ANALYSIS & PDE

Volume 6 No. 6 2013

msp

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2013 is US $160/year for the electronic version, and $310/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS AND PDE
Vol. 6, No. 6, 2013
dx.doi.org/10.2140/apde.2013.6.1243

A NEKHOROSHEV-TYPE THEOREM FOR THE NONLINEAR SCHRÖDINGER EQUATION ON THE TORUS

ERWAN FAOU AND BENOÎT GRÉBERT

We prove a Nekhoroshev type theorem for the nonlinear Schrödinger equation

$$iu_t = -\Delta u + V \star u + \partial_{\bar{u}} g(u, \bar{u}), \quad x \in \mathbb{T}^d,$$

where V is a typical smooth Fourier multiplier and g is analytic in both variables. More precisely, we prove that if the initial datum is analytic in a strip of width $\rho > 0$ whose norm on this strip is equal to ε, then if ε is small enough, the solution of the nonlinear Schrödinger equation above remains analytic in a strip of width $\rho/2$, with norm bounded on this strip by $C\varepsilon$ over a very long time interval of order $\varepsilon^{-\sigma|\ln \varepsilon|^\beta}$, where $0 < \beta < 1$ is arbitrary and $C > 0$ and $\sigma > 0$ are positive constants depending on β and ρ.

1. Introduction and statements

We consider the nonlinear Schrödinger equation

$$iu_t = -\Delta u + V \star u + \partial_{\bar{u}} g(u, \bar{u}), \quad x \in \mathbb{T}^d, \quad t \in \mathbb{R}, \tag{1-1}$$

where V is a smooth convolution potential and g is an analytic function on a neighborhood of the origin in $\mathbb{C}^2$ which has a zero of order at least 3 at the origin and satisfies $g(z, \bar{z}) \in \mathbb{R}$. In more standard models, the convolution term is replaced by a multiplicative potential. The use of a convolution potential makes the analysis of the resonances easier.

For instance, when

$$g(u, \bar{u}) = \frac{a}{p+1}|u|^{2p+2}$$

with $a \in \mathbb{R}$ and $p \in \mathbb{N}$, we recover the standard NLS equation $iu_t = -\Delta u + V \star u + a|u|^{2p}u$. Equation (1-1) is a Hamiltonian system associated with the Hamiltonian function

$$H(u, \bar{u}) = \int_{\mathbb{T}^d} \big(|\nabla u|^2 + (V \star u)\bar{u} + g(u, \bar{u})\big)\,\mathrm{d}x$$

and the complex symplectic structure $i\,\mathrm{d}u \wedge \mathrm{d}\bar{u}$.

This equation has been considered with Hamiltonian tools in [Bambusi and Grébert 2003; Eliasson and Kuksin 2010]. The first of these papers (see also [Bambusi and Grébert 2006; Bourgain 1996] for related results) contains a Birkhoff normal form theorem adapted to this equation and discusses dynamical consequences on the long time behavior of the solutions with small initial Cauchy data in Sobolev spaces.

MSC2010: 35B40, 35Q55, 37K55.
Keywords: Nekhoroshev theorem, nonlinear Schrödinger equation, normal forms.

More precisely, it is proved that for s sufficiently large, if the Sobolev norm of index s of the initial datum u_0 is sufficiently small (of order ε), then the Sobolev norm of index s of the solution is bounded by 2ε during a very long time (of order ε^{-r} with r arbitrary). In the second paper cited, Eliasson and Kuksin obtain a KAM theorem adapted to this equation. In particular, they prove that in a neighborhood of $u = 0$, many finite-dimensional invariant tori associated with the linear part of the equation are preserved by small Hamiltonian perturbations. In other words, (1-1) has many quasiperiodic solutions. In both cases, nonresonance conditions have to be imposed on the frequencies of the linear part, and thus on the potential V (these are not exactly the same in the two different cases).

Both results are related to the stability of the zero solution, which is an elliptic equilibrium of the linear equation. The first result establishes the stability for polynomials' times with respect to the size of the (small) initial datum, while the second proves the stability for all time of certain solutions. In the present work, we extend the technique of normal forms, establishing the stability of the solutions for times of order $\varepsilon^{-\sigma|\ln\varepsilon|^\beta}$ for some constants $\sigma > 0$ and $\beta < 1$, with ε being the size of the initial datum in an analytic space.

We now state our result more precisely. We assume that for $m > d/2$, $R > 0$, V belongs to the space

$$\mathcal{W}_m = \left\{ V(x) = \sum_{a\in\mathbb{Z}^d} w_a e^{ia\cdot x} \;\middle|\; v_a := \frac{w_a(1+|a|)^m}{R} \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \text{ for any } a \in \mathbb{Z}^d \right\}, \tag{1-2}$$

which we endow with the product probability measure. Here, for $a = (a_1, \dots, a_d) \in \mathbb{Z}^d$, we set $|a|^2 = a_1^2 + \cdots + a_d^2$.

For $\rho > 0$, we denote by $\mathcal{A}_\rho \equiv \mathcal{A}_\rho(\mathbb{T}^d; \mathbb{C})$ the space of functions ϕ that are analytic on the complex neighborhood of a d-dimensional torus $\mathbb{T}^d$ given by $I_\rho = \{x + iy \mid x \in \mathbb{T}^d, y \in \mathbb{R}^d \text{ and } |y| < \rho\}$ and continuous on the closure of this strip. We then denote by $|\cdot|_\rho$ the usual norm on $\mathcal{A}_\rho$:

$$|\phi|_\rho = \sup_{z\in I_\rho} |\phi(z)|.$$

We note that $(\mathcal{A}_\rho, |\cdot|_\rho)$ is a Banach space.

Our main result is a Nekhoroshev type theorem:

Theorem 1.1. *There exists a subset $\mathcal{V} \subset \mathcal{W}_m$ of full measure, such that for $V \in \mathcal{V}$, $\beta < 1$ and $\rho > 0$, the following holds: there exist $C > 0$ and $\varepsilon_0 > 0$ such that if*

$$u_0 \in \mathcal{A}_{2\rho} \quad \textit{and} \quad |u_0|_{2\rho} = \varepsilon \le \varepsilon_0,$$

then the solution of (1-1) with initial datum u_0 exists in $\mathcal{A}_{\rho/2}$ for times $|t| \le \varepsilon^{-\sigma_\rho|\ln\varepsilon|^\beta}$ and satisfies

$$|u(t)|_{\rho/2} \le C\varepsilon \quad \textit{for } |t| \le \varepsilon^{-\sigma_\rho|\ln\varepsilon|^\beta}, \tag{1-3}$$

with $\sigma_\rho = \min\{\frac{1}{10}, \frac{1}{2}\rho\}$. Furthermore, writing $u(t) = \sum_{k\in\mathbb{Z}^d} \xi_k(t)e^{ik\cdot x}$, we have

$$\sum_{k\in\mathbb{Z}^d} e^{\rho|k|}\big||\xi_k(t)| - |\xi_k(0)|\big| \le \varepsilon^{3/2} \quad \textit{for } |t| \le \varepsilon^{-\sigma_\rho|\ln\varepsilon|^\beta}. \tag{1-4}$$

Estimate (1-4) asserts that there is almost no variation of the actions[1].

In finite dimension n, the standard Nekhoroshev result [1977] controls the dynamic over times of order $\exp(\sigma/\varepsilon^{1/(\tau+1)})$ for some $\sigma > 0$ and $\tau > n + 1$ (see, for instance, [Benettin et al. 1985; Giorgilli and Galgani 1985; Pöschel 1993]), which is of course much better than $\varepsilon^{-\sigma|\ln\varepsilon|^{\beta}} = e^{\sigma|\ln\varepsilon|^{(1+\beta)}}$. Nevertheless, this standard result does not extend to the infinite-dimensional context. Actually, that the term $\varepsilon^{-1/(\tau+1)}$ in the exponential validity time can be replaced by $|\ln\varepsilon|^{(1+\beta)}$ at the limit $n \to \infty$ is good news!

To our knowledge, the only previous works in the direction of obtaining Nekhoroshev estimates for PDEs were obtained by Bambusi [1999a; 1999b]. However, the result in [Bambusi 1999a], which develops ideas expressed by Bourgain [1996], concerns a smaller set of functions made of entire analytic functions only, and nevertheless yields a weaker control on a large but finite number of modes.

The five main differences with the previous works on normal forms are:

- In the finite-dimensional case and in Bambusi's work, the central argument consists in optimizing the order of the Birkhoff normal form with respect to the size of the initial datum. Here we introduce a Fourier truncation and we optimize the order of the Birkhoff normal form *and* the order of the truncation.
- We prove in the Appendix that, generically with respect to V, the spectrum of $-\Delta + V\star$ satisfies a nonresonance condition much more efficient than the standard one (see Remark 2.7).
- We use ℓ^1-type norms to control the Fourier coefficients and the vector fields instead of the usual ℓ^2-type norms. Of course this choice does not allow us to work in Hilbert spaces and induces a slight loss of regularity each time the estimates are transposed from the Fourier space to the initial space of analytic functions. But it turns out that this choice simplifies the estimates on the vector fields (see Proposition 2.5 below and [Faou and Grébert 2011] for a similar framework in the context of numerical analysis).
- We use the zero momentum condition: in the Fourier space, the nonlinear term contains only monomials $z_{j_1} \dots z_{j_k}$ with $j_1 + \dots + j_k = 0$ (see Definition 2.4). This property allows us to control the largest index by the others.
- We notice that the Hamiltonian vector field of a monomial $z_{j_1} \dots z_{j_k}$ containing at least three Fourier modes z_ℓ with large indices ℓ induces a flow whose dynamics is controlled during a very long time in the sense that the dynamic almost excludes exchanges between high Fourier modes and low Fourier modes (see Proposition 2.11). In [Bambusi 2003; Bambusi and Grébert 2006], such terms were neglected since the vector field of a monomial containing at least three Fourier modes with large indices is small in *Sobolev norm* (but not in analytic norm), and thus will almost keep all the modes invariant. This more subtle analysis was also used in [Faou et al. 2010].

Our method could be generalized by considering not only zero momentum monomials but also monomials with finite or exponentially decreasing momentum. This would certainly allow us to consider a nonlinear Schrödinger equation with a multiplicative potential V and nonlinearities depending periodically

[1] Here the actions are the square of the modulus of the Fourier coefficients, $I_k = |\xi_k|^2$.

on x:

$$iu_t = -\Delta u + Vu + \partial_{\bar u} g(x,u,\bar u), \quad x \in \mathbb{T}^d.$$

Nevertheless, this generalization would generate a lot of technicalities and we prefer to focus in the present article on the simplicity of the arguments.

2. Setting and hypothesis

2A. ***Hamiltonian formalism.*** Equation (1-1) is a semilinear PDE locally well posed in the Sobolev space $H^s(\mathbb{T}^d)$ with $s > d/2$ (see, for instance, [Cazenave 2003]). Let u be a (local) solution of (1-1) and consider $(\xi,\eta) = (\xi_a,\eta_a)_{a\in\mathbb{Z}^d}$ the Fourier coefficients of u, $\bar u$

$$u(x) = \sum_{a\in\mathbb{Z}^d} \xi_a e^{ia\cdot x} \quad \text{and} \quad \bar u(x) = \sum_{a\in\mathbb{Z}^d} \eta_a e^{-ia\cdot x}. \tag{2-1}$$

A standard calculation shows that u is a solution in $H^s(\mathbb{T}^d)$ of (1-1) if and only if (ξ,η) is a solution in[2] $\ell^2_s \times \ell^2_s$ of the system

$$\begin{cases} \dot\xi_a = -i\omega_a\xi_a - i\dfrac{\partial P}{\partial \eta_a}, & a\in\mathbb{Z}^d, \\ \dot\eta_a = i\omega_a\eta_a - i\dfrac{\partial P}{\partial \xi_a}, & a\in\mathbb{Z}^d, \end{cases} \tag{2-2}$$

where the linear frequencies are given by $\omega_a = |a|^2 + v_a$. As in (1-2), the notation is $V = \sum v_a e^{ia\cdot x}$. The nonlinear part is given by

$$P(\xi,\eta) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} g\Big(\sum \xi_a e^{ia\cdot x}, \sum \eta_a e^{-ia\cdot x}\Big)\,\mathrm{d}x. \tag{2-3}$$

This system is Hamiltonian when endowing the set of pairs $(\xi_a,\eta_a)\in\mathbb{C}^{\mathbb{Z}^d}\times\mathbb{C}^{\mathbb{Z}^d}$ with the symplectic structure

$$i\sum_{a\in\mathbb{Z}^d} \mathrm{d}\xi_a\wedge\mathrm{d}\eta_a. \tag{2-4}$$

We define the set $\mathcal{Z} = \mathbb{Z}^d\times\{\pm1\}$. For $j=(a,\delta)\in\mathcal{Z}$, we define $|j| = |a|$ and we denote by $\bar j$ the index $(a,-\delta)$.

We identify a pair $(\xi,\eta)\in\mathbb{C}^{\mathbb{Z}^d}\times\mathbb{C}^{\mathbb{Z}^d}$ with $(z_j)_{j\in\mathcal{Z}}\in\mathbb{C}^{\mathcal{Z}}$ via the formula

$$j=(a,\delta)\in\mathcal{Z} \implies \begin{cases} z_j = \xi_a & \text{if } \delta = 1, \\ z_j = \eta_a & \text{if } \delta = -1. \end{cases} \tag{2-5}$$

By a slight abuse of notation, we often write $z = (\xi,\eta)$ to denote such an element.

For a given $\rho > 0$, we consider the Banach space $\mathcal{L}_\rho$ made of elements $z\in\mathbb{C}^{\mathcal{Z}}$ such that

$$\|z\|_\rho := \sum_{j\in\mathcal{Z}} e^{\rho|j|}|z_j| < \infty,$$

[2] As usual, $\ell^2_s = \big\{(\xi_a)_{a\in\mathbb{Z}^d} \mid \sum(1+|a|^{2s})|\xi_a|^2 < +\infty\big\}$.

using the symplectic form (2-4). We say that $z \in \mathcal{L}_\rho$ is *real* when $z_{\bar{j}} = \bar{z}_j$ for any $j \in \mathcal{Z}$. In this case, we write $z = (\xi, \bar{\xi})$ for some $\xi \in \mathbb{C}^{\mathbb{Z}^d}$. In this situation, we can associate with z the function u defined by (2-1).

The next lemma shows the relation with the space $\mathcal{A}_\rho$ defined above:

Lemma 2.1. *Let u be a complex valued function analytic on a neighborhood of $\mathbb{T}^d$, and let $(z_j)_{j\in\mathcal{Z}}$ be the sequence of its Fourier coefficients defined by (2-1) and (2-5). Then for all $\mu < \rho$, we have*

$$\text{if } u \in \mathcal{A}_\rho, \quad \text{then } z \in \mathcal{L}_\mu \quad \text{and} \quad \|z\|_\mu \le c_{\rho,\mu} |u|_\rho, \tag{2-6}$$

$$\text{if } z \in \mathcal{L}_\rho, \quad \text{then } u \in \mathcal{A}_\mu \quad \text{and} \quad |u|_\mu \le c_{\rho,\mu} \|z\|_\rho, \tag{2-7}$$

where $c_{\rho,\mu}$ is a constant depending on ρ and μ and the dimension d.

Proof. Assume that $u \in \mathcal{A}_\rho$. Then by using the Cauchy formula, we get $|z_j| \le |u|_\rho e^{-\rho|j|}$ for all $j \in \mathcal{Z}$. Hence, for $\mu < \rho$ we have

$$\|z\|_\mu \le |u|_\rho \sum_{j\in\mathcal{Z}} e^{(\mu-\rho)|j|} \le |u|_\rho \left(2 \sum_{n\in\mathbb{Z}} e^{\frac{(\mu-\rho)}{\sqrt{d}}|n|}\right)^d \le \left(\frac{2}{1 - e^{\frac{(\mu-\rho)}{\sqrt{d}}}}\right)^d |u|_\rho.$$

Conversely, assume that $z \in \mathcal{L}_\rho$. Then $|\xi_a| \le \|z\|_\rho e^{-\rho|a|}$ for all $a \in \mathbb{Z}^d$, and thus by (2-1), for all $x \in \mathbb{T}^d$ and $y \in \mathbb{R}^d$ with $|y| \le \mu$, we get

$$|u(x+iy)| \le \sum_{a\in\mathbb{Z}^d} |\xi_a| e^{|ay|} \le \|z\|_\rho \sum_{a\in\mathbb{Z}^d} e^{-(\rho-\mu)|a|} \le \left(\frac{2}{1 - e^{\frac{(\mu-\rho)}{\sqrt{d}}}}\right)^d \|z\|_\rho.$$

Hence, u is bounded on the strip I_μ. □

For a function F of $\mathscr{C}^1(\mathcal{L}_\rho, \mathbb{C})$, we define its Hamiltonian vector field by $X_F = J\nabla F$, where J is the symplectic operator on $\mathcal{L}_\rho$ induced by the symplectic form (2-4), $\nabla F(z) = (\partial F/\partial z_j)_{j\in\mathcal{Z}}$, and where by definition, for $j = (a, \delta) \in \mathbb{Z}^d \times \{\pm 1\}$ we set

$$\frac{\partial F}{\partial z_j} = \begin{cases} \dfrac{\partial F}{\partial \xi_a} & \text{if } \delta = 1, \\[2ex] \dfrac{\partial F}{\partial \eta_a} & \text{if } \delta = -1. \end{cases}$$

For two functions F and G, the Poisson bracket is (formally) defined as

$$\{F, G\} = \nabla F^T J \nabla G = i \sum_{a\in\mathbb{Z}^d} \frac{\partial F}{\partial \eta_a}\frac{\partial G}{\partial \xi_a} - \frac{\partial F}{\partial \xi_a}\frac{\partial G}{\partial \eta_a}. \tag{2-8}$$

We say that a Hamiltonian function H is *real* if $H(z)$ is real for all real z.

Definition 2.2. For a given $\rho > 0$, we denote by $\mathcal{H}_\rho$ the space of real Hamiltonians P satisfying

$$P \in \mathscr{C}^1(\mathcal{L}_\rho, \mathbb{C}) \quad \text{and} \quad X_P \in \mathscr{C}^1(\mathcal{L}_\rho, \mathcal{L}_\rho).$$

For F and G in $\mathcal{H}_\rho$, the formula (2-8) is well defined. With a given Hamiltonian function $H \in \mathcal{H}_\rho$, we associate the Hamiltonian system

$$\dot{z} = X_H(z) = J\nabla H(z),$$

which also reads

$$\dot{\xi}_a = -i\frac{\partial H}{\partial \eta_a} \quad \text{and} \quad \dot{\eta}_a = i\frac{\partial H}{\partial \xi_a}, \quad a \in \mathbb{Z}^d. \tag{2-9}$$

We define the local flow $\Phi^t_H(z)$ associated with the previous system (for an interval of times $t \geq 0$ depending a priori on the initial condition z). If $z = (\xi, \bar{\xi})$ and if H is real, the flow $(\xi^t, \eta^t) = \Phi^t_H(z)$ is also real; $\xi^t = \bar{\eta}^t$ for all t. Choosing the Hamiltonian given by

$$H(\xi, \eta) = \sum_{a\in\mathbb{Z}^d} \omega_a \xi_a \eta_a + P(\xi, \eta),$$

P being given by (2-3), we recover the system (2-2), that is, the expression of the NLS equation (1-1) in Fourier modes.

Remark 2.3. The quadratic Hamiltonian $H_0 = \sum_{a\in\mathbb{Z}^d} \omega_a \xi_a \eta_a$ corresponding to the linear part of (1-1) does not belong to $\mathcal{H}_\rho$. Nevertheless, it generates a flow which maps $\mathcal{L}_\rho$ into $\mathcal{L}_\rho$ explicitly given for all time t and for all indices a by $\xi_a(t) = e^{-i\omega_a t}\xi_k(0)$, $\eta_a(t) = e^{i\omega_a t}\eta_k(0)$. On the other hand, we will see that, in our setting, the nonlinearity P belongs to $\mathcal{H}_\rho$.

2B. ***Space of polynomials.*** In this subsection we define a class of polynomials on $\mathbb{C}^{\mathcal{Z}}$.

We first need more notations concerning multi-indices: letting $\ell \geq 2$ and $\boldsymbol{j} = (j_1, \dots, j_\ell) \in \mathcal{Z}^\ell$ with $j_i = (a_i, \delta_i)$, we define

- the monomial associated with $\boldsymbol{j}$
$$z_{\boldsymbol{j}} = z_{j_1} \dots z_{j_\ell};$$
- the momentum of $\boldsymbol{j}$
$$\mathcal{M}(\boldsymbol{j}) = a_1\delta_1 + \dots + a_\ell\delta_\ell, ; \tag{2-10}$$
- and the divisor associated with $\boldsymbol{j}$
$$\Omega(\boldsymbol{j}) = \delta_1\omega_{a_1} + \dots + \delta_\ell\omega_{a_\ell}, \tag{2-11}$$
where for $a \in \mathbb{Z}^d$, $\omega_a = |a|^2 + v_a$ are the frequencies of the linear part of (1-1).

We then define the set of indices with *zero momentum* by

$$\mathcal{I}_\ell = \{\boldsymbol{j} = (j_1, \dots, j_\ell) \in \mathcal{Z}^\ell \mid \mathcal{M}(\boldsymbol{j}) = 0\}. \tag{2-12}$$

On the other hand, we say that $\boldsymbol{j} = (j_1, \dots, j_\ell) \in \mathcal{Z}^\ell$ is *resonant*, and we write $\boldsymbol{j} \in \mathcal{N}_\ell$, if ℓ is even and $\boldsymbol{j} = \boldsymbol{i} \cup \bar{\boldsymbol{i}}$ for some choice of $\boldsymbol{i} \in \mathcal{Z}^{\ell/2}$. In particular, if $\boldsymbol{j}$ is resonant, then its associated divisor vanishes, $\Omega(\boldsymbol{j}) = 0$, and its associated monomials depend only on the actions

$$z_{\boldsymbol{j}} = z_{j_1} \dots z_{j_\ell} = \xi_{a_1}\eta_{a_1} \dots \xi_{a_{\ell/2}}\eta_{a_{\ell/2}} = I_{a_1} \dots I_{a_{\ell/2}},$$

where $I_a(z) = \xi_a \eta_a$ denotes the action associated with the index a for all $a \in \mathbb{Z}^d$.

Finally, if z is real, then $I_a(z) = |\xi_a|^2$, and for odd r, the resonant set $\mathcal{N}_r$ is empty.

Definition 2.4. For $k \geq 2$, a (formal) polynomial $P(z) = \sum a_{\boldsymbol{j}} z_{\boldsymbol{j}}$ belongs to $\mathcal{P}_k$ if P is real, of degree k, has a zero of order at least 2 in $z = 0$, and satisfies the following conditions:

- P contains only monomials having zero momentum (i.e., such that $\mathcal{M}(\boldsymbol{j}) = 0$ when $a_{\boldsymbol{j}} \neq 0$), and thus P reads

$$P(z) = \sum_{\ell=2}^{k} \sum_{\boldsymbol{j} \in \mathcal{I}_\ell} a_{\boldsymbol{j}} z_{\boldsymbol{j}} \tag{2-13}$$

  with the relation $a_{\bar{\boldsymbol{j}}} = \bar{a}_{\boldsymbol{j}}$.

- The coefficients $a_{\boldsymbol{j}}$ are bounded: $\sup_{\boldsymbol{j} \in \mathcal{I}_\ell} |a_{\boldsymbol{j}}| < +\infty$ for all $\ell = 2, \dots, k$.

We endow $\mathcal{P}_k$ with the norm

$$\|P\| = \sum_{\ell=2}^{k} \sup_{\boldsymbol{j} \in \mathcal{I}_\ell} |a_{\boldsymbol{j}}|. \tag{2-14}$$

The zero momentum assumption in Definition 2.4 is crucial to obtaining the following proposition:

Proposition 2.5. *Let $k \geq 2$ and $\rho > 0$. We have $\mathcal{P}_k \subset \mathcal{H}_\rho$, and for P a homogeneous polynomial of degree k in $\mathcal{P}_k$, we have the estimates*

$$|P(z)| \leq \|P\| \, \|z\|_\rho^k \tag{2-15}$$

and

$$\|X_P(z)\|_\rho \leq 2k \|P\| \, \|z\|_\rho^{k-1} \quad \text{for all } z \in \mathcal{L}_\rho. \tag{2-16}$$

Furthermore, for $P \in \mathcal{P}_k$ and $Q \in \mathcal{P}_\ell$, we have $\{P, Q\} \in \mathcal{P}_{k+\ell-2}$ and the estimate

$$\|\{P, Q\}\| \leq 2k\ell \|P\| \|Q\|. \tag{2-17}$$

Proof. Let

$$P(z) = \sum_{\boldsymbol{j} \in \mathcal{I}_k} a_{\boldsymbol{j}} z_{\boldsymbol{j}};$$

we have

$$|P(z)| \leq \|P\| \sum_{\boldsymbol{j} \in \mathcal{Z}^k} |z_{j_1}| \dots |z_{j_k}| \leq \|P\| \, \|z\|_{\ell^1}^k \leq \|P\| \|z\|_\rho^k,$$

and the first inequality (2-15) is proved.

To prove the second estimate, let $\ell \in \mathcal{Z}$; by using the zero momentum condition, we get

$$\left| \frac{\partial P}{\partial z_\ell} \right| \leq k \|P\| \sum_{\substack{\boldsymbol{j} \in \mathcal{Z}^{k-1} \\ \mathcal{M}(\boldsymbol{j}) = -\mathcal{M}(\ell)}} |z_{j_1} \dots z_{j_{k-1}}|.$$

Therefore

$$\|X_P(z)\|_\rho = \sum_{\ell\in\mathcal{Z}} e^{\rho|\ell|}\left|\frac{\partial P}{\partial z_\ell}\right| \le k\|P\| \sum_{\ell\in\mathcal{Z}} \sum_{\substack{\boldsymbol{j}\in\mathcal{Z}^{k-1}\\ \mathcal{M}(\boldsymbol{j})=-\mathcal{M}(\ell)}} e^{\rho|\ell|}|z_{j_1}\dots z_{j_{k-1}}|.$$

But if $\mathcal{M}(\boldsymbol{j}) = -\mathcal{M}(\ell)$, then

$$e^{\rho|\ell|} \le \exp\big(\rho(|j_1|+\dots+|j_{k-1}|)\big) \le \prod_{n=1,\dots,k-1} e^{\rho|j_n|}.$$

Hence, after summing in ℓ, we get[3]

$$\|X_P(z)\|_\rho \le 2k\|P\| \sum_{\boldsymbol{j}\in\mathcal{Z}^{k-1}} e^{\rho|j_1|}|z_{j_1}|\dots e^{\rho|j_{k-1}|}|z_{j_{k-1}}| \le 2k\|P\|\,\|z\|_\rho^{k-1},$$

which yields (2-16).

Assume now that P and Q are homogeneous polynomials of degrees k and ℓ respectively and with coefficients $a_{\boldsymbol{k}}$, $\boldsymbol{k}\in\mathcal{I}_k$ and $b_{\boldsymbol{\ell}}$, $\boldsymbol{\ell}\in\mathcal{I}_\ell$. It is clear that $\{P,Q\}$ is a monomial of degree $k+\ell-2$ satisfying the zero momentum condition. Furthermore, we can write

$$\{P,Q\}(z) = \sum_{\boldsymbol{j}\in\mathcal{I}_{k+\ell-2}} c_{\boldsymbol{j}}\, z_{\boldsymbol{j}},$$

where $c_{\boldsymbol{j}}$ is expressed as a sum of coefficients $a_{\boldsymbol{k}} b_{\boldsymbol{\ell}}$ for which there exists an $a\in\mathbb{Z}^d$ and $\epsilon\in\{\pm1\}$ such that

$$(a,\epsilon)\subset \boldsymbol{k}\in\mathcal{I}_k \quad\text{and}\quad (a,-\epsilon)\subset\boldsymbol{\ell}\in\mathcal{I}_\ell,$$

and such that if for instance $(a,\epsilon)=k_1$ and $(a,-\epsilon)=\ell_1$, we necessarily have $(k_2,\dots,k_k,\ell_2,\dots,\ell_\ell)=\boldsymbol{j}$. Hence, for a given $\boldsymbol{j}$, the zero momentum condition on $\boldsymbol{k}$ and on $\boldsymbol{\ell}$ determines the value of ϵa, which in turn determines two possible values of (ϵ,a).

This proves (2-17) for monomials. The extension to polynomials follows from the definition of the norm (2-14).

The last assertion and the fact that the Poisson bracket of two real Hamiltonian is real follow immediately from the definitions. □

2C. *Nonlinearity.* We assume that the nonlinearity g is analytic in a neighborhood of the origin in $\mathbb{C}^2$: There exist positive constants M and R_0 such that the Taylor expansion

$$g(v_1,v_2) = \sum_{k_1,k_2\ge0} \frac{1}{k_1!k_2!}\partial_{k_1}\partial_{k_2} g(0,0) v_1^{k_1} v_2^{k_2}$$

is uniformly convergent and bounded by M on the ball $|v_1|+|v_2|\le 2R_0$. Hence, formula (2-3) defines an analytic function P on the ball $\|z\|_\rho\le R_0$ in $\mathcal{L}_\rho$, and we have

$$P(z) = \sum_{k\ge0} P_k(z),$$

[3] Note that $\mathcal{M}(a,\delta) = \mathcal{M}(-a,-\delta)$, whence we get the coefficient 2.

where P_k for all $k \geq 0$ is a homogeneous polynomial given by

$$P_k = \sum_{k_1+k_2=k} \sum_{(\boldsymbol{a},\boldsymbol{b}) \in (\mathbb{Z}^d)^{k_1} \times (\mathbb{Z}^d)^{k_2}} p_{\boldsymbol{a},\boldsymbol{b}} \xi_{a_1} \dots \xi_{a_{k_1}} \eta_{b_1} \dots \eta_{b_{k_2}},$$

with

$$p_{\boldsymbol{a},\boldsymbol{b}} = \frac{1}{k_1!k_2!} \partial_{k_1} \partial_{k_2} g(0,0) \int_{\mathbb{T}^d} e^{i\mathcal{M}(\boldsymbol{a},\boldsymbol{b})\cdot x}\, \mathrm{d}x$$

and $\mathcal{M}(\boldsymbol{a},\boldsymbol{b}) = a_1 + \dots + a_{k_1} - b_1 - \dots - b_{k_2}$ the moment of $\xi_{a_1} \dots \xi_{a_{k_1}} \eta_{b_1} \dots \eta_{b_{k_2}}$. Therefore, it is clear that P_k satisfies the zero momentum condition, and thus $P_k \in \mathcal{P}_k$ for all $k \geq 0$. Furthermore, we have the estimate $\|P_k\| \leq M R_0^{-k}$ for all $k \geq 0$.

2D. *Nonresonance condition.* In order to control the divisors (2-11), we need to impose a nonresonance condition on the linear frequencies ω_a, $a \in \mathbb{Z}^d$.

For $r \geq 3$ and $\boldsymbol{j} = (j_1, \dots, j_r) \in \mathfrak{X}^r$, we define $\mu(\boldsymbol{j})$ as the third largest integer amongst $|j_1|, \dots, |j_r|$. We recall that the resonant set $\mathcal{N}_r$ is the set of multi-indices $\boldsymbol{j} \in \mathfrak{X}^r$ such that $\boldsymbol{j} = \boldsymbol{i} \cup \bar{\boldsymbol{i}}$ for some $\boldsymbol{i} \in \mathfrak{X}^{r/2}$.

Hypothesis 2.6. There exist $\gamma > 0$, $\nu \geq 1$ and $c_0 > 0$ such that for all $r \geq 3$ and for all nonresonant $\boldsymbol{j} \in \mathfrak{X}^r \backslash \mathcal{N}_r$, we have

$$|\Omega(\boldsymbol{j})| \geq \frac{\gamma c_0^r}{\mu(\boldsymbol{j})^{\nu r}}. \tag{2-18}$$

Remark 2.7. Classically, a nonresonance condition reads (see, for instance, [Bambusi and Grébert 2006]): for all $r \geq 3$, there exist $\gamma(r) > 0$ and $\nu(r) > 0$ such that for all nonresonant $\boldsymbol{j} \in \mathfrak{X}^r$, we have

$$|\Omega(\boldsymbol{j})| \geq \frac{\gamma(r)}{\mu(\boldsymbol{j})^{\nu(r)}}.$$

In Hypothesis 2.6, we make precise the dependence of γ and ν with respect to r. In particular, we impose that ν be linear: $\nu(r) = \nu r$. This is crucial to optimizing the choice of r as a function of ε in Section 3B.

Recall that for $V = \sum_{a \in \mathbb{Z}^d} w_a e^{ia\cdot x}$ in the space $\mathcal{W}_m$ defined in (1-2), the frequencies are

$$\omega_a = |a|^2 + w_a = |a|^2 + \frac{R v_a}{(1+|a|)^m}, \quad a \in \mathbb{Z}^d,$$

with $v_a \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ for all a. In the Appendix, we prove:

Proposition 2.8. *Fix $\gamma > 0$ small enough and $m > d/2$. There exist positive constants c_0 and ν depending only on m, R and d, and a set $F_\gamma \subset \mathcal{W}_m$ whose measure is larger than $1 - 4\gamma^{1/7}$, such that if $V \in F_\gamma$, then (2-18) holds true for all nonresonant $\boldsymbol{j} \in \mathfrak{X}^r$ and for all $r \geq 3$.*

Thus Hypothesis 2.6 is satisfied for all $V \in \mathcal{V}$, where

$$\mathcal{V} = \bigcup_{\gamma > 0} F_\gamma \tag{2-19}$$

is a subset of full measure in $\mathcal{W}_m$.

2E. *Normal forms.* We fix an index $N \geq 1$. For a fixed integer $k \geq 3$, we set

$$\mathcal{J}_k(N) = \{\boldsymbol{j} \in \mathcal{I}_k \mid \mu(\boldsymbol{j}) > N\}.$$

Definition 2.9. Let N be an integer. We say that a polynomial $Z \in \mathcal{P}_k$ is in N-normal form if it can be written

$$Z = \sum_{\ell=3}^{k} \sum_{\boldsymbol{j} \in \mathcal{N}_\ell \cup \mathcal{J}_\ell(N)} a_{\boldsymbol{j}} z_{\boldsymbol{j}}.$$

In other words, Z contains either monomials depending only on the actions or monomials whose indices $\boldsymbol{j}$ satisfy $\mu(\boldsymbol{j}) > N$, that is, monomials involving at least three modes with index greater than N.

We now motivate the introduction of this definition. First, we recall:

Lemma 2.10. *Let $f : \mathbb{R} \to \mathbb{R}_+$ be a continuous function and $y : \mathbb{R} \to \mathbb{R}_+$ a differentiable function satisfying the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) \leq 2f(t)\sqrt{y(t)} \quad \textit{for all } t \in \mathbb{R}.$$

Then we have the estimate

$$\sqrt{y(t)} \leq \sqrt{y(0)} + \int_0^t f(s)\,\mathrm{d}s \quad \textit{for all } t \in \mathbb{R}.$$

Proof. Let $\epsilon > 0$ and define $y_\epsilon = y + \epsilon$, a nonnegative function whose square root is differentiable. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{y_\epsilon(t)} \leq 2f(t)\frac{\sqrt{y(t)}}{\sqrt{y_\epsilon(t)}} \leq 2f(t),$$

and thus

$$\sqrt{y_\varepsilon(t)} \leq \sqrt{y_\epsilon(0)} + \int_0^t f(s)\,\mathrm{d}s.$$

The claim is proved by taking $\epsilon \to 0$. ☐

For a given number N and for $z \in \mathcal{L}_\rho$, we define

$$\mathsf{R}_\rho^N(z) = \sum_{|j|>N} e^{\rho|j|}|z_j|.$$

Notice that if $z \in \mathcal{L}_{\rho+\mu}$, then

$$\mathsf{R}_\rho^N(z) \leq e^{-\mu N}\|z\|_{\rho+\mu}. \tag{2-20}$$

Proposition 2.11. *Let $N \in \mathbb{N}$ and $k \geq 3$. Suppose that Z is a homogeneous polynomial of degree k in N-normal form. Let $z(t)$ be a real solution of the flow generated by the Hamiltonian $H_0 + Z$. Then we have*

$$\mathsf{R}_\rho^N(z(t)) \leq \mathsf{R}_\rho^N(z(0)) + 4k^3\|Z\| \int_0^t \mathsf{R}_\rho^N(z(s))^2 \|z(s)\|_\rho^{k-3}\,\mathrm{d}s \tag{2-21}$$

and

$$\|z(t)\|_\rho \leq \|z(0)\|_\rho + 4k^3\|Z\| \int_0^t \mathsf{R}_\rho^N(z(s))^2 \|z(s)\|_\rho^{k-3}\,\mathrm{d}s. \tag{2-22}$$

Proof. Fix $a \in \mathbb{Z}^d$ and let $I_a(t) = \xi_a(t)\eta_a(t)$ be the actions associated with the solution of the Hamiltonian system generated by $H_0 + Z$. Let us recall that as $z(t) = (\xi(t), \eta(t))$ is a real solution, we have $\xi_a(t) = \bar{\eta}_a(t)$ for all times where the solution is defined. Using (2-17) and $H_0 = H_0(I)$, we have

$$|e^{2\rho|a|}\dot{I}_a| = |e^{2\rho|a|}\{I_a, Z\}| \le 2k\|Z\|\,\big|e^{\rho|a|}\sqrt{I_a}\big|\Bigg(\sum_{\substack{\mathcal{M}(\boldsymbol{j})=\pm a\\ 2\text{ indices}>N}} e^{\rho|a|}|z_{j_1}\dots z_{j_{k-1}}|\Bigg).$$

Then using Lemma 2.10, we get

$$e^{\rho|a|}\sqrt{I_a(t)} \le e^{\rho|a|}\sqrt{I_a(0)} + 2k\|Z\|\int_0^t \Bigg(\sum_{\substack{\mathcal{M}(\boldsymbol{j})=\pm a\\ 2\text{ indices}>N}} e^{\rho|j_1|}|z_{j_1}|\dots e^{\rho|j_{k-1}|}|z_{j_{k-1}}|\Bigg)\,\mathrm{d}s. \tag{2-23}$$

Ordering the multi-indices such that $|j_1|$ and $|j_2|$ are the largest, and using the fact that $z(t)$ is real (and thus $|z_j| = \sqrt{I_a}$ for $j = (a, \pm 1) \in \mathcal{Z}$), we obtain, after summation in $|a| > N$,

$$\begin{aligned}\mathsf{R}^N_\rho(z(t)) &\le \mathsf{R}^N_\rho(z(0)) + 4k^3\|Z\|\int_0^t\Bigg(\sum_{\substack{|j_1|,|j_2|\ge N\\ j_3,\dots,j_{k-1}\in\mathcal{Z}}} e^{\rho|j_1|}|z_{j_1}|\dots e^{\rho|j_{k-1}|}|z_{j_{k-1}}|\Bigg)\,\mathrm{d}s\\ &\le \mathsf{R}^N_\rho(z(0)) + 4k^3\|Z\|\int_0^t \mathsf{R}^N_\rho(z(s))^2\|z(s)\|_\rho^{k-3}\,\mathrm{d}s.\end{aligned}$$

Inequality (2-22) is proved in the same way. □

Remark 2.12. These estimates will be central to the final bootstrap argument. Actually, as a consequence of Proposition 2.11, we have: if $z(t)$ is the solution of a Hamiltonian system in N-normal form with an initial datum z_0 satisfying $\|z_0\|_{2\rho} = \varepsilon$, then, as $\mathsf{R}^N_\rho(z_0) = \mathbb{O}(\varepsilon e^{-\rho N})$, Equations (2-21) and (2-22) guarantee that $\mathsf{R}^N_\rho(z(t))$ remains of order $\mathbb{O}(\varepsilon e^{-\rho N})$ and the norm of $z(t)$ remains of order ε over exponentially long time $t = \mathbb{O}(e^{\rho N})$.

The next result is an easy consequence of the nonresonance condition and of the definition of normal forms:

Proposition 2.13. *Assume that the nonresonance condition (2-18) is satisfied and let N be fixed. Let Q be a homogenous polynomial of degree k. Then the homological equation*

$$\{\chi, H_0\} - Z = Q \tag{2-24}$$

admits a polynomial solution (χ, Z) homogeneous of degree k, such that Z is in N-normal form, and such that

$$\|Z\| \le \|Q\| \quad \textit{and} \quad \|\chi\| \le \frac{N^{\nu k}}{\gamma c_0^k}\|Q\|. \tag{2-25}$$

Proof. Assume that $Q = \sum_{\boldsymbol{j}\in\mathcal{I}_k} Q_{\boldsymbol{j}} z_{\boldsymbol{j}}$ and seek $Z = \sum_{\boldsymbol{j}\in\mathcal{I}_k} Z_{\boldsymbol{j}} z_{\boldsymbol{j}}$ and $\chi = \sum_{\boldsymbol{j}\in\mathcal{I}_k} \chi_{\boldsymbol{j}} z_{\boldsymbol{j}}$ such that (2-24) is satisfied. Equation (2-24) can be written in terms of polynomial coefficients

$$i\Omega(\boldsymbol{j})\chi_{\boldsymbol{j}} - Z_{\boldsymbol{j}} = Q_{\boldsymbol{j}}, \quad \boldsymbol{j} \in \mathcal{I}_k,$$

where $\Omega(\boldsymbol{j})$ is given in (2-11). We then define

$$\begin{aligned} Z_{\boldsymbol{j}} &= Q_{\boldsymbol{j}}, \quad \chi_{\boldsymbol{j}} = 0 && \text{if } \boldsymbol{j} \in \mathcal{N}_k \text{ or } \mu(\boldsymbol{j}) > N, \\ Z_{\boldsymbol{j}} &= 0, \quad \chi_{\boldsymbol{j}} = \frac{Q_{\boldsymbol{j}}}{i\Omega(\boldsymbol{j})} && \text{if } \boldsymbol{j} \notin \mathcal{N}_k \text{ and } \mu(\boldsymbol{j}) \leq N. \end{aligned}$$

In view of (2-18), this leads to (2-25). □

3. Proof of the main theorem

3A. ***Recursive equation.*** We aim to construct a canonical transformation τ such that in the new variables, the Hamiltonian $H_0 + P$ is in normal form modulo a small remainder term. Using Lie transforms to generate τ, the problem can be written thus: Find a polynomial $\chi = \sum_{k=3}^{r} \chi_k$, a polynomial $Z = \sum_{k=3}^{r} Z_k$ in normal form, and a smooth Hamiltonian R satisfying $\partial^\alpha R(0) = 0$ for all $\alpha \in \mathbb{N}^{\mathcal{Z}}$ with $|\alpha| \leq r$, such that

$$(H_0 + P) \circ \Phi_\chi^1 = H_0 + Z + R. \tag{3-1}$$

Then the exponential estimate (1-3) will be obtained by optimizing the choice of r and N.

We recall that for χ and K two Hamiltonian functions, for all $k \geq 0$ we have

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}(K \circ \Phi_\chi^t) = \{\chi, \{\cdots\{\chi, K\}\cdot\}\}(\Phi_\chi^t) = (\mathrm{ad}_\chi^k K)(\Phi_\chi^t),$$

where $\mathrm{ad}_\chi K = \{\chi, K\}$. Also, if K, L are homogeneous polynomials of degrees k and ℓ, then $\{K, L\}$ is a homogeneous polynomial of degree $k + \ell - 2$. Therefore, by using Taylor's formula, we obtain

$$(H_0 + P) \circ \Phi_\chi^1 - (H_0 + P) = \sum_{k=0}^{r-3} \frac{1}{(k+1)!} \mathrm{ad}_\chi^k(\{\chi, H_0 + P\}) + \mathbb{O}_r, \tag{3-2}$$

where $\mathbb{O}_r$ stands for a smooth function R satisfying $\partial^\alpha R(0) = 0$ for all $\alpha \in \mathbb{N}^{\mathcal{Z}}$ with $|\alpha| \leq r$.

On the other hand, we know that for $\zeta \in \mathbb{C}$, the following relation holds:

$$\left(\sum_{k=0}^{r-3} \frac{B_k}{k!} \zeta^k\right)\left(\sum_{k=0}^{r-3} \frac{1}{(k+1)!} \zeta^k\right) = 1 + O(|\zeta|^{r-2}),$$

where B_k are the Bernoulli numbers defined by the expansion of the generating function $\dfrac{z}{e^z - 1}$. Therefore, defining the two differential operators

$$A_r = \sum_{k=0}^{r-3} \frac{1}{(k+1)!} \mathrm{ad}_\chi^k \quad \text{and} \quad B_r = \sum_{k=0}^{r-3} \frac{B_k}{k!} \mathrm{ad}_\chi^k,$$

we get

$$B_r A_r = \mathrm{Id} + C_r,$$

where C_r is a differential operator satisfying

$$C_r \mathbb{O}_3 = \mathbb{O}_r.$$

Applying B_r to the two sides of (3-2), we obtain

$$\{\chi, H_0 + P\} = B_r(Z - P) + \mathbb{O}_r.$$

Plugging the decompositions in homogeneous polynomials of χ, Z and P into this equation and equating the terms of same degree, we obtain after a straightforward calculation the recursive equations

$$\{\chi_m, H_0\} - Z_m = Q_m, \quad m = 3, \dots, r, \tag{3-3}$$

where

$$Q_m = -P_m + \sum_{k=3}^{m-1} \{P_{m+2-k}, \chi_k\} + \sum_{k=1}^{m-3} \frac{B_k}{k!} \sum_{\substack{\ell_1+\dots+\ell_{k+1}=m+2k \\ 3\le \ell_i \le m-k}} \mathrm{ad}_{\chi_{\ell_1}} \dots \mathrm{ad}_{\chi_{\ell_k}} (Z_{\ell_{k+1}} - P_{\ell_{k+1}}). \tag{3-4}$$

In the last sum, $\ell_i \le m-k$ as a consequence of $3 \le \ell_i$ and $\ell_1 + \dots + \ell_{k+1} = m + 2k$.

Once these recursive equations are solved, we define the remainder term as $R = (H_0 + P) \circ \Phi^1_\chi - H_0 - Z$. By construction, R is analytic on a neighborhood of the origin in $\mathcal{L}_\rho$ and $R = \mathbb{O}_r$. As a consequence, by Taylor's formula,

$$\begin{aligned} R = \sum_{m\ge r+1} \sum_{k=1}^{m-3} \frac{1}{k!} \sum_{\substack{\ell_1+\dots+\ell_k=m+2k \\ 3\le \ell_i \le r}} \mathrm{ad}_{\chi_{\ell_1}} \dots \mathrm{ad}_{\chi_{\ell_k}} H_0 \\ + \sum_{m\ge r+1} \sum_{k=0}^{m-3} \frac{1}{k!} \sum_{\substack{\ell_1+\dots+\ell_{k+1}=m+2k \\ 3\le \ell_1+\dots+\ell_k \le r \\ 3\le \ell_{k+1}}} \mathrm{ad}_{\chi_{\ell_1}} \dots \mathrm{ad}_{\chi_{\ell_k}} P_{\ell_{k+1}}. \end{aligned} \tag{3-5}$$

Lemma 3.1. *Assume that the nonresonance condition (2-18) is fulfilled for some constants γ, c_0, ν. Then there exists $C > 0$ such that for all r and N, and for $m = 3, \dots, r$, there exist homogeneous polynomials χ_m and Z_m of degree m, with Z_m in N-normal forms, which are solutions of the recursive equation (3-3) and satisfy*

$$\|\chi_m\| + \|Z_m\| \le (CmN^\nu)^{m^2}. \tag{3-6}$$

Proof. We define χ_m and Z_m by induction using Proposition 2.13. Note that (3-6) is clearly satisfied for $m = 3$, provided C is big enough. Estimate (2-25) yields

$$\gamma c_0^m N^{-\nu m} \|\chi_m\| + \|Z_m\| \le \|Q_m\|. \tag{3-7}$$

Using the definition (3-4) of the term Q_m and the estimate on the Bernoulli numbers, $|B_k| \le k! c^k$ for some $c > 0$, together with (2-17), which implies that for all $\ell \ge 3$, $\|\mathrm{ad}_{\chi_\ell} R\| \le 2m\ell \|R\|$ for any polynomial R of degree less than m, we have, for all $m \ge 3$,

$$\begin{aligned} \|Q_m\| \le \|P_m\| + 2 \sum_{k=3}^{m-1} k(m+2-k) \|P_{m+2-k}\| \, \|\chi_k\| \\ + 2 \sum_{k=1}^{m-3} (Cm)^k \sum_{\substack{\ell_1+\dots+\ell_{k+1}=m+2k \\ \le \ell_i \le m-k}} \ell_1 \|\chi_{\ell_1}\| \dots \ell_k \|\chi_{\ell_k}\| \, \|Z_{\ell_{k+1}} - P_{\ell_{k+1}}\|. \end{aligned} \tag{3-8}$$

for some constant C. Let us set $\beta_m = m(\|\chi_m\| + \|Z_m\|)$. Equation (3-7) implies that

$$\beta_m \leq (CN^\nu)^m m \|Q_m\|,$$

for some constant C independent of m.

Using that $\|P_m\| \leq MR_0^{-m}$ (see the end of Section 2D), we have that $\|P_m\|$ and $m\|P_m\|$ are uniformly bounded with respect to m. Hence, the previous inequality implies that

$$\beta_m \leq \beta_m^{(1)} + \beta_m^{(2)},$$

where

$$\beta_m^{(1)} = (CN^\nu)^m m\Bigl(1 + \sum_{k=3}^{m-1} \beta_k\Bigr) \tag{3-9}$$

and

$$\beta_m^{(2)} = N^{\nu m}(Cm)^{m-2} \sum_{k=1}^{m-3} \sum_{\substack{\ell_1+\cdots+\ell_{k+1}=m+2k \\ \leq \ell_i \leq m-k}} \beta_{\ell_1} \dots \beta_{\ell_k}(\beta_{\ell_{k+1}} + 1), \tag{3-10}$$

for some constant C depending on M, R_0, γ and c_0. It remains to prove that $\beta_m \leq (CmN^\nu)^{\delta m^2}$ by induction, for some constant δ. Again, this is true for $m = 3$ by adapting C if necessary. Thus, assume that $\beta_j \leq (CjN^\nu)^{j^2}$, $j = 3, \dots, m-1$. As soon as $C > 1$,

$$1 \leq (CmN^\nu)^{m^2} \quad \text{for all } m \geq 3, \tag{3-11}$$

so we get

$$\beta_m^{(1)} \leq (CN^\nu)^m m^{m+2} (CmN^\nu)^{(m-1)^2} \leq \tfrac{1}{2}(CmN^\nu)^{m^2}$$

as soon as $m \geq 3$ and provided $C > 2$.

Using (3-11) again and the induction hypothesis, we get

$$\beta_m^{(2)} \leq N^{\nu m}(Cm)^{m-2} \sum_{k=1}^{m-3} \sum_{\substack{\ell_1+\cdots+\ell_{k+1}=m+2k \\ 3\leq \ell_i \leq m-k}} \bigl(CN^\nu(m-k)\bigr)^{\ell_1^2+\cdots+\ell_{k+1}^2}.$$

The maximum of $\ell_1^2 + \cdots + \ell_{k+1}^2$ when $\ell_1 + \cdots + \ell_{k+1} = m + 2k$ and $3 \leq \ell_i \leq m-k$ is obtained for $\ell_1 = \cdots = \ell_k = 3$ and $\ell_{k+1} = m-k$ and its value is $(m-k)^2 + 9k$. Furthermore, the cardinality of $\{\ell_1 + \cdots + \ell_{k+1} = m+2k,\ 3 \leq \ell_i \leq m-k\}$ is smaller than m^{k+1}, and hence we obtain, for $m \geq 4$ and after adapting C if necessary,

$$\beta_m^{(2)} \leq \max_{k=1,\dots,m-3} N^{\nu m}(Cm)^{m-2} Cm^{k+2}\bigl(CN^\nu(m-k)\bigr)^{(m-k)^2+9k} \leq \tfrac{1}{2}(CmN^\nu)^{m^2}. \qquad \square$$

3B. *Normal form result.* For any $R_0 > 0$, we set $B_\rho(R_0) = \{z \in \mathcal{L}_\rho \mid \|z\|_\rho < R_0\}$.

Theorem 3.2. *Assume that P is analytic on a ball $B_\rho(R_0)$ for some $R_0 > 0$ and $\rho > 0$. Assume that the nonresonance condition (2-18) is satisfied, and let $\beta < 1$ and $M > 1$ be fixed. Then there exist constants $\varepsilon_0 > 0$ and $\sigma > 0$ such that for all $\varepsilon < \varepsilon_0$, there exist a polynomial χ, a polynomial Z in $N = |\ln\varepsilon|^{1+\beta}$ normal form, and a Hamiltonian R analytic on $B_\rho(M\varepsilon)$, such that*

$$(H_0 + P)\circ\Phi^1_\chi = H_0 + Z + R. \tag{3-12}$$

Furthermore, for all $z \in B_\rho(M\varepsilon)$,

$$\|X_Z(z)\|_\rho + \|X_\chi(z)\|_\rho \le 2\varepsilon^{3/2} \quad \text{and} \quad \|X_R(z)\|_\rho \le \varepsilon e^{-\frac{1}{4}|\ln\varepsilon|^{1+\beta}}. \tag{3-13}$$

Proof. Using Lemma 3.1, for all N and r, we can construct polynomial Hamiltonians

$$\chi(z) = \sum_{k=3}^{r} \chi_k(z) \quad \text{and} \quad Z(z) = \sum_{k=3}^{r} Z_k(z),$$

with Z in N-normal form, such that (3-12) holds with $R = \mathcal{O}_r$. Now for fixed $\varepsilon > 0$, we choose

$$N \equiv N(\varepsilon) = |\ln\varepsilon|^{1+\beta} \quad \text{and} \quad r \equiv r(\varepsilon) = |\ln\varepsilon|^{\beta}.$$

This choice is motivated by the necessity of a balance between Z and R in (3-12): The error induced by Z is controlled as in Remark 2.12, while the error induced by R is controlled by Lemma 3.1. By (3-6), we have

$$\begin{aligned}
\|\chi_k\| &\le (CkN^\nu)^{k^2} \le \exp\big(k(\nu k(1+\beta)\ln|\ln\varepsilon| + k\ln Ck)\big)\\
&\le \exp\big(k(\nu r(1+\beta)\ln|\ln\varepsilon| + r\ln Cr)\big)\\
&\le \exp\big(k|\ln\varepsilon|\big(\nu|\ln\varepsilon|^{\beta-1}(1+\beta)\ln|\ln\varepsilon| + |\ln\varepsilon|^{\beta-1}\ln C|\ln\varepsilon|^{\beta}\big)\big) \le \varepsilon^{-k/8},
\end{aligned} \tag{3-14}$$

as $\beta < 1$, and for $\varepsilon \le \varepsilon_0$ sufficiently small. Therefore, using Proposition 2.5, for $z \in B_\rho(M\varepsilon)$ we obtain

$$|\chi_k(z)| \le \varepsilon^{-k/8}(M\varepsilon)^k \le M^k\varepsilon^{7k/8},$$

and thus

$$|\chi(z)| \le \sum_{k\ge 3} M^k\varepsilon^{7k/8} \le \varepsilon^{3/2},$$

for ε small enough. Similarly, for all $k \le r$, we have

$$\|X_{\chi_k}(z)\|_\rho \le 2k\varepsilon^{-k/8}(M\varepsilon)^{k-1} \le 2kM^{k-1}\varepsilon^{7k/8-1}$$

and

$$\|X_\chi(z)\|_\rho \le \sum_{k\ge 3} 2kM^{k-1}\varepsilon^{7k/8-1} \le C\varepsilon^{-1}\varepsilon^{21/8} \le \varepsilon^{3/2},$$

for ε small enough. Similar bounds clearly hold for $Z = \sum_{k=3}^{r} Z_k$, which shows the first estimate in (3-13).

On the other hand, using $\mathrm{ad}_{\chi_{\ell_k}} H_0 = Z_{\ell_k} + Q_{\ell_k}$ (see (3-3)) and then using Lemma 3.1 and the definition of Q_m (see (3-4)), we get $\|\mathrm{ad}_{\chi_{\ell_k}} H_0\| \le (CkN^\nu)^{\ell_k{}^2} \le \varepsilon^{-\ell_k/8}$, where the last inequality proceeds as in (3-14). Thus, using (3-5), (3-14) and $\|P_{\ell_{k+1}}\| \le MR_0^{-\ell_{k+1}}$, we obtain by Proposition 2.5 that for $z \in B_\rho(M\varepsilon)$,

$$\|X_R(z)\|_\rho \le \sum_{m\ge r+1} \sum_{k=0}^{m-3} m(Cr)^{3m} \varepsilon^{-\frac{m+2k}{8}} \varepsilon^{m-1} \le \sum_{m\ge r+1} m^2 (Cr)^{3m} \varepsilon^{m/2} \le (Cr)^{3r} \varepsilon^{r/2}.$$

Therefore, since $r = |\ln \varepsilon|^\beta$, we get $\|X_R(z)\|_\rho \le \varepsilon\, e^{-\frac14 |\ln\varepsilon|^{1+\beta}}$ for $z \in B_\rho(M\varepsilon)$ and ε small enough. □

3C. *Bootstrap argument.* We are now in position to prove the main theorem of Section 1. It is a direct consequence of Theorem 3.2.

Let $u_0 \in \mathcal{A}_{2\rho}$ with $|u_0|_{2\rho} = \varepsilon$, and denote by $z(0)$ the corresponding sequence of its Fourier coefficients which belongs, by Lemma 2.1, to $\mathcal{L}_{(3/2)\rho}$ with $\|z(0)\|_{(3/2)\rho} \le (c_\rho/4)\varepsilon$ and

$$c_\rho = \frac{2^{d+2}}{(1-e^{-\rho/2\sqrt{d}})^d}.$$

Let $z(t)$ be the local solution in $\mathcal{L}_\rho$ of the Hamiltonian system associated with $H = H_0 + P$.

Let χ, Z and R be given by Theorem 3.2 with $M = c_\rho$ and let $y(t) = \Phi_\chi^1(z(t))$. We recall that since $\chi(z) = O(\|z\|^3)$, the transformation Φ_χ^1 is close to the identity: $\Phi_\chi^1(z) = z + O(\|z\|^2)$, and thus, for ε small enough, we have $\|y(0)\|_{(3/2)\rho} \le (c_\rho/2)\varepsilon$. In particular, as given in (2-20),

$$R_\rho^N(y(0)) \le \frac{c_\rho}{2}\varepsilon\, e^{-(\rho/2)N} \le \frac{c_\rho}{2}\varepsilon\, e^{-\sigma N},$$

where $\sigma = \sigma_\rho \le \rho/2$.

Let T_ε be the largest time T such that $R_\rho^N(y(t)) \le c_\rho\varepsilon\, e^{-\sigma N}$ and $\|y(t)\|_\rho \le c_\rho\varepsilon$ for all $|t| \le T$. By construction, we have

$$y(t) = y(0) + \int_0^t X_{H_0+Z}(y(s))\,ds + \int_0^t X_R(y(s))\,ds.$$

So using (2-21) for the first vector field and (3-13) for the second one, we get, for $|t| < T_\varepsilon$,

$$\begin{aligned} R_\rho^N(y(t)) &\le \tfrac12 c_\rho \varepsilon e^{-\sigma N} + 4|t| \sum_{k=3}^{r} \|Z_k\| k^3 (c_\rho\varepsilon)^{k-1} e^{-2\sigma N} + |t|\varepsilon e^{-\frac14 |\ln\varepsilon|^{1+\beta}} \\ &\le \Bigl(\tfrac12 + 4|t| \sum_{k=3}^{r} \|Z_k\| k^3 (c_\rho\varepsilon)^{k-2} e^{-\sigma N} + |t|\varepsilon e^{-\frac18 |\ln\varepsilon|^{1+\beta}}\Bigr) c_\rho\varepsilon e^{-\sigma N}, \end{aligned} \tag{3-15}$$

where in the last inequality we used $\sigma = \min\{\frac{1}{10}, \frac12\rho\}$ and $N = |\ln\varepsilon|^{1+\beta}$.

Using Lemma 3.1, we then verify that

$$R_\rho^N(y(t)) \le \bigl(\tfrac12 + C|t|\varepsilon\, e^{-\sigma N}\bigr) c_\rho \varepsilon e^{-\sigma N},$$

and thus, for ε small enough,

$$R_\rho^N(y(t)) \le c_\rho\varepsilon\, e^{-\sigma N} \quad \text{for all } |t| \le \min\{T_\varepsilon, e^{\sigma N}\}. \tag{3-16}$$

Similarly, we obtain

$$\|y(t)\|_\rho \le c_\rho \varepsilon \quad \text{for all } |t| \le \min\{T_\varepsilon, e^{\sigma N}\}. \tag{3-17}$$

In view of the definition of T_ε, inequalities (3-16) and (3-17) imply $T_\varepsilon \ge e^{\sigma N}$. In particular, $\|z(t)\|_\rho \le 2c_\rho \varepsilon$ for $|t| \le e^{\sigma N} = \varepsilon^{-\sigma |\ln \varepsilon|^\beta}$, and using (2-7), we finally obtain (1-3) with

$$C = \frac{2^{2d+5}}{(1-e^{-\rho/2\sqrt{d}})^{2d}}.$$

Estimate (1-4) is another consequence of the normal form result and Proposition 2.11. Actually, we use that the Fourier coefficients of $u(t)$ are given by $z(t)$, which is ε^2-close to $y(t)$, which in turn is almost invariant: in view of (2-23) and as in (3-15), we have

$$\sum_{j\in\mathbb{Z}} e^{\rho|j|}\big||y_j(t)|-|y_j(0)|\big| \le \left(4|t|\sum_{k=3}^{r}\|Z_k\|k^3(c_\rho\varepsilon)^{k-1}e^{-2\sigma N} + |t|\varepsilon e^{-\frac{1}{4}|\ln\varepsilon|^{1+\beta}}\right),$$

from which we deduce

$$\sum_{j\in\mathbb{Z}} e^{\rho|j|}\big||y_j(t)|-|y_j(0)|\big| \le |t|e^{-\sigma N},$$

and then (1-4).

Appendix: Proof of the nonresonance hypothesis

Instead of proving Proposition 2.8, we prove a slightly more general result. For a multi-index $\boldsymbol{j} \in \mathcal{Z}^r$, we define

$$N(\boldsymbol{j}) = \prod_{k=1}^{r}(1+|j_k|).$$

Proposition A.1. *Fix $\gamma > 0$ small enough and $m > d/2$. There exist positive constants C and ν depending only on m, R and d, and a set $F_\gamma \subset \mathcal{W}_m$ (see (1-2)) whose measure is larger than $1-4\gamma$, such that if $V \in F_\gamma$, then for any $r \ge 1$,*

$$\left|\Omega(\boldsymbol{j}) + \varepsilon_1\omega_{\ell_1} + \varepsilon_2\omega_{\ell_2}\right| \ge \frac{C^r\gamma^7}{N(\boldsymbol{j})^\nu} \tag{A-1}$$

for any $\boldsymbol{j} \in \mathcal{Z}^r$, any indices $\ell_1, \ell_2 \in \mathbb{Z}^d$, and any $\varepsilon_1, \varepsilon_2 \in \{0, 1, -1\}$ such that $(\boldsymbol{j}, (\ell_1, \varepsilon_1), (\ell_2, \varepsilon_2))$ is nonresonant[4].

In order to prove Proposition A.1, we first prove that $\Omega(\boldsymbol{j})$ cannot accumulate on $\mathbb{Z}$. Precisely, we have:

Lemma A.2. *Fix $\gamma > 0$ and $m > d/2$. There exist $0 < C < 1$ depending only on m, R and d, and a set $F'_\gamma \subset \mathcal{W}_m$ whose measure is larger than $1-4\gamma$, such that if $V \in F'_\gamma$, then for any $r \ge 1$,*

$$|\Omega(\boldsymbol{j}) - b| \ge \frac{C^r\gamma}{N(\boldsymbol{j})^{m+d+3}} \tag{A-2}$$

[4] The resonant set $\mathcal{N}_r$, $r \ge 2$, is defined in Section 2D.

for any nonresonant $\boldsymbol{j} \in \mathbb{Z}^r$ and for any $b \in \mathbb{Z}$.

Proof. Let $(\alpha_1, \dots, \alpha_r) \neq 0$ in $\mathbb{Z}^r$, $M > 0$ and $c \in \mathbb{R}$. The set

$$\mathscr{E}(\eta) = \left\{ x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^r \;\middle|\; \left| \sum_{i=1}^r \alpha_i x_i + c \right| < \eta \right\}$$

is a slice of thickness 2η of the hypercube $[-M, M]^r$ guided by the hyperplane $\{\sum_{i=1}^r \alpha_i x_i + c = 0\}$, whose normal α has a norm larger than 1. Since the largest diagonal in the hypercube $\left[-\frac{1}{2}, \frac{1}{2}\right]^r$ has a length equal to $\sqrt{r}$, we get that the base of the slice $\mathscr{E}(\eta)$ is included in a hyperdisc of dimension $r-1$ and radius $\frac{1}{2}\sqrt{r}$. Recall that the volume of a ball in $\mathbb{R}^m$ of radius ρ equals $\pi^{m/2}\rho^m / \Gamma(m/2+1)$. So we deduce that the volume of $\mathscr{E}(\eta)$ is smaller than[5]

$$2\eta\pi^{(r-1)/2} \frac{\left(\frac{1}{2}\sqrt{r}\right)^{r-1}}{\Gamma\left(\frac{r-1}{2}+1\right)} \leq 2\eta \frac{\left(\frac{1}{2}\sqrt{\pi r}\right)^{r-1}}{\left(\frac{r-1}{2}\right)!} \leq C^r \eta$$

for a constant C independent of r. Hence, given $\boldsymbol{j} = (a_i, \delta_i)_{i=1}^r \in \mathscr{Z}^r$ and $b \in \mathbb{Z}$, the Lebesgue measure of

$$\mathscr{X}_\eta := \left\{ x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^r \;\middle|\; \left| \sum_{i=1}^r \delta_i(|a_i|^2 + x_i) - b \right| < \eta \right\}$$

is smaller than $2\eta r^{\frac{r-1}{2}}$. Now consider the set (using the notation (1-2))

$$\{V \in \mathscr{W}_m \mid |\Omega(\boldsymbol{j}) - b| < \eta\} = \left\{ V \in \mathscr{W}_m \;\middle|\; \left| \sum_{i=1}^r \delta_i\left(|a_i|^2 + \frac{v_{a_i} R}{(1+|a_i|)^m}\right) - b \right| < \eta \right\}. \tag{A-3}$$

It is contained in the set of the V's such that $(Rv_{a_i}/(1+|a_i|)^m)_{i=1}^r \in \mathscr{X}_\eta$. Hence the measure of (A-3) is smaller than $R^{-r}N(\boldsymbol{j})^m C^r \eta$. To conclude the proof, we have to sum over all the possible $\boldsymbol{j}$'s and all the possible b's. Now for a given $\boldsymbol{j}$, if $|\Omega(\boldsymbol{j}) - b| \geq \eta$ with $\eta \leq 1$, then $|b| \leq 2N(\boldsymbol{j})^2$. So to guarantee (A-2) for all possible choices of $\boldsymbol{j}$, b and r, it suffices to remove from $\mathscr{W}_m$ a set of measure

$$4\gamma \sum_{\boldsymbol{j} \in \mathscr{Z}^r} \frac{C^r}{R^r N(\boldsymbol{j})^{m+3+d}} N(\boldsymbol{j})^{m+2} \leq 4\gamma \left[\frac{2C}{R} \sum_{\ell \in \mathbb{Z}^d} \frac{1}{(1+|\ell|)^{d+1}} \right]^r.$$

Choosing $C \leq \frac{1}{2} R \left(\sum_{\ell \in \mathbb{Z}^d} \frac{1}{(1+|\ell|)^{d+1}} \right)^{-1}$ proves the result. □

Proof of Proposition A.1. First of all, for $\varepsilon_1 = \varepsilon_2 = 0$, (A-1) is a direct consequence of Lemma A.2, choosing $\nu \geq m+d+3$, $\gamma \leq 1$ and $F_\gamma = F'_\gamma$ (recall that $r \geq 1$).

When $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = 0$, we will prove that for some constants C and ν, we have

$$\left|\Omega(\boldsymbol{j}) \pm \omega_{\ell_1}\right| \geq \frac{C^r \gamma}{N(\boldsymbol{j})^\nu}, \tag{A-4}$$

[5] We use the formula of the gamma function valid for even integers, but the asymptotic is the same in the odd case.

which implies inequality (A-1) for $\gamma \le 1$. Notice that $|\Omega(\boldsymbol{j})| \le N(\boldsymbol{j})^2$ and thus, if $|\ell_1| \ge 2N(\boldsymbol{j})$, (A-4) is always true. When $|\ell_1| \le 2N(\boldsymbol{j})$, using that $N(\boldsymbol{j},\ell) = N(\boldsymbol{j})(1+|\ell_1|)$, applying Lemma A.2 with $b=0$ and $V \in F'_\gamma = F_\gamma$, we get

$$\left|\Omega(\boldsymbol{j}) + \varepsilon_1\omega_{\ell_1}\right| = \left|\Omega(\boldsymbol{j},(\ell_1,\varepsilon_1))\right| \ge \frac{C^{r+1}\gamma}{N(\boldsymbol{j})^{m+d+3}(3N(\boldsymbol{j}))^{m+d+3}} \ge \frac{\tilde{C}^r\gamma}{N(\boldsymbol{j})^\nu},$$

with $\nu = 2(m+d+3)$ and $\tilde{C} = 2C^2/3^{m+d+3}$.

When $\varepsilon_1\varepsilon_2 = 1$, a similar argument yields an estimate of the form

$$\left|\Omega(\boldsymbol{j}) \pm (\omega_{\ell_1} + \omega_{\ell_2})\right| \ge \frac{C^r\gamma}{N(\boldsymbol{j})^\nu},$$

for some constants C, ν, and for $V \in F'_\gamma = F_\gamma$.

So it remains to establish an estimate of the form

$$\left|\Omega(\boldsymbol{j}) + \omega_{\ell_1} - \omega_{\ell_2}\right| \ge \frac{\tilde{C}^r\gamma^7}{N(\boldsymbol{j})^\nu}, \tag{A-5}$$

for some constant $\tilde{C}$ and $V \in F_\gamma$ to be defined. Assuming $|\ell_1| \le |\ell_2|$, we have

$$\left|\omega_{\ell_1} - \omega_{\ell_2} - \ell_1^2 + \ell_2^2\right| \le \left|\frac{R|v_{\ell_1}|}{(1+|\ell_1|)^m} - \frac{R|v_{\ell_2}|}{(1+|\ell_2|)^m}\right| \le \frac{R}{(1+|\ell_1|)^m},$$

for all v_{ℓ_1} and v_{ℓ_2} in $\left[-\frac{1}{2},\frac{1}{2}\right]$; see (1-2). Therefore, if $(1+|\ell_1|)^m \ge (2R/C^r\gamma)N(\boldsymbol{j})^{m+d+3}$, we obtain (A-5) directly from Lemma A.2 applied with $b = \ell_1^2 - \ell_2^2$ and choosing $\nu = m+d+3$, $\tilde{C} = C/2$ and $F_\gamma = F'_\gamma$.

Finally, assume $(1+|\ell_1|)^m \le (2R/C^r\gamma)N(\boldsymbol{j})^{m+d+3}$. Then taking into account $|\Omega(\boldsymbol{j})| \le N(\boldsymbol{j})^2$, inequality (A-5) is satisfied when $\ell_2^2 - \ell_1^2 \ge 2N(\boldsymbol{j})^2$. It remains to consider the case when

$$1+|\ell_1| \le 1+|\ell_2| \le \left[2\left(\frac{2R}{C^r\gamma}N(\boldsymbol{j})^{m+d+3}\right)^{2/m} + 4N(\boldsymbol{j})^2\right]^{1/2} \le 2\left(\frac{3R}{C^r\gamma}\right)^{1/m} N(\boldsymbol{j})^{\frac{m+d+3}{m}}.$$

Again we use Lemma A.2 to conclude that

$$\begin{aligned}\left|\Omega(\boldsymbol{j}) + \omega_{\ell_1} - \omega_{\ell_2}\right| &\ge \frac{C^{r+2}\gamma}{\left[N(\boldsymbol{j})(1+|\ell_1|)(1+|\ell_2|)\right]^{m+d+3}} \\ &\ge \frac{C^{r+2}\gamma\left(\dfrac{C^r\gamma}{3.2^m R}\right)^{\frac{m+d+3}{m}}}{N(\boldsymbol{j})^{m+d+3}N(\boldsymbol{j})^{2\frac{(m+d+3)^2}{m}}} \ge \frac{\tilde{C}^r\gamma^{4+3/m}}{N(\boldsymbol{j})^\nu},\end{aligned}$$

as $m > \dfrac{d}{2}$, and with $\nu = m+d+3+\dfrac{(m+d+3)^2}{m}$ and $\tilde{C} = \dfrac{C^{\frac{4m+d+3}{m}}}{3.2^m R}$. This last estimate implies (A-1). ∎

References

[Bambusi 1999a] D. Bambusi, "On long time stability in Hamiltonian perturbations of non-resonant linear PDEs", *Nonlinearity* **12**:4 (1999), 823–850. MR 2000m:37172 Zbl 0989.37073

[Bambusi 1999b] D. Bambusi, "Nekhoroshev theorem for small amplitude solutions in nonlinear Schrödinger equations", *Math. Z.* **230**:2 (1999), 345–387. MR 2000h:35146 Zbl 0928.35160

[Bambusi 2003] D. Bambusi, "Birkhoff normal form for some nonlinear PDEs", *Comm. Math. Phys.* **234**:2 (2003), 253–285. MR 2003k:37121 Zbl 1032.37051

[Bambusi and Grébert 2003] D. Bambusi and B. Grébert, "Forme normale pour NLS en dimension quelconque", *C. R. Math. Acad. Sci. Paris* **337**:6 (2003), 409–414. MR 2004i:35283 Zbl 1030.35143

[Bambusi and Grébert 2006] D. Bambusi and B. Grébert, "Birkhoff normal form for partial differential equations with tame modulus", *Duke Math. J.* **135**:3 (2006), 507–567. MR 2007j:37124 Zbl 1110.37057

[Benettin et al. 1985] G. Benettin, L. Galgani, and A. Giorgilli, "A proof of Nekhoroshev's theorem for the stability times in nearly integrable Hamiltonian systems", *Celestial Mech.* **37**:1 (1985), 1–25. MR 87i:58051 Zbl 0602.58022

[Bourgain 1996] J. Bourgain, "Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations", *Geom. Funct. Anal.* **6**:2 (1996), 201–230. MR 97f:35013 Zbl 0872.35007

[Cazenave 2003] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics **10**, Amer. Math. Soc., Providence, RI, 2003. MR 2004j:35266 Zbl 1055.35003

[Eliasson and Kuksin 2010] L. H. Eliasson and S. B. Kuksin, "KAM for the nonlinear Schrödinger equation", *Ann. of Math.* (2) **172**:1 (2010), 371–435. MR 2011g:37203 Zbl 1201.35177

[Faou and Grébert 2011] E. Faou and B. Grébert, "Hamiltonian interpolation of splitting approximations for nonlinear PDEs", *Found. Comput. Math.* **11**:4 (2011), 381–415. MR 2012e:65310 Zbl 1232.65176

[Faou et al. 2010] E. Faou, B. Grébert, and E. Paturel, "Birkhoff normal form for splitting methods applied to semilinear Hamiltonian PDEs, II: Abstract splitting", *Numer. Math.* **114**:3 (2010), 459–490. MR 2011a:65445 Zbl 1189.65299

[Giorgilli and Galgani 1985] A. Giorgilli and L. Galgani, "Rigorous estimates for the series expansions of Hamiltonian perturbation theory", *Celestial Mech.* **37**:2 (1985), 95–112. MR 87i:58056

[Nekhoroshev 1977] N. N. Nekhoroshev, "An exponential estimate of the time of stability of nearly integrable Hamiltonian systems", *Uspehi Mat. Nauk* **32**:6(198) (1977), 5–66, 287. In Russian; translated in *Russ. Math. Surveys* **32**:6 (1977), 1–65. MR 58 #18570 Zbl 0389.70028

[Pöschel 1993] J. Pöschel, "Nekhoroshev estimates for quasi-convex Hamiltonian systems", *Math. Z.* **213**:2 (1993), 187–216. MR 94m:58089 Zbl 0857.70009

Received 16 Feb 2011. Revised 23 Jan 2013. Accepted 28 Feb 2013.

ERWAN FAOU: Erwan.Faou@inria.fr
INRIA and ENS Cachan Bretagne, Avenue Robert Schumann, F-35170 Bruz, France

BENOÎT GRÉBERT: benoit.grebert@univ-nantes.fr
Laboratoire de Mathématiques Jean Leray, Université de Nantes, 2 Rue de la Houssiniere, F-44322 Nantes Cedex 3, France

ANALYSIS AND PDE
Vol. 6, No. 6, 2013
dx.doi.org/10.2140/apde.2013.6.1263

L^q BOUNDS ON RESTRICTIONS OF SPECTRAL CLUSTERS TO SUBMANIFOLDS FOR LOW REGULARITY METRICS

MATTHEW D. BLAIR

We prove L^q bounds on the restriction of spectral clusters to submanifolds in Riemannian manifolds equipped with metrics of $C^{1,\alpha}$ regularity for $0 \le \alpha \le 1$. Our results allow for Lipschitz regularity when $\alpha = 0$, meaning they give estimates on manifolds with boundary. When $0 < \alpha \le 1$, the scalar second fundamental form for a codimension 1 submanifold can be defined, and we show improved estimates when this form is negative definite. This extends results of Burq, Gérard, and Tzvetkov and Hu to manifolds with low regularity metrics.

1. Introduction

Let M be a compact, smooth manifold of dimension $n \ge 2$ equipped with Riemannian metric g of at least Lipschitz regularity. Let Δ_g denote the associated (negative) Laplace–Beltrami operator whose action in coordinates is given by the differential operator

$$\Delta_g f = \frac{1}{\sqrt{\det g_{kl}}} \sum_{i,j} \partial_i (g^{ij} \sqrt{\det g_{kl}}\, \partial_j f).$$

There exists an orthonormal basis $\{\phi_j\}_{j=1}^{\infty}$ of $L^2(M)$ consisting of eigenfunctions of Δ_g, which can be seen by passing to quadratic forms; see, for example, [Smith 2006a, Section 1]. We write the corresponding Helmholtz equation for ϕ_j as $(\Delta_g + \lambda_j^2)\phi_j = 0$ so that λ_j gives the frequency of vibration associated to ϕ_j.

Given $\lambda \ge 1$, we let Π_λ be the projection operator on $L^2(M)$ defined by $\Pi_\lambda f := \sum_{\lambda_j \in [\lambda, \lambda+1]} \langle f, \phi_j \rangle \phi_j$, where $\langle \cdot\,,\,\cdot \rangle$ denotes the usual L^2 inner product with respect to the Riemannian measure. We call functions f which are in the range of some Π_λ "spectral clusters". They form approximate eigenfunctions or quasimodes as $\|(\Delta_g + \lambda^2)\Pi_\lambda f\|_{L^2(M)} \le C\lambda \|f\|_{L^2(M)}$. Sogge [1988] proved that when g is a C^∞ metric, the following L^q bounds on the projections $\Pi_\lambda f$ are satisfied for $q \ge 2$:

$$\|\Pi_\lambda f\|_{L^q(M)} \le C\lambda^{\delta} \|f\|_{L^2(M)}, \tag{1-1}$$

where

$$\delta = \delta(q) = \max\Big(\frac{n-1}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big), n\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{1}{2}\Big).$$

The author was supported in part by the National Science Foundation grant DMS-1001529.
MSC2010: primary 35P99, 35R05, 42B37; secondary 35L15, 42C15, 42B20.
Keywords: eigenfunctions, L^p estimates, spectral cluster estimates, quasimodes, wave packets, folding singularities.

He also provided examples showing that the exponent $\delta(q)$ is the best possible for these approximate eigenfunctions. Since Π_λ is a projection operator, any L^q bound it satisfies implies L^q bounds on individual eigenfunctions. Determining when these bounds are sharp for subsequences of eigenfunctions is an area of active interest, though we do not examine this issue here.

H. Smith [2006b] proved that the bounds (1-1) are satisfied for $C^{1,1}$ metrics. The assumption of $C^{1,1}$ regularity is the lowest degree of continuity needed to ensure the uniqueness of geodesics on M. Since eigenfunctions naturally give rise to solutions to the wave equation, propagation of singularities suggests that this is a relevant consideration for the validity of such bounds. Indeed, [Smith and Sogge 1994; Smith and Tataru 2002] give examples of $C^{1,\alpha}$ metrics (Lipschitz when $\alpha = 0$) which give rise to spectral clusters $\Pi_\lambda f_\lambda = f_\lambda$ for each $\lambda \geq 1$ such that

$$\frac{\|f_\lambda\|_{L^q(M)}}{\|f_\lambda\|_{L^2(M)}} \geq c\lambda^{\frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)(1+\sigma)}, \quad \sigma = \frac{1-\alpha}{3+\alpha}, \tag{1-2}$$

showing that the bounds (1-1) cannot hold for $2 < q < 2(n+2(1+\alpha)^{-1})/(n-1)$. In each case, the cluster f_λ is highly concentrated in a tube about a curve segment of length 1 and diameter $\lambda^{-2/(3+\alpha)}$ (cf. (1-10) below). This shows that the family $\{f_\lambda\}_{\lambda\geq 1}$ exhibits a greater degree of concentration than Sogge's examples which saturate the bounds (1-1) when $2 < q \leq \frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)$ (they are concentrated in tubes with diameter $\lambda^{-1/2}$). Smith [2006a] showed positive results for any $C^{1,\alpha}$ metric, proving that the ratio on the left in (1-2) is always bounded above by $C\lambda^{\frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)(1+\sigma)}$ when $2 \leq q \leq 2(n+1)/(n-1)$. He also proved that the bound (1-1) holds when $q = \infty$. By interpolation, this shows (1-1) with a loss of σ/q derivatives when $2(n+1)/(n-1) \leq q \leq \infty$, though Koch, Smith, and Tataru [Koch et al. 2012] improved upon this.

In a similar vein, when $g \in C^\infty$, results of Burq, Gérard, and Tzvetkov [Burq et al. 2007], Hu [2009], and Reznikov [2004] show L^q bounds on the restriction of these spectral clusters to embedded submanifolds $P \subset M$ of the form

$$\|\Pi_\lambda f\|_{L^q(P)} \leq C\lambda^\delta \|f\|_{L^2(M)}, \quad q \geq 2, \tag{1-3}$$

where $\|\Pi_\lambda f\|_{L^q(P)}$ is taken to mean the L^q norm of the restriction $\Pi_\lambda f|_P$. In this case, $\delta = \delta(k,q)$ depends on the dimension of the submanifold k and on q. In particular, when $k = n-1$,

$$\delta = \max\left(\frac{n-1}{2} - \frac{n-1}{q}, \frac{n-1}{4} - \frac{n-2}{2q}\right),$$

that is,

$$\delta(n-1,q) = \begin{cases} \dfrac{n-1}{2} - \dfrac{n-1}{q}, & \text{if } \dfrac{2n}{n-1} \leq q \leq \infty, \\[2ex] \dfrac{n-1}{4} - \dfrac{n-2}{2q}, & \text{if } 2 \leq q \leq \dfrac{2n}{n-1}. \end{cases} \tag{1-4}$$

Otherwise, when $1 \leq k \leq n-2$,

$$\delta(k,q) = \frac{n-1}{2} - \frac{k}{q} \tag{1-5}$$

with the exception of $(k, q) = (n-2, 2)$, where there is a logarithmic loss for $\lambda \geq 2$, $\|\Pi_\lambda f\|_{L^2(P)} \leq C(\log \lambda)^{1/2}\lambda^{1/2}\|f\|_{L^2(M)}$. These bounds were proved in a semiclassical setting by Tacy [2010]. We also remark that the bound (1-3) in the case $k = n-1$, $q = 2$ was previously observed by Tataru [1998] as a consequence of the estimates in [Greenleaf and Seeger 1994]. As will be discussed in Section 2, these bounds provide an improvement over what would be obtained by trace theorems for Sobolev spaces.

One reason the bounds (1-1), (1-3) are of such great interest is that they illuminate the size and concentration properties of eigenfunctions. In particular, Smith's work on $C^{1,\alpha}$ metrics [2006a] is significant in that it addresses concentration phenomena in situations where the roughness of the metric means that geodesic curves may fail to be unique. It also led to the development of sharp bounds of the form (1-1) for the Dirichlet and Neumann Laplacians on compact Riemannian manifolds with boundary; see [Smith and Sogge 2007]. Indeed, one strategy for proving estimates in this context is to form the double of the manifold, essentially gluing two copies of the manifold along the boundary. While this eliminates the boundary, it gives rise to a metric of Lipschitz regularity; see, for example, [Blair et al. 2008, p. 420]. Hence any result on manifolds with Lipschitz metrics also applies to manifolds with boundary. At the same time, the bounds (1-3) when $n = 2$, $k = 1$ (curves in 2 dimensional manifolds) for $g \in C^\infty$ have garnered additional interest in recent works which relate improvements in these estimates to improvements in the inequalities in (1-1); see [Bourgain 2009; Sogge 2011; Ariturk 2011].

On the other hand, one of the notable aspects of [Burq et al. 2007] is that the authors showed an improvement on (1-3) when $n = 2$ and P is a curve with nonvanishing geodesic curvature. Specifically, they proved that

$$\|\Pi_\lambda f\|_{L^2(P)} \leq C\lambda^{1/6}\|f\|_{L^2(M)}. \tag{1-6}$$

This was then generalized to all dimensions by Hu [2009], who obtained the same bound for any codimension 1 submanifold with negative definite scalar second fundamental form (or positive definite, depending on the choice of normal vector). As before, these bounds also follow from an observation of Tataru [1998] based on known estimates of Hörmander [1985, 25.3]. The bound (1-6) can then be interpolated with (1-3) when $q = \frac{2n}{n-1}$ and $\delta = \frac{n-1}{2n}$ to show that the δ in (1-4) can be improved to

$$\delta = \frac{n-1}{3} - \frac{2n-3}{3q} \quad \text{when } 2 \leq q < \frac{2n}{n-1}.$$

These bounds thus speak to the concentration properties of eigenfunctions. When P is in some sense "far away" from containing geodesic segments, eigenfunctions have less tendency to concentrate near P. Hassell and Tacy [2012] proved bounds of this type in a semiclassical setting.

In the present work, we consider the development of the bounds (1-3) for $C^{1,\alpha}$ metrics with $0 \leq \alpha \leq 1$, allowing for Lipschitz regularity when $\alpha = 0$. As a corollary, we obtain bounds of this type (with a loss) for the Dirichlet and Neumann Laplacians on compact manifolds with boundary. Bounds of the form (1-3) when $n = 2$, $k = 1$ for manifolds with concave boundaries are due to Ariturk [2011], provided Dirichlet conditions are imposed. However, the presence of gliding rays when the manifold possesses a point of convexity within the boundary complicates matters considerably.

Theorem 1.1. *Suppose* $g \in C^{1,\alpha}$ *with* $0 \le \alpha \le 1$, *allowing for Lipschitz regularity when* $\alpha = 0$. *When* $k = n-1$ *and* $2 \le q \le 2n/(n-1)$, *we have, for* $\delta = (n-1)/4 - (n-2)/(2q)$,

$$\|\Pi_\lambda f\|_{L^q(P)} \le C\lambda^{\delta(1+\sigma)}\|f\|_{L^2(M)}, \quad \sigma = \frac{1-\alpha}{3+\alpha}. \tag{1-7}$$

Moreover, when $k = n-1$, $2n/(n-1) \le q \le \infty$ *or* $k \le n-2$, *we suppose that* $\delta = (n-1)/2 - k/q$ *and* $\delta + \sigma/q < 1+\alpha$ *with* σ *as above. In this case, the following bounds are satisfied*:

$$\|\Pi_\lambda f\|_{L^q(P)} \le C\lambda^{\delta+\sigma/q}\|f\|_{L^2(M)} \tag{1-8}$$

with C *replaced by* $C(\log\lambda)^{1/2}$ *when* $(k,q) = (n-2,2)$. *The admissibility condition on* δ, q *can be relaxed to* $\delta + \sigma/q \le 1+\alpha$ *when* $\alpha = 0$ *or* $\alpha = 1$.

Furthermore, we will show improvements akin to (1-6) when $0 < \alpha \le 1$. For these metrics the Christoffel symbols are well defined and continuous on M by the usual coordinate formula

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$

(with the summation convention in effect). Hence there is also a well defined Levi-Civita connection associated to the metric g on M, mapping C^1 vector fields to continuous vector fields with the usual properties. In particular, given a smooth, embedded, codimension 1 submanifold of P, the scalar second fundamental form is well-defined and if it is negative definite throughout P for a suitable choice of normal vector field, we shall call it "curved". We will see that in this case, the power of λ in (1-7) with $q = 2$ can be improved to $1/6 + \sigma/2$ (which can be seen as strictly less than $(1/4)(1+\sigma)$ when $\sigma < \frac{1}{3}$).

Theorem 1.2. *Suppose* $g \in C^{1,\alpha}$ *with* $0 < \alpha \le 1$, *and that* P *is a "curved" codimension* 1 *submanifold as defined above. Then the following bounds are satisfied*:

$$\|\Pi_\lambda f\|_{L^2(P)} \le C\lambda^{\frac{1}{6}+\frac{\sigma}{2}}\|f\|_{L^2(M)}, \quad \sigma = \frac{1-\alpha}{3+\alpha}. \tag{1-9}$$

Moreover, interpolating this bound with the $q = 2n/(n-1)$ *case of* (1-7) *yields an improvement of that estimate for* $2 \le q < 2n/(n-1)$.

Following [Smith 2006a], we will show that, for each theorem, the $0 \le \alpha < 1$ case follows from the $\alpha = 1$ case by rescaling methods. This involves dilating coordinates so that sets of diameter $\approx \lambda^{-\sigma}$ in P have diameter ≈ 1 in the new coordinates. Since the metric can be approximated by one with $C^{1,1}$ regularity here, the bounds from the $\alpha = 1$ case can then be applied. In the original coordinates, this then implies that the estimates (1-7), (1-8), (1-9) hold with $\sigma = 0$ over sets of diameter $\approx \lambda^{-\sigma}$. By incorporating the flux estimates from [Smith 2006a], it can then be seen that Theorems 1.1 and 1.2 follow by taking a sum over all such sets.

The bounds (1-1) for $C^{1,1}$ metrics in [Smith 2006b] (and those for manifolds with boundary in [Smith and Sogge 2007]) were proved by wave equation methods. Specifically, square function estimates are developed for solutions to the wave equation on these manifolds, bounding the $L^q(M)$ norm of the square function

$$x \mapsto \left(\int_0^1 |u(t,x)|^2\, dt\right)^{1/2}, \quad \text{where } (\partial_t^2 - \Delta_g)u = 0.$$

As will be seen below, the spectral clusters above naturally give rise to solutions to the wave equation, and these estimates imply bounds on the $\Pi_\lambda f$. Square function estimates were first proved in [Mockenhaupt et al. 1993] for smooth metrics, using that Fourier integral operators can be used to invert the equation. However, when $g \in C^{1,1}$, the roughness of the metric means that these methods are inapplicable, so a crucial development [Smith 2006b] was the construction of a suitable parametrix using wave packet methods. The resulting approximate solution operators can be thought of as generalized Fourier integral operators where the associated canonical relation satisfies the curvature condition in [Mockenhaupt et al. 1993].

We follow the same strategy here, essentially proving bounds on the $L^q(P)$ norm of the square function above. Once again, the roughness of the metric means that we are led to use wave packet methods to construct a parametrix. In this case, the canonical relations which arise naturally have folding singularities. In Theorem 1.1, the relation has a one-sided fold and in Theorem 1.2 the relation essentially has a two-sided fold. There is a significant body of work on $L^2 \to L^q$ bounds for Fourier integral operators with folding singularities; see [Greenleaf and Seeger 1994; Hörmander 1985; Melrose and Taylor 1985; Pan and Sogge 1990; Cuccagna 1997] (the first of which treats one-sided folds). A key technical development in the present work is that the operators arising from the wave packet transform satisfy the desired square function estimates in spite of the inapplicability of these results for Fourier integral operators. Nonetheless, the approach taken here is in part inspired by these works.

Notation. We use C^α to denote the Hölder class of order α. Moreover, $C^{1,\alpha}$ will denote the class of metrics or functions whose first derivative is in C^α, taking the contrived convention that Lipschitz regularity is allowable when $\alpha = 0$. In what follows, $X \lesssim Y$ will denote that $X \le CY$ for some implicit constant C which is in some sense uniform, though when used in decay estimates, it may depend on the order N. Similarly, $X \approx Y$ will denote that $X \lesssim Y$ and $Y \lesssim X$. We use d as the differential which carries scalar functions to covector fields and vectors into matrices in the natural way. Given a metric g under discussion, we let $\langle \cdot, \cdot \rangle_g$, $|\cdot|_g$ denote the inner product and length induced by the metric either in the tangent or cotangent space. Lastly, given a vector $x \in \mathbb{R}^n$, x' and x'' will typically denote a vector in $\mathbb{R}^l$, $l < n$, formed by taking a subcollection of the components of x. The nature of this subcollection may vary depending on the section.

Remark on admissibility conditions. The admissibility condition $\delta + \sigma/q < 1 + \alpha$ (with equality allowed when $\alpha = 0, 1$) arises in Section 2, where elliptic regularity is used to show that when a cluster $\Pi_\lambda f$ is considered in a coordinate system, the high frequency components (with respect to the Fourier transform) satisfy better bounds than those near frequency λ. However, it can be checked that the condition $\delta < \frac{1}{2}$ is always satisfied when $k = n-1$ and $2 \le q \le 2n/(n-1)$ and that $\delta < 5/6$ holds for sufficiently small $q > 2$ when $k = 2$, ensuring that, in many relevant cases, the admissibility condition is satisfied. On the other hand, Smith [2006a, p. 969] showed that the bound $\|\Pi_\lambda f\|_{L^\infty(M)} \lesssim \lambda^{(n-1)/2} \|f\|_{L^2(M)}$ holds whenever g is Lipschitz. The key observation here is that one can write $\Pi_\lambda f = \exp(-\lambda^{-2}\Delta_g)\Pi_\lambda \tilde{f}$ with $\|\Pi_\lambda \tilde{f}\|_{L^2(M)} \approx \|\Pi_\lambda f\|_{L^2(M)}$. The $L^\infty(M)$ bounds then follow by combining Saloff-Coste's Gaussian upper bounds [1992] on the heat kernel with Smith's $L^{2(n+1)/(n-1)}(M)$ bounds on $\Pi_\lambda f$.

However, the same argument gives the continuity of each $\Pi_\lambda f \in L^2(M)$ since the fixed time heat kernel is continuous on $M \times M$ (as observed in [Saloff-Coste 1992, Section 6]). Thus Smith's L^∞ bounds on spectral clusters imply L^∞ bounds on their restrictions and this can be interpolated with the $L^q(P)$ bounds for submanifolds of low codimension to see that, in many cases, the admissibility conditions can be relaxed. This also ensures that the restrictions are well-defined.

Remark on the optimality of (1-7). As noted above in (1-2), the examples in [Smith and Sogge 1994; Smith and Tataru 2002] show that the bounds from [Smith 2006a] establishing $L^q(M)$ bounds are sharp for small values of $q > 2$. We comment here that the same examples show that the bounds (1-7) in Theorem 1.1 are sharp as well. Indeed, the examples in [Smith and Sogge 1994] produce metrics of $C^{1,\alpha}$ regularity and associated spectral clusters f_λ which are concentrated in a tube of length 1 and diameter $\lambda^{-2/(3+\alpha)}$, that is, a set of the form

$$|x_1| \lesssim 1, \quad |(x_2, \dots, x_n)| \lesssim \lambda^{-2/(3+\alpha)}. \tag{1-10}$$

Therefore if we take P to be defined by $x_n = 0$, we see that the rapid decay outside of this set implies

$$\frac{\|f_\lambda\|_{L^q(P)}}{\|f_\lambda\|_{L^2(M)}} \approx \lambda^{\frac{2}{3+\alpha}\left(\frac{n-1}{2} - \frac{n-2}{q}\right)}.$$

However, $\frac{1}{2}(\sigma + 1) = 2/(3+\alpha)$, showing that the exponent simplifies to $\delta(1+\sigma)$ and hence the bound (1-7) is optimal.

2. Microlocal reductions

In this section, we will reduce the main theorems to proving square function estimates for frequency localized solutions to a hyperbolic pseudodifferential equation. We follow an approach due to Smith [2006a]; see also [Blair et al. 2008]. The needed reductions are fairly common to both theorems, so we begin by treating all cases at the same time. It is thus convenient to take the convention that $\delta(\sigma)$ is defined by taking the power of λ appearing in (1-7), (1-8), or (1-9), realizing that in all cases $\delta(0)$ denotes the power without loss of derivatives. Moreover, the admissibility conditions mean that if $\sigma > 0$ and $0 < \alpha < 1$, $\delta(\sigma) - 1 < \alpha$ (respectively $\delta(\sigma) - 1 \le \alpha$ when $\alpha = 0, 1$).

Throughout these preliminary reductions, we will make use of the fact that when $k < n$, we have the following embedding for traces in $\mathbb{R}^k \times \{0\}$, $\{0\} \in \mathbb{R}^{n-k}$:

$$H^{n/2-k/q}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^k \times \{0\}), \tag{2-1}$$

which can be seen by first applying Sobolev embedding on $\mathbb{R}^k \times \{0\}$, and then using the trace theorem for L^2 based Sobolev spaces. The estimates in Theorems 1.1 and 1.2 thus exhibit a gain relative to Sobolev embedding. The gain is largest when $q = 2$: a quarter or a third of a derivative when $k = n-1$, depending on whether the submanifold is curved, and half a derivative (up to a possible logarithmic correction) when $k \le n-2$.

It suffices to prove the main theorem for a spectral cluster f satisfying $f = \Pi_\lambda f$. We begin by observing that f satisfies the following bounds in Sobolev spaces defined by the spectral resolution of Δ_g

$$\|(\Delta_g+\lambda^2)f\|_{H^s(M)}+\|df\|_{H^s(M)}\lesssim\lambda^{s+1}\|f\|_{L^2(M)}.$$

It thus suffices to prove bounds on f of the form

$$\|f\|_{L^q(P)}\lesssim\sum_i\lambda^{\delta(\sigma)-1-s_i}(\lambda\|f\|_{H^{s_i}(M)}+\|df\|_{H^{s_i}(M)}+\|(\Delta_g+\lambda^2)f\|_{H^{s_i}(M)}) \tag{2-2}$$

where a sum is taken over a finite collection of $0\le s_i\le 1$.

Multiplication by any smooth bump function ψ preserves $H^1(M)$, and, by interpolation, $H^s(M)$ for any $s\in[0,1]$. Therefore, by taking a partition of unity on M, it suffices to prove (2-2) with f replaced by ψf, where ψ is supported in a suitable coordinate chart which intersects P. Specifically, we will take slice coordinates so that P is identified with $\mathbb{R}^k\times\{0\}$. Furthermore, by taking a sufficiently fine partition of unity and dilating coordinates, we may assume that for some c_0 sufficiently small,

$$\|g^{ij}-\delta_{ij}\|_{C^{1,\alpha}(\mathbb{R}^n)}\le c_0. \tag{2-3}$$

By elliptic regularity (see, for example, [Gilbarg and Trudinger 1983, Theorem 8.10, Theorem 9.11]) and interpolation, we have, for any g supported in this coordinate chart, $\|g\|_{H^s(M)}\approx\|g\|_{H^s(\mathbb{R}^n)}$ for $s\in[0,2]$. Next we observe that in coordinates within $\operatorname{supp}(\psi)$, f satisfies an equation of the form

$$gd^2f+\lambda^2f=w,\quad gd^2f=\sum_{1\le i,j\le n}g^{ij}\partial^2_{ij}f \tag{2-4}$$

where w is a sum consisting of $(\Delta_g+\lambda^2)f$ and products of the form $a\cdot\partial_j f$, with $a\in C^\alpha$ (or L^∞, $C^{0,1}$ when $\alpha=0,1$ respectively) in turn a product of functions of the form g^{ij}, $\sqrt{\det g_{ij}}$, or their first derivatives. Hence multiplication by these functions preserves $H^s(\mathbb{R}^n)$ for $s=0$ and $s\in[0,\alpha)$ when $\alpha>0$ (respectively $s\in[0,1]$ when $\alpha=1$) meaning that that, for any such s,

$$\|w\|_{H^s(\mathbb{R}^n)}\lesssim\|f\|_{H^s(M)}+\|df\|_{H^s(M)}+\|(\Delta_g+\lambda^2)f\|_{H^s(M)}.$$

Furthermore, elliptic regularity (see, for example, [Gilbarg and Trudinger 1983, Theorem 9.11]) also gives that

$$\|d^2f\|_{L^2(\mathbb{R}^n)}\lesssim\|\Delta_g f\|_{L^2(\mathbb{R}^n)}+\|df\|_{L^2(\mathbb{R}^n)}+\|f\|_{L^2(\mathbb{R}^n)}. \tag{2-5}$$

Moreover, when $\delta(\sigma)>1$ (which only occurs when $\alpha>0$), we have

$$\|[g^{ij},\langle D\rangle^{\delta(\sigma)-1}]\partial^2_{ij}f\|_{L^2(\mathbb{R}^n)}+\|[\partial_i g^{ij},\langle D\rangle^{\delta(\sigma)-1}]\partial_j f\|_{L^2(\mathbb{R}^n)}\lesssim\|df\|_{H^{\delta(\sigma)-1}(\mathbb{R}^n)}, \tag{2-6}$$

where $\langle D\rangle$ denotes the Fourier multiplier with symbol $(1+|\xi|^2)^{1/2}$. This means that we may replace L^2 by $H^{\delta(\sigma)-1}$ in (2-5). Indeed, the bound on the first term in (2-6) follows as a consequence of the Coifman–Meyer commutator theorem (see, for example, [Taylor 1991, Proposition 3.6B]) and the second follows since the admissibility condition on $\delta(\sigma)$ implies that multiplication by $\partial_i g^{ij}$ preserves $H^{\delta(\sigma)-1}(\mathbb{R}^n)$.

With this in mind, we define the following norm when $\delta(\sigma)\le 1$:

$$|||f|||:=\|f\|_{L^2(\mathbb{R}^n)}+\lambda^{-1}\|df\|_{L^2(\mathbb{R}^n)}+\lambda^{-2}\|d^2f\|_{L^2(\mathbb{R}^n)}+\lambda^{-1}\|w\|_{L^2(\mathbb{R}^n)}.$$

When $\delta(\sigma) > 1$, we define

$$|||f||| := \sum_{j=0}^{2} \lambda^{-j}\|d^j f\|_{L^2(\mathbb{R}^n)} + \lambda^{-1}\|w\|_{L^2(\mathbb{R}^n)} + \lambda^{-(\delta(\sigma)-1)}\Big(\sum_{j=0}^{2}\lambda^{-j}\|d^j f\|_{H^{\delta(\sigma)-1}(\mathbb{R}^n)} + \lambda^{-1}\|w\|_{H^{\delta(\sigma)-1}(\mathbb{R}^n)}\Big).$$

Given the observations above, it now suffices to show that

$$\|f\|_{L^q(\mathbb{R}^k\times\{0\})} \lesssim \lambda^{\delta(\sigma)}|||f|||. \tag{2-7}$$

Without loss of generality, we may assume that f is supported in a cube of sidelength 1 centered at the origin and that the metric is defined over a cube of sidelength 8 centered at the origin. Hence we may smoothly extend the metric g so that it is defined over all of $\mathbb{R}^n$ and is equal to the flat metric for $|x|$ sufficiently large without altering the equation for f. Given $r > 0$, we let $S_r = S_r(D)$ denote a Fourier multiplier which applies a smooth cutoff to frequencies $|\xi| \le r$ and define $\mathrm{g}_\lambda = S_{c^2\lambda}\mathrm{g}$ where $c > 0$ will be taken to be sufficiently small. Since

$$\|\mathrm{g}_\lambda - \mathrm{g}\|_{L^\infty} \lesssim \lambda^{-1}, \tag{2-8}$$

we may replace g by g_λ in (2-4) when $\delta(\sigma) \le 1$, as the error can be absorbed into the right-hand side of (2-7). The same holds when $1 < \delta(\sigma)$ is admissible, which can be seen by using the similar bound $\|\mathrm{g}_\lambda - \mathrm{g}\|_{C^\alpha} \lesssim \lambda^{-1}$ and the fact that multiplication by a C^α function preserves $H^{\delta(\sigma)-1}(\mathbb{R}^n)$.

We now write f as $f = f_{<\lambda} + f_\lambda + f_{>\lambda}$ where $f_{<\lambda} = S_{c\lambda}f$ and $f_{>\lambda} = f - S_{c^{-1}\lambda}f$. Observe that, when $s = 0$,

$$\big\|[S_{c\lambda}, \mathrm{g}_\lambda]\big\|_{H^s\to H^s} + \big\|[S_{c^{-1}\lambda}, \mathrm{g}_\lambda]\big\|_{H^s\to H^s} \lesssim \lambda^{-1}, \tag{2-9}$$

which follows from simple bounds on the kernel of the commutators. When $1 < \delta(\sigma)$ is admissible, the same holds with $s = \delta(\sigma) - 1$. Indeed, $\lambda S_{c\lambda}$ (and similarly $\lambda S_{c^{-1}\lambda}$) defines an operator in $S^1_{1,0}$, hence the symbolic calculus gives $[\lambda S_{c\lambda}, \mathrm{g}_\lambda] \in C^\alpha S^0_{1,0}$ (in the notation of [Taylor 1991]). The claim then follows by [Taylor 1991, Proposition 2.1D] or by commuting with derivatives when $\alpha = 1$. Defining $w_{<\lambda} := \mathrm{g}_\lambda d^2 f_{<\lambda} + \lambda^2 f_{<\lambda}$, $w_{>\lambda} := \mathrm{g}_\lambda d^2 f_{>\lambda} + \lambda^2 f_{>\lambda}$, we have

$$\|w_{<\lambda}\|_{H^s(\mathbb{R}^n)} + \|w_{>\lambda}\|_{H^s(\mathbb{R}^n)} \lesssim \lambda^{-1}\|d^2 f\|_{H^s(\mathbb{R}^n)} + \|w\|_{H^s(\mathbb{R}^n)} \tag{2-10}$$

for $s = 0$ and for $s = \delta(\sigma) - 1$ when the latter quantity is positive.

To bound $f_{<\lambda}$, $f_{>\lambda}$, we use arguments from [Smith 2006a, Corollary 5]. Since $\|\mathrm{g}_\lambda d^2 f_{<\lambda}\|_{L^2} \lesssim (c\lambda)^2\|f_{<\lambda}\|_{L^2}$, (2-1) and the equation give the stronger estimate

$$\|f_{<\lambda}\|_{L^q(\mathbb{R}^k\times\{0\})} \lesssim \lambda^{n/2-k/q}\|f_{<\lambda}\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{n/2-k/q-2}\|w_{<\lambda}\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{n/2-k/q-1}|||f|||.$$

For the high frequency term $f_{>\lambda}$, we use that, when $s \ge 0$,

$$\lambda^2\|f_{>\lambda}\|_{H^s(\mathbb{R}^n)} + \lambda\|df_{>\lambda}\|_{H^s(\mathbb{R}^n)} \lesssim c\|d^2 f_{>\lambda}\|_{H^s(\mathbb{R}^n)}.$$

This bound with $s = 0$ can be combined with elliptic regularity to obtain

$$\|d^2 f_{>\lambda}\|_{L^2(\mathbb{R}^n)} \lesssim \|w_{>\lambda}\|_{L^2(\mathbb{R}^n)}. \tag{2-11}$$

When $n/2-k/q \le 2$, (2-1) yields a gain of at least 1/2 of a derivative in the estimate for $f_{>\lambda}$. The case $n/2-k/q>2$ only arises when $\alpha>0$ and $\delta(\sigma)=(n-1)/2-k/q+\sigma/q$, and in this case we use (2-6) (with g_λ replacing g) to bootstrap the elliptic regularity estimate, which yields a similar gain for $f_{>\lambda}$, since

$$\|f_{>\lambda}\|_{H^{\delta(\sigma)+1}(\mathbb{R}^n)} \lesssim \|w_{>\lambda}\|_{H^{\delta(\sigma)-1}(\mathbb{R}^n)} \lesssim |||f|||.$$

We are now reduced to proving bounds on f_λ. Reasoning as in (2-9), we obtain $|||f_\lambda||| \lesssim |||f|||$. We now impose a further microlocal decomposition of the function, writing $f_\lambda = f_{\lambda,T}+f_{\lambda,N}$, where $\hat{f}_{\lambda,T}$ is localized to directions tangent to the submanifold and $\hat{f}_{\lambda,N}$ is localized to normal directions. Specifically, we write $f_{\lambda,N}=\sum_{j=k+1}^n f_{\lambda,j}$ where $f_{\lambda,j}$ is frequency localized to a set of the form

$$\operatorname{supp}(\widehat{f_{\lambda,j}}) \subset \{\xi : \lambda \approx |\xi|,\ |\xi_j| \gtrsim \varepsilon|(\xi_1,\dots,\xi_j,\xi_{j+1},\dots,\xi_n)|\},$$

with ε suitably small. Using (2-9) again, we have

$$\|\mathrm{g}_\lambda d^2 f_{\lambda,j}+\lambda^2 f_{\lambda,j}\|_{L^2(\mathbb{R}^n)} \lesssim |||f_\lambda|||. \tag{2-12}$$

With this in mind, the flux estimates of [Smith 2006a, p. 974] give

$$\|f_{\lambda,j}\|_{L^\infty_{x_j}L^2_{x'}} \lesssim |||f_\lambda||| \tag{2-13}$$

where x' denotes the vector consisting of every component in $\mathbb{R}^n$ but x_j. Combining this with the $n-1$ dimensional version of (2-1) on the hyperplane $x_j=0$, we have

$$\|f_{\lambda,j}\|_{L^q(\mathbb{R}^k\times\{0\})} \lesssim \lambda^{(n-1)/2-k/q}\|f_{\lambda,j}\|_{L^2(x_j=0)} \lesssim \lambda^{\delta(\sigma)}|||f_\lambda|||.$$

We now further decompose $f_{\lambda,T}$ as $f_{\lambda,T}=\sum_j f_{\lambda,\omega_j}$ where $\{\omega_j\}$ is a finite collection of unit vectors and $\operatorname{supp}(\widehat{f_{\lambda,\omega_j}})$ lies in a small conic set containing ω_j. Without loss of generality, it suffices to treat the case $\omega_j=-e_1=(-1,0,\dots,0)$. Recalling (2-12) and simplifying notation, it now suffices to prove $\|f_\lambda\|_{L^q(\mathbb{R}^k\times\{0\})} \lesssim \lambda^{\delta(\sigma)}|||f_\lambda|||$ for f_λ satisfying

$$\operatorname{supp}(\widehat{f_\lambda}) \subset \{\xi : |\xi/|\xi|-(-e_1)| \lesssim \varepsilon\}. \tag{2-14}$$

As a consequence of (2-13) with $x_j=x_1$ and Hölder's inequality, if S_R is a slab of the form $S_R=\{x : |x_1-r|\le R\}$ for some r,

$$\|f_\lambda\|_{L^2(S_R)} \lesssim R^{1/2}|||f_\lambda|||. \tag{2-15}$$

Set $\rho=(n-1)/2-k/q$ so that $\rho-\delta(0)=\delta(0)-1/q$ when $k=n-1$, $2\le q\le 2n/(n-1)$, and $\rho=\delta(0)$ in all other cases of Theorem 1.1. Given a cube Q_R of sidelength $R=\lambda^{-\sigma}$ which intersects $\mathbb{R}^k\times\{0\}$, we let Q_R^* denote its double, and also set $w_\lambda := \mathrm{g}_\lambda d^2 f_\lambda+\lambda^2 f_\lambda$. We claim that Theorem 1.1 now follows from the bound

$$\|f_\lambda\|_{L^q((\mathbb{R}^k\times\{0\})\cap Q_R)} \lesssim \lambda^{(1-\sigma)\delta(0)}R^{-\rho}\big(R^{-1/2}\|f_\lambda\|_{L^2(Q_R^*)}+R^{1/2}\lambda^{-1}\|w_\lambda\|_{L^2(Q_R^*)}\big). \tag{2-16}$$

Moreover, Theorem 1.2 will follow from taking $q=2$ and $\delta(0)=\frac{1}{6}$ here when P is curved (as $\rho=0$ in this case). Indeed, if these bounds hold, we may sum over the cubes Q_R contained in S_R which intersect

$\mathbb{R}^k \times \{0\}$ to obtain

$$\|f_\lambda\|_{L^q((\mathbb{R}^k\times\{0\})\cap S_R)} \lesssim \lambda^{(1-\sigma)\delta(0)+\sigma\rho}(R^{-1/2}\|f_\lambda\|_{L^2(S_R^*)} + R^{1/2}\lambda^{-1}\|w_\lambda\|_{L^2(S_R^*)}).$$

Recalling (2-15), the right-hand side is bounded by $\lambda^{(1-\sigma)\delta(0)+\sigma\rho}|||f_\lambda|||$. Given the previous observations on ρ, the desired bound on f_λ then follows by taking a sum over the $\mathbb{O}(R^{-1})$ slabs S_R in $|x_1| \le 3/4$ and the rapid decay property

$$|f_\lambda(x)| \lesssim (\lambda|x|)^{-N}\|f_\lambda\|_{L^2(\mathbb{R}^n)} \quad \text{for } \max_j |x_j| \ge \tfrac{3}{4}. \tag{2-17}$$

The latter is a consequence of our assumption that f is supported in a cube of sidelength 1 at the origin, which implies that f_λ is concentrated in a λ^{-1} neighborhood of this cube.

At this stage, we pause to remark on a useful feature of our metric when P is curved. Let N be a suitable unit normal vector field such that $\langle N, \partial_n\rangle > 0$. Observe that given any $n-1$ vector $(X^1, \dots, X^{n-1})$ such that $(X^1)^2 + \cdots + (X^{n-1})^2 = 1$, we may assume that, over P, the quantity

$$-\sum_{1\le i,j\le n-1} \langle N, \nabla_{\partial_i}\partial_j\rangle_{\mathrm{g}} X^i X^j \tag{2-18}$$

is uniformly bounded from above and below. Indeed, since $\partial_1, \dots, \partial_{n-1}$ span the tangent space to P, one just applies the assumption that P is curved to constant vector fields of the form $X^j\partial_j$ (with the summation convention in effect). Using that $\nabla_{\partial_i}\partial_j$ is the vector field $\Gamma^k_{ij}\partial_k$, we may use that $\langle N, \partial_k\rangle_{\mathrm{g}} \equiv 0$ on P for $k \ne n$ and that $\langle N, \partial_n\rangle_{\mathrm{g}}$ is bounded above to get that

$$-\sum_{1\le i,j\le n-1} \Gamma^n_{ij} X^i X^j \tag{2-19}$$

is uniformly bounded from above and below over P for all such $(X^1, \dots, X^{n-1})$. Using that $\|\mathrm{g}-\mathrm{g}_\lambda\|_{C^1} \lesssim \lambda^{-\alpha}$, the bounds also hold when the Christoffel symbols are taken with respect to g_λ.

Returning to the proof of (2-16), we dilate variables $x \mapsto Rx$, set $\mu := R\lambda$, and make the slight abuse of notation that $f_\mu(x) = f_\lambda(Rx)$. We will see that this reduces the general bounds to those without a loss of derivatives, and hence we will take $\delta = \delta(0)$ below. Indeed, rescaling the bound (2-16) gives

$$\|f_\mu\|_{L^q((\mathbb{R}^k\times\{0\})\cap Q)} \lesssim \mu^\delta(\|f_\mu\|_{L^2(Q^*)} + \mu^{-1}\|\mathrm{g}_\mu d^2 f_\mu + \mu^2 f_\mu\|_{L^2(Q^*)}). \tag{2-20}$$

When P is curved, rescaling yields the same with $q = 2$ and $\delta = (1+\beta)/6$ where $\beta = \sigma/(1-\sigma)$. Here Q is now a cube of sidelength 1, which we may take to be centered at the origin, and $\mathrm{g}_\mu(x) := \mathrm{g}_\lambda(Rx)$. We now have that if $\mathrm{g}_{\mu^{1/2}} := S_{c^2\mu^{1/2}}\mathrm{g}_\mu$, then (cf. (2-3))

$$\|\mathrm{g}_\mu - \mathrm{g}_{\mu^{1/2}}\|_{L^\infty} \lesssim c_0\mu^{-1}, \tag{2-21}$$

and we may replace g_μ by $\mathrm{g}_{\mu^{1/2}}$ in (2-20), since the error can be absorbed into the right-hand side. The metric $\mathrm{g}_{\mu^{1/2}}$ has C^2 regularity, namely,

$$\|\mathrm{g}^{ij}_{\mu^{1/2}} - \delta_{ij}\|_{C^2} \lesssim c_0 \quad \text{and} \quad \|\partial^\alpha \mathrm{g}^{ij}_{\mu^{1/2}}\|_{C^2} \le \mu^{1/2(|\alpha|-2)} \text{ for } |\alpha| \ge 2. \tag{2-22}$$

We pause again to discuss the effect of this dilation and regularization on the upper and lower bounds on (2-19) for curved metrics. For unit $n-1$ vectors $(X^1,\dots,X^{n-1})$, we now have

$$c_1 \le -\mu^{\beta} \sum_{1\le i,j\le n-1} \Gamma^n_{ij}(x) X^i X^j \lesssim c_0 \tag{2-23}$$

for $x \in P$. Here the Christoffel symbols can be taken with respect to the metric $\mathrm{g}_{\mu^{1/2}}$, since we now have (2-21) and $\|\mathrm{g}_\mu - \mathrm{g}_{\mu^{1/2}}\|_{C^{1,\alpha}} \lesssim \mu^{-1/2} \ll \mu^{-\beta}$. Moreover, by continuity, we may assume that if c_0 is chosen sufficiently small, then the inequality holds for all $x \in Q$ at the expense of decreasing c_1 slightly.

We will prove the bound (2-20) by wave equation methods. Let $u_\mu(t,x) = \cos(t\mu) f_\mu(x)$. It suffices to show that if $F_\mu = (\partial_t^2 - \mathrm{g}_{\mu^{1/2}} d^2) u_\mu$,

$$\|u_\mu\|_{L^q((\mathbb{R}^k\times\{0\})\cap Q; L^2(-\frac12,\frac12))} \lesssim \mu^{\delta}(\|u_\mu(0,\cdot)\|_{L^2(Q^*)} + \mu^{-1}\|F_\mu\|_{L^2((-1,1)\times Q^*)}).$$

Now let $\psi(t,x)$ denote a smooth cutoff that is identically 1 on $\left(-\frac12,\frac12\right)^{n+1}$ and supported in $\left(-\frac34,\frac34\right)^{n+1}$. Replacing u_μ by ψu_μ, and similarly for F_μ, it suffices to show that

$$\|u_\mu\|_{L^q(\mathbb{R}^k\times\{0\}; L^2(\mathbb{R}))} \lesssim \mu^{\delta}(\|u_\mu(0,\cdot)\|_{L^2(\mathbb{R}^n)} + \mu^{-1}\|F_\mu\|_{L^2(\mathbb{R}^{n+1})}), \tag{2-24}$$

since energy estimates bound the error terms which arise when commuting $(\partial_t^2 - \mathrm{g}_{\mu^{1/2}} d^2)$ with ψ. Next we let $\Gamma^\pm_\mu(\tau,\xi)$ be smooth cutoffs to regions of the form

$$\{(\tau,\xi) : \pm\tau \approx |\xi|, |\xi| \approx \mu, |\xi/|\xi| - (-e_1)| \lesssim \varepsilon\} \tag{2-25}$$

and supported in a slightly larger set. Let $u^\pm_\mu = \Gamma^\pm_\mu(D_{t,x}) u_\mu$. By [Smith 2006b, Lemma 2.3] and the localization of f_μ, we see that elliptic regularity and (2-1) yield an estimate on $u_\mu - u^+_\mu - u^-_\mu$ with a gain of at least half a derivative relative to the right-hand side of (2-24). It thus suffices to prove (2-24) with u_μ replaced by $u^\pm_\mu$. The proof of the bound follows in the next two sections.

3. General submanifolds

In this section, we prove (3-4) and hence Theorem 1.1. Recall that coordinates are chosen so that P is identified with $(y,0) \in \mathbb{R}^n$ with $y \in \mathbb{R}^k$, $0 \in \mathbb{R}^{n-k}$. In this section, we take the following notational conventions on coordinates in $\mathbb{R}^n$. The letters w, y, z denote vectors in $\mathbb{R}^k$, and given such a vector we let $\bar y$ denote the vector in $\mathbb{R}^n$ determined by $\bar y = (y,0)$. The letters x, ξ, v typically denote vectors in $\mathbb{R}^n$, and we often decompose such a vector as $x = (x_1, x', x'')$ where $x' = (x_2,\dots,x_k)$, $x'' = (x_{k+1},\dots,x_n)$.

We begin by showing that $u^\pm_\mu$ solves an equation which is hyperbolic in x_1. Given (2-22), we have that, for (τ,ξ) in the regions (2-25), $\mathrm{g}^{ij}_{\mu^{1/2}}\xi_i\xi_j - \tau^2$ defines a quadratic in ξ_1 with two real roots and hence we may write

$$\mathrm{g}^{ij}_{\mu^{1/2}}(x)\xi_i\xi_j - \tau^2 = \mathrm{g}^{11}_{\mu^{1/2}}(x)(\xi_1 + q^-(x,\tau,\xi'))(\xi_1 - q^+(x,\tau,\xi')) \tag{3-1}$$

with $q^\pm > 0$ and homogeneous of degree 1 for such (τ,ξ). We further regularize these symbols, taking $p^\pm(\cdot,\tau,\xi') = S_{c^2\mu^{1/2}} q^\pm(\cdot,\tau,\xi')$. By the elliptic regularity argument in [Smith 2006b, Lemma 2.4], the function u_μ satisfies

$$(-i\partial_{x_1} + p^\pm(x, D_{t,x'}))u_\mu^\pm = G_\mu^\pm, \tag{3-2}$$

with $\|G_\mu^\pm\|_{L^2(\mathbb{R}^{n+1})}$ bounded by the terms in parentheses on the right-hand side of (2-24). Moreover, akin to (2-17), we have the rapid decay property

$$|u_\mu^\pm(t,x)| \lesssim (\mu|(t,x)|)^{-N}\|u_\mu\|_{L^2(\mathbb{R}^{n+1})}, \quad \text{for } \max(|t|,|x_1|,\dots,|x_n|) \ge 1. \tag{3-3}$$

Thus, by energy estimates, it can be seen that

$$\|u_\mu^\pm\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|u_\mu(0,\cdot)\|_{L^2(\mathbb{R}^n)} + \mu^{-1}\|\partial_t u_\mu(0,\cdot)\|_{L^2(\mathbb{R}^n)} + \mu^{-1}\|G_\mu\|_{L^2(\mathbb{R}^{n+1})},$$

since the right-hand side is compactly supported. By (3-3), it suffices to show that

$$\|u_\mu^\pm\|_{L^q((-1,1)\times\mathbb{R}^{k-1}\times\{0\};L^2(\mathbb{R}))} \lesssim \mu^\delta(\|u_\mu^\pm\|_{L^2(\mathbb{R}^{n+1})} + \mu^{-1}\|G_\mu^\pm\|_{L^2(\mathbb{R}^{n+1})}). \tag{3-4}$$

It suffices to treat the term u_μ^-, as bounds on the u_μ^+ will follow from time reversal. Hence we suppress the superscripts on u_μ^-, G_μ^-, p^- below and assume the minus sign is taken when referencing (2-25).

It is convenient to change the roles of t and x_1 above, and correspondingly τ and ξ_1, treating (3-2) as an equation which is hyperbolic in t, rather than in x_1. As a consequence of (2-22), p is now a function of (t,x,ξ) (or more precisely (t,x',x'',ξ)) satisfying the bounds

$$\left|\partial_{x,t}^\gamma\partial_\xi^\beta\left(p(t,x,\xi) - \sqrt{\xi_1^2 - |(\xi',\xi'')|^2}\right)\right| \lesssim c_0, \quad |\gamma| \le 2, \tag{3-5}$$

for $|\xi| = 1$ in a cone of the form

$$\{\xi : -\xi_1 \gtrsim \varepsilon^{-1}|(\xi',\xi'')|\}, \tag{3-6}$$

and c_0 can be replaced by $c_0\mu^{-\beta}$ when $|\gamma| = 1$. Moreover, for ξ in the same set,

$$|\partial_{x,t}^\gamma p(t,x,\xi)| \lesssim \mu^{\frac{1}{2}(|\gamma|-2)}, \quad |\gamma| \ge 2. \tag{3-7}$$

By (3-3) and time translation, it suffices to prove that over the time interval $(0,1)$,

$$\|u_\mu\|_{L^q_{t,y'}L^2_{y_1}} \lesssim \mu^\delta(\|u_\mu\|_{L^\infty_t L^2_x} + \|G_\mu\|_{L^2_{t,x}})$$

where we understand the left-hand side to be

$$\left(\int_0^1\int_{\mathbb{R}^{k-1}\times\{0\}}\left(\int_{\mathbb{R}}|u_\mu(t,\bar{y})|^2\,dy_1\right)^{q/2}dy'\,dt\right)^{1/q}, \quad y' = (y_2,\dots,y_k),$$

and the $L^\infty_t L^2_x$ norm on the right-hand side as $L^\infty((0,1);L^2(\mathbb{R}^n))$. Moreover, since

$$p(t,x,D) - p^*(t,x,D) \in Op(S^0_{1,\frac{1}{2}}),$$

we may differentiate $\|u_\mu(t,\cdot)\|^2_{L^2_y}$ in t to obtain

$$\|u_\mu\|_{L^\infty_t L^2_y} \lesssim \|u_\mu\|_{L^2(\mathbb{R}^{n+1})} + \|G_\mu\|_{L^2(\mathbb{R}^{n+1})}.$$

Let the wave packet transform $T_\mu : \mathcal{S}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{2n})$ be defined by

$$T_\mu f(x,\xi) = \mu^{n/4} \int e^{-i\langle \xi, v-x\rangle} \phi(\mu^{1/2}(v-x)) f(v)\, dv$$

where ϕ is a real-valued, radial Schwartz function such that $\operatorname{supp}(\hat{\phi})$ is contained in the unit ball and normalized so that $\|\phi\|_{L^2} = (2\pi)^{-n/2}$. The normalization ensures that $T_\mu^* T_\mu$ is the identity on $L^2(\mathbb{R}^n)$ and hence $\|T_\mu f\|_{L^2(\mathbb{R}^{2n}_{x,\xi})} = \|f\|_{L^2(\mathbb{R}^n_z)}$. Let $g_\mu(x) := u_\mu(0,x)$ and let $\Theta_{r,t}(x,\xi)$ denote the time-r value of the integral curve determined by the Hamiltonian flow of p with $\Theta_{r,t}(x,\xi)|_{r=t} = (x,\xi)$. Given [Smith 2006b, Lemma 3.2, Lemma 3.3], we may write

$$(T_\mu u_\mu)(t,x,\xi) = T_\mu g_\mu(\Theta_{0,t}(x,\xi)) + \int_0^t \widetilde{G}_\mu(r, \Theta_{r,t}(x,\xi))\, dr \tag{3-8}$$

where $\widetilde{G}$ satisfies

$$\int_0^t \|\widetilde{G}_\mu(r,\cdot)\|_{L^2(\mathbb{R}^{2n}_{x,\xi})}\, dr \lesssim \|u_\mu\|_{L^\infty_t L^2_v} + \int_0^t \|G_\mu(r,\cdot)\|_{L^2(\mathbb{R}^n_v)}\, dr, \tag{3-9}$$

for $t \in (0,1)$. Indeed, these lemmas show that if H_p denotes the Hamiltonian vector field defined by p, then $T_\mu p(\cdot\,, D) - H_p T_\mu$ defines an operator bounded on L^2, and that (3-8) follows by solving the corresponding transport equation. Furthermore, given the frequency localization of $p(\cdot\,,\xi)$ and the compact support of ϕ, we may assume that uniformly in r, x, we have

$$\operatorname{supp}((T_\mu g_\mu)(x,\cdot)),\ \operatorname{supp}(\widetilde{G}(r,x,\cdot)) \subset \{\xi : |\xi| \approx \mu, -\xi_1 \gtrsim \varepsilon^{-1}|(\xi',\xi'')|\}. \tag{3-10}$$

Define the propagator

$$W\tilde{f}(t,y) = T_\mu^*(\tilde{f} \circ \Theta_{0,t})(\bar{y}),$$

and observe that, given (3-8), (3-9), it suffices to show that

$$\|W\tilde{f}\|_{L^q_{t,y'} L^2_{y_1}} \lesssim \mu^\delta \|\tilde{f}\|_{L^2_{x,\xi}} \tag{3-11}$$

with a $(\log\mu)^{1/2}$ loss when $(k,q) = (n-2,2)$. Let W_t denote the restricted operator $W_t\tilde{f}(y) = W\tilde{f}(r,y)|_{r=t}$. By duality, it suffices to see that, for functions $F(s,z)$,

$$\|WW^*F\|_{L^q_{t,y'} L^2_{y_1}} \lesssim \mu^{2\delta} \|F\|_{L^{q'}_{s,z'} L^2_{z_1}}. \tag{3-12}$$

To prove this, we will show that

$$\|W_t W_s^* h\|_{L^\infty_{y'} L^2_{y_1}} \lesssim \mu^{n-1}(1+\mu|t-s|)^{-(n-1)/2} \|h\|_{L^1_{y'} L^2_{y_1}}, \tag{3-13}$$

$$\|W_t W_s^* h\|_{L^2_y} \lesssim \mu^{n-k}(1+\mu|t-s|)^{-(n-k)/2} \|h\|_{L^2_y}. \tag{3-14}$$

When $k = n-2$ and $q = 2$, Young's inequality and (3-14) give (3-12) with the logarithmic loss. In all other cases with $k \le n-2$, we may interpolate (3-13) and (3-14) to obtain

$$\|W_t W_s^* h\|_{L^q_{y'} L^2_{y_1}} \lesssim \mu^{2(\frac{n-1}{2}-\frac{k-1}{q})}(1+\mu|t-s|)^{-(\frac{n-1}{2}-\frac{k-1}{q})} \|h\|_{L^{q'}_{y'} L^2_{y_1}} \tag{3-15}$$

and use that $(1+|s|)^{-((n-1)/2-(k-1)/q)} \in L^{q/2}(\mathbb{R})$ to get (3-12). The same argument works when $k = n-1$ and $2n/(n-1) < q < \infty$. To handle the remaining cases when $k = n-1$, we use that

$$\mu^{2(\frac{n-1}{2}-\frac{n-2}{q})}(1+\mu|t-s|)^{-(\frac{n-1}{2}-\frac{n-2}{q})} \lesssim \mu^{\frac{n-1}{2}-\frac{n-2}{q}}|t-s|^{-\frac{n-1}{2}+\frac{n-2}{q}}.$$

Hence (3-12) follows from the Hardy–Littlewood–Sobolev inequality when $q = 2n/(n-1)$. When $2 \le q < 2n/(n-1)$, the right-hand side is in $L^{q/2}_{\mathrm{loc}}$ and Young's inequality gives (3-12).

In what follows, we will denote the integral kernel of $W_t W_s^*$ as $K_{t,s}(y,z)$. The bound (3-13) follows from the proofs of the bounds [Smith 2006b, (3.5); Smith and Sogge 2007, (5.4), (7.2)]. Those works establish the uniform inequality

$$\int |K_{t,s}(y,z)|\,dy_1 + \int |K_{t,s}(y,z)|\,dz_1 \lesssim \mu^{n-1}(1+\mu|t-s|)^{-(n-1)/2}.$$

It thus suffices to prove (3-14). Using that $(x,\xi) \mapsto \Theta_{r_1,r_2}(x,\xi)$ defines a diffeomorphism which preserves $dx \wedge d\xi$, the kernel of $W_t W_s^*$ can be realized as (cf. [Smith and Sogge 2007, p. 127])

$$K_{t,s}(y,z) = \mu^{n/2} \int e^{i\langle \xi, \bar{z}-x\rangle - i\langle \xi_{s,t}, \bar{y}-x_{s,t}\rangle} \phi(\mu^{1/2}(\bar{z}-x))\phi(\mu^{1/2}(\bar{y}-x_{s,t}))\Gamma(\xi)\,dx\,d\xi \tag{3-16}$$

with $(x_{s,t}, \xi_{s,t})$ abbreviating $(x_{s,t}(x,\xi), \xi_{s,t}(x,\xi))$. Here Γ is a cutoff supported in a region of the form appearing in (3-10), which may be inserted since we are only interested in functions $\tilde{f}$ satisfying that condition.

Before proceeding, we observe bounds on the bicharacteristic flow of p.

Theorem 3.1. *Suppose $(x,\xi) \in \mathbb{R}^{2n}$ with ξ in the set defined by (3-6). Let $\Theta_{t,s}(x,\xi)$ be as in (3-8), that is, $\Theta_{t,s}(x,\xi)|_{t=s} = (x,\xi)$ and*

$$\partial_s x_{t,s}(x,\xi) = d_\xi p(s, \Theta_{t,s}(x,\xi)), \quad \partial_s \xi_{t,s}(x,\xi) = -d_x p(s, \Theta_{t,s}(x,\xi)). \tag{3-17}$$

Then, for $t, s \in [0,1]$, first partials of $x_{t,s}(x,\xi)$, $\xi_{t,s}(x,\xi)$ in x, ξ satisfy

$$|d_x x_{t,s} - I| + |d_x \xi_{t,s}| \lesssim c_0|t-s|, \tag{3-18}$$

$$|d_\xi x_{t,s}(x,\xi) - \int_t^s d_\xi d_\xi p(\Theta_{r,t}(x,\xi))\,dr| + |d_\xi \xi_{t,s}(x,\xi) - I| \lesssim c_0|t-s|^2. \tag{3-19}$$

Proof. Differentiating the equations (3-17) gives

$$\partial_r \begin{bmatrix} dx_{t,r} \\ d\xi_{t,r} \end{bmatrix} = M(r, x_{t,r}, \xi_{t,r}) \begin{bmatrix} dx_{t,r} \\ d\xi_{t,r} \end{bmatrix}, \quad \text{where } M = \begin{bmatrix} d_x d_\xi p & d_\xi d_\xi p \\ -d_\xi d_x p & d_x d_x p \end{bmatrix}.$$

By Gronwall's inequality and the bounds (3-5), we have

$$|d_x x_{t,r} - I| + |d_x \xi_{t,r}| \lesssim 1, \quad |d_\xi x_{t,r}| + |d_\xi \xi_{t,r} - I| \lesssim 1,$$

and substituting these bounds back into the integral equation for $dx_{t,r}, d\xi_{t,r}$ implies the theorem. □

This type of argument can also be used to bound higher order derivatives of $x_{t,s}, \xi_{t,s}$, see, for example, (4-10) below. Such bounds are used in the proof of the next theorem. It is due to [Smith and Sogge

2007, Theorem 5.4], which obtains bounds on $K_{t,s}$ under the assumption that Γ is a smooth cutoff to a (possibly) smaller set.

Theorem 3.2. *Suppose $\bar{\theta} = \min(1, \mu^{-1/2}|t-s|^{-1/2})$ and the smooth cutoff Γ in (3-16) is supported in a set contained in (3-10) of the form*

$$\operatorname{supp}(\Gamma) \subset \{\xi : |\xi/|\xi| - \eta| \lesssim \bar{\theta}\} \tag{3-20}$$

for some unit vector $\eta \in \mathbb{S}^{n-1}$. Let $(x_{t,s}, \nu_{t,s}) = \Theta_{t,s}(\bar{z}, \eta)$. Then $K_{t,s}$ satisfies the pointwise bounds

$$|K_{t,s}(y,z)| \lesssim \mu^n \bar{\theta}^{n-1}(1 + \mu\bar{\theta}|\bar{y} - x_{t,s}| + \mu|\langle \nu_{t,s}, \bar{y} - x_{t,s}\rangle|)^{-N}. \tag{3-21}$$

Observing that $\mu^{n-k}(1+\mu|t-s|)^{-(n-k)/2} \approx \min(\mu^{n-k}, \mu^{(n-k)/2}|t-s|^{-(n-k)/2})$, we begin treating the case $|t-s| \le \mu^{-1}$, that is, the case where the first quantity is smaller. In this case, we apply (3-21) in Theorem 3.2 with $\bar{\theta} = 1$ and $\eta = -e_1$ to obtain

$$|K_{t,s}(y,z)| \lesssim \mu^n(1 + \mu|\bar{y} - x_{t,s}(z, -e_1)|)^{-N},$$

which gives the first half of (3-22) below. Making the measure-preserving change of variables

$$(x, \xi) \mapsto (x_{t,s}(x,\xi), \xi_{t,s}(x,\xi))$$

in (3-16), we may reverse the roles of y and z in Theorem 3.2 to obtain an analogous bound which yields

$$\int |K_{t,s}(y,z)|\,dy + \int |K_{t,s}(y,z)|\,dz \lesssim \mu^{n-k} \tag{3-22}$$

(strictly speaking, the change of variables replaces $\Gamma(\xi)$ by $\Gamma(\xi_{t,s}(x,\xi))$, but this does not change the validity of the bounds in Theorem 3.2).

It now suffices to treat the more involved case where $\mu^{-1} < |t-s| \le 1$, and for the remainder of this section we assume $t, s \in [0,1]$ are two fixed values satisfying this condition. Using the notation suggested by Theorem 3.2, we set $\bar{\theta} = \mu^{-1/2}|t-s|^{-1/2}$ so that $\mu\bar{\theta}^2|t-s| = 1$. Using a partition of unity, we take a decomposition $K_{t,s} = \sum_j K^j$ where K^j is defined by replacing Γ in (3-16) by a smooth cutoff Γ_j, with Γ_j supported in a set of the form $|\xi/|\xi| - \eta^j| \lesssim \bar{\theta}$ and $\{\eta^j\}$ is a collection of unit vectors in the cone $\{-\xi_1 \gtrsim \varepsilon^{-1}|(\xi', \xi'')|\}$ separated by a distance of at least $\approx \bar{\theta}^{-1}$. In particular, we may assume that, for fixed j,

$$\sum_l (1 + \bar{\theta}^{-1}|\eta^j - \eta^l|)^{-(n+1)} \lesssim 1. \tag{3-23}$$

Let T_j be the operator defined by $(T_j h)(y) = \int K^j(y,z)h(z)\,dz$ and observe that, since $|(\nu^j)_1| \approx 1$, (3-21) in Theorem 3.2 with $\eta = \eta^j$ gives

$$\int |K^j(y,z)|\,dy \lesssim \mu^{n-k}\bar{\theta}^{n-k}.$$

By the same symmetry argument used in (3-22), we now have

$$\|T_j h\|_{L^2_{\bar{y}}} \lesssim \mu^{n-k}\bar{\theta}^{n-k}\|h\|_{L^2_{\bar{y}}} = \mu^{(n-k)/2}|t-s|^{-(n-k)/2}\|h\|_{L^2_{\bar{y}}}$$

(though in what follows, it is convenient to express the bounds in terms of μ, $\bar{\theta}$).

We claim that there exists a constant C such that if $\bar\theta^{-1}|\eta^j-\eta^l|\ge C$, then

$$\|T_l^*T_j\|_{L^2\to L^2}+\|T_lT_j^*\|_{L^2\to L^2}\lesssim \mu^{2(n-k)}\bar\theta^{2(n-k)}(1+\bar\theta^{-1}|\eta^j-\eta^l|)^{-N}.$$

Since $W_tW_s^*=\sum_j T_j$, Cotlar's lemma then implies (3-14). Furthermore, we focus on the bound for $T_l^*T_j$, as a symmetric argument yields the bound on $T_lT_j^*$. Set

$$J_{j,l}(z,w)=\int \overline{K^l(y,z)}K^j(y,w)\,dy.$$

We will show that, for $\bar\theta^{-1}|\eta^j-\eta^l|\ge C$,

$$|J_{j,l}(z,w)|\lesssim \mu^{2n-k}\bar\theta^{2n-1-k}(1+\mu\bar\theta|z-w|+\mu|\langle\eta^l,\bar z-\bar w\rangle|+\bar\theta^{-1}|\eta^j-\eta^l|)^{-N}. \tag{3-24}$$

The proof of (3-24) varies based on whether $|(\eta_1^j-\eta_1^l,\dots,\eta_k^j-\eta_k^l)|\ge|(\eta^j-\eta^l)''|$ or the opposite inequality holds. In the first case, we write

$$J_{j,l}(z,w)=\mu^{n/2}\iint\left(\int e^{i\langle\xi,\bar y-x\rangle-i\langle\tilde\xi,\bar y-\tilde x\rangle}\phi(\mu^{1/2}(\bar y-x))\phi(\mu^{1/2}(\bar y-\tilde x))\,dy\right)\times\psi(z,w,x,\xi,\tilde x,\tilde\xi)\Gamma_j(\xi)\Gamma_l(\tilde\xi)\,dx\,d\xi\,d\tilde x\,d\tilde\xi \tag{3-25}$$

where $\tilde x,\tilde\xi$ denote the variables in the integral defining K_l and ψ is a function independent of y. The y integral in parentheses is a constant multiple of

$$\int e^{i\tilde\psi}\hat\phi(\mu^{-1/2}((\zeta_1,\zeta',\zeta'')-\xi))\hat\phi(\mu^{-1/2}((\zeta_1,\zeta',\tilde\xi'')-\tilde\xi))\,d\zeta_1d\zeta'd\zeta''d\tilde\xi'' \tag{3-26}$$

where $\tilde\psi$ is some real-valued phase function. Since $\operatorname{supp}(\hat\phi)$ is contained in the unit ball and

$$2|(\eta_1^j-\eta_1^l,\dots,\eta_k^j-\eta_k^l)|\ge|\eta^j-\eta^l|,$$

this integral vanishes if $\bar\theta^{-1}|\eta^l-\eta^j|\ge C$, as this implies that $|(\xi_1-\tilde\xi_1,\dots,\xi_k-\tilde\xi_k)|\gtrsim C\mu\bar\theta\ge C\mu^{1/2}$.

We now turn to the case where $|(\eta^j)''-(\eta^l)''|\ge|(\eta_1^j-\eta_1^l,\dots,\eta_k^j-\eta_k^l)|$. In this case, we use (3-21) in Theorem 3.2 to bound $|K_l|$, $|K_j|$ individually. After some minor manipulations, this yields

$$\begin{aligned}|J_{j,l}(z,w)|&\\ \lesssim\mu^{2n}\bar\theta^{2(n-1)}\int&\big(1+\mu\bar\theta|\bar y-x_{t,s}(\bar w,\eta^j)|+\mu\bar\theta|\bar y-x_{t,s}(\bar z,\eta^l)|+\mu|\langle\nu_{t,s}(\bar z,\eta^l),\bar y-x_{t,s}(\bar z,\eta^l)\rangle|\big)^{-6N}\\ &\times(1+\mu|\langle\nu_{t,s}(\bar w,\eta^j),\bar y-x_{t,s}(\bar w,\eta^j)\rangle-\langle\nu_{t,s}(\bar z,\eta^l),\bar y-x_{t,s}(\bar z,\eta^l)\rangle|)^{-N}\,dy\end{aligned} \tag{3-27}$$

We take $3N$ of the powers in the first factor of the integrand on the right and claim that up to implicit constants, it is bounded above by

$$(1+\mu\bar\theta|z-w|+\bar\theta^{-1}|\eta^j-\eta^l|)^{-3N}. \tag{3-28}$$

To see this, first observe that the $3N$ powers from the integrand are dominated by

$$(1+\mu\bar\theta|x_{t,s}(\bar z,\eta^l)-x_{t,s}(\bar w,\eta^j)|+64\mu\bar\theta|x''_{t,s}(\bar z,\eta^l)-x''_{t,s}(\bar w,\eta^j)|)^{-3N}.$$

By the bounds (3-18), (3-19) in Theorem 3.1, we have

$$|x_{t,s}(\bar z,\eta^l)-x_{t,s}(\bar w,\eta^j)|\geq \tfrac{3}{4}|z-w|-2|t-s|\,|\eta^l-\eta^j| \tag{3-29}$$

provided c_0 and ε are taken sufficiently small. Next we use that

$$\bigl|x''_{t,s}(\bar w,\eta^j)-x''_{t,s}(\bar z,\eta^l)\bigr|\geq \bigl|x''_{t,s}(\bar w,\eta^j)-x''_{t,s}(\bar w,\eta^l)\bigr|-\bigl|x''_{t,s}(\bar w,\eta^l)-x''_{t,s}(\bar z,\eta^l)\bigr|.$$

To bound the second term on the right, we use that as a consequence of (3-18) the $(n-k)\times n$ matrix $d_x x''_{t,s}$ satisfies

$$\bigl|d_x x''_{t,s}-[0\quad I_{n-k}]\bigr|\lesssim c_0|t-s|.$$

Recalling that $\bar w=(w,0)$, $\bar z=(z,0)$, this gives

$$\bigl|x''_{t,s}(\bar w,\eta^l)-x''_{t,s}(\bar z,\eta^l)\bigr|\lesssim c_0|t-s|\,|z-w|.$$

We now use (3-5), (3-19) to get that $d_\xi x''_{t,s}(x,\xi)$ is the $(n-k)\times n$ block matrix

$$(s-t)(\xi_1^2-|(\xi',\xi'')|^2)^{-3/2}\bigl[\xi_1\xi''\quad -\xi''(\xi')^T\quad -\bigl((\xi_1^2-|(\xi',\xi'')|^2)I_{n-k}+\xi''(\xi'')^T\bigr)\bigr]$$

plus an error term which is $\mathcal{O}(c_0|t-s|)$. Here ξ'' is taken to be a column vector. Since $|(\eta^l-\eta^j)''|\geq \frac{1}{2}|\eta^l-\eta^j|$ and $|(\xi',\xi'')|\lesssim\varepsilon|\xi_1|$, we have

$$|x''_{t,s}(\bar w,\eta^j)-x''_{t,s}(\bar w,\eta^l)|\geq \tfrac{1}{8}|t-s|\,|\eta^l-\eta^j|.$$

In summary, for some uniform constant M,

$$64|x''_{t,s}(\bar w,\eta^j)-x''_{t,s}(\bar z,\eta^l)|\geq 8|t-s|\,|\eta^l-\eta^j|-Mc_0|t-s|\,|z-w|. \tag{3-30}$$

By taking c_0 sufficiently small, the negative term in (3-30) can be absorbed by the first term in (3-29) and vice versa, which shows (3-28).

We now turn to the second factor in the integrand of (3-27). The triangle inequality gives

$$\mu\bigl|\langle\nu_{t,s}(\bar w,\eta^j),\bar y-x_{t,s}(\bar w,\eta^j)\rangle-\langle\nu_{t,s}(\bar z,\eta^l),\bar y-x_{t,s}(\bar z,\eta^l)\rangle\bigr|\geq\mu|\langle\eta^j,\bar z-\bar w\rangle|-E$$

with

$$E=\mu\bigl|\nu_{t,s}(\bar z,\eta^l)-\nu_{t,s}(\bar w,\eta^j)\bigr|\,\bigl|\bar y-x_{t,s}(\bar z,\eta^l)\bigr|+\mu\bigl|\langle\nu_{t,s}(\bar w,\eta^j),x_{t,s}(\bar z,\eta^l)-x_{t,s}(\bar w,\eta^j)\rangle-\langle\eta^j,\bar z-\bar w\rangle\bigr|.$$

We claim that

$$E\lesssim(\mu\bar\theta)^2|\bar y-x_{t,s}(\bar z,\eta^l)|^2+\bar\theta^{-2}|\eta^j-\eta^l|^2+(\mu\bar\theta)^2|z-w|^2+1. \tag{3-31}$$

The error induced by E can thus be absorbed by $2N$ of the powers in (3-28) and $2N$ of the powers in the first factor in (3-27). This concludes the proof of (3-24), as the remaining N powers of the first factor in (3-27) can be used to integrate in y.

To bound the first term in E, we use the geometric-arithmetic mean inequality and observe that the bounds on $d_x\xi_{t,s}$, $d_\xi\xi_{t,s}$ in Theorem 3.1 give

$$\bar\theta^{-1}\bigl|\nu_{t,s}(\bar z,\eta^l)-\nu_{t,s}(\bar w,\eta^j)\bigr|\lesssim\bar\theta^{-1}|z-w|+\bar\theta^{-1}|\eta^j-\eta^l|.$$

Since $\bar\theta^{-1} \le \mu\bar\theta$ when $|t-s| \le 1$, this is seen to be bounded by the right-hand side of (3-31). Using that $\mu\bar\theta^2|s-t| = 1$ and $\mu \le (\mu\bar\theta)^2$, the rest of (3-31) follows from

$$\left|\langle v_{t,s}(\bar w, \eta^j), x_{t,s}(\bar z, \eta^l) - x_{t,s}(\bar w, \eta^j)\rangle - \langle \eta^j, \bar z - \bar w\rangle\right| \lesssim |z-w|^2 + \bar\theta^2|s-t|,$$

which can be seen by differentiating the expression on the left in s; see [Smith and Sogge 2007, p. 133].

