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STABILIZATION FOR THE SEMILINEAR WAVE EQUATION WITH GEOMETRIC CONTROL CONDITION

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In this article, we prove the exponential stabilization of the semilinear wave equation with a damping effective in a zone satisfying the geometric control condition only. The nonlinearity is assumed to be subcritical, defocusing and analytic. The main novelty compared to previous results is the proof of a unique continuation result in large time for some undamped equation. The idea is to use an asymptotic smoothing effect proved by Hale and Raugel in the context of dynamical systems. Then, once the analyticity in time is proved, we apply a unique continuation result with partial analyticity due to Robbiano, Zuily, Tataru and Hörmander. Some other consequences are also given for the controllability and the existence of a compact attractor.

Dans cet article, on prouve la décroissance exponentielle de l'équation des ondes semilinéaires avec un amortissement actif dans une zone satisfaisant seulement la condition de contrôle géométrique. La nonlinéarité est supposée sous-critique, défocalisante et analytique. La principale nouveauté par rapport aux résultats précédents est la preuve d'un résultat de prolongement unique en grand temps pour une solution non amortie. L'idée est d'utiliser un effet régularisant asymptotique prouvé par Hale et Raugel dans le contexte des systèmes dynamiques. Ensuite, une fois l'analyticité en temps prouvée, on applique un théorème de prolongement unique avec analyticité partielle dû à Robbiano, Zuily, Tataru et Hörmander. Des applications à la contrôlabilité et à l'existence d'attracteur global compact pour l'équation des ondes sont aussi données.

1. Introduction

In this article, we consider the semilinear damped wave equation

$$\begin{cases} \Box u+\gamma(x)\partial_t u+\beta u+f(u)=0 & (t,x)\in\mathbb{R}_+\times\Omega,\\ u(t,x)=0 & (t,x)\in\mathbb{R}_+\times\partial\Omega,\\ (u,\partial_t u)=(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega), \end{cases} \tag{1-1}$$

where $\Box=\partial_{tt}^2-\Delta$, with Δ being the Laplace–Beltrami operator with Dirichlet boundary conditions. The domain Ω is a connected $\mathscr{C}^\infty$ three-dimensional Riemannian manifold with boundaries, which is either:

(i) Compact.

(ii) A compact perturbation of $\mathbb{R}^3$, that is $\mathbb{R}^3\setminus D$, where D is a bounded smooth domain, endowed with a smooth metric equal to the euclidean one outside of a ball.

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(iii) A manifold with periodic geometry (cylinder, $\mathbb{R}^3$ with a periodic metric, ...).

The nonlinearity $f \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ is assumed to be defocusing, energy subcritical and such that 0 is an equilibrium point. More precisely, we assume that there exists $C > 0$ such that

$$f(0) = 0, \quad sf(s) \geq 0, \quad |f(s)| \leq C(1+|s|)^p, \quad |f'(s)| \leq C(1+|s|)^{p-1}, \tag{1-2}$$

with $1 \leq p < 5$.

We assume $\beta \geq 0$ to be such that $\Delta - \beta$ is a negative-definite operator, that is that we have a Poincaré inequality $\int_\Omega |\nabla u|^2 + \beta |u|^2 \geq C \int_\Omega |u|^2$ with $C > 0$. In particular, this may require $\beta > 0$ if $\partial\Omega = \varnothing$ or if Ω is unbounded.

The damping $\gamma \in L^\infty(\Omega)$ is a nonnegative function. We assume that there exist an open set $\omega \subset \Omega$, $\alpha \in \mathbb{R}$, $x_0 \in \Omega$ and $R \geq 0$ such that

$$\Omega \setminus B(x_0, R) \subset \omega \qquad \text{and} \qquad \gamma(x) \geq \alpha > 0 \quad \text{for all } x \in \omega. \tag{1-3}$$

Moreover, we assume that ω satisfies the geometric control condition introduced in [Rauch and Taylor 1974; Bardos et al. 1992]:

(GCC) There exists $L > 0$ such that any generalized geodesic of Ω of length L meets the set ω where the damping is effective.

The associated energy $E \in \mathscr{C}^0(X, \mathbb{R}_+)$ is given by

$$E(u) := E(u, \partial_t u) = \frac{1}{2}\int_\Omega (|\partial_t u|^2 + |\nabla u|^2 + \beta |u|^2) + \int_\Omega V(u), \tag{1-4}$$

where $V(u) = \int_0^u f(s)ds$. Due to assumption (1-2) and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, this energy is well defined and, if u solves (1-1), we have, at least formally,

$$\partial_t E(u(t)) = -\int_\Omega \gamma(x)|\partial_t u(x,t)|^2 \, dx \leq 0. \tag{1-5}$$

The system is therefore dissipative. We are interested in the exponential decay of the energy of the nonlinear damped wave equation (1-1), that is, the property:

(ED) For any $E_0 \geq 0$, there exist $K > 0$ and $\lambda > 0$ such that, for all solutions u of (1-1) with $E(u(0)) \leq E_0$,

$$E(u(t)) \leq K e^{-\lambda t} E(u(0)) \quad \text{for all } t \geq 0.$$

Property (ED) means that the damping term $\gamma \partial_t u$ stabilizes any solution of (1-1) to zero, which is an important property from the dynamical and control points of view.

Our main theorem is as follows.

Theorem 1.1. *Assume that the damping γ satisfies (1-3) and the geometric control condition* (GCC)*. If f is real analytic and satisfies (1-2), then the exponential decay property* (ED) *holds.*

Theorem 1.1 applies for nonlinearities f that are globally analytic. Of course, the nonlinearities $f(u) = |u|^{p-1}u$ are not analytic if $p \notin \{1, 3\}$, but we can replace these usual nonlinearities by similar ones as $f(u) = (u/\operatorname{th}(u))^{p-1}u$, which are analytic for all $p \in [1, 5)$. Note that the estimates (1-2) are only required for $s \in \mathbb{R}$, so that they do not imply that f is a polynomial. Moreover, we can show that (ED) holds in fact for almost all the nonlinearities f satisfying (1-2), including nonanalytic ones.

More precisely, we set

$$\mathfrak{C}^1(\mathbb{R}) = \{f \in \mathscr{C}^1(\mathbb{R}) \mid \text{there exist } C > 0 \text{ and } p \in [1,5) \text{ such that (1-2) holds}\} \tag{1-6}$$

and endow this set with the Whitney topology (or any other reasonable topology). We recall that the Whitney topology is the topology generated by the neighborhoods

$$\mathcal{N}_{f,\delta} = \big\{g \in \mathfrak{C}^1(\mathbb{R}) \mid \max\big(|f(u)-g(u)|, |f'(u)-g'(u)|\big) < \delta(u) \text{ for all } u \in \mathbb{R}\big\}, \tag{1-7}$$

where f is any function in $\mathfrak{C}^1(\mathbb{R})$ and δ is any positive continuous function. The set $\mathfrak{C}^1(\mathbb{R})$ is a Baire space, which means that any generic set, that is, any set containing a countable intersection of open and dense sets, is dense in $\mathfrak{C}^1(\mathbb{R})$ (see Proposition 7.1). The Baire property ensures that the genericity of a set in $\mathfrak{C}^1(\mathbb{R})$ is a good notion for “the set contains almost all nonlinearities f”.

Theorem 1.2. *Assume that the damping γ satisfies (1-3) and the geometric control condition* (GCC). *There exists a generic set $\mathfrak{G} \subset \mathfrak{C}^1(\mathbb{R})$ such that the exponential decay property* (ED) *holds for all $f \in \mathfrak{G}$.*

The statements of both theorems lead to some remarks.

• Of course, our results and their proofs should easily extend to any space dimension $d \geq 3$ if the exponent p of the nonlinearity satisfies $p < (d+2)/(d-2)$.

• Actually, it may be possible to get $\lambda > 0$ in (ED) uniform with respect to the size of the data. We can take for instance $\lambda = \tilde{\lambda} - \varepsilon$, where $\tilde{\lambda}$ is the decay rate of the linear equation. The idea is that once we know the existence of a decay rate, we know that the solution is close to zero for a large time. Then, for small solutions, the nonlinear term can be neglected to get almost the same decay rate as the linear equation. We refer for instance to [Laurent et al. 2010] in the context of KdV equation. Notice that the possibility to get the same result with a constant K independent of E_0 is an open problem.

• The assumption on β is important to ensure some coercivity of the energy and to preclude the spatially constant functions to be undamped solutions for the linear equation. It has been proved in [Dehman and Gérard 2002] for $\mathbb{R}^3$ and in [Laurent 2011] for a compact manifold that exponential decay can fail without this term β.

• The geometric control condition is known to be not only sufficient but also necessary for the exponential decay of the linear damped equation. The proof of the optimality uses some sequences of solutions which are asymptotically concentrated outside of the damping region. We can use the same idea in our nonlinear stabilization context. First, the observability for a certain time eventually large is known to be equivalent to the exponential decay of the energy. This was for instance noticed in [Dehman and Gérard 2002, Proposition 2] in a similar context; see also Proposition 2.5 of this paper. Then we take as initial data the same sequence that would give a counterexample for the linear observability. The linearizability property (see [Gérard 1996]) allows to obtain that the nonlinear solution is asymptotically close to the linear one. This contradicts the observability property for the nonlinear solution as it does for the linear case. Hence the geometric control condition is also necessary for the exponential decay of the nonlinear equation.

• Our geometrical hypotheses on Ω may look strange, however they are only assumed for sake of simplicity. In fact, our results should apply more generally for any smooth manifold with bounded geometry, that is, such Ω that can be covered by a set of $\mathscr{C}^\infty$ charts $\alpha_i : U_i \mapsto \alpha_i(U_i) \subset \mathbb{R}^3$ such that $\alpha_i(U_i)$ is equal either to $B(0,1)$ or to $B_+(0,1) = \{x \in B(0,1), x_1 > 0\}$ (in the case with boundaries) and such that, for any $r \geq 0$ and $s \in [1,\infty]$, the $W^{r,s}$ norm of a function u in $W^{r,s}(\Omega,\mathbb{R})$ is equivalent to the norm $\left(\sum_{i\in\mathbb{N}} \|u \circ \alpha_i^{-1}\|^s_{W^{r,s}(\alpha_i(U_i))}\right)^{1/s}$.

The stabilization property (ED) for Equation (1-1) has been studied in [Haraux 1985a; Zuazua 1990; 1991; Dehman 2001] for $p < 3$. For $p \in [3,5)$, our main reference is the work of Dehman, Lebeau and Zuazua [Dehman et al. 2003]. This work is mainly concerned with the stabilization problem previously described on the Euclidean space $\mathbb{R}^3$ with flat metric and stabilization active outside of a ball. The main purpose of this paper is to extend their result to a nonflat geometry where multiplier methods cannot be used or do not give the optimal result with respect to the geometry. Other stabilization results for the nonlinear wave equation can be found in [Aloui et al. 2011] and the references therein. Some works have been done in the difficult critical case $p = 5$; we refer to [Dehman and Gérard 2002; Laurent 2011].

The proofs in these articles use three main ingredients:

(i) The exponential decay of the linear equation, which is equivalent to the geometric control condition (GCC).

(ii) A more or less involved compactness argument.

(iii) A unique continuation result implying that $u \equiv 0$ is the unique solution of

$$\begin{cases} \Box u + \beta u + f(u) = 0, & \\ \partial_t u = 0 & \text{on } [-T,T]\times\omega. \end{cases} \tag{1-8}$$

The results are mainly of the type "geometric control condition" plus "unique continuation" implies "exponential decay". This type of implication is even stated explicitly in some related works for the nonlinear Schrödinger equation [Dehman et al. 2006; Laurent 2010].

In the subcritical case $p < 5$, the less understood point is the unique continuation property (iii). In the previous works as [Dehman et al. 2003], the authors use unique continuation results based on Carleman estimates. The resulting geometric assumptions are not very natural and are stronger than (GCC). Indeed, the unique continuation was often proved with some Carleman estimates that required some strong geometric conditions. For instance for a flat metric, the usual geometric assumption that appear are often of "multiplier type" that is ω is a neighborhood of $\{x \in \partial\Omega \mid (x - x_0)\cdot n(x) > 0\}$ which are known to be stronger than the geometric control condition (see [Miller 2002] for a discussion about the links between these assumptions). Moreover, on curved spaces, this type of condition often needs to be checked by hand in each situation, which is mostly impossible.

Our main improvement in this paper is the proof of unique continuation in infinite time under the geometric control condition only. We show that, if the nonlinearity f is analytic (or generic), then one can use the result of Robbiano and Zuily [1998] to obtain a unique continuation property (iii) for infinite time $T = +\infty$ with the geometric control condition (GCC) only.

The central argument of the proof of our main result, Theorem 1.1, is the unique continuation property of [Robbiano and Zuily 1998] (see Section 3). This result applies for solutions u of (1-8) being smooth in space and analytic in time. If f is analytic, then the solutions of (1-1) are of course not necessarily analytic in time since the damped wave equations are not smoothing in finite time. However, the damped wave equations admit an asymptotic smoothing effect, i.e., are smoothing in infinite time. Hale and Raugel [2003] have shown that, for compact trajectories, this asymptotic smoothing effect also concerns the analyticity (see Section 5). In other words, combining [Robbiano and Zuily 1998] and [Hale and Raugel 2003] shows that the unique solution of (1-8) is $u \equiv 0$ if f is analytic and if $T = +\infty$. This combination has already been used by dynamicists for $p < 3$ (Hale and Raugel, private communication; [Joly 2007]).

