, \quad M = \Big[\frac{\delta}{\bar{\varepsilon}_0}\Big], \quad \bar{\varepsilon} = \bar{\varepsilon}_0^{21/20} \quad \text{and} \quad \bar{H} = \{y = -M\bar{\varepsilon}\},$$

then we can construct barriers $\bar{\rho}_{\bar{\varepsilon}}$ and $\underline{\rho}_{\bar{\varepsilon}}$ such that

$$\bar{\rho}_{\bar{\varepsilon}} \leq w_{\bar{\varepsilon}}(x) \leq \underline{\rho}_{\bar{\varepsilon}}, \tag{32}$$

with their corresponding Neumann boundary conditions on H:

$$\frac{\partial}{\partial y}\bar{\rho}_{\bar{\varepsilon}}, \quad \frac{\partial}{\partial y}\underline{\rho}_{\bar{\varepsilon}} = \mu_{1/M}(g_{\tilde{k}}) + O(\bar{\delta}_0^{\alpha_0}) \quad \text{and} \quad \bar{H} \cap I_k, \tag{33}$$

where their respective derivative is taken as a limit from the region $\{-1 \leq y < -\bar{\delta}_0\}$.

8. Now we proceed to estimate the averaging behavior of u^ε away from the Neumann boundary. By (21) of Lemmas 4.4 and 4.6,

$$\left|\mu_{1/N}(g_{\tilde{k}}) - \mu_{1/M}(g_{\tilde{k}})\right| < \Lambda\left(\frac{1}{N}\right) + \left(\frac{1}{M}\right)^\beta, \tag{34}$$

where $\Lambda\left(\frac{1}{N}\right) = CN^{-\alpha/2}$. Let us write $\mu_{1/N}(g_{\tilde{k}}) = \mu_{\tilde{k},N}$, and let h and $\bar{h}$ respectively solve

$$\begin{cases} F(D^2h) = 0 & \text{in } \{-1 \leq y \leq -N\varepsilon\}, \\ h = 1 & \text{on } \{y = -1\}, \\ \dfrac{\partial h}{\partial \nu} = \mu_{\tilde{k},N} & \text{on } H \cap I_k, \end{cases}$$

and

$$\begin{cases} F(D^2\bar{h}) = 0 & \text{in } \{-1 \leq y \leq -M\bar{\varepsilon}\}, \\ \bar{h} = 1 & \text{on } \{y = -1\}, \\ \dfrac{\partial \bar{h}}{\partial \nu} = \mu_{\tilde{k},M} & \text{on } \bar{H} \cap I_k. \end{cases}$$

Let $\mu(h)$ and $\mu(\bar{h})$ be the respective slope of linear approximation for h and $\bar{h}$.

Then it follows from (34) that if $\delta_0 \sim N\varepsilon$ and $\bar{\delta}_0 \sim M\bar{\varepsilon}$ are sufficiently small,

$$|\mu(h) - \mu(\bar{h})| < C\left(m\left(\frac{1}{N}\right) + \left(\frac{1}{M}\right)^\beta\right). \tag{35}$$

Lastly, observe that by (31) and (32), there exists $0 < \gamma < 1$ such that

$$|\mu(w_\varepsilon) - \mu(h)| < C\delta^\gamma \quad \text{and} \quad |\mu(w_{\bar{\varepsilon}}) - \mu(\bar{h})| < C\delta^\gamma.$$

The above inequalities and (35) yield

$$|\mu(w_\varepsilon) - \mu(w_{\bar{\varepsilon}})| < C\left(\delta^\gamma + m\left(\frac{1}{N}\right) + \left(\frac{1}{M}\right)^\beta\right).$$

Then we conclude from (26) that

$$|\mu(u_\varepsilon) - \mu(u_{\bar{\varepsilon}})| < C\left(\delta^\gamma + m\left(\frac{1}{N}\right) + \left(\frac{1}{M}\right)^\beta\right). \tag{36}$$

9. Lastly, we estimate the rate of convergence of $\mu(u_\varepsilon)$ to $\mu(\nu_1)$ as $\varepsilon \to 0$. The claim is that

$$|\mu(\nu_1)-\mu(u_\varepsilon)|\le C\big(\varepsilon_0^\beta+\varepsilon_0^{21\alpha/200}+\varepsilon_0^{1/20}\big).$$

We will argue similarly as in the proof of Lemma 4.2(ii). Let us define v^ε, the linear approximation of u_ε, as in (P_{μ_j}) of page 960, where the reference function u_j is replaced by u_ε.

Recall that $\Omega_1=\{y:-1\le y\cdot\nu_1\le 0\}$. We define

$$\tilde\Omega_1:=\Omega_1\cap\{y:y\cdot\nu_1\le -N\varepsilon\delta^{-1}\nu_1\}$$

and $L:=\partial\Omega_1-N\varepsilon\delta^{-1}\nu_1$. For any given $x_0\in L$ and for any $x\in L$, there exists $y\in\mathbb{R}^2$ such that $|x-y|\le N\varepsilon m$, $x_0-y=0 \bmod \varepsilon\mathbb{Z}^2$, and

$$\operatorname{dist}(y,L)\le\varepsilon|\nu_1-e_2|=\varepsilon\varepsilon_0.$$

(Recall that $m=\left[\frac{1}{\delta}\right]+1$.) Then by arguing as in (15), for $x\in L$,

$$|u_\varepsilon(x_0)-u_\varepsilon(x)|\le C\varepsilon_0^\beta+C(N\varepsilon\delta^{-1})^\alpha(N\varepsilon m)^\alpha\le C(\varepsilon_0^\beta+\varepsilon^{\alpha/10}).$$

Hence, due to the comparison principle (Theorem 2.2) applied to u_ε and v_ε in the domain $\tilde\Omega_1$, we obtain

$$|u_\varepsilon-v_\varepsilon|\le C(\varepsilon_0^\beta+\varepsilon^{\alpha/10}+N\varepsilon\delta^{-1})=C(\varepsilon_0^\beta+\varepsilon_0^{21\alpha/200}+\varepsilon_0^{1/20}). \tag{37}$$

Following the proof of (21) using (37) instead of (13), we conclude

$$|\mu(u_\varepsilon)-\mu(\nu_1)|\le C(\varepsilon_0^\beta+\varepsilon_0^{21\alpha/200}+\varepsilon_0^{1/20})\le\delta.$$

Parallel arguments apply to ν_2. Combining the above inequality with (36),

$$|\mu(\nu_1)-\mu(\nu_2)|\le C\Big(\delta^\gamma+m\Big(\frac{1}{N}\Big)+\Big(\frac{1}{M}\Big)^\beta\Big).$$

Since N and M grow to infinity as ε and $\bar\varepsilon$ go to zero, the above inequality proves the lemma.

10. For the general dimensions $n>2$, let us define

$$g_i(x_1,\dots,x_{n-1},x_n)=g_i(x_1,\dots,x_{n-1})=g(x_1,\dots,x_{n-1},\delta(i-1))$$

for $i=0,1,\dots,m:=[\delta^{-1}]$. Let us also define

$$I_{k_1,k_2,\dots,k_{n-1}}:=\big[(k_1-1)N\varepsilon,k_1N\varepsilon\big]\times\cdots\times\big[(k_{n-1}-1)N\varepsilon,k_{n-1}N\varepsilon\big]\times\mathbb{R}.$$

Then parallel arguments as in steps 1–9 would apply to yield the proposition in $\mathbb{R}^n$. □

Remark 4.8. The proof breaks down for $F=F(D^2u,x/\varepsilon)$, since the idea of perturbing the problem by tilting the Neumann boundary and its boundary data, that is, the approximation of u_η by w_η in step 3, does not apply if the inside operator also depends on x/ε.

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SUNHI CHOI: schoi@math.arizona.edu
Department of Mathematics, University of Arizona, Tucson, AZ 85721, United States

INWON C. KIM: ikim@math.ucla.edu
Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90095, United States

KI-AHM LEE: kiahm@math.snu.ac.kr
School of Mathematical Sciences, Seoul National University, Seoul 151-747, South Korea

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LONG-TIME ASYMPTOTICS FOR TWO-DIMENSIONAL EXTERIOR FLOWS WITH SMALL CIRCULATION AT INFINITY

THIERRY GALLAY AND YASUNORI MAEKAWA

We consider the incompressible Navier–Stokes equations in a two-dimensional exterior domain Ω, with no-slip boundary conditions. Our initial data are of the form $u_0 = \alpha\Theta_0 + v_0$, where Θ_0 is the Oseen vortex with unit circulation at infinity and v_0 is a solenoidal perturbation belonging to $L^2(\Omega)^2 \cap L^q(\Omega)^2$ for some $q \in (1,2)$. If $\alpha \in \mathbb{R}$ is sufficiently small, we show that the solution behaves asymptotically in time like the self-similar Oseen vortex with circulation α. This is a global stability result, in the sense that the perturbation v_0 can be arbitrarily large, and our smallness assumption on the circulation α is independent of the domain Ω.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth exterior domain, namely an unbounded connected open subset of the Euclidean plane with a smooth compact boundary $\partial\Omega$. We consider the free motion of an incompressible viscous fluid in Ω, with no-slip boundary conditions on $\partial\Omega$. The evolution is governed by the Navier–Stokes equations

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = \Delta u - \nabla p, \quad \operatorname{div} u = 0 & \text{for } x \in \Omega, \quad t > 0, \\ u(x,t) = 0 & \text{for } x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1}$$

where $u(x,t) \in \mathbb{R}^2$ denotes the velocity of a fluid particle at point $x \in \Omega$ and time $t > 0$, and $p(x,t)$ is the pressure in the fluid at the same point. For simplicity, both the kinematic viscosity and the density of the fluid have been normalized to 1. The initial velocity field $u_0 : \Omega \to \mathbb{R}^2$ is assumed to be divergence-free and tangent to the boundary on $\partial\Omega$.

If the initial velocity u_0 belongs to the energy space

$$L^2_\sigma(\Omega) = \{u \in L^2(\Omega)^2 \mid \operatorname{div} u = 0 \text{ in } \Omega,\ u\cdot n = 0 \text{ on } \partial\Omega\},$$

where n denotes the unit normal on $\partial\Omega$, then it is known that system (1) has a unique global solution $u \in C^0([0,\infty); L^2_\sigma(\Omega)) \cap C^1((0,\infty); L^2_\sigma(\Omega)) \cap C^0((0,\infty); H^1_0(\Omega)^2 \cap H^2(\Omega)^2)$, which satisfies the energy equality

$$\frac{1}{2}\|u(\cdot,t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u(\cdot,s)\|^2_{L^2(\Omega)}\,ds = \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} \quad \text{for all } t > 0.$$

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This global well-posedness result was first established by Leray [1933] in the particular case where $\Omega = \mathbb{R}^2$, and subsequently extended to more general domains, including exterior domains, by various authors [Leray 1934; Ladyženskaja 1959; Lions and Prodi 1959; Kato and Fujita 1962; Fujita and Kato 1964; Kozono and Ogawa 1993b]. It is also known that the kinetic energy $\frac{1}{2}\|u(\cdot,t)\|^2_{L^2(\Omega)}$ converges to zero as $t \to \infty$ [Masuda 1984; Borchers and Miyakawa 1992; Kozono and Ogawa 1993b], and precise decay rates can be obtained under additional assumptions on the initial data [Kozono and Ogawa 1993a; He and Miyakawa 2006; Bae and Jin 2006].

In two-dimensional fluid mechanics, however, the assumption that the velocity field u be square integrable is quite restrictive, because it implies (if $u = 0$ on $\partial\Omega$) that the associated vorticity field $\omega = \partial_1 u_2 - \partial_2 u_1$ has zero mean over Ω; see [Majda and Bertozzi 2002, Section 3.1.3]. In many important examples, this condition is not satisfied and the kinetic energy of the flow is therefore infinite. For instance, when $\Omega = \mathbb{R}^2$, the Navier–Stokes equations (1) have a family of explicit self-similar solutions of the form $u(x,t) = \alpha\Theta(x,t)$, $p(x,t) = \alpha^2\Pi(x,t)$, where $\alpha \in \mathbb{R}$ is a parameter and

$$\Theta(x,t) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}\Big(1 - e^{-\frac{|x|^2}{4(1+t)}}\Big), \qquad \nabla\Pi(x,t) = \frac{x}{|x|^2}|\Theta(x,t)|^2. \tag{2}$$

Here and in the sequel, if $x = (x_1, x_2) \in \mathbb{R}^2$, we define $x^\perp = (-x_2, x_1)$ and $|x|^2 = x_1^2 + x_2^2$. The solution (2) is called the *Lamb–Oseen vortex* with circulation α. Remark that $|\Theta(x,t)| = \mathbb{O}(|x|^{-1})$ as $|x| \to \infty$, so that $\Theta(\cdot,t) \notin L^2(\mathbb{R}^2)^2$, and that the circulation at infinity of the vector field Θ is equal to 1, in the sense that $\oint_{|x|=R} \Theta_1 \,\mathrm{d}x_1 + \Theta_2 \,\mathrm{d}x_2 \to 1$ as $R \to \infty$. The corresponding vorticity distribution

$$\Xi(x,t) = \partial_1\Theta_2(x,t) - \partial_2\Theta_1(x,t) = \frac{1}{4\pi(1+t)} e^{-\frac{|x|^2}{4(1+t)}} \tag{3}$$

has a constant sign and satisfies $\int_{\mathbb{R}^2} \Xi(x,t)\,\mathrm{d}x = 1$ for all $t \geq 0$. Oseen's vortex plays an important role in the dynamics of the Navier–Stokes equations in $\mathbb{R}^2$, because it describes the long-time asymptotics of all solutions whose vorticity distribution is integrable. This result was first proved in [Giga and Kambe 1988] for small solutions, and subsequently in [Carpio 1994] for large solutions with small circulation. The general case was finally settled in [Gallay and Wayne 2005]. It is worth mentioning that all these results were obtained using the vorticity formulation of the Navier–Stokes equations.