4. Curved submanifolds

In this section, we prove the bound (2-24) with $q = 2$, $\delta = \frac{1}{6}(1+\beta)$ which implies Theorem 1.2. In contrast to the previous section, it will be more convenient to work with an equation which is hyperbolic in t rather than in x_1. To this end, we simply set $q^\pm(x,\xi) = \pm\left(\sum_{i,j} g^{ij}(x)\xi_i\xi_j\right)^{1/2}$ and $p^\pm(\cdot,\xi) = S_{c2\mu^{1/2}}q^\pm(\cdot,\xi)$. As a consequence, we vary the notational conventions slightly so that if $x \in \mathbb{R}^n$, we denote $x' = (x_1,\dots,x_{n-1}) \in \mathbb{R}^{n-1}$ so that $x = (x', x_n)$. All other conventions will carry over as before.

Following reductions similar to the previous section, it suffices to show that

$$\|u_\mu^\pm\|_{L^2((0,1)\times\mathbb{R}^{n-1}\times\{0\})} \lesssim \mu^{(1/6)(1+\beta)}(\|u_\mu^\pm\|_{L^2(\mathbb{R}^{n+1})} + \mu^{-1}\|G_\mu^\pm\|_{L^2(\mathbb{R}^{n+1})})$$

where $G_\mu^\pm = (-i\partial_t + p^\pm(x,D))u_\mu^\pm$. As before, it suffices to treat the u_μ^-, so we suppress the superscripts.

The wave packet transform from above can also be used here, and after following the initial reductions in Section 3, it suffices to show that the propagator

$$W\tilde f(t,y) = T_\mu^*(\tilde f\circ\Theta_{0,t})(\bar y) = \mu^{n/4}\int e^{i\langle \xi_{t,0}(x,\xi), \bar y - x_{t,0}(x,\xi)\rangle}\phi(\mu^{1/2}(\bar y - x_{t,0}(x,\xi)))\tilde f(x,\xi)\,dx\,d\xi$$

satisfies

$$\|W\tilde f\|_{L^2_{t,y}} \lesssim \mu^{(1/6)(1+\beta)}\|\tilde f\|_{L^2_{x,\xi}}, \quad \beta = \frac{\sigma}{1-\sigma} < \tfrac{1}{2}. \tag{4-1}$$

where $\tilde f$ is supported in a region of the form $\{\xi : |\xi| \approx \mu, |\xi_1/|\xi| - (-e_1)| \lesssim \varepsilon\}$. In this section, the map $\Theta_{t,s}$ is determined by the new value of p and hence $\Theta_{t,s} = \Theta_{t-s,0}$. Given (3-3), we may assume (t,y) are restricted to $(0,1)\times(-1,1)^{n-1}$, that is, we bound the $L^2((0,1)\times(-1,1)^{n-1})$ norm of $W\tilde f$. We now exploit the property (2-23).

Lemma 4.1. *Let $(x(t), v(t))$ be a solution to the geodesic equation in tangent space*

$$\frac{dx_k}{dt} = v^k(t), \quad \frac{dv^k}{dt} = -v^i(t)v^j(t)\Gamma^k_{ij}(x(t)) \tag{4-2}$$

relative to the Christoffel symbols defined by $g_{\mu^{1/2}}$ (with the summation convention in effect). Suppose further that $(x(t), v(t))$ is defined for $t \in [-1,1]$ and that the geodesic has unit speed in that $|v(t)|_{g_{\mu^{1/2}}} \equiv 1$. If $v(t)$ further satisfies $|v^n(t)| \lesssim \varepsilon$, where ε is sufficiently small, then there exists a uniform constant c_1 such that the n-th component of the velocity satisfies

$$c_1\mu^{-\beta}|t| \le v^n(t) - v^n(0) \lesssim c_0\mu^{-\beta}|t|. \tag{4-3}$$

Furthermore, the difference between $x_n(t)$ *and its linearization about* 0 *satisfies*

$$|x_n(t) - x_n(0) - v^n(0)t| \lesssim c_0 \mu^{-\beta} |t|^2. \tag{4-4}$$

Proof. If ε is sufficiently small relative to the c_1 appearing in (2-23), then $-v^i(t)v^j(t)\Gamma^n_{ij}(x(t))$ is uniformly bounded from above and below. Adjusting the constant c_1, the bound (4-3) is thus a consequence of the integral equations arising from (4-2). The integral equation for $x_n(t)$ similarly gives Equation (4-4). □

Recall that solutions to (4-2) are naturally associated to curves $(x(t), \xi(t))$ in the cotangent bundle by the identification $v^k(t) = \mathrm{g}^{kl}_{\mu^{1/2}}(x(t))\xi_l(t)$. The curves in phase space are solutions to the Hamiltonian equations

$$\frac{dx}{dt} = d_\xi H, \quad \frac{d\xi}{dt} = -d_x H, \quad H(x,\xi) = \tfrac{1}{2}\mathrm{g}^{ij}_{\mu^{1/2}}\xi_i\xi_j.$$

With this in mind, we define $a(x,\xi) = g^{nm}(x)\xi_m = \partial_{\xi_n} H$, where, again, the summation convention is used. If $(x_{t,s}(x,\xi), \xi_{t,s}(x,\xi))$ were integral curves of the Hamiltonian vector field determined by $q = \sqrt{\mathrm{g}^{ij}\xi_i\xi_j}$, we would have $a(x_{t,s}, \xi_{t,s}) = |\xi|_{\mathrm{g}_{\mu^{1/2}}} v_n(s-t)$, where $v_n(r)$ is the n-th component of the velocity vector in (4-2) at time r with initial data satisfying

$$x_k(0) = x_k, \quad v^k(0) = (\mathrm{g}^{kl}_{\mu^{1/2}}(x)\xi_l)/|\xi|_{\mathrm{g}_{\mu^{1/2}}}, \quad |v(0)|_{\mathrm{g}_{\mu^{1/2}}} = 1.$$

However, in the solution operator W under consideration, the $(x_{t,s}, \xi_{t,s})$ are integral curves of the Hamiltonian vector field determined by $p(\cdot,\xi) = S_{c^2\mu^{1/2}} q(\cdot,\xi)$. Given the bounds

$$|\partial^\gamma_\xi (p-q)(x,\xi)| \lesssim \mu^{-1}, \quad |\partial^\gamma_x (p-q)(x,\xi)| \lesssim c_0\mu^{1/2},$$

valid for $|\xi| \approx \mu$, we can use Gronwall's inequality to approximate the integral curves of $d_\xi p \cdot d_x - d_x p \cdot d_\xi$ by those of $d_\xi q \cdot d_x - d_x q \cdot d_\xi$ and deduce that, for $|\xi| \approx \mu$,

$$a(x_{t,s}(x,\xi), \xi_{t,s}(x,\xi)) = |\xi|_{\mathrm{g}_{\mu^{1/2}}} v_n(t-s) + \mathcal{O}(\mu^{1/2}|t-s|) \tag{4-5}$$

where $v_n(t-s)$ is as before. By the same tack, (4-4) gives that for $(x_{t,s})_n = \langle x_{t,s}, e_n\rangle$,

$$|(x_{t,s})_n(x,\xi) - x_n - |\xi|^{-1}_{\mathrm{g}_{\mu^{1/2}}} a(x,\xi)(t-s)| \lesssim c_0\mu^{-\beta}|t-s|^2 + \mu^{-1/2}|t-s|. \tag{4-6}$$

Let N_μ, n_μ be integers such that $N_\mu \approx \log_2(\mu^{(1/3)(1+\beta)})$, $n_\mu \approx \log_2(\mu^\beta)$ and take a smooth partition of unity $\{\Gamma_j(r)\}_{j=n_\mu}^{N_\mu}$ on $\mathbb{R}$ satisfying

$$\begin{aligned}
\operatorname{supp}(\Gamma_{n_\mu}) &\subset \{r \in \mathbb{R} : |r| \geq \mu 2^{-n_\mu - 2}\},\\
\operatorname{supp}(\Gamma_j) &\subset \{r \in \mathbb{R} : |r| \in [\mu 2^{-j-2}, \mu 2^{-j+2}]\}, \quad n_\mu < j < N_\mu,\\
\operatorname{supp}(\Gamma_{N_\mu}) &\subset \{r \in \mathbb{R} : |r| \leq \mu 2^{-N_\mu + 2}\}.
\end{aligned}$$

For each $n_\mu \leq j \leq N_\mu$, we define

$$W^j \tilde f(t,y) = \mu^{n/4} \int e^{i\langle \xi_{t,0}, \bar y - x_{t,0}\rangle} \phi(\mu^{1/2}(\bar y - x_{t,0}))\Gamma_j(a(x_{t,0},\xi_{t,0}))\tilde f(x,\xi)\,dx\,d\xi$$

and, as before, we let $W_t^j \tilde{f}(y) = W^j \tilde{f}(r, y)|_{r=t}$. It suffices to show that

$$\|W^j \tilde{f}\|_{L^2_{t,y}} \lesssim 2^{j/2} \|\tilde{f}\|_{L^2_{x,\xi}}. \tag{4-7}$$

When $\beta = 0$, the decomposition above is consistent with earlier treatments of FIOs whose canonical relations possess a two-sided fold; see, for example, [Cuccagna 1997]. Indeed, for an FIO determined by the classical Lax parametrix, the singularities of the right projection of the canonical relation are determined by $a(x_{t,0}, \xi_{t,0}) = 0$ and it is effective to take dyadic decomposition in $a(x_{t,0}, \xi_{t,0})/\mu$ in scales $1 \geq 2^{-j} \geq \mu^{-1/3}$. For $\beta > 0$, scaling considerations relating to the dilation of variables $x \mapsto \lambda^{-\sigma} x$ in Section 2 then suggest that the dyadic scales should not be finer than $\mu^{-(1/3)(1+\beta)}$. In our circumstance, we can view the splitting of $|a(x_{t,0}, \xi_{t,0})|/\mu$ into scales less than and greater than $\mu^{-\frac{1}{3}(1+\beta)}$ as a decomposition into tangential and nontangential momenta, respectively. It can be seen that this threshold gives the largest scale at which our estimate for tangential momenta (4-18) is effective. At the same time, restricting nontangential momenta to scales at least this size allows us to achieve an appreciable gain in the bounds for W^j by using the linear approximation of phase space transport in (4-23) below. The selection of n_μ is more technical; its choice is based on the fact that, for $|a(x, \xi)|/\mu \geq \mu^{-\beta}$, the $(\xi_{t,0})_n$ component of the Hamiltonian flow can be linearized over a unit time scale.

Let ω_n be the unit vector pointing in the direction of $(\mathrm{g}^{n1}(\bar{z}), \dots, \mathrm{g}^{nn}(\bar{z}))$ and B denote the projection matrix onto the subspace orthogonal to ω_n. Given the decomposition above, we will need to consider the following class of integrals more general than those in Theorem 3.2:

$$\begin{aligned} K_{t,s}(y, z) = \mu^{n/2} \iint e^{i\langle \xi, \bar{z}-x\rangle - i\langle \xi_{t,s}, \bar{y}-x_{t,s}\rangle} \phi(\mu^{1/2}(\bar{z}-x))\phi(\mu^{1/2}(\bar{y}-x_{t,s})) \\ \times \tilde{\Gamma}(\xi)\Gamma_j(a(x,\xi))\Gamma_j(a(x_{t,s}, \xi_{t,s}))\, dx\, d\xi \end{aligned} \tag{4-8}$$

where Γ_j is defined as above with $n_\mu \leq j \leq N_\mu$ and

$$\operatorname{supp}(\tilde{\Gamma}) \subset \{\xi : |\xi| \approx \mu, |\xi_1/|\xi| - (-e_1)| \lesssim \varepsilon, |B\xi/|B\xi| - \eta| \lesssim \bar{\theta}\}, \tag{4-9}$$

for some unit vector η orthogonal to ω_n. In particular, if $\bar{\theta} = 1$, $W_t^j (W_s^j)^*$ takes this form. Our first task is to observe a generalization of Theorem 3.2.

Theorem 4.2. *Suppose $\bar{\theta} = \min(1, \mu^{-1/2}|t-s|^{-1/2}) \geq 2^{-j}$ and $K_{t,s}(y, z)$ is defined by (4-8), (4-9). Let ζ denote a fixed vector in the ξ-support of $\tilde{\Gamma}(\cdot)\Gamma_j(a(\bar{z}, \cdot))$ and $w_{t,s} = x_{t,s}(\bar{z}, \zeta)$, $\nu_{t,s} = \xi_{t,s}(\bar{z}, \zeta)/|\xi_{t,s}(\bar{z}, \zeta)|$. Then $K_{t,s}(y, z)$ satisfies the bounds*

$$|K_{t,s}(y, z)| \lesssim \mu^n \bar{\theta}^{n-2} 2^{-j} \big(1 + \mu\bar{\theta}|B \cdot (\bar{y} - w_{t,s})| + \mu|\langle \nu_{t,s}, \bar{y} - x_{t,s}\rangle|\big)^{-N}.$$

Proof. The proof is only a slight modification of the argument in [Smith and Sogge 2007, p. 152] and hence we only outline the significant differences. Indeed, the only alteration is that, in our case, $a(x, \xi)$ replaces ξ_n and the factor $\Gamma_j(a(x_{t,s}, \xi_{t,s}))$ is also present. Let $\omega_1, \dots, \omega_n$ be an orthonormal basis on $\mathbb{R}^n$

containing ω_n. We then define the following vector fields, which preserve the phase in (4-8):

$$L_0 = \frac{1 - i(\langle \xi, \bar{z} - x\rangle - \langle \xi_{t,s}, \bar{y} - x_{t,s}\rangle)\langle \xi, d_\xi\rangle}{1 + |\langle \xi, \bar{z} - x\rangle - \langle \xi_{t,s}, \bar{y} - x_{t,s}\rangle|^2},$$

$$L_k = \frac{1 - i(\mu\bar{\theta})^2\langle \omega_k, \bar{z} - x - d_\xi \xi_{t,s}\cdot(\bar{y} - x_{t,s})\rangle\langle \omega_k, d_\xi\rangle}{1 + \mu^2\bar{\theta}^2|\langle \omega_k, \bar{z} - x - d_\xi \xi_{t,s}\cdot(\bar{y} - x_{t,s})\rangle|^2}, \quad 1 \le k \le n-1.$$

We define L_n analogously to L_k above with ω_n replacing ω_k and 2^{-j} replacing $\bar{\theta}$. The idea is to integrate by parts in (4-8) using these vector fields. We display the following bounds on the derivatives of $\Theta_{t,s}(x,\xi)$ in x, ξ [Smith and Sogge 2007, (5.6), (5.7), (5.11), (5.12)]:

$$\begin{aligned}
&|d_x^2 x_{t,s}| \lesssim \langle \mu^{1/2}|t-s|\rangle, && |d_x^2 \xi_{t,s}| \lesssim \mu^{1/2},\\
&|d_x d_\xi x_{t,s}| \lesssim |t-s|\langle \mu^{1/2}|t-s|\rangle, && |d_x d_\xi \xi_{t,s}| \lesssim \langle \mu^{1/2}|t-s|\rangle,\\
&|d_\xi^k x_{t,s}| + |d_\xi^k \xi_{t,s}| \lesssim |t-s|\langle \mu^{1/2}|t-s|\rangle^{k-1}, \quad k \ge 2, &&\\
&|(\xi\cdot d_\xi)^j(\mu\bar{\theta}d_\xi)^\alpha \mu^{3/2}\bar{\theta} d_\xi x_{t,s}| \lesssim 1, && |(\xi\cdot d_\xi)^j(\mu\bar{\theta}d_\xi)^\alpha \mu\bar{\theta}\langle d_\xi \xi_{t,s}, \bar{y} - x_{t,s}\rangle| \lesssim \langle \mu^{1/2}|\bar{y} - x_{t,s}|\rangle,
\end{aligned} \tag{4-10}$$

where the last one is valid for $j + |\alpha| \ge 1$. these bounds were used [Smith and Sogge 2007] to prove Theorem 3.2 above and the aforementioned estimates.

Here the first crucial matter is to observe that the result of applying powers of the differential operators $\langle \xi, d_\xi\rangle$ and $\mu\bar{\theta}\langle \omega_k, d_\xi\rangle$ for $k = 1, \dots, n-1$ to $\Gamma_j^{(i)}(a(x,\xi))$, $\Gamma_j^{(i)}(a(x_{t,s}, \xi_{t,s}))$ is dominated by the other factors in the integrand. Powers of $\langle \xi, d_\xi\rangle$ are easily handled by homogeneity. Differentiating $\Gamma_j^{(i)}$ yields a gain of $\mu^{-1}2^j$, while derivatives of $\bar{\theta}2^j a(x,\xi)$ in the direction of ω_k are

$$\bar{\theta}2^j\langle \omega_k, d_\xi\rangle a(x,\xi) = \bar{\theta}2^j\langle \omega_k, \mathrm{g}^{nm}(x) - \mathrm{g}^{nm}(\bar{z})\rangle.$$

Since $\bar{\theta}2^j \lesssim \mu^{\frac{1}{3}(1+\beta)} \ll \mu^{1/2}$, this is dominated by $\mu^{1/2}|x - \bar{z}|$.

For $\Gamma_j^{(i)}(a(x_{t,s}, \xi_{t,s}))$, first consider a single power of $\bar{\theta}2^j\langle \omega_k, d_\xi\rangle$ on $a(x_{t,s}, \xi_{t,s})$

$$\begin{aligned}
\bar{\theta}2^j&\langle \omega_k, d_\xi\rangle a(x_{t,s}(x,\xi), \xi_{t,s}(x,\xi))\\
&= \bar{\theta}2^j\langle \omega_k, d_\xi\rangle(\mathrm{g}^{nm}(x_{t,s})(\xi_{t,s})_m)\\
&= \bar{\theta}2^j(d_x \mathrm{g}^{nm}(x_{t,s})\cdot\langle \omega_k, d_\xi x_{t,s}\rangle(\xi_{t,s})_m + \mathrm{g}^{nm}(x_{t,s})\langle \omega_k, d_\xi(\xi_{t,s})_m\rangle).
\end{aligned} \tag{4-11}$$

The first term on the right is bounded as $|d_\xi x_{t,s}(x,\xi)||\xi_{t,s}| \lesssim |t-s|$ and $\bar{\theta}2^j|t-s| \ll 1$. For the second, we rewrite the sum in m as

$$(\mathrm{g}^{nm}(x_{t,s}) - \mathrm{g}^{nm}(\bar{z}))\langle \omega_k, d_\xi(\xi_{t,s})_m\rangle + \mathrm{g}^{nm}(\bar{z})\langle \omega_k, d_\xi(\xi_{t,s})_m - e_m\rangle.$$

The second term is $\mathcal{O}(|t-s|)$ and can be dominated as before. For the first term we use that

$$|x_{t,s}(x,\xi) - \bar{z}| \le |x_{t,s}(x,\xi) - x| + |x - \bar{z}|.$$

The first term here is $\mathbb{O}(|t-s|)$ and the second can be treated as above. For higher derivatives of (4-11), we simply use homogeneity and (4-10) to see that the result of applying l additional powers of $\bar{\theta}2^j\langle\omega_k, d_\xi\rangle$ is bounded by $(\bar{\theta}2^j)^{l+1}|t-s|\mu^{-l/2} \ll 1$.

Integration by parts using $L_0, \dots, L_n$ gives that $|K_{t,s}(y,z)|$ is dominated by

$$\mu^{n/2}\iint (1+\mu^{1/2}|\bar{z}-x|+\mu^{1/2}|\bar{y}-x_{t,s}|)^{-N}(1+\mu\bar{\theta}|B\cdot(\bar{z}-x-d_\xi\xi_{t,s}\cdot(\bar{y}-x_{t,s}))| \\ +\mu 2^j|\langle\omega_n, \bar{z}-x-d_\xi\xi_{t,s}\cdot(\bar{y}-x_{t,s})\rangle|+|\langle\xi,\bar{z}-x\rangle-\langle\xi_{t,s},\bar{y}-x_{t,s}\rangle|)^{-N}\,d\xi dx \quad (4\text{-}12)$$

and we may assume that the values of ξ are restricted to $\xi \in \operatorname{supp}(\widetilde{\Gamma}(\cdot)\Gamma_j(a(x,\cdot)))$.

Now observe that if ξ is such a vector and $\tilde{\xi}$, $\tilde{\zeta}$ are vectors in the direction of ξ, ζ normalized so that $|B\tilde{\xi}| = |B\tilde{\zeta}| = 1$, then

$$|B(\tilde{\xi}-\tilde{\zeta})| \lesssim \bar{\theta}, \quad |(I-B)(\tilde{\xi}-\tilde{\zeta})| \lesssim |\bar{z}-x|+2^{-j}. \quad (4\text{-}13)$$

The first of the two inequalities is evident from the support condition on $\widetilde{\Gamma}$ and the second follows by observing that $|B\xi|, |B\zeta| \approx \mu$ and

$$\mathrm{g}^{nm}(\bar{z})\tilde{\xi}_m-\mathrm{g}^{nm}(\bar{z})\tilde{\zeta}_m = (\mathrm{g}^{nm}(\bar{z})-\mathrm{g}^{nm}(x))\tilde{\xi}_m+\mathrm{g}^{nm}(x)\tilde{\xi}_m-\mathrm{g}^{nm}(\bar{z})\tilde{\zeta}_m$$

Given (4-13), the proof of [Smith and Sogge 2007, (5.13)] goes through with only minor adjustments. Hence (4-12) is further dominated by

$$\mu^{n/2}\iint (1+\mu\bar{\theta}|B\cdot d_\xi\xi_{t,s}\cdot(\bar{y}-w_{t,s})|+\mu 2^{-j}|\langle\omega_n, d_\xi\xi_{t,s}\cdot(\bar{y}-w_{t,s})\rangle| \\ +|\langle\xi_{t,s},\bar{y}-w_{t,s}\rangle|)^{-N}(1+\mu^{1/2}|\bar{z}-x|+\mu^{1/2}|\bar{y}-x_{t,s}|)^{-N}\,d\xi dx \quad (4\text{-}14)$$

where ξ values are restricted as before. Observe that since $\mu^{1/2} \ll \mu 2^{-j} \le \mu\bar{\theta}$ and $d_\xi\xi_{t,s}$ is invertible, the middle two terms in the first factor dominate $\mu^{1/2}|\bar{y}-w_{t,s}|$.

We next see that we may replace $\xi_{t,s}$ by $\xi_{t,s}(\bar{z},\zeta)$ in the expression $\langle\xi_{t,s},\bar{y}-w_{t,s}\rangle$. Without loss of generality, we may assume that $|B\xi| = |B\zeta|$. We note that

$$|\langle\xi_{t,s}(x,\zeta)-\xi_{t,s}(\bar{z},\zeta),\bar{y}-w_{t,s}\rangle| \lesssim \mu|x-\bar{z}||\bar{y}-x_{t,s}|.$$

It now remains to bound $|\langle\xi_{t,s}(x,\xi)-\xi_{t,s}(x,\zeta),\bar{y}-w_{t,s}\rangle|$. We thus write

$$\xi_{t,s}(x,\xi)-\xi_{t,s}(x,\zeta) = (\xi-\zeta)\cdot d_\xi\xi_{t,s}+\mathbb{O}(|\xi-\zeta|^2\mu^{-1/2}|t-s|).$$

For the first term here, note that

$$(\xi-\zeta)\cdot d_\xi\xi_{t,s} = (\xi-\zeta)\cdot B\cdot d_\xi\xi_{t,s}+(\xi-\zeta)\cdot(I-B)\cdot d_\xi\xi_{t,s}.$$

Since B is an orthogonal projection and

$$|B\cdot(\xi-\zeta)| \lesssim \mu\bar{\theta}, \quad |(I-B)\cdot(\xi-\zeta)| \lesssim \mu 2^{-j}+\mu|x-\bar{z}|,$$

the error induced by the first term here is dominated by the other terms in the integrand in (4-14). We then use $\mu\bar{\theta}^2|t-s| \le 1$ to bound the error term similarly.

Also, replacing $d_\xi \xi_{t,s}$ by the identity matrix in (4-14) yields an acceptable error, as it is bounded by

$$\mu\bar{\theta}|t-s|\,|\bar{y}-x_{t,s}| \lesssim \mu^{1/2}|\bar{y}-x_{t,s}|.$$

Finally, for each x, the region of integration in ξ can be restricted to a set of volume $\approx \mu^n\bar{\theta}^{n-2}2^{-j}$, which is enough to conclude the proof. $\square$

Note that, by (2-22), we may assume that the difference between B and projection onto the first $n-1$ coordinates yields an error which is no more than $\mathcal{O}(c_0)$. Moreover, since $|\nu_{t,s}-e_1| \lesssim \varepsilon + c_0$, we have, as a consequence of this theorem, that

$$\int |K_{t,s}(y,z)|\,dy \lesssim \mu 2^{-j}. \tag{4-15}$$

We now begin the proof of (4-7) when $j = N_\mu$, claiming there exists $\tilde{c}_1$ such that

$$W_t^{N_\mu}(W_s^{N_\mu})^* = 0 \quad \text{whenever } |t-s| \ge \tilde{c}_1\mu^\beta 2^{-N_\mu}. \tag{4-16}$$

To see this, recall that the kernel of $W_t^{N_\mu}(W_s^{N_\mu})^*$ is given by an integral of the form (4-8) with $\bar{\theta} = 1$. Since $\mu^{-\beta} \gg \mu^{-1/2}$, by (4-3) and (4-5), there exists a constant $\tilde{c}_1$, inversely proportional to c_1 above, such that $|a(x_{t,s}(x,\xi),\xi_{t,s}(x,\xi))| \ge \mu 2^{-N_\mu+2}$ whenever $\mu^{-\beta}|t-s| \ge \tilde{c}_1 2^{-N_\mu}$ and $\xi \in \operatorname{supp}(\Gamma_{N_\mu})$.

Turning to the case $|t-s| \le \tilde{c}_1\mu^\beta 2^{-N_\mu}$, take a collection of unit vectors η^i orthogonal to ω_n and mutually separated by a distance $\approx \bar{\theta}$ so that (3-23) holds. Now write $K_{t,s} = \sum_i K_i(y,z)$ where each K_i is defined as in (4-8) with η replaced by η^i. Next observe that $|\eta^j-\eta^l| \lesssim |(\eta^j-\eta^l)'|$, which can be seen by noting that the linear map which projects the subspace orthogonal to ω_n onto its first $n-1$ components is invertible and depends continuously on z. An adjustment of the almost orthogonality argument in (3-26) thus shows that the operators $T_l^*T_j$, $T_lT_j^*$ vanish if $\bar{\theta}^{-1}|\eta^i-\eta^j| \ge C$ for some large C. Observe that

$$\int |K_i(y,z)|\,dy + \int |K_i(y,z)|\,dz \lesssim \mu 2^{-N_\mu}. \tag{4-17}$$

But the first half of this is a consequence of (4-15) and the second half follows by symmetry and the same bound. Indeed the theorem applies here, as our assumption on $|t-s|$ means that $\bar{\theta} \gtrsim \mu^{-(1/2)(1+\beta)}2^{(1/2)N_\mu} \approx 2^{-(3/2)N_\mu+(1/2)N_\mu} = 2^{-N_\mu}$. The bound (4-7) now follows by duality, since Young's inequality in t,s gives

$$\|W^{N_\mu}(W^{N_\mu})^*\|_{L^2_{s,z}\to L^2_{t,y}} \lesssim \mu 2^{-N_\mu}\cdot\mu^\beta 2^{-N_\mu} \approx 2^{N_\mu}. \tag{4-18}$$

For $n_\mu \le j < N_\mu$, we take a partition of unity over $\mathbb{R}^{n-1}$, $\sum_l \chi(y-l) \equiv 1$ such that the sum is taken over $l \in \mathbb{Z}^{n-1}$ and $\operatorname{supp}(\chi) \subset [-1,1]^{n-1}$. Use this to define

$$\chi_l(y) := \chi(\mu^{-\beta}2^j y - l) \quad \text{and} \quad W^{j,l}\tilde{f}(t,y) := \chi_l(y)W^j\tilde{f}(t,y)$$

and we consider only those l such that $\operatorname{supp}(\chi_l)$ intersects $(-1,1)^n$. By the support properties of χ, we may take C sufficiently large so that $(W^{j,m})^*W^{j,l}$ vanishes whenever $|l-m| \ge C$. We next claim that we can take C so that

$$\|W^{j,l}(W^{j,m})^*\|_{L^2\to L^2} \lesssim \mu^{-N} \quad \text{whenever } |l-m|\geq C. \tag{4-19}$$

Since there is at most $\mathcal{O}(\mu^{(n-1)/3})$ of the $W^{j,l}$, the estimate (4-7) on W^j will follow by Cotlar's lemma and Young's inequality provided we can show

$$\|W_t^{j,l}(W_s^{j,l})^* h\|_{L_y^2} \lesssim \mu 2^{-j}(1+\mu 2^{-2j}|t-s|)^{-2}\|h\|_{L_y^2}. \tag{4-20}$$

In order to show (4-19), we can write the kernel of the operator, denoted by $K_{t,s}^{l,m}(y,z)$, as the product of $\chi_l(y)\chi_m(z)$ with an integral of the form (4-8) with $\bar\theta=1$. Given the compact support of $K_{t,s}^{l,m}$ in y and z it suffices to show that this integral is dominated by μ^{-N} for any N. Similar to the $j=N_\mu$ case, if $\xi\in\operatorname{supp}(\Gamma_j)$ and $|t-s|\geq \tilde c_1\mu^\beta 2^{-j}$ for some $\tilde c_1$ depending only on c_1, then $\Gamma_j(a(x_{t,s},\xi_{t,s}))=0$, meaning the kernel vanishes for such t,s. When $|t-s|\leq \tilde c_1\mu^\beta 2^{-j}$, we use that

$$|x_{t,s}(\bar z,\xi)-x_{t,s}(x,\xi)|\lesssim |\bar z - x|$$

to dominate the integral in (4-8) simply by

$$\mu^{n/2}\iint (1+\mu^{1/2}|\bar z-x|+\mu^{1/2}|\bar y - x_{t,s}(\bar z,\xi)|)^{-2N}\widetilde\Gamma(\xi)\Gamma_j(a(x,\xi))\,dx\,d\xi.$$

Using the elementary estimate $|x_{t,s}(\bar z,\xi)-\bar z|\leq 2|t-s|$, we see that if $|l-m|\geq 2^4\tilde c_1$,

$$|\bar y - x_{t,s}(\bar z,\xi)|\geq |y-z|-2|t-s|\geq \mu^\beta 2^{-j-2}|l-m|-2\tilde c_1\mu^\beta 2^{-j}\geq \mu^\beta 2^{-j}\geq \mu^{-1/3}$$

and hence $\mu^{1/2}|\bar z - x_{t,s}(\bar y,\xi)|\gtrsim \mu^{1/6}$. This implies the desired bound on $K_{t,s}^{l,m}(y,z)$.

We now turn to (4-20). It suffices to restrict attention to $|t-s|\leq \tilde c_1\mu^\beta 2^{-j}$, though this does not play a crucial role in the argument. First consider the case where t,s satisfy $|t-s|\leq \mu^{-1}2^{2j}$. We begin by observing that a slight adjustment of the almost orthogonality argument in (3-26) and preceding (4-18) allows us to assume that the kernel $K_{t,s}^{l,l}(y,z)$ of $W_t^{j,l}(W_s^{j,l})^*$ is the product of $\chi_l(y)\chi_l(z)$ and an integral of the form (4-8) with $\bar\theta=\min(1,\mu^{-1/2}|t-s|^{-1/2})$. Indeed, reasoning as in (3-25), we are lead to consider the integral

$$\int e^{i\langle \xi,\bar y - x\rangle - i\langle\tilde\xi,\bar y-\tilde x\rangle}\phi(\mu^{1/2}(\bar y - x))\phi(\mu^{1/2}(\bar y-\tilde x))\chi_l^2(y)\,dy.$$

While this integral does not vanish when $\mu^{1/2}\ll\mu\bar\theta\leq|\xi-\tilde\xi|$, we may bound its absolute value by $C_N\mu^{-N}$ for any N, which is just as effective. Indeed, we may take the Fourier transform similarly to (3-26) and, since the Fourier transform of χ_l^2 is concentrated (though not localized) in a ball of radius $\mu^{-\beta}2^j\leq\mu^{1/3}\ll\mu^{1/2}$, the rapid decay in μ follows. We now conclude (4-20) for $|t-s|\leq\mu^{-1}2^{2j}$ by applying (4-15) and reasoning analogously to (4-17).

To show (4-20) when $|t-s|>\mu^{-1}2^{2j}$, we take the decomposition used in Section 3, writing the kernel $K_{t,s}^{l,l}=\sum_i K_i$ with K_i defined by replacing the Γ in (3-16) by a smooth cutoff $\widetilde\Gamma_{j,i}$ to a region of the form

$$\{\xi\in\operatorname{supp}(\widetilde\Gamma(\cdot)\Gamma_j(a(x,\cdot)) : -\xi_1\approx\mu, |\xi/|\xi|-\eta^i|\lesssim\bar\theta\},\quad \bar\theta=\mu^{-1/2}|t-s|^{-1/2}$$

where $\eta_i \in \mathbb{S}^{n-1}$. As before, we assume that η^i are separated so that (3-23) holds. The estimates (3-21) in Theorem 3.2 give

$$|K_i(y;z)| \lesssim \mu^n \bar\theta^{n-1}\big(1+\mu\bar\theta\,|\bar y - x^i_{t,s}| + \mu\,|\langle \nu^i_{t,s}, \bar y - x^i_{t,s}\rangle|\big)^{-N} \tag{4-21}$$

with $x^i_{t,s} = x_{t,s}(\bar z, \eta^i)$. We will show that

$$|\bar y - x^i_{t,s}| \gtrsim 2^{-j}|t-s|. \tag{4-22}$$

Together with our assumption on t, s, this gives $\mu^{1/2}2^{-j}|t-s|^{1/2} \lesssim \mu\bar\theta|\bar y - x^i_{t,s}|$, and hence this additional decay and the almost orthogonality arguments above can be integrated into the proof of (3-24) to obtain

$$\|W_t^{j,l}(W_s^{j,l})^*\|_{L^2\to L^2} \lesssim \mu\bar\theta(1+\mu 2^{-2j}|t-s|)^{-2} \le \mu 2^{-j}(1+\mu 2^{-2j}|t-s|)^{-2}.$$

To show (4-22), first consider t, s satisfying $\mu^{-1}2^{2j} < |t-s| \le \mu^\beta 2^{-j+3}$ (note that this is nontrivial when $2^j < 2\mu^{\frac{1}{3}(1+\beta)} \approx 2^{N_\mu}$, a relevant consequence of the $j < N_\mu$ threshold discussed above). We may assume c_0 in (2-22) is sufficiently small and use a linear approximation of the n-th component of $x_{t,s}(\bar z, \eta^i)$ in (4-6) to obtain

$$|(x_{t,s})_n(\bar z, \eta^i)| \gtrsim 2^{-j}|t-s|, \tag{4-23}$$

since the n-th component of $\bar z$ vanishes. Indeed, over this time scale, the error term is smaller than the linearization.

Now assume that $|t-s| \ge \mu^\beta 2^{-j+3}$. Taking ε, c_0 sufficiently small, we obtain

$$|(x_{t,s})_1(\bar z, \eta^i) - z_1| = \left|\int_s^t \partial_{\xi_1} p(r, \Theta_{r,s}(\bar z, \eta^i))\,dr\right| \ge \tfrac{1}{2}|t-s|$$

Since $y, z \in \operatorname{supp}(\chi_l)$ we obtain $|y-z| \le 2^{-j+1}\mu^\beta \le \frac{1}{4}|t-s|$ and hence we have the stronger bound

$$|(x_{t,s})_1(\bar z, \eta^i) - y_1| \ge \tfrac{1}{2}|t-s| - |z_1 - y_1| \ge \tfrac{1}{4}|t-s|.$$

Acknowledgements

The author thanks the anonymous referee for pointing out a significant error in an earlier draft of the paper and for other helpful suggestions.

References

[Ariturk 2011] S. Ariturk, "Concentration of eigenfunctions near a concave boundary", *Comm. Partial Differential Equations* **36**:11 (2011), 1881–1918. MR 2012k:58047 Zbl 1235.58019

[Blair et al. 2008] M. D. Blair, H. F. Smith, and C. D. Sogge, "On multilinear spectral cluster estimates for manifolds with boundary", *Math. Res. Lett.* **15**:3 (2008), 419–426. MR 2009d:58044 Zbl 1146.58019

[Bourgain 2009] J. Bourgain, "Geodesic restrictions and L^p-estimates for eigenfunctions of Riemannian surfaces", pp. 27–35 in *Linear and complex analysis*, edited by A. Alexandrov et al., Amer. Math. Soc. Transl. Ser. 2 **226**, Amer. Math. Soc., Providence, RI, 2009. MR 2011b:58066 Zbl 1189.58015

[Burq et al. 2007] N. Burq, P. Gérard, and N. Tzvetkov, "Restrictions of the Laplace–Beltrami eigenfunctions to submanifolds", *Duke Math. J.* **138**:3 (2007), 445–486. MR 2008f:58029 Zbl 1131.35053

[Cuccagna 1997] S. Cuccagna, "L^2 estimates for averaging operators along curves with two-sided k-fold singularities", *Duke Math. J.* **89**:2 (1997), 203–216. MR 99f:58199 Zbl 0908.47050

[Gilbarg and Trudinger 1983] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der Math. Wissenschaften **224**, Springer, Berlin, 1983. MR 86c:35035 Zbl 0562.35001

[Greenleaf and Seeger 1994] A. Greenleaf and A. Seeger, "Fourier integral operators with fold singularities", *J. Reine Angew. Math.* **455** (1994), 35–56. MR 95h:58130 Zbl 0799.42008

[Hassell and Tacy 2012] A. Hassell and M. Tacy, "Semiclassical L^p estimates of quasimodes on curved hypersurfaces", *J. Geom. Anal.* **22**:1 (2012), 74–89. MR 2868958 Zbl 06112524

[Hörmander 1985] L. Hörmander, *The analysis of linear partial differential operators, IV: Fourier integral operators*, Grundlehren der Math. Wissenschaften **275**, Springer, Berlin, 1985. MR 87d:35002b Zbl 0612.35001

[Hu 2009] R. Hu, "L^p norm estimates of eigenfunctions restricted to submanifolds", *Forum Math.* **21**:6 (2009), 1021–1052. MR 2011a:35364 Zbl 1187.35147

[Koch et al. 2012] H. Koch, H. Smith, and D. Tataru, "Sharp L^p bounds on spectral clusters for Lipschitz metrics", preprint, 2012. arXiv 1207.2417

[Melrose and Taylor 1985] R. B. Melrose and M. E. Taylor, "Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle", *Adv. in Math.* **55**:3 (1985), 242–315. MR 86m:35095 Zbl 0591.58034

[Mockenhaupt et al. 1993] G. Mockenhaupt, A. Seeger, and C. D. Sogge, "Local smoothing of Fourier integral operators and Carleson–Sjölin estimates", *J. Amer. Math. Soc.* **6**:1 (1993), 65–130. MR 93h:58150 Zbl 0776.58037

[Pan and Sogge 1990] Y. Pan and C. D. Sogge, "Oscillatory integrals associated to folding canonical relations", *Colloq. Math.* **60/61**:2 (1990), 413–419. MR 92e:58210 Zbl 0746.58076

[Reznikov 2004] A. Reznikov, "Norms of geodesic restrictions on hyperbolic surfaces and representation theory", preprint, 2004. arXiv math/0403437

[Saloff-Coste 1992] L. Saloff-Coste, "Uniformly elliptic operators on Riemannian manifolds", *J. Differential Geom.* **36**:2 (1992), 417–450. MR 93m:58122 Zbl 0735.58032

[Smith 2006a] H. F. Smith, "Sharp $L^2 \to L^q$ bounds on spectral projectors for low regularity metrics", *Math. Res. Lett.* **13**:5-6 (2006), 967–974. MR 2008f:35068 Zbl 05121811

[Smith 2006b] H. F. Smith, "Spectral cluster estimates for $C^{1,1}$ metrics", *Amer. J. Math.* **128**:5 (2006), 1069–1103. MR 2009i:35061 Zbl 05071301

[Smith and Sogge 1994] H. F. Smith and C. D. Sogge, "On Strichartz and eigenfunction estimates for low regularity metrics", *Math. Res. Lett.* **1**:6 (1994), 729–737. MR 95h:35156 Zbl 0832.35018

[Smith and Sogge 2007] H. F. Smith and C. D. Sogge, "On the L^p norm of spectral clusters for compact manifolds with boundary", *Acta Math.* **198**:1 (2007), 107–153. MR 2008d:58026 Zbl 1189.58017

[Smith and Tataru 2002] H. F. Smith and D. Tataru, "Sharp counterexamples for Strichartz estimates for low regularity metrics", *Math. Res. Lett.* **9**:2-3 (2002), 199–204. MR 2003e:35173 Zbl 1003.35075

[Sogge 1988] C. D. Sogge, "Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds", *J. Funct. Anal.* **77**:1 (1988), 123–138. MR 89d:35131 Zbl 0641.46011

[Sogge 2011] C. D. Sogge, "Kakeya–Nikodym averages and L^p-norms of eigenfunctions", *Tohoku Math. J.* (2) **63**:4 (2011), 519–538. MR 2872954 Zbl 1234.35156

[Tacy 2010] M. Tacy, "Semiclassical L^p estimates of quasimodes on submanifolds", *Comm. Partial Differential Equations* **35**:8 (2010), 1538–1562. MR 2011m:58044 Zbl 1205.35352

[Tataru 1998] D. Tataru, "On the regularity of boundary traces for the wave equation", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **26**:1 (1998), 185–206. MR 99e:35129 Zbl 0932.35136

[Taylor 1991] M. E. Taylor, *Pseudodifferential operators and nonlinear PDE*, Progress in Mathematics **100**, Birkhäuser, 1991. MR 92j:35193 Zbl 0746.35062

Received 1 Mar 2012. Revised 6 Aug 2012. Accepted 20 Dec 2012.

MATTHEW D. BLAIR: blair@math.unm.edu
Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, USA

ANALYSIS AND PDE
Vol. 6, No. 6, 2013
dx.doi.org/10.2140/apde.2013.6.1289

FROM THE LAPLACIAN WITH VARIABLE MAGNETIC FIELD TO THE ELECTRIC LAPLACIAN IN THE SEMICLASSICAL LIMIT

NICOLAS RAYMOND

We consider a twisted magnetic Laplacian with Neumann condition on a smooth and bounded domain of $\mathbb{R}^2$ in the semiclassical limit $h \to 0$. Under generic assumptions, we prove that the eigenvalues admit a complete asymptotic expansion in powers of $h^{1/4}$.

1. Introduction and main results

Let Ω be an open bounded and simply connected subset of $\mathbb{R}^2$ with smooth boundary. Let us consider a smooth vector potential $\boldsymbol{A}$ such that $\beta = \nabla \times \boldsymbol{A} > 0$ on $\overline{\Omega}$ and a a smooth and positive function on $\overline{\Omega}$. We are interested in estimating the eigenvalues $\lambda_n(h)$ of the operator $P_{h,\boldsymbol{A}} = (ih\nabla + \boldsymbol{A})a(ih\nabla + \boldsymbol{A})$ whose domain is given by

$$\operatorname{Dom}(P_{h,\boldsymbol{A}}) = \left\{\psi \in L^2(\Omega) : (-ih\nabla + \boldsymbol{A})a(-ih\nabla + \boldsymbol{A})\psi \in L^2(\Omega) \text{ and } (-ih\nabla + \boldsymbol{A})\psi \cdot \nu = 0 \text{ on } \partial\Omega\right\}.$$

The corresponding quadratic form, denoted by $Q_{h,\boldsymbol{A}}$, is defined on $H^1(\Omega)$ by

$$Q_{h,\boldsymbol{A}}(\psi) = \int_\Omega a(x)|(-ih\nabla + \boldsymbol{A})\psi|^2\,dx.$$

By gauge invariance, it is standard that the spectrum of $P_{h,\boldsymbol{A}}$ depends on the magnetic field $\beta = \nabla \times \boldsymbol{A}$, but not on the potential $\boldsymbol{A}$ itself.

Motivation and presentation of the problem.

Motivation and context. Before stating our main result, we should briefly describe the context and the motivations of this paper. As much in 2D as in 3D, the magnetic Laplacian, corresponding to the case when $a = 1$, appears in the theory of superconductivity when studying the third critical field H_{C_3} that appears after the linearization of the Ginzburg–Landau functional (see, for instance, [Lu and Pan 1999; 2000; Fournais and Helffer 2010]). It turns out that H_{C_3} can be related to the lowest eigenvalue of the magnetic Laplacian in the regime $h \to 0$.

In fact, the case which is mainly investigated in the literature is the case when the magnetic field is constant. In 2D, the two-terms asymptotics is done in the case of the disk by Bauman, Phillips and Tang in [Bauman et al. 1998] (see also [Bernoff and Sternberg 1998; del Pino et al. 2000]) and is generalized by Helffer and Morame [2001] to smooth and bounded domains. The asymptotic expansion at any order of

MSC2010: 35J10, 35P15.
Keywords: semiclassical analysis, magnetic field, normal form.

all the lowest eigenvalues is proved by Fournais and Helffer [2006]. In 3D, one can mention the celebrated paper [Helffer and Morame 2004], which gives the two-terms asymptotics of the first eigenvalue.

When the magnetic field is variable (and $a = 1$), fewer results are known. In 2D, [Lu and Pan 1999] provides a one-term asymptotics of the lowest eigenvalue, and [Raymond 2009] gives the two-term asymptotics under generic assumptions (we can also mention [Helffer and Kordyukov 2011], which deals with the case without boundary and provides a full asymptotic expansion of the eigenvalues). In 3D, for the one-term asymptotics, one can mention [Lu and Pan 2000], and for a three-terms asymptotics upper bound, [Raymond 2010] (see also [Raymond 2012], where a complete asymptotics is proved for a toy model).

Here we consider a twist factor $a > 0$. As we will see, the presence of a (which is maybe not the main point of this paper) will not complicate the philosophy of the analysis, even if it will lead us to use generalizations of the Feynman–Hellmann theorems (such generalizations were introduced by physicists to analyze the anisotropic Ginzburg–Landau functional; see [Doria and de Andrade 1996]). In fact, this additional term obliges us to have a more synthetic sight of the structure of the magnetic Laplacian. The motivation to add this term comes from [Chapman et al. 1995], where the authors deal with the anisotropic Ginzburg–Landau functional (which is an effective mass model). We can also refer to [Alama et al. 2010], where closely related problems appear. Moreover, we will see that the quantity to minimize to get the lowest energy is the function $a\beta$, so that this situation recalls what happens in 3D in [Lu and Pan 2000; Raymond 2010] and where the three-terms asymptotics is still not established.

Under generic assumptions, we will prove in this paper that the eigenvalues $\lambda_n(h)$ admit complete asymptotic expansions in powers of $h^{1/4}$.

Heuristics. Let us discuss the heuristics a little bit, to understand the problem. Let us fix a point $x_0 \in \overline{\Omega}$. If $x_0 \in \Omega$ and if we approximate the vector potential $\boldsymbol{A}$ by its linear part, we can locally write the magnetic Laplacian as

$$a(x_0)(h^2 D_x^2 + (hD_y - \beta(x_0)x)^2) + \text{lower-order terms}.$$

The lowest eigenvalue can be computed after a Fourier transform with respect to y and a translation with respect to x (which reduces to a 1D harmonic oscillator); it provides an eigenvalue $a(x_0)\beta(x_0)h$. If $x_0 \in \partial\Omega$, and considering the standard boundary coordinates (s,t) ($t > 0$ being the distance to the boundary and s the curvilinear coordinate), we get the approximation

$$h^2 D_t^2 + (hD_s - \beta(x_0)t)^2 + \text{lower-order terms}.$$

The shape of this formal approximation invites us to recall basic properties of the de Gennes operator.

The de Gennes operator. For $\xi \in \mathbb{R}$, we consider the Neumann realization H_ξ in $L^2(\mathbb{R}_+)$ associated with the operator

$$-\frac{d^2}{dt^2} + (t-\xi)^2, \quad \operatorname{Dom}(H_\xi) = \{u \in B^2(\mathbb{R}_+) : u'(0) = 0\}. \tag{1-1}$$

One knows (see [Dauge and Helffer 1993]) that it has compact resolvent and that its lowest eigenvalue is denoted by $\mu(\xi)$; the associated L^2-normalized and positive eigenstate is denoted by $u_\xi = u(\cdot\,,\xi)$ and is in the Schwartz class. The function $\xi \mapsto \mu(\xi)$ admits a unique minimum, say at $\xi = \xi_0$, and we let

$$\Theta_0 = \mu(\xi_0), \quad C_1 = \frac{u_{\xi_0}^2(0)}{3}. \tag{1-2}$$

Let us also recall identities established in [Bernoff and Sternberg 1998]. For $k \in \mathbb{N}$, we let

$$M_k = \int_{t>0} (t-\xi_0)^k |u_{\xi_0}(t)|^2 \, dt,$$

and we have

$$M_0 = 1, \quad M_1 = 0, \quad M_2 = \tfrac{1}{2}\Theta_0, \quad M_3 = \tfrac{1}{2}C_1, \quad \tfrac{1}{2}\mu''(\xi_0) = 3C_1\sqrt{\Theta_0}. \tag{1-3}$$

Main result. Let us introduce the general assumptions under which we will work throughout this paper. As already mentioned, the natural invariant associated with the operator is the function $a\beta$. We will assume that

$$\Theta_0 \min_{\partial\Omega} a(x)\beta(x) < \min_{\Omega} a(x)\beta(x) \tag{1-4}$$

and that

$$x \in \partial\Omega \mapsto a(x)\beta(x) \text{ admits a unique and nondegenerate minimum at } x_0. \tag{1-5}$$

Remark 1.1. Assumption (1-4) is automatically satisfied when the magnetic field is constant (and is sometimes called the surface superconductivity condition), and Assumption (1-5) excludes the case of constant magnetic field. Therefore, our generic assumption deals with a complementary situation analyzed in [Fournais and Helffer 2006], that is, the situation with a generically variable magnetic field.

Let us state our first rough estimate of the n-th eigenvalue $\lambda_n(h)$ of $P_{h,\mathbf{A}}$ that we will prove in this paper:

Proposition 1.2. *Under Assumptions (1-4) and (1-5), for all $n \geq 1$, we have*

$$\lambda_n(h) = \Theta_0 h a(x_0)\beta(x_0) + O(h^{5/4}). \tag{1-6}$$

From this proposition, we see that the asymptotics of $\lambda_n(h)$ is related to local properties of $P_{h,\mathbf{A}}$ near the point of the boundary x_0. That is why we are led to introduce the standard system of local coordinates (s,t) near x_0, where t is the distance to the boundary and s the curvilinear coordinate on the boundary (see (2-1)). We denote by $\Phi : (s,t) \mapsto x$ the corresponding local diffeomorphism. We write the Taylor expansions

$$\tilde{a}(s,t) = a(\Phi(s,t)) = 1 + a_1 s + a_2 t + a_{11}s^2 + a_{12}st + a_{22}t^2 + O(|s|^3 + |t|^3) \tag{1-7}$$

and

$$\tilde{\beta}(s,t) = \beta(\Phi(s,t)) = 1 + b_1 s + b_2 t + b_{11}s^2 + b_{12}st + b_2 t^2 + O(|s|^3 + |t|^3), \tag{1-8}$$

where we have assumed the normalization

$$a(x_0) = \beta(x_0) = 1. \tag{1-9}$$

Let us translate the generic assumptions (1-4) and (1-5). The critical point condition becomes

$$a_1 = -b_1, \tag{1-10}$$

and the nondegeneracy property can be reformulated as

$$b_{11} + a_1 b_1 + a_{11} = a_{11} + b_{11} - a_1^2 = \alpha > 0. \tag{1-11}$$

We can now state the main result of this paper:

Theorem 1.3. *We assume (1-4) and (1-5) and the normalization condition (1-9). For all $n \geq 1$, there exist a sequence $(\gamma_{n,j})_{j\geq 0}$ and $h_0 > 0$ such that for all $h \in (0, h_0)$, we have*

$$\lambda_n(h) \underset{h\to 0}{\sim} h \sum_{j\geq 0} \gamma_{n,j} h^{j/4}.$$

Moreover, we have, for all $n \geq 1$,

$$\gamma_{n,0} = \Theta_0, \quad \gamma_{n,1} = 0, \quad \gamma_{n,2} = C(k_0, a_2, b_2) + (2n-1)\left(\frac{\alpha\Theta_0\mu''(\xi_0)}{2}\right)^{1/2},$$

with

$$C(k_0, a_2, b_2) = -C_1 k_0 + \frac{3C_1}{2} a_2 + \left(\frac{C_1}{2} + \xi_0\Theta_0\right) b_2.$$

Comments about the main theorem. Let us first notice that Theorem 1.3 completes the one of Fournais and Helffer [2006, Theorem 1.1] dealing with a constant magnetic field (see also [Fournais and Helffer 2006, Remark 1.2], where the variable magnetic field case is left as an open problem).

It turns out that Theorem 1.3 generalizes [Raymond 2009, Theorem 1.7]. Moreover, as a consequence of the asymptotics of the eigenvalues (which are simple for h small enough), we also get the corresponding asymptotics for the eigenfunctions. These eigenfunctions are approximated (in the L^2 sense) by the power series, which we will use as quasimodes (see (2-10)). In particular, the eigenfunctions are approximated by functions in the form

$$u_{\xi_0}(h^{-1/2}t)g(h^{-1/4}s),$$

where g is a renormalized Hermite function.

As we will see in the proof, the construction of appropriate trial functions can give a hint of the natural scales of the problem ($h^{1/2}$ with respect to t and $h^{1/4}$ with respect to s). Nevertheless, as far as we know, there are no structural explanations in the literature of the double scales phenomena related to the magnetic Laplacian.

In this paper, we will explain how, thanks to conjugations of the magnetic Laplacian (by explicit unitary transforms in the spirit of Egorov's theorem; see [Egorov 1971; Robert 1987; Martinez 2002]), we can reduce the study to an electric Laplacian which is in the Born–Oppenheimer form (see [Combes et al. 1981; Martinez 1989]). The main point of the Born–Oppenheimer approximation is that it naturally involves two different scales (related to the so-called slow and fast variables).

As we recalled at the beginning of the introduction, many papers deal with the two or three first terms of $\lambda_1(h)$ and do not analyze $\lambda_n(h)$ (for $n \geq 2$); see, for instance, [Helffer and Morame 2004; Raymond 2009]. One could think that it is just a technical extension. But, as can be seen in [Fournais and Helffer 2006] (see also [Dombrowski and Raymond 2013]), the difficulty of the extension relies on the microlocalization properties of the operator: The authors have to combine a very fine analysis using pseudodifferential calculus (to catch the a priori behavior of the eigenfunctions with respect to a phase variable) and the

Grušin reduction machinery [1972]. Let us emphasize that these microlocalization properties are one of the deepest features of the magnetic Laplacian and are often found at the core of proofs (see, for instance, [Helffer and Morame 2004, Sections 11.2 and 13.2; Fournais and Helffer 2006, Sections 5 and 6]). We will see how we can avoid the introduction of the pseudodifferential (or abstract functional) calculus. In fact, we will also avoid the Grušin formalism by keeping only the main idea behind it: We can use the true eigenfunctions as quasimodes for the first-order approximation of $P_{h,\boldsymbol{A}}$ and deduce a tensorial structure for the eigenfunctions.

In our investigation, we will introduce successive changes of variables and unitary transforms, such as changes of gauge and weighted Fourier transforms (which are all associated with canonical transformations of the symbol). By doing this, we will reduce the symbol of the operator (or, equivalently, reduce the quadratic form), thanks to the a priori localization estimates. By gathering all these transforms, one would obtain a Fourier integral operator which transforms (modulo lower-order terms) the magnetic Laplacian into an electric Laplacian in the Born–Oppenheimer form. For this normal form, we can prove Agmon estimates with respect to a phase variable. These estimates involve, for the normal form, strong microlocalization estimates, and spare us, for instance, the multiple commutator estimates needed in [Fournais and Helffer 2006, Section 5].

Scheme of the proof. Let us now describe the scheme of the proof. In Section 2, we perform a construction of quasimodes and quasieigenvalues thanks to a formal expansion in power series of the operator. This analysis relies on generalizations of the Feynman–Hellmann formula and of the virial theorem, which were already introduced in [Raymond 2010], and which are an alternative to the Grušin approach used in [Fournais and Helffer 2006]. Then we use the spectral theorem to infer the existence of a spectrum near each constructed power series. In Section 3, we prove a rough lower bound for the lowest eigenvalues and deduce Agmon estimates with respect to the variable t, which provide a localization of the lowest eigenfunctions in a neighborhood of the boundary of size $h^{1/2}$. In Section 4 , we improve the lower bound of Section 3 and deduce a localization of size $h^{1/4}$ with respect to the tangential coordinate s. In Section 5, we prove a lower bound for $Q_{h,\boldsymbol{A}}$ thanks to the definition of "magnetic coordinates," and we reduce the study to a model operator (in the Born–Oppenheimer form) for which we are able to estimate the spectral gap between the lowest eigenvalues.

2. Accurate construction of quasimodes

This section is devoted to the proof of the following theorem:

Theorem 2.1. *For all $n \geq 1$, there exists a sequence $(\gamma_{n,j})_{j\geq 0}$ such that, for all $J \geq 0$, there exist $h_0 > 0$, $C > 0$ such that*

$$d\left(h\sum_{j=0}^{J}\gamma_{n,j}h^{j/4}, \sigma(P_{h,\boldsymbol{A}})\right) \leq Ch^{(J+1)/4}.$$

Moreover, we have, for all $n \geq 1$:

$$\gamma_{n,0} = \Theta_0, \quad \gamma_{n,1} = 0, \quad \gamma_{n,2} = C(k_0, a_2, b_2) + (2n-1)\left(\frac{(a_{11}+b_{11}-a_1^2)\Theta_0\mu''(\xi_0)}{2}\right)^{1/2}.$$

The proof of Theorem 2.1 is based on a construction of quasimodes for $P_{h,\boldsymbol{A}}$ localized near x_0.

Local coordinates (s,t). We use the local coordinates (s,t) near $x_0=(0,0)$, where $t(x)=d(x,\partial\Omega)$ and $s(x)$ is the tangential coordinate of x. We choose a parametrization of the boundary:

$$\gamma:\mathbb{R}/(|\partial\Omega|\mathbb{Z})\to\partial\Omega.$$

Let $\nu(s)$ be the unit vector normal to the boundary, pointing inward at the point $\gamma(s)$. We choose the orientation of the parametrization γ to be counterclockwise, so that

$$\det(\gamma'(s),\nu(s))=1.$$

The curvature $k(s)$ at the point $\gamma(s)$ is given in this parametrization by

$$\gamma''(s)=k(s)\nu(s).$$

The map Φ defined by

$$\Phi:\mathbb{R}/(|\partial\Omega|\mathbb{Z})\times\,]0,t_0[\,\to\Omega,\qquad (s,t)\mapsto\gamma(s)+t\nu(s) \tag{2-1}$$

is clearly a diffeomorphism, when t_0 is sufficiently small, with image

$$\Phi\big(\mathbb{R}/(|\partial\Omega|\mathbb{Z})\times\,]0,t_0[\big)=\{x\in\Omega\mid d(x,\partial\Omega)<t_0\}=\Omega_{t_0}.$$

We let

$$\tilde{A}_1(s,t)=(1-tk(s))\boldsymbol{A}(\Phi(s,t))\cdot\gamma'(s),\qquad \tilde{A}_2(s,t)=\boldsymbol{A}(\Phi(s,t))\cdot\nu(s),\qquad \tilde{\beta}(s,t)=\beta(\Phi(s,t)),$$

and we get

$$\partial_s\tilde{A}_2-\partial_t\tilde{A}_1=(1-tk(s))\tilde{\beta}(s,t).$$

The quadratic form becomes

$$Q_{h,\boldsymbol{A}}(\psi)=\int\tilde{a}(1-tk(s))\big|(-ih\partial_t+\tilde{A}_2)\psi\big|^2+\tilde{a}(1-tk(s))^{-1}\big|(-ih\partial_s+\tilde{A}_1)\psi\big|^2\,ds\,dt.$$

In a (simply connected) neighborhood of $(0,0)$, we can choose a gauge such that

$$\tilde{A}_1(s,t)=-\int_{t_1}^{t}(1-t'k(s))\tilde{\beta}(s,t')\,dt',\qquad \tilde{A}_2=0. \tag{2-2}$$

The operator in the coordinates (s,t). Near x_0 and using a suitable gauge (see (2-2)), we are led to construct quasimodes for the operator

$$\mathscr{L}(s,-ih\partial_s;t,-ih\partial_t)=\\ -h^2(1-tk(s))^{-1}\partial_t(1-tk(s))\tilde{a}\partial_t+(1-tk(s))^{-1}(-ih\partial_s+\tilde{A})(1-tk(s))^{-1}\tilde{a}(-ih\partial_s+\tilde{A}),$$

where (see (1-8))

$$\tilde{A}(s,t)=(t-\xi_0h^{1/2})+b_1s(t-\xi_0h^{1/2})+(b_2-k_0)\frac{t^2}{2}+b_{11}s^2(t-\xi_0h^{1/2})+O(|t|^3+|st^2|).$$

Let us now perform the scaling

$$s=h^{1/4}\sigma\quad\text{and}\quad t=h^{1/2}\tau.$$

The operator becomes

$$\mathscr{L}(h) = \mathscr{L}\big(h^{1/4}\sigma, -ih^{3/4}\partial_\sigma; h^{1/2}\tau, -ih^{1/2}\partial_\tau\big).$$

We can formally write $\mathscr{L}(h)$ as a power series:

$$\mathscr{L}(h) \sim h \sum_{j\geq 0} \mathscr{L}_j h^{j/4},$$

where

$$\mathscr{L}_0 = -\partial_\tau^2 + (\tau-\xi_0)^2, \tag{2-3}$$

$$\begin{aligned}\mathscr{L}_1 &= -a_1\sigma\partial_\tau^2 - 2i\partial_\sigma(\tau-\xi_0) + a_1(\tau-\xi_0)^2\sigma + 2b_1\sigma\tau(\tau-\xi_0)\\ &= a_1\sigma H_{\xi_0} - 2i\partial_\sigma(\tau-\xi_0) + 2b_1\sigma(\tau-\xi_0)^2,\end{aligned} \tag{2-4}$$

$$\begin{aligned}\mathscr{L}_2 = -a_2\tau\partial_\tau^2 &- a_2\partial_\tau + k_0\partial_\tau + 2k_0\tau(\tau-\xi_0)^2 + a_2\tau(\tau-\xi_0)^2\\ &+ (b_2-k_0)\tau^2(\tau-\xi_0) - ia_1(\tau-\xi_0)\\ &+ \sigma^2\big(a_{11}H_{\xi_0} - a_1^2(\tau-\xi_0)^2 + 2b_{11}(\tau-\xi_0)^2\big)\\ &- \partial_\sigma^2 - 2ia_1(\tau-\xi_0)\sigma\partial_\sigma + ia_1(\tau-\xi_0)\partial_\sigma\sigma.\end{aligned} \tag{2-5}$$

The aim is now to define good quasimodes for $\mathscr{L}(h)$. Before starting the construction, we shall recall in the next subsection a few formulas coming from perturbation theory.

Feynman–Hellmann and virial formulas. For $\rho > 0$ and $\xi \in \mathbb{R}$, let us introduce the Neumann realization on $\mathbb{R}_+$ of

$$H_{\rho,\xi} = -\rho^{-1}\partial_\tau^2 + (\rho^{1/2}\tau - \xi)^2.$$

By scaling, we observe that $H_{\rho,\xi}$ is unitarily equivalent to H_ξ and that $H_{1,\xi} = H_\xi$ (the corresponding eigenfunction is $u_{1,\xi} = u_\xi$). The form domain of $H_{\rho,\xi}$ is $B^1(\mathbb{R}_+)$ and is independent from ρ and ξ so that the family $(H_{\rho,\xi})_{\rho>0,\xi\in\mathbb{R}}$ is a holomorphic family of type (B) (see [Kato 1966, p. 395]). The lowest eigenvalue of $H_{\rho,\xi}$ is $\mu(\xi)$ and we will denote by $u_{\rho,\xi}$ the corresponding normalized eigenfunction:

$$u_{\rho,\xi}(\tau) = \rho^{1/4} u_\xi(\rho^{1/2}\tau).$$

Since u_ξ satisfies the Neumann condition, we observe that $\partial_\rho^m\partial_\xi^n u_{\rho,\xi}$ also satisfies it. In order to lighten the notation, when it is not ambiguous we will write H for $H_{\rho,\xi}$, u for $u_{\rho,\xi}$, and μ for $\mu(\xi)$.

The main idea is now to take derivatives of

$$Hu = \mu u \tag{2-6}$$

with respect to ρ and ξ. Taking the derivative with respect to ρ and ξ, we get the following proposition:

Proposition 2.2. *We have*

$$(H-\mu)\partial_\xi u = 2(\rho^{1/2}\tau - \xi)u + \mu'(\xi)u \tag{2-7}$$

and

$$(H-\mu)\partial_\rho u = -\rho^{-2}\partial_\tau^2 - \xi\rho^{-1}(\rho^{1/2}\tau-\xi) - \rho^{-1}\tau(\rho^{1/2}\tau-\xi)^2. \tag{2-8}$$

Moreover, we get

$$(H-\mu)(Su) = Xu, \tag{2-9}$$

where

$$X = -\frac{\xi}{2}\mu'(\xi) + \rho^{-1}\partial_\tau^2 + (\rho^{1/2}\tau - \xi)^2$$

and

$$S = -\frac{\xi}{2}\partial_\xi - \rho\partial_\rho.$$

Proof. Taking the derivatives with respect to ξ and ρ of (2-6), we get

$$(H-\mu)\partial_\xi u = \mu'(\xi)u - \partial_\xi Hu$$

and

$$(H-\mu)\partial_\rho u = -\partial_\rho H.$$

We have $\partial_\xi H = -2(\rho^{1/2}\tau - \xi)$ and $\partial_\rho H = \rho^{-2}\partial_\rho^2 + \rho^{-1/2}\tau(\rho^{1/2}\tau - \xi)$. □

Taking $\rho = 1$ and $\xi = \xi_0$ in (2-7), we deduce, with the Fredholm alternative:

Corollary 2.3. *We have*

$$(H_{\xi_0} - \mu(\xi_0))v_{\xi_0} = 2(t-\xi_0)u_{\xi_0},$$

with

$$v_{\xi_0} = (\partial_\xi u_\xi)_{|\xi=\xi_0}.$$

Moreover, we have

$$\int_{\tau>0} (\tau-\xi_0)u_{\xi_0}^2\,d\tau = 0.$$

Corollary 2.4. *We have, for all $\rho > 0$,*

$$\int_{\tau>0} (\rho^{1/2}\tau - \xi_0)u_{\rho,\xi_0}^2\,d\tau = 0$$

and

$$\int_{\tau>0} (\tau-\xi_0)(\partial_\rho u)_{\rho=1,\xi=\xi_0} u\,d\tau = -\frac{\xi_0}{4}.$$

Corollary 2.5. *We have*

$$(H_{\xi_0} - \mu(\xi_0))S_0 u = (\partial_\tau^2 + (\tau-\xi_0)^2)u_{\xi_0},$$

where

$$S_0 u = -(\partial_\rho u_{\rho,\xi})_{|\rho=1,\xi=\xi_0} - \frac{\xi_0}{2}v_{\xi_0}.$$

Moreover, we have

$$\|\partial_\tau u_{\xi_0}\|^2 = \|(\tau-\xi_0)u_{\xi_0}\|^2 = \frac{\Theta_0}{2}.$$

The next three propositions deal with the second derivatives of (2-6) with respect to ξ and ρ.

Proposition 2.6. *We have*

$$(H_\xi - \mu(\xi))w_{\xi_0} = 4(\tau-\xi_0)v_{\xi_0} + (\mu''(\xi_0)-2)u_{\xi_0},$$

with

$$w_{\xi_0} = (\partial_\xi^2 u_\xi)_{|\xi=\xi_0}.$$

Moreover, we have

$$\int_{\tau>0} (\tau-\xi_0)v_{\xi_0}u_{\xi_0}\,d\tau = \frac{2-\mu''(\xi_0)}{4}.$$

Proof. Taking the derivative of (2-7) with respect to ξ (with $\rho=1$), we get

$$(H_\xi - \mu(\xi))\partial_\xi^2 u_\xi = 2\mu'(\xi)\partial_\xi u_\xi + 4(\tau-\xi)\partial_\xi u_\xi + (\mu''(\xi)-2)u_\xi.$$

It remains to take $\xi=\xi_0$ and to write the Fredholm alternative. □

Proposition 2.7. *We have*

$$(H-\mu)(\partial_\rho^2 u)_{\rho=1,\xi=\xi_0} = -2(\partial_\tau^2+(\tau-\xi_0)^2)(\partial_\rho u)_{\rho=1,\xi=\xi_0} - 2\xi_0(\tau-\xi_0)(\partial_\rho u)_{\rho=1,\xi=\xi_0} + \left(2\partial_\tau^2 - \frac{\xi_0\tau}{2}\right)u_{\xi_0}$$

and

$$\left\langle(\partial_\tau^2+(\tau-\xi_0)^2)(\partial_\rho u)_{\rho=1,\xi=\xi_0}, u_{\xi_0}\right\rangle = -\frac{\Theta_0}{2}.$$

Proof. We just have to take the derivative of (2-8) with respect to ρ and $\rho=1$, $\xi=\xi_0$. To get the second identity, we use the Fredholm alternative, Corollaries 2.4 and 2.5. □

Taking the derivative of (2-9) with respect to ρ, we find:

Lemma 2.8. *We have*

$$(H-\mu)(\partial_\rho Su)_{\rho=1,\xi=\xi_0} = (-\partial_\tau^2+\tau(\tau-\xi_0))u_{\xi_0} - (\partial_\rho H)_{\rho=1,\xi=\xi_0}(S_0u) + (\partial_\tau^2+(\tau-\xi_0)^2)(\partial_\rho u)_{\rho=1,\xi=\xi_0}$$

and

$$\left\langle(\partial_\rho H)_{\rho=1,\xi=\xi_0}(S_0u), u\right\rangle = \frac{\Theta_0}{2}.$$

Lemma 2.9. *We have*

$$\langle(\tau-\xi_0)S_0u, u_{\xi_0}\rangle = \frac{\xi_0}{8}\mu''(\xi_0).$$

Proof. We have

$$\mu'(\xi) = -2\int_{\tau>0}(\rho^{1/2}\tau-\xi)u_{\rho,\xi}^2\,d\tau$$

and

$$S_0\mu' = -2\int_{\tau>0} S_0(\rho^{1/2}\tau-\xi)u_{\xi_0}^2\,d\tau - 4\int_{\tau>0}(\tau-\xi_0)S_0u\,u_{\xi_0}\,d\tau.$$ □

Combining Lemmas 2.8 and 2.9, we deduce:

Proposition 2.10. *We have*

$$\left\langle(-\partial_\tau^2-(\tau-\xi_0)^2)S_0u, u_{\xi_0}\right\rangle = -\frac{\Theta_0}{2} + \frac{\Theta_0}{8}\mu''(\xi_0).$$

Proposition 2.11. *We have*

$$\left\langle (\partial_\tau^2 + (\tau-\xi_0)^2) v_{\xi_0}, u_{\xi_0} \right\rangle = \frac{\xi_0 \mu''(\xi_0)}{4}.$$

Proof. We take the derivative of (2-7) with respect to ρ (after having fixed $\xi = \xi_0$):

$$(H-\mu)(\partial_\xi u)_{\xi=\xi_0} = 2(\rho^{1/2}\tau - \xi_0) u_{\rho,\xi_0}.$$

We deduce

$$(H-\mu)(\partial_\rho \partial_\xi u)_{\rho=1,\xi=\xi_0} = -(\partial_\rho H)_{\rho=1,\xi=\xi_0} v_{\xi_0} + \tau u_{\xi_0} + 2(\tau-\xi_0)(\partial_\rho u)_{\rho=1,\xi=\xi_0}.$$

The Fredholm alternative provides

$$\left\langle (\partial_\tau^2 + \tau(\tau-\xi_0)) v_{\xi_0}, u_{\xi_0} \right\rangle = \xi_0 + 2\left\langle (\tau-\xi_0)(\partial_\rho u)_{\rho=1,\xi=\xi_0}, u_{\xi_0} \right\rangle = \frac{\xi_0}{2},$$

where we have used Corollary 2.4. □

We have now the elements to perform an accurate construction of quasimodes.

Construction. We look for quasimodes expressed as power series,

$$\psi \sim \sum_{j \geq 0} \psi_j h^{j/4},$$

and eigenvalues,

$$\lambda \sim h \sum_{j \geq 0} \lambda_j h^{j/4},$$

so that, in the sense of formal series,

$$\mathscr{L}(h)\psi \sim \lambda\psi.$$

Term in h. We consider the equation

$$(\mathscr{L}_0 - \lambda_0)\psi_0 = 0.$$

We are led to take $\lambda_0 = \Theta_0$ and $\psi_0(\sigma,\tau) = f_0(\sigma) u_{\xi_0}(\tau)$.

Term in $h^{5/4}$. We want to solve the equation

$$(\mathscr{L}_0 - \Theta_0)\psi_1 = \lambda_1 \psi_0 - \mathscr{L}_1 \psi_0.$$

We have, using that $b_1 = -a_1$ and by Proposition 2.2,

$$(\mathscr{L}_0 - \Theta_0)\left(\psi_1 - i f_0'(\sigma) v_{\xi_0} - a_1 \sigma f_0(\sigma) S_0 u\right) = \lambda_1 u_{\xi_0}.$$

This implies that $\lambda_1 = 0$, and we take

$$\psi_1(\sigma,\tau) = i f_0'(\sigma) v_{\xi_0} + a_1 \sigma f_0(\sigma) S_0 u + f_1(\sigma) u_{\xi_0}(\tau),$$

f_0 and f_1 being to determine.

Term in $h^{3/2}$. We consider the equation

$$(\mathscr{L}_0-\Theta_0)\psi_2=\lambda_2\psi_0-\mathscr{L}_1\psi_1-\mathscr{L}_2\psi_0.$$

Let us rewrite this equation by using the expression of ψ_1:

$$(\mathscr{L}_0-\Theta_0)\psi_2=\lambda_2\psi_0-\mathscr{L}_1\big(if_0'(\sigma)v_{\xi_0}+a_1\sigma f_0(\sigma)S_0u\big)-\mathscr{L}_1(f_1(\sigma)u_{\xi_0})-\mathscr{L}_2\psi_0.$$

With Proposition 2.2, we deduce

$$(\mathscr{L}_0-\Theta_0)\big(\psi_2-if_1'(\sigma)v_{\xi_0}-a_1\sigma f_1(\sigma)S_0u\big)=\lambda_2\psi_0-\mathscr{L}_1\big(if_0'(\sigma)v_{\xi_0}+a_1\sigma f_0(\sigma)S_0u\big)-\mathscr{L}_2\psi_0.$$

We take the partial scalar product (with respect to τ) of the right-hand side with u_{ξ_0}, and we get the equation

$$\big\langle\mathscr{L}_1\big(if_0'(\sigma)v_{\xi_0}+a_1\sigma f_0(\sigma)S_0u\big)+\mathscr{L}_2\psi_0,u_{\xi_0}\big\rangle_\tau=\lambda_2 f_0.$$

This equation can be written in the form

$$\big(AD_\sigma^2+B_1\sigma D_\sigma+B_2D_\sigma\sigma+C\sigma^2+D\big)f_0=\lambda_2 f_0.$$

Terms in D_σ^2. Let us first analyze $\langle\mathscr{L}_2u_{\xi_0},u_{\xi_0}\rangle$. It is easy to see that this term is 1. Let us then analyze $\langle\mathscr{L}_1\psi_1,u_{\xi_0}\rangle$. With Proposition 2.6, we deduce that this term is $-2\langle(\tau-\xi_0)v_{\xi_0}u_{\xi_0}\rangle=(\mu''(\xi_0)/2)-1$. We get $A=\mu''(\xi_0)/2>0$.

Terms in σ^2. Let us collect the terms of $\langle\mathscr{L}_2u_{\xi_0},u_{\xi_0}\rangle$. We get

$$\Theta_0a_{11}+2b_{11}\langle(\tau-\xi_0)^2u_{\xi_0},u_{\xi_0}\rangle-a_1^2\langle(\tau-\xi_0)^2u_{\xi_0},u_{\xi_0}\rangle.$$

With Corollary 2.5, this term is equal to

$$\Theta_0a_{11}+\Theta_0b_{11}-\frac{\Theta_0}{2}a_1^2.$$

Let us analyze the terms coming from $\langle\mathscr{L}_1\psi_1,u_{\xi_0}\rangle$. We obtain the term

$$a_1^2\big\langle(-\partial_\tau^2-(\tau-\xi_0)^2)S_0u,u_{\xi_0}\big\rangle=-\frac{\Theta_0}{2}a_1^2+\Theta_0\frac{\mu''(\xi_0)}{8}a_1^2,$$

where we have used Proposition 2.10. Thus, we have

$$C=\Theta_0a_{11}+\Theta_0b_{11}-\Theta_0a_1^2+\frac{\Theta_0}{8}\mu''(\xi_0)a_1^2>0.$$

Terms in σD_σ. This term only comes from $\langle\mathscr{L}_1\psi_1,u_{\xi_0}\rangle$. It is equal to

$$a_1\big\langle(\partial_\tau^2+(\tau-\xi_0)^2)v_{\xi_0},u_{\xi_0}\big\rangle=a_1\frac{\xi_0\mu''(\xi_0)}{4},$$

where we have used Proposition 2.11.

Terms in $D_\sigma\sigma$. This term is

$$2a_1\langle(\tau-\xi_0)S_0u,u_{\xi_0}\rangle=a_1\frac{\xi_0\mu''(\xi_0)}{4},$$

where we have applied Lemma 2.9.

Value of D. We have:

$$D = \big\langle\big(-a_2\tau\partial_\tau^2 - a_2\partial_\tau + k_0\partial_\tau + 2k_0\tau(\tau-\xi_0)^2 + a_2\tau(\tau-\xi_0)^2\big)u_{\xi_0}, u_{\xi_0}\big\rangle + \big\langle\big((b_2-k_0)\tau^2(\tau-\xi_0) - ia_1(\tau-\xi_0)\big)u_{\xi_0}, u_{\xi_0}\big\rangle.$$

Using the relations (1-3) and the definition of C_1 given in (1-2), we get

$$D = C(k_0, a_2, b_2).$$

Let us introduce the quadratic form, which is fundamental in the analysis. We let

$$\mathfrak{Q}(\sigma,\eta) = \frac{\mu''(\xi_0)}{2}\eta^2 + a_1\frac{\xi_0\mu''(\xi_0)}{4}\eta\sigma + a_1\frac{\xi_0\mu''(\xi_0)}{4}\sigma\eta + \Theta_0\left(a_{11}+b_{11}-a_1^2+a_1^2\frac{\mu''(\xi_0)}{8}\right)\sigma^2.$$

Lemma 2.12. *$\mathfrak{Q}$ is definite and positive.*

Proof. We notice that $\mu''(\xi_0) > 0$ and $a_{11}+b_{11}-a_1^2+a_1^2(\mu''(\xi_0)/8) > 0$. The determinant is given by

$$\Theta_0\frac{\mu''(\xi_0)}{2}\left(a_{11}+b_{11}-a_1^2+a_1^2\frac{\mu''(\xi_0)}{8}\right) - a_1^2\frac{\Theta_0\mu''(\xi_0)^2}{16} = \frac{\Theta_0\mu''(\xi_0)}{2}(a_{11}+b_{11}-a_1^2) > 0. \qquad \square$$

We immediately deduce that $\mathfrak{Q}(\sigma, -i\partial_\sigma)$ is unitarily equivalent to a harmonic oscillator and that the increasing sequence of its eigenvalues is given by

$$\left\{(2n+1)\left(\frac{\Theta_0\mu''(\xi_0)}{2}(a_{11}+b_{11}-a_1^2)\right)^{1/2}\right\}_{n\in\mathbb{N}}.$$

The compatibility equation becomes

$$\mathfrak{Q}(\sigma, D_\sigma)f_0 = (\lambda_2 - D)f_0.$$

Thus, we choose λ_2 such that $\lambda_2 - D$ is in the spectrum of $\mathfrak{Q}(\sigma, D_\sigma)$ and we take for f_0 the corresponding normalized eigenfunction (which is in the Schwartz class). For that choice of f_0, we can consider the unique solution $\psi_2^\perp$ (which is in the Schwartz class) of

$$(\mathscr{L}_0 - \Theta_0)\psi_2^\perp = \lambda_2\psi_0 - \mathscr{L}_1\big(if_0'(\sigma)v_{\xi_0} + a_1\sigma f_0(\sigma)S_0u\big) - \mathscr{L}_2\psi_0$$

satisfying $\langle\psi_2^\perp, u_{\xi_0}\rangle = 0$. It follows that ψ_2 is in the form

$$\psi_2 = \psi_2^\perp(\sigma,\tau) + if_1'(\sigma)v_{\xi_0} + a_1\sigma f_1(\sigma)S_0u + f_2(\sigma)u_{\xi_0},$$

where f_1 and f_2 are still to be determined.

Higher-order terms. Let $N \geq 2$. Let us assume that, for $0 \leq j \leq N-2$, the functions ψ_j are determined and belong to the Schwartz class. Moreover, let us also assume that, for $j = N-1, N$, we can write

$$\psi_j(\sigma,\tau) = \psi_j^\perp(\sigma,\tau) + if_{j-1}'(\sigma)v_{\xi_0} + a_1\sigma f_{j-1}(\sigma)S_0u + f_j(\sigma)u_{\xi_0},$$

where the $(\psi_j^\perp)_{j=N-1,N}$ and f_{N-2} are determined functions in the Schwartz class and the $(f_j)_{j=N-1,N}$ are not determined. Finally, we also assume that the $(\lambda_j)_{0\leq j\leq N}$ are determined. We notice that this recursion assumption is satisfied for $N = 2$. Let us write the equation of order $N+1$:

$$(\mathscr{L}_0 - \Theta_0)\psi_{N+1} = \lambda_{N+1}\psi_0 - \mathscr{L}_1\psi_N + (\lambda_2 - \mathscr{L}_2)\psi_{N-1} - \mathscr{L}_{N+1}\psi_0 + \sum_{j=1}^{N-2}(\lambda_{N+1-j} - \mathscr{L}_{N+1-j})\psi_j.$$

This equation takes the form

$$(\mathscr{L}_0 - \Theta_0)\psi_{N+1} = \lambda_{N+1}\psi_0 - \mathscr{L}_1\psi_N + (\lambda_2 - \mathscr{L}_2)\,\psi_{N-1} + F_N(\sigma,\tau),$$

where F_N is a determined function in the Schwartz class by the recursion assumption. By Proposition 2.2, we can rewrite

$$\begin{aligned}(\mathscr{L}_0 - \Theta_0)&\big(\psi_{N+1} - i f_N'(\sigma)v_{\xi_0} - a_1\sigma f_N(\sigma)S_0 u\big)\\ &= \lambda_{N+1}\psi_0 - \mathscr{L}_1\big(\psi_N^{\perp}(\sigma,\tau) + i f_{N-1}'(\sigma)v_{\xi_0} + a_1\sigma f_{N-1}(\sigma)S_0 u\big) + (\lambda_2 - \mathscr{L}_2)\psi_{N-1} + F_N(\sigma,\tau)\\ &= \lambda_{N+1}\psi_0 - \mathscr{L}_1\big(i f_{N-1}'(\sigma)v_{\xi_0} + a_1\sigma f_{N-1}(\sigma)S_0 u\big) + (\lambda_2 - \mathscr{L}_2)(f_{N-1}u_{\xi_0}) + G_N(\sigma,\tau),\end{aligned}$$

where G_N is a determined function of the Schwartz class. We now write the Fredholm condition. The same computation as previously leads to an equation in the form

$$\mathfrak{Q}(\sigma, -i\partial_\sigma) f_{N-1} = \big(\lambda_2 - C(a_2, b_2, k_0)\big) f_{N-1} + \lambda_{N+1} f_0 + g_N(\sigma),$$

with $g_N = \langle G_N, u_{\xi_0}\rangle_\tau$. This can be rewritten as

$$\big(\mathfrak{Q}(\sigma, -i\partial_\sigma) - (\lambda_2 - C(a_2, b_2, k_0))\big) f_{N-1} = g_N(\sigma) + \lambda_{N+1} f_0.$$

The Fredholm condition applied to this equation provides $\lambda_{N+1} = -\langle g_N, f_0\rangle_\sigma$ and a unique solution f_{N-1} in the Schwartz class such that $\langle f_{N-1}, f_0\rangle_\sigma = 0$. For this choice of f_{N-1} and λ_{N+1}, we can consider the unique solution $\psi_{N+1}^{\perp}$ (in the Schwartz class) such that

$$\begin{aligned}(\mathscr{L}_0 - \Theta_0)&\psi_{N+1}^{\perp}\\ &= \lambda_{N+1}\psi_0 - \mathscr{L}_1\big(\psi_N^{\perp}(\sigma,\tau) + i f_{N-1}'(\sigma)v_{\xi_0} + a_1\sigma f_{N-1}(\sigma)S_0 u\big) + (\lambda_2 - \mathscr{L}_2)\psi_{N-1} + F_N(\sigma,\tau).\end{aligned}$$

This leads us to take

$$\psi_{N+1} = \psi_{N+1}^{\perp} + i f_N'(\sigma)v_{\xi_0} + a_1\sigma f_N(\sigma)S_0 u + f_{N+1}u_{\xi_0}.$$

This ends the proof of the recursion. Thus, we have constructed two sequences $(\lambda_j)_j$ and $(\psi_j)_j$ which depend on n (through the choice of f_0). Let us write $\lambda_{n,j}$ for λ_j and $\psi_{n,j}$ for ψ_j to emphasize this dependence.

Conclusion: proof of Theorem 2.1. Let us consider a smooth cutoff function χ_0 near x_0. For $n \geq 1$ and $J \geq 0$, we let

$$\psi_h^{[n,J]}(x) = \chi_0(x)\sum_{j=0}^{J}\psi_{n,j}\big(h^{-1/4}s(x), h^{-1/2}t(x)\big)h^{j/4} \tag{2-10}$$

and

$$\lambda_h^{[n,J]} = \sum_{j=0}^{J}\lambda_{n,j}h^{j/4}.$$

Using the fact that the ψ_j are in the Schwartz class, we get

$$\big\|\big(P_{h,\mathbf{A}}-\lambda_h^{[n,J]}\big)\psi_h^{[n,J]}\big\| \le C(n,J)h^{(J+1)/4}\|\psi_h^{[n,J]}\|.$$

Thanks to the spectral theorem, we deduce Theorem 2.1.

3. Rough lower bound and consequence

This section is devoted to establishing a rough lower bound for $\lambda_n(h)$. In particular, we give the first term of the asymptotics and deduce the so-called normal Agmon estimates, which are rather standard (see, for instance, [Helffer and Morame 2001; Fournais and Helffer 2006; Raymond 2009]).

A first lower bound. We now aim at proving a lower bound:

Proposition 3.1. *We have*

$$\lambda_n(h) \ge \Theta_0 h a(x_0)\beta(x_0) - Ch^{5/4}.$$

Proof. We use a partition of unity with balls D_j of size $r = h^\rho$, satisfying

$$\sum_j \chi_j^2 = 1 \quad \text{and} \quad \sum_j \|\nabla\chi_j\|^2 \le Cr^{-2} = Ch^{-2\rho}. \tag{3-1}$$

The so-called IMS formula (see [Cycon et al. 1987]) provides

$$Q_{h,\mathbf{A}}(\psi) = \sum_j Q_{h,\mathbf{A}}(\chi_j\psi) - h^2\sum_j\int_\Omega a\|\nabla\chi_j\|^2|\psi|^2\,dx,$$

and thus

$$Q_{h,\mathbf{A}}(\psi) \ge \sum_j Q_{h,\mathbf{A}}(\chi_j\psi) - Ch^{2-2\rho}\|\psi\|^2.$$

In each ball, we approximate a by a constant:

$$Q_{h,\mathbf{A}}(\chi_j\psi) \ge (a(x_j)-Ch^\rho)\|(-ih\nabla+\mathbf{A})(\chi_j\psi)\|^2.$$

If D_j does not intersect the boundary, then

$$\|(-ih\nabla+\mathbf{A})(\chi_j\psi)\|^2 \ge h\int_\Omega \beta(x)|\chi_j\psi|^2\,dx.$$

We deduce

$$Q_{h,\mathbf{A}}(\chi_j\psi) \ge \big(a(x_j)\beta(x_j)h - Ch^{1+\rho}\big)\|\chi_j\psi\|^2.$$

If D_j intersects the boundary, we can assume that its center is on the boundary, and we write in the local coordinates (up to a change of gauge):

$$Q_{h,\mathbf{A}}(\chi_j\psi) \ge (1-Ch^\rho)\int \tilde{a}\big(h^2|\partial_t(\chi_j\psi)|^2 + |(-ih\partial_s+\tilde{A}_1)(\chi_j\psi)|^2\big)\,ds\,dt.$$

We deduce

$$Q_{h,\mathbf{A}}(\chi_j\psi) \ge (1-Ch^\rho)(a(x_j)-Ch^\rho)\int h^2|\partial_t(\chi_j\psi)|^2 + |(-ih\partial_s+\tilde{A}_1)(\chi_j\psi)|^2\,ds\,dt.$$

We approximate A_1 by its linear approximation A_1^{lin}, and we have

$$\begin{aligned}\int h^2|\partial_t(\chi_j\psi)|^2+|(-ih\partial_s+\tilde{A}_1)(\chi_j\psi)|^2\,ds\,dt\\ &\geq(1-\varepsilon)\int h^2|\partial_t(\chi_j\psi)|^2+|(-ih\partial_s+\tilde{A}_1^{\mathrm{lin}})(\chi_j\psi)|^2\,ds\,dt-C\varepsilon^{-1}\int|x-x_j|^4|\chi_j\psi|^2\,dx\\ &\geq\big((1-\varepsilon)\Theta_0\beta(x_j)h-C\varepsilon^{-1}h^{4\rho}\big)\|\chi_j\psi\|^2.\end{aligned}$$

To optimize the remainder, we choose $\varepsilon=h^{2\rho-1/2}$. Then we take $\rho=\frac{3}{8}$, and the conclusion follows. □

Normal Agmon estimates: localization in t. We now prove the following (weighted) localization estimates:

Proposition 3.2. *Let us consider a smooth cutoff function χ supported in a fixed neighborhood of the boundary. Let $(\lambda_n(h),\psi_h)$ be an eigenpair of $P_{h,\mathbf{A}}$. For all $\delta\geq 0$, there exist ε_0, $C\geq 0$ and h_0 such that, for $h\in(0,h_0)$,*

$$\|e^{\varepsilon_0 t(x)h^{-1/2}+\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\|^2\leq C\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\|^2,$$

$$Q_{h,\mathbf{A}}\big(e^{\varepsilon_0 t(x)h^{-1/2}+\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big)\leq Ch\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\|^2.$$

Proof. The proof is based on a technique of Agmon (see, for instance, [Agmon 1982; 1985; Helffer 1988]). Let us recall the IMS formula; we have, for an eigenpair $(\lambda_n(h),\psi_h)$,

$$Q_{h,\mathbf{A}}(e^{\Phi}\psi_h)=\lambda_n(h)\|e^{\Phi}\psi_h\|^2+h^2\|a^{1/2}\nabla\Phi e^{\Phi}\psi_h\|^2.$$

We take

$$\Phi=\varepsilon_0 t(x)h^{-1/2}+\delta\chi(x)|s(x)|h^{-1/4},\tag{3-2}$$

where χ is a smooth cutoff function supported near the boundary and where $s:\partial\Omega\mapsto(-|\partial\Omega|/2,|\partial\Omega|/2)$ is the curvilinear coordinate such that $s(x_0)=0$. We use a partition of unity χ_j as in (3-1), but with balls of radius $Rh^{1/2}$ with R large enough (the x_j denote the centers), and we get

$$\sum_j\big(Q_{h,\mathbf{A}}(\chi_j e^{\Phi}\psi_h)-\lambda_n(h)\|\chi_j e^{\Phi}\psi_h\|^2-CR^{-2}h-h^2\|\chi_j a^{1/2}\nabla\Phi e^{\Phi}\psi_h\|^2\big)\leq 0.$$

We now distinguish between the balls intersecting the boundary (bnd) and the others (int). For the interior balls, we have the lower bound, for $\eta>0$ and h small enough,

$$Q_{h,\mathbf{A}}(\chi_j e^{\Phi}\psi_h)\geq\big(a(x_j)\beta(x_j)h-Ch^{3/2}\big)\|\chi_j e^{\Phi}\psi_h\|^2.$$

For the boundary balls, we have

$$Q_{h,\mathbf{A}}(\chi_j e^{\Phi}\psi_h)\geq\big(\Theta_0 a(x_j)\beta(x_j)h-Ch^{3/2}\big)\|\chi_j e^{\Phi}\psi_h\|^2.$$

Let us now split the sum:

$$\sum_{j\ \mathrm{int}}\int\big(a(x_j)\beta(x_j)h-\Theta_0 a(x_0)\beta(x_0)h-Ch^{3/2}-CR^{-2}h-Ch^2\|\nabla\Phi\|^2\big)|\chi_j e^{\Phi}\psi_h|^2\,dx\\ \leq Ch\sum_{j\ \mathrm{bnd}}\|\chi_j e^{\Phi}\psi_h\|^2.$$

With (3-2), we can notice that

$$\|\nabla\Phi\|^2 \leq C(\varepsilon_0^2 h^{-1} + \delta^2 h^{-1/2}).$$

Taking R large enough and ε_0 and h small enough and using (1-5), we get the existence of $c > 0$ such that

$$a(x_j)\beta(x_j)h - \Theta_0 a(x_0)\beta(x_0)h - Ch^{3/2} - CR^{-2}h - Ch^2\|\nabla\Phi\|^2 \geq ch.$$

We deduce

$$c\sum_{j\ \text{int}} \|\chi_j e^{\Phi}\psi_h\|^2 \leq C\sum_{j\ \text{bnd}} \|\chi_j e^{\Phi}\psi_h\|^2.$$

Due to support considerations, we can write

$$C\sum_{j\ \text{bnd}} \|\chi_j e^{\Phi}\psi_h\|^2 \leq \tilde{C}\sum_{j\ \text{bnd}} \big\|\chi_j e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2.$$

Thus, we infer

$$\|e^{\Phi}\psi_h\|^2 \leq \tilde{C}\big\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2.$$

We deduce that

$$\sum_j Q_{h,\mathbf{A}}(\chi_j e^{\Phi}\psi_h) \leq Ch\big\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2,$$

and thus

$$Q_{h,\mathbf{A}}(e^{\Phi}\psi_h) \leq Ch\big\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2. \qquad \square$$

Corollary 3.3. *Let $\eta \in \left(0, \frac{1}{2}\right]$. Let $(\lambda_n(h), \psi_h)$ be an eigenpair of $P_{h,\mathbf{A}}$. For all $\delta \geq 0$, there exist ε_0, $C \geq 0$ and h_0 such that, for $h \in (0, h_0)$,*

$$\big\|\chi_{h,\eta}e^{\varepsilon_0 t(x)h^{-1/2}+\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2 \leq C\big\|\chi_{h,\eta}e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2,$$
$$Q_{h,\mathbf{A}}\big(\chi_{h,\eta}e^{\varepsilon_0 t(x)h^{-1/2}+\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big) \leq Ch\big\|\chi_{h,\eta}e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2,$$

where $\chi_{h,\eta}(x) = \hat{\chi}(t(x)h^{-1/2+\eta})$, and with $\hat{\chi}$ a smooth cutoff function being 1 near 0.

Proof. With Proposition 3.2, we have

$$\big\|\chi_{h,\eta}e^{\varepsilon_0 t(x)h^{-1/2}+\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2 \leq C\big\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2.$$

We can write

$$\big\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2 = \big\|\chi_{h,\eta}e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2 + \big\|\sqrt{1-\chi_{h,\eta}}e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2.$$

Using Proposition 3.2, we have the estimate

$$\begin{aligned}\big\|\sqrt{1-\chi_{h,\eta}}e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2 &= \big\|\sqrt{1-\chi_{h,\eta}}e^{-\varepsilon_0 t(x)h^{-1/2}}e^{\chi(x)\varepsilon_0 t(x)h^{-1/2}+\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2\\ &= O(h^{\infty})\big\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2.\end{aligned}$$

The IMS formula provides

$$Q_{h,\mathbf{A}}(e^{\Phi}\psi_h) = Q_{h,\mathbf{A}}(\chi_{h,\eta}e^{\Phi}\psi_h) + Q_{h,\mathbf{A}}\big(\sqrt{1-\chi_{h,\eta}}e^{\Phi}\psi_h\big) + O(h^{1+2\eta})\|e^{\Phi}\psi_h\|^2. \qquad \square$$

Corollary 3.4. *Let $\eta \in \left(0, \frac{1}{2}\right]$. Let $(\lambda_n(h), \psi_h)$ be an eigenpair of $P_{h,\mathbf{A}}$. For all $\delta \geq 0$, there exist ε_0, $C \geq 0$ and h_0 such that, for $h \in (0, h_0)$,*

$$\left\| \chi_{h,\eta} e^{\varepsilon_0 t(x) h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4}} (-ih\partial_s + \tilde{A}_1)\psi_h \right\|^2 \leq Ch \left\| \chi_{h,\eta} e^{\delta\chi(x)|s(x)|h^{-1/4}} \psi_h \right\|^2,$$
$$\left\| \chi_{h,\eta} e^{\varepsilon_0 t(x) h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4}} (-ih\partial_t + \tilde{A}_2)\psi_h \right\|^2 \leq Ch \left\| \chi_{h,\eta} e^{\delta\chi(x)|s(x)|h^{-1/4}} \psi_h \right\|^2.$$

4. Order of the second term: localization in s

It is well-known that the order of the second term in the asymptotics of $\lambda_n(h)$ is closely related to localization properties of the corresponding eigenfunctions. The aim of this section is to establish such properties. Let us mention that similar estimates were proved in [Raymond 2009] through a technical analysis. Here we give a less technical proof using a very rough functional calculus.

Proposition 4.1. *Under the generic assumptions, there exist $C > 0$ and $h_0 > 0$ such that for $h \in (0, h_0)$,*

$$\lambda_n(h) \geq \Theta_0 a(x_0)\beta(x_0)h - Ch^{3/2}.$$

Moreover, for all $\delta \geq 0$, there exist $C > 0$ and $h_0 > 0$ such that for $h \in (0, h_0)$,

$$\int e^{2\delta\chi(x)|s|h^{-1/4}} |\psi|^2 \, ds \, dt \leq C\|\psi\|^2.$$

Proof. Let us recall the so-called IMS formula (see, for instance, [Cycon et al. 1987]); we have, for an eigenpair $(\lambda_n(h), \psi)$,

$$Q_{h,\mathbf{A}}(e^{\Phi}\psi) - \lambda_n(h)\|e^{\Phi}\psi\|^2 - h^2\|a^{1/2}\nabla\Phi e^{\Phi}\psi\|^2 = 0.$$

We take

$$\Phi = \delta\chi(x)|s(x)|h^{-1/4}, \quad \text{with } \delta \geq 0. \tag{4-1}$$

The idea is now to prove a suitable lower bound for $Q_{h,\mathbf{A}}$. We use a partition of unity (χ_j) (see (3-1)) with balls of radius $h^{1/4}$ and centers (s_j, t_j). We get the lower bound

$$Q_{h,\mathbf{A}}(e^{\Phi}\psi) \geq \sum_j Q_{h,\mathbf{A}}(\psi_j) - Ch^{3/2}\|e^{\Phi}\psi\|^2,$$

where

$$\psi_j = \chi_j e^{\Phi}\psi,$$

and we deduce

$$\sum_j Q_{h,\mathbf{A}}(\psi_j) - Ch^{3/2}\|\psi_j\|^2 - \lambda_n(h)\|\psi_j\|^2 \leq 0, \tag{4-2}$$

since we have, thanks to (4-1), $\|\nabla\Phi\|^2 \leq Ch^{-1/2}$.

Interior balls. Considering the balls not intersecting the boundary, we get (see the proof of Proposition 3.1):

$$\sum_{j \text{ int}} Q_{h,\mathbf{A}}(\psi_j) \geq \sum_{j \text{ int}} \left(a(x_j)\beta(x_j)h - Ch^{5/4}\right)\|\psi_j\|^2.$$

Using Assumption (1-4), we deduce

$$\sum_{j \text{ int}} Q_{h,\mathbf{A}}(\psi_j) \geq \sum_{j \text{ int}} \left(\Theta_0 a(x_0)\beta(x_0)h - Ch^{5/4}\right)\|\psi_j\|^2. \tag{4-3}$$

Boundary balls. Let us consider the j such that D_j intersects the boundary (we can assume that its center is $(s_j, 0)$). Using first the normal Agmon estimates, we have the lower bound

$$\sum_{j \text{ bnd}} Q_{h,\mathbf{A}}(\psi_j) \geq \sum_{j \text{ bnd}} \int \tilde{a}\big(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2\big)\, ds\, dt - Ch^{3/2}\|e^{\Phi}\psi\|^2,$$

where we have used the IMS formula to get

$$\sum_{j \text{ bnd}} \int t\tilde{a}\big(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2\big)\, ds\, dt$$
$$\leq C \int_{0<t<t_0} t\tilde{a}\big(|(-ih\partial_t + \tilde{A}_2)e^{\Phi}\psi|^2 + |(ih\partial_s + \tilde{A}_1)e^{\Phi}\psi|^2\big)\, ds\, dt + Ch^{3/2}\|e^{\Phi}\psi\|^2.$$

Using again the normal estimates (see Corollaries 3.3 and 3.4) and also the size of the balls, we get

$$\sum_{j \text{ bnd}} Q_{h,\mathbf{A}}(\psi_j) \geq \sum_{j \text{ bnd}} \int \tilde{a}_j^{\text{lin}}\big(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2\big)\, ds\, dt - Ch^{3/2}\|e^{\Phi}\psi\|^2, \tag{4-4}$$

where

$$\tilde{a}_j^{\text{lin}} = a_j + (s - s_j)\partial_s\tilde{a}(x_j).$$

Let us fix $S_0 > 0$ to distinguish between the balls whose centers are close to $x_0 = (0, 0)$ and the others.

Case $|s_j| \geq S_0$. Let us consider the boundary balls such that $|s_j| \geq S_0$. Using the size of the balls, we get the lower bound

$$\int \tilde{a}_j^{\text{lin}}\big(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2\big)\, ds\, dt \geq \big(\Theta_0 a(x_j)\beta(x_j)h - Ch^{5/4}\big)\|\psi_j\|^2$$
$$\geq \Theta_0(1+\varepsilon)a(x_0)\beta(x_0)h\|\psi_j\|^2, \tag{4-5}$$

where $\varepsilon > 0$ only depends on S_0, β, a and Ω.

Case $|s_j| \leq S_0$. Let us consider the boundary balls such that $|s_j| \leq S_0$. In each ball, we can use a new gauge so that

$$\sum_{\substack{j \text{ bnd} \\ |s_j| \leq S_0}} \int \tilde{a}_j^{\text{lin}}\big(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2\big)\, ds\, dt$$
$$= \sum_{\substack{j \text{ bnd} \\ |s_j| \leq S_0}} \int \tilde{a}_j^{\text{lin}}\big(|h\partial_t\psi_j|^2 + |(ih\partial_s + \tilde{A}_1^{\text{new}})\psi_j|^2\big)\, ds\, dt,$$

where $\tilde{A}_1^{\text{new}}$ (we omit the dependence on j) satisfies

$$|\tilde{A}_1^{\text{new}} - t\tilde{\beta}_j^{\text{lin}}| \leq C(t|s - s_j|^2 + t^2),$$

with

$$\tilde{\beta}_j^{\text{lin}} = \tilde{\beta}_j + \partial_s\tilde{\beta}(x_j)(s - s_j).$$

We obtain, thanks to the (weighted) estimates of Agmon,

$$\sum_{\substack{j \text{ bnd}\\ |s_j|\le S_0}} \int \tilde{a}_j^{\mathrm{lin}}\big(|h\partial_t\psi_j|^2+|(ih\partial_s+\tilde{A}_1^{\mathrm{new}})\psi_j|^2\big)\,ds\,dt \\ \ge (1-h^{1/2})\sum_{\substack{j \text{ bnd}\\ |s_j|\le S_0}} \int \tilde{a}_j^{\mathrm{lin}}\big(h^2|\partial_t\psi_j|^2+|(ih\partial_s+t\tilde{\beta}_j^{\mathrm{lin}})\psi_j|^2\big)\,ds\,dt - Ch^{3/2}\|e^{\Phi}\psi\|^2. \quad (4\text{-}6)$$

In each ball, we use the change of variables (which is a scaling with respect to τ depending on σ)

$$\sigma = s \quad \text{and} \quad \tau = \{\tilde{\beta}_j^{\mathrm{lin}}\}^{1/2}t.$$

We can write

$$\partial_t = \{\tilde{\beta}_j^{\mathrm{lin}}\}^{1/2}\partial_\tau \quad \text{and} \quad \partial_s = \partial_\sigma + \partial_s(\{\tilde{\beta}_j^{\mathrm{lin}}\}^{1/2})\partial_\tau$$

and

$$ds\,dt = \{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau.$$

We obtain

$$\int \tilde{a}_j^{\mathrm{lin}}\big(h^2|\partial_t\psi_j|^2+|(ih\partial_s+t\tilde{\beta}_j^{\mathrm{lin}})\psi_j|^2\big)\,ds\,dt \\ \ge (1-h^{1/2})\int \tilde{a}_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\big(h^2|\partial_t\hat{\psi}_j|^2+|\big(ih\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\partial_\sigma+\tau\big)\hat{\psi}_j|^2\big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau \\ -Ch^{3/2}\int|\tau\partial_\tau\hat{\psi}_j|^2\,d\sigma\,d\tau, \quad (4\text{-}7)$$

where $\hat{\psi}_j$ denotes ψ_j in the coordinates (σ,τ). With the normal Agmon estimates (see Corollaries 3.3 and 3.4), we have

$$\sum_{\substack{j \text{ bnd}\\ |s_j|\le S_0}} \int |\tau\partial_\tau\hat{\psi}_j|^2\,d\sigma\,d\tau \le C\|e^{\Phi}\psi\|^2.$$

We must now obtain an appropriate lower bound for

$$\int \tilde{a}_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\big(h^2|\partial_t\hat{\psi}_j|^2+|\big(ih\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\partial_\sigma+\tau\big)\hat{\psi}_j|^2\big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau.$$

This is the end of the following lemma.

Lemma 4.2. *We have*

$$\int \tilde{a}_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\big(h^2|\partial_t\hat{\psi}_j|^2+\big|\big(ih\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\partial_\sigma+\tau\big)\hat{\psi}_j\big|^2\big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau \\ \ge h\Theta_0\int\Big(a(x_0)\beta(x_0)+\frac{\alpha}{4}\sigma^2\Big)|\hat{\psi}_j|^2\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau - Ch^2\|\hat{\psi}_j\|^2.$$

Proof. We can notice that the Dirichlet realization on $(-\tilde{S}_0,\tilde{S}_0)$ of $D_\sigma\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}$ is self-adjoint on $L^2(\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}d\sigma)$. Thus, we shall commute D_σ and $\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}$ and control the remainder due to the commutator.

Notation 4.3. Henceforth, $\partial_\sigma(f)$ will denote the derivative of the function f, whereas $\partial_\sigma f$ will denote the composition of the differentiation ∂_σ with the multiplication by f.

We can write

$$\begin{aligned}\int \tilde{a}_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\Big(h^2|\partial_t\hat{\psi}_j|^2+\big|\big(ih\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\partial_\sigma+\tau\big)\hat{\psi}_j\big|^2\Big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau\\ =\int \tilde{a}_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}h^2|\partial_t\hat{\psi}_j|^2\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau\\ +\int a_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\big|\big(ih\partial_\sigma\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}-ih\partial_\sigma(\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2})+\tau\big)\hat{\psi}_j\big|^2\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau.\end{aligned}$$

We can estimate the double product:

$$\begin{aligned}2h\Re\Big(\int a_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\big(ih\partial_\sigma\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}+\tau\big)\hat{\psi}_j i\partial_\sigma\big(\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\big)\overline{\hat{\psi}_j}\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau\Big)\\ =-2h^2\Re\Big(\int a_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\partial_\sigma\big(\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\big)\partial_\sigma\big(\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\hat{\psi}_j\big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\overline{\hat{\psi}_j}\,d\sigma\,d\tau\Big)\\ =-h^2\int a_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\partial_\sigma\big(\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\big)\partial_\sigma\Big(\big|\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\hat{\psi}_j\big|^2\Big)\,d\sigma\,d\tau=O(h^2)\|\psi_j\|^2,\end{aligned}$$

where we have used an integration by parts for the last estimate. We deduce

$$\begin{aligned}\int \tilde{a}_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\Big(h^2|\partial_t\hat{\psi}_j|^2+\big|\big(ih\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\partial_\sigma+\tau\big)\hat{\psi}_j\big|^2\Big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau\\ \geq\int \tilde{a}_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\Big(h^2|\partial_t\hat{\psi}_j|^2+\big|\big(ih\partial_\sigma\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}+\tau\big)\hat{\psi}_j\big|^2\Big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau-Ch^2\|\psi_j\|^2.\end{aligned}\tag{4-8}$$

For S_0 small enough, we have, using the nondegeneracy, for s such that $|s|\leq\tilde{S}_0$ (with $\tilde{S}_0$ slightly bigger than S_0),

$$\tilde{a}_j^{\mathrm{lin}}(s)\tilde{\beta}_j^{\mathrm{lin}}(s)\geq a(x_0)\beta(x_0)+\frac{\alpha}{4}|s|^2.$$

Let us analyze the integral:

$$\begin{aligned}\int\big|\sigma\big(ih\partial_\sigma\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}+\tau\big)\hat{\psi}_j\big|^2\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau\\ =\int\big|\big(ih\partial_\sigma\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}+\tau\big)\sigma\hat{\psi}_j-ih\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\hat{\psi}_j\big|^2\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau.\end{aligned}$$

We must estimate the double product:

$$\begin{aligned}2\Re\int\Big(\big(ih\partial_\sigma\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}+\tau\big)\sigma\hat{\psi}_j ih\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\overline{\hat{\psi}_j}\Big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau\\ =-2h^2\Re\int\Big(\partial_\sigma\big(\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\sigma\hat{\psi}_j\big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\overline{\hat{\psi}_j}\Big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau\\ =-h^2\int\partial_\sigma\big|\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\hat{\psi}_j\big|^2\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau+O(h^2)\|\hat{\psi}_j\|^2=O(h^2)\|\hat{\psi}_j\|^2.\end{aligned}$$

We infer:

$$\int \tilde{a}_j^{\text{lin}}\tilde{\beta}_j^{\text{lin}}\Big(h^2|\partial_t\hat{\psi}_j|^2+\big|\big(ih\partial_\sigma\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}+\tau\big)\hat{\psi}_j\big|^2\Big)\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}\,d\sigma\,d\tau$$
$$\geq a(x_0)\beta(x_0)\int\Big(h^2|\partial_t\hat{\psi}_j|^2+\big|\big(ih\partial_\sigma\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}+\tau\big)\hat{\psi}_j\big|^2\Big)\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}\,d\sigma\,d\tau$$
$$+\frac{\alpha}{4}\int\Big(h^2|\partial_t(\sigma\hat{\psi}_j)|^2+\big|\big(ih\partial_\sigma\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}+\tau\big)\sigma\hat{\psi}_j\big|^2\Big)\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}\,d\sigma\,d\tau-Ch^2\|\hat{\psi}_j\|^2.$$

We recall that, for all $\xi\in\mathbb{R}$,

$$\int\Big(h^2|\partial_t\phi|^2+\big|(\tau-h\xi-\xi_0h^{1/2})\phi\big|^2\Big)\,d\tau\geq h\mu(\xi_0+h^{1/2}\xi)\|\phi\|^2\geq\Theta_0h\|\phi\|^2.$$

We infer with the functional calculus:

$$\int \tilde{a}_j^{\text{lin}}\tilde{\beta}_j^{\text{lin}}\Big(h^2|\partial_t\hat{\psi}_j|^2+\big|\big(ih\partial_\sigma\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}+\tau-\xi_0h^{1/2}\big)\hat{\psi}_j\big|^2\Big)\{\hat{\beta}_j^{\text{lin}}\}^{-1/2}\,d\sigma\,d\tau$$
$$\geq h\Theta_0\int\Big(a(x_0)\beta(x_0)+\frac{\alpha}{4}\sigma^2\Big)|\hat{\psi}_j|^2\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}\,d\sigma\,d\tau-Ch^2\|\hat{\psi}_j\|^2. \quad (4\text{-}9)$$

This concludes the proof. □

Lower bound for $\lambda_n(h)$. If we take $\delta=0$, we deduce, with (4-2)–(4-7) and Lemma 4.2,

$$\lambda_n(h)\|\psi\|^2\geq\sum_j\Theta_0ha(x_0)\beta(x_0)\int|\psi_j|^2\,dx-Ch^{3/2}\|\psi\|^2.$$

Tangential Agmon estimate. Gathering the estimates (4-3), (4-5), (4-7) and Lemma 4.2, we deduce the existence of $c>0$ such that

$$\sum_{\substack{j\text{ bnd}\\|s_j|\leq S_0}}\Big(\Theta_0h\int\Big(a(x_0)\beta(x_0)+\frac{\alpha}{4}s^2\Big)|\psi_j|^2\,ds\,dt-\Theta_0h\|\psi_j\|^2-Ch^{3/2}\|\psi_j\|^2\Big)$$
$$+\sum_{\substack{j\text{ bnd}\\|s_j|\geq S_0}}ch\|\psi_j\|^2+\sum_{j\text{int}}ch\|\psi_j\|^2\leq0$$

and

$$\sum_{\substack{j\text{ bnd}\\2C_0h^{1/4}\leq|s_j|\leq s_0}}\Big(\Theta_0h\int\frac{\alpha}{4}s^2|\psi_j|^2\,ds\,dt-Ch^{3/2}\|\psi_j\|^2\Big)\leq Ch^{3/2}\|\psi_0\|^2\leq Ch^{3/2}\|\psi\|^2.$$

Taking C_0 large enough, we infer

$$\sum_{\substack{j\text{ bnd}\\2C_0h^{1/4}\leq|s_j|\leq s_0}}\|\psi_j\|^2\leq C\|\psi\|^2,$$

so that

$$\sum_{\substack{j\text{ bnd}\\|s_j|\leq s_0}}\|\psi_j\|^2\leq C\|\psi\|^2\quad\text{and}\quad\sum_j\|\psi_j\|^2=\|e^{\Phi}\psi\|^2\leq C\|\psi\|^2.$$ □

Let us write an immediate corollary (see Corollaries 3.3 and 3.4).

Corollary 4.4. *Let* $(\eta_1, \eta_2) \in \left(0, \frac{1}{2}\right] \times \left(0, \frac{1}{4}\right]$. *Let* $(\lambda_n(h), \psi_h)$ *be an eigenpair of* $P_{h,\mathbf{A}}$. *For all* $(k,l) \in \mathbb{N}$, *there exist* $C \geq 0$ *and* $h_0 > 0$ *such that, for* $h \in (0, h_0)$,

$$\left\| \chi_{h,\eta_1,\eta_2} s^k t^l \psi_h \right\|^2 \leq C h^{k/2} h^l \|\psi_h\|^2,$$
$$\left\| \chi_{h,\eta_1,\eta_2} s^k t^l (-ih\partial_s + \tilde{A}_1)\psi_h \right\|^2 \leq C h h^{k/2} h^l \|\psi_h\|^2,$$
$$\left\| \chi_{h,\eta_1,\eta_2} s^k t^l (-ih\partial_t + \tilde{A}_2)\psi_h \right\|^2 \leq C h h^{k/2} h^l \|\psi_h\|^2,$$

where $\chi_{h,\eta_1,\eta_2}(x) = \hat{\chi}(t(x)h^{-1/2+\eta_1})\hat{\chi}(s(x)h^{-1/4+\eta_2})$. *Moreover, we have*

$$\left\| (1-\chi_{h,\eta_1,\eta_2}) s^k t^l \psi_h \right\|^2 = O(h^\infty)\|\psi_h\|^2,$$
$$\left\| (1-\chi_{h,\eta_1,\eta_2}) s^k t^l (-ih\partial_s + \tilde{A}_1)\psi_h \right\|^2 = O(h^\infty)\|\psi_h\|^2,$$
$$\left\| (1-\chi_{h,\eta_1,\eta_2}) s^k t^l (-ih\partial_t + \tilde{A}_2)\psi_h \right\|^2 = O(h^\infty)\|\psi_h\|^2.$$

Remark 4.5. In the following, each reference to the "estimates of Agmon" will be a reference to this last corollary. Moreover, at some point, the localization ideas behind Section 3 and 4, which are summarized in the last corollary, follow from the general philosophy developed in the last decade (an improvement of the approximation of the eigenvalues provides an improvement of localization and conversely). In the next section, we will strongly use these a priori estimates.

5. Unitary transforms and the Born–Oppenheimer approximation

We use a cutoff function χ_h near x_0 with support or order $h^{1/4-\tilde{\eta}}$ with $\tilde{\eta} > 0$. For all $N \geq 1$, let us consider L^2-normalized eigenpairs $(\lambda_n(h), \psi_{n,h})_{1 \leq n \leq N}$ such that $\langle \psi_{n,h}, \psi_{m,h} \rangle = 0$ when $n \neq m$. We consider the N dimensional space defined by

$$\mathfrak{E}_N(h) = \operatorname*{span}_{1 \leq n \leq N} \tilde{\psi}_{n,h}, \quad \text{where } \tilde{\psi}_{n,h} = \chi_h \psi_{n,h}.$$

Remark 5.1. The estimates of Agmon of Corollary 4.4 are satisfied by all the elements of $\mathfrak{E}_N(h)$.

We can notice that, with the estimates of Agmon, for all $\tilde{\psi} \in \mathfrak{E}_N(h)$,

$$Q_{h,\mathbf{A}}(\tilde{\psi}) \leq \lambda_N(h)\|\tilde{\psi}\|^2 + O(h^\infty)\|\tilde{\psi}\|^2. \tag{5-1}$$

In the following subsection, we provide a lower bound for $Q_{h,\mathbf{A}}$ on $\mathfrak{E}_N(h)$.

Remark 5.2. Let us underline the main spirit of this section. We are going to use successive canonical transformations of the symbol of our operator (change of variable, change of gauge, weighted Fourier transform) or, equivalently, of the associated quadratic form. In the spirit of Egorov's theorem, all these transformations will give rise to different remainders which can be treated thanks to the a priori localization estimates. Then, after conjugations by these successive unitary transforms, we will reduce the analysis to one of an electric Laplacian in the Born–Oppenheimer form.

Choice of gauge and new coordinates: a first lower bound. On the support of χ_h, we use a gauge such that $\tilde{A}_2 = 0$ and

$$|\tilde{A}_1 - \tilde{A}_1^{\mathrm{app}}| \le C(t^3 + |s|t^2 + |s|^2 t),$$

where

$$\tilde{A}_1^{\mathrm{app}} = t(1 + b_1 s + b_{11} s^2) - \xi_0 \hat{b}(s)^{1/2} h^{1/2} + \frac{\hat{b}_2}{2} t^2 = t\hat{b}(s) - \xi_0 \hat{b}(s)^{1/2} h^{1/2} + \frac{\hat{b}_2}{2} t^2,$$

where $\hat{b}_2 = b_2 - k_0$. We also let

$$\tilde{a}^{\mathrm{app}}(s,t) = 1 + a_1 s + a_{11} s^2 + a_2 t = \hat{a}(s) + a_2 t.$$

Moreover, in this neighborhood of (0, 0), we introduce new coordinates:

$$\tau = t(\hat{b}(s))^{1/2}, \quad \sigma = s. \tag{5-2}$$

In particular, we get

$$\partial_t = (\hat{b}(\sigma))^{1/2} \partial_\tau, \quad \partial_s = \partial_\sigma + \tfrac{1}{2} \hat{b}^{-1} \partial_s \hat{b} \tau \partial_\tau$$

and

$$ds\, dt = \hat{b}^{-1/2}\, d\sigma\, d\tau.$$

To simplify the notation, we let $p = \hat{b}^{-1/2}$. We will also use the change of variable

$$\check{\sigma} = \int_0^\sigma \frac{1}{p(u)}\, du = f(\sigma)$$

so that $L^2(p\, d\sigma)$ becomes $L^2(\check{p}^2\, d\check{\sigma})$.

This subsection is devoted to the proof of the following lower bound of $Q_{h,\mathbf{A}}$ on $\mathfrak{E}_N(h)$.

Proposition 5.3. *There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and all $\tilde{\psi} \in \mathfrak{E}_N(h)$,*

$$Q_{h,\mathbf{A}}(\tilde{\psi}) \ge \check{Q}_{h,\mathrm{app}}(\check{\psi}) - C h^{3/2+1/4} \|\tilde{\psi}\|^2, \tag{5-3}$$

where

$$\begin{aligned}\check{Q}_{h,\mathrm{app}}(\check{\psi}) = &\int (1 + a_2\tau)(1 - \tau k_0) |h\partial_\tau \check{\psi}|^2 \check{p}^2\, d\check{\sigma}\, d\tau \\ &+ \int (1 + a_2\tau)(1 - \tau k_0)^{-1} \left| \left(ih\check{p}^{-1} \partial_{\check{\sigma}} \check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau \right) \check{\psi} \right|^2 \check{p}^2\, d\check{\sigma}\, d\tau \\ &+ h\alpha\Theta_0 \int \check{\sigma}^2 |\check{\psi}|^2 \check{p}^2\, d\check{\sigma}\, d\tau,\end{aligned}$$

where $\check{\psi}$ denotes $\tilde{\psi}$ in the coordinates $(\check{\sigma}, \tau)$.

In order to prove Proposition 5.3, we will need this lemma:

Lemma 5.4. *There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and all $\tilde{\psi} \in \mathfrak{E}_N(h)$,*

$$Q_{h,\mathbf{A}}(\tilde{\psi}) \ge \hat{Q}_{h,\mathrm{app}}(\hat{\psi}) - C h^{3/2+1/4} \|\tilde{\psi}\|^2,$$

where

$$\hat{Q}_{h,\mathrm{app}}(\hat{\psi})$$
$$= \int m_2(\sigma,\tau)|h\partial_\tau\hat{\psi}|^2\hat{b}^{-1/2}\,d\sigma\,d\tau + \int m_1(\sigma,\tau)\Big|\Big(h\Xi+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}\,d\sigma\,d\tau,$$

with

$$\Xi = i\partial_\sigma\hat{b}^{-1/2},\quad m_1(\sigma,\tau)=(1+\alpha\sigma^2)(1+a_2\tau)(1-\tau k_0)^{-1},\quad m_2(\sigma,\tau)=(1+\alpha\sigma^2)(1+a_2\tau)(1-\tau k_0),$$

and where $\hat{\psi}$ denotes $\tilde{\psi}$ in the coordinates (σ,τ).

Proof. We have

$$Q_{h,\mathbf{A}}(\tilde{\psi}) = \int \tilde{a}\big(1-tk(s)\big)\big|(-ih\partial_t+\tilde{A}_2)\tilde{\psi}\big|^2+\tilde{a}\big(1-tk(s)\big)^{-1}\big|(ih\partial_s+\tilde{A}_1)\tilde{\psi}\big|^2\,ds\,dt.$$

Thanks to the normal and tangential Agmon estimates, we get

$$Q_{h,\mathbf{A}}(\tilde{\psi}) \geq \int \tilde{a}(1-tk_0)h^2|\partial_t\tilde{\psi}|^2+\tilde{a}(1-tk_0)^{-1}\big|(ih\partial_s+\tilde{A}_1)\tilde{\psi}\big|^2\,ds\,dt - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

The Agmon estimates imply

$$Q_{h,\mathbf{A}}(\tilde{\psi}) \geq \int \tilde{a}^{\mathrm{app}}(1-tk_0)h^2|\partial_t\tilde{\psi}|^2+\tilde{a}^{\mathrm{app}}(1-tk_0)^{-1}\big|(ih\partial_s+\tilde{A}_1^{\mathrm{app}})\tilde{\psi}\big|^2\,ds\,dt - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

We get

$$Q_{h,\mathbf{A}}(\tilde{\psi}) \geq \int \hat{a}(1+a_2 t)\Big((1-tk_0)h^2|\partial_t\tilde{\psi}|^2+(1-tk_0)^{-1}\big|\big(ih\partial_s+\tilde{A}_1^{\mathrm{app}}\big)\tilde{\psi}\big|^2\Big)\,ds\,dt - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

With the coordinates (σ,τ), we obtain

$$\int \hat{a}(1+a_2 t)(1-tk_0)h^2|\partial_t\tilde{\psi}|^2+(1+a_2 t)(1-tk_0)^{-1}\big|(ih\partial_s+\tilde{A}_1^{\mathrm{app}})\tilde{\psi}\big|^2\,ds\,dt \geq \hat{Q}_h(\hat{\psi})-Ch^{3/2+1/4}\|\tilde{\psi}\|^2,$$

where

$$\hat{Q}_h(\hat{\psi}) = \int \tilde{m}_2(\sigma,\tau)|h\partial_\tau\hat{\psi}|^2\hat{b}^{-1/2}\,d\sigma\,d\tau$$
$$+\int \tilde{m}_1(\sigma,\tau)\Big|\Big(h\hat{b}^{-1/2}i\partial_\sigma+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\hat{b}^{-1/2}-h\frac{\partial_\sigma\hat{b}}{2\hat{b}^{3/2}}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}\,d\sigma\,d\tau,$$

where

$$\tilde{m}_1(\sigma,\tau)=\hat{a}\hat{b}(1+a_2\tau)(1-\tau k_0)^{-1},\qquad \tilde{m}_2(\sigma,\tau)=\hat{a}\hat{b}(1+a_2\tau)(1-\tau k_0).$$

With the estimates of Agmon, we can simplify the quadratic form modulo lower-order terms:

$$\begin{aligned}\hat{Q}_h(\hat{\psi}) \geq \int \tilde{m}_2(\sigma,\tau)|h\partial_\tau\hat{\psi}|^2\hat{b}^{-1/2}\,d\sigma\,d\tau\\ +\int \tilde{m}_1(\sigma,\tau)\Big|\Big(h\hat{b}^{-1/2}i\partial_\sigma+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}\,d\sigma\,d\tau\\ -Ch^{3/2+1/4}\|\tilde{\psi}\|^2.\end{aligned}$$

We recall that $\hat{a}\hat{b}=1+\alpha\sigma^2+O(|\sigma|^3)$, so that with the estimates of Agmon we infer

$$\begin{aligned}\hat{Q}_h(\hat{\psi}) \geq \int m_2(\sigma,\tau)|h\partial_\tau\hat{\psi}|^2\hat{b}^{-1/2}\,d\sigma\,d\tau\\ +\int m_1(\sigma,\tau)\Big|\Big(h\hat{b}^{-1/2}i\partial_\sigma+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}\,d\sigma\,d\tau\\ -Ch^{3/2+1/4}\|\tilde{\psi}\|^2.\end{aligned}$$

We now want to replace $\hat{b}^{-1/2}i\partial_\sigma$ by $i\partial_\sigma\hat{b}^{-1/2}$, which is self-adjoint on $L^2(\hat{b}^{-1/2}d\sigma d\tau)$. Writing a commutator, we get

$$\begin{aligned}&\int m_1(\sigma,\tau)\Big|\Big(h\hat{b}^{-1/2}i\partial_\sigma+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}d\sigma d\tau\\ &\quad=\int m_1(\sigma,\tau)\Big|\Big(hi\partial_\sigma\hat{b}^{-1/2}-ih(\partial_\sigma\hat{b}^{-1/2})+\tau-\xi_0h^{1/2}+\frac{b_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}\,d\sigma\,d\tau.\end{aligned}$$

Let us consider the double product

$$\begin{aligned}&2h\Re\Big(\int m_1(\sigma,\tau)\Big(hi\partial_\sigma\hat{b}^{-1/2}+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\,i(\partial_\sigma\hat{b}^{-1/2})\overline{\hat{\psi}}\hat{b}^{-1/2}d\sigma\,d\tau\Big)\\ &\quad=2h\Re\Big(\int m_1(\sigma,\tau)\Big(hi\partial_\sigma\hat{b}^{-1/2}-h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\,i(\partial_\sigma\hat{b}^{-1/2})\overline{\hat{\psi}}\hat{b}^{-1/2}\,d\sigma\,d\tau\Big)\\ &\quad=-2h^2\Re\int m_1(\sigma,\tau)\Big(\partial_\sigma(\hat{b}^{-1/2}\hat{\psi})(\partial_\sigma\hat{b}^{-1/2})\overline{\hat{\psi}}\hat{b}^{-1/2}\,d\sigma\,d\tau\Big)+O(h^2)\|\hat{\psi}\|^2,\end{aligned}$$

where we have used the normal Agmon estimates. We deduce that

$$\begin{aligned}&2\Re\Big(\int m_1(\sigma,\tau)\Big(hi\partial_\sigma\hat{b}^{-1/2}+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\,i(\partial_\sigma\hat{b}^{-1/2})\overline{\hat{\psi}}\hat{b}^{-1/2}\,d\sigma\,d\tau\Big)\\ &\qquad=-h^2\int m_1(\sigma,\tau)(\partial_\sigma\hat{b}^{-1/2})\partial_\sigma|\hat{b}^{-1/2}\hat{\psi}|^2\,d\sigma\,d\tau+O(h^2)\|\hat{\psi}\|^2\\ &\qquad=O(h^2)\|\hat{\psi}\|^2.\end{aligned}$$

This implies

$$\int m_1(\sigma,\tau)\Big|\Big(h\hat{b}^{-1/2}i\partial_\sigma+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}\,d\sigma\,d\tau$$
$$\geq\int m_1(\sigma,\tau)\Big|\Big(hi\partial_\sigma\hat{b}^{-1/2}+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}\,d\sigma\,d\tau-Ch^2\|\hat{\psi}\|^2. \qquad \square$$

Proof of Proposition 5.3. We use Lemma 5.4. In the coordinates $(\check{\sigma},\tau)$, we have

$$\hat{Q}_{h,\mathrm{app}}(\hat{\psi})=\int m_2(f^{-1}(\check{\sigma}),\tau)|h\partial_\tau\check{\psi}|^2\check{p}^2\,d\check{\sigma}\,d\tau$$
$$+\int m_1(f^{-1}(\check{\sigma}),\tau)\Big|\Big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\check{\psi}\Big|^2\check{p}^2\,d\check{\sigma}\,d\tau,$$

where

$$m_1(f^{-1}(\check{\sigma}),\tau)=\big(1+\alpha f^{-1}(\check{\sigma})^2\big)(1+a_2\tau)(1-\tau k_0)^{-1},$$
$$m_2(f^{-1}(\check{\sigma}),\tau)=\big(1+\alpha f^{-1}(\check{\sigma})^2\big)(1+a_2\tau)(1-\tau k_0).$$

We notice that $f^{-1}(\check{\sigma})=\check{\sigma}+O(|\check{\sigma}|^2)$, so we can use the estimates of Agmon to get

$$\hat{Q}_{h,\mathrm{app}}(\hat{\psi})\geq\int m_2(\check{\sigma},\tau)|h\partial_\tau\check{\psi}|^2\check{p}^2\,d\check{\sigma}\,d\tau$$
$$+\int m_1(\check{\sigma},\tau)\Big|\Big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\check{\psi}\Big|^2\check{p}^2\,d\check{\sigma}\,d\tau-Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

This inequality can be rewritten as

$$\hat{Q}_{h,\mathrm{app}}(\hat{\psi})\geq\check{Q}_{h,\mathrm{app},1}(\check{\psi})+\check{Q}_{h,\mathrm{app},2}(\check{\psi})-Ch^{3/2+1/4}\|\tilde{\psi}\|^2,$$

where

$$\check{Q}_{h,\mathrm{app},1}(\check{\psi})=\int(1+a_2\tau)(1-\tau k_0)|h\partial_\tau\check{\psi}|^2\check{p}^2\,d\check{\sigma}\,d\tau$$
$$+\int(1+a_2\tau)(1-\tau k_0)^{-1}\Big|\Big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\check{\psi}\Big|^2\check{p}^2\,d\check{\sigma}\,d\tau$$

and

$$\check{Q}_{h,\mathrm{app},2}(\check{\psi})=\int(1+a_2\tau)(1-\tau k_0)|h\partial_\tau(\check{\sigma}\check{\psi})|^2\check{p}^2\,d\check{\sigma}\,d\tau$$
$$+\int(1+a_2\tau)(1-\tau k_0)^{-1}\Big|\check{\sigma}\Big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\check{\psi}\Big|^2\check{p}^2\,d\check{\sigma}\,d\tau.$$

Reduction of $\check{Q}_{h,\mathrm{app},2}(\check{\psi})$. By the estimates of Agmon, we have

$$\check{Q}_{h,\mathrm{app},2}(\check{\psi})\geq\int|h\partial_\tau(\check{\sigma}\check{\psi})|^2\check{p}^2\,d\check{\sigma}\,d\tau$$
$$+\int\Big|\check{\sigma}\Big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\check{\psi}\Big|^2\check{p}^2\,d\check{\sigma}\,d\tau-Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

Moreover, we get

$$\int \Big|\check{\sigma}\Big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\check{\psi}\Big|^2\check{p}^2 d\check{\sigma}\,d\tau \geq \int \big|\check{\sigma}\big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0 h^{1/2}\big)\check{\psi}\big|^2\check{p}^2 d\check{\sigma}\,d\tau - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

Let us analyze $\int |\check{\sigma}(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0 h^{1/2})\check{\psi}|^2\check{p}^2\,d\check{\sigma}\,d\tau$. We have

$$\int \big|\check{\sigma}\big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0 h^{1/2}\big)\check{\psi}\big|^2\check{p}^2\,d\check{\sigma}\,d\tau = \int \big|\big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0 h^{1/2}\big)\check{\sigma}\check{\psi}-ih\check{\psi}\big|^2\check{p}^2\,d\check{\sigma}\,d\tau.$$

The double product is

$$2\Re\Big(\int \big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0 h^{1/2}\big)\check{\sigma}\check{\psi}\,ih\overline{\check{\psi}}\,\check{p}^2\,d\check{\sigma}\,d\tau\Big) = -2h^2\Re\Big(\int (\check{p}^{-1}\partial_{\check{\sigma}}\check{p})\check{\sigma}\check{\psi}\overline{\check{\psi}}\,\check{p}^2\,d\check{\sigma}\,d\tau\Big).$$

But we have

$$2\Re\Big(\int \partial_{\check{\sigma}}(\check{\sigma}\check{p}\check{\psi})\overline{\check{p}\check{\psi}}\,d\check{\sigma}\,d\tau\Big) = 2\Re\Big(\int \check{p}\check{\psi}\,\overline{\check{p}\check{\psi}}\,d\check{\sigma}\,d\tau\Big)+\int \check{\sigma}\partial_{\check{\sigma}}|\check{p}\check{\psi}|^2\,d\check{\sigma}\,d\tau$$

and

$$\int \check{\sigma}\partial_{\check{\sigma}}|\check{p}\check{\psi}|^2\,d\check{\sigma}\,d\tau = -\int |\check{p}\check{\psi}|^2\,d\check{\sigma}\,d\tau.$$

Gathering the estimates, we obtain the lower bound:

$$\hat{Q}_{h,\mathrm{app}}(\hat{\psi}) \geq \check{Q}_{h,\mathrm{app}}(\check{\psi}) - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

A weighted Fourier transform: toward a model operator. We now define the unitary transform which diagonalizes the self-adjoint operator $\check{p}^{-1}D_{\check{\sigma}}\check{p}$ (for completeness, one should extend $\check{p}$ by 1 away from a neighborhood of 0). As we will see, with the coordinate $\check{\sigma}$, this transform admits a nice expression.

Weighted Fourier transform. Let us now introduce the weighted Fourier transform $\mathscr{F}_{\check{p}}$:

$$(\mathscr{F}_{\check{p}}\psi)(\lambda) = \int_{\mathbb{R}} e^{-i\lambda\check{\sigma}}\psi(\check{\sigma})\check{p}(\check{\sigma})\,d\check{\sigma} = \mathscr{F}(\check{p}\psi).$$

We observe that $\mathscr{F}_{\check{p}} : L^2(\mathbb{R}, \check{p}^2 d\check{\sigma}) \to L^2(\mathbb{R}, d\lambda)$ is unitary. Standard computations provide

$$\mathscr{F}_{\check{p}}((\check{p}^{-1}D_{\check{\sigma}}\check{p})\psi) = \lambda\mathscr{F}_{\check{p}}(\psi) \quad \text{and} \quad \mathscr{F}_{\check{p}}(\check{\sigma}\psi) = -D_\lambda\mathscr{F}_{\check{p}}(\psi).$$

Proposition 5.5. *There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and all $\tilde{\psi} \in \mathfrak{E}_N(h)$,*

$$\check{Q}_{h,\mathrm{app}}(\check{\psi}) \geq \int (1+a_2\tau)(1-\tau k_0)|hD_\tau\check{\phi}|^2\,d\lambda\,d\tau + \int (1+a_2\tau)(1-\tau k_0)^{-1}\Big|\Big(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\Big)\check{\phi}\Big|^2\,d\lambda\,d\tau + h\alpha\Theta_0\int |D_\lambda\check{\phi}|^2\,d\lambda\,d\tau - Ch^{3/2+1/4}\|\tilde{\psi}\|^2,$$

where $\check{\phi} = e^{-ib_1/2h(-h\lambda\tau^2/2+\tau^3/3-\xi_0 h^{1/2}\tau^2/2+(b_2/8)\tau^4)}\mathscr{F}_{\check{p}}(\check{\psi})$.

Proof. We have

$$\check{Q}_{h,\mathrm{app}}(\check{\psi}) = \int (1+a_2\tau)(1-\tau k_0)|h\partial_\tau\check{\varphi}|^2\,d\lambda\,d\tau + \int (1+a_2\tau)(1-\tau k_0)^{-1}\left|\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\right)\check{\varphi}\right|^2 d\lambda\,d\tau + h\alpha\Theta_0\int |D_\lambda\check{\varphi}|^2\,d\lambda\,d\tau,$$

where $\check{\varphi} = \mathscr{F}_{\check{p}}(\check{\psi})$. With the normal estimates, we can write

$$\begin{aligned}\int (1+a_2\tau)(1-\tau k_0)^{-1}&\left|\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\right)\check{\varphi}\right|^2 d\lambda\,d\tau\\
&\geq \int (1+a_2\tau)(1-\tau k_0)^{-1}\left|\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\right)\check{\varphi}\right|^2 d\lambda\,d\tau\\
&\quad - b_1\Re\left(\int (1+a_2\tau)(1-\tau k_0)^{-1}\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\right)\check{\varphi}\overline{\tau h D_\tau\check{\varphi}}\,d\lambda\,d\tau\right)\\
&\geq \int (1+a_2\tau)(1-\tau k_0)^{-1}\left|\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\right)\check{\varphi}\right|^2 d\lambda\,d\tau\\
&\quad - b_1\Re\left(\int\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\right)\check{\varphi}\overline{\tau h D_\tau\check{\varphi}}\,d\lambda\,d\tau\right) - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.\end{aligned}$$

Completing a square and using the normal Agmon estimates to control the additional terms, we get

$$\begin{aligned}\check{Q}_{h,\mathrm{app}}(\check{\psi}) &\geq \int (1+a_2\tau)(1-\tau k_0)\left|\left(hD_\tau-\frac{b_1}{2}\tau\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\right)\right)\check{\varphi}\right|^2 d\lambda\,d\tau\\
&+\int (1+a_2\tau)(1-\tau k_0)^{-1}\left|\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\right)\check{\varphi}\right|^2 d\lambda\,d\tau + h\alpha\Theta_0\int |D_\lambda\check{\varphi}|^2\,d\lambda\,d\tau - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.\end{aligned}$$

We now change the gauge by letting

$$\check{\varphi} = e^{ib_1/2h(-h\lambda\tau^2/2+\tau^3/3-\xi_0 h^{1/2}\tau^2/2+(b_2/8)\tau^4)}\check{\phi}.$$

We deduce

$$\check{Q}_{h,\mathrm{app}}(\check{\psi}) \geq \int (1+a_2\tau)(1-\tau k_0)|hD_\tau\check{\phi}|^2\,d\lambda\,d\tau + \int (1+a_2\tau)(1-\tau k_0)^{-1}\left|\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\right)\check{\phi}\right|^2 d\lambda\,d\tau + h\alpha\Theta_0\int\left|D_\lambda\big(e^{-i\lambda b_1\tau^2/4}\check{\phi}\big)\right|^2 d\lambda\,d\tau - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

Finally we write

$$\begin{aligned}\int \left|D_\lambda\left(e^{-i\lambda b_1\tau^2/4}\check{\phi}\right)\right|^2 d\lambda\, d\tau &= \int \left|D_\lambda\check{\phi} - \frac{b_1}{4}\tau^2\check{\phi}\right|^2 d\lambda\, d\tau \\ &\geq \int |D_\lambda\check{\phi}|^2\, d\lambda\, d\tau - C\|\tau^2\check{\phi}\|\|D_\lambda\check{\phi}\| \\ &\geq \int |D_\lambda\check{\phi}|^2\, d\lambda\, d\tau - C\|\tau^2\check{\psi}\|\|D_\lambda\check{\phi}\|.\end{aligned}$$

In addition, we notice that

$$\|D_\lambda\check{\phi}\| \leq C\left(\|\check{\sigma}\check{\psi}\| + \|\tau^2\check{\psi}\|\right) \leq Ch^{1/4}\|\tilde{\psi}\|. \qquad \square$$

In order to get a good model operator, we shall add a cutoff function with respect to τ. Let $\eta \in \left(0, \frac{1}{100}\right)$. Let χ be a cutoff function such that

$$\chi(t) = 1 \text{ for } |t| \leq 1, \qquad 0 \leq \chi \leq 1, \qquad \operatorname{supp}\chi \subset [-2, 2].$$

We define

$$l(x) = x\chi(h^\eta x).$$

Applying the normal Agmon estimates, we have:

Proposition 5.6. *There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and all $\tilde{\psi} \in \mathfrak{E}_N(h)$,*

$$\begin{aligned}\check{Q}_{h,\mathrm{app}}(\check{\psi}) \geq &\int \left(1 + a_2 h^{1/2} l(h^{-1/2}\tau)\right)\left(1 - h^{1/2} l(h^{-1/2}\tau)k_0\right)|hD_\tau\check{\phi}|^2\, d\lambda\, d\tau \\ &+ \int \left(1 + a_2 h^{1/2} l(h^{-1/2}\tau)\right)\left(1 - h^{1/2} l(h^{-1/2}\tau)k_0\right)^{-1}\left|\left(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2} h l(h^{-1/2}\tau)^2\right)\check{\phi}\right|^2 d\lambda\, d\tau \\ &+ h\alpha\Theta_0 \int |D_\lambda\check{\phi}|^2\, d\lambda\, d\tau - Ch^{3/2+1/4}\|\tilde{\psi}\|^2,\end{aligned}$$

where $\check{\phi} = e^{-ib_1/2h\left(-h\lambda\tau^2/2 + \tau^3/3 - \xi_0 h^{1/2}\tau^2/2 + (b_2/8)\tau^4\right)}\mathscr{F}_{\check{p}}(\check{\psi})$.

Remark 5.7. In particular, we have reduced the analysis to an electric Laplacian (with curvature terms), which has essentially the Born–Oppenheimer form (see our recent work [Bonnaillie-Noël et al. 2012], where a similar and simpler model appears). To see this more precisely, let us adopt a heuristical point of view. If we forget the different terms due to curvature, the operator which appears is in the form

$$h\alpha\Theta_0 D_\lambda^2 + h^2 D_\tau^2 + (-h\lambda + \tau - \xi_0 h^{1/2})^2.$$

After the rescaling $\lambda = h^{-1/4}\tilde{\lambda}$, $\tau = h^{1/2}x$, we get

$$h\left(h^{1/2}\alpha\Theta_0 D_{\tilde{\lambda}}^2 + D_x^2 + (-h^{1/4}\tilde{\lambda} - x - \xi_0)^2\right).$$

Therefore we are led to analyze a problem which is semiclassical with respect to just one variable. At some point (that we will justify at the end of this section), we can reduce the study to

$$h\big(h^{1/2}\alpha\Theta_0 D_{\tilde{\lambda}}^2+\mu(\xi_0+h^{1/4}\tilde{\lambda})\big),$$

and then (Taylor expansion)

$$h\left(h^{1/2}\alpha\Theta_0 D_{\tilde{\lambda}}^2+\Theta_0+\frac{\mu''(\xi_0)}{2}h^{1/2}\tilde{\lambda}^2\right).$$

Finally we recognize the harmonic oscillator, whose spectrum is well-known.

A simpler model in the Born–Oppenheimer spirit. We introduce the rescaled quadratic form:

$$\begin{aligned}Q_{\eta,h}(\varphi)=&\int\big(1+a_2h^{1/2}l(x)\big)\big(1-l(x)k_0h^{1/2}\big)|\partial_x\varphi|^2\,d\lambda\,dx\\&+\int\big(1+a_2l(x)h^{1/2}\big)\big(1-l(x)k_0h^{1/2}\big)^{-1}\left|\left(x-\xi_0+h^{1/2}\lambda+\frac{\hat{b}_2}{2}l(x)^2h^{1/2}\right)\varphi\right|^2\,d\lambda\,dx\\&+\alpha\Theta_0\int|D_\lambda\varphi|^2\,d\lambda\,dx.\end{aligned}$$

We recall that $\hat{b}_2=b_2-k_0$. We will denote by $H_{\eta,h}$ its corresponding Friedrichs extension. We will denote by $\nu_n(Q_{\eta,h})$ the sequence of its Rayleigh quotients. For each λ, we will need to consider the quadratic form

$$\begin{aligned}q_{\lambda,\eta,h}(\varphi)=&\int\big(1+a_2h^{1/2}l(x)\big)\big(1-l(x)k_0h^{1/2}\big)\,|\partial_x\varphi|^2\,dx\\&+\int\big(1+a_2l(x)h^{1/2}\big)\big(1-l(x)k_0h^{1/2}\big)^{-1}\left|\left(x-\xi_0+h^{1/2}\lambda+\frac{\hat{b}_2}{2}l(x)^2h^{1/2}\right)\varphi\right|^2\,dx,\end{aligned}$$

whose domain is $B^1(\mathbb{R}^+)$. We denote by $\nu_j(q_{\lambda,\eta,h})$ the increasing sequence of the eigenvalues of the associated operator. The main proposition of this subsection is the following:

Proposition 5.8. *For all $n\geq 1$, there exist $h_0>0$ and $C>0$ such that, for $h\in(0,h_0)$:*

$$\nu_n(Q_{\eta,h})\geq\Theta_0+\left(C(k_0,a_2,b_2)+(2n-1)\sqrt{\frac{\alpha\mu''(\xi_0)\Theta_0}{2}}\right)h^{1/2}-Ch^{1/2+1/8}.$$

With Propositions 5.6 and 5.3, inequality (5-1), and the min-max principle, we first deduce the size of the spectral gap between the lowest eigenvalues of $P_{h,\boldsymbol{A}}$. Then, with Theorem 2.1, we deduce Theorem 1.3.

Elementary properties of the spectrum. This subsection is devoted to basic properties of the spectrum of $Q_{\eta,h}$. The following proposition provides a lower bound for $\nu_1(q_{\lambda,\eta,h})$.

Proposition 5.9. *There exist positive constants C, c_0, M and h_0 such that if $h\in(0,h_0)$, then:*

(1) *If $|\lambda|\geq Mh^{-1/4-\eta}$, then*

$$\nu_1(q_{\lambda,\eta,h})\geq\Theta_0+c_0\min(1,\lambda^2h).$$

(2) *If* $|\lambda| \leq Mh^{-1/4-\eta}$, *then*

$$\nu_1(q_{\lambda,\eta,h}) \geq \Theta_0 + C(k_0, a_2, b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h - Ch^{3/4-3\eta},$$

where $C(k_0, a_2, b_2)$ *is given in Theorem 1.3.*

Proof. The proof is left to the reader as an adaptation of [Fournais and Helffer 2010, Proposition 5.2.1]. □

Let us now prove a lower bound for the essential spectrum of $H_{\eta,h}$.

Proposition 5.10. *There exist* $h_0 > 0$ *and* $\tilde{c}_0 > 0$ *such that, if* $h \in (0, h_0)$, *then*

$$\inf \sigma_{\mathrm{ess}}(Q_{\eta,h}) \geq \Theta_0 + \tilde{c}_0.$$

Proof. Let $\phi \in \mathrm{Dom}(Q_{\eta,h})$ such that $\mathrm{supp}(\phi) \subset \mathbb{R}^2_+ \setminus [-\tilde{R}, \tilde{R}]^2$. Let us use a partition of unity $\chi_{1,R}^2 + \chi_{2,R}^2 = 1$ such that $\chi_{1,R}(x) = \chi_1(R^{-1}x)$ and where χ_1 is a smooth cutoff function being 1 near 0. We have

$$Q_{\eta,h}(\phi) \geq Q_{\eta,h}(\chi_{1,R}\phi) + Q_{\eta,h}(\chi_{2,R}\phi) - CR^{-2}\|\phi\|^2.$$

For $R \geq 2h^{-\eta}$, we have (the metrics becomes flat and we can compare with a problem in $\mathbb{R}^2$)

$$Q_{\eta,h}(\chi_{2,R}\phi) \geq \|\chi_{2,R}\phi\|^2.$$

We have

$$Q_{\eta,h}(\chi_{1,R}\phi) \geq \int_{\mathbb{R}^2_+} \nu_1(q_{\lambda,\eta,h})|\chi_{1,R}\phi|^2 + \alpha\Theta_0|D_\lambda(\chi_{1,R}\phi)|^2\, dx\, d\lambda.$$

Taking $h \in (0, h_0)$ (where h_0 is given by Proposition 5.9) and $\tilde{R} \geq h^{-1/2}$, we infer

$$Q_{\eta,h}(\chi_{1,R}\phi) \geq \int_{\mathbb{R}^2_+} (\Theta_0 + c_0)|\chi_{1,R}\phi|^2\, dx\, d\lambda.$$

This implies that

$$Q_{\eta,h}(\phi) \geq \big(\min(1, \Theta_0 + c_0) - Ch^{2\eta}\big)\|\phi\|^2.$$

The conclusion follows from a Persson's lemma-like argument (see [Persson 1960; Fournais and Helffer 2010, Appendix B.3]). □

The following proposition provides an upper bound for the lowest eigenvalues of $H_{\eta,h}$.