One of the main interests of this paper is the use of arguments coming from both the dynamical study and the control theory of the damped wave equations. The reader familiar with the control theory could find interesting the use of the asymptotic smoothing effect to get unique continuation property with smooth solutions. The one familiar with the dynamical study of PDEs could be interested in the use of Strichartz estimates to deal with the case $p \in [3, 5)$. The main part of the proof of Theorem 1.1 is written with arguments coming from the dynamical study of PDEs. They are simpler than the corresponding ones of control theory, but far less accurate since they do not give any estimation for the time of observability. Anyway, such accuracy is not important here since we use the unique continuation property for (1-8) with $T = +\infty$. We briefly recall in Section 8 how these propagation of compactness and regularity properties could have been proved with some arguments more usual in the control theory.

Moreover, we give two applications of our results in both contexts of control theory and dynamical systems. First, as it is usual in control theory, some results of stabilization can be coupled with local control theorems to provide global controllability in large time.

Theorem 1.3. *Assume that f satisfies the conditions of Theorem 1.1 or belongs to the generic set $\mathfrak{G}$ defined by Theorem 1.2. Let $R_0 > 0$ and ω satisfying the geometric control condition. Then there exists $T > 0$ such that for any (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1)$ in $H_0^1(\Omega) \times L^2(\Omega)$ with*

$$\|(u_0, u_1)\|_{H^1\times L^2} \le R_0 \quad \textit{and} \quad \|(\tilde{u}_0, \tilde{u}_1)\|_{H^1\times L^2} \le R_0$$

there exists $g \in L^\infty([0, T], L^2(\Omega))$ supported in $[0, T] \times \omega$ such that the unique strong solution of

$$\begin{cases} \Box u + \beta u + f(u) = g & \text{on } [0, T] \times \Omega, \\ (u(0), \partial_t u(0)) = (u_0, u_1), \end{cases}$$

satisfies $(u(T), \partial_t u(T)) = (\tilde{u}_0, \tilde{u}_1)$.

The second application of our results concerns the existence of a compact global attractor. A compact global attractor is a compact set, which is invariant by the flow of the PDE and which attracts the bounded sets. The existence of such an attractor is an important dynamical property because it roughly says that the dynamics of the PDE may be reduced to dynamics on a compact set, which is often finite-dimensional. See [Hale 1988; Raugel 2002] for reviews of this concept. Theorems 1.1 and 1.2 show that $\{0\}$ is a global attractor for the damped wave equation (1-1). Of course, it is possible to obtain a more complex attractor by considering an equation of the type

$$\begin{cases} \partial_{tt}^2 u+\gamma(x)\partial_t u=\Delta u-\beta u-f(x,u) & (x,t)\in\Omega\times\mathbb{R}_+,\\ u(x,t)=0 & (x,t)\in\partial\Omega\times\mathbb{R}_+,\\ (u,\partial_t u)=(u_0,u_1)\in H_0^1\times L^2, \end{cases} \tag{1-9}$$

where $f\in C^\infty(\Omega\times\mathbb{R},\mathbb{R})$ is real analytic with respect to u and satisfies the following properties. There exist $C>0$, $p\in[1,5)$ and $R>0$ such that for all $(x,u)\in\Omega\times\mathbb{R}$,

$$|f(x,u)|\le C(1+|u|)^p,\quad |f_x'(x,u)|\le C(1+|u|)^p,\quad |f_u'(x,u)|\le C(1+|u|)^{p-1}, \tag{1-10}$$

$$x\in\partial\Omega \quad\Longrightarrow\quad f(x,0)=0, \tag{1-11}$$

$$(x\notin B(x_0,R)\quad\text{or}\quad |u|\ge R)\quad\Longrightarrow\quad f(x,u)u\ge 0, \tag{1-12}$$

where x_0 denotes a fixed point of the manifold.

Theorem 1.4. *Assume f is as above. Then the dynamical system generated by (1-9) in $H_0^1(\Omega)\times L^2(\Omega)$ is gradient and admits a compact global attractor $\mathcal{A}$.*

Of course, we would get the same result for f in a generic set similar to the one of Theorem 1.2.

We begin this paper by setting our main notations and recalling the basic properties of Equation (1-1) in Section 2. We recall the unique continuation property of Robbiano and Zuily in Section 3, whereas Sections 4 and 5 are concerned by the asymptotic compactness and the asymptotic smoothing effect of the damped wave equation. The proofs of our main results, Theorem 1.1 and 1.2, are given in Sections 6 and 7, respectively. An alternative proof, using more usual arguments from control theory, is sketched in Section 8. Finally, Theorems 1.3 and 1.4 are discussed in Section 9.

2. Notations and basic properties of the damped wave equation

In this paper, we use the following notations:

$$U=(u,u_t),\qquad F=(0,f),\qquad A=\begin{pmatrix}0 & \mathrm{Id}\\ \Delta-\beta & -\gamma\end{pmatrix}.$$

In this setting, (1-1) becomes

$$\partial_t U(t)=AU(t)+F(U).$$

We set $X=H_0^1(\Omega)\times L^2(\Omega)$ and for $s\in[0,1]$, we denote by X^s the space

$$X^s=D\big((-\Delta+\beta)^{(s+1)/2}\big)\times D\big((-\Delta+\beta)^{s/2}\big)=(H^{1+s}(\Omega)\cap H_0^1(\Omega))\times H_0^s(\Omega).$$

Notice that $X^0=X$ and $X^1=D(A)$ (even if γ is only in L^∞).

We recall that E denotes the energy defined by (1-4). We also emphasize that (1-2) and the invertibility of $\Delta-\beta$ implies that a set is bounded in X if and only if its energy E is bounded. Moreover, for all $E_0\ge 0$, there exists $C>0$ such that

$$E(u,v)\le E_0\quad\text{for all }\ (u,v)\in X\qquad\Longrightarrow\qquad \frac{1}{C}\,\|(u,v)\|_X^2\le E(u,v)\le C\,\|(u,v)\|_X^2\,. \tag{2-1}$$

To simplify some statements in the proofs, we assume without loss of generality that $3 < p < 5$. This will avoid some meaningless statements with negative Lebesgue exponents since $p = 3$ is the exponent where Strichartz estimates are not necessary and can be replaced by Sobolev embeddings.

We recall that Ω is endowed with a metric g. We denote by d the distance on Ω defined by

$$d(x, y) = \inf\{l(c) \mid c \in \mathscr{C}^\infty([0, 1], \Omega) \text{ with } c(0) = x \text{ and } c(1) = y\},$$

where $l(c)$ is the length of the path c according to the metric g. A ball $B(x, R)$ in Ω is naturally defined by

$$B(x, R) = \{y \in \Omega, d(x, y) < R\}.$$

For instance, if $\Omega = \mathbb{R}^3 \setminus B_{\mathbb{R}^3}(0, 1)$, the distance between $(0, 0, 1)$ and $(0, 0, -1)$ is π (and not 2) and the ball $B((0, 0, 1), \pi)$ has nothing to do with the classical ball $B_{\mathbb{R}^3}((0, 0, 1), \pi)$ of $\mathbb{R}^3$.

Cauchy problem. The global existence and uniqueness of solutions of the subcritical wave equation (1-1) with $\gamma \equiv 0$ has been studied in [Ginibre and Velo 1985; 1989]. Their method also applies for $\gamma \neq 0$ since this term is linear and well defined in the energy space X. Moreover, their argument to prove uniqueness also yields the continuity of the solutions with respect to the initial data.

The central argument is the use of Strichartz estimates.

Theorem 2.1 (Strichartz estimates). *Let $T > 0$ and (q, r) satisfy*

$$\frac{1}{q} + \frac{3}{r} = \frac{1}{2}, \qquad q \in [7/2, +\infty]. \tag{2-2}$$

There exists $C = C(T, q) > 0$ such that for every $G \in L^1([0, T], L^2(\Omega))$ and every $(u_0, u_1) \in X$, the solution u of

$$\begin{cases} \Box u + \gamma(x)\partial_t u = G(t), \\ \quad (u, \partial_t u)(0) = (u_0, u_1), \end{cases}$$

satisfies the estimate

$$\|u\|_{L^q([0,T],L^r(\Omega))} \le C\left(\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|G\|_{L^1([0,T],L^2(\Omega))}\right).$$

The result was stated in the Euclidean space $\mathbb{R}^3$ by Strichartz [1977] and Ginibre and Velo with $q \in (2, +\infty]$. Kapitanskiĭ [1990] extended the result to variable coefficients. On a bounded domain, the first estimates were proved by Burq, Lebeau and Planchon [Burq et al. 2008] for $q \in [5, +\infty]$ and extended to a larger range by Blair, Smith and Sogge in [Blair et al. 2009]. Note that, thanks to the counterexamples of Ivanovici [2012], we know that we cannot expect some Strichartz estimates in the full range of exponents in the presence of boundaries.

From these results, we deduce the estimates for the damped wave equation by absorption for T small enough. We can iterate the operation in a uniform number of steps. Actually, for the purpose of the semilinear wave equation, it is sufficient to consider the Strichartz estimate $L^{2p/(p-3)}([0, T], L^{2p}(\Omega))$, which gives $u^p \in L^{2/(p-3)}([0, T], L^2(\Omega)) \subset L^1([0, T], L^2(\Omega))$ because $1 < 2/(p-3) < +\infty$.

Theorem 2.2 (Cauchy problem). *Let f satisfy (1-2). Then for any $(u_0, u_1) \in X = H_0^1(\Omega) \times L^2(\Omega)$ there exists a unique solution $u(t)$ of the subcritical damped wave equation (1-1). Moreover, this solution is defined for all $t \in \mathbb{R}$ and its energy $E(u(t))$ is nonincreasing in time.*

For any $E_0 \geq 0$, $T \geq 0$ and (q, r) satisfying (2-2), there exists a constant C such that if u is a solution of (1-1) with $E(u(0)) \leq E_0$, then

$$\|u\|_{L^q([0,T],L^r(\Omega))} \leq C \left(\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)}\right).$$

In addition, for any $E_0 \geq 0$ and $T \geq 0$, there exists a constant C such that if u and $\tilde{u}$ are two solutions of (1-1) with $E(u(0)) \leq E_0$ and $E(\tilde{u}(0)) \leq E_0$, then

$$\sup_{t\in[-T,T]} \|(u, \partial_t u)(t) - (\tilde{u}, \partial_t \tilde{u})(t)\|_X \leq C\|(u, \partial_t u)(0) - (\tilde{u}, \partial_t \tilde{u})(0)\|_X.$$

Proof. The existence and uniqueness for small times is a consequence of the Strichartz estimates and of the subcriticality of the nonlinearity; see [Ginibre and Velo 1989]. The solution can be globalized backward and forward in time thanks to the energy estimates (1-5) for smooth solutions. Indeed,

$$E(t) \leq E(s) + C \int_t^s E(\tau)\, d\tau,$$

and thus Gronwall inequality for $t \leq s$ and the decay of energy for $t \geq s$ show that the energy does not blow up in finite time. This allows us to extend the solution for all times since the energy controls the norm of the space X by (2-1).

For the uniform continuity estimate, we notice that $w = u - \tilde{u}$ is solution of

$$\begin{cases} \Box w + \beta w + \gamma(x)\partial_t w = -w g(u, \tilde{u}), \\ (w, \partial_t w)(0) = (u, \partial_t u)(0) - (\tilde{u}, \partial_t \tilde{u})(0), \end{cases}$$

where $g(s, \tilde{s}) = \int_0^1 f'(s + \tau(\tilde{s} - s))\, d\tau$ fulfills $|g(s, \tilde{s})| \leq C(1 + |s|^{p-1} + |\tilde{s}|^{p-1})$. Let $q = 2p/(p-3)$, then the Strichartz and Hölder estimates give

$$\begin{aligned} \|(w, \partial_t w)(t)\|_{L^\infty([0,T],X)\cap L^q([0,T],L^{2p})} &\leq C\|(w, \partial_t w)(0)\|_X + C\|wg(u, \tilde{u})\|_{L^1([0,T],L^2)} \\ &\leq C\|(w, \partial_t w)(0)\|_X + CT\|w\|_{L^\infty([0,T],L^2)} \\ &\quad + T^\theta \|w\|_{L^q([0,T],L^{2p})} \left(\|u\|^{p-1}_{L^q([0,T],L^{2p})} + \|\tilde{u}\|^{p-1}_{L^q([0,T],L^{2p})}\right) \end{aligned}$$

with $\theta = (5 - p)/2 > 0$. We get the expected result for T small enough by absorption since we already know a uniform bound (depending on E_0) for the Strichartz norms of u and $\tilde{u}$. Then we iterate the operation to get the result for large T. $\square$

Exponential decay of the linear semigroup. In this paper, we will strongly use the exponential decay for the linear semigroup in the case where γ may vanish but satisfies the geometric assumptions of this paper. In this case, (1-3) enables us to control the decay of energy outside a large ball and the geometric control condition (GCC) enables to control the energy trapped in this ball.

Proposition 2.3. *Assume that $\gamma \in L^\infty(\Omega)$ satisfies (1-3) and (GCC). There exist two positive constants C and λ such that*

$$|||e^{At}|||_{\mathcal{L}(X^s)} \le Ce^{-\lambda t} \quad \text{for all } s \in [0,1] \text{ and all } t \ge 0.$$

The exponential decay of the damped wave equation under the geometric control condition is well known since the works of Rauch and Taylor [1974] on a compact manifold and Bardos, Lebeau and Rauch [Bardos et al. 1988; 1992] on a bounded domain. Yet we did not find any reference for unbounded domains ([Aloui and Khenissi 2002; Khenissi 2003] concern unbounded domains but local energy only). It is noteworthy that the decay of the linear semigroup in unbounded domains seems not to have been extensively studied for the moment.