In the case of an exterior domain $\Omega \subset \mathbb{R}^2$, much less is known about infinite-energy solutions, mainly because the vorticity formulation is not convenient anymore due to the boundary conditions. A general existence result was established in [Kozono and Yamazaki 1995], who proved that system (1) is globally well-posed for initial data u_0 in the weak L^2 space $L^{2,\infty}_\sigma(\Omega)$, provided that the local singularity of u_0 in $L^{2,\infty}$ is sufficiently small. In what follows, we consider initial data of the form

$$u_0 = \alpha\chi\Theta_0 + v_0, \tag{4}$$

where $\Theta_0(x) = \Theta(x,0)$ is Oseen's vortex at time $t = 0$, and $\chi : \mathbb{R}^2 \to [0,1]$ is a smooth, radially symmetric cut-off function such that $\chi = 0$ on a neighborhood of $\mathbb{R}^2 \setminus \Omega$ and $\chi(x) = 1$ when $|x|$ is sufficiently large. For any $\alpha \in \mathbb{R}$ and any $v_0 \in L^2_\sigma(\Omega)$, Theorem 4 in [Kozono and Yamazaki 1995] asserts that the Navier–Stokes equation (1) has a global solution with initial data (4), which is unique in an appropriate class. However, little is known about the long-time behavior of this solution, and in particular there is no a priori estimate which guarantees that the $L^{2,\infty}$ norm of u remains bounded for all times.

Very recently, a first result concerning the long-time behavior of solutions of (1) with initial data of the form (4) was obtained by Iftimie, Karch, and Lacave:

Theorem 1.1 [Iftimie et al. 2011]. *Let $\Omega \subset \mathbb{R}^2$ be a smooth exterior domain whose complement $\mathbb{R}^2 \setminus \Omega$ is a connected set in $\mathbb{R}^2$. For any $v_0 \in L^2_\sigma(\Omega)$, there exists a constant $\epsilon = \epsilon(v_0, \Omega) > 0$ such that, for all $\alpha \in [-\epsilon, \epsilon]$, the solution of (1) with initial data (4) satisfies*

$$\lim_{t\to\infty} t^{\frac{1}{2}-\frac{1}{p}} \|u(\cdot,t) - \alpha\Theta(\cdot,t)\|_{L^p(\Omega)} = 0 \quad \text{for all } p \in (2,\infty). \tag{5}$$

Moreover, there exists $\epsilon_0 = \epsilon_0(\Omega) > 0$ such that $\epsilon \geq \epsilon_0$ if $\|v_0\|_{L^2} \leq \epsilon_0$.

Theorem 1.1 shows that solutions of (1) which are finite-energy perturbations of Oseen's vortex $\alpha\Theta_0$ behave asymptotically in time like the self-similar Oseen vortex $\alpha\Theta(x,t)$, provided that the circulation at infinity, α, is sufficiently small, depending on the size of the initial perturbation. The conclusion holds in particular when both the circulation α and the finite-energy perturbation v_0 are small, so that Theorem 1.1 extends to exterior domains the result of [Giga and Kambe 1988]. For large solutions, however, the assumption that α be small depending on v_0 is very restrictive. The goal of the present paper is to prove the following result, which reaches a conclusion similar to that of Theorem 1.1 under different assumptions on the initial data:

Theorem 1.2. *Fix $q \in (1,2)$, and let $\mu = 1/q - 1/2$. There exists a constant $\epsilon = \epsilon(q) > 0$ such that, for any smooth exterior domain $\Omega \subset \mathbb{R}^2$ and for all initial data of the form (4) with $|\alpha| \leq \epsilon$ and $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, the solution of the Navier–Stokes equations (1) satisfies*

$$\|u(\cdot,t) - \alpha\Theta(\cdot,t)\|_{L^2(\Omega)} + t^{1/2}\|\nabla u(\cdot,t) - \alpha\nabla\Theta(\cdot,t)\|_{L^2(\Omega)} = \mathcal{O}(t^{-\mu}), \tag{6}$$

as $t \to +\infty$.

Here, we also suppose that the circulation at infinity is small, and we assume in addition that the initial perturbation belongs to $L^2_\sigma(\Omega) \cap L^q(\Omega)^2$ for some $q < 2$. Unlike in Theorem 1.1, the limiting case $q = 2$ is not included, and the proof shows that $\epsilon(q) = \mathcal{O}(\sqrt{2-q})$ as $q \to 2$. However, there is absolutely no restriction on the size of the perturbation v_0; hence Theorem 1.2 establishes a *global stability property* for the Lamb–Oseen vortices (with small circulation) in two-dimensional exterior domains. In this sense, our result can be considered as a generalization to exterior domains of the work of Carpio [1994], although our proof relies on completely different ideas. On the other hand, since our perturbations decay faster at infinity (in space) than those considered by Iftimie, Karch, and Lacave, we are able to show that the difference $u(x,t) - \alpha\Theta(x,t)$ converges rapidly to zero, like an inverse power of time, as $t \to \infty$. In particular, using (6) and elementary interpolation, we obtain the estimate

$$\sup_{t>0} t^{\frac{1}{q}-\frac{1}{p}} \|u(\cdot,t) - \alpha\Theta(\cdot,t)\|_{L^p(\Omega)} < \infty \quad \text{for all } p \in [2,\infty),$$

which improves (5) since $q < 2$.

At this point, it is useful to mention that the assumption that u_0 can be decomposed as in (4) for some $\alpha \in \mathbb{R}$ and some $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$ is automatically satisfied if we suppose that the initial vorticity $\omega_0 = \operatorname{curl} u_0$ is sufficiently localized. Indeed, let us assume for simplicity that u_0 vanishes on the boundary $\partial\Omega$. For $1 \le p < \infty$, we denote by $\dot W^{1,p}_{0,\sigma}(\Omega)$ the completion with respect to the norm $u \mapsto \|\nabla u\|_{L^p}$ of the space of all smooth, divergence-free vector fields with compact support in Ω. Using this notation, we have the following result:

Proposition 1.3. *Fix $q \in (1,2)$. Assume that u_0 belongs to $\dot W^{1,p}_{0,\sigma}(\Omega)$ for some $p \in [1,2)$, and that the associated vorticity $\omega_0 = \operatorname{curl} u_0$ satisfies*

$$\int_\Omega (1+|x|^2)^m |\omega_0(x)|^2 \,\mathrm{d}x < \infty \tag{7}$$

for some $m > 2/q$. If we define $\alpha = \int_\Omega \omega_0(x)\,\mathrm{d}x$, then u_0 can be decomposed as in (4) for some $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$. In particular, if $|\alpha| \le \epsilon$, the conclusion of Theorem 1.2 holds.

For completeness, we give a short proof of Proposition 1.3 in the Appendix. Returning to the discussion of Theorem 1.2, we emphasize that the smallness condition on the circulation α is independent of the domain Ω, which can be an arbitrary multiply connected exterior domain. In fact, the proof will show that the optimal constant $\epsilon(q)$ is entirely determined by quantities that appear in the evolution equation for the perturbation of Oseen's vortex in the whole plane $\mathbb{R}^2$. Note that Oseen vortices are known to be globally stable for all values of the circulation α when $\Omega = \mathbb{R}^2$ [Gallay and Wayne 2005], but in that particular case one can use the vorticity equation to obtain precise information on the solutions of (1). The reader who is not interested in precise convergence rates could consider the following variant of Theorem 1.2, where the condition on the circulation is totally explicit:

Corollary 1.4. *There exists a universal constant $\epsilon_* \ge 4.956$ such that, if $|\alpha| < \epsilon_*$ and if*

$$v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$$

for all $q \in (1,2)$, the solution of the Navier–Stokes equations (1) with initial data (4) satisfies

$$\|u(\cdot,t) - \alpha\Theta(\cdot,t)\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

The rest of this paper is devoted to the proof of Theorem 1.2, which is quite different from that of Theorem 1.1 in [Iftimie et al. 2011]. In the preliminary section (Section 2), we collect various estimates on the truncated Oseen vortex $\chi\Theta$, which can be verified by direct calculations. In Section 3, following the classical approach of [Fujita and Kato 1964], we prove the existence of a unique global solution of (1) for small initial data of the form (4), and we obtain the asymptotics (6) for small solutions. To deal with large solutions, we derive in Section 4 a "logarithmic energy estimate", which shows that the energy norm of the perturbation v has at most a logarithmic growth as $t \to \infty$. This is the key new ingredient, which we use as a substitute for the classical energy inequality when $\alpha \neq 0$. Exploiting this estimate and our assumption that $v_0 \in L^q(\Omega)^2$, we control in Section 5 the evolution of a fractional primitive of v, and we deduce that the perturbation $v(\cdot,t)$ converges to zero in energy norm, at least along a sequence of times. Thus we can eventually use the results of Section 3, and the conclusion follows.

2. The truncated Oseen vortex

Fix $\rho \ge 1$ large enough so that $\{x \in \mathbb{R}^2 \mid |x| \ge \rho\} \subset \Omega$. Let $\chi(x) = \tilde{\chi}(x/\rho)$, where $\tilde{\chi} \in C^\infty(\mathbb{R}^2)$ is a radially symmetric cut-off function satisfying $\tilde{\chi}(x) = 0$ when $|x| \le 1$, $\tilde{\chi}(x) = 1$ when $|x| \ge 2$, and $0 \le \tilde{\chi}(x) \le 1$ for all $x \in \mathbb{R}^2$. We define the *truncated Oseen vortex* (with unit circulation) as follows:

$$u^\chi(x,t) = \chi(x)\Theta(x,t) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}\Big(1 - e^{-\frac{|x|^2}{4(1+t)}}\Big)\chi(x), \quad x \in \mathbb{R}^2,\ t \ge 0. \tag{8}$$

Since χ is radially symmetric and $\operatorname{supp}\chi \subset \{x \in \mathbb{R}^2 \mid |x| \ge \rho\} \subset \Omega$, it is clear that $u^\chi(x,t)$ is a smooth divergence-free vector field which vanishes in a neighborhood of $\mathbb{R}^2 \setminus \Omega$. Let $\omega^\chi = \partial_1 u_2^\chi - \partial_2 u_1^\chi$ be the corresponding vorticity field, namely

$$\omega^\chi(x,t) = \chi(x)\Xi(x,t) + \frac{1}{2\pi}\frac{1}{|x|^2}\Big(1 - e^{-\frac{|x|^2}{4(1+t)}}\Big)x \cdot \nabla\chi(x), \tag{9}$$

where $\Xi(x,t)$ is defined in (3). Since $u^\chi(x,t) = \Theta(x,t)$ whenever $|x| \ge 2\rho$, the circulation of u^χ at infinity is equal to 1, so that $\int_{\mathbb{R}^2} \omega^\chi \,\mathrm{d}x = 1$. Moreover, a direct calculation shows that

$$(u^\chi \cdot \nabla)u^\chi = \frac{1}{2}\nabla|u^\chi|^2 + (u^\chi)^\perp\omega^\chi = -\frac{x}{|x|^2}|u^\chi|^2; \tag{10}$$

hence there exists a radially symmetric function $p^\chi(x,t)$ such that $-\nabla p^\chi = (u^\chi \cdot \nabla)u^\chi$. This shows that $P(u^\chi \cdot \nabla)u^\chi = 0$, where P denotes the Leray–Hopf projection in Ω, namely the orthogonal projection in $L^2(\Omega)^2$ onto the subspace $L^2_\sigma(\Omega)$.

The following elementary estimates will be useful:

Lemma 2.1. (i) *For any $p \in (2,\infty]$, there exists a constant $a_p > 0$ such that*

$$\|u^\chi(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le \frac{a_p}{(1+t)^{\frac{1}{2}-\frac{1}{p}}}, \quad t \ge 0. \tag{11}$$

(ii) *For any $p \in (1,\infty]$, there exists a constant $b_p > 0$ such that*

$$\|\nabla u^\chi(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le \frac{b_p}{(1+t)^{1-\frac{1}{p}}}, \quad t \ge 0. \tag{12}$$

(iii) *For all $t, s \ge 0$, we have*

$$\|u^\chi(\cdot,t) - u^\chi(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} \le \frac{1}{4\pi}\left|\log\frac{1+t}{1+s}\right|. \tag{13}$$

(iv) *There exists a constant $\kappa_1 > 0$ such that, for all $t, s \ge 0$,*

$$\|\nabla u^\chi(\cdot,t) - \nabla u^\chi(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} \le \kappa_1\left|\frac{1}{1+t} - \frac{1}{1+s}\right|. \tag{14}$$

Moreover all constants a_p, b_p, and κ_1 are independent of ρ, and hence of the domain Ω.

Proof. By (8) we have

$$u^\chi(x,t) = \chi(x)\Theta(x,t) = \frac{\chi(x)}{\sqrt{1+t}}\Theta_0\Big(\frac{x}{\sqrt{1+t}}\Big),$$

where $\Theta_0(x) = \Theta(x,0)$. Since $0 \le \chi \le 1$ and $\Theta_0 \in L^p(\mathbb{R}^2)^2$ for all $p > 2$, we find

$$\|u^\chi(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le \frac{1}{\sqrt{1+t}}\Big\|\Theta_0\Big(\frac{\cdot}{\sqrt{1+t}}\Big)\Big\|_{L^p(\mathbb{R}^2)} = \frac{\|\Theta_0\|_{L^p(\mathbb{R}^2)}}{(1+t)^{\frac{1}{2}-\frac{1}{p}}}, \quad t \ge 0.$$

This proves (11).