Proposition 5.11. *For all* $M \geq 1$, *there exist* $h_0 > 0$, $C > 0$ *such that for all* $1 \leq n \leq M$:

$$\nu_n(Q_{\eta,h}) \leq h^{-1}\lambda_n(h) + O(h^\infty).$$

Proof. This is a consequence of (5-1) together with the lower bounds of Propositions 5.3 and 5.6 and the min-max principle (see for instance [Reed and Simon 1978]). □

Remark 5.12. For h small enough, we deduce that there are at least M eigenvalues below $\Theta_0 + \tilde{c}_0$. Let us consider the first M eigenvalues $\nu_n(Q_{\eta,h})$ below $\Theta_0 + \tilde{c}_0$. With Theorem 2.1, we deduce that, for all $M \geq 1$, there exist $h_0 > 0$ and $C(M) > 0$ such that, for $1 \leq n \leq M$,

$$0 \leq \nu_n(Q_{\eta,h}) - \Theta_0 \leq C(M)h^{1/2}.$$

For $1 \leq n \leq M$, let us consider a normalized eigenfunction $f_{n,\eta,h}$ associated to $\nu_n(Q_{\eta,h})$ so that $f_{n,\eta,h}$ and $f_{m,\eta,h}$ are orthogonal if $n \neq m$. Let us introduce:

$$\mathfrak{F}_M(h) = \operatorname{span}_{1\leq j\leq M}(f_{j,\eta,h}).$$

Agmon estimates. First, let us state Agmon estimates with respect to x.

Proposition 5.13. *There exist $h_0 > 0$, $\varepsilon_0 > 0$, $C > 0$ such that, for all $f \in \mathfrak{F}_M(h)$,*

$$\int_{\mathbb{R}^2_+} e^{\varepsilon_0 x} |f|^2 \, dx \, d\lambda \leq C\|f\|^2.$$

Proof. Let us use a partition of unity, $\chi_{1,R}^2 + \chi_{2,R}^2 = 1$, with $R \geq h^{-\eta}$. We take $\Phi = \varepsilon_0 \chi(x/r)|x|$. This IMS formula implies (with $f = f_{n,\eta,h}$)

$$Q_{\eta,h}(\chi_{1,R}e^{\Phi} f) + Q_{\eta,h}(\chi_{2,R}e^{\Phi} f) - C\varepsilon_0^2\|e^{\Phi} f\|^2 - \nu_n(Q_{\eta,h})\|e^{\Phi} f\|^2 \leq 0.$$

We recall that

$$Q_{\eta,h}(\chi_{2,R}e^{\Phi} f) \geq \|\chi_{2,R}e^{\Phi} f\|^2$$

and that

$$Q_{\eta,h}(\chi_{1,R}e^{\Phi} f) \geq \int \nu_1(q_{\lambda,\eta,h})|\chi_{1,R}e^{\Phi} f|^2 \, dx \, d\lambda.$$

On the one hand, we have

$$Q_{\eta,h}(\chi_{2,R}e^{\Phi} f) - C\varepsilon_0^2\|\chi_{2,R}e^{\Phi} f\|^2 - (\Theta_0 + Ch^{1/2})\|\chi_{2,R}e^{\Phi} f\|^2 \geq (1 - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2})\|\chi_{2,R}e^{\Phi} f\|^2.$$

On the other hand, we get

$$\begin{aligned} &Q_{\eta,h}(\chi_{1,R}e^{\Phi} f) - C\varepsilon_0^2\|\chi_{1,R}e^{\Phi} f\|^2 - (\Theta_0 + Ch^{1/2})\|\chi_{1,R}e^{\Phi} f\|^2 \\ &\qquad \geq \int \big(\nu_1(q_{\eta,\lambda,h}) - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2}\big)|\chi_{1,R}e^{\Phi} f|^2 \, dx \, d\lambda. \end{aligned}$$

When $|\lambda| \geq Mh^{-1/4-\eta}$, we have

$$\nu_1(q_{\eta,\lambda,h}) - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2} \geq -C\varepsilon_0^2 - Ch^{1/2}.$$

When $|\lambda| \leq Mh^{-1/4}$, we have

$$\nu_1(q_{\eta,\lambda,h}) - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2} \geq -C\varepsilon_0^2 - \tilde{C}h^{1/2}.$$

If h and ε_0 are small enough, we deduce that

$$(1 - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2})\|\chi_{2,R}e^{\Phi} f\|^2 \leq C\|\chi_{1,R}e^{\Phi} f\|^2,$$

so that

$$\|\chi_{2,R}e^{\Phi} f\|^2 \leq \tilde{C}\|f\|^2 \quad \text{and} \quad \|e^{\Phi} f\|^2 \leq \hat{C}\|f\|^2,$$

where $\tilde{C}$ and $\hat{C}$ are independent from r. It remains to make $r \to +\infty$ and apply the Fatou lemma. Finally, it is easy to extend the inequality to $f \in \mathfrak{F}_M(h)$. ☐

Then, we will need Agmon estimates with respect to λ:

Proposition 5.14. *There exist $h_0 > 0$, $C > 0$ such that, for all $f \in \mathfrak{F}_M(h)$,*

$$\int_{\mathbb{R}^2_+} e^{2h^{1/4}|\lambda|}|f|^2\,dx\,d\lambda \le C\|f\|^2 \tag{5-4}$$

and

$$\int_{\mathbb{R}^2_+} e^{2h^{1/4}|\lambda|}|D_\lambda f|^2\,dx\,d\lambda \le Ch^{1/2}\|f\|^2. \tag{5-5}$$

Remark 5.15. Heuristically, these estimates with respect to λ correspond to the phase space localization of [Fournais and Helffer 2006, Section 5].

Proof. We take $f = f_{j,\eta,h}$ and use the IMS formula (with $\Phi = h^{1/4}\chi(r^{-1}|\lambda|)|\lambda|$) to get

$$Q_{\eta,h}(e^\Phi f) \le \nu_j(Q_{\eta,h})\|e^\Phi f\|^2 + C\|\nabla\Phi e^\Phi f\|^2 \le \big(\Theta_0 + C(M)h^{1/2} + Ch^{1/2}\big)\|e^\Phi f\|^2.$$

We recall that

$$Q_{\eta,h}(e^\Phi f) \ge \int_{\mathbb{R}^2_+} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2 + \alpha\Theta_0|D_\lambda(e^\Phi f)|^2\,dx\,d\lambda \ge \int_{\mathbb{R}^2_+} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2\,dx\,d\lambda.$$

We have, for all $D > 0$,

$$\int_{\mathbb{R}^2_+} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2\,dx\,d\lambda = \int_{|\lambda|\le Dh^{-1/4}} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2\,dx\,d\lambda + \int_{|\lambda|\ge Dh^{-1/4}} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2\,dx\,d\lambda.$$

Moreover, we get

$$\int_{|\lambda|\ge Mh^{-1/4-\eta}} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2\,dx\,d\lambda \ge \int_{|\lambda|\ge Mh^{-1/4-\eta}} \big(\Theta_0 + c_0\min(1,h\lambda^2)\big)|e^\Phi f|^2\,dx\,d\lambda$$

and

$$\begin{aligned}\int_{Dh^{-1/4}\le|\lambda|\le Mh^{-1/4-\eta}} &\nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2\,dx\,d\lambda \\ &\ge \int_{Dh^{-1/4}\le|\lambda|\le Mh^{-1/4-\eta}} \left(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h - Ch^{3/4-3\eta}\right)|e^\Phi f|^2\,dx\,d\lambda.\end{aligned}$$

This leads to

$$\int_{|\lambda|\ge Dh^{-1/4}} \big(c_1\min(1,h\lambda^2) - \tilde{C}h^{1/2} - C\alpha^2 h^{1/2}\big)|e^\Phi f|^2\,dx\,d\lambda \le \tilde{C}h^{1/2}\int_{|\lambda\le Dh^{-1/4}} |f|^2\,d\lambda\,dx.$$

It remains to take D large enough, and we get (5-4). Then we have

$$\int_{\mathbb{R}^2_+} \big(\nu_1(q_{\lambda,\eta,h}) - \Theta_0\big)|e^\Phi f|^2 + \alpha\Theta_0|D_\lambda(e^\Phi f)|^2\,dx\,d\lambda \le Ch^{1/2}\|f\|^2.$$

But we notice that

$$\int_{\mathbb{R}^2_+}\big(\nu_1(q_{\lambda,\eta,h})-\Theta_0\big)|e^{\Phi}f|^2\,dx\,d\lambda$$
$$\geq \int_{Dh^{-1/4}\leq|\lambda|\leq Mh^{-1/4-\eta}}\left(C(k_0,a_2,b_2)h^{1/2}+\frac{\mu''(\xi_0)}{2}\lambda^2 h-Ch^{3/4-3\eta}\right)|e^{\Phi}f|^2\,dx\,d\lambda$$
$$+\int_{|\lambda\leq Dh^{-1/4}}\left(C(k_0,a_2,b_2)h^{1/2}+\frac{\mu''(\xi_0)}{2}\lambda^2 h-Ch^{3/4-3\eta}\right)|e^{\Phi}f|^2\,d\lambda\,dx.$$

Taking D larger, we get

$$\int_{Dh^{-1/4}\leq|\lambda|\leq Mh^{-1/4-\eta}}\left(\frac{\mu''(\xi_0)}{2}\lambda^2 h-Ch^{1/2}-Ch^{3/4-3\eta}\right)|e^{\Phi}f|^2\,dx\,d\lambda\geq 0.$$

Moreover, we have

$$\left|\int_{|\lambda\leq Dh^{-1/4}}\left(C(k_0,a_2,b_2)h^{1/2}+\frac{\mu''(\xi_0)}{2}\lambda^2 h-Ch^{3/4-3\eta}\right)|e^{\Phi}f|^2\,d\lambda\,dx\right|\leq Ch^{1/2}\|f\|^2. \qquad \square$$

Approximations of eigenvectors by tensor products. Let us define the quadratic form q_0 with domain $B^1(\mathbb{R}_+)\otimes L^2(\mathbb{R})$:

$$q_0(\varphi)=Q_0(\varphi)-\Theta_0\|\varphi\|^2=\int_{\mathbb{R}^2_+}|\partial_x\varphi|^2+|(x-\xi_0)\varphi|^2-\Theta_0|\varphi|^2\,dx\,d\lambda.$$

The Friedrichs extension of q_0 is the operator $H_{\xi_0}\otimes\mathrm{Id}_{L^2(\mathbb{R})}$. We also define the Feshbach–Grušin projection on the kernel of $H_{\xi_0}\otimes\mathrm{Id}_{L^2(\mathbb{R})}$:

$$\Pi_0\varphi=\langle\varphi,u_{\xi_0}\rangle_x u_{\xi_0}(x).$$

The next proposition states an approximation result for the elements of $\mathfrak{F}_M(h)$ (which behave as tensor products):

Proposition 5.16. *For all $M\geq 1$, there exist $h_0>0$ and $C>0$ such that we have, for all $f\in\mathfrak{F}_M(h)$,*

$$\|f-\Pi_0 f\|_{L^2}+\|\partial_x(f-\Pi_0 f)\|_{L^2}+\|x(f-\Pi_0 f)\|_{L^2}\leq Ch^{1/8}\|f\|, \tag{5-6}$$
$$\|(\lambda f-\Pi_0\lambda f\|_{L^2}+\|\partial_x(\lambda f-\Pi_0\lambda f)\|_{L^2}+\|x(\lambda f-\Pi_0\lambda f)\|_{L^2}\leq Ch^{-1/8}\|f\|, \tag{5-7}$$
$$\|(\partial_\lambda f-\Pi_0\partial_\lambda f\|_{L^2}+\|\partial_x(\partial_\lambda f-\Pi_0\partial_\lambda f)\|_{L^2}+\|x(\partial_\lambda f-\Pi_0\partial_\lambda f)\|_{L^2}\leq Ch^{3/8}\|f\|. \tag{5-8}$$

In particular, Π_0 is an isomorphism from $\mathfrak{F}_M(h)$ onto its range.

Proof. We take $f=f_{j,\eta,h}$. By definition, we have

$$H_{\eta,h}f=\nu_j(Q_{\eta,h})f. \tag{5-9}$$

Approximation of f. We deduce

$$Q_{\eta,h}(f) = \nu_j(Q_{\eta,h})\|f\|^2 \leq (\Theta_0 + Ch^{1/2})\|f\|^2.$$

We have

$$Q_{\eta,h}(f) \geq (1 - Ch^{1/2-\eta}) \int_{\mathbb{R}^2_+} |\partial_x f|^2 + \left|\left(x - \xi_0 + h^{1/2}\lambda + h^{1/2}\frac{\hat{b}_2}{2}l(x)^2\right)f\right|^2 dx\,d\lambda.$$

Moreover, we get (using the estimates of Agmon), for all $\varepsilon \in (0, 1)$:

$$\int_{\mathbb{R}^2_+} |\partial_x f|^2 + \left|\left(x - \xi_0 + h^{1/2}\lambda + h^{1/2}\frac{b_2}{2}l(x)^2\right)f\right|^2 dx\,d\lambda \geq (1-\varepsilon)Q_0(f) - C\varepsilon^{-1}h^{1/2}\|f\|^2.$$

Taking $\varepsilon = h^{1/4}$, we deduce

$$q_0(f) \leq Ch^{1/4}\|f\|^2.$$

We deduce (5-6).

Approximation of λf. We multiply (5-9) by λ and take the scalar product with λf:

$$Q_{\eta,h}(\lambda f) \leq (\Theta_0 + Ch^{1/2})\|\lambda f\|^2 + \left|\langle [H_{\eta,h}, \lambda]f, \lambda f\rangle\right|.$$

Thus, it follows that

$$Q_{\eta,h}(\lambda f) \leq (\Theta_0 + Ch^{1/2})\|\lambda f\|^2 + \alpha\Theta_0|\langle D_\lambda f, \lambda f\rangle| \leq \Theta_0\|\lambda f\|^2 + C\|f\|^2.$$

We get

$$Q_{\eta,h}(\lambda f) \geq (1 - Ch^{1/2-\eta})\big((1-\varepsilon)Q_0(\lambda f) - C\varepsilon^{-1}\|f\|^2\big).$$

We take $\varepsilon = h^{1/4}$ to deduce

$$q_0(\lambda f) \leq Ch^{-1/4}\|f\|^2.$$

We infer (5-7).

Approximation of $D_\lambda f$. We take the derivative of (5-9) with respect to λ and take the scalar product with $\partial_\lambda f$:

$$Q_{\eta,h}(\partial_\lambda f) \leq (\Theta_0 + Ch^{1/2})\|\partial_\lambda f\|^2 + \left|\langle [H_{\eta,h}, \partial_\lambda]f, \partial_\lambda f\rangle\right|.$$

The estimates of Agmon give

$$\left|\langle [H_{\eta,h}, \partial_\lambda]f, \partial_\lambda f\rangle\right| \leq Ch^{3/4}\|f\|^2.$$

We have

$$Q_{\eta,h}(\partial_\lambda f) \geq (1 - Ch^{1/2-\eta})\big((1-\varepsilon)Q_0(\partial_\lambda f) - C\varepsilon^{-1}h\|f\|^2\big).$$

We take $\varepsilon = h^{1/4}$ and deduce

$$q_0(\partial_\lambda f) \leq Ch^{3/4}\|f\|^2.$$

We infer (5-8). □

Conclusion: proof of Proposition 5.8. For all $f \in \mathfrak{F}_M(h)$, we have the lower bound

$$\begin{aligned} Q_{\eta,h}(f) &\geq \int_{\mathbb{R}^2_+} \nu_1(q_{\lambda,\eta,h})|f|^2 + \alpha\Theta_0|D_\lambda f|^2\,dx\,d\lambda \\ &\geq \int_{\mathbb{R}^2_+}\Big(\nu_1(q_{\lambda,\eta,h}) - \Big(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\Big)\Big)|f|^2\,dx\,d\lambda \\ &\quad + \int_{\mathbb{R}^2_+}\Big(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\Big)|f|^2\,dx\,d\lambda + \alpha\Theta_0|D_\lambda f|^2\,dx\,d\lambda. \end{aligned}$$

We now estimate

$$\begin{aligned} &\int_{\mathbb{R}^2_+}\Big(\nu_1(q_{\lambda,\eta,h}) - \Big(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\Big)\Big)|f|^2\,dx\,d\lambda \\ &\qquad = \int_{|\lambda|\geq Mh^{-1/4-\eta}}\Big(\nu_1(q_{\lambda,\eta,h}) - \Big(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\Big)\Big)|f|^2\,dx\,d\lambda \\ &\qquad\quad + \int_{|\lambda|\leq Mh^{-1/4-\eta}}\Big(\nu_1(q_{\lambda,\eta,h}) - \Big(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\Big)\Big)|f|^2\,dx\,d\lambda. \end{aligned}$$

Moreover, we get

$$\begin{aligned} &\int_{|\lambda|\geq Mh^{-1/4-\eta}}\Big(\nu_1(q_{\lambda,\eta,h}) - \Big(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\Big)\Big)|f|^2\,dx\,d\lambda \\ &\qquad \geq \int_{|\lambda|\geq Mh^{-1/4-\eta}} -\Big(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\Big)|f|^2\,dx\,d\lambda = O(h^\infty)\|f\|^2, \end{aligned}$$

where the last estimate is a consequence of the estimates of Agmon. Then we get

$$\int_{|\lambda|\leq Mh^{-1/4-\eta}}\Big(\nu_1(q_{\lambda,\eta,h}) - \Big(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\Big)\Big)|f|^2\,dx\,d\lambda \geq -Ch^{3/4-3\eta}\|f\|^2.$$

We deduce

$$\begin{aligned} Q_{\eta,h}(f) \geq \int_{\mathbb{R}^2_+}\Big(C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\Big)\,|f|^2\,dx\,d\lambda + \alpha\Theta_0|D_\lambda f|^2\,dx\,d\lambda \\ + \Theta_0\|f\|^2 - Ch^{3/4-3\eta}\|f\|^2. \end{aligned}$$

We now use Proposition 5.16 to get

$$\begin{aligned} Q_{\eta,h}(f) \geq \int_{\mathbb{R}^2_+}\Big(C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\Big)\,|\Pi_0 f|^2\,dx\,d\lambda + \alpha\Theta_0|D_\lambda \Pi_0 f|^2\,dx\,d\lambda \\ + \Theta_0\|f\|^2 - Ch^{1/2+1/8}\|\Pi_0 f\|^2. \end{aligned}$$

But we notice that for all $f \in \mathfrak{F}_M(h)$,

$$Q_{\eta,h}(f) \leq \nu_M(Q_{\eta,h})\|f\|^2,$$

and thus:

$$\int_{\mathbb{R}^2_+} \left(C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) |\Pi_0 f|^2 \, dx \, d\lambda + \alpha \Theta_0 |D_\lambda \Pi_0 f|^2 \, dx \, d\lambda$$
$$\leq \big(\nu_M(Q_{\eta,h}) - \Theta_0 \big) \|f\|^2 + C h^{1/2+1/8} \|\Pi_0 f\|^2$$
$$\leq \big(\nu_M(Q_{\eta,h}) - \Theta_0 \big) \|\Pi_0 f\|^2 + \tilde{C} h^{1/2+1/8} \|\Pi_0 f\|^2.$$

The conclusion follows from the min-max principle.

Acknowledgments

The author thanks San Vũ Ngọc for his comments, which have improved the presentation of the paper.

References

[Agmon 1982] S. Agmon, *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*, Mathematical Notes **29**, Princeton University Press, 1982. MR 85f:35019 Zbl 0503.35001

[Agmon 1985] S. Agmon, "Bounds on exponential decay of eigenfunctions of Schrödinger operators", pp. 1–38 in *Schrödinger operators* (Como, 1984), Lecture Notes in Math. **1159**, Springer, Berlin, 1985. MR 87i:35157 Zbl 0583.35027

[Alama et al. 2010] S. Alama, L. Bronsard, and B. Galvão-Sousa, "Thin film limits for Ginzburg–Landau with strong applied magnetic fields", *SIAM J. Math. Anal.* **42**:1 (2010), 97–124. MR 2010m:35498 Zbl 1210.35242

[Bauman et al. 1998] P. Bauman, D. Phillips, and Q. Tang, "Stable nucleation for the Ginzburg–Landau system with an applied magnetic field", *Arch. Rational Mech. Anal.* **142**:1 (1998), 1–43. MR 99g:58040 Zbl 0922.35157

[Bernoff and Sternberg 1998] A. Bernoff and P. Sternberg, "Onset of superconductivity in decreasing fields for general domains", *J. Math. Phys.* **39**:3 (1998), 1272–1284. MR 99a:82099 Zbl 1056.82523

[Bonnaillie-Noël et al. 2012] V. Bonnaillie-Noël, M. Dauge, N. Popoff, and N. Raymond, "Discrete spectrum of a model Schrödinger operator on the half-plane with Neumann conditions", *Z. Angew. Math. Phys.* **63**:2 (2012), 203–231. MR 2912345

[Chapman et al. 1995] S. J. Chapman, Q. Du, and M. D. Gunzburger, "On the Lawrence–Doniach and anisotropic Ginzburg–Landau models for layered superconductors", *SIAM J. Appl. Math.* **55**:1 (1995), 156–174. MR 95m:82120 Zbl 0819.35133

[Combes et al. 1981] J.-M. Combes, P. Duclos, and R. Seiler, "The Born–Oppenheimer approximation", pp. 185–212 Plenum, New York, 1981.

[Cycon et al. 1987] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, Springer, Berlin, 1987. MR 88g:35003 Zbl 0619.47005

[Dauge and Helffer 1993] M. Dauge and B. Helffer, "Eigenvalues variation, I: Neumann problem for Sturm–Liouville operators", *J. Differential Equations* **104**:2 (1993), 243–262. MR 94j:47097 Zbl 0784.34021

[Dombrowski and Raymond 2013] N. Dombrowski and N. Raymond, "Semiclassical analysis with vanishing magnetic fields", *J. Spectr. Theory* **3**:3 (2013), 423–464. MR 3073418

[Doria and de Andrade 1996] M. M. Doria and S. C. B. de Andrade, "Virial theorem for the anisotropic Ginzburg–Landau theory", *Phys. Rev. B* **53** (1996), 3440–3454.

[Egorov 1971] J. V. Egorov, "Canonical transformations and pseudodifferential operators", *Trudy Moskov. Mat. Obšč.* **24** (1971), 3–28. In Russian. MR 50 #14371 Zbl 0228.35079

[Fournais and Helffer 2006] S. Fournais and B. Helffer, "Accurate eigenvalue asymptotics for the magnetic Neumann Laplacian", *Ann. Inst. Fourier* (*Grenoble*) **56**:1 (2006), 1–67. MR 2007g:35198 Zbl 1097.47020

[Fournais and Helffer 2010] S. Fournais and B. Helffer, *Spectral methods in surface superconductivity*, Progress in Nonlinear Differential Equations and their Applications **77**, Birkhäuser, Boston, 2010. MR 2662319 Zbl 05366495

[Grušin 1972] V. V. Grušin, "Hypoelliptic differential equations and pseudodifferential operators with operator-valued symbols", *Mat. Sb. (N.S.)* **88(130)** (1972), 504–521. MR 47 #5427

[Helffer 1988] B. Helffer, *Semi-classical analysis for the Schrödinger operator and applications*, Lecture Notes in Mathematics **1336**, Springer, Berlin, 1988. MR 90c:81043 Zbl 0647.35002

[Helffer and Kordyukov 2011] B. Helffer and Y. A. Kordyukov, "Semiclassical spectral asymptotics for a two-dimensional magnetic Schrödinger operator: the case of discrete wells", pp. 55–78 in *Spectral theory and geometric analysis*, Contemp. Math. **535**, Amer. Math. Soc., Providence, RI, 2011. MR 2560751 Zbl 1218.58017

[Helffer and Morame 2001] B. Helffer and A. Morame, "Magnetic bottles in connection with superconductivity", *J. Funct. Anal.* **185**:2 (2001), 604–680. MR 2002m:81051 Zbl 1078.81023

[Helffer and Morame 2004] B. Helffer and A. Morame, "Magnetic bottles for the Neumann problem: curvature effects in the case of dimension 3 (general case)", *Ann. Sci. École Norm. Sup.* (4) **37**:1 (2004), 105–170. MR 2006g:35249 Zbl 1057.35061

[Kato 1966] T. Kato, *Perturbation theory for linear operators*, Grundlehren der mathematischen Wissenschaften **132**, Springer, New York, 1966. MR 34 #3324 Zbl 0148.12601

[Lu and Pan 1999] K. Lu and X.-B. Pan, "Eigenvalue problems of Ginzburg–Landau operator in bounded domains", *J. Math. Phys.* **40**:6 (1999), 2647–2670. MR 2001c:35167 Zbl 0943.35058

[Lu and Pan 2000] K. Lu and X.-B. Pan, "Surface nucleation of superconductivity in 3-dimensions", *J. Differential Equations* **168**:2 (2000), 386–452. MR 2002b:82069 Zbl 0972.35152

[Martinez 1989] A. Martinez, "Développements asymptotiques et effet tunnel dans l'approximation de Born–Oppenheimer", *Ann. Inst. H. Poincaré Phys. Théor.* **50**:3 (1989), 239–257. MR 91e:81022 Zbl 0689.35063

[Martinez 2002] A. Martinez, *An introduction to semiclassical and microlocal analysis*, Springer, New York, 2002. MR 2003b:35010 Zbl 0994.35003

[Persson 1960] A. Persson, "Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator", *Math. Scand.* **8** (1960), 143–153. MR 24 #A3412 Zbl 0145.14901

[del Pino et al. 2000] M. del Pino, P. L. Felmer, and P. Sternberg, "Boundary concentration for eigenvalue problems related to the onset of superconductivity", *Comm. Math. Phys.* **210**:2 (2000), 413–446. MR 2001k:35231 Zbl 0982.35077

[Raymond 2009] N. Raymond, "Sharp asymptotics for the Neumann Laplacian with variable magnetic field: case of dimension 2", *Ann. Henri Poincaré* **10**:1 (2009), 95–122. MR 2010e:35199 Zbl 1210.81034

[Raymond 2010] N. Raymond, "On the semiclassical 3D Neumann Laplacian with variable magnetic field", *Asymptot. Anal.* **68**:1-2 (2010), 1–40. MR 2675335 Zbl 1200.35202

[Raymond 2012] N. Raymond, "Semiclassical 3D Neumann Laplacian with variable magnetic field: a toy model", *Comm. Partial Differential Equations* **37**:9 (2012), 1528–1552. MR 2969489 Zbl 06102472

[Reed and Simon 1978] M. Reed and B. Simon, *Methods of modern mathematical physics, IV: Analysis of operators*, Academic Press, New York, 1978. MR 58 #12429c Zbl 0401.47001

[Robert 1987] D. Robert, *Autour de l'approximation semi-classique*, Progress in Mathematics **68**, Birkhäuser, Boston, 1987. MR 89g:81016 Zbl 0621.35001

Received 5 May 2012. Revised 2 Aug 2012. Accepted 18 Oct 2012.

NICOLAS RAYMOND: nicolas.raymond@univ-rennes1.fr
Institut de recherche mathématique de Rennes, Université de Rennes 1, 35042 Rennes Cedex, France

ANALYSIS AND PDE
Vol. 6, No. 6, 2013
dx.doi.org/10.2140/apde.2013.6.1327

msp

STABILITY AND INSTABILITY FOR SUBSONIC TRAVELING WAVES OF THE NONLINEAR SCHRÖDINGER EQUATION IN DIMENSION ONE

DAVID CHIRON

We study the stability/instability of the subsonic traveling waves of the nonlinear Schrödinger equation in dimension one. Our aim is to propose several methods for showing instability (use of the Grillakis–Shatah–Strauss theory, proof of existence of an unstable eigenvalue via an Evans function) or stability. For the latter, we show how to construct in a systematic way a Liapounov functional for which the traveling wave is a local minimizer. These approaches allow us to give a complete stability/instability analysis in the energy space including the critical case of the kink solution. We also treat the case of a cusp in the energy-momentum diagram.

1. Introduction

This paper is a continuation of our previous work [Chiron 2012], where we consider the one-dimensional nonlinear Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} + \partial_x^2 \Psi + \Psi f(|\Psi|^2) = 0. \tag{NLS}$$

This equation appears as a relevant model in condensed matter physics: Bose–Einstein condensation and superfluidity (see [Roberts and Berloff 2001; Ginzburg and Pitaevskiĭ 1958; Gross 1963; Abid et al. 2003]); nonlinear optics (see, for instance, the survey [Kivshar and Luther-Davies 1998]). Several nonlinearities may be encountered in physical situations: $f(\varrho) = \pm\varrho$ gives rise to the focusing/defocusing cubic NLS; $f(\varrho) = 1 - \varrho$ to the so-called Gross–Pitaevskii equation; $f(\varrho) = -\varrho^2$ (see [Kolomeisky et al. 2000] for Bose–Einstein condensates); more generally a pure power; the "cubic-quintic" NLS (see [Barashenkov and Panova 1993]), where

$$f(\varrho) = -\alpha_1 + \alpha_3\varrho - \alpha_5\varrho^2$$

and α_1, α_3 and α_5 are positive constants such that f has two positive roots; and in nonlinear optics, we may take (see [Kivshar and Luther-Davies 1998])

$$f(\varrho) = -\alpha\varrho^\nu - \beta\varrho^{2\nu}, \quad f(\varrho) = -\frac{\varrho_0}{2}\left(\frac{1}{(1+\frac{1}{\varrho_0})^\nu} - \frac{1}{(1+\frac{\varrho}{\varrho_0})^\nu}\right), \quad f(\varrho) = -\alpha\varrho\left(1 + \gamma\tanh\frac{\varrho^2-\varrho_0^2}{\sigma^2}\right), \tag{1}$$

MSC2010: 35B35, 35J20, 35Q40, 35Q55, 35C07.
Keywords: traveling wave, nonlinear Schrödinger equation, Gross–Pitaevskii equation, stability, Evans function, Liapounov functional.

where α, β, γ, ν, $\sigma > 0$ are given constants (the second one, for instance, takes into account saturation effects), etc. As a consequence, as in [Chiron 2012], we shall consider a rather general nonlinearity f, with f of class $\mathscr{C}^2$. In the context of Bose–Einstein condensation or nonlinear optics, the natural condition at infinity appears to be

$$|\Psi|^2 \to r_0^2 \quad \text{as } |x| \to +\infty,$$

where $r_0 > 0$ is such that $f(r_0^2) = 0$.

For solutions Ψ of (NLS) which do not vanish, we may use the Madelung transform

$$\Psi = A \exp(i\phi)$$

and rewrite (NLS) as an hydrodynamical system with an additional quantum pressure

$$\begin{cases} \partial_t A + 2\partial_x \phi \partial_x A + A\partial_x^2 \phi = 0, \\ \partial_t \phi + (\partial_x \phi)^2 - f(A^2) - \dfrac{\partial_x^2 A}{A} = 0 \end{cases} \quad \text{or} \quad \begin{cases} \partial_t \rho + 2\partial_x(\rho u) = 0, \\ \partial_t u + 2u\partial_x u - \partial_x(f(\rho)) - \partial_x\left(\dfrac{\partial_x^2(\sqrt{\rho})}{\sqrt{\rho}}\right) = 0, \end{cases} \tag{2}$$

with $(\rho, u) \equiv (A^2, \partial_x \phi)$. When neglecting the quantum pressure and linearizing this Euler system around the particular trivial solution $\Psi = r_0$ (or $(A, u) = (r_0, 0)$), we obtain the free wave equation

$$\begin{cases} \partial_t \bar{A} + r_0 \partial_x \bar{U} = 0, \\ \partial_t \bar{U} - 2r_0 f'(r_0^2)\partial_x \bar{A} = 0, \end{cases}$$

with associated speed of sound

$$\mathfrak{c}_s \equiv \sqrt{-2r_0^2 f'(r_0^2)} > 0,$$

provided f satisfies the defocusing assumption $f'(r_0^2) < 0$ (that is, the Euler system is hyperbolic in the region $\rho \simeq r_0^2$), which we will assume throughout the paper. Concerning the rigorous justification of the free wave regime for the Gross–Pitaevskii equation (in arbitrary dimension), see [Béthuel et al. 2010]. The speed of sound $\mathfrak{c}_s$ enters in a crucial way in the question of existence of traveling waves for (NLS) with modulus tending to r_0 at infinity (see, e.g., [Chiron 2012]).

The nonlinear Schrödinger equation formally preserves the energy

$$E(\psi) \equiv \int_{\mathbb{R}} |\partial_x \psi|^2 + F(|\psi|^2)\, dx,$$

where $F(\varrho) \equiv \int_{\varrho}^{r_0^2} f$. Since

$$F(\varrho) \sim \frac{\mathfrak{c}_s^2}{8r_0^2}(\varrho - r_0^2)^2 \sim \frac{\mathfrak{c}_s^2}{2}(\sqrt{\varrho} - r_0)^2$$

when $\varrho \to r_0^2$, it follows that the natural energy space turns out to be the space

$$\mathscr{X} \equiv \left\{\psi \in L^\infty(\mathbb{R}),\ \partial_x \psi \in L^2(\mathbb{R}),\ |\psi| - r_0 \in L^2(\mathbb{R})\right\} \subset \mathscr{C}_b(\mathbb{R}, \mathbb{C}),$$

endowed with the distance

$$d_{\mathscr{X}}(\psi, \tilde{\psi}) \equiv \|\partial_x \psi - \partial_x \tilde{\psi}\|_{L^2(\mathbb{R})} + \||\psi| - |\tilde{\psi}|\|_{L^2(\mathbb{R})} + |\psi(0) - \tilde{\psi}(0)|.$$

The Cauchy problem was shown to be locally well posed in the Zhidkov space $\{\psi \in L^\infty(\mathbb{R}),\ \partial_x\psi \in L^2(\mathbb{R})\}$ by P. Zhidkov [2001] (see also the work by C. Gallo [2004]). For global well-posedness results, see [Gallo 2008; Gérard 2008]. More precisely, the local well-posedness we shall use is the following.

Theorem 1 [Zhidkov 2001; Gallo 2004]. *Let $\Psi^{\text{in}} \in \mathscr{X}$. Then, there exists $T_* > 0$ and a unique solution Ψ to (NLS) such that $\Psi_{|t=0} = \Psi^{\text{in}}$ and $\Psi - \Psi^{\text{in}} \in \mathscr{C}([0,T_*), H^1(\mathbb{R}))$. Moreover, $E(\Psi(t))$ does not depend on t.*

The other quantity formally conserved by the Schrödinger flow, due to the invariance by translation, is the momentum. The momentum is not easy to define in dimension one for maps that vanish somewhere (see [Béthuel et al. 2008a; 2008b]). However, if ψ does not vanish, we have a lifting $\psi = Ae^{i\phi}$, and then the correct definition of the momentum is given by [Kivshar and Yang 1994]

$$P(\psi) \equiv \int_{\mathbb{R}} \langle i\psi|\partial_x\psi\rangle\left(1-\frac{r_0^2}{|\psi|^2}\right)dx = \int_{\mathbb{R}}(A^2-r_0^2)\partial_x\phi\,dx,$$

where $\langle\cdot|\cdot\rangle$ denotes the real scalar product in $\mathbb{C}$. We define

$$\mathscr{X}_{\text{hy}} \equiv \{v \in \mathscr{X},\ \inf_{\mathbb{R}}|v| > 0\},$$

which is the open subset of $\mathscr{X}$ in which we have lifting and where the hydrodynamical formulation (2) of (NLS) is possible through the Madelung transform. It turns out that, if the initial datum belongs to $\mathscr{X}_{\text{hy}}$, the solution of (NLS) provided by Theorem 1 remains in $\mathscr{X}_{\text{hy}}$ for small times, and that the momentum is indeed conserved on this time interval (see [Gallo 2004]).

1A. ***The traveling waves and energy-momentum diagrams.*** The traveling waves with speed of propagation c are special solutions of (NLS) of the form

$$\Psi(t,x) = U(x-ct).$$

The profile U has then to solve the ODE

$$\partial_x^2 U + Uf(|U|^2) = ic\partial_x U \tag{TW$_c$}$$

together with the condition $|U(x)| \to r_0$ as $x \to \pm\infty$. These particular solutions play an important role in the long-time dynamics of (NLS) with nonzero condition at infinity. Possibly conjugating (TW$_c$), we see that we may assume that $c \geq 0$ without loss of generality. Moreover, we shall restrict ourselves to traveling waves which belong to the energy space $\mathscr{X}$ (so that $|U| \to r_0$ at $\pm\infty$ by the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow \mathscr{C}_0(\mathbb{R},\mathbb{C}) \equiv \{h \in \mathscr{C}(\mathbb{R},\mathbb{C}),\ \lim_{\pm\infty} h = 0\}$). For traveling waves $U_c \in \mathscr{X}$ that do not vanish in $\mathbb{R}$, hence that may be lifted to $U_c = A_c e^{i\phi_c}$, the ODE (TW$_c$) can be transformed (see, e.g., [Chiron 2012]) into the system

$$\partial_x\phi_c = \frac{c}{2}\cdot\frac{\eta_c}{\eta_c+r_0^2}, \quad 2\partial_x^2\eta_c + \mathscr{V}_c'(\eta_c) = 0, \quad \text{with } \eta_c \equiv A_c^2 - r_0^2,$$

and where the function $\mathscr{V}_c$ is related to f by the formula

$$\mathscr{V}_c(\xi) \equiv c^2\xi^2 - 4(r_0^2+\xi)F(r_0^2+\xi).$$

To a nontrivial traveling wave U_c is associated (see [ibid.]) some $\xi_c \geq -r_0^2$ such that $\mathcal{V}_c(\xi_c) = 0 \neq \mathcal{V}'_c(\xi_c)$ and $\mathcal{V}_c$ is negative between ξ_c and $-r_0^2$, and η_c varies between 0 and ξ_c; that is, $\{\inf_{\mathbb{R}} |U_c|, \sup_{\mathbb{R}} |U_c|\} = \{r_0, \sqrt{r_0^2 + \xi_c}\}$. Moreover, the only traveling wave solution (if it exists) that vanishes somewhere is for $c = 0$ and is called the kink: it is an odd solution (up to a space translation) and then $\xi_0 = 0$.

We have also seen in [ibid.] that any traveling wave in $\mathcal{X}$ with speed $c > \mathfrak{c}_s$ is constant, and also that any nonconstant traveling wave in $\mathcal{X}$ of speed $c_* \in (0, \mathfrak{c}_s)$ belongs to a unique (up to the natural invariances: phase factor and translation) local branch $c \mapsto U_c$ defined for c close to c_*.

In [ibid.], we have investigated the qualitative behaviors of the traveling waves for (NLS) with nonzero condition at infinity for a general nonlinearity f. A particular attention has been payed in [ibid.] to the transonic limit, where we have an asymptotic behavior governed by the Korteweg–de Vries or the generalized Korteweg–de Vries equation. In order to illustrate the very different situations we may encounter when we allow a general nonlinearity f, we give now some energy-momentum diagrams we have obtained (one is taken from the appendix in [Chiron and Scheid 2012], where we have performed numerical simulations in dimension two for the model cases we have studied in [Chiron 2012]):

- The Gross–Pitaevskii nonlinearity: $f(\varrho) = 1 - \varrho$ (see Figure 1).
- A cubic-quintic-septic nonlinearity: $f(\varrho) = -(\varrho - 1) + \frac{3}{2}(\varrho - 1)^2 - \frac{3}{2}(\varrho - 1)^3$ (see Figure 2).
- A cubic-quintic-septic nonlinearity: $f(\varrho) \equiv -4(\varrho-1) - 36(\varrho-1)^3$ or $f(\varrho) \equiv -4(\varrho-1) - 60(\varrho-1)^3$. For these two nonlinearities, the graph of E and P vs. speed c is given in Figure 3, but the (E, P) diagrams are, respectively, those in Figure 4.
- A cubic-quintic-septic nonlinearity: $f(\varrho) \equiv -\frac{1}{2}(\varrho - 1) + \frac{3}{4}(\varrho - 1)^2 - 2(\varrho - 1)^3$ (see Figure 5).
- A degenerate case: $f(\varrho) \equiv -2(\varrho-1) + 3(\varrho-1)^2 - 4(\varrho-1)^3 + 5(\varrho-1)^4 - 6(\varrho-1)^5$ (see Figure 6).
- A perturbation of the previous degenerate case: $f(\varrho) \equiv -2(\varrho-1) + (3-10^{-3})(\varrho-1)^2 - 4(\varrho-1)^3 + 5(\varrho-1)^4 - 6(\varrho-1)^5$ (see Figure 7).
- A saturated NLS: $f(\varrho) \equiv \exp((1-\varrho)/\varrho_0) - 1$ with $\varrho_0 = 0.4$ (see Figure 8).
- Another saturated NLS: $f(\varrho) \equiv \frac{1}{2}\varrho_0\big(1/(1+\varrho/\varrho_0)^2 - 1/(1+1/\varrho_0)^2\big)$, with $\varrho_0 = 0.08$ (see Figure 9).
- The cubic-quintic nonlinearity: $f(\varrho) \equiv -(\varrho - 1) - 3(\varrho - 1)^2$ (see Figure 10).

Through the study (in [Chiron 2012]) of these model cases, we have shown that, if the energy-momentum diagram is well-known for the Gross–Pitaevskii equation, the qualitative properties of the traveling wave solutions can *not* be easily deduced from the global shape of the nonlinearity f. In particular, even if we restrict ourselves to smooth and decreasing nonlinearities (as is the Gross–Pitaevskii one), we see that we may have a great variety of behaviors: multiplicity of solutions, branches with diverging energy and momentum, nonexistence of traveling waves for some $c_0 \in (0, \mathfrak{c}_s)$, branches of solutions that cross, existence of sonic traveling waves, transonic limit governed by the mKdV or more generally by the gKdV solitary wave equation instead of the usual KdV one, existence of cusps, etc.

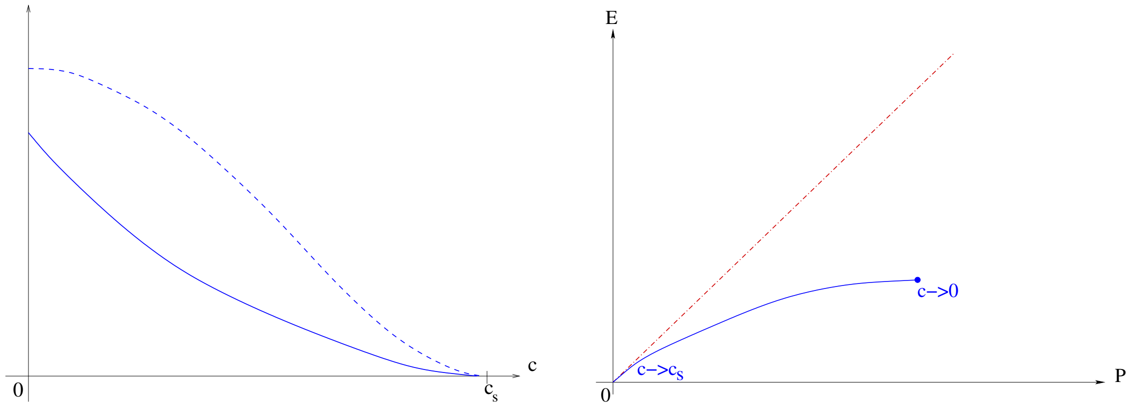


Figure 1. (a) Energy (dashed curve) and momentum (full curve) vs. speed; (b) (E, P) diagram.

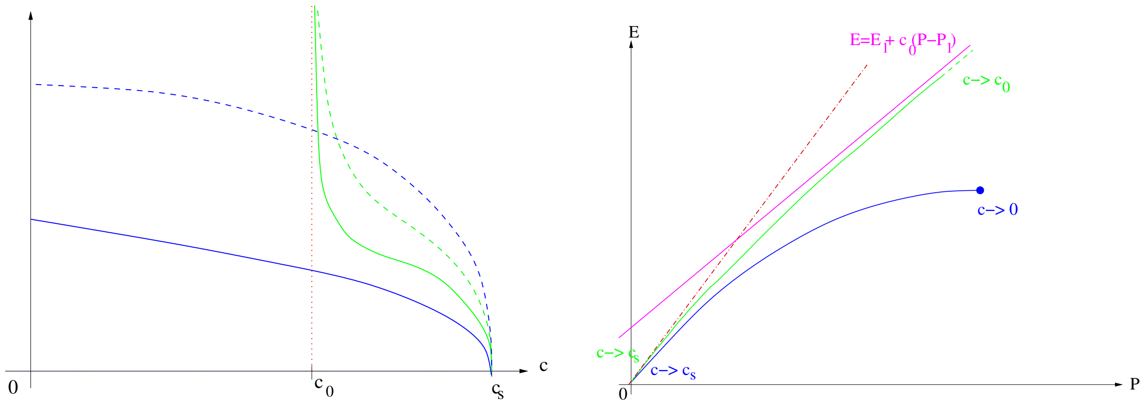


Figure 2. (a) Energy (dashed curve) and momentum (full curve) vs. speed; (b) (E, P) diagram.

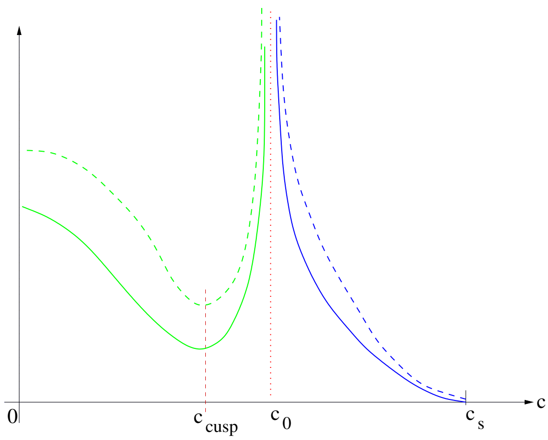


Figure 3. Energy (dashed curve) and momentum (full curve) vs. speed.

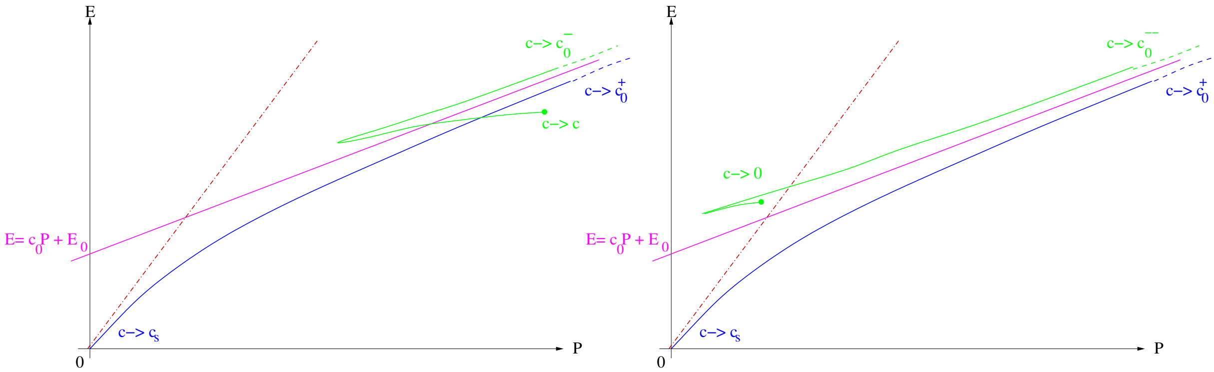


Figure 4. The two (E, P) diagrams.

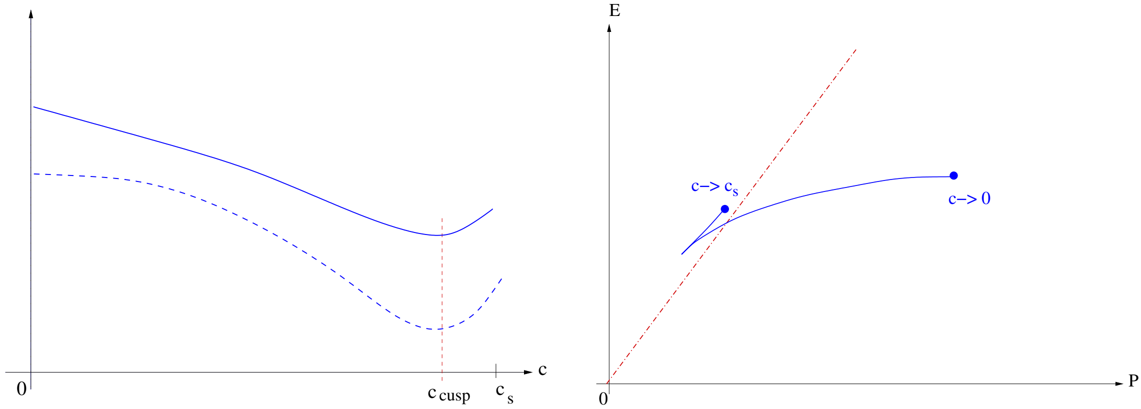


Figure 5. (a) Energy (dashed curve) and momentum (full curve) vs. speed; (b) (E, P) diagram.

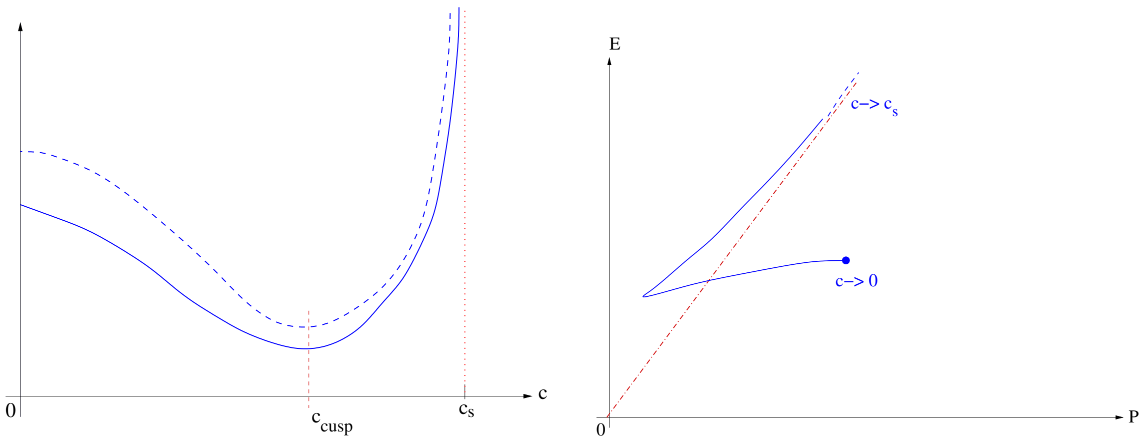


Figure 6. (a) Energy (dashed curve) and momentum (full curve) vs. speed; (b) (E, P) diagram.

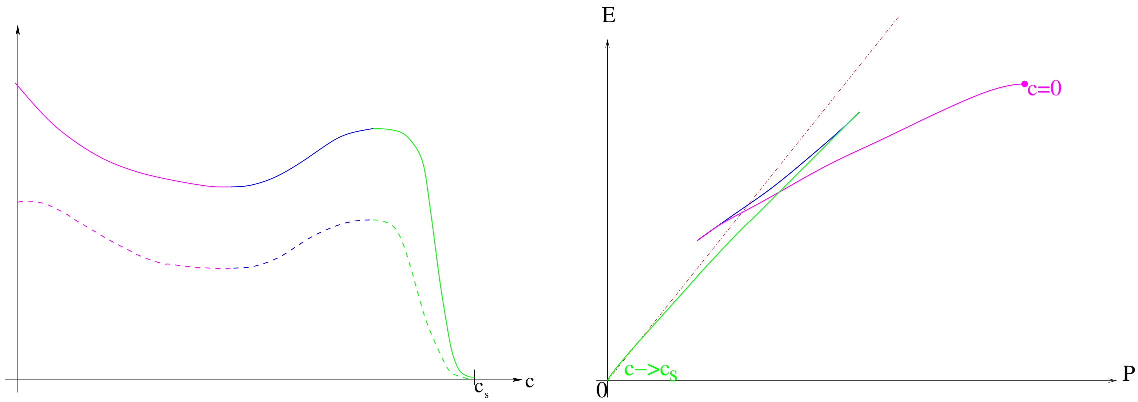


Figure 7. (a) Energy (dashed curve) and momentum (full curve) vs. speed; (b) qualitative (E, P) diagram.

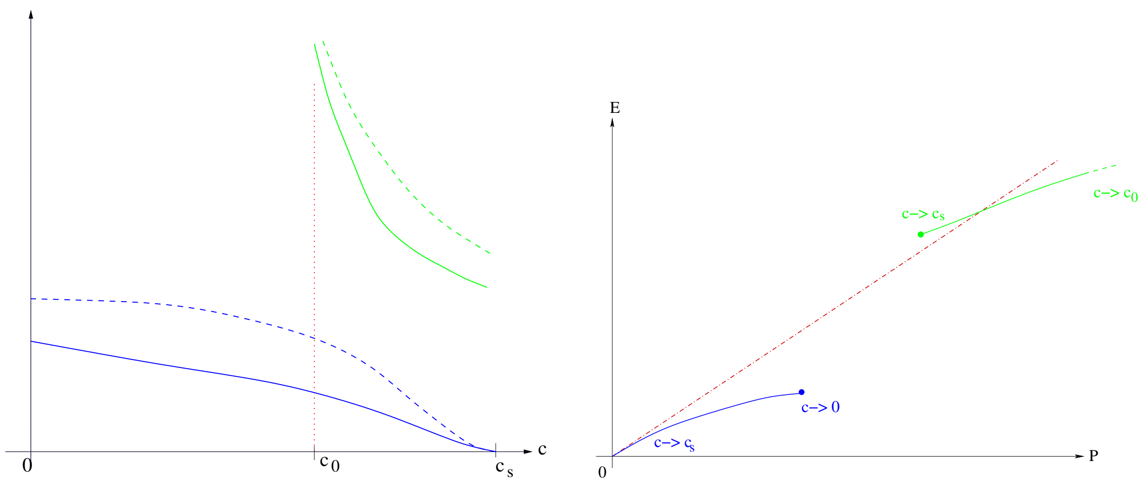


Figure 8. (a) Energy (dashed curve) and momentum (full curve) vs. speed; (b) (E, P) diagram.

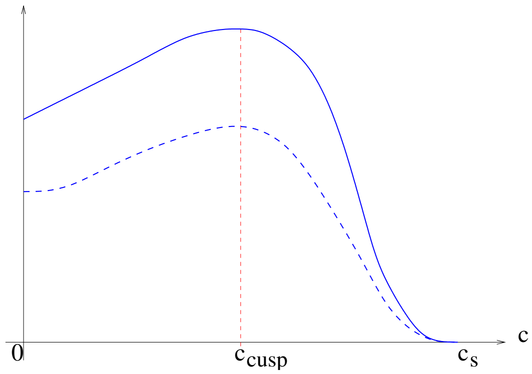
We investigate now the behavior at infinity of the nontrivial traveling waves, which depends on whether $c = \mathfrak{c}_s$ or not. We denote by $\mathbb{N}$ the set of nonnegative integers and $\mathbb{N}^*$ the set of positive integers. We consider for $m \in \mathbb{N}$ the following assumption:

$(\mathcal{A}_m)$ f is of class $\mathscr{C}^{m+3}$ near r_0^2. Moreover, for $1 \le j < m+2$, we have

$$\frac{f^{(j)}(r_0^2)}{(j+1)!} r_0^{2j} = (-1)^{j+1} \frac{\mathfrak{c}_s^2}{4} \quad \text{but} \quad \frac{f^{(m+2)}(r_0^2)}{(m+3)!} r_0^{2(m+2)} \neq (-1)^{m+3} \frac{\mathfrak{c}_s^2}{4}$$

(note that, for $j = 1$, equality always holds by definition of the speed of sound $\mathfrak{c}_s = \sqrt{-2r_0^2 f'(r_0^2)}$).

Proposition 2. *Let $U_c \in \mathscr{Z}$ be a nonconstant traveling wave of speed $0 \le c \le \mathfrak{c}_s$.*



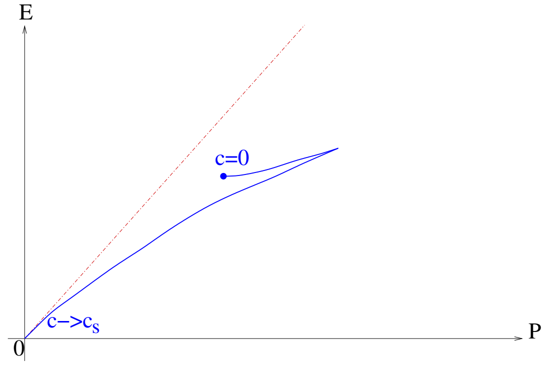
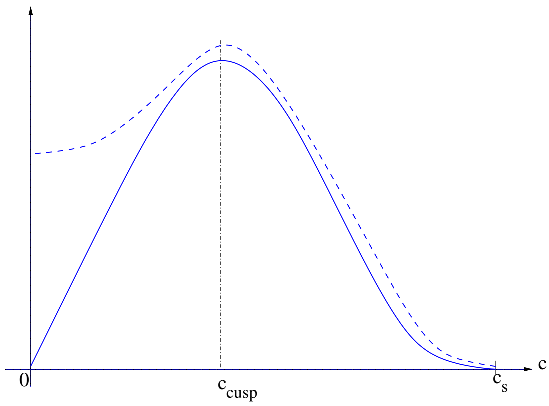


Figure 9. (a) Energy (dashed curve) and momentum (full curve), (b) (E, P) diagram.



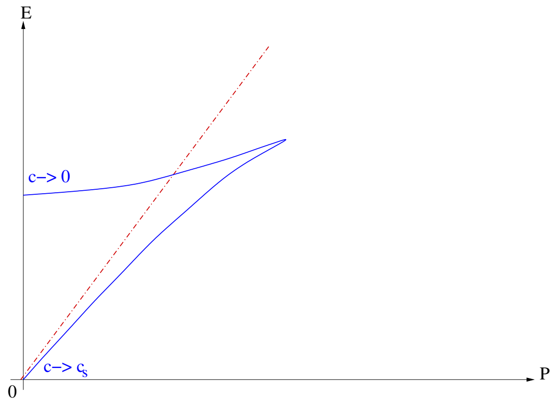


Figure 10. (a) Energy (dashed curve) and momentum (full curve), (b) (E, P) diagram.

(i) *If* $c = 0$, *then there exists* $\phi_0 \in \mathbb{R}$ *such that* $\mathrm{e}^{i\phi_0} U_0$ *is a real-valued function and there exist two real constants* $M_0 \neq 0$ (*depending only on* f *and* ξ_0) *and* x_0 *such that, as* $x \to \pm\infty$,

$$\mathrm{e}^{i\phi_0} U_0(x) \mp r_0 \sim M_0 \exp(-\mathfrak{c}_s |x - x_0|) \text{ if } \xi_0 = -r_0^2, \quad \mathrm{e}^{i\phi_0} U_0(x) - r_0 \sim M_0 \exp(-\mathfrak{c}_s |x - x_0|) \text{ if } \xi_0 \neq -r_0^2.$$

(ii) *If* $0 < c < \mathfrak{c}_s$, *then* U_c *does not vanish, and hence can be lifted*: $U_c = A_c \mathrm{e}^{i\phi_c}$. *Furthermore, there exist four real constants* M_c, Θ_c (*depending only on* f, c *and* ξ_c), x_0 *and* ϕ_0 *such that, as* $x \to \pm\infty$,

$$|U_c(x)|^2 - r_0^2 = \eta_c(x) \sim \frac{2r_0^2}{c} \partial_x \phi(x) \sim M_c \exp\Big(-\sqrt{\mathfrak{c}_s^2 - c^2}|x - x_0|\Big),$$

and

$$\phi(x) - \phi_0 \mp \Theta_c \sim -\mathrm{sgn}(x) \frac{c M_c}{2r_0^2 \sqrt{\mathfrak{c}_s^2 - c^2}} \exp\Big(-\sqrt{\mathfrak{c}_s^2 - c^2}|x - x_0|\Big).$$

(iii) *If* $c = \mathfrak{c}_s$ *then* $U_{\mathfrak{c}_s}$ *does not vanish, and hence can be lifted*: $U_{\mathfrak{c}_s} = A_{\mathfrak{c}_s} e^{i\phi_{\mathfrak{c}_s}}$. *We assume that there exists* $m \in \mathbb{N}$ *such that* $(\mathcal{A}_m)$ *is satisfied and define*

$$\Lambda_m \equiv \frac{4}{r_0^{2(m+1)}} \left[\frac{r_0^{2(m+2)}}{(m+3)!} f^{(m+2)}(r_0^2) + (-1)^{m+2} \frac{\mathfrak{c}_s^2}{4} \right] \neq 0.$$

Then, we have, as $x \to \pm\infty$,

$$|U_{\mathfrak{c}_s}(x)|^2 - r_0^2 = \eta_{\mathfrak{c}_s}(x) \sim \frac{2r_0^2}{\mathfrak{c}_s} \partial_x \phi(x) \sim \operatorname{sgn}(\xi_{\mathfrak{c}_s}) \left(\frac{4}{(m+1)^2 |\Lambda_m| x^2} \right)^{\frac{1}{m+1}}$$

and

$$\phi(x) \sim \frac{\mathfrak{c}_s \operatorname{sgn}(\xi_{\mathfrak{c}_s})}{2r_0^2} \left(\frac{4}{(m+1)^2 |\Lambda_m|} \right)^{\frac{1}{m+1}} \begin{cases} \operatorname{sgn}(x) \ln|x| & \text{if } m = 1, \\ \dfrac{m+1}{m-1} \operatorname{sgn}(x) |x|^{\frac{m-1}{m+1}} & \text{if } m \geq 2, \end{cases}$$

and, if $m = 0$, *there exist* $\Theta_{\mathfrak{c}_s} \in \mathbb{R}$ *and* $\phi_0 \in \mathbb{R}$ *such that*

$$\phi(x) - \phi_0 \mp \Theta_{\mathfrak{c}_s} \sim \operatorname{sgn}(\xi_{\mathfrak{c}_s}) \frac{2\mathfrak{c}_s}{r_0^2 |\Lambda_0| x}.$$

In particular, since we impose $U_{\mathfrak{c}_s} \in \mathscr{Z}$, *we must have* $m \in \{0, 1, 2\}$.

For the Gross–Pitaevskii nonlinearity ($f(\varrho) = 1 - \varrho$), we may compute explicitly the traveling waves for $0 < c < \mathfrak{c}_s = \sqrt{2}$ (see [Tsuzuki 1971; Béthuel et al. 2008a]):

$$U_c(x) = \sqrt{\frac{2 - c^2}{2}} \tanh x \frac{\sqrt{2 - c^2}}{2} - i \frac{c}{\sqrt{2}},$$

up to the invariances of the problem: translations and multiplications by a phase factor. On this explicit formula, the decay of the phase and modulus can be checked. In particular, as $x \to \pm\infty$, we have

$$U_c(x) \to \pm \sqrt{1 - \frac{c^2}{\mathfrak{c}_s}} - i \frac{c}{\mathfrak{c}_s}.$$

Remark 3. In the above statements, the constants ϕ_0 and x_0 reflect the gauge and translation invariance. In the spirit of the model cases proposed in [Chiron 2012], for

$$f(\varrho) \equiv -2(\varrho - 1) + 3(\varrho - 1)^2 - 4(\varrho - 1)^3 + 5(\varrho - 1)^4 - 12(\varrho - 1)^5,$$

we obtain a smooth decreasing nonlinearity tending to $-\infty$ at $+\infty$ (thus qualitatively similar to the Gross–Pitaevskii nonlinearity) for which we have $r_0 = 1$, $\mathfrak{c}_s = 2$, and $\mathscr{V}_{\mathfrak{c}_s}(\xi) = -4\xi^4 - 8\xi^5$. For this nonlinearity f, there exists a nontrivial sonic traveling wave of infinite energy (corresponding to $\xi_{\mathfrak{c}_s} = -1/2$), since $m = 3$.

The aim of this paper is to investigate the stability of the traveling waves for the one-dimensional NLS. We recall the definition of orbital stability in a metric space $(\mathscr{X}, d_{\mathscr{X}})$ for which we have a local in time existence result.

Definition 4. Let $0 \le c \le \mathfrak{c}_s$ and $U_c \in \mathcal{Z}$ be a nontrivial traveling wave of speed c. We say that U_c is orbitally stable in $(\mathcal{X}, d_{\mathcal{X}})$, where $\mathcal{X} \subset \mathcal{Z}$, if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for any initial datum $\Psi^{\rm in} \in \mathcal{X}$ such that $d_{\mathcal{X}}(\Psi^{\rm in}, U_c) \le \delta$, any solution Ψ to (NLS) with initial datum $\Psi^{\rm in}$ is global in $\mathcal{X}$ and

$$\sup_{t\ge 0} \inf_{\substack{y\in\mathbb{R}\\ \theta\in\mathbb{R}}} d_{\mathcal{X}}(\Psi(t), \mathrm{e}^{i\theta} U_c(\cdot - y)) \le \epsilon.$$

In the sequel, U_c will always stand for a nontrivial traveling wave, and we freeze the translation invariance by imposing that $|U_c|$ is even. Moreover, the solutions of (NLS) we consider will always be those given by Theorem 1.

1B. *Stability and instability in the case* $0 < c < \mathfrak{c}_s$.

1B1. *Stability for the hydrodynamical and the energy distances.* The first stability result for the traveling waves for (NLS) with nonzero condition at infinity is due to Z. Lin [2002]. The analysis relies on the hydrodynamical form of (NLS), which is valid for solutions that never vanish. The advantage is to work with a fixed functional space since $(\eta, u) = (A^2 - r_0^2, \partial_x \phi) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, whereas the traveling waves have a limit $r_0 \mathrm{e}^{\pm i\Theta_c}$ (up to a phase factor) at $\pm\infty$ depending on the speed c. Lin's result establishes rigorously the stability criterion found in [Bogdan et al. 1989; Barashenkov 1996].

Theorem 5 [Lin 2002]. *Assume that $0 < c_* < \mathfrak{c}_s$ is such that there exists a nontrivial traveling wave U_{c_*}. Then, there exists some small $\sigma > 0$ such that U_{c_*} belongs to a locally unique continuous branch of nontrivial traveling waves U_c defined for $c_* - \sigma \le c \le c_* + \sigma$.*

(i) *Assume*

$$\frac{dP(U_c)}{dc}_{|c=c_*} < 0.$$

Then, $U_{c_*} = A_* \mathrm{e}^{i\phi_*}$ is orbitally stable in the sense that, for any $\epsilon > 0$, there exists $\delta > 0$ such that, if $\Psi^{\rm in} = A^{\rm in} \mathrm{e}^{i\phi^{\rm in}} \in \mathcal{Z}$ satisfies

$$\|A^{\rm in} - A_*\|_{H^1(\mathbb{R})} + \|\partial_x \phi^{\rm in} - \partial_x \phi_*\|_{L^2(\mathbb{R})} \le \delta,$$

then the solution Ψ to (NLS) such that $\Psi_{|t=0} = \Psi^{\rm in}$ never vanishes, can be lifted to $\Psi = A\mathrm{e}^{i\phi}$, and we have

$$\sup_{t\ge 0} \inf_{y\in\mathbb{R}} \bigl\{ \|A(t) - A_*(\cdot - y)\|_{H^1(\mathbb{R})} + \|\partial_x \phi(t) - \partial_x \phi_*(\cdot - y)\|_{L^2(\mathbb{R})} \bigr\} \le \epsilon.$$

(ii) *Assume*

$$\frac{dP(U_c)}{dc}_{|c=c_*} > 0.$$

Then, $U_{c_*} = A_* \mathrm{e}^{i\phi_*}$ is orbitally unstable in the sense that there exists $\epsilon > 0$ such that, for any $\delta > 0$, there exists $\Psi^{\rm in} = A^{\rm in} \mathrm{e}^{i\phi^{\rm in}} \in \mathcal{Z}$ verifying

$$\|A^{\rm in} - A_*\|_{H^1(\mathbb{R})} + \|\partial_x \phi^{\rm in} - \partial_x \phi_*\|_{L^2(\mathbb{R})} \le \delta,$$

but such that, if Ψ denotes the solution to (NLS) with $\Psi_{|t=0} = \Psi^{\text{in}}$, then there exists $t > 0$ such that Ψ does not vanish on the time interval $[0, t]$ but

$$\inf_{y\in\mathbb{R}} \big\{ \|A(t) - A_*(\cdot - y)\|_{H^1(\mathbb{R})} + \|\partial_x\phi(t) - \partial_x\phi_*(\cdot - y)\|_{L^2(\mathbb{R})} \big\} \geq \epsilon.$$

By the one-dimensional Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow \mathscr{C}_0(\mathbb{R})$, it is clear that, since U_{c_*} does not vanish in $\mathbb{R}$, by imposing $\| |\Psi^{\text{in}}| - |U_{c_*}| \|_{H^1(\mathbb{R})} = \|A^{\text{in}} - A_*\|_{H^1(\mathbb{R})}$ small, Ψ^{in} does not vanish in $\mathbb{R}$ and thus can be lifted.

Remark 6. We point out that [Gallo 2004] fills two gaps in the proof from [Lin 2002]: the first one concerns the local in time existence for the hydrodynamical system (see (15) in Section 3C) and the second one is about the conservation of the energy and the momentum. Furthermore, we make two additional remarks on the proof from [Lin 2002] in Section 3C.

Theorem 5 is stability or instability in the open set $\mathscr{X}_{\text{hy}} \subset \mathscr{X}$ for the hydrodynamical distance

$$d_{\text{hy}}(\psi, \tilde{\psi}) \equiv \|A - \tilde{A}\|_{H^1(\mathbb{R})} + \|\partial_x\phi - \partial_x\tilde{\phi}\|_{L^2(\mathbb{R})} + \left|\arg\left(\frac{\psi(0)}{\tilde{\psi}(0)}\right)\right|, \quad \psi = A e^{i\phi}, \quad \tilde{\psi} = \tilde{A} e^{i\tilde{\phi}},$$

which is not the energy distance. Here, $\arg : \mathbb{C}^* \to (-\pi, +\pi]$ is the principal argument. For the stability, it suffices to consider the phase $\theta \in \mathbb{R}$ such that $\arg\big(\Psi(t)/(e^{i\theta} U_{c_*}(\cdot - y))\big)$ is zero at $x = 0$, where y is the translation parameter. For the instability, the phase $\theta \in \mathbb{R}$ does not matter. The result of [Lin 2002] is based on the application of the Grillakis–Shatah–Strauss theory [Grillakis et al. 1987] (see also [Bona et al. 1987; Souganidis and Strauss 1990]) to the hydrodynamical formulation of (NLS) (see Section 3C). One difficulty is to overcome the fact that the Hamiltonian operator ∂_x is not onto.

On the energy-momentum diagrams, the stability can be checked either on the graphs of E and P with respect to c, or on the concavity of the curve $P \mapsto E$. Indeed, we have seen in [Chiron 2012] that the so-called Hamilton group relation

$$c = \frac{dE}{dP}, \quad \text{or} \quad \frac{dE(U_c)}{dc} = c\frac{dP(U_c)}{dc},$$

holds, where the derivative is computed on the local branch. Therefore,

$$\frac{d^2E}{dP^2} = \frac{d}{dP}\frac{dE}{dP} = \frac{dc}{dP}.$$

This means that we have stability when $P \mapsto E$ is concave, that is, $d^2E/dP^2 < 0$, and instability if $P \mapsto E$ is convex, i.e., $d^2E/dP^2 > 0$.

Actually, the proof of [Grillakis et al. 1987; Lin 2002] provides an explicit control, as shown in the following lemma.

Lemma 7. *Under the assumptions of Theorem 5 and in the case* (i) *of stability, we have, provided $d_{\text{hy}}(\Psi^{\text{in}}, U_{c_*})$ is small enough,*

$$\sup_{t\geq 0}\inf_{y\in\mathbb{R}} \big\{ \|A(t) - A_*(\cdot - y)\|_{H^1(\mathbb{R})} + \|\partial_x\phi(t) - \partial_x\phi_*(\cdot - y)\|_{L^2(\mathbb{R})} \big\}$$
$$\leq K\sqrt{|E(\Psi^{\text{in}}) - E(U_{c_*})| + |P(\Psi^{\text{in}}) - P(U_{c_*})|}, \quad (3)$$

as well as the control

$$\sup_{t\geq 0}\inf_{\substack{y\in\mathbb{R}\\ \theta\in\mathbb{R}}} d_{\mathrm{hy}}(\Psi(t),\mathrm{e}^{i\theta}U_{c_*}(\cdot-y))\leq K d_{\mathrm{hy}}(\Psi^{\mathrm{in}},U_{c_*}). \tag{4}$$

Remark 8. The second estimate (4) is not a simple consequence of the control (3), but relies on a comparison to U_c for some c close to c_* instead of a comparison to U_{c_*} (this idea has also been used in [Weinstein 1986]). It follows that, in the definition of stability for U_{c_*}, one can take $\delta=\mathbb{O}(\varepsilon)$.

Let us stress that Z. Lin's result (Theorem 5) is given in terms of the hydrodynamical distance d_{hy}, which is not the energy distance $d_{\mathfrak{E}}$. As a matter of fact, the Madelung transform

$$\mathcal{M}:(\mathfrak{E}_{\mathrm{hy}},d_{\mathfrak{E}})\ni U\mapsto\left(\eta,u,\frac{U(0)}{|U(0)|}\right)\in H^1(\mathbb{R},\mathbb{R})\times L^2(\mathbb{R},\mathbb{R})\times\mathbb{S}^1,$$

where $U=A\mathrm{e}^{i\phi}$, $\eta=A^2-r_0^2$ and $u=\partial_x\phi$, is not so well behaved.

Lemma 9. (i) *The mapping $\mathcal{M}:(\mathfrak{E}_{\mathrm{hy}},d_{\mathfrak{E}})\to H^1(\mathbb{R},\mathbb{R})\times L^2(\mathbb{R},\mathbb{R})\times\mathbb{S}^1$ is an homeomorphism.*

(ii) *There exists $\phi_*\in\mathscr{C}^2(\mathbb{R},\mathbb{R})$ such that $\partial_x\phi_*\in L^2(\mathbb{R})$ and a sequence $(\phi_n)_{n\geq 1}$ of functions in $H^1(\mathbb{R},\mathbb{R})$ such that, when $n\to+\infty$,*

$$0<d_{\mathrm{hy}}(\mathrm{e}^{i\phi_*},\mathrm{e}^{i\phi_*}\mathrm{e}^{i\phi_n})\to 0\quad\textit{but}\quad\frac{d_{\mathfrak{E}}(\mathrm{e}^{i\phi_*},\mathrm{e}^{i\phi_*}\mathrm{e}^{i\phi_n})}{d_{\mathrm{hy}}(\mathrm{e}^{i\phi_*},\mathrm{e}^{i\phi_*}\mathrm{e}^{i\phi_n})}\to+\infty.$$

Therefore, $\mathcal{M}^{-1}$ is not locally Lipschitz continuous in general. However, for the stability issues, we compare the $d_{\mathfrak{E}}$ and the d_{hy} distances to some fixed traveling wave U_*, which enjoys some nice decay properties at infinity. Let us now stress the link between the two distances d_{hy} and $d_{\mathfrak{E}}$ in this case.

Lemma 10. *Let $0\leq c_*\leq\mathfrak{c}_s$ and assume that $U_*\in\mathfrak{E}$ is a nonconstant traveling wave with speed c_* that does not vanish. If $c_*=\mathfrak{c}_s$, we further assume that assumption $(\mathcal{A}_0)$ is satisfied. Then, there exists some constants K and $\delta>0$, depending only on U_*, such that, for any $\psi\in\mathfrak{E}$ verifying $d_{\mathfrak{E}}(\psi,U_*)\leq\delta$, we have*

$$\frac{1}{K}d_{\mathrm{hy}}(\psi,U_*)\leq d_{\mathfrak{E}}(\psi,U_*)\leq K d_{\mathrm{hy}}(\psi,U_*).$$

An immediate corollary of Lemma 10 is that Theorem 5 is also a stability/instability result in the energy distance. If one wishes for only a stability/instability result, it is sufficient to invoke the fact that the mapping $\mathcal{M}$ is an homeomorphism. However, the use of Lemma 10 provides a stronger explicit control similar to the one obtained in Lemma 7 (see (3)). In particular, in the definition of stability for U_{c_*} in $(\mathfrak{E},d_{\mathfrak{E}})$, one can take $\delta=\mathbb{O}(\varepsilon)$.

Corollary 11. *Assume that $0<c_*<\mathfrak{c}_s$ is such that there exists a nontrivial traveling wave U_{c_*}. Then, there exists some small $\sigma>0$ such that U_{c_*} belongs to a locally unique continuous branch of nontrivial traveling waves U_c defined for $c_*-\sigma\leq c\leq c_*+\sigma$.*

(i) *If $(dP(U_c)/dc)_{|c=c_*}<0$, then $U_{c_*}=A_*\mathrm{e}^{i\phi_*}$ is orbitally stable in $(\mathfrak{E},d_{\mathfrak{E}})$. Furthermore, if $\Psi(t)$ is the (global) solution to (NLS) with initial datum Ψ^{in}, then we have, for some constant K depending* *only on* U_{c_*} *and provided* $d_{\mathfrak{X}}(\Psi^{\mathrm{in}}, U_{c_*})$ *is sufficiently small,*

$$\sup_{t\geq 0} \inf_{\substack{y\in\mathbb{R}\\ \theta\in\mathbb{R}}} d_{\mathfrak{X}}(\Psi(t), \mathrm{e}^{i\theta} U_{c_*}(\cdot - y)) \leq K\sqrt{|E(\Psi^{\mathrm{in}}) - E(U_{c_*})| + |P(\Psi^{\mathrm{in}}) - P(U_{c_*})|},$$

as well as the control

$$\sup_{t\geq 0} \inf_{\substack{y\in\mathbb{R}\\ \theta\in\mathbb{R}}} d_{\mathfrak{X}}(\Psi(t), \mathrm{e}^{i\theta} U_{c_*}(\cdot - y)) \leq K d_{\mathfrak{X}}(\Psi^{\mathrm{in}}, U_{c_*}).$$

(ii) *If* $(dP(U_c)/dc)_{|c=c_*} > 0$, *then* $U_{c_*} = A_* \mathrm{e}^{i\phi_*}$ *is orbitally unstable in* $(\mathfrak{X}, d_{\mathfrak{X}})$.

For the Gross–Pitaevskii nonlinearity ($f(\varrho) = 1-\varrho$), the stability (for the energy distance $d_{\mathfrak{X}}$) of the traveling waves with speed $0 < c < \mathfrak{c}_s$ was proved by F. Béthuel, P. Gravejat and J.-C. Saut [Béthuel et al. 2008a] through the variational characterization that these solutions are minimizers of the energy under the constraint of fixed momentum. However, in view of the energy momentum diagrams in Section 1A, this constraint minimization approach can not be used in the general setting we consider here. Indeed, this method provides only stability, but there may exist unstable traveling waves. Moreover, it follows from the proof of Theorem 5 that stable waves are local minimizers of the energy at fixed momentum but not necessarily global minimizers. Finally, we emphasize that the spectral methods allow us to derive an explicit (Lipschitz) control in case of stability.

1B2. *Stability via a Liapounov functional.* Another way to prove the orbital stability is to find a Liapounov functional. By Liapounov functional, we mean a functional which is conserved by the (NLS) flow and for which the traveling wave U_c is a local minimum (for instance, a critical point with second derivative $\geq \delta\,\mathrm{Id}$ for some $\delta > 0$). Such a Liapounov functional always exists in the Grillakis–Shatah–Strauss theory when $(dP(U_c)/dc)_{|c=c_*} < 0$, as shown by Theorem 26 in Appendix A. Its direct application to our problem leads us to define the functional in $\mathfrak{X}_{\mathrm{hy}}$

$$\mathscr{L}(\psi) \equiv E(\psi) - c_* P(\psi) + \frac{M}{2}(P(\psi) - P(U_{c_*}))^2,$$

where M is some positive parameter. It turns out that $\mathscr{L}$ is such a Liapounov functional when M is sufficiently large. Since the proof relies on the Grillakis–Shatah–Strauss framework, we have to work in the hydrodynamical variables. However, by Lemma 10, we recover the case of the energy distance.

Theorem 12. *Assume that, for some* $c_* \in (0, \mathfrak{c}_s)$ *and* $\sigma > 0$ *small,* $(0, \mathfrak{c}_s) \supset [c_* - \sigma, c_* + \sigma] \ni c \mapsto U_c \in \mathfrak{X}$ *is a continuous branch of nontrivial traveling waves with* $(dP(U_c)/dc)_{|c=c_*} < 0$. *If*

$$M > \frac{1}{-\frac{dP(U_c)}{dc}_{|c=c_*}} > 0,$$

there exist $\epsilon > 0$ *and* K, *depending only on* U_{c_*}, *such that, for any* $\psi \in \mathfrak{X}$ *with*

$$\inf_{y\in\mathbb{R},\,\theta\in\mathbb{R}} d_{\mathrm{hy}}(\psi, \mathrm{e}^{i\theta} U_{c_*}(\cdot - y)) \leq \epsilon,$$

we have

$$\inf_{\substack{y\in\mathbb{R}\\ \theta\in\mathbb{R}}} d^2_{\mathrm{hy}}(\psi, \mathrm{e}^{i\theta} U_{c_*}(\cdot - y)) \le K(\mathscr{L}(\psi) - \mathscr{L}(U_{c_*})),$$

and analogously with d_{hy} *replaced by* $d_{\mathfrak{X}}$. *Consequently,* $U_{c_*} = A_* \mathrm{e}^{i\phi_*}$ *is orbitally stable in* $(\mathfrak{X}_{\mathrm{hy}}, d_{\mathrm{hy}})$ *and in* $(\mathfrak{X}, d_{\mathfrak{X}})$. *Furthermore, if* $\Psi(t)$ *is the (global) solution to* (NLS) *with initial datum* Ψ^{in}, *then we have*

$$\sup_{t\ge 0} \inf_{\substack{y\in\mathbb{R}\\ \theta\in\mathbb{R}}} d_{\mathrm{hy}}(\Psi(t), \mathrm{e}^{i\theta} U_{c_*}(\cdot - y)) \le K\sqrt{\mathscr{L}(\Psi^{\mathrm{in}}) - \mathscr{L}(U_{c_*})} \le K d_{\mathrm{hy}}(\Psi^{\mathrm{in}}, U_{c_*}),$$

provided $d_{\mathrm{hy}}(\Psi^{\mathrm{in}}, U_{c_*})$ *is sufficiently small, and analogously with* d_{hy} *replaced by* $d_{\mathfrak{X}}$.

For the traveling waves for (NLS) in dimension one, this type of Liapounov functional appears for the first time in the paper by I. Barashenkov [1996]. However, there, the problem is treated directly on the wave function Ψ, whereas the correct proof holds on the hydrodynamical variables, in particular because of the gauge invariance $(\theta, \Psi) \mapsto \mathrm{e}^{i\theta}\Psi$. For instance, that work suggests that we have stability for H^1 perturbations, whereas it holds only for perturbations in the energy space. Finally, we fill some gaps in the proof of [Barashenkov 1996].

1B3. *Instability via the existence of an unstable eigenvalue.* In the Grillakis–Shatah–Strauss theory [Grillakis et al. 1987], the instability is not shown by proving the existence of a unstable eigenvalue for the linearized (NLS) and then a nonlinear instability result (see however [Grillakis et al. 1990] when the Hamiltonian skew-adjoint operator is onto). There exist, however, some general results that prove the existence of unstable eigenvalues. For the instability of bound states for (NLS) (and also for the nonlinear Klein–Gordon equation), that is, solutions of the form $\mathrm{e}^{i\omega t} U_\omega(x)$, M. Grillakis [1988] shows that the condition $d/d\omega\big(\int_{\mathbb{R}^d} |U_\omega|^2\big)_{|\omega=\omega_*} > 0$ is sufficient for the existence of such an unstable eigenvalue. However, the proof relies on the fact that the bound states are real-valued functions (up to a phase factor) and it is not clear whether it extends to the case of traveling waves we are studying. Indeed, since we have to work in hydrodynamical variables in order to have a fixed functional space, the linearized operator does not have (for $c \neq 0$) the structure required for the application of [Grillakis 1988]. Another general result is due to O. Lopes [2002] but it assumes that the linearized equation can be solved using a semigroup. This is not the case for our problem once it is written in hydrodynamical variables (see below). Finally, Z. Lin [2008] proposes an alternative approach for the existence of unstable eigenvalues. The method has the advantage of allowing pseudodifferential equations (like the Benjamin–Ono equation). However, the results are given for three model equations involving a scalar unknown, and it is not clear whether the proof can be extended to the case of systems.

The linearization of (NLS) near the traveling wave U_{c_*} in the frame moving with speed c_* is

$$i\frac{\partial\psi}{\partial t} - ic_*\partial_x\psi + \partial_x^2\psi + \psi f(|U_{c_*}|^2) + 2\langle\psi|U_{c_*}\rangle f'(|U_{c_*}|^2)U_{c_*} = 0, \tag{5}$$

and, thus, searching for exponentially growing modes $\psi(t,x) = \mathrm{e}^{\lambda t} w(x)$ leads to the eigenvalue problem

$$i\lambda w - ic_*\partial_x w + \partial_x^2 w + w f(|U_{c_*}|^2) + 2\langle w|U_{c_*}\rangle f'(|U_{c_*}|^2)U_{c_*} = 0, \tag{6}$$

with $\mathrm{Re}(\lambda) > 0$ and $w \neq 0$. For one-dimensional problems, the linear instability is commonly shown through the use of Evans functions (see the classical paper [Pego and Weinstein 1992] and also the review article [Sandstede 2002]). For our problem, we look for an unstable eigenvalue for the equation written in hydrodynamical variables; namely we look for exponentially growing solutions $(\underline{\eta}, \underline{u})$ of the linear problem (written in the moving frame)

$$\begin{cases} \partial_t \underline{\eta} - c_* \partial_x \underline{\eta} + 2\partial_x((r_0^2 + \eta_*)\underline{u} + \underline{\eta} u_*) = 0, \\ \partial_t \underline{u} - c_* \partial_x \underline{u} + 2\partial_x(u_* \underline{u}) - \partial_x(f'(r_0^2 + \eta_*)\underline{\eta}) \\ \qquad - \partial_x \left\{ \dfrac{1}{2\sqrt{r_0^2 + \eta_*}} \partial_x^2 \left(\dfrac{\underline{\eta}}{\sqrt{r_0^2 + \eta_*}} \right) - \dfrac{\underline{\eta} \partial_x^2 \left(\sqrt{r_0^2 + \eta_*} \right)}{2(r_0^2 + \eta_*)^{3/2}} \right\} = 0, \end{cases} \tag{7}$$

where (η_*, u_*) is the reference solution. The advantage is here again to work with a fixed functional space in variables (η, u). Due to the term

$$\partial_x \left\{ \frac{1}{2\sqrt{r_0^2 + \eta_*}} \partial_x^2 \left(\frac{\underline{\eta}}{\sqrt{r_0^2 + \eta_*}} \right) \right\},$$

this equation can not be solved using a semigroup, except in the trivial case where η_* is constant; hence the result of [Lopes 2002] does not apply. However, system (7) is a particular case of the Euler–Korteweg system for capillary fluids (see [Benzoni-Gavage 2010a] for a survey on this model). We may then use a linear instability result already shown for the Euler–Korteweg system with the Evans function method, as in work by K. Zumbrun [2008] for a simplified system, and more recently by S. Benzoni-Gavage [2010b] for the complete Euler–Korteweg system.

Theorem 13. *Assume that, for some $c_* \in (0, \mathfrak{c}_s)$ and $\sigma > 0$ small,*

$$(0, \mathfrak{c}_s) \supset [c_* - \sigma, c_* + \sigma] \ni c \mapsto U_c \in \mathscr{X}$$

is a continuous branch of nontrivial traveling waves with

$$\frac{dP(U_c)}{dc}_{|c=c_*} > 0.$$

Then, there exists exactly one unstable eigenvalue $\gamma_0 \in \{\mathrm{Re} > 0\}$ for (6) and $\gamma_0 \in (0, +\infty)$; that is, (NLS) is (spectrally) linearly unstable.

Once we have shown the existence of an unstable eigenvalue for the linearized NLS equation (5), we can prove a nonlinear instability result as in [Henry et al. 1982; de Bouard 1995]. Note that, here, we no longer work in the hydrodynamical variables, where the high-order derivatives involve nonlinear terms, but on the semilinear NLS equation.

Corollary 14. *Under the assumptions of Theorem 13, U_{c_*} is unstable in $U_{c_*} + H^1(\mathbb{R}, \mathbb{C})$ (endowed with the natural H^1 distance): there exists ϵ such that, for any $\delta > 0$, there exists $\Psi^{\mathrm{in}} \in U_{c_*} + H^1(\mathbb{R})$ such that $\|\Psi^{\mathrm{in}} - U_{c_*}\|_{H^1(\mathbb{R})} \leq \delta$, but, if $\Psi \in U_{c_*} + \mathscr{C}([0, T^*), H^1(\mathbb{R}))$ denotes the maximal solution of (NLS), then there exists $0 < t < T^*$ such that $\|\Psi(t) - U_{c_*}\|_{H^1(\mathbb{R})} \geq \epsilon$.*

Since the proof is very similar to the one in [Henry et al. 1982; de Bouard 1995], we omit it. We may actually prove a stronger instability result, since the above one is not proved by tracking the exponentially growing mode. In [Di Menza and Gallo 2007], a spectral mapping theorem is shown and used to show the nonlinear instability by tracking this exponentially growing mode, which is a natural mechanism of instability. In Appendix B, we show that this spectral mapping theorem holds for a wide class of Hamiltonian equations. The direct application of Corollary B.6 in Appendix B gives the following nonlinear instability result.

Corollary 15. *We make the assumptions of Theorem 13, so that there exists an unstable eigenmode* $(\gamma_0, w) \in (0, +\infty) \times H^1(\mathbb{R})$, $\|w\|_{H^1} = 1$. *There exists* $M > 0$ *such that, for any solution* $\psi \in \mathscr{C}(\mathbb{R}_+, H^1(\mathbb{R}, \mathbb{C}))$ *of the linearized equation (5), we have the growth estimate of the semigroup*

$$\|\psi(t)\|_{H^1(\mathbb{R})} \leq M \mathrm{e}^{\gamma_0 t} \|\psi(0)\|_{H^1(\mathbb{R})} \quad \text{for all } t \geq 0.$$

Moreover, U_{c_*} *has also the following instability property: there exist* $K > 0$, $\delta > 0$ *and* $\varepsilon_0 > 0$, *such that, for any* $0 < \delta < \delta_0$, *the solution* $\Psi(t)$ *to (NLS) with initial datum* $\Psi^{\mathrm{in}} = U_{c_*} + \delta w \in U_{c_*} + H^1(\mathbb{R})$ *exists at least on* $[0, \gamma_0^{-1} \ln(2\varepsilon_0/\delta)]$ *and satisfies*

$$\|\Psi(t) - U_{c_*} - \delta \mathrm{e}^{\gamma_0 t} w\|_{H^1(\mathbb{R})} \leq K \delta^2 \mathrm{e}^{2\gamma_0 t}.$$

In particular, for $t = \gamma_0^{-1} \ln(2\varepsilon_0/\delta)$ *and* $\epsilon \equiv \varepsilon_0/K$, *we have*

$$\inf_{y \in \mathbb{R}} \| |\Psi(t)| - |U_{c_*}|(\cdot - y)\|_{L^2(\mathbb{R})} \geq \epsilon \quad \text{and} \quad \inf_{y \in \mathbb{R}} \| |\Psi(t)| - |U_{c_*}|(\cdot - y)\|_{L^\infty(\mathbb{R})} \geq \epsilon,$$

which implies

$$\inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} \|\Psi(t) - \mathrm{e}^{i\theta} U_{c_*}(\cdot - y)\|_{H^1(\mathbb{R})} \geq \epsilon$$

as well as

$$\inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} d_{\mathrm{hy}}(\Psi(t), \mathrm{e}^{i\theta} U_{c_*}(\cdot - y)) \geq \epsilon \quad \text{and} \quad \inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} d_{\mathfrak{X}}(\Psi(t), \mathrm{e}^{i\theta} U_{c_*}(\cdot - y)) \geq \epsilon.$$

With the above result, we then show the nonlinear instability also in the energy space, and thus recover the instability result of Z. Lin but this time by tracking the unstable growing mode.

1B4. *Instability at a cusp.* In this section, we investigate the question of stability in the degenerate case $dP/dc = 0$. In [Grillakis et al. 1987] (see also [Grillakis et al. 1990]), a stability result for the wave of speed c_* is shown when the action $c \mapsto S(c) = E(U_c) - cP(U_c)$ (on the local branch) is such that, for instance, $d^2S/dc^2 = -dP/dc$ is positive for $c \neq c_*$ but vanishes for $c = c_*$. In the energy-momentum diagrams of Section 1A, the situation is different since dP/dc changes sign *at* the cusps, or, equivalently, the action $c \mapsto S(c) = E(U_c) - cP(U_c)$ (on the local branch) changes its concavity *at* the cusp. A. Comech and D. Pelinovsky [2003] show that, for the nonlinear Schrödinger equation, a bound state associated with a cusp in the energy-charge diagram is unstable. The proof relies on a careful analysis of the linearized equation, which is spectrally stable, but linearly unstable (with polynomial growth for the linear problem). A similar technique was used by A. Comech, S. Cuccagna and D. Pelinovsky [2007] for the generalized

Korteweg–de Vries equation. Then, M. Ohta [2011] also proved the nonlinear instability of these "bound states" using a Liapounov functional as in [Grillakis et al. 1987]. However, in [Ohta 2011], it is assumed that $J = T'(0)$ and that J is onto, which are both not true here (and there are further restrictions due to the introduction of an intermediate Hilbert space). M. Maeda [2012] has extended the above instability result, removing some assumptions in [Ohta 2011]. We show the instability of traveling waves associated with a cusp in the energy-momentum diagram in the generic case where $d^2P/dc^2 \neq 0$. Our approach follows the lines of [Maeda 2012], but with some modifications since our problem does not fit exactly the general framework of this paper. In particular, we can not find naturally a space "Y", and some functions appearing in the proof do not lie in the range of the skew-adjoint operator ∂_x involved in the Hamiltonian formalism. We overcome this difficulty using an approximation argument (similar to the one used in [Lin 2002]).

Theorem 16. *Assume that, for some $c_* \in (0, \mathfrak{c}_s)$ and $\sigma > 0$ small, $(0, \mathfrak{c}_s) \supset [c_* - \sigma, c_* + \sigma] \ni c \mapsto U_c \in \mathfrak{X}$ is a continuous branch of nontrivial traveling waves with*

$$\frac{dP(U_c)}{dc}_{|c=c_*} = 0 \neq \frac{d^2P(U_c)}{dc^2}_{|c=c_*},$$

and assume in addition that f is of class $\mathscr{C}^2$. Then, U_{c_*} is orbitally unstable in $(\mathfrak{X}, d_{\mathfrak{X}})$.

1C. *Stability in the case $c = 0$.*

1C1. *Instability for the bubbles.* When $c = 0$, we have two types of stationary waves: the bubbles, when $\xi_0 > -r_0^2$, are even functions (up to a translation) that do not vanish, and the kinks, when $\xi_0 = -r_0^2$, are odd functions (up to a translation). The instability of stationary bubbles has been shown by A. de Bouard [1995] (and is true even in higher dimension). The proof there relies on the proof of the existence of an unstable eigenvalue for the linearized NLS, and then the proof of a nonlinear instability result. An alternative proof of the linear instability of the bubbles is given in [Pelinovsky and Kevrekidis 2008, Theorem 3.11(ii)].

Theorem 17 [de Bouard 1995]. *Assume that there exists a bubble, that is, a nontrivial stationary ($c = 0$) wave U_0 which does not vanish. Then, U_0 is (linearly and nonlinearly) unstable in $U_0 + H^1(\mathbb{R})$ (endowed with the natural H^1 metric); that is, there exists ϵ such that, for any $\delta > 0$, there exists $\Psi^{\text{in}} \in U_0 + H^1(\mathbb{R})$ such that $\|\Psi^{\text{in}} - U_0\|_{H^1(\mathbb{R})} \leq \delta$, but, if $\Psi \in U_0 + \mathscr{C}([0, T^*), H^1(\mathbb{R}))$ denotes the maximal solution of (NLS), then there exists $0 < t < T^*$ such that $\|\Psi(t) - U_0\|_{H^1(\mathbb{R})} \geq \epsilon$.*

Actually, in the same way that Corollary 15 is a better instability result than Corollary 14, we have the following stronger instability result, which is a direct consequence of Corollary B.6 in Appendix B.

Proposition 18. *Assume that there exists a bubble, that is, a nontrivial stationary ($c = 0$) wave U_0 which does not vanish. Then, U_0 is (nonlinearly) unstable in $U_0 + H^1(\mathbb{R})$, $(\mathfrak{X}, d_{\mathfrak{X}})$ and $(\mathfrak{X}_{\text{hy}}, d_{\text{hy}})$ in the same sense as in Corollary 15.*

Finally, we would like to emphasize that we may recover the instability result for bubbles from the proof of Theorem 5, relying on the hydrodynamical form of (NLS), which holds true here since bubbles do not vanish. Our result holds in the energy space and for the hydrodynamical distance.

Theorem 19. *Assume that there exists a bubble, that is, a nontrivial stationary* ($c = 0$) *wave* U_0 *which does not vanish. Then, there exists some small* $\sigma > 0$ *such that* U_0 *belongs to a locally unique continuous branch of nontrivial traveling waves* U_c *defined for* $0 \le c \le \sigma$. *Then,* $c \mapsto P(U_c)$ *has a derivative at* $c = 0$,

$$\frac{dP(U_c)}{dc}_{|c=0} > 0$$

and $U_0 = A_* e^{i\phi_*}$ *is orbitally unstable for the distances* $d_{\mathscr{X}}$ *and* d_{hy}.

Proof. We give a proof based on the argument of [Lin 2002], which is possible since U_0 is a bubble, hence does not vanish, and the spectral decomposition used there still holds when $c = 0$. Moreover, it is clear that the mapping $c \mapsto (\eta_c, u_c) \in H^1 \times L^2$ is smooth up to $c = 0$, using the uniform exponential decay at infinity near $c = 0$ and arguing as in [Chiron 2012]. Therefore, it suffices to show that $(dP(U_c)/dc)_{|c=0} > 0$. From the expression of the momentum given in [ibid., Subsection 1.2], we have, for $0 \le c \le \sigma$,

$$P(U_c) = c\,\mathrm{sgn}(\xi_c) \int_0^{\xi_c} \frac{\xi^2}{r_0^2 + \xi} \frac{d\xi}{\sqrt{-\mathscr{V}_c(\xi)}} = c \left| \int_0^{\xi_0} \frac{\xi^2}{r_0^2 + \xi} \frac{d\xi}{\sqrt{-\mathscr{V}_0(\xi)}} \right| + o(c)$$

since $\xi_0 > -r_0^2$. Indeed, we are allowed to pass to the limit in the integral once it is written with the change of variables $\xi = t\xi_c$:

$$\int_0^{\xi_c} \frac{\xi^2}{r_0^2 + \xi} \frac{d\xi}{-\mathscr{V}_c(\xi)} = \int_0^1 \frac{\xi_c^3 t^2}{r_0^2 + t\xi_c} \frac{dt}{\sqrt{-\mathscr{V}_c(t\xi_c)}},$$

since $\xi_0 > -r_0^2$. Therefore,

$$\frac{dP(U_c)}{dc}_{|c=0} = \left| \int_0^{\xi_0} \frac{\xi^2}{r_0^2 + \xi} \frac{d\xi}{\sqrt{-\mathscr{V}_0(\xi)}} \right| > 0$$

since $\xi_0 \neq 0$ (U_0 is not trivial). The conclusion follows then from the proof of Theorem 5. □

When we know that $(dP(U_c)/dc)_{|c=0} > 0$, we may also use the Evans function as in Theorem 13 to show the existence of an unstable eigenmode. However, due to the fact that the kink U_0 is real-valued, we can use the arguments in [de Bouard 1995; Pelinovsky and Kevrekidis 2008].

1C2. *Stability analysis for the kinks.* We now turn to the case of the kinks ($\xi_0 = -r_0^2$ and U_0 is odd up to a translation). Since U_0 vanishes at the origin, the hydrodynamical form of (NLS) can not be used. The stability of the kink has attracted several recent works. L. Di Menza and C. Gallo [2007] have investigated the linear stability through the Vakhitov–Kolokolov function VK, defined by

$$\mathrm{VK}(\lambda) \equiv \int_{\mathbb{R}} \big([-\partial_x^2 - f(U_0^2) - \lambda]^{-1}(\partial_x U_0)\big)(\partial_x U_0)\, dx,$$

where U_0 is the kink, for $\lambda \in (\lambda_*, 0)$ for some $\lambda_* < 0$. They show that the Vakhitov–Kolokolov function VK has a limit VK_0 when $\lambda \to 0^-$. If $\mathrm{VK}_0 > 0$, then the linearization of (NLS) around the kink has an unstable real positive eigenvalue. When $\mathrm{VK}_0 < 0$, the linearization of (NLS) around the kink has a spectrum included in $i\mathbb{R}$ (spectral stability). Note that the approach of [Lin 2002] (extending [Grillakis et al. 1987]) does not give directly unstable eigenvalues in the case $dP/dc > 0$. Recently, the link between the quantity dP/dc and the sign of VK_0 has been given by D. Pelinovsky and P. Kevrekidis [2008] (proof of Lemma 3.10 there, the factor $\sqrt{2}$ coming from the coefficients of the NLS equation in [Pelinovsky and Kevrekidis 2008]):

$$2\sqrt{2}\,\mathrm{VK}_0 = \lim_{c\to 0} \frac{dP(U_c)}{dc}, \tag{8}$$

and they also prove, in a different way from [Di Menza and Gallo 2007], that we have spectral stability when $\lim_{c\to 0} dP/dc < 0$ and existence of an unstable eigenvalue (in $\mathbb{R}_+^*$) if $\lim_{c\to 0} dP/dc > 0$. It is shown in [Pelinovsky and Kevrekidis 2008] that the limit $\lim_{c\to 0} dP(U_c)/dc$ does exist. Actually, it is proved there that the function $[0, c_0) \ni c \mapsto P(U_c)$ is of class $\mathscr{C}^1$ and that the derivative at $c = 0$ is also given by (see (8))

$$\begin{aligned}\lim_{c\to 0} \frac{dP(U_c)}{dc} &= 2\sqrt{2}\,\mathrm{VK}_0 = 2\sqrt{2} \lim_{\lambda\to 0^-} \int_{\mathbb{R}} \big([-\partial_x^2 - f(U_0^2) - \lambda]^{-1}(\partial_x U_0)\big)(\partial_x U_0)\,dx \\ &= 2\sqrt{2} \int_{\mathbb{R}} \mathrm{Im}\Big(\frac{\partial U_c}{\partial c}_{|c=0}\Big)\partial_x U_0\,dx. \end{aligned}\tag{9}$$

Our next lemma gives an explicit formula of the expression (9), involving only the nonlinearity f.

Lemma 20. *Assume that U_0 is a kink. Then, there exists $c_0 \in (0, \mathfrak{c}_s)$ such that U_0 belongs to the (locally) unique branch $[0, c_0) \ni c \mapsto U_c \in \mathscr{X}$. Moreover, $P(U_c) \to r_0^2\pi$ as $c \to 0$ and the continuous extension $[0, c_0) \ni c \mapsto P(U_c)$ has a derivative at $c = 0$ given by*

$$\frac{dP(U_c)}{dc}_{|c=0} = -\frac{8r_0^3}{3\sqrt{F(0)}} + \frac{1}{2}\int_0^{r_0^2} \frac{(\varrho - r_0^2)^2}{\varrho^{3/2}}\Big(\frac{1}{\sqrt{F(\varrho)}} - \frac{1}{\sqrt{F(0)}}\Big)\,d\varrho.$$

The advantage of the formula given in Lemma 20 compared to (9) is that it allows a direct computation of $(dP(U_c)/dc)_{|c=0}$ when f is known, which does not require computing numerically U_0 and $(\partial U_c/\partial c)_{|c=0}$. For instance, it is quite well adapted to the stability analysis as in [Fakau and Karval'u 2009]. Let us observe that it may happen that a kink is unstable (see [Kivshar and Krolikowski 1995; Di Menza and Gallo 2007]).

In the case of linear instability, [Di Menza and Gallo 2007] shows that, then, nonlinear instability holds. Actually, a stronger result is proved there, showing that the L^∞ norm (and not only the H^1 norm) does not remain small.

Theorem 21 [Di Menza and Gallo 2007]. *Assume that there exists a kink, that is, a nontrivial stationary ($c = 0$) wave U_0 vanishing somewhere, and satisfying $(dP(U_c)/dc)_{|c=0} > 0$. Then, U_0 is (linearly and nonlinearly) unstable in the sense that there exists ϵ such that, for any $\delta > 0$, there exists $\Psi^{\mathrm{in}} \in U_0 + H^1(\mathbb{R})$ such that $\|\Psi^{\mathrm{in}} - U_0\|_{H^1(\mathbb{R})} \le \delta$, but, if $\Psi \in U_0 + \mathscr{C}([0, T^*), H^1(\mathbb{R}))$ denotes the maximal solution of (NLS), then there exists $0 < t < T^*$ such that $\|\Psi(t) - U_0\|_{L^\infty(\mathbb{R})} \ge \epsilon$.*

The proof in [Di Menza and Gallo 2007] relies on the tracking of the exponentially growing eigenmode. One may actually improve slightly the result as this was done in Corollary 15. As a matter of fact, this was the result in Theorem 21 that has motivated us for Corollary 15.

We focus now on the nonlinear stability issue when there is linear (spectral) stability, that is, when $(dP(U_c)/dc)_{|c=0} < 0$. Concerning the Gross–Pitaevskii nonlinearity ($f(\varrho) = 1 - \varrho$), for which we have $(dP(U_c)/dc)_{|c=0} < 0$, we quote two papers on this question. The first one is the work of P. Gérard and Z. Zhang [2009] where the stability is shown by inverse scattering, hence in a space of functions sufficiently decaying at infinity. The analysis then relies on the integrability of the one-dimensional GP equation. The other work is by F. Béthuel, P. Gravejat, J.-C. Saut and D. Smets [Béthuel et al. 2008b]. They prove the orbital stability of the kink of the Gross–Pitaevskii equation by showing that the kink is a global minimizer of the energy under the constraint that a variant of the momentum is fixed (recall that the definition of the momentum has to be clarified for an arbitrary function in the energy space), and that the corresponding minimizing sequences are compact (up to space translations and phase factors). In this approach, it is crucial (see [Béthuel et al. 2008a; 2008b]) that $E_{\text{kink}} < \mathfrak{c}_s P_{\text{kink}} = \mathfrak{c}_s r_0^2 \pi$ in order to prevent the dichotomy case for the minimizing sequences. However, since the energy of the kink is equal to

$$E_{\text{kink}} = 4 \int_{-r_0^2}^{0} \frac{F(r_0^2 + \xi)}{\sqrt{-\mathscr{V}_0(\xi)}} \, d\xi = 2 \int_{-r_0^2}^{0} \sqrt{\frac{F(r_0^2 + \xi)}{r_0^2 + \xi}} \, d\xi = 2 \int_0^{r_0^2} \sqrt{\frac{F(\varrho)}{\varrho}} \, d\varrho,$$

whereas its momentum is always equal to $r_0^2 \pi$, it is clear that the condition $E_{\text{kink}} < \mathfrak{c}_s P_{\text{kink}} = \mathfrak{c}_s r_0^2 \pi$ does not hold in general, as shown in the following example.

Example. For $\kappa \geq 0$, consider

$$f(\varrho) \equiv 1 - \varrho + \kappa (1 - \varrho)^3,$$

which is smooth and decreases to $-\infty$ as the Gross–Pitaevskii nonlinearity. We have $r_0 = 1$, $\mathfrak{c}_s = \sqrt{2}$, $F(\varrho) = (1 - \varrho)^2/2 + \kappa (1 - \varrho)^4/4$ and

$$E_{\text{kink}} = 2 \int_0^{r_0^2} \sqrt{\frac{F(\varrho)}{\varrho}} \, d\varrho = 2 \int_0^{r_0^2} \sqrt{\frac{2(1-\varrho)^2 + \kappa (1-\varrho)^4}{4\varrho}} \, d\varrho > \mathfrak{c}_s r_0^2 \pi = \pi \sqrt{2}$$

for κ large (the left-hand side tends to $+\infty$), and numerical computations show that it is the case for $\kappa \geq 14$. Furthermore, Lemma 20 gives

$$\sqrt{F(0)} \frac{dP(U_c)}{dc}_{|c=0} = -\frac{8}{3} + \frac{1}{2} \int_0^1 \frac{(\varrho - 1)^2}{\varrho^{3/2}} \left(\sqrt{\frac{F(0)}{F(\varrho)}} - 1 \right) d\varrho. \tag{10}$$

Since $F(0)/F(\varrho) = (2 + \kappa)/(2(\varrho - 1)^2 + \kappa (\varrho - 1)^4)$, it can be easily checked that the right-hand side of (10) is a decreasing function of κ tending to

$$-\frac{8}{3} + \frac{1}{2} \int_0^1 \frac{(\varrho - 1)^2}{\varrho^{3/2}} \left(\frac{1}{(\varrho - 1)^2} - 1 \right) d\varrho = -1$$

when $\kappa \to +\infty$ (by monotone convergence). In particular, for any $\kappa \geq 0$, we have $(dP(U_c)/dc)_{|c=0} < 0$; that is, the kink is always (linearly) stable. The energy-momentum diagram for this type of nonlinearity with κ large is as in the right part of Figure 4 (the left part correspond to κ smaller).

In comparison with the constraint minimization approach as in [Béthuel et al. 2008a; 2008b], which allows us to establish a global minimization result, the spectral methods as in [Grillakis et al. 1987; Lin 2002] allow us to put forward locally minimizing properties, which turn out to be useful for the stability analysis in dimension one.

In the stability analysis of the kink, one issue is the definition of the momentum P, which was up to now given only for maps in $\mathscr{X}_{\text{hy}}$, that is, for maps that never vanish, but the kink vanishes at the origin. In [Béthuel et al. 2008b], the notion of momentum was extended to the whole energy space $\mathscr{X}$, hence including maps vanishing somewhere, as a quantity defined mod 2π, and was called "untwisted momentum". This notion will be useful for our stability result.

Lemma 22 [Béthuel et al. 2008b]. *If $\psi \in \mathscr{X}$, the limit*

$$\mathfrak{P}(\psi) \equiv \lim_{R\to+\infty}\left[\int_{-R}^{+R} \langle i\psi|\partial_x\psi\rangle\,dx - r_0^2\big(\arg(\psi(+R)) - \arg(\psi(-R))\big)\right]$$

exists in $\mathbb{R}/(2\pi r_0^2\mathbb{Z})$. The mapping $\mathfrak{P}:\mathscr{X}\to\mathbb{R}/(2\pi r_0^2\mathbb{Z})$ is continuous and, if $\psi\in\mathscr{X}$ satisfies $\inf_{\mathbb{R}}|\psi|>0$ (i.e., $\psi\in\mathscr{X}_{\text{hy}}$), then $\mathfrak{P}(\psi) = P(\psi) \mod 2\pi r_0^2$. Finally, if $\Psi\in\mathscr{C}([0,T),\mathscr{X})$ is a solution to (NLS), then $\mathfrak{P}(\Psi(t))$ does not depend on t.

Proof. For the sake of completeness, we recall the proof of [Béthuel et al. 2008b]. Let $\psi\in\mathscr{X}$ and let us verify the Cauchy criterion. Since $|\psi|\to r_0>0$ at $\pm\infty$, we have a lifting $\psi = A_\pm e^{i\phi_\pm}$ in $(-\infty,-R_0)$ and in $(+R_0,+\infty)$ for some R_0 sufficiently large. For $R'>R>R_0$, we thus have in $\mathbb{R}/(2\pi r_0^2\mathbb{Z})$

$$\begin{aligned}
&\left[\int_{-R'}^{+R'} \langle i\psi|\partial_x\psi\rangle\,dx - r_0^2\big(\arg(\psi(+R')) - \arg(\psi(-R'))\big)\right]\\
&\quad-\left[\int_{-R}^{+R} \langle i\psi|\partial_x\psi\rangle\,dx - r_0^2\big(\arg(\psi(+R)) - \arg(\psi(-R))\big)\right]\\
&\qquad= \int_R^{R'} \langle i\psi|\partial_x\psi\rangle\,dx + \int_{-R'}^{-R} \langle i\psi|\partial_x\psi\rangle\,dx\\
&\qquad\qquad - r_0^2\big(\arg(\psi(R')) - \arg(\psi(R))\big) + r_0^2\big(\arg(\psi(-R')) - \arg(\psi(-R))\big)\\
&\qquad= \int_R^{R'} A_+^2\partial_x\phi_+\,dx + \int_{-R'}^{-R} A_-^2\partial_x\phi_-\,dx - r_0^2\big(\phi_+(R')-\phi_+(R)\big) + r_0^2\big(\phi_-(-R')-\phi_-(-R)\big)\\
&\qquad= \int_R^{R'} (A_+^2 - r_0^2)\partial_x\phi_+\,dx + \int_{-R'}^{-R} (A_-^2-r_0^2)\partial_x\phi_-\,dx.
\end{aligned}$$

The absolute value of each term is $\le K\int_{\pm x\ge\pm R}|\partial_x\psi|^2 + (|\psi|-r_0)^2\,dx$ and thus tends to zero if $R\to+\infty$. Thus, $\mathfrak{P}(\psi)$ is well-defined. The proof of the continuity follows the same lines, and allows us to show that $\mathfrak{P}$ is actually locally Lipschitz continuous. Let $\psi\in\mathscr{X}$. If $\tilde\psi\in\mathscr{X}$ and $d_{\mathscr{X}}(\tilde\psi,\psi)$ is small enough, we have $\||\tilde\psi|-|\psi|\|_{L^\infty}$ as small as we want. In particular, if $R_0>0$ is large enough so that $|\psi|\ge 3r_0/4$ for $|x|\ge R$, we have $|\tilde\psi|\ge r_0/2$ for $|x|\ge R_0$. As a consequence, writing $\psi = A_\pm e^{i\phi_\pm}$ and

$\tilde{\psi} = \tilde{A}_\pm e^{i\tilde{\phi}_\pm}$ in $(-\infty, -R_0)$ and in $(+R_0, +\infty)$, we have, in $\mathbb{R}/(2\pi r_0^2\mathbb{Z})$ and for $R > R_0$,

$$\begin{aligned}
&\Big[\int_{-R}^{+R} \langle i\psi|\partial_x\psi\rangle\,dx - r_0^2\big(\arg(\psi(+R)) - \arg(\psi(-R))\big)\Big] \\
&\quad - \Big[\int_{-R}^{+R} \langle i\tilde{\psi}|\partial_x\tilde{\psi}\rangle\,dx - r_0^2\big(\arg(\tilde{\psi}(+R)) - \arg(\tilde{\psi}(-R))\big)\Big] \\
&= \int_{-R_0}^{+R_0} \langle i(\psi-\tilde{\psi})|\partial_x\psi\rangle + \langle i\tilde{\psi}|\partial_x(\psi-\tilde{\psi})\rangle\,dx + \int_{R_0}^{R} A_+^2\partial_x\phi_+ - \tilde{A}_+^2\partial_x\tilde{\phi}_+\,dx - r_0^2\big(\phi_+(+R) - \tilde{\phi}_+(+R)\big) \\
&\quad + \int_{-R}^{-R_0} A_-^2\partial_x\phi_- - \tilde{A}_-^2\partial_x\tilde{\phi}_-\,dx + r_0^2\big(\phi_-(-R) - \tilde{\phi}_-(-R)\big) \\
&= \int_{-R_0}^{+R_0} \langle i(\psi-\tilde{\psi})|\partial_x\psi\rangle + \langle i\tilde{\psi}|\partial_x(\psi-\tilde{\psi})\rangle\,dx + r_0^2\big(\phi_+(+R_0) - \tilde{\phi}_+(+R_0)\big) + r_0^2\big(\phi_-(-R_0) - \tilde{\phi}_-(-R_0)\big) \\
&\quad + \int_{R_0}^{R} (A_+^2 - r_0^2)\partial_x\phi_+ - (\tilde{A}_+^2 - r_0^2)\partial_x\tilde{\phi}_+\,dx + \int_{-R}^{-R_0} (A_-^2 - r_0^2)\partial_x\phi_- - (\tilde{A}_-^2 - r_0^2)\partial_x\tilde{\phi}_-\,dx.
\end{aligned}$$

We now estimate all the terms. For the last line, we use the Cauchy–Schwarz inequality to get $|\int_{R_0}^{R}(A_+^2 - r_0^2)\partial_x\phi_+\,dx| \le K(\psi)\|A_+ - r_0\|_{L^2(\mathbb{R})}\|A_+\partial_x\phi_+\|_{L^2(\mathbb{R})} \le K(\psi)d_{\mathfrak{X}}(\psi,\tilde{\psi})$, and similarly for the other terms. Moreover, using that $(\psi-\tilde{\psi})(x) = (\psi-\tilde{\psi})(0) + \int_0^x \partial_x(\psi-\tilde{\psi})$, we get, by the Cauchy–Schwarz inequality, $\|\psi-\tilde{\psi}\|_{\mathscr{C}^0([-R_0,+R_0])} \le |(\psi-\tilde{\psi})(0)| + \sqrt{R_0}\|\partial_x\psi - \partial_x\tilde{\psi}\|_{L^2(\mathbb{R})} \le K(R_0)d_{\mathfrak{X}}(\psi,\tilde{\psi})$. Thus, the terms of the second line can be estimated by $K(\psi,R_0)d_{\mathfrak{X}}(\psi,\tilde{\psi})$, and those of the first line can also be bounded by $K(\psi,R_0)d_{\mathfrak{X}}(\psi,\tilde{\psi})$. Passing to the limit as $R \to +\infty$ then gives

$$|\mathfrak{P}(\psi) - \mathfrak{P}(\tilde{\psi}) \bmod 2\pi r_0^2| \le K(\psi,R_0)d_{\mathfrak{X}}(\psi,\tilde{\psi}).$$

This completes the proof for the definition of $\mathfrak{P}$. To show that $\mathfrak{P}$ is constant under the (NLS) flow, we use that $\Psi \in \Psi(0) + \mathscr{C}([0,T),H^1)$ and the approximation by smoother solutions (see Proposition 1 in [Béthuel et al. 2008b]). □

For the stability of the kink, we can no longer use the Grillakis–Shatah–Strauss theory applied to the hydrodynamical formulation of (NLS), since the kink vanishes at the origin. Therefore, it is natural to consider the Liapounov functional $\mathscr{L}$ introduced in Section 1B2, which becomes, in the stationary case $c = 0$,

$$\mathscr{L}(\psi) = E(\psi) + \frac{M}{2}(P(\psi) - P(U_0))^2.$$

Since the momentum P is not well-defined in $\mathfrak{X}$, we have to replace it by the untwisted momentum $\mathfrak{P}$, which is defined modulo $2\pi r_0^2$. Consequently, it is natural to define the functional in $\mathfrak{X}$

$$\mathscr{K}(\psi) \equiv E(\psi) + 2Mr_0^4\sin^2\frac{\mathfrak{P}(\psi) - r_0^2\pi}{2r_0^2},$$

which is well-defined and continuous in $\mathfrak{X}$ since $\sin^2$ is π-periodic. In addition, $\mathscr{K}$ is conserved by the (NLS) flow as E and $\mathfrak{P}$.

Theorem 23. *Assume that there exists a kink, that is, a nontrivial stationary* ($c = 0$) *wave* U_0 *which is odd. Assume also that*

$$\frac{dP(U_c)}{dc}_{|c=0} < 0.$$

Then, there exists some small $\mu_* > 0$ *such that* U_0 *is a* local *minimizer of* $\mathscr{K}$. *More precisely, defining*

$$\mathscr{V}_{\mu_*} \equiv \{\psi \in \mathfrak{X},\ \inf_{\mathbb{R}}|\psi| < \mu_*\},$$

we have, for any $\psi \in \mathscr{V}_{\mu_*} \setminus \{\mathrm{e}^{i\theta}U_0(\cdot - y), \theta \in \mathbb{R}, y \in \mathbb{R}\}$,

$$\mathscr{K}(\psi) > \mathscr{K}(U_0) = E(U_0).$$

The crucial point in this result is to prove that the functional $\mathscr{K}(\psi)$ controls the infimum $\inf_{\mathbb{R}}|\psi|$. As we shall see in the proof (Section 6B), the key idea is to study the infimum of the functional $\mathscr{K}$ at fixed $\inf_{\mathbb{R}}|\psi|$ (small), and then to prove (see Proposition 6.2) that, for $\psi \in \mathscr{V}_{\mu_*}$, there holds, for some constant K depending only on f,

$$\mathscr{K}(\psi) \geq \mathscr{K}(U_0) + \frac{(\inf_{\mathbb{R}}|\psi|)^2}{K}.$$

This will be achieved by a fine analysis of some minimizing sequences. From this locally minimizing property of the kink when $(dP(U_c)/dc)_{|c=0} < 0$, we infer its orbital stability, provided we can prove some compactness on the minimizing sequences. A main step here is the control on $\inf_{\mathbb{R}}|\psi|$. Our method allows to infer a control on the distance of the solution to (NLS) to the orbit of the kink, but it is much weaker than those obtained by spectral methods in Lemma 7 or Corollary 11 for instance.

Theorem 24. *Assume that there exists a kink, that is, an odd nontrivial stationary* ($c = 0$) *wave* U_0, *and that* $(dP(U_c)/dc)_{|c=0} < 0$. *Then,* U_0 *is orbitally stable in* $(\mathfrak{X}, d_{\mathfrak{X}})$. *Moreover, if* $\Psi(t)$ *is the* (*global*) *solution to* (NLS) *with initial datum* Ψ^{in}, *we have the control*

$$\sup_{t\geq 0} \inf_{\substack{y\in\mathbb{R}\\ \theta\in\mathbb{R}}} d_{\mathfrak{X}}(\Psi(t), \mathrm{e}^{i\theta}U_0(\cdot - y)) \leq K \sqrt[8]{\mathscr{K}(\Psi^{\mathrm{in}}) - E(U_0)} \leq K \sqrt[4]{d_{\mathfrak{X}}(\Psi^{\mathrm{in}}, U_0)}$$

provided that the right-hand side is sufficiently small.

This result settles the nonlinear stability under the condition $(dP(U_c)/dc)_{|c=0} < 0$ for a general nonlinearity f. In particular, it may be applied to the nonlinearity f given in the example above. It shows that the stability of the kink holds with $\delta = \mathcal{O}(\varepsilon^4)$. We do not claim that the exponent $1/8$ is optimal.

For a complete study of the stability of the traveling waves, it would remain to investigate the case of the sonic ($c = \mathfrak{c}_s$) traveling waves (when they exist). The methods we have developed do not apply directly, and we give in Section 7 some of the difficulties associated with this critical situation.

2. Decay at infinity (proof of Proposition 2)

For simplicity, we shall define

$$\mathscr{V}(\xi) \equiv \mathscr{V}_{\mathfrak{c}_s}(\xi) = \mathfrak{c}_s^2\xi^2 - 4(r_0^2 + \xi)F(r_0^2 + \xi).$$

We freeze the invariance by translation by imposing $|U_c|$ (hence also $\partial_x\phi$) even, so that we can use the formulas in [Chiron 2012]. In particular, it suffices to show the asymptotics for $x \to +\infty$: the case $x \to -\infty$ follows by symmetry. We start with the proof of case (iii). Under assumption $(\mathcal{A}_m)$ and since $F' = -f$, we infer the Taylor expansion

$$\begin{aligned}
\mathcal{V}(\xi) &= \mathfrak{c}_s^2\xi^2 + 4(r_0^2+\xi)\Big(\frac{1}{2!}f'(r_0^2)\xi^2 + \cdots + \frac{1}{(m+2)!}f^{(m+1)}(r_0^2)\xi^{m+2} \\
&\qquad\qquad + \frac{1}{(m+3)!}f^{(m+2)}(r_0^2)\xi^{m+3} + \mathcal{O}(\xi^{m+4})\Big) \\
&= \frac{4r_0^2}{(m+3)!}f^{(m+2)}(r_0^2)\xi^{m+3} + \frac{4}{(m+2)!}f^{(m+1)}(r_0^2)\xi^{m+3} + \mathcal{O}(\xi^{m+4}) \\
&= \frac{4}{r_0^{2(m+1)}}\Big[\frac{r_0^{2(m+2)}}{(m+3)!}f^{(m+2)}(r_0^2) + (-1)^{m+2}\frac{\mathfrak{c}_s^2}{4}\Big]\xi^{m+3} + \mathcal{O}(\xi^{m+4}) = \Lambda_m\xi^{m+3} + \mathcal{O}(\xi^{m+4}),
\end{aligned}$$

since, when $(\mathcal{A}_m)$ holds, all the terms $\mathcal{O}(\xi^{m+2})$ cancel out. The coefficient Λ_m is not zero by assumption. Note that the existence of a nontrivial sonic wave, which depends on the global behavior of $\mathcal{V}$, imposes that $\Lambda_m\xi^{m+3} < 0$ when ξ is small and has the sign of $\xi_{\mathfrak{c}_s}$. Therefore, from the formula (following from the Hamiltonian equation $2\partial_x^2\eta_c + \mathcal{V}'(\eta_c) = 0$; see [Chiron 2012] for example)

$$x = -\operatorname{sgn}(\xi_{\mathfrak{c}_s})\int_{\xi_{\mathfrak{c}_s}}^{\eta_{\mathfrak{c}_s}(x)} \frac{d\xi}{\sqrt{-\mathcal{V}(\xi)}}$$

and since we have, as $\eta \to 0$ (with the sign of $\xi_{\mathfrak{c}_s}$),

$$\begin{aligned}
\int_{\xi_{\mathfrak{c}_s}}^{\eta}\frac{d\xi}{\sqrt{-\mathcal{V}(\xi)}} &= \int_{\xi_{\mathfrak{c}_s}}^{\eta}\frac{d\xi}{\sqrt{-\Lambda_m\xi^{m+3}}} + \int_{\xi_{\mathfrak{c}_s}}^{\eta}\frac{\mathcal{V}(\xi)-\Lambda_m\xi^{m+3}}{\sqrt{-\mathcal{V}(\xi)}\sqrt{-\Lambda_m\xi^{m+3}}\big[\sqrt{-\mathcal{V}(\xi)}+\sqrt{-\Lambda_m\xi^{m+3}}\big]}\,d\xi \\
&= -\frac{2\operatorname{sgn}(\xi_{\mathfrak{c}_s})}{m+1}\Big(\frac{1}{\sqrt{-\Lambda_m\eta^{m+1}}} - \frac{1}{\sqrt{-\Lambda_m\xi_{\mathfrak{c}_s}^{m+1}}}\Big) + \begin{cases}\mathcal{O}(1) & \text{if } m=0,\\ \mathcal{O}(|\ln|\eta||) & \text{if } m=1,\\ \mathcal{O}(\eta^{-\frac{m-1}{2}}) & \text{if } m\ge 2\end{cases}
\end{aligned}$$

(here, we use that the last integrand is $\mathcal{O}(\xi^{-(m+1)/2})$ as $\xi \to 0$), it follows that, as $x \to +\infty$,

$$\eta_{\mathfrak{c}_s}(x) = \operatorname{sgn}(\xi_{\mathfrak{c}_s})\Big(\frac{4}{(m+1)^2|\Lambda_m|}\Big)^{\frac{1}{m+1}}\frac{1}{x^{\frac{2}{m+1}}} + \begin{cases}\mathcal{O}(1/x^3) & \text{if } m=0,\\ \mathcal{O}(\ln(x)/x^2) & \text{if } m=1,\\ \mathcal{O}(1/x^{\frac{4}{m+1}}) & \text{if } m\ge 2.\end{cases}$$

This shows the asymptotics for the modulus, or $\eta_{\mathfrak{c}_s}$. The asymptotic expansion for $\partial_x\phi_{\mathfrak{c}_s}$ is easily deduced from the equation on the phase $2\partial_x\phi_{\mathfrak{c}_s} = \mathfrak{c}_s\eta_{\mathfrak{c}_s}/(r_0^2+\eta_{\mathfrak{c}_s})$, and the phase $\phi_{\mathfrak{c}_s}$ is then computed by integration, which completes the proof of case (iii).

The proof of (ii) is easier. Indeed, in this case, the function $\mathcal{V}$ has the expansion

$$\mathcal{V}(\xi) = \mathfrak{c}_s^2\xi^2 - 4(r_0^2+\xi)F(r_0^2+\xi) = \mathcal{O}(\xi^3);$$

hence

$$\mathcal{V}_c(\xi) = \mathcal{V}(\xi) - (\mathfrak{c}_s^2 - c^2)\xi^2 = -(\mathfrak{c}_s^2 - c^2)\xi^2 + \mathcal{O}(\xi^3).$$

As a consequence, the result follows from the expansion, for $\eta \to 0$,

$$\begin{aligned}\int_{\xi_c}^{\eta} \frac{d\xi}{\sqrt{-\mathcal{V}(\xi)}} &= \int_{\xi_c}^{\eta} \frac{d\xi}{\sqrt{(\mathfrak{c}_s^2 - c^2)\xi^2}} + \int_{\xi_c}^{\eta} \frac{\mathcal{V}(\xi)}{\sqrt{-\mathcal{V}_c(\xi)}\sqrt{(\mathfrak{c}_s^2 - c^2)\xi^2}\big[\sqrt{-\mathcal{V}_c(\xi)} + \sqrt{(\mathfrak{c}_s^2 - c^2)\xi^2}\big]}\, d\xi \\ &= \operatorname{sgn}(\xi_c)\frac{\ln(\eta/\xi_c)}{\sqrt{\mathfrak{c}_s^2 - c^2}} + \int_{\xi_c}^{0} \frac{\mathcal{V}(\xi)}{\sqrt{-\mathcal{V}_c(\xi)}\sqrt{(\mathfrak{c}_s^2 - c^2)\xi^2}\big[\sqrt{-\mathcal{V}_c(\xi)} + \sqrt{(\mathfrak{c}_s^2 - c^2)\xi^2}\big]}\, d\xi + \mathcal{O}(\eta)\end{aligned}$$

since the integrand for the last integral is continuous at $\xi = 0$. This yields the desired expansion for the modulus:

$$\begin{aligned}\eta_c(x) &= \xi_c \exp\Big(-x\sqrt{\mathfrak{c}_s^2 - c^2} - \int_{\xi_c}^{0} \frac{\mathcal{V}(\xi)}{\sqrt{-\xi^2\mathcal{V}_c(\xi)}\big[\sqrt{-\mathcal{V}_c(\xi)} + \sqrt{(\mathfrak{c}_s^2 - c^2)\xi^2}\big]}\, d\xi\Big) + \mathcal{O}\big[\exp\big(-2x\sqrt{\mathfrak{c}_s^2 - c^2}\big)\big] \\ &= M_c \exp\big(-x\sqrt{\mathfrak{c}_s^2 - c^2}\big) + \mathcal{O}\big[\exp\big(-2x\sqrt{\mathfrak{c}_s^2 - c^2}\big)\big],\end{aligned}$$

with

$$M_c \equiv \xi_c \exp\Big(-\int_{\xi_c}^{0} \frac{\mathcal{V}(\xi)}{\sqrt{-\xi^2\mathcal{V}_c(\xi)}\big[\sqrt{-\mathcal{V}_c(\xi)} + \sqrt{(\mathfrak{c}_s^2 - c^2)\xi^2}\big]}\, d\xi\Big) \neq 0,$$

and hence for the phase by similar computations to those above. The proof of case (i) is similar, separating the case $\xi_0 = -r_0^2$ of the kink (even solution) from the case $\xi_0 \neq -r_0^2$ of the bubble (odd solution), and is omitted. □

3. Stability results deduced from the hydrodynamical formulation of (NLS)

3A. ***Proof of Lemma 9.*** **(i) The mapping $\mathcal{M}$ is an homeomorphism.** Let $\psi = A e^{i\phi}$, $(\psi_n = A_n e^{i\phi_n})_n \in \mathfrak{X}$ such that $\psi_n \to \psi$ for d_{hy}. Then, $A_n - A \to 0$ in H^1, $\partial_x\phi_n \to \partial_x\phi$ in L^2 and we may assume (possibly adding some multiple of 2π to ϕ_n), that $\phi_n(0) \to \phi(0)$. We write, using the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ for the second-to-last line,

$$\begin{aligned}&d_{\mathfrak{X}}(\psi_n, \psi) \\ &= \|\partial_x\psi_n - \partial_x\psi\|_{L^2} + \|\,|\psi_n| - |\psi|\,\|_{L^2} + |\psi_n(0) - \psi(0)| \\ &= \|e^{i\phi_n}\partial_x A_n + iA_n e^{i\phi_n}\partial_x\phi_n - e^{i\phi}\partial_x A - iAe^{i\phi}\partial_x\phi\|_{L^2} + \|A_n - A\|_{L^2} + |A_n(0)e^{i\phi_n(0)} - A(0)e^{i\phi(0)}| \\ &\leq \|(e^{i\phi_n} - e^{i\phi})\partial_x A\|_{L^2} + \|e^{i\phi}(\partial_x A_n - \partial_x A)\|_{L^2} + \|(A_n - A)e^{i\phi_n}\partial_x\phi_n\|_{L^2} + \|A(e^{i\phi_n} - e^{i\phi})\partial_x\phi\|_{L^2} \\ &\quad + \|Ae^{i\phi_n}(\partial_x\phi - \partial_x\phi_n)\|_{L^2} + \|A_n - A\|_{L^2} + |(A_n(0) - A(0))e^{i\phi_n(0)}| + |A(0)(e^{i\phi_n(0)} - e^{i\phi(0)})| \\ &\leq \|(e^{i\phi_n} - e^{i\phi})\partial_x A\|_{L^2} + K\|A_n - A\|_{H^1} + \|(A_n - A)\|_{H^1}\|\partial_x\phi_n\|_{L^2} + \|A\|_{L^\infty}\|(e^{i\phi_n} - e^{i\phi})\partial_x\phi\|_{L^2} \\ &\quad + \|A\|_{L^\infty}\|\partial_x\phi - \partial_x\phi_n\|_{L^2} + \|A\|_{L^\infty}|e^{i\phi_n(0)} - e^{i\phi(0)}| \qquad (11) \\ &= \|(e^{i\phi_n} - e^{i\phi})\partial_x A\|_{L^2} + \|A\|_{L^\infty}\|(e^{i\phi_n} - e^{i\phi})\partial_x\phi\|_{L^2} + o_{n\to+\infty}(1),\end{aligned}$$

from the convergences we have. Now observe that

$$\phi_n(x) = \phi_n(0) + \int_0^x \partial_x\phi_n(t)\,dt \to \phi(0) + \int_0^x \partial_x\phi(t)\,dt = \phi(0)$$

pointwise; hence, by the dominated convergence theorem, $\|(\mathrm{e}^{i\phi_n} - \mathrm{e}^{i\phi})\partial_x A\|_{L^2} \to 0$, and similarly for the other term. Therefore, $d_{\mathfrak{X}}(\psi_n, \psi) \to 0$ as wished.

Let now $\psi = A\mathrm{e}^{i\phi}$, $(\psi_n = A_n\mathrm{e}^{i\phi_n})_n \in \mathfrak{X}$ such that $\psi_n \to \psi$ for $d_{\mathfrak{X}}$. Then, $A_n - A = |\psi_n| - |\psi| \to 0$ in L^2, $\partial_x\psi_n \to \partial_x\psi$ in L^2 and $\psi_n(0) \to \psi(0)$. Since $|\cdot|$ is 1-Lipschitz continuous, we infer for the modulus

$$\|\partial_x A_n - \partial_x A\|_{L^2} = \|\partial_x|\psi_n| - \partial_x|\psi|\,\|_{L^2} \le \|\partial_x\psi_n - \partial_x\psi\|_{L^2}.$$

Moreover, $\psi_n(0) \to \psi(0)$ and this implies $\arg(\psi_n(0)/\psi(0)) \to 0$. Therefore, it suffices to show that $\partial_x\phi_n \to \partial_x\phi$ in L^2. We use the formula $A^2\partial_x\phi = \langle i\psi|\partial_x\psi\rangle$, which yields

$$\begin{aligned}\partial_x\phi_n - \partial_x\phi &= \frac{\langle i\psi_n|\partial_x\psi_n\rangle}{A_n^2} - \frac{\langle i\psi|\partial_x\psi\rangle}{A^2}\\ &= \langle i\psi_n|\partial_x\psi_n\rangle\Big(\frac{1}{A_n^2} - \frac{1}{A^2}\Big) + \frac{\langle i\psi_n|\partial_x(\psi-\psi_n)\rangle}{A^2} - \frac{\langle i(\psi-\psi_n)|\partial_x\psi\rangle}{A^2};\end{aligned}$$

hence

$$\begin{aligned}&\|\partial_x\phi_n - \partial_x\phi\|_{L^2}\\ &\le \frac{\|\psi_n\|_{L^\infty}\|A\|_{L^\infty}}{(\inf_{\mathbb{R}} A^2)(\inf_{\mathbb{R}} A_n^2)}\|A_n - A\|_{L^2} + \frac{\|\psi_n\|_{L^\infty}}{\inf_{\mathbb{R}} A^2}\|\partial_x\psi - \partial_x\psi_n\|_{L^2} + \frac{1}{\inf_{\mathbb{R}} A^2}\|\,|\psi_n - \psi|\partial_x\psi\|_{L^2}. \quad (12)\end{aligned}$$

The first two terms tend to zero as $n \to +\infty$. For the last term, we use here again the dominated convergence theorem since $\psi_n(x) = \psi_n(0) + \int_0^x \partial_x\psi_n(t)\,dt \to \psi(0) + \int_0^x \partial_x\psi(t)\,dt = \psi(0)$ pointwise. This concludes the proof of (i).

Proof of (ii). Let us define $\phi_* : \mathbb{R} \to \mathbb{R}$ by $\phi_*(x) \equiv \frac{1}{2}(\ln x)^2 \mathbf{1}_{x\ge 1}$. Then, straightforward computations give $\partial_x\phi_*(x) = ((\ln x)/x)\mathbf{1}_{x\ge1} \in L^2(\mathbb{R})$ and, for $X \ge \mathrm{e}$, by monotonicity of $\partial_x\phi_*$,

$$\int_X^{2X} (\partial_x\phi_*)^2\,dx \ge X\frac{\ln^2(2X)}{(2X)^2} \ge \frac{(\ln X)^2}{4X}. \quad (13)$$

We now consider $\phi_n : \mathbb{R} \to \mathbb{R}$ defined by $\phi_n(x) = 0$ if $x \le 0$ or $x \ge 3n\pi$, $\phi_n(x) = x/n$ if $0 \le x \le n\pi$, $\phi_n(x) = \pi$ if $n\pi \le x \le 2n\pi$ and $\phi_n(x) = 3\pi - x/n$ if $2n\pi \le x \le 3n\pi$. Then, we easily obtain

$$d_{\mathrm{hy}}(\mathrm{e}^{i\phi_*}, \mathrm{e}^{i\phi_* + i\phi_n}) = \|\partial_x\phi_n\|_{L^2} = \sqrt{2 \times \frac{\pi n}{n^2}} = \sqrt{\frac{2\pi}{n}} \to 0.$$

Moreover,

$$d_{\mathfrak{X}}(\mathrm{e}^{i\phi_*}, \mathrm{e}^{i\phi_*+i\phi_n}) = \|\partial_x\phi_*\mathrm{e}^{i\phi_*} - (\partial_x\phi_* + \partial_x\phi_n)\mathrm{e}^{i\phi_*+i\phi_n}\|_{L^2} \ge \|\partial_x\phi_*(\mathrm{e}^{i\phi_n} - 1)\|_{L^2} - \|\partial_x\phi_n\|_{L^2},$$

and, by our choice of ϕ_n and using (13),

$$\|\partial_x\phi_*(\mathrm{e}^{i\phi_n} - 1)\|_{L^2}^2 \ge \int_{n\pi}^{2n\pi} 4(\partial_x\phi_*)^2\,dx \ge \frac{(\ln X)^2}{X}\Big|_{X=n\pi} \sim \frac{(\ln n)^2}{n\pi}.$$

Since $(\ln n)/\sqrt{n\pi} \gg \sqrt{2\pi/n} = d_{\mathrm{hy}}(\mathrm{e}^{i\phi_*}, \mathrm{e}^{i\phi_*+i\phi_n})$, it follows that, as wished,

$$d_{\mathfrak{X}}(\mathrm{e}^{i\phi_*}, \mathrm{e}^{i\phi_*+i\phi_n}) \geq \frac{\ln n}{\sqrt{n\pi}}(1+o(1)) \gg \sqrt{\frac{2\pi}{n}} = d_{\mathrm{hy}}(\mathrm{e}^{i\phi_*}, \mathrm{e}^{i\phi_*+i\phi_n}).$$

We do not know whether the mapping $\mathcal{M}$ is locally Lipschitz, but it is probably not.

3B. ***Proof of Lemma 10.*** Note first that, since U_* does not vanish, if δ is sufficiently small and $d_{\mathfrak{X}}(\psi, U_*) \leq \delta$, then $\| |\psi| - |U_*| \|_{L^\infty} \leq (1/2) \inf_{\mathbb{R}} |U_*|$; hence $|\psi| \geq (1/2)\inf_{\mathbb{R}} |U_*| > 0$ in $\mathbb{R}$; thus ψ does not vanish, may be lifted to $\psi = A\exp(i\phi)$, and we may further assume $\phi(0) - \phi_*(0) \in (-\pi, +\pi]$. In (11), we can easily check that the terms leading to the "$o(1)$" are indeed controlled by $K(\psi)d_{\mathrm{hy}}(\psi_n, \psi)$. In other words, we have

$$d_{\mathfrak{X}}(\psi, U_*) \leq \|(\mathrm{e}^{i\phi} - \mathrm{e}^{i\phi_*})\partial_x A_*\|_{L^2} + \|A_*\|_{L^\infty} \|(\mathrm{e}^{i\phi} - \mathrm{e}^{i\phi_*})\partial_x \phi_*\|_{L^2} + K(U_*)d_{\mathrm{hy}}(\psi, U_*),$$

provided $d_{\mathrm{hy}}(\psi, U_*)$ is small enough. In order to bound the two remaining terms, we write, for $x \in \mathbb{R}$,

$$\phi(x) - \phi_*(x) = \phi(0) - \phi_*(0) + \int_0^x \partial_x\phi(y) - \partial_x\phi_*(y)\, dy,$$

which implies, using that $\mathbb{R} \ni \theta \mapsto \mathrm{e}^{i\theta}$ is 1-Lipschitz and the Cauchy–Schwarz inequality,

$$|1 - \mathrm{e}^{i(\phi_*(x) - \phi(x))}| \leq |\phi(0) - \phi_*(0)| + \sqrt{|x|}\, \|u - u_*\|_{L^2}. \tag{14}$$

Consequently,

$$\|(\mathrm{e}^{i\phi} - \mathrm{e}^{i\phi_*})\partial_x A_*\|_{L^2} \leq |\phi(0) - \phi_*(0)|\, \|\partial_x A_*\|_{L^2} + \|u - u_*\|_{L^2} \left\|\sqrt{|x|}\partial_x A_*\right\|_{L^2}$$

and

$$\|(\mathrm{e}^{i\phi} - \mathrm{e}^{i\phi_*})\partial_x \phi_*\|_{L^2} \leq |\phi(0) - \phi_*(0)|\, \|\partial_x \phi_*\|_{L^2} + \|u - u_*\|_{L^2} \left\|\sqrt{|x|}\partial_x \phi_*\right\|_{L^2}.$$

Both terms are $\leq K(U_*)d_{\mathrm{hy}}(\psi, U_*)$. Indeed, $U_* \in \mathfrak{X}$ is a traveling wave; hence A_*, $\partial_x A_*$, $\partial_x \phi_*$ are bounded functions which decay at infinity exponentially if $0 \leq c < \mathfrak{c}_s$ (cf. part (i) or (ii) of Proposition 2). If $c = \mathfrak{c}_s$, since assumption $(\mathcal{A}_0)$ is satisfied, we invoke Proposition 2(iii), which ensures that $\partial_x \phi_*$ and $\partial_x A_*$ decay at the rate $\mathcal{O}(|x|^{-2})$ ($\partial_x A_*$ decays faster actually). Therefore, $\sqrt{|x|}\partial_x \phi_* \in L^2$. Gathering these estimates provides

$$d_{\mathfrak{X}}(\psi, U_*) \leq K(U_*)d_{\mathrm{hy}}(\psi, U_*).$$

On the other hand, from (12) and the estimate $\|A - A_*\|_{H^1} \leq d_{\mathfrak{X}}(\psi, U_*)$ (see the proof of (i)), we infer

$$d_{\mathrm{hy}}(\psi, U_*) \leq K(U_*)d_{\mathfrak{X}}(\psi, U_*) + \frac{1}{\inf_{\mathbb{R}} A^2} \| |\psi - U_*| \partial_x U_* \|_{L^2}.$$

Using here again the estimate $|\psi(x) - U_*(x)| \leq |\phi(0) - \phi_*(0)| + \sqrt{|x|}\, \|\partial_x \psi - U_*\|_{L^2}$, we deduce

$$d_{\mathrm{hy}}(\psi, U_*) \leq K(U_*)d_{\mathfrak{X}}(\psi, U_*).$$

The proof is complete. $\square$

3C. ***Two remarks on the proof of Theorem 5.*** We would like to point out two minor points concerning the proof of Theorem 5 by Z. Lin. We recall that the proof of [Lin 2002] relies on the Grillakis–Shatah–Strauss theory [Grillakis et al. 1987] once we have written (NLS) under the hydrodynamical form (2), defining $\psi = A\mathrm{e}^{i\phi}$, $(\rho, u) \equiv (|\psi|^2 = A^2, \partial_x\phi)$:

$$\begin{cases} \partial_t\rho + 2\partial_x(\rho u) = 0, \\ \partial_t u + 2u\partial_x u - \partial_x(f(\rho)) - \partial_x\left(\dfrac{\partial_x^2(\sqrt{\rho})}{\sqrt{\rho}}\right) = 0, \end{cases}$$

or, more precisely, with $\eta \equiv \rho - r_0^2 = |\psi|^2 - r_0^2$ and denoting by $\delta E/\delta\eta$, $\delta E/\delta u$ the variational derivatives,

$$\frac{\partial}{\partial t}\begin{pmatrix}\eta \\ u\end{pmatrix} = J\begin{pmatrix}\dfrac{\delta E}{\delta\eta} \\ \dfrac{\delta E}{\delta u}\end{pmatrix}, \quad J \equiv \begin{pmatrix}0 & \partial_x \\ \partial_x & 0\end{pmatrix}. \tag{15}$$

We first remark that the scalar product in the Hilbert space $X = H^1 \times L^2$ can not be $((\eta, u), (\tilde{\eta}, \tilde{u}))_{H^1\times L^2} = \int_{\mathbb{R}} \eta\tilde{\eta} + u\tilde{u}\, dx$ as used in [Lin 2002], but the natural one is $((\eta, u), (\tilde{\eta}, \tilde{u}))_{H^1\times L^2} = \int_{\mathbb{R}} \eta\tilde{\eta} + \partial_x\eta\partial_x\tilde{\eta} + u\tilde{u}\, dx$. This requires us to make some minor changes in the proof, especially not to identify $(H^1)^*$ with H^1. For instance, a linear mapping B is associated with the momentum through the formula

$$P_{\mathrm{hy}}(\eta, u) \equiv \int_{\mathbb{R}} \eta u\, dx = \tfrac{1}{2}(B(\eta, u), (\eta, u))_{H^1\times L^2} \quad \text{with } B \equiv \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$$

for the (nonhilbertian) scalar product $((\eta, u), (\tilde{\eta}, \tilde{u}))_{H^1\times L^2} = \int_{\mathbb{R}} \eta\tilde{\eta} + u\tilde{u}\, dx$. The correct definition is actually

$$P_{\mathrm{hy}}(\eta, u) = \int_{\mathbb{R}} \eta u\, dx = \tfrac{1}{2}\langle B(\eta, u), (\eta, u)\rangle_{X^*, X} \quad \text{with } B \equiv \begin{pmatrix}0 & \iota^* \\ \iota & 0\end{pmatrix},$$

where $\iota : H^1 \hookrightarrow L^2$ is the canonical injection. As already mentioned in Section 1B, the two points in the proof of [Lin 2002] that have been completed in [Gallo 2004] are that: Lin uses a local in time existence for the hydrodynamical system (15) in $H^1 \times L^2$, and not only in $\{\rho \in L^\infty, \partial_x\rho \in L^2\} \times L^2$; and that the energy and the momentum are indeed conserved for the local solution if the initial datum does not vanish.