We give a proof of Proposition 2.3 using a microlocal defect measure as done in [Lebeau 1996; Burq 1997a] (see also [Burq and Gérard 1997] for the proof of the necessity). The only difference with respect to these results is that the manifold that we consider may be unbounded. Since a microlocal defect measure only reflects the local propagation, we thus have to use the property of equipartition of the energy to deal with the energy at infinity and to show a propagation of compactness (see [Dehman et al. 2003] for the flat case).

Lemma 2.4. *Let $T > L$, where L is given by (GCC). Assume that $(U_{n,0}) \subset X$ is a bounded sequence, which weakly converges to 0 and assume that $U_n(t) = (u_n(t), \partial_t u_n(t)) = e^{At}U_{n,0}$ satisfies*

$$\int_0^T \int_\Omega \gamma(x)|\partial_t u_n|^2 \to 0. \tag{2-3}$$

Then $(U_{n,0})$ converges to 0 strongly in X.

Proof. Let μ be a microlocal defect measure associated to (u_n) (see [Gérard 1991; Tartar 1990; Burq 1997b] for the definition). Note that (2-3) implies that μ can also be associated to the solution of the wave equation without damping, so the weak regularity of γ is not problematic for the propagation and we get that μ is concentrated on $\{\tau^2 - |\xi|_x^2 = 0\}$, where (τ, ξ) are the dual variables of (t, x). Moreover, (2-3) implies that $\gamma\tau^2\mu = 0$ and so $\mu \equiv 0$ on $S^*(]0, T[\times\omega)$. Then, by using the propagation of the measure along the generalized bicharacteristic flow of Melrose–Sjöstrand and the geometric control condition satisfied by ω, we obtain $\mu \equiv 0$ everywhere. We do not give more details about propagation of microlocal defect measures and refer to the Appendix of [Lebeau 1996] or Section 3 of [Burq 1997b] (see also [Gérard and Leichtnam 1993] for some close propagation results in a different context). Since $\mu \equiv 0$, we know that

$$U_n \to 0 \quad \text{on } H^1 \times L^2(]0, T[\times B(x_0, R))$$

for every $R > 0$.

To finish the proof, we need the classical equipartition of the energy to get the convergence to 0 in the whole manifold Ω. Since γ is uniformly positive outside a ball $B(x_0, R)$, (2-3) and the previous arguments imply that

$$\partial_t u_n \to 0 \quad \text{in } L^2([0, T] \times \Omega).$$

Let $\varphi \in C_0^\infty(]0,T[)$ with $\varphi \geq 0$ and $\varphi(t) = 1$ for $t \in [\varepsilon, T-\varepsilon]$. We multiply the equation by $\varphi(t)u_n$ and we obtain

$$0 = -\iint_{[0,T]\times\Omega} \varphi(t)|\partial_t u_n|^2 - \iint_{[0,T]\times\Omega} \varphi'(t)\partial_t u_n u_n + \iint_{[0,T]\times\Omega} \varphi(t)|\nabla u_n|^2 + \iint_{[0,T]\times\Omega} \varphi(t)\beta|u_n|^2 + \iint_{[0,T]\times\Omega} \varphi(t)\gamma(x)\partial_t u_n u_n.$$

The L^2 norm of $u_n(t)$ is bounded, while $\partial_t u_n \to 0$ in $L^2([0,T]\times\Omega)$, so the first, second and fifth terms converge to zero. Then the above equation yields

$$\iint_{[0,T]\times\Omega} \varphi(t)\left(\beta|u_n|^2 + |\nabla u_n|^2\right) \to 0.$$

Finally, notice that the energy identity $\|U_{n,0}\|_X^2 = \|U_n(t)\|_X^2 + \int_0^T \int_\Omega \gamma(x)|\partial_t u_n|^2$ shows that

$$\iint_{[0,T]\times\Omega} \varphi(t)\left(\beta|u_n|^2 + |\nabla u_n|^2\right) \sim \|U_{n,0}\|_X^2 \int_0^T \varphi(t),$$

and thus that $\|U_{n,0}\|_X$ goes to zero. □

Proof of Proposition 2.3. Once Lemma 2.4 is established, the proof follows the arguments of the classical case, where Ω is bounded. We briefly recall them.

We first treat the case $s = 0$. As in Proposition 2.5, the exponential decay of the energy is equivalent to the observability estimate, that is, the existence of $C > 0$ and $T > 0$ such that, for any trajectory $U(t) = e^{At}U_0$ in X,

$$\int_0^T \int_\Omega \gamma(x)|\partial_t u|^2 \geq C\|U(0)\|_X^2. \tag{2-4}$$

We argue by contradiction: Assume that (2-4) does not hold for any positive T and C. Then there exists a sequence of initial data $U_n(0)$ with $\|U_n(0)\|_X = 1$ and such that

$$\int_0^n \int_\Omega \gamma(x)|\partial_t u_n(t,x)|^2 dt dx \to 0 \quad \text{as } n \to +\infty,$$

where $(u_n, \partial_t u_n)(t) = U_n(t) = e^{At}U_n(0)$. Let $\widetilde{U}_n = U_n(n/2 + \cdot)$. We have

$$\int_{-n/2}^{n/2} \int_\Omega \gamma(x)|\partial_t \tilde{u}_n(t,x)|^2 dt dx \to 0 \quad \text{as } n \to +\infty,$$

and, for any $t \in [-n/2, n/2]$,

$$\|\widetilde{U}_n(t)\|_X^2 = \|\widetilde{U}_n(-n/2)\|_X^2 - \int_{-n/2}^t \int_\Omega \gamma(x)|\partial_t \tilde{u}_n(s,x)|^2 ds dx \to 1 \quad \text{as } n \to +\infty.$$

We can thus assume that $U_n(0)$ converges to $U_\infty(0) \in X$, weakly in X. For any $T > 0$, $U_n(t)$ and $\partial_t U_n(t)$ are bounded in $L^\infty([-T,T], X)$ and $L^\infty([-T,T], L^2(\Omega) \times H^{-1}(\Omega))$, respectively. Thus, by using Ascoli's theorem, we may also assume that $U_n(t)$ strongly converges to $U_\infty(t)$ in $L^\infty([-T,T], L^2(K)\times H^{-1}(K))$, where K is any compact of Ω. Hence $(u_\infty, \partial_t u_\infty)(t) = U_\infty(t) = e^{At}U_\infty(0)$ is a solution of

$$\begin{cases} \Box u_\infty + \beta u_\infty = 0 & \text{on } \mathbb{R}\times\Omega, \\ \partial_t u_\infty = 0 & \text{on } \mathbb{R}\times\omega. \end{cases} \tag{2-5}$$

in $L^2 \times H^{-1}$. Since $U_\infty(0) \in X$ belongs to X, we deduce that, in fact, $U_\infty(t)$ solves (2-5) in X.

To finish the proof of Proposition 2.3, we have to show that $U_\infty \equiv 0$. Indeed, applying Lemma 2.4, we would get that U_n converges strongly to 0, which contradicts the hypothesis $\|U_n(0)\|_X = 1$. Note that $U_\infty \equiv 0$ is a direct consequence of a unique continuation property as in Corollary 3.2. However, Corollary 3.2 requires Ω to be smooth, whereas Proposition 2.3 could be more general. Therefore, we recall another classical argument to show that $U_\infty \equiv 0$.

Denote by N the set of functions $U_\infty(0) \in X$ satisfying (2-5), which is obviously a linear subspace of X. We will prove that $N = \{0\}$. Since $\gamma(x)|\partial_t u_\infty|^2 \equiv 0$ for functions u_∞ in N and since N is a closed subspace, Lemma 2.4 shows that any weakly convergent subsequence of N is in fact strongly convergent. By the Riesz theorem, N is therefore finite-dimensional. For any $t \in \mathbb{R}$, e^{tA} applies N into itself and thus $A_{|N}$ is a bounded linear operator. Assume that $N \neq \{0\}$, then $A_{|N}$ admits an eigenvalue λ with eigenvector $Y = (y_0, y_1) \in N$. This means that $y_1 = \lambda y_0$ and that $(\Delta - \beta)y_0 = \lambda^2 y_0$. Moreover, we know that $y_1 = 0$ on ω and so, if $\lambda \neq 0$, that $y_0 = 0$ on ω. This implies $y_0 \equiv 0$ by the unique continuation property of elliptic operators. Finally, if $\lambda = 0$, we have $(\Delta - \beta)y_0 = 0$ and $y_0 = 0$, because, by assumption, $\Delta - \beta$ is a negative definite operator.

So we have proved $N = \{0\}$ and therefore $U_\infty = 0$, that is, $\widetilde{U}_n(0)$ converges to 0 weakly in X. We can then apply Lemma 2.4 on any interval $[-n/2, -n/2 + T]$, where L is the time in the geometric control condition (GCC) and obtain a contradiction to $\|U_n(0)\|_X = 1$.

Let us now consider the cases $s \in (0, 1]$. The basic semigroup properties (see [Pazy 1983]) show that, if $U \in X^1 = D(A)$, then $e^{At}U$ belongs to $D(A)$ and

$$\begin{aligned} \|e^{At}U\|_{X^1} = \|Ae^{At}U\|_X + \|e^{At}U\|_X &= \|e^{At}AU\|_X + \|e^{At}U\|_X \\ &\le Ce^{-\lambda t}\left(\|AU\|_X + \|U\|_X\right) = Ce^{-\lambda t}\|U\|_{D(A)}. \end{aligned}$$

This shows Proposition 2.3 for $s = 1$. Notice that we do not have to require any regularity for γ to obtain this result. Then Proposition 2.3 for $s \in (0, 1)$ follows by interpolating between the cases $s = 0$ and $s = 1$ (see [Tartar 2007]). □

First nonlinear exponential decay properties. Theorem 2.2 shows that the energy E is nonincreasing along the solutions of (1-1). The purpose of this paper is to obtain the exponential decay of this energy in the sense of property (ED) stated above. We first recall the well-known criterion for exponential decay.

Proposition 2.5. *The exponential decay property* (ED) *holds if and only if there exist T and C such that*

$$E(u(0)) \le C\big(E(u(0)) - E(u(T))\big) = C\iint_{[0,T]\times\Omega} \gamma(x)|\partial_t u(x,t)|^2\,dt\,dx \tag{2-6}$$

for all solutions u of (1-1) *with $E(u(0)) \le E_0$.*

Proof. If (ED) holds then (2-6) holds for T large enough since $E(u(0))-E(u(T))\geq(1-Ke^{-\lambda T})E(u(0))$. Conversely, if (2-6) holds, using $E(u(T))\leq E(u(0))$, we get $E(u(T))\leq C/(C+1)E(u(0))$ and thus $E(u(kT))\leq(C/(C+1))^k E(u(0))$. Using again the decay of the energy to fill the gaps $t\in(kT,(k+1)T)$, this shows that (ED) holds. ☐

First, we prove exponential decay in the case of positive damping, which will be helpful to study what happens outside a large ball since (1-3) is assumed in the whole paper. Note that the fact that $-\Delta+\beta$ is positive is necessary to avoid for instance the constant undamped solutions.

Proposition 2.6. *Assume that $\omega=\Omega$, that is that $\gamma(x)\geq\alpha>0$ everywhere. Then* (ED) *holds.*

Proof. We recall here the classical proof. We introduce a modified energy

$$\widetilde{E}(u)=\int_\Omega \tfrac{1}{2}(|\partial_t u|^2+|\nabla u|^2+\beta|u|^2)+V(u)+\varepsilon u\partial_t u$$

with $\varepsilon>0$. Since $\int_\Omega|\nabla u|^2+\beta|u|^2$ controls $\|u\|^2_{L^2}$, $\widetilde{E}$ is equivalent to E for ε small enough and it is sufficient to obtain the exponential decay of the auxiliary energy $\widetilde{E}$. Using $\gamma\geq\alpha>0$ and $uf(u)\geq 0$, a direct computation shows for ε small enough that

$$\begin{aligned}\widetilde{E}(u(T))-\widetilde{E}(u(0))&=\int_0^T\int_\Omega -\gamma(x)|\partial_t u|^2+\varepsilon|\partial_t u|^2+\varepsilon\gamma(x)u\partial_t u-\varepsilon(|\nabla u|^2+\beta|u|^2)-\varepsilon uf(u)\\&\leq -C\int_0^T\|(u,\partial_t u)\|^2_{H^1\times L^2}\leq -C\int_0^T\widetilde{E}(t)\,dt\leq -CT\widetilde{E}(T),\end{aligned}$$

where $C>0$ is a constant that may change from line to line. Thus, $\widetilde{E}(u(0))-\widetilde{E}(u(T))\geq CT\widetilde{E}(u(T))$ with $CT>0$ and therefore $\widetilde{E}(u(0))\geq\mu\widetilde{E}(u(T))$ with $\mu>1$. As in the proof of Proposition 2.3, this last property implies the exponential decay of $\widetilde{E}$ and thus the one of E. ☐

3. A unique continuation result for equations with partially holomorphic coefficients

Comparatively to previous articles on the stabilization of the damped wave equations as [Dehman et al. 2003], one of the main novelties of this paper is the use of a unique continuation theorem requiring partially analyticity of the coefficients, but very weak geometrical assumptions as shown in Corollary 3.2. We use here the following result of Robbiano and Zuily [1998]. This result has also been proved independently by Hörmander [1997] and has been generalized by Tataru [1999]. Note that the idea of using partial analyticity for unique continuation was introduced by Tataru [1995] but it requires some global analyticity assumptions that are not fulfilled in our case. All these results use very accurate microlocal analysis and hold in a much more general framework than the one of the wave equation. However, for sake of simplicity, we restrict the statement to this case.