Similarly, we have $\partial_i u^\chi = \chi\partial_i\Theta + (\partial_i\chi)\Theta$ for $i = 1,2$. As $\partial_i\Theta_0 \in L^p(\mathbb{R}^2)^2$ for all $p > 1$, we obtain as before

$$\|\chi\partial_i\Theta(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le \frac{1}{1+t}\Big\|\partial_i\Theta_0\Big(\frac{\cdot}{\sqrt{1+t}}\Big)\Big\|_{L^p(\mathbb{R}^2)} = \frac{\|\partial_i\Theta_0\|_{L^p(\mathbb{R}^2)}}{(1+t)^{1-\frac{1}{p}}}, \quad t \ge 0. \tag{15}$$

On the other hand, the function $\partial_i\chi$ is supported in the annulus

$$D = \{x \in \mathbb{R}^2 \mid \rho \le |x| \le 2\rho\},$$

and satisfies $|\partial_i\chi(x)| \le C\rho^{-1}$ for some $C > 0$ independent of ρ. Moreover, it follows from (2) that

$$|\Theta(x,t)| \le \frac{1}{2\pi}\min\Big(\frac{1}{|x|}, \frac{|x|}{4(1+t)}\Big), \quad x \in \mathbb{R}^2,\ t \ge 0;$$

hence

$$|(\partial_i\chi(x))\Theta(x,t)| \le C\min\Big(\frac{1}{\rho^2}, \frac{1}{1+t}\Big)\mathbf{1}_D(x), \quad x \in \mathbb{R}^2,\ t \ge 0,$$

where $\mathbf{1}_D$ is the characteristic function of D. Taking the L^p norm of both sides, we thus obtain

$$\|(\partial_i\chi)\Theta(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le C\rho^{2/p}\min\Big(\frac{1}{\rho^2}, \frac{1}{1+t}\Big) \le \frac{C}{(1+t)^{1-\frac{1}{p}}}, \quad t \ge 0. \tag{16}$$

Combining (15) and (16), we arrive at (12).

To prove (13), we observe that

$$\begin{aligned}\|u^\chi(\cdot,t) - u^\chi(\cdot,s)\|_{L^2(\mathbb{R}^2)}^2 &\le \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\frac{1}{|x|^2}\Big(e^{-\frac{|x|^2}{4(1+t)}} - e^{-\frac{|x|^2}{4(1+s)}}\Big)^2\,\mathrm{d}x \\ &= \frac{1}{2\pi}\log\Big\{\frac{1}{2}\sqrt{\frac{1+t}{1+s}} + \frac{1}{2}\sqrt{\frac{1+s}{1+t}}\Big\} \le \frac{1}{4\pi}\Big|\log\frac{1+t}{1+s}\Big|\end{aligned}$$

for all $t, s \ge 0$. Finally, using (9), we find

$$\omega^\chi(x,t) - \omega^\chi(x,s) = \chi(x)\big(\Xi(x,t) - \Xi(x,s)\big) - \frac{x\cdot\nabla\chi(x)}{2\pi|x|^2}\Big(e^{-\frac{|x|^2}{4(1+t)}} - e^{-\frac{|x|^2}{4(1+s)}}\Big).$$

Thus $\|\nabla u^\chi(\cdot,t)-\nabla u^\chi(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} = \|\omega^\chi(\cdot,t)-\omega^\chi(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} \le \big(J_1(t,s)^{1/2}+J_2(t,s)^{1/2}\big)^2$, where

$$\begin{aligned} J_1(t,s) &= \int_{\mathbb{R}^2} \chi(x)^2\big(\Xi(x,t)-\Xi(x,s)\big)^2\,\mathrm{d}x \le \int_{\mathbb{R}^2}\big(\Xi(x,t)-\Xi(x,s)\big)^2\,\mathrm{d}x \\ &= \frac{1}{8\pi}\left\{\frac{1}{1+t}+\frac{1}{1+s}-\frac{4}{t+s+2}\right\} \le \frac{1}{8\pi}\left|\frac{1}{1+t}-\frac{1}{1+s}\right|, \end{aligned}$$

and

$$\begin{aligned} J_2(t,s) &= \int_{\mathbb{R}^2} \frac{|\nabla\chi(x)|^2}{4\pi^2|x|^2}\Big(e^{-\frac{|x|^2}{4(1+t)}}-e^{-\frac{|x|^2}{4(1+s)}}\Big)^2\,\mathrm{d}x \le C\rho^{-4}\int_D \Big(e^{-\frac{|x|^2}{4(1+t)}}-e^{-\frac{|x|^2}{4(1+s)}}\Big)^2\,\mathrm{d}x \\ &\le C\rho^{-2}\sup_{x\in D}\Big|e^{-\frac{|x|^2}{4(1+t)}}-e^{-\frac{|x|^2}{4(1+s)}}\Big| \le C\left|\frac{1}{1+t}-\frac{1}{1+s}\right|. \end{aligned}$$

We thus obtain (14), which is the desired estimate. For later use, we also observe that $J_2(t,s)$ can be bounded by $C\rho^2\big(\frac{1}{1+t}-\frac{1}{1+s}\big)^2$, for some $C>0$ independent of ρ. Since $\rho\ge 1$, this gives the alternative estimate

$$\|\nabla u^\chi(\cdot,t)-\nabla u^\chi(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} \le \frac{1}{8\pi}\left|\frac{1}{1+t}-\frac{1}{1+s}\right| + C\rho^2\left|\frac{1}{1+t}-\frac{1}{1+s}\right|^{3/2}, \tag{17}$$

which will be used in Section 4. This concludes the proof of Lemma 2.1. □

The truncated Oseen vortex is not a solution of the Navier–Stokes equation, and therefore we need to control the remainder term $R^\chi = \Delta u^\chi - \partial_t u^\chi = (\Delta\chi)\Theta + 2(\nabla\chi\cdot\nabla)\Theta$, which has the explicit expression

$$R^\chi(x,t) = \Theta(x,t)\Delta\chi(x) + 2\frac{x\cdot\nabla\chi(x)}{|x|^2}\big(x^\perp\Xi(x,t)-\Theta(x,t)\big). \tag{18}$$

Lemma 2.2. *There exists a constant $\kappa_2>0$ (independent of ρ) such that, for any $p\in[1,\infty]$,*

$$\|R^\chi(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le \frac{\kappa_2\rho^{\frac{2}{p}-1}}{1+t}, \qquad t\ge 0. \tag{19}$$

Moreover, for any vector field $u\in H^1_{\mathrm{loc}}(\mathbb{R}^2)^2$, we have

$$\left|\int_{\mathbb{R}^2} R^\chi(x,t)\cdot u(x)\,\mathrm{d}x\right| \le \frac{\kappa_2\rho}{1+t}\|\nabla u\|_{L^2(D)}, \qquad t\ge 0, \tag{20}$$

where $D=\{x\in\mathbb{R}^2 \mid \rho\le|x|\le 2\rho\}$.

Proof. It is clear from (18) that $|R^\chi(x,t)|\le C\rho^{-1}(1+t)^{-1}\mathbf{1}_D(x)$ for all $x\in\mathbb{R}^2$ and all $t\ge 0$, and (19) follows immediately. Moreover, we have $R^\chi(x,t)=x^\perp Q^\chi(x,t)$ for some radially symmetric scalar function $Q(x,t)$; hence $R^\chi(\cdot,t)$ has zero mean over the annulus D. If $u\in H^1_{\mathrm{loc}}(\mathbb{R}^2)^2$ and if we denote by $\bar u$ the average of u over D, the Poincaré–Wirtinger inequality implies

$$\left|\int_{\mathbb{R}^2} R^\chi(x,t)\cdot u(x)\,\mathrm{d}x\right| = \left|\int_D R^\chi(x,t)\cdot(u(x)-\bar u)\,\mathrm{d}x\right| \le C\rho\|R^\chi(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|\nabla u\|_{L^2(D)},$$

and using (19) with $p=2$ we obtain (20). □

3. Asymptotic behavior of small solutions

Given $\alpha \in \mathbb{R}$, we consider solutions of (1) of the form

$$u(x,t) = \alpha u^\chi(x,t) + v(x,t), \quad p(x,t) = \alpha^2 p^\chi(x,t) + q(x,t), \tag{21}$$

where $u^\chi(x,t)$ is the truncated Oseen vortex (8) and p^χ is the associated pressure. The perturbation $v(x,t)$ satisfies the no-slip boundary condition and the equation

$$\partial_t v + \alpha(u^\chi \cdot \nabla)v + \alpha(v \cdot \nabla)u^\chi + (v \cdot \nabla)v = \Delta v + \alpha R^\chi - \nabla q, \quad \operatorname{div} v = 0, \tag{22}$$

where R^χ is given by (18). If we apply the Leray–Hopf projection P and use the fact that $PR^\chi = R^\chi$, we obtain the equivalent system

$$\partial_t v + \alpha P\big((u^\chi \cdot \nabla)v + (v \cdot \nabla)u^\chi\big) + P(v \cdot \nabla)v = -Av + \alpha R^\chi, \tag{23}$$

where $A = -P\Delta$ is the Stokes operator, which is selfadjoint and nonnegative in $L^2_\sigma(\Omega)$ with domain $D(A) = L^2_\sigma(\Omega) \cap H^1_0(\Omega)^2 \cap H^2(\Omega)^2$; see [Constantin and Foias 1988].

In this section, we fix some initial time $t_0 \geq 0$ and prove the existence of global solutions to (23) with small initial data $v_0 = v(\cdot, t_0)$ in the energy space. The integral equation associated with (23) is

$$v(t) = S(t-t_0)v_0 + \int_{t_0}^t S(t-s)\big\{\alpha R^\chi(s) - P(v(s)\cdot\nabla)v(s) - \alpha P\big((u^\chi(s)\cdot\nabla)v(s) + (v(s)\cdot\nabla)u^\chi(s)\big)\big\}\,\mathrm{d}s, \tag{24}$$

where $v(t) \equiv v(\cdot, t)$ and $S(t) = \exp(-tA)$ is the Stokes semigroup. For $p \in (1,\infty)$, we denote by $L^p_\sigma(\Omega)$ the closure in $L^p(\Omega)^2$ of the set of all smooth divergence-free vector fields with compact support in Ω. We then have the following standard estimates:

Proposition 3.1. *The Stokes operator $-A$ generates an analytic semigroup of contractions in $L^2_\sigma(\Omega)$. Moreover, for each $t > 0$ the operator $S(t) = \exp(-tA)$ extends to a bounded linear operator from $L^q_\sigma(\Omega)$ into $L^2_\sigma(\Omega)$ for $1 < q \leq 2$, and there exists a constant $C = C(q) > 0$ (independent of Ω) such that*

$$t^{\frac{1}{q}-\frac{1}{2}}\|S(t)v_0\|_{L^2(\Omega)} + t^{\frac{1}{q}}\|\nabla S(t)v_0\|_{L^2(\Omega)} \leq C\|v_0\|_{L^q(\Omega)}, \quad t > 0, \tag{25}$$

for all $v_0 \in L^q_\sigma(\Omega)$. In particular, we can take $C = 2$ in (25) if $q = 2$.

Since A is selfadjoint and nonnegative, it is clear that $\{S(t)\}_{t\geq 0}$ is an analytic semigroup of contractions in $L^2_\sigma(\Omega)$. In particular, we have $\|S(t)v_0\|_{L^2} \leq \|v_0\|_{L^2}$ and $t^{1/2}\|\nabla S(t)v_0\|_{L^2} = t^{1/2}\|A^{1/2}S(t)v_0\|_{L^2} \leq \|v_0\|_{L^2}$ for all $t > 0$ if $v_0 \in L^2_\sigma(\Omega)$. On the other hand, general $L^q - L^p$ estimates for $S(t)$ were established in [Borchers and Varnhorn 1993; Dan and Shibata 1999a; 1999b; Kozono and Yamazaki 1995; Maremonti and Solonnikov 1997], but the corresponding constants depend a priori on the domain Ω. The fact that (25) holds with C independent of Ω was already observed in [Borchers and Miyakawa 1992; Kozono and Ogawa 1993a]. For the reader's convenience, we reproduce the proof of (25) in Section 5 below.

The main result of this section is this:

Proposition 3.2. *Fix $\mu \in (0, 1/2)$. There exist positive constants K_0, δ, V_Ω, and T_Ω such that, if $t_0 \ge T_\Omega$, if $|\alpha| \le \delta$, and if $\|v_0\|_{L^2(\Omega)} \le V_\Omega$, then the perturbation equation (23) has a unique global solution $v \in C^0([t_0, \infty); L^2_\sigma(\Omega))$ such that*

$$\sup_{t \ge t_0} \|v(t)\|_{L^2(\Omega)} + \sup_{t > t_0} (t-t_0)^{\frac{1}{2}} \|\nabla v(t)\|_{L^2(\Omega)} \le 4\|v_0\|_{L^2(\Omega)} + K_0 \rho^{\frac{1}{2}} |\alpha| (1+t_0)^{-\frac{1}{4}}. \tag{26}$$

Here K_0 and δ are independent of Ω. In addition, if

$$M := \sup_{\tau > 0} \tau^{\mu} \|S(\tau) v_0\|_{L^2(\Omega)} + \sup_{\tau > 0} \tau^{\mu + \frac{1}{2}} \|\nabla S(\tau) v_0\|_{L^2(\Omega)} < \infty, \tag{27}$$

then

$$\sup_{t > t_0} (t-t_0)^{\mu} \|v(t)\|_{L^2(\Omega)} + \sup_{t > t_0} (t-t_0)^{\mu + \frac{1}{2}} \|\nabla v(t)\|_{L^2(\Omega)} \le 2M + C_\Omega |\alpha|, \tag{28}$$

for some $C_\Omega > 0$ depending on Ω.