The second point is that, in the proof of stability (Theorem 3.5 in [Grillakis et al. 1987]), it is used that, if $\mathcal{U} \in X$ and $(\mathcal{U}_n)_{n\in\mathbb{N}} \in X$ is a sequence such that $E(\mathcal{U}_n) \to E(\mathcal{U})$ and $P_{\mathrm{hy}}(\mathcal{U}_n) \to P_{\mathrm{hy}}(\mathcal{U})$, then there exists a sequence $(\tilde{\mathcal{U}}_n)_{n\in\mathbb{N}} \in X$ such that $\mathcal{U}_n - \tilde{\mathcal{U}}_n \to 0$ in X, $E(\tilde{\mathcal{U}}_n) \to E(\mathcal{U})$ and $P_{\mathrm{hy}}(\tilde{\mathcal{U}}_n) = P_{\mathrm{hy}}(\mathcal{U})$. In the context of bound states, the existence of such a sequence $(\tilde{\mathcal{U}}_n)_{n\in\mathbb{N}} \in X$ follows by simple scaling in space, since then the momentum or charge is simply $\int_{\mathbb{R}^d} \mathcal{U}_n^2\, dx$. However, for the one-dimensional traveling waves for (NLS), the momentum P is scaling invariant. We do not know if the existence of such a sequence holds in a general framework, but, for the problem we are studying, we can rely on the following lemma, which is an adaptation of Lemma 6 in [Béthuel et al. 2008a] (see also lemma in [Béthuel et al. 2008b]).

Lemma 3.1. *There exist* $\mathfrak{p}_0 > 0$ *and* $K > 0$, *depending only on* f, *such that, for any* $\mathfrak{p} \in (-\mathfrak{p}_0, +\mathfrak{p}_0)$ *and* $\mu \in \mathbb{R}$ *with* $|\mu| \leq |\mathfrak{p}|$, *there exists* $w = a\mathrm{e}^{i\varphi} \in H^1([0, 1/(2|\mathfrak{p}|)), \mathbb{C})$ *verifying*

$$w(0) = w\left(\frac{1}{2|\mathfrak{p}|}\right), \quad |w(0)| = r_0 + \mu, \quad \int_0^{1/(2|\mathfrak{p}|)} (a^2 - r_0^2)\partial_x\varphi \, dx = \mathfrak{p},$$
$$\int_0^{1/(2|\mathfrak{p}|)} |\partial_x w|^2 + F(|w|^2) \, dx \leq K|\mathfrak{p}|.$$

Proof. If $\mathfrak{p} = 0$, we simply take $w = r_0$. We then assume $0 < \mathfrak{p} < \mathfrak{p}_0$, since the case $-\mathfrak{p}_0 < \mathfrak{p} < 0$ will follow by complex conjugation. We then define, for some small δ to be determined later,

$$w(x) \equiv \sqrt{r_0^2 - \delta + 2\mathfrak{p}(1 - |8\mathfrak{p}x - 1|)_+} \exp[i(1 - |4\mathfrak{p}x - 1|)_+] = a\mathrm{e}^{i\varphi}.$$

It is clear that $w \in H^1([0, 1/(2\mathfrak{p})], \mathbb{C})$ and that $w(0) = w(1/(2\mathfrak{p})) = \sqrt{r_0^2 - \delta}$; thus $|w(0)| = r_0 + \mu$ provided we choose $\delta = -\mu^2 - 2r_0\mu = \mathcal{O}(|\mu|)$. Moreover, since the phase φ has compact support $[0, 1/(2\mathfrak{p})]$,

$$\begin{aligned}\int_0^{1/(2\mathfrak{p})} (a^2 - r_0^2)\partial_x\varphi \, dx &= \int_0^{1/(2\mathfrak{p})} \{-\delta + 2\mathfrak{p}(1 - |8\mathfrak{p}x - 1|)_+\}\partial_x(1 - |4\mathfrak{p}x - 1|)_+ \, dx \\ &= 2\mathfrak{p}\int_0^{1/(2\mathfrak{p})} (1 - |8\mathfrak{p}x - 1|)_+ \partial_x(1 - |4\mathfrak{p}x - 1|)_+ \, dx \\ &= 2\mathfrak{p}\int_0^{1/(2\mathfrak{p})} (1 - |8\mathfrak{p}x - 1|)_+ \partial_x(1 - |4\mathfrak{p}x - 1|)_+ \, dx.\end{aligned}$$

For the last integral, the first factor is equal to 0 if $x \geq 1/(4\mathfrak{p})$ and the second factor is equal to $4\mathfrak{p}$ when $x \leq 1/(4\mathfrak{p})$. Hence, direct computation gives

$$\int_0^{1/(2\mathfrak{p})} (a^2 - r_0^2)\partial_x\varphi \, dx = 2\mathfrak{p}\int_0^{1/(4\mathfrak{p})} (1 - |8\mathfrak{p}x - 1|)_+ \times 4\mathfrak{p} \, dx = \mathfrak{p}.$$

For the energy part, notice first that

$$|a^2 - r_0^2| = |-\delta + 2\mathfrak{p}(1 - |8\mathfrak{p}x - 1|)_+| \leq |\delta| + 2\mathfrak{p}_0$$

is as small as we want if $|\delta|$ and $\mathfrak{p}_0$ are chosen sufficiently small. Therefore,

$$F(|w|^2) \leq K(a^2 - r_0^2)^2.$$

By simple computations, we have

$$\begin{aligned}&\int_0^{1/(2\mathfrak{p})} |\partial_x w|^2 + F(|w|^2) \, dx \\ &\quad\leq K\int_0^{1/(2\mathfrak{p})} \mathfrak{p}^2|\partial_x(1 - |8\mathfrak{p}x - 1|)_+|^2 + |\partial_x(1 - |4\mathfrak{p}x - 1|)_+|^2 + (-\delta + 2\mathfrak{p}(1 - |8\mathfrak{p}x - 1|)_+)^2 \, dx \\ &\quad\leq K\mathfrak{p}^3 + K\mathfrak{p} + K\frac{\delta^2 + \mathfrak{p}^2}{\mathfrak{p}} \leq K\mathfrak{p}\end{aligned}$$

since $\delta = \mathcal{O}(|\mu|) = \mathcal{O}(\mathfrak{p})$, which concludes the proof. ☐

We then consider a sequence $\mathcal{U}_n = (\eta_n, u_n) \in X = H^1 \times L^2$ and show the existence of the desired sequence $\tilde{\mathcal{U}}_n = (\tilde{\eta}_n, \tilde{u}_n) \in X$. We recall that $\mathcal{U}_n$ (respectively, $\mathcal{U}$) is associated with a mapping $\psi_n \in \mathcal{Z}$ (respectively, U_*) that does not vanish. We have $P_{\mathrm{hy}}(\mathcal{U}_n) = P(\psi_n) \to P(U_*)$; thus, for n large enough, $|P(\psi_n) - P(U_*)| \leq \mathfrak{p}_0$. For n fixed, we now pick $R_n > 0$ large enough so that

$$\int_{R_n}^{+\infty} |\partial_x \psi_n|^2 + (|\psi_n| - r_0)^2 \, dx \leq [P(\psi_n) - P(U_*)]^2.$$

In particular, by the Sobolev embedding,

$$\big||\psi_n|(R_n) - r_0\big| \leq \||\psi_n| - r_0\|_{L^\infty([R_n,+\infty))} \leq \sqrt{\int_{R_n}^{+\infty} |\partial_x |\psi_n||^2 + (|\psi_n| - r_0)^2 \, dx} \leq |P(\psi_n) - P(U_*)|.$$

We are now in position to apply (for n large) Lemma 3.1 with $(\mathfrak{p}, \mu) = (P(U_*) - P(\psi_n), |\psi_n|(R_n) - r_0)$. This provides the mapping $w_n \in H^1([0, 1/(2|\mathfrak{p}|)), \mathbb{C})$. Since $|\psi_n|(R_n) - r_0 \to 0$, for n large enough, there exists $\theta_n \in \mathbb{R}$ such that $\psi_n(R_n) = \mathrm{e}^{i\theta_n} |\psi_n|(R_n) = \mathrm{e}^{i\theta_n}(r_0 + \mu) = \mathrm{e}^{i\theta_n} w_n(0)$. We then consider the mapping $\tilde{\psi}_n \in \mathcal{Z}$ defined by

$$\tilde{\psi}_n(x) \equiv \begin{cases} \psi_n(x) & \text{if } x \leq R_n, \\ \mathrm{e}^{i\theta_n} w_n(x - R_n) & \text{if } R_n \leq x \leq R_n + \dfrac{1}{2|P(\psi_n) - P(U_*)|}, \\ \psi_n\left(x - \dfrac{1}{2|P(\psi_n) - P(U_*)|}\right) & \text{if } x \geq R_n + \dfrac{1}{2|P(\psi_n) - P(U_*)|}. \end{cases}$$

From the construction of w_n and the phase factor θ_n, $\tilde{\psi}_n$ is well-defined and continuous. It is clear that

$$P(\tilde{\psi}_n) = P(\psi_n) + \int_0^{1/(2\mathfrak{p})} (a_n^2 - r_0^2) \partial_x \varphi_n \, dx = P(\psi_n) + \mathfrak{p} = P(U_*)$$

for every (large) n, and that

$$\begin{aligned} E(\tilde{\psi}_n) &= E(\psi_n) + \int_0^{1/(2\mathfrak{p})} |\partial_x w|^2 + F(|w|^2) \, dx \\ &= E(U_*) + o(1) + \mathcal{O}(|\mathfrak{p}|) = E(U_*) + o(1) + \mathcal{O}(|P(U_*) - P(\psi_n)|) \end{aligned}$$

converges to $P(U_*)$ as $n \to +\infty$. Denoting by $\tilde{\mathcal{U}}_n \in X$ the hydrodynamical expression of $\tilde{\psi}_n$, it remains to show that $\mathcal{U}_n - \tilde{\mathcal{U}}_n \to 0$ in $X = H^1 \times L^2$. We thus compute, with the definition of $\tilde{\psi}_n$,

$$\begin{aligned} \|\mathcal{U}_n - \tilde{\mathcal{U}}_n\|_X^2 &= \int_{R_n}^{+\infty} \big|\partial_x |\psi_n| - \partial_x |\tilde{\psi}_n|\big|^2 + (|\psi_n| - |\tilde{\psi}_n|)^2 + (u_n - \tilde{u}_n)^2 \, dx \\ &\leq 2 \int_{R_n}^{+\infty} |\partial_x |\psi_n||^2 + |\partial_x |\tilde{\psi}_n||^2 + (|\psi_n| - r_0)^2 + (|\tilde{\psi}_n| - r_0)^2 + u_n^2 + \tilde{u}_n^2 \, dx \\ &\leq 4K \int_{R_n}^{+\infty} |\partial_x \psi_n|^2 + (|\psi_n| - r_0)^2 \, dx + 2 \int_0^{1/(2|P(U_*) - P(\psi_n)|)} |\partial_x w_n|^2 + (|w_n|^2 - r_0^2)^2 \, dx \\ &\leq 4K[P(\psi_n) - P(U_*)]^2 + K|P(\psi_n) - P(U_*)| \to 0. \end{aligned}$$

For the second-to-last inequality, we have used that, for $|x| \geq R_n$, ψ_n has modulus uniformly close to r_0; hence $|\partial_x|\psi_n||^2 + u_n^2 \leq K|\partial_x \psi_n|^2$. Note that the construction still holds for the energy distance, the computations being similar.

3D. *Proof of Lemma 7.* **Proof of estimate (3).** Instead of concluding the stability proof as in [Grillakis et al. 1987], we can notice that we have actually the bound

$$E_{\mathrm{hy}}(\mathcal{U}) - E_{\mathrm{hy}}(\mathcal{U}_{c*}) \geq \frac{1}{K} \inf_{y\in\mathbb{R}} \|\mathcal{U} - \mathcal{U}_{c*}(\cdot - y)\|^2 \tag{16}$$

as soon as $P(U_{c*}) = P_{\mathrm{hy}}(\mathcal{U}_{c*}) = P_{\mathrm{hy}}(\mathcal{U})$ and $\mathcal{U} \in \mathcal{O}_\varepsilon \equiv \{\mathcal{V} \in X,\ \inf_{y\in\mathbb{R}} \|\mathcal{V} - \mathcal{U}_{c*}(\cdot - y)\|_X < \varepsilon\}$ for some small ε. If Ψ^{in} does not have momentum equal to $P_{\mathrm{hy}}(\mathcal{U}_{c*})$, we use Lemma 3.1 to infer that there exists $\tilde{\Psi}(t)$, with momentum equal to $P_{\mathrm{hy}}(\mathcal{U}_{c*}) = P(U_{c*})$, and such that $E(\tilde{\Psi}(t)) - E(\Psi(t)) = \mathcal{O}(|P(\Psi(t)) - P(U_{c*})|)$ and $d_{\mathrm{hy}}(\Psi(t), \tilde{\Psi}(t)) \leq \mathcal{O}(\sqrt{|P(\Psi(t)) - P(U_{c*})|})$. Therefore, for $t \geq 0$, denoting by $\Psi_{\mathrm{hy}}(t) \in X$ and $\tilde{\Psi}_{\mathrm{hy}}(t) \in X$ the hydrodynamical variables for Ψ and $\tilde{\Psi}(t)$,

$$\begin{aligned}
\inf_{y\in\mathbb{R}} \|\Psi_{\mathrm{hy}}(t) - \mathcal{U}_{c*}(\cdot - y)\| &\leq \inf_{y\in\mathbb{R}} \big[\|\tilde{\Psi}_{\mathrm{hy}}(t) - \mathcal{U}_{c*}(\cdot - y)\| + \|\Psi_{\mathrm{hy}}(t) - \tilde{\Psi}_{\mathrm{hy}}(t)\|\big] \\
&\leq \sqrt{K}\sqrt{E(\tilde{\Psi}(t)) - E(U_{c*})} + \mathcal{O}\big(\sqrt{|P(\Psi(t)) - P(U_{c*})|}\big) \\
&\leq K\Big[\sqrt{|E(\Psi(t)) - E(U_{c*})| + |P(\Psi^{\mathrm{in}}) - P(U_{c*})|} + \sqrt{|P(\Psi^{\mathrm{in}}) - P(U_{c*})|}\Big],
\end{aligned}$$

which yields (3).

The above estimate is optimal when $P(\Psi^{\mathrm{in}}) = P(U_{c*})$ since U_{c*} is a critical point of the action $E - c_* P$. This bound shows that, in the definition of stability, one has to take $\delta = \mathcal{O}(\varepsilon^2)$ in general. The estimate (3) shows that one can actually take $\delta = \mathcal{O}(\varepsilon)$.

Proof of estimate (4). The point is to compare $\Psi(t)$ to U_c with $c \simeq c_*$ such that $P(U_c) = P(\Psi^{\mathrm{in}})$ instead of comparing to U_{c*}. In other words, we replace $\tilde{\Psi}(t)$ by U_c. Note first that, since $(dP/dc)_{|c=c_*} < 0$, there exists, by the implicit function theorem, such a $c \simeq c_*$. We then proceed as follows. Let $\Psi^{\mathrm{in}} \in \mathcal{Z}$ be close to U_{c*}. Then, there exists $c = c(\Psi^{\mathrm{in}}) \simeq c_*$ such that $P(U_c) = P(\Psi^{\mathrm{in}})$. Moreover, since $(dP/dc)_{|c=c_*} \neq 0$, it follows

$$\begin{aligned}
\|\mathcal{U}_c - \mathcal{U}_{c*}\| \leq K|c - c_*| \leq K|P(U_c) - P(U_{c*})| &= K|P(\Psi^{\mathrm{in}}) - P(U_{c*})| \\
&\leq K\|\Psi^{\mathrm{in}}_{\mathrm{hy}} - \mathcal{U}_{c*}\| \leq K d_{\mathrm{hy}}(\Psi^{\mathrm{in}}, U_{c*}).
\end{aligned} \tag{17}$$

From (16), we have

$$E_{\mathrm{hy}}(\mathcal{U}) - E_{\mathrm{hy}}(\mathcal{U}_c) \geq \frac{1}{K} \inf_{y\in\mathbb{R}} \|\mathcal{U} - \mathcal{U}_c(\cdot - y)\|^2$$

as soon as $P_{\mathrm{hy}}(\mathcal{U}) = P_{\mathrm{hy}}(\mathcal{U}_c)$. The fact that the constant K can be taken to be uniform with respect to c for c close to c_* comes directly from the proof in [Grillakis et al. 1987]. Therefore, for $t \geq 0$,

$$\begin{aligned}
\inf_{y\in\mathbb{R}} \|\Psi_{\mathrm{hy}}(t) - \mathcal{U}_{c*}(\cdot - y)\| &\leq \inf_{y\in\mathbb{R}} \big[\|\Psi_{\mathrm{hy}}(t) - \mathcal{U}_c(\cdot - y)\| + \|\mathcal{U}_c(\cdot - y) - \mathcal{U}_{c*}(\cdot - y)\|\big] \\
&\leq \sqrt{K}\sqrt{E(\Psi(t)) - E(U_c)} + \mathcal{O}(|P(\Psi^{\mathrm{in}}) - P(U_{c*})|).
\end{aligned}$$

Using that $P(\Psi(t)) = P_{\rm hy}(\Psi_{\rm hy}(t)) = P_{\rm hy}(\mathcal{U}_c)$ and that $\mathcal{U}_c$ is a critical point of the action $E_{\rm hy} - cP_{\rm hy}$, we infer $E(\Psi(t)) - E(\mathcal{U}_c) = [E_{\rm hy} - cP_{\rm hy}](\Psi_{\rm hy}^{\rm in}) - [E_{\rm hy} - cP_{\rm hy}](\mathcal{U}_c) = \mathcal{O}(\|\Psi_{\rm hy}^{\rm in} - \mathcal{U}_c\|^2)$. Consequently,

$$\begin{aligned}\inf_{y\in\mathbb{R}} \|\Psi_{\rm hy}(t) - \mathcal{U}_{c_*}(\cdot - y)\| &\leq K\big(\|\Psi_{\rm hy}^{\rm in} - \mathcal{U}_c\| + \|\Psi_{\rm hy}^{\rm in} - \mathcal{U}_{c_*}\|\big)\\ &\leq K d_{\rm hy}(\Psi^{\rm in}, U_{c_*}) + K\|\mathcal{U}_c - \mathcal{U}_{c_*}\| \leq K d_{\rm hy}(\Psi^{\rm in}, U_{c_*}),\end{aligned}$$

by (17). This gives (4).

4. Instability result for cusps: Proof of Theorem 16

In this section, we set $\mathcal{F}_c \equiv E_{\rm hy} - cP_{\rm hy}$ and we assume

$$-\frac{d^2\mathcal{F}_c(U_c)}{dc^2}_{|c=c_*} = \frac{dP(U_c)}{dc}_{|c=c_*} = 0 \quad \text{and} \quad 0 \neq \ddot{P}_* \equiv \frac{d^2P(U_c)}{dc^2}_{|c=c_*} = -\frac{d^3\mathcal{F}_c(U_c)}{dc^3}_{|c=c_*}.$$

The approach is reminiscent of the proof of [Maeda 2012]. Several modifications are necessary since, for the skew-adjoint operator $J = \partial_x$, we can not find the required Hilbert space Y. More degenerate cases can probably be considered as in [Maeda 2012].

We shall denote by $\mathbb{I} : X \to X^*$ and $\mathbb{I}_{H^1} : H^1 \to (H^1)^*$ the Riesz isomorphisms and define $\mathcal{U} = (\eta, u)^t \in X = H^1(\mathbb{R}, \mathbb{R}) \times L^2(\mathbb{R}, \mathbb{R})$ and $H \equiv L^2(\mathbb{R}, \mathbb{R}) \times L^2(\mathbb{R}, \mathbb{R})$, endowed with its canonical scalar product. They are the corresponding Hilbert spaces needed in [ibid.]. We consider the symmetric matrix

$$\mathbb{B} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is such that $\mathbb{B}^2 = \mathrm{Id}_2$ and $2P_{\rm hy}(\mathcal{U}) = (\mathbb{B}\mathcal{U}, \mathcal{U})_H$.

Our assumption $(dP(U_c)/dc)_{|c=c_*} = 0 \neq (d^2P(U_c)/dc^2)_{|c=c_*}$ will simplify a little the computations in [Maeda 2012]. The functions η_1 and η_2 used there become now

$$\eta_1(\gamma) = \mathcal{F}_{c_*+\gamma}(\mathcal{U}_{c_*+\gamma}) - \mathcal{F}_{c_*}(\mathcal{U}_{c_*}) - \gamma \frac{d\mathcal{F}_c(U_c)}{dc}_{|c=c_*} \sim -\frac{\gamma^3}{6}\ddot{P}_*$$

and

$$\eta_2(\gamma) = \frac{d\eta_1}{d\gamma} = -P(U_{c_*+\gamma}) + P(U_{c_*}) \sim -\frac{\gamma^2}{2}\ddot{P}_*.$$

In order to clarify the dualities used by Maeda, we provide some elements of the proof adapted to our context.

Lemma 4.1. *There exists $\gamma_0 > 0$ small and $\sigma : (-\gamma_0, +\gamma_0) \to \mathbb{R}$ with $\sigma(\gamma) \sim -\gamma^2\ddot{P}_*/(2\|\mathcal{U}_*\|_H^2)$ and such that, for any $\gamma \in (-\gamma_0, +\gamma_0)$,*

$$P_{\rm hy}(\mathcal{U}_{c_*+\gamma} + \sigma(\gamma)\mathbb{B}\mathcal{U}_{c_*+\gamma}) = P_{\rm hy}(\mathcal{U}_*).$$

Proof. We have

$$\begin{aligned}P_{\rm hy}(\mathcal{U}_{c_*+\gamma} + \sigma\mathbb{B}\mathcal{U}_{c_*+\gamma}) &= \tfrac{1}{2}(\mathbb{B}\mathcal{U}_{c_*+\gamma} + \sigma\mathcal{U}_{c_*+\gamma}, \mathcal{U}_{c_*+\gamma} + \sigma\mathbb{B}\mathcal{U}_{c_*+\gamma})_H\\ &= P_{\rm hy}(\mathcal{U}_{c_*+\gamma}) + \sigma\|\mathcal{U}_{c_*+\gamma}\|_H^2 + \sigma^2 P_{\rm hy}(\mathcal{U}_{c_*+\gamma}).\end{aligned}$$

Since $\|\mathcal{U}_*\|_H^2 \neq 0$, the conclusion follows from an easy implicit function argument near $\sigma = \gamma = 0$. In [Maeda 2012], the linear mapping B is seen from X to X^*, but, here, there is no confusion in defining $\mathcal{U}_{c_*+\gamma} + \sigma \mathbb{B} \mathcal{U}_{c_*+\gamma} \in H = L^2 \times L^2$. □

We define, for $\gamma \in (-\gamma_0, +\gamma_0)$,

$$\mathcal{W}(\gamma) \equiv \mathcal{U}_{c_*+\gamma} + \sigma(\gamma)\mathbb{B}\mathcal{U}_{c_*+\gamma},$$

which then satisfies $P_{\mathrm{hy}}(\mathcal{W}(\gamma)) = P_{\mathrm{hy}}(\mathcal{U}_*)$ by construction.

Lemma 4.2. *As $\gamma \to 0$, we have $\mathcal{F}_{c_*}(\mathcal{W}(\gamma)) - \mathcal{F}_{c_*}(\mathcal{U}_{c_*}) \sim -(\gamma^3/6)\ddot{P}_*$.*

Proof. Using that $\mathcal{F}'_{c_*+\gamma}(\mathcal{U}_{c_*+\gamma}) = 0$, $P_{\mathrm{hy}}(\mathcal{W}(\gamma)) = P_{\mathrm{hy}}(\mathcal{U}_{c_*}) = -(d\mathcal{F}_c(U_c)/dc)_{|c=c_*}$ and $\sigma(\gamma) = \mathcal{O}(\gamma^2)$, we have by the Taylor expansion

$$\begin{aligned}\mathcal{F}_{c_*}(\mathcal{W}(\gamma)) - \mathcal{F}_{c_*}(\mathcal{U}_{c_*}) &= \mathcal{F}_{c_*+\gamma}(\mathcal{U}_{c_*+\gamma} + \sigma(\gamma)\mathbb{B}\mathcal{U}_{c_*+\gamma}) - \mathcal{F}_{c_*}(\mathcal{U}_{c_*}) + \gamma P_{\mathrm{hy}}(\mathcal{W}(\gamma))\\ &= \mathcal{F}_{c_*+\gamma}(\mathcal{U}_{c_*+\gamma}) - \mathcal{F}_{c_*}(\mathcal{U}_{c_*}) - \gamma \frac{d\mathcal{F}_c(U_c)}{dc}_{|c=c_*} + \mathcal{O}(\gamma^4) \sim -\frac{\gamma^3}{6}\ddot{P}_*,\end{aligned}$$

as wished. □

We recall that we have defined the tubular neighborhood $\mathcal{O}_\varepsilon = \{\mathcal{V} \in X,\ \inf_{y\in\mathbb{R}} \|\mathcal{V} - \mathcal{U}_*(\cdot - y)\|_X < \varepsilon\}$.

Lemma 4.3. *For $\varepsilon > 0$ small enough, there exist four $\mathcal{C}^1$ mappings $\bar{\gamma}$, α, $\bar{y} : \mathcal{O}_\varepsilon \to \mathbb{R}$ and $\vartheta : \mathcal{O}_\varepsilon \to X$, satisfying, for $\mathcal{U} \in \mathcal{O}_\varepsilon$,*

$$\mathcal{U}(\cdot - \bar{y}(\mathcal{U})) = \mathcal{W}(\bar{\gamma}(\mathcal{U})) + \vartheta(\mathcal{U}) + \alpha(\mathcal{U})\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}$$

and the orthogonality relations

$$(\vartheta(\mathcal{U}), \partial_x \mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})_H = (\vartheta(\mathcal{U}), [\partial_c \mathcal{U}_c]_{|c=c_*+\bar{\gamma}(\mathcal{U})})_H = (\vartheta(\mathcal{U}), \mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})_H = 0.$$

Finally, $\mathbb{I}^{-1}\bar{\gamma}' \in H^2 \times H^1$ and $\mathbb{I}_{H^1}^{-1}\partial\bar{\gamma}/\partial\eta \in H^4$.

Proof. We consider the mapping $G : X \times \mathbb{R} \times (-\gamma_0, +\gamma_0) \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$G(\mathcal{U}, y, \gamma, \alpha) \equiv \begin{pmatrix} (\mathcal{U}(\cdot - y) - \mathcal{W}(\gamma) - \alpha\mathbb{B}\mathcal{U}_{c_*+\gamma}, \partial_x \mathcal{U}_{c_*+\gamma})_H \\ (\mathcal{U}(\cdot - y) - \mathcal{W}(\gamma) - \alpha\mathbb{B}\mathcal{U}_{c_*+\gamma}, [\partial_c \mathcal{U}_c]_{|c=c_*+\gamma})_H \\ (\mathcal{U}(\cdot - y) - \mathcal{W}(\gamma) - \alpha\mathbb{B}\mathcal{U}_{c_*+\gamma}, \mathbb{B}\mathcal{U}_{c_*+\gamma})_H \end{pmatrix}.$$

Then $G(\mathcal{U}_*, 0, 0, 0) = 0$ since $\mathcal{W}(0) = \mathcal{U}_*$. In order to show that G is of class $\mathcal{C}^1$, we have to pay attention to the translation term $\mathcal{U}(\cdot - y)$, since differentiation in y requires $\mathcal{U} \in H^1 \times H^1$ whereas we only have $\mathcal{U} \in X = H^1 \times L^2$. It thus suffices to write

$$G(\mathcal{U}, y, \gamma, \alpha) = \begin{pmatrix} (\mathcal{U}, \partial_x \mathcal{U}_{c_*+\gamma}(\cdot + y))_H - (\mathcal{W}(\gamma) + \alpha\mathbb{B}\mathcal{U}_{c_*+\gamma}, \partial_x \mathcal{U}_{c_*+\gamma})_H \\ (\mathcal{U}, [\partial_c \mathcal{U}_c]_{|c=c_*+\gamma}(\cdot + y))_H - (\mathcal{W}(\gamma) + \alpha\mathbb{B}\mathcal{U}_{c_*+\gamma}, [\partial_c \mathcal{U}_c]_{|c=c_*+\gamma})_H \\ (\mathcal{U}, \mathbb{B}\mathcal{U}_{c_*+\gamma}(\cdot + y))_H - (\mathcal{W}(\gamma) + \alpha\mathbb{B}\mathcal{U}_{c_*+\gamma}, \mathbb{B}\mathcal{U}_{c_*+\gamma})_H \end{pmatrix}$$

to see that G is indeed of class $\mathscr{C}^1$ on $X \times \mathbb{R} \times (-\gamma_0, +\gamma_0) \times \mathbb{R}$ since $c \mapsto \mathscr{U}_c$ is smooth. Moreover, using that $\partial_\gamma \mathscr{W}_{|\gamma=0} = [\partial_c \mathscr{U}_c]_{|c=c_*}$, we infer

$$\frac{\partial G}{\partial(y,\gamma,\alpha)}(\mathscr{U}_*, 0,0,0) = \begin{pmatrix} (\mathscr{U}_*, \partial_x^2 \mathscr{U}_*)_H & -([\partial_c \mathscr{U}_c]_{|c=c_*}, \partial_x \mathscr{U}_*)_H & -(\mathbb{B}\mathscr{U}_*, \partial_x \mathscr{U}_*)_H \\ -(\mathscr{U}_*, \partial_x [\partial_c \mathscr{U}_c]_{|c=c_*})_H & -\|[\partial_c \mathscr{U}_c]_{|c=c_*}\|_H^2 & -(\mathbb{B}\mathscr{U}_*, [\partial_c \mathscr{U}_c]_{|c=c_*})_H \\ -(\mathscr{U}_*, \mathbb{B}\partial_x \mathscr{U}_*)_H & -([\partial_c \mathscr{U}_c]_{|c=c_*}, \mathbb{B}\mathscr{U}_*)_H & -\|\mathbb{B}\mathscr{U}_*\|_H^2 \end{pmatrix}.$$

At this stage, the argument in [Maeda 2012] is to use that

$$([\partial_c \mathscr{U}_c]_{|c=c_*}, \partial_x \mathscr{U}_*)_H = -(\partial_x [\partial_c \mathscr{U}_c]_{|c=c_*}, \mathscr{U}_*)_H = 0,$$

which is assumption 2(iii) there. This equality holds true for us since we have chosen $\mathscr{U}_c$ even for any c (close to c_*). Furthermore, $(\mathscr{U}_*, \partial_x^2 \mathscr{U}_*)_H = -\|\partial_x \mathscr{U}_*\|_H^2$ by integration by parts, $(\mathbb{B}\mathscr{U}_*, \partial_x \mathscr{U}_*)_H = (\mathscr{U}_*, \mathbb{B}\partial_x \mathscr{U}_*)_H = 0$ since $\mathbb{B}\partial_x = J$ is skew-adjoint, and $(\mathbb{B}\mathscr{U}_*, [\partial_c \mathscr{U}_c]_{|c=c_*})_H = \partial_c [P_{\mathrm{hy}}(\mathscr{U}_c)]_{|c=c_*} = 0$ by hypothesis. Therefore,

$$\frac{\partial G}{\partial(y,\gamma,\alpha)}(\mathscr{U}_*, 0,0,0) = \begin{pmatrix} -\|\partial_x \mathscr{U}_{c_*}\|_H^2 & 0 & 0 \\ 0 & -\|[\partial_c \mathscr{U}_c]_{|c=c_*}\|_H^2 & 0 \\ 0 & 0 & -\|\mathbb{B}\mathscr{U}_*\|_H^2 \end{pmatrix}$$

is invertible; thus the implicit function theorem provides three real-valued functions $\underline{y}$, $\underline{\gamma}$ and $\underline{\alpha}$, defined near $\mathscr{U}_*$ (in X) and with $\underline{y}(\mathscr{U}_*) = \underline{\gamma}(\mathscr{U}_*) = \underline{\alpha}(\mathscr{U}_*) = 0$, such that $G(\mathscr{U}, \underline{y}(\mathscr{U}), \underline{\gamma}(\mathscr{U}), \underline{\alpha}(\mathscr{U})) = 0$. These functions are extended to $\mathbb{O}_\varepsilon$ (for ε small enough) by the formulas $\bar{y}(\mathscr{U}) \equiv \underline{y}(\mathscr{U}(\cdot - y)) + y$, $\bar{\gamma}(\mathscr{U}) \equiv \underline{\gamma}(\mathscr{U}(\cdot - y))$ and $\alpha(\mathscr{U}) \equiv \underline{\alpha}(\mathscr{U}(\cdot - y))$ for any $y \in \mathbb{R}$ such that $\mathscr{U}(\cdot - y)$ lies in the neighborhood of $\mathscr{U}_*$ where $\underline{y}$, $\underline{\gamma}$ and $\underline{\alpha}$ are defined. Consequently, the mapping

$$\vartheta(\mathscr{U}) \equiv \mathscr{U}(\cdot - \bar{y}(\mathscr{U})) - \mathscr{W}(\bar{\gamma}(\mathscr{U})) - \bar{\alpha}(\mathscr{U})\mathbb{B}\mathscr{U}_{c_*+\bar{\gamma}(\mathscr{U})}$$

is orthogonal in H to $\partial_x \mathscr{U}_{c_*+\bar{\gamma}(\mathscr{U})}$, $[\partial_c \mathscr{U}_c]_{|c=c_*+\bar{\gamma}(\mathscr{U})}$ and $\mathbb{B}\mathscr{U}_{c_*+\bar{\gamma}(\mathscr{U})}$, as desired. Since f is assumed of class $\mathscr{C}^2$, we have $\mathscr{U}_c \in H^4$ and the regularities $\mathbb{I}^{-1}\bar{\gamma}' \in H^2 \times H^1$ and $\mathbb{I}_{H^1}^{-1} \partial\bar{\gamma}/\partial\eta \in H^4$ follow easily. □

Remark 4.4. We would like to point out that, in [ibid., Lemma 3], it is claimed that "$w(u)$" is orthogonal to "$\partial_\omega \phi_{\omega+\Lambda(u)}$" (we refer to the notations there). However, since "$T(\theta(u)) - \Psi(\Lambda(u))$" is already orthogonal to "$\partial_\omega \phi_{\omega+\Lambda(u)}$" by construction, this is equivalent to "$\langle B\phi_{\omega+\Lambda(u)}, \partial_\omega \phi_{\omega+\Lambda(u)}\rangle = 0$", or "$\partial_{\omega'}[Q(\phi_{\omega'})] = 0$" at "$\omega' = \omega + \Lambda(u)$". We have not understood why this should happen since, in general, for the function $\omega' \mapsto Q(\phi_{\omega'})$, the point ω is the only local critical point. For this reason, we have added a component to the original mapping G in [ibid.]. Let us observe that, then, Lemma 3 in [ibid.] uses the assumption "$d''(\omega) = 0$". On the other hand, the derivative of G in [ibid.] assumes "$u \in D(T'(0))$", for otherwise the expression "$G_{1,1}(u, \theta, \Lambda) = \langle T'(0)T(\theta)u, T'(0)\phi_{\omega+\Lambda}\rangle$", for instance, is meaningless. We have therefore given some details showing clearly the smoothness of G.

We now prove a lemma which shows that the quadratic functional associated with $\mathscr{F}_*''$ gives a good control on $\vartheta(\mathscr{U})$ thanks to the orthogonality conditions on this function. This result is in the spirit of Lemma 7 in [Ohta 2011].

Lemma 4.5. *There exist* $0 < \gamma_1 \le \gamma_0$ *and* $K_0 > 0$ *such that, if* $\gamma \in (-\gamma_1, +\gamma_1)$ *and if* $\vartheta \in X$ *satisfies*

$$(\vartheta, \partial_x \mathcal{U}_{c_*+\gamma})_H = (\vartheta, [\partial_c \mathcal{U}_c]_{|c=c_*+\gamma})_H = (\vartheta, \mathbb{B}\mathcal{U}_{c_*+\gamma})_H = 0,$$

then $\langle \mathcal{F}''_{c_*+\gamma}(\mathcal{U}_{c_*+\gamma})\vartheta, \vartheta\rangle_{X^*,X} \ge K_0 \|\vartheta\|_X^2$.

Proof. As a first step, we prove that, if $\vartheta \in X$ satisfies $\vartheta \neq 0$,

$$(\vartheta, \partial_x \mathcal{U}_*)_H = (\vartheta, [\partial_c \mathcal{U}_c]_{|c=c_*})_H = (\vartheta, \mathbb{B}\mathcal{U}_*)_H = 0,$$

then $\langle \mathcal{F}''_*(\mathcal{U}_*)\vartheta, \vartheta\rangle_{X^*,X} > 0$. Indeed, assume that $\langle \mathcal{F}''_*(\mathcal{U}_*)\vartheta, \vartheta\rangle_{X^*,X} \le 0$. Let $\chi \in X$ be a negative eigenvector of $\mathcal{F}''_*$. We claim that we can not have $((\vartheta, \chi)_H, ([\partial_c \mathcal{U}_c]_{|c=c_*}, \chi)_H) = (0,0)$. Otherwise, $(\vartheta, \chi)_H = 0$ implies that ϑ is L^2-orthogonal to χ, which is the eigenvector associated with the only negative eigenvalue $-\mu_0$ of $\mathcal{F}''_*$ seen as an unbounded operator on L^2; thus $\langle \mathcal{F}''_*(\mathcal{U}_*)\vartheta, \vartheta\rangle_{X^*,X} \ge 0$, and, since we assume equality, this means that ϑ belongs to the kernel of $\mathcal{F}''_*(\mathcal{U}_*)$, which is spanned by $\mathcal{U}_* = \partial_x \mathcal{U}_*$, but the condition $(\vartheta, \partial_x \mathcal{U}_*)_H = 0$ then implies $\vartheta = 0$, a contradiction. Therefore, there exists $(a,b) \in \mathbb{R}^2$ such that $(a,b) \neq (0,0)$ and $(a[\partial_c \mathcal{U}_c]_{|c=c_*} + b\vartheta, \chi)_H = 0$. The nonzero vector $p \equiv a[\partial_c \mathcal{U}_c]_{|c=c_*} + b\vartheta$ then satisfies $(p, \chi)_H = 0$ and $(p, J\mathcal{U}_*)_H = a([\partial_c \mathcal{U}_c]_{|c=c_*}, J\mathcal{U}_*)_H + b(\vartheta, J\mathcal{U}_*)_H = 0$, so that $\langle \mathcal{F}''_*(\mathcal{U}_*)p, p\rangle_{X^*,X} > 0$. Here, we have used once again that $([\partial_c \mathcal{U}_c]_{|c=c_*}, J\mathcal{U}_*)_H = 0$ since the left vector is an even function and the right vector an odd function. However, in view of the equality $\langle \mathcal{F}''_*(\mathcal{U}_*)[\partial_c \mathcal{U}_c]_{|c=c_*}, \phi\rangle_{X^*,X} = (\mathbb{B}\mathcal{U}_*, \phi)_H$, valid for any $\phi \in X$ (which follows from differentiation of $E'_{\mathrm{hy}}(\mathcal{U}_c) = cP'_{\mathrm{hy}}(\mathcal{U}_c) = c(\mathbb{B}\mathcal{U}_c, \cdot)_H$ at $c = c_*$), we have

$$\langle \mathcal{F}''_*(\mathcal{U}_*)[\partial_c \mathcal{U}_c]_{|c=c_*}, \vartheta\rangle_{X^*,X} = (\mathbb{B}[\partial_c \mathcal{U}_c]_{|c=c_*}, \vartheta)_H = 0.$$

As a consequence,

$$\begin{aligned} 0 < \langle \mathcal{F}''_*(\mathcal{U}_*)p, p\rangle_{X^*,X} &= a^2 \langle \mathcal{F}''_*(\mathcal{U}_*)[\partial_c \mathcal{U}_c]_{|c=c_*}, [\partial_c \mathcal{U}_c]_{|c=c_*}\rangle_{X^*,X} + b^2 \langle \mathcal{F}''_*(\mathcal{U}_*)\vartheta, \vartheta\rangle_{X^*,X} \\ &= a^2 (\mathbb{B}\mathcal{U}_*, [\partial_c \mathcal{U}_c]_{|c=c_*})_H + b^2 \langle \mathcal{F}''_*(\mathcal{U}_*)\vartheta, \vartheta\rangle_{X^*,X} = b^2 \langle \mathcal{F}''_*(\mathcal{U}_*)\vartheta, \vartheta\rangle_{X^*,X}, \end{aligned}$$

since $(\mathbb{B}\mathcal{U}_*, [\partial_c \mathcal{U}_c]_{|c=c_*})_H = \partial_c [P_{\mathrm{hy}}(\mathcal{U}_c)]_{|c=c_*} = 0$ in our situation. We reach a contradiction since the right-hand side is supposed ≤ 0.

We now prove the lemma by contradiction, and then assume that there exist sequences $(\vartheta_n)_{n\ge 1} \in X$ and $(\gamma_n)_{n \ge 1} \in (0, \gamma_0)$ such that $\gamma_n \to 0$, $\|\vartheta_n\|_X^2 = 1$ and

$$(\vartheta_n, \partial_x \mathcal{U}_{c_*+\gamma_n})_H = (\vartheta_n, [\partial_c \mathcal{U}_c]_{|c=c_*+\gamma_n})_H = (\vartheta_n, \mathbb{B}\mathcal{U}_{c_*+\gamma_n})_H = 0, \tag{18}$$

but $\langle \mathcal{F}''_{c_*+\gamma_n}(\mathcal{U}_{c_*+\gamma_n})\vartheta_n, \vartheta_n\rangle_{X^*,X} \to 0$. Possibly passing to a subsequence, we may assume the existence of some $\vartheta = (\zeta, \upsilon) \in X$ such that $\vartheta_n \equiv (\zeta_n, \upsilon_n) \rightharpoonup \vartheta$ in $X = H^1 \times L^2$. We then show the lower semicontinuity of $\langle \mathcal{F}''_*(\mathcal{U}_*)\vartheta, \vartheta\rangle_{X^*,X}$. This is roughly a verification of part of assumption (A3) in [Ohta 2011], used in Lemma 7 there. By the compact Sobolev embedding, we may assume $\zeta_n \to \zeta$ in $L^\infty_{\mathrm{loc}}(\mathbb{R})$.

A straightforward computation gives

$$\langle \mathcal{F}''_{c_*+\gamma}(\mathcal{U}_{c_*+\gamma})\vartheta, \vartheta\rangle_{X^*,X} = \int_{\mathbb{R}} \frac{(\partial_x \zeta)^2}{2(r_0^2+\eta_{c_*+\gamma})} - \frac{\partial_x\zeta\partial_x\eta_{c_*+\gamma}}{(r_0^2+\eta_{c_*+\gamma})^2} + \frac{\zeta^2(\partial_x\eta_{c_*+\gamma})^2}{4(r_0^2+\eta_{c_*+\gamma})^3} + 2(r_0^2+\eta_{c_*+\gamma})\upsilon^2 + 2(2u_{c_*+\gamma}-(c_*+\gamma))\upsilon\zeta - f'(r_0^2+\eta_{c_*+\gamma})\zeta^2\,dx.$$

Since $r_0^2+\eta_{c_*+\gamma_n}$ remains bounded away from zero uniformly and $\eta_{c_*+\gamma_n} \to \eta_{c_*}$ in $W^{1,\infty}(\mathbb{R}) \cap H^1(\mathbb{R})$ as $n \to +\infty$, the weak convergence $\zeta_n \rightharpoonup \zeta$ in H^1 implies

$$\int_{\mathbb{R}} \frac{(\partial_x \zeta)^2}{2(r_0^2+\eta_{c_*})} - \frac{\partial_x\zeta\partial_x\eta_{c_*}}{(r_0^2+\eta_{c_*})^2} + \frac{\zeta^2(\partial_x\eta_{c_*})^2}{4(r_0^2+\eta_{c_*})^3}\,dx \leq \varliminf_{n\to+\infty} \int_{\mathbb{R}} \frac{(\partial_x \zeta_n)^2}{2(r_0^2+\eta_{c_*+\gamma_n})} - \frac{\partial_x\zeta_n\partial_x\eta_{c_*+\gamma_n}}{(r_0^2+\eta_{c_*+\gamma_n})^2} + \frac{\zeta_n^2(\partial_x\eta_{c_*+\gamma_n})^2}{4(r_0^2+\eta_{c_*+\gamma_n})^3}\,dx. \quad (19)$$

For the remaining terms, we write, for some $R > 0$ to be determined later,

$$\begin{aligned}\int_{\mathbb{R}} 2(r_0^2+\eta_{c_*+\gamma_n})\upsilon_n^2 &+ 2(2u_{c_*+\gamma_n}-(c_*+\gamma_n))\upsilon_n\zeta_n - f'(r_0^2+\eta_{c_*+\gamma_n})\zeta_n^2\,dx \\ &= \int_{\mathbb{R}} 2\Big[(r_0^2+\eta_{c_*+\gamma_n})^{1/2}\upsilon_n + \frac{(2u_{c_*+\gamma_n}-(c_*+\gamma_n))\zeta_n}{2(r_0^2+\eta_{c_*+\gamma_n})^{1/2}}\Big]^2 dx \\ &\quad + \int_{|x|\leq R} + \int_{|x|\geq R} \frac{1}{2}\Big(-\frac{(2u_{c_*+\gamma_n}-(c_*+\gamma_n))^2}{r_0^2+\eta_{c_*+\gamma_n}} - 2f'(r_0^2+\eta_{c_*+\gamma_n})\Big)\zeta_n^2\,dx.\end{aligned}$$

For the first integral, we may use that $(\zeta_n, \upsilon_n) \rightharpoonup (\zeta, \upsilon)$ in $L^2 \times L^2$ and the fact that $(\eta_{c_*+\gamma_n}, u_{c_*+\gamma_n})$ converges to (η_*, u_*) uniformly to deduce

$$(r_0^2+\eta_{c_*+\gamma_n})^{1/2}\upsilon_n + \frac{(2u_{c_*+\gamma_n}-(c_*+\gamma_n))\zeta_n}{2(r_0^2+\eta_{c_*+\gamma_n})^{1/2}} \rightharpoonup (r_0^2+\eta_*)^{1/2}\upsilon + \frac{(2u_*-c_*)\zeta}{2(r_0^2+\eta_*)^{1/2}} \quad \text{in } L^2; \quad (20)$$

hence,

$$\int_{\mathbb{R}} 2\Big[(r_0^2+\eta_*)^{1/2}\upsilon + \frac{(2u_*-c_*)\zeta}{2(r_0^2+\eta_*)^{1/2}}\Big]^2 dx \leq \varliminf_{n\to+\infty} \int_{\mathbb{R}} 2\Big[(r_0^2+\eta_{c_*+\gamma_n})^{1/2}\upsilon_n + \frac{(2u_{c_*+\gamma_n}-(c_*+\gamma_n))\zeta_n}{2(r_0^2+\eta_{c_*+\gamma_n})^{1/2}}\Big]^2 dx. \quad (21)$$

Since $\zeta_n \to \zeta$ in $L^\infty([-R,+R])$ and $(u_{c_*+\gamma_n}, \eta_{c_*+\gamma_n}) \to (u_*, \eta_*)$ uniformly, it follows that

$$\int_{|x|\leq R} \frac{1}{2}\Big(-\frac{(2u_*-c_*)^2}{r_0^2+\eta_*} - 2f'(r_0^2+\eta_*)\Big)\zeta^2\,dx = \lim_{n\to+\infty} \int_{|x|\leq R} \frac{1}{2}\Big(-\frac{(2u_{c_*+\gamma_n}-(c_*+\gamma_n))^2}{r_0^2+\eta_{c_*+\gamma_n}} - 2f'(r_0^2+\eta_{c_*+\gamma_n})\Big)\zeta_n^2\,dx.$$

For the last integral, we have to use the decay at infinity of $\eta_{c_*+\gamma}$ and $u_{c_*+\gamma}$ uniformly for $|\gamma|$ small.

This gives

$$-\frac{(2u_{c_*+\gamma_n}-(c_*+\gamma_n))^2}{r_0^2+\eta_{c_*+\gamma_n}}-2f'(r_0^2+\eta_{c_*+\gamma_n})\to\frac{\mathfrak{c}_s^2-c_*^2}{r_0^2}$$

as $|x|\to+\infty$, uniformly in n. Since $0<c_*<\mathfrak{c}_s$, there exist some small $\delta>0$ and some $R>0$ large such that, for any n and any x with $|x|\geq R$,

$$-\frac{(2u_{c_*+\gamma_n}-(c_*+\gamma_n))^2}{r_0^2+\eta_{c_*+\gamma_n}}-2f'(r_0^2+\eta_{c_*+\gamma_n})\geq\delta.$$

In particular, since $\zeta_n\rightharpoonup\zeta$ in L^2,

$$\mathbf{1}_{|x|\geq R}\Big(-\frac{(2u_{c_*+\gamma_n}-(c_*+\gamma_n))^2}{r_0^2+\eta_{c_*+\gamma_n}}-2f'(r_0^2+\eta_{c_*+\gamma_n})\Big)^{1/2}\zeta_n\rightharpoonup\mathbf{1}_{|x|\geq R}\Big(-\frac{(2u_*-c_*)^2}{r_0^2+\eta_*}-2f'(r_0^2+\eta_*)\Big)^{1/2}\zeta$$

in L^2; thus

$$\int_{|x|\geq R}\frac{1}{2}\Big(-\frac{(2u_*-c_*)^2}{r_0^2+\eta_*}-2f'(r_0^2+\eta_*)\Big)\zeta^2\,dx \\ \leq\varliminf_{n\to+\infty}\int_{|x|\geq R}\frac{1}{2}\Big(-\frac{(2u_{c_*+\gamma_n}-(c_*+\gamma_n))^2}{r_0^2+\eta_{c_*+\gamma_n}}-2f'(r_0^2+\eta_{c_*+\gamma_n})\Big)\zeta_n^2\,dx. \quad (22)$$

Combining these three $\varliminf$ inequalities, we deduce

$$\langle\mathscr{F}_*''(\mathfrak{U}_*)\vartheta,\vartheta\rangle_{X^*,X}\leq\varliminf_{n\to+\infty}\langle\mathscr{F}''_{c_*+\gamma_n}(\mathfrak{U}_{c_*+\gamma_n})\vartheta_n,\vartheta_n\rangle_{X^*,X}=0. \quad (23)$$

Turning back to our sequence (ϑ_n,γ_n), we may pass to the limit in (18):

$$(\vartheta,\partial_x\mathfrak{U}_*)_H=(\vartheta,[\partial_c\mathfrak{U}_c]_{|c=c_*})_H=(\vartheta,\mathbb{B}\mathfrak{U}_*)_H=0.$$

Comparing with (23), we deduce from our first claim that $\vartheta=0$. This means that we must have equality in all the above $\varliminf$ inequalities. In particular, the weak convergence (22) is actually strong; thus $\zeta_n\to\zeta=0$ in $L^2(\mathbb{R})$ (the strong convergence in $\{|x|\leq R\}$ being already known since $\zeta_n\to\zeta$ in $L^\infty_{\mathrm{loc}}(\mathbb{R})$). Going back to the equality in (19) thus provides $\partial_x\zeta_n\to\partial_x\zeta=0$ in $L^2(\mathbb{R})$, since $r_0^2+\eta_{c_*+\gamma_n}$ remains uniformly bounded away from zero, and by weak convergence,

$$0=\int_{\mathbb{R}}\frac{\zeta^2(\partial_x\eta_{c_*})^2}{4(r_0^2+\eta_{c_*})^3}\,dx=\lim_{n\to+\infty}\int_{\mathbb{R}}\frac{\partial_x\zeta_n\partial_x\eta_{c_*+\gamma_n}}{(r_0^2+\eta_{c_*+\gamma_n})^2}\,dx.$$

Finally, the equality in (21) means that (20) is actually a strong convergence; that is, $\upsilon_n\to\upsilon=0$ in L^2 since $\zeta_n\to\zeta$ in L^2. The contradiction then follows: $1=\|\vartheta_n\|_X^2=\|\zeta_n\|_{L^2}^2+\|\partial_x\zeta_n\|_{L^2}^2+\|\upsilon_n\|_{L^2}^2\to0.$ □

Remark 4.6. This result is also Lemma 7 in [Maeda 2012], and is said to be Lemma 7 in [Ohta 2011]. However, the hypotheses of Lemma 7 in [Ohta 2011] are not satisfied, and in particular assumption (B3) there. It is natural to believe that this assumption is satisfied is most physical situations, but it is not clear whether it always holds true in the general framework of [Maeda 2012] without further hypothesis.

The next lemma provides a control for $\alpha(\mathfrak{U})$.

Lemma 4.7. *Assume $\varepsilon > 0$ small enough. Then, there exists $K > 0$ such that, for any $\mathcal{U} \in \mathcal{O}_\varepsilon$ satisfying $P_{\mathrm{hy}}(\mathcal{U}) = P_{\mathrm{hy}}(\mathcal{U}_*)$, we have*

$$|\alpha(\mathcal{U})| \le K\big(\bar{\gamma}^2(\mathcal{U})\|\vartheta(\mathcal{U})\|_X + \|\vartheta(\mathcal{U})\|_X^2\big).$$

Proof. It is the same as in [Maeda 2012, Lemma 8], but we give it for completeness. We expand and use that $\mathbb{B}^2 = \mathrm{Id}_2$ and the definition $\mathcal{W}(\gamma) \equiv \mathcal{U}_{c_*+\gamma} + \sigma(\gamma)\mathbb{B}\mathcal{U}_{c_*+\gamma}$ for the second line:

$$\begin{aligned}
P_{\mathrm{hy}}(\mathcal{U}_*) &= P_{\mathrm{hy}}(\mathcal{U}) = P_{\mathrm{hy}}(\mathcal{U}(\cdot - \bar{y}(\mathcal{U}))) = P_{\mathrm{hy}}\big(\mathcal{W}(\bar{\gamma}(\mathcal{U})) + \vartheta(\mathcal{U}) + \alpha(\mathcal{U})\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}\big) \\
&= P_{\mathrm{hy}}(\mathcal{W}(\bar{\gamma}(\mathcal{U}))) + P_{\mathrm{hy}}(\vartheta(\mathcal{U})) + \alpha^2(\mathcal{U})P_{\mathrm{hy}}(\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}) + \alpha(\mathcal{U})(\mathbb{B}\vartheta(\mathcal{U}), \mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})_H \\
&\qquad + (\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}, \vartheta(\mathcal{U}))_H + \alpha(\mathcal{U})(\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}, \mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})_H \\
&\qquad + \sigma(\bar{\gamma}(\mathcal{U}))(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}, \vartheta(\mathcal{U}))_H + \sigma(\bar{\gamma}(\mathcal{U}))\alpha(\mathcal{U})(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}, \mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})_H.
\end{aligned}$$

Since $P_{\mathrm{hy}}(\mathcal{W}(\bar{\gamma}(\mathcal{U}))) = P_{\mathrm{hy}}(\mathcal{U}_*)$, we infer

$$-\alpha(\mathcal{U})[\|\mathcal{U}_*\|_H^2 + o(1)] = \sigma(\bar{\gamma}(\mathcal{U}))(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}, \vartheta(\mathcal{U}))_H + P_{\mathrm{hy}}(\vartheta(\mathcal{U}))$$

and the conclusion follows since $\sigma(\gamma) = \mathcal{O}(\gamma^2)$ by Lemma 4.1. □

Now, we give a lemma useful to estimate $\vartheta(\mathcal{U})$.

Lemma 4.8. *Assume $\varepsilon > 0$ small enough. Then, there exists $K > 0$ such that, for any $\mathcal{U} \in \mathcal{O}_\varepsilon$ satisfying $P_{\mathrm{hy}}(\mathcal{U}) = P_{\mathrm{hy}}(\mathcal{U}_*)$ and $\mathcal{F}_*(\mathcal{U}) - \mathcal{F}_*(\mathcal{U}_*) < 0$, we have*

$$\|\vartheta(\mathcal{U})\|_X^2 \le K|\bar{\gamma}(\mathcal{U})|^3.$$

In particular, $|\alpha(\mathcal{U})| \le K|\bar{\gamma}(\mathcal{U})|^3$.

Proof. It is the same as in [Maeda 2012, Lemma 9]. Note first that the last assertion is a direct consequence of the first one and Lemma 4.7. Next, we argue by contradiction and assume that there exists a sequence $\mathcal{U}_n \to \mathcal{U}_*$ in X such that $\mathcal{F}_*(\mathcal{U}_n) - \mathcal{F}_*(\mathcal{U}_*) < 0$ and $\|\vartheta(\mathcal{U}_n)\|_X^2 \gg |\bar{\gamma}(\mathcal{U}_n)|^3$. For simplicity, we define $\bar{y}_n = \bar{y}(\mathcal{U}_n)$, $\vartheta_n = \vartheta(\mathcal{U}_n)$, $\alpha_n = \alpha(\mathcal{U}_n)$. Then, by Lemma 4.7, we have $|\alpha_n| \le K(\bar{\gamma}_n^2\|\vartheta_n\|_X + \|\vartheta_n\|_X^2) \le K(\|\vartheta_n\|_X^{7/3} + \|\vartheta_n\|_X^2) = \mathcal{O}(\|\vartheta_n\|_X^2)$. Therefore, by the Taylor expansion and Lemma 4.3, we have

$$\begin{aligned}
\mathcal{F}_*(\mathcal{U}_n) - \mathcal{F}(\mathcal{U}_*) &= \mathcal{F}_*(\mathcal{U}_n(\cdot - \bar{y}_n)) - \mathcal{F}(\mathcal{U}_*) = \mathcal{F}_*(\mathcal{W}(\bar{\gamma}_n) + \vartheta_n + \alpha_n\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}_n}) - \mathcal{F}(\mathcal{U}_*) \\
&= \mathcal{F}_*(\mathcal{W}(\bar{\gamma}_n)) - \mathcal{F}(\mathcal{U}_*) + \langle \mathcal{F}'_*(\mathcal{W}(\bar{\gamma}_n)), \vartheta_n + \alpha_n\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}_n}\rangle_{X^*,X} \\
&\qquad + \tfrac{1}{2}\langle \mathcal{F}''_*(\mathcal{W}(\bar{\gamma}_n))\vartheta_n, \vartheta_n\rangle_{X^*,X} + o(\|\vartheta(\mathcal{U}_n)\|_X^2).
\end{aligned} \tag{24}$$

However, by Lemma 4.2, $\mathcal{F}_{c_*}(\mathcal{W}(\gamma)) - \mathcal{F}_{c_*}(\mathcal{U}_{c_*}) = \mathcal{O}(|\gamma|^3)$, and, since $\mathcal{F}'_*(\mathcal{W}(\gamma)) = \mathcal{F}'_*(\mathcal{W}(0)) + o(1) = \mathcal{F}'_*(\mathcal{U}_*) + o(1) = o(1)$, we have $\langle \mathcal{F}'_*(\mathcal{W}(\bar{\gamma}_n)), \alpha_n\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}_n}\rangle_{X^*,X} = o(\|\vartheta_n\|_X^2)$. Furthermore, using $\mathcal{F}'_* = \mathcal{F}'_{c_*+\bar{\gamma}_n} + \bar{\gamma}_n\mathbb{B}$, the third orthogonality condition in Lemma 4.3 and that $\sigma(\gamma) = \mathcal{O}(\gamma^2)$, we deduce

$$\begin{aligned}
\langle \mathcal{F}'_*(\mathcal{W}(\bar{\gamma}_n)), \vartheta_n\rangle_{X^*,X} &= \langle \mathcal{F}'_{c_*+\bar{\gamma}_n}(\mathcal{W}(\bar{\gamma}_n)), \vartheta_n\rangle_{X^*,X} + \bar{\gamma}_n(\mathbb{B}\mathcal{W}(\bar{\gamma}_n), \vartheta_n)_H \\
&= \langle \mathcal{F}'_{c_*+\bar{\gamma}_n}(\mathcal{U}_{c_*+\bar{\gamma}_n}) + \sigma(\bar{\gamma}_n)\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}_n}, \vartheta_n\rangle_{X^*,X} + \bar{\gamma}_n\sigma(\bar{\gamma}_n)(\mathcal{U}_{c_*+\bar{\gamma}_n}, \vartheta_n)_H \\
&= \mathcal{O}(\bar{\gamma}_n^2\|\vartheta_n\|_X) + \mathcal{O}(|\bar{\gamma}_n|^3\|\vartheta_n\|_X) = o(\|\vartheta_n\|_X^{7/3}) = o(\|\vartheta_n\|_X^2).
\end{aligned}$$

For the last line, we have used another Taylor expansion with $\mathscr{F}'_{c_*+\bar{\gamma}_n}(\mathscr{U}_{c_*+\bar{\gamma}_n}) = 0$. Finally, Lemma 4.5 yields $\langle \mathscr{F}''_{c_*+\bar{\gamma}_n}(\mathscr{U}_{c_*+\bar{\gamma}_n})\vartheta_n, \vartheta_n\rangle_{X^*,X} \geq K_0\|\vartheta_n\|_X^2$. Reporting these expansions in (24) yields

$$\mathscr{F}_*(\mathscr{U}_n) - \mathscr{F}(\mathscr{U}_*) \geq \frac{K_0}{4}\|\vartheta_n\|_X^2 + o(\|\vartheta_n\|_X^2) \geq \frac{K_0}{8}\|\vartheta_n\|_X^2$$

for n sufficiently large, which contradicts our assumption. □

We now need to find an extension of the functionals "A" and "P" used in [Maeda 2012] (and also in [Ohta 2011]). In these works, these functionals are built on what should be here "$J^{-1}\partial_c\mathscr{U}_c = \mathbb{B}\partial_x^{-1}\partial_c\mathscr{U}_c$", but, unfortunately, $\partial_c\phi_c$ does not have vanishing integral over $\mathbb{R}$ (for instance, $\partial_c\eta_c$ has constant sign). We rely instead on a construction of a suitable approximation of "$J^{-1}\partial_c\mathscr{U}_c$". A similar construction is used in [Lin 2002].

Lemma 4.9. *For any $0 < \kappa < 1$, there exists a $\mathscr{C}^2$ mapping $\Upsilon_\kappa : (-\gamma_1, +\gamma_1) \to X$ such that, for any $\gamma \in (-\gamma_1, +\gamma_1)$, $\Upsilon_\kappa(\gamma) \in H^2 \times H^1$ is an odd function verifying*

$$\|J\Upsilon_\kappa(\gamma) - [\partial_c\mathscr{U}_c]_{|c=c_*+\gamma}\|_X \leq \kappa.$$

Proof. We fix an even function $\Theta_0 \in \mathscr{C}_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}}\Theta_0\,dx = 1$. For $T > 0$ to be fixed later, but independent of γ and κ, we set $t_\kappa \equiv T/\kappa^2 > 0$ and

$$\Upsilon_\kappa(\gamma)(x) \equiv \mathbb{B}\int_0^x\Big[[\partial_c\mathscr{U}_c]_{|c=c_*+\gamma}(y) - \frac{1}{t_\kappa}\Theta_0\Big(\frac{y}{t_\kappa}\Big)\int_{\mathbb{R}}[\partial_c\mathscr{U}_c]_{|c=c_*+\gamma}(z)\,dz\Big]\,dy.$$

It is clear that $\Upsilon_\kappa(\gamma) \in \mathscr{C}^1(\mathbb{R})$ and that, since $J = \partial_x\mathbb{B}$ and $\mathbb{B}^2 = \mathrm{Id}_2$,

$$J\Upsilon_\kappa(\gamma) - [\partial_c\mathscr{U}_c]_{|c=c_*+\gamma} = \frac{1}{t_\kappa}\Theta_0\Big(\frac{\cdot}{t_\kappa}\Big)\int_{\mathbb{R}}[\partial_c\mathscr{U}_c]_{|c=c_*+\gamma}(z)\,dz.$$

In particular,

$$\|J\Upsilon_\kappa(\gamma) - \partial_c[\mathscr{U}_c]_{|c=c_*+\gamma}\|_X^2 = \Big[\frac{1}{t_\kappa}\|\Theta_0\|_{L^2}^2 + \frac{1}{t_\kappa^3}\|\partial_x\Theta_0\|_{L^2}^2\Big]\Big(\int_{\mathbb{R}}[\partial_c\mathscr{U}_c]_{|c=c_*+\gamma}(z)\,dz\Big)^2 \leq \kappa^2$$

if we choose $T = T(c_*, \mathscr{U}_*, \Theta_0) > 0$ sufficiently large and γ_1 smaller if necessary. Moreover, $\Upsilon_\kappa(\gamma)$ is odd since $\mathscr{U}_c$ and Θ_0 are even. In addition, the even function

$$y \mapsto [\partial_c\mathscr{U}_c]_{|c=c_*+\gamma}(y) - \frac{1}{t_\kappa}\Theta_0\Big(\frac{y}{t_\kappa}\Big)\int_{\mathbb{R}}[\partial_c\mathscr{U}_c]_{|c=c_*+\gamma}(z)\,dz$$

decays exponentially at infinity (since Θ_0 has compact support and $\partial_c\mathscr{U}_c$ decays exponentially), and has zero integral (since Θ_0 has integral equal to one); hence

$$\Upsilon_\kappa(\gamma)(x) = -\mathbb{B}\int_x^{+\infty}\Big[[\partial_c\mathscr{U}_c]_{|c=c_*+\gamma}(y) - \frac{1}{t_\kappa}\Theta_0\Big(\frac{y}{t_\kappa}\Big)\int_{\mathbb{R}}[\partial_c\mathscr{U}_c]_{|c=c_*+\gamma}(z)\,dz\Big]\,dy$$

and decays exponentially at infinity. It follows easily from these two equalities that $\gamma \mapsto \Upsilon_\kappa(\gamma) \in L^2 \times L^2$ is well-defined and continuous; hence also $\gamma \mapsto \Upsilon_\kappa(\gamma) \in H^2 \times H^1$. By the same type of arguments,

$$\frac{\partial \Upsilon_\kappa}{\partial \gamma}(\gamma)(x) = \mathbb{B} \int_0^x \left[[\partial_c^2 \mathcal{U}_c]_{|c=c_*+\gamma}(y) - \frac{1}{t_\kappa} \Theta_0\left(\frac{y}{t_\kappa}\right) \int_{\mathbb{R}} [\partial_c^2 \mathcal{U}_c]_{|c=c_*+\gamma}(z)\, dz \right] dy$$

is well-defined and is a continuous function of γ with values in $H^2 \times H^1$, and similarly for the second derivative. □

We now define, in the tubular neighborhood $\mathcal{O}_\varepsilon$ of $\mathcal{U}_*$, the functional (corresponding to "A" in [Maeda 2012])

$$\Omega_\kappa(\mathcal{U}) \equiv (\mathcal{U}(\cdot - \bar{y}(\mathcal{U})), \Upsilon_\kappa(\bar{\gamma}(\mathcal{U})))_H = (\mathcal{U}, \Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})))_H$$

depending on $\kappa \in (0,1)$, which will be determined later. The first properties of Ω_κ are given below.

Lemma 4.10. *For any $0 < \kappa < 1$, $\Omega_\kappa : \mathcal{O}_\varepsilon \to \mathbb{R}$ is of class $\mathscr{C}^1$. In addition, there exists some bounded mapping $\mathcal{N}_{\bar{\gamma}} : \mathcal{O}_\varepsilon \to X$ such that, if $\Psi_{\rm hy} \in \mathscr{C}^1([0,T), X)$ is a solution to (15) that remains in $\mathcal{O}_\varepsilon$, then*

$$\frac{d}{dt}\Omega_\kappa(\Psi_{\rm hy}(t)) = \Xi_\kappa(\Psi_{\rm hy}(t)),$$

where $\Xi_\kappa : \mathcal{O}_\varepsilon \to \mathbb{R}$ is defined by

$$\Xi_\kappa(\mathcal{U}) \equiv -\big\langle \mathcal{F}'_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{U}), \big\{ J\Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})) + (\mathcal{U}, \partial_\gamma \Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})))_H \mathcal{N}_{\bar{\gamma}}(\mathcal{U}) \big\} \big\rangle_{X^*,X}.$$

Proof. The fact that Ω_κ is of class $\mathscr{C}^1$ follows directly from the second expression and the fact that $\bar{y}$ and $\bar{\gamma}$ are $\mathscr{C}^1$ (in [Maeda 2012, formula (3.11)], the same remark as for the smoothness of G after Lemma 4.3 holds, since it requires "$u \in D(T'(0))$"). If $\Psi_{\rm hy} = (\eta, u) \in \mathscr{C}^1([0,T), X)$ is a solution to (15) that remains in $\mathcal{O}_\varepsilon$, we therefore have, defining $\bar{\gamma}(t) = \bar{\gamma}(\Psi_{\rm hy}(t))$ and $\bar{y}(t) = \bar{y}(\Psi_{\rm hy}(t))$,

$$\begin{aligned} \frac{d}{dt}\Omega_\kappa(\Psi_{\rm hy}(t)) &= (\partial_t \Psi_{\rm hy}(t), \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)))_H \\ &\quad + (\Psi_{\rm hy}(t), \partial_x \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)))_H \langle \bar{y}'(\Psi_{\rm hy}(t)), \partial_t \Psi_{\rm hy}(t)\rangle_{X^*,X} \\ &\quad + (\Psi_{\rm hy}(t), \partial_\gamma \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)))_H \langle \bar{\gamma}'(\Psi_{\rm hy}(t)), \partial_t \Psi_{\rm hy}(t)\rangle_{X^*,X}. \end{aligned} \tag{25}$$

We now observe that the invariance of Ω_κ by translation provides by differentiation the equality, for $\mathcal{U} \in \mathcal{O}_\varepsilon$,

$$\begin{aligned} 0 = \frac{d}{dy}\Omega_\kappa(\mathcal{U}(\cdot - y))_{|y=0} &= (\mathcal{U}, \partial_x \Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})))_H \\ &= (\mathbb{B}\mathcal{U}, J\Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})))_H = \langle P'_{\rm hy}(\mathcal{U}), J\Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U}))\rangle_{X^*,X}. \end{aligned} \tag{26}$$

In particular, the second term in (25) vanishes. In addition, since $\Psi_{\rm hy} = (\eta, u) \in \mathscr{C}^1([0,T), X)$ is a solution to (15) that remains in $\mathbb{O}_\varepsilon$, we have, denoting by $\delta E_{\rm hy}/\delta\Psi$ the variational derivative,

$$\begin{aligned}
&\big(\partial_t \Psi_{\rm hy}(t), \Upsilon_\kappa(\bar\gamma(t))(\cdot + \bar y(t))\big)_H \\
&= \left(J\frac{\delta E_{\rm hy}}{\delta \Psi_{\rm hy}}(\Psi_{\rm hy}(t)), \Upsilon_\kappa(\bar\gamma(t))(\cdot + \bar y(t))\right)_H \\
&= -\left(\frac{\delta E_{\rm hy}}{\delta \Psi_{\rm hy}}(\Psi_{\rm hy}(t)), J\Upsilon_\kappa(\bar\gamma(t))(\cdot + \bar y(t))\right)_H = -\big\langle E'_{\rm hy}(\Psi_{\rm hy}(t)), J\Upsilon_\kappa(\bar\gamma(t))(\cdot + \bar y(t))\big\rangle_{X^*,X} \\
&= -\langle \mathscr{F}'_{c_*+\bar\gamma(t)}(\Psi_{\rm hy}(t)), J\Upsilon_\kappa(\bar\gamma(t))(\cdot+\bar y(t))\rangle_{X^*,X} - (c_*+\bar\gamma(t))\big\langle P'_{\rm hy}(\Psi_{\rm hy}(t)), J\Upsilon_\kappa(\bar\gamma(t))(\cdot+\bar y(t))\big\rangle_{X^*,X} \\
&= -\langle \mathscr{F}'_{c_*+\bar\gamma(t)}(\Psi_{\rm hy}(t)), J\Upsilon_\kappa(\bar\gamma(t))(\cdot + \bar y(t))\rangle_{X^*,X},
\end{aligned}$$

by (26). In addition, since $\delta P_{\rm hy}/\delta\Psi_{\rm hy} = \mathbb{B}\Psi_{\rm hy}$ and $J\mathbb{B} = \partial_x$,

$$\begin{aligned}
&\langle \bar\gamma'(\Psi_{\rm hy}(t)), \partial_t\Psi_{\rm hy}(t)\rangle_{X^*,X} \\
&\qquad = \left\langle \bar\gamma'(\Psi_{\rm hy}(t)), J\frac{\delta E_{\rm hy}}{\delta\Psi_{\rm hy}}(\Psi_{\rm hy}(t))\right\rangle_{X^*,X} \\
&\qquad = \left\langle \bar\gamma'(\Psi_{\rm hy}(t)), J\frac{\delta \mathscr{F}_{c_*+\bar\gamma(\mathscr{U})}}{\delta\Psi_{\rm hy}}(\Psi_h y(t))\right\rangle_{X^*,X} + (c_* + \bar\gamma(\mathscr{U}))\langle\bar\gamma'(\Psi_{\rm hy}(t)), \partial_x\Psi_{\rm hy}(t)\rangle_{X^*,X}.
\end{aligned}$$

The second term vanishes since $\bar\gamma$ is invariant by translation (by definition; see the proof of Lemma 4.3). As a consequence,

$$\begin{aligned}
&\langle \bar\gamma'(\Psi_{\rm hy}(t)), \partial_t\Psi_{\rm hy}(t)\rangle_{X^*,X} \\
&\quad = \left(J\frac{\delta\mathscr{F}_{c_*+\bar\gamma(t)}}{\delta\Psi_{\rm hy}}(\Psi_{\rm hy}(t)), \mathbb{I}^{-1}\bar\gamma'(\Psi_{\rm hy}(t))\right)_X = -\left(\frac{\delta\mathscr{F}_{c_*+\bar\gamma(t)}}{\delta\Psi_{\rm hy}}(\Psi_{\rm hy}(t)), J\mathbb{I}^{-1}\bar\gamma'(\Psi_{\rm hy}(t))\right)_X \\
&\quad = -\left(\frac{\delta\mathscr{F}_{c_*+\bar\gamma(t)}}{\delta\Psi_{\rm hy}}(\Psi_{\rm hy}(t)), J\mathbb{I}^{-1}\bar\gamma'(\Psi_{\rm hy}(t))\right)_H - \left(\partial_x\frac{\delta\mathscr{F}_{c_*+\bar\gamma(t)}}{\delta\eta}(\Psi_{\rm hy}(t)), \partial_x J\mathbb{I}_{H^1}^{-1}\frac{\partial\bar\gamma}{\partial\eta}(\Psi_{\rm hy}(t))\right)_{L^2}.
\end{aligned}$$

The first term is simply $-\langle\mathscr{F}'_{c_*+\bar\gamma(\mathscr{U})}(\Psi_{\rm hy}(t)), J\mathbb{I}^{-1}\bar\gamma'(\Psi_{\rm hy}(t))\rangle_{X^*,X}$. We then define $\mathscr{N}_{\bar\gamma} : \mathbb{O}_\varepsilon \to X$ by $\mathscr{N}_{\bar\gamma}(\mathscr{U}) \equiv J\mathbb{I}^{-1}\bar\gamma'(\mathscr{U}) - (\partial_x^2 J\mathbb{I}_{H^1}^{-1}(\delta\bar\gamma/\delta\eta)(\mathscr{U}), 0) \in X = H^1 \times L^2$ (see the regularity shown for $\bar\gamma'$ in Lemma 4.3), so that integration by parts yields

$$\langle\bar\gamma'(\Psi_{\rm hy}(t)), \partial_t\Psi_{\rm hy}(t)\rangle_{X^*,X} = -\langle\mathscr{F}'_{c_*+\bar\gamma(\mathscr{U})}(\Psi_{\rm hy}(t)), \mathscr{N}_{\bar\gamma}(\Psi_{\rm hy}(t))\rangle_{X^*,X}.$$

Inserting these relations into (25) then gives

$$\begin{aligned}
&\frac{d}{dt}\Omega_\kappa(\Psi_{\rm hy}(t)) \\
&= -\big\langle\mathscr{F}'_{c_*+\bar\gamma(t)}(\Psi_{\rm hy}(t)), \big\{J\Upsilon_\kappa(\bar\gamma(t))(\cdot + \bar y(t)) + (\Psi_{\rm hy}(t), \partial_\gamma\Upsilon_\kappa(\bar\gamma(t))(\cdot + \bar y(t)))_H\mathscr{N}_{\bar\gamma}(\Psi_{\rm hy}(t))\big\}\big\rangle_{X^*,X},
\end{aligned}$$

which is the desired equality.

If $\Psi_{\mathrm{hy}} \in \mathscr{C}^0([0,T), X)$ is just a continuous in time solution to (15) that remains in $\mathbb{O}_\varepsilon$, then the integrated relation

$$\Omega_\kappa(\Psi_{\mathrm{hy}}(t)) = \Omega_\kappa(\Psi_{\mathrm{hy}}^{\mathrm{in}}) + \int_0^t \Xi_\kappa(\Psi_{\mathrm{hy}}(\tau))\, d\tau$$

holds, as can be seen by using the continuity of the flow and the approximation of such a solution by smoother ones (see [Gallo 2004]). ☐

We now compute the asymptotics of $\Xi_\kappa(\mathscr{W}(\gamma))$ for $\gamma \to 0$ and small κ.

Lemma 4.11. *We have*

$$\Xi_\kappa(\mathscr{W}(\gamma)) = -\frac{\gamma^2 \ddot{P}_*}{2} + o_{(\gamma,\kappa)\to(0,0)}(\gamma^2).$$

Proof. The proof follows the one of Lemma 5 in [Maeda 2012]. As a first step, notice that $\bar{\gamma}(\mathscr{W}(\gamma)) = \gamma$, $\bar{y}(\mathscr{W}(\gamma)) = 0$, as can be seen from the equality $G(\mathscr{W}(\gamma), 0, 0, 0) = 0$ and the local uniqueness of the solution to $G = 0$. Therefore, since $\mathscr{F}'_{c_*+\gamma}(\mathscr{U}_{c_*+\gamma}) = 0$ and $\sigma(\gamma) \sim -\gamma^2 \ddot{P}_*/(2\|\mathscr{U}_*\|_H^2)$,

$$\begin{aligned}\mathscr{F}'_{c_*+\bar{\gamma}(\mathscr{W}(\gamma))}(\mathscr{W}(\gamma)) = \mathscr{F}'_{c_*+\gamma}(\mathscr{U}_{c_*+\gamma} + \sigma(\gamma)\mathbb{B}\mathscr{U}_{c_*+\gamma}) &= \sigma(\gamma)\mathscr{F}''_{c_*+\gamma}(\mathscr{U}_{c_*+\gamma})[\mathbb{B}\mathscr{U}_{c_*+\gamma}] + o_{\gamma\to 0}(\gamma^2)\\ &= -\frac{\gamma^2 \ddot{P}_*}{2\|\mathscr{U}_*\|_H^2}\mathscr{F}''_*(\mathscr{U}_*)[\mathbb{B}\mathscr{U}_*] + o_{\gamma\to 0}(\gamma^2).\end{aligned}$$

In addition, since $\mathscr{U}_c$ is even and $\Upsilon_\kappa(\gamma)$ is odd, we deduce

$$\big(\mathscr{W}(\gamma)(\cdot + \bar{y}(\mathscr{W}(\gamma))), \partial_\gamma \Upsilon_\kappa(\gamma)\big)_H = \big(\mathscr{U}_{c_*+\gamma} + \sigma(\gamma)\mathbb{B}\mathscr{U}_{c_*+\gamma}, \partial_\gamma \Upsilon_\kappa(\gamma)\big)_H = 0.$$

Consequently,

$$\Xi_\kappa(\mathscr{W}(\gamma)) = \frac{\gamma^2 \ddot{P}_*}{2\|\mathscr{U}_*\|_H^2}\langle \mathscr{F}''_*(\mathscr{U}_*)[\mathbb{B}\mathscr{U}_*], J\Upsilon_\kappa(\gamma)\rangle_{X^*,X} + o_{\gamma\to 0}(\gamma^2),$$

where "$o_{\gamma\to 0}(\gamma^2)$" does not depend on κ. Moreover, by Lemma 4.9, $\|J\Upsilon_\kappa(\gamma) - [\partial_c \mathscr{U}_c]_{|c=c_*+\gamma}\|_X \le \kappa$ independently of $\gamma \in (-\gamma_1, +\gamma_1)$; hence

$$\begin{aligned}\Xi_\kappa(\mathscr{W}(\gamma)) &= \frac{\gamma^2 \ddot{P}_*}{2\|\mathscr{U}_*\|_H^2}\langle \mathscr{F}''_*(\mathscr{U}_*)[\mathbb{B}\mathscr{U}_*], [\partial_c \mathscr{U}_c]_{|c=c_*+\gamma}\rangle_{X^*,X} + o_{(\gamma,\kappa)\to(0,0)}(\gamma^2)\\ &= \frac{\gamma^2 \ddot{P}_*}{2\|\mathscr{U}_*\|_H^2}\langle \mathscr{F}''_*(\mathscr{U}_*)[\mathbb{B}\mathscr{U}_*], [\partial_c \mathscr{U}_c]_{|c=c_*}\rangle_{X^*,X} + o_{(\gamma,\kappa)\to(0,0)}(\gamma^2).\end{aligned}$$

Finally, using once again the equality (for $\phi \in X$) $\langle \mathscr{F}''_*(\mathscr{U}_*)[\partial_c \mathscr{U}_c]_{|c=c_*}, \phi\rangle_{X^*,X} = (\mathbb{B}\mathscr{U}_*, \phi)_H$ and that $\mathscr{F}''_*$ is self-adjoint, we infer

$$\langle \mathscr{F}''_*(\mathscr{U}_*)[\mathbb{B}\mathscr{U}_*], [\partial_c \mathscr{U}_c]_{|c=c_*}\rangle_{X^*,X} = \langle \mathscr{F}''_*(\mathscr{U}_*)([\partial_c \mathscr{U}_c]_{|c=c_*}), \mathbb{B}\mathscr{U}_*\rangle_{X^*,X} = \|\mathbb{B}\mathscr{U}_*\|_H^2 = \|\mathscr{U}_*\|_H^2,$$

and reporting this into the previous expression gives the result. ☐

We now compute the asymptotics of Ξ_κ for more general functions.

Lemma 4.12. *Let $\varepsilon > 0$ be small enough. If $\mathcal{U} \in \mathcal{O}_\varepsilon$ satisfies $P_{\mathrm{hy}}(\mathcal{U}) = P_{\mathrm{hy}}(\mathcal{U}_*)$ and $\mathcal{F}_*(\mathcal{U}) - \mathcal{F}_*(\mathcal{U}_*) < 0$, then, we have*

$$\Xi_\kappa(\mathcal{U}) = -\frac{\bar{\gamma}^2(\mathcal{U})\ddot{P}_*}{2} + o(\bar{\gamma}^2(\mathcal{U}))$$

uniformly for $0 < \kappa \leq |\bar{\gamma}(\mathcal{U})|^3$.

Proof. First, we may apply Lemma 4.8 and infer that $\|\vartheta(\mathcal{U})\|_X^2 + |\alpha(\mathcal{U})| = \mathcal{O}(|\bar{\gamma}^3(\mathcal{U})|)$. We write $\Xi_\kappa(\mathcal{U}) = \Xi_\kappa(\mathcal{U}(\cdot - \bar{y}(\mathcal{U}))) = \Xi_\kappa(\mathcal{W}(\bar{\gamma}(\mathcal{U})) + \vartheta(\mathcal{U}) + \alpha(\mathcal{U})\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}) = \Xi_\kappa(\mathcal{W}(\bar{\gamma}(\mathcal{U})) + \vartheta(\mathcal{U})) + \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3)$ and, recalling the expression

$$\Xi_\kappa(\mathcal{U}) = -\big\langle \mathcal{F}'_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{U}), \big\{ J\Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})) + (\mathcal{U}, \partial_\gamma \Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})))_H \mathcal{N}_{\bar{\gamma}}(\mathcal{U})\big\}\big\rangle_{X^*,X},$$

we wish to make a Taylor expansion. First, note that

$$\begin{aligned}\mathcal{F}'_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{U}) &= \mathcal{F}'_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{W}(\bar{\gamma}(\mathcal{U}))) + \mathcal{F}''_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{W}(\bar{\gamma}(\mathcal{U})))[\vartheta(\mathcal{U})] + \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3)\\ &= \mathcal{F}'_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{W}(\bar{\gamma}(\mathcal{U}))) + \mathcal{F}''_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})[\vartheta(\mathcal{U})] + \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3);\end{aligned}$$

hence, since $\mathcal{F}'_{c_*+\bar{\gamma}}(\mathcal{U}_{c_*+\bar{\gamma}}) = 0$ and (Lemma 4.1) $\sigma(\gamma) = \mathcal{O}(\gamma^2)$, we have $\mathcal{W}(\bar{\gamma}) = \mathcal{U}_{c_*+\bar{\gamma}} + \mathcal{O}(\gamma^2)$; thus

$$\begin{aligned}\Xi_\kappa(\mathcal{U}) - \Xi_\kappa(\mathcal{W}(\bar{\gamma}(\mathcal{U}))) = \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3) - \Big\langle &\mathcal{F}''_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})[\vartheta(\mathcal{U})],\\ &\big\{ J\Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})) + (\mathcal{U}, \partial_\gamma \Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})))_H \mathcal{N}_{\bar{\gamma}}(\mathcal{U})\big\}\Big\rangle_{X^*,X}.\end{aligned}$$

Now, in the bracket term, we may replace $\mathcal{U}$ by $\mathcal{W}(\bar{\gamma}(\mathcal{U})) + \mathcal{O}(\|\vartheta(\mathcal{U})\|_X)$ (since $\|\vartheta(\mathcal{U})\|_X^2 = \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3)$). By the computations of Lemma 4.11 and the equalities $\bar{\gamma}(\mathcal{W}(\gamma)) = \gamma$, $\bar{y}(\mathcal{W}(\gamma)) = 0$, this gives

$$\begin{aligned}\Xi_\kappa(\mathcal{U}) - \Xi_\kappa(\mathcal{W}(\bar{\gamma}(\mathcal{U}))) &= \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3) - \big\langle \mathcal{F}''_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})[\vartheta(\mathcal{U})], J\Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))\big\rangle_{X^*,X}\\ &= \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3) - \big\langle \mathcal{F}''_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})[\vartheta(\mathcal{U})], \{\partial_c \mathcal{U}_c\}_{|c=c_*+\bar{\gamma}(\mathcal{U})}\big\rangle_{X^*,X} + \mathcal{O}(\kappa)\\ &= \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3) - \big\langle \mathcal{F}''_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})[\{\partial_c \mathcal{U}_c\}_{|c=c_*+\bar{\gamma}(\mathcal{U})}], \vartheta(\mathcal{U})\big\rangle_{X^*,X} + \mathcal{O}(\kappa)\end{aligned}$$

using Lemma 4.9 and the self-adjointness of $\mathcal{F}''_{c_*+\bar{\gamma}(\mathcal{U})}$. Choosing $0 < \kappa \leq |\bar{\gamma}(\mathcal{U})|^3$ and from the equality (for $\phi \in X$) $\langle \mathcal{F}''_c(\mathcal{U}_c)[\partial_c \mathcal{U}_c], \phi\rangle_{X^*,X} = (\mathbb{B}\mathcal{U}_c, \phi)_H$, we infer

$$\Xi_\kappa(\mathcal{U}) - \Xi_\kappa(\mathcal{W}(\bar{\gamma}(\mathcal{U}))) = \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3) - (\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}, \vartheta(\mathcal{U}))_H = \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3),$$

by the orthogonality condition in Lemma 4.5. Inserting the expansion of $\Xi_\kappa(\mathcal{W}(\gamma))$ given in Lemma 4.11 yields the conclusion. □

Proof of Theorem 16. We have to show that there exists $\varepsilon > 0$ such that, for any $\delta > 0$, we can choose an initial datum at distance $\leq \delta$ from $\mathcal{U}_*$ but that escapes from $\mathcal{O}_\varepsilon$. Since $\mathcal{W}(\gamma) \to \mathcal{U}_*$ in X, we shall take the initial datum to be $\mathcal{W}(\gamma)$ for some small γ, and denote by $\Psi_{\mathrm{hy}}(t)$ the corresponding solution. In view of Lemma 4.2, we have $\mathcal{F}_*(\mathcal{W}(\gamma)) - \mathcal{F}_*(\mathcal{U}_*) \sim -\gamma^3 \ddot{P}_*/6$; hence we can choose γ with the sign of $\ddot{P}_* \neq 0$ so that

$$\mathcal{F}_*(\mathcal{W}(\gamma)) - \mathcal{F}_*(\mathcal{U}_*) \sim -|\gamma|^3 |\ddot{P}_*|/6 < 0.$$

We now assume that $\Psi_{\rm hy}(t)$ is globally defined and remains in $\mathcal{O}_\varepsilon$, where ε is as in Lemma 4.8. By conservation of energy and momentum and the construction of $\mathcal{W}(\gamma)$, we deduce $P_{\rm hy}(\Psi_{\rm hy}(t)) = P_{\rm hy}(\mathcal{W}(\gamma)) = P_{\rm hy}(\mathcal{U}_*)$, and $\mathcal{F}_*(\Psi_{\rm hy}(t)) - \mathcal{F}_*(\mathcal{U}_*) = \mathcal{F}_*(\mathcal{W}(\gamma)) - \mathcal{F}_*(\mathcal{U}_*) < 0$. The first step is to have a control on $\bar\gamma(t) \equiv \bar\gamma(\Psi_{\rm hy}(t))$. We define $\alpha(t) = \alpha(\Psi_{\rm hy}(t))$, $\bar\gamma(t) = \bar\gamma(\Psi_{\rm hy}(t))$ and $\vartheta(t) = \vartheta(\Psi_{\rm hy}(t))$. Applying Lemma 4.8, we obtain $\|\vartheta(t)\|_X^2 + |\alpha(t)| = \mathcal{O}(|\bar\gamma^3(t)|)$. In addition, Lemma 4.2 and a Taylor expansion give

$$\begin{aligned}
&\mathcal{F}_*(\Psi_{\rm hy}(t)) - \mathcal{F}_*(\mathcal{U}_*)\\
&= \mathcal{F}_*(\mathcal{W}(\bar\gamma(t)) + \vartheta(t) + \alpha(t)\mathbb{B}\mathcal{U}_{c_*+\bar\gamma(t)}) - \mathcal{F}_*(\mathcal{U}_*)\\
&= \mathcal{F}_*(\mathcal{W}(\bar\gamma(t))) - \mathcal{F}_*(\mathcal{U}_*) + \langle \mathcal{F}_*'(\mathcal{W}(\bar\gamma(t))), \vartheta(t)\rangle_{X^*,X} + \tfrac12 \langle \mathcal{F}_*''(\mathcal{W}(\bar\gamma(t)))\vartheta(t), \vartheta(t)\rangle_{X^*,X} + o(|\bar\gamma^3(t)|)\\
&= -\frac{\bar\gamma^3(t)\ddot P_*}{6} + \langle \mathcal{F}_*'(\mathcal{W}(\bar\gamma(t))), \vartheta(t)\rangle_{X^*,X} + \tfrac12 \langle \mathcal{F}_*''(\mathcal{W}(\bar\gamma(t)))\vartheta(t), \vartheta(t)\rangle_{X^*,X} + o(|\bar\gamma^3(t)|),
\end{aligned}$$

where we have used that $\mathcal{F}_*'(\mathcal{W}(\bar\gamma(t)) = o(1)$ (for the terms involving $\alpha(t)$) and Lemma 4.2. Furthermore, by the orthogonality relations in Lemma 4.3 and using that $\sigma(\gamma) = \mathcal{O}(\gamma^2)$ and $\mathcal{F}_c'(\mathcal{U}_c) = 0$, we have

$$\begin{aligned}
\langle \mathcal{F}_*'(\mathcal{W}(\bar\gamma(t))), \vartheta(t)\rangle_{X^*,X} &= \langle \mathcal{F}'_{c_*+\bar\gamma(t)}(\mathcal{W}(\bar\gamma(t))), \vartheta(t)\rangle_{X^*,X} + \bar\gamma(t)(\mathbb{B}\mathcal{W}(\bar\gamma(t))), \vartheta(t))_H\\
&= \langle \mathcal{F}'_{c_*+\bar\gamma(t)}(\mathcal{U}_{c_*+\bar\gamma(t)} + \sigma(\bar\gamma(t)))\mathbb{B}\mathcal{U}_{c_*+\bar\gamma(t)}, \vartheta(t)\rangle_{X^*,X} = \mathcal{O}(|\bar\gamma^{7/2}(t)|).
\end{aligned}$$

In addition, by Lemma 4.5, the second-to-last term is $\geq K_0\|\vartheta(t)\|_X^2/2$. As a consequence, by conservation of $\mathcal{F}_*(\Psi_{\rm hy}(t))$, we infer, for small γ,

$$0 > -|\gamma|^3|\ddot P_*|/3 > \mathcal{F}_*(\mathcal{W}(\gamma)) - \mathcal{F}_*(\mathcal{U}_*) = \mathcal{F}_*(\Psi_{\rm hy}(t)) - \mathcal{F}_*(\mathcal{U}_*) \geq -\frac{\bar\gamma^3(t)\ddot P_*}{6} + o(|\bar\gamma^3(t)|),$$

In particular, this forces $\bar\gamma(t)$ to always be of the sign of $\ddot P_*$ and to satisfy $|\bar\gamma(t)| \geq |\gamma|/2$ (provided ε and γ are small enough).

Since, now, we have a good upper bound for $|\bar\gamma(t)|$, we can choose $\kappa = \kappa(\gamma) \equiv \gamma^3/8$, which is such that, for any $t \geq 0$, $\kappa \leq |\bar\gamma(t)|^3$. In particular, we can apply Lemma 4.12 and get

$$\Xi_\kappa(\Psi_{\rm hy}(t)) = -\frac{\bar\gamma(t)^2\ddot P_*}{2} + o(\bar\gamma(t)^2).$$

With this choice $\kappa = \kappa(\gamma)$, we deduce from Lemma 4.10 that

$$\frac{d}{dt}\Omega_{\kappa(\gamma)}(\Psi_{\rm hy}(t)) = \Xi_\kappa(\Psi_{\rm hy}(t)) = -\frac{\bar\gamma(t)^2\ddot P_*}{2} + o(\bar\gamma(t)^2).$$

Since $|\bar\gamma(t)| \geq |\gamma|/2$, it follows that, when $\ddot P_* < 0$ (the case $\ddot P_* > 0$ is analogous),

$$\frac{d}{dt}\Omega_{\kappa(\gamma)}(\Psi_{\rm hy}(t)) \geq -\frac{\gamma^2\ddot P_*}{8} > 0;$$

hence $\Omega_{\kappa(\gamma)}(\Psi_{\rm hy}(t))$ is unbounded as t goes to $+\infty$. However, by definition of Ω_κ, we have by the Cauchy–Schwarz inequality $|\Omega_{\kappa(\gamma)}(\mathcal{U})| \leq \|\mathcal{U}\|_H\|\Upsilon_{\kappa(\gamma)}\|_H \leq C(\gamma)$ for $\mathcal{U} \in \mathcal{O}_\varepsilon$. We have reached a contradiction. The proof of Theorem 16 is complete. $\square$

5. The linear instability ($0 < c_* < c_s$)

5A. ***Proof of Theorem 13.*** **Existence of at least one unstable eigenvalue.** The proof of the existence of at least one unstable eigenvalue relies on the Evans function technique, as in [Zumbrun 2008; Benzoni-Gavage 2010b]. We shall actually use Theorem 1 in [Benzoni-Gavage 2010b] when observing (see, e.g., [Benzoni-Gavage 2010a]) that the Euler–Korteweg system

$$\begin{cases} \partial_t \rho + 2\partial_x(\rho u) = 0, \\ \partial_t u + 2u\partial_x u - \partial_x(f(\rho)) - \partial_x\big(K(\rho)\partial_x^2\rho + \frac{1}{2}K'(\rho)(\partial_x\rho)^2\big) = 0, \end{cases} \tag{EK}$$

where $K : (0, +\infty) \to (0, +\infty)$ is the (smooth enough) capillarity, reduces to (2) (where, we recall, $\Psi = A\mathrm{e}^{i\phi}$, $\rho = A^2$ and $u = \partial_x\phi$); namely,

$$\begin{cases} \partial_t \rho + 2\partial_x(\rho u) = 0, \\ \partial_t u + 2u\partial_x u - \partial_x(f(\rho)) - \partial_x\left(\dfrac{\partial_x^2(\sqrt{\rho})}{\sqrt{\rho}}\right) = 0, \end{cases}$$

for the capillarity $K(\varrho) = 1/(2\varrho)$, as can be shown by straightforward computations. The associated eigenvalue problem in the moving frame is

$$\begin{cases} \lambda\zeta - c_*\partial_x\zeta + 2\partial_x((r_0^2 + \eta_*)\upsilon + \zeta u_*) = 0, \\ \lambda\upsilon - c_*\partial_x\upsilon + 2\partial_x(u_*\upsilon) - \partial_x(f'(r_0^2+\eta_*)\zeta) \\ \qquad\qquad - \partial_x\left\{\dfrac{1}{2\sqrt{r_0^2+\eta_*}}\partial_x^2\left(\dfrac{\zeta}{\sqrt{r_0^2+\eta_*}}\right) - \dfrac{\zeta\partial_x^2\big(\sqrt{r_0^2+\eta_*}\big)}{2(r_0^2+\eta_*)^{3/2}}\right\} = 0. \end{cases} \tag{27}$$

The link with the original eigenvalue problem (6) is done through the formula

$$w = U_*\left(\frac{\zeta}{2} + i\int_{-\infty}^x \upsilon\right), \tag{28}$$

since this corresponds to

$$\Psi = U_{c_*} + \psi = U_{c_*} + \mathrm{e}^{\lambda t}w(x) = (A_{c_*} + \mathrm{e}^{\lambda t}\zeta(x))\exp\left(i\phi_{c_*} + i\mathrm{e}^{\lambda t}\int_{-\infty}^x \upsilon\right).$$

Notice indeed that the second equation in (27) gives $\int_{\mathbb{R}} \upsilon\, dx = 0$. It then follows from Theorem 1 in [Benzoni-Gavage 2010b] that, under the assumption $(dP(U_c)/dc)_{|c=c_*} > 0$, there exists at least one unstable eigenvalue $\gamma_0 \in (0, +\infty)$.

Existence of at most one unstable eigenvalue. The fact that there exists at most one unstable eigenvalue follows from arguments as in [Benzoni-Gavage et al. 2005, Appendix B] and is a direct consequence of Theorem 3.1 in [Pego and Weinstein 1992], that we recall now.

Theorem 25 [Pego and Weinstein 1992]. *Let $\mathcal{J}$ and $\mathcal{L}$ be two operators on a real Hilbert space X, with $\mathcal{L}$ self-adjoint and $\mathcal{J}$ skew-symmetric. Then, the number of eigenvalues, counting algebraic multiplicities, of $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}$ in the right half-plane $\{\mathrm{Re} > 0\}$ is less than or equal to the number of negative eigenvalues of $\mathcal{L}$, counting multiplicities.*

In order to apply this result to our problem, let us write the eigenvalue problem (27) under the form

$$\lambda \begin{pmatrix} \zeta \\ \upsilon \end{pmatrix} = -\partial_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathscr{L} \begin{pmatrix} \zeta \\ \upsilon \end{pmatrix},$$

where $\mathcal{M}$ is the self-adjoint Sturm–Liouville operator

$$\mathcal{M} \equiv -f'(r_0^2+\eta_*) - \frac{1}{2\sqrt{r_0^2+\eta_*}} \partial_x^2 \Big(\frac{\cdot}{\sqrt{r_0^2+\eta_*}} \Big) + \frac{\partial_x^2(\sqrt{r_0^2+\eta_*})}{2(r_0^2+\eta_*)^{3/2}}$$

(which is bounded from below) on $\mathscr{H} \equiv L^2 \times L^2$ and with

$$\mathscr{L} \equiv \begin{pmatrix} \mathcal{M} & 2u_* - c_* \\ 2u_* - c_* & 2(r_0^2+\eta_*) \end{pmatrix}.$$

We are in the setting of Theorem 25 with $\mathcal{J} = -\partial_x \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ skew-symmetric and $\mathscr{L}$ self-adjoint. We thus show that $\mathscr{L}$ has at most one negative eigenvalue. Since $r_0^2+\eta_*$ remains bounded away from zero, it is clear that, for $\sigma < 0$ and (ζ, υ) given, $\mathscr{L}(\zeta, \upsilon)^t = \sigma(\zeta, \upsilon)^t$ if and only if

$$\mathcal{M}^\dagger \zeta - \frac{(c_* - 2u_*)^2}{2(r_0^2+\eta_*)} \cdot \frac{\sigma}{2(r_0^2+\eta_*) - \sigma} \zeta = \sigma \zeta, \quad \text{with } \mathcal{M}^\dagger \equiv \mathcal{M} - \frac{(c_* - 2u_*)^2}{2(r_0^2+\eta_*)}, \tag{29}$$

since we may express υ in terms of ζ with the second equation. We observe that the translation invariance shows that $\partial_x(\eta_*, u_*)^t$ belongs to the kernel of $\mathscr{L}$; that is, using once again the relation $2u_c = 2\partial_x \phi_c = c\eta_c/(\eta_c + r_0^2)$, $\mathcal{M}^\dagger \partial_x \eta_* = 0$. Furthermore, $\mathcal{M}^\dagger$ has the same continuous spectrum as its constant coefficient limit as $x \to \pm\infty$, namely

$$-\frac{1}{2r_0^2}\partial_x^2 + \frac{\mathfrak{c}_s^2 - c_*^2}{2r_0^2};$$

that is, $\sigma_{\text{ess}}(\mathcal{M}^\dagger) = [\mathfrak{c}_s^2 - c_*^2, +\infty) \subset (0, +\infty)$, since $0 < c_* < \mathfrak{c}_s$. Since $\partial_x \eta_*$ has exactly one zero (at $x = 0$), it follows from standard Sturm–Liouville theory that $\mathcal{M}^\dagger$ has precisely one negative eigenvalue $\mu < 0$ and that the second eigenvalue is 0. Taking the scalar product with (29) yields

$$\langle \mathcal{M}^\dagger \zeta, \zeta \rangle_{L^2} - \int_{\mathbb{R}} \frac{\sigma (c_* - 2u_*)^2 \zeta^2}{2(r_0^2+\eta_*)[2(r_0^2+\eta_*) - \sigma]}\, dx = \sigma \|\zeta\|_{L^2}^2.$$

Now, for $s \le 0$, we consider the self-adjoint operator

$$\mathcal{M}_s^\dagger \equiv \mathcal{M}^\dagger - \frac{(c_* - 2u_*)^2}{2(r_0^2+\eta_*)} \cdot \frac{s}{2(r_0^2+\eta_*) - s}.$$

Clearly, $\mathcal{M}_{s=0}^\dagger = \mathcal{M}^\dagger$, $\sigma_{\text{ess}}(\mathcal{M}_s^\dagger) \subset [\mathfrak{c}_s^2 - c_*^2, +\infty) \subset (0, +\infty)$, and $\mathbb{R}_- \ni s \mapsto \mathcal{M}_s^\dagger$ is decreasing. Let us assume now that the self-adjoint operator $\mathscr{L}$ has at least two negative eigenvalues. Then, we denote by $\sigma_1 < \sigma_2 < 0$ the two smallest eigenvalues of $\mathscr{L}$ (necessarily simple), and ζ_1, ζ_2 two associated eigenvectors. Since $\mathscr{L}$ is self-adjoint, $\langle \zeta_1, \zeta_2 \rangle_{L^2} = 0$. Furthermore, $\langle \mathcal{M}_{s=\sigma_1}^\dagger \zeta_1, \zeta_1 \rangle_{L^2} = \sigma_1 \|\zeta_2\|_{L^2}^2 < 0$; hence, by monotonicity, $\langle \mathcal{M}_s^\dagger \zeta_2, \zeta_2 \rangle_{L^2} < 0$ for any $\sigma_1 \le s \le 0$. Therefore, $\mathcal{M}_s^\dagger$ has at least one negative eigenvalue for $\sigma_1 \le s \le 0$. We denote by $\lambda_{\min}(s)$ the smallest eigenvalue of $\mathcal{M}_s^\dagger$. Then, $\lambda_{\min}(s=0) = \mu < 0$

and $\lambda_{\min}$ decreases in $[\sigma_1, 0]$. Moreover, we may choose a positive eigenvector ζ_s for the eigenvalue $\lambda_1(s)$, with $\zeta_1 = \zeta_{\sigma_1}$. Since $\sigma(\mathcal{M}^\dagger) \cap \mathbb{R}_+ = \{\mu, 0\}$, it follows from the monotonicity that, for any $\sigma_1 \le s < 0$, we have $\sigma(\mathcal{M}_s^\dagger) \cap \mathbb{R}_- = \{\lambda_{\min}(s)\}$. When $s = \sigma_2 \in (\sigma_1, 0)$, we then have $\sigma_2 \in \sigma(\mathcal{M}_{s=\sigma_2}^\dagger) \cap \mathbb{R}_-$, and thus $\sigma_2 = \lambda_{\min}(\sigma_2)$, which implies that we may choose $\zeta_2 > 0$ without loss of generality. Similarly, if $s = \sigma_2$, we see that we may choose $\zeta_2 > 0$. We obtain a contradiction since then $\langle \zeta_1, \zeta_2 \rangle_{L^2} > 0$ and thus ζ_1 and ζ_2 cannot be orthogonal in L^2. We have thus shown that $\mathcal{L}$ has at most one negative eigenvalue, and then Theorem 25 shows that $\mathcal{J}\mathcal{L}$ has at most one eigenvalue in $\{\mathrm{Re} > 0\}$, as wished.

5B. *Resolvent and semigroup estimates (proof of Corollary 15).* In this section, we drop the "$*$" for the traveling wave we are considering. When linearizing the NLS equation in the moving frame with speed c, we obtain

$$i\frac{\partial \psi}{\partial t} - ic\partial_x \psi + \partial_x^2 \psi + \psi f(|U|^2) + 2\langle \psi, U\rangle f'(|U|^2)U = 0, \tag{30}$$

or

$$\begin{aligned}\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} &= \begin{pmatrix} c\partial_x - 2f'(|U|^2)U_1U_2 & -\partial_x^2 - f(|U|^2) - 2f'(|U|^2)U_2^2 \\ \partial_x^2 + f(|U|^2) + 2f'(|U|^2)U_1^2 & c\partial_x + 2f'(|U|^2)U_1U_2\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\\ &= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix} -\partial_x^2 - f(|U|^2) - 2f'(|U|^2)U_1^2 & -c\partial_x - 2f'(|U|^2)U_1U_2 \\ c\partial_x - 2f'(|U|^2)U_1U_2 & -\partial_x^2 - f(|U|^2) - 2f'(|U|^2)U_2^2\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}.\end{aligned}$$

We wish to show that this linear equation can be solved using a continuous semigroup. In order to handle later the nonlinear terms, we work in $H^1(\mathbb{R}, \mathbb{C}^2)$ instead of $L^2(\mathbb{R}, \mathbb{C}^2)$. Therefore, we consider the unbounded operator $\mathcal{A} : D(\mathcal{A}) = H^3(\mathbb{R}, \mathbb{C}^2) \subset H^1(\mathbb{R}, \mathbb{C}^2) \to H^1(\mathbb{R}, \mathbb{C}^2)$ on $H^1(\mathbb{R}, \mathbb{C}^2)$ defined by

$$\mathcal{A} \equiv \begin{pmatrix} c\partial_x - 2f'(|U|^2)U_1U_2 & -\partial_x^2 - f(|U|^2) - 2f'(|U|^2)U_2^2 \\ \partial_x^2 + f(|U|^2) + 2f'(|U|^2)U_1^2 & c\partial_x + 2f'(|U|^2)U_1U_2\end{pmatrix}.$$

It follows easily that, for $\psi = \binom{\psi_1}{\psi_2} \in H^1(\mathbb{R}, \mathbb{C}^2)$,

$$\begin{aligned}&\mathrm{Re}(\langle \mathcal{A}\psi | \psi\rangle_{H^1(\mathbb{R},\mathbb{C}^2)})\\ &\quad= \mathrm{Re}\big(\langle -2f'(|U|^2)U_1U_2\psi_1, \psi_1\rangle_{H^1(\mathbb{R},\mathbb{C})} + \langle -2f'(|U|^2)U_1U_2\psi_2, \psi_2\rangle_{H^1(\mathbb{R},\mathbb{C})}\\ &\qquad\quad + \langle [f(|U|^2) + 2f'(|U|^2)U_1^2]\psi_1, \psi_2\rangle_{H^1(\mathbb{R},\mathbb{C})} - \langle [f(|U|^2) + 2f'(|U|^2)U_1^2]\psi_2, \psi_1\rangle_{H^1(\mathbb{R},\mathbb{C})}\big)\\ &\quad\le K\|\psi\|^2_{H^1(\mathbb{R},\mathbb{C}^2)}.\end{aligned}$$

Moreover, the spectrum of $\mathcal{A}$ is included in the half-space $\{\mathrm{Re} \le \sigma_0\}$; hence $\mathcal{A}$ generates a continuous semigroup $e^{t\mathcal{A}}$ on $H^1(\mathbb{R}, \mathbb{C}^2)$.

In order to estimate the growth of the semigroup $e^{t\mathcal{A}}$ on $H^1(\mathbb{R}, \mathbb{C}^2)$, we could try to use the same approach as [Di Menza and Gallo 2007], which relies on the proof of the spectral mapping theorem in [Gesztesy et al. 2000]. However, our situation is slightly different since, in these studies, the reference solution is real-valued (it is a bound state in [Gesztesy et al. 2000] and the kink in [Di Menza and Gallo 2007]). Therefore, $U_2 = 0$ and $\mathcal{A}$ has no diagonal term, and the system is much more decoupled than in our situation. As a matter of fact, it is not very clear whether the arguments of [Gesztesy et al. 2000] carry over to our problem. We thus have chosen to use the general approach given in Appendix B. We thus verify the assumptions of Theorem B.5 (see also Corollary B.6) there, which are easy: $\mathscr{A}$ generates a semigroup in $H^1(\mathbb{R},\mathbb{C}^2)$ and the spectrum of $\mathscr{A}$ is of the form $i\mathbb{R}\cup\{-\gamma_0,+\gamma_0\}$, where $i\mathbb{R}$ is the essential spectrum and $\pm\gamma_0$ two simple eigenvalues. Moreover, the eigenvector associated with γ_0 belongs to $H^3(\mathbb{R},\mathbb{C}^2)=D(J)$. Therefore, Theorem B.5 in Appendix B applies and the growth estimate for the linearized problem follows. For the nonlinear instability result, we argue as for Corollary B.6 in Appendix B, since the manifold $\mathfrak{M}=\{|U_*|(\cdot-y),\ y\in\mathbb{R}\}$ is transverse to the curve $\sigma\mapsto|U_*+\sigma w|$ in $r_0+H^1(\mathbb{R})$. Indeed, it follows from (28) that $|U_*+\sigma w|=A_*+\sigma\zeta+\mathbb{O}_{H^1}(\sigma^2)$. Assume that $\zeta=\alpha\partial_x|U_*|$, with $\alpha\in\mathbb{R}$. Then, integration of the first equation of (27) provides

$$\lambda(|U_*|-r_0)-c_*\partial_x|U_*|+2((r_0^2+\eta_*)\upsilon+u_*\zeta)=0;$$

hence, using that $|U_*|=\sqrt{r_0^2+\eta_*}$ and the equality $2u_*=c\eta_*/(r_0^2+\eta_*)$, we infer

$$\upsilon+\alpha\left\{\lambda\frac{|U_*|-r_0}{r_0^2+\eta_*}+\frac{c_*r_0^2}{4(r_0^2+\eta_*)^{3/2}}\partial_x\eta_*\right\}=0.$$

Since $\int_{\mathbb{R}}\upsilon=0$ and $|U_*|-r_0$ has constant sign in $\mathbb{R}$, integrating over $\mathbb{R}$ then implies $\alpha=0$, which in turn yields $\zeta=\upsilon=0$ and $w_*=0$, a contradiction. Consequently, $\zeta\notin\mathbb{R}\partial_x|U_*|$ and the manifold $\mathfrak{M}=\{|U_*|(\cdot-y),\ y\in\mathbb{R}\}$ is indeed transverse to the curve $\sigma\mapsto|U_*+\sigma w|$ in $r_0+H^1(\mathbb{R})$.

6. Stability analysis for the kink ($c=0$)

6A. ***Proof of Lemma 20.*** Let us recall that the momentum $P(U_c)$, for $c>0$, has the expression

$$P(U_c)=c\int_{\xi_c}^0\frac{\xi^2}{r_0^2+\xi}\frac{d\xi}{\sqrt{-\mathscr{V}_c(\xi)}},$$

since $\operatorname{sgn}(\xi_c)=-1$. Therefore, we decompose $P(U_c)$ with two integrals:

$$P(U_c)=c\int_{\xi_c}^0\frac{\xi^2}{r_0^2+\xi}\frac{d\xi}{\sqrt{-\mathscr{V}_c'(\xi_c)(\xi-\xi_c)}}+c\int_{\xi_c}^0\frac{\xi^2}{r_0^2+\xi}\left(\frac{1}{\sqrt{-\mathscr{V}_c(\xi)}}-\frac{1}{\sqrt{-\mathscr{V}_c'(\xi_c)(\xi-\xi_c)}}\right)d\xi.\quad(31)$$

Using the change of variables $\xi=t\xi_c$, the second integral in (31) is equal to

$$\begin{aligned}\xi_c^3\int_1^0&\frac{t^2}{r_0^2+t\xi_c}\left(\frac{1}{\sqrt{-\mathscr{V}_c(t\xi_c)}}-\frac{1}{\sqrt{-\xi_c\mathscr{V}_c'(\xi_c)(t-1)}}\right)dt\\&=(-r_0^2)^3\int_1^0\frac{t^2}{r_0^2-tr_0^2}\left(\frac{1}{\sqrt{-\mathscr{V}_0(-tr_0^2)}}-\frac{1}{\sqrt{-4r_0^2F(0)(t-1)}}\right)dt+o_{c\to0}(1)\\&=\int_{-r_0^2}^0\frac{\xi^2}{r_0^2+\xi}\left(\frac{1}{\sqrt{-\mathscr{V}_0(\xi)}}-\frac{1}{\sqrt{4F(0)(\xi+r_0^2)}}\right)d\xi+o_{c\to0}(1).\end{aligned}$$

The passage to the limit $c\to0$ being justified by the dominated convergence theorem since the absolute value of the integrand is $\le Kt$ for $0\le t\le 1/2$ for small c and for $1/2\le t\le1$, since $\xi_c>-r_0^2$,

$r_0^2 + t\xi_c \geq r_0^2(1-t)$ and hence is equal to

$$
\begin{aligned}
&= \left| \frac{t^2}{r_0^2 + t\xi_c} \cdot \frac{\mathcal{V}_c(t\xi_c) - \xi_c \mathcal{V}'_c(\xi_c)(t-1)}{\sqrt{-\mathcal{V}_c(t\xi_c)}\sqrt{-\xi_c \mathcal{V}'_c(\xi_c)(t-1)}\big[\sqrt{-\mathcal{V}_c(t\xi_c)} + \sqrt{-\xi_c \mathcal{V}'_c(\xi_c)(t-1)}\big]} \right| \\
&\leq K \frac{(1-t)^2}{(1-t)\sqrt{1-t}\sqrt{1-t}\sqrt{1-t}} = \frac{K}{\sqrt{1-t}} \in L^1((1/2,1)).
\end{aligned}
$$

Furthermore, letting $\xi = \xi_c + (r_0^2 + \xi_c)t^2$, $t \geq 0$, the first integral in (31) is equal to

$$
\begin{aligned}
&\frac{1}{\sqrt{r_0^2+\xi_c}} \int_0^{\sqrt{-\xi_c/(r_0^2+\xi_c)}} \frac{(\xi_c + (r_0^2+\xi_c)t^2)^2}{1+t^2} \cdot \frac{2\,dt}{\sqrt{-\mathcal{V}'_c(\xi_c)}} \\
&\quad = \frac{2}{\sqrt{r_0^2+\xi_c}\sqrt{-\mathcal{V}'_c(\xi_c)}} \left\{ r_0^4 \left[\frac{\pi}{2} - \arctan \sqrt{\frac{r_0^2+\xi_c}{-\xi_c}} \right] - 2r_0^2(r_0^2+\xi_c)\sqrt{\frac{-\xi_c}{r_0^2+\xi_c}} \right. \\
&\qquad\qquad \left. + (r_0^2+\xi_c)^2 \left[\sqrt{\frac{-\xi_c}{r_0^2+\xi_c}} + \frac{1}{3}\sqrt{\frac{-\xi_c}{r_0^2+\xi_c}}^{\,3} \right] \right\},
\end{aligned}
$$

by direct computation. Since $\xi_c \simeq -r_0^2$ is a simple zero of $\mathcal{V}_c(\xi) = c^2\xi^2 - 4(r_0^2+\xi)F(r_0^2+\xi)$, we have

$$
\xi_c = -r_0^2 + \frac{c^2 r_0^4}{4F(0)} + \frac{c^4 r_0^6}{4F(0)}\left(\frac{r_0^2 f(0)}{F(0)} - 2\right) + o_{c\to 0}(c^4) = -r_0^2 + \frac{c^2 r_0^4}{4F(0)} + \mathcal{O}_{c\to 0}(c^4);
$$

thus

$$
-\mathcal{V}'_c(\xi_c) = 4F(0) + \mathcal{O}_{c\to 0}(c^2) \quad \text{and} \quad \frac{2}{\sqrt{r_0^2+\xi_c}\sqrt{-\mathcal{V}'_c(\xi_c)}} = \frac{2}{r_0^2 c} + \mathcal{O}_{c\to 0}(c).
$$

As a consequence, the first integral in (31) is equal to

$$
\frac{r_0^2\pi}{c} + \left\{ -\frac{r_0^3}{\sqrt{F(0)}} - \frac{2r_0^3}{\sqrt{F(0)}} + \frac{r_0^3}{3\sqrt{F(0)}} \right\} + \mathcal{O}_{c\to 0}(c) = \frac{r_0^2\pi}{c} - \frac{8r_0^3}{3\sqrt{F(0)}} + \mathcal{O}_{c\to 0}(c).
$$

Gathering these two relations, we obtain

$$
P(U_c) = r_0^2\pi + c\left\{ -\frac{8r_0^3}{3\sqrt{F(0)}} + \int_{-r_0^2}^0 \frac{\xi^2}{r_0^2+\xi}\left(\frac{1}{\sqrt{-\mathcal{V}_0(\xi)}} - \frac{1}{\sqrt{4F(0)(\xi+r_0^2)}} \right) d\xi \right\} + o_{c\to 0}(c),
$$

as wished.

6B. *Proof of Theorem 23*. Since we have a kink solution U_0 for $c = 0$, this implies that $\mathcal{V}_0(\xi) = -4(r_0^2+\xi)F(r_0^2+\xi)$ is negative in $(-r_0^2, 0)$ and that $-r_0^2$ is a simple zero of $\mathcal{V}_0$; that is, $F(0) > 0$. Then, $F > 0$ in $[0, r_0^2)$ and

$$
F(\varrho) \simeq (\mathfrak{c}_s^2/(4r_0^2))(\varrho - r_0^2)^2
$$

for $\varrho \to r_0^2$, and it follows that there exists $K_0 > 0$ such that

$$
F(\varrho) \geq \frac{1}{K_0}(\varrho - r_0^2)^2.
$$

We consider for $\mu \geq 0$ the quantity

$$\mathcal{K}_{\min}(\mu) \equiv \inf\big\{\mathcal{K}(u),\, u \in \mathcal{Z},\, \inf_{\mathbb{R}} |u| = \mu\big\}.$$

The study of $\mathcal{K}_{\min}(0)$ is easy.

Proposition 6.1. *We have*

$$\mathcal{K}_{\min}(0) = E(U_0).$$

More precisely, for any $U \in \mathcal{Z}$,

$$E(U) \geq 4 \int_{\inf_{\mathbb{R}} |U|}^{r_0} \sqrt{F(s^2)}\, ds \quad \text{and} \quad E(U_0) = 4 \int_0^{r_0} \sqrt{F(s^2)}\, ds.$$

Finally, if $U \in \mathcal{Z}$, $\inf_{\mathbb{R}} |U| = 0$ and $\mathcal{K}(U) = E(U_0)$, then there exist $y \in \mathbb{R}$ and $\theta \in \mathbb{R}$ such that $U = \mathrm{e}^{i\theta} U_0(\cdot - y)$.

Proof. Taking U_0 as a comparison map, we see that $\mathcal{K}_{\min}(0) \leq E(U_0)$. Moreover, if $U \in \mathcal{Z}$ and $\inf_{\mathbb{R}} |U| = \mu \geq 0$, we may assume, up to a translation, that $\mu = |U|(0)$. Then, defining

$$G(r) \equiv 2 \int_{r_0}^{r} \sqrt{F(s^2)}\, ds,$$

we have the inequalities

$$\begin{aligned}
\int_0^{+\infty} |\partial_x U|^2 + F(|U|^2)\, dx &\geq \int_0^{+\infty} |\partial_x |U||^2 + F(|U|^2)\, dx \geq 2 \int_0^{+\infty} \big|\sqrt{F(|U|^2)}\partial_x |U|\big|\, dx \\
&= \int_0^{+\infty} |\partial_x [G(|U|)]|\, dx \geq \left| \int_0^{+\infty} \partial_x [G(|U|)]\, dx \right| \\
&= \big|G(|U|(+\infty)) - G(|U|(0))\big| = |G(r_0) - G(\mu)| = 2 \int_\mu^{r_0} \sqrt{F(s^2)}\, ds.
\end{aligned}$$

Arguing similarly in $(-\infty, 0)$, we get

$$E(U) \geq 4 \int_\mu^{r_0} \sqrt{F(s^2)}\, ds.$$

For the kink U_0, which is real-valued, we have the first integral $|\partial_x U_0|^2 = F(U_0^2)$; hence, using the change of variables $s = U_0(x)$,

$$E(U_0) = 4 \int_0^{+\infty} F(U_0^2)\, dx = 4 \int_0^{r_0} \sqrt{F(s^2)}\, ds.$$

If $\mu = 0$, we have then $E(U) \geq E(U_0)$; hence $\mathcal{K}(U) \geq E(U) \geq E(U_0)$ as wished.

Assume finally that $U \in \mathcal{Z}$ satisfies $\inf_{\mathbb{R}} |U| = 0$ and $\mathcal{K}(U) = E(U_0)$. Then $\mu = 0$ and all the above inequalities are equalities. In particular, we must have $|\partial_x U| = |\partial_x |U||$ and equality in $|\partial_x |U||^2 + F(|U|^2) \geq 2|\sqrt{F(|U|^2)}\partial_x |U||$, which means that $|\partial_x |U|| = \sqrt{F(|U|^2)}$. Combining this ODE with the condition $|U|(0) = 0$, we see that $|U| = |U_0|$, since $|U_0|$ solves $\partial_x U_0 = \sqrt{F(U_0^2)}$. Finally, the fact that

$|\partial_x U| = |\partial_x |U||$ implies that the phase is constant in $(-\infty, 0)$ and in $(0, +\infty)$: there exist two constants $\theta_\pm \in \mathbb{R}$ satisfying $U(x) = \mathrm{e}^{i\theta_\pm}|U_0|(x)$ for $\pm x \geq 0$. Therefore, $\mathfrak{P}(U) = r_0^2(\theta_+ - \theta_-) \bmod 2\pi r_0^2$, and then

$$E(U_0) = \mathcal{H}(u) = E(U_0) + 2Mr_0^4 \sin^2 \frac{\theta_+ - \theta_- - \pi}{2}$$

implies $\theta_+ - \theta_- = \pi \bmod 2\pi$; that is, $U = \mathrm{e}^{i\theta_+}U_0$ in $\mathbb{R}$, which is the desired result. □

We recall the expansion $P(U_s) = r_0^2\pi + s\dot{P}_0 + o(s)$ as $s \to 0$, where $\dot{P}_0 \equiv (dP(U_s)/ds)_{|s=0}$. From the Hamilton group relation $dE(U_s)/ds = s\,dP(U_s)/ds$, we also infer by integration $E(U_s) = E(U_0) + \frac{1}{2}s^2\dot{P}_0 + o(s^2)$. As a first step, we define the small parameter $\mu_* > 0$. The key point is to prove the following result.

Proposition 6.2. *There exist some constant $K > 0$ and a small $\mu_* > 0$ such that, for any $0 < \mu \leq \mu_*$,*

$$\mathcal{H}_{\min}(\mu) = \inf\{\mathcal{H}(U),\, U \in \mathcal{Z},\, \inf_{\mathbb{R}} |u| = \mu\} \geq E(U_0) + \frac{\mu^2}{K}.$$

Proof. Notice first that, for $c > 0$ small, there exists U_c traveling wave of speed c, and that $\inf_{\mathbb{R}} |U_c| = \sqrt{r_0^2 + \xi_c}$ with ξ_c a smooth function in c such that $\xi_c = -r_0^2 + c^2r_0^4/(4F(0)) + \mathcal{O}(c^4)$; hence $\inf_{\mathbb{R}} |U_c| = cr_0^2/(2\sqrt{F(0)}) + \mathcal{O}(c^2)$ and is smooth. Therefore, there exists, for $0 \leq \mu \leq \mu_*$ small, a unique σ_μ, with $\sigma_\mu = 2\mu\sqrt{F(0)}/r_0^2 + \mathcal{O}(\mu^2)$, such that $\mu = \inf_{\mathbb{R}} |U_{\sigma_\mu}|$. In particular, taking U_{σ_μ} as a comparison map in $\mathcal{H}_{\min}(\mu)$, we have

$$\begin{aligned}
\mathcal{H}_{\min}(\mu) \leq \mathcal{H}(U_{\sigma_\mu}) &= E(U_{\sigma_\mu}) + 2Mr_0^4 \sin^2 \frac{P(U_{\sigma_\mu}) - r_0^2\pi}{2r_0^2} \\
&= E(U_0) + \frac{\sigma_\mu^2}{2}\dot{P}_0 + o(\sigma_\mu^2) + 2Mr_0^4 \sin^2 \frac{\sigma_\mu\dot{P}_0 + o(\sigma_\mu)}{2r_0^2} \\
&= E(U_0) + \frac{\sigma_\mu^2}{2}(\dot{P}_0 + M\dot{P}_0^2) + o(\sigma_\mu^2).
\end{aligned}$$

In particular, it follows that, for some positive constant K and for μ_* small enough,

$$\mathcal{H}_{\min}(\mu) \leq E(U_0) + K\mu^2 \leq \tfrac{11}{10}E(U_0). \tag{32}$$

Consider now c small, a bounded open interval (x_-, x_+) and η a solution to the Newton equation

$$2\partial_x^2\eta + \mathcal{V}_c'(\eta) = 0$$

in (x_-, x_+), with $\partial_x\eta(x_+) \leq 0 \leq \partial_x\eta(x_-)$, $\eta(x_+) \leq -r_0^2 + \mu_*^2$ and $\eta(x_-) \leq -r_0^2 + \mu_*^2$. As $c \to 0$, $\mathcal{V}_c$ converges to $\mathcal{V}_0$ in $\mathcal{C}^1([-r_0^2, 0])$. Moreover, $\mathcal{V}_0$ is negative in $(-r_0^2, 0)$ and has a simple zero at $-r_0^2$. Therefore, if c and $\mu_* > 0$ are sufficiently small, we must have $\int_{x_-}^{x_+} F(r_0^2 + \eta)\,dx \geq \frac{1}{2}\int_{\mathbb{R}} F(U_0^2)\,dx$. Consequently, if $v = A\mathrm{e}^{i\varphi}$ solves (TW$_c$) on a bounded interval (x_-, x_+), satisfies $2\partial_x\varphi = c\eta/(r_0^2 + \eta)$ $(\eta \equiv A^2 - r_0^2)$ and if $|v|$ is $\leq \mu_*$ at x_+ and at x_-, with $\partial_x|v|(x_+) \leq 0 \leq \partial_x|v|(x_-)$, then

$$\int_{x_-}^{x_+} |\partial_x v|^2 + F(|v|^2)\,dx \geq \tfrac{1}{2}E(U_0). \tag{33}$$

Here, we use that the Newton equation on the modulus $|V|$ actually holds true in (x_-, x_+). Since $F > 0$ in $[0, r_0^2)$ and $F(\varrho) \simeq r_0^2(\varrho - r_0^2)^2$ when $\varrho \to r_0^2$, there exist $K > 0$ and $\kappa > 0$ such that $F(\varrho) \geq (\varrho - r_0^2)^2/K$ for $0 \leq \varrho \leq r_0^2(1+\kappa)^2$. Hence, if $\inf_{\mathbb{R}} |v| \geq \mu > 0$, then

$$|P(v)| \leq \frac{K}{\mu} E(v). \tag{34}$$

Moreover, arguing as in the proof of Proposition 6.1, we show that there exists $\varkappa > 0$ such that, if $U \in \mathcal{Z}$ and $|U|$ takes values $\leq \mu_*$ and $\geq r_0(1+\kappa)$, then

$$E(U) \geq E(U_0)(1+\varkappa).$$

In particular, since $\mathcal{K}_{\min}(\mu) \leq E(U_0) + \mathcal{O}(\mu^2)$, we may choose μ_* sufficiently small so that, if $U \in \mathcal{Z}$ and $\mathcal{K}(U) \leq \mathcal{K}_{\min}(\mu) + \mu_*$, then $|U| \leq r_0(1+\kappa)$. This means that, for the mappings we are considering, $F(\varrho) \geq (\varrho - r_0^2)^2/K$.

Step 1: Construction of a suitable minimizing sequence. There exists a sequence $(V_n)_{n\geq 0}$ in $\mathcal{Z}$ such that $\inf_{\mathbb{R}} |V_n| = \mu = |V_n|(0)$, $V_n = A_n e^{i\phi_n}$, $P(V_n) \in [0, \pi r_0^2]$,

$$2A_n^2 \partial_x \phi_n = c_n(A_n^2 - r_0^2), \quad c_n \equiv M r_0^2 \sin \frac{r_0^2\pi - P(V_n)}{2r_0^2} \geq 0 \quad \text{and} \quad \lim_{n\to+\infty} \mathcal{K}(V_n) = \mathcal{K}_{\min}(\mu).$$

Since $\mu > 0$, the maps V we consider may be lifted to $V = Ae^{i\phi}$. Therefore (with $u = \partial_x \phi$),

$$\mathcal{K}_{\min}(\mu) = \inf\left\{ \int_{\mathbb{R}} (\partial_x A)^2 + F(A^2)\, dx + \inf\left\{ \int_{\mathbb{R}} A^2 u^2\, dx + 2M r_0^4 \sin^2 \frac{\int_{\mathbb{R}} (A^2 - r_0^2) u\, dx - r_0^2\pi}{2r_0^2}, \right.\right.$$
$$\left.\left. u \in L^2(\mathbb{R}, \mathbb{R}) \right\},\ A \in r_0 + H^1(\mathbb{R}, \mathbb{R}),\ \inf_{\mathbb{R}} A = \mu \right\}. \tag{35}$$

The infimum in u may be written

$$\inf_{p\in\mathbb{R}} \inf\left\{ \int_{\mathbb{R}} A^2 u^2\, dx + 2M r_0^4 \sin^2 \frac{p - r_0^2\pi}{2r_0^2},\ u \in L^2(\mathbb{R}, \mathbb{R}) \text{ s.t. } \int_{\mathbb{R}} (A^2 - r_0^2) u\, dx = p \right\}.$$

For each $p \in \mathbb{R}$, we minimize in u a quadratic functional on an affine hyperplane, with minimizer given by

$$u_p = p \left(\int_{\mathbb{R}} \frac{(A^2 - r_0^2)^2}{A^2}\, dx \right)^{-1} \frac{A^2 - r_0^2}{A^2}.$$

As a consequence, the infimum in u in (35) is

$$\inf_{p\in\mathbb{R}} \left[\int_{\mathbb{R}} A^2 u_p^2\, dx + 2M r_0^4 v \sin^2 \frac{p - r_0^2\pi}{2r_0^2} \right] = \inf_{p\in\mathbb{R}} \left[p^2 \left(\int_{\mathbb{R}} \frac{(A^2 - r_0^2)^2}{A^2}\, dx \right)^{-1} + 2M r_0^4 \sin^2 \frac{p - \pi r_0^2}{2r_0^2} \right].$$

It is clear that this last infimum is achieved only for p inside $[-\pi r_0^2, +\pi r_0^2]$. Indeed, the second term is $2\pi r_0^2$-periodic, and, if $p > \pi r_0^2$, then $p - 2\pi r_0^2$ is a better competitor. Moreover, the function $p \mapsto \sin^2((p - \pi r_0^2)/(2r_0^2))$ is continuous and even; hence we may consider some $p \in [0, \pi r_0^2]$ (depending on A), which is a minimizer for this last infimum. The corresponding u_p is then a minimizer for the infimum in u in (35). Writing that

$$\frac{d}{dp}\left[p^2\left(\int_{\mathbb{R}}\frac{(A^2-r_0^2)^2}{A^2}\,dx\right)^{-1}+2Mr_0^4\sin^2\frac{p-\pi r_0^2}{2r_0^2}\right] = 2p\left(\int_{\mathbb{R}}\frac{(A^2-r_0^2)^2}{A^2}\,dx\right)^{-1}+2Mr_0^2\sin\frac{p-\pi r_0^2}{2r_0^2}=0,$$

we deduce the relations

$$2A^2u=c(A^2-r_0^2),\quad c\equiv Mr_0^2\sin\frac{p-\pi r_0^2}{2r_0^2}.$$

We conclude by considering a minimizing sequence (A_n) in (35), and translating in space so that $\inf_{\mathbb{R}} A_n=\mu=|A_n|(0)$.

Since $F\ge 0$ in $\mathbb{R}_+$, we have

$$\int_{\mathbb{R}}|\partial_x V_n|^2\,dx\le \mathcal{K}(V_n)\le \tfrac{12}{10}E(U_0)$$

for n large. Therefore, by the compact Sobolev embedding $H^1([-R,+R])\hookrightarrow L^\infty([-R,+R])$, we may assume, up to a possible subsequence, that there exists $V\in H^1_{\mathrm{loc}}(\mathbb{R})$ such that, for any $R>0$, $V_n\rightharpoonup V$ in $H^1([-R,+R])$ and $V_n\to V$ uniformly on $[-R,+R]$. Moreover, by lower semicontinuity and Fatou's lemma, $E(V)\le \varliminf_{n\to+\infty}E(V_n)$. Since $|V_n|\ge\mu>0$ in $\mathbb{R}$, we have $|V|\ge\mu>0$ in $\mathbb{R}$ and thus a lifting $V=A\mathrm{e}^{i\phi}$. Furthermore, $\inf_{\mathbb{R}}A_n=\mu=|V_n|(0)$; hence $\inf_{\mathbb{R}}A=\mu=|V|(0)$. We also know that $P(V_n)\in[0,r_0^2\pi]$ for all n; hence we may assume, up to another subsequence, that $P(V_n)$ converges to some $P_\infty\in[0,r_0^2\pi]$. We also set

$$c\equiv\lim_{n\to+\infty}c_n=Mr_0^2\sin\frac{P_\infty-\pi r_0^2}{2r_0^2}.$$

In view of Step 1, and the convergence $A_n\to A$ uniformly on any compact interval $[-R,+R]$, it follows that

$$2A^2\partial_x\phi=c(A^2-r_0^2)\quad\text{and}\quad \partial_x\phi_n\to\partial_x\phi\quad\text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}). \tag{36}$$

Note that

$$\int_{\mathbb{R}}|\partial_x V|^2+\frac{1}{K}(|V|^2-r_0^2)^2\,dx\le E(V)<+\infty;$$

hence $|V|\to r_0$ at $\pm\infty$. In particular, there exist $-\infty<R_-\le 0\le R_+<+\infty$ such that $|V|>\mu$ in $(-\infty,R_-)$ and in $(R_+,+\infty)$ and $|V|(R_\pm)=\mu$.

Step 2. There exist $-\infty<z_-\le 0\le z_+<+\infty$ such that

$$A(x)=A_c(x-R_++z_+)\quad\text{for } x\ge R_+\quad\text{and}\quad A(x)=A_c(x-R_-+z_-)\quad\text{for } x\le R_-.$$

We work for $x\ge R_+$, the other case being similar. We consider $\chi\in\mathcal{C}^1_c((R_+,+\infty),\mathbb{C})$, $t\in\mathbb{R}$ small such that $V_n^t\equiv v_n+t\chi$ satisfies $|V_n^t|>\mu$ in $(R_+,+\infty)$. This is possible since $\inf_{\mathrm{Supp}(\chi)}|V_n|>\mu$. Then,

$|V_n^t| \geq \mu$ in $\mathbb{R}$ and $|V_n^t|(0) = \mu$; hence V_n^t is then a comparison map for $\mathcal{K}_{\min}(\mu)$, and, in view of the equality $P(V_n^t) = P(V_n) + 2t \int_{R_+}^{+\infty} \langle i\partial_x V_n | \chi\rangle \, dx + \mathcal{O}(t^2)$, it follows that

$$\begin{aligned}\mathcal{K}_{\min}(\mu) &\leq \mathcal{K}(V_n^t)\\ &= \mathcal{K}_{\min}(\mu) + o_{n\to+\infty}(1) + 2t \int_{R_+}^{+\infty} \langle \partial_x V_n, \partial_x \chi\rangle \, dx + t^2 \int_{R_+}^{+\infty} |\partial_x \chi|^2 \, dx\\ &\quad - 2t \int_{R_+}^{+\infty} f(|V_n|^2)\langle V_n, \chi\rangle \, dx + Mt \sin \frac{P(V_n) - \pi r_0^2}{r_0^2} \int_{R_+}^{+\infty} \langle i\partial_x V_n, \chi\rangle \, dx + \mathcal{O}_{t\to 0}(t^2).\end{aligned}$$

Letting $n \to +\infty$ and using the weak and strong convergences for V_n, we infer

$$\begin{aligned}0 \leq 2t \int_{R_+}^{+\infty} \langle \partial_x V, \partial_x \chi\rangle \, dx - 2t \int_{R_+}^{+\infty} f(|V|^2)\langle V, \chi\rangle \, dx&\\ - Mt \sin \frac{r_0^2\pi - P_\infty}{2r_0^2} \int_{R_+}^{+\infty} \langle i\partial_x V, \chi\rangle \, dx + \mathcal{O}_{t\to 0}(t^2).&\end{aligned}$$

Dividing by $t \neq 0$ and letting $t \to 0^+$ and then $t \to 0^-$, we deduce that V solves (TW$_c$) in $(R_+, +\infty)$ and V has finite energy. Moreover, $|V|(R_+) = \mu$ is small; thus $V = e^{i\theta_+} U_c(\cdot - R_+ + z_+)$ in $(R_+, +\infty)$ for some constants z_+ and θ_+, and the speed c is such that $\inf_{\mathbb{R}} A_c = \sqrt{r_0^2 + \xi_c} \leq \mu$; hence $c \leq \sigma(\mu) \leq K\mu$. Since $|V|$ has finite energy in $\mathbb{R}$ and solves (TW$_c$) in $(R_+, +\infty)$, V is $\mathcal{C}^1$ in $[R_+, +\infty)$. Moreover, $|V|$ reaches a minimum at $x = R_+$; thus we must have $\partial_x^+|V|(R_+) \geq 0$, which imposes $z_+ \geq 0$. Note that, A_c being even, it is possible to translate V so that $R \equiv R_+ = -R_-$ and $z \equiv z_+ = -z_-$. Observe that $\mu = A_c(z) \geq A_0(z)$; hence $z \leq K\mu$. This yields

$$\int_{|x|\geq R} |\partial_x V|^2 + F(|V|^2) \, dx = \int_{|x|\geq z} |\partial_x U_c|^2 + F(|U_c|^2) \, dx \geq E(U_0) - K\mu. \tag{37}$$

In particular, we deduce from (32)

$$2RF(\mu^2) \leq \int_{|x|\leq R} |\partial_x V|^2 + F(|V|^2) \, dx \leq K\mu;$$

hence $R \leq K\mu$ for μ small ($F(0) > 0$).

Step 3. We prove that $A = \mu$ in $(R_-, R_+) = (-R, +R)$.

Indeed, if it is not the case, there exists a bounded interval (x_-, x_+) such that $A = |V| > \mu$ in (x_-, x_+) and $|V|(x_\pm) = \mu$, with $\partial_x|V|(x_+) \leq 0 \leq \partial_x|V|(x_-)$. Therefore, we can make perturbations of the amplitude A_n localized in (x_-, x_+). Hence, arguing as in Step 2, we see that, then, V solves (TW$_c$) in (x_-, x_+), with $2A^2\partial_x\phi = c(A^2 - r_0^2)$ and $|V|(x_\pm) = \mu$, $\partial_x|V|(x_+) \leq 0 \leq \partial_x|V|(x_-)$. We then are in position to apply (33), yielding

$$\int_{x_-}^{x_+} |\partial_x V|^2 + F(|V|^2) \, dx \geq \tfrac{1}{2} E(U_0),$$

but the combination with (37) provides

$$\tfrac{11}{10}E(U_0) \geq \mathcal{K}_{\min}(\mu) \geq \int_{x_-}^{x_+} |\partial_x V|^2 + F(|V|^2)\,dx + \int_{|x|\geq R} |\partial_x V|^2 + F(|V|^2)\,dx$$
$$\geq \tfrac{1}{2}E(U_0) + E(U_0) - K\mu_* = \tfrac{3}{2}E(U_0) - K\mu_*,$$

which is not possible if μ_* is sufficiently small.

Step 4. We have $R = 0$ or ($z = 0$ and $c = \sigma_\mu$).

Indeed, assume $R > 0$, and consider $\zeta \in \mathscr{C}_c^1((0,+\infty),\mathbb{R})$, $\zeta \geq 0$, $t \geq 0$ and $V_n^t \equiv (A_n + t\zeta)e^{i\phi_n}$, so that $|V_n^t| = A_n + t\zeta \geq \mu$ in $\mathbb{R}$. Since $R > 0$, we actually have $\inf_{\mathbb{R}} |V_n^t| = \mu$ and V_n^t is a comparison map for $\mathcal{K}_{\min}(\mu)$. Arguing as before, we thus have

$$\mathcal{K}_{\min}(\mu) \leq \mathcal{K}(V_n^t) = \mathcal{K}_{\min}(\mu) + o_{n\to+\infty}(1) + 2t\int_0^{+\infty} \partial_x A_n \partial_x \zeta\,dx + t^2\int_0^{+\infty} (\partial_x\zeta)^2\,dx$$
$$+ 2t\int_0^{+\infty} A_n\zeta(\partial_x\phi_n)^2\,dx + t^2\int_0^{+\infty} \zeta^2(\partial_x\phi_n)^2\,dx - 2t\int_0^{+\infty} f(A_n^2)A_n\zeta\,dx$$
$$+ Mr_0^2 t \sin\frac{P(V_n) - r_0^2\pi}{r_0^2}\int_0^R 2A_n\zeta\partial_x\phi_n\,dx + \mathcal{O}_{t\to 0}(t^2).$$

By (36), we may pass to the limit as $n \to +\infty$ in all the terms and deduce

$$0 \leq 2t\int_0^{+\infty} \partial_x A\partial_x\zeta\,dx + 2t\int_0^{+\infty} A\zeta(\partial_x\phi)^2\,dx - 2t\int_0^{+\infty} f(A^2)A\zeta\,dx - 2ct\int_0^{+\infty} A\zeta\partial_x\phi\,dx + \mathcal{O}_{t\to 0}(t^2).$$

At this stage, we see the relevance of taking a minimizing sequence as chosen in Step 1, since it allows us to pass to the limit in the nonlinear terms involving $\partial_x\phi_n$. As a consequence, using (36),

$$-\partial_x^2 A - Af(A^2) + \frac{c^2}{4}\frac{(A^2 - r_0^2)^2}{A^3} \geq 0$$

in the distributional sense in $(0,+\infty)$. The term $-Af(A^2) + \frac{1}{4}c^2(A^2 - r_0^2)^2/A^3$ is continuous in $\mathbb{R}$. However, since $A(x) = \mu$ for $0 \leq x \leq R$ and $A(x) = A_c(x - R + z)$ for $x \geq R$, we infer $-\partial_x^2 A = -\partial_x A_c(z)\delta_{x=R}$ plus a piecewise continuous function in the distributional sense in $(0,+\infty)$. Since $\partial_x A_c(z) \geq 0$ (recall that $z \geq 0$), this forces $\partial_x A_c(z) = 0$; that is, $z = 0$. Consequently, $\mu = |V|(R) = A(R) = A_c(z) = A_c(0)$ and then $c = \sigma_\mu$.

In the next step, we take into account the loss in the weak convergence $V_n \rightharpoonup V$.

Step 5. There exists $K > 0$ such that

$$E_\sharp \geq \frac{P_\sharp}{K}, \quad \text{where } E_\sharp \equiv \lim_{n\to+\infty} E(V_n) - E(V) \geq 0, \quad P_\sharp \equiv \lim_{n\to+\infty} P(V_n) - P(V) = P_\infty - P(V).$$

Let $\epsilon > 0$ be fixed but small, and pick some $X > 0$ large so that

$$\left|E(V) - \int_{|x|\leq X} |\partial_x V|^2 + F(|V|^2)\,dx\right| \leq \epsilon, \qquad \left|P(V) - \int_{|x|\leq X} (A^2 - r_0^2)u\,dx\right| \leq \epsilon.$$

We claim that there exists some small $\bar{\mu} > 0$, independent of ϵ, such that $|V_n| \geq \bar{\mu}$ for $|x| \geq X$ and n large. Indeed, otherwise, we may argue as in Step 3 and show, as in the beginning of the proof there, that $\int_{|x|\geq X} |\partial_x V_n|^2 + F(|V_n|^2)\,dx \geq \frac{1}{2}E(U_0)$. This is not possible since

$$\tfrac{12}{10}E(U_0) \geq \varliminf_{n\to+\infty} E(V_n) \geq \tfrac{1}{2}E(U_0) + \int_{|x|\leq X} |\partial_x V|^2 + F(|V|^2)\,dx \geq \tfrac{1}{2}E(U_0) + E(V) - \epsilon,$$

and $E(V)$ is close to $E(U_0)$ as $\mu \to 0$. Therefore, as for (34),

$$\left|\int_{|x|\geq X} (A_n^2 - r_0^2)u_n\,dx\right| \leq \frac{K}{\bar{\mu}} \int_{|x|\geq X} |\partial_x V_n|^2 + F(|V_n|^2)\,dx.$$

Consequently,

$$\begin{aligned}
&E(V_n) - E(V)\\
&\geq \int_{|x|\leq X} |\partial_x V_n|^2 + F(|V_n|^2)\,dx - \int_{|x|\leq X} |\partial_x V|^2 + F(|V|^2)\,dx + \int_{|x|\geq X} |\partial_x V_n|^2 + F(|V_n|^2)\,dx - \epsilon\\
&\geq \int_{|x|\leq X} |\partial_x V_n|^2 + F(|V_n|^2)\,dx - \int_{|x|\leq X} |\partial_x V|^2 + F(|V|^2)\,dx + \frac{\bar{\mu}}{K}\left|\int_{|x|\geq X} (A_n^2 - r_0^2)u_n\,dx\right| - \epsilon.
\end{aligned}$$

Passing to the liminf and using the weak convergence in $[-X, +X]$, we infer

$$\varliminf_{n\to+\infty} E(V_n) - E(V) \geq \frac{\bar{\mu}}{K} \varliminf_{n\to+\infty} \left|P(V_n) - \int_{|x|\leq X} (A_n^2 - r_0^2)u_n\,dx\right| - \epsilon.$$

However, (36) implies

$$\int_{|x|\leq X} (A_n^2 - r_0^2)u_n\,dx \to \int_{|x|\leq X} (A^2 - r_0^2)u\,dx,$$

so that

$$E_\sharp \geq \frac{\bar{\mu}}{K}\left|P_\infty - \int_{|x|\leq X} (A^2 - r_0^2)u\,dx\right| - \epsilon \geq \frac{\bar{\mu}}{K}|P_\infty - P(V)| - \left(1 + \frac{\bar{\mu}}{K}\right)\epsilon = \frac{\bar{\mu}}{K}|P_\sharp| - \left(1 + \frac{\bar{\mu}}{K}\right)\epsilon.$$

Letting $\epsilon \to 0$, the conclusion follows.

Step 6. There exists $K > 0$ such that, if $R > 0$, then

$$\mathcal{H}_{\min}(\mu) \geq E(U_0) + \frac{\mu^2}{K}.$$

We recall the expansion $P(U_s) = r_0^2\pi + s\dot{P}_0 + o(s)$ as $s \to 0$, where $\dot{P}_0 \equiv (dP(U_s)/ds)_{|s=0}$. From the Hamilton group relation $dE(U_s)/ds = s\,dP(U_s)/ds$, we also infer by integration $E(U_s) = E(U_0) + \frac{1}{2}s^2\dot{P}_0 + o(s^2)$. On the other hand, by definition of c_n,

$$\begin{aligned}
2Mr_0^4 \sin^2 \frac{P(V_n) - r_0^2\pi}{2r_0^2} &= Mr_0^4\left[1 - \cos\frac{P(V_n) - r_0^2\pi}{r_0^2}\right]\\
&= Mr_0^4\left[1 - \sqrt{1 - \sin^2\frac{P(V_n) - r_0^2\pi}{r_0^2}}\right] = Mr_0^4\left[1 - \sqrt{1 - \frac{c_n^2}{M^2}}\right]
\end{aligned}$$

for n large. Here, we have used that $Mc_n = \sin((r_0^2\pi - P(V_n))/r_0^2) \to Mc \in [0, K\mu_*]$ (cf. Step 2); thus $\cos((r_0^2\pi - P(V_n))/r_0^2) \geq 0$, for, otherwise, we would have, by Proposition 6.1,

$$\begin{aligned}\mathcal{K}(V_n) = E(V_n) + 2Mr_0^4 \sin^2 \frac{P(V_n) - r_0^2\pi}{2r_0^2} &\geq E(U_0) - K\mu + Mr_0^4\Big[1 + \sqrt{1 - \frac{c_n^2}{M^2}}\Big] \\ &\geq E(U_0) - K\mu_* + 2Mr_0^4 + \mathcal{O}(\mu_*^2),\end{aligned}$$

but this contradicts (32) if μ_* is sufficiently small.