Theorem 3.1. *Let $d\geq 1$, $(x_0,t_0)\in\mathbb{R}^d\times\mathbb{R}$ and let $\mathcal{U}$ be a neighborhood of (x_0,t_0). Let $(A_{i,j}(x,t))_{i,j=1,\dots,d}$, $b(x,t)$, $(c_i(x,t))_{i=1,\dots,d})$ and $d(x,t)$ be bounded coefficients in $\mathscr{C}^\infty(\mathcal{U},\mathbb{R})$. Let v be a strong solution of*

$$\partial^2_{tt}v=\operatorname{div}(A(x,t)\nabla v)+b(x,t)\partial_t v+c(x,t)\nabla v+d(x,t)v,\qquad (x,t)\in\mathcal{U}\subset\mathbb{R}^d\times\mathbb{R}.\tag{3-1}$$

Let $\varphi\in\mathscr{C}^2(\mathcal{U},\mathbb{R})$ such that $\varphi(x_0,t_0)=0$ and $(\nabla\varphi,\partial_t\varphi)(x,t)\neq 0$ for all $(x,t)\in\mathcal{U}$. Assume that:

(i) *The coefficients* A, b, c *and* d *are analytic in time.*

(ii) $A(x_0, t_0)$ *is a symmetric positive definite matrix.*

(iii) *The hypersurface* $\{(x,t) \in \mathcal{U}, \varphi(x,t) = 0\}$ *is not characteristic at* (x_0, t_0), *that is, that we have* $|\partial_t \varphi(x_0, t_0)|^2 \neq \langle \nabla\varphi(x_0, t_0) \mid A(x_0, t_0)\nabla\varphi(x_0, t_0)\rangle$.

(iv) $v \equiv 0$ *in* $\{(x,t) \in \mathcal{U}, \varphi(x,t) \leq 0\}$.

Then $v \equiv 0$ *in a neighborhood of* (x_0, t_0).

Proof. We only have to show that Theorem 3.1 is a direct translation of Theorem A of [Robbiano and Zuily 1998] in the framework of the wave equation. To use the notations of [ibid.], we let x_a be the time variable and x_b the space variable and we set $(x_0, t_0) = x^0 = (x_b^0, x_a^0)$. Equation (3-1) corresponds to the differential operator

$$P = \xi_a^2 - {}^t\xi_b A(x_b, x_a)\xi_b - b(x_b, x_a)\xi_a - c(x_b, x_a)\xi_b - d(x_b, x_a)$$

with principal symbol $p_2 = \xi_a^2 - {}^t\xi_b A(x_b, x_a)\xi_b$.

All the statements of Theorem 3.1 are obvious translations of Theorem A of [ibid.], except maybe for the fact that hypothesis (iii) implies the hypothesis of pseudoconvexity of [ibid.]. We compute $\{p_2, \varphi\} = 2\xi_a\varphi_a' - 2{}^t\xi_b A(x_a, x_b)\varphi_b'$. Let us set $\zeta = (x_a^0, x_b^0, i\varphi_a'(x^0), \xi_b + i\varphi_b'(x^0))$, then $\{p_2, \varphi\}(\zeta) = 0$ if and only if

$$i(\varphi_a'(x^0))^2 - i\,{}^t\varphi_b'(x^0))A(x^0)\varphi_b'(x^0) - {}^t\,\xi_b A(x^0)\varphi_b'(x^0) = 0.$$

This is possible only if $(\varphi_a'(x^0))^2 = {}^t\varphi_b'(x^0)A(x^0)\varphi_b'(x^0)$, that is if the hypersurface $\varphi = 0$ is characteristic at (x_0, t_0). Thus, if this hypersurface is not characteristic, then the pseudoconvexity hypothesis of Theorem A of [ibid.] holds. □

The previous theorem allows us to prove some unique continuation result with some optimal time and geometric assumptions. This allows us to prove unique continuation where the geometric condition is only, roughly speaking, that we do not contradict the finite speed of propagation.

Corollary 3.2. *Let* $T > 0$ (*or* $T = +\infty$) *and let* b, $(c_i)_{i=1,2,3}$ *and* d *be coefficients in* $\mathscr{C}^\infty(\Omega \times [0, T], \mathbb{R})$. *Assume moreover that* b, c *and* d *are analytic in time and that* v *is a strong solution of*

$$\partial_{tt}^2 v = \Delta v + b(x,t)\partial_t v + c(x,t)\nabla v + d(x,t)v, \quad (x,t) \in \Omega \times (-T, T). \tag{3-2}$$

Let $\mathbb{O}$ *be a nonempty open subset of* Ω *and assume that* $v(x,t) = 0$ *in* $\mathbb{O} \times (-T, T)$. *Then* $v(x, 0) \equiv 0$ *in* $\mathbb{O}_T = \{x_0 \in \Omega, d(x_0, \mathbb{O}) < T\}$.

As consequences:

(a) *If* $T = +\infty$, *then* $v \equiv 0$ *everywhere.*

(b) *If* $v \equiv 0$ *in* $\mathbb{O} \times (-T, T)$ *and* $\overline{\mathbb{O}}_T = \Omega$, *then* $v \equiv 0$ *everywhere.*

Proof. Since Ω is assumed to be connected, both consequences are obvious from the first statement.

Let x_0 be given such that $d(x_0, \mathbb{O}) < T$. There is a point $x_* \in \mathbb{O}$ linked to x_0 by a smooth curve of length $l < T$ that stays away from the boundary. We introduce a sequence of balls $B(x_0, r), \ldots, B(x_K, r)$ with $r \in (0, T/K)$, $x_{k-1} \in B(x_k, r)$ and $x_K = x_*$, such that $B(x_k, r)$ stays away from the boundary and is

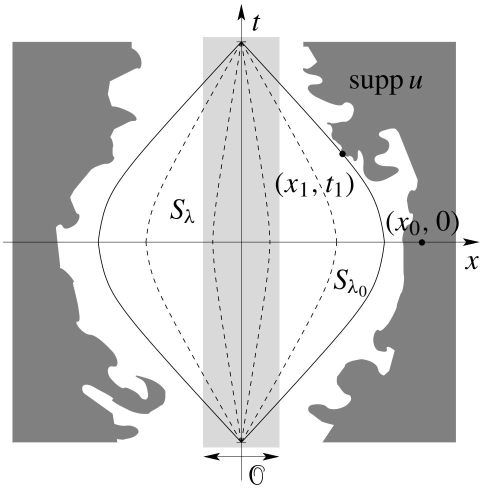

Figure 1. The proof of Corollary 3.2.

small enough such that it is diffeomorphic to an open set of $\mathbb{R}^3$ via the exponential map. Note that such a sequence of balls exists because the smooth curve linking x_0 to x_K is compact and of length smaller than T. We also notice that it is sufficient to prove Corollary 3.2 in each ball $B(x_k, r)$. Indeed, this would enable us to apply Corollary 3.2 in $B(x_K, r) \times (-T, T)$ to obtain that v vanishes in a neighborhood of x_{K-1} for $t \in (-T+r, T-r)$ and then to apply it recursively in $B(x_{K-1}, r) \times (-T+r, T-r)$, ..., $B(x_1, r) \times (-T+(K-1)r, T-(K-1)r)$ to obtain that $v(x_0, 0) = 0$.

From now on, we assume that $x_0 \in B(x_*, r)$ and that v vanishes in a neighborhood $\mathcal{O}$ of x_* for $t \in (-r, r)$. Since $d(x_0, x_*) < r$, we can introduce a nonnegative function $h \in \mathscr{C}^\infty([-r, r], \mathbb{R})$ such that $h(0) > d(x_0, x_*)$, $h(\pm r) = 0$ and $|h'(t)| < 1$ for all $t \in [-r, r]$. We set $\mathcal{U} = B(x_*, r) \times (-r, r)$ and for any $\lambda \in [0, 1]$, we define

$$\varphi_\lambda(x, t) = d(x, x_*)^2 - \lambda h(t)^2.$$

Since r is assumed to be smaller than the radius of injectivity of the exponential map, φ_λ is a smooth well-defined function. We prove Corollary 3.2 by contradiction. Assume that $v(x_0, 0) \neq 0$. We denote by V_λ the volume $\{(x, t) \in \mathcal{U}, \varphi_\lambda(x, t) \leq 0\}$. We notice that $V_{\lambda_1} \subset V_{\lambda_2}$ if $\lambda_1 < \lambda_2$, that for small λ, V_λ is included in $\mathcal{O} \times (-r, r)$ where v vanishes, and that V_1 contains $(x_0, 0)$ where v does not vanish. Thus

$$\lambda_0 = \sup\{\lambda \in [0, 1] : v(x, t) = 0 \text{ for all} (x, t) \in V_\lambda\}$$

is well defined and belongs to $(0, 1)$. For t close to $-r$ or r, $h(t)$ is small and the section $\{x, (x, t) \in V_{\lambda_0}\}$ of V_{λ_0} is contained in $\mathcal{O}$ where v vanishes. Therefore, by compactness, the hypersurface $S_{\lambda_0} = \partial V_{\lambda_0}$ must touch the support of v at some point $(x_1, t_1) \in \mathcal{U}$ (see Figure 1).

In local coordinates, Δ can be written as $\operatorname{div}(A(x)\nabla .) + c(x) \cdot \nabla$. Moreover,

$$\langle \nabla\varphi_\lambda | A \nabla\varphi_\lambda \rangle = |\nabla_g d(\cdot, x_*)|_g^2 = 1,$$

where the index g means that the gradient and norm are taken according to the metric. Therefore, the hypersurface S_{λ_0} is noncharacteristic at (x_1, t_1) in the sense of hypothesis (iii) of Theorem 3.1 since $|\partial_t \varphi_\lambda(x, t)| = |\lambda h'(t_1)| < 1$. Thus, we can apply Theorem 3.1 with $\varphi = \varphi_{\lambda_0}$ at the point (x_1, t_1), mapping everything in the three-dimensional Euclidean frame via the exponential chart. We get that v must vanish in a neighborhood of (x_1, t_1). This is obviously a contradiction since (x_1, t_1) has been taken in the support of v. □

4. Asymptotic compactness

As soon as t is positive, a solution $u(t)$ of a parabolic PDE becomes smooth and stays in a compact set. The smoothing effect in finite time of course fails for the damped wave equations. However, these PDEs admit in some sense a smoothing effect in infinite time. This effect is called *asymptotic compactness* if one is interested in extracting asymptotic subsequences as in Proposition 4.3, or *asymptotic smoothness* if one uses the regularity of globally bounded solutions as in Proposition 4.4. For the reader interested in these notions, we refer to [Hale 1988]. The proof of this asymptotic smoothing effect is based on the variation of constant formula $U(t) = e^{At}U_0 + \int_0^t e^{A(t-s)} F(U(s))ds$ and two properties:

- The exponential decay of the linear group (Proposition 2.3), which implies that the linear part $e^{At}U_0$ asymptotically disappears.
- The regularity of the nonlinearity F implying the compactness of the nonlinear term $\int_0^t e^{A(t-s)} F(U(s))ds$ (Corollary 4.2 below). Note that the subcriticality of f is the key point of this property and that our arguments cannot be extended as they stand to the critical case $p = 5$.

The purpose of this section is to prove some compactness and regularity results about undamped solutions as (1-8). Note that these results could also have been obtained with a more "control theoretic" proof (see Section 8 for a sketch of the alternative proof) based on propagation results or observability estimates. Here, we have chosen to give a different one using asymptotic regularization, which is more common in dynamical systems. The spirit of the proof remains quite similar: we prove that the nonlinearity is more regular than it seems *a priori* and use some properties of the damped linear equation.

Regularity of the nonlinearity. Since f is subcritical, it is shown in [Dehman et al. 2003] that the nonlinear term of (1-1) yields a gain of smoothness.

Theorem 4.1 (Dehman, Lebeau and Zuazua [2003]). *Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^3, \mathbb{R})$, $R > 0$ and $T > 0$. Let $s \in [0, 1)$ and let $\varepsilon = \min(1 - s, (5 - p)/2, (17 - 3p)/14) > 0$ with p and f as in (1-2). There exist (q, r) satisfying (2-2) and $C > 0$ such that the following property holds: If $v \in L^\infty([0, T], H^{1+s}(\mathbb{R}^3))$ is a function with finite Strichartz norms $\|v\|_{L^q([0,T],L^r(\mathbb{R}^3))} \le R$, then $\chi(x) f(v) \in L^1([0, T], H^{s+\varepsilon}(\mathbb{R}^3))$ and moreover*

$$\|\chi(x) f(v)\|_{L^1([0,T],H^{s+\varepsilon}(\mathbb{R}^3))} \le C\|v\|_{L^\infty([0,T],H^{1+s}(\mathbb{R}^3))}.$$

The constant C depends only on χ, s, T, (q, r), R and the constant in estimate (1-2).

Theorem 4.1 is a copy of Theorem 8 of [Dehman et al. 2003], except for two points.