Proof. We follow the classical approach of [Fujita and Kato 1964]. Given $t_0 \ge 0$, we introduce the Banach space $X = \{v \in C^0([t_0, \infty); L^2_\sigma(\Omega)) \cap C^0((t_0, \infty); H^1_0(\Omega)^2) \mid \|v\|_X < \infty\}$, equipped with the norm

$$\|v\|_X = \sup_{t \ge t_0} \|v(t)\|_{L^2} + \sup_{t > t_0} (t-t_0)^{\frac{1}{2}} \|\nabla v(t)\|_{L^2}.$$

If $v_0 \in L^2_\sigma(\Omega)$, we define $\bar v(t) = S(t-t_0) v_0$ for $t \ge t_0$. In view of (25), we have $\bar v \in X$ and $\|\bar v\|_X \le 2\|v_0\|_{L^2}$. On the other hand, given any $v \in X$ we define, for $t \ge t_0$,

$$(Fv)(t) = \int_{t_0}^{t} S(t-s)\big(\alpha R^{\chi}(s) + \alpha G_1^{v}(s) + G_2^{v}(s)\big)\,\mathrm{d}s = \alpha F_0(t) + \alpha (F_1 v)(t) + (F_2 v)(t),$$

where $G_1^v(s) = -P(u^{\chi}(s)\cdot\nabla) v(s) - P(v(s)\cdot\nabla) u^{\chi}(s)$ and $G_2^v(s) = -P(v(s)\cdot\nabla)v(s)$. We shall show that F maps X into X, and that there exist positive constants C_1, C_2, $C_{3,\Omega}$ (independent of t_0) such that

$$\|Fv\|_X \le C_1 \rho^{\frac{1}{2}} |\alpha| (1+t_0)^{-\frac{1}{4}} + |\alpha| C_2 \|v\|_X + C_{3,\Omega} \|v\|_X^2, \tag{29}$$

$$\|Fv - F\tilde v\|_X \le |\alpha| C_2 \|v - \tilde v\|_X + C_{3,\Omega}(\|v\|_X + \|\tilde v\|_X)\|v - \tilde v\|_X, \tag{30}$$

for all v, $\tilde v \in X$.

To prove (29), we estimate separately the contributions of F_0, F_1, and F_2. First, using (25) with $q = 4/3$, we obtain for $t > t_0$:

$$\|F_0(t)\|_{L^2} + (t-t_0)^{\frac{1}{2}} \|\nabla F_0(t)\|_{L^2} \le C \int_{t_0}^{t} \left(\frac{1}{(t-s)^{\frac{1}{4}}} + \frac{(t-t_0)^{\frac{1}{2}}}{(t-s)^{\frac{3}{4}}} \right) \|R^{\chi}(s)\|_{L^{\frac{4}{3}}}\,\mathrm{d}s, \tag{31}$$

and from Lemma 2.2 we know that $\|R^{\chi}(s)\|_{L^{4/3}} \le C\rho^{1/2}(1+s)^{-1}$ for all $s \ge 0$. It follows that $\|F_0\|_X \le C_1 \rho^{1/2}(1+t_0)^{-1/4}$ for some $C_1 > 0$ independent of t_0 and Ω. In a similar way, we find

$$\|(F_2 v)(t)\|_{L^2} + (t-t_0)^{\frac{1}{2}} \|\nabla (F_2 v)(t)\|_{L^2} \le C \int_{t_0}^{t} \left(\frac{1}{(t-s)^{\frac{1}{4}}} + \frac{(t-t_0)^{\frac{1}{2}}}{(t-s)^{\frac{3}{4}}} \right) \|G_2^{v}(s)\|_{L^{\frac{4}{3}}}\,\mathrm{d}s. \tag{32}$$

Using the fact that the Leray–Hopf projection is a bounded operator in $L^{4/3}(\Omega)^2$, whose norm depends a priori on Ω, we estimate

$$\|G_2^v(s)\|_{L^{\frac{4}{3}}} \le C_\Omega \|v(s)\|_{L^4} \|\nabla v(s)\|_{L^2} \le C_\Omega \|v(s)\|_{L^2}^{\frac{1}{2}} \|\nabla v(s)\|_{L^2}^{\frac{3}{2}} \le \frac{C_\Omega \|v\|_X^2}{(s-t_0)^{\frac{3}{4}}},$$

for all $s > t_0$. It follows that $\|F_2 v\|_X \le C_{3,\Omega}\|v\|_X^2$, where $C_{3,\Omega} > 0$ is independent of t_0. Finally, to bound F_1, we proceed in a slightly different way in order to obtain a constant C_2 that does not depend on Ω. Observing that $G_1^v(s) = -A^{1/2}A^{-1/2}P\,\mathrm{div}(u^\chi \otimes v + v \otimes u^\chi)(s)$, and that $\|A^{1/2}v\|_{L^2} = \|\nabla v\|_{L^2}$ for all $v \in L^2_\sigma(\Omega)) \cap H^1_0(\Omega)^2$, we can use (25) with $q = 2$ to obtain

$$\|(F_1 v)(t)\|_{L^2} \le \int_{t_0}^t (t-s)^{-\frac{1}{2}} \big\|A^{-1/2}P\,\mathrm{div}(u^\chi \otimes v + v \otimes u^\chi)(s)\big\|_{L^2}\,\mathrm{d}s. \tag{33}$$

Similarly, the quantity $(t-t_0)^{\frac{1}{2}}\|\nabla(F_1 v)(t)\|_{L^2}$ can be bounded by

$$\int_{t_0}^{\frac{t+t_0}{2}} \frac{(t-t_0)^{\frac{1}{2}}}{t-s} \big\|A^{-1/2}P\,\mathrm{div}(u^\chi \otimes v + v \otimes u^\chi)(s)\big\|_{L^2}\,\mathrm{d}s + \int_{\frac{t+t_0}{2}}^t \frac{(t-t_0)^{\frac{1}{2}}}{(t-s)^{\frac{1}{2}}} \|G_1^v(s)\|_{L^2}\,\mathrm{d}s. \tag{34}$$

Since $A^{-1/2}P\,\mathrm{div}$ defines a bounded operator from $L^2(\Omega)^4$ into $L^2_\sigma(\Omega)$ whose norm is less than or equal to 1 (see [Sohr 2001, Lemma III-2-6-1]), we have from (11)

$$\big\|A^{-1/2}P\,\mathrm{div}(u^\chi \otimes v + v \otimes u^\chi)(s)\big\|_{L^2} \le 2\|u^\chi(s)v(s)\|_{L^2} \le 2a_\infty(1+s)^{-\frac{1}{2}}\|v\|_X.$$

Moreover, using (11) and (12) we find

$$\|G_1^v(s)\|_{L^2} \le \|u^\chi(s)\nabla v(s)\|_{L^2} + \|v(s)\nabla u^\chi(s)\|_{L^2} \le \frac{a_\infty \|v\|_X}{(1+s)^{\frac{1}{2}}(s-t_0)^{\frac{1}{2}}} + \frac{b_\infty\|v\|_X}{1+s}.$$

Inserting these estimates into (33) and (34), we obtain $\|F_1 v\|_X \le C_2\|v\|_X$ for some $C_2 > 0$ independent of t_0 and Ω. Since $Fv = \alpha F_0 + \alpha F_1 v + F_2 v$, this concludes the proof of (29), and the Lipschitz bound (30) is established in exactly the same way.

Now let $B_r = \{v \in X \mid \|v\|_X \le r\}$, where $r > 0$ is small enough so that $4rC_{3,\Omega} \le 1$. If we assume that $4|\alpha|C_2 \le 1$, $8\|v_0\|_{L^2} \le r$, and $4C_1\rho^{1/2}|\alpha|(1+t_0)^{-1/4} \le r$, the estimates above imply that the map $v \mapsto \bar{v} + Fv$ leaves the closed ball B_r invariant and is a strict contraction in B_r. By construction, the unique fixed point of that map in B_r is the desired solution of (24). This proves the existence part of Proposition 3.2 with

$$K_0 = 2C_1, \quad \delta = \frac{1}{4C_2}, \quad V_\Omega = \frac{1}{32C_{3,\Omega}}, \quad T_\Omega = \left(\frac{4C_1C_{3,\Omega}\rho^{\frac{1}{2}}}{C_2}\right)^4.$$

In a second step, we assume that (27) holds for some $\mu \in (0, 1/2)$. Given any $T > t_0$, we define

$$\mathscr{E}_T = \sup_{t_0 \le t \le T} (t-t_0)^\mu \|v(t)\|_{L^2} + \sup_{t_0 < t \le T} (t-t_0)^{\mu+\frac{1}{2}} \|\nabla v(t)\|_{L^2},$$

where v is the solution of (24) constructed in the previous step. Our goal is to show that $\mathscr{E}_T$ is uniformly bounded by a constant which does not depend on T. Since $v(t) = S(t-t_0)v_0 + (Fv)(t)$, we have

$$\mathscr{E}_T \le M + \sup_{t_0 \le t \le T} (t-t_0)^{\mu} \|(Fv)(t)\|_{L^2} + \sup_{t_0 < t \le T} (t-t_0)^{\mu+\frac{1}{2}} \|\nabla (Fv)(t)\|_{L^2}, \tag{35}$$

where M is defined in (27). To estimate the last two terms, we proceed as above. Let $p \in (1,2)$ be such that $1/p > \mu + 1/2$, and define $q \in (2,\infty)$ by the relation $1/q = 1/p - 1/2$. As in (31) and (32), we have

$$(t-t_0)^{\mu} \|F_0(t)\|_{L^2} + (t-t_0)^{\mu+\frac{1}{2}} \|\nabla F_0(t)\|_{L^2} \le C \int_{t_0}^{t} \left(\frac{(t-t_0)^{\mu}}{(t-s)^{\frac{1}{q}}} + \frac{(t-t_0)^{\mu+\frac{1}{2}}}{(t-s)^{\frac{1}{p}}} \right) \|R^{\chi}(s)\|_{L^p} \, ds,$$

$$(t-t_0)^{\mu} \|(F_2 v)(t)\|_{L^2} + (t-t_0)^{\mu+\frac{1}{2}} \|\nabla (F_2 v)(t)\|_{L^2} \le C \int_{t_0}^{t} \left(\frac{(t-t_0)^{\mu}}{(t-s)^{\frac{1}{q}}} + \frac{(t-t_0)^{\mu+\frac{1}{2}}}{(t-s)^{\frac{1}{p}}} \right) \|G_2^{v}(s)\|_{L^p} \, ds,$$

for $t \in (t_0, T]$. Moreover $\|R^{\chi}(s)\|_{L^p} \le C\rho^{\frac{2}{p}-1}(1+s)^{-1}$ and

$$\|P(v(s)\cdot\nabla)v(s)\|_{L^p} \le C_{\Omega} \|v(s)\|_{L^q} \|\nabla v(s)\|_{L^2} \le C_{\Omega} \|v(s)\|_{L^2}^{\frac{2}{q}} \|\nabla v(s)\|_{L^2}^{2-\frac{2}{q}} \le \frac{C_{\Omega} \|v\|_X \mathscr{E}_T}{(s-t_0)^{\mu+1-\frac{1}{q}}}$$

for all $s \in (t_0, T]$. The term involving $F_1 v$ is estimated as in (33) and (34), and we find

$$(t-t_0)^{\mu} \|(F_1 v)(t)\|_{L^2} \le C \int_{t_0}^{t} \frac{(t-t_0)^{\mu} \mathscr{E}_T}{(t-s)^{\frac{1}{2}} (1+s)^{\frac{1}{2}} (s-t_0)^{\mu}} \, ds,$$

$$(t-t_0)^{\mu+\frac{1}{2}} \|\nabla (F_1 v)(t)\|_{L^2} \le C \int_{t_0}^{\frac{t+t_0}{2}} \frac{(t-t_0)^{\mu+\frac{1}{2}} \mathscr{E}_T}{(t-s)(1+s)^{\frac{1}{2}} (s-t_0)^{\mu}} \, ds$$
$$+ C \int_{\frac{t+t_0}{2}}^{t} \frac{(t-t_0)^{\mu+\frac{1}{2}}}{(t-s)^{\frac{1}{2}}} \left(\frac{\mathscr{E}_T}{(1+s)^{\frac{1}{2}} (s-t_0)^{\mu+\frac{1}{2}}} + \frac{\mathscr{E}_T}{(1+s)(s-t_0)^{\mu}} \right) ds.$$

If we insert these estimates into (35), we obtain after elementary calculations

$$\mathscr{E}_T \le M + \tilde{C}_1 \rho^{\frac{2}{p}-1} |\alpha| (1+t_0)^{-\frac{1}{p}+\mu+\frac{1}{2}} + \tilde{C}_2 |\alpha| \mathscr{E}_T + \tilde{C}_{3,\Omega} \|v\|_X \mathscr{E}_T, \tag{36}$$

for some positive constants $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_{3,\Omega}$ independent of T and t_0. Now, taking δ and V_{Ω} smaller and T_{Ω} larger if needed, we can ensure that $\tilde{C}_2|\alpha| + \tilde{C}_{3,\Omega}\|v\|_X \le 1/2$. Then (36) implies that

$$\mathscr{E}_T \le 2M + 2 \frac{\tilde{C}_1 \rho^{\frac{2}{p}-1} |\alpha|}{(1+t_0)^{\frac{1}{p}-\mu-\frac{1}{2}}}$$

for all $T > t_0$, and (28) follows. This concludes the proof. □

Remark 3.3. The proof of Proposition 3.2 can be modified in a classical way [Fujita and Kato 1964; Brezis 1994] to yield the following local existence result. For any $\alpha \in \mathbb{R}$, any $t_0 \ge 0$, and any $v_0 \in L^2_{\sigma}(\Omega)$, there exists $T = T(\alpha, v_0, \Omega) > 0$ such that (23) has a unique solution $v \in C^0([t_0, t_0+T]; L^2_{\sigma}(\Omega)) \cap C^0((t_0, t_0+T]; H^1_0(\Omega)^2)$ satisfying $v(t_0) = v_0$; moreover, any upper bound on $|\alpha| + \|v_0\|_{H^1}$ gives a lower bound on the local existence time T. In our formulation of Proposition 3.2, smallness conditions were imposed on α and v_0 to ensure global existence, and the assumption on the initial time t_0 guarantees that the smallness condition on α is independent of the domain Ω.