We assume $R > 0$, so that, by Step 4, $z = 0$ and $c = \sigma_\mu$. We recall $\sigma_\mu = 2\mu\sqrt{F(0)}/r_0^2 + \mathcal{O}(\mu^2) \sim 2\mu\sqrt{F(0)}/r_0^2$. By definition of $E_\sharp$, one has

$$\begin{aligned}E_\sharp + E(V) + Mr_0^4\Big[1 - \sqrt{1 - \frac{c^2}{M^2}}\Big] &\leq \varliminf_{n\to+\infty} E(V_n) + \lim_{n\to+\infty} 2Mr_0^4 \sin^2 \frac{P(V_n) - r_0^2\pi}{2r_0^2} \\ &= \varliminf_{n\to+\infty} \mathcal{K}(V_n) = \mathcal{K}_{\min}(\mu)\end{aligned}$$

since (V_n) is minimizing for $\mathcal{K}_{\min}(\mu)$. Moreover, from the expression of V, we have (for $R > 0$)

$$E(V) = E(U_{\sigma_\mu}) + 2R\Big[\frac{\sigma_\mu^2(r_0^2 - \mu^2)^2}{4\mu^2} + F(\mu^2)\Big] \quad \text{and} \quad P(V) = P(U_{\sigma_\mu}) + R\sigma_\mu \frac{(r_0^2 - \mu^2)^2}{\mu^2}.$$

Furthermore, we have $P_\sharp = P_\infty - P(V)$ and $c = Mr_0^2 \sin((r_0^2\pi - P_\infty)/r_0^2)$ with $P_\infty \in [0, r_0^2\pi]$ and $\cos((r_0^2\pi - P_\infty)/r_0^2) \geq 0$; thus

$$P_\sharp = P_\infty - P(V) = r_0^2\pi - r_0^2 \arcsin\Big(\frac{c}{Mr_0^2}\Big) - P(U_{\sigma_\mu}) - R\sigma_\mu \frac{(r_0^2 - \mu^2)^2}{\mu^2}.$$

Combining this with the expansions of $E(U_\sigma)$ and $P(U_\sigma)$ gives

$$\begin{aligned}\mathcal{K}_{\min}(\mu) &\geq E(U_0) + E_\sharp + \frac{\sigma_\mu^2}{2}\dot{P}_0 + o(\sigma_\mu^2) + M\Big[1 - \sqrt{1 - \frac{\sigma_\mu^2}{M^2}}\Big] + 2R\Big[\frac{\sigma_\mu^2(r_0^2 - \mu^2)^2}{4\mu^2} + F(\mu^2)\Big] \\ &\geq E(U_0) + \frac{|P_\sharp|}{K} + \frac{\sigma_\mu^2}{2}\Big[\dot{P}_0 + \frac{1}{M}\Big] + o(\mu^2) + 4RF(0) \\ &\geq E(U_0) + \frac{1}{K}\Big|r_0^2 \arcsin(\sigma_\mu/(Mr_0^2)) + \sigma_\mu\dot{P}_0 + R\sigma\frac{(r_0^2 - \mu^2)^2}{\mu^2} + o(\sigma_\mu)\Big| + \frac{\sigma_\mu^2}{2}\Big[\dot{P}_0 + \frac{1}{M}\Big] \\ &\qquad + o(\mu^2) + 4RF(0) \\ &\geq E(U_0) + \frac{1}{K}\Big|\frac{\sigma_\mu}{M} + \sigma_\mu\dot{P}_0 + R\sigma_\mu\frac{(r_0^2 - \mu^2)^2}{\mu^2} + o(\sigma_\mu)\Big| + \frac{\sigma_\mu^2}{2}\Big[\dot{P}_0 + \frac{1}{M}\Big] + o(\mu^2) + 4RF(0) \\ &\geq E(U_0) + \frac{\sigma_\mu}{K}\Big|\dot{P}_0 + \frac{1}{M} + R\frac{(r_0^2 - \mu^2)^2}{\mu^2}\Big| + \frac{\sigma_\mu^2}{2}\Big[\dot{P}_0 + \frac{1}{M}\Big] + 4RF(0) + o(\mu^2).\end{aligned}$$

The right-hand side is a continuous piecewise affine function of R (the "o" does not depend on R). Since $\sigma_\mu(r_0^2 - \mu^2)^2/(K\mu^2) \simeq 1/\mu \gg 4F(0)$ and $\dot{P}_0 + 1/M < 0$ (since $M > -\dot{P}_0^{-1}$ by hypothesis), it follows that the right-hand side is a function of R which is decreasing in $[0, R_0(\mu)]$ and increasing in $[R_0(\mu), +\infty)$, with

$$R_0(\mu) \equiv -\left(\dot{P}_0 + \frac{1}{M}\right)\frac{\mu^2}{(r_0^2-\mu^2)^2} \sim -\left(\dot{P}_0 + \frac{1}{M}\right)\frac{\mu^2}{r_0^4} > 0.$$

Therefore, using once again that $\sigma_\mu^2 \sim 4\mu^2 F(0)/r_0^4$,

$$\begin{aligned}\mathcal{K}_{\min}(\mu) &\geq E(U_0) + \frac{\sigma_\mu^2}{2}\left[\dot{P}_0 + \frac{1}{M}\right] + 4R_0(\mu)F(0) + o(\mu^2)\\ &= E(U_0) + \left[\dot{P}_0 + \frac{1}{M}\right]\frac{2\mu^2 F(0)}{r_0^4} - \left[\dot{P}_0 + \frac{1}{M}\right]\frac{4\mu^2 F(0)}{r_0^4} + o(\mu^2)\\ &= E(U_0) - \mu^2\frac{2F(0)}{r_0^4}\left[\dot{P}_0 + \frac{1}{M}\right] + o(\mu^2).\end{aligned}$$

In view of our hypothesis $\dot{P}_0 + 1/M < 0$, we infer that

$$\mathcal{K}_{\min}(\mu) \geq E(U_0) + \frac{\mu^2}{K}$$

for μ_* sufficiently small and some positive constant K, as wished. If the assumption $\dot{P}_0 + 1/M < 0$ is not satisfied, but, if $\dot{P}_0 + 1/M > 0$ for instance, then the function of R above is increasing in $[0, +\infty)$, with minimum value achieved at $R = 0$ and equal to

$$E(U_0) + \frac{\sigma_\mu^2}{2}\left[\dot{P}_0 + \frac{1}{M}\right] + o(\mu^2) = E(U_0) + \frac{2\mu^2 F(0)}{r_0^4}\left[\dot{P}_0 + \frac{1}{M}\right] + o(\mu^2) \geq E(U_0) + \frac{\mu}{K}.$$

We then would have concluded a stronger estimate, which is actually in contradiction with (32); hence we are necessarily in the case $R > 0$. The assumption $\dot{P}_0 + 1/M < 0$ is however crucial for the last step.

Step 7. We assume $\dot{P}_0 + 1/M < 0$. Then, for μ_* sufficiently small, the case $R = 0$ does not occur.

We argue in a similar way, but, since $R = 0$, the expressions for $E(V)$ and $P(V)$ are given by

$$E(V) = E(U_c) - 4\int_0^z F(|U_c|^2)\,dx \quad \text{and} \quad P(V) = P(U_c) - 2\int_0^z \frac{c}{2}\frac{(r_0^2 - A_c^2)^2}{A_c^2}\,dx.$$

Here, we have used that $|\partial_x U_c|^2 = F(|U_c|^2)$ since U_c solves (TW$_c$). Combining this here again with the expansions of $E(U_c)$ and $P(U_c)$ gives, using that $0 \leq c \leq K\mu$,

$$\begin{aligned}\mathcal{K}_{\min}(\mu) &\geq E(U_0) + E_\sharp + \frac{c^2}{2}\dot{P}_0 + o(c^2) + M\left[1 - \sqrt{1 - \frac{c^2}{M^2}}\right] - 4\int_0^z F(|U_c|^2)\,dx\\ &\geq E(U_0) + \frac{|P_\sharp|}{K} + \frac{c^2}{2}\left[\dot{P}_0 + \frac{1}{M}\right] - 4zF(0) + o(\mu^2)\\ &\geq E(U_0) + \frac{1}{K}\left|\arcsin(c/M) + c\dot{P}_0 - c\int_0^z \frac{(r_0^2 - A_c^2)^2}{A_c^2}\,dx + o(c)\right| + \frac{c^2}{2}\left[\dot{P}_0 + \frac{1}{M}\right] + o(\mu^2) - 4zF(0).\end{aligned}$$

Following the lines of the proof of Lemma 20, we have

$$c\int_0^z \frac{(r_0^2-A_c^2)^2}{A_c^2}\,dx = 2\arctan\sqrt{\frac{\mu^2}{r_0^2+\xi_c}} + \mathcal{O}(\mu^2). \tag{38}$$

Indeed, noticing that $A_c = \mathcal{O}(\mu)$ in $[0,z]$ with $z \le K\mu$, we write, expanding the square,

$$\int_0^z \frac{(r_0^2-A_c^2)^2}{A_c^2}\,dx = \int_0^z \frac{r_0^4}{A_c^2} - 2 + A_c^2\,dx = \int_0^z \frac{r_0^4}{A_c^2}\,dx + \mathcal{O}(\mu).$$

Then, using the change of variable $\xi = \eta_c(x)$,

$$\begin{aligned}
\int_0^z \frac{(r_0^2-A_c^2)^2}{A_c^2}\,dx &= \int_{\xi_c}^{\mu^2-r_0^2} \frac{r_0^4}{(r_0^2+\xi)\sqrt{-\mathcal{V}_c(\xi)}}\,d\xi + \mathcal{O}(\mu)\\
&= \int_{\xi_c}^{\mu^2-r_0^2} \frac{r_0^4}{(r_0^2+\xi)\sqrt{-\mathcal{V}_c'(\xi_c)(\xi-\xi_c)}}\,d\xi\\
&\quad + \int_{\xi_c}^{\mu^2-r_0^2} \frac{r_0^4}{(r_0^2+\xi)}\left(\frac{1}{\sqrt{-\mathcal{V}_c(\xi)}} - \frac{1}{\sqrt{-\mathcal{V}_c'(\xi_c)(\xi-\xi_c)}}\right)d\xi + \mathcal{O}(\mu)\\
&= \frac{2}{c}\arctan\sqrt{\frac{\mu^2}{r_0^2+\xi_c}-1} + \mathcal{O}(\mu), \qquad (39)
\end{aligned}$$

by computations similar to those for the proof of Lemma 20. This proves (38). Therefore,

$$\begin{aligned}
&\mathcal{K}_{\min}(\mu)\\
&\ge E(U_0) + \frac{1}{K}\left|c\left[\dot P_0 + \frac{1}{M}\right] - 2\arctan\sqrt{\frac{\mu^2}{r_0^2+\xi_c}-1} + o(c)\right| + \frac{c^2}{2}\left[\dot P_0 + \frac{1}{M}\right] + o(\mu^2) - 4zF(0). \qquad (40)
\end{aligned}$$

By (32), the left-hand side is $\le E(U_0) + K\mu^2$. Since $\dot P_0 + 1/M < 0$, $c \le K\mu$, $z \le K\mu$ and $F(0) > 0$, this implies

$$\left|c\left[\dot P_0 + \frac{1}{M}\right] - 2\arctan\sqrt{\frac{\mu^2}{r_0^2+\xi_c}} + o(c)\right| \le K\mu;$$

thus

$$\arctan\sqrt{\frac{\mu^2}{r_0^2+\xi_c}-1} \le K\mu,$$

and, finally, for μ_* small enough,

$$0 \le \frac{\mu^2}{r_0^2+\xi_c} - 1 \le K\mu^2.$$

Combining this with the equality $r_0^2 + \xi_c = c^2 r_0^4/(4F(0)) + \mathcal{O}(c^4)$ seen during the proof of Lemma 20, we infer

$$c = \frac{2\sqrt{F(0)}}{r_0^2}\mu + \mathcal{O}(\mu^2).$$

In particular, going back to (39) and since, for $0 \le x \le z$,

$$r_0^2 + \xi_c = A_c^2(0) \le A_c^2(x) \le A_c^2(z) = \mu^2,$$

this implies

$$\frac{z r_0^4}{\mu^2} \le \int_0^z \frac{r_0^4}{A_c^2} \le \frac{2}{c} \arctan \sqrt{\frac{\mu^2}{r_0^2 + \xi_c} - 1} + \mathcal{O}(\mu) \le \frac{K\mu}{c} + K\mu \le K,$$

which provides (since $c \approx \mu$)

$$z \le K\mu^2.$$

Inserting this into (40) and keeping in mind that the left-hand side is $\le E(U_0) + K\mu^2$, we deduce

$$c\left[\dot{P}_0 + \frac{1}{M}\right] - 2\arctan \sqrt{\frac{\mu^2}{r_0^2 + \xi_c} - 1} = o(\mu).$$

However, since $\arctan \sqrt{\mu^2/(r_0^2 + \xi_c) - 1} \ge 0$, this gives

$$o(\mu) \le c\left[\dot{P}_0 + \frac{1}{M}\right] \sim \frac{2\mu\sqrt{F(0)}}{r_0^2}\left[\dot{P}_0 + \frac{1}{M}\right],$$

yielding a contradiction for small μ since we have $\dot{P}_0 + 1/M < 0$ by assumption. Therefore, the case $R = 0$ does not occur for sufficiently small μ_*. If we had $\dot{P}_0 + 1/M > 0$, we would not have been able to show that $\mathcal{K}_{\min}(\mu)$ gives a control on μ.

The proof of Proposition 6.2 is complete. ☐

Proof of Theorem 23. Let $U \in \mathcal{V}_{\mu_*}$. If $\mu \equiv \inf_{\mathbb{R}} |U| > 0$, then Proposition 6.2 gives $\mathcal{K}(U) \ge E(U_0) + \mu^2/K > E(U_0) = \mathcal{K}(U_0)$. If $\inf_{\mathbb{R}} |U| = 0$, we deduce from Proposition 6.1 that $\mathcal{K}(U) \ge E(U_0) + 2M r_0^4 \sin^2((\mathfrak{P}(U) - \pi r_0^2) r_0^2)$. Hence $\mathcal{K}(U) > E(U_0)$ except if $\mathcal{K}(U) = E(U_0)$. From the study of the equality case in Proposition 6.1, it follows that $U \in \{e^{i\theta} U_0(\cdot - y),\ y \in \mathbb{R}, \theta \in \mathbb{R}\}$, as claimed. ☐

6C. *Proof of Theorem 24.* As a first step, we shall need a quantified version of Proposition 6.1.

Proposition 6.3. *There exist $\epsilon_0 > 0$ and $K > 0$, depending only on f, such that, for any $U \in \mathfrak{X}$ verifying*

$$\mathcal{K}(U) - E(U_0) \le \epsilon_0 \quad \textit{and} \quad \inf_{\mathbb{R}} |U| \le \epsilon_0,$$

we have

$$\inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} d_{\mathfrak{X}}(U, e^{i\theta} U_0(\cdot - y)) \le K\Big(\mathcal{K}(U) - E(U_0) + \inf_{\mathbb{R}} |U|\Big)^{1/4}.$$

Proof. First, we translate the problem in space so that $\mu \equiv \inf_{\mathbb{R}} |U| = |U|(0)$ and shall choose the phase factor later. We follow the lines of the proof of Proposition 6.1 and actually get (writing $U = Ae^{i\phi}$ locally in $\{|U|>0\}$)

$$\begin{aligned}
&\int_0^{+\infty} |\partial_x U|^2 + F(|U|^2)\,dx \\
&= \int_0^{+\infty} \mathbf{1}_{|U|>0} A^2(\partial_x\phi)^2\,dx + \int_0^{+\infty} \big|\partial_x |U|\big|^2 + F(|U|^2)\,dx \\
&= \int_0^{+\infty} \mathbf{1}_{|U|>0} A^2(\partial_x\phi)^2\,dx + \int_0^{+\infty} \Big[\sqrt{F(|U|^2)} - \big|\partial_x|U|\big|\Big]^2\,dx + 2\int_0^{+\infty} \Big|\sqrt{F(|U|^2)}\partial_x|U|\Big|\,dx \\
&\geq \int_0^{+\infty} \mathbf{1}_{|U|>0} A^2(\partial_x\phi)^2\,dx + \int_0^{+\infty} \Big[\sqrt{F(|U|^2)} - \big|\partial_x|U|\big|\Big]^2\,dx + 2\int_\mu^{r_0} \sqrt{F(s^2)}\,ds.
\end{aligned}$$

Arguing similarly in $(-\infty,0)$, we get

$$E(U) \geq E(U_0) + \int_{\mathbb{R}} \mathbf{1}_{|U|>0} A^2(\partial_x\phi)^2\,dx + \int_{\mathbb{R}} \Big[\sqrt{F(|U|^2)} - |\partial_x|U||\Big]^2\,dx - 4\int_0^{\mu} \sqrt{F(s^2)}\,ds. \quad (41)$$

The gradient of the phase is controlled using (41). We shall now estimate the modulus part. Let us define $A \equiv |U|$ and

$$h \equiv \partial_x A - \sqrt{F(A^2)},$$

for which we have, by (41),

$$\|h\|^2_{L^2(\mathbb{R})} \leq E(U) - E(U_0) + 4\int_0^{\mu} \sqrt{F(s^2)}\,ds \leq E(U) - E(U_0) + K\mu. \quad (42)$$

Recall that U_0 satisfies $(\partial_x U_0)^2 = F(U_0^2)$ in $\mathbb{R}$; hence $\partial_x U_0 = \sqrt{F(U_0^2)}$ in $\mathbb{R}_+$. Setting $\Theta \equiv A - |U_0|$, we infer

$$\partial_x \Theta = \sqrt{F(A^2)} - \sqrt{F(U_0^2)} + h \quad \text{in } \mathbb{R}_+.$$

We set, for $x \geq 0$,

$$G(x,\theta) \equiv \sqrt{F((U_0(x)+\theta)^2)} - \sqrt{F(U_0^2(x))} + \frac{U_0(x) f(U_0^2(x))\theta}{\sqrt{F(U_0^2(x))}}.$$

Since U_0 satisfies $\partial_x^2 U_0 + U_0(x) f(U_0^2(x)) = 0$ and $\partial_x U_0 = \sqrt{F(U_0^2(x))}$ in $\mathbb{R}_+$, it follows that

$$G(x,\theta) = \sqrt{F((U_0(x)+\theta)^2)} - \sqrt{F(U_0^2(x))} - \frac{\partial_x^2 U_0(x)}{\partial_x U_0(x)}\theta.$$

Moreover, by the Taylor expansion, we infer the existence of $K>0$ and $\theta_0>0$ such that, for $|\theta| \leq \theta_0$, $x \in \mathbb{R}_+$,

$$|G(x,\theta)| \leq K\theta^2.$$

The estimate is clearly uniform in view of the exponential decay of $\partial_x U_0$ at infinity. Therefore,

$$\partial_x \Theta = \frac{\partial_x^2 U_0(x)}{\partial_x U_0(x)}\Theta + G(x,\Theta) + h(x). \quad (43)$$

We view this ODE as a linear ODE with source term $G(x, \Theta(x)) + h(x)$. Since $\partial_x U_0$ solves the homogeneous equation, we infer, from Duhamel's formula and the fact that $\Theta(0) = A(0) - U_0(0) = |U(0)| = \mu$, that, for $x \geq 0$,

$$\Theta(x) = \mu + \partial_x U_0(x) \int_0^x \frac{G(z, \Theta(z)) + h(z)}{\partial_x U_0(z)}\, dz. \tag{44}$$

We shall prove that this equation implies that, if μ and $\|h\|_{L^2(\mathbb{R}_+)}$ are sufficiently small, then

$$\|\Theta\|_{L^2(\mathbb{R}_+)} \leq K(\|h\|_{L^2(\mathbb{R}_+)} + \mu). \tag{45}$$

We assume $\mu < \theta_0/2$. Note that, since U_0 is a kink, we have the decays given in Proposition 2. Hence, there exist two positive constants K_1 and K_2 such that

$$\frac{\mathrm{e}^{-\mathfrak{c}_s x}}{K_1} \leq \partial_x U_0(x) \leq K_2 \mathrm{e}^{-\mathfrak{c}_s x} \quad \text{for all } x \in \mathbb{R}_+.$$

In particular, if $|\Theta(x)| \leq \theta_0$ in the interval $[0, R]$, then (44) implies, for $x \in [0, R]$,

$$\begin{aligned} |\Theta(x)| &\leq \mu + K_1 K_2 \mathrm{e}^{-\mathfrak{c}_s x} \int_0^x \mathrm{e}^{\mathfrak{c}_s z} \big[K \|\Theta\|_{L^\infty([0,R])} |\Theta(z)| + |h|(z)\big]\, dz \\ &\leq \mu + \frac{K K_1 K_2}{\mathfrak{c}_s} \|\Theta\|^2_{L^\infty([0,R])} + \frac{K_1 K_2}{\sqrt{2\mathfrak{c}_s}} \|h\|_{L^2(\mathbb{R}_+)} \end{aligned}$$

by the Cauchy–Schwarz inequality. We thus choose $\|h\|_{L^2(\mathbb{R}_+)} + \mu$ sufficiently small so that

$$4\left(\mu + \frac{K_1 K_2}{\sqrt{2\mathfrak{c}_s}} \|h\|_{L^2(\mathbb{R})}\right) \leq \tilde{\theta}_0 \equiv \min\left\{\theta_0, \frac{\mathfrak{c}_s}{2 K K_1 K_2}\right\}.$$

Then, we consider the set $\mathcal{R}$ of all $R > 0$ such that $|\Theta(x)| \leq \tilde{\theta}_0$ in the interval $[0, R]$. Since $\Theta \in H^1(\mathbb{R}, \mathbb{C})$ is continuous by the Sobolev embedding and $|\Theta(0)| = \mu < \tilde{\theta}_0$, $\mathcal{R} \neq \varnothing$ and is closed in $\mathbb{R}_+^*$. Moreover, the above estimate shows that, for $R \in \mathcal{R}$,

$$\|\Theta\|_{L^\infty([0,R])} \leq \mu + \frac{K K_1 K_2}{\mathfrak{c}_s} \|\Theta\|^2_{L^\infty([0,R])} + \frac{K_1 K_2}{\sqrt{2\mathfrak{c}_s}} \|h\|_{L^2(\mathbb{R}_+)},$$

which gives

$$\|\Theta\|_{L^\infty([0,R])} \left(1 - \frac{K K_1 K_2}{\mathfrak{c}_s} \|\Theta\|_{L^\infty([0,R])}\right) \leq \mu + \frac{K_1 K_2}{\sqrt{2\mathfrak{c}_s}} \|h\|_{L^2(\mathbb{R}_+)},$$

and then

$$\|\Theta\|_{L^\infty([0,R])} \leq 2\left[\mu + \frac{K_1 K_2}{\sqrt{2\mathfrak{c}_s}} \|h\|_{L^2(\mathbb{R})}\right] \leq \frac{\tilde{\theta}_0}{2} < \tilde{\theta}_0. \tag{46}$$

Consequently, $\mathcal{R}$ is open in $\mathbb{R}_+^*$. By connexity, $\mathcal{R} = \mathbb{R}_+^*$, proving (45). In what follows, we assume $\|h\|_{L^2(\mathbb{R}_+)} + \mu$ is sufficiently small so that $\|\Theta\|_{L^\infty} \leq \tilde{\theta}_0$; thus $|G(x, \Theta)| \leq K\Theta^2$. In particular,

$$|\Theta(x)| \leq \mu + K_1 K_2 \int_0^x \mathrm{e}^{-\mathfrak{c}_s (x-z)} \big[K \|\Theta\|_{L^\infty([0,R])} |\Theta(z)| + |h|(z)\big]\, dz.$$

For $R > 0$ to be determined later, we then deduce from classical convolution estimates that

$$\|\Theta\|_{L^2([0,R])} \leq \mu\sqrt{R} + K_3\|\Theta\|_{L^\infty(\mathbb{R}_+)}\|\Theta\|_{L^2([0,R])} + K_3\|h\|_{L^2(\mathbb{R}_+)}.$$

Imposing that $\|h\|_{L^2(\mathbb{R}_+)} + \mu$ be smaller if necessary, we may assume that

$$K_3\|\Theta\|_{L^\infty(\mathbb{R}_+)} \leq K_3K(\|h\|_{L^2(\mathbb{R}_+)} + \mu) \leq \tfrac{1}{2},$$

so that we get

$$\|\Theta\|_{L^2([0,R])} \leq K_4\big(\mu\sqrt{R} + \|h\|_{L^2(\mathbb{R}_+)}\big).$$

Reporting this into (43) provides

$$\|\partial_x\Theta\|^2_{L^2([0,R])} \leq K_5(\mu^2R + \|h\|^2_{L^2(\mathbb{R}_+)}).$$

Arguing similarly in $[-R, 0]$ and using (42), we obtain an H^1 estimate for Θ in $[-R, R]$:

$$\|\Theta\|^2_{H^1([-R,R])} \leq K_6\big(E(U) - E(U_0) + \mu^2R + \mu\big). \tag{47}$$

We now turn to the estimate in $\{|x| \geq R\}$. For that purpose, we write

$$\begin{aligned}\int_{|x|\geq R}(\partial_x|U|)^2+\frac{1}{K}(|U|^2-r_0^2)^2\,dx\leq E(U)-E(U_0)+\int_{|x|\geq R}(\partial_xU_0)^2+F(U_0^2)\,dx\\ -\int_{|x|\leq R}(\partial_x|U|)^2+F(|U|^2)\,dx+\int_{|x|\leq R}(\partial_xU_0)^2+F(U_0^2)\,dx.\end{aligned} \tag{48}$$

Since U_0 decays exponentially (see Proposition 2), it follows that

$$\int_{|x|\geq R}|\partial_xU_0|^2 + F(U_0^2)\,dx \leq K\mathrm{e}^{-\mathfrak{c}_sR}.$$

Furthermore, by integration by parts,

$$\begin{aligned}&-\int_{|x|\leq R}(\partial_x|U|)^2+F(|U|^2)\,dx+\int_{|x|\leq R}(\partial_xU_0)^2+F(U_0^2)\,dx\\ &=-\int_{|x|\leq R}2\partial_xU_0\partial_x\Theta-2U_0f(U_0^2)\Theta\,dx-\int_{|x|\leq R}(\partial_x\Theta)^2+F([U_0+\Theta]^2)-F(U_0^2)-2U_0F'(U_0^2)\Theta\,dx\\ &\leq\int_{|x|\leq R}2\Theta[\partial_x^2U_0+U_0f(U_0^2)]\,dx-2\Theta(+R)\partial_xU_0(+R)+2\Theta(-R)\partial_xU_0(-R)+K\|\Theta\|^2_{H^1([-R,+R])}\\ &\leq K\mathrm{e}^{-\mathfrak{c}_sR}+K\big(E(U)-E(U_0)+\mu^2R+\mu\big).\end{aligned}$$

For the second-to-last line, we have used that $\theta \mapsto F([U_0+\theta]^2) - F(U_0^2) - 2U_0F'(U_0^2)\theta$ is $\mathcal{O}(\theta^2)$ as $\theta \to 0$ and, for the last line, that U_0 solves $\partial_x^2U_0 + U_0f(U_0^2) = 0$, the exponential decay of ∂_xU_0 and the uniform bound on Θ. Reporting these estimates into (48) provides

$$\begin{aligned}\|\Theta\|^2_{H^1(\{|x|\geq R\})} &= \int_{|x|\geq R} (\partial_x|U| - \partial_x|U_0|)^2 + (|U|-|U_0|)^2\,dx\\ &\leq 2\int_{|x|\geq R} (\partial_x|U|)^2 + (\partial_x|U_0|)^2 + (|U|-r_0)^2 + (|U_0|-r_0)^2\,dx\\ &\leq K[E(U)-E(U_0)+\mathrm{e}^{-\mathfrak{c}_s R} + \mu^2 R + \mu].\end{aligned}$$

Combining this with (47), we deduce that, for any $R>0$, we have

$$\|\Theta\|^2_{H^1(\mathbb{R})} \leq K[E(U)-E(U_0)+\mathrm{e}^{-\mathfrak{c}_s R} + \mu^2 R + \mu].$$

We then choose $R=\mu^{-1}$ if $\mu>0$ or $R\to+\infty$ if $\mu=0$, and get

$$\|\Theta\|_{H^1(\mathbb{R})} \leq K\sqrt{E(U)-E(U_0)+\mu}.$$

Notice that, if $f'<0$ everywhere, then we may give a quick proof of the above estimate, since, using here again integration by parts and that $\partial_x^2 U_0 + U_O f(U_O^2)=0$, we may deduce that

$$E(U)-E(U_0) \geq -4\mu\partial_x U_0(0) + \int_0^{+\infty} (\partial_x\Theta)^2\,dx + \int_{\mathbb{R}} F((U_0+\Theta)^2) - F(U_0^2) - 2U_0\Theta F'(U_0^2)\,dx,$$

and, since $f'<0$, $F((U_0+\theta)^2)-F(U_0^2)-2U_0\theta F'(U_0^2)\geq \theta^2/K$ by the Taylor expansion, providing the desired H^1 bound on Θ.

Observe now that

$$\mathcal{K}(U)-E(U_0)\geq E(U)-E(U_0)\geq \int_{\mathbb{R}} \mathbf{1}_{|U|>0}[A\partial_x\phi]^2\,dx;$$

hence

$$\begin{aligned}\|\partial_x U - \partial_x U_0\|_{L^2(\mathbb{R})} &= \|\partial_x(|U_0|+\Theta)\mathrm{e}^{i\phi}\mathbf{1}_{|U|>0} + i\mathbf{1}_{|U|>0}A\partial_x\phi\mathrm{e}^{i\phi} - \partial_x U_0\|_{L^2(\mathbb{R})}\\ &\leq \|\mathrm{e}^{i\phi}\mathbf{1}_{|U|>0}\partial_x|U_0| - \partial_x U_0\|_{L^2(\mathbb{R})} + \|\mathbf{1}_{|U|>0}A\partial_x\phi\|_{L^2(\mathbb{R})} + \|\Theta\|_{L^2(\mathbb{R})}\\ &\leq \|\mathrm{e}^{i\phi}\mathbf{1}_{|U|>0}\partial_x|U_0| - \partial_x U_0\|_{L^2(\mathbb{R})} + K[\mathcal{K}(U)-E(U_0)+\mu]^{1/2}.\end{aligned} \tag{49}$$

We distinguish now the cases $\mu=0$ and $\mu>0$, and begin with the assumption $\mu>0$. Then, we have a global lifting $U=A\mathrm{e}^{i\phi}$ and

$$\begin{aligned}d_{\mathfrak{X}}(U,U_0) &= \|\partial_x U - \partial_x U_0\|_{L^2(\mathbb{R})} + \|\,|U|-|U_0|\,\|_{L^2(\mathbb{R})} + |U(0)-U_0(0)|\\ &= \|\partial_x U - \partial_x U_0\|_{L^2(\mathbb{R})} + \|\Theta\|_{L^2(\mathbb{R})} + \mu\\ &\leq \|\mathrm{e}^{i\phi}\partial_x|U_0| - \partial_x U_0\|_{L^2(\mathbb{R})} + K[\mathcal{K}(U)-E(U_0)+\mu]^{1/2}.\end{aligned}$$

Now, we notice that

$$\begin{aligned}\|\mathrm{e}^{i\phi}\partial_x|U_0| - \partial_x U_0\|^2_{L^2(\mathbb{R})} &= 2\int_{\mathbb{R}}\big[(\partial_x U_0)^2 - \partial_x U_0\partial_x|U_0|\cos\phi\big]\,dx\\ &= 2\int_0^{+\infty}(\partial_x U_0)^2(1-\cos\phi)\,dx + 2\int_{-\infty}^0 (\partial_x U_0)^2(1+\cos\phi)\,dx\end{aligned} \tag{50}$$

and that

$$\mathcal{H}(U) - E(U_0) \geq 2Mr_0^4 \sin^2 \frac{\mathfrak{P}(U) - r_0^2\pi}{2r_0^2} \geq \frac{1}{K}(P(U) - r_0^2\pi \mod 2\pi r_0^2)^2. \tag{51}$$

We define $\delta = (\mathcal{H}(U) - E(U_0) + \mu)^{1/4}$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left|\int_{|x|\geq\delta}(A^2 - r_0^2)\partial_x\phi \, dx\right| &\leq \frac{K}{\inf_{|x|\geq\delta} A}\left(\int_{|x|\geq\delta}(A^2 - r_0^2)^2 \, dx\right)^{1/2}\left(\int_{|x|\geq\delta}(A\partial_x\phi)^2 \, dx\right)^{1/2} \\ &\leq \frac{K}{\inf_{|x|\geq\delta} A}\big(E(U) - E(U_0) + \mu\big)^{1/2}.\end{aligned}$$

Inserting this into (51) gives

$$\begin{aligned}\left|\int_{|x|\leq\delta}(A^2 - r_0^2)\partial_x\phi \, dx - r_0^2\pi \mod 2\pi r_0^2\right| &\leq K\left[(\mathcal{H}(U) - E(U_0))^{1/2} + \frac{1}{\inf_{|x|\geq\delta} A}\big(E(U) - E(U_0) + \mu\big)^{1/2}\right] \\ &\leq \frac{K}{\inf_{|x|\geq\delta} A}\big(\mathcal{H}(U) - E(U_0) + \mu\big)^{1/2}.\end{aligned}$$

In addition, by the Cauchy–Schwarz inequality,

$$\left|\int_{|x|\leq\delta} A^2\partial_x\phi \, dx\right| \leq \sqrt{2\delta}(\sup_{|x|\leq\delta} A)\big(\mathcal{H}(U) - E(U_0) + \mu\big)^{1/2}.$$

Consequently,

$$\begin{aligned} & r_0^2\big|\phi(+\delta) - \phi(-\delta) - \pi \mod 2\pi\big| \\ &\quad\leq \left|\int_{|x|\leq\delta}(A^2 - r_0^2)\partial_x\phi \, dx - r_0^2\pi \mod 2\pi r_0^2\right| + \sqrt{2\delta}(\sup_{|x|\leq\delta} A)\big(\mathcal{H}(U) - E(U_0) + \mu\big)^{1/2} \\ &\quad\leq \left[\frac{K}{\inf_{|x|\geq\delta} A} + \sqrt{2\delta}(\sup_{|x|\leq\delta} A)\right]\big(\mathcal{H}(U) - E(U_0) + \mu\big)^{1/2}. \end{aligned} \tag{52}$$

From our choice $\delta = (\mathcal{H}(U) - E(U_0) + \mu)^{1/4} \ll 1$ and since $\|\Theta\|_{L^\infty(\mathbb{R})} \leq K(\mathcal{H}(U) - E(U_0) + \mu)^{1/2} = \mathcal{O}(\delta^2)$, we infer $\inf_{|x|\geq\delta} A \geq \inf_{|x|\geq\delta}|U_0| - \|\Theta\|_{L^\infty(\mathbb{R})} \geq \delta/K$. Similarly, we have $\sup_{|x|\leq\delta} A \leq \sup_{|x|\leq\delta}|U_0| + \|\Theta\|_{L^\infty(\mathbb{R})} \leq K\delta$. Reporting this into (52) yields

$$\big|\phi(+\delta) - \phi(-\delta) - \pi \mod 2\pi\big| \leq K\delta.$$

We now freeze the gauge invariance by imposing $\phi(+\delta) = 0$. Note that then $\phi(-\delta) = \pi + \mathcal{O}(\delta)$. Furthermore, since $\phi(+\delta) = 0$,

$$\int_{|x|\geq\delta}(\partial_x\phi)^2 \, dx \leq \frac{K}{(\inf_{|x|\geq\delta} A)^2}\int_{|x|\geq\delta} A^2(\partial_x\phi)^2 \, dx \leq \frac{K}{\delta^2}\delta^4 = K\delta^2,$$

which implies, for $x \geq \delta$,

$$|1 - \cos\phi(x)| \leq |1 - \cos\phi(0)| + \left|\int_\delta^x \partial_x\phi \sin\phi\right| \leq K\delta\sqrt{x}$$

and, similarly, since $\cos\phi(-\delta) = \cos(\pi + \mathcal{O}(\delta)) = -1 + \mathcal{O}(\delta^2)$, for $x \le -\delta$,

$$|1 + \cos\phi(x)| \le K\delta\sqrt{|x|}.$$

We turn back to (50) and infer

$$\begin{aligned}\|e^{i\phi}\partial_x|U_0| - \partial_x U_0\|^2_{L^2(\mathbb{R})} &\le K\delta + 2\int_\delta^{+\infty}(\partial_x U_0)^2(1-\cos\phi)\,dx + 2\int_{-\infty}^{-\delta}(\partial_x U_0)^2(1+\cos\phi)\,dx\\ &\le K\delta + K\delta\int_{\mathbb{R}}(\partial_x U_0)^2\sqrt{|x|}\,dx = K\delta.\end{aligned}$$

Inserting these estimates in (49), it follows that

$$d_{\mathfrak{X}}(U, U_0) \le K\delta.$$

We now turn to the case $\mu = 0$. Without loss of generality, we may assume that $|U| > 0$ in $(-\infty, 0)$ (since $|U| \to r_0 > 0$ at $\pm\infty$), and let $\ell \ge 0$ be such that $|U|(\ell) = 0$ and $|U| > 0$ in $(\ell, +\infty)$. We first estimate ℓ by writing that

$$|U_0|(\ell) = |U|(\ell) + \Theta(\ell) = \Theta(\ell) \le \|\Theta\|_{L^\infty(\mathbb{R})} \le K\big(\mathcal{H}(U) - E(U_0) + \mu\big)^{1/2} = K\delta^2;$$

thus $\ell \le K\delta^2$. Moreover, we have two local liftings $U = Ae^{i\phi_+}$ in $[\ell, +\infty)$ and $U = Ae^{i\phi_-}$ in $(-\infty, 0)$. Going back to (49), we then deduce

$$\begin{aligned}&d_{\mathfrak{X}}(U, U_0)\\ &\quad\le \|e^{i\phi_-}\partial_x|U_0| - \partial_x U_0\|_{L^2(-\infty,0)} + \|e^{i\phi_+}\partial_x|U_0| - \partial_x U_0\|_{L^2(\ell,+\infty)} + K\delta + K[\mathcal{H}(U) - E(U_0)]^{1/2}.\end{aligned}$$

Arguing as for the case $\mu > 0$, we obtain $|U| = A \ge \delta/K$ in $[\ell+\delta, +\infty)$ and in $(-\infty, -\delta)$. By definition of $\mathfrak{P}$, we have

$$\mathfrak{P}(U) = \int_{-\delta}^{\ell+\delta}\langle iU|\partial_x U\rangle + \int_{\ell+\delta}^{+\infty}(A^2 - r_0^2)\partial_x\phi_+\,dx + r_0^2\phi_+(\ell+\delta) + \int_{-\infty}^{-\delta}(A^2 - r_0^2)\partial_x\phi_-\,dx - r_0^2\phi_+(-\delta)$$

in $\mathbb{R}/(2\pi r_0^2\mathbb{Z})$; hence the same arguments as in the case $\mu > 0$ provide

$$\big|\phi_+(\ell+\delta) - \phi_+(-\delta) - \pi \mod 2\pi\big| \le K\delta,$$

since the integral $\int_{-\delta}^{\ell+\delta}\langle iU|\partial_x U\rangle$ is bounded by $K\sqrt{\delta}$ by the Cauchy–Schwarz inequality. Imposing $\phi_+(\ell+\delta)$ for the gauge invariance, we infer $1 - \cos(\phi_+(\ell+\delta)) = 0$ and $\phi_+(-\delta) = \pi + \mathcal{O}(\sqrt{\delta}) \mod 2\pi$; hence $1 + \cos(\phi_-(-\delta)) = \mathcal{O}(\delta)$. Therefore, we conclude as before that

$$d_{\mathfrak{X}}(U, U_0) \le K\delta,$$

which finishes the proof of the proposition. □

In order to prove Theorem 24, we use Proposition 6.2, which provides

$$\mathcal{H}(U) \ge E(U_0) + \frac{1}{K}(\inf_{\mathbb{R}}|U|)^2;$$

thus

$$\mu = \inf_{\mathbb{R}} |U| \leq K \sqrt{\mathcal{H}(U) - E(U_0)}.$$

Inserting this bound in Proposition 6.3 then gives

$$d_{\mathfrak{X}}(U, U_0) \leq K\big[\mathcal{H}(U) - E(U_0) + K \sqrt{\mathcal{H}(U) - E(U_0)}\big]^{1/4} \leq K \sqrt[8]{\mathcal{H}(U) - E(U_0)},$$

and the proof is complete.

7. About the stability analysis for the sonic waves ($c = \mathfrak{c}_s$)

We have left aside in our study the case of the sonic waves ($c = \mathfrak{c}_s$), but would like to say a few words on the difficulties associated with this critical case.

We note that, if there exists a sonic nontrivial traveling wave, it does not vanish; hence we may use the hydrodynamical formulation (15) of (NLS) as in [Lin 2002]. The point is that the Sturm–Liouville operator (see [Lin 2002, Section 4])

$$L \equiv -\frac{\partial}{\partial x}\left(\frac{1}{4(r_0^2 - \eta)}\frac{\partial}{\partial x}\right) + q(x),$$

with

$$q(x) \equiv \frac{(\partial_x \eta)^2}{4(r_0^2 - \eta)^3} - \frac{\partial}{\partial x}\left(\frac{\partial_x \eta}{4(r_0^2 - \eta)^2}\right) - \frac{1}{2} f'(r_0^2 - \eta) - \frac{c^2 r_0^4}{4(r_0^2 - \eta)^3},$$

has, by Weyl's theorem, essential spectrum $\sigma_{\text{ess}}(L) = [0, +\infty)$ when $c = \mathfrak{c}_s$. Indeed, we know from Proposition 2 that $\eta_{\mathfrak{c}_s}$ and its derivatives tend to zero at infinity; hence, as $x \to \pm\infty$, $q(x) \to -\frac{1}{2} f'(r_0^2) - c^2/(4r_0^2) = 0$ since $c^2 = \mathfrak{c}_s^2 = -2r_0^2 f'(r_0^2)$. Therefore, there does not exist $\delta > 0$ such that $\langle Hp, p\rangle \geq \delta \|p\|^2$ for any p orthogonal to the subspace spanned by the negative and the zero eigenvalue, and thus the Grillakis–Shatah–Strauss theory does not apply.

In the case $(dP/dc)_{|c=\mathfrak{c}_s} < 0$, where it is natural to expect stability, a natural thing would be to try to work with the functional

$$\mathcal{L}(\psi) \equiv E(\psi) - \mathfrak{c}_s P(\psi) + \frac{M}{2}(P(\psi) - P(U_{\mathfrak{c}_s}))^2$$

and to follow the lines of the proof of Theorem 23. Indeed, the spectral analysis shall not give positive definiteness of the Hessian due to presence of essential spectrum down to 0. Therefore, we may study $\mathcal{L}$ at fixed $\mu = \inf_{\mathbb{R}} |\psi|$ close to $\inf_{\mathbb{R}} |U_{\mathfrak{c}_s}|$. When $0 < c_* < \mathfrak{c}_s$ and $(dP/dc)_{|c=c_*} \neq 0$, the infimum of $|U_c|$ contains a neighborhood of $\inf_{\mathbb{R}} |U_{c_*}|$ for c close to c_*. For $c_* = \mathfrak{c}_s$, this is no longer the case: we have only a one-sided neighborhood of $\inf_{\mathbb{R}} |U_{\mathfrak{c}_s}|$. It is plausible that the study for μ in this one-sided neighborhood of $\inf_{\mathbb{R}} |U_{\mathfrak{c}_s}|$ can be done as in the proof of Theorem 23, but, for the remaining values of μ, we have to find a sharp ansatz, which is not very easy to find.

Furthermore, for the linear instability which is expected if $(dP/dc)_{|c=\mathfrak{c}_s} > 0$, let us mention the following point. For the eigenvalue problem studied in [Benzoni-Gavage 2010b], the characteristic equation for the constant coefficient limit at infinity, namely

$$r^4 - (\mathfrak{c}_s^2 - c_*^2) r^2 - 2c_* \lambda r + \lambda^2 = 0,$$

becomes, when $c_* = \mathfrak{c}_s$,

$$r^4 - 2\mathfrak{c}_s \lambda r + \lambda^2 = 0. \tag{53}$$

The behavior of the roots for small λ is then different from the case $0 < c_* < \mathfrak{c}_s$. Indeed, there exists a root $\sim \lambda/(2\mathfrak{c}_s)$ for $\lambda \to 0$, and, for the three other roots, we use the variable $r = \sqrt[3]{\lambda} z$, which transforms $r^4 - 2\mathfrak{c}_s \lambda r + \lambda^2 = 0$ into $z^4 - 2\mathfrak{c}_s z + \lambda^{2/3} = 0$. This last equation has, for $\lambda \to 0$, three roots $\sim j^k \sqrt[3]{2\mathfrak{c}_s}$, where $j = \mathrm{e}^{2i\pi/3}$ and $k = 0, 1, 2$. In particular, (53) has three roots $\sim j^k \sqrt[3]{2\mathfrak{c}_s \lambda}$, $k = 0, 1, 2$. The value $\lambda = 0$ is then a branching point, and we shall have a smooth problem not in λ but in $\sqrt[3]{\lambda}$. Since analyticity is not necessary for our purpose, we may define an Evans function $\tilde{\mathbb{D}}$ in $\mathbb{R}_+$, smooth, and such that, for $\lambda > 0$, $\tilde{\mathbb{D}}(\sqrt[3]{\lambda}) = 0$ if and only if λ is an unstable eigenvalue for (27). Another difficulty comes from the fact that it will be difficult to find an analytic extension of the Evans function $\tilde{\mathbb{D}}$ near 0 since, by Proposition 2, for $c_* = \mathfrak{c}_s$, u_* and η_* decay only at an algebraic rate and not an exponential rate. Consequently, we can not use the gap lemma of [Gardner and Zumbrun 1998] and [Kapitula and Sandstede 1998]. Finally, as a straightforward computation shows, the stable and unstable subspaces for the eigenvalue problem are transverse for $\lambda > 0$ but their continuous extensions at $\lambda = 0$ have a nontrivial intersection. Therefore, both stability and instability require some further analysis, and the situation is then much more delicate than the one studied in Section 5A.

Appendix A. Construction of a Liapounov functional in the stable case in the Grillakis–Shatah–Strauss framework

We work with the notations of [Grillakis et al. 1987], and recall them briefly. We consider a Hamiltonian equation in a real Hilbert space $\mathscr{X}$, with scalar product $(\cdot,\cdot)_{\mathscr{X}}$, under the form

$$\frac{\partial u}{\partial t} = JE'(u), \tag{$\mathscr{H}$}$$

where $J : \mathscr{X}^* \to \mathscr{X}$ is a closed linear operator with dense domain and skew-symmetric. Assume that T is a $\mathscr{C}_0$-group of unitary operators in $\mathscr{X}$ generated by $T'(0)$, which is skew-adjoint and with dense domain, and that E is invariant by T; that is, $E(T(s)u) = E(u)$ for any $s \in \mathbb{R}$, $u \in \mathscr{X}$. Assume moreover that $T(s)J = JT(-s)^*$ for any $s \in \mathbb{R}$ and that there exists $B : \mathscr{X} \to \mathscr{X}^*$, linear and bounded, such that $B^* = B$ and JB is an extension of $T'(0)$. We then set

$$Q(u) \equiv \tfrac{1}{2} \langle Bu, u \rangle_{\mathscr{X}^*, \mathscr{X}}.$$

The basic assumptions of [Grillakis et al. 1987] are the following ones.

Assumption 1 (existence of solutions). For any $r > 0$ there exists $t_* > 0$, depending only on r, such that, for any $u^{\mathrm{in}} \in \mathscr{X}$, there exists a $u \in \mathscr{C}((-t_*, t_*), \mathscr{X})$ with $u(0) = u^{\mathrm{in}}$ solution of ($\mathscr{H}$) in the sense that, for any $\varphi \in D(J) \subset \mathscr{X}^*$,

$$\frac{d}{dt}\langle u(t), \varphi\rangle_{\mathscr{X}^*,\mathscr{X}} = -\langle E'(u(t)), J\varphi\rangle_{\mathscr{X}^*,\mathscr{X}} \quad \text{in } \mathscr{D}'((-t_*, t_*)),$$

and verifying $E(u(t)) = E(u^{\text{in}})$ and $Q(u(t)) = Q(u^{\text{in}})$ for $t \in (-t_*, t_*)$.

Assumption 2 (existence of "bound states"). There exists an interval $\Omega \subset \mathbb{R}$, not reduced to a singleton, and a mapping $\Omega \ni \omega \mapsto \phi_\omega \in \mathscr{X}$ of class $\mathscr{C}^1$ such that, for any $\omega \in \Omega$,

$$E'(\phi_\omega) = \omega Q'(\phi_\omega), \quad \phi_\omega \in D(T'(0)^3) \cap D(JIT'(0)^2), \quad T'(0)\phi_\omega \neq 0.$$

Assumption 3 (spectral decomposition). For each $\omega \in \Omega$, the operator $H_\omega \equiv E''(\phi_\omega) - \omega Q''(\phi_\omega) : \mathscr{X} \to \mathscr{X}^*$ has its kernel spanned by $T'(0)\phi_\omega$, has one negative simple eigenvalue and the rest of its spectrum is positive and bounded away from zero.

Under Assumption 2, we consider some $\omega_* \in \Omega$ and the associated bound state ϕ_{ω_*}, and then define, for $M > 0$, the functional

$$\mathscr{L}_{\omega_*}(u) \equiv E(u) - \omega_* Q(u) + \frac{M}{2}(Q(u) - Q(\phi_{\omega_*}))^2.$$

It is clear that ϕ_{ω_*} is a critical point of $\mathscr{L}_{\omega_*}$: $\mathscr{L}'(\phi_{\omega_*}) = E'(\phi_{\omega_*}) - \omega_* Q'(\phi_{\omega_*}) = 0$. We denote by

$$\Lambda \equiv \mathscr{L}''_{\omega_*}(\phi_{\omega_*}) = H_{\omega_*} + M\langle Q'(\phi_{\omega_*}), \cdot\rangle_{\mathscr{X}^*,\mathscr{X}} Q'(\phi_{\omega_*})$$

its second derivative, which is a self-adjoint operator. The main result of this appendix is the following.

Theorem 26. *We make Assumptions 2 and 3 and suppose that the operator $\langle Q'(\phi_{\omega_*}), \cdot\rangle_{\mathscr{X}^*,\mathscr{X}} Q'(\phi_{\omega_*})$ is a compact perturbation of H_{ω_*}. If $(dQ(\phi_\omega)/d\omega)_{|\omega=\omega_*} < 0$ and*

$$M > \frac{1}{-\dfrac{dQ(\phi_\omega)}{d\omega}_{|\omega=\omega_*}},$$

there exists $\delta > 0$ such that

$$\langle \Lambda v, v\rangle \geq \delta\|v\|^2 \quad \textit{for all } v \in X \quad \textit{s.t. } (v, T'(0)\phi_{\omega_*})_{\mathscr{X}} = 0.$$

In particular, for any $u \in X$ with $\inf_{s\in\mathbb{R}} \|u - T(s)\phi_{\omega_*}\|^2 \leq \epsilon$, we have

$$\inf_{s\in\mathbb{R}} \|u - T(s)\phi_{\omega_*}\|^2 \leq \frac{2}{\delta}(\mathscr{L}(u) - \mathscr{L}(\phi_{\omega_*})).$$

Therefore, when Assumption 1 is moreover satisfied, the (global) solution $u(t)$ to $(\mathscr{H})$ with initial datum u^{in} satisfies

$$\sup_{t\in\mathbb{R}} \inf_{s\in\mathbb{R}} \|u(t) - T(s)\phi_{\omega_*}\|^2 \leq \frac{2}{\delta}(\mathscr{L}(u^{\text{in}}) - \mathscr{L}(\phi_{\omega_*})) \leq K\|u^{\text{in}} - \phi_{\omega_*}\|^2,$$

provided the right-hand side is sufficiently small.

We point out that the condition that the operator $\langle Q'(\phi_{\omega_*}), \cdot\rangle_{\mathscr{X}^*,\mathscr{X}} Q'(\phi_{\omega_*})$ is a compact perturbation of H_{ω_*} is not very restrictive, since, in many cases coming from PDEs, it involves less derivatives than H_{ω_*} and $Q'(\phi_{\omega_*})$ tends to zero at spatial infinity.

This type of Liapounov functional has been used in [Barashenkov 1996] to prove that the traveling waves of (NLS) in dimension one are stable when $dP/dc < 0$. The proof follows basically the one in [Barashenkov 1996], but some points have to be clarified. The interest of this type of Liapounov functional is that the saddle point ϕ_{ω_*} is now a nondegenerate local minimum for $\mathscr{L}_{\omega_*}$. This is a great advantage for numerical simulation of the "bound states", since a gradient flow method on $\mathscr{L}_{\omega_*}$ can be used. This approach has been used, with a very similar functional, by N. Papanicolaou and P. Spathis [1999] for the numerical simulation of the traveling waves for a planar ferromagnets model. In the same spirit, in [Chiron and Scheid 2012], we also use a gradient flow method on this type of functional for the numerical simulation of the traveling waves for (NLS) in two dimensions.

Proof of Theorem 26. Recall that the spectrum of H_{ω_*} is, by Assumption 3, such that $-\lambda_*^2 \in \sigma(H_{\omega_*})$, $0 \in \sigma(H_{\omega_*})$ and $\sigma(H_{\omega_*}) \setminus \{-\lambda_*^2, 0\} \subset [\delta, +\infty)$ for some $\delta > 0$. Since we assume that $\langle Q'(\phi_{\omega_*}), \cdot\rangle Q'(\phi_{\omega_*})$ is a compact perturbation of H_{ω_*}, the essential spectrum of Λ is the same as the one of H_{ω_*}, and hence is included in $[\delta, +\infty)$. Furthermore, $0 \in \sigma(H_{\omega_*})$ and $\ker(H_{\omega_*}) = \mathbb{R}T'(0)\phi_{\omega_*}$ by Assumption 3. Since $Q'(\phi_{\omega_*}) = B\phi_{\omega_*}$ and JB is an extension of $T'(0)$, we have that $\langle Q'(\phi_{\omega_*}), T'(0)\phi_{\omega_*}\rangle_{\mathscr{X}^*,\mathscr{X}} = \langle B\phi_{\omega_*}, JB\phi_{\omega_*}\rangle_{\mathscr{X}^*,\mathscr{X}} = 0$; hence $\Lambda(T'(0)\phi_{\omega_*}) = 0$. Noticing that $\langle Q'(\phi_{\omega_*}), \cdot\rangle_{\mathscr{X}^*,\mathscr{X}} Q'(\phi_{\omega_*})$ is a nonnegative operator, we infer that $\ker(\Lambda) = \ker(H_{\omega_*}) = \mathbb{R}T'(0)\phi_{\omega_*}$ is one-dimensional. Therefore, it suffices to show that Λ has no eigenvalues in $(-\infty, 0)$. As we have seen that $\langle Q'(\phi_{\omega_*}), \cdot\rangle_{X^*,X} Q'(\phi_{\omega_*})$ is a nonnegative operator, we deduce that $\sigma(\Lambda) \subset [-\lambda_*^2, +\infty)$. Let us first show that $-\lambda_*^2 \notin \sigma(\Lambda)$ by contradiction. If $-\lambda_*^2$ is an eigenvalue of Λ, then there exists $v \in X$, $v \neq 0$, such that $0 = (\Lambda + \lambda_*^2)v = (H + \lambda_*^2)v + M\langle Q'(\phi_{\omega_*}), v\rangle_{\mathscr{X}^*,\mathscr{X}} Q'(\phi_{\omega_*})$. Taking the duality product with v yields $0 = \langle (H_{\omega_*} + \lambda_*^2)v, v\rangle_{\mathscr{X}^*,\mathbb{B}X} + M\langle Q'(\phi_{\omega_*}), v\rangle^2_{\mathscr{X}^*,\mathscr{X}}$. Since the two terms in the sum are nonnegative, this implies $\langle Q'(\phi_{\omega_*}), v\rangle_{\mathscr{X}^*,\mathscr{X}} = 0$ and $\langle (H_{\omega_*} + \lambda_*^2)v, v\rangle_{\mathscr{X}^*,\mathscr{X}} = 0$, which in turn implies $v \in \ker(H_{\omega_*} + \lambda_*^2) = \mathbb{R}\chi$ (here, χ is a negative eigenvector of H_{ω_*} for the eigenvalue $-\lambda_*^2 < 0$). As a consequence, we must have $\langle Q'(\phi_{\omega_*}), \chi\rangle_{\mathscr{X}^*,\mathscr{X}} = 0$. On the other hand, differentiating the equality $E'(\phi_\omega) - \omega Q'(\phi_\omega) = 0$ at $\omega = \omega_*$ yields $Q'(\phi_{\omega_*}) = H_{\omega_*}\phi'$, where $\phi' \equiv (d\phi/d\omega)_{|\omega=\omega_*}$. Thus we must have $0 = \langle H_{\omega_*}\phi', \chi\rangle_{\mathscr{X}^*,\mathscr{X}} = \langle H_{\omega_*}\chi, \phi'\rangle_{\mathscr{X}^*,\mathscr{X}} = -\lambda_*^2(\chi, \phi')$. Therefore, ϕ' is orthogonal to χ and this gives $\langle H_{\omega_*}\phi', \phi'\rangle_{\mathscr{X}^*,\mathscr{X}} \geq 0$. However, this is not possible if $(dQ(\phi_\omega)/d\omega)_{|\omega=\omega_*} < 0$, since $(dQ(\phi_\omega)/d\omega)_{|\omega=\omega_*} = -\langle H_{\omega_*}\phi', \phi'\rangle_{\mathscr{X}^*,\mathscr{X}}$. As a consequence, if λ is a negative element of the spectrum of Λ, then $-\lambda_*^2 < \lambda < 0$ and λ is an eigenvalue: there exists $v \in X$ such that $v \neq 0$ and

$$\lambda v = \Lambda v = H_{\omega_*}v + M\langle Q'(\phi_{\omega_*}), v\rangle_{\mathscr{X}^*,\mathscr{X}} Q'(\phi_{\omega_*}).$$

Since $-\lambda_*^2 < \lambda < 0$, we then infer

$$v = -M\langle Q'(\phi_{\omega_*}), v\rangle_{\mathscr{X}^*,\mathscr{X}} (H_{\omega_*} - \lambda)^{-1} Q'(\phi_{\omega_*}). \tag{A-1}$$

Since $v \neq 0$, we can not have $\langle Q'(\phi_{\omega*}), v\rangle_{\mathscr{X}^*,\mathscr{X}} = 0$. Then, taking the scalar product of (A-1) with $\mathbb{I}^{-1}Q'(\phi_{\omega*})$ (here, $\mathbb{I} : \mathscr{X} \to \mathscr{X}^*$ is the usual Riesz isomorphism) gives

$$g(\lambda) = 0, \quad \text{where } g(t) \equiv 1 + M\big((H_{\omega*} - t)^{-1}Q'(\phi_{\omega*}), \mathbb{I}^{-1}Q'(\phi_{\omega*})\big)_{\mathscr{X}}, \quad -\lambda_*^2 < t < 0.$$

It is clear that g is smooth in $(-\lambda_*^2, 0)$ and that

$$g'(t) = M\big((H_{\omega*} - t)^{-2}Q'(\phi_{\omega*}), \mathbb{I}^{-1}Q'(\phi_{\omega*})\big)_{\mathscr{X}} = M\|(H_{\omega*} - t)^{-1}Q'(\phi_{\omega*})\|_{\mathscr{X}}^2 > 0.$$

We now study the limit of g at 0^-. Let us recall that $H_{\omega*}\phi' = Q'(\phi_{\omega*})$ and that we have already seen that $\langle Q'(\phi_{\omega*}), T'(0)\phi_{\omega*}\rangle_{\mathscr{X}^*,\mathscr{X}} = 0$; i.e., $\mathbb{I}^{-1}Q'(\phi_{\omega*})$ is orthogonal to $\ker(H_{\omega*})$. Therefore, as $t \to 0^-$,

$$\big((H_{\omega*} - t)^{-1}Q'(\phi_{\omega*}), \mathbb{I}^{-1}Q'(\phi_{\omega*})\big) \to (\phi', \mathbb{I}^{-1}Q'(\phi_{\omega*})) = \langle Q'(\phi_{\omega*}), \phi'\rangle_{\mathscr{X}^*,\mathscr{X}} = \frac{dQ(\phi_\omega)}{d\omega}_{|\omega=\omega_*}$$

and thus

$$g(t) \to 1 + M\frac{dQ(\phi_\omega)}{d\omega}_{|\omega=\omega_*} \quad \text{as } t \to 0^-.$$

Since $(dQ(\phi_\omega)/d\omega)_{|\omega=\omega_*} < 0$ by hypothesis, it follows that, if $M > -1/(dQ(\phi_\omega)/d\omega)_{|\omega=\omega_*} > 0$, the function g increases in $(-\lambda_*^2, 0)$ and tends to some negative limit at 0^-. In particular, g is negative; hence we can not have $g(\lambda) = 0$ with $\lambda \in (-\lambda_*^2, 0)$. We have therefore shown that the spectrum of Λ consists in a simple eigenvalue 0 with eigenspace spanned by $T'(0)\phi_{\omega*}$ and the rest of the spectrum is positive and bounded away from 0. This concludes the proof. □

We would like to point out the fact that, in the proof of [Barashenkov 1996], $-\lambda_*^2 \notin \sigma(\Lambda)$ was not shown, the kernel of Λ was not studied and the essential spectrum was not considered. Moreover, the functional spaces are not given; hence we do not know for which perturbations stability holds.

Appendix B. From linear to nonlinear instability

We still consider in this appendix an abstract Hamiltonian equation in the framework of [Grillakis et al. 1987]

$$\frac{\partial u}{\partial t} = JE'(u) \tag{$\mathscr{H}$}$$

on the real Hilbert space $\mathscr{X}$, with scalar product $(\cdot,\cdot)_{\mathscr{X}}$. Here $E : \mathscr{X} \to \mathbb{R}$ is of class $\mathscr{C}^2$ and $J : \mathscr{X}^* \to \mathscr{X}$ is a closed linear operator with dense domain and skew-symmetric in the sense that $(u, Jw)_{\mathscr{X}} = -\langle w, Ju\rangle_{\mathscr{X}^*,\mathscr{X}}$ for $u \in \mathscr{X}$, $w \in \mathscr{X}^*$.

We assume that there exists a $\mathscr{C}_0$-group T of unitary operators in $\mathscr{X}$ generated by $T'(0)$, which is skew-adjoint and with dense domain, and that E is invariant by T; that is, $E(T(\omega)u) = E(u)$ for any $\omega \in \mathbb{R}$, $u \in \mathscr{X}$. Assume moreover that $T(\omega)J = JT(-\omega)^*$ for any $\omega \in \mathbb{R}$ and that there exists $B : \mathscr{X} \to \mathscr{X}^*$, linear and bounded, such that $B^* = B$ and JB is an extension of $T'(0)$. We then set

$$Q(u) \equiv \tfrac{1}{2}\langle Bu, u\rangle_{\mathscr{X}^*,\mathscr{X}},$$

which is invariant by the flow ($\mathcal{H}$) (see [Grillakis et al. 1987]). By "bound state", we mean a particular solution U of ($\mathcal{H}$) of the form $U(t) = T(\omega t)\phi$ for some $\omega \in \mathbb{R}$ and where $\phi \in \mathcal{X}$, $\phi \neq 0$. In other words, $E'(\phi) = \omega Q'(\phi)$.

There exist an open interval $\Omega \subset \mathbb{R}$, not reduced to a singleton, and a mapping $\Omega \ni \omega \mapsto \phi_\omega \in X$ of class $\mathscr{C}^1$ such that, for any $\omega \in \Omega$,

$$E'(\phi_\omega) = \omega Q'(\phi_\omega), \quad \phi_\omega \in D(T'(0)^3) \cap D(JIT'(0)^2), \quad T'(0)\phi_\omega \neq 0.$$

The solution $U(t) = T(\omega t)\phi$ is said to be stable in $\mathcal{X}$ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that any solution to ($\mathcal{H}$) with initial datum $u^{\text{in}} \in B_{\mathcal{X}}(\phi, \delta)$ is global in time and remains in $B_{\mathcal{X}}(\phi, \varepsilon)$ for $t \geq 0$. Otherwise, it is said to be unstable. This supposes some knowledge of the Cauchy problem for ($\mathcal{H}$) (at least existence of solutions). If we are given some Banach space $\mathcal{Y} \supset \mathcal{X}$ with continuous imbedding $\mathcal{X} \hookrightarrow \mathcal{Y}$, we may also say that the solution $U(t) = T(\omega t)\phi$ is said to be stable from $\mathcal{X}$ to $\mathcal{Y}$ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that any solution to ($\mathcal{H}$) with initial datum $u^{\text{in}} \in B_{\mathcal{X}}(\phi, \delta)$ remains in $B_{\mathcal{Y}}(\phi, \varepsilon)$ for $t \geq 0$. Clearly, a solution stable in $\mathcal{X}$ is precisely a solution stable from $\mathcal{X}$ to $\mathcal{X}$, and is also stable from $\mathcal{X}$ to $\mathcal{Y}$; hence instability from $\mathcal{X}$ to $\mathcal{Y}$ is a stronger statement that instability in $\mathcal{X}$.

In our framework, the notion of orbital stability is more relevant. Let us consider $\mathbb{G}$ a group and $\mathbb{T} : \mathbb{R} \times \mathbb{G} \to \mathscr{GL}_c(\mathcal{X})$ a unitary representation of $\mathbb{R} \times \mathbb{G}$ on $\mathcal{X}$, extending $T : \mathbb{R} \to \mathcal{X}$ and leaving E and Q invariant. Then, $U(t) = T(\omega t)\phi$ is said to be orbitally stable in $\mathcal{X}$ (for the group $\mathbb{G}$) if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that any solution to ($\mathcal{H}$) with initial datum $u^{\text{in}} \in B(\phi, \delta)$ is global in time and remains in $\bigcup_{(\omega,g) \in \mathbb{R} \times \mathbb{G}} B(\mathbb{T}(\omega, g)\phi, \varepsilon)$ for $t \geq 0$. We may also define orbital instability from $\mathcal{X}$ to $\mathcal{Y} \supset \mathcal{X}$ in a natural way.

In [Grillakis et al. 1987; 1990], a general framework for the stability analysis for the "bound state" has been given. In particular, the nonlinear orbital instability is proved in [Grillakis et al. 1987] through the construction of a Liapounov-type functional. However, this method does not give a clear understanding neither of how we get farther from the "bound state", nor on which timescale it occurs.

The need for allowing an additional group of invariances $\mathbb{G}$ can be seen in the case of bound state solutions, that is $U(t) = \mathrm{e}^{i\omega t}\phi_\omega$, to the nonlinear Schrödinger equation

$$i\partial_t \Psi + \Delta\Psi + \Psi f(|\Psi|^2) = 0, \tag{NLS}$$

or the nonlinear Klein–Gordon equation in $\mathbb{R}^d$

$$\partial_t^2 \Psi = \Delta\Psi + \Psi f(|\Psi|^2), \tag{NLKG}$$

since, then, the invariance by translation in space must be taken into account in the definition of orbital stability, and we are in a case where $\mathbb{G} = \mathbb{R}^d$ acts naturally by translation. The translations are taken into account in [Cazenave and Lions 1982]. In [Grillakis et al. 1987; 1990], the notion of orbital stability is for $\mathbb{G}$ trivial. It is clear from the definition that orbital stability for $\mathbb{G} = \{0\}$ implies orbital stability for arbitrary $\mathbb{G}$. For the instability in the nonlinear Schrödinger equation or the nonlinear Klein–Gordon equation, [Grillakis et al. 1987] and [Shatah and Strauss 1985] work with radial H^1 functions. The fact that this also implies the orbital instability with the action of $\mathbb{G} = \mathbb{R}^d$ by translations follows immediately from the fact that for any $\theta \in [0, 2\pi]$ the manifold $\mathfrak{M}_\theta \equiv \{e^{i\theta}\phi(\cdot - y),\ y \in \mathbb{R}^d\}$ is orthogonal to $H^1_{\mathrm{rad}}(\mathbb{R}^d)$.

For the stability analysis of a "bound state" $U(t) = T(\omega_* t)(\phi_{\omega_*})$, it is natural to consider the linearization of $(\mathscr{H})$ near ϕ. More precisely, we linearize according to the ansatz $u(t) = T(\omega_* t)(\phi_{\omega_*} + v(t))$, so that the "bound state" becomes stationary. The linearized problem then becomes

$$\frac{\partial v}{\partial t} = J(E''(\phi) - \omega Q''(\phi))v = \mathscr{J}\mathscr{L}v, \tag{$\mathscr{H}_{\mathrm{lin}}$}$$

where, $\mathbb{I} : \mathscr{X} \to \mathscr{X}^*$ denoting the Riesz isomorphism, $\mathscr{J} \equiv J\mathbb{I} : \mathscr{X} \to \mathscr{X}$ is skew-adjoint.

The purpose of this appendix is to give a general result, for Hamiltonian equations, showing that linear instability implies nonlinear (orbital) instability. By linear instability, we mean that the complexification of $[\mathscr{J}\mathscr{L}]_{\mathbb{C}}$ has at least one eigenvalue in the right half-space $\{\mathrm{Re} > 0\}$. The argument follows ideas from the works of F. Rousset and N. Tzvetkov [2008; 2009].

Showing the existence of an unstable eigenvalue can be done through various techniques: see [Grillakis et al. 1990] (in the framework of [Grillakis et al. 1987] when J is onto), [Grillakis 1988] (assuming a special structure of the Hamiltonian equation); for uses of the Vakhitov–Kolokolov function, see [de Bouard 1995], [Di Menza and Gallo 2007] or [Pelinovsky and Kevrekidis 2008]. When J is not onto, we quote [Lopes 2002]. For one-dimensional partial differential equations, one may also use the Evans function (see the survey [Sandstede 2002]) as in [Pego and Weinstein 1992; Gardner and Zumbrun 1998; Kapitula and Sandstede 1998; Zumbrun 2008]. The paper [Lin 2008] proposes another approach which allows treating pseudodifferential equations, such as the BBM equation, the Benjamin–Ono equation, regularized Boussinesq equations, the intermediate long wave equation, etc.

In order to pass from linear to nonlinear instability, the following result is standard. We refer to the paper by D. Henry, J. Perez and W. Wreszinski [Henry et al. 1982]. It can also be found in [Grillakis 1988; Shatah and Strauss 2000].

Theorem B.1 [Henry et al. 1982; Grillakis 1988; Shatah and Strauss 2000]. *We assume that $\mathscr{A}$ generates a continuous semigroup on X and that $\sigma(\mathscr{A})$ meets the right half-space $\{\mathrm{Re} > 0\}$. We assume moreover that $F : X \to X$ is locally Lipschitz continuous and satisfies, for some $\alpha > 0$, $\|F(v)\|_X = \mathbb{O}(\|v\|_X^{1+\alpha})$ as $v \to 0$. Then, the solution $\phi = 0$ is unstable for the equation $\partial_t v = \mathscr{A}v + F(v)$.*

In [Shatah and Strauss 2000], it is claimed that an orbital instability result can also be established. Theorem B.1 shows nonlinear instability without assuming that the equation is Hamiltonian. However, if $(\mathscr{H}_{\mathrm{lin}})$ can be solved using a semigroup, it does not give the growth of its norm. Moreover, it does not say that, if the initial datum is in a most unstable direction, that is, an eigendirection of $\mathscr{A}$ corresponding to an eigenvalue of maximal positive real part (plus the complex conjugate if necessary), then one can track the exponential growth of the solution. In particular, it does not explain the mechanism of instability and does not give any information on the timescale on which one see the instability. For instance, some strong instability results are shown by proving blow-up in finite time (see [Berestycki and Cazenave 1981]), but the instability due to an exponentially growing mode holds on a much smaller timescale.

We wish to provide here some results clarifying the instability mechanism by tracking the exponentially growing mode.

A spectral mapping theorem for linearized Hamiltonian equations. When we want to prove a nonlinear instability result from a linear instability one, we need some information on the growth of the semigroup $\mathcal{J}\mathcal{L}$, when such a semigroup $\mathrm{e}^{t\mathcal{J}\mathcal{L}}$ exists, which we shall assume in this appendix. The growth estimate on $\mathrm{e}^{t\mathcal{J}\mathcal{L}}$ relies classically on the following spectral mapping result due to J. Prüss [1984], which generalizes the work of L. Gearhart [1978].

Theorem B.2 [Prüss 1984]. *Let X be a complex Hilbert space and $\mathcal{A}$ an unbounded operator on X which generates a continuous semigroup $\mathrm{e}^{t\mathcal{A}}$ on X. For $t \in (0, +\infty)$, we have*

$$\sigma(\mathrm{e}^{t\mathcal{A}}) \setminus \{0\} = \Big\{\mathrm{e}^{\lambda t},\ \textit{either}\ \Big(\lambda + \frac{2i\pi}{t}\mathbb{Z}\Big) \cap \sigma(\mathcal{A}) \neq \varnothing,\ \textit{or}\ \sup_{k\in\mathbb{Z}} \Big\| \Big(\mathcal{A} - \lambda - \frac{2i\pi k}{t}\Big)^{-1} \Big\|_{\mathcal{L}_c(X)} = +\infty \Big\}.$$

The following result is an immediate corollary.

Corollary B.3. *Let X be a complex Hilbert space and $\mathcal{A}$ an unbounded operator on X which generates a continuous semigroup $\mathrm{e}^{t\mathcal{A}}$ on X. Assume that, for any $\gamma \in \mathbb{R}^*$, we have*

$$\limsup_{|\tau|\to+\infty} \|(\mathcal{A} - \gamma - i\tau)^{-1}\|_{\mathcal{L}_c(X)} < +\infty,$$

and that there exists $\vartheta_0 \in [0, +\infty)$ such that $\sigma_{\mathrm{ess}}(\mathcal{A}) = \{i\vartheta,\ \vartheta \in \mathbb{R},\ |\vartheta| \geq \vartheta_0\}$. Then, for any $t \in (0, +\infty)$, the spectral mapping holds: $\sigma(\mathrm{e}^{t\mathcal{A}}) \setminus \{0\} = \mathrm{e}^{t\sigma(\mathcal{A})}$.

Proof. Since $\sigma_{\mathrm{ess}}(\mathcal{A}) = \{i\vartheta,\ \vartheta \in \mathbb{R},\ |\vartheta| \geq \vartheta_0\}$, we have $\mathbb{S}^1 \subset \mathrm{e}^{t\sigma(\mathcal{A})} \subset \sigma(\mathrm{e}^{t\mathcal{A}})$. If $\lambda \in \mathbb{C}$ does not have modulus one, then note that, when $(\lambda + (2i\pi/t)\mathbb{Z}) \cap \sigma(\mathcal{A}) = \varnothing$, the supremum for $k \in \mathbb{Z}$ in Theorem B.2 can be $+\infty$ only when $|k| \to +\infty$, and we conclude with our hypothesis. □

The fact that we exclude 0 in the spectral mapping theorem just comes from the fact that we consider a semigroup and not a group. However, in most Hamiltonian PDEs, we have time reversibility and we have actually a continuous group and not only a semigroup. In most cases, we work with $A : D(A) \subset Y \to Y$ where Y is a real Hilbert space, thus for applying Theorem B.2 or Corollary B.3 we have to consider, as usual, the complexified operator $A_{\mathbb{C}} : D(A_{\mathbb{C}}) \equiv D(A) \oplus iD(A) \subset Y_{\mathbb{C}} \equiv Y \oplus iY \to Y_{\mathbb{C}}$ defined by $A_{\mathbb{C}}(u + iv) = Au + iAv$.

It seems that the first time Theorem B.2 is used to prove a growth estimate on a semigroup was by T. Kapitula and B. Sandstede [1998]. Later, F. Gesztesy et al. [2000] also used this result for bound states for (NLS). The bounds on the resolvent in [Kapitula and Sandstede 1998] were proved using the particular structure of the linearized operator. In [Gesztesy et al. 2000], the computations are more involved and rely on suitable kernel estimates of some Hilbert–Schmidt operators. The same type of estimates have also been used in [Di Menza and Gallo 2007].

The main objective of this appendix is to provide a generalization of these results to a wide class of Hamiltonian equations. Indeed, the approaches in [Kapitula and Sandstede 1998; Gesztesy et al. 2000] seem specific to the problem. In addition, it is not clear whether the computations in [Gesztesy et al.

2000; Di Menza and Gallo 2007] can be extended to other types of equations. In particular, in [Chiron 2012] and in the present paper, we have a situation similar to the one studied in [Di Menza and Gallo 2007], namely traveling wave solutions to a nonlinear Schrödinger equation with nonzero condition at infinity, but, for nonzero propagation speeds, the traveling wave is not real-valued (as it is in [Di Menza and Gallo 2007] for stationary waves or for bound state solutions), and the block diagonal structure of the linearized Hamiltonian disappears. An additional difficulty is that, in [Chiron 2012] and the present work, the limits of the traveling waves at $+\infty$ and $-\infty$ differ.

The proof we give is based on ideas from [Rousset and Tzvetkov 2008; 2009] and makes very few spectral assumptions on $\mathscr{L}$.

Assumption A. The spectrum of $\mathscr{L}$ consists in a finite number (possibly zero) of nonpositive eigenvalues $-\mu_1, \dots, -\mu_q$ in $(-\infty, 0]$, each one with finite multiplicity, and the rest of the spectrum is positive and bounded away from 0. Furthermore, for any $1 \le k \le q$, we have $\ker(\mathscr{L} + \mu_k) \subset D(\mathscr{J})$ and $\mathscr{J}[\ker(\mathscr{L} + \mu_k)] \subset D(\mathscr{L})$. Finally, there exists $\vartheta_0 \in [0, +\infty)$ such that $\sigma_{\mathrm{ess}}(\mathscr{J}\mathscr{L}) = \{i\vartheta,\ \vartheta \in \mathbb{R},\ |\vartheta| \ge \vartheta_0\}$.

The first hypothesis on the location of the spectrum of $\mathscr{L}$ is quite weak, since it is satisfied when $\mathscr{L}$ is bounded from below and has essential spectrum positive and bounded away from zero. Indeed, if $\delta > 0$ is such that $\sigma_{\mathrm{ess}}(\mathscr{L}) \subset [2\delta, +\infty)$, then the eigenvalues of $\mathscr{L}$ in $(-\infty, \delta]$ are isolated, of finite multiplicity, and are bounded from below by assumption. The second hypothesis $\ker(\mathscr{L} + \mu_k) \subset D(\mathscr{L}\mathscr{J})$ is a regularity assumption on the eigenvectors.

Let us recall that Theorem 25 ensures that the number of eigenvalues (with algebraic multiplicities) of $\mathscr{J}\mathscr{L}$ in the right half-space $\{\mathrm{Re} > 0\}$ is less than or equal to the number of negative eigenvalues of $\mathscr{L}$, and hence is finite under Assumption A. Let us now state our main result, the proof of which is given starting on page 1413.

Theorem B.4. *We make Assumption A and suppose that $\mathscr{J}\mathscr{L}$ generates a continuous semigroup. Then, for any $t \in (0, +\infty)$, the spectral mapping holds*: $\sigma(\mathrm{e}^{t[\mathscr{J}\mathscr{L}]_{\mathbb{C}}}) \setminus \{0\} = \mathrm{e}^{t\sigma([\mathscr{J}\mathscr{L}]_{\mathbb{C}})}$. *Furthermore, defining*

$$\gamma_0 \equiv \sup\bigl\{\mathrm{Re}(\lambda),\ \lambda \in \sigma([\mathscr{J}\mathscr{L}]_{\mathbb{C}}) \cap \{\mathrm{Re} \ge 0\}\bigr\} \in [0, +\infty),$$

for any $\beta > 0$, there exists $M(\beta) > 0$ such that, for any $t \ge 0$, we have

$$\|\mathrm{e}^{t\mathscr{J}\mathscr{L}}\|_{\mathscr{L}_c(\mathscr{X})} \le M(\beta)\mathrm{e}^{(\gamma_0+\beta)t}.$$

Assume in addition $\gamma_0 > 0$ and define

$$m \equiv \max\bigl\{\textit{algebraic multiplicity of } \lambda,\ \lambda \in \sigma([\mathscr{J}\mathscr{L}]_{\mathbb{C}}) \textit{ s.t. } \mathrm{Re}\,\lambda = \gamma_0\bigr\} \in \mathbb{N}^*.$$

Then, there exists $M_0 > 0$ such that, for any $t \ge 0$, we have

$$\|\mathrm{e}^{t\mathscr{J}\mathscr{L}}\|_{\mathscr{L}_c(\mathscr{X})} \le M_0(1+t)^{m-1}\mathrm{e}^{\gamma_0 t}.$$

In particular, Theorem B.4 provides a very simple proof of the spectral mapping theorem used in [Gesztesy et al. 2000; Di Menza and Gallo 2007]. Indeed, the self-adjoint operator $\mathscr{L}$ involved in these papers is block diagonal:

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix},$$

and both $\mathcal{L}_1$, $\mathcal{L}_2$ have at most two nonnegative eigenvalues. More generally, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed self-adjoint operators on X verifying Assumption A and if $\mathcal{N} : X \to X$ is a linear bounded operator which is compact with respect to $\mathcal{L}_1$ and $\mathcal{L}_2$, then the self-adjoint operator

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & \mathcal{N} \\ \mathcal{N}^* & \mathcal{L}_2 \end{pmatrix}$$

also satisfies Assumption A. Indeed, $\mathcal{L}$ is bounded from below (since $\mathcal{N}$ is bounded) and its essential spectrum is $\sigma_{\mathrm{ess}}(\mathcal{L}_1) \cup \sigma_{\mathrm{ess}}(\mathcal{L}_2) \subset [\delta, +\infty)$ for some positive δ, since $\mathcal{N}$ is compact with respect to $\mathcal{L}_1$ and $\mathcal{L}_2$. In [Kapitula and Sandstede 1998, Section 7.1; Georgiev and Ohta 2012, Proposition 10], a spectral mapping theorem is used for such an operator. In [Kapitula and Sandstede 1998], the specific algebra of the problem was used, and for [Georgiev and Ohta 2012], the proof relies on the arguments in [Gesztesy et al. 2000], but here again, in both cases, we may use Theorem B.4 to show the same result.

Passing from linear to nonlinear instability.

Semilinear type models. We start with a classical result for "semilinear" equations, proved on page 1416.

Theorem B.5. *Let X be a real Hilbert space, and consider an evolution equation of the form*

$$\frac{dv}{dt} = \mathcal{A}v + \Phi(v),$$

where $\Phi : X \to X$ is a locally Lipschitz mapping satisfying $\Phi(v) = \mathcal{O}(\|v\|_X^2)$ as $v \to 0$ and $\mathcal{A}$ is a linear operator which generates a semigroup. We assume that $\mathcal{A}_{\mathbb{C}} : D(\mathcal{A}_{\mathbb{C}}) \subset X_{\mathbb{C}} \to X_{\mathbb{C}}$ has an unstable eigenvalue in the right half-plane $\{\mathrm{Re} > 0\}$ and a finite number of eigenvalues in $\{\mathrm{Re} > 0\}$. We define

$$\gamma_0 \equiv \sup\bigl\{\mathrm{Re}(\mu),\ \mu \in \sigma([\mathcal{J}\mathcal{L}]_{\mathbb{C}}) \cap \{\mathrm{Re} > 0\}\bigr\} \in (0, +\infty)$$

and fix $\lambda \in \sigma(\mathcal{A}_{\mathbb{C}})$ with $\mathrm{Re}(\lambda) = \gamma_0$ and an associated eigenvector $w_{\mathbb{C}} \in D(\mathcal{A}_{\mathbb{C}})$ such that $\|\mathrm{Re}(w_{\mathbb{C}})\|_X = 1$. Assume furthermore that there exist $0 \le \beta < \gamma_0$ and $M_0 > 0$ such that

$$\|\mathrm{e}^{t\mathcal{A}}\|_{\mathcal{L}_c(X)} \le M_0 \mathrm{e}^{(\gamma_0+\beta)t}.$$

Then, 0 is an unstable solution. More precisely, there exist $K > 0$, $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, for any $0 < \delta < \delta_0$, the solution v with initial datum $v^{\mathrm{in}} = \delta\,\mathrm{Re}(w_{\mathbb{C}}) \in D(\mathcal{A})$ exists at least on $[0, \ln(2\varepsilon_0/\delta)/\gamma_0]$ and satisfies, for $0 \le t \le \ln(2\varepsilon_0/\delta)/\gamma_0$,

$$\|v(t) - \delta\,\mathrm{Re}(\mathrm{e}^{t\lambda} w_{\mathbb{C}})\|_X \le K\delta^2 \mathrm{e}^{2t\gamma_0} \quad \textit{and} \quad \|v(t)\|_X \ge \delta \mathrm{e}^{t\gamma_0} - K\delta^2\mathrm{e}^{2t\gamma_0}.$$

In particular, for $0 < \varepsilon < \varepsilon_0$, we see the instability for $t = (1/\gamma_0)\ln(2\varepsilon/\delta)$. If Y is a Banach space containing X and with continuous imbedding $X \hookrightarrow Y$, the trivial solution 0 is also unstable from X to Y.

Let us observe that it is always possible to choose the (complex) eigenvector w so that $\mathrm{Re}(w_{\mathbb{C}}) \neq 0$ since, for any $\theta \in \mathbb{R}$, $\mathrm{e}^{i\theta} w$ is also an eigenvector. The following corollary deals with the orbital instability. We recall that, under Assumption A, $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}$ has a finite number of eigenvalues in $\{\mathrm{Re} > 0\}$.

Corollary B.6. *We make Assumption A and suppose that $\mathcal{J}\mathcal{L}$ generates a continuous semigroup. Let $\mathcal{Y}$ be a Banach space containing $\mathcal{X}$ and with continuous imbedding $\mathcal{X} \hookrightarrow \mathcal{Y}$. Assume moreover that $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}$ has at least one eigenvalue in $\{\mathrm{Re} > 0\}$ and choose $\lambda \in \mathbb{C}$ with*

$$\mathrm{Re}(\lambda) = \gamma_0 \equiv \max\big\{\mathrm{Re}(\mu),\ \mu \in \sigma([\mathcal{J}\mathcal{L}]_{\mathbb{C}}) \cap \{\mathrm{Re} > 0\}\big\} \in (0, +\infty)$$

and $w_{\mathbb{C}} \in D(\mathcal{A}_{\mathbb{C}})$ an associated eigenvector such that $\|\mathrm{Re}(w_{\mathbb{C}})\|_{\mathcal{X}} = 1$. We assume moreover that $\mathfrak{M} \equiv \{\mathbb{T}(\omega, g)\phi_{\omega_*},\ \omega \in \mathbb{R},\ g \in \mathbb{G}\}$ is a $\mathscr{C}^1$ submanifold of $\mathcal{X}$. We finally suppose that the equation ($\mathcal{H}$) is semilinear in the sense that there exists $\Phi : \mathcal{X} \to \mathcal{X}$ locally Lipschitz continuous such that $\Phi(v) = \mathcal{O}(\|v\|^2_{\mathcal{X}})$ as $v \to 0$ and

$$J(E' - \omega_* Q')(\phi_{\omega_*} + v) = J(E'' - \omega_* Q'')(\phi_{\omega_*})[v] + \Phi(v).$$

Then, there exist $K > 0$, $\varepsilon_0 > 0$ and $\delta_0 > 0$, depending only on $\mathrm{Re}(w_{\mathbb{C}})$ and $\mathfrak{M}$, with the following properties. For any $0 < \delta < \delta_0$, the solution u to ($\mathcal{H}$) with initial datum $u^{\mathrm{in}} = \phi_{\omega_*} + \delta\,\mathrm{Re}(w_{\mathbb{C}}) \in D(\mathcal{A})$ exists at least on $[0, \ln(2\varepsilon_0/\delta)/\gamma_0]$ and satisfies, for $0 \le t \le \ln(2\varepsilon_0/\delta)/\gamma_0$,

$$\mathrm{dist}_{\mathcal{Y}}(u(t), \mathfrak{M}) \ge \frac{\delta}{K}\mathrm{e}^{t\gamma_0} - K\delta^2\mathrm{e}^{2t\gamma_0}.$$

In particular, the "bound state" solution $T(\omega_* t)\phi_{\omega_*}$ is nonlinearly orbitally unstable from $\mathcal{X}$ to $\mathcal{Y}$ and, for $0 < \varepsilon < \varepsilon_0/K$, we see the instability for $t = (1/\gamma_0)\ln(2K\varepsilon/\delta)$.

In [Henry et al. 1982], a similar assertion is made for the orbital instability in the remark after Theorem 2 there, but with $\mathcal{Y} = \mathcal{X}$. For applications to PDEs, the space $\mathcal{X}$ may be a Sobolev space H^s, and $\mathcal{Y}$ a space like L^2 or L^∞ for instance. The framework of [Grillakis et al. 1987] is the single energy space (for instance H^1), but an instability result established by tracking exponentially growing modes allows proving instability from the regular space $\mathcal{X}$ (H^1) to the nonregular space $\mathcal{Y}$ (L^2 or L^∞). Here, we may obtain instability in L^2.

Remark B.7. In the framework of [ibid.], where a Liapounov-type functional is used, it follows that the instability is seen for a time at most equal to $K\varepsilon/\delta^2$, where K is some positive constant. This timescale is much larger than the natural one $(1/\gamma_0)\ln(2K\varepsilon/\delta)$.

Some applications. We may apply our result to the nonlinear Schrödinger equation

$$i\partial_t\Psi + \Delta\Psi + \Psi f(|\Psi|^2) = 0, \tag{NLS}$$

or the nonlinear Klein–Gordon equation

$$\partial_t^2\Psi = \Delta\Psi + \Psi f(|\Psi|^2) \tag{NLKG}$$

in $\mathbb{R}^d$. We shall consider a nonlinearity f at least $\mathscr{C}^1$, so that we are in the framework of [ibid.].

• A bound state solution for these two equations is a particular solution of the form $U(t) = \mathrm{e}^{i\omega t}\phi_\omega$. The instability is in general linked to the fact that

$$\frac{d}{d\omega}\int_{\mathbb{R}^d} |\phi_\omega|^2\,dx < 0 \quad \text{for (NLS)}, \quad \text{resp.} \quad \frac{d}{d\omega}\left(\omega\int_{\mathbb{R}^d} |\phi_\omega|^2\,dx\right) < 0 \quad \text{for (NLKG)}.$$

The existence of at least one unstable eigenvalue has been shown under this assumption by [Grillakis 1988] for radial bound states with an arbitrary number of nodes and in [Grillakis et al. 1990] for radial ground states. Corollary B.6 may be applied with $\mathscr{X} = H^s(\mathbb{R}^d)$, where $s \in \mathbb{N}$, $s > d/2$ and assuming that the nonlinearity satisfies $f \in \mathscr{C}^{s+2}$, and $\mathscr{Y} = L^2(\mathbb{R}^d)$ or $L^\infty(\mathbb{R}^d)$. The result in [Mizumachi 2006] shows the instability of linearly unstable bound states for (NLS) (in dimension $d = 2$) with $f(\varrho) = \varrho^{(p-1)/2}$ by showing the exponential growth of an unstable eigenmode. Our result gives a simple proof of this result, but restricted to the sufficiently smooth cases, namely p an odd integer or $p > 5+2s > 5+d$. For nonsmooth nonlinearities, the situation is more delicate (see [Mizumachi 2006]). An alternative approach is to combine Strichartz estimates with the growth estimate on the semigroup $\mathrm{e}^{t\mathscr{J}\mathscr{L}}$ given in Theorem B.4, as in [Georgiev and Ohta 2012].

• Corollary B.6 also applies to the discrete nonlinear Schrödinger equation

$$i\partial_t\Psi_n + \varepsilon(\Psi_{n+1} - 2\Psi_n + \Psi_{n-1}) + \Psi_n f(|\Psi_n|^2) = 0 \quad \text{for all } n \in \mathbb{Z}, \tag{DNLS}$$

as studied in [Melvin et al. 2008] with the saturated nonlinearity $f(\varrho) = \beta/(1+\varrho)$, $\beta > 0$ (existence of traveling wave solution) and in [Fitrakis et al. 2007] (defocusing cubic DNLS, i.e., $f(\varrho) = -\beta\varrho$ for some $\beta > 0$). The numerical analysis in [Fitrakis et al. 2007] shows the existence of linearly unstable bound state solutions. The traveling wave solutions numerically obtained in [Melvin et al. 2008] are linearly stable, but it may happen that, for other nonlinearities f, some are linearly unstable.

Quasilinear PDEs. For quasilinear problems, we shall not make restrictions on the smoothness of the nonlinearity. The result relies on the strategy of E. Grenier [2000] and [Rousset and Tzvetkov 2008; 2009]. We consider the evolution equation

$$\frac{du}{dt} = J(L_0 u + \nabla F(u)) \tag{E}$$

for $u : \mathbb{R}^d \to \mathbb{R}^\nu$, where $F \in \mathscr{C}^\infty(\mathbb{R}^\nu, \mathbb{R})$, with the following hypotheses. The operator J is a Fourier multiplier, skew-symmetric on L^2, into and with domain containing H^1. There exists $\sigma > 0$ such that the operator L_0 is a Fourier multiplier with domain containing $H^{2\sigma}$, symmetric and having a self-adjoint realization on $L^2(\mathbb{R}^d, \mathbb{R}^\nu)$. Moreover, for some $C > 0$, the operator L_0 satisfies

$$\frac{1}{C}\|u\|_{H^\sigma}^2 \le (L_0 u, u)_{L^2} \le C\|u\|_{H^\sigma}^2.$$

The framework proposed in [Rousset and Tzvetkov 2008] was for L_0 coercive in H^1; that is, $\sigma = 1$. For the examples below, we shall have $\sigma = 1/2$ or $\sigma = 2$, which requires very few modifications to the proof of [ibid.]. We still assume that, for some group $\mathbb{G}$, there exists a unitary representation of $\mathbb{G}$ on $\mathscr{X}$, $\mathbb{T} : \mathbb{G} \to \mathscr{GL}_c(\mathscr{X})$, leaving the equation (E) invariant.

We consider a stationary solution of the evolution equation (E), that is, some $Q \in H^\infty(\mathbb{R}^d, \mathbb{R}^\nu)$ such that $L_0 Q + \nabla F(Q) = 0$. We are interested in the stability of this solution. We assume that the commutator $[J, \nabla^2 F(Q)]$ is bounded in L^2, which is the case when J is bounded in L^2 or when $d = 1$ and $J = \partial_x$. We suppose that, for the problem

$$\frac{\partial u}{\partial t} = J(L_0 u + \nabla F(u^a + u) - \nabla F(u^a) + G),$$

where u^a is smooth, bounded as well as its derivatives and $G \in \mathscr{C}(\mathbb{R}, H^s)$ for every s, we have local well-posedness for s large enough: there exists a time $T > 0$ and a unique solution in $\mathscr{C}([0, T], H^s)$. We moreover assume that, for some continuous nondecreasing function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ with $\kappa(0) = 0$, the tame estimate

$$|(\partial_x^\alpha J\{\nabla^2 F(w + v)[v]\}, \partial_x^\alpha v)_{L^2}| \le \kappa(\|w\|_{W^{s+1,\infty}} + \|v\|_{H^s})\|v\|_{H^s}^2,$$

with $|\alpha| \le s$, holds true. In order to control high-order derivatives, we finally require that, for s large enough, there exist a self-adjoint operator $\mathbb{M}_s$ and C_s such that

$$|(\mathbb{M}_s u, v)_{L^2}| \le C_s \|u\|_{H^s}\|v\|_{H^s}, \quad (\mathbb{M}_s u, u)_{L^2} \ge \|u\|_{H^s}^2 - C_s \|u\|_{H^{s-\min(\sigma,1)}}^2$$

and

$$\mathrm{Re}(JLu, \mathbb{M}_s u)_{L^2} \le C_s \|u\|_{H^s}\|u\|_{H^{s-\min(\sigma,1)}}$$

(for a criterion which ensures the existence of such a multiplier, see Lemma 5.1 in [ibid.]).

Adapting the strategy of [Rousset and Tzvetkov 2008; 2009], we may deduce the following result. Since the proof is very similar, we omit it.

Theorem B.8. *We make the above assumptions and assume moreover that $L_0 + \nabla^2 F(Q)$ satisfies Assumption A in L^2. We assume furthermore that $[J(L_0 + \nabla^2 F(Q))]_{\mathbb{C}}$ has an unstable eigenvalue in the right half-plane $\{\mathrm{Re} > 0\}$, define*

$$\gamma_0 \equiv \sup\big\{\mathrm{Re}(\lambda),\ \lambda \in \sigma([J(L_0 + \nabla^2 F(Q))]_{\mathbb{C}}) \cap \{\mathrm{Re} > 0\}\big\} \in (0, +\infty)$$

and fix $\lambda \in \sigma([J(L_0 + \nabla^2 F(Q))]_{\mathbb{C}})$ satisfying $\mathrm{Re}(\lambda) = \gamma_0$ and an associated eigenvector $w_{\mathbb{C}} \in D([J(L_0 + \nabla^2 F(Q))]_{\mathbb{C}})$ such that $\|\mathrm{Re}(w_{\mathbb{C}})\|_{H^s} = 1$. There exists $s_0 \in \mathbb{N}$ such that, if $s \ge s_0$, Q is nonlinearly unstable from H^s to L^2 and to L^∞: there exist $K > 0$, $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, for any $0 < \delta < \delta_0$, the H^s solution u to (E) with initial datum $u^{\mathrm{in}} = Q + \delta\,\mathrm{Re}(w_{\mathbb{C}}) \in H^s$ exists at least on $[0, \ln(2\varepsilon_0/\delta)/\gamma_0]$ and satisfies, for $0 \le t \le \ln(2\varepsilon_0/\delta)/\gamma_0$,

$$\|u(t) - Q - \delta\,\mathrm{Re}(\mathrm{e}^{t\lambda} w_{\mathbb{C}})\|_{H^s} \le K\delta^2 \mathrm{e}^{2t\gamma_0};$$

hence

$$\|u(t) - Q\|_{L^2} \ge \delta \mathrm{e}^{t\gamma_0} - K\delta^2\mathrm{e}^{2t\gamma_0} \quad \textit{and} \quad \|u(t) - Q\|_{L^\infty} \ge \delta \mathrm{e}^{t\gamma_0} - K\delta^2\mathrm{e}^{2t\gamma_0}.$$

If, in addition, $\mathfrak{M} \equiv \{\mathbb{T}(g)Q,\ g \in \mathbb{G}\}$ is a $\mathscr{C}^1$ submanifold of H^s, then we also have

$$\mathrm{dist}_{L^2}(u(t), \mathfrak{M}) \ge K\delta \mathrm{e}^{t\gamma_0} - K\delta^2\mathrm{e}^{2t\gamma_0} \quad \textit{and} \quad \mathrm{dist}_{L^\infty}(u(t), \mathfrak{M}) \ge K\delta \mathrm{e}^{t\gamma_0} - K\delta^2\mathrm{e}^{2t\gamma_0}.$$

In particular, for $0 < \varepsilon < \varepsilon_0/K$, *we see the nonlinear orbital instability for* $t = (1/\gamma_0)\ln(2K\varepsilon/\delta)$.

Some applications to nonlinear dispersive wave equations. Some model quasilinear equations are given by wave equations (in one space dimension) such as the generalized Korteweg–de Vries equation

$$\partial_t u + \partial_x(f(u)) + \partial_x^3 u = 0, \tag{gKdV}$$

the generalized regularized Korteweg–de Vries equation, also called Benjamin–Bona–Mahony equation or Peregrine equation when $f(u) = u^2/2$,

$$\partial_t u + \partial_x u + \partial_x(f(u)) - \partial_t\partial_x^2 u = 0, \tag{gBBM}$$

the generalized regularized Boussinesq equation

$$\partial_t^2 u - \partial_x^2 u - \partial_x^2(f(u)) - \partial_t^2\partial_x^2 u = 0. \tag{grBsq}$$

Each of these equations admits a nontrivial solitary wave solution $u(t, x) = U_c(x - ct)$ for c in $(0, +\infty)$, $(1, +\infty)$ and $(-\infty, -1) \cup (1, +\infty)$, respectively. For these solitary wave solutions, the momentum is, respectively,

$$P(U_c) = \int_{\mathbb{R}} U_c^2\,dx = \|U_c\|_{L^2}^2, \quad P(U_c) = \int_{\mathbb{R}} U_c^2 + (\partial_x U_c)^2\,dx, \quad P(U_c) = c\int_{\mathbb{R}} U_c^2 + (\partial_x U_c)^2\,dx.$$

The existence of exactly one unstable eigenvalue has been shown with the use of an Evans function by R. Pego and M. Weinstein [1992] for these three equations under the condition $dP(U_c)/dc < 0$. Lopes [2002] also gives a linear instability result. Equations (gBBM) and (grBsq) turn out to be semilinear due to the regularization effect. Indeed, they may be written

$$\partial_t u + (1-\partial_x^2)^{-1}\partial_x u + (1-\partial_x^2)^{-1}\partial_x(f(u)) = 0, \quad \partial_t^2 u - (1-\partial_x^2)^{-1}\partial_x^2 u - (1-\partial_x^2)^{-1}\partial_x^2(f(u)) = 0.$$

Therefore, Corollary B.6 applies to these two models and this shows the nonlinear instability when linear instability holds.

In [Lin 2008], some generalizations of the equations (gKdV), (gBBM) and (grBsq) have been proposed that take into account pseudodifferential operators. These are, respectively,

$$\partial_t u + \partial_x(f(u)) - \partial_x \mathcal{M} u = 0, \tag{I}$$

$$\partial_t u + \partial_x u + \partial_x(f(u)) + \partial_t \mathcal{M} u = 0 \tag{II}$$

and

$$\partial_t^2 u - \partial_x^2 u - \partial_x^2(f(u)) + \partial_t^2 \mathcal{M} u = 0. \tag{III}$$

Here, $\mathcal{M}$ is a Fourier multiplier of symbol $\hat{\mathcal{M}}$: $\widehat{\mathcal{M}w} = \hat{\mathcal{M}}\hat{w}$ (here, $\hat{\cdot}$ denotes the Fourier transform). We assume $\hat{\mathcal{M}} \geq 0$ (otherwise, see [ibid.]). When $\mathcal{M} = -\partial_x^2$, these equations reduce to (gKdV), (gBBM) and (grBsq), respectively. The Benjamin–Ono equation ($\hat{\mathcal{M}} = |\xi|$), the Smith equation ($\hat{\mathcal{M}} = \sqrt{1+\xi^2} - 1$) and the intermediate long wave (or Whitham) equation ($\hat{\mathcal{M}} = \xi/\tanh(\xi H) - 1/H$, for some constant $H > 0$)

are common models of dispersive wave equations that are of type (I). We refer to [ibid.] for references on these models and the existence of solitary waves. The associated momentum is

$$P_{\mathrm{I}}(U_c) = \int_{\mathbb{R}} U_c^2\,dx = \|U_c\|_{L^2}^2, \quad P_{\mathrm{II}}(U_c) = \|(1+\mathcal{M})^{1/2}U_c\|_{L^2}^2, \quad P_{\mathrm{III}}(U_c) = c\|(1+\mathcal{M})^{1/2}U_c\|_{L^2}^2.$$

For these models, Evans function type arguments do not work since we no longer have a differential equation (it is nonlocal). The paper [ibid.] proposes a different approach than the Evans function technique for establishing the existence of unstable eigenvalues. However, it is not completely clear whether this method extends easily to the case of systems such as the Euler–Korteweg system (EK) (given at the beginning of Section 5A).

Theorem B.9 [Lin 2008]. *We consider one of the equations (I), (II) or (III) with f of class $\mathscr{C}^1$ satisfying $f(0)=f'(0)=0$ and $|f(u)| \gg |u|$ for $|u| \to +\infty$. We assume moreover that $\hat{\mathcal{M}}$ is even, nonnegative, and satisfies, for some $m \geq 1$, $0 < \underline{\lim}_{+\infty}\, \hat{\mathcal{M}}(\xi)/\xi^m \leq \overline{\lim}_{+\infty}\, \hat{\mathcal{M}}(\xi)/\xi^m < \infty$. Assume that $c \mapsto \phi_c = U_c(x-ct)$ is a $\mathscr{C}^1$ branch of traveling wave solution to (I), (II) or (III) with $U_c \in H^{m/2}(\mathbb{R})$ defined near c_* and suppose that the linearized operator $\mathscr{L}$ has exactly one negative eigenvalue, that $\ker \mathscr{L}$ is spanned by $\partial_x U_{c_*}$ and that $(dP(U_c)/dc)_{|c=c_*} < 0$. Then, U_{c_*} is linearly unstable.*

It is not easy to determine whether the hypotheses of Theorem B.9 hold true when $\mathcal{M}$ is not a (differential) Sturm–Liouville operator. See however [Albert 1992] on this question. It is clear that, if the assumptions of Theorem B.9 are satisfied, then Assumption A is also satisfied. As for the (gBBM) and the (grBsq) equations, the equations (II) and (III) turn out to be semilinear; thus we may prove nonlinear orbital instability by applying Corollary B.6.

The Kawahara equation (or fifth-order KdV equation)

$$\partial_t u + \partial_x(f(u)) + \alpha\partial_x^3 u + \beta\partial_x^5 u = 0, \tag{K}$$

with $\alpha, \beta \neq 0$ two real constants, is another relevant dispersive model. For this equation, it may happen that the linearized equation around the solitary wave has more than one negative eigenvalue, in which case [Grillakis et al. 1987; 1990; Lopes 2002; Lin 2008] do not give a clear necessary and sufficient condition for stability. T. Bridges and G. Derks [2002] give a sufficient condition for linear instability for solitary wave solutions, but also for other types of traveling solutions. This condition is probably not necessary since it may happen that there exist at least two unstable eigenvalues, or two complex conjugate eigenvalues.

Instead of stating a general result for nonlinear orbital instability, we shall consider several model cases on which we will verify the hypotheses of Theorem B.8, in particular the question of the existence of the multiplier $\mathbb{M}_s$.

Proposition B.10. *We consider the equation (I), namely*

$$\partial_t u + \partial_x(f(u)) - \partial_x \mathcal{M} u = 0,$$

with f of class $\mathscr{C}^1$ satisfying $f(0)=f'(0)=0$ and $|f(u)| \gg |u|$ for $|u| \to +\infty$. We assume that $\hat{\mathcal{M}}$ is one of the following functions:

$$-\xi^2 \quad (\mathit{KdV}); \quad \xi^4+\alpha\xi^2 \quad (\mathit{Kawahara}); \quad |\xi| \quad (\mathit{Benjamin–Ono});$$
$$\frac{\xi}{\tanh(\xi H)} - \frac{1}{H} \quad (\mathit{intermediate\ long\ wave}); \quad \sqrt{1+\xi^2}-1 \quad (\mathit{Smith}).$$

There exists $s_0>0$ such that, if there exists $c \in \mathbb{R}$ such that (I) has a nontrivial solitary wave $U_c \in L^2$ which is linearly unstable, then, for any $s \geq s_0$, it is also nonlinearly unstable from H^s to H^s, to L^2 and to L^∞.

By application of Theorem B.8, we are thus able to show the nonlinear instability from H^s to L^2 or L^∞ by tracking the exponentially growing mode (this question was left open in [Lin 2008] and also in [Lopes 2002]). In particular, we obtain the L^2 nonlinear instability of the linearly unstable solitary waves for these models.

Proof. All the assumptions for Theorem B.8 for these types of models are satisfied in Section 8.1 in [Rousset and Tzvetkov 2008], except the existence of the multiplier $\mathbb{M}_s$.

For the KdV equation, where $\sigma=1$, we shall take (for $s \geq 2$ an integer)

$$\mathbb{M}_s \equiv (-1)^s \partial_x^{2s} + \frac{1+2s}{3}(-1)^{s-1}\partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\cdot\},$$

as the computations from [ibid., Section 8.1] show. For the Kawahara equation, with $\sigma=2$, we take (for $s \geq 4$ an integer)

$$\mathbb{M}_s \equiv (-1)^s \partial_x^{2s} + \frac{1+2s}{5}(-1)^{s-1}\partial_x^{s-2}\{f'(Q)\partial_x^{s-2}\cdot\}$$

and, since the computations are very similar, we omit them. For the Benjamin–Ono equation, we have $\hat{\mathcal{M}}(\xi)=|\xi|$ and $\sigma=1/2$, and we will then have to deal with pseudodifferential operators which are Fourier multipliers with homogeneous symbol. For this type of operator, we shall need some commutator estimates. We denote by $\mathscr{F}(w)$ or $\hat{w}$ the Fourier transform of w, and $\mathscr{H}$ the Fourier multiplier with symbol $-i\,\mathrm{sgn}(\xi)$ (this is the Hilbert transform).

Lemma B.11. (i) *Let $h \in L^\infty(\mathbb{R})$ with $\mathscr{F}(\mathcal{M}^{1/2}h) \in L^1(\mathbb{R})$ (for instance, $h \in H^\sigma(\mathbb{R})$ for some $\sigma>1$). Then, there exists $C>0$ such that, for any $v \in H^{1/2}(\mathbb{R})$,*

$$\|\mathcal{M}^{\frac12}(hv) - h\mathcal{M}^{\frac12}v\|_{L^2(\mathbb{R})} \leq C\|v\|_{L^2(\mathbb{R})}.$$

(ii) *Let $h \in L^\infty(\mathbb{R})$ with $\mathscr{F}(\mathcal{M}^{3/2}h) \in L^1(\mathbb{R})$ (for instance, $h \in H^\sigma(\mathbb{R})$ for some $\sigma>2$). Then, there exists $C>0$ such that, for any $v \in H^{3/2}(\mathbb{R})$,*

$$\left\|\mathcal{M}^{\frac32}\{hv\} - h\mathcal{M}^{\frac32}v - \tfrac32[\partial_x h]\mathcal{M}^{\frac12}\mathscr{H}v\right\|_{L^2(\mathbb{R})} \leq C\|v\|_{L^2(\mathbb{R})}.$$

(iii) *Let $h \in L^\infty(\mathbb{R})$ with $\mathscr{F}(\partial_x\mathcal{M}^{1/2}h) \in L^1(\mathbb{R})$ (for instance, $h \in H^\sigma(\mathbb{R})$ for some $\sigma>2$). Then, there exists $C>0$ such that, for any $v \in H^{3/2}(\mathbb{R})$,*

$$\left\|\partial_x\mathcal{M}^{\frac12}\{hv\} - h\partial_x\mathcal{M}^{\frac12}v - \tfrac32[\partial_x h]\mathcal{M}^{\frac12}v\right\|_{L^2(\mathbb{R})} \leq C\|v\|_{L^2(\mathbb{R})}.$$

Proof. We have

$$\mathscr{F}(\mathcal{M}^{\frac{1}{2}}(hv) - h\mathcal{M}^{\frac{1}{2}}v)(\xi) = \int_{\mathbb{R}} |\xi|^{\frac{1}{2}}\hat{h}(\xi-\zeta)\hat{v}(\zeta)\,d\zeta - \int_{\mathbb{R}} |\zeta|^{\frac{1}{2}}\hat{h}(\xi-\zeta)\hat{v}(\zeta)\,d\zeta.$$

Using the inequality $\big||\xi|^{1/2} - |\zeta|^{1/2}\big| \le C|\xi-\zeta|^{1/2}$, we thus obtain

$$\big|\mathscr{F}(\mathcal{M}^{\frac{1}{2}}(hv) - h\mathcal{M}^{\frac{1}{2}}v)(\xi)\big| \le C\int_{\mathbb{R}} |\xi-\zeta|^{\frac{1}{2}}|\hat{h}(\xi-\zeta)|\cdot|\hat{v}(\zeta)|\,d\zeta = C\{|\mathscr{F}(\mathcal{M}^{\frac{1}{2}}h)| * |\hat{v}|\}(\xi)$$

and we conclude with the classical convolution estimate $L^1 * L^2 \subset L^2$. This argument does not provide the sharpest bound in h, since it involves $\|\mathscr{F}(\mathcal{M}^{1/2}h)\|_{L^1}$, whereas the use of paradifferential calculus will use only $\|h\|_{\mathscr{C}^{1/2}}$. However, we shall to use this refinement here.

The starting point for the second inequality is

$$\big||\xi|^{\frac{3}{2}} - |\zeta|^{\frac{3}{2}} - \tfrac{3}{2}|\zeta|^{\frac{1}{2}}\operatorname{sgn}(\zeta)(\xi-\zeta)\big| \le C|\xi-\zeta|^{\frac{3}{2}}.$$

Using the homogeneity $\xi = \theta\zeta$, this is a direct consequence of the easy inequality

$$\Big||\theta|^{\frac{3}{2}} - 1 - \frac{3}{2}(\theta-1)\Big| \le C|\theta-1|^{\frac{3}{2}}.$$

Therefore,

$$\begin{aligned}
&\big|\mathscr{F}\big(\mathcal{M}^{\frac{3}{2}}\{hv\} - h\mathcal{M}^{\frac{3}{2}}v - \tfrac{3}{2}[\partial_x h]\mathcal{M}^{\frac{1}{2}}\mathscr{H}v\big)(\xi)\big| \\
&\quad = \Big|\int_{\mathbb{R}} |\xi|^{\frac{3}{2}}\hat{h}(\xi-\zeta)\hat{v}(\zeta)\,d\zeta - \int_{\mathbb{R}} |\zeta|^{\frac{3}{2}}\hat{h}(\xi-\zeta)\hat{v}(\zeta)\,d\zeta - \int_{\mathbb{R}} \tfrac{3}{2}|\zeta|^{\frac{1}{2}}\operatorname{sgn}(\zeta)(\xi-\zeta)\hat{h}(\xi-\zeta)\hat{v}(\zeta)\,d\zeta\Big| \\
&\quad \le C\int_{\mathbb{R}} |\xi-\zeta|^{\frac{3}{2}}|\hat{h}(\xi-\zeta)|\cdot|\hat{v}(\zeta)|\,d\zeta \\
&\quad = C|\mathscr{F}(\mathcal{M}^{\frac{3}{2}}h)| * |\hat{v}|,
\end{aligned}$$

and we conclude as before. For the third inequality, we argue in a similar way with the estimate

$$\big|i\xi|\xi|^{\frac{1}{2}} - i\zeta|\zeta|^{\frac{1}{2}} - i\tfrac{3}{2}|\zeta|^{\frac{1}{2}}(\xi-\zeta)\big| \le C|\xi-\zeta|^{\frac{3}{2}}.$$

The proof is complete. □

For the Benjamin–Ono equation, $\hat{\mathcal{M}}(\xi) = |\xi|$, $\sigma = 1/2$ and the index s will be half an integer: $s \in \mathbb{N}/2$. Therefore, we set $s = [s] + \{s\}$, with $[s]$ integer and $\{s\} \in \{0; 1/2\}$. Let us define, for $s \in \mathbb{N}/2$, $s \ge 1$,

$$\mathbb{M}_s \equiv \begin{cases} (-1)^s\partial_x^{2s} + \gamma_s\mathcal{M}^{\frac{1}{2}}\partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}\cdot\} & \text{if } \{s\} = 0, \\ (-1)^{[s]}\partial_x^{2[s]}\mathcal{M} + \gamma_s\partial_x^{[s]}\{f'(Q)\partial_x^{[s]}\cdot\} & \text{if } \{s\} = \frac{1}{2}, \end{cases}$$

for some real constant γ_s to be determined later. It is clear that $\mathbb{M}_s$ is self-adjoint on L^2 and that there exists $C_s > 0$ such that

$$|(\mathbb{M}_s u, v)_{L^2}| \le C_s\|u\|_{H^s}\|v\|_{H^s} \quad \text{and} \quad (\mathbb{M}_s u, u)_{L^2} \ge \|u\|_{H^s}^2 - C_s\|u\|_{H^{s-\frac{1}{2}}}^2.$$

To verify the assumptions for the multiplier $\mathbb{M}_s$, it remains to study $\mathrm{Re}(J(L_0+\nabla^2 F(Q))u, \mathbb{M}_s u)_{L^2}$. When $\{s\}=0$, i.e., $s\in\mathbb{N}$, this quantity is

$$\begin{aligned}
&\mathrm{Re}(\partial_x(\mathcal{M}+c+f'(Q))u, \mathbb{M}_s u)_{L^2}\\
&\quad= \mathrm{Re}(\partial_x \mathcal{M}u, (-1)^s\partial_x^{2s}u)_{L^2} + \gamma_s \,\mathrm{Re}(\partial_x\mathcal{M}u, \mathcal{M}^{\frac12}\partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac12}u\})_{L^2}\\
&\qquad + \mathrm{Re}(\partial_x[f'(Q)u], (-1)^s\partial_x^{2s}u)_{L^2} + \gamma_s\,\mathrm{Re}(\partial_x[f'(Q)u], \mathcal{M}^{\frac12}\partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac12}u\})_{L^2}\\
&\qquad + c\,\mathrm{Re}(\partial_x u, \mathbb{M}_s u)_{L^2}. \qquad\qquad \text{(B-1)}
\end{aligned}$$

By skew-adjointness, the first and last scalar products are zero. By integration by parts and the Leibniz formula, we deduce, since $Q\in H^\infty$,

$$\begin{aligned}
&\mathrm{Re}(\partial_x[f'(Q)u], (-1)^s\partial_x^{2s}u)_{L^2}\\
&\qquad = \mathrm{Re}(\partial_x^{s+1}[f'(Q)u], \partial_x^s u)_{L^2}\\
&\qquad \le \mathrm{Re}(f'(Q)\partial_x^{s+1}u, \partial_x^s u)_{L^2} + (s+1)\,\mathrm{Re}(\partial_x[f'(Q)]\partial_x^s u, \partial_x^s u)_{L^2} + C_s\|u\|_{H^s}\|u\|_{H^{s-1}}\\
&\qquad \le \left(s+\tfrac12\right)\mathrm{Re}(\partial_x[f'(Q)]\partial_x^s u, \partial_x^s u)_{L^2} + C_s\|u\|_{H^s}\|u\|_{H^{s-1}}.
\end{aligned}$$

Similarly, using the easy estimates $\|\mathcal{M}^{1/2}v\|_{L^2}\le K\|v\|_{H^{1/2}}$ and $\|hv\|_{H^{1/2}}\le C(h)\|v\|_{H^{1/2}}$ for $h\in L^\infty$ with $\mathcal{F}(\mathcal{M}^{1/2}h)\in L^1$ (this is an immediate consequence of Lemma B.11),

$$\begin{aligned}
&\gamma_s\,\mathrm{Re}(\partial_x[f'(Q)u], \mathcal{M}^{\frac12}\partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac12}u\})_{L^2}\\
&\qquad = \gamma_s(-1)^{s-1}\,\mathrm{Re}(\mathcal{M}^{\frac12}\partial_x^s[f'(Q)u], f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac12}u)_{L^2}\\
&\qquad \le \gamma_s(-1)^{s-1}\,\mathrm{Re}(\mathcal{M}^{\frac12}[f'(Q)\partial_x^s u], f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac12}u)_{L^2} + C\|u\|^2_{H^{s-\frac12}}.
\end{aligned}$$

Using Lemma B.11, we deduce $\|\mathcal{M}^{1/2}[f'(Q)\partial_x^s u] - f'(Q)\mathcal{M}^{1/2}\partial_x^s u\|_{L^2}\le C(Q)\|u\|_{H^s}$; thus

$$\begin{aligned}
&\gamma_s\,\mathrm{Re}(\partial_x[f'(Q)u], \mathcal{M}^{\frac12}\partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac12}u\})_{L^2}\\
&\qquad \le \gamma_s(-1)^{s-1}\,\mathrm{Re}(f'(Q)\partial_x^s\mathcal{M}^{\frac12}u, f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac12}u)_{L^2} + C\|u\|_{H^s}\|u\|_{H^{s-\frac12}}\\
&\qquad = \frac{\gamma_s}{2}(-1)^s\,\mathrm{Re}(\partial_x[f'(Q)]\partial_x^{s-1}\mathcal{M}^{\frac12}u, f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac12}u)_{L^2} + C\|u\|_{H^s}\|u\|_{H^{s-\frac12}}\\
&\qquad \le C\|u\|^2_{H^{s-\frac12}} + C\|u\|_{H^s}\|u\|_{H^{s-\frac12}} \le C\|u\|_{H^s}\|u\|_{H^{s-\frac12}}.
\end{aligned}$$

We now turn to the term

$$\gamma_s\,\mathrm{Re}(\partial_x\mathcal{M}u, \mathcal{M}^{\frac12}\partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac12}u\})_{L^2} = \gamma_s(-1)^{s-1}\,\mathrm{Re}(\partial_x^s u, \mathcal{M}^{\frac32}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac12}u\})_{L^2}.$$

Using Lemma B.11, we write

$$\begin{aligned}
\left\|\mathcal{M}^{\frac32}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac12}u\} - f'(Q)\partial_x^{s-1}\mathcal{M}^2 u - \tfrac32\partial_x[f'(Q)]\mathcal{M}^{\frac12}\mathcal{H}\{\partial_x^{s-1}\mathcal{M}^{\frac12}u\}\right\|_{L^2} &\le C(Q)\|\partial_x^{s-1}\mathcal{M}^{\frac12}u\|_{L^2}\\
&\le C(Q)\|u\|_{H^{s-\frac12}},
\end{aligned}$$

which implies

$$\begin{aligned}
&\gamma_s \operatorname{Re}(\partial_x \mathcal{M} u, \mathcal{M}^{\frac{1}{2}} \partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}} u\})_{L^2} \\
&\qquad \leq \gamma_s(-1)^{s-1}\operatorname{Re}(\partial_x^s u, f'(Q)\partial_x^{s-1}\mathcal{M}^2 u)_{L^2} + \tfrac{3}{2}\gamma_s(-1)^{s-1}\operatorname{Re}(\partial_x^s u, \partial_x[f'(Q)]\mathcal{M}^{\frac{1}{2}}\mathcal{H}\{\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}} u\})_{L^2} \\
&\qquad\qquad + C\|u\|_{H^s}\|u\|_{H^{s-\frac{1}{2}}}.
\end{aligned}$$

Noticing that $\mathcal{M}^2 = -\partial_x^2$ and $\mathcal{M}^{1/2}\mathcal{H}\partial_x^{s-1}\mathcal{M}^{1/2} = \partial_x^{s-1}\mathcal{M}\mathcal{H} = -\partial_x^s$ (since $\mathcal{M}\mathcal{H}$ has symbol equal to $-i\xi$), we infer

$$\begin{aligned}
&\gamma_s \operatorname{Re}(\partial_x \mathcal{M} u, \mathcal{M}^{\frac{1}{2}} \partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}} u\})_{L^2} \\
&\qquad \leq \gamma_s(-1)^{s}\operatorname{Re}(\partial_x^s u, f'(Q)\partial_x^{s+1} u)_{L^2} + \tfrac{3}{2}\gamma_s(-1)^{s}\operatorname{Re}(\partial_x^s u, \partial_x[f'(Q)]\partial_x^s u)_{L^2} + C\|u\|_{H^s}\|u\|_{H^{s-\frac{1}{2}}} \\
&\qquad = \gamma_s(-1)^{s}\operatorname{Re}(\partial_x^s u, \partial_x[f'(Q)]\partial_x^s u)_{L^2} + C\|u\|_{H^s}\|u\|_{H^{s-\frac{1}{2}}}
\end{aligned}$$

by integration by parts.

Reporting these estimates into (B-1), we infer

$$\begin{aligned}
&\operatorname{Re}(\partial_x(\mathcal{M} + c + f'(Q))u, \mathbb{M}_s u)_{L^2} \\
&\qquad \leq \left(s + \tfrac{1}{2}\right)\operatorname{Re}(\partial_x[f'(Q)]\partial_x^s u, \partial_x^s u)_{L^2} + \gamma_s(-1)^s\operatorname{Re}(\partial_x^s u, \partial_x[f'(Q)]\partial_x^s u)_{L^2} + C\|u\|_{H^s}\|u\|_{H^{s-\frac{1}{2}}}.
\end{aligned}$$

Therefore, the choice

$$\gamma_s \equiv (-1)^{s-1}\left(s + \tfrac{1}{2}\right)$$

provides the desired control

$$\operatorname{Re}(\partial_x(\mathcal{M} + c + f'(Q))u, \mathbb{M}_s u)_{L^2} \leq C\|u\|_{H^s}\|u\|_{H^{s-\frac{1}{2}}}.$$

When $\{s\} = 1/2$, the computations are similar: (B-1) becomes now

$$\begin{aligned}
&\operatorname{Re}(\partial_x(\mathcal{M} + c + f'(Q))u, \mathbb{M}_s u)_{L^2} \\
&\qquad = \operatorname{Re}(\partial_x \mathcal{M} u, (-1)^{[s]}\partial_x^{2[s]}\mathcal{M} u)_{L^2} + \gamma_s \operatorname{Re}(\partial_x \mathcal{M} u, \partial_x^{[s]}\{f'(Q)\partial_x^{[s]} u\})_{L^2} \\
&\qquad\qquad + \operatorname{Re}(\partial_x[f'(Q)u], (-1)^{[s]}\partial_x^{2[s]}\mathcal{M} u)_{L^2} + \gamma_s \operatorname{Re}(\partial_x[f'(Q)u], \partial_x^{[s]}\{f'(Q)\partial_x^{[s]} u\})_{L^2} \\
&\qquad\qquad + c\operatorname{Re}(\partial_x u, \mathbb{M}_s u)_{L^2}, \qquad \text{(B-2)}
\end{aligned}$$

and the first and last scalar products still vanish. Moreover, by integration by parts and the Leibniz formula, we deduce, since $Q \in H^\infty$,

$$\begin{aligned}
\gamma_s \operatorname{Re}(\partial_x[f'(Q)u], \partial_x^{[s]}\{f'(Q)\partial_x^{[s]} u\})_{L^2} &= \gamma_s(-1)^{[s]}\operatorname{Re}(\partial_x^{[s]+1}[f'(Q)u], f'(Q)\partial_x^{[s]} u)_{L^2} \\
&\leq \gamma_s(-1)^{[s]}\operatorname{Re}(f'(Q)\partial_x^{[s]+1} u, f'(Q)\partial_x^{[s]} u)_{L^2} + C\|u\|_{H^{[s]}}^2 \\
&\leq \gamma_s(-1)^{[s]-1}\operatorname{Re}(\partial_x[f'(Q)]\partial_x^{[s]} u, f'(Q)\partial_x^{[s]} u)_{L^2} + C\|u\|_{H^{[s]}}^2 \\
&\leq C\|u\|_{H^{[s]}}^2 = C\|u\|_{H^{s-\frac{1}{2}}}^2.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\mathrm{Re}(\partial_x[f'(Q)u], (-1)^{[s]}\partial_x^{2[s]}\mathcal{M}u)_{L^2}\\
&\qquad = \mathrm{Re}(\mathcal{M}^{\frac12}\partial_x^{[s]+1}[f'(Q)u], \partial_x^{[s]}\mathcal{M}^{\frac12}u)_{L^2}\\
&\qquad \le \mathrm{Re}(\partial_x\mathcal{M}^{\frac12}\{f'(Q)\partial_x^{[s]}u\}, \partial_x^{[s]}\mathcal{M}^{\frac12}u)_{L^2}\\
&\qquad\qquad + [s]\,\mathrm{Re}(\partial_x\mathcal{M}^{\frac12}\{\partial_x[f'(Q)]\partial_x^{[s]-1}u\}, \partial_x^{[s]}\mathcal{M}^{\frac12}u)_{L^2} + C\|u\|_{H^{[s]-\frac12}}\|u\|_{H^{[s]+\frac12}}.
\end{aligned}$$

For the second scalar product, we write, by Lemma B.11,

$$\begin{aligned}
&\mathrm{Re}(\partial_x\mathcal{M}^{\frac12}\{\partial_x[f'(Q)]\partial_x^{[s]-1}u\}, \partial_x^{[s]}\mathcal{M}^{\frac12}u)_{L^2}\\
&\qquad = \mathrm{Re}(\mathcal{M}^{\frac12}\{\partial_x^2[f'(Q)]\partial_x^{[s]-1}u\}, \partial_x^{[s]}\mathcal{M}^{\frac12}u)_{L^2} + \mathrm{Re}(\mathcal{M}^{\frac12}\{\partial_x[f'(Q)]\partial_x^{[s]}u\}, \partial_x^{[s]}\mathcal{M}^{\frac12}u)_{L^2}\\
&\qquad \le C\|u\|_{H^{[s]-\frac12}}\|u\|_{H^{[s]+\frac12}} + \mathrm{Re}(\partial_x[f'(Q)]\partial_x^{[s]}\mathcal{M}^{\frac12}u, \partial_x^{[s]}\mathcal{M}^{\frac12}u)_{L^2} + C\|u\|_{H^{[s]}}\|u\|_{H^{[s]+\frac12}}\\
&\qquad \le \mathrm{Re}(\partial_x[f'(Q)]\partial_x^{[s]}\mathcal{M}^{\frac12}u, \partial_x^{[s]}\mathcal{M}^{\frac12}u)_{L^2} + C\|u\|_{H^{s-\frac12}}\|u\|_{H^s}.
\end{aligned}$$

For the first scalar product, we use Lemma B.11 once again:

$$\begin{aligned}
&\mathrm{Re}(\partial_x\mathcal{M}^{\frac12}\{f'(Q)\partial_x^{[s]}u\}, \partial_x^{[s]}\mathcal{M}^{\frac12}u)_{L^2}\\
&\quad \le \mathrm{Re}(f'(Q)\partial_x\mathcal{M}^{\frac12}\partial_x^{[s]}u, \partial_x^{[s]}\mathcal{M}^{\frac12}u)_{L^2} + \tfrac32\mathrm{Re}(\partial_x[f'(Q)]\partial_x^{[s]}\mathcal{M}^{\frac12}u, \partial_x^{[s]}\mathcal{M}^{\frac12}u)_{L^2} + C\|u\|_{H^{s-\frac12}}\|u\|_{H^s}\\
&\quad \le \mathrm{Re}(\partial_x[f'(Q)]\partial_x^{[s]}\mathcal{M}^{\frac12}u, \partial_x^{[s]}\mathcal{M}^{\frac12}u)_{L^2} + C\|u\|_{H^{s-\frac12}}\|u\|_{H^s}.
\end{aligned}$$

As a consequence, since $[s] = s - \frac12$,

$$\mathrm{Re}(\partial_x[f'(Q)u], (-1)^{[s]}\partial_x^{2[s]}\mathcal{M}u)_{L^2} \le \big(s+\tfrac12\big)\,\mathrm{Re}(\partial_x[f'(Q)]\partial_x^{[s]}\mathcal{M}^{\frac12}u, \partial_x^{[s]}\mathcal{M}^{\frac12}u)_{L^2} + C\|u\|_{H^{s-\frac12}}\|u\|_{H^s}.$$

We turn finally to the term

$$\gamma_s\,\mathrm{Re}(\partial_x\mathcal{M}u, \partial_x^{[s]}\{f'(Q)\partial_x^{[s]}u\})_{L^2} = \gamma_s(-1)^{[s]}\,\mathrm{Re}(\partial_x^{[s]}\mathcal{M}^{\frac12}u, \partial_x\mathcal{M}^{\frac12}\{f'(Q)\partial_x^{[s]}u\})_{L^2},$$

and infer, by Lemma B.11,

$$\begin{aligned}
\gamma_s\mathrm{Re}(\partial_x\mathcal{M}u,\partial_x^{[s]}\{f'(Q)\partial_x^{[s]}u\})_{L^2} &\le \gamma_s(-1)^{[s]}\mathrm{Re}(\partial_x^{[s]}\mathcal{M}^{\frac12}u, f'(Q)\partial_x\mathcal{M}^{\frac12}\partial_x^{[s]}u)_{L^2}\\
&\qquad + \tfrac32\gamma_s(-1)^{[s]}\mathrm{Re}(\partial_x^{[s]}\mathcal{M}^{\frac12}u, \partial_x[f'(Q)]\mathcal{M}^{\frac12}\partial_x^{[s]}u)_{L^2} + C\|u\|_{H^{s-\frac12}}\|u\|_{H^s}\\
&= \gamma_s(-1)^{[s]}\mathrm{Re}(\partial_x^{[s]}\mathcal{M}^{\frac12}u, \partial_x[f'(Q)]\mathcal{M}^{\frac12}\partial_x^{[s]}u)_{L^2} + C\|u\|_{H^{s-\frac12}}\|u\|_{H^s}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathrm{Re}(\partial_x(\mathcal{M}+c+f'(Q))u, \mathbb{M}_s u)_{L^2}\\
&\qquad \le \big(s+\tfrac12+\gamma_s(-1)^{[s]}\big)\,\mathrm{Re}(\partial_x^{[s]}\mathcal{M}^{\frac12}u, \partial_x[f'(Q)]\mathcal{M}^{\frac12}\partial_x^{[s]}u)_{L^2} + C\|u\|_{H^{s-\frac12}}\|u\|_{H^s};
\end{aligned}$$

hence choosing $\gamma_s \equiv (-1)^{[s]-1}(s+\frac12)$ gives the result.

It remains to study the cases of the intermediate long wave equation and the Smith equation, for which $\hat{\mathcal{M}}$ is, respectively,

$$\frac{\xi}{\tanh(\xi H)} - \frac{1}{H}, \quad \sqrt{1+\xi^2} - 1.$$

We denote by $\mathcal{M}_0$ the operator with symbol $|\xi|$ (the one of the Benjamin–Ono equation), and define $\mathbb{M}_s$ as for the Benjamin–Ono case (hence with "$\mathcal{M}$"$= \mathcal{M}_0$). We observe that, in both cases, $\tilde{\mathcal{M}} \equiv \mathcal{M} - \mathcal{M}_0$ is bounded on L^2. Indeed, its symbol is continuous in $\mathbb{R}$ and, for $\xi \to \pm\infty$,

$$\hat{\mathcal{M}}(\xi) = \frac{\xi}{\tanh(\xi H)} - \frac{1}{H} = \frac{\xi}{\operatorname{sgn}(\xi) + \mathcal{O}(\mathrm{e}^{-2|\xi|H})} - \frac{1}{H} = |\xi| - \frac{1}{H} + \mathcal{O}(|\xi|\mathrm{e}^{-2|\xi|H})$$

and

$$\hat{\mathcal{M}}(\xi) = \sqrt{1+\xi^2} - 1 = |\xi|\sqrt{1+\xi^{-2}} - 1 = |\xi| - 1 + \mathcal{O}(|\xi|^{-1}),$$

respectively. In the quantity $\operatorname{Re}(\partial_x(\mathcal{M} + c + f'(Q))u, \mathbb{M}_s u)_{L^2}$, we then have to bound from above the extra term $\operatorname{Re}(\partial_x(\tilde{\mathcal{M}}u), \mathbb{M}_s u)_{L^2}$; that is (using the skew-adjointness for the higher-order derivatives in $\mathbb{M}_s$),

$$\begin{aligned}
&\operatorname{Re}(\partial_x(\tilde{\mathcal{M}}u), \gamma_s \mathcal{M}_0^{\frac12}\partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}_0^{\frac12}u\})_{L^2} \\
&\qquad\qquad = \gamma_s(-1)^{s-1}\operatorname{Re}(\partial_x^s\mathcal{M}_0^{\frac12}(\tilde{\mathcal{M}}u), f'(Q)\partial_x^{s-1}\mathcal{M}_0^{\frac12}u)_{L^2} \quad \text{if } \{s\} = 0; \\
&\operatorname{Re}(\partial_x(\tilde{\mathcal{M}}u), \gamma_s\partial_x^{[s]}\{f'(Q)\partial_x^{[s]}u\})_{L^2} = \gamma_s(-1)^{[s]}\operatorname{Re}(\partial_x^{[s]+1}(\tilde{\mathcal{M}}u), f'(Q)\partial_x^{[s]}u)_{L^2} \quad \text{if } \{s\} = \tfrac12.
\end{aligned}$$

We then note that, in both cases, one may actually split $\tilde{\mathcal{M}} = \mathcal{M} - \mathcal{M}_0 = \tilde{\mathcal{M}}_c + \tilde{\mathcal{M}}_h$, where $\tilde{\mathcal{M}}_c$ is the multiplication by $-1/H$ (respectively, -1) and $\tilde{\mathcal{M}}_h$ has a symbol which is continuous in $\mathbb{R}$ and $\mathcal{O}(|\xi|^{-1})$ at infinity, so that $\tilde{\mathcal{M}}_h$ is bounded from H^σ to $H^{\sigma+1}$ if $\sigma \ge 0$. Therefore, when $\{s\} = 0$, we easily get

$$\begin{aligned}
&\operatorname{Re}(\partial_x(\tilde{\mathcal{M}}u), \gamma_s\mathcal{M}_0^{\frac12}\partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}_0^{\frac12}u\})_{L^2} \\
&= \gamma_s(-1)^{s-1}\operatorname{Re}(\partial_x^s\mathcal{M}_0^{\frac12}(\tilde{\mathcal{M}}_c u), f'(Q)\partial_x^{s-1}\mathcal{M}_0^{\frac12}u)_{L^2} + \gamma_s(-1)^{s-1}\operatorname{Re}(\partial_x^s\mathcal{M}_0^{\frac12}(\tilde{\mathcal{M}}_h u), f'(Q)\partial_x^{s-1}\mathcal{M}_0^{\frac12}u)_{L^2} \\
&\le \tfrac12\gamma_s(-1)^s\operatorname{Re}(\partial_x^{s-1}\mathcal{M}_0^{\frac12}u, \tilde{\mathcal{M}}_c\partial_x[f'(Q)]\partial_x^{s-1}\mathcal{M}_0^{\frac12}u)_{L^2} + C\|u\|^2_{H^{s-\frac12}} \le C\|u\|^2_{H^{s-\frac12}}
\end{aligned}$$

and similarly when $\{s\} = 1/2$. Therefore, the estimate

$$\operatorname{Re}(\partial_x(\mathcal{M} + c + f'(Q))u, \mathbb{M}_s u)_{L^2} \le C\|u\|_{H^{s-\frac12}}\|u\|_{H^s}$$

remains true for the intermediate long wave equation and the Smith equation. The proof of Proposition B.10 is thus completed by applying Theorem B.9. □

We now turn to the deferred proofs of Theorem B.4, Theorem B.5, and Corollary B.6.

Proof of Theorem B.4. We shall prove the resolvent estimate required in Corollary B.3. Let us consider $\lambda = \gamma + i\tau \in \mathbb{C}$ with $\gamma \neq 0$ and the resolvent equation $(\mathcal{J}\mathcal{L} - \lambda)v = \Sigma$, or

$$(\gamma + i\tau)v = \mathcal{J}\mathcal{L}(v) - \Sigma. \tag{B-3}$$

By hypothesis, the essential spectrum of $\mathcal{J}\mathcal{L}$ is of the form $i[\mathbb{R}\setminus(-\vartheta_0,+\vartheta_0)]$. Moreover, we have seen that $\mathcal{J}\mathcal{L}$ has a finite number of eigenvalues in the half-space $\{\mathrm{Re}>0\}$; hence, for $|\tau|\geq\tau_0$ sufficiently large, we know that there exists a unique solution v to (B-3). By taking the scalar product with $\mathcal{L}(v)$, we deduce the conservation law

$$\gamma(v,\mathcal{L}(v))_{\mathcal{X}}=-\mathrm{Re}(\Sigma,\mathcal{L}(v))_{\mathcal{X}}. \tag{B-4}$$

By our assumption, there exist a finite (possibly empty) number of eigenvalues in $(-\infty,0]$, $(-\mu_1,\dots,-\mu_q)$, each one of finite multiplicity. For any $1\leq k\leq q$, we fix an orthonormal basis $(\chi_{k,\ell})_{1\leq\ell\leq n_k}$ of the eigenspace $\ker(\mathcal{L}+\mu_k)$. By Assumption A, any eigenvector $\chi_{k,\ell}$ is smooth in the sense that $\chi_{k,\ell}\in D(\mathcal{J})$ and $\mathcal{J}\chi_{k,\ell}\in D(\mathcal{L})$.

We then make a spectral orthogonal decomposition

$$v=\sum_{k=1}^{q}\sum_{\ell=1}^{n_k}\alpha_{k,\ell}\chi_{k,\ell}+v_+,$$

where $\mathcal{L}(\chi_{k,\ell})=\mu_k\chi_{k,\ell}$ and $(v_+,\mathcal{L}(v_+))_{\mathcal{X}}\geq\delta\|v_+\|_{\mathcal{X}}^2$ for some positive δ. In the double sum, we have a finite number (independent of v) of terms. Inserting this into (B-4) yields

$$|\gamma|\delta\|v_+\|_{\mathcal{X}}^2\leq|\gamma|\delta(v_+,\mathcal{L}(v_+))_{\mathcal{X}}\leq\delta\Big[|\mathrm{Re}(\Sigma,\mathcal{L}(v))_{\mathcal{X}}|+\sum_{k,\ell}\mu_k|\alpha_{k,\ell}|^2\Big]\leq K\|\Sigma\|_{\mathcal{X}}\|v\|_{\mathcal{X}}+K\sum_{k,\ell}|\alpha_{k,\ell}|^2.$$

Using the inequality $ab\leq\varepsilon a^2+b^2/(4\varepsilon)$ with $a=\|v\|_{\mathcal{X}}$, $b=K\|\Sigma\|_{\mathcal{X}}$ and $\varepsilon=|\gamma|\delta/2$, the equality $\|v\|_{\mathcal{X}}^2=\|v_+\|_{\mathcal{X}}^2+\sum_{k,\ell}|\alpha_{k,\ell}|^2$ and incorporating the term $|\gamma|\delta\|v_+\|_{\mathcal{X}}^2/2$ in the left-hand side, we infer

$$\frac{|\gamma|\delta}{2}\|v_+\|_{\mathcal{X}}^2\leq K'\sum_{k,\ell}|\alpha_{k,\ell}|^2+K''\|\Sigma\|_{\mathcal{X}}^2. \tag{B-5}$$

On the other hand, since $\chi_{k,\ell}\in D(\mathcal{J})$ and $\mathcal{J}\chi_{k,\ell}\in D(\mathcal{L})$ by Assumption A, taking the scalar product of (B-1) with $\chi_{k,\ell}$ provides

$$(\gamma+i\tau)\alpha_{k,\ell}=-(v,\mathcal{L}\mathcal{J}\chi_{k,\ell})_{\mathcal{X}}-(\Sigma,\chi_{k,\ell})_{\mathcal{X}}.$$

Consequently,

$$(|\gamma|+|\tau|)|\alpha_{k,\ell}|\leq K_{k,\ell}\|v\|_{\mathcal{X}}+K\|\Sigma\|_{\mathcal{X}};$$

thus

$$(|\gamma|+|\tau|)^2\sum_{k,\ell}|\alpha_{k,\ell}|^2\leq K_0\|v\|_{\mathcal{X}}^2+K\|\Sigma\|_{\mathcal{X}}^2=K_0\sum_{k,\ell}|\alpha_{k,\ell}|^2+K\|v_+\|_{\mathcal{X}}^2+K\|\Sigma\|_{\mathcal{X}}^2,$$

which implies, if $|\tau|\geq 1+\sqrt{K_0}-|\gamma|$,

$$\sum_{k,\ell}|\alpha_{k,\ell}|^2\leq K\frac{\|v_+\|_{\mathcal{X}}^2+\|\Sigma\|_{\mathcal{X}}^2}{(|\gamma|+|\tau|)^2-K_0}.$$

Reporting this into (B-5) gives

$$\frac{|\gamma|\delta}{2}\|v_+\|_{\mathscr{X}}^2 \le K'K\frac{\|v_+\|_{\mathscr{X}}^2+\|\Sigma\|_{\mathscr{X}}^2}{(|\gamma|+|\tau|)^2-K_0}+K''\|\Sigma\|_{\mathscr{X}}^2.$$

If $|\tau| \ge 1+\sqrt{K_0+4KK'/|\gamma|\delta}-|\gamma|$, we deduce

$$\frac{|\gamma|\delta}{4}\|v_+\|_{\mathscr{X}}^2 \le \left(K''+\frac{K'K}{(|\gamma|+|\tau|)^2-K_0}\right)\|\Sigma\|_{\mathscr{X}}^2 \le K_1\|\Sigma\|_{\mathscr{X}}^2,$$

and it follows that

$$\|v\|_{\mathscr{X}}^2=\|v_+\|_{\mathscr{X}}^2+\sum_{k,\ell}|\alpha_{k,\ell}|^2 \le K_2\|\Sigma\|_{\mathscr{X}}^2,$$

where K_2 does not depend on $|\tau|$ (large enough), as wished.

The proof of the first semigroup estimate then follows easily; see, for instance, Proposition 2 in [Prüss 1984].

Proof of the semigroup estimate when $\gamma_0 > 0$. Here, we assume $\gamma_0 > 0$. As a consequence, the spectrum of $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}$ is of the form $\sigma_{\mathrm{s}} \cup \sigma_{\mathrm{u}}$, where $\sigma_{\mathrm{ess}}([\mathcal{J}\mathcal{L}]_{\mathbb{C}}) \subset \sigma_{\mathrm{s}} \subset \{\mathrm{Re} \le 0\}$ and $\varnothing \ne \sigma_{\mathrm{u}} \subset \{\mathrm{Re} > 0\}$ consists in a finite number of eigenvalues of finite algebraic multiplicities. Therefore, we may define (see, e.g., [Kato 1976; Hislop and Sigal 1996]) the spectral Riesz projection

$$\mathbb{P} \equiv \frac{1}{2i\pi}\int_{\Gamma}([\mathcal{J}\mathcal{L}]_{\mathbb{C}}-z)^{-1}\,dz,$$

where Γ is any simple (positively oriented) closed curve enclosing σ_{u}. As a consequence, $\mathbb{P}$ is bounded, commutes with $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}$ on $D([\mathcal{J}\mathcal{L}]_{\mathbb{C}})$ and satisfies $\sigma([\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P})=\sigma_{\mathrm{u}}$, $\sigma([\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\mathrm{Id}-\mathbb{P}))=\sigma_{\mathrm{s}}$. Moreover, $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P}$ is bounded, and hence generates a continuous semigroup, $\mathrm{e}^{t[\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P}}$, given by the exponential series

$$\mathrm{e}^{t[\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P}}=\sum_{n=0}^{+\infty}\frac{t^n([\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P})^n}{n!}.$$

In addition, $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\mathrm{Id}-\mathbb{P}) = [\mathcal{J}\mathcal{L}]_{\mathbb{C}} - [\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P}$ also generates a continuous semigroup and we have $\mathrm{e}^{t[\mathcal{J}\mathcal{L}]_{\mathbb{C}}}=\mathrm{e}^{t\mathcal{J}\mathcal{L}\mathbb{P}}\mathrm{e}^{t[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\mathrm{Id}-\mathbb{P})}$.

The semigroup generated by the bounded operator $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P}$ is easily analyzed. We shall now apply the spectral mapping theorem of J. Prüss (Theorem B.2) to $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\mathrm{Id}-\mathbb{P})$ in order to control the growth of its norm. By Corollary B.3, it suffices to estimate its resolvent $[[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\mathrm{Id}-\mathbb{P})-(\gamma+i\tau)]^{-1}$ for large $|\tau|$ (note that $\sigma([\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\mathrm{Id}-\mathbb{P}))=\sigma_{\mathrm{s}} \subset \{\mathrm{Re} \le 0\}$). If $\Sigma \in \mathscr{X}_{\mathbb{C}}$ and $|\tau|$ is large, it is clear that the solution $u \in \mathscr{X}_{\mathbb{C}}$ to $[[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\mathrm{Id}-\mathbb{P})-(\gamma+i\tau)]u=\Sigma$ is given by

$$u=[[\mathcal{J}\mathcal{L}]_{\mathbb{C}}-(\gamma+i\tau)]^{-1}(\mathrm{Id}-\mathbb{P})\Sigma-\frac{1}{\gamma+i\tau}\mathbb{P}\Sigma;$$

thus, for $|\tau|$ large,

$$\big\|[[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\mathrm{Id}-\mathbb{P})-(\gamma+i\tau)]^{-1}\big\|_{\mathscr{L}_c(\mathscr{X}_{\mathbb{C}})} \le \big\|[[\mathcal{J}\mathcal{L}]_{\mathbb{C}}-(\gamma+i\tau)]^{-1}\big\|_{\mathscr{L}_c(\mathscr{X}_{\mathbb{C}})}\|\mathrm{Id}-\mathbb{P}\|_{\mathscr{L}_c(\mathscr{X})}+\frac{1}{|\gamma+i\tau|}\|\mathbb{P}\|_{\mathscr{L}_c(\mathscr{X}_{\mathbb{C}})}$$

is bounded. Consequently, by Theorem B.2 and since $\sigma_s \subset \{\mathrm{Re} \leq 0\}$, $\sigma(\mathrm{e}^{t[\mathscr{J}\mathscr{L}]_{\mathbb{C}}(\mathrm{Id}-\mathbb{P})}) = \mathrm{e}^{t\sigma([\mathscr{J}\mathscr{L}]_{\mathbb{C}}(\mathrm{Id}-\mathbb{P}))} = \mathrm{e}^{t\sigma_s} \subset \bar{D}(0,1)$. It follows that, for any $\epsilon > 0$, there exists $K_\epsilon > 0$ such that

$$\|\mathrm{e}^{t[\mathscr{J}\mathscr{L}]_{\mathbb{C}}(\mathrm{Id}-\mathbb{P})}\|_{\mathscr{L}_c(\mathscr{X}_{\mathbb{C}})} \leq K_\epsilon \mathrm{e}^{\epsilon t} \quad \text{for all } t \geq 0.$$

Since $\mathrm{e}^{t[\mathscr{J}\mathscr{L}]_{\mathbb{C}}\mathbb{P}}$ is given by the exponential series, we also have the optimal estimate

$$\|\mathrm{e}^{t[\mathscr{J}\mathscr{L}]_{\mathbb{C}}\mathbb{P}}\|_{\mathscr{L}_c(\mathscr{X}_{\mathbb{C}})} \leq K_0(1+t)^{m-1}\mathrm{e}^{\gamma_0 t} \quad \text{for all } t \geq 0$$

by definition of m. We conclude by taking $\epsilon = \gamma_0/2$ for instance.