First, we would like to apply the result to a solution v of the damped wave equation on a manifold possibly with boundaries, where not all Strichartz exponents are available. This leads to the constraint $q \geq \frac{7}{2}$ for the Strichartz exponents (q, r) of (2-2) (see Theorem 2.2). In the proof of Theorem 8 of [ibid.], the useful Strichartz estimate corresponds to $r = 3(p-1)/(1-\varepsilon)$ and $q = 2(p-1)/(p-3+2\varepsilon)$ and it is required that $q \geq p-1$, which yields $\varepsilon \leq (5-p)/2$. In this paper, we require also that $q \geq 7/2$, which yields in addition $\varepsilon \leq (17-3p)/14$. Notice that $p < 5$ and thus both bounds are positive.

The second difference is that, in [ibid.] the function f is assumed to be of class $\mathscr{C}^3$ and to satisfy

$$|f''(u)| \leq C(1+|u|)^{p-2} \quad \text{and} \quad |f^{(3)}(u)| \leq C(1+|u|)^{p-3} \tag{4-1}$$

in addition of (1-2). Since Theorem 4.1 concerns the $L^1(H^{s'})$ norm of $\chi(x)f(v)$ for $s' = s+\varepsilon \leq 1$, we can omit assumption (4-1). Actually, we make the assumption $\varepsilon \leq 1-s$ which is not present in [ibid.] and a careful study of their proof shows that (1-2) is not necessary under that assumption.

Indeed, let $\tilde{f}(u) = \mathrm{th}^3(u)|u|^p$. The function $\tilde{f}$ is of class $\mathscr{C}^3$ and satisfies (1-2) and (4-1). Hence Theorem 8 of [ibid.] can be applied to $\tilde{f}$ and we can bound the $L^1(H^{s'})$ norm of $\tilde{f}$ as in Theorem 4.1. On the other hand, we notice that

$$|\tilde{f}(u)| \underset{\pm\infty}{\sim} |u|^p, \qquad \tilde{f}'(u) \underset{\pm\infty}{\sim} p|u|^{p-1}, \qquad \tilde{f}'(u) \geq 0.$$

Then, since f satisfies (1-2), there exists $C > 0$ such that $|f(u)| \leq C(1+|\tilde{f}(u)|)$ and $|f'(u)| \leq C(1+\tilde{f}'(u))$. Thus, if we assume that $v > u$ to fix the notations,

$$\begin{aligned}
|f(v)-f(u)| &\leq (v-u)\int_0^1 |f'(u+\tau(v-u))|\,d\tau \\
&\leq C(v-u) + C(v-u)\int_0^1 \tilde{f}'(u+\tau(v-u))\,d\tau \\
&\leq C(v-u) + C(\tilde{f}(v)-\tilde{f}(u)) \leq C|v-u| + C|\tilde{f}(v)-\tilde{f}(u)|.
\end{aligned}$$

For $0 < s < 1$, using the above inequalities and the definition of the $H^{s'}$ norm as

$$\|\chi f(u)\|^2_{H^{s'}} = \|\chi f(u)\|^2_{L^2} + \iint_{\mathbb{R}^6} \frac{|\chi(x)f(u(x)) - \chi(y)f(u(y))|^2}{|x-y|^{2s'}}\,dx\,dy,$$

we obtain

$$\|\chi f(u)\|_{L^1(H^{s'})} \leq C\|u\|_{L^\infty(H^1)} + C\|\tilde{\chi}\tilde{f}(u)\|_{L^1(H^{s'})},$$

where $\tilde{\chi}$ is another cut-off function with larger support. Hence for $0 < s < 1$, the conclusion of Theorem 4.1 holds not only for $\tilde{f}$ but also for f. If $s' = 1$, we just apply the chain rule and the proof is easier.

Note that the above arguments show that the constant C depends on f through estimate (1-2) only. Notice in addition that since f is only $\mathscr{C}^1$, we cannot expect $\chi f(v)$ to be more regular than H^1 and that is why we also assume $\varepsilon \leq 1-s$.

In this paper, we use a generalization of Theorem 8 of [Dehman et al. 2003] for noncompact manifolds with boundaries.

Corollary 4.2. *Let $R > 0$ and $T > 0$. Let $s \in [0, 1)$ and let $\varepsilon = \min(1-s, (5-p)/2, (17-3p)/14) > 0$ with p as in (1-2). There exist (q, r) satisfying (2-2) and $C > 0$ such that the following property holds: If $v \in L^\infty([0, T], H^{1+s}(\Omega) \cap H^1_0(\Omega))$ is a function with finite Strichartz norms $\|v\|_{L^q([0,T],L^r(\Omega))} \leq R$, then $f(v) \in L^1([0, T], H_0^{s+\varepsilon}(\Omega))$ and moreover*

$$\|f(v)\|_{L^1([0,T],H_0^{s+\varepsilon}(\Omega))} \leq C\|v\|_{L^\infty([0,T],H^{s+1}(\Omega)\cap H^1_0(\Omega))}.$$

The constant C depends only on Ω, (q, r), R and the constant in estimate (1-2).

Proof. Since we assumed that Ω has a bounded geometry in the sense that Ω is either compact or a compact perturbation of a manifold with periodic metric, Ω can be covered by a set of $\mathscr{C}^\infty$ charts $\alpha_i : U_i \to \alpha_i(U_i) \subset \mathbb{R}^3$ such that $\alpha_i(U_i)$ is equal either to $B(0, 1)$ or to $B_+(0, 1) = \{x \in B(0, 1), x_1 > 0\}$ and such that, for any $s \geq 0$ the norm of a function $u \in H^s(\Omega)$ is equivalent to the norm

$$\left(\sum_{i\in\mathbb{N}} \|u \circ \alpha_i^{-1}\|^2_{H^s(\alpha_i(U_i))}\right)^{1/2}.$$

Moreover, the Strichartz norm $L^q([0, T], L^r(\alpha_i(U_i))$ of $v \circ \alpha_i^{-1}$ is uniformly controlled from above by the Strichartz norm $L^q([0, T], L^r(U_i))$ of v, which is bounded by R.

Therefore, it is sufficient to prove that Corollary 4.2 holds for Ω being either $B(0, 1)$ or $B_+(0, 1)$. Say that $\Omega = B_+(0, 1)$, the case $\Omega = B(0, 1)$ being simpler. To apply Theorem 4.1, we extend v in a neighborhood of $B_+(0, 1)$ as follows. For $x \in B_+(0, 2)$, we use the radial coordinates $x = (r, \sigma)$ and we set

$$\tilde{v}(x) = \tilde{v}(r, \sigma) = 5v(1-r, \sigma) - 20v(1-r/2, \sigma) + 16v(1-r/4, \sigma).$$

Then, for $x = (x_1, x_2, x_3) \in B_-(0, 2)$, we set

$$\tilde{v}(x) = 5v(-x_1, x_2, x_3) - 20v(-x_1/2, x_2, x_3) + 16v(-x_1/4, x_2, x_3).$$

Notice that $\tilde{v}$ is an extension of v in $B(0, 2)$, which preserves the $\mathscr{C}^2$ regularity, and that the H^s norm for $s \leq 2$ as well as the Strichartz norms of $\tilde{v}$ are controlled by the corresponding norms of v. Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^3)$ be a cut-off function such that $\chi \equiv 1$ in $B_+(0, 1)$ and $\chi \equiv 0$ outside $B(0, 2)$. Applying Theorem 4.1 to $\chi(x) f(\chi(x)\tilde{v})$ yields a control of $\|f(v)\|_{L^1([0,T],H^{s+\varepsilon}(B_+(0,1)))}$ by $\|v\|_{L^\infty([0,T],H^{s+1}(\Omega))}$. Finally, notice that $f(0) = 0$ and thus the Dirichlet boundary condition on v naturally implies the one on $f(v)$. □

Asymptotic compactness and regularization effect. As explained in the beginning of this section, using the Duhamel formula $U(t) = e^{At}U_0 + \int_0^t e^{A(t-s)} F(U(s))ds$ and Corollary 4.2, we obtain two propositions related to the asymptotic smoothing effect of the damped wave equations.

Proposition 4.3. *Let $f \in \mathscr{C}^1(\mathbb{R})$ satisfy (1-2), let (u_0^n, u_1^n) be a sequence of initial data which is bounded in $X = H^1_0(\Omega) \times L^2(\Omega)$ and let (u_n) be the corresponding solutions of the damped wave equation (1-1). Let $(t_n) \in \mathbb{R}$ be a sequence of times such that $t_n \to +\infty$ when n goes to $+\infty$.*

Then there exist subsequences $(u_{\psi(n)})$ and $(t_{\psi(n)})$ and a global solution u_∞ of (1-1) such that

$$(u_{\psi(n)}, \partial_t u_{\psi(n)})(t_{\psi(n)} + .) \to (u_\infty, \partial_t u_\infty)(\cdot) \quad \text{in } \mathscr{C}^0([-T, T], X) \quad \text{for all } T > 0.$$

Proof. We use the notations of Section 2. Due to the equivalence between the norm of X and the energy given by (2-1) and the fact that the energy is decreasing in time, we know that $U_n(t)$ is uniformly bounded in X with respect to n and $t \geq 0$. So, up to taking a subsequence, it weakly converges to a limit $U_\infty(0)$ which gives a global solution U_∞. We notice that, due to the continuity of the Cauchy problem with respect to the initial data stated in Theorem 2.2, it is sufficient to show that $U_{\psi(n)}(t_{\psi(n)}) \to U_\infty(0)$ for some subsequence $\psi(n)$. We have

$$\begin{aligned} U_n(t_n) &= e^{At_n}U_n(0) + \int_0^{t_n} e^{sA}F(U_n(t_n-s))\,ds \\ &= e^{At_n}U_n(0) + \sum_{k=0}^{\lfloor t_n\rfloor -1} e^{kA}\int_0^1 e^{sA}F(U_n(t_n-k-s))\,ds + \int_{\lfloor t_n\rfloor}^{t_n} e^{sA}F(U_n(t_n-s))\,ds \\ &= e^{At_n}U_n(0) + \sum_{k=0}^{\lfloor t_n\rfloor -1} e^{kA}I_{k,n} + I_n. \end{aligned} \tag{4-2}$$

Theorem 2.2 shows that the Strichartz norms $\|u_n(t_n-k-\,.\,)\|_{L^q([0,1],L^r(\Omega))}$ are uniformly bounded since the energy of U_n is uniformly bounded. Therefore Corollary 4.2 and Proposition 2.3 show that the terms $I_{n,k} = \int_0^1 e^{sA}F(U_n(t_n-k-s))ds$, as well as I_n, are bounded by some constant M in $H^{1+\varepsilon}(\Omega)\times H^\varepsilon(\Omega)$ uniformly in n and k. Using Proposition 2.3 again and summing up, we get that the last terms of (4-2) are bounded in $H^{1+\varepsilon}(\Omega)\times H^\varepsilon(\Omega)$ uniformly in n by

$$\left\| \sum_{k=0}^{\lfloor t_n\rfloor -1} e^{kA}I_{k,n} + I_n \right\|_{X^\varepsilon} \leq \sum_{k=0}^{\lfloor t_n\rfloor -1} Ce^{-\lambda k}M + M \leq M\left(1+\frac{C}{1-e^{-\lambda}}\right).$$

Moreover, Proposition 2.3 shows that $e^{At_n}U_n(0)$ goes to zero in X when n goes to $+\infty$. Therefore, by a diagonal extraction argument and the Rellich theorem, we can extract a subsequence $U_{\psi(n)}(t_{\psi(n)})$ that converges to $U_\infty(0)$ in $H^1_0(B)\times L^2(B)$ for all bounded set B of Ω.

To finish the proof of Proposition 4.3, we have to show that this convergence holds in fact in X and not only locally. Let $\eta > 0$ be given. Let $T > 0$ and let $\widetilde{U}_n$ be the solution of (1-1) with $\widetilde{U}_n(0) = U_n(t_n - T)$ and with γ being replaced by $\tilde{\gamma}$, where $\tilde{\gamma}(x) \equiv \gamma(x)$ for large x and $\tilde{\gamma} \geq \alpha > 0$ everywhere. By Proposition 2.6, $\|\widetilde{U}_n(T)\|_X \leq \eta$ if T is chosen sufficiently large and if n is large enough so that $t_n - T > 0$. Since the information propagates at finite speed in the wave equation, $U_n(t_n) \equiv \widetilde{U}_n(T)$ outside a large enough bounded set and thus $U_{\psi(n)}(t_{\psi(n)})$ has a X norm smaller than η outside this bounded set. On the other hand, we can assume that the norm of $U_\infty(0)$ is also smaller than η outside the bounded set. Then, choosing n large enough, $\|U_{\psi(n)}(t_{\psi(n)}) - U_\infty(0)\|_X$ becomes smaller than 3η. □

The trajectories U^∞ appearing in Proposition 4.3 are trajectories which are bounded in X for all times $t \in \mathbb{R}$. The following result shows that these special trajectories are more regular than the usual trajectories of the damped wave equation.

Proposition 4.4. *Let $f \in \mathscr{C}^1(\mathbb{R})$ satisfying (1-2) and let $E_0 \geq 0$. There exists a constant M such that if u is a solution of (1-1) that exists for all times $t \in \mathbb{R}$ and satisfies $\sup_{t\in\mathbb{R}} E(u(t)) \leq E_0$, then*

$t \mapsto U(t) = (u(t), \partial_t u(t))$ is continuous from $\mathbb{R}$ into $D(A)$ and

$$\sup_{t\in\mathbb{R}} \|(u(t), \partial_t u(t))\|_{D(A)} \le M.$$

In addition, M depends only on E_0 and the constants in (1-2).