4. A logarithmic energy estimate

In this section, we establish our key estimate for large solutions of (23) in the energy space. Fix $\alpha \in \mathbb{R}$, $v_0 \in L^2_\sigma(\Omega)$, and let $v \in C^0([0,T]; L^2_\sigma(\Omega)) \cap C^0((0,T]; H^1_0(\Omega)^2)$ be a solution of (23) with initial data $v(0) = v_0$; see Remark 3.3. We first derive a crude bound on v using a classical energy estimate. Multiplying both sides of (23) by v and integrating by parts over Ω, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{L^2} + \|\nabla v(t)\|^2_{L^2} = \alpha\big\langle v(t), R^\chi(t)\big\rangle - \alpha\big\langle v(t), (v(t)\cdot\nabla)u^\chi(t)\big\rangle, \tag{37}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $L^2_\sigma(\Omega)$, so that $\|\cdot\|_{L^2} = \langle\cdot,\cdot\rangle^{1/2}$. Using (20), we easily obtain

$$|\alpha\langle v(t), R^\chi(t)\rangle| \le \frac{\kappa_2\rho|\alpha|}{1+t}\|\nabla v(t)\|_{L^2} \le \frac{\eta}{2}\|\nabla v(t)\|^2_{L^2} + \frac{\kappa_2^2\rho^2\alpha^2}{2\eta(1+t)^2},$$

for any $\eta \in (0,1]$. Moreover, applying (12) with $p=\infty$, we see that

$$\big|\big\langle v(t), (v(t)\cdot\nabla)u^\chi(t)\big\rangle\big| \le \frac{b_\infty}{1+t}\|v(t)\|^2_{L^2}.$$

We thus obtain the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{L^2} + (2-\eta)\|\nabla v(t)\|^2_{L^2} \le \frac{2b_\infty|\alpha|}{1+t}\|v(t)\|^2_{L^2} + \frac{\kappa_2^2\rho^2\alpha^2}{\eta(1+t)^2}, \quad 0 < t \le T.$$

Using Gronwall's lemma, we deduce that

$$\|v(t)\|^2_{L^2} + (2-\eta)\int_{t_0}^t \|\nabla v(s)\|^2_{L^2}\,\mathrm{d}s \le \left(\frac{1+t}{1+t_0}\right)^{2b_\infty|\alpha|}\left(\|v(t_0)\|^2_{L^2} + \frac{\kappa_2^2\rho^2\alpha^2}{\eta(1+t_0)}\right), \tag{38}$$

for $0 \le t_0 < t \le T$.

We shall see that estimate (38) is pessimistic for large times, but it already implies that the solutions of (23) in the energy space $L^2_\sigma(\Omega)$ are global. Indeed, (38) shows that the norm $\|v(t)\|_{L^2}$ grows at most polynomially in time, and it is then straightforward to establish a similar result for $\|\nabla v(t)\|_{L^2}$. In particular, the H^1 norm of $v(t)$ cannot blow up in finite time, and using Remark 3.3 we conclude that all solutions of (23) in $L^2_\sigma(\Omega)$ are global.

The aim of this section is to establish the following "logarithmic energy estimate", which improves (38) for large times.

Proposition 4.1. *There exists a constant $K_1 > 0$ (independent of Ω) such that, for any $\alpha \in \mathbb{R}$ and any $v_0 \in L^2_\sigma(\Omega)$, the solution of (23) with initial data v_0 satisfies, for all $t \ge 1$,*

$$\|v(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla v(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \le K_1\big(\|v_0\|^2_{L^2(\Omega)} + \alpha^2\log(1+t) + D_{\alpha,\rho}\big), \tag{39}$$

where $D_{\alpha,\rho} = \alpha^2\log(1+|\alpha|) + \alpha^2\rho^2$.

Proof. As in (38), we introduce here a parameter $\eta \in (0,1]$, which will be used in Section 5 below to specify the optimal smallness condition on the circulation α and prove Corollary 1.4. The reader who is not interested in optimal constants should set $\eta = 1$ everywhere.

Given any $\tau \geq 0$, we define

$$\tilde{v}(x,t) = u(x,t) - \alpha u^{\chi}(x,t+\tau) = v(x,t) + \alpha\big(u^{\chi}(x,t) - u^{\chi}(x,t+\tau)\big), \tag{40}$$

for all $x \in \Omega$ and all $t > 0$. Then $\tilde{v}$ satisfies (23) where $u^{\chi}(x,t)$ and $R^{\chi}(x,t)$ are replaced by $u^{\chi}(x,t+\tau)$ and $R^{\chi}(x,t+\tau)$, respectively. Proceeding exactly as above, we thus obtain the following energy estimate:

$$\|\tilde{v}(t)\|_{L^2}^2 + (2-\eta)\int_0^t \|\nabla\tilde{v}(s)\|_{L^2}^2 \,\mathrm{d}s \leq \left(\frac{1+t+\tau}{1+\tau}\right)^{2b_\infty|\alpha|}\left(\|\tilde{v}(0)\|_{L^2}^2 + \frac{\kappa_2^2\rho^2\alpha^2}{\eta(1+\tau)}\right), \tag{41}$$

for all $t > 0$. Now, we fix $t \geq 1$ and choose $\tau = Nt - 1$, where

$$N = N_{\alpha,\eta} = \max\left(1, \frac{2b_\infty|\alpha|}{\log(1+\eta)}\right).$$

This choice implies that

$$\left(\frac{1+t+\tau}{1+\tau}\right)^{2b_\infty|\alpha|} = \left(1+\frac{1}{N}\right)^{2b_\infty|\alpha|} \leq 1+\eta.$$

On the other hand, using (13), (40), we find

$$\|v(t)\|_{L^2}^2 \leq (1+\eta)\|\tilde{v}(t)\|_{L^2}^2 + \frac{1+\eta}{\eta}\alpha^2\|u^{\chi}(t) - u^{\chi}(t+\tau)\|_{L^2}^2 \leq (1+\eta)\|\tilde{v}(t)\|_{L^2}^2 + \frac{\alpha^2}{2\pi\eta}\log(N+1),$$

$$\|\tilde{v}(0)\|_{L^2}^2 \leq \frac{1+\eta}{\eta}\|v_0\|_{L^2}^2 + (1+\eta)\alpha^2\|u^{\chi}(0) - u^{\chi}(\tau)\|_{L^2}^2 \leq \frac{2}{\eta}\|v_0\|_{L^2}^2 + \frac{(1+\eta)\alpha^2}{4\pi}\log(Nt).$$

Similarly, using (17), we find

$$\begin{aligned}\int_0^t \|\nabla v(s)\|_{L^2}^2 \,\mathrm{d}s &\leq 2\int_0^t \|\nabla\tilde{v}(s)\|_{L^2}^2 \,\mathrm{d}s + 2\alpha^2\int_0^t \|\nabla u^{\chi}(s) - \nabla u^{\chi}(s+\tau)\|_{L^2}^2 \,\mathrm{d}s \\ &\leq 2\int_0^t \|\nabla\tilde{v}(s)\|_{L^2}^2 \,\mathrm{d}s + \frac{\alpha^2}{4\pi}\log(1+t) + C\rho^2\alpha^2.\end{aligned}$$

Thus, it follows from (41) that

$$\|v(t)\|_{L^2}^2 \leq \frac{(1+\eta)^3\alpha^2}{4\pi}\log t + \frac{C}{\eta}\big(\|v_0\|_{L^2}^2 + \alpha^2\log(N+1) + \alpha^2\rho^2\big), \tag{42}$$

$$\int_0^t \|\nabla v(s)\|_{L^2}^2 \,\mathrm{d}s \leq \frac{(1+\eta)^3\alpha^2}{2\pi}\log(1+t) + \frac{C}{\eta}\big(\|v_0\|_{L^2}^2 + \alpha^2\rho^2\big) + C\alpha^2\log N, \tag{43}$$

for some universal constant $C > 0$. Setting $\eta = 1$ and using the definition of N, we see that (39) follows from (42), (43). $\square$

5. Estimate for a fractional primitive of the velocity field

In this final section, we consider the solution of (23) with initial data $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, for some fixed $q \in (1,2)$, and we define $\mu = 1/q - 1/2 \in (0, 1/2)$. If A is the Stokes operator in $L^2_\sigma(\Omega)$, we recall that A is selfadjoint and nonnegative in $L^2_\sigma(\Omega)$, so that the fractional power A^β can be defined for all $\beta > 0$. The following result shows that the range of A^μ contains the (dense) subspace $L^2_\sigma(\Omega) \cap L^q(\Omega)^2$.

Lemma 5.1 [Borchers and Miyakawa 1992; Kozono and Ogawa 1993a]. *Let $q \in (1,2)$ and $\mu = 1/q - 1/2$. For all $v \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, there exists a unique $w \in D(A^\mu) \subset L^2_\sigma(\Omega)$ such that $v = A^\mu w$. Moreover, there exists a constant $C = C(q) > 0$ (independent of v and Ω) such that $\|w\|_{L^2(\Omega)} \le C\|v\|_{L^q(\Omega)}$.*

Remark 5.2. If v, w are as in Lemma 5.1, we define $w = A^{-\mu} v$. The fact that inequality $\|w\|_{L^2(\Omega)} \le C\|v\|_{L^q(\Omega)}$ holds with a constant C independent of the domain Ω follows directly from the proof given in [Kozono and Ogawa 1993a, Lemmas 2.1 and 2.2].

As a first application of Lemma 5.1, we give a short proof of inequality (25), which was used in Section 3.

Proof of Proposition 3.1. It is sufficient to prove (25) for $1 < q < 2$. Let $\mu = 1/q - 1/2$, and let $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$. By Lemma 5.1, there exists a unique $w_0 \in D(A^\mu)$ such that $v_0 = A^\mu w_0$. Thus

$$\|S(t)v_0\|_{L^2(\Omega)} = \|A^\mu S(t) w_0\|_{L^2(\Omega)} \le t^{-\mu}\|w_0\|_{L^2(\Omega)} \le C t^{-\mu}\|v_0\|_{L^q(\Omega)},$$

with C depending only on q. The estimate for the first derivative is proved in the same way, since $\|\nabla S(t)v_0\|_{L^2(\Omega)} = \|A^{\mu+1/2} S(t) w_0\|_{L^2(\Omega)}$. This proves (25) for all $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, and the general case follows by a density argument. □

Let $v \in C^0([0,\infty); L^2_\sigma(\Omega)) \cap C^0((0,\infty); H^1_0(\Omega)^2)$ be the solution of (23) with initial data v_0, which was constructed in Sections 3 and 4. Since $v_0 \in L^q_\sigma(\Omega)$ by assumption, it is rather straightforward to verify that $v(t) \in L^q_\sigma(\Omega)$ for all $t > 0$. Thus, by Lemma 5.1, we can define $w(t) = A^{-\mu} v(t)$ for all $t > 0$. This quantity solves the equation

$$\partial_t w + Aw + \alpha F_\mu(u^\chi, v) + \alpha F_\mu(v, u^\chi) + F_\mu(v,v) = \alpha A^{-\mu} R^\chi, \tag{44}$$

where $F_\mu(u,v)$ is the bilinear term formally defined by

$$F_\mu(u,v) = A^{-\mu} P(u\cdot\nabla)v. \tag{45}$$

We refer to [Kozono and Ogawa 1993a, Section 2] for a rigorous definition and a list of properties of the bilinear map F_μ. Our goal here is to establish the following estimate:

Proposition 5.3. *There exist positive constants K_2 and c (independent of Ω) such that, for any $\alpha \in \mathbb{R}$ and any solution v of (23) with initial data $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, the function $w(t) = A^{-\mu} v(t)$ satisfies, for all $t \ge 1$,*

$$\|w(t)\|^2_{L^2} + \int_0^t \|\nabla w(s)\|^2_{L^2}\,\mathrm{d}s \le K_2(1+t)^{c\alpha^2} \exp\bigl(K_2(\|v_0\|^2_{L^2} + D_{\alpha,\rho})\bigr)\bigl(\|v_0\|^2_{L^q} + \rho^2\alpha^2\bigr), \tag{46}$$

where $D_{\alpha,\rho} = \alpha^2 \log(1+|\alpha|) + \alpha^2\rho^2$.