Proof of Theorem B.5. Since $\mathscr{A}$ generates a continuous semigroup, v is a solution to $\partial_t v = \mathscr{A}v + \Phi(v)$ if and only if it is a mild solution:

$$v(t) = \mathrm{e}^{t\mathscr{A}}v^{\mathrm{in}} + \int_0^t \mathrm{e}^{(t-\tau)\mathscr{A}}\Phi(v(\tau))\,d\tau.$$

There exists $r_0 > 0$ such that $\|\Phi(v)\|_X \leq M\|v\|_X^2$ if $\|v\|_X \leq r_0$. We choose $v^{\mathrm{in}} = \delta\,\mathrm{Re}\,w$, where $\|\mathrm{Re}\,w\|_X = 1$ and w is an eigenvector for the eigenvalue λ, and write the solution under the form $v = \mathrm{e}^{t\mathscr{A}}v^{\mathrm{in}} + \tilde{v} = \mathrm{Re}(\mathrm{e}^{t\lambda}w) + \tilde{v}$. If $\lambda \in \mathbb{R}$, we can choose $w \in D(\mathscr{A}) \subset D(\mathscr{A}_{\mathbb{C}})$. Then,

$$\tilde{v}(t) = \int_0^t \mathrm{e}^{(t-\tau)\mathscr{A}}\Phi(\delta\,\mathrm{Re}(\mathrm{e}^{t\lambda}w) + \tilde{v}(\tau))\,d\tau.$$

Let us define $r_1 \equiv \min\bigl(r_0, (\gamma_0-\beta)/(2MM_0)\bigr)$ and let $T > 0$ be the maximal time such that $T < \ln(r/(2\delta))/\gamma_0$ and $\|\tilde{u}(\tau)\|_X < r_1/2$ in $[0,T)$, where $0 < r < r_1$ will be determined later. We shall work for $0 \leq t < T$, so that $\|\delta\,\mathrm{Re}(\mathrm{e}^{t\lambda}w) + \tilde{v}(\tau)\|_X < \delta\mathrm{e}^{t\gamma_0} + r_1/2 \leq r_1 \leq r_0$. Then,

$$\begin{aligned}
\|\tilde{v}(t)\| &\leq \int_0^t \|\mathrm{e}^{(t-\tau)\mathscr{A}}\|_{\mathscr{L}_c(X)} M\|\delta\,\mathrm{Re}(\mathrm{e}^{\tau\lambda}w) + \tilde{v}(\tau)\|^2\,d\tau \\
&\leq 2M_0M\int_0^t \mathrm{e}^{(\gamma_0+\beta)(t-\tau)}\bigl(\delta^2\mathrm{e}^{2\tau\gamma_0} + \|\tilde{v}(\tau)\|^2\bigr)\,d\tau \\
&\leq \frac{2M_0M}{\gamma_0-\beta}\delta^2\mathrm{e}^{2t\gamma_0} + r_1M_0M\int_0^t \mathrm{e}^{(\gamma_0+\beta)(t-\tau)}\|\tilde{v}(\tau)\|_X\,d\tau,
\end{aligned}$$

since $\beta < \gamma_0$. Applying now the Gronwall inequality to $\mathrm{e}^{-(\gamma_0+\beta)t}\|\tilde{v}(t)\|_X$ then gives, since $M_0Mr_1 < \gamma_0 - \beta$,

$$\|\tilde{u}(t)\|_X \leq \left[\frac{2M_0M}{\gamma_0-\beta} + \frac{2r_1M_0^2M^2}{(\gamma_0-\beta)(\gamma_0-\beta-r_1M_0M)}\right]\delta^2\mathrm{e}^{2t\gamma_0} = K\delta^2\mathrm{e}^{2t\gamma_0}.$$

We now choose $r \equiv \sqrt{r_1/K}$, so that the right-hand side is $\leq Kr^2/4 < r_1/2$, and this implies that u exists at least on $[0, \ln(r/(2\delta))/\gamma_0]$. In addition, for $0 \leq t < T$,

$$\|u(t)\|_X \geq \delta\mathrm{e}^{t\gamma_0} - \|\tilde{u}(t)\|_X \geq \delta\mathrm{e}^{t\gamma_0} - K\delta^2\mathrm{e}^{2t\gamma_0},$$

as desired. We conclude choosing $\varepsilon_0 > 0$ so small that $2\varepsilon_0 - K\varepsilon_0^2 \geq \varepsilon_0$.

Proof of Corollary B.6. We pick some $0 < \beta < \gamma_0$ (for instance $\beta = \gamma_0/2$) in order to have the semigroup estimate required in Theorem B.5. The solution $u(t) = T(\omega_* t)(\phi_{\omega_*} + v(t))$ satisfies, for $0 \le t \le \gamma_0^{-1} \ln(2\varepsilon_0/\delta)$,

$$\|v(t)\|_{\mathscr{X}} = \|T(-\omega_* t)u(t) - (\phi_{\omega_*} + \delta \operatorname{Re}(\mathrm{e}^{t\lambda} w))\|_{\mathscr{X}} \le K\delta^2 \mathrm{e}^{2t\gamma_0}.$$

Hence, $T(-\omega_* t)u(t)$ remains at distance $\le K\varepsilon_0$ from $\phi_{\omega_*} \in \mathfrak{M}$ and therefore

$$\operatorname{dist}_{\mathscr{X}}(u(t), \mathfrak{M}) \ge \operatorname{dist}_{\mathscr{X}}(\delta \operatorname{Re}(\mathrm{e}^{t\lambda} w), \mathfrak{M} - \phi) - K\delta^2 \mathrm{e}^{2t\gamma_0}.$$

Assume $\lambda \in \mathbb{R}$. Then, we observe that the straight line $\mathbb{R} \ni \theta \mapsto \theta w$ is transverse to the tangent space $\mathrm{T}_\phi \mathfrak{M}$ of the manifold $\mathfrak{M}$, since w is an eigenvector of $\mathscr{J}\mathscr{L}$ for $\lambda \ne 0$, and hence does not belong to the kernel of $\mathscr{L}$. Therefore, $\operatorname{dist}_{\mathscr{X}}(\theta w, \mathfrak{M} - \phi) \ge |\theta|/K_1$ for small $|\theta|$. Thus,

$$\operatorname{dist}_{\mathscr{X}}(u(t), \mathfrak{M}) \ge \frac{1}{K_1} \delta \mathrm{e}^{t\lambda} - K\delta^2 \mathrm{e}^{2t\gamma_0}.$$

If $\lambda \in \mathbb{C} \backslash \mathbb{R}$, the equation $[\mathscr{J}\mathscr{L}]_{\mathbb{C}}(w) = \lambda w$ splits as $\mathscr{J}\mathscr{L}(\operatorname{Re} w) = \operatorname{Re}(\lambda) \operatorname{Re} w - \operatorname{Im}(\lambda) \operatorname{Im} w$ and $\mathscr{J}\mathscr{L}(\operatorname{Im} w) = \operatorname{Im}(\lambda) \operatorname{Re} w + \operatorname{Re}(\lambda) \operatorname{Im} w$. Therefore, $\operatorname{Re} w$ and $\operatorname{Im} w$ do not belong to $\ker(\mathscr{L})$. Consequently, the surface $\mathbb{C} \ni \theta \mapsto \operatorname{Re}(\theta w)$ is transverse to the tangent space $\mathrm{T}_\phi \mathfrak{M}$ of the manifold $\mathfrak{M}$, and we conclude as before that

$$\operatorname{dist}_{\mathscr{X}}(u(t), \mathfrak{M}) \ge \frac{1}{K_1} \delta \mathrm{e}^{t\gamma_0} - K\delta^2 \mathrm{e}^{2t\gamma_0}.$$

Acknowledgement

We acknowledge the support of the French ANR (Agence Nationale de la Recherche) under Grant ANR JC ArDyPitEq.

References

[Abid et al. 2003] M. Abid, C. Huepe, S. Metens, C. Nore, C. T. Pham, L. S. Tuckerman, and M. E. Brachet, "Gross–Pitaevskii dynamics of Bose–Einstein condensates and superfluid turbulence", *Fluid Dynam. Res.* **33**:5-6 (2003), 509–544. MR 2004k:82115 Zbl 1060.76528

[Albert 1992] J. P. Albert, "Positivity properties and stability of solitary-wave solutions of model equations for long waves", *Comm. Partial Differential Equations* **17**:1-2 (1992), 1–22. MR 93d:35140 Zbl 0782.35064

[Barashenkov 1996] I. V. Barashenkov, "Stability criterion for dark solitons", *Phys. Rev. Lett.* **77**:7 (1996), 1193–1197.

[Barashenkov and Panova 1993] I. V. Barashenkov and E. Panova, "Stability and evolution of the quiescent and travelling solitonic bubbles", *Physica D* **69**:1-2 (1993), 114–134. Zbl 0791.35126

[Benzoni-Gavage 2010a] S. Benzoni-Gavage, "Propagating phase boundaries and capillary fluids", Lecture notes, Centro Internazionale per la Ricerca Matematica, 2010, Available at http://www.science.unitn.it/cirm/Benzoni-course.pdf.

[Benzoni-Gavage 2010b] S. Benzoni-Gavage, "Spectral transverse instability of solitary waves in Korteweg fluids", *J. Math. Anal. Appl.* **361**:2 (2010), 338–357. MR 2011f:35267 Zbl 1180.35451

[Benzoni-Gavage et al. 2005] S. Benzoni-Gavage, R. Danchin, S. Descombes, and D. Jamet, "Structure of Korteweg models and stability of diffuse interfaces", *Interfaces Free Bound.* **7**:4 (2005), 371–414. MR 2006h:35227 Zbl 1106.35056

[Berestycki and Cazenave 1981] H. Berestycki and T. Cazenave, "Instabilité des états stationnaires dans les équations de Schrödinger et de Klein–Gordon non linéaires", *C. R. Acad. Sci. Paris Sér. I Math.* **293**:9 (1981), 489–492. MR 84f:35120 Zbl 0492.35010

[Béthuel et al. 2008a] F. Béthuel, P. Gravejat, and J.-C. Saut, "Existence and properties of travelling waves for the Gross–Pitaevskii equation", pp. 55–103 in *Stationary and time dependent Gross–Pitaevskii equations* (Vienna, 2006), edited by A. Farina and J.-C. Saut, Contemp. Math. **473**, Amer. Math. Soc., Providence, RI, 2008. MR 2011a:35484 Zbl 1216.35132

[Béthuel et al. 2008b] F. Béthuel, P. Gravejat, J.-C. Saut, and D. Smets, "Orbital stability of the black soliton for the Gross–Pitaevskii equation", *Indiana Univ. Math. J.* **57**:6 (2008), 2611–2642. MR 2010d:35333 Zbl 1171.35012

[Béthuel et al. 2010] F. Béthuel, R. Danchin, and D. Smets, "On the linear wave regime of the Gross–Pitaevskii equation", *J. Anal. Math.* **110** (2010), 297–338. MR 2012c:35409 Zbl 1202.35296

[Bogdan et al. 1989] M. Bogdan, A. Kovalev, and A. Kosevich, "Stability criterion in imperfect Bose gas", *Fiz. Nizk. Temp.* **15**:5 (1989), 511–513. In Russian.

[Bona et al. 1987] J. L. Bona, P. E. Souganidis, and W. A. Strauss, "Stability and instability of solitary waves of Korteweg–de Vries type", *Proc. Roy. Soc. London Ser. A* **411**:1841 (1987), 395–412. MR 88m:35128 Zbl 0648.76005

[de Bouard 1995] A. de Bouard, "Instability of stationary bubbles", *SIAM J. Math. Anal.* **26**:3 (1995), 566–582. MR 96i:35120 Zbl 0823.35017

[Bridges and Derks 2002] T. J. Bridges and G. Derks, "Linear instability of solitary wave solutions of the Kawahara equation and its generalizations", *SIAM J. Math. Anal.* **33**:6 (2002), 1356–1378. MR 2003h:35220 Zbl 1011.35117

[Cazenave and Lions 1982] T. Cazenave and P.-L. Lions, "Orbital stability of standing waves for some nonlinear Schrödinger equations", *Comm. Math. Phys.* **85**:4 (1982), 549–561. MR 84i:81015 Zbl 0513.35007

[Chiron 2012] D. Chiron, "Travelling waves for the nonlinear Schrödinger equation with general nonlinearity in dimension one", *Nonlinearity* **25**:3 (2012), 813–850. MR 2887994 Zbl 06030687

[Chiron and Scheid 2012] D. Chiron and C. Scheid, "Travelling waves for the nonlinear Schrödinger equation with general nonlinearity in dimension two", preprint, 2012, Available at http://math.unice.fr/∼chiron/fichierspdf/Endim2.pdf. Zbl 06030687

[Comech and Pelinovsky 2003] A. Comech and D. Pelinovsky, "Purely nonlinear instability of standing waves with minimal energy", *Comm. Pure Appl. Math.* **56**:11 (2003), 1565–1607. MR 2005h:37176 Zbl 1072.35165

[Comech et al. 2007] A. Comech, S. Cuccagna, and D. E. Pelinovsky, "Nonlinear instability of a critical traveling wave in the generalized Korteweg–de Vries equation", *SIAM J. Math. Anal.* **39**:1 (2007), 1–33. MR 2008f:35336 Zbl 1136.35077

[Di Menza and Gallo 2007] L. Di Menza and C. Gallo, "The black solitons of one-dimensional NLS equations", *Nonlinearity* **20**:2 (2007), 461–496. MR 2007k:35455 Zbl 1128.35095

[Fakau and Karval'u 2009] M. Fakau and M. I. Karval'u, "Stability of dark screening solitons in photorefractive media", *Teoret. Mat. Fiz.* **160**:1 (2009), 49–58. In Russian; translated in *Theoret. and Math. Phys.* **160**:1 (2009), 917–924. MR 2011c:35534 Zbl 1179.78093

[Fitrakis et al. 2007] E. P. Fitrakis, P. G. Kevrekidis, H. Susanto, and D. J. Frantzeskakis, "Dark solitons in discrete lattices: saturable versus cubic nonlinearities", *Phys. Rev. E* (3) **75**:6 (2007), 066608, 12. MR 2008i:37174

[Gallo 2004] C. Gallo, "Schrödinger group on Zhidkov spaces", *Adv. Differential Equations* **9**:5-6 (2004), 509–538. MR 2006d.35255 Zbl 1103.35093

[Gallo 2008] C. Gallo, "The Cauchy problem for defocusing nonlinear Schrödinger equations with non-vanishing initial data at infinity", *Comm. Partial Differential Equations* **33**:4-6 (2008), 729–771. MR 2009c:35438 Zbl 1156.35086

[Gardner and Zumbrun 1998] R. A. Gardner and K. Zumbrun, "The gap lemma and geometric criteria for instability of viscous shock profiles", *Comm. Pure Appl. Math.* **51**:7 (1998), 797–855. MR 99c:35152 Zbl 0933.35136

[Gearhart 1978] L. Gearhart, "Spectral theory for contraction semigroups on Hilbert space", *Trans. Amer. Math. Soc.* **236** (1978), 385–394. MR 57 #1191 Zbl 0326.47038

[Georgiev and Ohta 2012] V. Georgiev and M. Ohta, "Nonlinear instability of linearly unstable standing waves for nonlinear Schrödinger equations", *J. Math. Soc. Japan* **64**:2 (2012), 533–548. MR 2916078 Zbl 1253.35158

[Gérard 2008] P. Gérard, "The Gross–Pitaevskii equation in the energy space", pp. 129–148 in *Stationary and time dependent Gross–Pitaevskii equations* (Vienna, 2006), edited by A. Farina and J.-C. Saut, Contemp. Math. **473**, Amer. Math. Soc., Providence, RI, 2008. MR 2010e:35259 Zbl 1166.35373

[Gérard and Zhang 2009] P. Gérard and Z. Zhang, "Orbital stability of traveling waves for the one-dimensional Gross–Pitaevskii equation", *J. Math. Pures Appl.* (9) **91**:2 (2009), 178–210. MR 2011f:35322 Zbl 1232.35152

[Gesztesy et al. 2000] F. Gesztesy, C. K. R. T. Jones, Y. Latushkin, and M. Stanislavova, "A spectral mapping theorem and invariant manifolds for nonlinear Schrödinger equations", *Indiana Univ. Math. J.* **49**:1 (2000), 221–243. MR 2001g:37144 Zbl 0969.35123

[Ginzburg and Pitaevskiĭ 1958] V. L. Ginzburg and L. P. Pitaevskiĭ, "On the theory of superfluidity", *Soviet Physics. JETP* **7**:5 (1958), 858–861. MR 21 #4663

[Grenier 2000] E. Grenier, "On the nonlinear instability of Euler and Prandtl equations", *Comm. Pure Appl. Math.* **53**:9 (2000), 1067–1091. MR 2001i:76056 Zbl 1048.35081

[Grillakis 1988] M. Grillakis, "Linearized instability for nonlinear Schrödinger and Klein–Gordon equations", *Comm. Pure Appl. Math.* **41**:6 (1988), 747–774. MR 89m:35192 Zbl 0632.70015

[Grillakis et al. 1987] M. Grillakis, J. Shatah, and W. Strauss, "Stability theory of solitary waves in the presence of symmetry, I", *J. Funct. Anal.* **74**:1 (1987), 160–197. MR 88g:35169 Zbl 0656.35122

[Grillakis et al. 1990] M. Grillakis, J. Shatah, and W. Strauss, "Stability theory of solitary waves in the presence of symmetry, II", *J. Funct. Anal.* **94**:2 (1990), 308–348. MR 92a:35135 Zbl 0711.58013

[Gross 1963] E. P. Gross, "Hydrodynamics of a superfluid condensate", *J. Math. Phys.* **4**:2 (1963), 195–207.

[Henry et al. 1982] D. B. Henry, J. F. Perez, and W. F. Wreszinski, "Stability theory for solitary-wave solutions of scalar field equations", *Comm. Math. Phys.* **85**:3 (1982), 351–361. MR 83m:35131 Zbl 0546.35062

[Hislop and Sigal 1996] P. D. Hislop and I. M. Sigal, *Introduction to spectral theory, with applications to Schrödinger operators*, Applied Mathematical Sciences **113**, Springer, New York, 1996. MR 98h:47003 Zbl 0855.47002

[Kapitula and Sandstede 1998] T. Kapitula and B. Sandstede, "Stability of bright solitary-wave solutions to perturbed nonlinear Schrödinger equations", *Phys. D* **124**:1-3 (1998), 58–103. MR 99h:35199 Zbl 0935.35150

[Kato 1976] T. Kato, *Perturbation theory for linear operators*, 2nd ed., Grundlehren der Mathematischen Wissenschaften **132**, Springer, Berlin, 1976. MR 53 #11389 Zbl 0342.47009

[Kivshar and Krolikowski 1995] Y. S. Kivshar and W. Krolikowski, "Instabilities of dark solitons", *Optics Letters* **20**:14 (1995), 1527–1529.

[Kivshar and Luther-Davies 1998] Y. S. Kivshar and B. Luther-Davies, "Dark optical solitons: physics and applications", *Physics Reports* **298**:2-3 (1998), 81–197.

[Kivshar and Yang 1994] Y. S. Kivshar and X. Yang, "Perturbation-induced dynamics of dark solitons", *Phys. Rev. E* (3) **49**:2 (1994), 1657–1670. MR 96k:78007

[Kolomeisky et al. 2000] E. B. Kolomeisky, T. J. Newman, J. P. Straley, and X. Qi, "Low-dimensional Bose liquids: beyond the Gross–Pitaevskii approximation", *Phys. Rev. Lett.* **85**:8 (2000), 1146–1149.

[Lin 2002] Z. Lin, "Stability and instability of traveling solitonic bubbles", *Adv. Differential Equations* **7**:8 (2002), 897–918. MR 2003d:35243 Zbl 1033.35117

[Lin 2008] Z. Lin, "Instability of nonlinear dispersive solitary waves", *J. Funct. Anal.* **255**:5 (2008), 1191–1224. MR 2010m.35456 Zbl 1157.35096

[Lopes 2002] O. Lopes, "A linearized instability result for solitary waves", *Discrete Contin. Dyn. Syst.* **8**:1 (2002), 115–119. MR 2002j:35264 Zbl 1001.35108

[Maeda 2012] M. Maeda, "Stability of bound states of Hamiltonian PDEs in the degenerate cases", *J. Funct. Anal.* **263**:2 (2012), 511–528. MR 2923422 Zbl 1244.35008

[Melvin et al. 2008] T. R. O. Melvin, A. R. Champneys, P. G. Kevrekidis, and J. Cuevas, "Travelling solitary waves in the discrete Schrödinger equation with saturable nonlinearity: existence, stability and dynamics", *Phys. D* **237**:4 (2008), 551–567. MR 2009h:35405 Zbl 1167.35448

[Mizumachi 2006] T. Mizumachi, "A remark on linearly unstable standing wave solutions to NLS", *Nonlinear Anal.* **64**:4 (2006), 657–676. MR 2007f:35273 Zbl 1091.35093

[Ohta 2011] M. Ohta, "Instability of bound states for abstract nonlinear Schrödinger equations", *J. Funct. Anal.* **261**:1 (2011), 90–110. MR 2012g:37134 Zbl 1228.34092

[Papanicolaou and Spathis 1999] N. Papanicolaou and P. N. Spathis, "Semitopological solitons in planar ferromagnets", *Nonlinearity* **12**:2 (1999), 285–302. MR 99k:82076 Zbl 0938.35183

[Pego and Weinstein 1992] R. L. Pego and M. I. Weinstein, "Eigenvalues, and instabilities of solitary waves", *Philos. Trans. Roy. Soc. London Ser. A* **340**:1656 (1992), 47–94. MR 93g:35115 Zbl 0776.35065

[Pelinovsky and Kevrekidis 2008] D. E. Pelinovsky and P. G. Kevrekidis, "Dark solitons in external potentials", *Z. Angew. Math. Phys.* **59**:4 (2008), 559–599. MR 2009i:35306 Zbl 1157.35089

[Prüss 1984] J. Prüss, "On the spectrum of C_0-semigroups", *Trans. Amer. Math. Soc.* **284**:2 (1984), 847–857. MR 85f:47044 Zbl 0572.47030

[Roberts and Berloff 2001] P. Roberts and N. Berloff, "The nonlinear Schrödinger equation as a model of superfluid helium", pp. 235–257 in *Quantized vortex dynamics and superfluid turbulence*, edited by C. F. Barenghi et al., Lecture Notes in Physics **571**, Springer, Berlin, 2001. Zbl 0994.82105

[Rousset and Tzvetkov 2008] F. Rousset and N. Tzvetkov, "Transverse nonlinear instability of solitary waves for some Hamiltonian PDE's", *J. Math. Pures Appl.* (9) **90**:6 (2008), 550–590. MR 2011a:35473 Zbl 1159.35063

[Rousset and Tzvetkov 2009] F. Rousset and N. Tzvetkov, "Transverse nonlinear instability for two-dimensional dispersive models", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**:2 (2009), 477–496. MR 2010j:35475 Zbl 1169.35374

[Sandstede 2002] B. Sandstede, "Stability of travelling waves", pp. 983–1055 in *Handbook of dynamical systems, II*, edited by B. Fiedler, North-Holland, Amsterdam, 2002. MR 2004e:37121 Zbl 1056.35022

[Shatah and Strauss 1985] J. Shatah and W. Strauss, "Instability of nonlinear bound states", *Comm. Math. Phys.* **100**:2 (1985), 173–190. MR 87b:35159 Zbl 0603.35007

[Shatah and Strauss 2000] J. Shatah and W. Strauss, "Spectral condition for instability", pp. 189–198 in *Nonlinear PDE's, dynamics and continuum physics* (South Hadley, MA, 1998), edited by J. Bona et al., Contemp. Math. **255**, Amer. Math. Soc., Providence, RI, 2000. MR 2001c:34127 Zbl 0960.47033

[Souganidis and Strauss 1990] P. E. Souganidis and W. A. Strauss, "Instability of a class of dispersive solitary waves", *Proc. Roy. Soc. Edinburgh Sect. A* **114**:3-4 (1990), 195–212. MR 92a:35143 Zbl 0713.35108

[Tsuzuki 1971] T. Tsuzuki, "Nonlinear waves in the Pitaevskii–Gross equation", *J. Low Temp. Phys.* **4**:4 (1971), 441–457.

[Weinstein 1986] M. I. Weinstein, "Lyapunov stability of ground states of nonlinear dispersive evolution equations", *Comm. Pure Appl. Math.* **39**:1 (1986), 51–67. MR 87f:35023 Zbl 0594.35005

[Zhidkov 2001] P. E. Zhidkov, *Korteweg–de Vries and nonlinear Schrödinger equations: qualitative theory*, Lecture Notes in Mathematics **1756**, Springer, Berlin, 2001. MR 2002h:35285 Zbl 0987.35001

[Zumbrun 2008] K. Zumbrun, "A sharp stability criterion for soliton-type propagating phase boundaries in Korteweg's model", *Z. Anal. Anwend.* **27**:1 (2008), 11–30. MR 2009h:35391 Zbl 1187.35229

Received 25 Jun 2012. Revised 4 Sep 2012. Accepted 28 Feb 2013.

DAVID CHIRON: chiron@unice.fr
Laboratoire J.A. Dieudonné, Université de Nice-Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 02, France

ANALYSIS AND PDE
Vol. 6, No. 6, 2013
dx.doi.org/10.2140/apde.2013.6.1421

msp

SEMICLASSICAL MEASURES FOR INHOMOGENEOUS SCHRÖDINGER EQUATIONS ON TORI

NICOLAS BURQ

The purpose of this note is to investigate the high-frequency behavior of solutions to linear Schrödinger equations. More precisely, Bourgain (1997) and Anantharaman and Macià (2011) proved that any weak-$*$ limit of the square density of solutions to the time-dependent homogeneous Schrödinger equation is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R} \times \mathbb{T}^d$. The contribution of this article is that the same result automatically holds for nonhomogeneous Schrödinger equations, which allows for abstract potential type perturbations of the Laplace operator.

1. Introduction

In this note we are interested in understanding the high-frequency behavior of solutions of linear Schrödinger equations on tori, $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Consider a sequence of initial data $(u_{0,n})$, bounded in $L^2(\mathbb{T}^d)$ and denote by (u_n) the sequence of solutions to the Schrödinger equation and by (ν_n) their concentration measures given by

$$u_n = e^{it\Delta} u_{0,n}, \qquad \nu_n = |u_n|^2(t, x)\, dt\, dx.$$

The sequence ν_n on $\mathbb{R}_t \times \mathbb{T}^d$ is bounded (in mass) on any time interval $(0, T)$ by $T \sup_n \|u_{0,n}\|^2_{L^2(\mathbb{T}^d)}$. The following result was proved in [Bourgain 1997, Remark, page 108] and later, using a completely different approach that follows a more geometric path, in [Anantharaman and Macià 2011, Theorem 1]. (See also [Jakobson 1997; Macià 2011; Burq and Zworski 2004; 2005;Aïssiou et al. 2011] for related works.)

Theorem 1. *Any weak-$*$ limit of the sequence (ν_n) is absolutely continuous with respect to the Lebesgue measure $dt\, dx$ on $\mathbb{R}_t \times \mathbb{T}^d$.*

Remark 1.1. Actually, in [Anantharaman and Macià 2011] a more precise description of the possible limits is given and the result is proved in the case of Schrödinger operators $\Delta + V(t, x)$, if $V \in L^\infty(\mathbb{R}_t \times \mathbb{T}^2)$ is also continuous except possibly on a set of (spacetime) Lebesgue measure 0.

The purpose of this note is to show that the result in Theorem 1 extends to the case of solutions to the nonhomogeneous Schrödinger equation, and, consequently, to the case of Schrödinger operators $\Delta + V$ where $V \in L^1_{\mathrm{loc}}(\mathbb{R}_t; \mathscr{L}(L^2(\mathbb{T}^d)))$ (we also give as an illustration an application to a simple nonlinear equation). Let us emphasize that our approach uses no particular property of the Laplace operator on tori

The author was partially supported by the Agence Nationale de la Recherche, project NOSEVOL, 2011 BS01019 01.
MSC2010: 35LXX.
Keywords: defect-measures, Schrödinger equations.

other than selfadjointness (to get L^2 bounds for the time evolution) and the fact that Theorem 1 holds, which is used as a black box, and establishes an abstract link between the study of weak-$*$ limits of solutions of the homogeneous and inhomogeneous Schrödinger equations.

2. Inhomogeneous Schrödinger equations

Definition 2.1. Let $T > 0$. For any sequence (u_n) bounded in $L^2((0,T)\times\mathbb{T}^d)$, we say that the sequence (u_n) satisfies property (AC_T) if any weak-$*$ limit ν of (ν_n) is absolutely continuous with respect to the Lebesgue measure on $(0,T)\times\mathbb{T}^d$.

Theorem 2. *Let $(u_{n,0})$ and (f_n) be two sequences bounded in $L^2(\mathbb{T}^d)$ and $L^1_{\mathrm{loc}}(\mathbb{R}_t; L^2(\mathbb{T}^d))$, respectively. Let u_n be the solution of*

$$(i\partial_t+\Delta)u_n = f_n, \quad u_n|_{t=0} = u_{n,0}, \quad u_n = e^{it\Delta}u_{n,0} + \frac{1}{i}\int_0^t e^{i(t-s)\Delta} f_n(s)\,ds.$$

Then, for any $T > 0$, the sequence (u_n), which is clearly bounded in $L^2((0,T)\times\mathbb{T}^2)$ by

$$T^{1/2}\sup_n\left(\|u_{n,0}\|_{L^2(\mathbb{T}^d)} + \|f_n\|_{L^1((0,T);L^2(\mathbb{T}^d))}\right),$$

satisfies property (AC_T).

Corollary 2.2. *Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}_t; \mathcal{L}(L^2(\mathbb{T}^2)))$ (for example, V can be a potential in $L^1_{\mathrm{loc}}(\mathbb{R}_t; L^\infty(\mathbb{T}^2))$ acting by pointwise multiplication). For any sequence $(u_{n,0})_{n\in\mathbb{N}}$ bounded in $L^2(\mathbb{T}^2)$, let (u_n) be the sequence of the unique solutions in $C^0(\mathbb{R}; L^2(\mathbb{T}^2))$ of*

$$(i\partial_t + \Delta + V(t))u_n = 0, \quad u_n|_{t=0} = u_{n,0}.$$

Then the sequence (u_n) satisfies the property (AC_T) for any $T > 0$.

Indeed, since

$$\frac{d}{dt}\|u_n\|^2_{L^2(\mathbb{T}^d)} = 2\Re(\partial_t u, u)_{L^2(\mathbb{T}^d)} = 2\Re(i\Delta u + iVu, u)_{L^2(\mathbb{T}^d)} = -2\Im(Vu,u)_{L^2(\mathbb{T}^d)},$$

by Gronwall's inequality, we obtain

$$\|u_n(t)\|^2_{L^2(\mathbb{T}^d)} \le \|u_{n,0}\|^2_{L^2(\mathbb{T}^d)} e^{\int_0^t \|V(s)\|_{\mathcal{L}(L^2(\mathbb{T}^d))}\,ds},$$

and, consequently, the sequence $(f_n) = (-V(t)u_n)$ is clearly bounded in $L^1_{\mathrm{loc}}(\mathbb{R}_t; L^2(\mathbb{T}^d))$ and we can apply Theorem 2.

Remark 2.3. Any time independent $V \in \mathcal{L}(L^2(\mathbb{T}^d))$ satisfies the assumptions above, and, consequently, if (u_n) is a sequence of L^2 normalized eigenfunctions of $\Delta + V$, it follows from Corollary 2.2 that any weak-$*$ limit of $|u_n|^2(x)\,dx$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^d$. The proof we present below seems to be intrinsically time-dependent. However, it would be interesting to obtain a proof of this result avoiding the detour via the study of the time-dependent Schrödinger equation.

Proof of Theorem 2. If (u_n) satisfies property (AC_T), then the sequence (u_n+v_n) satisfies property (AC_T) if and only if the sequence (v_n) satisfies property (AC_T). This is because if $|u_n|^2\,dt\,dx$ and $|v_n|^2\,dt\,dx$ converge weakly to ν and μ, respectively, then, according to the Cauchy–Schwarz inequality, any weak-$*$ limit of $|u_n+v_n|^2\,dt\,dx$ is absolutely continuous with respect to $\nu+\mu$. The following result shows that the set of sequences satisfying property (AC_T) is closed in some weak-strong topology.

Lemma 2.4. *Consider (u_n) bounded in $L^2((0,T)\times\mathbb{T}^2)$. Assume that there exists for any $k\in\mathbb{N}$ a sequence $(u_n^{(k)})_{n\in\mathbb{N}}$ such that*

(1) *for any k, the sequence $(u_n^{(k)})_{n\in\mathbb{N}}$ satisfies property (AC_T);*

(2) *the sequences $(u_n^{(k)})_{n\in\mathbb{N}}$ are approximating the sequence (u_n) in the sense that*

$$\lim_{k\to+\infty}\limsup_{n\to+\infty}\|u_n-u_n^{(k)}\|_{L^2((0,T)\times\mathbb{T}^2)}=0. \tag{2-1}$$

Then the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies property (AC_T).

Proof. Indeed, for any $\epsilon>0$, let k_0 be such that, for any $k\ge k_0$,

$$\limsup_n\|u_n-u_{n,k}\|_{L^2((0,T)\times\mathbb{T}^2)}<\epsilon.$$

Then, if ν and $\nu^{(k)}$ are weak-$*$ limits of the sequences $(u_n)_{n\in\mathbb{N}}$ and $(u_n^{(k)})_{n\in\mathbb{N}}$, respectively, associated to the same subsequence $n_p\to+\infty$, we have, for any $f\in C^0((0,T)\times\mathbb{T}^2)$ and large n,

$$\begin{aligned}\int_{(0,T)\times\mathbb{T}^2}|u_{n_p}|^2\chi\,dx\,dt&\le\int_{(0,T)\times\mathbb{T}^2}2(|u_{n_p}-u_{n_p}^{(k)}|^2+|u_{n_p}^{(k)}|^2)\,dx\,dt\\&\le 2\epsilon^2+2\int_{(0,T)\times\mathbb{T}^2}2|u_{n_p}^{(k)}|^2)\chi\,dx\,dt.\end{aligned}\tag{2-2}$$

Passing to the limit $p\to+\infty$, we obtain

$$\langle\nu,\chi\rangle\le 2\epsilon^2+2\langle\nu^{(k)},\chi\rangle.$$

On the other hand, according to the Riesz theorem (see, for example, [Rudin 1987, Theorem 2.14]), the measures ν, $\nu^{(k)}$ which are defined on the Borelian σ-algebra, $\mathcal{M}$, are *regular*, and, consequently,

$$\begin{aligned}&\forall E\in\mathcal{M},\ \ \nu(E)=\sup_{F\text{closed},\,F\subset E}\nu(U)=\inf_{U\text{open},\,E\subset U}\nu(U),\\&\forall E\in\mathcal{M},\ \ \nu^{(k)}(E)=\sup_{F\text{closed},\,F\subset E}\nu^{(k)}(U)=\inf_{U\text{open},\,E\subset U}\nu^{(k)}(U).\end{aligned}\tag{2-3}$$

For any $E\in\mathcal{M}$, taking $F_p\subset E$ and $E\subset O_p$ such that

$$\lim_{p\to+\infty}\nu(F_p)=\nu(E),\qquad\lim_{p\to+\infty}\nu^{(k)}(O_p)=\nu^{(k)}(E)$$

and $\chi_p\in C_0((0,1)\times\mathbb{T}^d;[0,1])$ is equal to 1 on F_p and supported in O_p, we obtain, according to (2-2),

$$\nu(E)\le 2\epsilon^2+2\nu^{(k)}(E).$$

Now consider E a subset of $(0,T)\times\mathbb{T}^d$-Lebesgue measure 0. Since by assumption $\nu^{(k)}$ is absolutely continuous with respect to the Lebesgue measure, we have $\nu^{(k)}(E)=0$, and hence $\nu(E)\le 2\epsilon^2$. Consequently, since $\epsilon>0$ can be taken arbitrarily small, we have $\nu(E)=0$, which proves that ν is also absolutely continuous with respect to the Lebesgue measure. □

We come back to the proof of Theorem 2 and fix $T>0$. According to Duhamel's formula,

$$u_n=e^{it\Delta}u_{0,n}+\frac{1}{i}\int_0^t e^{i(t-s)\Delta}f_n(s)\,ds.$$

According to the remark above, since we know that the sequence $(e^{it\Delta}u_{0,n})$ satisfies property (AC_T), it is enough to prove that the sequence $(v_n)=(\int_0^t e^{i(t-s)}f_n(s)\,ds)$ satisfies property (AC_T). The key point of the analysis is that if instead of v_n we had

$$\tilde{v}_n=\int_0^T e^{i(t-s)\Delta}Vu_n(s)\,ds=e^{it\Delta}g_n,\quad g_n=\int_0^T e^{-is\Delta}Ve^{is(\Delta+V)}u_{n,0}(s)\,ds,$$

we could conclude using Theorem 1, because $\tilde{v}_n$ is a solution to the homogeneous Schrödinger equation with initial data the bounded sequence (g_n). To pass from $\tilde{v}_n$ to v_n, we adapt an idea borrowed from harmonic analysis (the Christ–Kiselev Lemma [2001]) in the simple form written in [Burq and Planchon 2006] (see also [Burq 2011]). Here the idea is to show that the sequence (v_n) can be approximated by other sequences $(v_n^{(k)})$ in the sense of (2-1) (actually, we get a stronger convergence, as we can replace the lim sup in (2-1) by a sup), where each $(v_n^{(k)})$ is a finite sum of solutions of the homogeneous Schrödinger equation, properly truncated in time, and hence satisfy property (AC_T). Let

$$\|f_n\|_{L^1((0,T);L^2(\mathbb{T}^2))}=c_n\le C.$$

We decompose the interval $(0,T)$ into dyadic pieces on which the $L^1((0,T);L^2(\mathbb{T}^d))$-norm of f_n is equal to $2^{-q}c_n$. For this, we recursively construct (on the index $q\in\mathbb{N}$) certain sequences $(t_{p,q,n})_{q\in\mathbb{N},\ p=1,\dots,2^q}$ such that

- $0=t_{0,q,n}<t_{1,q,n}<\cdots<t_{2^q,q,n}=T$,
- $\|f_n\|_{L^1((t_{p,q,n},t_{p+1,q,n});L^2(\mathbb{T}^2))}=2^{-q}c_n$,
- $t_{2p,q,n}=t_{p,q-1,n}$ for any $p=0,\dots,2^{q-1}$.

Notice that if the function

$$G_n:t\in[0,T]\mapsto\|f_n\|_{L^1((0,t);L^2(\mathbb{T}^d))}\in[0,c_n]$$

is strictly increasing, the points $t_{p,q,n}$ are uniqeuly determined by the relation $G_n(t_{p,q,n})=p2^{-q}c_n$, and the last condition above is automatic. In the general case, the function G_n (which is clearly nondecreasing) can have some flat parts, and, consequently, the points $t_{p,q,n}$ may not be unique anymore. The last condition above ensures that the choice made at step $q+1$ is consistent with the choice made at step q. For $j=0,\dots,2^q-1$, let

$$I_{j,q,n}=[t_{2j,q,n},t_{2j+1,q,n}[,\quad J_{j,q,n}=[t_{2j+1,q,n},t_{2j+2,q,n}[,\quad Q_{j,q,n}=J_{j,q,n}\times I_{j,q,n}.$$

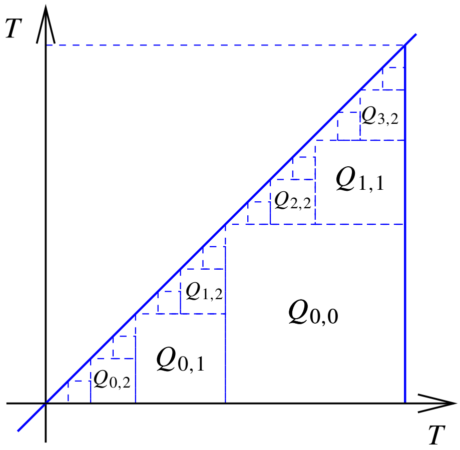


Figure 1. Decomposition of a triangle as a union of disjoint squares.

Notice that

$$\{((t,s)\in[0,T[^2;\, s\le t\} = \bigsqcup_{q=0}^{+\infty}\bigsqcup_{j=0}^{2^q-1} Q_{j,q,n} \Rightarrow 1_{s\le t} = \sum_{q=0}^{+\infty}\sum_{j=0}^{2^q-1} 1_{Q_{j,q,n}}(t,s).$$

Now (if we are able to prove that the series in q converges) we have

$$\begin{aligned} v_n &= \int_0^t e^{i(t-s)\Delta} f_n(s)\,ds = \int_0^T 1_{s\le t} e^{i(t-s)\Delta} f_n(s)\,ds \\ &= \sum_{q=0}^{+\infty}\sum_{j=0}^{2^q-1} 1_{t\in J_{j,q,n}} \int_0^T e^{i(t-s)\Delta} 1_{s\in I_{j,q,n}} f_n(s)\,ds = \sum_{q=0}^{+\infty}\sum_{j=0}^{2^q-1} 1_{t\in J_{j,q,n}} e^{it\Delta} g_{j,q,n}\,ds, \end{aligned} \tag{2-4}$$

with

$$\begin{gathered} g_{j,q,n}(x) = \int_0^T e^{-is\Delta} 1_{s\in I_{j,q,n}} f_n(s)\,ds = \int_{t_{2j,q,n}}^{t_{2j+1,q,n}} e^{-is\Delta} f_n(s)\,ds, \\ \|g_{j,q,n}\|_{L^2(\mathbb{T}^d)} \le \|f_n\|_{L^1((t_{2j,q,n},t_{2j+1,q,n}T);L^2(\mathbb{T}^d))} = 2^{-q}c_n. \end{gathered} \tag{2-5}$$

Let

$$v_n^{(k)} = \sum_{q=0}^{k}\sum_{j=0}^{2^q-1} 1_{t\in J_{j,q,n}} e^{it\Delta} g_{j,q,n}\,ds.$$

Noticing that if a sequence (w_n) satisfies property(AC_T), then, for any sequences $0\le t_{1,n} < t_{2,n}\le T$, the sequence $(1_{t\in(t_{1,n},t_{2,n})}w_n)$ satisfies property(AC_T), we see that for any $k\in\mathbb{N}$, the sequence $(v_n^{(k)})$ satisfies property (AC_T). On the other hand, since for $j\ne j'$, $1_{t\in J_{j,q,n}}$ and $1_{t\in J_{j',q,n}}$ have disjoint supports, we get, according to (2-5),

$$\begin{aligned} \Big\| \sum_{j=0}^{2^q-1} 1_{t\in J_{j,q,n}} e^{it\Delta} g_{j,q,n} \Big\|_{L^\infty((0,T);L^2(\mathbb{T}^d))} &\le \sup_{0\le j\le 2^q-1} \|1_{t\in J_{j,q,n}} e^{it\Delta} g_{j,q,n}\|_{L^\infty((0,T);L^2(\mathbb{T}^d))} \\ &\le \sup_{0\le j\le 2^q-1} \|g_{j,q,n}\|_{L^2(\mathbb{T}^d))} \le 2^{-q}c_n. \end{aligned} \tag{2-6}$$

As a consequence, we get that the series (2-4) is convergent and

$$\|v_n - v_n^{(k)}\|_{L^2((0,T)\times\mathbb{T}^d)} \le \sqrt{T} c_n 2^{-k} \le C 2^{-k},$$

which, according to Lemma 2.4, concludes the proof of Theorem 2. □

3. An illustration

We consider here the nonlinear Schrödinger equation

$$(i\partial_t + \Delta)u + V(u,t)u = 0 \quad \text{on } \mathbb{T}^d, \quad u|_{t=0} = 0 \tag{3-1}$$

where the function $z \in \mathbb{C} \mapsto V(z,t)z \in \mathbb{C}$ is globally Lipschitz with respect to the z variable, with a time-integrable Lipschitz constant; that is, there exists $C \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $C(t) > 0$ for all t and

$$|V(z,t)z - V(z',t)z'| \le C(t)|z - z'| \quad \text{for all } z, z' \in \mathbb{C}.$$

Notice, for example, that the choice $V(u,t) = |u|^2/(1+\epsilon|u|^2)$ satisfies these assumptions for any $\epsilon > 0$.

Proposition 3.1. *For any $u_0 \in L^2(\mathbb{T}^d)$, there exists a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{T}^d))$ to (3-1). Furthermore, there exists a continuous increasing function, $F(t)$, such that, for any $u_0 \in L^2(\mathbb{T}^d)$, the solution u satisfies*

$$\|u\|_{L^2(\mathbb{T}^d)}(t) \le F(t)\|u_0\|_{L^2(\mathbb{T}^d)}. \tag{3-2}$$

Corollary 3.2. *For any sequence of initial data $(u_{0,n})$ bounded in $L^2(\mathbb{T}^d)$, the sequence (u_n) of solutions to (3-1) satisfies*

$$\|V(u_n,t)u_n\|_{L^2(\mathbb{T}^d)} \le C(t)\|u_n\|_{L^\infty((0,t);L^2(\mathbb{T}^d))} \le C(t)f(t)\|u_{0,n}\|_{L^2(\mathbb{T}^d)} \in L^1_{\mathrm{loc}}(\mathbb{R}_t),$$

and, consequently, the sequence (u_n) satisfies property (AC_T) for any $T > 0$.

Proof of Proposition 3.1. Let

$$K : u \in L^\infty((0,T); L^2(\mathbb{T}^d)) \mapsto e^{it\Delta}u_0 + \frac{1}{i}\int_0^t e^{i(t-s)}(V(u(s),s)u(s))\,ds.$$

We have

$$\begin{aligned} \|K(u) - e^{it\Delta}u_0\|_{L^\infty((0,T);L^2(\mathbb{T}^d))} &\le \int_0^T C(s)\,ds\,\|u\|_{L^\infty((0,T);L^2(\mathbb{T}^d))}, \\ \|K(u) - K(v)\|_{L^\infty((0,T);L^2(\mathbb{T}^d))} &\le \int_0^T C(s)\,ds\,\|u - v\|_{L^\infty((0,T);L^2(\mathbb{T}^d))}. \end{aligned} \tag{3-3}$$

We obtain that the map K has a unique fixed point on the ball centered on $e^{it\Delta}u_0$ with radius $\|u_0\|_{L^2(\mathbb{T}^d)}$ in $L^\infty((0,T); L^2(\mathbb{T}^d))$, as soon as $\int_0^T C(s)\,ds \le \frac{1}{2}$. This proves the local existence claim. To obtain existence on any time interval $[0,\widetilde{T}]$, we write $[0,\widetilde{T}] = \bigcup_{j=1}^N [t_j, t_{j+1}]$, where we choose t_j recursively such that $\int_{t_j}^{t_{j+1}} C(s)\,ds \le \frac{1}{2}$. Taking $\int_{t_j}^{t_{j+1}} C(s)\,ds = \frac{1}{2}$ for all $j < N-1$ gives the bound

$$N \le 1 + 2\int_0^{\widetilde{T}} C(s)\,ds. \tag{3-4}$$

Then applying the first step recursively gives a solution on $[0, \widetilde{T}]$ that, according to (3-4), satisfies

$$\|u\|_{L^2(\mathbb{T}^d)}(\widetilde{T}) \leq 2^N \|u_0\|_{L^2(\mathbb{T}^d)} \leq 2^{1+2\int_0^t C(s)\,ds} \|u_0\|_{L^2(\mathbb{T}^d)}.$$

The uniqueness claim in Proposition 3.1 follows now from standard methods. □

Acknowledgements

I would like to thank P. Gérard for suggesting the application in Section 3.

References

[Aïssiou et al. 2011] T. Aïssiou, D. Jakobson, and F. Macià, "Uniform estimates for the solutions of the Schrödinger equation on the torus and regularity of semiclassical measures", preprint, 2011. arXiv 1110.6521

[Anantharaman and Macià 2011] N. Anantharaman and F. Macià, "The dynamics of the Schrödinger flow from the point of view of semiclassical measures", preprint, 2011. arXiv 1102.0907

[Bourgain 1997] J. Bourgain, "Analysis results and problems related to lattice points on surfaces", pp. 85–109 in *Harmonic analysis and nonlinear differential equations* (Riverside, CA, 1995), edited by M. Lapidus et al., Contemp. Math. **208**, Amer. Math. Soc., Providence, RI, 1997. MR 99c:42012 Zbl 0884.42007

[Burq 2011] N. Burq, "Large-time dynamics for the one-dimensional Schrödinger equation", *Proc. Roy. Soc. Edinburgh Sect. A* **141**:2 (2011), 227–251. MR 2012f:35499 Zbl 1226.35072

[Burq and Planchon 2006] N. Burq and F. Planchon, "Smoothing and dispersive estimates for 1D Schrödinger equations with BV coefficients and applications", *J. Funct. Anal.* **236**:1 (2006), 265–298. MR 2007b:35276 Zbl pre05037260

[Burq and Zworski 2004] N. Burq and M. Zworski, "Geometric control in the presence of a black box", *J. Amer. Math. Soc.* **17**:2 (2004), 443–471. MR 2005d:47085 Zbl 1050.35058

[Burq and Zworski 2005] N. Burq and M. Zworski, "Bouncing ball modes and quantum chaos", *SIAM Rev.* **47**:1 (2005), 43–49. MR 2006d:81111 Zbl 1072.81022

[Christ and Kiselev 2001] M. Christ and A. Kiselev, "Maximal functions associated to filtrations", *J. Funct. Anal.* **179**:2 (2001), 409–425. MR 2001i:47054 Zbl 0974.47025

[Jakobson 1997] D. Jakobson, "Quantum limits on flat tori", *Ann. of Math.* (2) **145**:2 (1997), 235–266. MR 99e:58194 Zbl 0874.58088

[Macià 2011] F. Macià, "The Schrödinger flow in a compact manifold: high-frequency dynamics and dispersion", pp. 275–289 in *Modern aspects of the theory of partial differential equations*, edited by M. Ruzhansky and J. Wirth, Oper. Theory Adv. Appl. **216**, Birkhäuser/Springer Basel AG, Basel, 2011. MR 2858875

[Rudin 1987] W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987. MR 88k:00002 Zbl 0925.00005

Received 19 Sep 2012. Accepted 12 Dec 2012.

NICOLAS BURQ: nicolas.burq@math.u-psud.fr
Mathématiques, Université Paris Sud, Bâtiment 425, 91405 Orsay Cedex, France
and
UMR 8628 du CNRS and Ecole Normale Supérieure, 45 rue d'Ulm, 75005 Paris Cedex 05, France

ANALYSIS AND PDE
Vol. 6, No. 6, 2013
dx.doi.org/10.2140/apde.2013.6.1429

DECAY OF VISCOUS SURFACE WAVES WITHOUT SURFACE TENSION IN HORIZONTALLY INFINITE DOMAINS

YAN GUO AND IAN TICE

We consider a viscous fluid of finite depth below the air, occupying a three-dimensional domain bounded below by a fixed solid boundary and above by a free moving boundary. The fluid dynamics are governed by the gravity-driven incompressible Navier–Stokes equations, and the effect of surface tension is neglected on the free surface. The long-time behavior of solutions near equilibrium has been an intriguing question since the work of Beale (1981).

This is the second in a series of three papers by the authors that answers the question. Here we consider the case in which the free interface is horizontally infinite; we prove that the problem is globally well-posed and that solutions decay to equilibrium at an algebraic rate. In particular, the free interface decays to a flat surface.

Our framework utilizes several techniques, which include

(1) a priori estimates that utilize a “geometric” reformulation of the equations;
(2) a two-tier energy method that couples the boundedness of high-order energy to the decay of low-order energy, the latter of which is necessary to balance out the growth of the highest derivatives of the free interface;
(3) control of both negative and positive Sobolev norms, which enhances interpolation estimates and allows for the decay of infinite surface waves.

Our decay estimates lead to the construction of global-in-time solutions to the surface wave problem.

1. Introduction

Formulation of the equations in Eulerian coordinates. We consider a viscous, incompressible fluid evolving in a moving domain

$$\Omega(t) = \{y \in \Sigma \times \mathbb{R} \mid -b < y_3 < \eta(y_1, y_2, t)\}. \tag{1-1}$$

Here we assume that $\Sigma = \mathbb{R}^2$. The lower boundary of $\Omega(t)$ is assumed to be rigid and given, but the upper boundary is a free surface that is the graph of the unknown function $\eta : \Sigma \times \mathbb{R}^+ \to \mathbb{R}$. We assume that $b > 0$ is a fixed constant, so that the lower boundary is flat. For each t, the fluid is described by its velocity

Y. Guo was supported in part by NSF grant 0603815 and Chinese NSF grant 10828103. I. Tice was supported by an NSF Postdoctoral Research Fellowship. Both authors were partially supported by the Beijing International Mathematical Research Center.
MSC2010: primary 35Q30, 35R35, 76D03; secondary 35B40, 76E17.
Keywords: Navier–Stokes equations, free boundary problems, global existence.

and pressure functions $(u, p) : \Omega(t) \to \mathbb{R}^3 \times \mathbb{R}$. We require that (u, p, η) satisfy the gravity-driven incompressible Navier–Stokes equations in $\Omega(t)$ for $t > 0$:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u & \text{in } \Omega(t), \\ \operatorname{div} u = 0 & \text{in } \Omega(t), \\ \partial_t \eta = u_3 - u_1 \partial_{y_1} \eta - u_2 \partial_{y_2} \eta & \text{on } \{y_3 = \eta(y_1, y_2, t)\}, \\ (pI - \mu \mathbb{D}(u))\nu = g\eta\nu & \text{on } \{y_3 = \eta(y_1, y_2, t)\}, \\ u = 0 & \text{on } \{y_3 = -b\} \end{cases} \tag{1-2}$$

for ν the outward-pointing unit normal on $\{y_3 = \eta\}$, I the 3×3 identity matrix, $(\mathbb{D}u)_{ij} = \partial_i u_j + \partial_j u_i$ the symmetric gradient of u, $g > 0$ the strength of gravity, and $\mu > 0$ the viscosity. The tensor $(pI - \mu \mathbb{D}(u))$ is known as the viscous stress tensor. The third equation in (1-2) implies that the free surface is advected with the fluid. Note that in (1-2) we have shifted the gravitational forcing to the boundary and eliminated the constant atmospheric pressure, p_{atm}, in the usual way, by adjusting the actual pressure $\bar{p}$ according to $p = \bar{p} + gy_3 - p_{\mathrm{atm}}$.

The problem is augmented with initial data (u_0, η_0) satisfying certain compatibility conditions, which for brevity we will not write now. We will assume that $\eta_0 > -b$ on Σ.

Without loss of generality, we may assume that $\mu = g = 1$. Indeed, a standard scaling argument allows us to scale so that $\mu = g = 1$, at the price of multiplying b by a positive constant. This means that, up to renaming b, we arrive at the above problem with $\mu = g = 1$.

The problem (1-2) possesses a natural physical energy. For sufficiently regular solutions, we have an energy evolution equation that expresses how the change in physical energy is related to the dissipation:

$$\frac{1}{2}\int_{\Omega(t)} |u(t)|^2 + \frac{1}{2}\int_{\Sigma} |\eta(t)|^2 + \frac{1}{2}\int_0^t \int_{\Omega(s)} |\mathbb{D}u(s)|^2 \, ds = \frac{1}{2}\int_{\Omega(0)} |u_0|^2 + \frac{1}{2}\int_{\Sigma} |\eta_0|^2. \tag{1-3}$$

The first two integrals constitute the kinetic and potential energies, while the third constitutes the dissipation. The structure of this energy evolution equation is the basis of the energy method we will use to analyze (1-2).

Geometric form of the equations. In order to work in a fixed domain, we want to flatten the free surface via a coordinate transformation. We will not use a Lagrangian coordinate transformation, but rather a flattening transformation introduced by Beale [1984]. To this end, we consider the fixed domain

$$\Omega := \{x \in \Sigma \times \mathbb{R} \mid -b < x_3 < 0\}, \tag{1-4}$$

for which we will write the coordinates as $x \in \Omega$. We think of Σ as the upper boundary of Ω, and write $\Sigma_b := \{x_3 = -b\}$ for the lower boundary. We continue to view η as a function on $\Sigma \times \mathbb{R}^+$. We define

$$\bar{\eta} := \mathcal{P}\eta = \text{harmonic extension of } \eta \text{ into the lower half space}, \tag{1-5}$$

where $\mathcal{P}\eta$ is defined by (A-17). The harmonic extension $\bar{\eta}$ allows us to flatten the coordinate domain via the mapping

$$\Omega \ni x \mapsto (x_1, x_2, x_3 + \bar{\eta}(x, t)(1 + x_3/b)) =: \Phi(x, t) = (y_1, y_2, y_3) \in \Omega(t). \tag{1-6}$$

Note that $\Phi(\Sigma, t) = \{y_3 = \eta(y_1, y_2, t)\}$ and $\Phi(\cdot, t)|_{\Sigma_b} = Id_{\Sigma_b}$, that is, Φ maps Σ to the free surface and keeps the lower surface fixed. We have

$$\nabla\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A & B & J \end{pmatrix} \quad \text{and} \quad \mathscr{A} := (\nabla\Phi^{-1})^T = \begin{pmatrix} 1 & 0 & -AK \\ 0 & 1 & -BK \\ 0 & 0 & K \end{pmatrix} \tag{1-7}$$

for

$$\begin{aligned} &A = \partial_1 \bar{\eta}\tilde{b}, && B = \partial_2 \bar{\eta}\tilde{b}, \\ &J = 1 + \bar{\eta}/b + \partial_3 \bar{\eta}\tilde{b}, && K = J^{-1}, \\ &\tilde{b} = (1 + x_3/b). \end{aligned} \tag{1-8}$$

Here $J = \det \nabla\Phi$ is the Jacobian of the coordinate transformation.

If η is sufficiently small (in an appropriate Sobolev space), the mapping Φ is a diffeomorphism. This allows us to transform the problem to one on the fixed spatial domain Ω for $t \geq 0$. In the new coordinates, the PDE (1-2) becomes

$$\begin{cases} \partial_t u - \partial_t \bar{\eta}\tilde{b}K\partial_3 u + u \cdot \nabla_{\mathscr{A}} u - \Delta_{\mathscr{A}} u + \nabla_{\mathscr{A}} p = 0 & \text{in } \Omega, \\ \operatorname{div}_{\mathscr{A}} u = 0 & \text{in } \Omega, \\ S_{\mathscr{A}}(p, u)\mathscr{N} = \eta\mathscr{N} & \text{on } \Sigma, \\ \partial_t \eta = u \cdot \mathscr{N} & \text{on } \Sigma, \\ u = 0 & \text{on } \Sigma_b, \\ u(x, 0) = u_0(x), \eta(x', 0) = \eta_0(x'). \end{cases} \tag{1-9}$$

Here we have written the differential operators $\nabla_{\mathscr{A}}$, $\operatorname{div}_{\mathscr{A}}$, and $\Delta_{\mathscr{A}}$ with their actions given by $(\nabla_{\mathscr{A}} f)_i := \mathscr{A}_{ij}\partial_j f$, $\operatorname{div}_{\mathscr{A}} X := \mathscr{A}_{ij}\partial_j X_i$, and $\Delta_{\mathscr{A}} f = \operatorname{div}_{\mathscr{A}} \nabla_{\mathscr{A}} f$ for appropriate f and X; for $u \cdot \nabla_{\mathscr{A}} u$ we mean $(u \cdot \nabla_{\mathscr{A}} u)_i := u_j \mathscr{A}_{jk}\partial_k u_i$. We have also written $\mathscr{N} := -\partial_1\eta e_1 - \partial_2\eta e_2 + e_3$ for the nonunit normal to $\{y_3 = \eta(y_1, y_2, t)\}$, and we write $S_{\mathscr{A}}(p, u) = (pI - \mathbb{D}_{\mathscr{A}} u)$ for the stress tensor, where I is the 3×3 identity matrix and $(\mathbb{D}_{\mathscr{A}} u)_{ij} = \mathscr{A}_{ik}\partial_k u_j + \mathscr{A}_{jk}\partial_k u_i$ is the symmetric $\mathscr{A}$-gradient. Note that if we extend $\operatorname{div}_{\mathscr{A}}$ to act on symmetric tensors in the natural way, $\operatorname{div}_{\mathscr{A}} S_{\mathscr{A}}(p, u) = \nabla_{\mathscr{A}} p - \Delta_{\mathscr{A}} u$ for vector fields satisfying $\operatorname{div}_{\mathscr{A}} u = 0$.

Recall that $\mathscr{A}$ is determined by η through the relation (1-7). This means that all of the differential operators in (1-9) are connected to η, and hence to the geometry of the free surface. This geometric structure is essential to our analysis, as it allows us to control high-order derivatives that would otherwise be out of reach.

Beale's nondecay theorem. Many authors have considered problems similar to (1-2), both with and without viscosity and surface tension [Bae 2011; Beale 1981; 1984; Beale and Nishida 1985; Germain et al. 2009; Hataya 2009; Lannes 2005; Nishida et al. 2004; Solonnikov 1977; Sylvester 1990; Tani and Tanaka 1995; Wu 1997; 1999; 2009; 2011]. We refer the reader to the introduction of [Guo and Tice 2013b] for a more thorough discussion of how these results relate to ours.

Beale [1981] developed a local existence theory for the problem (1-2) in Lagrangian coordinates, where the unknowns are replaced with $v = u \circ \zeta$, $q = p \circ \zeta$ for ζ the Lagrangian flow map, which satisfies $\partial_t \zeta = v$. The result showed that (roughly speaking), given $v_0 \in H^{r-1}$ for $r \in (3, 7/2)$, there exists a unique solution v on a time interval $(0, T)$, with T depending on v_0, such that $v \in L^2 H^r \cap H^{r/2} L^2$. A second local existence theorem was then proved for small data near equilibrium. It showed that for any fixed $0 < T < \infty$, there exists a collection of data small enough that a unique solution exists on $(0, T)$.

The second result suggests that solutions should exist globally in time for small data. If global solutions do exist, it is natural to expect the free surface to decay to 0 as $t \to \infty$. However, the third result [Beale 1981] was a nondecay theorem that showed that a "reasonable" extension to small-data global well-posedness with decay of the free surface fails. Among other things, the theorem's hypotheses require that

$$
\begin{aligned}
v &\in L^1([0,\infty); H^r(\Omega)) \quad \text{for } r \in (3, 7/2), \\
\zeta_3|_\Sigma &\in L^2([0,\infty); L^2(\Sigma)), \\
v(x,0) &= 0, \quad \zeta(x,0) = x + \varepsilon\Theta(x), \\
\lim_{t\to\infty} \zeta_3|_\Sigma &= 0,
\end{aligned}
\tag{1-10}
$$

where Ω is given by (1-4), $\zeta(x, 0)$ is the flow map that gives the geometry of the initial fluid domain, Θ is a specially chosen function satisfying certain conditions, and $\varepsilon > 0$ is a small parameter. Note that the third line in (1-10) implies that the system is initially close to equilibrium, and the fourth line implies that the free surface decays to 0 as $t \to \infty$.

The proof of the nondecay theorem, which is a reductio ad absurdum, hinges on the special conditions imposed on the map Θ and the fact that $v \in L^1 H^r$. In the discussion of this result, Beale pointed out that it does not imply the nonexistence of global-in-time solutions, but rather that establishing global-in-time results requires stronger or different hypotheses than those imposed in the nondecay theorem.

The nondecay theorem raises two intriguing questions. First, is viscosity alone capable of producing global well-posedness? Second, if global solutions exist, do they decay as $t \to \infty$? Our main result answers both questions in the affirmative. In order to avoid the applicability of the nondecay theorem, we must show why its hypotheses are not satisfied. We would like to highlight three crucial ways in which we do this. The first and most obvious is that we work in a different coordinate system and within a different functional framework. In particular this requires higher regularity of the initial data and imposes more compatibility conditions than are satisfied by the data in the nondecay theorem.

Second, we will find (see (1-21)) that u decays according to $\|u(t)\|_2^2 \le C/(1+t)^{1+\lambda}$ for $\lambda \in (0, 1)$. This is not sufficiently rapid to guarantee that u belongs to the space $L^1([0, \infty); H^2(\Omega))$, which is in violation of the first line of (1-10), a key assumption in the nondecay result. Technically, our u is in Eulerian coordinates, but if we formally identify u with v, we see the difficulty clearly: we cannot integrate the equation $\partial_t \zeta = v$ to obtain ζ as $t \to \infty$, which means that we cannot make sense of the fourth equation in (1-10). One of the advantages of the Eulerian and geometric formulations is that the free surface function η may be analyzed without regard to what is happening to the entire flow map ζ in Ω.

Third, we find that η decays in time according to $\|\eta(t)\|_0^2 \le C/(1+t)^\lambda$ for $\lambda \in (0,1)$. This is not fast enough to guarantee that η is in $L^2([0,\infty); L^2(\Sigma))$. If we identify η with $\zeta_3|_\Sigma$, we see that we cannot guarantee that the second condition in (1-10) holds.

The above decay rates should be compared to those in the problem with surface tension (see the discussion on page 1442), which in general allows for faster decay to equilibrium. In this context, [Beale and Nishida 1985] showed that the decay estimates $\|u(t)\|_2^2 \le C/(1+t)^2$ and $\|\eta(t))\|_0^2 \le C/(1+t)$ are sharp. As such, we should not expect $u \in L^1 H^2$ or $\eta \in L^2 L^2$ in our problem.

Local well-posedness. The a priori estimates we develop in this paper are done in different coordinates and in a different functional framework from those used in [Beale 1981]. As such, we need a local well-posedness theory for (1-9) in our framework. We proved this in Theorem 1.1 of our companion paper [Guo and Tice 2013b]. Since we will need the result here, we record it now.

In order to state our result, we must explain our notation for Sobolev spaces and norms. We take $H^k(\Omega)$ and $H^k(\Sigma)$ for $k \ge 0$ to be the usual Sobolev spaces. When we write norms we suppress the H and Ω or Σ. When we write $\|\partial_t^j u\|_k$ and $\|\partial_t^j p\|_k$ we always mean that the space is $H^k(\Omega)$, and when we write $\|\partial_t^j \eta\|_k$ we always mean that the space is $H^k(\Sigma)$.

In the following we write ${}_0H^1(\Omega) := \{u \in H^1(\Omega) \mid u|_{\Sigma_b} = 0\}$ and

$$\mathscr{X}_T = \{u \in L^2([0,T]; {}_0H^1(\Omega)) \mid \operatorname{div}_{\mathscr{A}(t)} u(t) = 0 \text{ for a.e. } t\}. \tag{1-11}$$

The compatibility conditions for the initial data are the natural ones that would be satisfied for solutions in our functional framework. They are cumbersome to write, so we do not record them here. We refer the reader to [Guo and Tice 2013b] for their precise definition.

Theorem 1.1. *Let $N \ge 3$ be an integer. Assume that u_0 and η_0 satisfy the bound $\|u_0\|_{4N}^2 + \|\eta_0\|_{4N+1/2}^2 < \infty$ as well as the appropriate compatibility conditions. There exist $\delta_0, T_0 \in (0,1)$ such that if*

$$0 < T \le T_0 \min\left\{1, \frac{1}{\|\eta_0\|_{4N+1/2}^2}\right\}, \tag{1-12}$$

and $\|u_0\|_{4N}^2 + \|\eta_0\|_{4N}^2 \le \delta_0$, there exists a unique solution (u, p, η) to (1-9) on the interval $[0,T]$ that achieves the initial data. The solution obeys the estimates

$$\begin{aligned}
\sum_{j=0}^{2N} \sup_{0\le t\le T} \|\partial_t^j u\|_{4N-2j}^2 + \sum_{j=0}^{2N} \sup_{0\le t\le T} \|\partial_t^j \eta\|_{4N-2j}^2 + \sum_{j=0}^{2N-1} \sup_{0\le t\le T} \|\partial_t^j p\|_{4N-2j-1}^2 \\
+ \int_0^T \left(\sum_{j=0}^{2N} \|\partial_t^j u\|_{4N-2j+1}^2 + \sum_{j=0}^{2N-1} \|\partial_t^j p\|_{4N-2j}^2 \right) + \|\partial_t^{2N+1} u\|_{(\mathscr{X}_T)^*}^2 \\
+ \int_0^T \left(\|\eta\|_{4N+1/2}^2 + \|\partial_t \eta\|_{4N-1/2}^2 + \sum_{j=2}^{2N+1} \|\partial_t^j \eta\|_{4N-2j+5/2}^2 \right) \\
\le C(\|u_0\|_{4N}^2 + \|\eta_0\|_{4N}^2 + T\|\eta_0\|_{4N+1/2}^2)
\end{aligned} \tag{1-13}$$

and

$$\sup_{0\le t\le T} \|\eta\|_{4N+1/2}^2 \le C(\|u_0\|_{4N}^2 + (1+T)\|\eta_0\|_{4N+1/2}^2) \tag{1-14}$$

for a universal constant $C > 0$. The solution is unique among functions that achieve the initial data and for which the sum of the first three sums in (1-13) is finite. Moreover, η is such that the mapping $\Phi(\cdot\,, t)$, defined by (1-6), is a C^{4N-2} diffeomorphism for each $t \in [0, T]$.

Remark 1.2. All of the computations involved in the a priori estimates that we develop in this paper are justified by Theorem 1.1 and a specialization of it, Theorem 10.7, that we prove later. In this sense, Theorem 1.1 is a necessary ingredient in the global analysis of (1-9).

Main result. Sylvester [1990] and Tani and Tanaka [1995] studied the existence of small-data global-in-time solutions via the parabolic regularity method pioneered by Beale [1981] and Solonnikov [1977]. The papers make no claims about the decay of the solutions. It has been pointed out in the literature that the proofs in [Sylvester 1990; Tani and Tanaka 1995] are incomplete, so, to our knowledge, the existence of global solutions is still an open question. An interesting feature of our analysis, as described in detail later, is that our construction of global-in-time solutions is predicated on the decay of the solutions, that is, the decay is a necessary ingredient in global existence.

To state our global well-posedness result, we must first define various energies and dissipations. The exact form of some of the energies is too complicated to write out here, so we will neglect doing so, referring to the proper definitions later in the paper (pages 1450–1452). We assume that $\lambda \in (0, 1)$ is a fixed constant and we define $\mathcal{I}_\lambda u$ according to (A-7) and $\mathcal{I}_\lambda \eta$ according to (A-8). The high-order energy is

$$\mathcal{E}_{10} := \|\mathcal{I}_\lambda u\|_0^2 + \sum_{j=0}^{10} \|\partial_t^j u\|_{20-2j}^2 + \sum_{j=0}^{9} \|\partial_t^j p\|_{19-2j}^2 + \|\mathcal{I}_\lambda \eta\|_0^2 + \sum_{j=0}^{10} \|\partial_t^j \eta\|_{20-2j}^2, \tag{1-15}$$

and the high-order dissipation rate is

$$\begin{aligned}\mathcal{D}_{10} := \|\mathcal{I}_\lambda u\|_1^2 + \sum_{j=0}^{10} \|\partial_t^j u\|_{21-2j}^2 + \|\nabla p\|_{19}^2 + \sum_{j=1}^{9} \|\partial_t^j p\|_{20-2j}^2 \\ + \|D\eta\|_{20-3/2}^2 + \|\partial_t \eta\|_{20-1/2}^2 + \sum_{j=2}^{11} \|\partial_t^j \eta\|_{20-2j+5/2}^2.\end{aligned} \tag{1-16}$$

We write the high-order spatial derivatives of η as

$$\mathcal{F}_{10} := \|\eta\|_{20+1/2}^2. \tag{1-17}$$

We define the low-order energies $\mathcal{E}_{7,1}$ and $\mathcal{E}_{7,2}$ according to (2-52) and (2-53) with $n = 7$. Here the index m in $\mathcal{E}_{7,m}$ is a "minimal derivative" count that is included in order to improve decay rates in our estimates. Finally, we define the total energy

$$\mathcal{G}_{10}(t) = \sup_{0\le r\le t} \mathcal{E}_{10}(r) + \int_0^t \mathcal{D}_{10}(r)\,dr + \sum_{m=1}^{2} \sup_{0\le r\le t} (1+r)^{m+\lambda} \mathcal{E}_{7,m}(r) + \sup_{0\le r\le t} \frac{\mathcal{F}_{10}(r)}{(1+r)}. \tag{1-18}$$

Notice that the low-order terms $\mathcal{E}_{7,m}$ are weighted, so bounds on $\mathcal{G}_{10}$ yield decay estimates for $\mathcal{E}_{7,m}$.

Theorem 1.3. *Suppose the initial data* (u_0, η_0) *satisfy the compatibility conditions of Theorem 1.1. There exists a* $\kappa > 0$ *such that if* $\mathscr{E}_{10}(0) + \mathscr{F}_{10}(0) < \kappa$, *there exists a unique solution* (u, p, η) *to (1-9) on the interval* $[0, \infty)$ *that achieves the initial data. The solution obeys the estimate*

$$\mathscr{G}_{10}(\infty) \leq C_1(\mathscr{E}_{10}(0) + \mathscr{F}_{10}(0)) < C_1\kappa, \tag{1-19}$$

where $C_1 > 0$ *is a universal constant. For any* $0 \leq \rho < \lambda$, *we have*

$$\sup_{t\geq 0}\big[(1+t)^{2+\rho}\|u(t)\|^2_{C^2(\Omega)}\big] \leq C(\rho)(\mathscr{E}_{10}(0) + \mathscr{F}_{10}(0)) < C(\rho)\kappa, \tag{1-20}$$

for $C(\rho) > 0$ *a constant depending on* ρ. *Also,*

$$\begin{aligned}\sup_{t\geq 0}\bigg[(1+t)^{1+\lambda}\|u(t)\|^2_2 + (1+t)^{1+\lambda}\|\eta(t)\|^2_{L^\infty} + \sum_{j=0}^{1}(1+t)^{j+\lambda}\|D^j\eta(t)\|^2_0\bigg] &\leq C(\mathscr{E}_{10}(0) + \mathscr{F}_{10}(0)) \\ &< C\kappa \end{aligned} \tag{1-21}$$

for a universal constant $C > 0$.

Remark 1.4. In our companion paper [Guo and Tice 2013a], where we analyze (1-9) in horizontally periodic domains, we require η_0 to satisfy the "zero average condition"

$$\int_\Sigma \eta_0 = 0. \tag{1-22}$$

For the horizontally periodic problem, this condition propagates in time (see Lemma 2.7, a variant of which holds in the periodic case), from which one sees that (1-22) is a necessary condition for decay in L^2 or L^∞. It also serves as an obstacle to applying Beale's nondecay theorem since the conditions that the map Θ in (1-10) must satisfy are incompatible with (1-22). For a complete discussion, we refer to [Guo and Tice 2013a].

In the present case, the bound $\mathscr{E}_{10}(0) < \kappa$ requires, in particular, that the initial data satisfy $\|\mathscr{I}_\lambda\eta_0\|^2_0 < \infty$. This condition can be viewed as a sort of weak version of the zero average condition in the infinite case. To see this, note that if η_0 is sufficiently nice, say $L^1(\Sigma)$, then

$$0 = \int_\Sigma \eta_0 \iff \hat{\eta}_0(0) = 0, \tag{1-23}$$

for $\hat{\cdot}$ the Fourier transform. This means that the zero average condition is equivalent to requiring that $\hat{\eta}_0$ vanishes at the origin. We enforce a weak version of this by requiring that $\mathscr{I}_\lambda\eta_0 \in L^2(\Sigma) = H^0(\Sigma)$, which requires that $|\xi|^{-2\lambda}|\hat{\eta}_0(\xi)|^2$ is integrable near $\xi = 0$. Since $\lambda < 1$, this does not require $\hat{\eta}_0(0) = 0$, but it does prevent $|\hat{\eta}_0|$ from being "too big" at the origin. Note that the condition $\mathscr{I}_\lambda\eta_0 \in L^2$ is more general than (1-22).

Remark 1.5. The decay estimates (1-20) and (1-21) do not follow directly from the decay of $\mathscr{E}_{7,1}(t)$ and $\mathscr{E}_{7,2}(t)$ implied by (1-19). Rather, they are deduced via auxiliary arguments, employing (1-19).

Remark 1.6. The decay of $\|u(t)\|^2_2$ given in (1-21) is not fast enough to guarantee that u belongs to $L^1([0, \infty); H^2(\Omega))$. Even if we could take $\lambda = 1$, we would still get logarithmic blow-up of the L^1H^2 norm.

Remark 1.7. The function η is sufficiently small to guarantee that the mapping $\Phi(\cdot, t)$, defined in (1-6), is a diffeomorphism for each $t \geq 0$. As such, we may change coordinates to $y \in \Omega(t)$ to produce a global-in-time, decaying solution to (1-2).

Remark 1.8. Later in the paper, we let $N \geq 3$ be an integer and perform our analysis in terms of estimates at the $2N$ and $N+2$ levels; we take $N = 5$ in the present case to get the 10 and 7 appearing above. This is not optimal. With somewhat more work, we can improve our results to $N = 4$ with the restriction that $\lambda \in (3/5, 1)$. It is likely that this can be further improved by adjusting the scheme from $2N$ and $N+2$ to something slightly different. We have sacrificed optimality in order to simplify the presentation and make our "two-tier energy method" clearer. The first tier is at the level $2N$ and the second at the level $N+2$, which is meant to be roughly half of the first tier. The extra $+2$ is added to aid in applying some Sobolev embeddings.

Remark 1.9. It was established in [Castro et al. 2011; 2012] that solutions to inviscid free boundary problems, starting from smooth initial data, can develop finite-time splash singularities. Given this, it is reasonable to expect that a generic large-data version of Theorem 1.3 does not hold.

The proof of Theorem 1.3 is completed in Section 11. We now present a summary of the principal difficulties we encounter in our analysis as well as a sketch of the key ideas used in our proof.

Principal difficulties. In the study of the unforced incompressible Navier–Stokes equations in a fixed bounded domain with no-slip boundary conditions, it is natural to use the energy method to prove that solutions decay in time. Indeed, for sufficiently smooth solutions one may prove an analogue of (1-3) that relates the natural energy and dissipation:

$$\partial_t \mathscr{E} + \mathscr{D} := \partial_t \int_\Omega \frac{|u(t)|^2}{2} + \frac{1}{2} \int_\Omega |\mathbb{D}u(t)|^2 = 0. \tag{1-24}$$

Korn's inequality allows us to control $C\mathscr{E}(t) \leq \mathscr{D}(t)$ for a constant $C > 0$ independent of time, which shows that the dissipation is stronger than the energy. From this and Gronwall's lemma we may immediately deduce that the energy $\mathscr{E}$ decays exponentially in time and that we have the estimate $\mathscr{E}(t) \leq \mathscr{E}(0) \exp(-Ct)$.

If one seeks to similarly use the energy method to obtain decay estimates for solutions to (1-2), one encounters a fundamental obstacle that may already be observed in the differential form of (1-3)

$$\partial_t \left(\int_{\Omega(t)} \frac{|u(t)|^2}{2} + \int_\Sigma \frac{|\eta(t)|^2}{2} \right) + \frac{1}{2} \int_{\Omega(t)} |\mathbb{D}u(t)|^2 = 0. \tag{1-25}$$

The difficulty is that the dissipation provides no direct control of the η-term in the energy. As such, we must resort to using the equations (1-2) to try to control $\|\eta(t)\|_0$ in terms of $\|\mathbb{D}u(t)\|_0$. From (1-2) we see that there are only two available routes: solving for η in the fourth equation, or using the third equation, which is the kinetic transport equation. If we pursue the first route, we must be able to control

$$\|p(t)\|^2_{H^0(\Sigma)} + \|\mathbb{D}u(t)\nu \cdot \nu\|^2_{H^0(\Sigma)} \lesssim \|\mathbb{D}u(t)\|^2_{H^0(\Omega(t))}, \tag{1-26}$$

which is not possible. If instead we pursue the second route, we must estimate η as a solution to the kinematic transport equation. Such an estimate (see Lemma A.9) only allows us to estimate $\|\eta(t)\|_0$ in terms of $\int_0^t \|\mathbb{D}u(s)\|_0 \, ds$. That is, transport estimates do not provide control of the η-part of the energy in terms of the "instantaneous" dissipation, but rather in terms of the "cumulative" integrated dissipation. From this we see that in our problem the dissipation is actually weaker than the energy, so we cannot argue as above to deduce exponential decay.

We might hope that we could avoid this problem by working with a high-regularity energy method, but we will always encounter the same type of problem as above. Regardless of the level of regularity in the energy, the instantaneous dissipation is always weaker than the instantaneous energy, which prevents us from deducing exponential decay of the energy. Instead we pursue a strategy similar to one employed in [Strain and Guo 2006] for another problem where the dissipation is weaker than the energy. We first show that high-order energies are bounded by using an integrated version of (1-25) for derivatives of the solution. Then we consider a low-order energy and show that an equation of the form (1-25) holds, that is, $\partial_t \mathscr{E}_{\text{low}} + C\mathscr{D}_{\text{low}} \le 0$. Now, instead of trying to estimate (1-26) for low-order derivatives, we instead interpolate between low-order derivatives and high-order derivatives, which are bounded. Instead of an estimate $C\mathscr{E}_{\text{low}} \le \mathscr{D}_{\text{low}}$, we must prove one of the form $C\mathscr{E}_{\text{low}}^{1+\theta} \le \mathscr{D}_{\text{low}}$ for some $\theta > 0$. We can then use this to derive the differential inequality $\partial_t \mathscr{E}_{\text{low}} + C\mathscr{E}_{\text{low}}^{1+\theta} \le 0$, which can be integrated to see that $\mathscr{E}_{\text{low}}(t) \lesssim \mathscr{E}_{\text{low}}(0)/(1+t)^{1/\theta}$. We would then find that the low-order energy decays algebraically in time rather than exponentially.

To complete this program, we must overcome a pair of intertwined difficulties. First, to close the high-order energy estimates with, say $\|u\|_{4N+1}^2$ for an integer $N \ge 0$ in the dissipation, we have to control η in $H^{4N+1/2}$. The only option for this is to again appeal to estimates for solutions to the transport equation, which say (roughly speaking) that

$$\sup_{0\le t\le T} \|\eta\|_{4N+1/2}^2 \le C \exp\left(C \int_0^T \|Du(t)\|_{H^2(\Sigma)}\, dt \right) \left[\|\eta_0\|_{4N+1/2}^2 + T \int_0^T \|u(t)\|_{4N+1}^2\, dt \right]. \tag{1-27}$$

Without knowing a priori that u decays, the right side of this estimate has the potential to grow at the rate of $(1+T)e^{C\sqrt{T}}$. Even if u decays rapidly, the right side can still grow like $(1+T)$. This growth is potentially disastrous in closing the high-order, global-in-time estimates. To manage the growth, we must identify a special decaying term that always appears in products with the highest derivatives of η. If the special term decays quickly enough, we can hope to balance the growth and close the high-order estimates. Due to the growth in (1-27), we believe that it is not possible to construct global-in-time solutions without also deriving a decay result.

This leads us to the second difficulty in this program. The decay rate of the special term is dictated by the decay rate of the low-order energy, so we must make sure that the low-order energy decays sufficiently quickly. This amounts to making the constant $\theta > 0$ appearing in the interpolation estimates above sufficiently small. We must then carefully choose the terms that will appear in the low-order and high-order energies in order to keep θ small enough. It turns out that this requires us to enforce a minimal derivative count in the low-order energy, that is, only terms with m derivatives or more are allowed. It also requires us to extend the high-order energy to include estimates of negative horizontal derivatives up to order $\lambda \in (0, 1)$. Then $\theta = \theta(m, \lambda)$, and only by taking $m = 2$, $\lambda > 0$ can we make θ small enough to achieve the desired decay rate.

The resolution of these intertwined difficulties requires a delicate and involved analysis. We now sketch some of the techniques we will employ.

Horizontal energy evolution estimates. In order to use the natural energy structure of the problem (given in Eulerian coordinates by (1-3)) to study high-order derivatives, we can only apply derivatives that do not break the structure of the boundary condition $u = 0$ on Σ_b. Since Σ_b is flat, any differential operator $\partial^\alpha = \partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ is allowed. We apply these operators for various choices of α and sum the resulting energy evolution equations. After estimating the nonlinear terms that appear from differentiating (1-9), we are eventually led to evolution equations for these "horizontal" energies and dissipations, $\bar{\mathscr{E}}_{10}$, $\bar{\mathscr{D}}_{10}$, $\bar{\mathscr{E}}_{7,m}$, and $\bar{\mathscr{D}}_{7,m}$ for $m = 1, 2$ (see (2-45) and (2-47)–(2-49) for precise definitions). Here we write bars to indicate "horizontal" derivatives. Roughly speaking, at high-order we have the estimate

$$\bar{\mathscr{E}}_{10}(t) + \int_0^t \bar{\mathscr{D}}_{10}(r)\,dr \lesssim \mathscr{E}_{10}(0) + \int_0^t (\mathscr{E}_{10}(r))^\theta \mathscr{D}_{10}(r)\,dr + \int_0^t \sqrt{\mathscr{D}_{10}(r)\mathscr{K}(r)\mathscr{F}_{10}(r)}\,dr, \tag{1-28}$$

where $\mathscr{K}$ is of the form

$$\mathscr{K} = \|\nabla u\|_{C^1}^2 + \|Du\|_{H^2(\Sigma)}^2, \tag{1-29}$$

and $\theta > 0$; and at low-order we have

$$\partial_t \bar{\mathscr{E}}_{7,m} + \bar{\mathscr{D}}_{7,m} \lesssim \mathscr{E}_{10}^\theta \mathscr{D}_{7,m}, \tag{1-30}$$

where $\mathscr{D}_{7,m}$ is the low-order dissipation. Notice that the product $\mathscr{K}\mathscr{F}_{10}$ in (1-28) multiplies low-order norms of u against the highest-order norm of η. Technically, the estimate (1-28) also involves $\mathscr{I}_\lambda u$ and $\mathscr{I}_\lambda \eta$ in addition to horizontal derivatives. For the moment let us ignore these terms and continue with the discussion of our energy method. We will discuss $\mathscr{I}_\lambda$ in detail below.

The actual derivation of bounds like (1-28)–(1-30) is delicate and depends crucially on the geometric structure of the equations given in (1-9). Indeed, if we attempted to rewrite (1-9) as a perturbation of the usual constant-coefficient Navier–Stokes equations, we would fail to achieve the estimate (1-28) because we would be unable to control the interaction between $\partial_t^{10} p$ and $\operatorname{div} \partial_t^{10} u$, the latter of which does not vanish in the geometric form of the equations.

Comparison estimates. The next step in the analysis is to replace the horizontal energies and dissipations with the full energies and dissipations. We prove that there is a universal $0 < \delta < 1$ such that if $\mathscr{E}_{10} \le \delta$, then

$$\mathscr{E}_{10} \lesssim \bar{\mathscr{E}}_{10}, \quad \mathscr{D}_{10} \lesssim \bar{\mathscr{D}}_{10} + \mathscr{K}\mathscr{F}_{10}, \quad \mathscr{E}_{7,m} \lesssim \bar{\mathscr{E}}_{7,m}, \quad \mathscr{D}_{7,m} \lesssim \bar{\mathscr{D}}_{7,m}. \tag{1-31}$$

This estimate is extremely delicate and can only be obtained by carefully using the structure of the equations (1-9). We make use of every bit of information from the boundary conditions and the vorticity equations to establish it. There are two structural components of the estimates that are of such importance that we mention them now. First, the equation $\operatorname{div}_{\mathscr{A}} u = 0$ allows us to write $\partial_3 u_3 = -(\partial_1 u_1 + \partial_2 u_2) + G^2$ for some quadratic nonlinearity G^2. This allows us to "trade" a vertical derivative of u_3 for horizontal derivatives of u_1 and u_2, an indispensable trick in our analysis. Second, the interaction between the parabolic scaling of u ($\partial_t u \sim \Delta u$) and the transport scaling of η ($\partial_t \eta \sim u_3|_\Sigma$) allows us to gain regularity for the temporal derivatives of η in the dissipation, and it also gives us control of $\partial_t^{11}\eta$, which is one more time derivative than appears in the energy.

Two-tier energy method. Suppose we know that

$$\mathcal{K}(r) \le \frac{\delta}{(1+r)^{2+\gamma}} \tag{1-32}$$

for some $0 < \delta < 1$ and $\gamma > 0$. Since η satisfies a transport equation, we may use Lemma A.9 to derive an estimate of the form

$$\sup_{0\le r\le t} \mathcal{F}_{10}(r) \lesssim \exp\biggl(C\int_0^t \sqrt{\mathcal{K}(r)}\,dr\biggr)\biggl[\mathcal{F}_{10}(0) + t\int_0^t \mathcal{D}_{10}(r)\,dr\biggr]. \tag{1-33}$$

Although the right side of this equation could potentially blow up exponentially in time, the decay of $\mathcal{K}$ in (1-32) implies that

$$\sup_{0\le r\le t} \mathcal{F}_{10}(r) \lesssim \mathcal{F}_{10}(0) + t\int_0^t \mathcal{D}_{10}(r)\,dr. \tag{1-34}$$

Note that $\gamma > 0$ in (1-32) is essential; we would not be able to tame the exponential term in (1-33) without it, and then (1-34) would not hold. This estimate allows for $\mathcal{F}_{10}(t)$ to grow linearly in time, but in the product $\mathcal{K}(r)\mathcal{F}_{10}(r)$ that appears in (1-28), we can use the decay of $\mathcal{K}$ to balance this growth. Then if $\sup_{0\le r\le t}\mathcal{E}_{10}(r) \le \delta$ with δ small enough, we can combine (1-28), (1-31), (1-32), and (1-34) to get the estimate

$$\sup_{0\le r\le t} \mathcal{E}_{10}(r) + \int_0^t \mathcal{D}_{10}(r)\,dr \lesssim \mathcal{E}_{10}(0) + \mathcal{F}_{10}(0). \tag{1-35}$$

This highlights the first step of our two-tier energy method: the decay of low-order terms (that is, $\mathcal{K}$) can balance the growth of $\mathcal{F}_{10}$, yielding boundedness of the high-order terms. In order to close this argument, we must use a second step: the boundedness of the high-order terms implies the decay of low-order terms, and in particular the decay of $\mathcal{K}$.

To obtain this decay, we combine (1-30) and (1-31) to see that

$$\partial_t\bar{\mathcal{E}}_{7,m} + \tfrac{1}{2}\mathcal{D}_{7,m} \le 0 \tag{1-36}$$

if $\mathcal{E}_{10} \le \delta$ for δ small enough. If we could show that $\bar{\mathcal{E}}_{7,m} \lesssim \mathcal{D}_{7,m}$, this estimate would yield exponential decay of $\bar{\mathcal{E}}_{7,m}$ and $\mathcal{E}_{7,m}$. An inspection of $\bar{\mathcal{E}}_{7,m}$ and $\mathcal{D}_{7,m}$ (see (2-45) and (2-51)) shows that $\mathcal{D}_{7,m}$ can control every term in $\bar{\mathcal{E}}_{7,m}$ except $\|\eta\|_0^2$ (and $\|\partial_t\eta\|_0^2$ when $m = 2$). In a sense, this means that exponential decay fails precisely because the dissipation fails to control η at the lowest order. In lieu of $\bar{\mathcal{E}}_{7,m} \lesssim \mathcal{D}_{7,m}$, we interpolate between $\mathcal{E}_{10}$ (which can control all the lowest-order terms of η) and $\mathcal{D}_{7,m}$:

$$\bar{\mathcal{E}}_{7,m} \lesssim \mathcal{E}_{10}^{1/(m+\lambda+1)}\mathcal{D}_{7,m}^{(m+\lambda)/(m+\lambda+1)}. \tag{1-37}$$

Combining (1-36) with (1-37) and the boundedness of $\mathcal{E}_{10}$ in terms of the data, (1-35), then allows us to deduce that

$$\partial_t\bar{\mathcal{E}}_{7,m} + \frac{C}{(\mathcal{E}_{10}(0)+\mathcal{F}_{10}(0))^{1/(m+\lambda)}}(\bar{\mathcal{E}}_{7,m})^{1+1/(m+\lambda)} \le 0. \tag{1-38}$$

Integrating this differential inequality and employing some auxiliary estimates then leads us to the bound

$$\mathscr{E}_{7,m}(t) \lesssim \bar{\mathscr{E}}_{7,m}(t) \lesssim \frac{\mathscr{E}_{10}(0) + \mathscr{F}_{10}(0)}{(1+t)^{m+\lambda}}. \tag{1-39}$$

We thus use the boundedness of high-order terms to deduce the decay of low-order terms, completing the second step of the two-tier energy estimates.

Negative Sobolev estimates via $\mathcal{I}_\lambda$. Notice that the decay rate in (1-39) is enhanced by $\lambda \in (0, 1)$. As we will see below, the parameter $\gamma > 0$ in the decay of $\mathscr{K}$, given in (1-32), is determined by the rate $m+\lambda$. If we were to set $\lambda = 0$, we would not get $\gamma > 0$ and we would be unable to balance the growth of $\mathscr{F}_{10}$. Estimates (1-34) and (1-35) would fail, and we would be unable to close our estimates. We thus see the necessity of introducing the "negative Sobolev" estimates via the horizontal Riesz potential $\mathcal{I}_\lambda$.

The difficulty then is that we must apply the nonlocal operator $\mathcal{I}_\lambda$ to a nonlinear PDE and then study the evolution of $\mathcal{I}_\lambda u$ and $\mathcal{I}_\lambda \eta$. The flatness of the lower boundary Σ_b is essential here, since it allows us to have $\mathcal{I}_\lambda u = 0$ on Σ_b. This means that the operator $\mathcal{I}_\lambda$ does not break the boundary conditions, and we can use the natural energy structure to include $\|\mathcal{I}_\lambda u\|_0^2$ and $\|\mathcal{I}_\lambda \eta\|_0^2$ in the energy and $\|\mathcal{I}_\lambda u\|_1^2$ in the dissipation. To close the estimates for these terms, we must be able to estimate $\mathcal{I}_\lambda$ acting on various nonlinearities in terms of $\mathscr{E}_{10}^\theta \mathscr{D}_{10}$ for some $\theta > 0$. These estimates turn out to be rather delicate, and we must again employ almost all of the structure of the equations and boundary conditions in order to derive them. They are also responsible for the constraint $\lambda < 1$. For $\lambda \geq 1$, the nonlinear estimates would not work as we need them to. In general, for quadratic nonlinearities in dimension n, we expect to restrict $\lambda < n/2$.

We should point out that, a priori, we do not know that $\mathcal{I}_\lambda u(t)$ or $\mathcal{I}_\lambda \eta(t)$ even make sense for $t > 0$, since this is not provided by Theorem 1.1. To show that these terms are well-defined, which then justifies applying $\mathcal{I}_\lambda$ to the equations, we must actually prove a specialization of the local well-posedness theorem that includes the boundedness of $\mathcal{I}_\lambda u$, $\mathcal{I}_\lambda p$, and $\mathcal{I}_\lambda \eta$. We do this in Theorem 10.7.

Interpolation estimates and minimal derivative counts. The negative Sobolev estimates alone do not close the overall estimates in our two-tier energy method. To do that, we must verify that $\mathscr{K}$ decays as in (1-32) for some $\gamma > 0$. An inspection of $\mathscr{E}_{7,m}$ shows that we cannot directly control $\mathscr{K} \lesssim \mathscr{E}_{7,m}$ for either $m = 1$ or $m = 2$, so we must resort to an interpolation argument. We show that through interpolation it is actually possible to control $\mathscr{K} \lesssim \mathscr{E}_{7,1}$, but the $\mathscr{E}_{7,1}$ only decays like $(1+t)^{-1-\lambda}$, which is not fast enough for (1-32). The energy $\mathscr{E}_{7,2}$ decays at a faster rate, but we cannot show that $\mathscr{K} \lesssim \mathscr{E}_{7,2}$. Instead, we show that if $\mathscr{E}_{7,2}(t) \leq \varepsilon(1+t)^{-2-\lambda}$, then

$$\mathscr{K} \lesssim \mathscr{E}_{7,2}^{(8+2\lambda)/(8+4\lambda)} \lesssim \varepsilon^{(8+2\lambda)/(8+4\lambda)} \frac{1}{(1+t)^{2+\lambda/2}}, \tag{1-40}$$

so that, after renaming $\delta = C\varepsilon^{(8+2\lambda)/(8+4\lambda)}$ and $\gamma = \lambda/2 > 0$, we find that (1-32) does hold.

The parameters m and λ interact in an important way. The decay rate increases with m and with λ. As mentioned above, we are technically constrained to $\lambda < 1$, so we must increase m to 2 in order to hit the target decay rate in (1-32). It is tempting, then, to consider abandoning the $\mathcal{I}_\lambda$ operators and simply use a third energy with $m \geq 3$, which should decay like $(1+t)^{-m}$. However, if one were to do this for any $m \geq 3$, one would find that there is a corresponding decrease in the interpolation power: $\mathcal{H} \lesssim \mathcal{E}_{7,m}^{\theta(m)}$, where $\theta(m)$ decreases with m in such a way that $m\theta(m) \leq 2$, so that (1-32) would fail. We thus see that the negative estimates are not just a convenience, but rather a necessity.

The derivation of (1-40) is delicate, requiring a two-step bootstrap process to iteratively improve the interpolation powers. We again crucially make use of the structure of the equations and boundary conditions. We extensively interpolate between our negative Sobolev estimates and our positive Sobolev estimates. The utility of the negative estimates is quite clear here: the interpolation powers improve when we interpolate with negative derivatives (as opposed to say, no derivatives).

To complete the proof of (1-40), we crucially use an estimate for $\mathcal{I}_1 \partial_t \eta$. This corresponds to $\lambda = 1$, so we are not able to apply $\mathcal{I}_1 \partial_t$ to the equations to obtain the estimate. Rather, the estimate comes for free from the transport equation for η, which allows us to write $\partial_t \eta = -\partial_1 U_1 - \partial_2 U_2$ for $U_i \in H^1$. In our analysis of the horizontally periodic problem [Guo and Tice 2013a], where we can take $\Sigma = \mathbb{T}^2$, this identity and (1-22) give rise to a Poincaré inequality $\|\eta(t)\|_0^2 \lesssim \|D\eta(t)\|_0^2$ for $t \geq 0$, which is crucial in our analysis there. From this we see that the estimate for $\mathcal{I}_1 \partial_t \eta$ is of analytic importance for the problem (1-2).

The interpolation of negative and positive Sobolev estimates provides a completely new tool in the study of time decay in dissipative PDE problems in the whole (or semi-infinite) space. For the viscous surface wave problem, a particular advantage of the negative-positive method is that, unlike the usual $L^p - L^q$ machinery, our norms are preserved along the time evolution. We anticipate that this method will prove useful in the analysis of other dissipative equations.

Remark 1.10. After the completion of this paper we became aware of [Hataya and Kawashima 2009], which is an announcement of a decay result for the viscous surface wave problem in horizontally infinite domains. The paper provides a terse sketch of their proposed proof that employs a modification of the Beale–Solonnikov parabolic framework, which is a framework completely different from ours. Full details of the proof are promised in forthcoming work, but to our knowledge no such work has appeared in the literature to date. From the information provided in the sketch, it is unclear to us how the decay rates involved, none of which are faster than $1/(1+t)^2$ for any norm-squared of the velocity field, are sufficiently rapid to balance the growth of the highest derivatives of η. In particular, it is not clear to us how their method can provide control of $\mathcal{H}$ as in (1-32), which we need to close the transport estimate (1-33) and to control the growth of $\mathcal{F}_{10}$ in (1-28) and (1-31).

Comparison to the periodic problem. We proved in [Guo and Tice 2013a] the analogue of Theorem 1.3 for horizontally periodic domains. In this context we take $N \geq 3$ to be an integer and consider energies and dissipations $\mathcal{E}_{2N}$, $\mathcal{D}_{2N}$, $\mathcal{F}_{2N}$, and $\mathcal{G}_{2N}$; these are modifications of what we use here (with $N = 5$) that include temporal derivatives up to order $2N$. See that paper for the precise definitions. By increasing N, we can achieve arbitrarily fast algebraic rates for the solutions, which we identify as "almost exponential decay."

In order to compare with Theorem 1.3, we record a version of the periodic result now.

Theorem 1.11. *Suppose the initial data* (u_0, η_0) *satisfy the compatibility conditions of Theorem 1.1 and* η_0 *satisfies the zero average condition (1-22). Let* $N \geq 3$ *be an integer. There exists a* $0 < \kappa = \kappa(N)$ *such* *that if* $\mathscr{E}_{2N}(0) + \mathscr{F}_{2N}(0) < \kappa$, *there exists a unique solution* (u, p, η) *to* (1-9) *on the interval* $[0, \infty)$ *that achieves the initial data. The solution obeys the estimates*

$$\mathscr{G}_{2N}(\infty) \le C_1(\mathscr{E}_{2N}(0) + \mathscr{F}_{2N}(0)) < C_1\kappa, \tag{1-41}$$

$$\sup_{t\ge 0}(1+t)^{4N-8}[\|u(t)\|^2_{2N+4} + \|\eta(t)\|^2_{2N+4}] \le C_1(\mathscr{E}_{2N}(0) + \mathscr{F}_{2N}(0)) < C_1\kappa, \tag{1-42}$$

where $C_1 > 0$ *is a universal constant.*

Remark 1.12. A key difference between the periodic result, Theorem 1.11, and the nonperiodic result, Theorem 1.3, is that in the periodic case, increasing N also increases the decay rate. No such gain is possible in the nonperiodic case, which is why we specialize to the case $N = 5$ there. In the periodic case, we do not use the same type of interpolation arguments that we use in the infinite case. This allows us to relax to $N \ge 3$.

Remark 1.13. Hataya [2009] studied the periodic problem with a flat bottom. Using the Beale–Solonnikov parabolic theory [Beale 1981; 1984; Solonnikov 1977], it was shown that

$$\int_0^\infty (1+t)^2\|u(t)\|^2_{r-1}\,dt + \sup_{t\ge 0}(1+t)^2\|\eta(t)\|^2_{r-2} < \infty \tag{1-43}$$

for $r \in (5, 11/2)$. Our result on the periodic problem is an improvement of this in two important ways. First, we establish faster decay rates by working in a higher regularity context. Second, we allow for a more general non-flat bottom geometry (see [Guo and Tice 2013a] for details).

Comparison to the case with surface tension. If the effect of surface tension is included at the air-fluid free interface, the formulation of the PDE must be changed. Surface tension is modeled by modifying the fourth equation in (1-2) to be

$$(pI - \mu\mathbb{D}(u))\nu = g\eta\nu - \sigma H\nu, \tag{1-44}$$

where $H = \partial_i(\partial_i\eta/\sqrt{1+|D\eta|^2})$ is the mean curvature of the surface $\{y_3 = \eta(t)\}$ and $\sigma > 0$ is the surface tension.

Beale [1984] proved small-data global well-posedness for the problem with surface tension in horizontally infinite domains. The flattened coordinate system we employ was introduced in [Beale 1984] and used in place of Lagrangian coordinates. However, Beale employed a change of unknown velocities that is more complicated than just a coordinate change. Well-posedness was demonstrated with $u \in L^2H^r$ and $\eta \in L^2H^{r+1/2}$, given that $u_0 \in H^{r-1/2}$, $\eta_0 \in H^r$ are sufficiently small for $r \in (3, 7/2)$. In this context it is understood that surface tension leads to the decay of certain modes, thereby aiding global existence.

Beale and Nishida [1985] studied the asymptotic properties of the solutions constructed in [Beale 1984]. They showed that if $\eta_0 \in L^1(\Sigma)$, then

$$\sup_{t\ge 0}(1+t)^2\|u(t)\|^2_2 + \sup_{t\ge 0}\sum_{j=1}^{2}(1+t)^{1+j}\|D^j\eta(t)\|^2_0 < \infty, \tag{1-45}$$

and that this decay rate is optimal. Taking $\lambda \approx 1$ in our Theorem 1.3, the estimates (1-21) yield almost the same decay rates.

Nishida, Teramoto, and Yoshihara [Nishida et al. 2004] showed that in horizontally periodic domains with surface tension and a flat bottom, if η_0 has zero average, there exists a $\gamma > 0$ such that

$$\sup_{t\geq 0} e^{\gamma t}[\|u(t)\|_2^2 + \|\eta(t)\|_3^2] < \infty. \tag{1-46}$$

In this case, (1-44) gives a third way of estimating η in terms of the dissipation; using this, it is possible to show that the dissipation is stronger than the energy. Thus, if surface tension is added in the periodic case, fully exponential decay is possible, whereas without surface tension we only recover algebraic decay of arbitrary order in Theorem 1.11.

The comparison of these two results with ours establishes a nice contrast between the surface tension and non-surface tension cases. Without surface tension we can recover "almost" the same decay rate as in the case with surface tension. This shows that viscosity is the basic decay mechanism and that the effect of surface tension serves to enhance the decay rate.

Definitions and terminology. We now mention some of the definitions, bits of notation, and conventions that we will use throughout the paper.

Einstein summation and constants. We employ the Einstein convention of summing over repeated indices for vector and tensor operations. Throughout the paper $C > 0$ will denote a generic constant that can depend on the parameters of the problem, N, and Ω, but does not depend on the data, etc. We refer to such constants as "universal." They are allowed to change from one inequality to the next. When a constant depends on a quantity z we write $C = C(z)$ to indicate this. We employ the notation $a \lesssim b$ to mean that $a \leq Cb$ for a universal constant $C > 0$.

Norms. We write $H^k(\Omega)$ with $k \geq 0$ and $H^s(\Sigma)$ with $s \in \mathbb{R}$ for the usual Sobolev spaces. We typically write $H^0 = L^2$; the exception to this is when we use $L^2([0, T]; H^k)$ notation to indicate the space of square-integrable functions with values in H^k.

To avoid notational clutter, we avoid writing $H^k(\Omega)$ or $H^k(\Sigma)$ in our norms and typically write only $\|\cdot\|_k$. Since we do this for functions defined on both Ω and Σ, this presents some ambiguity. We avoid this by adopting two conventions. First, we assume that functions have natural spaces on which they "live." For example, the functions u, p, and $\bar{\eta}$ live on Ω, while η itself lives on Σ. As we proceed in our analysis, we will introduce various auxiliary functions; the spaces they live on will always be clear from the context. Second, whenever the norm of a function is computed on a space different from the one in which it lives, we will explicitly write the space. This typically arises when computing norms of traces onto Σ of functions that live on Ω.

Derivatives. We write $\mathbb{N} = \{0, 1, 2, \ldots\}$ for the collection of nonnegative integers. When using space-time differential multi-indices, we write $\mathbb{N}^{1+m} = \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m)\}$ to emphasize that the 0-index term is related to temporal derivatives. For just spatial derivatives we write $\mathbb{N}^m$. For $\alpha \in \mathbb{N}^{1+m}$ we write $\partial^\alpha = \partial_t^{\alpha_0} \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}$. We define the parabolic counting of such multi-indices by writing $|\alpha| = 2\alpha_0 + \alpha_1 + \cdots + \alpha_m$. We write Df for the horizontal gradient of f, that is, $Df = \partial_1 f e_1 + \partial_2 f e_2$, while ∇f denotes the usual full gradient.

For a given norm $\|\cdot\|$ and integers $k, m \geq 0$, we introduce the following notation for sums of spatial derivatives:

$$\|D_m^k f\|^2 := \sum_{\substack{\alpha\in\mathbb{N}^2 \\ m\leq|\alpha|\leq k}} \|\partial^\alpha f\|^2 \quad \text{and} \quad \|\nabla_m^k f\|^2 := \sum_{\substack{\alpha\in\mathbb{N}^3 \\ m\leq|\alpha|\leq k}} \|\partial^\alpha f\|^2. \tag{1-47}$$

The convention we adopt in this notation is that D refers to only "horizontal" spatial derivatives, while ∇ refers to full spatial derivatives. For space-time derivatives we add bars to our notation:

$$\|\bar{D}_m^k f\|^2 := \sum_{\substack{\alpha\in\mathbb{N}^{1+2} \\ m\leq|\alpha|\leq k}} \|\partial^\alpha f\|^2 \quad \text{and} \quad \|\bar{\nabla}_m^k f\|^2 := \sum_{\substack{\alpha\in\mathbb{N}^{1+3} \\ m\leq|\alpha|\leq k}} \|\partial^\alpha f\|^2. \tag{1-48}$$

When $k = m \geq 0$, we write

$$\|D^k f\|^2 = \|D_k^k f\|^2, \quad \|\nabla^k f\|^2 = \|\nabla_k^k f\|^2, \quad \|\bar{D}^k f\|^2 = \|\bar{D}_k^k f\|^2, \quad \|\bar{\nabla}^k f\|^2 = \|\bar{\nabla}_k^k f\|^2. \tag{1-49}$$

We allow for composition of derivatives in this counting scheme in a natural way; for example, we write

$$\|DD_m^k f\|^2 = \|D_m^k Df\|^2 = \sum_{\substack{\alpha\in\mathbb{N}^2 \\ m\leq|\alpha|\leq k}} \|\partial^\alpha Df\|^2 = \sum_{\substack{\alpha\in\mathbb{N}^2 \\ m+1\leq|\alpha|\leq k+1}} \|\partial^\alpha f\|^2 = \|D_{m+1}^{k+1} f\|^2. \tag{1-50}$$

Plan of paper. Throughout the paper we assume that $N \geq 5$ and $\lambda \in (0, 1)$ are both fixed. Notice that Theorem 1.3 is phrased with the choice $N = 5$.

In Section 2 we prove some preliminary lemmas and we define the energies and dissipations. In Section 3 we perform our bootstrap interpolation argument to control various quantities in terms of $\mathcal{E}_{N+2,m}$ and $\mathcal{D}_{N+2,m}$. In Section 4 we present estimates of the nonlinear forcing terms G^i (as defined in (2-24)–(2-31)) and some other nonlinearities. In Section 5 we use the geometric form of the equations to estimate the evolution of the highest-order temporal derivatives. We also analyze the natural (no derivatives) energy in this context. Section 6 concerns similar energy evolution estimates for the other horizontal derivatives. For these we employ the linear perturbed framework with the G^i forcing terms. In Section 7 we assemble the estimates of Sections 5 and 6 into unified estimates. Section 8 concerns the comparison estimates, where we show how to estimate the full energies and dissipations in terms of their horizontal counterparts. Section 9 combines all of the analysis of Sections 3–8 into our a priori estimates for solutions to (1-9). Section 10 concerns a specialized version of the local well-posedness theorem that includes the boundedness of $\mathcal{I}_\lambda$ terms. Finally, in Section 11 we record our global well-posedness and decay result, proving Theorem 1.3.

Below, in (2-58), we will define the total energy $\mathcal{G}_{2N}$ that we use in the global well-posedness analysis. For the purposes of deriving our a priori estimates, we assume throughout Sections 3–9 that solutions to (1-9) are given on the interval $[0, T]$ and that $\mathcal{G}_{2N}(T) \leq \delta$ for $0 < \delta < 1$ as small as in Lemma 2.6, so that its conclusions hold. This also means that $\mathcal{E}_{2N}(t) \leq 1$ for $t \in [0, T]$. We should remark that Theorem 1.1 does not produce solutions that necessarily satisfy $\mathcal{G}_{2N}(T) < \infty$. All of the terms in $\mathcal{G}_{2N}(T)$ are controlled by Theorem 1.1 except those involving the Riesz operator: $\|\mathcal{I}_\lambda u\|_0^2$, $\|\mathcal{I}_\lambda \eta\|_0^2$, and $\int_0^T \|\mathcal{I}_\lambda u(t)\|_1^2 \, dt$. To guarantee that these terms are well-defined, we must prove a specialized version of the local well-posedness result, Theorem 10.7. In principle, we should record this before the a priori estimates, but the technique we use to control the $\mathcal{I}_\lambda$ terms is based on one we develop for the a priori estimates, so we present the theorem in Section 10 after the a priori estimates. Note that the bounds of Theorem 10.7 control more than just $\mathcal{G}_{2N}(T)$ (in particular, $\partial_t^{2N+1}u$, $\partial_t^{2N}p$, and $\mathcal{I}_\lambda p$), and the extra control it provides guarantees that all of the calculations used in the a priori estimates are justified.

2. Preliminaries for the a priori estimates

In this section we present some preliminary results that we use in our a priori estimates. We first record some useful properties of the matrix $\mathcal{A}$. Then we present two forms of equations similar to (1-9) and describe the corresponding energy evolution structure. Afterward we record some useful lemmas.

Properties of $\mathcal{A}$. The following lemma records some of the properties of the matrix $\mathcal{A}$ that will be used throughout the paper.

Lemma 2.1. *Let $\mathcal{A}$ be defined by (1-7).*

(1) *For each $j = 1, 2, 3$ we have $\partial_k(J\mathcal{A}_{jk}) = 0$.*

(2) *$\mathcal{A}_{ij} = \delta_{ij} + \delta_{j3}Z_i$ for δ_{ij}, the Kronecker delta, and $Z = -AKe_1 - BKe_2 + (K-1)e_3$.*

(3) *On Σ we have $J\mathcal{A}e_3 = \mathcal{N}$, while on Σ_b we have that $J\mathcal{A}e_3 = e_3$.*

Proof. The first and second items may be verified by a simple computation. The first part of the third item holds since $\tilde{b} = 1$ on Σ, which means that $J\mathcal{A}e_3 = -Ae_1 - Be_2 + e_3 = -\partial_1\bar{\eta}e_1 - \partial_2\bar{\eta}e_2 + e_3 = -\partial_1\eta e_1 - \partial_2\eta e_2 + e_3 = \mathcal{N}$ on Σ. The second part of the third item follows similarly, since $\tilde{b} = 0$ on Σ_b. $\square$

Geometric form. We now give a linear formulation of the PDE (1-9) in its geometric form. Suppose that η, u are known and that $\mathcal{A}$, $\mathcal{N}$, J, etc. are given in terms of η as usual ((1-7), etc). We then consider the linear equation for (v, q, ζ) given by

$$\begin{cases} \partial_t v - \partial_t\bar{\eta}\tilde{b}K\partial_3 v + u\cdot\nabla_{\mathcal{A}}v + \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(q, v) = F^1 & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}} v = F^2 & \text{in } \Omega, \\ S_{\mathcal{A}}(q, v)\mathcal{N} = \zeta\mathcal{N} + F^3 & \text{on } \Sigma, \\ \partial_t\zeta - \mathcal{N}\cdot v = F^4 & \text{on } \Sigma, \\ v = 0 & \text{on } \Sigma_b. \end{cases} \tag{2-1}$$

Now we record the natural energy evolution equation associated to solutions (v, q, ζ) of the geometric form equations (2-1).

Lemma 2.2. *Suppose that u and η are solutions to (1-9). Suppose (v, q, ζ) solve (2-1). Then*

$$\partial_t\left(\frac{1}{2}\int_\Omega J|v|^2 + \frac{1}{2}\int_\Sigma|\zeta|^2\right) + \frac{1}{2}\int_\Omega J|\mathbb{D}_{\mathcal{A}}v|^2 = \int_\Omega J(v\cdot F^1 + qF^2) + \int_\Sigma -v\cdot F^3 + \zeta F^4. \tag{2-2}$$

Proof. We multiply the i-th component of the first equation of (2-1) by Jv_i, sum over i, and integrate over Ω to find that

$$\mathrm{I} + \mathrm{II} = \mathrm{III} \tag{2-3}$$

for

$$\mathrm{I} = \int_\Omega \partial_t v_i J v_i - \partial_t \bar{\eta} \tilde{b} \partial_3 v_i v_i + u_j \mathscr{A}_{jk} \partial_k v_i J v_i, \tag{2-4}$$

$$\mathrm{II} = \int_\Omega \mathscr{A}_{jk} \partial_k S_{ij}(q, v) J v_i, \quad \mathrm{III} = \int_\Omega F^1 \cdot v J. \tag{2-5}$$

In order to integrate by parts in I, II we will utilize the geometric identity $\partial_k(J\mathscr{A}_{ik}) = 0$ for each i, which is proved in Lemma 2.1.

Then

$$\mathrm{I} = \partial_t \int_\Omega \frac{|v|^2 J}{2} + \int_\Omega -\frac{|v|^2 \partial_t J}{2} - \partial_t \bar{\eta} \tilde{b} \partial_3 \frac{|v|^2}{2} + u_j \partial_k \Big(J \mathscr{A}_{jk} \frac{|v|^2}{2} \Big) =: \mathrm{I}_1 + \mathrm{I}_2. \tag{2-6}$$

Since $\tilde{b} = 1 + x_3/b$, an integration by parts and an application of the boundary condition $v = 0$ on Σ_b reveals that

$$\begin{aligned} \mathrm{I}_2 &= \int_\Omega -\frac{|v|^2 \partial_t J}{2} - \partial_t \bar{\eta} \tilde{b} \partial_3 \frac{|v|^2}{2} + u_j \partial_k \Big(J \mathscr{A}_{jk} \frac{|v|^2}{2} \Big) \\ &= \int_\Omega -\frac{|v|^2 \partial_t J}{2} + \frac{|v|^2}{2} \Big(\frac{\partial_t \bar{\eta}}{b} + \tilde{b} \partial_t \partial_3 \bar{\eta} \Big) - \int_\Omega \partial_k u_j J \mathscr{A}_{jk} \frac{|v|^2}{2} + \frac{1}{2} \int_\Sigma -\partial_t \eta |v|^2 + u_j J \mathscr{A}_{jk} e_3 \cdot e_k |v|^2. \end{aligned} \tag{2-7}$$

It is straightforward to verify that $\partial_t J = \partial_t \bar{\eta}/b + \tilde{b} \partial_t \partial_3 \bar{\eta}$ in Ω and that $J \mathscr{A}_{jk} e_3 \cdot e_k = \mathscr{N}_j$ on Σ. Then since u, η satisfy $\partial_k u_j \mathscr{A}_{jk} = 0$ and $\partial_t \eta = u \cdot \mathscr{N}$, we have $\mathrm{I}_2 = 0$. Hence

$$\mathrm{I} = \partial_t \int_\Omega \frac{|v|^2 J}{2}. \tag{2-8}$$

A similar integration by parts shows that

$$\begin{aligned} \mathrm{II} &= \int_\Omega -\mathscr{A}_{jk} S_{ij}(q, v) J \partial_k v_i + \int_\Sigma J \mathscr{A}_{j3} S_{ij}(q, v) v_i \\ &= \int_\Omega -q \mathscr{A}_{ik} \partial_k v_i J + J \frac{|\mathbb{D}_{\mathscr{A}} v|^2}{2} + \int_\Sigma S_{ij}(q, v) \mathscr{N}_j v_i, \end{aligned} \tag{2-9}$$

so that (2-1) implies

$$\mathrm{II} = \int_\Omega -q J F^2 + J \frac{|\mathbb{D}_{\mathscr{A}} v|^2}{2} + \int_\Sigma \zeta \mathscr{N} \cdot v + v \cdot F^3. \tag{2-10}$$

But (2-1) also implies that

$$\int_\Sigma \zeta \mathscr{N} \cdot v = \int_\Sigma \zeta (\partial_t \zeta - F^4) = \partial_t \int_\Sigma \frac{|\zeta|^2}{2} + \int_\Sigma -\zeta F^4, \tag{2-11}$$

which means

$$\mathrm{II} = \int_\Omega -q J F^2 + J \frac{|\mathbb{D}_{\mathscr{A}} v|^2}{2} + \partial_t \int_\Sigma \frac{|\zeta|^2}{2} + \int_\Sigma -\zeta F^4 + v \cdot F^3. \tag{2-12}$$

Now (2-2) follows from (2-3), (2-8), and (2-12). □

Remark 2.3. In our analysis we will apply Lemma 2.2 with $v=\partial^\alpha u$, $q=\partial^\alpha p$, and $\zeta=\partial^\alpha \eta$ for $\partial^\alpha=\partial_t^{\alpha_0}$ with $\alpha_0 \le 2N$. In the case $\alpha_0=2N$ we do not know that $\partial_t^{2N} p$ is well-defined. However, as is verified in Theorem 4.3 of [Guo and Tice 2013b], the result of Lemma 2.2 holds in this case when integrated in time, with the understanding that the $q=\partial_t^{2N} p$ term is integrated by parts in time.

In order to utilize (2-1), we apply the differential operator $\partial^\alpha=\partial_t^{\alpha_0}$ to (1-9). The resulting equations are (2-1) for $v=\partial^\alpha u$, $q=\partial^\alpha p$, and $\zeta=\partial^\alpha\eta$, where

$$F^1=F^{1,1}+F^{1,2}+F^{1,3}+F^{1,4}+F^{1,5}+F^{1,6} \tag{2-13}$$

for

$$F_i^{1,1}=\sum_{0<\beta<\alpha} C_{\alpha,\beta}\partial^\beta(\partial_t\bar{\eta}\tilde{b}K)\partial^{\alpha-\beta}\partial_3 u_i+\sum_{0<\beta\le\alpha} C_{\alpha,\beta}\partial^{\alpha-\beta}\partial_t\bar{\eta}\partial^\beta(\tilde{b}K)\partial_3 u_i, \tag{2-14}$$

$$F_i^{1,2}=-\sum_{0<\beta\le\alpha} C_{\alpha,\beta}(\partial^\beta(u_j\mathcal{A}_{jk})\partial^{\alpha-\beta}\partial_k u_i+\partial^\beta\mathcal{A}_{ik}\partial^{\alpha-\beta}\partial_k p), \tag{2-15}$$

$$F_i^{1,3}=\sum_{0<\beta\le\alpha} C_{\alpha,\beta}\partial^\beta\mathcal{A}_{j\ell}\partial^{\alpha-\beta}\partial_\ell(\mathcal{A}_{im}\partial_m u_j+\mathcal{A}_{jm}\partial_m u_i), \tag{2-16}$$

$$F_i^{1,4}=\sum_{0<\beta<\alpha} C_{\alpha,\beta}\mathcal{A}_{jk}\partial_k(\partial^\beta\mathcal{A}_{i\ell}\partial^{\alpha-\beta}\partial_\ell u_j+\partial^\beta\mathcal{A}_{j\ell}\partial^{\alpha-\beta}\partial_\ell u_i), \tag{2-17}$$

$$F_i^{1,5}=\partial^\alpha\partial_t\bar{\eta}\tilde{b}K\partial_3 u_i, \quad \text{and} \quad F_i^{1,6}=\mathcal{A}_{jk}\partial_k(\partial^\alpha\mathcal{A}_{i\ell}\partial_\ell u_j+\partial^\alpha\mathcal{A}_{j\ell}\partial_\ell u_i). \tag{2-18}$$

In these equations, the terms $C_{\alpha,\beta}$ are constants that depend on α and β. The term $F^2=F^{2,1}+F^{2,2}$ for

$$F^{2,1}=-\sum_{0<\beta<\alpha} C_{\alpha,\beta}\partial^\beta\mathcal{A}_{ij}\partial^{\alpha-\beta}\partial_j u_i \quad \text{and} \quad F^{2,2}=-\partial^\alpha\mathcal{A}_{ij}\partial_j u_i. \tag{2-19}$$

We write $F^3=F^{3,1}+F^{3,2}$ for

$$F^{3,1}=-\sum_{0<\beta\le\alpha} C_{\alpha,\beta}\partial^\beta D\eta(\partial^{\alpha-\beta}\eta-\partial^{\alpha-\beta}p), \tag{2-20}$$

$$F_i^{3,2}=\sum_{0<\beta\le\alpha} C_{\alpha,\beta}(\partial^\beta(\mathcal{N}_j\mathcal{A}_{im})\partial^{\alpha-\beta}\partial_m u_j+\partial^\beta(\mathcal{N}_j\mathcal{A}_{jm})\partial^{\alpha-\beta}\partial_m u_i). \tag{2-21}$$

Finally,

$$F^4=-\sum_{0<\beta\le\alpha} C_{\alpha,\beta}\partial^\beta D\eta\cdot\partial^{\alpha-\beta}u. \tag{2-22}$$

Perturbed linear form. Writing the equations in the form (1-9) is more faithful to the geometry of the free boundary problem, but it is inconvenient for many of our a priori estimates. This stems from the fact that if we want to think of the coefficients of the equations for u, p as being frozen for a fixed free boundary given by η, the underlying linear operator has nonconstant coefficients. This makes it unsuitable for applying differential operators.

To get around this problem, in many parts of the paper we will analyze the PDE in a different formulation, which looks like a perturbation of the linearized problem. The utility of this form of the equations lies in the fact that the linear operators have constant coefficients. The equations in this form are

$$\begin{cases} \partial_t u + \nabla p - \Delta u = G^1 & \text{in } \Omega, \\ \operatorname{div} u = G^2 & \text{in } \Omega, \\ (pI - \mathbb{D}u - \eta I)e_3 = G^3 & \text{on } \Sigma, \\ \partial_t \eta - u_3 = G^4 & \text{on } \Sigma, \\ u = 0 & \text{on } \Sigma_b. \end{cases} \tag{2-23}$$

Here we have written

$$G^1 = G^{1,1} + G^{1,2} + G^{1,3} + G^{1,4} + G^{1,5}$$

for

$$G_i^{1,1} = (\delta_{ij} - \mathscr{A}_{ij})\partial_j p, \tag{2-24}$$

$$G_i^{1,2} = u_j \mathscr{A}_{jk} \partial_k u_i, \tag{2-25}$$

$$G_i^{1,3} = [K^2(1 + A^2 + B^2) - 1]\partial_{33} u_i - 2AK\partial_{13} u_i - 2BK\partial_{23} u_i, \tag{2-26}$$

$$G_i^{1,4} = [-K^3(1 + A^2 + B^2)\partial_3 J + AK^2(\partial_1 J + \partial_3 A) + BK^2(\partial_2 J + \partial_3 B) - K(\partial_1 A + \partial_2 B)]\partial_3 u_i, \tag{2-27}$$

$$G_i^{1,5} = \partial_t \bar{\eta}(1 + x_3/b)K\partial_3 u_i; \tag{2-28}$$

G^2 is the function

$$G^2 = AK\partial_3 u_1 + BK\partial_3 u_2 + (1 - K)\partial_3 u_3, \tag{2-29}$$

and G^3 is the vector

$$\begin{aligned} G^3 := \partial_1 \eta \begin{pmatrix} p - \eta - 2(\partial_1 u_1 - AK\partial_3 u_1) \\ -\partial_2 u_1 - \partial_1 u_2 + BK\partial_3 u_1 + AK\partial_3 u_2 \\ -\partial_1 u_3 - K\partial_3 u_1 + AK\partial_3 u_3 \end{pmatrix} \\ + \partial_2 \eta \begin{pmatrix} -\partial_2 u_1 - \partial_1 u_2 + BK\partial_3 u_1 + AK\partial_3 u_2 \\ p - \eta - 2(\partial_2 u_2 - BK\partial_3 u_2) \\ -\partial_2 u_3 - K\partial_3 u_2 + BK\partial_3 u_3 \end{pmatrix} + \begin{pmatrix} (K-1)\partial_3 u_1 - AK\partial_3 u_3 \\ (K-1)\partial_3 u_2 - BK\partial_3 u_3 \\ 2(K-1)\partial_3 u_3 \end{pmatrix}. \end{aligned} \tag{2-30}$$

Finally,

$$G^4 = -D\eta \cdot u. \tag{2-31}$$

Remark 2.4. The appearance of the term $(p - \eta)$ in the first two rows of the first two vectors in the definition of G^3 can cause some technical problems later when we attempt to estimate G^3. Notice though, that according to (2-23), we may write

$$\begin{aligned} (p - \eta) &= 2\partial_3 u_3 + G^3 \cdot e_3 \\ &= \partial_1 \eta(-\partial_1 u_3 - K\partial_3 u_1 + AK\partial_3 u_3) + \partial_2 \eta(-\partial_2 u_3 - K\partial_3 u_2 + BK\partial_3 u_3) + 2K\partial_3 u_3 \end{aligned} \tag{2-32}$$

on Σ. We may then replace the appearances of $(p - \eta)$ in (2-30) with the right side of (2-32).

At several points in our analysis we will need to localize (2-23) by multiplying by a cutoff function. This leads us to consider the energy evolution for a minor modification of (2-23).

Lemma 2.5. *Suppose* (v, q, ζ) *solve*

$$\begin{cases} \partial_t v + \nabla q - \Delta v = \Phi^1 & \text{in } \Omega, \\ \operatorname{div} v = \Phi^2 & \text{in } \Omega, \\ (qI - \mathbb{D}v)e_3 = a\zeta e_3 + \Phi^3 & \text{on } \Sigma, \\ \partial_t \zeta - v_3 = \Phi^4 & \text{on } \Sigma, \\ v = 0 & \text{on } \Sigma_b, \end{cases} \tag{2-33}$$

where either $a = 0$ *or* $a = 1$. *Then*

$$\partial_t \left(\frac{1}{2} \int_\Omega |v|^2 + \frac{1}{2} \int_\Sigma a|\zeta|^2 \right) + \frac{1}{2} \int_\Omega |\mathbb{D}v|^2 = \int_\Omega v \cdot (\Phi^1 - \nabla \Phi^2) + q\Phi^2 + \int_\Sigma -v \cdot \Phi^3 + a\zeta \Phi^4. \tag{2-34}$$

Proof. We may rewrite the first equation in (2-33) as $\partial_t v + \operatorname{div}(qI - \mathbb{D}v) = \Phi^1 - \nabla\Phi^2$. We then take the inner-product of this equation with v and integrate over Ω to find

$$\partial_t \int_\Omega \frac{|v|^2}{2} - \int_\Omega (qI - \mathbb{D}v) : \nabla v + \int_\Sigma (qI - \mathbb{D}v)e_3 \cdot v = \int_\Omega v \cdot (\Phi^1 - \nabla \Phi^2). \tag{2-35}$$

We then use the second equation in (2-33) to compute

$$\int_\Omega -(qI - \mathbb{D}v) : \nabla v = \int_\Omega -q \operatorname{div} v + \frac{|\mathbb{D}v|^2}{2} = \int_\Omega -q\Phi^2 + \frac{|\mathbb{D}v|^2}{2}. \tag{2-36}$$

The boundary conditions in (2-33) provide the equality

$$\int_\Sigma (qI - \mathbb{D}v)e_3 \cdot v = \int_\Sigma a\zeta v_3 + v \cdot \Phi^3 = \partial_t \int_\Sigma a\frac{|\zeta|^2}{2} + \int_\Sigma -a\zeta\Phi^4 + v \cdot \Phi^3. \tag{2-37}$$

Combining (2-35)–(2-37) then yields (2-34). □

Some initial lemmas. The following result is useful for removing the appearance of J factors.

Lemma 2.6. *There exists a universal* $0 < \delta < 1$ *such that if* $\|\eta\|_{5/2}^2 \le \delta$, *then*

$$\|J - 1\|_{L^\infty}^2 + \|A\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \le \tfrac{1}{2} \quad \text{and} \quad \|K\|_{L^\infty}^2 + \|\mathscr{A}\|_{L^\infty}^2 \lesssim 1. \tag{2-38}$$

Proof. According to the definitions of A, B, J given in (1-8) and Lemma A.5, we may bound

$$\|J - 1\|_{L^\infty}^2 + \|A\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \lesssim \|\bar\eta\|_3^2 \lesssim \|\eta\|_{5/2}^2. \tag{2-39}$$

Then if δ is sufficiently small, we find that the first inequality in (2-38) holds. As a consequence, $\|K\|_{L^\infty}^2 + \|\mathscr{A}\|_{L^\infty}^2 \lesssim 1$, which is the second inequality in (2-38). □

We now compute $\partial_t \eta$ in terms of a pair of auxiliary functions, U_1 and U_2, defined on Σ. In our analysis later in the paper u and η will always be sufficiently smooth to justify the calculations in the next lemma, and $U_i \in H^1(\Sigma)$ always holds.

Lemma 2.7. *For* $i = 1, 2$, *define* $U_i : \Sigma \to \mathbb{R}$ *by*

$$U_i(x') = \int_{-b}^0 J(x', x_3) u_i(x', x_3)\, dx_3. \tag{2-40}$$

Then $\partial_t \eta = -\partial_1 U_1 - \partial_2 U_2$ *on* Σ *for solutions to* (1-9).

Proof. Let $\varphi \in \mathcal{S}(\Sigma)$, the Schwartz class. On Σ we know from Lemma 2.1 that $u \cdot \mathcal{N} = u \cdot (J\mathcal{A}e_3) = J\mathcal{A}^T u \cdot e_3 = J\mathcal{A}^T u \cdot \nu$, where $\nu = e_3$ is the unit normal to Σ. We may use the equation for $\partial_t \eta$ in (1-9) and the divergence theorem to compute

$$\begin{aligned}\int_\Sigma \partial_t \eta \varphi &= \int_\Sigma (-u_1 \partial_1 \eta - u_2 \partial_2 \eta + u_3)\varphi = \int_\Sigma \varphi J \mathcal{A}_{ij} u_i \nu_j = \int_\Omega \partial_j (\varphi J \mathcal{A}_{ij} u_i) \\ &= \int_\Omega \partial_j \varphi J \mathcal{A}_{ij} u_i + \varphi \partial_j (J\mathcal{A}_{ij}) u_i + \varphi J \mathcal{A}_{ij} \partial_j u_i = \int_\Omega \partial_j \varphi J \mathcal{A}_{ij} u_i, \end{aligned} \tag{2-41}$$

where the last equality follows from the geometric identity $\partial_j (J\mathcal{A}_{ij}) = 0$ (see Lemma 2.1) and the equation $\mathcal{A}_{ij}\partial_j u_i = 0$, which is the second equation in (1-9). According to Lemma 2.1, we may write $\mathcal{A}_{ij} = \delta_{ij} + \delta_{j3} Z_i$ for δ_{ij}, the Kronecker delta, and $Z = -AKe_1 - BKe_2 + (K-1)e_3$. Then

$$\int_\Omega \partial_j \varphi J \mathcal{A}_{ij} u_i = \int_\Omega \partial_j \varphi J u_i (\delta_{ij} + \delta_{j3} Z_i) = \int_\Omega \partial_i \varphi J u_i + \int_\Omega \partial_3 \varphi J u_i Z_i = \int_\Omega \partial_i \varphi J u_i, \tag{2-42}$$

since $\partial_3 \varphi = 0$, a consequence of the fact that $\varphi = \varphi(x_1, x_2)$ is independent of x_3. Again because φ depends only on $(x_1, x_2) = x' \in \Sigma$, we may write

$$\int_\Omega \partial_i \varphi J u_i = \int_\Sigma \partial_i \varphi(x') \int_{-b}^0 J(x', x_3) u_i(x', x_3)\, dx_3 dx' = \int_\Sigma \partial_i \varphi(x') U_i(x')\, dx'. \tag{2-43}$$

Now we chain together (2-41), (2-42), and (2-43) and integrate by parts to deduce that

$$\int_\Sigma \partial_t \eta \varphi = \int_\Sigma -\varphi \partial_i U_i. \tag{2-44}$$

Since this holds for any $\varphi \in \mathcal{S}(\Sigma)$, we then have that $\partial_t \eta = -\partial_i U_i$. □

Energies and dissipations. Below we define the energies and dissipations we will use in our analysis. We state them in general in terms of two integers $n, m \in \mathbb{N}$ with $n \geq m$. In our actual analysis we will take $n = 2N$ and $n = N + 2$ for $N \geq 5$ and $m = 1, 2$. Recall that we employ the derivative conventions described on page 1443. We define the horizontal instantaneous energy with minimal derivative count m (or just horizontal energy, for short) by

$$\bar{\mathcal{E}}_{n,m} := \|\bar{D}_m^{2n-1} u\|_0^2 + \|D\bar{D}^{2n-1} u\|_0^2 + \|\sqrt{J}\partial_t^n u\|_0^2 + \|\bar{D}_m^{2n}\eta\|_0^2. \tag{2-45}$$

Here the first three terms are split in this manner for the technical convenience of adding the $\sqrt{J}$ term to only the highest temporal derivative.

Remark 2.8. In light of Lemma 2.6, we see that $\bar{\mathcal{E}}_{n,m}$ satisfies

$$\tfrac{1}{2}(\|\bar{D}_m^{2n} u\|_0^2 + \|\bar{D}_m^{2n}\eta\|_0^2) \leq \bar{\mathcal{E}}_{n,m} \leq \tfrac{3}{2}(\|\bar{D}_m^{2n} u\|_0^2 + \|\bar{D}_m^{2n}\eta\|_0^2). \tag{2-46}$$

We define the horizontal dissipation rate with minimal derivative count m (horizontal dissipation) by

$$\bar{\mathcal{D}}_{n,m} := \|\bar{D}_m^{2n} \mathbb{D} u\|_0^2. \tag{2-47}$$

Let $\mathcal{I}_\lambda$ be defined by (A-7)–(A-8). The horizontal energy without a minimal derivative restriction is

$$\bar{\mathcal{E}}_n := \|\mathcal{I}_\lambda u\|_0^2 + \|\bar{D}_0^{2n} u\|_0^2 + \|\mathcal{I}_\lambda \eta\|_0^2 + \|\bar{D}_0^{2n} \eta\|_0^2, \tag{2-48}$$

and the horizontal dissipation without a minimal derivative restriction is

$$\bar{\mathcal{D}}_n := \|\mathbb{D}\mathcal{I}_\lambda u\|_0^2 + \|\bar{D}_0^{2n} \mathbb{D} u\|_0^2. \tag{2-49}$$

In addition to the horizontal energy and dissipation, we must also define full energies and dissipations, which involve full derivatives. We write the full energy as

$$\mathcal{E}_n := \|\mathcal{I}_\lambda u\|_0^2 + \sum_{j=0}^{n} \|\partial_t^j u\|_{2n-2j}^2 + \sum_{j=0}^{n-1} \|\partial_t^j p\|_{2n-2j-1}^2 + \|\mathcal{I}_\lambda \eta\|_0^2 + \sum_{j=0}^{n} \|\partial_t^j \eta\|_{2n-2j}^2, \tag{2-50}$$

and we define the full dissipation rate by

$$\mathcal{D}_n := \|\mathcal{I}_\lambda u\|_1^2 + \sum_{j=0}^{n} \|\partial_t^j u\|_{2n-2j+1}^2 + \|\nabla p\|_{2n-1}^2 + \sum_{j=1}^{n-1} \|\partial_t^j p\|_{2n-2j}^2 + \|D\eta\|_{2n-3/2}^2 + \|\partial_t \eta\|_{2n-1/2}^2 + \sum_{j=2}^{n+1} \|\partial_t^j \eta\|_{2n-2j+5/2}^2. \tag{2-51}$$

Remark 2.9. The energy $\mathcal{E}_n$ controls $\|\eta\|_{2n}^2 \asymp \|\eta\|_0^2 + \|D\eta\|_{2n-1}^2$, while the dissipation $\mathcal{D}_n$ controls only $\|D\eta\|_{2n-3/2}^2$. The failure of $\mathcal{D}_n$ to control $\|\eta\|_0^2$ and this half derivative deficit in $D\eta$ are key difficulties that we must overcome in our analysis. However, $\mathcal{D}_n$ controls more temporal derivatives of η than $\mathcal{E}_n$ does. A similar discrepancy exists in the fact that $\mathcal{E}_n$ controls $\|p\|_{2n-1}^2$ while $\mathcal{D}_n$ controls only $\|\nabla p\|_{2n-1}^2$.

We define a similar energy with a minimal derivative count of one by

$$\mathcal{E}_{n,1} := \bar{\mathcal{E}}_{n,1} + \|\nabla^2 u\|_{2n-2}^2 + \sum_{j=1}^{n} \|\partial_t^j u\|_{2n-2j}^2 + \|\nabla p\|_{2n-2}^2 + \sum_{j=1}^{n-1} \|\partial_t^j p\|_{2n-2j-1}^2 + \|D\eta\|_{2n-1}^2 + \sum_{j=1}^{n} \|\partial_t^j \eta\|_{2n-2j}^2, \tag{2-52}$$

and with a minimal derivative count of two by

$$\mathcal{E}_{n,2} := \bar{\mathcal{E}}_{n,2} + \|\nabla^3 u\|_{2n-3}^2 + \sum_{j=1}^{n} \|\partial_t^j u\|_{2n-2j}^2 + \|\nabla^2 p\|_{2n-3}^2 + \sum_{j=1}^{n-1} \|\partial_t^j p\|_{2n-2j-1}^2 + \|D^2\eta\|_{2n-2}^2 + \sum_{j=1}^{n} \|\partial_t^j \eta\|_{2n-2j}^2. \tag{2-53}$$

Similarly, the dissipation with a minimal derivative count of one is

$$\mathcal{D}_{n,1} := \bar{\mathcal{D}}_{n,1} + \|\nabla^3 u\|_{2n-2}^2 + \sum_{j=1}^{n} \|\partial_t^j u\|_{2n-2j+1}^2 + \|\nabla^2 p\|_{2n-2}^2 + \sum_{j=1}^{n-1} \|\partial_t^j p\|_{2n-2j}^2 + \|D^2\eta\|_{2n-5/2}^2 + \|\partial_t \eta\|_{2n-1/2}^2 + \sum_{j=2}^{n+1} \|\partial_t^j \eta\|_{2n-2j+5/2}^2, \tag{2-54}$$

while the dissipation with a minimal derivative count of two is

$$\mathcal{D}_{n,2} := \bar{\mathcal{D}}_{n,2} + \|\nabla^4 u\|_{2n-3}^2 + \sum_{j=1}^{n} \|\partial_t^j u\|_{2n-2j+1}^2 + \|\nabla^3 p\|_{2n-3}^2 + \|\partial_t \nabla p\|_{2n-3}^2 \\ + \sum_{j=2}^{n-1} \|\partial_t^j p\|_{2n-2j}^2 + \|D^3\eta\|_{2n-7/2}^2 + \|D\partial_t \eta\|_{2n-3/2}^2 + \sum_{j=2}^{n+1} \|\partial_t^j \eta\|_{2n-2j+5/2}^2. \quad (2\text{-}55)$$

Note that, by definition, $\mathcal{E}_{n,m} \geq \bar{\mathcal{E}}_{n,m}$ and $\mathcal{D}_{n,m} \geq \bar{\mathcal{D}}_{n,m}$. In all of these definitions, the index n counts the highest number of time derivatives used. Notice that $\mathcal{E}_{n,m}$ and $\mathcal{D}_{n,m}$ are subject to the same sorts of discrepancies described in Remark 2.9.

Certain norms of η and u will play a special role in our analysis; we write

$$\mathcal{F}_{2N} := \|\eta\|_{4N+1/2}^2, \quad (2\text{-}56)$$

$$\mathcal{K} := \|\nabla u\|_{L^\infty}^2 + \|\nabla^2 u\|_{L^\infty}^2 + \sum_{i=1}^{2} \|Du_i\|_{H^2(\Sigma)}^2. \quad (2\text{-}57)$$

Note that the regularity of u will always be sufficiently high for the L^∞ norms in $\mathcal{K}$ to be considered as $C^0(\bar{\Omega})$ norms, where $\bar{\Omega}$ is the closure of Ω. Finally, we define the total energy we will use in our analysis:

$$\mathcal{G}_{2N}(t) := \sup_{0\leq r\leq t} \mathcal{E}_{2N}(r) + \int_0^t \mathcal{D}_{2N}(r)\,dr + \sum_{m=1}^{2} \sup_{0\leq r\leq t} (1+r)^{m+\lambda} \mathcal{E}_{N+2,m}(r) + \sup_{0\leq r\leq t} \frac{\mathcal{F}_{2N}(r)}{(1+r)}. \quad (2\text{-}58)$$

Some initial estimates. We have the following lemma that constrains N.

Lemma 2.10. *If $N \geq 4$, then, for $m = 1, 2$, we have $\mathcal{E}_{N+2,m} \lesssim \mathcal{E}_{2N}$ and $\mathcal{D}_{N+2,m} \lesssim \mathcal{E}_{2N}$.*

Proof. The proof follows by simply comparing the definitions of these terms. □

Now we present an estimate of $\mathcal{I}_1 \partial_t \eta$.

Lemma 2.11. *We have the estimate $\|\mathcal{I}_1 \partial_t \eta\|_0^2 \lesssim \|u\|_0^2 \leq \mathcal{E}_{2N}$.*

Proof. According to Lemma 2.7, we have $\partial_t \eta = -\partial_i U_i$, where U_i, $i = 1, 2$, is defined in the lemma. It is easy to see that $U_i \in H^1(\Sigma)$. Taking the Fourier transform and writing $U = (U_1, U_2)$, we find that

$$\|\mathcal{I}_1 \partial_t \eta\|_0^2 = \int_\Sigma |\xi|^{-2} |\widehat{\partial_t \eta}(\xi)|^2 d\xi \lesssim \int_\Sigma |\xi|^{-2} |\xi \cdot \widehat{U}(\xi)|^2 d\xi \lesssim \int_\Sigma |\widehat{U}(\xi)|^2 d\xi = \|U\|_{H^0(\Sigma)}^2. \quad (2\text{-}59)$$

However, Hölder's inequality and Lemma 2.6 imply that $\|U\|_{H^0(\Sigma)} \lesssim \|J\|_{L^\infty} \|u\|_0 \lesssim \|u\|_0$, so the desired estimate follows. □

3. Interpolation estimates at the $N+2$ level

Initial interpolation estimates for η, $\bar{\eta}$, u and ∇p. The fact that $\mathcal{E}_{N+2,m}$ and $\mathcal{D}_{N+2,m}$, $m = 1, 2$, have a minimal count of derivatives creates numerous problems when we try to estimate terms with fewer derivatives in terms of $\mathcal{E}_{N+2,m}$ and $\mathcal{D}_{N+2,m}$. Our way around this is to interpolate between $\mathcal{E}_{N+2,m}$

(or $\mathfrak{D}_{N+2,m}$) and $\mathscr{E}_{2N}$. In the next few pages (through page 1467) we will prove various interpolation inequalities of the form

$$\|X\|^2 \lesssim (\mathscr{E}_{N+2,m})^\theta (\mathscr{E}_{2N})^{1-\theta} \quad \text{and} \quad \|X\|^2 \lesssim (\mathfrak{D}_{N+2,m})^\theta (\mathscr{E}_{2N})^{1-\theta}, \tag{3-1}$$

where $\theta \in (0, 1]$, X is some quantity, and $\|\cdot\|$ is some norm (usually either H^0 or L^∞).

In the interest of brevity, we record these estimates in tables that only list the value of θ in the estimate. Before each table we will tell which norms are being considered and give a rough summary of the terms X that appear in the table. For example, we might write "the following table encodes the power in the $H^0(\Sigma)$ and $H^0(\Omega)$ interpolation estimates for η and $\bar{\eta}$ and their derivatives," before the following table.

X	$\mathscr{E}_{N+2,1}$	$\mathfrak{D}_{N+2,1} \sim \mathscr{E}_{N+2,2}$	$\mathfrak{D}_{N+2,2}$
$\eta, \bar{\eta}$	θ_1	θ_2	θ_3
$D\eta, \nabla\bar{\eta}$	θ_4	θ_5	θ_6

We understand this to mean that

$$\|\eta\|_0^2 \lesssim (\mathscr{E}_{N+2,1})^{\theta_1} (\mathscr{E}_{2N})^{1-\theta_1}, \quad \|\eta\|_0^2 \lesssim (\mathfrak{D}_{N+2,1})^{\theta_2} (\mathscr{E}_{2N})^{1-\theta_2}, \quad \|\eta\|_0^2 \lesssim (\mathscr{E}_{N+2,2})^{\theta_2} (\mathscr{E}_{2N})^{1-\theta_2} \tag{3-2}$$

and

$$\|\eta\|_0^2 \lesssim (\mathfrak{D}_{N+2,2})^{\theta_3} (\mathscr{E}_{2N})^{1-\theta_3}, \quad \|\nabla\bar{\eta}\|_{H^0(\Omega)}^2 \lesssim (\mathscr{E}_{N+2,1})^{\theta_4} (\mathscr{E}_{2N})^{1-\theta_4},$$
$$\|\nabla\bar{\eta}\|_{H^0(\Omega)}^2 \lesssim (\mathfrak{D}_{N+2,1})^{\theta_5} (\mathscr{E}_{2N})^{1-\theta_5}, \tag{3-3}$$

etc. When we write $\mathfrak{D}_{N+2,1} \sim \mathscr{E}_{N+2,2}$ in a table, it means that θ is the same when interpolating between $\mathfrak{D}_{N+2,1}$ and $\mathscr{E}_{2N}$ and between $\mathscr{E}_{N+2,2}$ and $\mathscr{E}_{2N}$. When we write multiple entries for X, we mean that the same interpolation estimates hold for each item listed. Often, we will have a θ appearing in a table of the form $\theta = 1/(1+r)$. When we write this, we mean that the desired interpolation inequality holds with this θ for any fixed $r \in (0, 1)$, and the constant in the inequality then depends on r.