Proof. We use a bootstrap argument. For any $t \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$U(t) = e^{nA}U(t-n) + \sum_{k=0}^{n-1} e^{kA} \int_0^1 e^{sA} F(U(t-k-s))\,ds.$$

Using Proposition 2.3, when n goes to $+\infty$, we get

$$U(t) = \sum_{k=0}^{+\infty} e^{kA} \int_0^1 e^{sA} F(U(t-k-s))\,ds. \tag{4-3}$$

Moreover, arguing exactly as in the proof of Proposition 4.3, we show that Proposition 2.3 and Corollary 4.2 imply that (4-3) also holds in X^ε. Hence, $U(t)$ is uniformly bounded in X^ε. Then, using again Proposition 2.3 and Corollary 4.2, (4-3) also holds in $X^{2\varepsilon}$, and so on. Repeating the arguments and noting that, until the last step, ε only depends on p, we obtain that $U(t)$ is uniformly bounded in $X^1 = D(A)$.

Since the constant C of Corollary 4.2 only depends on f through estimate (1-2), the same holds for the bound M here. □

Proposition 4.5. *The Sobolev embedding $H^2(\Omega) \hookrightarrow \mathscr{C}^0(\Omega)$ holds and there exists a constant $\mathcal{K}$ such that*

$$\sup_{x\in\Omega} |u(x)| \le \mathcal{K}\|u\|_{H^2} \quad \text{for all } u \in H^2(\Omega).$$

In particular, the solution u in the statement of Proposition 4.4 belongs to $\mathscr{C}^0(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and

$$\sup_{(x,t)\in\overline{\Omega}\times\mathbb{R}} |u(x,t)| \le \mathcal{K}M.$$

Proof. Proposition 4.5 follows directly from the fact that Ω has a bounded geometry and from the classical Sobolev embedding $H^2 \hookrightarrow \mathscr{C}^0$ in the ball $B(0, 1)$ of $\mathbb{R}^3$. □

5. Smoothness and uniqueness of nondissipative complete solutions

In this section, we consider only a nondissipative complete solution, that is, a solution u^* existing for all times $t \in \mathbb{R}$ for which the energy E is constant. In other words, $u^*(t)$ solves

$$\begin{cases} \partial^2_{tt} u^* = \Delta u^* - \beta u^* - f(u^*) & (x,t) \in \Omega \times \mathbb{R}, \\ u^*(x,t) = 0 & (x,t) \in \partial\Omega \times \mathbb{R}, \\ \partial_t u^*(x,t) = 0 & (x,t) \in \operatorname{supp}\gamma \times \mathbb{R}. \end{cases} \tag{5-1}$$

Since the energy E is not dissipated by $u^*(t)$, we can write $E(u^*)$ instead of $E(u^*(t))$. Yet, an interesting fact that will be used several times in the sequel is that such u^* is, at the same time, solution of both damped and undamped equations.

The purpose of this section is:

- First show that u^* is analytic in time and smooth in space. The central argument is to use a theorem of J. K. Hale and G. Raugel [2003].
- Then use the unique continuation result of L. Robbiano and C. Zuily stated in Corollary 3.2 to show that u^* is necessarily an equilibrium point of (1-1).
- Finally show that the assumption $sf(s) \geq 0$ implies that $u^* \equiv 0$.

We point out that the first two steps are valid and very helpful in a more general framework than the one of our paper.

Smoothness and partial analyticity of u^*. First we recall here the result of Section 2.2 of [Hale and Raugel 2003], adapting the statement to suit our notations:

Theorem 5.1. *Let Y be a Banach space. Let $P_n \in \mathcal{L}(Y)$ be a sequence of continuous linear maps and let $Q_n = \mathrm{Id} - P_n$. Let $A : D(A) \to Y$ be the generator of a continuous semigroup e^{tA} and let $G \in \mathscr{C}^1(Y)$. We assume that V is a complete mild solution in Y of*

$$\partial_t V(t) = AV(t) + G(V(t)) \quad \text{for all } t \in \mathbb{R}.$$

We further assume that:

(i) *$\{V(t), t \in \mathbb{R}\}$ is contained in a compact set K of Y.*

(ii) *For any $y \in Y$, $P_n y$ converges to y when n goes $+\infty$ and (P_n) and (Q_n) are sequences of $\mathcal{L}(Y)$ bounded by K_0.*

(iii) *The operator A splits as $A = A_1 + B_1$, where B_1 is bounded and A_1 commutes with P_n.*

(iv) *There exist M and $\lambda > 0$ such that $\|e^{At}\|_{\mathcal{L}(Y)} \leq Me^{-\lambda t}$ for all $t \geq 0$.*

(v) *G is analytic in the ball $B_Y(0, r)$, where r is such that $r \geq 4K_0 \sup_{t\in\mathbb{R}} \|V(t)\|_Y$. More precisely, there exists $\rho > 0$ such that G can be extended to an holomorphic function of $B_Y(0, r) + iB_Y(0, \rho)$.*

(vi) *$\{DG(V(t))V_2 \,|t \in \mathbb{R}, \|V_2\|_Y \leq 1\}$ is a relatively compact set of Y.*

Then the solution $V(t)$ is analytic from $t \in \mathbb{R}$ into Y.

More precisely, Theorem 5.1 is Theorem 2.20 (which relates to Theorem 2.12) of [Hale and Raugel 2003] applied with hypotheses (H3mod) and (H5).

Proposition 4.4 shows that u^* is continuous in both space and time variables. We apply Theorem 5.1 to show that because f is analytic, u^* is also analytic with respect to the time.

Proposition 5.2. *Let $f \in \mathscr{C}^1(\mathbb{R})$ satisfying (1-2) and let $E_0 \geq 0$. Let $\mathcal{K}$ and M be the constants given by Propositions 4.4 and 4.5. Assume that f is analytic in $[-4\mathcal{K}M, 4\mathcal{K}M]$. Then for any nondissipative complete solution $u^*(t)$ solving (5-1) and satisfying $E(u^*) \leq E_0$, $t \mapsto u^*(\cdot, t)$ is analytic from $\mathbb{R}$ into X^α with $\alpha \in (1/2, 1)$. In particular, for all $x \in \Omega$, $u^*(x, t)$ is analytic with respect to the time.*

Proof. Theorem 5.1 uses strongly some compactness properties. Therefore, we need to truncate our solution to apply the theorem on a bounded domain (of course, this is not necessary and easier if Ω is already bounded).

Let $\chi \in \mathscr{C}_0^\infty(\overline{\Omega})$ be such that $\partial\chi/\partial\nu = 0$ on $\partial\Omega$, $\chi \equiv 1$ in $\{x \in \Omega, \gamma(x) = 0\}$ and $\operatorname{supp}\chi$ is included in a smooth bounded subdomain $\mathbb{O}$ of Ω. Since Proposition 4.4 shows that $u^* \in \mathscr{C}^0(\mathbb{R}, D(A))$ and since u^* is constant with respect to the time in $\operatorname{supp}\gamma$, $(1-\chi)u^*$ is obviously analytic from $\mathbb{R}$ into $D(A)$. It remains to obtain the analyticity of χu^*.

In this proof, the damping γ needs to be more regular than just $L^\infty(\Omega)$. We replace γ by a damping $\tilde{\gamma} \in C^\infty(\Omega)$, which has the same geometrical properties (GCC) and (1-3) and which vanishes where γ does. Notice that $\gamma\partial_t u^* \equiv 0 \equiv \tilde{\gamma}\partial_t u^*$, therefore replacing γ by $\tilde{\gamma}$ has no consequences here.

Let $v = \chi u^*$, we have

$$\begin{cases} \partial_{tt}^2 v + \tilde{\gamma}(x)\partial_t v = \Delta v - \beta v + g(x, v) & (x, t) \in \mathbb{O} \times \mathbb{R}_+, \\ v(x, t) = 0 & (x, t) \in \partial\mathbb{O} \times \mathbb{R}_+, \end{cases} \tag{5-2}$$

with $g(x, v) = -\chi(x) f(v + (1-\chi)u^*(x)) - 2(\nabla\chi\, \nabla u^*)(x) - (u^*\Delta\chi)(x)$. We apply Theorem 5.1 with the following setting: Let $Y = X^\alpha = H^{1+\alpha}(\mathbb{O}) \cap H_0^1(\mathbb{O}) \times H_0^\alpha(\mathbb{O})$ with $\alpha \in (1/2, 1)$. Let $V = (v, \partial_t v)$ and let $G(v) = (0, g(., v))$. We set

$$A = A_1 + B_1 = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta - \beta & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\tilde{\gamma} \end{pmatrix}.$$

Let $(\lambda_k)_{k\geq 1}$ be the negative eigenvalues of the Laplacian operator on $\mathbb{O}$ with Dirichlet boundary conditions and let (φ_k) be corresponding eigenfunctions. We set P_n to be the canonical projections of X on the subspace generated by $((\varphi_k, 0))_{k=1,\ldots,n}$ and $((0, \varphi_k))_{k=1,\ldots,n}$.

To finish the proof of Proposition 5.2, we only have to check that the hypotheses of Theorem 5.1 hold.

The trajectory V is compact since we know by Proposition 4.4 that it is bounded in X^1, which gives (i).

Hypothesis (ii) and (iii) hold with $K_0 = 1$ by construction of P_n and because B_1 is bounded in Y since $\tilde{\gamma}$ belongs to $C^\infty(\Omega)$. Hypothesis (iv) follows from Proposition 2.3.

We recall that $u^*(x, \cdot)$ is constant outside $\chi^{-1}(1)$ and belongs locally to $H^{1+\alpha}$ since $u^* \in D(A)$. Therefore, the terms $(1-\chi)u^*(x)$, $\nabla\chi\, \nabla u^*$ and $u^*\Delta\chi$ appearing in the definition of g are in H^1. Moreover, they satisfy Dirichlet boundary condition on $\partial\Omega$ since $u^* \equiv 0$ and $\partial_\nu\chi \equiv 0$ there. Of course, they also satisfy Dirichlet boundary condition on the other parts of $\partial\mathbb{O}$ since $\chi \equiv 0$ outside $\mathbb{O}$. Notice that $\alpha > 1/2$ and thus $H^{1+\alpha}(\mathbb{O}) \cap H_0^1(\mathbb{O})$ is an algebra included in $\mathscr{C}^0$. Therefore (1-2) shows that G is of class $\mathscr{C}^1$ in the bounded sets of Y. Since $u \in [-4\mathscr{K}M, 4\mathscr{K}M] \mapsto f(u) \in \mathbb{R}$ is analytic, it can be extended to a holomorphic function in $[-4\mathscr{K}M, 4\mathscr{K}M] + i[-\rho, \rho]$ for small $\rho > 0$. Using again the embedding $H^{1+\alpha}(\mathbb{O}) \hookrightarrow \mathscr{C}^0(\mathbb{O})$ and the definitions of $\mathscr{K}$ and M, we deduce that (v) holds.

Finally, for $V_2 = (v_2, \partial_t v_2)$ with $\|V_2\|_Y \leq 1$, $DG(V(t))V_2 = (0, -\chi(x) f'(v(t) + (1-\chi)u^*(x))v_2)$ is relatively compact in Y since $v(t)$ is bounded in $H^2 \cap H_0^1$ due to Proposition 4.4 and therefore $v_2 \in H^{1+\alpha} \mapsto \chi(x) f'(v(t) + (1-\chi)u^*(x))v_2 \in H^\alpha$ is a compact map. This yields (vi). $\square$

Once the time-regularity of u^* is proved, the space-regularity follows directly.

Proposition 5.3. *Let f and u^* be as in Proposition 5.2. Then $u^* \in \mathscr{C}^\infty(\Omega \times \mathbb{R})$.*

Proof. Proposition 5.2 shows that u^* and all its time-derivatives belong to X^α with $\alpha \in (1/2, 1)$. Due to the Sobolev embeddings, this implies that any time-derivative of u^* is Hölder continuous. Writing

$$\Delta u^* = \partial_{tt}^2 u^* + \beta u^* + f(u^*) \tag{5-3}$$

and using the local elliptic regularity properties (see [Miranda 1970] and the references therein), we get that u^* is locally of class $\mathscr{C}^{2,\lambda}$ in space for some $\lambda \in (0, 1)$. Thus, u^* is of class $\mathscr{C}^{2,\lambda}$ in both time and space. Then we can use a bootstrap argument in (5-3) to show that u^* is of class $\mathscr{C}^{2k,\lambda}$ for all $k \in \mathbb{N}$. □

Identification of u^*. The smoothness and the partial analyticity of u^* shown in Propositions 5.2 and 5.3 enable us to use the unique continuation result of [Robbiano and Zuily 1998].

Proposition 5.4. *Let f and u^* be as in Proposition 5.2. Then u^* is constant in time, i.e., u^* is an equilibrium point of the damped wave equation (1-1).*

Proof. Setting $v = \partial_t u^*$, we get $\partial_{tt}^2 v = \Delta v - \beta v - f'(u^*)v$. Propositions 5.2 and 5.3 show that u^* is smooth and analytic with respect to the time and moreover $v \equiv 0$ in $\operatorname{supp} \gamma$. Thus, the unique continuation result stated in Corollary 3.2 yields $v \equiv 0$ everywhere. □

The sign assumption on f directly implies that 0 is the only possible equilibrium point of (1-1).