Proof. Taking the scalar product of both sides of (44) with w, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{L^2}^2 + \|A^{1/2}w(t)\|_{L^2}^2 + \alpha\big\langle F_\mu(u^\chi(t), v(t)), w(t)\big\rangle + \alpha\big\langle F_\mu(v(t), u^\chi(t)), w(t)\big\rangle + \big\langle F_\mu(v(t), v(t)), w(t)\big\rangle = \alpha\big\langle A^{-\mu}R^\chi(t), w(t)\big\rangle. \quad (47)$$

We recall that $\|A^{1/2}w\|_{L^2} = \|\nabla w\|_{L^2}$ for all $w \in D(A^{1/2}) = L^2_\sigma(\Omega) \cap H^1_0(\Omega)^2$. To bound the other terms, we observe that

$$|\langle F_\mu(u^\chi, v), w\rangle| = |\langle (u^\chi\cdot\nabla)v, A^{-\mu}w\rangle| = |\langle (u^\chi\cdot\nabla)A^{-\mu}w, v\rangle| \le \|u^\chi\|_{L^\infty}\|A^{\frac{1}{2}-\mu}w\|_{L^2}\|v\|_{L^2}$$
$$= \|u^\chi\|_{L^\infty}\|A^{\frac{1}{2}-\mu}w\|_{L^2}\|A^\mu w\|_{L^2} \le \|u^\chi\|_{L^\infty}\|A^{1/2}w\|_{L^2}\|w\|_{L^2},$$

where in the last inequality we used the interpolation inequality for fractional powers of A. The same argument shows that $|\langle F_\mu(v, u^\chi), w\rangle| \le \|u^\chi\|_{L^\infty}\|A^{1/2}w\|_{L^2}\|w\|_{L^2}$. In a similar way, we find

$$|\langle F_\mu(v, v), w\rangle| = |\langle (v\cdot\nabla)v, A^{-\mu}w\rangle| = |\langle (v\cdot\nabla)A^{-\mu}w, v\rangle| \le \|v\|_{L^4}^2\|A^{\frac{1}{2}-\mu}w\|_{L^2}$$
$$\le C_*^2\|\nabla v\|_{L^2}\|v\|_{L^2}\|A^{\frac{1}{2}-\mu}w\|_{L^2} \le C_*^2\|\nabla v\|_{L^2}\|A^{1/2}w\|_{L^2}\|w\|_{L^2},$$

where $C_* > 0$ is the best constant of Gagliardo–Nirenberg's inequality

$$\|f\|_{L^4(\mathbb{R}^2)} \le C_*\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\nabla f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}. \quad (48)$$

Finally, since $|\langle A^{-\mu}R^\chi, w\rangle| = |\langle R^\chi, A^{-\mu}w\rangle| \le \kappa_2\rho(1+t)^{-1}\|A^{\frac{1}{2}-\mu}w\|_{L^2}$ by (20), we can use interpolation and Young's inequality to obtain

$$|\alpha\langle A^{-\mu}R^\chi, w\rangle| \le \frac{\kappa_2\rho|\alpha|}{1+t}\|A^{1/2}w\|_{L^2}^{1-2\mu}\|w\|_{L^2}^{2\mu} \le \frac{\eta}{4}\|A^{1/2}w\|_{L^2}^2 + \frac{\|w\|_{L^2}^2}{2(1+t)^{\gamma_1}} + \frac{C_\eta\rho^2\alpha^2}{2(1+t)^{\gamma_2}},$$

for some exponents $\gamma_1, \gamma_2 > 1$ satisfying $\gamma_2 + 2\mu\gamma_1 = 2$. Here $\eta \in (0, 1]$ is as in the proof of Proposition 4.1, and $C_\eta > 0$ denotes a constant depending only on η. Inserting all these estimates into (47), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + 2\|\nabla w\|_{L^2}^2 \le 2H\|\nabla w\|_{L^2}\|w\|_{L^2} + \frac{\eta}{2}\|\nabla w\|_{L^2}^2 + \frac{\|w\|_{L^2}^2}{(1+t)^{\gamma_1}} + \frac{C_\eta\rho^2\alpha^2}{(1+t)^{\gamma_2}}, \quad (49)$$

where $H = 2|\alpha|\|u^\chi\|_{L^\infty} + C_*^2\|\nabla v\|_{L^2}$.

To exploit (49), we apply Young's inequality again and obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + \eta\|\nabla w\|_{L^2}^2 \le \left(\frac{H^2}{2-3\eta/2} + \frac{1}{(1+t)^{\gamma_1}}\right)\|w\|_{L^2}^2 + \frac{C_\eta\rho^2\alpha^2}{(1+t)^{\gamma_2}},$$

which can be integrated using Gronwall's lemma. The result is

$$\|w(t)\|_{L^2}^2 + \eta\int_0^t \|\nabla w(s)\|_{L^2}^2\,\mathrm{d}s \le C\exp\left(\frac{\Phi(t)}{1-3\eta/4}\right)\big(\|w_0\|_{L^2}^2 + C_\eta\rho^2\alpha^2\big), \quad t \ge 0, \quad (50)$$

where $\Phi(t) = \frac{1}{2}\int_0^t H(s)^2\,\mathrm{d}s$ and C is a positive constant depending only on γ_1, γ_2. It remains to estimate the quantity $\Phi(t)$ in (50). Using (11) with $p = \infty$, the logarithmic energy estimate (43), and Minkowski's inequality, we find

$$\begin{aligned}
2\Phi(t) = \int_0^t H(s)^2\,\mathrm{d}s &\le \int_0^t \left\{\frac{2|\alpha|a_\infty}{(1+s)^{1/2}} + C_*^2\|\nabla v(s)\|_{L^2}\right\}^2 \mathrm{d}s \\
&\le \left\{|\alpha|\log(1+t)^{1/2}\left(2a_\infty + \frac{C_*^2(1+\eta)^{\frac{3}{2}}}{\sqrt{2\pi}}\right) + C_\eta(\|v_0\|_{L^2} + D_{\alpha,\rho}^{1/2})\right\}^2 \\
&\le 2C_0(1+\eta)^4\alpha^2\log(1+t) + C_\eta(\|v_0\|_{L^2}^2 + D_{\alpha,\rho}), \quad t \ge 1, \qquad (51)
\end{aligned}$$

where $D_{\alpha,\rho} = \alpha^2\log(1+|\alpha|) + \alpha^2\rho^2$ and

$$C_0 = \frac{1}{2}\left(2a_\infty + \frac{C_*^2}{\sqrt{2\pi}}\right)^2. \qquad (52)$$

If we now replace (51) into (50) and set $\eta = 1$, we obtain (46) since $\|w_0\|_{L^2} \le C\|v_0\|_{L^q}$ by Lemma 5.1. This concludes the proof. □

Corollary 5.4. *Under the assumptions of Proposition 5.3, there exists a positive constant K depending on Ω, α, and $\|v_0\|_{L^2\cap L^q}$ such that, for any $T \ge 2$, there exists a time $t \in [T/2, T]$ for which*

$$\|v(t)\|_{L^2(\Omega)}^2 \le K(1+t)^{c\alpha^2 - 2\mu}. \qquad (53)$$

Proof. Fix $T \ge 2$. In view of (46), there exists a time $t \in [T/2, T]$ such that

$$\|\nabla w(t)\|_{L^2}^2 \le \frac{2}{T}\int_{T/2}^T \|\nabla w(s)\|_{L^2}^2\,\mathrm{d}s \le \frac{2}{T}C(1+T)^{c\alpha^2} \le 2^{c\alpha^2+2}C(1+t)^{c\alpha^2-1},$$

where C depends on ρ, α, and $\|v_0\|_{L^2\cap L^q}$. Moreover, $\|w(t)\|_{L^2}^2 \le C(1+t)^{c\alpha^2}$ by (46). Thus, using the interpolation inequality $\|v(t)\|_{L^2} = \|A^\mu w(t)\|_{L^2} \le \|\nabla w(t)\|_{L^2}^{2\mu}\|w(t)\|_{L^2}^{1-2\mu}$, we obtain (53). □

Proof of Theorem 1.2. Fix $q \in (1,2)$, and assume that $\epsilon > 0$ is small enough so that $c\epsilon^2 < 2\mu$, where $\mu = 1/q - 1/2$ and c is as in Proposition 5.3. We also suppose that $\epsilon \le \delta$, where $\delta > 0$ is as in Proposition 3.2. Given $\alpha \in [-\epsilon, \epsilon]$ and $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, let $v \in C^0([0,\infty); L^2_\sigma(\Omega)) \cap C^0((0,\infty); H^1_0(\Omega)^2)$ be the solution of (23) with initial data $v(0) = v_0$, which was constructed in Sections 3 and 4. In view of (53), since $c\alpha^2 < 2\mu$, we can take $t_0 > 0$ large enough (depending on Ω, α, and v_0) so that $\|v(t_0)\|_{L^2} \le V_\Omega$, where V_Ω is as in Proposition 3.2. Moreover, since $v(t_0) = A^\mu w(t_0)$ for some $w(t_0) \in L^2_\sigma(\Omega)$, we have

$$\sup_{\tau>0} \tau^\mu \|S(\tau)v_0\|_{L^2} + \sup_{\tau>0} \tau^{\mu+\frac{1}{2}}\|\nabla S(\tau)v_0\|_{L^2} \le C\|w(t_0)\|_{L^2} < \infty.$$

Applying Proposition 3.2, we conclude that the solution v of (23) satisfies (28), namely

$$\|u(\cdot,t) - \alpha u^\chi(\cdot,t)\|_{L^2} + t^{1/2}\|\nabla u(\cdot,t) - \alpha\nabla u^\chi(\cdot,t)\|_{L^2(\Omega)} = \mathcal{O}(t^{-\mu}), \qquad (54)$$

as $t \to \infty$. But $\|u^\chi - \Theta\|_{L^2} + \|\nabla u^\chi - \nabla\Theta\|_{L^2} \le C(1+t)^{-1}$ for all $t \ge 0$; hence (6) follows from (54). □

Proof of Corollary 1.4. The proof of Proposition 5.3 shows that the constant c in (46), (53) satisfies $c \le C_0(1 + \mathcal{O}(\eta))$, where C_0 is defined in (52) and $\eta \in (0,1]$ can be chosen arbitrarily small. On the other hand, since by assumption $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$ for all $q \in (1,2)$, we can take $\mu = 1/q - 1/2$

arbitrarily close to $1/2$. Thus, if we assume that $|\alpha| < \epsilon_* = C_0^{-1/2}$, we see that the condition $c\alpha^2 < 2\mu$ can be fulfilled by an appropriate choice of η and μ. Now, take $t \geq 2$ and let $t_0 \in [t/2, t]$ be the time defined in Corollary 5.4, for which $\|v(t_0)\|_{L^2}^2 \leq K(1+t_0)^{c\alpha^2-2\mu}$. Using (38) with $\eta = 1$, we conclude

$$\|v(t)\|_{L^2}^2 \leq C\left(\frac{1+t}{1+t_0}\right)^{2b_\infty|\alpha|}\left(\|v(t_0)\|_{L^2}^2 + (1+t_0)^{-1}\right) \leq C(1+t)^{c\alpha^2-2\mu} \xrightarrow[t\to\infty]{} 0,$$

which is the desired result. Here the constant $C > 0$ depends on α, ρ, and v_0, but not on t. To estimate ϵ_*, we use (52) and observe that $a_\infty = \|\Theta_0\|_{L^\infty} \approx 0.050784$. Moreover, the optimal constant in the Gagliardo–Nirenberg inequality (48) satisfies $C_*^4 \leq 2/(3\pi)$; see [Del Pino and Dolbeault 2002]. Using these values, we find $C_0 \leq 0.0407108$; hence $\epsilon_* = C_0^{-1/2} \geq 4.95616$. Finally, it was kindly pointed out to us by Jean Dolbeault that the optimal constant C_* can be computed numerically: $C_* \approx 0.6430$. This yields the approximate value $\epsilon_* \approx 5.306$. $\square$

Appendix: Proof of Proposition 1.3

We recall the following characterization of the space $\dot{W}_{0,\sigma}^{1,p}(\Omega)$ for $1 \leq p < 2$:

$$\dot{W}_{0,\sigma}^{1,p}(\Omega) = \left\{u \in L^{\frac{2p}{2-p}}(\Omega)^2 \;\middle|\; \|\nabla u\|_{L^p} < \infty,\ u = 0 \text{ on } \partial\Omega,\ \operatorname{div} u = 0 \text{ in } \Omega\right\} \tag{A-1}$$

(see, e.g., [Galdi 1994, Chapter III.5]). Here ∇u and $\operatorname{div} u$ denote weak derivatives of u, and the condition "$u = 0$ on $\partial\Omega$" means that the boundary trace of u, which is well defined because $\nabla u \in L^p(\Omega)^4$, vanishes.

Given $u_0 \in \dot{W}_{0,\sigma}^{1,p}(\Omega)$ satisfying (7), we define $u : \mathbb{R}^2 \to \mathbb{R}^2$ and $\omega : \mathbb{R} \to \mathbb{R}^2$ by

$$u(x) = \begin{cases} u_0(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega, \end{cases} \qquad \omega(x) = \begin{cases} \omega_0(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Since $u = 0$ on $\partial\Omega$, we have $\nabla u \in L^p(\mathbb{R}^2)^4$ and $\partial_1 u_2 - \partial_2 u_1 = \omega \in L^p(\mathbb{R}^2)$. Moreover (7) implies that $\omega \in L^2(m)$ for some $m > 2/q > 1$, where

$$L^2(m) = \left\{\omega \in L^2(\mathbb{R}^2) \;\middle|\; \int_{\mathbb{R}^2} (1+|x|^2)^m |\omega(x)|^2 \,\mathrm{d}x < \infty\right\}.$$

Thus, using Hölder's inequality, it is easy to verify that $\omega \in L^1(\mathbb{R}^2)$, so that we can define

$$\alpha = \int_{\mathbb{R}^2} \omega(x)\,\mathrm{d}x = \int_\Omega \omega_0(x)\,\mathrm{d}x.$$

Moreover, using the Biot–Savart formula in $\mathbb{R}^2$ and the fact that $u \in L^{2p/(2-p)}(\mathbb{R}^2)^2$, we obtain the equality

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)\,\mathrm{d}y = \frac{1}{2\pi}\int_\Omega \frac{(x-y)^\perp}{|x-y|^2}\omega_0(y)\,\mathrm{d}y, \tag{A-2}$$

for almost all $x \in \mathbb{R}^2$. We emphasize at this point that the representation (A-2) is *not* what is usually called the Biot–Savart law in the domain Ω, because the velocity field defined by (A-2) for an arbitrary vorticity $\omega_0 \in L^1(\Omega)$ will not, in general, be tangent to the boundary on $\partial\Omega$. However, if we start from a velocity field u_0 that vanishes on $\partial\Omega$, the argument above shows that (A-2) holds with $\omega_0 = \operatorname{curl} u_0$. We refer to [Iftimie et al. 2003] for a more detailed discussion of the Biot–Savart law in a two-dimensional exterior domain.