We must record estimates for too many choices of X to allow us to write the full details of each estimate. However, most of the estimates are straightforward, so in our proofs we will frequently present only a sketch of how to obtain them, providing details only for the most delicate estimates. The terms we estimate are often linear combinations of several terms, each of which would get a different interpolation power. When this occurs, we will record the lowest power achieved by a term in the sum. According to Lemma 2.10, this is justified by the estimate

$$\begin{aligned}\mathscr{E}_{2N}^{1-\theta}\mathscr{E}_{N+2,m}^{\theta} + \mathscr{E}_{2N}^{1-\kappa}\mathscr{E}_{N+2,m}^{\kappa} &= \mathscr{E}_{2N}^{1-\theta}\mathscr{E}_{N+2,m}^{\theta} + \mathscr{E}_{2N}^{1-\kappa}\mathscr{E}_{N+2,m}^{\kappa-\theta}\mathscr{E}_{N+2,m}^{\theta} \\ &\lesssim \mathscr{E}_{2N}^{1-\theta}\mathscr{E}_{N+2,m}^{\theta} + \mathscr{E}_{2N}^{1-\kappa}\mathscr{E}_{2N}^{\kappa-\theta}\mathscr{E}_{N+2,m}^{\theta} \lesssim \mathscr{E}_{2N}^{1-\theta}\mathscr{E}_{N+2,m}^{\theta}\end{aligned} \tag{3-4}$$

for $0 \le \theta \le \kappa \le 1$. A similar estimate holds with $\mathscr{E}_{N+2,m}$ replaced by $\mathfrak{D}_{N+2,m}$. It may happen that in estimating a product of two or more terms, we end up with estimates of the form

$$\|X\|^2 \lesssim (\mathscr{E}_{N+2,m})^{\theta_1} (\mathscr{E}_{2N})^{1-\theta_1} (\mathscr{E}_{N+2,m})^{\theta_2} (\mathscr{E}_{2N})^{1-\theta_2} \tag{3-5}$$

with $\theta_1+\theta_2>1$. In this case, Lemma 2.10 again allows us to bound

$$\|X\|^2 \lesssim (\mathcal{E}_{N+2,m})^1(\mathcal{E}_{N+2,m})^{\theta_1+\theta_2-1}(\mathcal{E}_{2N})^{2-\theta_1-\theta_2} \lesssim \mathcal{E}_{N+2,m}\mathcal{E}_{2N} \le \mathcal{E}_{N+2,m}, \tag{3-6}$$

where we have used the bound $\mathcal{E}_{2N} \le 1$. It might also happen that (3-5) occurs with $\theta_1 < 1$ and $\theta_2 = 1/(1+r)$, in which case we always understand that r is chosen so that $\theta_1+\theta_2 = 1$.

Now that our notation is explained, we turn to the estimates themselves We begin with estimates of η.

Lemma 3.1. *The following table encodes the power in the $L^\infty(\Sigma)$ and $L^\infty(\Omega)$ interpolation estimates for η and $\bar{\eta}$ and their derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$\eta, \bar{\eta}$	$(\lambda+1)/(\lambda+1+r)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$
$D\eta, \nabla\bar{\eta}$	1	$(\lambda+2)/(\lambda+2+r)$	$(\lambda+2)/(\lambda+3)$
$D^2\eta, \nabla^2\bar{\eta}$	1	1	$(\lambda+3)/(\lambda+3+r)$
$D^3\eta, \nabla^3\bar{\eta}$	1	1	1
$\partial_t\eta, \partial_t\bar{\eta}$	1	1	$2/(2+r)$
$D\partial_t\eta, \nabla\partial_t\bar{\eta}$	1	1	1

The following table encodes the power in the $H^0(\Sigma)$ and $H^0(\Omega)$ interpolation estimates for η and $\bar{\eta}$ and their derivatives.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$\eta, \bar{\eta}$	$\lambda/(\lambda+1)$	$\lambda/(\lambda+2)$	$\lambda/(\lambda+3)$
$D\eta, \nabla\bar{\eta}$	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$
$D^2\eta, \nabla^2\bar{\eta}$	1	1	$(\lambda+2)/(\lambda+3)$
$D^3\eta, \nabla^3\bar{\eta}$	1	1	1
$\partial_t\eta, \partial_t\bar{\eta}$	1	1	$1/2$
$D\partial_t\eta, \nabla\partial_t\bar{\eta}$	1	1	1

Proof. The estimates follow directly from the Sobolev embeddings and Lemmas A.6 and A.7, using the bounds $\|\mathcal{I}_\lambda\eta\|_0^2 \le \mathcal{E}_{2N}$ and $\|\mathcal{I}_1\partial_t\eta\|_0^2 \lesssim \mathcal{E}_{2N}$, the latter of which is a consequence of Lemma 2.11. □

Now we record some estimates involving u.

Lemma 3.2. Table 3.1(a) *encodes the power in the $L^\infty(\Omega)$ and $L^\infty(\Sigma)$ interpolation estimates for u and its derivatives.*

Table 3.1(b) *encodes the power in the $H^0(\Omega)$ interpolation estimates for u and its derivatives.*

Table 3.1(c) *encodes the power in some improved $L^\infty(\Sigma)$ interpolation estimates for u and its tangential derivatives on Σ. Here we restrict to $r \in (0, 1/2)$.*

Proof. The estimates of the first two tables follow directly from Sobolev embeddings and Lemmas A.8 and A.13. For the $L^\infty(\Sigma)$ estimates of the last table, we use $r \in [0, 1/2)$ in (A-34) of Lemma A.7 along with trace estimates and Lemma A.13 to bound

(a)

X	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1} \sim \mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
u	$1/(1+r)$	$1/2$	$1/3$
Du	1	$2/(2+r)$	$2/3$
∇u	$1/(1+r)$	$1/2$	$1/3$
D^2u	1	1	$1/(1+r)$
$D\nabla u$	1	$2/(2+r)$	$2/3$
$\nabla^2 u$	1	$1/(1+r)$	$1/2$
$\nabla^3 u$	1	1	$1/(1+r)$
$\nabla^4 u$	1	1	1
$\partial_t u$	1	1	1

(b)

X	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
u	$\lambda/(\lambda+1)$	$\lambda/(\lambda+1)$	$\lambda/(\lambda+2)$	$\lambda/(\lambda+2)$
Du	1	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$
D^2u	1	1	1	1
$\nabla D^2 u$	1	1	1	1
$\partial_t u$	1	1	1	1

(c)

X	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
u	$1/(1+r)$	$1/(1+r)$	$1/2$	$1/2$
Du	1	$2/(2+r)$	$2/(2+r)$	$2/(2+r)$

Table 3.2. Tables for Lemma 3.2.

$$\|u\|^2_{L^\infty(\Sigma)} \lesssim (\|u\|^2_{H^0(\Sigma)})^{(s+r-1)/(s+r)} (\|D^s u\|_{H^r(\Sigma)})^{1/(s+r)} \lesssim (\|u\|_1^2)^{(s+r-1)/(s+r)} (\|D^s u\|_1^2)^{1/(s+r)}$$
$$\lesssim (\|u\|_1^2)^{(s+r-1)/(s+r)} (\|D^s \nabla u\|_0^2)^{1/(s+r)}. \quad (3\text{-}7)$$

For $\mathscr{E}_{N+2,1}$ and $\mathscr{D}_{N+2,1}$ we choose $s=1$ and $r \in (0, 1/2)$, while for $\mathscr{E}_{N+2,2}$ and $\mathscr{D}_{N+2,m}$ we choose $s=2$ and $r=0$. In both cases, $\|u\|_1^2 \le \mathscr{E}_{2N}$ and $\|D^s \nabla u\|_0^2 \le \mathscr{E}_{N+2,m}$. A similar argument works for the Du estimates in $L^\infty(\Sigma)$. □

Now we estimate ∇p in L^∞.

Lemma 3.3. *The following table encodes the power in the $L^\infty(\Omega)$ interpolation estimates for derivatives of p.*

X	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1} \sim \mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
∇p	1	$1/(1+r)$	$1/2$
$\nabla^2 p$	1	1	$1/(1+r)$
$\partial_t p$	1	1	$1/(1+r)$
$\nabla^3 p$	1	1	1
$\partial_t \nabla p$	1	1	1

Proof. The estimates follow directly from the Sobolev embeddings and Lemma A.8. ☐

Interpolation estimates for G^i, $i = 1, 2, 3, 4$. Now that we have some preliminary estimates for u, η, $\bar{\eta}$, and ∇p (plus some of their derivatives), we can estimate the G^i forcing terms defined in (2-24)–(2-31).

Lemma 3.4. *The following table encodes the power in the $L^\infty(\Omega)$ interpolation estimates for $G^{1,i}$, $i = 1, \ldots, 5$ and G^1 and their spatial derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$G^{1,1}$	1	1	$(3\lambda+5)/(2\lambda+6)$
$\nabla G^{1,1}$	1	1	1
$G^{1,2}$	1	1	$2/3$
$DG^{1,2}$	1	1	1
$\nabla G^{1,2}$	1	1	$2/3$
$G^{1,3}$	1	1	$(3\lambda+5)/(2\lambda+6)$
$\nabla G^{1,3}$	1	1	1
$G^{1,4}$	1	1	1
$\nabla G^{1,4}$	1	1	1
$G^{1,5}$	1	1	1
$\nabla G^{1,5}$	1	1	1
G^1	1	1	$2/3$
DG^1	1	1	1
∇G^1	1	1	$2/3$

The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for $G^{1,i}$, $i = 1, \ldots, 5$ and G^1 and their spatial derivatives.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$G^{1,1}$	1	1	1	$(3\lambda+3)/(2\lambda+6)$
$\nabla G^{1,1}$	1	1	1	$(3\lambda+5)/(2\lambda+6)$
$G^{1,2}$	1	$(3\lambda+1)/(2\lambda+2)$	$(3\lambda+2)/(2\lambda+4)$	$(4\lambda+2)/(3\lambda+6)$
$DG^{1,2}$	1	1	1	$(5\lambda+4)/(3\lambda+6)$
$G^{1,3}$	1	1	1	$(3\lambda+3)/(2\lambda+6)$
$\nabla G^{1,3}$	1	1	1	$(3\lambda+5)/(2\lambda+6)$
$G^{1,4}$	1	1	1	$(4\lambda+6)/(3\lambda+9)$
$DG^{1,4}$	1	1	1	1
$G^{1,5}$	1	1	1	$5/6$
$\nabla G^{1,5}$	1	1	1	1
G^1	1	$(3\lambda+1)/(2\lambda+2)$	$(3\lambda+2)/(2\lambda+4)$	$(4\lambda+2)/(3\lambda+6)$
DG^1	1	1	1	$(5\lambda+4)/(3\lambda+6)$

Proof. The definitions of $G^{1,i}$ show that these terms are linear combinations of products of two or more terms that can be estimated in either L^∞ or H^0 by using Sobolev embeddings and Lemmas 3.1, 3.2, and 3.3. For the L^∞ table we estimate products using the usual algebra of L^∞: $\|XY\|_{L^\infty} \le \|X\|_{L^\infty}\|Y\|_{L^\infty}$. For the H^0 table, we estimate products with both

$$\|XY\|_0^2 \le \|X\|_0^2 \|Y\|_{L^\infty}^2 \quad \text{and} \quad \|XY\|_0^2 \le \|Y\|_0^2 \|X\|_{L^\infty}^2, \tag{3-8}$$

and then take the larger value of θ produced by these two bounds.

The interpolation powers recorded in the above tables have been determined using the full structure of the $G^{1,i}$, $i = 1, \dots, 5$, as defined in (2-24)–(2-31). However, for each $G^{1,i}$, $i = 1, \dots, 5$, it is possible to identify a "principal term" that has the same essential structure as the term in $G^{1,i}$ that determines the interpolation powers appearing in the tables. For the sake of clarity we record these principal terms now:

$$G^{1,1} \sim \bar{\eta}\nabla p, \quad G^{1,2} \sim u \cdot \nabla u, \quad G^{1,3} \sim \bar{\eta}\partial_3^2 u, \quad G^{1,4} \sim \partial_3\bar{\eta}\partial_3 u, \quad G^{1,5} \sim \tilde{b}\partial_t\bar{\eta}\partial_3 u. \qquad \square$$

Now we estimate G^2.

Lemma 3.5. *The following table encodes the power in the $L^\infty(\Omega)$ and $L^\infty(\Sigma)$ interpolation estimates for G^2 and its spatial derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^2	1	1	$(4\lambda+6)/(3\lambda+9)$
DG^2	1	1	1
∇G^2	1	1	$(3\lambda+5)/(2\lambda+6)$
$\nabla^2 G^2$	1	1	1

The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for G^2 and its spatial derivatives.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^2	1	$(3\lambda+2)/(2\lambda+4)$	$(4\lambda+3)/(3\lambda+9)$
DG^2	1	1	$(4\lambda+6)/(3\lambda+9)$
∇G^2	1	1	$(3\lambda+3)/(2\lambda+6)$
$\nabla^2 G^2$	1	1	$(3\lambda+5)/(2\lambda+6)$

Proof. The estimates may be derived as in Lemma 3.4, so we only record the principal term in G^2. For these estimates, $G^2 \sim \bar{\eta}\partial_3 u_3$. $\square$

Now we record G^3 estimates.

Lemma 3.6. *The following table encodes the power in the $L^\infty(\Sigma)$ interpolation estimates for G^3 and its spatial derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^3	1	1	$(4\lambda+6)/(3\lambda+9)$
DG^3	1	1	1
D^2G^3	1	1	1

The following table encodes the power in the $H^0(\Sigma)$ interpolation estimates for G^3 and its spatial derivatives.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^3	1	$(3\lambda+2)/(2\lambda+4)$	$(4\lambda+3)/(3\lambda+9)$
DG^3	1	1	$(4\lambda+6)/(3\lambda+9)$
D^2G^3	1	1	1

Proof. Recall that by Remark 2.4, we may remove the appearance of $(p-\eta)$ in G^3. This allows us to perform the estimates of G^3 terms as in Lemmas 3.4 and 3.5. The principal term may be identified as $G^3 \sim \eta\partial_3 u$. □

Now we record G^4 estimates.

Lemma 3.7. *The following table encodes the power in the $L^\infty(\Sigma)$ interpolation estimates for G^4 and its spatial derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^4	1	1	1
DG^4	1	1	1
D^2G^4	1	1	1

The following table encodes the power in the $H^0(\Sigma)$ interpolation estimates for G^4 and its spatial derivatives.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^4	1	1	$(3\lambda+5)/(2\lambda+6)$
DG^4	1	1	1
D^2G^4	1	1	1

Proof. The estimates again work as in Lemmas 3.4–3.6. In this case there is no need to identify the principal term, since $G^4 = -D\eta \cdot u$ is already in a simple form. □

Improved estimates for u, ∇p. Now we will use the structure of the equations (2-23) to improve our estimates for u, ∇p, etc. Our first estimate is for Dp. It constitutes an improvement of our existing L^∞ estimate, Lemma 3.3, as well as a first H^0 estimate.

Lemma 3.8. *The following table encodes the power in an $L^\infty(\Omega)$ interpolation estimate.*

	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
Dp	1	$1/(1+r)$	$(\lambda+2)/(\lambda+3)$

The following table encodes the power in an $H^0(\Omega)$ interpolation estimate.

	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
Dp	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$

Proof. In order to record the proof of both the H^0 and L^∞ estimates at the same time, we will generically write $\|\cdot\|$ to refer to either the $H^0(\Omega)$ or $L^\infty(\Omega)$ norm. Similarly, we will write $\|\cdot\|_\Sigma$ to refer to the $H^0(\Sigma)$ or $L^\infty(\Sigma)$ norm. The starting point is an application of Lemma A.10 to bound

$$\|Dp\|^2 \lesssim \|Dp\|_\Sigma^2 + \|\partial_3 Dp\|^2. \tag{3-9}$$

We will estimate both terms on the right-hand side in order to prove the lemma.

In order to estimate Dp on Σ we utilize the boundary conditions in (2-23) to write

$$\partial_i p = \partial_i \eta + 2\partial_i \partial_3 u_3 + \partial_i (G^3 \cdot e_3) \tag{3-10}$$

for $i = 1, 2$. From this we easily see that

$$\|Dp\|_\Sigma^2 \lesssim \|D\eta\|_\Sigma^2 + \|DG^3\|_\Sigma^2 + \|D\partial_3 u_3\|_\Sigma^2. \tag{3-11}$$

The first two terms may be estimated with Lemmas 3.1 and 3.6, but we must further exploit the structure of the equations in order to control the last term. For the H^0 estimate we use trace theory and the second equation in (2-23),

$$\partial_3 u_3 = G^2 - \partial_1 u_1 - \partial_2 u_2, \tag{3-12}$$

to see that

$$\|D\partial_3 u_3\|_{H^0(\Sigma)}^2 \lesssim \|D\partial_3 u_3\|_1^2 \lesssim \|DG^2\|_1^2 + \|D^2 u\|_1^2. \tag{3-13}$$

Since $D^2 u = 0$ on Σ_b, we may use Lemma A.13 to bound

$$\|D^2 u\|_1^2 \lesssim \|\nabla D^2 u\|_0^2, \tag{3-14}$$

so that, upon replacing in the previous inequality, we find

$$\|D\partial_3 u_3\|_{H^0(\Sigma)}^2 \lesssim \|DG^2\|_0^2 + \|D\nabla G^2\|_0^2 + \|D^2 \nabla u\|_0^2. \tag{3-15}$$

For the corresponding L^∞ estimate we again use (3-12) to bound

$$\|D\partial_3 u_3\|_{L^\infty(\Sigma)}^2 \lesssim \|DG^2\|_{L^\infty(\Sigma)}^2 + \|D^2 u\|_{L^\infty(\Sigma)}^2. \tag{3-16}$$

By Lemma A.13 we know that $\|D^2 u\|_{L^\infty(\Sigma)}^2 \lesssim \|\nabla D^2 u\|_{L^\infty(\Omega)}^2$. On the other hand, $DG^2 \in C^0(\bar{\Omega})$ (this may be verified using the Sobolev embeddings and Theorem 4.2), so that $\|DG^2\|_{L^\infty(\Sigma)}^2 \le \|DG^2\|_{L^\infty(\Omega)}^2$. We may then replace these to arrive at the bound

$$\|D\partial_3 u_3\|_{L^\infty(\Sigma)}^2 \lesssim \|DG^2\|_{L^\infty(\Omega)}^2 + \|\nabla D^2 u\|_{L^\infty(\Omega)}^2. \tag{3-17}$$

Then, from (3-15) and (3-17), we know that

$$\|D\partial_3 u_3\|_\Sigma^2 \lesssim \|DG^2\|^2 + \|D\nabla G^2\|^2 + \|D^2 \nabla u\|^2. \tag{3-18}$$

Combining (3-11) with (3-18) yields

$$\|Dp\|_{\Sigma}^2 \lesssim \|D\eta\|_{\Sigma}^2 + \|DG^3\|_{\Sigma}^2 + \|DG^2\|^2 + \|D\nabla G^2\|^2 + \|D^2\nabla u\|^2. \tag{3-19}$$

We may then employ Lemmas 3.1, 3.2, 3.3, 3.5, and 3.6 to derive the interpolation power for $\|Dp\|_{\Sigma}^2$; we record this power in the following table. Both the L^∞ and H^0 powers are determined by $D\eta$, but the L^∞ estimate only improves the result of Lemma 3.3 for $\mathfrak{D}_{N+2,2}$.

	$\mathscr{E}_{N+2,1}$	$\mathfrak{D}_{N+2,1} \sim \mathscr{E}_{N+2,2}$	$\mathfrak{D}_{N+2,2}$
$\|Dp\|_{L^\infty(\Sigma)}^2$	1	$1/(1+r)$	$(\lambda+2)/(\lambda+3)$
$\|Dp\|_{H^0(\Sigma)}^2$	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$

Now we will estimate the term $\|\partial_3 Dp\|^2$. For this we use (2-23) to write

$$\partial_i\partial_3 p = \partial_i[(\partial_1^2+\partial_2^2-\partial_t)u_3 + \partial_3^2 u_3 + G^1\cdot e_3] \tag{3-20}$$

for $i=1,2$. Again using (3-12), we may write

$$\partial_i\partial_3^2 u_3 = \partial_i\partial_3(G^2 - \partial_1 u_1 - \partial_2 u_2). \tag{3-21}$$

Combining these two equations then shows that

$$\|D\partial_3 p\|^2 \lesssim \|D^3 u\|^2 + \|D^2\nabla u\|^2 + \|D\partial_t u\|^2 + \|DG^1\|^2 + \|D\nabla G^2\|^2. \tag{3-22}$$

We may then employ Lemmas 3.2, 3.3, 3.4, and 3.5 to derive the interpolation power for $\|D\partial_3 p\|^2$; we record this power in the following table. The H^0 powers are determined by DG^1, but note that the L^∞ estimate does not improve the result of Lemma 3.3.

	$\mathscr{E}_{N+2,1}$	$\mathfrak{D}_{N+2,1} \sim \mathscr{E}_{N+2,2}$	$\mathfrak{D}_{N+2,2}$
$\|D\partial_3 p\|_{L^\infty}^2$	1	1	$1/(1+r)$
$\|D\partial_3 p\|_0^2$	1	1	$(5\lambda+4)/(3\lambda+6)$

Now we return to (3-9) and employ our estimates of $\|Dp\|_{\Sigma}^2$ and $\|D\partial_3 p\|^2$ to deduce the desired interpolation powers for $\|Dp\|^2$. Notice that we may also combine (3-9) with (3-19) and (3-22) for the estimate

$$\|Dp\|^2 \lesssim \|D\eta\|_{\Sigma}^2 + \|D\partial_t u\|^2 + \|D^3 u\|^2 + \|D^2\nabla u\|^2 + \|DG^1\|^2 + \|DG^2\|^2 + \|D\nabla G^2\|^2 + \|DG^3\|_{\Sigma}^2. \tag{3-23}$$

This concludes the proof. □

With this lemma in hand, we can now derive improved estimates for u.

Proposition 3.9. *The following table encodes the improved power in the $L^\infty(\Omega)$ interpolation estimate for u and its derivatives.*

	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1} \sim \mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
u	1	$1/(1+r)$	$2/3$
$\partial_3 u_i,\ i=1,2$	1	$1/(1+r)$	$2/3$
$\partial_3 u_3$	1	$2/(2+r)$	$2/3$
∇u	1	$1/(1+r)$	$2/3$
$\nabla^2 u$	1	$1/(1+r)$	$2/3$

The following table encodes the power in the $H^0(\Omega)$ interpolation estimate for u and its derivatives.

	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
u	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$
$\partial_3 u_i,\ i=1,2$	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$
$\partial_3 u_3$	1	$(3\lambda+2)/(2\lambda+4)$	$(3\lambda+2)/(2\lambda+4)$	$(4\lambda+3)/(3\lambda+9)$
Du	1	1	$(2\lambda+3)/(2\lambda+4)$	$(\lambda+2)/(\lambda+3)$
∇u	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$
$D\nabla u$	1	1	$(2\lambda+3)/(2\lambda+4)$	$(\lambda+2)/(\lambda+3)$
$D\partial_3 u_3$	1	1	1	$(4\lambda+6)/(3\lambda+9)$
$\nabla\partial_3 u_3$	1	1	$(2\lambda+3)/(2\lambda+4)$	$(3\lambda+3)/(2\lambda+6)$
$\nabla^2 u$	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$

The following table encodes the improved power in the $L^\infty(\Omega)$ interpolation estimate for ∇p.

	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1} \sim \mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
∇p	1	$2/(2+r)$	$2/3$

The following table encodes the power in the $H^0(\Omega)$ interpolation estimate for derivatives of p.

	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
$\partial_3 p$	1	$(3\lambda+1)/(2\lambda+2)$	$(3\lambda+2)/(2\lambda+4)$	$(4\lambda+2)/(3\lambda+6)$
∇p	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$

Proof. As in Lemma 3.8 we will write $\|\cdot\|$ and $\|\cdot\|_\Sigma$ to refer to both the H^0 and L^∞ norms on Ω and Σ, respectively. We divide the proof into several steps, beginning with estimates of ∇u. With these established, we can extend to estimates of u, $D\nabla u$, Du, $D\partial_3 u_3$, and $\nabla\partial_3 u_3$ by employing Poincaré's inequality and interpolation. This in turn leads to estimates for $\partial_3 p$ and $\nabla^2 u$.

Step 1: Estimates of ∇u. To begin the ∇u estimates, we split the components of ∇u into those involving x_1, x_2 derivatives and those involving x_3 derivatives. Indeed, we have

$$\|\nabla u\|^2 \lesssim \|Du\|^2 + \|\partial_3 u_3\|^2 + \sum_{i=1}^{2} \|\partial_3 u_i\|^2. \tag{3-24}$$

Lemma 3.2 provides an estimate of Du, but not of $\partial_3 u$, so we must use the structure of the equations (2-23) to estimate the latter two terms.

To estimate $\partial_3 u_3$ we use the second equation in (2-23) to bound

$$\|\partial_3 u_3\|^2 \lesssim \|G^2\|^2 + \|Du\|^2. \tag{3-25}$$

Then Lemmas 3.2 and 3.5 provide interpolation estimates of G^2 and Du and hence the estimates of $\partial_3 u_3$ listed in the tables. The Du term determines the power for L^∞, while the power is determined by G^2 for H^0.

To estimate $\partial_3 u_i$ for $i = 1, 2$, we first apply Lemma A.10 to get

$$\|\partial_3 u_i\|^2 \lesssim \|\partial_3 u_i\|_\Sigma^2 + \|\partial_3^2 u_i\|^2. \tag{3-26}$$

For the first term on the right, we use the third equation in (2-23) to bound

$$\|\partial_3 u_i\|_\Sigma^2 \lesssim \|Du_3\|_\Sigma^2 + \|G^3\|_\Sigma^2. \tag{3-27}$$

Since $Du = 0$ on Σ_b, we can use trace theory, Lemma A.13, and the equation $\operatorname{div} u = G^2$ for

$$\|Du_3\|_\Sigma^2 \lesssim \|\nabla Du_3\|^2 \lesssim \|D^2 u\|^2 + \|DG^2\|^2. \tag{3-28}$$

For the second term on the right side of (3-26), we use (2-23) to bound

$$\|\partial_3^2 u_i\|^2 \lesssim \|\partial_t u\|^2 + \|D^2 u\|^2 + \|Dp\|^2 + \|G^1\|^2. \tag{3-29}$$

We may then combine estimates (3-26)–(3-29) to deduce that

$$\|\partial_3 u_i\|^2 \lesssim \|\partial_t u\|^2 + \|D^2 u\|^2 + \|Dp\|^2 + \|G^1\|^2 + \|DG^2\|^2 + \|G^3\|_\Sigma^2. \tag{3-30}$$

Now we use Lemmas 3.2, 3.4–3.6, and 3.8 to find the interpolation powers for $\partial_3 u_i$, $i = 1, 2$, listed in the tables. For L^∞ the power is determined by Dp for $\mathcal{E}_{N+2,1}$, $\mathcal{E}_{N+2,2}$, and $\mathcal{D}_{N+2,1}$ and by G^1 for $\mathcal{D}_{N+2,2}$, while for H^0 the power is determined by Dp.

With estimates for Du, $\partial_3 u_3$, and $\partial_3 u_i$ for $i = 1, 2$ in hand, we return to (3-24) to derive the estimates for ∇u listed in the tables. For both the L^∞ and H^0 estimates the power is determined by $\partial_3 u_i$, $i = 1, 2$.

Step 2: Extensions to estimates of u, $D\nabla u$, $D\partial_3 u_3$, and $\nabla\partial_3 u_3$. Now we apply Lemma A.13 to control u in terms of ∇u:

$$\|u\|^2 \lesssim \|\nabla u\|^2. \tag{3-31}$$

Our estimates for ∇u then provide the estimates for u listed in the tables.

We now turn to $D\nabla u$. Clearly $\|D\nabla u\|_0^2$ is controlled by both $\mathcal{E}_{N+2,1}$ and $\mathcal{D}_{N+2,1}$, which yields the powers of 1 in the tables. An application of (A-38) from the Appendix with $\lambda = 0$, $q = 1$, and $s = 1$ shows that

$$\|D\nabla u\|_0^2 \lesssim (\|\nabla u\|_0^2)^{1/2} (\|D^2 \nabla u\|_0^2)^{1/2}. \tag{3-32}$$

We employ this in conjunction with our estimate for ∇u and the estimate of $D^2\nabla u$ from Lemma 3.2 to get the interpolation powers for $D\nabla u$ listed in the tables for $\mathscr{E}_{N+2,2}$ and $\mathscr{D}_{N+2,2}$. The estimates for Du listed in the tables follow immediately from the estimates for $D\nabla u$ via Poincaré:

$$\|Du\|^2 \lesssim \|D\nabla u\|^2. \tag{3-33}$$

In order to estimate $D\partial_3 u_3$ and $\nabla\partial_3 u_3$ in H^0 we use that $\operatorname{div} u = G^2$ for

$$\|\nabla\partial_3 u_3\|_0^2 \lesssim \|\nabla G^2\|_0^2 + \|D\nabla u\|_0^2, \tag{3-34}$$

$$\|D\partial_3 u_3\|_0^2 \lesssim \|DG^2\|_0^2 + \|D^2 u\|_0^2. \tag{3-35}$$

Then our estimate for $D\nabla u$ and Lemmas 3.2 and 3.5 yield the estimates listed in the tables. For $\nabla\partial_3 u_3$ the power is determined by $D\nabla u$ for $\mathscr{E}_{N+2,1}$, $\mathscr{D}_{N+2,1}$, $\mathscr{E}_{N+2,2}$ and by ∇G^2 for $\mathscr{D}_{N+2,2}$. For $D\partial_3 u_3$ the power is determined by DG^2.

Step 3: Estimates of $\partial_3 p$ and ∇p. Lemma 3.8 provides estimates for Dp, so to complete an estimate for ∇p we only need to consider $\partial_3 p$. For this we again use (2-23) to bound

$$\|\partial_3 p\|^2 \lesssim \|\partial_3^2 u_3\|^2 + \|D^2 u\|^2 + \|\partial_t u\|^2 + \|G^1\|^2. \tag{3-36}$$

This and (3-34) then imply that

$$\|\partial_3 p\|^2 \lesssim \|D\nabla u\|^2 + \|D^2 u\|^2 + \|\partial_t u\|^2 + \|G^1\|^2 + \|\nabla G^2\|^2, \tag{3-37}$$

and we may use Lemmas 3.2, 3.4, and 3.5 along with our new $D\nabla u$ estimate to determine the powers in the tables for $\partial_3 p$. In the L^∞ estimate the power is determined by $D\nabla u$, and in the H^0 estimate the power is determined by G^1. Then the estimates for ∇p follow by comparing the Dp estimates of Lemma 3.8 to the $\partial_3 p$ estimates.

Step 4: Estimates of $\nabla^2 u$. Finally we consider $\nabla^2 u$, which we decompose according to x_1, x_2, and x_3 derivatives:

$$\|\nabla^2 u\|^2 \lesssim \|D^2 u\|^2 + \|D\nabla u\|^2 + \|\partial_3^2 u_3\|^2 + \sum_{i=1}^{2} \|\partial_3^2 u_i\|^2. \tag{3-38}$$

According to our bounds (3-29) and (3-34), we may replace this with

$$\|\nabla^2 u\|^2 \lesssim \|\partial_t u\|^2 + \|D^2 u\|^2 + \|D\nabla u\|^2 + \|Dp\|^2 + \|G^1\|^2 + \|\nabla G^2\|^2. \tag{3-39}$$

Then Lemmas 3.2, 3.4, 3.5, and 3.8 with our new estimate of $D\nabla u$ provide the estimates in the table for $\nabla^2 u$. For L^∞ the power is determined by Dp for $\mathscr{E}_{N+2,1}$, $\mathscr{E}_{N+2,2}$, and $\mathscr{D}_{N+2,1}$ and by G^1 for $\mathscr{D}_{N+2,2}$, while for H^0 it is determined by Dp. □

Bootstrapping: first iteration. We now use the improved estimates of Lemma 3.8 and Proposition 3.9 to improve the estimates of G^i, $i = 1, \ldots, 4$, recorded in Lemmas 3.4–3.7. We will only record the improvements for the $H^0(\Omega)$ estimates.

Lemma 3.10. *The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for $G^{1,i}$, $i = 1, \dots, 5$, and G^1 and their spatial derivatives.*

X	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
$G^{1,1}$	1	1	1	$(5\lambda+6)/(3\lambda+9)$
$\nabla G^{1,1}$	1	1	1	1
$G^{1,2}$	1	1	1	1
$\nabla G^{1,2}$	1	1	1	1
$G^{1,3}$	1	1	1	$(5\lambda+6)/(3\lambda+9)$
$\nabla G^{1,3}$	1	1	1	1
$G^{1,4}$	1	1	1	1
$\nabla G^{1,4}$	1	1	1	1
$G^{1,5}$	1	1	1	1
$\nabla G^{1,5}$	1	1	1	1
G^1	1	1	1	$(5\lambda+6)/(3\lambda+9)$
∇G^1	1	1	1	1

Proof. We perform the estimates as in Lemma 3.4, except that now we use the improved interpolation estimates of Lemma 3.8 and Proposition 3.9. □

We now record the G^2 estimates.

Lemma 3.11. *The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for G^2 and its spatial derivatives.*

X	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
G^2	1	1	1	$(7\lambda+6)/(3\lambda+9)$
DG^2	1	1	1	1
∇G^2	1	1	1	$(5\lambda+5)/(2\lambda+6)$
$\nabla^2 G^2$	1	1	1	1

Proof. We perform the estimates as in Lemma 3.5, except that now we use the improved interpolation estimates of Proposition 3.9, in particular the distinct estimates for $\partial_3 u_3$ and $\partial_3 u_i$, $i = 1, 2$. These are crucial since in G^2 the term $\partial_3 u_i$ is multiplied by a derivative of $\bar{\eta}$ but $\partial_3 u_3$ is multiplied by $\bar{\eta}$ itself. This means that for the present interpolation estimates we may identify the principal term in G^2 as $G^2 \sim \bar{\eta}\partial_3 u_3 + \partial_1\bar{\eta}\partial_3 u_1 + \partial_2\bar{\eta}\partial_3 u_2$. □

We now record the G^3 estimates. We omit the proof since it follows that of Lemma 3.6, using the improved estimates of Lemma 3.8 and Proposition 3.9.

Lemma 3.12. *The following table encodes the power in the $H^0(\Sigma)$ interpolation estimates for G^3 and its spatial derivatives.*

X	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
G^3	1	1	1	$(5\lambda+6)/(3\lambda+9)$
DG^3	1	1	1	$(5\lambda+6)/(3\lambda+9)$
D^2G^3	1	1	1	1

We now record the G^4 estimates. We again omit the proof.

Lemma 3.13. *The following table encodes the power in the $H^0(\Sigma)$ interpolation estimates for G^4 and its spatial derivatives.*

X	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
G^4	1	1	1	1
DG^4	1	1	1	1
D^2G^4	1	1	1	1

The improved estimates for G^i, $i = 1, \dots, 4$, allow us to improve the H^0 estimates of Proposition 3.9.

Theorem 3.14. *The following table encodes the power in the $H^0(\Omega)$ interpolation estimate for u and its derivatives.*

	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
u	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$
$\partial_3 u_3$	1	1	$(2\lambda+3)/(2\lambda+4)$	$(\lambda+2)/(\lambda+3)$
Du	1	1	$(2\lambda+3)/(2\lambda+4)$	$(\lambda+2)/(\lambda+3)$
∇u	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$
$D\nabla u$	1	1	$(2\lambda+3)/(2\lambda+4)$	$(\lambda+2)/(\lambda+3)$
$\nabla\partial_3 u_3$	1	1	$(2\lambda+3)/(2\lambda+4)$	$(\lambda+2)/(\lambda+3)$
$\nabla^2 u$	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$

The following table encodes the power in the $H^0(\Omega)$ interpolation estimate for derivatives of p.

	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
$\partial_3 p$	1	1	$(2\lambda+3)/(2\lambda+4)$	$(\lambda+2)/(\lambda+3)$
∇p	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$

Proof. The powers are the same as those listed in Proposition 3.9 except for $\partial_3 u_3$, $\nabla\partial_3 u_3$, and $\partial_3 p$.

To arrive at the $\partial_3 p$ estimates, we again employ the estimate (3-37) of Proposition 3.9, except that now we use Lemmas 3.10 and 3.11 for estimates of G^1 and ∇G^2 and Proposition 3.9 for the estimate of $D\nabla u$. The terms $\partial_t u$ and $D^2 u$ are still estimated with Lemma 3.2. The power in the $\partial_3 p$ estimate is determined by $D\nabla u$.

For the $\partial_3 u_3$ terms, we employ the equation $\operatorname{div} u = G^2$ to bound

$$\|\partial_3 u_3\|^2 \lesssim \|G^2\|^2 + \|Du\|^2 \quad \text{and} \quad \|\nabla\partial_3 u_3\|^2 \lesssim \|\nabla G^2\|^2 + \|D\nabla u\|^2. \tag{3-40}$$

The estimates of $\partial_3 u_3$ and $\nabla\partial_3 u_3$ in the table follow from these bounds and Lemmas 3.9 and 3.11, with the power of the former determined by Du and that of the latter determined by $D\nabla u$. ☐

Bootstrapping: second iteration. We now use the improved estimates of Theorem 3.14 to improve the estimates of G^i, $i = 1, 2$, recorded in Lemmas 3.10–3.11. We once again omit the proof.

Theorem 3.15. *The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for $G^{1,i}$, $i = 1, \dots, 5$, and G^1 and their spatial derivatives.*

X	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
$G^{1,1}$	1	1	1	$(2\lambda+2)/(\lambda+3)$
$\nabla G^{1,1}$, $\nabla^2 G^{1,1}$	1	1	1	1
$G^{1,2}$, $\nabla G^{1,2}$, $\nabla^2 G^{1,2}$	1	1	1	1
$G^{1,3}$	1	1	1	$(2\lambda+2)/(\lambda+3)$
$\nabla G^{1,3}$, $\nabla^2 G^3$	1	1	1	1
$G^{1,4}$, $\nabla G^{1,4}$, $\nabla^2 G^{1,4}$	1	1	1	1
$G^{1,5}$, $\nabla G^{1,5}$, $\nabla^2 G^{1,5}$	1	1	1	1
G^1	1	1	1	$(2\lambda+2)/(\lambda+3)$
∇G^1, $\nabla^2 G^1$	1	1	1	1

The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for G^2 and its spatial derivatives.

X	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
G^2, ∇G^2, $\nabla^2 G^2$	1	1	1	1

Now we make final improvements to our estimates.

Proposition 3.16. *The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for $D\partial_3 u_i$ for $i = 1, 2$.*

X	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
$D\partial_3 u_i$, $i = 1, 2$	1	1	1	$(\lambda+2)/(\lambda+3)$

The following table encodes the power in an $H^2(\Sigma)$ estimates for Du_i for $i = 1, 2$.

X	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
Du_i, $i = 1, 2$	1	1	1	$(\lambda+2)/(\lambda+3)$

The following table encodes the power in the improved $H^0(\Sigma)$ interpolation estimates for $\partial_t \eta$.

X	$\mathscr{E}_{N+2,1}$	$\mathscr{D}_{N+2,1}$	$\mathscr{E}_{N+2,2}$	$\mathscr{D}_{N+2,2}$
$\partial_t \eta$	1	1	1	$(\lambda+2)/(\lambda+3)$

Proof. We may argue as in the derivation of (3-23) of Lemma 3.8 to bound

$$\|D^2 p\|^2 \lesssim \|D^2\eta\|^2+\|D^2\partial_t u\|^2+\|D^4 u\|^2+\|D^3\nabla u\|^2+\|D^2 G^1\|^2+\|D^2 G^2\|^2+\|D^2\nabla G^2\|^2+\|D^2 G^3\|_{\Sigma}^2. \quad (3\text{-}41)$$

We may also argue as in the derivation of (3-30) of Proposition 3.9 to bound

$$\|D\partial_3 u_i\|^2 \lesssim \|D\partial_t u\|^2+\|D^3 u\|^2+\|D^2 p\|^2+\|DG^1\|^2+\|DG^2\|^2+\|DG^3\|_{\Sigma}^2 \quad (3\text{-}42)$$

for $i=1,2$. Combining (3-41) and (3-42) and employing Theorems 3.14 and 3.15 and Lemmas 3.12 and 3.13, we then find the $H^0(\Omega)$ estimates for $D\partial_3 u_i$, $i=1,2$, listed in the table. The power is determined by $D^2\eta$.

We now turn to the $\|Du_i\|_{H^2(\Sigma)}^2$ estimate for $i=1,2$. We employ trace theory and the Poincaré inequality to bound

$$\|Du_i\|_{H^0(\Sigma)}^2 \lesssim \|D\partial_3 u_i\|_0^2 \quad\text{and}\quad \|D^3 u_i\|_{H^0(\Sigma)}^2 \lesssim \|D^3\partial_3 u_i\|_0^2, \quad (3\text{-}43)$$

and then we utilize our new estimate for $D\partial_3 u_i$ to deduce the $H^2(\Sigma)$ estimates listed in the table. The power is determined by $D\partial_3 u_i$ since $D^3\partial_3 u_i$ has four derivatives and hence has a power of 1.

Finally, for the $\partial_t\eta$ estimate we use (2-23), trace theory, and Lemma A.13 to bound

$$\|\partial_t\eta\|_{H^0(\Sigma)}^2 \lesssim \|u_3\|_{H^0(\Sigma)}^2+\|G^4\|_{H^0(\Sigma)}^2 \lesssim \|\nabla u_3\|_0^2+\|G^4\|_{H^0(\Sigma)}^2. \quad (3\text{-}44)$$

Then Theorem 3.14 and Lemma 3.13 provide the $\partial_t\eta$ estimate for $\mathcal{D}_{N+2,2}$ listed in the table, with the power determined by ∇u_3; the estimates for $\mathcal{E}_{N+2,1}$, $\mathcal{E}_{N+2,2}$, $\mathcal{D}_{N+2,1}$ come from Lemma 3.1. □

Now we record an interpolation estimate for $\mathcal{K}$, as defined by (2-57).

Lemma 3.17. *We have* $\mathcal{K} \lesssim \mathcal{E}_{N+2,2}^{(8+2\lambda)/(8+4\lambda)}$.

Proof. By definition, $\mathcal{K}=\|\nabla u\|_{L^\infty}^2+\|\nabla^2 u\|_{L^\infty}^2+\sum_{i=1}^2\|Du_i\|_{H^2(\Sigma)}^2$. We may now use the $H^2(\Sigma)$ interpolation estimate of Proposition 3.16 and the L^∞ interpolation estimate of Proposition 3.9 with $r=2\lambda/(4+\lambda)$ to bound $\mathcal{K}\lesssim\mathcal{E}_{N+2,2}^{2/(2+r)}$. The choice of r implies that $2/(2+r)=(8+2\lambda)/(8+4\lambda)$, and the result follows. □

Estimates at the high end. Our analysis so far in Section 3 has dealt with the problems associated with estimating terms involving fewer derivatives than appear in $\mathcal{E}_{N+2,m}$, $\mathcal{D}_{N+2,m}$. We now turn to the problem of estimating terms involving more derivatives than are controlled by $\mathcal{D}_{N+2,m}$. We accomplish such an estimate by interpolating between $\mathcal{D}_{N+2,m}$ and $\mathcal{E}_{2N}$, which controls more derivatives since $N\geq 5$. Fortunately, the only term we must concern ourselves with is $D^{2N+4}\eta$, and to simplify things we will only estimate it in terms of $\mathcal{D}_{N+2,2}$. This suffices since $\mathcal{D}_{N+2,2}\lesssim\mathcal{D}_{N+2,1}$.

Lemma 3.18. *We have the estimate*

$$\|D^{2N+4}\eta\|_{1/2}^2+\|\nabla^{2N+5}\bar\eta\|_0^2 \lesssim (\mathcal{E}_{2N})^{2/(4N-7)}(\mathcal{D}_{N+2,2})^{(4N-9)/(4N-7)}. \quad (3\text{-}45)$$

Proof. According to Lemma A.5, with $q = 2N+5$, we may bound

$$\|\nabla^{2N+5}\bar{\eta}\|_0^2 \lesssim \|\eta\|^2_{\dot{H}^{2N+9/2}(\Sigma)} \lesssim \|D^{2N+4}\eta\|^2_{1/2}, \tag{3-46}$$

so it suffices to prove (3-45) with only the $D^{2N+4}\eta$ term on the left side. To prove this, we will use a standard Sobolev interpolation inequality:

$$\|f\|_s \lesssim \|f\|_{s-r}^{q/(r+q)} \|f\|_{s+q}^{r/(r+q)} \tag{3-47}$$

for $s, q > 0$ and $0 \le r \le s$. Applying this to $f = D^3\eta$ with $s = 2N+3/2$, $r = 1$, and $q = 2N-9/2$, we find that

$$\|D^{2N+4}\eta\|_{1/2} \le \|D^3\eta\|_{2N+3/2} \lesssim \|D^3\eta\|_{2N+1/2}^{(4N-9)/(4N-7)} \|D^3\eta\|_{4N-3}^{2/(4N-7)}. \tag{3-48}$$

The desired inequality then follows by squaring and using the definitions of $\mathscr{E}_{2N}$ and $\mathscr{D}_{N+2,2}$. □

Our next result utilizes Lemma 3.18 to estimate products such as $uD^{2N+4}\eta$.

Lemma 3.19. *Let $P = P(K, \eta, D\eta)$ be a polynomial in $K, \eta, D\eta$. Then there exists a $\theta > 0$ such that*

$$\|(D^{2N+4}\eta)u\|^2_{H^{1/2}(\Sigma)} + \|(D^{2N+4}\eta)P\nabla u\|^2_{H^{1/2}(\Sigma)} \lesssim \mathscr{E}_{2N}^{\theta}\mathscr{D}_{N+2,2}. \tag{3-49}$$

Let $Q = Q(K, \tilde{b}, \bar{\eta}, \nabla\bar{\eta})$ be a polynomial in $K, \tilde{b}, \bar{\eta}, \nabla\bar{\eta}$. Then there exists a $\theta > 0$ such that

$$\|(\nabla^{2N+5}\bar{\eta})Q\nabla u\|_0^2 \lesssim \mathscr{E}_{2N}^{\theta}\mathscr{D}_{N+2,2}. \tag{3-50}$$

Proof. According to the bound (A-2) of Lemma A.1, we may bound

$$\begin{aligned}\|(D^{2N+4}\eta)u\|^2_{H^{1/2}(\Sigma)} &+ \|(D^{2N+4}\eta)P\nabla u\|^2_{H^{1/2}(\Sigma)} \\ &\lesssim \|D^{2N+4}\eta\|^2_{H^{1/2}(\Sigma)}\|u\|^2_{H^2(\Sigma)} + \|D^{2N+4}\eta\|^2_{H^{1/2}(\Sigma)}\|P\nabla u\|^2_{H^2(\Sigma)}.\end{aligned} \tag{3-51}$$

Trace theory and Lemma A.13 (both u and D^2u vanish on Σ_b) imply that

$$\begin{aligned}\|u\|^2_{H^2(\Sigma)} + \|\nabla u\|^2_{H^2(\Sigma)} &\lesssim \|u\|^2_{H^0(\Sigma)} + \|D^2u\|^2_{H^0(\Sigma)} + \|\nabla u\|^2_{H^0(\Sigma)} + \|D^2\nabla u\|^2_{H^0(\Sigma)} \\ &\lesssim \|\nabla u\|_0^2 + \|D^2\nabla u\|_0^2 + \|\nabla^2 u\|_0^2 + \|\nabla^2 D^2 u\|_0^2,\end{aligned} \tag{3-52}$$

but then an application of Theorem 3.14 to all the terms on the right side shows that

$$\|u\|^2_{H^2(\Sigma)} + \|\nabla u\|^2_{H^2(\Sigma)} \lesssim (\mathscr{D}_{N+2,2})^{(1+\lambda)/(3+\lambda)}. \tag{3-53}$$

It is easy to see, based on the terms controlled by $\mathscr{E}_{2N}$ and the Sobolev embeddings, that $\|P\|^2_{C^2(\Sigma)} \lesssim 1 + \mathscr{E}_{2N} \lesssim 1$. We may then combine this with (3-53) and the easy bound $\|fg\|^2_{H^2(\Sigma)} \lesssim \|f\|^2_{H^2(\Sigma)}\|g\|^2_{C^2(\Sigma)}$ to deduce that

$$\|u\|^2_{H^2(\Sigma)} + \|P\nabla u\|^2_{H^2(\Sigma)} \lesssim \|u\|^2_{H^2(\Sigma)} + \|\nabla u\|^2_{H^2(\Sigma)} \lesssim (\mathscr{D}_{N+2,2})^{(1+\lambda)/(3+\lambda)}. \tag{3-54}$$

Then this bound, (3-51), and Lemma 3.18 imply that

$$\|(D^{2N+4}\eta)u\|^2_{H^{1/2}(\Sigma)} + \|(D^{2N+4}\eta)P\nabla u\|^2_{H^{1/2}(\Sigma)} \lesssim \mathscr{E}_{2N}^{\theta}\mathscr{D}_{N+2,2}^{\kappa} \tag{3-55}$$

for some $\theta > 0$ and for

$$\kappa = \frac{4N-9}{4N-7} + \frac{\lambda+1}{\lambda+3} \geq \frac{4N-9}{4N-7} + \frac{1}{3} = \frac{16N-34}{12N-21} \geq 1, \tag{3-56}$$

since $N \geq 4$. Since $\mathfrak{D}_{N+2,2} \lesssim \mathcal{E}_{2N} \leq 1$, we may bound $\mathfrak{D}^{\kappa}_{N+2,2} \lesssim \mathfrak{D}_{N+2,2}$ in (3-55), which then yields (3-49).

To derive (3-50), we first bound

$$\|(\nabla^{2N+5}\bar{\eta}) Q \nabla u\|_0^2 \leq \|\nabla^{2N+5}\bar{\eta}\|_0^2 \|\nabla u\|_{L^\infty}^2 \|Q\|_{L^\infty}^2. \tag{3-57}$$

The first term on the right is controlled with Lemma 3.18. The second term satisfies

$$\|\nabla u\|_{L^\infty}^2 \lesssim (\mathfrak{D}_{N+2,2})^{2/3} \tag{3-58}$$

by virtue of the L^∞ estimates of Proposition 3.9. The third term satisfies $\|Q\|_{L^\infty}^2 \lesssim 1 + \mathcal{E}_{2N} \lesssim 1$ by Sobolev embeddings and the definition of $\mathcal{E}_{2N}$. The estimate (3-50) follows by combining these bounds as above. □

4. Nonlinear estimates

Estimates of G^i at the $N+2$ level. We now provide estimates of G^i, defined by (2-24)–(2-31), in terms of $\mathcal{E}_{N+2,m}$ and $\mathfrak{D}_{N+2,m}$. Recall that, for sums of space-time derivatives, we use the notation $\bar{D}^k_m$ and $\bar{\nabla}^k_m$, as described on page 1443.

Theorem 4.1. *Let $m \in \{1, 2\}$. Then there exists a $\theta > 0$ such that*

$$\|\bar{\nabla}_m^{2(N+2)-2} G^1\|_0^2 + \|\bar{\nabla}_0^{2(N+2)-2} G^2\|_1^2 + \|\bar{D}_m^{2(N+2)-2} G^3\|_{1/2}^2 + \|\bar{D}_0^{2(N+2)-2} G^4\|_{1/2}^2 \lesssim \mathcal{E}_{2N}^{\theta} \mathcal{E}_{N+2,m} \tag{4-1}$$

and

$$\|\bar{\nabla}_m^{2(N+2)-1} G^1\|_0^2 + \|\bar{\nabla}_0^{2(N+2)-1} G^2\|_1^2 + \|\bar{D}_m^{2(N+2)-1} G^3\|_{1/2}^2 + \|\bar{D}_0^{2(N+2)-1} G^4\|_{1/2}^2 + \|\bar{D}^{2(N+2)-2} \partial_t G^4\|_{1/2}^2 \lesssim \mathcal{E}_{2N}^{\theta} \mathfrak{D}_{N+2,m}. \tag{4-2}$$

Proof. The estimates of these nonlinearities are fairly routine to derive: we note that all terms are quadratic or of higher order; then we apply the differential operator and expand using the Leibniz rule; each term in the resulting sum is also at least quadratic, and we estimate one term in H^k ($k = 0$, $1/2$, or 1 depending on G^i) and the other term in L^∞ or H^m for m depending on k, using Sobolev embeddings, trace theory, and Lemmas A.1 and A.5–A.8. The derivative count in the differential operators is chosen in order to allow estimation by $\mathcal{E}_{N+2,m}$ in (4-1) and by $\mathfrak{D}_{N+2,m}$ in (4-2). There is only one difficulty that arises. Because $\mathcal{E}_{N+2,m}$ and $\mathfrak{D}_{N+2,m}$ involve minimal derivative counts, there may be terms in the sum $\partial^\alpha G^i$ that cannot be directly estimated. To handle these terms, we invoke the interpolation results of Theorems 3.14 and 3.16 and Proposition 3.9, as well as the specialized interpolation results of Lemma 3.19. A detailed proof of the estimates is quite lengthy, so for the sake of brevity we present only a sketch.

Let $\alpha \in \mathbb{N}^{1+3}$ with $m \leq |\alpha| \leq 2(N+2) - 2$ and consider $\partial^\alpha G^1$. Since G^1 involves ∇p and $\partial^\beta u$, $\partial^\beta \bar{\eta}$ with $|\beta| \leq 2$, we find that $\partial^\alpha G^1$ involves at most (with parabolic counting) $2(N+2) - 1$ derivatives of p, and at most $2(N+2)$ derivatives of u and $\bar{\eta}$. We have that G^1 is a linear combination of at least quadratic terms, and as such, so is $\partial^\alpha G^1$. Let us consider a generic term in the sum $\partial^\alpha G^1$, which we write as XY with X of the form $\partial^\beta u$ or $\partial^\beta \bar{\eta}$ with $|\beta| \le 2(N+2)$ or else $\partial^\beta p$ with $|\beta| \le 2(N+2)-1$, and Y a polynomial in lower-order derivatives. If $|\beta|$ is sufficiently large with respect to m, the minimal derivative count is exceeded and we may estimate $\|X\|_0^2 \lesssim \mathscr{E}_{N+2,m}$. It is easy to verify, using Sobolev embeddings and Lemmas A.1 and A.5–A.8, that we always have $\|Y\|_{L^\infty}^2 \lesssim \mathscr{E}_{2N}^\theta$ for some $\theta > 0$. Then

$$\|XY\|_0^2 \le \|X\|_0^2 \|Y\|_{L^\infty}^2 \lesssim \mathscr{E}_{N+2,m}\mathscr{E}_{2N}^\theta. \tag{4-3}$$

On the other hand, if $|\beta|$ is not large, we must resort to interpolation, using Theorems 3.14 and 3.16 and Proposition 3.9. In this case, it can be verified that we always get estimates of the form $\|X\|_0^2 \lesssim (\mathscr{E}_{2N})^{1-\theta_1}(\mathscr{E}_{N+2,m})^{\theta_1}$ and $\|Y\|_{L^\infty}^2 \lesssim (\mathscr{E}_{2N})^{\theta_2}(\mathscr{E}_{N+2,m})^{\theta_3}$ with $\theta_1 \in (0,1]$, $\theta_2, \theta_3 \ge 0$, and $\theta_1 + \theta_3 \ge 1$, so that

$$\|XY\|_0^2 \le \|X\|_0^2 \|Y\|_{L^\infty}^2 \lesssim \mathscr{E}_{N+2,m}\mathscr{E}_{2N}^\theta \tag{4-4}$$

for some $\theta > 0$. This analysis works for every XY appearing in $\partial^\alpha G^1$, so

$$\|\overline{\nabla}_m^{2(N+2)-2} G^1\|_0^2 \lesssim \mathscr{E}_{N+2,m}\mathscr{E}_{2N}^\theta \tag{4-5}$$

for some $\theta > 0$. It can then be verified, through a straightforward but lengthy analysis like that used above, that all of the estimates in (4-1) hold. We note, though, that in order to estimate the G^3 terms, we must use Remark 2.4 to remove the appearance of $(p-\eta)$ in G^3.

Now we sketch the proof of the estimates in (4-2). We may argue as above to estimate all terms that arise in $\partial^\alpha G^i$ with two exceptions: terms involving $\nabla^{2N+5}\bar{\eta}$ on Ω or $D^{2N+4}\eta$ on Σ. These always have the form of the terms estimated in Lemma 3.19, so we may use that lemma for estimates in terms of $\mathscr{E}_{2N}^\theta \mathscr{D}_{N+2,2}$, which suffice for (4-2) since $\mathscr{D}_{N+2,2} \lesssim \mathscr{D}_{N+2,1}$. Then (4-2) follows by combining the estimates of the exceptional terms with the estimates of the terms as above. □

Estimates of G^i at the $2N$ level. Now we derive estimates for the nonlinear G^i terms, defined by (2-24)–(2-31), at the $2N$ level. Recall that, for sums of space-time derivatives, we use the notation $\bar{D}_m^k$ and $\overline{\nabla}_m^k$, as described on page 1443.

Theorem 4.2. *Let $m \in \{1,2\}$. Then there exists a $\theta > 0$ such that*

$$\|\overline{\nabla}_0^{4N-2} G^1\|_0^2 + \|\overline{\nabla}_0^{4N-2} G^2\|_1^2 + \|\bar{D}_0^{4N-2} G^3\|_{1/2}^2 + \|\bar{D}_0^{4N-2} G^4\|_{1/2}^2 \lesssim \mathscr{E}_{2N}^{1+\theta}, \tag{4-6}$$

$$\begin{aligned}\|\overline{\nabla}_0^{4N-2} G^1\|_0^2 + \|\overline{\nabla}_0^{4N-2} G^2\|_1^2 + \|\bar{D}_0^{4N-2} G^3\|_{1/2}^2 + \|\bar{D}_0^{4N-2} G^4\|_{1/2}^2 + \|\overline{\nabla}^{4N-3}\partial_t G^1\|_0^2 \\ + \|\overline{\nabla}^{4N-3}\partial_t G^2\|_1^2 + \|\bar{D}^{4N-3}\partial_t G^3\|_{1/2}^2 + \|\bar{D}^{4N-2}\partial_t G^4\|_{1/2}^2 \lesssim \mathscr{E}_{2N}^\theta \mathscr{D}_{2N},\end{aligned} \tag{4-7}$$

and

$$\|\nabla^{4N-1} G^1\|_0^2 + \|\nabla^{4N-1} G^2\|_1^2 + \|D^{4N-1} G^3\|_{1/2}^2 + \|D^{4N-1} G^4\|_{1/2}^2 \lesssim \mathscr{E}_{2N}^\theta \mathscr{D}_{2N} + \mathscr{K}\mathscr{F}_{2N}. \tag{4-8}$$

Proof. As explained in the proof of Theorem 4.1, the estimates are routine and lengthy, so we present only a sketch. The estimates in (4-6) are straightforward since $\mathscr{E}_{2N}$ has no minimal derivative restrictions. They may be derived using Sobolev embeddings, trace theory, and Lemmas A.1, A.5, and the L^∞ estimates of Lemma A.6.

The only terms with minimal derivatives in $\mathfrak{D}_{2N}$ are $D\eta$ and ∇p. The latter presents no problem, since, owing to Remark 2.4, p itself never appears in any of the G^i terms. The former may be dealt with by using Lemmas A.6 and A.7 to produce interpolation estimates of $\bar\eta$ and η in terms of $D\eta$. Whenever interpolation is needed to estimate these terms, there are always other terms multiplying them that allow for the recovery of a power of 1 on $\mathfrak{D}_{2N}$. Using these estimates with Sobolev embeddings, trace theory, and Lemmas A.1, A.5, and A.6 then yields (4-7).

We now turn to the derivation of (4-8). Consider $\partial^\alpha G^i$ with $|\alpha| = 4N-1$ and $\alpha_0 = 0$, that is, purely spatial derivatives, and expand $\partial^\alpha G^i$ using the Leibniz rule. With two exceptions, we may argue as in the derivation of (4-7) to estimate the desired norms of all of the resulting terms by $\mathcal{E}_{2N}^\theta \mathfrak{D}_{2N}$ for $\theta > 0$. The exceptional terms are ones involving either $\nabla^{4N+1}\bar\eta$ in Ω or $D^{4N}\eta$ on Σ. We will now show how to estimate the exceptional terms with $\mathcal{K}\mathcal{F}_{2N}$, as defined by (2-57) and (2-56). Identifying the product structure $\mathcal{K}\mathcal{F}_{2N}$ is one of the key difficulties in our analysis.

In $\nabla^{4N-1}G^1$ there are terms of the form $\partial^\beta\bar\eta Q\partial^\gamma u$, with

$$Q = Q(A, B, J, K, \nabla A, \nabla B, \nabla J), \tag{4-9}$$

a polynomial, and $\beta, \gamma \in \mathbb{N}^3$ with $|\beta| = 4N+1$ and $|\gamma| = 1$. To estimate such a term, we use Lemma A.5 to bound

$$\|\nabla^{4N+1}\bar\eta\|_0^2 \lesssim \|D^{4N+1/2}\eta\|_0^2 \lesssim \mathcal{F}_{2N}. \tag{4-10}$$

Sobolev embeddings imply that $\|Q\|_{L^\infty}^2 \lesssim 1 + \mathcal{E}_{2N}^\theta \lesssim 1$ for some $\theta > 0$, so

$$\|\partial^\beta\bar\eta Q\partial^\gamma u\|_0^2 \lesssim \|\nabla^{4N+1}\bar\eta\|_0^2 \|\nabla u\|_{L^\infty}^2 \|Q\|_{L^\infty}^2 \lesssim \|D^{4N+1/2}\eta\|_0^2 \|\nabla u\|_{L^\infty}^2 \lesssim \mathcal{F}_{2N}\mathcal{K}. \tag{4-11}$$

This estimate then yields the G^1 estimate in (4-8).

In $\nabla^{4N-1}G^2$ there are terms of the form $\partial^\beta\bar\eta Q\partial^\gamma u$ with $Q = Q(A, B, K)$, a polynomial, and $\beta, \gamma \in \mathbb{N}^3$ with $|\beta| = 4N$, $|\gamma| = 1$. Again, Sobolev embeddings imply that $\|Q\|_{C^1(\Omega)}^2 \lesssim 1 + \mathcal{E}_{2N}^\theta \lesssim 1$, so

$$\begin{aligned}
\|\partial^\beta\bar\eta Q\partial^\gamma u\|_1^2 &\lesssim \|Q\|_{C^1(\Omega)}^2 \|\partial^\beta\bar\eta\partial^\gamma u\|_1^2 \lesssim \|\partial^\beta\bar\eta\partial^\gamma u\|_0^2 + \|\partial^\beta\bar\eta\nabla\partial^\gamma u\|_0^2 + \|\nabla\partial^\beta\bar\eta\partial^\gamma u\|_0^2 \\
&\lesssim \|\nabla^{4N}\bar\eta\|_0^2 \|\nabla u\|_{C^1(\Omega)}^2 + \|\nabla^{4N+1}\bar\eta\|_0^2 \|\nabla u\|_{L^\infty}^2 \\
&\lesssim \|\eta\|_{4N-1/2}^2 \|\nabla u\|_3^2 + \mathcal{K}\mathcal{F}_{2N} \lesssim \mathcal{E}_{2N}\mathfrak{D}_{2N} + \mathcal{K}\mathcal{F}_{2N},
\end{aligned} \tag{4-12}$$

where again we have used Lemma A.5 and Sobolev embeddings. This estimate yields the G^2 estimate in (4-8).

In $D^{4N-1}G^3$ there are terms of the form $\partial^\beta\eta Q\partial^\gamma u$, where $\beta \in \mathbb{N}^2$ with $|\beta| = 4N$, $\gamma \in \mathbb{N}^3$ with $|\gamma| = 1$, and Q is a term for which we can estimate $\|Q\|_{C^1(\Sigma)}^2 \lesssim 1 + \mathcal{E}_{2N}^\theta \lesssim 1$. Then Lemma A.2 implies that

$$\|\partial^\beta\eta Q\partial^\gamma u\|_{H^{1/2}(\Sigma)}^2 \lesssim \|\partial^\beta\eta\|_{1/2}^2 \|Q\partial^\gamma u\|_{C^1}^2 \lesssim \|\eta\|_{4N+1/2}^2 \|Q\|_{C^1}^2 \|\nabla u\|_{C^1(\Sigma)}^2 \lesssim \mathcal{F}_{2N}\mathcal{K}, \tag{4-13}$$

where in the last inequality we have used $\|\nabla u\|_{C^1(\Sigma)}^2 \lesssim \mathcal{K}$, which follows since ∇u and $\nabla^2 u$ are continuous on the closure of Ω. This estimate yields the G^3 estimate in (4-8).

In $D^{4N-1}G^4$ the exceptional terms are of the form $\partial^\beta \eta u_i$, where $\beta \in \mathbb{N}^2$ with $|\beta| = 4N$ and $i = 1, 2$. Then Lemma A.1 implies that

$$\|\partial^\beta \eta u_1\|^2_{H^{1/2}(\Sigma)} \lesssim \|\partial^\beta \eta\|^2_{1/2} \|u_i\|^2_{H^2(\Sigma)} \lesssim \mathcal{F}_{2N}\mathcal{K}. \tag{4-14}$$

This estimate yields the G^4 estimate in (4-8). □

Estimates of other nonlinearities. The next result provides estimates for $\mathcal{I}_\lambda G^i$ and its derivatives.

Proposition 4.3. *We have*

$$\|\mathcal{I}_\lambda G^1\|_2^2 + \|\mathcal{I}_\lambda \partial_t G^1\|_0^2 + \|\mathcal{I}_\lambda G^2\|_2^2 + \|\mathcal{I}_\lambda \partial_t G^2\|_0^2 \lesssim \mathcal{E}_{2N} \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}, \tag{4-15}$$

$$\|\mathcal{I}_\lambda G^3\|_1^2 + \|\mathcal{I}_\lambda G^4\|_1^2 \lesssim \mathcal{E}_{2N} \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}, \tag{4-16}$$

$$\|\mathcal{I}_\lambda G^4\|_0^2 \lesssim \mathcal{D}_{2N}^2. \tag{4-17}$$

Proof. For each $i = 1, 2$ and for $\alpha \in \mathbb{N}^{1+3}$ such that $|\alpha| \le 2$, we can write $\partial^\alpha G^i = P^i_\alpha Q^i_\alpha$, where P^i_α is polynomial in the terms $\partial^\beta \tilde{b}$, $\partial^\beta K$, $\partial^\beta \bar{\eta}$, and $\partial^\beta u$ for $\beta \in \mathbb{N}^{1+3}$ with $|\beta| \le 4$, and Q^i_α is linear in the terms $\partial^\beta \nabla u$, $\partial^\beta \nabla^2 u$, and $\partial^\beta \nabla p$ for $|\beta| \le 2$. Then we may employ the bound (A-9) of Lemma A.3 to see that

$$\|\partial^\alpha \mathcal{I}_\lambda G^i\|_0^2 \lesssim \|P^i_\alpha\|_0^2 (\|Q^i_\alpha\|_1^2)^\lambda (\|D Q^i_\alpha\|_1^2)^{1-\lambda}. \tag{4-18}$$

It is then easily verified, using the Sobolev embedding, Lemmas A.1 and A.5–A.6, and the fact that $\mathcal{E}_{2N} \le 1$, that

$$\|P^i_\alpha\|_0^2 \lesssim \mathcal{E}_{2N} \quad \text{and} \quad \|Q^i_\alpha\|_2^2 \lesssim \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}, \tag{4-19}$$

which, together with (4-18), implies (4-15).

For $i = 3, 4$ and $\alpha \in \mathbb{N}^2$ such that $|\alpha| \le 1$, we may similarly decompose $\partial^\alpha G^i = P^i_\alpha Q^i_\alpha$. When $i = 3$ we must also employ Remark 2.4 to replace the $p - \eta$ term. We then argue as above, employing the bound (A-10) of Lemma A.3 as well as trace estimates, to deduce (4-16). The bound (4-17) also follows from Lemma A.3 and trace estimates, since

$$\|\mathcal{I}_\lambda G^4\|_0^2 \lesssim \|u\|^2_{H^0(\Sigma)} (\|D\eta\|_0^2)^\lambda (\|D^2\eta\|_0^2)^{1-\lambda} \lesssim \mathcal{D}_{2N} \mathcal{D}_{2N}^\lambda \mathcal{D}_{2N}^{1-\lambda} = \mathcal{D}_{2N}^2. \qquad \square$$

Now we provide some further estimates of product terms that will be useful later when we analyze the energy evolution for $\mathcal{I}_\lambda u$ and $\mathcal{I}_\lambda \eta$.

Lemma 4.4. *Let A, B, K be as defined in (1-8). We have*

$$\|\mathcal{I}_\lambda[(AK)\partial_3 u_1 + (BK)\partial_3 u_2]\|_0^2 + \sum_{i=1}^2 \|\mathcal{I}_\lambda[u \partial_i K]\|_0^2 \lesssim \mathcal{D}_{2N}^2 \tag{4-20}$$

and

$$\|\mathcal{I}_\lambda[(1-K)u]\|_0^2 \lesssim (\mathcal{E}_{2N})^{1/(1+\lambda)} (\mathcal{D}_{2N})^{(1+2\lambda)/(1+\lambda)}. \tag{4-21}$$

Also, if G^2 is as defined in (2-29), then

$$\|\mathcal{I}_\lambda[(1-K)G^2]\|_0^2 \lesssim \mathcal{E}_{2N} \mathcal{D}_{2N}^2. \tag{4-22}$$

Proof. We apply Lemma A.3, treating the AK, BK, $\partial_i K$ terms as f and the u, ∇u terms as g, to bound

$$\|\mathcal{I}_\lambda[(AK)\partial_3 u_1 + (BK)\partial_3 u_2]\|_0^2 + \sum_{i=1}^{2} \|\mathcal{I}_\lambda[u\partial_i K]\|_0^2 \lesssim (\|AK\|_0^2 + \|BK\|_0^2 + \|DK\|_0^2)\|u\|_3^2. \tag{4-23}$$

From Lemma 2.6, the fact that $\partial_i K = -K^2 \partial_i J$, and Lemma A.5, we know that

$$\|AK\|_0^2 + \|BK\|_0^2 + \|DK\|_0^2 \lesssim \|\nabla\bar{\eta}\|_1^2 \lesssim \|D\eta\|_1^2 \le \mathcal{D}_{2N}. \tag{4-24}$$

Then, since $\|u\|_3^2 \le \mathcal{D}_{2N}$, we know that (4-20) holds.

Now, since $1-K = K(J-1)$, we can again use Lemmas A.3 and 2.6 to see that

$$\|\mathcal{I}_\lambda[(1-K)u]\|_0^2 \lesssim \|K(1-J)\|_0^2\|u\|_2^2 \lesssim \|\bar{\eta}\|_1^2\|u\|_2^2. \tag{4-25}$$

To control $\bar{\eta}$ we use Lemmas A.5 and A.7 to bound

$$\begin{aligned}\|\bar{\eta}\|_1^2 \lesssim \|\eta\|_0^2 + \|D\eta\|_0^2 &\lesssim (\|\mathcal{I}_\lambda\eta\|_0^2)^{1/(1+\lambda)}(\|D\eta\|_0^2)^{\lambda/(1+\lambda)} + (\|D\eta\|_0^2)^{1/(1+\lambda)}(\|D\eta\|_0^2)^{\lambda/(1+\lambda)}\\ &\lesssim (\mathcal{E}_{2N})^{1/(1+\lambda)}(\mathcal{D}_{2N})^{\lambda/(1+\lambda)}.\end{aligned} \tag{4-26}$$

Then (4-21) follows from these two estimates and the fact that $\|u\|_2^2 \le \mathcal{D}_{2N}$.

For the estimate of the $(1-K)G^2$ term, we once more use Lemma A.3 to see that

$$\|\mathcal{I}_\lambda[(1-K)G^2]\|_0^2 \lesssim \|G^2\|_0^2\|1-K\|_2^2. \tag{4-27}$$

By differentiating the equation $JK = 1$, we may compute the derivatives of K in terms of the derivatives of J; this allows us to bound, by virtue of Lemmas 2.6 and A.5,

$$\|1-K\|_2^2 \lesssim \|\bar{\eta}\|_3^2 \lesssim \|\eta\|_{5/2}^2 \lesssim \|\eta\|_0^2 + \|D\eta\|_{3/2}^2. \tag{4-28}$$

Then we may argue as in (4-26) to estimate the right side of this inequality, and we deduce that

$$\|1-K\|_2^2 \lesssim (\mathcal{E}_{2N})^{1/(1+\lambda)}(\mathcal{D}_{2N})^{\lambda/(1+\lambda)}. \tag{4-29}$$

On the other hand, from the definition of G^2 in (2-29), we see that

$$\|G^2\|_0^2 \lesssim \|\nabla u\|_0^2(\|\bar{\eta}\|_{L^\infty}^2 + \|\nabla\bar{\eta}\|_{L^\infty}^2). \tag{4-30}$$

We estimate the L^∞ norms by using (A-25) of Lemma A.6 first with $q = 0$, $s = 1$, $r = \lambda^2 + \lambda$ and then with $q = 1$, $s = 1$, $r = \lambda^2 + 2\lambda$ to see that

$$\begin{aligned}\|\bar{\eta}\|_{L^\infty}^2 + \|\nabla\bar{\eta}\|_{L^\infty}^2 &\lesssim (\|\mathcal{I}_\lambda\eta\|_0^2)^{\lambda/(\lambda+1)}(\|D\eta\|_0^2)^{1/(\lambda+1)} + (\|\mathcal{I}_\lambda\eta\|_0^2)^{\lambda/(\lambda+1)}(\|D^2\eta\|_0^2)^{1/(\lambda+1)}\\ &\le (\mathcal{E}_{2N})^{\lambda/(\lambda+1)}(\mathcal{D}_{2N})^{1/(\lambda+1)}.\end{aligned} \tag{4-31}$$

Then, since $\|\nabla u\|_0^2 \le \mathcal{D}_{2N}$, we have

$$\|G^2\|_0^2 \lesssim (\mathcal{E}_{2N})^{\lambda/(\lambda+1)}(\mathcal{D}_{2N})^{1+1/(\lambda+1)}, \tag{4-32}$$

which yields (4-22) when combined with (4-27) and (4-29). $\square$

Now we provide an estimate of $\partial_t^j \mathcal{A}$ when $j = 2N+1$ and when $j = N+3$.

Lemma 4.5. *Let $\mathcal{A}$ be given by (1-7). We have*

$$\|\partial_t^{2N+1}\mathcal{A}\|_0^2 \lesssim \mathfrak{D}_{2N}, \tag{4-33}$$

while for $m = 1, 2$,

$$\|\partial_t^{N+3}\mathcal{A}\|_0^2 \lesssim \mathfrak{D}_{N+2,m}. \tag{4-34}$$

Proof. We will only prove (4-33); the bound (4-34) follows from similar analysis. Since $\|\partial_t^{2N+1}\eta\|_{1/2}^2 \le \mathfrak{D}_{2N}$ and temporal derivatives commute with the Poisson integral, we may employ Lemma A.5 to bound

$$\|\partial_t^{2N+1}\bar{\eta}\|_1^2 = \|\partial_t^{2N+1}\bar{\eta}\|_0^2 + \|\nabla\partial_t^{2N+1}\bar{\eta}\|_0^2 \lesssim \|\partial_t^{2N+1}\eta\|_{1/2}^2 \le \mathfrak{D}_{2N}. \tag{4-35}$$

From this we easily deduce that

$$\|\partial_t^{2N+1}J\|_0^2 + \|\partial_t^{2N+1}K\|_0^2 \lesssim \mathfrak{D}_{2N}. \tag{4-36}$$

This, the previous bound, and the Sobolev embeddings then imply (4-33) since the components of $\mathcal{A}$ are either unity, K, $-\partial_1\bar{\eta}\tilde{b}K$, or $-\partial_2\bar{\eta}\tilde{b}K$. □

5. Energy evolution using the geometric form

Estimates of the perturbations when $\partial^\alpha = \partial_t^{\alpha_0}$ is applied to (1-9). We now present estimates of the perturbations F^i, defined by (2-13)–(2-22) when $\partial^\alpha = \partial_t^{2N}$.

Theorem 5.1. *Let $\partial^\alpha = \partial_t^{2N}$ and let F^1, F^2, F^3, F^4 be defined by (2-13)–(2-22). Then*

$$\|F^1\|_0^2 + \|\partial_t(JF^2)\|_0^2 + \|F^3\|_0^2 + \|F^4\|_0^2 \lesssim \mathcal{E}_{2N}\mathfrak{D}_{2N}. \tag{5-1}$$

Proof. We first consider the F^1 estimate. Each term in the sums that define F^1 is at least quadratic. It is straightforward to see that each such term can be written in the form XY, where X involves fewer temporal derivatives than Y, and we may use the usual Sobolev embeddings and Lemmas A.1 and A.5 along with the definitions of $\mathcal{E}_{2N}$ and $\mathfrak{D}_{2N}$ (given in (2-50) and (2-51), respectively) to estimate

$$\|X\|_{L^\infty}^2 \lesssim \mathcal{E}_{2N} \quad \text{and} \quad \|Y\|_0^2 \lesssim \mathfrak{D}_{2N}. \tag{5-2}$$

Then $\|XY\|_0^2 \le \|X\|_{L^\infty}^2\|Y\|_0^2 \lesssim \mathcal{E}_{2N}\mathfrak{D}_{2N}$, and the F^1 estimate in (5-1) follows by summing. A similar argument, also employing trace estimates, yields the F^3 and F^4 estimates in (5-1). Note though, that to estimate the $\beta = \alpha$ term in $F^{3,1}$ we use Remark 2.4 to replace $(p - \eta)$.

The same analysis also works for $\partial_t(JF^{2,1})$ and shows that $\|\partial_t(JF^{2,1})\|_0^2 \lesssim \mathcal{E}_{2N}\mathfrak{D}_{2N}$. To handle $\partial_t(JF^{2,2})$ we must also be able to estimate $\|\partial_t^{2N+1}\mathcal{A}\|_0^2 \lesssim \mathfrak{D}_{2N}$, but this is possible due to Lemma 4.5. Then a similar splitting into L^∞ and H^0 estimates shows that $\|\partial_t(JF^{2,2})\|_0^2 \lesssim \mathcal{E}_{2N}\mathfrak{D}_{2N}$, and then the $\partial_t(JF^2)$ estimate in (5-1) follows since $F^2 = F^{2,1} + F^{2,2}$. □

We now present estimates for these perturbations when $\partial^\alpha = \partial_t^{N+2}$.

Theorem 5.2. *Let $\partial^\alpha = \partial_t^{N+2}$ and let F^1, F^2, F^3, F^4 be defined by (2-13)–(2-22). Then, for $m = 1, 2$, we have*

$$\|F^1\|_0^2 + \|\partial_t(JF^2)\|_0^2 + \|F^3\|_0^2 + \|F^4\|_0^2 \lesssim \mathscr{E}_{2N}\mathscr{D}_{N+2,m}. \tag{5-3}$$

Also, if $N \geq 3$, there exists a $\theta > 0$ such that

$$\|F^2\|_0^2 \lesssim \mathscr{E}_{2N}^\theta \mathscr{E}_{N+2,m} \tag{5-4}$$

for $m = 1, 2$.

Proof. The proof of (5-3) is essentially the same as that of Theorem 5.1. For the F^1, F^3, and F^4 estimates we note that each term in their definition is of the form XY where X involves fewer temporal derivatives than Y, which involves at least two temporal derivatives. We estimate $\|X\|_{L^\infty}^2 \lesssim \mathscr{E}_{2N}$ and $\|Y\|_0^2 \lesssim \mathscr{D}_{N+2,m}$ and then sum to get (5-3). Note that since Y involves at least two temporal derivatives, there is no problem estimating it in terms of $\mathscr{D}_{N+2,m}$. The $\partial_t(JF^2)$ estimate works similarly, except we must also use the bound (4-34) from Lemma 4.5. Note also that in estimating the $\beta = \alpha$ term in $F^{3,1}$, we must employ Remark 2.4 to remove $(p - \eta)$.

We now turn to the proof of (5-4). Recall that $F^2 = F^{2,1} + F^{2,2}$, as defined in (2-19). Since the sum in $F^{2,1}$ runs over $1 \leq \beta \leq N+1$, we may bound

$$\|F^{2,1}\|_0^2 \lesssim \sum_{1\leq\beta\leq N+1} \|\partial_t^\beta \mathscr{A}\|_{L^\infty}^2 \|\partial_t^{N+2-\beta} u\|_1^2 \lesssim \sum_{1\leq\beta\leq N+1} \mathscr{E}_{2N} \|\partial_t^{N+2-\beta} u\|_{2(N+2)-2(N+2-\beta)}^2$$
$$\lesssim \mathscr{E}_{2N}\mathscr{E}_{N+2,m}. \tag{5-5}$$

For $F^{2,2}$, a calculation reveals that

$$F^{2,2} = -\partial_t^{N+2}\mathscr{A}_{ij}\partial_j u_i = -\partial_t^{N+2}\mathscr{A}_{i3}\partial_3 u_i = \partial_t^{N+2}(\partial_1\bar{\eta}\tilde{b}K)\partial_3 u_1 + \partial_t^{N+2}(\partial_2\bar{\eta}\tilde{b}K)\partial_3 u_2 - \partial_t^{N+2}K\partial_3 u_3. \tag{5-6}$$

We may use the L^∞ interpolation estimate of Proposition 3.9 to bound $\|\partial_3 u_i\|_{L^\infty}^2 \lesssim \mathscr{E}_{N+2,m}$ for $i = 1, 2$ and $m = 1, 2$, which then implies that

$$\|\partial_t^{N+2}(\partial_1\bar{\eta}\tilde{b}K)\partial_3 u_1 + \partial_t^{N+2}(\partial_2\bar{\eta}\tilde{b}K)\partial_3 u_2\|_0^2 \lesssim \mathscr{E}_{2N}\mathscr{E}_{N+2,m} \tag{5-7}$$

if we estimate $\partial_3 u_i$ in L^∞ and the ∂_t^{N+1} terms in H^0. On the other hand, the relation $JK = 1$ (recall the definition in (1-8)), the Leibniz rule, and Lemma A.5 imply that

$$\|\partial_t^{N+2}K\|_0^2 \lesssim \sum_{1\leq\gamma\leq N+2} \|\partial_t^\gamma J\|_0^2 \lesssim \sum_{1\leq\gamma\leq N+2} \|\partial_t^\gamma \bar{\eta}\|_1^2 \lesssim \sum_{1\leq\gamma\leq N+2} \|\partial_t^\gamma \eta\|_{1/2}^2$$
$$= \sum_{1\leq\gamma\leq N+1} \|\partial_t^\gamma \eta\|_{1/2}^2 + \|\partial_t^{N+2}\eta\|_{1/2}^2 \lesssim \mathscr{E}_{N+2,m} + \|\partial_t^{N+2}\eta\|_{1/2}^2. \tag{5-8}$$

To handle the last term we must use the standard Sobolev interpolation (3-47) with $s = r = 1/2$ and $q = 2N - 9/2$:

$$\|\partial_t^{N+2}\eta\|_{1/2}^2 \lesssim (\|\partial_t^{N+2}\eta\|_0^2)^\kappa (\|\partial_t^{N+2}\eta\|_{2N-4}^2)^{1-\kappa} \lesssim (\mathscr{E}_{N+2,m})^\kappa (\mathscr{E}_{2N})^{1-\kappa} \tag{5-9}$$

for $\kappa = (4N-9)/(4N-8)$. Then

$$\|\partial_t^{N+2} K \partial_3 u_3\|_0^2 \le \|\partial_t^{N+2} K\|_0^2 \|\partial_3 u_3\|_{L^\infty}^2 \lesssim \mathcal{E}_{N+2,m} \|\partial_3 u_3\|_{L^\infty}^2 + (\mathcal{E}_{N+2,m})^\kappa (\mathcal{E}_{2N})^{1-\kappa} \|\partial_3 u_3\|_{L^\infty}^2. \tag{5-10}$$

For the first term on the right we bound $\|\partial_3 u_3\|_{L^\infty}^2 \lesssim \mathcal{E}_{2N}$, and for the second we use the L^∞ interpolation bound of Proposition 3.9 with $r = 1/2$, so that $2/(2+r) = 4/5 \ge 1-\kappa$ and $\|\partial_3 u_3\|_{L^\infty}^2 \lesssim \mathcal{E}_{N+2,m}^{2/(2+r)} \lesssim \mathcal{E}_{N+2,m}^{1-\kappa}$. Then these estimates and (5-10) imply that

$$\|\partial_t^{N+2} K \partial_3 u_3\|_0^2 \lesssim \mathcal{E}_{N+2,m} (\mathcal{E}_{2N})^{1-\kappa}. \tag{5-11}$$

We then combine (5-6), (5-7), and (5-11) to see that

$$\|F^{2,2}\|_0^2 \lesssim \mathcal{E}_{N+2,m} (\mathcal{E}_{2N})^{1-\kappa}. \tag{5-12}$$

Then the estimate (5-4) follows from (5-5) and (5-12). □

Energy evolution with the highest and lowest count of temporal derivatives. We now show the time-integrated evolution estimate for $2N$ temporal derivatives.

Proposition 5.3. *There exists a $\theta > 0$ such that*

$$\|\partial_t^{2N} u(t)\|_0^2 + \|\partial_t^{2N} \eta(t)\|_0^2 + \int_0^t \|\mathbb{D} \partial_t^{2N} u\|_0^2 \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2} + \int_0^t \mathcal{E}_{2N}^\theta \mathcal{D}_{2N}. \tag{5-13}$$

Proof. We apply $\partial^\alpha = \partial_t^{2N}$ to (1-9). Then $v = \partial_t^{2N} u$, $q = \partial_t^{2N} p$, and $\zeta = \partial_t^{2N} \eta$ solve (2-1) with F^i, $i = 1, 2, 3, 4$, given by (2-13)–(2-22). Applying Lemma 2.2 (and Remark 2.3) to these functions and then integrating in time from 0 to t gives

$$\begin{aligned}
&\frac{1}{2}\int_\Omega J |\partial_t^{2N} u(t)|^2 + \frac{1}{2}\int_\Sigma |\partial_t^{2N} \eta(t)|^2 + \frac{1}{2}\int_0^t \int_\Omega J |\mathbb{D}_{\mathcal{A}} \partial_t^{2N} u|^2 \\
&= \frac{1}{2}\int_\Omega J |\partial_t^{2N} u(0)|^2 + \frac{1}{2}\int_\Sigma |\partial_t^{2N} \eta(0)|^2 + \int_0^t \int_\Omega J(\partial_t^{2N} u \cdot F^1 + \partial_t^{2N} p F^2) + \int_0^t \int_\Sigma -\partial_t^{2N} u \cdot F^3 + \partial_t^{2N} \eta F^4.
\end{aligned} \tag{5-14}$$

Here, because of Remark 2.3, we understand that this formula actually holds with

$$\int_0^t \int_\Omega \partial_t^{2N} p J F^2 := -\int_0^t \int_\Omega \partial_t^{2N-1} p \partial_t (J F^2) + \int_\Omega (\partial_t^{2N-1} p J F^2)(t) - \int_\Omega (\partial_t^{2N-1} p J F^2)(0). \tag{5-15}$$

We will estimate all of the terms involving F^i on the right side of this equation.

We begin with the F^1 term. According to Theorem 5.1 and Lemma 2.6, we may bound

$$\int_0^t \int_\Omega J \partial_t^{2N} u \cdot F^1 \le \int_0^t \|\partial_t^{2N} u\|_0 \|J\|_{L^\infty} \|F^1\|_0 \lesssim \int_0^t \sqrt{\mathcal{D}_{2N}} \sqrt{\mathcal{E}_{2N} \mathcal{D}_{2N}} = \int_0^t \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{2N}. \tag{5-16}$$

Similarly, we use Theorem 5.1 and trace theory to handle the F^3 and F^4 terms:

$$\begin{aligned}
\int_0^t \int_\Sigma -\partial_t^{2N} u \cdot F^3 + \partial_t^{2N} \eta F^4 &\le \int_0^t \|\partial_t^{2N} u\|_{H^0(\Sigma)} \|F^3\|_0 + \|\partial_t^{2N} \eta\|_0 \|F^4\|_0 \\
&\lesssim \int_0^t (\|\partial_t^{2N} u\|_1 + \|\partial_t^{2N} \eta\|_0) \sqrt{\mathcal{E}_{2N} \mathcal{D}_{2N}} \lesssim \int_0^t \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{2N}.
\end{aligned} \tag{5-17}$$

According to Theorem 5.1 we may estimate

$$-\int_0^t \int_\Omega \partial_t^{2N-1} p \partial_t (J F^2) \lesssim \int_0^t \|\partial_t^{2N-1} p\|_0 \|\partial_t (J F^2)\|_0 \lesssim \int_0^t \sqrt{\mathscr{D}_{2N}} \sqrt{\mathscr{E}_{2N} \mathscr{D}_{2N}} = \int_0^t \sqrt{\mathscr{E}_{2N}} \mathscr{D}_{2N}. \tag{5-18}$$

On the other hand, it is easy to verify using the Sobolev embeddings that

$$\int_\Omega (\partial_t^{2N-1} p J F^2)(t) - \int_\Omega (\partial_t^{2N-1} p J F^2)(0) \lesssim \mathscr{E}_{2N}(0) + (\mathscr{E}_{2N}(t))^{3/2}. \tag{5-19}$$

Hence

$$\int_0^t \int_\Omega \partial_t^{2N} p J F^2 \lesssim \mathscr{E}_{2N}(0) + (\mathscr{E}_{2N}(t))^{3/2} + \int_0^t \sqrt{\mathscr{E}_{2N}} \mathscr{D}_{2N}. \tag{5-20}$$

Now we combine (5-16), (5-17), and (5-20) to deduce that

$$\frac{1}{2} \int_\Omega J |\partial_t^{2N} u(t)|^2 + \frac{1}{2} \int_\Sigma |\partial_t^{2N} \eta(t)|^2 + \frac{1}{2} \int_0^t \int_\Omega J |\mathbb{D}_{\mathscr{A}} \partial_t^{2N} u|^2 \\ \lesssim \mathscr{E}_{2N}(0) + (\mathscr{E}_{2N}(t))^{3/2} + \int_0^t \sqrt{\mathscr{E}_{2N}} \mathscr{D}_{2N}. \tag{5-21}$$

We now seek to replace $J|\mathbb{D}_{\mathscr{A}} \partial_t^{2N} u|^2$ with $|\mathbb{D} \partial_t^{2N} u|^2$ and $J|\partial_t^{2N} u(t)|^2$ with $|\partial_t^{2N} u(t)|^2$ in (5-21). To this end, we write

$$J|\mathbb{D}_{\mathscr{A}} \partial_t^{2N} u|^2 = |\mathbb{D} \partial_t^{2N} u|^2 + (J-1)|\mathbb{D} \partial_t^{2N} u|^2 + J(\mathbb{D}_{\mathscr{A}} \partial_t^{2N} u + \mathbb{D} \partial_t^{2N} u) : (\mathbb{D}_{\mathscr{A}} \partial_t^{2N} u - \mathbb{D} \partial_t^{2N} u) \tag{5-22}$$

and estimate the last three terms on the right side. For the last term we note that

$$(\mathbb{D}_{\mathscr{A}} \partial_t^{2N} u \pm \mathbb{D} \partial_t^{2N} u)_{ij} = (\mathscr{A}_{ik} \pm \delta_{ik}) \partial_k \partial_t^{2N} u_j + (\mathscr{A}_{jk} \pm \delta_{jk}) \partial_k \partial_t^{2N} u_i, \tag{5-23}$$

so that Sobolev embeddings and Lemma A.5 provide the bounds

$$|\mathbb{D}_{\mathscr{A}} \partial_t^{2N} u - \mathbb{D} \partial_t^{2N} u| \lesssim \sqrt{\mathscr{E}_{2N}} |\nabla \partial_t^{2N} u| \quad \text{and} \quad |\mathbb{D}_{\mathscr{A}} \partial_t^{2N} u + \mathbb{D} \partial_t^{2N} u| \lesssim (1 + \sqrt{\mathscr{E}_{2N}}) |\nabla \partial_t^{2N} u|. \tag{5-24}$$

We then get

$$\int_0^t \int_\Omega |J(\mathbb{D}_{\mathscr{A}} \partial_t^{2N} u + \mathbb{D} \partial_t^{2N} u) : (\mathbb{D}_{\mathscr{A}} \partial_t^{2N} u - \mathbb{D} \partial_t^{2N} u)| \\ \lesssim \int_0^t (\sqrt{\mathscr{E}_{2N}} + \mathscr{E}_{2N}) \int_\Omega |\nabla \partial_t^{2N} u|^2 \lesssim \int_0^t \sqrt{\mathscr{E}_{2N}} \mathscr{D}_{2N}. \tag{5-25}$$

Similarly,

$$\int_0^t \int_\Omega |J-1| |\mathbb{D} \partial_t^{2N} u|^2 \lesssim \int_0^t \sqrt{\mathscr{E}_{2N}} \mathscr{D}_{2N} \quad \text{and} \quad \int_\Omega |J-1| |\partial_t^{2N} u(t)|^2 \lesssim (\mathscr{E}_{2N}(t))^{3/2}. \tag{5-26}$$

We may then use (5-22) and (5-25)–(5-26) to replace in (5-21) and derive the bound (5-13). □

Now we prove a similar result for when ∂_t^{N+2} is applied. This time, however, we do not want an inequality that is integrated in time, so we are forced to introduce an error term involving $\partial_t^{N+1} p$.

Proposition 5.4. *Let F^2 be given by (2-19) with $\partial^\alpha = \partial_t^{N+2}$. Then*

$$\partial_t\left(\|\sqrt{J}\partial_t^{N+2}u\|_0^2 + \|\partial_t^{N+2}\eta\|_0^2 - 2\int_\Omega J\partial_t^{N+1}pF^2\right) + \|\mathbb{D}\partial_t^{N+2}u\|_0^2 \lesssim \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2,m}. \tag{5-27}$$

Proof. We apply $\partial^\alpha = \partial_t^{N+2}$ to (1-9). Then $v = \partial_t^{N+2}u$, $q = \partial_t^{N+2}p$, and $\zeta = \partial_t^{N+2}\eta$ solve (2-1) with F^i, $i = 1, 2, 3, 4$, given by (2-13)–(2-22). Applying Lemma 2.2 to these functions gives

$$\begin{aligned}\partial_t\left(\frac{1}{2}\int_\Omega J|\partial_t^{N+2}u|^2 + \frac{1}{2}\int_\Sigma |\partial_t^{N+2}\eta|^2\right) + \frac{1}{2}\int_\Omega J|\mathbb{D}_{\mathscr{A}}\partial_t^{N+2}u|^2 \\ = \int_\Omega J(\partial_t^{N+2}u\cdot F^1 + \partial_t^{N+2}pF^2) + \int_\Sigma -\partial_t^{N+2}u\cdot F^3 + \partial_t^{N+2}\eta F^4.\end{aligned} \tag{5-28}$$

We will estimate all of the terms involving F^i on the right side of this equation as in Proposition 5.3.

We begin with the F^1 term. According to Theorem 5.2 and Lemma 2.6, we may bound

$$\int_\Omega J\partial_t^{N+2}u\cdot F^1 \le \|\partial_t^{N+2}u\|_0\|J\|_{L^\infty}\|F^1\|_0 \lesssim \sqrt{\mathscr{D}_{N+2,m}}\sqrt{\mathscr{E}_{2N}\mathscr{D}_{N+2,m}} = \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2,m}. \tag{5-29}$$

Similarly, we use Theorem 5.2 and trace theory to handle the F^3 and F^4 terms:

$$\begin{aligned}\int_\Sigma -\partial_t^{N+2}u\cdot F^3 + \partial_t^{N+2}\eta F^4 &\le \|\partial_t^{N+2}u\|_{H^0(\Sigma)}\|F^3\|_0 + \|\partial_t^{N+2}\eta\|_0\|F^4\|_0 \\ &\lesssim (\|\partial_t^{N+2}u\|_1 + \|\partial_t^{N+2}\eta\|_0)\sqrt{\mathscr{E}_{2N}\mathscr{D}_{N+2,m}} \lesssim \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2,m}.\end{aligned} \tag{5-30}$$

For the term $\partial_t^{N+2}pF^2$, there is one more time derivative on p than can be controlled by $\mathscr{D}_{N+2,m}$. We are then forced to pull out a time derivative:

$$\int_\Omega \partial_t^{N+2}pJF^2 = \partial_t\int_\Omega \partial_t^{N+1}pJF^2 - \int_\Omega \partial_t^{N+1}p\partial_t(JF^2). \tag{5-31}$$

Then, according to Theorem 5.2, we may estimate

$$-\int_\Omega \partial_t^{N+1}p\partial_t(JF^2) \le \|\partial_t^{N+1}p\|_0\|\partial_t(JF^2)\|_0 \lesssim \sqrt{\mathscr{D}_{N+2,m}}\sqrt{\mathscr{E}_{2N}\mathscr{D}_{N+2,m}} = \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2,m}. \tag{5-32}$$

Hence

$$\int_0^t\int_\Omega \partial_t^{2N}pJF^2 \lesssim \partial_t\int_\Omega \partial_t^{N+1}pJF^2 + \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2,m}. \tag{5-33}$$

Now we combine (5-28)–(5-30) and (5-33) to deduce that

$$\partial_t\left(\frac{1}{2}\int_\Omega J|\partial_t^{N+2}u|^2 + \frac{1}{2}\int_\Sigma|\partial_t^{N+2}\eta|^2 - \int_\Omega\partial_t^{N+1}pJF^2\right) + \frac{1}{2}\int_\Omega J|\mathbb{D}_{\mathscr{A}}\partial_t^{N+2}u|^2 \lesssim \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2,m}. \tag{5-34}$$

We may argue as in (5-22)–(5-26) of Proposition 5.3 to show that

$$\frac{1}{2}\int_\Omega |\mathbb{D}\partial_t^{N+2}u|^2 \lesssim \frac{1}{2}\int_\Omega J|\mathbb{D}_{\mathscr{A}}\partial_t^{N+2}u|^2 + \sqrt{\mathscr{E}_{2N}}\mathscr{D}_{N+2,m}. \tag{5-35}$$

Then (5-27) follows from (5-34) and (5-35). □

Finally, we record the basic energy estimate when no derivatives are applied.

Proposition 5.5. *We have*

$$\partial_t\left(\frac{1}{2}\int_\Omega J|u|^2+\frac{1}{2}\int_\Sigma |\eta|^2\right)+\frac{1}{2}\int_\Omega J|\mathbb{D}_{\mathcal{A}}u|^2=0. \tag{5-36}$$

In particular,

$$\|u(t)\|_0^2+\|\eta(t)\|_0^2+\int_0^t \|\mathbb{D}u\|_0^2 \lesssim \mathcal{E}_{2N}(0)+\int_0^t \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{5-37}$$

Proof. Setting $v=u$, $q=p$, $\zeta=\eta$, and $F^i=0$ for $i=1,2,3,4$ in Lemma 2.2 yields (5-36). We may argue as in (5-22)–(5-26) of Proposition 5.3 to estimate

$$\frac{1}{2}\int_\Omega |\mathbb{D}u|^2 \lesssim \frac{1}{2}\int_\Omega J|\mathbb{D}_{\mathcal{A}}u|^2+\sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{5-38}$$

Similarly, Lemma 2.6 allows us to estimate

$$\tfrac{1}{4}\int_\Omega |u|^2 \le \frac{1}{2}\int_\Omega J|u|^2. \tag{5-39}$$

Now we may integrate (5-36) in time from 0 to t and use these two estimates to derive (5-37). □

6. Energy evolution in the perturbed linear form

Energy evolution for horizontal derivatives. We now estimate how the evolution of the horizontal energy is coupled to the horizontal dissipation and the full energy and dissipation. Recall that $\mathcal{F}_{2N}$ is as defined in (2-56) and $\mathcal{K}$ is as defined in (2-57).

Lemma 6.1. *Let $\alpha\in\mathbb{N}^2$ be such that $|\alpha|=4N$, that is, let ∂^α be $4N$ spatial derivatives in the x_1, x_2 directions. Let G^4 be as defined by (2-31). Then*

$$\left|\int_\Sigma \partial^\alpha\eta\,\partial^\alpha G^4\right| \lesssim \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}+\sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}. \tag{6-1}$$

Proof. Throughout the proof β will always denote an element of $\mathbb{N}^2$, and we will write

$$Df\cdot\partial^\beta u=\partial_1 f\,\partial^\beta u_1+\partial_2 f\,\partial^\beta u_2$$

for a function f defined on Σ. Then by the Leibniz rule, we have

$$-\partial^\alpha G^4=\partial^\alpha(D\eta\cdot u)=D\partial^\alpha\eta\cdot u+\sum_{\substack{0<\beta\le\alpha\\|\beta|=1}} C_{\alpha,\beta}D\partial^{\alpha-\beta}\eta\cdot\partial^\beta u+\sum_{\substack{0<\beta\le\alpha\\|\beta|\ge 2}} C_{\alpha,\beta}D\partial^{\alpha-\beta}\eta\cdot\partial^\beta u \tag{6-2}$$

for constants $C_{\alpha,\beta}$ depending on α and β. We will analyze each of the three terms on the right separately.

For the first term, we integrate by parts to see that

$$\int_\Sigma \partial^\alpha\eta\, D\partial^\alpha\eta\cdot u=\frac{1}{2}\int_\Sigma D|\partial^\alpha\eta|^2\cdot u=-\frac{1}{2}\int_\Sigma \partial^\alpha\eta\,\partial^\alpha\eta(\partial_1 u_1+\partial_2 u_2). \tag{6-3}$$

This then allows us to use (A-3) of Lemma A.1 to bound

$$\begin{aligned}\left|\int_\Sigma \partial^\alpha \eta D\partial^\alpha \eta \cdot u\right| &\lesssim \|\partial^\alpha \eta\|_{1/2} \|\partial^\alpha \eta(\partial_1 u_1 + \partial_2 u_2)\|_{H^{-1/2}(\Sigma)} \\ &\lesssim \|\eta\|_{4N+1/2} \|\partial^\alpha \eta\|_{-1/2} \|\partial_1 u_1 + \partial_2 u_2\|_{H^2(\Sigma)} \\ &\lesssim \|\eta\|_{4N+1/2} \|D\eta\|_{4N-3/2} \|\partial_1 u_1 + \partial_2 u_2\|_{H^2(\Sigma)} \le \sqrt{\mathcal{F}_{2N}\mathcal{D}_{2N}\mathcal{K}}. \end{aligned} \tag{6-4}$$

Similarly, for the second term we estimate

$$\begin{aligned}\left|\int_\Sigma \partial^\alpha \eta \sum_{\substack{0<\beta\le\alpha \\ |\beta|=1}} C_{\alpha,\beta} D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u\right| &\lesssim \|D^{4N}\eta\|_{1/2} \|D^{4N}\eta\|_{-1/2} \sum_{i=1}^2 \|Du_i\|_{H^2(\Sigma)} \\ &\lesssim \|\eta\|_{4N+1/2} \|D\eta\|_{4N-3/2} \sum_{i=1}^2 \|Du_i\|_{H^2(\Sigma)} \le \sqrt{\mathcal{F}_{2N}\mathcal{D}_{2N}\mathcal{K}}. \end{aligned} \tag{6-5}$$

For the third term we first note that $\|\partial^\alpha\eta\|_{-1/2} \lesssim \|D\eta\|_{4N-3/2} \le \sqrt{\mathcal{D}_{2N}}$, which allows us to bound

$$\left|\int_\Sigma \partial^\alpha \eta D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u\right| \le \|\partial^\alpha \eta\|_{-1/2} \|D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u\|_{H^{1/2}(\Sigma)} \lesssim \sqrt{\mathcal{D}_{2N}} \|D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u\|_{H^{1/2}(\Sigma)}. \tag{6-6}$$

We estimate the last term on the right using Lemma A.1 and trace theory, but in different ways depending on $|\beta|$:

$$\begin{aligned}\|D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u\|_{H^{1/2}(\Sigma)} &\lesssim \begin{cases} \|D\partial^{\alpha-\beta}\eta\|_{1/2} \|\partial^\beta u\|_{H^2(\Sigma)} & \text{for } 2 \le |\beta| \le 2N, \\ \|D\partial^{\alpha-\beta}\eta\|_2 \|\partial^\beta u\|_{H^{1/2}(\Sigma)} & \text{for } 2N+1 \le |\beta| \le 4N \end{cases} \\ &\lesssim \begin{cases} \|D\eta\|_{4N-3/2} \|u\|_{2N+3} & \text{for } 2 \le |\beta| \le 2N, \\ \|D\eta\|_{2N+1} \|u\|_{4N+1} & \text{for } 2N+1 \le |\beta| \le 4N, \end{cases} \end{aligned} \tag{6-7}$$

so that $\|D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u\|_{H^{1/2}(\Sigma)} \lesssim \sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}}$ for all $0 < \beta \le \alpha$ with $|\beta| \ge 2$. Hence

$$\left|\int_\Sigma \partial^\alpha \eta \sum_{\substack{0<\beta\le\alpha \\ |\beta|\ge 2}} C_{\alpha,\beta} D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u\right| \lesssim \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}} = \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{6-8}$$

The estimate (6-1) then follows from (6-4), (6-5), and (6-8). □

Now we prove an estimate for horizontal derivatives up to order $2N$, excluding $\partial^\alpha = \partial_t^{2N}$ and no derivatives. Recall that we use the conventions for sums of derivatives described on page 1443.

Proposition 6.2. *Suppose that $\alpha \in \mathbb{N}^{1+2}$ is such that $\alpha_0 \le 2N-1$ and $1 \le |\alpha| \le 4N$. Then there exists a $\theta > 0$ such that*

$$\partial_t\left(\frac{1}{2}\int_\Omega |\partial^\alpha u|^2 + \frac{1}{2}\int_\Sigma |\partial^\alpha \eta|^2\right) + \frac{1}{2}\int_\Omega |\mathbb{D}\partial^\alpha u|^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}, \tag{6-9}$$

and, *in particular*,

$$\|\bar{D}_1^{4N-1}u(t)\|_0^2+\|D\bar{D}^{4N-1}u(t)\|_0^2+\|\bar{D}_1^{4N-1}\eta(t)\|_0^2+\|D\bar{D}^{4N-1}\eta(t)\|_0^2 \\ +\int_0^t \|\bar{D}_1^{4N-1}\mathbb{D}u\|_0^2+\|D\bar{D}^{4N-1}\mathbb{D}u\|_0^2 \lesssim \bar{\mathcal{E}}_{2N}(0)+\int_0^t \mathcal{E}_{2N}^\theta\mathcal{D}_{2N}+\sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}. \quad (6\text{-}10)$$

Proof. Let $\alpha\in\mathbb{N}^{1+2}$ satisfy $\alpha_0\le 2N-1$ and $1\le|\alpha|\le 4N$. Note that the constraint on α_0 implies that we do not exceed the number of temporal derivatives of p that we can control. An application of Lemma 2.5 to $v=\partial^\alpha u$, $q=\partial^\alpha p$, $\zeta=\partial^\alpha\eta$ with $\Phi^1=\partial^\alpha G^1$, $\Phi^2=\partial^\alpha G^2$, $\Phi^3=\partial^\alpha G^3$, $\Phi^4=\partial^\alpha G^4$, and $a=1$ reveals that

$$\partial_t\left(\frac{1}{2}\int_\Omega|\partial^\alpha u|^2+\frac{1}{2}\int_\Sigma|\partial^\alpha\eta|^2\right)+\frac{1}{2}\int_\Omega|\mathbb{D}\partial^\alpha u|^2 \\ =\int_\Omega \partial^\alpha u\cdot(\partial^\alpha G^1-\nabla\partial^\alpha G^2)+\partial^\alpha p\,\partial^\alpha G^2+\int_\Sigma -\partial^\alpha u\cdot\partial^\alpha G^3+\partial^\alpha\eta\,\partial^\alpha G^4. \quad (6\text{-}11)$$

Assume initially that $1\le|\alpha|\le 4N-1$. Then according to the estimates (4-7) and (4-8) of Theorem 4.2 and the definition of $\mathcal{D}_{2N}$, we have

$$\left|\int_\Omega \partial^\alpha u\cdot(\partial^\alpha G^1-\nabla\partial^\alpha G^2)+\partial^\alpha p\,\partial^\alpha G^2\right|\le\|\partial^\alpha u\|_0(\|\partial^\alpha G^1\|_0+\|\partial^\alpha G^2\|_1)+\|\partial^\alpha p\|_0\|\partial^\alpha G^2\|_0 \\ \lesssim\sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}^\theta\mathcal{D}_{2N}+\mathcal{K}\mathcal{F}_{2N}}\lesssim\mathcal{E}_{2N}^\kappa\mathcal{D}_{2N}+\sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}, \quad (6\text{-}12)$$

where in the last equality we have written $\kappa=\theta/2$ for $\theta>0$ the number provided by Theorem 4.2. Similarly, we may use Theorem 4.2 along with the trace estimate $\|\partial^\alpha u\|_{H^0(\Sigma)}\lesssim\|\partial^\alpha u\|_1\le\sqrt{\mathcal{D}_{2N}}$ to get

$$\left|\int_\Sigma -\partial^\alpha u\cdot\partial^\alpha G^3+\partial^\alpha\eta\,\partial^\alpha G^4\right|\le\|\partial^\alpha u\|_{H^0(\Sigma)}\|\partial^\alpha G^3\|_0+\|\partial^\alpha\eta\|_0\|\partial^\alpha G^4\|_0 \\ \lesssim\sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}^\theta\mathcal{D}_{2N}+\mathcal{K}\mathcal{F}_{2N}}\lesssim\mathcal{E}_{2N}^\kappa\mathcal{D}_{2N}+\sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}. \quad (6\text{-}13)$$

Now assume that $|\alpha|=4N$. Since $\alpha_0\le 2N-1$, we may write $\alpha=\beta+(\alpha-\beta)$ for some $\beta\in\mathbb{N}^2$ with $|\beta|=1$, that is, ∂^α involves at least one spatial derivative. Since $|\alpha-\beta|=4N-1$, we can then integrate by parts and use (4-7) and (4-8) of Theorem 4.2 to see that

$$\left|\int_\Omega \partial^\alpha u\cdot(\partial^\alpha G^1-\nabla\partial^\alpha G^2)\right|=\left|\int_\Omega \partial^{\alpha+\beta}u\cdot(\partial^{\alpha-\beta}G^1-\nabla\partial^{\alpha-\beta}G^2)\right| \\ \le\|\partial^{\alpha+\beta}u\|_0\big(\|\partial^{\alpha-\beta}G^1\|_0+\|\partial^{\alpha-\beta}G^2\|_1\big)\le\|\partial^\alpha u\|_1\big(\|\bar{\nabla}^{4N-1}G^1\|_0+\|\bar{\nabla}^{4N-1}G^2\|_1\big) \\ \lesssim\sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}^\theta\mathcal{D}_{2N}+\mathcal{K}\mathcal{F}_{2N}}\lesssim\mathcal{E}_{2N}^\kappa\mathcal{D}_{2N}+\sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}. \quad (6\text{-}14)$$

For the pressure term we do not need to integrate by parts; Theorem 4.2 provides the estimate

$$\left|\int_\Omega \partial^\alpha p\,\partial^\alpha G^2\right|\le\|\partial^\alpha p\|_0\|\partial^{\alpha-\beta}\partial^\beta G^2\|_0\le\|\partial^\alpha p\|_0\|\bar{\nabla}^{4N-1}G^2\|_1 \\ \lesssim\sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}^\theta\mathcal{D}_{2N}+\mathcal{K}\mathcal{F}_{2N}}\lesssim\mathcal{E}_{2N}^\kappa\mathcal{D}_{2N}+\sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}. \quad (6\text{-}15)$$

Next, we integrate by parts, employ Theorem 4.2, and use the trace estimate $H^1(\Omega) \hookrightarrow H^{1/2}(\Sigma)$ to get

$$
\begin{aligned}
\left| \int_\Sigma \partial^\alpha u \cdot \partial^\alpha G^3 \right| = \left| \int_\Sigma \partial^{\alpha+\beta} u \cdot \partial^{\alpha-\beta} G^3 \right| &\le \|\partial^{\alpha+\beta} u\|_{H^{-1/2}(\Sigma)} \|\partial^{\alpha-\beta} G^3\|_{1/2} \\
&\lesssim \|\partial^\alpha u\|_{H^{1/2}(\Sigma)} \|\bar{D}^{4N-1} G^3\|_{1/2} \lesssim \|\partial^\alpha u\|_1 \|\bar{D}^{4N-1} G^3\|_{1/2} \\
&\lesssim \sqrt{\mathfrak{D}_{2N}} \sqrt{\mathcal{E}^\theta_{2N} \mathfrak{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}} \lesssim \mathcal{E}^\kappa_{2N} \mathfrak{D}_{2N} + \sqrt{\mathfrak{D}_{2N} \mathcal{K}\mathcal{F}_{2N}}.
\end{aligned}
\tag{6-16}
$$

For the term $\partial^\alpha \eta \partial^\alpha G^4$ we must split into two cases: $\alpha_0 \ge 1$ and $\alpha_0 = 0$. In the former case, there is at least one temporal derivative in ∂^α, so $\|\partial^\alpha \eta\|_{1/2} \le \sqrt{\mathfrak{D}_{2N}}$, and hence Theorem 4.2 allows us to bound

$$
\begin{aligned}
\left| \int_\Sigma \partial^\alpha \eta \partial^\alpha G^4 \right| = \left| \int_\Sigma \partial^{\alpha+\beta} \eta \partial^{\alpha-\beta} G^4 \right| &\le \|\partial^{\alpha+\beta} \eta\|_{-1/2} \|\partial^{\alpha-\beta} G^4\|_{1/2} \lesssim \|\partial^\alpha \eta\|_{1/2} \|\bar{D}^{4N-1} G^4\|_{1/2} \\
&\lesssim \sqrt{\mathfrak{D}_{2N}} \sqrt{\mathcal{E}^\theta_{2N} \mathfrak{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}} \lesssim \mathcal{E}^\kappa_{2N} \mathfrak{D}_{2N} + \sqrt{\mathfrak{D}_{2N} \mathcal{K}\mathcal{F}_{2N}}.
\end{aligned}
\tag{6-17}
$$

In the latter case, $\alpha_0 = 0$, so that ∂^α involves only spatial derivatives; in this case we use Lemma 6.1 to bound

$$
\left| \int_\Sigma \partial^\alpha \eta \partial^\alpha G^4 \right| \lesssim \sqrt{\mathcal{E}_{2N}} \mathfrak{D}_{2N} + \sqrt{\mathfrak{D}_{2N} \mathcal{K}\mathcal{F}_{2N}}.
\tag{6-18}
$$

Now, in light of (6-11)–(6-18), we know that (6-9) holds. The bound (6-10) follows by applying (6-9) to all $1 \le |\alpha| \le 4N$ with $\alpha_0 \le 2N-1$, summing, and integrating in time from 0 to t. □

Our next result provides some preliminary interpolation estimates for G^2 and G^4 in terms of $\mathfrak{D}_{N+2,m}$, as defined in (2-54) and (2-55), but with a power greater than 1.

Lemma 6.3. *Let G^4 be as defined in (2-31). We have the estimate*

$$
\|D^{2N+3} G^4\|^2_{1/2} \lesssim (\mathfrak{D}_{N+2,2})^{1+2/(4N-7)}.
\tag{6-19}
$$

Also, there exists a $\theta > 0$ such that

$$
\|D G^4\|^2_0 \lesssim \mathcal{E}^\theta_{2N} (\mathfrak{D}_{N+2,1})^{1+1/(\lambda+2)} \quad \text{and} \quad \|\bar{D}^2 G^4\|^2_0 \lesssim \mathcal{E}^\theta_{2N} (\mathfrak{D}_{N+2,2})^{1+1/(\lambda+3)}.
\tag{6-20}
$$

Finally,

$$
\|D G^2\|^2_{L^1} \lesssim \mathcal{E}^\theta_{2N} (\mathfrak{D}_{N+2,1})^{1+\lambda/(\lambda+2)} \quad \text{and} \quad \|\bar{D}^2 G^2\|^2_{L^1} \lesssim \mathcal{E}^\theta_{2N} (\mathfrak{D}_{N+2,2})^{1+\lambda/(\lambda+3)}.
\tag{6-21}
$$

Proof. Let $\alpha \in \mathbb{N}^2$ be such that $|\alpha| = 2(N+2)-1$. The Leibniz rule, Lemma A.1, and trace theory imply

$$
\begin{aligned}
\|\partial^\alpha G^4\|_{1/2} &\lesssim \sum_{\substack{\beta \le \alpha \\ |\beta| \le N+2}} \|D \partial^\beta \eta\|_2 \|\partial^{\alpha-\beta} u\|_{H^{1/2}(\Sigma)} + \sum_{\substack{\beta \le \alpha \\ N+3 \le |\beta| \le 2N+3}} \|D \partial^\beta \eta\|_{1/2} \|\partial^{\alpha-\beta} u\|_{H^2(\Sigma)} \\
&\lesssim \|D\eta\|_{N+4} \|D^{2N+3}_{N+1} u\|_1 + \|D^3 \eta\|_{2(N+2)-5/2} \|u\|_{H^{N+2}(\Sigma)}.
\end{aligned}
\tag{6-22}
$$

Trace theory, Poincaré's inequality, the $H^0(\Omega)$ interpolation result for ∇u of Theorem 3.14, and the fact that $\|D^{N+2}u\|_1^2 \le \min\{\mathscr{E}_{2N}, \mathscr{D}_{N+2,2}\}$ imply that

$$\begin{aligned}\|u\|^2_{H^{N+2}(\Sigma)} &\lesssim \|u\|^2_{H^0(\Sigma)} + \|D^{N+2}u\|^2_{H^0(\Sigma)} \lesssim \|\nabla u\|_0^2 + \|D^{N+2}u\|_1^2 \\ &\lesssim \mathscr{D}_{N+2,2}^{(\lambda+1)/(\lambda+3)} + (\mathscr{E}_{2N})^{2/(\lambda+3)}(\mathscr{D}_{N+2,2})^{(\lambda+1)/(\lambda+3)} \lesssim \mathscr{D}_{N+2,2}^{(\lambda+1)/(\lambda+3)}. \end{aligned} \tag{6-23}$$

Let us now choose q so that

$$\frac{\lambda+1}{\lambda+3} + \frac{q}{q+1} = 1 + \frac{2}{4N-7}. \tag{6-24}$$

Since $N \ge 5$ and $\lambda \in (0, 1)$, we may find such a $q = q(\lambda)$ with $dq(\lambda)/d\lambda \le 0$ for $\lambda \in (0, 1)$:

$$q = \frac{8N+2\lambda-8}{4N(1+\lambda)-9\lambda-13} \in \left[\frac{8N-6}{8N-22}, \frac{8N-8}{4N-13}\right] \subset [1, 2N-9/2]. \tag{6-25}$$

Using this q, $r = 1$, and $s = 2(N+2)-5/2$ in the standard Sobolev interpolation inequality (3-47), we find that

$$\begin{aligned}\|D^3\eta\|^2_{2(N+2)-5/2} &\lesssim (\|D^3\eta\|^2_{2(N+2)-7/2})^{q/(1+q)}(\|D^3\eta\|^2_{2(N+2)-5/2+q})^{1/(1+q)} \\ &\lesssim (\mathscr{D}_{N+2,2})^{q/(1+q)}(\mathscr{E}_{2N})^{1/(1+q)} \lesssim (\mathscr{D}_{N+2,2})^{q/(1+q)}. \end{aligned} \tag{6-26}$$

Now (6-23), (6-26), and the choice of q imply that

$$\|D^3\eta\|^2_{2(N+2)-5/2}\|u\|^2_{H^{N+2}(\Sigma)} \lesssim (\mathscr{D}_{N+2,2})^{1+2/(4N-7)}. \tag{6-27}$$

The fact that $\|D^3\eta\|^2_{N+2} \le \min\{\mathscr{E}_{2N}, \mathscr{D}_{N+2,2}\}$ and the $H^0(\Sigma)$ interpolation result for $D\eta$ of Lemma 3.1 imply that

$$\begin{aligned}\|D\eta\|^2_{N+4} &\lesssim \|D\eta\|_0^2 + \|D^3\eta\|^2_{N+2} \\ &\lesssim \mathscr{D}_{N+2,2}^{(\lambda+1)/(\lambda+3)} + (\|D^3\eta\|^2_{N+2})^{2/(\lambda+3)}(\|D^3\eta\|^2_{N+2})^{(\lambda+1)/(\lambda+3)} \\ &\le \mathscr{D}_{N+2,2}^{(\lambda+1)/(\lambda+3)} + (\mathscr{E}_{2N})^{2/(\lambda+3)}(\mathscr{D}_{N+2,2})^{(\lambda+1)/(\lambda+3)} \lesssim \mathscr{D}_{N+2,2}^{(\lambda+1)/(\lambda+3)}. \end{aligned} \tag{6-28}$$

On the other hand, using the same q as above, we have

$$\begin{aligned}\|D^{2N+3}_{N+1}u\|_1^2 &= (\|D^{2N+3}_{N+1}u\|_1^2)^{q/(q+1)}(\|D^{2N+3}_{N+1}u\|_1^2)^{1/(q+1)} \\ &\lesssim (\mathscr{D}_{N+2,2})^{q/(1+q)}(\mathscr{E}_{2N})^{1/(1+q)} \le (\mathscr{D}_{N+2,2})^{q/(1+q)}. \end{aligned} \tag{6-29}$$

Then (6-28) and (6-29) imply that

$$\|D\eta\|^2_{N+4}\|D^{2N+3}_{N+1}u\|_1^2 \lesssim (\mathscr{D}_{N+2,2})^{1+2/(4N-7)}. \tag{6-30}$$

We then combine (6-22), (6-27), and (6-30) to deduce (6-19).

We now turn to the proof of the bounds (6-20) and (6-21). The bounds (6-20) may be deduced by applying an operator ∂^α with $\alpha \in \mathbb{N}^{1+2}$ satisfying either $|\alpha| = 1$ or $|\alpha| = 2$ to G^4, and then estimating the resulting products with one norm taken in H^0 and the others in L^∞, employing the H^0 and L^∞ interpolation estimates for η, u and their derivatives recorded in Lemma 3.1, Proposition 3.9, and Theorem 3.14. The bounds (6-21) may be deduced similarly except that at least two terms in the resulting products must be estimated in H^0 to deduce the resulting L^1 bounds. This presents no problem since G^2 is a linear combination of products of two or more terms. □

With this lemma in place, we may record the estimates for the evolution of the energy at the $N+2$ level.

Proposition 6.4. *Suppose that $m \in \{1, 2\}$ and $\alpha \in \mathbb{N}^{1+2}$ is such that $\alpha_0 \le N+1$ and $m \le |\alpha| \le 2(N+2)$. Then there exists a $\theta > 0$ such that*

$$\partial_t(\|\partial^\alpha u\|_0^2 + \|\partial^\alpha \eta\|_0^2) + \|\mathbb{D}\partial^\alpha u\|_0^2 \lesssim \mathscr{E}_{2N}^\theta \mathscr{D}_{N+2,m}. \tag{6-31}$$

In particular,

$$\partial_t(\|\bar{D}_m^{2N+3}u\|_0^2+\|D\bar{D}^{2N+3}u\|_0^2+\|\bar{D}_m^{2N+3}\eta\|_0^2+\|D\bar{D}^{2N+3}\eta\|_0^2)+\|\bar{D}_m^{2N+3}\mathbb{D}u\|_0^2+\|D\bar{D}^{2N+3}\mathbb{D}u\|_0^2 \lesssim \mathscr{E}_{2N}^\theta \mathscr{D}_{N+2,m}. \tag{6-32}$$

Proof. For $m \in \{1, 2\}$ and $\alpha \in \mathbb{N}^{1+2}$ such that $\alpha_0 \le N+1$ and $m \le |\alpha| \le 2(N+2)$, we argue as in Proposition 6.2 to deduce that (6-11) holds. Let X_α denote the right side of (6-11) for our range of α. To bound X_α, we break to three cases.

If $m+1 \le |\alpha| \le 2(N+2)-1$ or $|\alpha| = 2(N+2)$ with $1 \le \alpha_0 \le N+1$, we know from trace theory and the definitions of $\mathscr{D}_{N+2,m}$ that

$$\|\partial^\alpha u\|_0^2 + \|\partial^\alpha p\|_0^2 + \|\partial^\alpha u\|_{H^{1/2}(\Sigma)}^2 + \|\partial^\alpha \eta\|_{1/2}^2 \lesssim \mathscr{D}_{N+2,m}. \tag{6-33}$$

This allows us to argue as in Proposition 6.2, employing Theorem 4.1 in place of Theorem 4.2, to bound

$$|X_\alpha| \lesssim \mathscr{E}_{2N}^\theta \mathscr{D}_{N+2,m} \tag{6-34}$$

for some $\theta > 0$.

Now consider $|\alpha| = 2(N+2)$ with $\alpha_0 = 0$. In this case we know from the definitions (2-54) and (2-55) that there is a deficit of half a derivative that prevents us from bounding $\|\partial^\alpha \eta\|_{1/2}^2 \lesssim \mathscr{D}_{N+2,m}$, but we may still estimate

$$\|\partial^\alpha u\|_1^2 + \|\partial^\alpha p\|_0^2 + \|\partial^\alpha u\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathscr{D}_{N+2,m}. \tag{6-35}$$

We may then argue as in Proposition 6.2, integrating by parts and using these bounds as well as those from Theorem 4.1 to show that the first, second, and third integrals in the definition of X_α are bounded by $\mathscr{E}_{2N}^\theta \mathscr{D}_{N+2,m}$. For the fourth integral, we control $\|\partial^\alpha \eta\|_{1/2}^2$ through the interpolation estimate of Lemma 3.18:

$$\|\partial^\alpha \eta\|_{1/2}^2 \le \|D^{2N+4}\eta\|_{1/2}^2 \lesssim (\mathscr{E}_{2N})^{2/(4N-7)} (\mathscr{D}_{N+2,2})^{(4N-9)/(4N-7)}. \tag{6-36}$$

Then we may integrate by parts with $\alpha = \beta + (\alpha - \beta)$, $|\beta| = 1$ and employ this estimate along with (6-19) of Lemma 6.3 to see that

$$\begin{aligned}\left|\int_\Sigma \partial^\alpha \eta \partial^\alpha G^4\right| &= \left|\int_\Sigma \partial^{\alpha+\beta} \eta \partial^{\alpha-\beta} G^4\right| \le \|\partial^{\alpha+\beta}\eta\|_{-1/2}\|\partial^{\alpha-\beta}G^4\|_{1/2} \lesssim \|\partial^\alpha \eta\|_{1/2}\|D^{2N+3}G^4\|_{1/2} \\ &\lesssim \sqrt{(\mathscr{E}_{2N})^{2/(4N-7)}(\mathscr{D}_{N+2,2})^{(4N-9)/(4N-7)}}\sqrt{(\mathscr{D}_{N+2,2})^{1+2/(4N-7)}} \\ &= (\mathscr{E}_{2N})^{1/(4N-7)}\mathscr{D}_{N+2,2} \le (\mathscr{E}_{2N})^{1/(4N-7)}\mathscr{D}_{N+2,m}. \end{aligned} \tag{6-37}$$

Hence, when $|\alpha| = 2(N+2)$ with $\alpha_0 = 0$, there is a $\theta > 0$ such that

$$|X_\alpha| \lesssim \mathscr{E}_{2N}^\theta \mathscr{D}_{N+2,m}. \tag{6-38}$$

Finally, we consider the case of $|\alpha| = m$ for $m = 1, 2$. In this case we only know that

$$\|\partial^\alpha u\|_1^2 + \|\partial^\alpha u\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathscr{D}_{N+2,m}, \tag{6-39}$$

so only the first and third integrals of X_α may be handled directly as above to be bounded by $\mathscr{E}_{2N}^\theta \mathscr{D}_{N+2,m}$. For the fourth term in X_α we first use the $H^0(\Sigma)$ interpolation results of Lemma 3.1 and Proposition 3.16 to bound

$$\|D\eta\|_0^2 \lesssim (\mathscr{D}_{N+2,1})^{(\lambda+1)/(\lambda+2)} \quad \text{and} \quad \|D^2\eta\|_0^2 + \|\partial_t \eta\|_0^2 \lesssim (\mathscr{D}_{N+2,2})^{(\lambda+2)/(\lambda+3)}. \tag{6-40}$$

Then by (6-20) of Lemma 6.3, we know that

$$\begin{aligned}\left|\int_\Sigma \partial^\alpha \eta \partial^\alpha G^4\right| &\le \|\partial^\alpha \eta\|_0 \|\partial^\alpha G^4\|_0 \\ &\lesssim \begin{cases} \sqrt{(\mathscr{D}_{N+2,1})^{(\lambda+1)/(\lambda+2)}}\sqrt{\mathscr{E}_{2N}^\theta(\mathscr{D}_{N+2,1})^{1+1/(\lambda+2)}} & \text{for } m=1, \\ \sqrt{(\mathscr{D}_{N+2,2})^{(\lambda+2)/(\lambda+3)}}\sqrt{\mathscr{E}_{2N}^\theta(\mathscr{D}_{N+2,2})^{1+1/(\lambda+3)}} & \text{for } m=2 \end{cases} \\ &\le \mathscr{E}_{2N}^{\theta/2}\mathscr{D}_{N+2,m}. \end{aligned} \tag{6-41}$$

For the second term in X_α we first use the L^∞ interpolation estimates of Lemma 3.3 with $r = \lambda/2$ when $m = 1$ and with $r = \lambda/3$ when $m = 2$ to bound

$$\|Dp\|_{L^\infty}^2 \lesssim (\mathscr{D}_{N+2,1})^{2/(\lambda+2)} \quad \text{and} \quad \|D^2 p\|_{L^\infty}^2 + \|\partial_t p\|_{L^\infty}^2 \lesssim (\mathscr{D}_{N+2,2})^{3/(\lambda+3)}. \tag{6-42}$$

Then, by (6-21) of Lemma 6.3, we know that

$$\begin{aligned}\left|\int_\Omega \partial^\alpha p \partial^\alpha G^2\right| &\le \|\partial^\alpha p\|_{L^\infty} \|\partial^\alpha G^2\|_{L^1} \\ &\lesssim \begin{cases} \sqrt{(\mathscr{D}_{N+2,1})^{2/(\lambda+2)}}\sqrt{\mathscr{E}_{2N}^\theta(\mathscr{D}_{N+2,1})^{1+\lambda/(\lambda+2)}} & \text{for } m=1, \\ \sqrt{(\mathscr{D}_{N+2,2})^{3/(\lambda+3)}}\sqrt{\mathscr{E}_{2N}^\theta(\mathscr{D}_{N+2,2})^{1+\lambda/(\lambda+3)}} & \text{for } m=2 \end{cases} \\ &\le \mathscr{E}_{2N}^{\theta/2}\mathscr{D}_{N+2,m}. \end{aligned} \tag{6-43}$$

Hence, when $|\alpha| = m$ for $m = 1, 2$, we also have

$$|X_\alpha| \lesssim \mathscr{E}_{2N}^\theta \mathscr{D}_{N+2,m}. \tag{6-44}$$

Now, by (6-34), (6-38), and (6-44), we know that (6-31) holds. The bound (6-32) follows by summing (6-31) over the specified range of α. □

Energy evolution for $\mathcal{I}_\lambda u$ and $\mathcal{I}_\lambda \eta$. Before we can analyze the energy evolution for $\mathcal{I}_\lambda u$ and $\mathcal{I}_\lambda \eta$, we must first prove a lemma that provides control of $\mathcal{I}_\lambda p$.

Lemma 6.5. *We have*

$$\|\mathcal{I}_\lambda p\|_0^2 \lesssim \mathcal{E}_{2N}, \tag{6-45}$$

$$\|\mathcal{I}_\lambda Dp\|_0^2 \lesssim (\mathcal{E}_{2N})^{\lambda/(1+\lambda)}(\mathcal{D}_{2N})^{1/(1+\lambda)}. \tag{6-46}$$

Proof. Let $\alpha \in \mathbb{N}^2$ be such that $|\alpha| \in \{0, 1\}$. We may apply Lemma A.10 to see that

$$\|\partial^\alpha \mathcal{I}_\lambda p\|_0^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda p\|_{H^0(\Sigma)}^2 + \|\partial_3 \partial^\alpha \mathcal{I}_\lambda p\|_0^2. \tag{6-47}$$

In order to estimate each term on the right, we will use the structure of (2-23). Indeed, using the boundary condition, we find that

$$\|\partial^\alpha \mathcal{I}_\lambda p\|_{H^0(\Sigma)}^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda \eta\|_0^2 + \|\partial^\alpha \mathcal{I}_\lambda \partial_3 u_3\|_{H^0(\Sigma)}^2 + \|\partial^\alpha \mathcal{I}_\lambda G^3\|_0^2. \tag{6-48}$$

Trace theory and the divergence equation in (2-23) allow us to bound

$$\|\partial^\alpha \mathcal{I}_\lambda \partial_3 u_3\|_{H^0(\Sigma)}^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda \partial_3 u_3\|_1^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda G^2\|_1^2 + \|\partial^\alpha \mathcal{I}_\lambda Du\|_1^2 \lesssim \|\mathcal{I}_\lambda Du\|_2^2 + \|\mathcal{I}_\lambda G^2\|_2^2, \tag{6-49}$$

regardless of whether $|\alpha| = 0$ or 1. To estimate this $\mathcal{I}_\lambda Du$ term we apply Lemmas A.4 and A.13 to get

$$\|\mathcal{I}_\lambda Du\|_2^2 \lesssim \sum_{k=1}^{2} \|\mathcal{I}_\lambda D\nabla^k u\|_0^2 \lesssim \sum_{k=1}^{2} (\|\nabla^k u\|_0^2)^\lambda (\|D\nabla^k u\|_0^2)^{1-\lambda} \lesssim \|u\|_3^2. \tag{6-50}$$

By chaining together the bounds (6-48)–(6-50) and employing the G^i estimates of Proposition 4.3, we deduce that

$$\|\partial^\alpha \mathcal{I}_\lambda p\|_{H^0(\Sigma)}^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda \eta\|_0^2 + \|u\|_3^2 + \mathcal{E}_{2N} \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}. \tag{6-51}$$

Now we estimate $\partial_3 \partial^\alpha \mathcal{I}_\lambda p$ by using the first equation in (2-23) to bound

$$\|\partial^\alpha \mathcal{I}_\lambda \partial_3 p\|_0^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda \partial_t u_3\|_0^2 + \|\partial^\alpha \mathcal{I}_\lambda D^2 u\|_0^2 + \|\partial^\alpha \mathcal{I}_\lambda \partial_3^2 u_3\|_0^2 + \|\partial^\alpha \mathcal{I}_\lambda G^1\|_0^2. \tag{6-52}$$

When $|\alpha| = 1$, we can use Lemma A.4 to see that

$$\|\partial^\alpha \mathcal{I}_\lambda \partial_t u_3\|_0^2 \lesssim \|\mathcal{I}_\lambda D\partial_t u_3\|_0^2 \lesssim (\|\partial_t u_3\|_0^2)^\lambda (\|D\partial_t u_3\|_0^2)^{1-\lambda} \le \|\partial_t u\|_1^2. \tag{6-53}$$

When $|\alpha| = 0$, we cannot use Lemma A.4 directly, so we first use Lemma A.11 and the divergence equation in (2-23), and then use Lemma A.4:

$$\|\mathcal{I}_\lambda \partial_t u_3\|_0^2 \lesssim \|\partial_3 \mathcal{I}_\lambda \partial_t u_3\|_0^2 = \|\mathcal{I}_\lambda \partial_t \partial_3 u_3\|_0^2 \lesssim \|\mathcal{I}_\lambda \partial_t G^2\|_0^2 + \|\mathcal{I}_\lambda D\partial_t u\|_0^2 \lesssim \|\mathcal{I}_\lambda \partial_t G^2\|_0^2 + \|\partial_t u\|_1^2. \tag{6-54}$$

Then (6-53) and (6-54) imply that, regardless of whether $|\alpha| = 0$ or 1, we may bound

$$\|\partial^\alpha \mathcal{I}_\lambda \partial_t u_3\|_0^2 \lesssim \|\mathcal{I}_\lambda \partial_t G^2\|_0^2 + \|\partial_t u\|_1^2. \tag{6-55}$$

The term $\partial^\alpha \mathcal{I}_\lambda D^2 u$ may be estimated as in (6-50):

$$\|\partial^\alpha \mathcal{I}_\lambda D^2 u\|_0^2 \lesssim \|u\|_3^2. \tag{6-56}$$

To estimate the term $\partial^\alpha \mathcal{I}_\lambda \partial_3^2 u_3$, we again use the divergence equation to bound

$$\|\partial^\alpha \mathcal{I}_\lambda \partial_3^2 u_3\|_0^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda \partial_3 G^2\|_0^2 + \|\partial^\alpha \mathcal{I}_\lambda \partial_3 D u\|_0^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda \partial_3 G^2\|_0^2 + \|u\|_3^2, \tag{6-57}$$

where in the second inequality we have again argued as in (6-50). Then (6-52) and (6-55)–(6-57), together with Proposition 4.3, imply that

$$\|\partial^\alpha \mathcal{I}_\lambda \partial_3 p\|_0^2 \lesssim \|u\|_3^2 + \|\partial_t u\|_1^2 + \mathcal{E}_{2N} \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}. \tag{6-58}$$

The estimates (6-51) and (6-58) may be combined with (6-47) to show that

$$\|\partial^\alpha \mathcal{I}_\lambda p\|_0^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda \eta\|_0^2 + \|u\|_3^2 + \|\partial_t u\|_1^2 + \mathcal{E}_{2N} \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}. \tag{6-59}$$

When $|\alpha| = 0$ we bound the first three terms on the right side of (6-59) by $\mathcal{E}_{2N}$ and use the fact that $\mathcal{E}_{2N}^2 \le \mathcal{E}_{2N} \le 1$ to deduce (6-45). When $|\alpha| = 1$, we first use Lemma A.7 with $q = 1-\lambda$ and $s = \lambda$ to bound

$$\begin{aligned}\|\partial^\alpha \mathcal{I}_\lambda \eta\|_0^2 \le \|D\mathcal{I}_\lambda \eta\|_0^2 \lesssim \|D^{1-\lambda}\eta\|_0^2 &\lesssim (\|\mathcal{I}_\lambda \eta\|_0^2)^{\lambda/(1+\lambda)} (\|D\eta\|_0^2)^{1/(1+\lambda)} \\ &\lesssim (\mathcal{E}_{2N})^{\lambda/(1+\lambda)} (\mathcal{D}_{2N})^{1/(1+\lambda)},\end{aligned} \tag{6-60}$$

where, in the second inequality, $D^{1-\lambda}$ denotes the usual fractional derivative of order $1-\lambda$. Then we use the fact that $\mathcal{E}_{2N} \le 1$ to bound

$$\begin{aligned}\mathcal{E}_{2N} \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\} &\le (\min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\})^{\lambda/(1+\lambda)} (\min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\})^{1/(1+\lambda)} \\ &\le (\mathcal{E}_{2N})^{\lambda/(1+\lambda)} (\mathcal{D}_{2N})^{1/(1+\lambda)}.\end{aligned} \tag{6-61}$$

Similarly, since $\|u\|_3^2 + \|\partial_t u\|_1^2 \le \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}$, we have

$$\|u\|_3^2 + \|\partial_t u\|_1^2 \le (\mathcal{E}_{2N})^{\lambda/(1+\lambda)} (\mathcal{D}_{2N})^{1/(1+\lambda)}. \tag{6-62}$$

We then combine (6-59) with (6-60)–(6-62) to deduce (6-46). □

Our next lemma provides a bound for the integral of the product $\mathcal{I}_\lambda p \mathcal{I}_\lambda G^2$. The estimate is essential to analyzing the energy evolution of $\mathcal{I}_\lambda u$ and $\mathcal{I}_\lambda \eta$.

Lemma 6.6. *Let G^2 be given by (2-29). We have*

$$\left| \int_\Omega \mathcal{I}_\lambda p \mathcal{I}_\lambda G^2 \right| \lesssim \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{2N}. \tag{6-63}$$

Proof. We begin by writing

$$\int_\Omega \mathcal{I}_\lambda p \mathcal{I}_\lambda G^2 = \mathrm{I} + \mathrm{II} \tag{6-64}$$

for

$$\mathrm{I} := \int_\Omega \mathcal{I}_\lambda p \mathcal{I}_\lambda [(AK)\partial_3 u_1 + (BK)\partial_3 u_2] \quad \text{and} \quad \mathrm{II} := \int_\Omega \mathcal{I}_\lambda p \mathcal{I}_\lambda [(1-K)\partial_3 u_3]. \tag{6-65}$$

The term I is straightforward to estimate because of the bounds (4-20) of Lemma 4.4 and (6-45) of Lemma 6.5:

$$|\mathrm{I}| \le \|\mathcal{J}_\lambda p\|_0 \|\mathcal{J}_\lambda[(AK)\partial_3 u_1 + (BK)\partial_3 u_2]\|_0 \lesssim \sqrt{\mathcal{E}_{2N}}\,\mathcal{D}_{2N}. \tag{6-66}$$

To estimate the term II, we must first use the divergence equation in (2-23) to rewrite

$$(1-K)\partial_3 u_3 = (1-K)[G^2 - \partial_1 u_1 - \partial_2 u_2], \tag{6-67}$$

so that

$$\mathrm{II} = \int_\Omega \mathcal{J}_\lambda p \mathcal{J}_\lambda[(1-K)G^2] - \int_\Omega \mathcal{J}_\lambda p \mathcal{J}_\lambda[(1-K)(\partial_1 u_1 + \partial_2 u_2)] =: \mathrm{II}_1 + \mathrm{II}_2. \tag{6-68}$$

For the term II_1 we use the estimates (6-45) of Lemma 6.5 and (4-22) of Lemma 4.4 to bound

$$|\mathrm{II}_1| \le \|\mathcal{J}_\lambda p\|_0 \|\mathcal{J}_\lambda[(1-K)G^2]\|_0 \lesssim \sqrt{\mathcal{E}_{2N}}\sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}^2} = \mathcal{E}_{2N}\mathcal{D}_{2N}. \tag{6-69}$$

In order to control the term II_2 we first integrate by parts:

$$\mathrm{II}_2 = \int_\Omega \mathcal{J}_\lambda \partial_1 p \mathcal{J}_\lambda[(1-K)u_1] + \mathcal{J}_\lambda \partial_2 p \mathcal{J}_\lambda[(1-K)u_2] - \mathcal{J}_\lambda p \mathcal{J}_\lambda[u_1 \partial_1 K + u_2 \partial_2 K]. \tag{6-70}$$

Then we use Lemmas 6.5 and 4.4 to estimate

$$\begin{aligned} |\mathrm{II}_2| &\lesssim \|\mathcal{J}_\lambda Dp\|_0 \|\mathcal{J}_\lambda[(1-K)u]\|_0 + \|\mathcal{J}_\lambda p\|_0 \sum_{i=1}^2 \|\mathcal{J}_\lambda[u\partial_i K]\|_0^2 \\ &\lesssim \sqrt{(\mathcal{E}_{2N})^{\lambda/(1+\lambda)}(\mathcal{D}_{2N})^{1/(1+\lambda)}}\sqrt{(\mathcal{E}_{2N})^{1/(1+\lambda)}(\mathcal{D}_{2N})^{(1+2\lambda)/(1+\lambda)}} + \sqrt{\mathcal{E}_{2N}}\sqrt{\mathcal{D}_{2N}^2} \\ &\lesssim \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \end{aligned} \tag{6-71}$$

Since $\mathcal{E}_{2N} \le 1$, we can combine (6-69) and (6-71) to find that $|\mathrm{II}| \lesssim \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}$, which yields (6-63) when combined with (6-66). □

With these two lemmas in hand, we can now estimate how the energies of $\mathcal{J}_\lambda u$ and $\mathcal{J}_\lambda \eta$ evolve.

Proposition 6.7. *We have*

$$\partial_t\left(\frac{1}{2}\int_\Omega |\mathcal{J}_\lambda u|^2 + \frac{1}{2}\int_\Sigma |\mathcal{J}_\lambda \eta|^2\right) + \frac{1}{2}\int_\Omega |\mathbb{D}\mathcal{J}_\lambda u|^2 \lesssim \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{6-72}$$

In particular,

$$\frac{1}{2}\int_\Omega |\mathcal{J}_\lambda u(t)|^2 + \frac{1}{2}\int_\Sigma |\mathcal{J}_\lambda \eta(t)|^2 + \frac{1}{2}\int_0^t \int_\Omega |\mathbb{D}\mathcal{J}_\lambda u|^2 \lesssim \mathcal{E}_{2N}(0) + \int_0^t \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{6-73}$$

Proof. We apply $\mathcal{J}_\lambda$ to the equations (2-23) and then use Lemma 2.5 to see that

$$\begin{aligned} &\partial_t\left(\frac{1}{2}\int_\Omega |\mathcal{J}_\lambda u|^2 + \frac{1}{2}\int_\Sigma |\mathcal{J}_\lambda \eta|^2\right) + \frac{1}{2}\int_\Omega |\mathbb{D}\mathcal{J}_\lambda u|^2 \\ &\qquad = \int_\Omega \mathcal{J}_\lambda u \cdot (\mathcal{J}_\lambda G^1 - \nabla \mathcal{J}_\lambda G^2) + \mathcal{J}_\lambda p \mathcal{J}_\lambda G^2 + \int_\Sigma -\mathcal{J}_\lambda u \cdot \mathcal{J}_\lambda G^3 + \mathcal{J}_\lambda \eta \mathcal{J}_\lambda G^4. \end{aligned} \tag{6-74}$$

We will estimate each term on the right side of the equation. First we use trace theory and (4-15) and (4-16) of Proposition 4.3 to bound the first and third terms:

$$\left|\int_\Omega \mathcal{I}_\lambda u\cdot(\mathcal{I}_\lambda G^1-\nabla\mathcal{I}_\lambda G^2)\right|+\left|\int_\Sigma \mathcal{I}_\lambda u\cdot\mathcal{I}_\lambda G^3\right| \\ \lesssim \|\mathcal{I}_\lambda u\|_0(\|\mathcal{I}_\lambda G^1\|_0+\|\mathcal{I}_\lambda G^2\|_1)+\|\mathcal{I}_\lambda u\|_1\|\mathcal{I}_\lambda G^3\|_0 \lesssim \sqrt{\mathfrak{D}_{2N}}\sqrt{\mathcal{E}_{2N}\mathfrak{D}_{2N}}=\sqrt{\mathcal{E}_{2N}}\mathfrak{D}_{2N}. \quad (6\text{-}75)$$

For the third term we use Lemma 6.6 for

$$\left|\int_\Omega \mathcal{I}_\lambda p\mathcal{I}_\lambda G^2\right|\lesssim\sqrt{\mathcal{E}_{2N}}\mathfrak{D}_{2N}. \quad (6\text{-}76)$$

Finally, for the fourth term we use (4-17) of Proposition 4.3:

$$\int_\Sigma \mathcal{I}_\lambda\eta\mathcal{I}_\lambda G^4\le\|\mathcal{I}_\lambda\eta\|_0\|\mathcal{I}_\lambda G^4\|_0\lesssim\sqrt{\mathcal{E}_{2N}}\sqrt{\mathfrak{D}_{2N}^2}=\sqrt{\mathcal{E}_{2N}}\mathfrak{D}_{2N}. \quad (6\text{-}77)$$

The bound (6-72) follows by combining (6-74)–(6-77), and then (6-73) follows from (6-72) by integrating in time from 0 to t. □

7. Energy evolution estimates

We now assemble the estimates of the previous two sections into an estimate for the evolution of $\bar{\mathcal{E}}_{2N}$ and $\bar{\mathfrak{D}}_{2N}$.

Theorem 7.1. *There exists a $\theta>0$ such that*

$$\bar{\mathcal{E}}_{2N}(t)+\int_0^t\bar{\mathfrak{D}}_{2N}(r)\,dr \\ \lesssim\mathcal{E}_{2N}(0)+(\mathcal{E}_{2N}(t))^{3/2}+\int_0^t(\mathcal{E}_{2N}(r))^\theta\mathfrak{D}_{2N}(r)\,dr+\int_0^t\sqrt{\mathfrak{D}_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}\,dr. \quad (7\text{-}1)$$

Proof. The result follows by summing the estimates of Propositions 5.3, 5.5, 6.2, and 6.7 and recalling the definitions of $\bar{\mathcal{E}}_{2N}$ and $\bar{\mathfrak{D}}_{2N}$ given by (2-48) and (2-49), respectively. □

We can also assemble the estimates of the previous two sections into a similar estimate for the evolution of $\bar{\mathcal{E}}_{N+2,m}$ and $\bar{\mathfrak{D}}_{N+2,m}$.

Theorem 7.2. *Let F^2 be given by (2-19) with $\partial^\alpha=\partial_t^{N+2}$. There exists a $\theta>0$ such that*

$$\partial_t\left(\bar{\mathcal{E}}_{N+2,m}-2\int_\Omega J\partial_t^{N+1}pF^2\right)+\bar{\mathfrak{D}}_{N+2,m}\lesssim\mathcal{E}_{2N}^\theta\mathfrak{D}_{N+2,m}. \quad (7\text{-}2)$$

Proof. The result follows by summing the estimates of Propositions 5.4 and 6.4 and recalling the definitions of $\bar{\mathcal{E}}_{N+2,m}$ and $\bar{\mathfrak{D}}_{N+2,m}$ given by (2-45) and (2-47), respectively. □

8. Comparison results

We now prove a pair of estimates that compare the full dissipation and energy to the horizontal dissipation and energy. We show that, up to some error terms, the instantaneous energy $\mathcal{E}_{2N}$, (2-50), is comparable to the horizontal energy $\bar{\mathscr{E}}_{2N}$, (2-48), and that the dissipation rate $\mathscr{D}_{2N}$, (2-51), is comparable to the horizontal dissipation rate $\bar{\mathscr{D}}_{2N}$, (2-49). We also prove similar results for $\bar{\mathscr{E}}_{N+2,m}$ and $\bar{\mathscr{D}}_{N+2,m}$ defined by (2-45) and (2-47), respectively. To prove results for both $2N$ and $N+2$, we first prove general estimates involving $\mathscr{D}_n$ and $\mathscr{E}_n$, and then we specialize to the cases $n = N+2$ and $n = 2N$. The dissipation estimates are more involved, so we begin with them.

Dissipation. We first consider the dissipation rate.