Corollary 5.5. *Let $f \in \mathscr{C}^1(\mathbb{R})$ satisfying (1-2) and let $E_0 \geq 0$. Let $\mathscr{K}$ and M be the constants given by Propositions 4.4 and 4.5 and assume that f is analytic in $[-4\mathscr{K}M, 4\mathscr{K}M]$. Then the unique solution u^* of (5-1) with $E(u^*) \leq E_0$ is $u^* \equiv 0$.*

Proof. Due to Proposition 5.4, u^* is solution of $\Delta u^* - \beta u^* = f(u^*)$. By multiplying by u^* and integrating by parts, we obtain $\int_\Omega |\nabla u^*|^2 + \beta |u^*|^2 \, dx = -\int_\Omega u^* f(u^*) \, dx$, which is nonpositive due to assumption (1-2). Since $\beta \geq 0$ is such that $\Delta - \beta$ is negative definite, this shows that $u^* \equiv 0$. □

6. Proof of Theorem 1.1

Due to Proposition 2.5, Theorem 1.1 directly follows from the following result.

Proposition 6.1. *Let $f \in \mathscr{C}^1(\mathbb{R})$ satisfy (1-2) and let $E_0 \geq 0$. Let $\mathscr{K}$ and M be the constants given by Propositions 4.4 and 4.5. Assume that f is analytic in $[-4\mathscr{K}M, 4\mathscr{K}M]$ and that γ is as in Theorem 1.1. Then there exist $T > 0$ and $C > 0$ such that any u solution of (1-1) with $E(u)(0) \leq E_0$ satisfies*

$$E(u)(0) \leq C \iint_{[0,T]\times\Omega} \gamma(x) \, |\partial_t u|^2 \, dt dx.$$

Proof. We argue by contradiction: we assume that there exists a sequence (u_n) of solutions of (1-1) and a sequence of times (T_n) converging to $+\infty$ such that

$$\iint_{[0,T_n]\times\Omega} \gamma(x) \, |\partial_t u_n|^2 \, dt dx \leq \frac{1}{n} E(u_n)(0) \leq \frac{1}{n} E_0. \tag{6-1}$$

Set $\alpha_n = (E(u_n)(0))^{1/2}$. Since $\alpha \in [0, \sqrt{E_0}]$, we can assume that α_n converges to a limit α when n goes to $+\infty$. We distinguish two cases: $\alpha > 0$ and $\alpha = 0$.

First case. $\alpha_n \to \alpha > 0$. Notice that, due to (2-1), $\|(u_n, \partial_t u_n)(0)\|_X$ is uniformly bounded from above and from below by positive numbers. We set $u_n^* = u_n(T_n/2 + .)$. Due to the asymptotic compactness property stated in Proposition 4.3, we can assume that u_n^* converges to a solution u^* of (1-1) in $\mathscr{C}^0([-T, T], X)$ for all time $T > 0$. We notice that

$$E(u_n(0)) \geq E(u_n^*(0)) = E(u_n(0)) - \iint_{[0,T_n/2]\times\Omega} \gamma(x)\, |\partial_t u_n|^2 \geq (1 - 1/n) E(u_n(0))$$

and thus $E(u^*(0)) = \alpha^2 > 0$. Moreover, (6-1) shows that $\gamma \partial_t u_n^*$ converges to zero in $L^2([-T, T], L^2(\Omega))$ for any $T > 0$ and thus $\partial_t u^* \equiv 0$ in $\operatorname{supp} \gamma$. In other words, u^* is a nondissipative solution of (1-1), i.e., a solution of (5-1) with $E(u^*) = \alpha^2 \leq E_0$. Corollary 5.5 shows that $u^* \equiv 0$, which contradicts the positivity of $E(u^*(0))$.

Second case. $\alpha_n \to 0$. The assumptions on f allow to write $f(s) = f'(0)s + R(s)$ with

$$|R(s)| \leq C(|s|^2 + |s|^p) \quad \text{and} \quad |R'(s)| \leq C(|s| + |s|^{p-1}). \tag{6-2}$$

Let us make the change of unknown $w_n = u_n/\alpha_n$. Then w_n solves

$$\Box w_n + \gamma(x)\partial_t w_n + (\beta + f'(0)) w_n + \frac{1}{\alpha_n} R(\alpha_n w_n) = 0 \tag{6-3}$$

and

$$\iint_{[0,T_n]\times\Omega} \gamma(x)\, |\partial_t w_n|^2 \, dt dx \leq \frac{1}{n}. \tag{6-4}$$

Set $W_n = (w_n, \partial_t w_n)$. Due to the equivalence between norm and energy given by (2-1), the scaling $w_n = u_n/\alpha_n$ implies that $\|(w_n(0), \partial_t w_n(0))\|_X$ is uniformly bounded from above and from below by positive numbers. Moreover, (6-1) implies

$$\|W_n(t)\|_X = \frac{\|(U_n(t))\|_X}{\alpha_n} \geq C \frac{E(u_n(t))^{1/2}}{\alpha_n} \geq C \frac{(E(u_n)(0) - \alpha_n^2/n)^{1/2}}{\alpha_n} \geq \frac{C}{2} > 0 \tag{6-5}$$

for any $t \in [0, T_n]$ and n large enough.

We set $f_n = 1/\alpha_n R(u_n)$ and $F_n = (0, f_n)$. The stability estimate of Theorem 2.2 implies that $\|u_n\|_{L^q([k,k+1],L^r)} \leq C\alpha_n$ uniformly for $n, k \in \mathbb{N}$. In particular, combined with (6-2), this gives

$$\|f_n\|_{L^1([k,k+1],L^2)} = \left\| \frac{1}{\alpha_n} R(\alpha_n w_n) \right\|_{L^1([k,k+1],L^2)} \leq C(\alpha_n + \alpha_n^{p-1}).$$

We can argue as in Proposition 4.3 and write

$$W_n(T_n) = e^{\widetilde{A}T_n} W_n(0) + \sum_{k=0}^{\lfloor T_n \rfloor - 1} e^{\widetilde{A}(T_n - k)} \int_0^1 e^{-\widetilde{A}s} F_n(k+s)\, ds \\ + e^{\widetilde{A}(T_n - \lfloor T_n \rfloor)} \int_0^{T_n - \lfloor T_n \rfloor} e^{-\widetilde{A}s} F_n(\lfloor T_n \rfloor + s)\, ds, \tag{6-6}$$

where $\widetilde{A}$ is the modified damped wave operator

$$\widetilde{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta - \beta - f'(0) & -\gamma \end{pmatrix}.$$

Notice that $e^{\widetilde{A}t}$ decays exponentially, like e^{At} in Proposition 2.3, since (1-2) implies $f'(0) \geq 0$. By summing up as in Proposition 4.3, we get

$$\|W_n(T_n)\|_X \leq Ce^{-\lambda T_n} + C(\alpha_n + \alpha_n^{p-1}),$$

which goes to zero, in a contradiction with (6-5). □

As a direct consequence of Proposition 6.1, we obtain a unique continuation property for nonlinear wave equations. Notice that the time of observation T required for the unique continuation is not explicit. Thus, this result is not so convenient as a unique continuation property. But it may be useful for other nonlinear stabilization problems as $\Box u + \gamma(x) g(\partial_t u) + f(u) = 0$.

Corollary 6.2. *Let $f \in \mathscr{C}^1(\mathbb{R})$ satisfy (1-2) and let $E_0 \geq 0$. Assume that f is analytic in $\mathbb{R}$ and that ω is an open subset of Ω satisfying (GCC). Then there exist $T > 0$ such that the only solution u of*

$$\begin{cases} \Box u + \beta u + f(u) = 0 & \text{on } [-T, T] \times \Omega, \\ \partial_t u \equiv 0 & \text{on } [-T, T] \times \omega, \end{cases} \tag{6-7}$$

with $E(u)(0) \leq E_0$ is $u \equiv 0$.

Proof. Corollary 6.2 is a straightforward consequence of Proposition 6.1 since we can easily construct a smooth damping γ supported in ω and such that $\operatorname{supp} \gamma$ satisfies (GCC). We only have to remark that a solution u of (6-7) is also solution of (1-1). □

7. Proof of Theorem 1.2

Before starting the proof of Theorem 1.2 itself, we prove that $\mathfrak{C}^1(\mathbb{R})$ is a Baire space, that is, that any countable intersection of open dense sets is dense. This legitimizes the genericity in $\mathfrak{C}^1(\mathbb{R})$ as a good notion of large subsets of $\mathfrak{C}^1(\mathbb{R})$. We recall that $\mathfrak{C}^1(\mathbb{R})$ is defined by (1-6) and endowed by the Whitney topology, the open sets of which are generated by the neighborhoods $\mathcal{N}_{f,\delta}$ defined by (1-7).

Proposition 7.1. *The space $\mathfrak{C}^1(\mathbb{R})$ endowed with the Whitney topology is a Baire space.*

Proof. The set $\mathfrak{C}^1(\mathbb{R})$ is not an open set of $\mathscr{C}^1(\mathbb{R})$, and neither a submanifold. It is a closed subset of $\mathscr{C}^1(\mathbb{R})$, but $\mathscr{C}^1(\mathbb{R})$ endowed with the Whitney topology is not a completely metrizable space, since it is not even metrizable (the neighborhoods of a function f are not generated by a countable subset of them). Therefore, we have to go back to the basic proof of Baire property as in [Golubitsky and Guillemin 1973].

Let $\mathcal{U}$ be an open set of $\mathfrak{C}^1(\mathbb{R})$ and let $(\mathcal{O}_n)_{n\in\mathbb{N}}$ be a sequence of open dense sets of $\mathfrak{C}^1(\mathbb{R})$. By density, there exists a function $f_0 \in \mathfrak{C}^1(\mathbb{R})$ in $\mathcal{U} \cap \mathcal{O}_0$ and by openness, there exists a positive continuous function δ_0 such that the neighborhood $\mathcal{N}_{f_0,\delta_0}$ is contained in $\mathcal{U} \cap \mathcal{O}_0$. By choosing δ_0 small enough, one can also assume that $\mathcal{N}_{f_0,2\delta_0} \subset \mathcal{U} \cap \mathcal{O}_0$ and that $\sup_{u\in\mathbb{R}} |\delta_0(u)| \leq 1/2^0$. By recursion, one constructs similar balls $\mathcal{N}_{f_n,\delta_n} \subset \mathcal{N}_{f_{n-1},\delta_{n-1}} \subset \mathcal{N}_{f_0,\delta_0}$ such that $\mathcal{N}_{f_n,2\delta_n} \subset \mathcal{U} \cap \mathcal{O}_n$ and that $\sup_{u\in\mathbb{R}} |\delta_n(u)| \leq 1/2^n$. Since $\mathscr{C}^1([-m, m], \mathbb{R})$ endowed with the uniform convergence topology is a complete metric space, the sequence (f_i) converges to a function $f \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ uniformly in any compact set of $\mathbb{R}$. By construction, the limit f satisfies

$$\max\big(|f(u) - f_n(u)|, |f'(u) - f_n'(u)|\big) \leq \delta_n(u) < 2\delta_n(u) \quad \text{for all } n \in \mathbb{N} \text{ and all } u \in \mathbb{R}, \tag{7-1}$$

as well as $f(0)=0$ and $uf(u)\geq 0$ since any f_n satisfies (1-2). Moreover, there exist $C>0$ and $p\in[1,5)$ such that f_0 satisfies

$$|f_0(u)|\leq C(1+|u|)^p \quad \text{and} \quad |f_0'(u)|\leq C(1+|u|)^{p-1}. \tag{7-2}$$

Since $\max(|f(u)-f_0(u)|,|f'(u)-f_0'(u)|)\leq \delta_0(u)\leq 1$, f also satisfies (7-2) with a constant $C'=C+1$. Therefore, f satisfies (1-2) and thus belongs to $\mathfrak{C}^1(\mathbb{R})$. In addition, f satisfying (7-1) and $\mathcal{N}_{f_n,2\delta_n}$ being contained in $\mathcal{U}\cap\mathcal{O}_n$, we get $f\in\mathcal{U}\cap\mathcal{O}_n$ for all n. This shows that $\cap_{n\in\mathbb{N}}\mathcal{O}_n$ intersects any open set $\mathcal{U}$ and therefore is dense in $\mathfrak{C}^1(\mathbb{R})$. □

Proof of Theorem 1.2. We denote by $\mathfrak{G}_n$ the set of functions $f\in\mathfrak{C}^1(\mathbb{R})$ such that the exponential decay property (ED) holds for $E_0=n$. Obviously, $\mathfrak{G}=\cap_{n\in\mathbb{N}}\mathfrak{G}_n$ and hence it is sufficient to prove that $\mathfrak{G}_n$ is an open dense subset of $\mathfrak{C}^1(\mathbb{R})$. We sketch here the main arguments to prove this last property.

<u>$\mathfrak{G}_n$ is a dense subset.</u> Let $\mathcal{N}$ be a neighborhood of $f_0\in\mathfrak{C}^1(\mathbb{R})$. Up to choosing $\mathcal{N}$ smaller, we can assume that the constant in (1-2) is independent of $f\in\mathcal{N}$. Due to Propositions 4.4 and 4.5, there exist constants $\mathcal{K}$ and M such that, for all $f\in\mathcal{N}$, all the global nondissipative trajectories u of (1-1) with $E(u)\leq n$ are such that $\|u\|_{L^\infty(\Omega\times\mathbb{R})}\leq\mathcal{K}M$. We claim that we can choose $f\in\mathcal{N}$ as close to f_0 as wanted such that f is analytic on $[-4\mathcal{K}M,4\mathcal{K}M]$ and still satisfies (1-2). Then Proposition 6.1 shows that f satisfies (ED) with $E_0=n$, i.e., that $f\in\mathfrak{G}_n$.