Now, we decompose

$$u(x) = \alpha u^\chi(x,0) + v(x), \quad \omega(x) = \alpha\omega^\chi(x,0) + w(x), \quad x \in \mathbb{R}^2,$$

where u^χ, ω^χ are defined in (8), (9). By construction, we have $w \in L^2(m)$ and $\int_{\mathbb{R}^2} w \, \mathrm{d}x = 0$. Applying [Gallay and Wayne 2002, Proposition B.1], we deduce that the corresponding velocity field v, which is obtained from w via the Biot–Savart law in $\mathbb{R}^2$, satisfies

$$\int_{\mathbb{R}^2} (1+|x|^2)^{\frac{mr}{2}-1} |v(x)|^r \, \mathrm{d}x < \infty,$$

for all $r > 2$. Using Hölder's inequality again, we conclude that $v \in L^s(\mathbb{R}^2)^2$ for all $s > 2/m$; hence in particular $v \in L^2(\mathbb{R}^2)^2 \cap L^q(\mathbb{R}^2)^2$. Clearly $v(x) = 0$ for all $x \notin \Omega$; hence denoting by v_0 the restriction of v to Ω we obtain (4) with $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$. □

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THIERRY GALLAY: Thierry.Gallay@ujf-grenoble.fr
Institut Fourier, Université de Grenoble I, 100 rue des Maths, B.P. 74, 38402 Saint-Martin-d'Hères, France
http://www-fourier.ujf-grenoble.fr/~gallay

YASUNORI MAEKAWA: yasunori@math.kobe-u.ac.jp
Department of Mathematics, Graduate School of Science, Kobe University, 1-1 Rokkodai, Nada-ku, Kobe 657-8501, Japan
http://www.math.kobe-u.ac.jp/home-j/yasunori.html

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SECOND ORDER STABILITY FOR THE MONGE–AMPÈRE EQUATION AND STRONG SOBOLEV CONVERGENCE OF OPTIMAL TRANSPORT MAPS

GUIDO DE PHILIPPIS AND ALESSIO FIGALLI

The aim of this note is to show that Alexandrov solutions of the Monge–Ampère equation, with right-hand side bounded away from zero and infinity, converge strongly in $W^{2,1}_{\mathrm{loc}}$ if their right-hand sides converge strongly in L^1_{loc}. As a corollary, we deduce strong $W^{1,1}_{\mathrm{loc}}$ stability of optimal transport maps.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. In [De Philippis and Figalli 2013], we showed that convex Alexandrov solutions of

$$\begin{cases} \det D^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

with $0 < \lambda \le f \le \Lambda$, are $W^{2,1}_{\mathrm{loc}}(\Omega)$. More precisely, they were able to prove uniform interior $L \log L$-estimates for $D^2 u$. This result has also been improved in [De Philippis et al. 2013; Schmidt 2013], where it is actually shown that $u \in W^{2,\gamma}_{\mathrm{loc}}(\Omega)$ for some $\gamma = \gamma(n, \lambda, \Lambda) > 1$: more precisely, for any $\Omega' \Subset \Omega$,

$$\int_{\Omega'} |D^2 u|^\gamma \le C(n, \lambda, \Lambda, \Omega, \Omega'). \tag{1-2}$$

A question which naturally arises in view of the previous results is the following: choose a sequence of functions f_k with $\lambda \le f_k \le \Lambda$ which converges to f strongly in $L^1_{\mathrm{loc}}(\Omega)$, and denote by u_k and u the solutions of (1-1) corresponding to f_k and f, respectively. By the convexity of u_k and u and the uniqueness of solutions to (1-1), it is immediately deduced that $u_k \to u$ uniformly, and $\nabla u_k \to \nabla u$ in $L^p_{\mathrm{loc}}(\Omega)$ for any $p < \infty$. What can be said about the strong convergence of $D^2 u_k$? Due to the highly nonlinear character of the Monge–Ampère equation, this question is nontrivial. (Note that weak $W^{2,1}_{\mathrm{loc}}$ convergence is immediate by compactness, even under the weaker assumption that f_k converges to f weakly in $L^1_{\mathrm{loc}}(\Omega)$.)

The aim of this short note is to prove that strong convergence holds. Our main result is the following:

Theorem 1.1. *Let $\Omega_k \subset \mathbb{R}^n$ be a family of convex domains, and let $u_k : \Omega_k \to \mathbb{R}$ be convex Alexandrov solutions of*

$$\begin{cases} \det D^2 u_k = f_k & \text{in } \Omega_k, \\ u_k = 0 & \text{on } \partial\Omega_k, \end{cases} \tag{1-3}$$

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with $0 < \lambda \le f_k \le \Lambda$. *Assume that* Ω_k *converges to some convex domain* Ω *in the Hausdorff distance, and* $f_k \chi_{\Omega_k}$ *converges to* f *in* $L^1_{\mathrm{loc}}(\Omega)$. *Then, if* u *denotes the unique Alexandrov solution of*

$$\begin{cases} \det D^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for any $\Omega' \Subset \Omega$, *we have*

$$\|u_k - u\|_{W^{2,1}(\Omega')} \to 0 \quad \text{as } k \to \infty. \tag{1-4}$$

(*Obviously, since the functions* u_k *are uniformly bounded in* $W^{2,\gamma}(\Omega')$, *this gives strong convergence in* $W^{2,\gamma'}(\Omega')$ *for any* $\gamma' < \gamma$.)

As a consequence, we can prove the following stability result for optimal transport maps:

Theorem 1.2. *Let* $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ *be two bounded domains with* Ω_2 *convex, and let* f_k, g_k *be a family of probability densities such that* $0 < \lambda \le f_k, g_k \le \Lambda$ *inside* Ω_1 *and* Ω_2, *respectively. Assume that* $f_k \to f$ *in* $L^1(\Omega_1)$ *and* $g_k \to g$ *in* $L^1(\Omega_2)$, *and let* $T_k : \Omega_1 \to \Omega_2$ (*resp.* $T : \Omega_1 \to \Omega_2$) *be the* (*unique*) *optimal transport map for the quadratic cost sending* f_k *onto* g_k (*resp.* f *onto* g). *Then* $T_k \to T$ *in* $W^{1,\gamma'}_{\mathrm{loc}}(\Omega_1)$ *for some* $\gamma' > 1$.

We point out that, in order to prove (1-4) and the local $W^{1,1}$ stability of optimal transport maps, the interior $L \log L$-estimates from [De Philippis and Figalli 2013] are sufficient. Indeed, the $W^{2,\gamma}$-estimates are used just to improve the convergence from $W^{2,1}_{\mathrm{loc}}$ to $W^{2,\gamma'}_{\mathrm{loc}}$ with $\gamma' < \gamma$.

This paper is organized as follows: in the next section, we collect some notation and preliminary results. Then in Section 3 we prove Theorem 1.1, and in Section 4 we prove Theorem 1.2.

2. Notation and preliminaries

Given a convex function $u : \Omega \to \mathbb{R}$, we define its *Monge–Ampère measure* as

$$\mu_u(E) := |\partial u(E)| \quad \text{for all } E \subset \Omega \text{ Borel}$$

(see [Gutiérrez 2001, Theorem 1.1.13]), where

$$\partial u(E) := \bigcup_{x \in E} \partial u(x).$$

Here $\partial u(x)$ is the subdifferential of u at x, and $|F|$ denotes the Lebesgue measure of a set F. In case $u \in C^{1,1}_{\mathrm{loc}}$, by the area formula [Evans and Gariepy 1992, Paragraph 3.3], the following representation holds:

$$\mu_u = \det D^2 u \, dx.$$

The main property of the Monge–Ampère measure we are going to use is the following (see [Gutiérrez 2001, Lemmas 1.2.2 and 1.2.3]):

Proposition 2.1. *Let* $u_k : \Omega \to \mathbb{R}$ *be a sequence of convex functions converging locally uniformly to* u. *Then the associated Monge–Ampère measures* μ_{u_k} *converge to* μ_u *in duality with the space of continuous* *functions compactly supported in* Ω. *In particular,*

$$\mu_u(A) \le \liminf_{k\to\infty} \mu_{u_k}(A)$$

for any open set $A \subset \Omega$.

Given a Radon measure ν on $\mathbb{R}^n$ and a bounded convex domain $\Omega \subset \mathbb{R}^n$, we say that a convex function $u : \Omega \to \mathbb{R}$ is an *Alexandrov solution* of the Monge–Ampère equation

$$\det D^2 u = \nu \quad \text{in } \Omega$$

if $\mu_u(E) = \nu(E)$ for every Borel set $E \subset \Omega$.

If $v : \overline{\Omega} \to \mathbb{R}$ is a continuous function, we define its *convex envelope inside* Ω as

$$\Gamma_v(x) := \sup\{\ell(x) : \ell \le v \text{ in } \Omega,\ \ \ell \text{ affine}\}. \tag{2-1}$$

In case Ω is a convex domain and $v \in C^2(\Omega)$, it is easily seen that

$$D^2 v(x) \ge 0 \quad \text{for every } x \in \{v = \Gamma_v\} \cap \Omega \tag{2-2}$$

in the sense of symmetric matrices. Moreover, the following inequality between measures holds in Ω:

$$\mu_{\Gamma_v} \le \det D^2 v \mathbf{1}_{\{v=\Gamma_v\}}\, dx \tag{2-3}$$

(here $\mathbf{1}_E$ is the characteristic function of a set E).[1]

We recall that a continuous function v is said to be *twice differentiable* at x if there exists a (unique) vector $\nabla v(x)$ and a (unique) symmetric matrix $\nabla^2 v(x)$ such that

$$v(y) = v(x) + \nabla v(x)\cdot(y-x) + \tfrac{1}{2}\nabla^2 v(x)[y-x,\, y-x] + o(|y-x|^2).$$

In case v is twice differentiable at some point $x_0 \in \{v = \Gamma_v\}$, it is immediate to check that

$$\nabla^2 v(x_0) \ge 0. \tag{2-5}$$

[1] To see this, let us first recall that by [Gutiérrez 2001, Lemma 6.6.2], if $x_0 \in \Omega \setminus \{\Gamma_v = v\}$ and $a \in \partial\Gamma_v(x_0)$, then the convex set

$$\{x \in \Omega : \Gamma_v(x) = a\cdot(x-x_0) + \Gamma_v(x_0)\}$$

is nonempty and contains more than one point. In particular,

$$\partial\Gamma_v\big(\Omega\setminus\{\Gamma_v = v\}\big) \subset \{p \in \mathbb{R}^n : \text{there exist distinct } x, y \in \Omega \text{ such that } p \in \partial\Gamma_v(x)\cap\partial\Gamma_v(y)\}.$$

This last set is contained in the set of nondifferentiability of the convex conjugate of Γ_v, so it has zero Lebesgue measure (see [Gutiérrez 2001, Lemma 1.1.12]), and hence

$$\big|\partial\Gamma_v\big(\Omega\setminus\{\Gamma_v = v\}\big)\big| = 0. \tag{2-4}$$

Moreover, since $v \in C^1(\Omega)$, for any $x \in \{\Gamma_v = v\} \cap \Omega$, we have $\partial\Gamma_v(x) = \{\nabla v(x)\}$. Thus, using (2-4) and (2-2), for any open set $A \Subset \Omega$, we have

$$\mu_{\Gamma_v}(A) = \big|\partial\Gamma_v\big(A\cap\{\Gamma_v = v\}\big)\big| = \big|\nabla v\big(A\cap\{\Gamma_v = v\}\big)\big| \le \int_{A\cap\{\Gamma_v=v\}} |\det D^2 v| = \int_{A\cap\{\Gamma_v=v\}} \det D^2 v,$$

as desired. (The inequality above follows from the area formula in [Evans and Gariepy 1992, Paragraph 3.3.2] applied to the C^1 map ∇v.)

By the Alexandrov theorem, any convex function is twice differentiable almost everywhere (see, for instance, [Evans and Gariepy 1992, Paragraph 6.4]). In particular, (2-5) holds almost everywhere on $\{v = \Gamma_v\}$ whenever v is the difference of two convex functions.

Finally we recall that, in case $v \in W^{2,1}_{\mathrm{loc}}$, the pointwise Hessian of v coincides almost everywhere with its distributional Hessian [Evans and Gariepy 1992, Sections 6.3 and 6.4]. Since in the sequel we are going to deal with $W^{2,1}_{\mathrm{loc}}$ convex functions, we will use D^2u to denote both the pointwise and the distributional Hessian.

3. Proof of Theorem 1.1

We are going to use the following result:

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let $u, v : \overline{\Omega} \to \mathbb{R}$ be two continuous strictly convex functions such that $\mu_u = f\,dx$ and $\mu_v = g\,dx$, with $f, g \in L^1_{\mathrm{loc}}(\Omega)$. Then*

$$\mu_{\Gamma_{u-v}} \le \big(f^{1/n} - g^{1/n}\big)^n \mathbf{1}_{\{u-v=\Gamma_{u-v}\}}\,dx. \tag{3-1}$$

Proof. In case u, v are of class C^2 inside Ω, by (2-2) we have

$$0 \le D^2u(x) - D^2v(x) \quad \text{for every } x \in \{u - v = \Gamma_{u-v}\},$$

so using the monotonicity and the concavity of the function $\det^{1/n}$ on the cone of nonnegative symmetric matrices, we get

$$0 \le \det(D^2u - D^2v) \le \big((\det D^2u)^{1/n} - (\det D^2v)^{1/n}\big)^n \quad \text{on} \quad \{u - v = \Gamma_{u-v}\},$$

which, combined with (2-3), gives the desired result.

Now, for the general case, we consider a sequence of smooth uniformly convex domains Ω_k increasing to Ω and two sequences of smooth functions f_k and g_k converging respectively to f and g in $L^1_{\mathrm{loc}}(\Omega)$, and we solve

$$\begin{cases} \det D^2u_k = f_k & \text{in } \Omega_k, \\ u_k = u * \rho_k & \text{on } \partial\Omega_k, \end{cases} \qquad \begin{cases} \det D^2v_k = g_k & \text{in } \Omega_k, \\ v_k = v * \rho_k & \text{on } \partial\Omega_k, \end{cases}$$

where ρ_k is a smooth sequence of convolution kernels. In this way, both u_k and v_k are smooth on $\overline{\Omega}_k$ [Gilbarg and Trudinger 2001, Theorem 17.23], and $\|u_k - u\|_{L^\infty(\Omega_k)} + \|v_k - v\|_{L^\infty(\Omega_k)} \to 0$ as $k \to \infty$.[2] Hence, $\Gamma_{u_k - v_k}$ also converges locally uniformly to Γ_{u-v}. Moreover, it follows easily from the definition of a contact set that

$$\limsup_{k\to\infty} \mathbf{1}_{\{u_k - v_k = \Gamma_{u_k - v_k}\}} \le \mathbf{1}_{\{u-v=\Gamma_{u-v}\}}. \tag{3-2}$$

We now observe that the previous step applied to u_k and v_k gives

$$\mu_{\Gamma_{u_k - v_k}} \le \big((\det D^2u_k)^{1/n} - (\det D^2v_k)^{1/n}\big)^n \mathbf{1}_{\{u_k - v_k = \Gamma_{u_k - v_k}\}}\,dx.$$

Thus, letting $k \to \infty$ and taking into account Proposition 2.1 and (3-2), we obtain (3-1). □

[2] Indeed, it is easy to see that u_k and v_k converge uniformly to u and v, respectively, both on $\partial\Omega_k$ and in any compact subdomain of Ω. Then, using for instance a contradiction argument, one exploits the convexity of u_k (resp. v_k) and Ω_k and the uniform continuity of u (resp. v) to show that the convergence is actually uniform on the whole Ω_k.