To obtain this suitable function f, we proceed as follows. First, we set $a=4\mathcal{K}M$ and notice that it is sufficient to explain how we construct f in $[-a,a]$. Indeed, one can easily extend a perturbation f of f_0 in $[-a,a]$ satisfying $f(s)s\geq 0$ to a perturbation $\tilde{f}$ of f_0 in $\mathbb{R}$, equal to f_0 outside of $[-a-1,a+1]$ and such that $f(s)s\geq 0$ in $[-a-1,a+1]$. We construct f in $[-a,a]$ as follows. Since $f_0(s)s\geq 0$, we have that $f_0'(0)\geq 0$. We perturb f_0 to f_1 such that $f_1(0)=0$, $f_1'(s)\geq\varepsilon>0$ in a small interval $[-\eta,\eta]$ and $sf_1(s)\geq 2\varepsilon$ in $[-a,-\eta]\cup[\eta,a]$, where ε could be chosen as small as needed. Then we perturb f_1 to obtain a function f_2 which is analytic in $[-a,a]$ and satisfies $f_2'(s)>0$ in $[-\eta,\eta]$, $sf_2(s)\geq\varepsilon$ in $[-a,-\eta]\cup[\eta,a]$ and $|f_2(0)|<\varepsilon/a$. Finally, we set $f(s)=f_2(s)-f_2(0)$ and check that f is analytic and satisfies $sf(s)\geq 0$ in $[-a,a]$. Moreover, up to choosing ε very small, f is as close to f_0 as wanted.

<u>$\mathfrak{G}_n$ is an open subset.</u> Let $f_0\in\mathfrak{G}_n$. Proposition 2.3 shows the existence of a constant C and a time T such that for all solution u of (1-1),

$$E(u(0))\leq E_0 \quad\Longrightarrow\quad E(u(0))\leq C\int_0^T\int_\Omega \gamma(x)|\partial_t u(x,t)|^2dxdt. \tag{7-3}$$

The continuity of the trajectories in X with respect to $f\in\mathfrak{C}^1(\mathbb{R})$ is not difficult to obtain: using the strong control of f given by Whitney topology, the arguments are the same as the ones of the proof of the continuity with respect to the initial data, stated in Theorem 2.2. Thus, (7-3) holds also for any f in a neighborhood $\mathcal{N}$ of f_0, replacing the constant C by a larger one. Therefore, Proposition 2.3 shows that $\mathcal{N}\subset\mathfrak{G}_n$ and hence that $\mathfrak{G}_n$ is open. □

8. A proof of compactness and regularity with the usual arguments of control theory

In this section, we give an alternative proof of the compactness and regularity properties of Propositions 4.3 and 4.4. We only give its outline since it is redundant in light of the previous results of the article. Moreover, it is quite similar to the arguments of [Dehman et al. 2003]. Yet the arguments of this section are interesting because they do not require any asymptotic arguments and they show a regularization effect through an observability estimate with a finite time T, which can be explicit. However, for the moment, it seems impossible to obtain an analytic regularity similar to Proposition 5.2 with these kind of arguments.

Instead of using a Duhamel formula with an infinite interval of time $(-\infty, t)$ as in (4-3), the main idea is to use as a black box an observability estimate for T large enough, T being the time of geometric control condition,

$$\|U_0\|^2_{X^s} \leq C\|Be^{tA}U_0\|^2_{L^2([0,T],X^s)}, \tag{8-1}$$

where

$$A = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta - \beta & -\gamma \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma \end{pmatrix}.$$

The first aim is to prove that a solution of (5-1), globally bounded in energy, is also globally bounded in X^s for $s \in [0, 1]$. We proceed step by step. First, let us show that it is bounded in X^ε.

- We fix $T >$ large enough to get the observability estimate (8-1). By the existence theory on each $[t_0, t_0 + T]$, $u_{|[t_0,t_0+T]}$ is bounded in Strichartz norms, uniformly for $t_0 \in \mathbb{R}$. Since the nonlinearity is subcritical, Corollary 4.2 gives that $f(u)$ is globally bounded in $L^1([t_0, t_0 + T], H^{1+\varepsilon})$.
- We decompose the solution into its linear and nonlinear part by the Duhamel formula,

$$U(t) = e^{A(t-t_0)}U(t_0) + \int_{t_0}^{t} e^{A(t_0-\tau)} f(U(\tau))\,d\tau = U_{lin} + U_{Nlin}.$$

Since $f(u)$ is bounded in $L^1([t_0, t_0 + T], H^{1+\varepsilon})$, U_{Nlin} is uniformly bounded in $C([t_0, t_0 + T], X^\varepsilon)$.

- We will now use the linear observability estimate (8-1) with $s = \varepsilon$, applying it to U_{lin}:

$$\|U(t_0)\|^2_{X^\varepsilon} = \|U_{lin}(t_0)\|^2_{X^\varepsilon} \leq C\int_{t_0}^{t_0+T} \|\gamma(x)\partial_t u_{lin}\|^2_{H^\varepsilon}. \tag{8-2}$$

Then, using the triangular inequality, we get

$$\begin{aligned}\int_{t_0}^{t_0+T} \|\gamma(x)\partial_t u_{lin}\|^2_{H^\varepsilon} &\leq 2\int_{t_0}^{t_0+T} \|\gamma(x)\partial_t u\|^2_{H^\varepsilon} + 2\int_{t_0}^{t_0+T} \|\gamma(x)\partial_t u_{Nlin}\|^2_{H^\varepsilon} \\ &\leq 2\int_{t_0}^{t_0+T} \|\gamma(x)\partial_t u_{Nlin}\|^2_{H^\varepsilon} \leq C,\end{aligned}$$

where we have used that $\partial_t u \equiv 0$ on ω and that U_{Nlin} is bounded in $C([t_0, t_0 + T], X^\varepsilon)$. Combining this with (8-2) for any $t_0 \in \mathbb{R}$, we obtain that U is uniformly bounded in X^ε on $\mathbb{R}$.

Repeating the arguments, we show that u is bounded in $X^{2\varepsilon}$, $X^{3\varepsilon}$ and so on, until X^1. Similar ideas allow us to prove a theorem of propagation of compactness in finite time, replacing the asymptotic compactness property of Proposition 4.3.

As said above, an advantage of this method, compared to the one used in Propositions 4.3 and 4.4, is that it allows us to propagate the regularity or the compactness on some finite interval of fixed length. Yet, it seems that such propagation results are not available in the analytic setting. Indeed, it seems that, for nonlinear equations, the propagation of analytic regularity or of nullity in finite time is much harder to prove. We can for instance refer to the weaker (with respect to the geometry) result of Alinhac and Métivier [1984] or the negative result of Métivier [1993].

9. Applications

Control of the nonlinear wave equation. In this subsection, we give a short proof of Theorem 1.3, which states the global controllability of the nonlinear wave equation. The first step consists in a local control theorem.

Theorem 9.1 (local control). *Let ω satisfying the geometric control condition for a time T. Then there exists δ such that for any (u_0, u_1) in $H_0^1(\Omega) \times L^2(\Omega)$ with*

$$\|(u_0, u_1)\|_{H_0^1 \times L^2} \leq \delta$$

there exists $g \in L^\infty([0, T], L^2)$ supported in $[0, T] \times \omega$ such that the unique strong solution of

$$\begin{cases} \Box u + \beta u + f(u) = g & \text{on} \quad [0, T] \times \Omega, \\ (u(0), \partial_t u(0)) = (u_0, u_1), \end{cases}$$

satisfies $(u(T), \partial_t u(T)) = (0, 0)$.

Proof. The proof is exactly the same as that of Theorem 3 of [Dehman et al. 2003] or Theorem 3.2 of [Laurent 2011]. The main argument consists in seeing the problem as a perturbation of the linear controllability, which is known to be true in our setting. □

Now, as is very classical, we can combine the local controllability with our stabilization theorem to get global controllability.

Sketch of the proof of Theorem 1.3. In a first step, we choose as a control $g = -\gamma(x)\partial_t \tilde{u}$, where $\tilde{u}$ is solution of (1-1) with initial data (u_0, u_1). By uniqueness of solutions, we have $u = \tilde{u}$. Therefore, thanks to Theorem 1.1, for a large time T_1, only depending on R_0, we have $\|(u(T_1), \partial_t u(T_1))\|_{H^1 \times L^2} \leq \delta$. Then Theorem 9.1 allows to find a control that brings $(u(T_1), \partial_t u(T_1))$ to 0. In other words, we have found a control g supported in ω that brings (u_0, u_1) to 0. We obtain the same result for $(\tilde{u}_0, \tilde{u}_1)$ and conclude, by reversibility of the equation, that we can also bring 0 to $(\tilde{u}_0, \tilde{u}_1)$. □

Existence of a compact global attractor. In this subsection, we give the modification of the proofs of this paper necessary to get Theorem 1.4 about the existence of a global attractor.

The energy associated to (1-9) in $X = H_0^1(\Omega) \times L^2(\Omega)$ is given by

$$E(u, v) = \int_\Omega \tfrac{1}{2}(|\nabla u|^2 + |v|^2) + V(x, u)\, dx,$$

where $V(x, u) = \int_0^u f(x, \xi)\, d\xi$.

The existence of a compact global attractor for (1-9) is well known for the Sobolev subcritical case $p < 3$. The first proofs in this case go back to 1985 [Hale 1988; Haraux 1985b]; see [Raugel 2002] for other references. The case $p = 3$ as been studied in [Babin and Vishik 1992; Arrieta et al. 1992]. For $p \in (3, 5)$, Kapitanski [1995] proved the existence of a compact global attractor for (1-9) if Ω is a compact manifold without boundary and if $\gamma(x) = \gamma$ is a constant damping. Using the same arguments as in the proof of our main result, we can partially deal with the case $p \in (3, 5)$ with a localized damping $\gamma(x)$ and with unbounded manifold with boundaries.

Assume that f satisfies the assumption of Theorem 1.4. Then the arguments of this paper show the following properties.

(i) *The positive trajectories of bounded sets are bounded.* Indeed, (1-12) implies that for $x \notin B(x_0, R)$, we have $V(x, u) = \int_0^u f(x, \xi) d\xi \geq 0$. Moreover, for $x \in B(x_0, R)$, $V(x, \cdot)$ is nonincreasing on $(-\infty, -R)$ and nondecreasing on (R, ∞). Thus, $V(x, u)$ is bounded from below for $x \in B(x_0, R)$ and

$$E(u, v) \geq \tfrac{1}{2}\|(u, v)\|_X^2 + \operatorname{vol}(B(x_0, R)) \inf V \quad \text{for all } (u, v) \in X$$

The Sobolev embeddings $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ show that the bounded sets of X have a bounded energy. Since the energy E is nonincreasing along the trajectories of (1-9), we get that the trajectory of a bounded set is bounded.

(ii) *The dynamical system is asymptotically smooth.* The asymptotic compactness exactly corresponds to the statement of Proposition 4.3. Let us briefly explain why it can be extended to the case where f depends on x. The key point is the extension of Corollary 4.2. First notice that we assumed $f(x, 0) = 0$ on $\partial\Omega$ in order to guarantee the Dirichlet boundary condition for $f(x, u)$ if $u \in H_0^1(\Omega)$. Then it is not difficult to see that the discussion following Theorem 4.1 can be extended to the case f depending on x by using estimates (1-10). Corollary 4.2 follows then, except for a small change: since it is possible that $f(x, 0) \neq 0$ for some $x \in \Omega$, the conclusion of Corollary 4.2 should be replaced by

$$\|f(x, v)\|_{L^1([0,T], H_0^{s+\varepsilon}(\Omega))} \leq C\big(1 + \|v\|_{L^\infty([0,T], H^{s+1}(\Omega) \cap H_0^1(\Omega))}\big).$$

Then the proof of Proposition 4.3 is based on Corollary 4.2, the boundedness of the positive trajectories of bounded sets (both could be extended to the case where f depends on x as noticed above) and an application of Proposition 2.6 outside of a large ball. We conclude by noticing that, for x large, $f(x, u)u \geq 0$ and $\gamma(x) \geq \alpha > 0$ and thus Proposition 2.6 can still be applied exactly as in the proof of Proposition 4.3.

(iii) *The dynamical system generated by* (1-9) *is gradient.* That is, that the energy E is nonincreasing in time and is constant on a trajectory u if and only if u is an equilibrium point of (1-9). This last property is shown in Proposition 5.4 for f independent of x but can be easily generalized for $f = f(x, u)$. Notice that the proof of this property is the one where the analyticity of f is required since the unique continuation property of Section 3 is used. Finally, we remark that the gradient structure of (1-9) is interesting from the dynamical point of view since it implies that any trajectory $u(t)$ converges when t goes to $+\infty$ to the set of equilibrium points.

(iv) *The set of equilibrium points is bounded.* The argument is similar to the one of Corollary 5.5: if e is an equilibrium point of (1-9) then (1-12) implies that

$$\int \tfrac{1}{2}|\nabla e|^2 + \beta|e|^2 = -\int_\Omega f(x,e)e\,dx \le -\operatorname{vol}(B(x_0,R))\inf\{f(x,u)u \mid (x,u)\in\overline{\Omega}\times\mathbb{R}\},$$

where we have bounded $f(x,u)u$ from below exactly as we have done for $V(x,u)$ in (i).

It is well known (see [Hale 1988] or Theorem 4.6 of [Raugel 2002]) that properties (i)–(iv) yield the existence of a compact global attractor. Hence, we obtain the conclusion of Theorem 1.4.

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