Proof of Theorem 1.1. The L^1_{loc} convergence of u_k (resp. ∇u_k) to u (resp. ∇u) is easy and standard, so we focus on the convergence of the second derivatives.

Without loss of generality, we can assume that Ω' is convex, and that $\Omega' \Subset \Omega_k$ (since $\Omega_k \to \Omega$ in the Hausdorff distance, this is always true for k sufficiently large). Fix $\varepsilon \in (0, 1)$, let $\Gamma_{u-(1-\varepsilon)u_k}$ be the convex envelope of $u - (1-\varepsilon)u_k$ inside Ω' (see (2-1)), and define

$$A_k^\varepsilon := \bigl\{x \in \Omega' : u(x) - (1-\varepsilon)u_k(x) = \Gamma_{u-(1-\varepsilon)u_k}(x)\bigr\}.$$

Since $u_k \to u$ locally uniformly, $\Gamma_{u-(1-\varepsilon)u_k}$ converges uniformly to $\Gamma_{\varepsilon u} = \varepsilon u$ (as u is convex) inside Ω'. Hence, by applying Proposition 2.1 and (3-1) to u and $(1-\varepsilon)u_k$ inside Ω', we get that

$$\varepsilon^n \int_{\Omega'} f = \mu_{\Gamma_{\varepsilon u}}(\Omega') \le \liminf_{k\to\infty} \mu_{\Gamma_{u-(1-\varepsilon)u_k}}(\Omega') \le \liminf_{k\to\infty} \int_{\Omega'\cap A_k^\varepsilon} \bigl(f^{1/n} - (1-\varepsilon) f_k^{1/n}\bigr)^n.$$

We now observe that, since f_k converges to f in $L^1_{\mathrm{loc}}(\Omega)$, we have

$$\left| \int_{\Omega'\cap A_k^\varepsilon} \bigl(f^{1/n} - (1-\varepsilon) f_k^{1/n}\bigr)^n - \int_{\Omega'\cap A_k^\varepsilon} \varepsilon^n f \right| \le \int_{\Omega'} \bigl| \bigl(f^{1/n} - (1-\varepsilon) f_k^{1/n}\bigr)^n - \varepsilon^n f \bigr| \to 0$$

as $k \to \infty$. Hence, combining the two estimates above, we immediately get

$$\int_{\Omega'} f \le \liminf_{k\to\infty} \int_{\Omega'\cap A_k^\varepsilon} f,$$

or equivalently,

$$\limsup_{k\to\infty} \int_{\Omega'\setminus A_k^\varepsilon} f = 0.$$

Since $f \ge \lambda$ inside Ω (as a consequence of the fact that $f_k \ge \lambda$ inside Ω_k), this gives

$$\lim_{k\to\infty} |\Omega' \setminus A_k^\varepsilon| = 0 \quad \text{for all} \quad \varepsilon \in (0, 1). \tag{3-3}$$

We now recall that, by the results in [Caffarelli 1990; De Philippis and Figalli 2013; De Philippis et al. 2013; Schmidt 2013], both u and $(1-\varepsilon)u_k$ are strictly convex and belong to $W^{2,1}(\Omega')$. Hence we can apply (2-5) to deduce that

$$D^2 u - (1-\varepsilon) D^2 u_k \ge 0 \quad \text{almost everywhere on } A_k^\varepsilon.$$

In particular, by (3-3),

$$\bigl|\Omega' \setminus \{D^2 u \ge (1-\varepsilon) D^2 u_k\}\bigr| \to 0 \quad \text{as} \quad k \to \infty.$$

By a similar argument (exchanging the roles of u and u_k),

$$\bigl|\Omega' \setminus \{D^2 u_k \ge (1-\varepsilon) D^2 u\}\bigr| \to 0 \quad \text{as} \quad k \to \infty.$$

Hence, if we set $B_k^\varepsilon := \{x \in \Omega' : (1-\varepsilon) D^2 u_k \le D^2 u \le (1/(1-\varepsilon)) D^2 u_k\}$, we have

$$\lim_{k\to\infty} |\Omega' \setminus B_k^\varepsilon| = 0 \quad \text{for all} \quad \varepsilon \in (0, 1).$$

Moreover, by (1-2) applied to both u_k and u, we have[3]

$$\begin{aligned}\int_{\Omega'}|D^2u-D^2u_k| &= \int_{\Omega'\cap B_k^\varepsilon}|D^2u-D^2u_k|+\int_{\Omega'\setminus B_k^\varepsilon}|D^2u-D^2u_k|\\ &\le \frac{\varepsilon}{1-\varepsilon}\int_{\Omega'}|D^2u|+\|D^2u-D^2u_k\|_{L^\gamma(\Omega')}|\Omega'\setminus B_k^\varepsilon|^{1-1/\gamma}\\ &\le C\left(\frac{\varepsilon}{1-\varepsilon}+|\Omega'\setminus B_k^\varepsilon|^{1-1/\gamma}\right).\end{aligned}$$

Hence, first letting $k\to\infty$ and then sending $\varepsilon\to 0$, we obtain the desired result. □

4. Proof of Theorem 1.2

In order to prove Theorem 1.2, we will need the following lemma (note that for the next result we do not need to assume the convexity of the target domain):

Lemma 4.1. *Let $\Omega_1,\Omega_2\subset\mathbb{R}^n$ be two bounded domains, and let f_k, g_k be a family of probability densities such that $0<\lambda\le f_k, g_k\le\Lambda$ inside Ω_1 and Ω_2, respectively. Assume that $f_k\to f$ in $L^1(\Omega_1)$ and $g_k\to g$ in $L^1(\Omega_2)$, and let $T_k:\Omega_1\to\Omega_2$ (resp. $T:\Omega_1\to\Omega_2$) be the (unique) optimal transport map for the quadratic cost sending f_k onto g_k (resp. f onto g). Then*

$$\frac{f_k}{g_k\circ T_k}\to\frac{f}{g\circ T}\quad \text{in } L^1(\Omega_1).$$

Proof. By stability of optimal transport maps (see, for instance, [Villani 2009, Corollary 5.23]) and the fact that $f_k\ge\lambda$ (and so $f\ge\lambda$), we know that $T_k\to T$ in measure (with respect to Lebesgue) inside Ω.

We claim that $g\circ T_k\to g\circ T$ in $L^1(\Omega_1)$. Indeed, this is obvious if g is uniformly continuous (by the convergence in measure of T_k to T). In the general case, we choose $g_\eta\in C(\overline{\Omega_2})$ such that $\|g-g_\eta\|_{L^1(\Omega_2)}\le\eta$, and we observe that (recall that f_k, $f\ge\lambda$, g_k, $g\le\Lambda$, and that by the definition of transport maps, we have $T_\# f_k=g_k$, $T_\# f=g$)

$$\begin{aligned}\int_{\Omega_1}|g\circ T_k-g\circ T| &\le \int_{\Omega_1}|g_\eta\circ T_k-g_\eta\circ T|+\int_{\Omega_1}|g_\eta\circ T_k-g\circ T_k|\frac{f_k}{\lambda}+\int_{\Omega_1}|g_\eta\circ T-g\circ T|\frac{f}{\lambda}\\ &=\int_{\Omega_1}|g_\eta\circ T_k-g_\eta\circ T|+\int_{\Omega_2}|g_\eta-g|\frac{g_k}{\lambda}+\int_{\Omega_2}|g_\eta-g|\frac{g}{\lambda}\\ &\le\int_{\Omega_1}|g_\eta\circ T_k-g_\eta\circ T|+2\frac{\Lambda}{\lambda}\eta.\end{aligned}$$

Thus

$$\limsup_{k\to\infty}\int_{\Omega_1}|g\circ T_k-g\circ T|\le 2\frac{\Lambda}{\lambda}\eta,$$

and the claim follows by the arbitrariness of η.

---

[3] If instead of (1-2) we only had uniform $L\log L$ a priori estimates, in place of Hölder's inequality we could apply the elementary inequality $t\le\delta t\log(2+t)+e^{1/\delta}$ with $t=|D^2u-D^2u_k|$ inside $\Omega'\setminus B_k^\varepsilon$, and we would first let $k\to\infty$ and then send $\delta,\varepsilon\to 0$.

Since

$$\begin{aligned}\int_{\Omega_1} |g_k \circ T_k - g \circ T| &\le \int_{\Omega_1} |g_k \circ T_k - g \circ T_k| \frac{f_k}{\lambda} + \int_{\Omega_1} |g \circ T_k - g \circ T| \\ &= \int_{\Omega_2} |g_k - g| \frac{g_k}{\lambda} + \int_{\Omega_1} |g \circ T_k - g \circ T| \\ &\le \frac{\Lambda}{\lambda} \|g_k - g\|_{L^1(\Omega_2)} + \int_{\Omega_1} |g \circ T_k - g \circ T|,\end{aligned}$$

from the claim above we immediately deduce that also $g_k \circ T_k \to g \circ T$ in $L^1(\Omega_1)$.

Finally, since $g_k, g \ge \lambda$ and $f \le \Lambda$,

$$\begin{aligned}\int_{\Omega_1} \left| \frac{f_k}{g_k \circ T_k} - \frac{f}{g \circ T} \right| &\le \int_{\Omega_1} \left| \frac{f_k - f}{g_k \circ T_k} \right| + \int_{\Omega_1} f \left| \frac{1}{g_k \circ T_k} - \frac{1}{g \circ T} \right| \\ &\le \frac{1}{\lambda} \|f_k - f\|_{L^1(\Omega_1)} + \Lambda \int_{\Omega_1} \frac{|g_k \circ T_k - g \circ T|}{g_k \circ T_k \, g \circ T} \\ &\le \frac{1}{\lambda} \|f_k - f\|_{L^1(\Omega_1)} + \frac{\Lambda}{\lambda^2} \|g_k \circ T_k - g \circ T\|_{L^1(\Omega_1)},\end{aligned}$$

from which the desired result follows. □

Proof of Theorem 1.2. Since T_k are uniformly bounded in $W^{1,\gamma}(\Omega_1')$ for any $\Omega_1' \Subset \Omega$, it suffices to prove that $T_k \to T$ in $W^{1,1}_{\mathrm{loc}}(\Omega_1)$.

Fix $x_0 \in \Omega_1$ and $r > 0$ such that $B_r(x_0) \subset \Omega_1$. By compactness, it suffices to show that there is an open neighborhood $\mathcal{U}_{x_0}$ of x_0 such that $\mathcal{U}_{x_0} \subset B_r(x_0)$ and

$$\int_{\mathcal{U}_{x_0}} |T_k - T| + |\nabla T_k - \nabla T| \to 0.$$

It is well known [Caffarelli 1992] that T_k (resp. T) can be written as ∇u_k (resp. ∇u) for some strictly convex function $u_k : B_r(x_0) \to \mathbb{R}$ (resp. $u : B_r(x_0) \to \mathbb{R}$). Moreover, up to subtracting a constant from u_k (which will not change the transport map T_k), one may assume that $u_k(x_0) = u(x_0)$ for all $k \in \mathbb{N}$.

Since the functions $T_k = \nabla u_k$ are bounded (as they take values in the bounded set Ω_2), by classical stability of optimal maps (see for instance [Villani 2009, Corollary 5.23]) we get that $\nabla u_k \to \nabla u$ in $L^1_{\mathrm{loc}}(B_r(x_0))$. (Actually, if one uses [Caffarelli 1992], ∇u_k are locally uniformly Hölder maps, so they converge locally uniformly to ∇u.) Hence, to conclude the proof we only need to prove the convergence of $D^2 u_k$ to $D^2 u$ in a neighborhood of x_0.

To this aim, we observe that, by strict convexity of u, we can find a linear function $\ell(z) = a \cdot z + b$ such that the open convex set $Z := \{z : u(z) < u(x_0) + \ell(z)\}$ is nonempty and compactly supported inside $B_{r/2}(x_0)$. Hence, by the uniform convergence of u_k to u (which follows from the L^1_{loc} convergence of the gradients, the convexity of u_k and u, and the fact that $u_k(x_0) = u(x_0)$), and the fact that ∇u is transversal to ℓ on ∂Z, we get that $Z_k := \{z : u_k(z) < u_k(x_0) + \ell(z)\}$ are nonempty convex sets which converge in the Hausdorff distance to Z.

Moreover, by [Caffarelli 1992], the maps $v_k := u_k - \ell$ solve in the Alexandrov sense

$$\begin{cases} \det D^2 v_k = \dfrac{f_k}{g_k \circ T_k} & \text{in } Z_k, \\ v_k = 0 & \text{on } \partial Z_k \end{cases}$$

(here we used that the Monge–Ampère measures associated to v_k and u_k are the same). Therefore, thanks to Lemma 4.1, we can apply Theorem 1.1 to deduce that $D^2 u_k \to D^2 u$ in any relatively compact subset of Z, which concludes the proof. □

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GUIDO DE PHILIPPIS: guido.dephilippis@sns.it
Scuola Normale Superiore, p.za dei Cavalieri 7, I-56126 Pisa, Italy

ALESSIO FIGALLI: figalli@math.utexas.edu
Department of Mathematics, University of Texas at Austin, 1 University Station C1200, Austin, TX 78712, United States

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