Theorem 8.1. *Let $m \in \{1, 2\}$ and*

$$\mathscr{Y}_{n,m} := \|\bar{\nabla}_m^{2n-1} G^1\|_0^2 + \|\bar{\nabla}_0^{2n-1} G^2\|_1^2 + \|\bar{D}_m^{2n-1} G^3\|_{1/2}^2 + \|\bar{D}_0^{2n-1} G^4\|_{1/2}^2 + \|\bar{D}_0^{2n-2} \partial_t G^4\|_{1/2}^2. \tag{8-1}$$

If $m = 1$, then

$$\|\nabla^3 u\|_{2n-2}^2 + \sum_{j=1}^{n} \|\partial_t^j u\|_{2n-2j+1}^2 + \|\nabla^2 p\|_{2n-2}^2 + \sum_{j=1}^{n-1} \|\partial_t^j p\|_{2n-2j} + \|D^2\eta\|_{2n-5/2}^2 + \|\partial_t \eta\|_{2n-1/2}^2 + \sum_{j=2}^{n+1} \|\partial_t^j \eta\|_{2n-2j+5/2}^2 \lesssim \bar{\mathscr{D}}_{n,m} + \mathscr{Y}_{n,m}. \tag{8-2}$$

If $m = 2$, then

$$\|\nabla^4 u\|_{2n-3}^2 + \sum_{j=1}^{n} \|\partial_t^j u\|_{2n-2j+1}^2 + \|\nabla^3 p\|_{2n-3}^2 + \|\partial_t \nabla p\|_{2n-3}^2 + \sum_{j=2}^{n-1} \|\partial_t^j p\|_{2n-2j}^2 + \|D^3\eta\|_{2n-7/2}^2 + \|D\partial_t \eta\|_{2n-3/2}^2 + \sum_{j=2}^{n+1} \|\partial_t^j \eta\|_{2n-2j+5/2}^2 \lesssim \bar{\mathscr{D}}_{n,m} + \mathscr{Y}_{n,m}. \tag{8-3}$$

Proof. In this proof we must use a separate counting for spatial and temporal derivatives, so unlike elsewhere in the paper, we now only use $\alpha \in \mathbb{N}^2$ to refer to spatial derivatives. In order to compactly write our estimates, throughout the proof we write

$$\mathscr{Z} := \bar{\mathscr{D}}_{n,m} + \mathscr{Y}_{n,m}. \tag{8-4}$$

The proof is divided into several steps.

Step 1: application of Korn's inequality. Since any horizontal or temporal derivative of u vanishes on the lower boundary Σ_b, we may apply Lemma A.12 to derive the bound

$$\|\bar{D}_m^{2n} u\|_1^2 \lesssim \|\bar{D}_m^{2n} \mathbb{D} u\|_0^2 = \bar{\mathscr{D}}_{n,m}. \tag{8-5}$$

This $H^1(\Omega)$ bound will be more useful in what follows than an $H^0(\Omega)$ estimate of the symmetric gradient.

Step 2: initial estimates of the pressure and improvement of u estimates. Let $0 \le j \le n-1$ and $\alpha \in \mathbb{N}^2$ be such that

$$m \le 2j + |\alpha| \le 2n - 1. \tag{8-6}$$

Note that if $2j+|\alpha|=2n-1$, the condition $j\le n-1$ implies that $|\alpha|\ge 1$. This means that we are free to use (8-5) to bound

$$\|\partial^\alpha \partial_t^{j+1} u\|_0^2 \le \|\bar{D}_m^{2n} u\|_1^2 \lesssim \mathfrak{X}. \tag{8-7}$$

To extract further information, we apply the operator $\partial_t^j\partial^\alpha$ to the first two equations in (2-23) to find that

$$\partial^\alpha \partial_t^{j+1} u - \Delta \partial^\alpha \partial_t^j u + \nabla \partial^\alpha \partial_t^j p = \partial^\alpha \partial_t^j G^1, \tag{8-8}$$

$$\operatorname{div} \partial^\alpha \partial_t^j u = \partial^\alpha \partial_t^j G^2. \tag{8-9}$$

Because of the constraints on j,α given by (8-6), we may control

$$\|\partial^\alpha \partial_t^j G^1\|_0^2 + \|\partial^\alpha \partial_t^j G^2\|_1^2 \le \|\bar{D}_m^{2n-1} G^1\|_0^2 + \|\bar{D}_m^{2n-1} G^2\|_1^2 \le \mathfrak{X}. \tag{8-10}$$

We utilize the structure of (8-8)–(8-9) in conjunction with (8-7) and (8-10) to improve our estimates.

We will begin by utilizing (8-9) to control one of the terms in the third component of (8-8). We have

$$\partial^\alpha \partial_t^j (\partial_3 u_3) = \partial^\alpha \partial_t^j (-\partial_1 u_1 - \partial_2 u_2 + G^2), \tag{8-11}$$

so that (8-5) and (8-10) imply

$$\|\partial_3^2 \partial^\alpha \partial_t^j u_3\|_0^2 \lesssim \|\bar{D}_m^{2n} u\|_1^2 + \|\bar{D}_m^{2n-1} G^2\|_1^2 \lesssim \mathfrak{X}. \tag{8-12}$$

A further application of (8-5) to control $(\partial_1^2+\partial_2^2)\partial^\alpha\partial_t^j u_3$ then provides the estimate

$$\|\Delta \partial^\alpha \partial_t^j u_3\|_0^2 \lesssim \mathfrak{X}. \tag{8-13}$$

Applying the bounds (8-7), (8-10), and (8-13) to the third component of (8-8), we arrive at a partial bound for the pressure:

$$\|\partial_3 \partial^\alpha \partial_t^j p\|_0^2 \lesssim \mathfrak{X}. \tag{8-14}$$

It remains to control the terms $\partial_i\partial^\alpha\partial_t^j p$ and $\partial_3^2\partial^\alpha\partial_t^j u_i$ for $i=1,2$. To accomplish this, we employ an elliptic estimate of $\operatorname{curl} u =: \omega$. Taking the curl of (8-8) eliminates the pressure gradient and yields

$$\partial^\alpha \partial_t^{j+1} \omega = \Delta \partial^\alpha \partial_t^j \omega + \operatorname{curl}(\partial^\alpha \partial_t^j G^1). \tag{8-15}$$

We only need the first two components $\omega_1=\partial_2 u_3-\partial_3 u_2$, $\omega_2=\partial_3 u_1-\partial_1 u_3$, for which we use the Σ boundary condition in (2-23)

$$\partial_i u_3 + \partial_3 u_i = \mathbb{D} u e_3 \cdot e_i = -G^3 \cdot e_i \quad \text{for } i=1,2 \tag{8-16}$$

to derive the boundary conditions

$$\begin{cases} \omega_1 = 2\partial_2 u_3 + G^3\cdot e_2 & \text{on } \Sigma, \\ \omega_2 = -2\partial_1 u_3 - G^3 \cdot e_1 & \text{on } \Sigma. \end{cases} \tag{8-17}$$

No similar boundary condition is available on Σ_b, so we must resort to a localization using a cutoff function $\chi=\chi(x_3)$ given by $\chi\in C_c^\infty(\mathbb{R})$ with $\chi(x_3)=1$ for $x_3\in\Omega_1 := [-2b/3, 0]$ and $\chi(x_3)=0$ for $x_3\notin(-3b/4, 1/2)$.

The functions $\chi\omega_i$, $i = 1, 2$, satisfy

$$\Delta\partial^\alpha\partial_t^j(\chi\omega_i) = \chi(\partial^\alpha\partial_t^{j+1}\omega_i) + 2(\partial_3\chi)(\partial_3\partial^\alpha\partial_t^j\omega_i) + (\partial_3^2\chi)(\partial^\alpha\partial_t^j\omega_i) - \chi\,\mathrm{curl}(\partial^\alpha\partial_t^j G^1) \tag{8-18}$$

in Ω as well as the boundary conditions

$$\begin{cases} \partial^\alpha\partial_t^j(\chi\omega_1) = 2\partial_2\partial^\alpha\partial_t^j u_3 + \partial^\alpha\partial_t^j G^3\cdot e_2 & \text{on } \Sigma, \\ \partial^\alpha\partial_t^j(\chi\omega_2) = -2\partial_1\partial^\alpha\partial_t^j u_3 - \partial^\alpha\partial_t^j G^3\cdot e_1 & \text{on } \Sigma, \\ \partial^\alpha\partial_t^j(\chi\omega_1) = \partial^\alpha\partial_t^j(\chi\omega_2) = 0 & \text{on } \Sigma_b. \end{cases} \tag{8-19}$$

In order to employ an elliptic estimate of $\partial^\alpha\partial_t^j(\chi\omega_i)$, we must first prove two auxiliary estimates.

First we derive an estimate of the $H^{-1}(\Omega) = (H_0^1(\Omega))^*$ norm of each term on the right side of (8-18). Let $\varphi \in H_0^1(\Omega)$. When $\alpha \neq 0$, we may write $\alpha = \beta + (\alpha - \beta)$ with $|\beta| = 1$ and integrate by parts to bound

$$\left|\int_\Omega \varphi\chi\partial^\alpha\partial_t^{j+1}\omega_i\right| = \left|\int_\Omega \partial^\beta\varphi\chi\partial^{\alpha-\beta}\partial_t^{j+1}\omega_i\right| \le \|\varphi\|_1\|\chi\bar{D}_m^{2n}\omega_i\|_0, \tag{8-20}$$

since $2(j+1) + |\alpha - \beta| = 2j + |\alpha| + 1 \in [m+1, 2n]$. We may use (8-5) for

$$\|\chi\bar{D}_m^{2n}\omega_i\|_0^2 \lesssim \|\bar{D}_m^{2n}u\|_1^2 \lesssim \mathfrak{X}. \tag{8-21}$$

Chaining these inequalities together when $\alpha \neq 0$ and taking the supremum over all φ such that $\|\varphi\|_1 \le 1$, we get

$$\|\partial^\alpha\partial_t^{j+1}\omega_i\|_{H^{-1}}^2 \lesssim \mathfrak{X}. \tag{8-22}$$

A similar argument without an integration by parts shows that (8-22) is also true when $\alpha = 0$, since, in this case, the condition $j \le n-1$ implies that $m+2 \le 2(j+1) \le 2n$. Similarly, integrating by parts with ∂_3 in the dual-pairing, we may estimate the second term on the right side of (8-18):

$$\|2(\partial_3\chi)(\partial_3\partial^\alpha\partial_t^j\omega_i)\|_{H^{-1}}^2 \lesssim (\|\partial_3\chi\|_{L^\infty}^2 + \|\partial_3^2\chi\|_{L^\infty}^2)\|\bar{D}_m^{2n}\omega_i\|_0^2 \lesssim \|\bar{D}_m^{2n}u\|_1^2 \lesssim \mathfrak{X}. \tag{8-23}$$

The third term may be estimated without integration by parts in the dual-pairing:

$$\|(\partial_3^2\chi)(\partial^\alpha\partial_t^j\omega_i)\|_{H^{-1}}^2 \lesssim \|\partial_3^2\chi\|_{L^\infty}^2\|\bar{D}_m^{2n}\omega_i\|_0^2 \lesssim \|\bar{D}_m^{2n}u\|_1^2 \lesssim \mathfrak{X}. \tag{8-24}$$

The fourth term is estimated by integrating by parts with the curl operator and using (8-10):

$$\|\chi\,\mathrm{curl}(\partial^\alpha\partial_t^j G^1)\|_{H^{-1}}^2 \lesssim (\|\chi\|_{L^\infty}^2 + \|\partial_3\chi\|_{L^\infty}^2)\|\bar{D}_m^{2n-1}G^1\|_0^2 \lesssim \mathfrak{X}. \tag{8-25}$$

Combining these four estimates of the right side of (8-18) yields

$$\|\Delta\partial^\alpha\partial_t^j(\chi\omega_i)\|_{H^{-1}}^2 \lesssim \mathfrak{X} \quad \text{for } i = 1, 2. \tag{8-26}$$

Next, to complete the elliptic estimate of $\partial^\alpha\partial_t^j(\chi\omega_i)$, we also need $H^{1/2}(\Sigma)$ estimates for the boundary terms on the right side of the first two equations in (8-19). We may estimate the $\partial_i u_3$, $i = 1, 2$, terms with the embedding $H^1(\Omega) \hookrightarrow H^{1/2}(\Sigma)$:

$$\|\partial^\alpha\partial_t^j\partial_1 u_3\|_{H^{1/2}(\Sigma)}^2 + \|\partial^\alpha\partial_t^j\partial_2 u_3\|_{H^{1/2}(\Sigma)}^2 \lesssim \|\bar{D}_m^{2n}u\|_1^2 \lesssim \mathfrak{X}. \tag{8-27}$$

On the other hand, estimates of G^3 are already built into $\mathfrak{X}$:

$$\|\partial^\alpha \partial_t^j G^3\|_{1/2}^2 \le \|\bar{D}_m^{2n-1} G^3\|_{1/2}^2 \le \mathfrak{Y}_{n,m} \le \mathfrak{X}. \tag{8-28}$$

Since $\chi\omega_i = 0$ on Σ_b for $i = 1, 2$, we then deduce that

$$\|\partial^\alpha \partial_t^j (\chi\omega_i)\|_{H^{1/2}(\partial\Omega)}^2 \lesssim \mathfrak{X} \quad \text{for } i = 1, 2. \tag{8-29}$$

Now, according to (8-26), (8-29), standard elliptic estimates, and the fact that $\chi = 1$ on $\Omega_1 = [-2b/3, 0]$, we have

$$\|\partial^\alpha \partial_t^j \omega_i\|_{H^1(\Omega_1)}^2 \lesssim \|\partial^\alpha \partial_t^j (\chi\omega_i)\|_1^2 \lesssim \mathfrak{X} \quad \text{for } i = 1, 2. \tag{8-30}$$

We may then rewrite

$$\partial_3^2 \partial^\alpha \partial_t^j u_1 = \partial_3 \partial^\alpha \partial_t^j (\omega_2 + \partial_1 u_3) \quad \text{and} \quad \partial_3^2 \partial^\alpha \partial_t^j u_2 = \partial_3 \partial^\alpha \partial_t^j (\partial_2 u_3 - \omega_1) \tag{8-31}$$

and deduce from (8-30) and (8-5) that, for $i = 1, 2$, we have

$$\|\partial_3^2 \partial^\alpha \partial_t^j u_i\|_{H^0(\Omega_1)}^2 \lesssim \|\bar{D}_m^{2n} u_3\|_1^2 + \sum_{k=1}^{2} \|\partial^\alpha \partial_t^j \omega_k\|_{H^1(\Omega_1)}^2 \lesssim \mathfrak{X}. \tag{8-32}$$

We then apply this estimate along with (8-5) and (8-10) to the first two components of (8-8) to find that

$$\|\partial_i \partial^\alpha \partial_t^j p\|_{H^0(\Omega_1)}^2 \lesssim \mathfrak{X} \quad \text{for } i = 1, 2. \tag{8-33}$$

Now we sum the estimates (8-5), (8-12), (8-14), (8-32), and (8-33) over all $j \le n-1$ and $\alpha \in \mathbb{N}^2$ with $m \le 2j + |\alpha| \le 2n - 1$ to deduce that

$$\|\bar{D}_m^{2n-1} u\|_{H^2(\Omega_1)}^2 + \|\bar{D}_m^{2n-1} \nabla p\|_{H^0(\Omega_1)}^2 \lesssim \mathfrak{X}. \tag{8-34}$$

Step 3: bootstrapping, η estimates, and improved pressure estimates. Now we make use of Lemma 8.2 to bootstrap from (8-5) and (8-34) to

$$\|\nabla^{2+m} u\|_{H^{2n-m-1}(\Omega_1)}^2 + \|D^m u\|_{H^{2n-m+1}(\Omega_1)}^2 + \sum_{j=1}^{n} \|\partial_t^j u\|_{H^{2n-2j+1}(\Omega_1)}^2 + \|\nabla^{1+m} p\|_{H^{2n-m-1}(\Omega_1)}^2 + \sum_{j=1}^{n-1} \|\partial_t^j \nabla p\|_{H^{2n-2j-1}(\Omega_1)}^2 \lesssim \mathfrak{X}. \tag{8-35}$$

With this estimate in hand, we may derive some estimates for η on Σ by employing the boundary conditions of (2-23):

$$\eta = p - 2\partial_3 u_3 - G_3^3, \tag{8-36}$$

$$\partial_t \eta = u_3 + G^4. \tag{8-37}$$

Then (8-35) allows us to differentiate (8-36) to find that

$$\begin{aligned} \|D^{1+m}\eta\|_{2n-m-3/2}^2 &\lesssim \|D^{1+m} p\|_{H^{2n-m-3/2}(\Sigma)}^2 + \|D^{1+m}\partial_3 u_3\|_{H^{2n-m-3/2}(\Sigma)}^2 + \|D^{1+m} G^3\|_{2n-m-3/2}^2 \\ &\lesssim \|\nabla^{1+m} p\|_{H^{2n-m-1}(\Omega_1)}^2 + \|\nabla^{2+m} u\|_{H^{2n-m-1}(\Omega_1)}^2 + \|D_{m+1}^{2n-1} G^3\|_{1/2}^2 \lesssim \mathfrak{X}. \end{aligned} \tag{8-38}$$

Similarly, for $j = 2, \dots, n+1$, we may apply ∂_t^{j-1} to (8-37) and estimate

$$\begin{aligned} \|\partial_t^j \eta\|^2_{2n-2j+5/2} &\lesssim \|\partial_t^{j-1} u_3\|^2_{H^{2n-2j+5/2}(\Sigma)} + \|\partial_t^{j-1} G^4\|^2_{2n-2j+5/2} \\ &\lesssim \|\partial_t^{j-1} u\|^2_{H^{2n-2(j-1)+1}(\Omega_1)} + \|\partial_t^{j-1} G^4\|^2_{2n-2(j-1)+1/2} \lesssim \mathfrak{X}. \end{aligned} \tag{8-39}$$

It remains only to consider $\partial_t \eta$; in this case we must consider $m = 1$ and $m = 2$ separately. For $m = 1$, we again use (8-37) to see that

$$\|\partial_t \eta\|^2_{2n-1/2} \lesssim \|u_3\|^2_{H^{2n-1/2}(\Sigma)} + \|G^4\|^2_{2n-1/2} \lesssim \|u_3\|^2_{H^{2n-1/2}(\Sigma)} + \mathfrak{X}, \tag{8-40}$$

but now we use Lemma A.11, trace theory, and the second equation in (2-23) for the estimate

$$\begin{aligned} \|u_3\|^2_{H^{2n-1/2}(\Sigma)} &\lesssim \|u_3\|^2_{H^0(\Sigma)} + \|Du_3\|^2_{H^{2n-3/2}(\Sigma)} \lesssim \|\partial_3 u_3\|^2_{H^0(\Omega)} + \|Du_3\|^2_{H^{2n-1}(\Omega_1)} \\ &\lesssim \|G^2\|^2_0 + \|Du\|^2_0 + \|Du\|^2_{H^{2n-1}(\Omega_1)} \lesssim \mathfrak{X} \end{aligned} \tag{8-41}$$

by (8-10) and (8-35). Chaining (8-40)–(8-41) together implies that

$$\|\partial_t \eta\|^2_{2n-1/2} \lesssim \mathfrak{X} \quad \text{when } m = 1. \tag{8-42}$$

For $m = 2$, we differentiate (8-37) for the bound

$$\|D\partial_t \eta\|^2_{2n-3/2} \lesssim \|Du_3\|^2_{H^{2n-3/2}(\Sigma)} + \|DG^4\|^2_{2n-3/2} \lesssim \|Du_3\|^2_{H^{2n-3/2}(\Sigma)} + \mathfrak{X}, \tag{8-43}$$

but then the analogue of (8-41) is

$$\|Du_3\|^2_{H^{2n-3/2}(\Sigma)} \lesssim \|DG^2\|^2_0 + \|D^2 u\|^2_0 + \|D^2 u\|^2_{H^{2n-2}(\Omega_1)} \lesssim \mathfrak{X}. \tag{8-44}$$

Hence

$$\|D\partial_t \eta\|^2_{2n-3/2} \lesssim \mathfrak{X} \quad \text{when } m = 2. \tag{8-45}$$

Summing estimates (8-38), (8-39), (8-42), and (8-45) over $j = 0, \dots, n+1$ yields

$$\|D^2 \eta\|^2_{2n-5/2} + \|\partial_t \eta\|^2_{2n-1/2} + \sum_{j=2}^{n+1} \|\partial_t^j \eta\|^2_{2n-2j+5/2} \lesssim \mathfrak{X} \quad \text{for } m = 1, \tag{8-46}$$

$$\|D^3 \eta\|^2_{2n-7/2} + \|D\partial_t \eta\|^2_{2n-3/2} + \sum_{j=2}^{n+1} \|\partial_t^j \eta\|^2_{2n-2j+5/2} \lesssim \mathfrak{X} \quad \text{for } m = 2. \tag{8-47}$$

The η estimates (8-46)–(8-47) now allow us to improve our estimates of $\nabla \partial_t^j p$ to estimates for $\partial_t^j p$ for certain values of j. Indeed, for $j = m, \dots, n-1$ we may use Lemma A.10 and (8-36) to bound

$$\|\partial_t^j p\|^2_{H^0(\Omega_1)} \lesssim \|\partial_t^j \eta\|^2_0 + \|\partial_3 \partial_t^j u_3\|^2_{H^0(\Sigma)} + \|\partial_t^j G^3\|^2_0 + \|\partial_t^j \nabla p\|^2_{H^0(\Omega_1)} \lesssim \|\partial_t^j u_3\|^2_{H^2(\Omega_1)} + \mathfrak{X} \lesssim \mathfrak{X}. \tag{8-48}$$

This, (8-35), and (8-46)–(8-47) allow us to improve (8-35); when $m = 1$, we find that

$$\|\nabla^3 u\|^2_{H^{2n-2}(\Omega_1)} + \|Du\|^2_{H^{2n}(\Omega_1)} + \sum_{j=1}^{n} \|\partial_t^j u\|^2_{H^{2n-2j+1}(\Omega_1)} + \|\nabla^2 p\|^2_{H^{2n-2}(\Omega_1)} + \sum_{j=1}^{n-1} \|\partial_t^j p\|^2_{H^{2n-2j}(\Omega_1)} + \|D^2\eta\|^2_{2n-5/2} + \|\partial_t \eta\|^2_{2n-1/2} + \sum_{j=2}^{n+1} \|\partial_t^j \eta\|^2_{2n-2j+5/2} \lesssim \mathfrak{X}, \tag{8-49}$$

and when $m = 2$, we get the estimate

$$\|\nabla^4 u\|^2_{H^{2n-3}(\Omega_1)} + \|D^2 u\|^2_{H^{2n-1}(\Omega_1)} + \sum_{j=1}^{n} \|\partial_t^j u\|^2_{H^{2n-2j+1}(\Omega_1)} + \|\nabla^3 p\|^2_{H^{2n-3}(\Omega_1)} + \|\partial_t \nabla p\|^2_{H^{2n-3}(\Omega_1)} + \sum_{j=2}^{n-1} \|\partial_t^j p\|^2_{H^{2n-2j}(\Omega_1)} + \|D^3\eta\|^2_{2n-7/2} + \|D\partial_t \eta\|^2_{2n-3/2} + \sum_{j=2}^{n+1} \|\partial_t^j \eta\|^2_{2n-2j+5/2} \lesssim \mathfrak{X}. \tag{8-50}$$

Step 4: estimates in Ω_2. We now extend our estimates to the lower part of the domain, that is, $\Omega_2 := [-b, -b/3]$, by applying Lemma 8.3 to deduce that (8-97) holds when $m = 1$ and (8-98) holds when $m = 2$. We will now show that $\mathscr{X}_{n,m}$, defined by (8-96), can be controlled by $\mathfrak{X}$. The key to this is that, by construction, $\operatorname{supp}(\nabla\chi_2) \subset \Omega_1$, which implies that the H^1 and H^2 defined in the lemma satisfy $\operatorname{supp}(H^1) \cup \operatorname{supp}(H^2) \subset \Omega_1$. This allows us to use the estimates (8-49) in the case $m = 1$ and (8-50) in the case $m = 2$ to bound

$$\sum_{k=m+1}^{2n-1} \|\bar{D}^k H^1\|^2_{2n-k-1} + \|\bar{D}^k H^2\|^2_{2n-k} \lesssim \mathfrak{X}. \tag{8-51}$$

In order to estimate $\partial_t H^1 \cdot e_i$ for $i = 1, 2$, we note that it does not involve the pressure:

$$\partial_t H^1 \cdot e_i = -(\partial_3 \chi_2)\partial_3 \partial_t u_i - (\partial_3^2 \chi_2)\partial_t u_i. \tag{8-52}$$

Then we may again use (8-49)–(8-50) to see that

$$\sum_{i=1}^{2} \|\partial_t H^1 \cdot e_i\|^2_{2n-3} \lesssim \mathfrak{X}, \tag{8-53}$$

so that $\mathscr{X}_{n,m} \lesssim \mathfrak{X}$. Replacing in (8-97) and (8-98), we then find that

$$\|\nabla^3 u\|^2_{H^{2n-2}(\Omega_2)} + \sum_{j=1}^{n} \|\partial_t^j u\|^2_{H^{2n-2j+1}(\Omega_2)} + \|\nabla^2 p\|^2_{H^{2n-2}(\Omega_2)} + \sum_{j=1}^{n-1} \|\partial_t^j p\|^2_{H^{2n-2j}(\Omega_2)} \lesssim \mathfrak{X} \tag{8-54}$$

for $m = 1$, while, for $m = 2$,

$$\|\nabla^4 u\|^2_{H^{2n-3}(\Omega_2)} + \sum_{j=1}^{n} \|\partial_t^j u\|^2_{H^{2n-2j+1}(\Omega_2)} + \|\nabla^3 p\|^2_{H^{2n-3}(\Omega_2)} + \|\partial_t \nabla p\|^2_{H^{2n-3}(\Omega_2)} + \sum_{j=2}^{n-1} \|\partial_t^j p\|^2_{H^{2n-2j}(\Omega_2)} \lesssim \mathfrak{X}. \tag{8-55}$$

Step 5: synthesis and conclusion. To conclude, we note that $\Omega = \Omega_1 \cup \Omega_2$, which allows us to add the localized estimates (8-49) and (8-54) to deduce (8-2), and to add (8-50) to (8-55) to deduce (8-3). $\square$

We now present the key bootstrap estimate used in the proof of Theorem 8.1.

Lemma 8.2. *Let $\mathcal{Y}_{n,m}$ be defined by* (8-1) *and $\Omega_1 = [-2b/3, 0]$. Suppose that*

$$\|\bar{D}_m^{2n-2r+2} u\|^2_{H^{2r-1}(\Omega_1)} + \|\bar{D}_m^{2n-2r+1} u\|^2_{H^{2r}(\Omega_1)} + \|\bar{D}_m^{2n-2r+1} \nabla p\|^2_{H^{2r-2}(\Omega_1)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} \tag{8-56}$$

for an integer $r \in [1, \dots, n-(m+1)/2]$. Then

$$\|\bar{D}_m^{2n-2r} u\|^2_{H^{2r+1}(\Omega_1)} + \|\bar{D}_m^{2n-2r} \nabla p\|^2_{H^{2r-1}(\Omega_1)} + \|\bar{D}_m^{2n-2(r+1)+1} u\|^2_{H^{2r+2}(\Omega_1)} + \|\bar{D}_m^{2n-2(r+1)+1} \nabla p\|^2_{H^{2r}(\Omega_1)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m}. \tag{8-57}$$

Moreover, if (8-56) *holds with $r = 1$, then, for $m = 1, 2$, we have*

$$\|\nabla^{2+m} u\|^2_{H^{2n-m-1}(\Omega_1)} + \|D^m u\|^2_{H^{2n-m+1}(\Omega_1)} + \sum_{j=1}^{n} \|\partial_t^j u\|^2_{H^{2n-2j+1}(\Omega_1)} + \|\nabla^{1+m} p\|^2_{H^{2n-m-1}(\Omega_1)} + \sum_{j=1}^{n-1} \|\partial_t^j \nabla p\|^2_{H^{2n-2j-1}(\Omega_1)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m}. \tag{8-58}$$

Proof. Throughout the proof we write $\mathcal{Z} := \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m}$. We divide the proof into steps.

Step 1: Proof of (8-57). Let $\ell \in \{1, 2\}$ and take $0 \le j \le n-r$ and $\alpha \in \mathbb{N}^2$ such that

$$m \le 2j + |\alpha| \le 2n - 2r + 1 - \ell. \tag{8-59}$$

We apply the differential operator $\partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j$ to the first equation in (2-23) and split into separate equations for its third and first two components; after some rearrangement, these read

$$\partial_3^{2r-1+\ell} \partial^\alpha \partial_t^j p = -\partial_3^{2r-2+\ell} \partial^\alpha \partial_t^{j+1} u_3 + \Delta \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j u_3 + \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j G_3^1, \tag{8-60}$$

$$\Delta \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j u_i = \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^{j+1} u_i + \partial_i \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j p - \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j G_i^1 \tag{8-61}$$

for $i = 1, 2$. Notice that the constraints on r, j, $|\alpha|$ imply that $m \le |\alpha| + (2r-2+\ell) + 2j \le 2n-1$, so we may use the definition of $\mathcal{Y}_{n,m}$ in (8-1) to estimate

$$\|\partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j G^1\|_0^2 + \|\partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j G^2\|_1^2 \le \mathcal{Y}_{n,m} \le \mathcal{Z}. \tag{8-62}$$

Since $2r - 2 + \ell \ge 0$, we know that

$$\|\partial_3^{2r-2+\ell} \partial^\alpha \partial_t^{j+1} u\|^2_{H^0(\Omega_1)} \le \|\partial^\alpha \partial_t^{j+1} u\|^2_{H^{2r-2+\ell}(\Omega_1)}. \tag{8-63}$$

If $\ell = 2$ then $m \le |\alpha| + 2(j+1) \le 2n - 2r + 1$, so that

$$\|\partial^\alpha \partial_t^{j+1} u\|^2_{H^{2r-2+\ell}(\Omega_1)} = \|\partial^\alpha \partial_t^{j+1} u\|^2_{H^{2r}(\Omega_1)} \le \|\bar{D}_m^{2n-2r+1} u\|^2_{H^{2r}(\Omega_1)} \lesssim \mathcal{Z}. \tag{8-64}$$

On the other hand, if $\ell = 1$, then $m \le |\alpha| + 2(j+1) \le 2n - 2r + 2$, and hence

$$\|\partial^\alpha \partial_t^{j+1} u\|^2_{H^{2r-2+\ell}(\Omega_1)} = \|\partial^\alpha \partial_t^{j+1} u\|^2_{H^{2r-1}(\Omega_1)} \le \|\bar{D}_m^{2n-2r+2} u\|^2_{H^{2r-1}(\Omega_1)} \lesssim \mathcal{Z}. \tag{8-65}$$

Then, in either case,

$$\|\partial_3^{2r-2+\ell}\partial^\alpha\partial_t^{j+1}u\|^2_{H^0(\Omega_1)} \lesssim \mathfrak{X}. \tag{8-66}$$

We have written the equations (8-60)–(8-61) in this form so as to be able to employ the estimates (8-56), (8-62), and (8-66) to derive (8-57). We must consider the cases of $\ell = 1$ and $\ell = 2$ separately, starting with $\ell = 1$.

Let $\ell = 1$. According to the equation $\operatorname{div} u = G^2$ (the second of (2-23)), the constraint (8-59), and the bounds (8-56) and (8-62), we may estimate

$$\begin{aligned}\|\partial_3^{2r+1}\partial^\alpha\partial_t^j u_3\|^2_{H^0(\Omega_1)} &= \|\partial_3^{2r}\partial^\alpha\partial_t^j(G^2 - \partial_1 u_1 - \partial_2 u_2)\|^2_{H^0(\Omega_1)} \\ &\lesssim \|\partial_3^{2r-1}\partial^\alpha\partial_t^j G^2\|_1^2 + \|\partial^\alpha\partial_t^j(\partial_1 u_1 + \partial_2 u_2)\|^2_{H^{2r}(\Omega_1)} \lesssim \mathfrak{X},\end{aligned} \tag{8-67}$$

and hence (again using the constraint (8-59))

$$\|\Delta(\partial_3^{2r-1}\partial^\alpha\partial_t^j u_3)\|^2_{H^0(\Omega_1)} \lesssim \|\partial_3^{2r+1}\partial^\alpha\partial_t^j u_3\|^2_{H^0(\Omega_1)} + \|\partial_3^{2r-1}(\partial_1^2+\partial_2^2)\partial^\alpha\partial_t^j u_3\|^2_{H^0(\Omega_1)} \lesssim \mathfrak{X}. \tag{8-68}$$

We may then use (8-62), (8-66), and (8-68) in (8-60) for the pressure estimate

$$\|\partial_3^{2r}\partial^\alpha\partial_t^j p\|^2_{H^0(\Omega_1)} \lesssim \mathfrak{X}. \tag{8-69}$$

Turning now to the $i = 1, 2$ components, we note that, by (8-56) and the constraint (8-59),

$$\begin{aligned}&\|\partial_i\partial_3^{2r-1}\partial^\alpha\partial_t^j p\|^2_{H^0(\Omega_1)} + \|(\partial_1^2+\partial_2^2)\partial_3^{2r-1}\partial^\alpha\partial_t^j u_i\|^2_{H^0(\Omega_1)} \\ &\qquad\lesssim \|\bar{D}_m^{2n-2r+1}\nabla p\|^2_{H^{2r-2}(\Omega_1)} + \|\bar{D}_m^{2n-2r+1}u\|^2_{H^{2r}(\Omega_1)} \lesssim \mathfrak{X}\end{aligned} \tag{8-70}$$

for $i = 1, 2$. Plugging this, (8-62), and (8-66) into (8-61) then shows that

$$\|\partial_3^{2r+1}\partial^\alpha\partial_t^j u_i\|^2_{H^0(\Omega_1)} \lesssim \mathfrak{X} \quad \text{for } i = 1, 2. \tag{8-71}$$

Upon summing (8-67), (8-69), and (8-71) over $0 \le j \le n-r$ and α satisfying $m \le 2j + |\alpha| \le 2n-2r$, we deduce that

$$\|\partial_3^{2r+1}\bar{D}_m^{2n-2r}u\|^2_{H^0(\Omega_1)} + \|\partial_3^{2r}\bar{D}_m^{2n-2r}p\|^2_{H^0(\Omega_1)} \lesssim \mathfrak{X}. \tag{8-72}$$

Then, in light of (8-56) and (8-72), we have

$$\begin{aligned}\|\bar{D}_m^{2n-2r}u\|^2_{H^{2r+1}(\Omega_1)} + \|\bar{D}_m^{2n-2r}\nabla p\|^2_{H^{2r-1}(\Omega_1)} &\lesssim \|\bar{D}_m^{2n-2r+1}u\|^2_{H^{2r}(\Omega_1)} \\ &\quad + \|\bar{D}_m^{2n-2r+1}\nabla p\|^2_{H^{2r-2}(\Omega_1)} + \|\partial_3^{2r+1}\bar{D}_m^{2n-2r}u\|^2_{H^0(\Omega_1)} + \|\partial_3^{2r}\bar{D}_m^{2n-2r}p\|^2_{H^0(\Omega_1)} \lesssim \mathfrak{X}.\end{aligned} \tag{8-73}$$

In the case $\ell = 2$ we may argue as in the case $\ell = 1$, utilizing both (8-56) and (8-73) to derive the bound

$$\|\bar{D}_m^{2n-2r-1}u\|^2_{H^{2r+2}(\Omega_1)} + \|\bar{D}_m^{2n-2r-1}\nabla p\|^2_{H^{2r}(\Omega_1)} \lesssim \mathfrak{X}. \tag{8-74}$$

Then we may add (8-73) to (8-74) to deduce (8-57).

Step 2: The proof of (8-58), *part 1.* Now we turn to the proof of (8-58), assuming that (8-56) holds with $r=1$. By (8-57) we may iterate with $r=2$, $r=3$, etc., until

$$r=\begin{cases} n-1 & \text{if } m=1, \\ n-2 & \text{if } m=2, \end{cases} \quad \text{so that} \quad 2n-2(r+2)+1=\begin{cases} 1 & \text{if } m=1, \\ 3 & \text{if } m=2. \end{cases} \tag{8-75}$$

Summing the resulting bounds and adding (8-5) (to pick up the $\partial_t^n u$ term) yields the estimates

$$\|D^1 u\|^2_{H^{2n}(\Omega_1)}+\sum_{j=1}^{n}\|\partial_t^j u\|^2_{H^{2n-2j+1}(\Omega_1)}+\|D^1\nabla p\|^2_{H^{2n-2}(\Omega_1)}+\sum_{j=1}^{n-1}\|\partial_t^j\nabla p\|^2_{H^{2n-2j-1}(\Omega_1)}\lesssim \mathfrak{X} \tag{8-76}$$

in the case $m=1$, and

$$\begin{aligned}\|D_2^3 u\|^2_{H^{2n-2}(\Omega_1)}&+\|D_0^1\partial_t u\|^2_{H^{2n-2}(\Omega_1)}+\sum_{j=2}^{n}\|\partial_t^j u\|^2_{H^{2n-2j+1}(\Omega_1)} \\ &+\|D_2^3\nabla p\|^2_{H^{2n-4}(\Omega_1)}+\|D_0^1\partial_t\nabla p\|^2_{H^{2n-4}(\Omega_1)}+\sum_{j=2}^{n-1}\|\partial_t^j\nabla p\|^2_{H^{2n-2j-1}(\Omega_1)}\lesssim \mathfrak{X}\end{aligned} \tag{8-77}$$

in the case $m=2$.

Next, we improve the estimate (8-77). Let $0\le j$ and $\alpha\in\mathbb{N}^2$ be such that $2j+|\alpha|=2$, and apply the operator $\partial_3^{2n-3}\partial^\alpha\partial_t^j$ to the first equation of (2-23) and split into components as above to get

$$\partial_3^{2n-2}\partial^\alpha\partial_t^j p=-\partial_3^{2n-3}\partial^\alpha\partial_t^{j+1}u_3+\Delta\partial_3^{2n-3}\partial^\alpha\partial_t^j u_3+\partial_3^{2n-3}\partial^\alpha\partial_t^j G_3^1, \tag{8-78}$$

$$\Delta\partial_3^{2n-3}\partial^\alpha\partial_t^j u_i=\partial_3^{2n-3}\partial^\alpha\partial_t^{j+1}u_i+\partial_i\partial_3^{2n-3}\partial^\alpha\partial_t^j p-\partial_3^{2n-3}\partial^\alpha\partial_t^j G_i^1 \tag{8-79}$$

for $i=1,2$. We may then argue as above, utilizing (8-77), to deduce the bounds

$$\|\partial_3^{2n-1}\partial^\alpha\partial_t^j u_3\|^2_{H^0(\Omega_1)}+\|\partial_3^{2n-3}\partial^\alpha\partial_t^{j+1}u\|^2_{H^0(\Omega_1)}+\|D^2\partial_3^{2n-3}\partial^\alpha\partial_t^j u\|^2_{H^0(\Omega_1)}\lesssim \mathfrak{X}, \tag{8-80}$$

which, when combined with (8-78) and (8-79), imply that

$$\|\partial_3^{2n-2}\partial^\alpha\partial_t^j p\|^2_{H^0(\Omega_1)}+\|\partial_3^{2n-1}\partial^\alpha\partial_t^j u_i\|^2_{H^0(\Omega_1)}\lesssim \mathfrak{X} \tag{8-81}$$

for $i=1,2$. We may then use (8-80) and (8-81) with (8-77) to deduce that

$$\|D^2 u\|^2_{H^{2n-1}(\Omega_1)}+\sum_{j=1}^{n}\|\partial_t^j u\|^2_{H^{2n-2j+1}(\Omega_1)}+\|D^2\nabla p\|^2_{H^{2n-3}(\Omega_1)}+\sum_{j=1}^{n-1}\|\partial_t^j\nabla p\|^2_{H^{2n-2j-1}(\Omega_1)}\lesssim \mathfrak{X} \tag{8-82}$$

in the case $m=2$.

Step 3: The proof of (8-58), *part 2.* Now we claim that if for $m=1,2$ we have the inequality

$$\|D^m u\|^2_{H^{2n-m+1}(\Omega_1)}+\sum_{j=1}^{n}\|\partial_t^j u\|^2_{H^{2n-2j+1}(\Omega_1)}+\|D^m\nabla p\|^2_{H^{2n-m-1}(\Omega_1)}+\sum_{j=1}^{n-1}\|\partial_t^j\nabla p\|^2_{H^{2n-2j-1}(\Omega_1)}\lesssim \mathfrak{X}, \tag{8-83}$$

the inequality

$$\|\nabla^{2+m}u\|^2_{H^{2n-m-1}(\Omega_1)}+\|\nabla^{1+m}p\|^2_{H^{2n-m-1}(\Omega_1)}\lesssim \mathfrak{X} \tag{8-84}$$

also holds, which establishes the desired bound, (8-58), because of our inequalities (8-76) in the case $m=1$ and (8-82) in the case $m=2$. We begin the proof of the claim by noting that, since $2 \geq m$, we may use (8-83) to bound

$$\|\partial_3^m D^2 u\|^2_{H^{2n-m-1}(\Omega_1)} + \|\partial_3^{m-1} D\bar{D}_2^2 u\|^2_{H^{2n-m-1}(\Omega_1)} + \|\partial_3^{m-1} D^2 p\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathfrak{X}. \tag{8-85}$$

Now we let $|\alpha|=1$ and apply $\partial_3^m \partial^\alpha$ to the second equation of (2-23) to find that

$$\|\partial_3^{m+1}\partial^\alpha u_3\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \|\partial_3^m DG^2\|^2_{H^{2n-m-1}(\Omega_1)} + \|\partial_3^m D^2 u\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathfrak{X}. \tag{8-86}$$

Then we apply $\partial_3^{m-1}\partial^\alpha$ to the first equation of (2-23) to bound

$$\begin{aligned}\|\partial_3^m \partial^\alpha p\|^2_{H^{2n-m-1}(\Omega_1)} &\\ \lesssim \|\partial_3^{m+1}\partial^\alpha u_3\|^2_{H^{2n-m-1}(\Omega_1)} &+ \|\partial_3^{m-1}\partial^\alpha \bar{D}_2^2 u_3\|^2_{H^{2n-m-1}(\Omega_1)} + \|\partial_3^{m-1}\partial^\alpha G^1\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathfrak{X}\end{aligned} \tag{8-87}$$

and

$$\begin{aligned}\|\partial_3^{m+1}\partial^\alpha u_i\|^2_{H^{2n-m-1}(\Omega_1)} &\\ \lesssim \|\partial_3^{m-1}\partial^\alpha \bar{D}_2^2 u\|^2_{H^{2n-m-1}(\Omega_1)} &+ \|\partial_3^{m-1}\partial^\alpha Dp\|^2_{H^{2n-m-1}(\Omega_1)} + \|\partial_3^{m-1}\partial^\alpha G^1\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathfrak{X}\end{aligned} \tag{8-88}$$

for $i=1,2$. Summing (8-86)–(8-88) over all $|\alpha|=1$ then yields the inequality

$$\|\partial_3^{m+1} Du\|^2_{H^{2n-m-1}(\Omega_1)} + \|\partial_3^m Dp\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathfrak{X}. \tag{8-89}$$

Now we use (8-89) to improve to one more ∂_3 and one fewer horizontal derivative. We apply ∂_3^{m+1} to the second equation of (2-23) to find that

$$\|\partial_3^{m+2} u_3\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \|\partial_3^{m+1} G^2\|^2_{H^{2n-m-1}(\Omega_1)} + \|\partial_3^{m+1} Du\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathfrak{X}. \tag{8-90}$$

Then we apply ∂_3^m to the first equation of (2-23) to bound

$$\|\partial_3^{m+1} p\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \|\partial_3^{m+2} u_3\|^2_{H^{2n-m-1}(\Omega_1)} + \|\partial_3^m \bar{D}_2^2 u_3\|^2_{H^{2n-m-1}(\Omega_1)} + \|\partial_3^m G^1\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathfrak{X}, \tag{8-91}$$

$$\|\partial_3^{m+2} u_i\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \|\partial_3^m \bar{D}_2^2 u\|^2_{H^{2n-m-1}(\Omega_1)} + \|\partial_3^m Dp\|^2_{H^{2n-m-1}(\Omega_1)} + \|\partial_3^m G^1\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathfrak{X} \tag{8-92}$$

for $i=1,2$. Summing (8-90)–(8-92) then yields the inequality

$$\|\partial_3^{m+2} u\|^2_{H^{2n-m-1}(\Omega_1)} + \|\partial_3^{m+1} p\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathfrak{X}. \tag{8-93}$$

Finally, to complete the proof of the claim, we note that

$$\begin{aligned}\|\nabla^{2+m} u\|^2_{H^{2n-m-1}(\Omega_1)} + \|\nabla^{1+m} p\|^2_{H^{2n-m-1}(\Omega_1)} &\lesssim \|D^m u\|^2_{H^{2n-m+1}(\Omega_1)} + \|D^m \nabla p\|^2_{H^{2n-m-1}(\Omega_1)} \\ &\quad + \sum_{l=0}^{m-1} \|\partial_3^{m+2-\ell} D^\ell u\|^2_{H^{2n-m-1}(\Omega_1)} + \|\partial_3^{m+1-\ell} D^\ell p\|^2_{H^{2n-m-1}(\Omega_1)}.\end{aligned} \tag{8-94}$$

This and the bounds (8-83), (8-89), and (8-93) prove the claim. □

The following result allows for control of the dissipation rate in the lower domain.

Lemma 8.3. *Let $\chi_2 \in C_c^\infty(\mathbb{R})$ be such that $\chi_2(x_3) = 1$ for $x_3 \in \Omega_2 := [-b, -b/3]$ and $\chi_2(x_3) = 0$ for $x_3 \notin (-2b, -b/6)$. Let*

$$H^1 = \partial_3 \chi_2 (p e_3 - 2\partial_3 u) - (\partial_3^2 \chi_2) u \quad \text{and} \quad H^2 = \partial_3 \chi_2 u_3. \tag{8-95}$$

Define

$$\mathscr{X}_{n,m} = \sum_{k=m+1}^{2n-1} \|\bar{D}^k H^1\|_{2n-k-1}^2 + \|\bar{D}^k H^2\|_{2n-k}^2 + \sum_{i=1}^{2} \|\partial_t H^1 \cdot e_i\|_{2n-3}^2, \tag{8-96}$$

and let $\mathscr{Y}_{n,m}$ be as defined in (8-1). If $m = 1$, then

$$\|\nabla^3 u\|_{H^{2n-2}(\Omega_2)}^2 + \sum_{j=1}^{n} \|\partial_t^j u\|_{H^{2n-2j+1}(\Omega_2)}^2 + \|\nabla^2 p\|_{H^{2n-2}(\Omega_2)}^2 + \sum_{j=1}^{n-1} \|\partial_t^j p\|_{H^{2n-2j}(\Omega_2)}^2 \lesssim \bar{\mathscr{D}}_{n,m} + \mathscr{Y}_{n,m} + \mathscr{X}_{n,m}. \tag{8-97}$$

If $m = 2$, then

$$\|\nabla^4 u\|_{H^{2n-3}(\Omega_2)}^2 + \sum_{j=1}^{n} \|\partial_t^j u\|_{H^{2n-2j+1}(\Omega_2)}^2 + \|\nabla^3 p\|_{H^{2n-3}(\Omega_2)}^2 + \|\partial_t \nabla p\|_{H^{2n-3}(\Omega_2)}^2 + \sum_{j=2}^{n-1} \|\partial_t^j p\|_{H^{2n-2j}(\Omega_2)}^2 \lesssim \bar{\mathscr{D}}_{n,m} + \mathscr{Y}_{n,m} + \mathscr{X}_{n,m}. \tag{8-98}$$

Proof. When we localize with χ_2, we find that $\chi_2 u$ and $\chi_2 p$ solve

$$\begin{cases} -\Delta(\chi_2 u) + \nabla(\chi_2 p) = -\partial_t(\chi_2 u) + \chi_2 G^1 + H^1 & \text{in } \Omega, \\ \operatorname{div}(\chi_2 u) = \chi_2 G^2 + H^2 & \text{in } \Omega, \\ ((\chi_2 p)I - \mathbb{D}(\chi_2 u)) e_3 = 0 & \text{on } \Sigma, \\ \chi_2 u = 0 & \text{on } \Sigma_b. \end{cases} \tag{8-99}$$

Let $0 \le j \le n-1$ and $\alpha \in \mathbb{N}^2$ be such that

$$m + 1 \le |\alpha| + 2j \le 2n - 1. \tag{8-100}$$

Then we may apply Lemma A.14 and use the definition of $\mathscr{Y}_{n,m}$ given in (8-1) to see that

$$\begin{aligned} \|\partial^\alpha \partial_t^j (\chi_2 u)\|_{2n-|\alpha|-2j+1}^2 + \|\partial^\alpha \partial_t^j (\chi_2 p)\|_{2n-|\alpha|-2j}^2 &\lesssim \|\partial^\alpha \partial_t^{j+1} (\chi_2 u)\|_{2n-|\alpha|-2(j+1)+1}^2 + \|\partial^\alpha \partial_t^j (\chi_2 G^1 + H^1)\|_{2n-|\alpha|-2j-1}^2 \\ &\qquad + \|\partial^\alpha \partial_t^j (\chi_2 G^2 + H^2)\|_{2n-|\alpha|-2j}^2 \\ &\lesssim \|\partial^\alpha \partial_t^{j+1} (\chi_2 u)\|_{2n-|\alpha|-2(j+1)+1}^2 + \mathscr{Y}_{n,m} + \mathscr{X}_{n,m}. \end{aligned} \tag{8-101}$$

We first use estimate (8-101) and a finite induction to arrive at initial estimates for $\chi_2 u$ and $\chi_2 p$; we then use the structure of the equations (2-23) to improve these estimates.

Our finite induction will be performed on $\ell \in [1, 2n - m - 1]$ with $|\alpha| + 2j = 2n - \ell$, starting with the first two initial values, $\ell = 1$ and $\ell = 2$. We use the definition of $\bar{\mathscr{D}}_{n,m}$ given in (2-47) and Lemma A.12

in conjunction with the bounds on j, $|\alpha|$ given in (8-100) to see that

$$\|\partial^\alpha \partial_t^{j+1}(\chi_2 u)\|_0^2 \lesssim \|\partial^\alpha \partial_t^{j+1} u\|_0^2 \lesssim \bar{\mathfrak{D}}_{n,m}. \tag{8-102}$$

Then (8-101) with $|\alpha| + 2j = 2n - 1 = 2n - \ell$ implies that

$$\|\partial^\alpha \partial_t^j(\chi_2 u)\|_2^2 + \|\partial^\alpha \partial_t^j(\chi_2 p)\|_1^2 \lesssim \|\partial^\alpha \partial_t^{j+1}(\chi_2 u)\|_0^2 + \mathfrak{Y}_{n,m} + \mathfrak{X}_{n,m} \lesssim \bar{\mathfrak{D}}_{n,m} + \mathfrak{Y}_{n,m} + \mathfrak{X}_{n,m}. \tag{8-103}$$

Applying this bound for all α and j satisfying $|\alpha| + 2j = 2n - 1$ and summing, we find

$$\|\bar{D}^{2n-1}(\chi_2 u)\|_2^2 + \|\bar{D}^{2n-1}(\chi_2 p)\|_1^2 \lesssim \bar{\mathfrak{D}}_{n,m} + \mathfrak{Y}_{n,m} + \mathfrak{X}_{n,m}. \tag{8-104}$$

When $\ell = 2$ and $|\alpha| + 2j = 2n - \ell = 2n - 2$, a similar application of Lemma A.12 implies

$$\|\partial^\alpha \partial_t^{j+1}(\chi_2 u)\|_1^2 \lesssim \bar{\mathfrak{D}}_{n,m} \tag{8-105}$$

so that

$$\|\partial^\alpha \partial_t^j(\chi_2 u)\|_3^2 + \|\partial^\alpha \partial_t^j(\chi_2 p)\|_2^2 \lesssim \|\partial^\alpha \partial_t^{j+1}(\chi_2 u)\|_1^2 + \mathfrak{Y}_{n,m} + \mathfrak{X}_{n,m} \lesssim \bar{\mathfrak{D}}_{n,m} + \mathfrak{Y}_{n,m} + \mathfrak{X}_{n,m}. \tag{8-106}$$

This may be summed over $2j + |\alpha| = 2n - 2$ for the estimate

$$\|\bar{D}^{2n-2}(\chi_2 u)\|_3^2 + \|\bar{D}^{2n-2}(\chi_2 p)\|_2^2 \lesssim \bar{\mathfrak{D}}_{n,m} + \mathfrak{Y}_{n,m} + \mathfrak{X}_{n,m}. \tag{8-107}$$

Then (8-104) and (8-107) imply that

$$\|\bar{D}^{2n-1}(\chi_2 u)\|_2^2 + \|\bar{D}^{2n-2}(\chi_2 u)\|_3^2 + \|\bar{D}^{2n-1}(\chi_2 p)\|_1^2 + \|\bar{D}^{2n-2}(\chi_2 p)\|_2^2 \lesssim \bar{\mathfrak{D}}_{n,m} + \mathfrak{Y}_{n,m} + \mathfrak{X}_{n,m}. \tag{8-108}$$

Now suppose that the inequality

$$\sum_{\ell=1}^{\ell_0} \|\bar{D}^{2n-\ell}(\chi_2 u)\|_{\ell+1}^2 + \|\bar{D}^{2n-\ell}(\chi_2 p)\|_\ell^2 \lesssim \bar{\mathfrak{D}}_{n,m} + \mathfrak{Y}_{n,m} + \mathfrak{X}_{n,m} \tag{8-109}$$

holds for $2 \le \ell_0 < 2n - m - 1$. We claim that (8-109) holds with ℓ_0 replaced by $\ell_0 + 1$. Suppose $|\alpha| + 2j = 2n - (\ell_0 + 1)$ and apply (8-101) to see that

$$\|\partial^\alpha \partial_t^j(\chi_2 u)\|_{\ell_0+2}^2 + \|\partial^\alpha \partial_t^j(\chi_2 p)\|_{\ell_0+1}^2 \lesssim \|\partial^\alpha \partial_t^{j+1}(\chi_2 u)\|_{\ell_0}^2 + \mathfrak{Y}_{n,m} + \mathfrak{X}_{n,m} \lesssim \bar{\mathfrak{D}}_{n,m} + \mathfrak{Y}_{n,m} + \mathfrak{X}_{n,m}, \tag{8-110}$$

where in the last inequality we have invoked (8-109) with

$$|\alpha| + 2(j+1) = 2n - (\ell_0 + 1) + 2 = 2n - (\ell_0 - 1).$$

This proves the claim, so, by finite induction, the bound (8-109) holds for all $\ell_0 = 2, \dots, 2n - m - 1$. Choosing $\ell_0 = 2n - m - 1$ yields the estimate

$$\sum_{\ell=1}^{2n-m-1} \|\bar{D}^{2n-\ell}(\chi_2 u)\|_{\ell+1}^2 + \|\bar{D}^{2n-\ell}(\chi_2 p)\|_\ell^2 \lesssim \bar{\mathfrak{D}}_{n,m} + \mathfrak{Y}_{n,m} + \mathfrak{X}_{n,m}, \tag{8-111}$$

which implies, by virtue of the fact that $\chi_2 = 1$ on Ω_2, that

$$\sum_{k=m+1}^{2n-1} \|\bar{D}^k u\|^2_{H^{2n-k+1}(\Omega_2)} + \|\bar{D}^k p\|^2_{H^{2n-k}(\Omega_2)} = \sum_{\ell=1}^{2n-m-1} \|\bar{D}^{2n-\ell} u\|^2_{H^{\ell+1}(\Omega_2)} + \|\bar{D}^{2n-\ell} p\|^2_{H^{\ell}(\Omega_2)} \lesssim \bar{\mathfrak{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8-112}$$

Now we will improve the estimate (8-112) by using the equations (2-23), considering the cases $m = 1, 2$ separately. Let $m = 1$. Since $m + 1 = 2$, the bound (8-112) already covers all temporal derivatives of order 1 to $n - 1$. Since $\|\partial_t^n u\|_1^2$ is already controlled in $\bar{\mathfrak{D}}_{n,m}$, we must only improve spatial derivatives. First note that (8-112) implies that

$$\|\partial_3 \bar{D}^2 u\|^2_{H^{2n-2}(\Omega_2)} + \|D^2 p\|^2_{H^{2n-2}(\Omega_2)} \lesssim \bar{\mathfrak{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8-113}$$

Then we may apply the operator $\partial_3 D$ to the divergence equation in (2-23) to bound

$$\|\partial_3^2 D u_3\|^2_{H^{2n-2}(\Omega_2)} \lesssim \|\partial_3 D G^2\|^2_{H^{2n-2}(\Omega_2)} + \|\partial_3 D^2 u\|^2_{H^{2n-2}(\Omega_2)} \lesssim \bar{\mathfrak{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8-114}$$

Then applying the operator D to the first equation in (2-23) implies that

$$\begin{aligned} \|\partial_3 D p\|^2_{H^{2n-2}(\Omega_2)} + \|\partial_3^2 D u_i\|^2_{H^{2n-2}(\Omega_2)} &\lesssim \|D G^1\|^2_{H^{2n-2}(\Omega_2)} + \|D^2 p\|^2_{H^{2n-2}(\Omega_2)} + \|D \bar{D}_2^2 u\|^2_{H^{2n-2}(\Omega_2)} + \|\partial_3^2 D u_3\|^2_{H^{2n-2}(\Omega_2)} \\ &\lesssim \bar{\mathfrak{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m} \end{aligned} \tag{8-115}$$

for $i = 1, 2$. We can then iterate this process, applying ∂_3^2 to the divergence equation, then ∂_3 to the first equation in (2-23), and using all of the bounds derived from the previous step, to deduce that

$$\|\partial_3^2 p\|^2_{H^{2n-2}(\Omega_2)} + \|\partial_3^3 u\|^2_{H^{2n-2}(\Omega_2)} \lesssim \bar{\mathfrak{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8-116}$$

Combining (8-113)–(8-116) yields the estimate

$$\|\nabla^3 u\|^2_{H^{2n-2}(\Omega_2)} + \|\nabla^2 p\|^2_{H^{2n-2}(\Omega_2)} \lesssim \bar{\mathfrak{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}, \tag{8-117}$$

which together with (8-112) and the bound $\|\partial_t^n u\|^2_{H^1(\Omega_2)} \le \|\partial_t^n u\|_1^2 \lesssim \bar{\mathfrak{D}}_{n,m}$ implies (8-97).

In the case $m = 2$, we can argue as in the case $m = 1$ to control the spatial derivatives. That is, we first control $\partial_3 D^3 u$, $D^3 p$, then iteratively apply operators with an increasing number of ∂_3 powers to arrive at the bound

$$\|\nabla^4 u\|^2_{H^{2n-3}(\Omega_2)} + \|\nabla^3 p\|^2_{H^{2n-3}(\Omega_2)} \lesssim \bar{\mathfrak{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8-118}$$

Since $m + 1 = 3$ it remains to control $\partial_t u$ and $\partial_t \nabla p$. For the latter we apply $\partial_3 \partial_t$ to the divergence equation and use (8-1) and (8-112) to bound

$$\|\partial_3^2 \partial_t u_3\|^2_{H^{2n-3}(\Omega_2)} \lesssim \|\partial_3 \partial_t G^2\|^2_{H^{2n-3}(\Omega_2)} + \|\partial_3 \partial_t D u\|^2_{H^{2n-3}(\Omega_2)} \lesssim \bar{\mathfrak{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8-119}$$

Then applying ∂_t to the third component of the first equation in (2-23) shows that

$$\begin{aligned}\|\partial_3\partial_t p\|^2_{H^{2n-3}(\Omega_2)} &\lesssim \|\partial_t G^1\|^2_{H^{2n-3}(\Omega_2)} + \|\partial_t \bar{D}^2 u\|^2_{H^{2n-3}(\Omega_2)} + \|\partial_3^2\partial_t u_3\|^2_{H^{2n-3}(\Omega_2)} \\ &\lesssim \bar{\mathfrak{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m},\end{aligned} \tag{8-120}$$

which in turn implies that

$$\|\nabla\partial_t p\|^2_{H^{2n-3}(\Omega_2)} \lesssim \|\partial_3\partial_t p\|^2_{H^{2n-3}(\Omega_2)} + \|D\partial_t p\|^2_{H^{2n-3}(\Omega_2)} \lesssim \bar{\mathfrak{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8-121}$$

We may control $\partial_t u_3$ by applying ∂_t to the divergence equation in (2-23) to find that

$$\|\partial_3\partial_t u_3\|^2_{H^{2n-2}(\Omega_2)} \lesssim \|\partial_t G^2\|^2_{H^{2n-2}(\Omega_2)} + \|\bar{D}^3 u\|^2_{H^{2n-2}(\Omega_2)} \lesssim \bar{\mathfrak{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}, \tag{8-122}$$

but then, since $\partial_t u_3 = 0$ on Σ, we can use Poincaré's inequality (Lemma A.13) to bound

$$\begin{aligned}\|\partial_t u_3\|^2_{H^{2n-1}(\Omega_2)} &\lesssim \|\partial_t u_3\|^2_{H^0(\Omega_2)} + \|\nabla\partial_t u_3\|^2_{H^{2n-2}(\Omega_2)} \lesssim \|\nabla\partial_t u_3\|^2_{H^{2n-2}(\Omega_2)} \\ &\lesssim \|\partial_3\partial_t u_3\|^2_{H^{2n-2}(\Omega_2)} + \|\bar{D}^3 u_3\|^2_{H^{2n-2}(\Omega_2)} \lesssim \bar{\mathfrak{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}.\end{aligned} \tag{8-123}$$

Control of the terms $\partial_t u_i$, $i = 1, 2$, is slightly more delicate; for it we appeal to the first of the localized equations (8-99) rather than (2-23). The reason for this is that using (8-99) will allow us to control $\partial_3^2\partial_t(\chi_2 u_i)$ in all of Ω, giving us control of $\partial_t(\chi_2 u_i)$ in all of Ω via Poincaré and hence control of $\partial_t u_i$ in Ω_2. If instead we used (2-23), control of $\partial_3^2\partial_t u_i$ in Ω_2 would not yield the desired control of $\partial_t u_i$ in Ω_2 because we could not apply Poincaré's inequality. We apply ∂_t to the $i = 1, 2$ components of the first localized equation in (8-99) and use (8-111) to see that

$$\begin{aligned}&\|\partial_3^2\partial_t(\chi_2 u_i)\|^2_{H^{2n-3}(\Omega)} \\ &\quad\lesssim \|\partial_t H^1\cdot e_i\|^2_{H^{2n-3}(\Omega)} + \|\chi_2\partial_t G^1\|^2_{H^{2n-3}(\Omega)} + \|\partial_t D(\chi_2 p)\|^2_{H^{2n-3}(\Omega)} + \|\partial_t \bar{D}^2(\chi_2 u)\|^2_{H^{2n-3}(\Omega)} \\ &\quad\lesssim \bar{\mathfrak{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}.\end{aligned} \tag{8-124}$$

Now, since $\partial_t(\chi_2 u_i)$ and $\partial_3\partial_t(\chi_2 u_i)$ both vanish in an open set near Σ, we may apply Poincaré's inequality twice and use (8-124) to find that

$$\|\partial_t u_i\|^2_{H^{2n-1}(\Omega_2)} \lesssim \|\partial_t(\chi_2 u_i)\|^2_{H^{2n-1}(\Omega)} \lesssim \|\partial_3^2\partial_t(\chi_2 u_i)\|^2_{H^{2n-3}(\Omega)} \lesssim \bar{\mathfrak{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8-125}$$

To conclude the analysis for $m = 2$, we sum (8-112), (8-118), (8-121), (8-123), (8-125), and the bound $\|\partial_t^n u\|^2_{H^1(\Omega_2)} \le \|\partial_t^n u\|^2_1 \lesssim \bar{\mathfrak{D}}_{n,m}$ to derive (8-98). □

Instantaneous energy. Now we estimate the instantaneous energy. The proof is based on an argument very similar to the one used in the proof of Lemma 8.3. Recall that $\bar{\mathcal{E}}_{n,m}$ is defined by (2-45).

Theorem 8.4. *Define*

$$\mathcal{W}_{n,m} = \|\bar{\nabla}_m^{2n-2} G^1\|_0^2 + \|\bar{\nabla}_0^{2n-2} G^2\|_1^2 + \|\bar{D}_m^{2n-2} G^3\|_{1/2}^2 + \|\bar{D}_0^{2n-2} G^4\|_{1/2}^2. \tag{8-126}$$

If $m = 1$, then

$$\|\nabla^2 u\|_{2n-2}^2 + \sum_{j=1}^{n} \|\partial_t^j u\|_{2n-2j}^2 + \|\nabla p\|_{2n-2}^2 + \sum_{j=1}^{n-1} \|\partial_t^j p\|_{2n-2j-1}^2 + \|D\eta\|_{2n-1}^2 + \sum_{j=1}^{n} \|\partial_t^j \eta\|_{2n-2j}^2 \lesssim \bar{\mathscr{E}}_{n,m} + \mathscr{W}_{n,m}. \quad (8\text{-}127)$$

If $m = 2$, then

$$\|\nabla^3 u\|_{2n-3}^2 + \sum_{j=1}^{n} \|\partial_t^j u\|_{2n-2j}^2 + \|\nabla^2 p\|_{2n-3}^2 + \sum_{j=1}^{n-1} \|\partial_t^j p\|_{2n-2j-1}^2 + \|D^2\eta\|_{2n-2}^2 + \sum_{j=1}^{n} \|\partial_t^j \eta\|_{2n-2j}^2 \lesssim \bar{\mathscr{E}}_{n,m} + \mathscr{W}_{n,m}. \quad (8\text{-}128)$$

Proof. The proof is quite similar to that of Lemma 8.3, so we do not fill in all of the details. Throughout the proof we employ the notation $\mathscr{Z} := \bar{\mathscr{E}}_{n,m} + \mathscr{W}_{n,m}$.

Let $0 \le j \le n-1$ and $\alpha \in \mathbb{N}^2$ satisfy $m \le |\alpha| + 2j \le 2n-2$. To begin, we utilize the equations (2-23) with the elliptic estimate Lemma A.14 to bound

$$\|\partial^\alpha \partial_t^j u\|_{2n-|\alpha|-2j}^2 + \|\partial^\alpha \partial_t^j p\|_{2n-|\alpha|-2j-1}^2 \lesssim \|\partial^\alpha \partial_t^{j+1} u\|_{2n-|\alpha|-2j-2}^2 + \|\partial^\alpha \partial_t^j G^1\|_{2n-|\alpha|-2j-2}^2 + \|\partial^\alpha \partial_t^j G^2\|_{2n-|\alpha|-2j-1}^2 + \|\partial^\alpha \partial_t^j \eta\|_{2n-|\alpha|-2j-3/2}^2 + \|\partial^\alpha \partial_t^j G^3\|_{2n-|\alpha|-2j-3/2}^2. \quad (8\text{-}129)$$

The constraints on j, α allow us to bound

$$\|\partial^\alpha \partial_t^j G^1\|_{2n-|\alpha|-2j-2}^2 + \|\partial^\alpha \partial_t^j G^2\|_{2n-|\alpha|-2j-1}^2 + \|\partial^\alpha \partial_t^j G^3\|_{2n-|\alpha|-2j-3/2}^2 \lesssim \mathscr{W}_{n,m}, \quad (8\text{-}130)$$

and similarly

$$\|\partial^\alpha \partial_t^j \eta\|_{2n-|\alpha|-2j-3/2}^2 \lesssim \bar{\mathscr{E}}_{n,m}, \quad (8\text{-}131)$$

so that (8-129)–(8-131) imply that

$$\|\partial^\alpha \partial_t^j u\|_{2n-|\alpha|-2j}^2 + \|\partial^\alpha \partial_t^j p\|_{2n-|\alpha|-2j-1}^2 \lesssim \mathscr{Z} + \|\partial^\alpha \partial_t^{j+1} u\|_{2n-|\alpha|-2j-2}^2. \quad (8\text{-}132)$$

As in Lemma 8.3, we argue with a finite induction on $\ell \in [2, 2n-m]$, beginning with $\ell = 2, 3$. When $\ell = 2$ and $|\alpha| + 2j = 2n - 2 = 2n - \ell$, the definition of $\bar{\mathscr{E}}_{n,m}$ implies that

$$\|\partial^\alpha \partial_t^{j+1} u\|_0^2 \lesssim \bar{\mathscr{E}}_{n,m}, \quad (8\text{-}133)$$

which may be inserted into (8-132) for

$$\|\partial^\alpha \partial_t^j u\|_2^2 + \|\partial^\alpha \partial_t^j p\|_1^2 \lesssim \mathscr{Z}. \quad (8\text{-}134)$$

Summing over all α and j satisfying $|\alpha| + 2j = 2n-2$ shows that

$$\|\bar{D}^{2n-2} u\|_2^2 + \|\bar{D}^{2n-2} p\|_1^2 \lesssim \mathscr{Z}. \quad (8\text{-}135)$$

For $\ell = 3$ we note that $|\alpha| + 2j = 2n - 3$ implies that $j \le n - 2$, so that $|\alpha| \ge 1$. This allows us to write $\alpha = (\alpha - \beta) + \beta$ for $|\beta| = 1$ and to use (8-135) to see that

$$\|\partial^\alpha \partial_t^{j+1} u\|_1^2 \le \|\partial^{\alpha-\beta} \partial_t^{j+1} u\|_2^2 \le \|\bar{D}^{2n-2} u\|_2^2 \lesssim \mathfrak{X}. \tag{8-136}$$

Then we can plug this into (8-132) for each $|\alpha| + 2j = 2n - 3$ and sum to arrive at the bound

$$\|\bar{D}^{2n-3} u\|_3^2 + \|\bar{D}^{2n-3} p\|_2^2 \lesssim \mathfrak{X}. \tag{8-137}$$

Now we may use finite induction as in (8-109)–(8-112) of Lemma 8.3 to ultimately deduce the estimate

$$\sum_{k=m}^{2n-2} \|\bar{D}^k u\|_{2n-k}^2 + \|\bar{D}^k p\|_{2n-k-1}^2 = \sum_{\ell=2}^{2n-m} \|\bar{D}^{2n-\ell} u\|_\ell^2 + \|\bar{D}^{2n-\ell} p\|_{\ell-1}^2 \lesssim \mathfrak{X}. \tag{8-138}$$

Now we improve the estimate (8-138) by utilizing the structure of the equations (2-23), again arguing as in Lemma 8.3. The energy bound (8-138) in the case $m = 2$ is structurally similar to the bound (8-112) for the dissipation in the case $m = 1$, so we may argue as in (8-113)–(8-116), differentiating the equations (2-23) (with obvious modifications to the Sobolev indices and number of derivatives applied) and bootstrapping until we arrive at the bound

$$\|\nabla^3 u\|_{2n-3}^2 + \|\nabla^2 p\|_{2n-3}^2 \lesssim \mathfrak{X}. \tag{8-139}$$

Then (8-138), (8-139), and the bound $\|\partial_t^n u\|_0^2 \le \bar{\mathcal{E}}_{n,m}$ imply the bound (8-128).

In the case $m = 1$ we apply ∂_3 to the divergence equation in (2-23) to see that

$$\|\partial_3^2 u_3\|_{2n-2}^2 \lesssim \|\partial_3 G^2\|_{2n-2}^2 + \|\partial_3 D u\|_{2n-2}^2 \lesssim \mathfrak{X}. \tag{8-140}$$

We then use the first equation in (2-23) to bound

$$\|\partial_3 p\|_{2n-2}^2 + \sum_{i=1}^{2} \|\partial_3^2 u_i\|_{2n-2}^2 \lesssim \|G^1\|_{2n-2}^2 + \|\bar{D}^2 u\|_{2n-2}^2 + \|\partial_3^2 u_3\|_{2n-2}^2 + \|D p\|_{2n-2}^2 \lesssim \mathfrak{X}. \tag{8-141}$$

Then (8-138), (8-140), and (8-141) imply that

$$\|\nabla^2 u\|_{2n-2}^2 + \|\nabla p\|_{2n-2}^2 \lesssim \mathfrak{X}, \tag{8-142}$$

which, when added to (8-138) and the bound $\|\partial_t^n u\|_0^2 \le \bar{\mathcal{E}}_{n,m}$, yields (8-127). □

Specialization: estimates at the $2N$ and $N + 2$ levels. We now specialize the general results contained in Theorems 8.1 and 8.4 to the specific case of $n = 2N$ with no minimal derivative restriction, and to the case $n = N + 2$ with minimal derivative count $m = 1, 2$.

Theorem 8.5. *There exists a $\theta > 0$ such that*

$$\mathcal{D}_{2N} \lesssim \bar{\mathcal{D}}_{2N} + \mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \tag{8-143}$$

Proof. We apply Theorem 8.1 with $n = 2N$ and $m = 1$ to see that (8-2) holds. Theorem 4.2 provides an estimate of $\mathcal{Y}_{2N,1}$, as defined in (8-1):

$$\mathcal{Y}_{2N,1} \lesssim \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N} \tag{8-144}$$

for some $\theta > 0$. We may then use this in (8-2) to find that

$$\begin{aligned} \|\nabla^3 u\|_{4N-2}^2 + \sum_{j=1}^{2N} \|\partial_t^j u\|_{4N-2j+1}^2 + \|\nabla^2 p\|_{4N-2}^2 + \sum_{j=1}^{2N-1} \|\partial_t^j p\|_{4N-2j}^2 \\ + \|D^2\eta\|_{4N-5/2}^2 + \|\partial_t \eta\|_{4N-1/2}^2 + \sum_{j=2}^{2N+1} \|\partial_t^j \eta\|_{4N-2j+5/2}^2 \lesssim \bar{\mathcal{D}}_{2N} + \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \end{aligned} \tag{8-145}$$

We can improve the estimate for u in (8-145) by using the fact that $\bar{\mathcal{D}}_{2N}$ does not have a minimal derivative count. Indeed, by the definition (2-49) and Lemma A.12, we know that

$$\|\mathcal{I}_{\lambda} u\|_1^2 + \|u\|_1^2 \lesssim \bar{\mathcal{D}}_{2N}. \tag{8-146}$$

Now, since Ω satisfies the uniform cone property, we can apply Corollary 4.16 of [Adams 1975] to bound

$$\|u\|_{4N+1}^2 \lesssim \|u\|_0^2 + \|\nabla^{4N+1} u\|_0^2 \lesssim \|u\|_1^2 + \|\nabla^3 u\|_{4N-2}^2. \tag{8-147}$$

Then (8-145)–(8-147) imply that

$$\|\mathcal{I}_{\lambda} u\|_1^2 + \|u\|_{4N+1}^2 \lesssim \bar{\mathcal{D}}_{2N} + \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \tag{8-148}$$

We can use this improved estimate of u to improve the estimate of p by employing the first equation of (2-23) to bound

$$\|\nabla p\|_{4N-1}^2 \lesssim \|\partial_t u\|_{4N-1}^2 + \|\Delta u\|_{4N-1}^2 + \|G^1\|_{4N-1}^2. \tag{8-149}$$

The bounds (8-145) and (8-148) imply that

$$\|\partial_t u\|_{4N-1}^2 + \|\Delta u\|_{4N-1}^2 \lesssim \bar{\mathcal{D}}_{2N} + \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}, \tag{8-150}$$

while (4-7)–(4-8) of Theorem 4.2 imply that

$$\|G^1\|_{4N-1}^2 \lesssim \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \tag{8-151}$$

Hence (8-148)–(8-151) combine to show that

$$\|\nabla p\|_{4N-1}^2 \lesssim \bar{\mathcal{D}}_{2N} + \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \tag{8-152}$$

Finally, we improve the estimate for η. We use the boundary condition on Σ of (2-23) to bound

$$\begin{aligned} \|D\eta\|_{4N-3/2}^2 &\lesssim \|Dp\|_{H^{4N-3/2}(\Sigma)}^2 + \|D\partial_3 u_3\|_{H^{4N-3/2}(\Sigma)}^2 + \|DG^3\|_{4N-3/2}^2 \\ &\lesssim \|Dp\|_{4N-1}^2 + \|D\partial_3 u_3\|_{4N-1}^2 + \|DG^3\|_{4N-3/2}^2 \lesssim \bar{\mathcal{D}}_{2N} + \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \end{aligned} \tag{8-153}$$

In the last inequality we have used (8-148), (8-152), and Theorem 4.2. Now (8-143) follows from (8-145), (8-148), (8-152), and (8-153). □

Now we perform a similar analysis for the energy at the $2N$ level.

Theorem 8.6. *There exists a $\theta > 0$ such that*

$$\mathcal{E}_{2N} \lesssim \bar{\mathcal{E}}_{2N} + \mathcal{E}_{2N}^{1+\theta}. \tag{8-154}$$

Proof. We apply Theorem 8.4 with $n = 2N$ and $m = 1$ to see that (8-127) holds. Theorem 4.2 provides an estimate of $\mathcal{W}_{2N,1}$, as defined by (8-126):

$$\mathcal{W}_{2N,1} \lesssim \mathcal{E}_{2N}^{1+\theta} \tag{8-155}$$

for some $\theta > 0$. Replacing in (8-127) shows that

$$\|\nabla^2 u\|_{4N-2}^2 + \sum_{j=1}^{2N} \|\partial_t^j u\|_{4N-2j}^2 + \|\nabla p\|_{4N-2}^2 + \sum_{j=1}^{2N-1} \|\partial_t^j p\|_{4N-2j-1}^2 + \|D\eta\|_{4N-1}^2 + \sum_{j=1}^{2N} \|\partial_t^j \eta\|_{4N-2j}^2 \lesssim \bar{\mathcal{E}}_{2N} + \mathcal{E}_{2N}^{1+\theta}. \tag{8-156}$$

The definition of $\bar{\mathcal{E}}_{2N}$ implies that

$$\|\mathcal{I}_\lambda u\|_0^2 + \|u\|_0^2 + \|\mathcal{I}_\lambda \eta\|_0^2 + \|\eta\|_0^2 \le \bar{\mathcal{E}}_{2N}. \tag{8-157}$$

We may then sum the previous two bounds and employ Corollary 4.16 of [Adams 1975] as in the proof of Theorem 8.5 to find that

$$\|\mathcal{I}_\lambda u\|_0^2 + \sum_{j=0}^{2N} \|\partial_t^j u\|_{4N-2j}^2 + \|\nabla p\|_{4N-2}^2 + \sum_{j=1}^{2N-1} \|\partial_t^j p\|_{4N-2j-1}^2 + \|\mathcal{I}_\lambda \eta\|_0^2 + \sum_{j=0}^{2N} \|\partial_t^j \eta\|_{4N-2j}^2 \lesssim \bar{\mathcal{E}}_{2N} + \mathcal{E}_{2N}^{1+\theta}. \tag{8-158}$$

It remains only to estimate $\|p\|_{4N-1}^2$; since Lemma A.10 implies that

$$\|p\|_{4N-1}^2 \lesssim \|p\|_0^2 + \|\nabla p\|_{4N-2}^2 \lesssim \|p\|_{H^0(\Sigma)}^2 + \|\nabla p\|_{4N-2}^2, \tag{8-159}$$

it suffices to estimate $\|p\|_{H^0(\Sigma)}^2$. We do this by using the boundary condition in (2-23), trace theory, and estimate (4-6) of Theorem 4.2:

$$\|p\|_{H^0(\Sigma)}^2 \lesssim \|\eta\|_0^2 + \|G^3\|_0^2 + \|\partial_3 u_3\|_{H^0(\Sigma)}^2 \lesssim \|\eta\|_0^2 + \|u\|_{4N}^2 + \mathcal{E}_{2N}^{1+\theta}. \tag{8-160}$$

Then the estimate (8-154) easily follows from (8-158)–(8-160). □

We now consider the dissipation at the $N+2$ level.

Theorem 8.7. *For $m = 1, 2$ there exists a $\theta > 0$ such that*

$$\mathcal{D}_{N+2,m} \lesssim \bar{\mathcal{D}}_{N+2,m} + \mathcal{E}_{2N}^{\theta} \mathcal{D}_{N+2,m}. \tag{8-161}$$

Proof. We apply Theorem 8.1 with $n = N+2$ to see that (8-2) holds for $m = 1$ and (8-3) holds for $m = 2$. Theorem 4.1 provides an estimate for $\mathcal{Y}_{N+2,m}$, as defined by (8-1):

$$\mathcal{Y}_{N+2,m} \lesssim \mathcal{E}_{2N}^{\theta} \mathcal{D}_{N+2,m} \tag{8-162}$$

for some $\theta > 0$. The bound (8-161) follows from using this in (8-2)–(8-3). $\square$

We now consider the energy at the $N+2$ level.

Theorem 8.8. *For $m = 1, 2$ there exists a $\theta > 0$ such that*

$$\mathscr{E}_{N+2,m} \lesssim \bar{\mathscr{E}}_{N+2,m} + \mathscr{E}_{2N}^{\theta} \mathscr{E}_{N+2,m}. \tag{8-163}$$

Proof. We apply Theorem 8.4 with $n = N+2$ to see that (8-127) holds when $m = 1$ and (8-128) holds when $m = 2$. Theorem 4.1 provides an estimate for $\mathscr{W}_{N+2,m}$, as defined by (8-126):

$$\mathscr{W}_{N+2,m} \lesssim \mathscr{E}_{2N}^{\theta} \mathscr{E}_{N+2,m} \tag{8-164}$$

for some $\theta > 0$. The bound (8-163) follows from using this in (8-127)–(8-128). $\square$

9. A priori estimates

In this section we will combine the energy evolution estimates and the comparison estimates to derive a priori estimates for the total energy, $\mathscr{G}_{2N}$, defined by (2-58).

Estimates involving $\mathscr{F}_{2N}$ and $\mathscr{K}$. Recall that $\mathscr{F}_{2N}$ is defined by (2-56) and $\mathscr{K}$ is defined by (2-57). We begin with an estimate for $\mathscr{F}_{2N}$.

Lemma 9.1. *There exists a universal $C > 0$ such that*

$$\sup_{0 \le r \le t} \mathscr{F}_{2N}(r) \lesssim \exp\left(C \int_0^t \sqrt{\mathscr{K}(r)}\, dr \right) \left[\mathscr{F}_{2N}(0) + t \int_0^t (1 + \mathscr{E}_{2N}(r)) \mathscr{D}_{2N}(r)\, dr + \left(\int_0^t \sqrt{\mathscr{K}(r) \mathscr{F}_{2N}(r)}\, dr \right)^2 \right]. \tag{9-1}$$

Proof. Throughout this proof we write $u = \tilde{u} + u_3 e_3$, that is, we write $\tilde{u}$ for the part of u parallel to Σ. Then η solves the transport equation $\partial_t \eta + \tilde{u} \cdot D\eta = u_3$ on Σ. We may then use Lemma A.9 with $s = 1/2$ to estimate

$$\sup_{0 \le r \le t} \|\eta(r)\|_{1/2} \le \exp\left(C \int_0^t \|D\tilde{u}(r)\|_{H^{3/2}(\Sigma)}\, dr \right) \left[\|\eta_0\|_{1/2} + \int_0^t \|u_3(r)\|_{H^{1/2}(\Sigma)}\, dr \right]. \tag{9-2}$$

By the definition of $\mathscr{K}$, (2-57), we may bound $\|D\tilde{u}(r)\|_{H^{3/2}(\Sigma)} \le \sqrt{\mathscr{K}(r)}$, but we may also use trace theory to bound $\|u_3(r)\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathscr{D}_{2N}(r)$. This allows us to square both sides of (9-2) and utilize Cauchy–Schwarz to deduce that

$$\sup_{0 \le r \le t} \|\eta(r)\|_{1/2}^2 \lesssim \exp\left(2C \int_0^t \sqrt{K(r)}\, dr \right) \left[\|\eta_0\|_{1/2}^2 + t \int_0^t \mathscr{D}_{2N}(r)\, dr \right]. \tag{9-3}$$

To go to higher regularity, we let $\alpha \in \mathbb{N}^2$ with $|\alpha| = 4N$. Then we apply the operator ∂^α to the equation $\partial_t \eta + \tilde{u} \cdot D\eta = u_3$ to see that $\partial^\alpha \eta$ solves the transport equation

$$\partial_t(\partial^\alpha \eta) + \tilde{u} \cdot D(\partial^\alpha \eta) = \partial^\alpha u_3 - \sum_{0 < \beta \le \alpha} C_{\alpha,\beta} \partial^\beta \tilde{u} \cdot D\partial^{\alpha-\beta} \eta =: G^\alpha \tag{9-4}$$

with the initial condition $\partial^\alpha \eta_0$. We may then apply Lemma A.9 with $s = 1/2$ to find that

$$\sup_{0\le r\le t} \|\partial^\alpha \eta(r)\|_{1/2} \le \exp\Big(C\int_0^t \|D\tilde{u}(r)\|_{H^{3/2}(\Sigma)}\,dr\Big)\Big[\|\partial^\alpha \eta_0\|_{1/2} + \int_0^t \|G^\alpha(r)\|_{1/2}\,dr\Big]. \tag{9-5}$$

We will now estimate $\|G^\alpha\|_{1/2}$.

For $\beta\in\mathbb{N}^2$ satisfying $2N+1\le|\beta|\le 4N$ we may apply (A-2) of Lemma A.1 with $s_1 = r = 1/2$ and $s_2 = 2$ to bound

$$\|\partial^\beta \tilde{u} D\partial^{\alpha-\beta}\eta\|_{1/2} \lesssim \|\partial^\beta \tilde{u}\|_{H^{1/2}(\Sigma)}\|D\partial^{\alpha-\beta}\eta\|_2. \tag{9-6}$$

This and trace theory then imply that

$$\sum_{\substack{0<\beta\le\alpha\\ 2N+1\le|\beta|\le 4N}} \|C_{\alpha,\beta}\partial^\beta\tilde{u}\cdot D\partial^{\alpha-\beta}\eta\|_{1/2} \lesssim \|D^{4N}_{2N+1}u\|_1\|D_1^{2N}\eta\|_2 \lesssim \sqrt{\mathcal{D}_{2N}\mathcal{E}_{2N}}. \tag{9-7}$$

On the other hand, if β satisfies $1\le|\beta|\le 2N$, we again use Lemma A.1 to bound

$$\|\partial^\beta \tilde{u} D\partial^{\alpha-\beta}\eta\|_{1/2} \lesssim \|\partial^\beta \tilde{u}\|_{H^2(\Sigma)}\|D\partial^{\alpha-\beta}\eta\|_{1/2}, \tag{9-8}$$

so that

$$\begin{aligned}\sum_{\substack{0<\beta\le\alpha\\ 1\le|\beta|\le 2N}} \|C_{\alpha,\beta}\partial^\beta\tilde{u}\cdot D\partial^{\alpha-\beta}\eta\|_{1/2} &\lesssim \|D_1^{2N}u\|_3\|D^{4N-1}_{2N+1}\eta\|_{1/2} + \|D\tilde{u}\|_{H^2(\Sigma)}\|D^{4N}\eta\|_{1/2}\\ &\lesssim \sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}} + \sqrt{\mathcal{K}\mathcal{F}_{2N}}.\end{aligned} \tag{9-9}$$

The only remaining term in G^α is $\partial^\alpha u_3$, which we estimate with trace theory:

$$\|\partial^\alpha u_3\|_{H^{1/2}(\Sigma)} \lesssim \|D^{4N}u_3\|_1 \lesssim \sqrt{\mathcal{D}_{2N}}. \tag{9-10}$$

We may then combine (9-7), (9-9), and (9-10) for

$$\|G^\alpha\|_{1/2} \lesssim (1+\sqrt{\mathcal{E}_{2N}})\sqrt{\mathcal{D}_{2N}} + \sqrt{\mathcal{K}\mathcal{F}_{2N}}. \tag{9-11}$$

Returning now to (9-5), we square both sides and employ (9-11) and our previous estimate of the term in the exponential to find that

$$\begin{aligned}&\sup_{0\le r\le t} \|\partial^\alpha\eta(r)\|^2_{1/2}\\ &\le \exp\Big(2C\int_0^t\sqrt{\mathcal{K}(r)}\,dr\Big)\Big[\|\partial^\alpha\eta_0\|^2_{1/2} + t\int_0^t(1+\mathcal{E}_{2N}(r))\mathcal{D}_{2N}(r)\,dr + \Big(\int_0^t\sqrt{\mathcal{K}(r)\mathcal{F}_{2N}(r)}\,dr\Big)^2\Big].\end{aligned} \tag{9-12}$$

Then the estimate (9-1) follows by summing (9-12) over all $|\alpha| = 4N$, adding the resulting inequality to (9-3), and using the fact that $\|\eta\|^2_{4N+1/2} \lesssim \|\eta\|^2_{1/2} + \|D^{4N}\eta\|^2_{1/2}$. □

Now we use this result and the $\mathcal{K}$ estimate of Lemma 3.17 to derive a stronger result.

Proposition 9.2. *Let $\mathcal{G}_{2N}$ be defined by (2-58). There exists a universal constant $0<\delta<1$ such that if $\mathcal{G}_{2N}(T)\le\delta$, then for all $0\le t\le T$,*

$$\sup_{0\le r\le t}\mathcal{F}_{2N}(r) \lesssim \mathcal{F}_{2N}(0) + t\int_0^t \mathcal{D}_{2N}. \tag{9-13}$$

Proof. Suppose $\mathcal{G}_{2N}(T) \le \delta \le 1$, for δ to be chosen later. Fix $0 \le t \le T$. Then, according to Lemma 3.17, we have $\mathcal{K} \lesssim \mathcal{E}_{N+2,2}^{(8+2\lambda)/(8+4\lambda)}$, which means that

$$\int_0^t \sqrt{\mathcal{K}(r)}\,dr \lesssim \int_0^t (\mathcal{E}_{N+2,2}(r))^{(8+2\lambda)/(16+8\lambda)}\,dr \le \delta^{(8+2\lambda)/(16+8\lambda)} \int_0^t \frac{1}{(1+r)^{1+\lambda/4}}\,dr$$
$$\le \delta^{(8+2\lambda)/(16+8\lambda)} \int_0^\infty \frac{1}{(1+r)^{1+\lambda/4}}\,dr = \frac{4}{\lambda}\delta^{(8+2\lambda)/(16+8\lambda)}. \tag{9-14}$$

Since $\delta \le 1$, this implies that for any constant $C > 0$,

$$\exp\left(C \int_0^t \sqrt{\mathcal{K}(r)}\,dr\right) \lesssim 1. \tag{9-15}$$

Similarly,

$$\left(\int_0^t \sqrt{\mathcal{K}(r)\mathcal{F}_{2N}(r)}\,dr\right)^2 \lesssim \left(\sup_{0\le r\le t} \mathcal{F}_{2N}(r)\right)\left(\int_0^t \sqrt{\mathcal{K}(r)}\,dr\right)^2$$
$$\lesssim \left(\sup_{0\le r\le t} \mathcal{F}_{2N}(r)\right)\delta^{(8+2\lambda)/(8+4\lambda)}. \tag{9-16}$$

Then (9-14)–(9-16) and Lemma 9.1 imply that

$$\sup_{0\le r\le t} \mathcal{F}_{2N}(r) \le C\left(\mathcal{F}_{2N}(0) + t\int_0^t \mathcal{D}_{2N}\right) + C\delta^{(8+2\lambda)/(8+4\lambda)}(\sup_{0\le r\le t} \mathcal{F}_{2N}(r)), \tag{9-17}$$

for some universal $C > 0$. Then if δ is small enough that $C\delta^{(8+2\lambda)/(8+4\lambda)} \le 1/2$, we may absorb the right-hand $\mathcal{F}_{2N}$ term onto the left and deduce (9-13). □

This bound on $\mathcal{F}_{2N}$ allows us to estimate the integral of $\mathcal{K}\mathcal{F}_{2N}$ and $\sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}$.

Corollary 9.3. *There exists a universal constant $0 < \delta < 1$ such that if $\mathcal{G}_{2N}(T) \le \delta$, then*

$$\int_0^t \mathcal{K}(r)\mathcal{F}_{2N}(r)\,dr \lesssim \delta^{(8+2\lambda)/(8+4\lambda)}\mathcal{F}_{2N}(0) + \delta^{(8+2\lambda)/(8+4\lambda)} \int_0^t \mathcal{D}_{2N}(r)\,dr \tag{9-18}$$

and

$$\int_0^t \sqrt{\mathcal{D}_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}\,dr \lesssim \mathcal{F}_{2N}(0) + \delta^{(8+2\lambda)/(16+8\lambda)} \int_0^t \mathcal{D}_{2N}(r)\,dr \tag{9-19}$$

for $0 \le t \le T$.

Proof. Let $\mathcal{G}_{2N}(T) \le \delta$ with δ as small as in Proposition 9.2, so that estimate (9-13) holds. Lemma 3.17 implies that

$$\mathcal{K}(r) \lesssim (\mathcal{E}_{N+2,2}(r))^{(8+2\lambda)/(8+4\lambda)} \lesssim \delta^{(8+2\lambda)/(8+4\lambda)} \frac{1}{(1+r)^{2+\lambda/2}}. \tag{9-20}$$

This and (9-13) then imply that

$$\frac{1}{\delta^{(8+2\lambda)/(8+4\lambda)}} \int_0^t \mathcal{K}(r)\mathcal{F}_{2N}(r)\,dr \lesssim \mathcal{F}_{2N}(0) \int_0^t \frac{dr}{(1+r)^{2+\lambda/2}} + \int_0^t \frac{r}{(1+r)^{2+\lambda/2}} \left(\int_0^r \mathcal{D}_{2N}(s)\,ds \right) dr$$
$$\lesssim \mathcal{F}_{2N}(0) \int_0^\infty \frac{dr}{(1+r)^{2+\lambda/2}} + \left(\int_0^t \mathcal{D}_{2N}(r)\,dr \right) \left(\int_0^\infty \frac{dr}{(1+r)^{1+\lambda/2}} \right)$$
$$\lesssim \mathcal{F}_{2N}(0) + \int_0^t \mathcal{D}_{2N}(r)\,dr, \tag{9-21}$$

which is estimate (9-18). The estimate (9-19) follows from (9-18), Cauchy–Schwarz, and the fact that $\delta \le 1$:

$$\int_0^t \sqrt{\mathcal{D}_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}\,dr$$
$$\le \left(\int_0^t \mathcal{D}_{2N}(r)\,dr \right)^{1/2} \left(\int_0^t \mathcal{K}(r)\mathcal{F}_{2N}(r)\,dr \right)^{1/2}$$
$$\lesssim \left(\int_0^t \mathcal{D}_{2N}(r)\,dr \right)^{1/2} \left(\delta^{(8+2\lambda)/(8+4\lambda)} \mathcal{F}_{2N}(0) \right)^{1/2} + \delta^{(8+2\lambda)/(16+8\lambda)} \int_0^t \mathcal{D}_{2N}(r)\,dr$$
$$\lesssim \mathcal{F}_{2N}(0) + \left(\delta^{(8+2\lambda)/(16+8\lambda)} + \delta^{(8+2\lambda)/(8+4\lambda)} \right) \int_0^t \mathcal{D}_{2N}(r)\,dr$$
$$\lesssim \mathcal{F}_{2N}(0) + \delta^{(8+2\lambda)/(16+8\lambda)} \int_0^t \mathcal{D}_{2N}(r)\,dr. \qquad \square$$

Boundedness at the 2N level. We now show bounds at the $2N$ level in terms of the initial data.

Theorem 9.4. *Let $\mathcal{G}_{2N}$ be defined by (2-58). There exists a universal constant $0 < \delta < 1$ such that if $\mathcal{G}_{2N}(T) \le \delta$, then*

$$\sup_{0 \le r \le t} \mathcal{E}_{2N}(r) + \int_0^t \mathcal{D}_{2N} + \sup_{0 \le r \le t} \frac{\mathcal{F}_{2N}(r)}{(1+r)} \lesssim \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) \tag{9-22}$$

for all $0 \le t \le T$.

Proof. Combining the energy evolution estimate of Theorem 7.1 with the comparison estimates of Theorems 8.5 and 8.6, we find that

$$\mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N}(r)\,dr \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{1+\theta} + \int_0^t (\mathcal{E}_{2N}(r))^\theta \mathcal{D}_{2N}(r)\,dr$$
$$+ \int_0^t \sqrt{\mathcal{D}_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}\,dr + \int_0^t \mathcal{K}(r)\mathcal{F}_{2N}(r)\,dr \tag{9-23}$$

for some $\theta > 0$. Let us assume initially that $\delta \le 1$ is as small as in Lemma 2.6, Proposition 9.2, and Corollary 9.3, so that their conclusions hold. We may estimate the last two integrals in (9-23) with Corollary 9.3, using the fact that $\delta \le 1$:

$$\int_0^t \sqrt{\mathcal{D}_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}\,dr + \int_0^t \mathcal{K}(r)\mathcal{F}_{2N}(r)\,dr \lesssim \mathcal{F}_{2N}(0) + \delta^{(8+2\lambda)/(16+8\lambda)} \int_0^t \mathcal{D}_{2N}(r)\,dr. \tag{9-24}$$

On the other hand, $\sup_{0\le r\le t}\mathscr{E}_{2N}(r)\le \mathscr{G}_{2N}(T)\le\delta$, so

$$(\mathscr{E}_{2N}(t))^{1+\theta}+\int_0^t(\mathscr{E}_{2N}(r))^{\theta}\mathscr{D}_{2N}(r)\,dr\le\delta^{\theta}\mathscr{E}_{2N}(t)+\delta^{\theta}\int_0^t\mathscr{D}_{2N}(r)\,dr. \tag{9-25}$$

We may then combine (9-23)–(9-25) and write

$$\psi=\min\{\theta,\,(8+2\lambda)/(16+8\lambda)\}>0 \tag{9-26}$$

to deduce the bound

$$\mathscr{E}_{2N}(t)+\int_0^t\mathscr{D}_{2N}(r)\,dr\le C\,(\mathscr{E}_{2N}(0)+\mathscr{F}_{2N}(0))+C\delta^{\theta}\mathscr{E}_{2N}(t)+C\delta^{\psi}\int_0^t\mathscr{D}_{2N}(r)\,dr \tag{9-27}$$

for a universal constant $C>0$. Then if δ is sufficiently small so that $C\delta^{\theta}\le 1/2$ and $C\delta^{\psi}\le 1/2$, we may absorb the last two terms on the right side of (9-27) into the left, which then yields the estimate

$$\sup_{0\le r\le t}\mathscr{E}_{2N}(r)+\int_0^t\mathscr{D}_{2N}(r)\,dr\lesssim\mathscr{E}_{2N}(0)+\mathscr{F}_{2N}(0). \tag{9-28}$$

We then use this and Proposition 9.2 to estimate

$$\begin{aligned}\sup_{0\le r\le t}\frac{\mathscr{F}_{2N}(r)}{(1+r)}&\lesssim\sup_{0\le r\le t}\frac{\mathscr{F}_{2N}(0)}{(1+r)}+\sup_{0\le r\le t}\frac{r}{(1+r)}\int_0^r\mathscr{D}_{2N}(s)\,ds\\&\lesssim\mathscr{F}_{2N}(0)+\int_0^t\mathscr{D}_{2N}(r)dr\lesssim\mathscr{E}_{2N}(0)+\mathscr{F}_{2N}(0).\end{aligned} \tag{9-29}$$

Then (9-22) follows by summing (9-28) and (9-29). □

Decay at the $N+2$ level. Before showing the decay estimates, we first need an interpolation result.

Proposition 9.5. *There exists a universal $0<\delta<1$ such that if $\mathscr{G}_{2N}(T)\le\delta$, then*

$$\mathscr{D}_{N+2,m}(t)\lesssim\bar{\mathscr{D}}_{N+2,m}(t),\qquad\mathscr{E}_{N+2,m}(t)\lesssim\bar{\mathscr{E}}_{N+2,m}(t) \tag{9-30}$$

and

$$\bar{\mathscr{E}}_{N+2,m}(t)\lesssim(\mathscr{E}_{2N}(t))^{1/(m+\lambda+1)}(\bar{\mathscr{D}}_{N+2,m}(t))^{(m+\lambda)/(m+\lambda+1)} \tag{9-31}$$

for $m=1,2$ and $0\le t\le T$.

Proof. The bound $\mathscr{G}_{2N}(T)\le\delta$ and Theorems 8.7 and 8.8 imply that

$$\mathscr{D}_{N+2,m}\le C\bar{\mathscr{D}}_{N+2,m}+C\mathscr{E}_{2N}^{\theta}\mathscr{D}_{N+2,m}\le C\bar{\mathscr{D}}_{N+2,m}+C\delta^{\theta}\mathscr{D}_{N+2,m} \tag{9-32}$$

and

$$\mathscr{E}_{N+2,m}\le C\bar{\mathscr{E}}_{N+2,m}+C\mathscr{E}_{2N}^{\theta}\mathscr{E}_{N+2,m}\le C\bar{\mathscr{E}}_{N+2,m}+C\delta^{\theta}\mathscr{E}_{N+2,m} \tag{9-33}$$

for constants $C>0$ and $\theta>0$. Then if δ is small enough so that $C\delta^{\theta}\le 1/2$, we may absorb the second term on the right side of (9-32) and (9-33) into the left to deduce the bounds in (9-30).

We now turn to the proof of (9-31). According to Remark 2.8, we have

$$\bar{\mathscr{E}}_{N+2,m}\lesssim\|\bar{D}_m^{2N+4}u\|_0^2+\|\bar{D}_m^{2N+4}\eta\|_0^2, \tag{9-34}$$

and by Lemma A.12, we also know that

$$\|\bar{D}_m^{2N+4} u\|_0^2 \lesssim \|\bar{D}_m^{2N+4} \mathbb{D} u\|_0^2 = \bar{\mathscr{D}}_{N+2,m}. \tag{9-35}$$

On the other hand, the definition of $\mathscr{D}_{N+2,m}$, given by (2-54) when $m = 1$ and (2-55) when $m = 2$, together with (9-30) implies that

$$\|\bar{D}_{m+1}^{2N+4} \eta\|_0^2 \le \mathscr{D}_{N+2,m} + \|D^{2N+4}\eta\|_0^2 \lesssim \bar{\mathscr{D}}_{N+2,m} + \|D^{2N+4}\eta\|_0^2. \tag{9-36}$$

We may then combine (9-34)–(9-36) to see that

$$\bar{\mathscr{E}}_{N+2,m} \lesssim \bar{\mathscr{D}}_{N+2,m} + \|\bar{D}^m \eta\|_0^2 + \|D^{2N+4}\eta\|_0^2. \tag{9-37}$$

We first estimate the last term in (9-37). The standard Sobolev interpolation inequality (3-47) with $s = 2N+3-m$, $r = 1/2$, and $q = 2N-4$ allows us to estimate

$$\begin{aligned} \|D^{2N+4}\eta\|_0^2 &\le \|D^{m+1}\eta\|_{2N+3-m}^2 \\ &\lesssim (\|D^{m+1}\eta\|_{2N+5/2-m}^2)^{(4N-8)/(4N-7)} (\|D^{m+1}\eta\|_{4N-m-1}^2)^{1/(4N-7)} \\ &\lesssim (\mathscr{D}_{N+2,m})^{(4N-8)/(4N-7)} (\mathscr{E}_{2N})^{1/(4N-7)}. \end{aligned} \tag{9-38}$$

Since $N \ge 3$, $m \in \{1, 2\}$, and $\lambda \in (0, 1)$, we have $(4N-8)/(4N-7) > (m+\lambda)/(m+\lambda+1)$. Then this bound, the estimate (9-38), and the bound $\mathscr{D}_{N+2,m} \lesssim \mathscr{E}_{2N}$ from Lemma 2.10 imply that

$$\|D^{2N+4}\eta\|_0^2 \lesssim (\mathscr{D}_{N+2,m})^{(m+\lambda)/(m+\lambda+1)} (\mathscr{E}_{2N})^{1/(m+\lambda+1)}. \tag{9-39}$$

Now we turn to the $D^m\eta$ term in (9-37). In the case $m = 1$ we use the H^0 interpolation estimates of Lemma 3.1 to bound

$$\|\bar{D}^m \eta\|_0^2 = \|D\eta\|_0^2 \lesssim (\mathscr{E}_{2N})^{1/(2+\lambda)} (\mathscr{D}_{N+2,1})^{(1+\lambda)/(2+\lambda)}. \tag{9-40}$$

In the case $m = 2$ we use the H^0 interpolation estimates of $D^2\eta$ from Lemma 3.1 and the H^0 estimate of $\partial_t \eta$ from Proposition 3.16 to bound

$$\|\bar{D}^m \eta\|_0^2 = \|D^2\eta\|_0^2 + \|\partial_t \eta\|_0^2 \lesssim (\mathscr{E}_{2N})^{1/(3+\lambda)} (\mathscr{D}_{N+2,2})^{(2+\lambda)/(3+\lambda)}. \tag{9-41}$$

Together, (9-40) and (9-41) may be written as

$$\|\bar{D}^m \eta\|_0^2 \lesssim (\mathscr{E}_{2N})^{1/(m+\lambda+1)} (\mathscr{D}_{N+2,m})^{(m+\lambda)/(m+\lambda+1)}. \tag{9-42}$$

Now, according to Lemma 2.10, we can bound

$$\bar{\mathscr{D}}_{N+2,m} \le \mathscr{D}_{N+2,m} \lesssim (\mathscr{E}_{2N})^{1/(m+\lambda+1)} (\mathscr{D}_{N+2,m})^{(m+\lambda)/(m+\lambda+1)}. \tag{9-43}$$

Then we use the estimates (9-39), (9-42), and (9-43) to bound the right side of (9-37); the bound (9-31) follows from the resulting inequality and (9-30). □

Now we show that the extra integral term appearing in Theorem 7.2 can essentially be absorbed into $\bar{\mathscr{E}}_{N+2,m}$.

Lemma 9.6. *Let F^2 be defined by (2-19) with $\partial^\alpha = \partial_t^{N+2}$. There exists a universal $0 < \delta < 1$ such that if $\mathcal{G}_{2N}(T) \le \delta$, then*

$$\tfrac{2}{3}\bar{\mathcal{E}}_{N+2,m}(t) \le \bar{\mathcal{E}}_{N+2,m}(t) - 2\int_\Omega J(t)\partial_t^{N+1} p(t) F^2(t) \le \tfrac{4}{3}\bar{\mathcal{E}}_{N+2,m}(t) \tag{9-44}$$

for all $0 \le t \le T$.

Proof. Suppose that δ is as small as in Proposition 9.5. Then we combine estimate (5-4) of Theorem 5.2, Lemma 2.6, and estimate (9-30) of Proposition 9.5 to see that

$$\|J\|_{L^\infty}\|\partial_t^{N+1} p\|_0 \|F^2\|_0 \lesssim \sqrt{\mathcal{E}_{N+2,m}}\sqrt{\mathcal{E}_{2N}^\theta \mathcal{E}_{N+2,m}} = \mathcal{E}_{2N}^{\theta/2}\mathcal{E}_{N+2,m} \lesssim \mathcal{E}_{2N}^{\theta/2}\bar{\mathcal{E}}_{N+2,m} \lesssim \delta^{\theta/2}\bar{\mathcal{E}}_{N+2,m} \tag{9-45}$$

for some $\theta > 0$. This estimate and Cauchy–Schwarz then imply that

$$\left|2\int_\Omega J\partial_t^{N+1} p F^2\right| \le 2\|J\|_{L^\infty}\|\partial_t^{N+1} p\|_0\|F^2\|_0 \le C\delta^{\theta/2}\bar{\mathcal{E}}_{N+2,m} \le \tfrac{1}{3}\bar{\mathcal{E}}_{N+2,m} \tag{9-46}$$

if δ is small enough. The bound (9-44) then follows easily from (9-46). □

Now we prove decay at the $N+2$ level.

Theorem 9.7. *Let $\mathcal{G}_{2N}$ be defined by (2-58). There exists a universal constant $0 < \delta < 1$ such that if $\mathcal{G}_{2N}(T) \le \delta$, then*

$$\sup_{0\le r\le t} (1+r)^{m+\lambda}\mathcal{E}_{N+2,m}(r) \lesssim \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) \tag{9-47}$$

for all $0 \le t \le T$ and for $m \in \{1, 2\}$.

Proof. Let δ be as small as in Lemma 2.6, Theorem 9.4, Proposition 9.5, and Lemma 9.6. Theorem 7.2 and the estimate (9-30) of Proposition 9.5 imply that

$$\partial_t\left(\bar{\mathcal{E}}_{N+2,m} - 2\int_\Omega J\partial_t^{N+1} p F^2\right) + \bar{\mathcal{D}}_{N+2,m} \le C\mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,m} \le C\delta^\theta \bar{\mathcal{D}}_{N+2,m} \le \tfrac{1}{2}\bar{\mathcal{D}}_{N+2,m} \tag{9-48}$$

if δ is small enough (here $\theta > 0$). On the other hand, Theorem 9.4, (9-31) of Proposition 9.5, and (9-44) of Lemma 9.6 imply that

$$\begin{aligned} 0 &\le \tfrac{2}{3}\bar{\mathcal{E}}_{N+2,m} \le \bar{\mathcal{E}}_{N+2,m} - 2\int_\Omega J\partial_t^{N+1} p F^2 \le \tfrac{4}{3}\bar{\mathcal{E}}_{N+2,m} \\ &\le C(\mathcal{E}_{2N})^{1/(m+\lambda+1)}(\bar{\mathcal{D}}_{N+2,m})^{(m+\lambda)/(m+\lambda+1)} \le C_0\mathcal{Z}_0^{1/(m+\lambda+1)}(\bar{\mathcal{D}}_{N+2,m})^{(m+\lambda)/(m+\lambda+1)} \end{aligned} \tag{9-49}$$

for all $0 \le t \le T$, where we have written $\mathcal{Z}_0 := \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)$, and C_0 is a universal constant which we may assume satisfies $C_0 \ge 1$. Let us write

$$h(t) = \bar{\mathcal{E}}_{N+2,m}(t) - 2\int_\Omega J(t)\partial_t^{N+1} p(t) F^2(t) \ge 0, \tag{9-50}$$

as well as

$$s = \frac{1}{m+\lambda} \quad \text{and} \quad C_1 = \frac{1}{2C_0^{1+s}\mathcal{Z}_0^s}. \tag{9-51}$$

In these three terms we should distinguish between the cases $m = 1$ and $m = 2$, but to avoid notational clutter we will abuse notation and only write $h(t)$, s, and C_1. We may then combine (9-48) with (9-49) and use our new notation to derive the differential inequality

$$\partial_t h(t) + C_1 (h(t))^{1+s} \leq 0 \tag{9-52}$$

for $0 \leq t \leq T$.

Since $h(t) \geq 0$, we may integrate (9-52) to find that, for any $0 \leq r \leq T$,

$$h(r) \leq \frac{h(0)}{[1+sC_1(h(0))^s r]^{1/s}}. \tag{9-53}$$

Notice that Remark 2.8 implies that $\bar{\mathcal{E}}_{N+2,m} \leq \frac{3}{2}\mathcal{E}_{2N}$. Then (9-49) implies that $h(0) \leq \frac{4}{3}\bar{\mathcal{E}}_{N+2,m}(0) \leq 2\mathcal{E}_{2N}(0) \leq 2\mathcal{Z}_0$, which in turn implies that

$$sC_1(h(0))^s = \frac{s}{2C_0^{1+s}}\left(\frac{h(0)}{\mathcal{Z}_0}\right)^s \leq \frac{s}{2C_0^{1+s}} 2^s = \frac{s}{C_0^{1+s}} 2^{s-1} \leq 1 \tag{9-54}$$

since $0 < s < 1$ and $C_0 \geq 1$. A simple computation shows that

$$\sup_{r \geq 0} \frac{(1+r)^{1/s}}{(1+Mr)^{1/s}} = \frac{1}{M^{1/s}} \tag{9-55}$$

when $0 \leq M \leq 1$ and $s > 0$. This, (9-53), and (9-54) then imply that

$$(1+r)^{1/s} h(r) \leq h(0) \frac{(1+r)^{1/s}}{[1+sC_1(h(0))^s r]^{1/s}} \leq h(0) \left(\frac{2C_0^{1+s}}{s}\right)^{1/s} \frac{\mathcal{Z}_0}{h(0)} = \left(\frac{2C_0^{1+s}}{s}\right)^{1/s} \mathcal{Z}_0. \tag{9-56}$$

Now we use (9-30) of Proposition 9.5 together with (9-49) to bound

$$\mathcal{E}_{N+2,m}(r) \lesssim \bar{\mathcal{E}}_{N+2,m}(r) \lesssim h(r) \quad \text{for } 0 \leq r \leq T. \tag{9-57}$$

The estimate (9-47) then follows from (9-56), (9-57), and the fact that

$$s = 1/(m+\lambda) \quad \text{and} \quad \mathcal{Z}_0 = \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0). \qquad \square$$

A priori estimates for $\mathcal{G}_{2N}$. We now collect the results of Theorems 9.4 and 9.7 into a single bound on $\mathcal{G}_{2N}$, as defined by (2-58). The estimate recorded specifically names the constant in the inequality with $C_1 > 0$ so that it can be referenced later.

Theorem 9.8. *There exists a universal $0 < \delta < 1$ such that if $\mathcal{G}_{2N}(T) \leq \delta$, then*

$$\mathcal{G}_{2N}(t) \leq C_1(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)) \tag{9-58}$$

for all $0 \leq t \leq T$, where $C_1 > 0$ is a universal constant.

Proof. Let δ be as small as in Theorems 9.4 and 9.7. Then the conclusions of the theorems hold, and we may sum them to deduce (9-58). $\square$

10. Specialized local well-posedness

Propagation of $\mathcal{I}_\lambda$ bounds. To prove Theorem 1.3, we will combine our a priori estimates, Theorem 9.8, with a local well-posedness result. Theorem 1.1 is not quite enough since it does not address the boundedness of $\|\mathcal{I}_\lambda u(t)\|_0^2$, $\|\mathcal{I}_\lambda \eta(t)\|_0^2$, and $\|\mathcal{I}_\lambda p(t)\|_0^2$ for $t > 0$. In order to prove these bounds, we first study the cutoff operators $\mathcal{I}_\lambda^m$, which we define now. Let $m \geq 1$ be an integer. For a function f defined on Ω, we define the cutoff Riesz potential $\mathcal{I}_\lambda^m f$ by

$$\mathcal{I}_\lambda^m f(x', x_3) = \int_{\{|\xi| \geq 1/m\}} \hat{f}(\xi, x_3) |\xi|^{-\lambda} e^{2\pi i x' \cdot \xi} d\xi, \tag{10-1}$$

where $\hat{}$ denotes the Fourier transform in the (x_1, x_2) variables. Similarly, for f defined on Σ, we set

$$\mathcal{I}_\lambda^m f(x') = \int_{\{|\xi| \geq 1/m\}} \hat{f}(\xi) |\xi|^{-\lambda} e^{2\pi i x' \cdot \xi} d\xi. \tag{10-2}$$

The operator $\mathcal{I}_\lambda^m$ is clearly bounded on $H^0(\Omega)$ and $H^0(\Sigma)$, which allows us to apply it to our solutions and then study the evolution of $\mathcal{I}_\lambda^m u$ and $\mathcal{I}_\lambda^m \eta$.

Before doing so, we record some estimates for terms involving $\mathcal{I}_\lambda^m$ that are analogous to the $\mathcal{I}_\lambda$ estimates in Propositions Proposition 4.3 and 6.7 and in Lemmas 4.4, 4.5, 6.5, 6.6, A.3 and A.4. We begin with the analogues of the last two lemmas, which were the starting point for our $\mathcal{I}_\lambda$ estimates.

Lemma 10.1. *If $\mathcal{I}_\lambda h \in H^0(\Omega)$, then $\|\mathcal{I}_\lambda^m h\|_0^2 \leq \|\mathcal{I}_\lambda h\|_0^2$. A similar estimate holds if $\mathcal{I}_\lambda h \in H^0(\Sigma)$. As a consequence, the results of Lemmas A.3 and A.4 hold with $\mathcal{I}_\lambda$ replaced by $\mathcal{I}_\lambda^m$ and with the constants in the inequalities independent of m.*

Proof. Suppose that $\mathcal{I}_\lambda h \in H^0(\Omega)$ for some h. Then, writing $\hat{}$ for the horizontal Fourier transform, we easily see that

$$\|\mathcal{I}_\lambda^m h\|_0^2 = \int_{-b}^0 \int_{\{|\xi| \geq 1/m\}} |\hat{h}(\xi, x_3)|^2 |\xi|^{-2\lambda} d\xi dx_3 \leq \|\mathcal{I}_\lambda h\|_0^2. \tag{10-3}$$

The corresponding estimate in case $\mathcal{I}_\lambda h \in H^0(\Sigma)$ follows similarly. Then the estimates of Lemmas A.3 and A.4 may be combined with these inequalities to replace $\mathcal{I}_\lambda$ with $\mathcal{I}_\lambda^m$. □

We do not want our estimates for $\mathcal{I}_\lambda^m$ to be given in terms of $\mathcal{E}_{2N}$ since this energy contains $\mathcal{I}_\lambda$ terms. Instead, we desire estimates in terms of a modified energy, which we write as

$$\mathfrak{E}_{2N} := \mathcal{E}_{2N} - \|\mathcal{I}_\lambda u\|_0^2 - \|\mathcal{I}_\lambda \eta\|_0^2. \tag{10-4}$$

Lemma 10.1 allows us prove the following modification of Proposition 4.3. The proof is a simple adaptation of the one for Proposition 4.3, and is thus omitted.

Proposition 10.2. *Assume that $\mathfrak{E}_{2N} \leq 1$. We have*

$$\|\mathcal{I}_\lambda^m G^1\|_1^2 + \|\mathcal{I}_\lambda^m G^2\|_2^2 + \|\mathcal{I}_\lambda^m \partial_t G^2\|_0^2 + \|\mathcal{I}_\lambda^m G^3\|_1^2 + \|\mathcal{I}_\lambda^m G^4\|_1^2 \lesssim \mathfrak{E}_{2N}^2. \tag{10-5}$$

Here the constant in the inequality does not depend on m.

We may similarly modify the proof of Lemma 4.4, removing the interpolation arguments and simply estimating with $\mathfrak{E}_{2N}$ instead. This provides us with the following lemma, whose proof we omit.

Lemma 10.3. *Assume that $\mathfrak{E}_{2N} \le 1$. We have*

$$\|\mathcal{I}_\lambda^m[(AK)\partial_3 u_1 + (BK)\partial_3 u_2]\|_0^2 + \sum_{i=1}^2 \|\mathcal{I}_\lambda^m[u\partial_i K]\|_0^2 \lesssim \mathfrak{E}_{2N}^2, \tag{10-6}$$

$$\|\mathcal{I}_\lambda^m[(1-K)u]\|_0^2 + \|\mathcal{I}_\lambda^m[(1-K)G^2]\|_0^2 \lesssim \mathfrak{E}_{2N}^2. \tag{10-7}$$

Here the constants in the inequalities do not depend on m.

Lemma 10.3 leads to a modification of Lemma 6.5.

Lemma 10.4. *Assume that $\mathfrak{E}_{2N} \le 1$. We have*

$$\|\mathcal{I}_\lambda^m p\|_0^2 \lesssim \|\mathcal{I}_\lambda^m \eta\|_0^2 + \mathfrak{E}_{2N} \quad \textit{and} \quad \|\mathcal{I}_\lambda^m Dp\|_0^2 \lesssim \mathfrak{E}_{2N}. \tag{10-8}$$

Here the constants in the inequalities do not depend on m.

Proof. We may argue as in Lemma 6.5, employing Lemma 10.1 in place of Lemmas A.3 and A.4 as well as Proposition 10.2 and Lemma 10.3 in place of Proposition 4.3 and Lemma 4.4, to deduce the estimate $\|\partial^\alpha \mathcal{I}_\lambda^m p\|_0^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda^m \eta\|_0^2 + \|u\|_3^2 + \|\partial_t u\|_1^2 + \mathfrak{E}_{2N}^2$ for $\alpha \in \mathbb{N}^2$ with $|\alpha| \in \{0, 1\}$. We may bound $\|u\|_3^2 + \|\partial_t u\|_1^2 \le \mathfrak{E}_{2N}$. When $|\alpha| = 1$ we use Lemma 10.1 to estimate $\|\partial^\alpha \mathcal{I}_\lambda^m \eta\|_0^2 \lesssim (\|\eta\|_0^2)^\lambda (\|D\eta\|_0^2)^{1-\lambda} \le \mathfrak{E}_{2N}$. The desired estimates then follow from these estimates and the fact that $\mathfrak{E}_{2N} \le 1$. □

In turn, Lemma 10.4 gives a variant of Lemma 6.6. The proof is an easy modification of that of Lemma 6.6, using the above $\mathcal{I}_\lambda^m$ results in place of $\mathcal{I}_\lambda$ results, and is thus omitted.

Lemma 10.5. *Assume that $\mathfrak{E}_{2N} \le 1$. We have*

$$\left| \int_\Omega \mathcal{I}_\lambda^m p \mathcal{I}_\lambda^m G^2 \right| \lesssim \mathfrak{E}_{2N} \|\mathcal{I}_\lambda^m \eta\|_0 + \mathfrak{E}_{2N}. \tag{10-9}$$

Here the constant in the inequality does not depend on m.

These results now allow us to study the boundedness of $\mathcal{I}_\lambda u$, etc. We first apply the operator $\mathcal{I}_\lambda^m$ to the equations (2-23), which is possible since $\mathcal{I}_\lambda^m$ is bounded on $H^0(\Omega)$ and $H^0(\Sigma)$. Then the energy evolution for $\mathcal{I}_\lambda^m u$ and $\mathcal{I}_\lambda^m \eta$ allows us to derive bounds for these quantities, which yield bounds for $\mathcal{I}_\lambda u$ and $\mathcal{I}_\lambda \eta$ after passing to the limit $m \to \infty$.

Proposition 10.6. *Suppose that (u, p, η) are solutions on the time interval $[0, T]$ and that $\|\mathcal{I}_\lambda u_0\|_0^2 + \|\mathcal{I}_\lambda \eta_0\|_0^2 < \infty$ and $\sup_{0\le t\le T} \mathfrak{E}_{2N}(t) \le 1$. Then*

$$\sup_{0\le t\le T} (\|\mathcal{I}_\lambda u(t)\|_0^2 + \|\mathcal{I}_\lambda p(t)\|_0^2 + \|\mathcal{I}_\lambda \eta(t)\|_0^2) + \int_0^T \|\mathcal{I}_\lambda u(t)\|_1^2 \, dt \lesssim e^T (\|\mathcal{I}_\lambda u_0\|_0^2 + \|\mathcal{I}_\lambda \eta_0\|_0^2) + e^T \sup_{0\le t\le T} \mathfrak{E}_{2N}(t). \tag{10-10}$$

Proof. Since $\mathcal{I}_\lambda^m$ is a bounded operator on $H^0(\Omega)$ and $H^0(\Sigma)$, we are free to apply it to the equations (2-23). After doing so, we use Lemma 2.5 to see that

$$\partial_t\left(\frac{1}{2}\int_\Omega |\mathcal{I}_\lambda^m u|^2 + \frac{1}{2}\int_\Sigma |\mathcal{I}_\lambda^m \eta|^2\right) + \frac{1}{2}\int_\Omega |\mathbb{D}\mathcal{I}_\lambda^m u|^2 \\ = \int_\Omega \mathcal{I}_\lambda^m u\cdot(\mathcal{I}_\lambda^m G^1 - \nabla \mathcal{I}_\lambda^m G^2) + \mathcal{I}_\lambda^m p \mathcal{I}_\lambda^m G^2 + \int_\Sigma -\mathcal{I}_\lambda^m u\cdot \mathcal{I}_\lambda^m G^3 + \mathcal{I}_\lambda^m \eta \mathcal{I}_\lambda^m G^4. \quad (10\text{-}11)$$

We will estimate each term on the right side of this equation. First, we use Cauchy–Schwarz and Proposition 10.2 to estimate the first and fourth terms:

$$\left|\int_\Omega \mathcal{I}_\lambda^m u\cdot(\mathcal{I}_\lambda^m G^1 - \nabla\mathcal{I}_\lambda^m G^2)\right| + \left|\int_\Sigma \mathcal{I}_\lambda^m \eta \mathcal{I}_\lambda^m G^4\right| \\ \le \|\mathcal{I}_\lambda^m u\|_0(\|\mathcal{I}_\lambda^m G^1\|_0 + \|\mathcal{I}_\lambda^m G^2\|_1) + \|\mathcal{I}_\lambda^m \eta\|_0 \|\mathcal{I}_\lambda^m G^4\|_0 \\ \le \tfrac{1}{2}\|\mathcal{I}_\lambda^m u\|_0^2 + \tfrac{1}{4}\|\mathcal{I}_\lambda^m \eta\|_0^2 + \tfrac{1}{2}(\|\mathcal{I}_\lambda^m G^1\|_0 + \|\mathcal{I}_\lambda^m G^2\|_1)^2 + \|\mathcal{I}_\lambda^m G^4\|_0^2 \\ \le \tfrac{1}{2}\|\mathcal{I}_\lambda^m u\|_0^2 + \tfrac{1}{4}\|\mathcal{I}_\lambda^m \eta\|_0^2 + C\mathfrak{E}_{2N}^2 \quad (10\text{-}12)$$

for $C > 0$ independent of m. For the second term we use Lemma 10.5 and Cauchy's inequality for

$$\left|\int_\Omega \mathcal{I}_\lambda^m p \mathcal{I}_\lambda^m G^2\right| \le C\|\mathcal{I}_\lambda^m \eta\|_0 \mathfrak{E}_{2N} + C\mathfrak{E}_{2N} \le \tfrac{1}{4}\|\mathcal{I}_\lambda^m \eta\|_0^2 + C(\mathfrak{E}_{2N} + \mathfrak{E}_{2N}^2), \quad (10\text{-}13)$$

where again $C > 0$ is independent of m. Finally, for the third term we use trace theory, Proposition 10.2, and Lemma A.12 to bound

$$\left|\int_\Sigma \mathcal{I}_\lambda^m u\cdot\mathcal{I}_\lambda^m G^3\right| \le \|\mathcal{I}_\lambda^m u\|_{H^0(\Sigma)}\|\mathcal{I}_\lambda^m G^3\|_0 \le C\|\mathcal{I}_\lambda^m u\|_1\|\mathcal{I}_\lambda^m G^3\|_0 \\ \le C\|\mathbb{D}\mathcal{I}_\lambda^m u\|_0 \mathfrak{E}_{2N} \le \tfrac{1}{4}\|\mathbb{D}\mathcal{I}_\lambda^m u\|_0^2 + C\mathfrak{E}_{2N}^2, \quad (10\text{-}14)$$

with $C > 0$ independent of m. Now we use (10-12)–(10-14) to estimate the right side of (10-11); after rearranging the resulting bound, we find that

$$\partial_t(\|\mathcal{I}_\lambda^m u\|_0^2 + \|\mathcal{I}_\lambda^m \eta\|_0^2) + \tfrac{1}{2}\|\mathbb{D}\mathcal{I}_\lambda^m u\|_0^2 \le \|\mathcal{I}_\lambda^m u\|_0^2 + \|\mathcal{I}_\lambda^m \eta\|_0^2 + C(\mathfrak{E}_{2N} + \mathfrak{E}_{2N}^2) \quad (10\text{-}15)$$

for a constant $C > 0$ that does not depend on m.

The inequality (10-15) may be viewed as the differential inequality

$$\partial_t \mathscr{E}_{\lambda,m} + \tfrac{1}{2}\mathscr{D}_{\lambda,m} \le \mathscr{E}_{\lambda,m} + C(\mathfrak{E}_{2N} + \mathfrak{E}_{2N}^2), \quad (10\text{-}16)$$

where we have written $\mathscr{E}_{\lambda,m} = \|\mathcal{I}_\lambda^m u\|_0^2 + \|\mathcal{I}_\lambda^m \eta\|_0^2$ and $\mathscr{D}_{\lambda,m} = \|\mathbb{D}\mathcal{I}_\lambda^m u\|_0^2$. Applying Gronwall's lemma to (10-16) and using the fact that $\mathfrak{E}_{2N}(t) \le 1$ then shows that

$$\mathscr{E}_{\lambda,m}(t) + \frac{1}{2}\int_0^t \mathscr{D}_{\lambda,m}(s)\,ds \le \mathscr{E}_{\lambda,m}(0)e^t + C\int_0^t e^{t-s}\mathfrak{E}_{2N}(s)\,ds \\ \le \mathscr{E}_{\lambda,m}(0)e^t + C(e^t - 1)\sup_{0\le s\le t}\mathfrak{E}_{2N}(s), \quad (10\text{-}17)$$

where again $C > 0$ is independent of m. It is a simple matter to verify, using the definitions of $\mathcal{I}_\lambda^m$ and $\mathcal{I}_\lambda$, Parseval's theorem for the Fourier transform in (x_1, x_2), and the monotone convergence theorem, that, as $m \to \infty$,

$$\mathcal{E}_{\lambda,m}(s) = \|\mathcal{I}_\lambda^m u(s)\|_0^2 + \|\mathcal{I}_\lambda^m \eta(s)\|_0^2 \to \|\mathcal{I}_\lambda u(s)\|_0^2 + \|\mathcal{I}_\lambda \eta(s)\|_0^2 \tag{10-18}$$

for both $s = 0$ and $s = t$, and

$$\int_0^t \mathcal{D}_{\lambda,m}(s)\, ds \to \int_0^t \|\mathbb{D}\mathcal{I}_\lambda u(s)\|_0^2\, ds. \tag{10-19}$$

Now, according to these two convergence results, we may pass to the limit $m \to \infty$ in (10-17); the resulting estimate and Lemma A.12 then imply that

$$\sup_{0\le t\le T} (\|\mathcal{I}_\lambda u(t)\|_0^2 + \|\mathcal{I}_\lambda \eta(t)\|_0^2) + \int_0^T \|\mathcal{I}_\lambda u(t)\|_1^2\, dt \lesssim (\|\mathcal{I}_\lambda u_0\|_0^2 + \|\mathcal{I}_\lambda \eta_0\|_0^2) e^T + (e^T - 1) \sup_{0\le t\le T} \mathfrak{E}_{2N}(t). \tag{10-20}$$

On the other hand, from Lemma 10.4, we know that

$$\|\mathcal{I}_\lambda^m p(t)\|_0^2 \lesssim \|\mathcal{I}_\lambda^m \eta(t)\|_0^2 + \mathfrak{E}_{2N}(t). \tag{10-21}$$

We may then argue as above, employing the monotone convergence theorem, to pass to the limit $m \to \infty$ in this estimate. We then find that

$$\sup_{0\le t\le T} \|\mathcal{I}_\lambda p(t)\|_0^2 \lesssim \sup_{0\le t\le T} \|\mathcal{I}_\lambda \eta(t)\|_0^2 + \sup_{0\le t\le T} \mathfrak{E}_{2N}(t). \tag{10-22}$$

The estimate (10-10) then follows by combining (10-20) and (10-22). □

Local well-posedness. We now record the specialized version of the local well-posedness theorem. We include estimates for $\mathcal{I}_\lambda u$, $\mathcal{I}_\lambda \eta$, and $\mathcal{I}_\lambda p$. We also separate estimates for $\mathcal{E}_{2N}$ and $\mathcal{D}_{2N}$ from estimates for $\mathcal{F}_{2N}$ and $\mathfrak{E}_{2N}$, the latter of which is defined by (10-4).

Theorem 10.7. *Suppose that initial data are given satisfying the compatibility conditions of Theorem 1.1 and $\|u(0)\|_{4N}^2 + \|\eta(0)\|_{4N+1/2}^2 + \|\mathcal{I}_\lambda u(0)\|_0^2 + \|\mathcal{I}_\lambda \eta(0)\|_0^2 < \infty$. Let $\varepsilon > 0$. There exists a $\delta_0 = \delta_0(\varepsilon) > 0$ and a*

$$T_0 = C(\varepsilon) \min\left\{1, \frac{1}{\|\eta(0)\|_{4N+1/2}^2}\right\} > 0, \tag{10-23}$$

where $C(\varepsilon) > 0$ is a constant depending on ε, such that if $0 < T \le T_0$ and $\|u(0)\|_{4N}^2 + \|\eta(0)\|_{4N}^2 \le \delta_0$, there exists a unique solution (u, p, η) to (1-9) on the interval $[0, T]$ that achieves the initial data. The solution obeys the estimates

$$\sup_{0\le t\le T} \mathcal{E}_{2N}(t) + \sup_{0\le t\le T} \|\mathcal{I}_\lambda p(t)\|_0^2 + \int_0^T \mathcal{D}_{2N}(t)\, dt + \|\partial_t^{2N+1} u\|_{(\mathcal{X}_T)^*}^2 \le C_2(\varepsilon + \|\mathcal{I}_\lambda u(0)\|_0^2 + \|\mathcal{I}_\lambda \eta(0)\|_0^2), \tag{10-24}$$

and

$$\sup_{0\le t\le T} \mathfrak{E}_{2N}(t) \le \varepsilon \quad \textit{and} \quad \sup_{0\le t\le T} \mathcal{F}_{2N}(t) \le C_2\mathcal{F}_{2N}(0)+\varepsilon \tag{10-25}$$

for $C_2 > 0$ *a universal constant. Here* $\mathfrak{E}_{2N}$ *is as defined by* (10-4) *and* $\mathscr{X}_T$ *is defined in* (1-11).

Proof. The result follows directly from Proposition 10.6 and Theorem 1.1. □

Remark 10.8. The finiteness of the terms in (10-24) and (10-25) justifies all of the computations leading to Theorem 9.8. In particular, it shows that $\partial_t^{2N+1}u$ and $\partial_t^{2N}p$ are well-defined.

Remark 10.9. We could have recorded a version of Theorem 10.7 in which ε is replaced by various terms depending on the initial data in (10-24)–(10-25). We have chosen to introduce the ε term for convenience in our proof of Theorem 11.2.

11. Global well-posedness and decay: proof of Theorem 1.3

In order to combine the local existence result, Theorem 10.7, with the a priori estimates of Theorem 9.8, we must be able to estimate $\mathcal{G}_{2N}$, defined by (2-58), in terms of the estimates given in (10-24) and (10-25). We record this estimate now.

Proposition 11.1. *Let* $\mathfrak{E}_{2N}$ *be as defined by* (10-4). *There exists a universal constant* $C_3 > 0$ *with the following properties.*

(1) *If* $0 \le T$, *we have the estimate*

$$\mathcal{G}_{2N}(T) \le \sup_{0\le t\le T} \mathcal{E}_{2N}(t) + \int_0^T \mathcal{D}_{2N}(t)\,dt + \sup_{0\le t\le T} \mathcal{F}_{2N}(t) + C_3(1+T)^{2+\lambda} \sup_{0\le t\le T} \mathfrak{E}_{2N}(t). \tag{11-1}$$

(2) *If* $0 < T_1 \le T_2$ *and* $\sup_{T_1\le t\le T_2} \|\eta(t)\|^2_{5/2} \le \delta$, *where* $\delta > 0$ *is as in Lemma 2.6, we have the estimate*

$$\mathcal{G}_{2N}(T_2) \le C_3\mathcal{G}_{2N}(T_1) + \sup_{T_1\le t\le T_2} \mathcal{E}_{2N}(t) + \int_{T_1}^{T_2} \mathcal{D}_{2N}(t)\,dt + \frac{1}{(1+T_1)} \sup_{T_1\le t\le T_2} \mathcal{F}_{2N}(t) \\ + C_3(T_2-T_1)^2(1+T_2)^{2+\lambda} \sup_{T_1\le t\le T_2} \mathfrak{E}_{2N}(t). \tag{11-2}$$

Proof. We begin with the proof of the estimate (11-2). The definition of $\mathcal{G}_{2N}(T_2)$ in (2-58) allows us to estimate

$$\mathcal{G}_{2N}(T_2) \\ \le \mathcal{G}_{2N}(T_1) + \sup_{T_1\le t\le T_2} \mathcal{E}_{2N}(t) + \int_{T_1}^{T_2} \mathcal{D}_{2N}(t)\,dt + \sup_{T_1\le t\le T_2} \frac{\mathcal{F}_{2N}(t)}{(1+t)} + \sum_{m=1}^{2} \sup_{T_1\le t\le T_2} ((1+t)^{m+\lambda}\mathcal{E}_{N+2,m}(t)). \tag{11-3}$$

Since $N \ge 3$, it is easy to verify that

$$\sum_{j=0}^{N+2} \|\partial_t^{j+1}u\|^2_{2(N+2)-2j} + \|\partial_t^{j}u\|^2_{2(N+2)-2j} + \|\partial_t^{j+1}\eta\|^2_{2(N+2)-2j} + \|\partial_t^{j}\eta\|^2_{2(N+2)-2j} \lesssim \mathfrak{E}_{2N} \tag{11-4}$$

and
$$\sum_{j=0}^{N+1} \|\partial_t^{j+1} p\|^2_{2(N+2)-2j-1} + \|\partial_t^j p\|^2_{2(N+2)-2j-1} \lesssim \mathfrak{E}_{2N}. \tag{11-5}$$

We will use (11-4), (11-5), and an integration argument to estimate the last term in (11-3).

For $j = 1, \dots, N+2$ and $m = 1, 2$ we may integrate $\partial_t[(1+t)^{(m+\lambda)/2}\partial_t^j u(t)]$ in time from T_1 to $t \in [T_1, T_2]$ and use the estimates in (11-4) to deduce the bound

$$\|(1+t)^{(m+\lambda)/2}\partial_t^j u(t)\|_{2N+4-2j} \le \|(1+T_1)^{(m+\lambda)/2}\partial_t^j u(T_1)\|_{2N+4-2j}$$
$$+ \int_{T_1}^{T_2} \left((1+s)^{(m+\lambda)/2}\|\partial_t^{j+1} u(s)\|_{2N+4-2j} + \frac{(m+\lambda)}{2}(1+s)^{(m+\lambda-2)/2}\|\partial_t^j u(s)\|_{2N+4-2j} \right) ds$$
$$\lesssim \sqrt{\mathcal{G}_{2N}(T_1)} + (T_2 - T_1)(1+T_2)^{1+\lambda/2}\sqrt{\sup_{T_1 \le t \le T_2} \mathfrak{E}_{2N}(t)}. \tag{11-6}$$

Squaring both sides of this, summing over $j = 1, \dots, N+2$, taking the supremum, and then summing over $m = 1, 2$ then yields the bound

$$\sum_{m=1}^{2} \sup_{T_1 \le t \le T_2} \left((1+t)^{m+\lambda} \sum_{j=1}^{N+2} \|\partial_t^j u(t)\|^2_{2(N+2)-2j} \right) \lesssim \mathcal{G}_{2N}(T_1) + (T_2-T_1)^2(1+T_2)^{2+\lambda} \sup_{T_1 \le t \le T_2} \mathfrak{E}_{2N}(t). \tag{11-7}$$

We may also integrate $\partial_t[(1+t)^{(m+\lambda)/2}\partial^\alpha u(t)]$ for $\alpha \in \mathbb{N}^3$ with $|\alpha| = m+1$ and argue as above, again employing the estimate (11-4), to deduce the bound (after summing over all such α)

$$\sum_{m=1}^{2} \sup_{T_1 \le t \le T_2} ((1+t)^{m+\lambda}\|\nabla^{m+1} u(t)\|^2_{2(N+2)-m-1}) \lesssim \mathcal{G}_{2N}(T_1) + (T_2-T_1)^2(1+T_2)^{2+\lambda} \sup_{T_1 \le t \le T_2} \mathfrak{E}_{2N}(t). \tag{11-8}$$

Similarly, we may integrate $\partial_t[(1+t)^{(m+\lambda)/2}\partial^\alpha u(t)]$ for $\alpha \in \mathbb{N}^{1+2}$ with $m \le |\alpha| \le 2N+4$, argue as above with (11-4), and then employ the bound $\|\bar{D}_m^{2N+4} u\|_0^2 \lesssim \bar{\mathcal{E}}_{N+2,m}$ from Remark 2.8 (which holds for $t \in [T_1, T_2]$ because of our assumption on the size of $\|\eta\|^2_{5/2}$), to deduce the bound (again after summing over all such α)

$$\sum_{m=1}^{2} \sup_{T_1 \le t \le T_2} ((1+t)^{m+\lambda}\|\bar{D}_m^{2N+4} u(t)\|_0^2) \lesssim \mathcal{G}_{2N}(T_1) + (T_2-T_1)^2(1+T_2)^{2+\lambda} \sup_{T_1 \le t \le T_2} \mathfrak{E}_{2N}(t). \tag{11-9}$$

Together, the estimates (11-7)–(11-9) account for all of the u terms appearing in $\mathcal{E}_{N+2,m}$, as defined in (2-52) for $m = 1$ and (2-53) for $m = 2$.

Now we turn to the terms in $\mathcal{E}_{N+2,m}$ involving η and p. We may use the η estimates in (11-4) and the p estimates in (11-5) in a trio of integration arguments like those used above in (11-7)–(11-9). These yield the estimates

$$\sum_{m=1}^{2} \sup_{T_1 \le t \le T_2} \left((1+t)^{m+\lambda} \left[\sum_{j=1}^{N+1} \|\partial_t^j p(t)\|^2_{2(N+2)-2j-1} + \sum_{j=1}^{N+2} \|\partial_t^j \eta(t)\|^2_{2(N+2)-2j} \right] \right)$$
$$\lesssim \mathcal{G}_{2N}(T_1) + (T_2-T_1)^2(1+T_2)^{2+\lambda} \sup_{T_1 \le t \le T_2} \mathfrak{E}_{2N}(t), \tag{11-10}$$

$$\sum_{m=1}^{2} \sup_{T_1\le t\le T_2} ((1+t)^{m+\lambda}[\|\nabla^m p(t)\|^2_{2(N+2)-m-1} + \|D^m \eta(t)\|^2_{2(N+2)-m}]) \\ \lesssim \mathcal{G}_{2N}(T_1) + (T_2-T_1)^2(1+T_2)^{2+\lambda} \sup_{T_1\le t\le T_2} \mathfrak{E}_{2N}(t), \quad (11\text{-}11)$$

and

$$\sum_{m=1}^{2} \sup_{T_1\le t\le T_2} ((1+t)^{m+\lambda}\|\bar{D}^{2N+4}_m \eta(t)\|^2_0) \lesssim \mathcal{G}_{2N}(T_1) + (T_2-T_1)^2(1+T_2)^{2+\lambda} \sup_{T_1\le t\le T_2} \mathfrak{E}_{2N}(t). \quad (11\text{-}12)$$

Now we sum (11-7)–(11-12) and use the bound $\bar{\mathcal{E}}_{N+2,m} \lesssim \|\bar{D}^{2N+4}_m u\|^2_0 + \|\bar{D}^{2N+4}_m \eta\|^2_0$ from Remark 2.8 to find that

$$\sum_{m=1}^{2} \sup_{T_1\le t\le T_2} ((1+t)^{m+\lambda}\mathcal{E}_{N+2,m}(t)) \lesssim \mathcal{G}_{2N}(T_1) + (T_2-T_1)^2(1+T_2)^{2+\lambda} \sup_{T_1\le t\le T_2} \mathfrak{E}_{2N}(t). \quad (11\text{-}13)$$

Then (11-2) follows from (11-3), (11-13), and the trivial bound

$$\sup_{T_1\le t\le T_2} \frac{\mathcal{F}_{2N}(t)}{(1+t)} \le \frac{1}{(1+T_1)} \sup_{T_1\le t\le T_2} \mathcal{F}_{2N}(t). \quad (11\text{-}14)$$

Now we turn to the proof of (11-1). It is easy to see that $\mathcal{E}_{N+2,m}(t) \lesssim \mathfrak{E}_{2N}(t)$, which leads us to the simple bound

$$\sum_{m=1}^{2} \sup_{0\le t\le T} ((1+t)^{m+\lambda}\mathcal{E}_{N+2,m}(t)) \lesssim (1+T)^{2+\lambda} \sup_{0\le t\le T} \mathfrak{E}_{2N}(t). \quad (11\text{-}15)$$

Then this, (11-14) with T_1 replaced by 0 and T_2 replaced by T, and the definition of $\mathcal{G}_{2N}$ in (2-58) imply (11-1). □

We now turn to our main result.

Theorem 11.2. *Suppose the initial data (u_0, η_0) satisfy the compatibility conditions of Theorem 1.1. Let $\mathcal{E}_{2N}$, $\mathcal{F}_{2N}$, and $\mathcal{G}_{2N}$ be defined by (2-50), (2-56), and (2-58), respectively. There exists a $\kappa > 0$ such that if $\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) < \kappa$, there exists a unique solution (u, p, η) to (1-9) on the interval $[0, \infty)$ that achieves the initial data. The solution obeys the estimate*

$$\mathcal{G}_{2N}(\infty) \le C_1(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)) < C_1\kappa, \quad (11\text{-}16)$$

where $C_1 > 0$ is given by Theorem 9.8.

Proof. Let $0 < \delta < 1$ and $C_1 > 0$ be the constants from Theorem 9.8, $C_2 > 0$ the constant from Theorem 10.7, and $C_3 > 0$ the constant from Proposition 11.1. According to (11-1) of Proposition 11.1, if a solution exists on the interval $[0, T]$ with $T < 1$ and obeys the estimates (10-24) and (10-25), then

$$\mathcal{G}_{2N}(T) \le C_2\kappa + \varepsilon[C_2 + 1 + C_3 2^{2+\lambda}]. \quad (11\text{-}17)$$

If ε is chosen so that the latter term in (11-17) equals $\delta/2$, we may choose κ sufficiently small that $C_2\kappa < \delta/2$ and $\kappa < \delta_0(\varepsilon)$ (with $\delta_0(\varepsilon)$ given by Theorem 10.7); then Theorem 10.7 provides a unique solution on $[0, T]$ obeying the estimates (10-24) and (10-25), and hence $\mathcal{G}_{2N}(T) \le \delta$. According to Remark 10.8, all of the computations leading to Theorem 9.8 are justified by the estimates (10-24) and (10-25).

Let us now define

$$\begin{aligned} T_*(\kappa) = \sup\{T > 0 \mid &\text{ for every choice of initial data satisfying the compatibility} \\ &\text{conditions and } \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) < \kappa, \text{ there exists a unique solution} \\ &\text{on } [0, T] \text{ that achieves the data and satisfies } \mathcal{G}_{2N}(T) \le \delta\}. \end{aligned} \tag{11-18}$$

By the above analysis, $T_*(\kappa)$ is well-defined and satisfies $T_*(\kappa) > 0$ if κ is small enough, that is, there is a $\kappa_1 > 0$ such that $T_* : (0, \kappa_1] \to (0, \infty]$. It is easily verified that T_* is nonincreasing on $(0, \kappa_1]$. Let us now set

$$\varepsilon = \frac{\delta}{3} \min\left\{\frac{1}{1+C_2}, \frac{1}{C_3}\right\} \tag{11-19}$$

and then define $\kappa_0 \in (0, \kappa_1]$ by

$$\kappa_0 = \min\left\{\frac{\delta}{3C_1(C_3+2C_2)}, \frac{\delta_0(\varepsilon)}{C_1}, \kappa_1\right\}, \tag{11-20}$$

where $\delta_0(\varepsilon)$ is given by Theorem 10.7 with ε given by (11-19). We claim that $T_*(\kappa_0) = \infty$. Once the claim is established, the proof of the theorem is complete, since then $T_*(\kappa) = \infty$ for all $0 < \kappa \le \kappa_0$.

Suppose, by way of contradiction, that $T_*(\kappa_0) < \infty$. We will show that solutions can actually be extended past $T_*(\kappa_0)$ and that these solutions satisfy $\mathcal{G}_{2N}(T_2) \le \delta$ for $T_2 > T_*(\kappa_0)$, contradicting the definition of $T_*(\kappa_0)$. We begin by extending the solutions. By the definition of $T_*(\kappa_0)$, we know that, for every $0 < T_1 < T_*(\kappa_0)$ and any choice of data satisfying the compatibility conditions and the bound $\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) < \kappa_0$, there exists a unique solution on $[0, T_1]$ that achieves the initial data and satisfies $\mathcal{G}_{2N}(T_1) \le \delta$. Then, by Theorem 9.8, we know that, actually,

$$\mathcal{G}_{2N}(T_1) \le C_1(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)) < C_1\kappa_0. \tag{11-21}$$

In particular, this and (11-20) imply that

$$\mathcal{E}_{2N}(T_1) + \frac{\mathcal{F}_{2N}(T_1)}{(1+T_1)} < C_1\kappa_0 \le \delta_0(\varepsilon) \quad \text{for all } 0 < T_1 < T_*(\kappa_0), \tag{11-22}$$

where ε is given by (11-19). We view $(u(T_1), p(T_1), \eta(T_1))$ as initial data for a new problem; since (u, p, η) are already solutions, they satisfy the compatibility conditions needed to use them as data. Then, since $\mathcal{E}_{2N}(T_1) < \delta_0(\varepsilon)$, we can use Theorem 10.7 with ε given by (11-19) to extend solutions to $[T_1, T_2]$ for any T_2 satisfying

$$0 < T_2 - T_1 \le T_0 = C(\varepsilon) \min\{1, \mathcal{F}_{2N}(T_1)^{-1}\}. \tag{11-23}$$

In light of (11-22), we may bound

$$\bar{T} := C(\varepsilon) \min\left\{1, \frac{1}{\delta_0(\varepsilon)(1+T_*(\kappa_0))}\right\} \le T_0. \tag{11-24}$$

Notice that $\overline{T}$ depends on ε (given by (11-19)) and $T_*(\kappa_0)$, but is independent of T_1. Let

$$\gamma = \min\left\{\overline{T},\, T_*(\kappa_0),\, \frac{1}{(1+2T_*(\kappa_0))^{1+\lambda/2}}\right\}, \tag{11-25}$$

and then let us choose $T_1 = T_*(\kappa_0) - \gamma/2$ and $T_2 = T_*(\kappa_0) + \gamma/2$. The choice of γ implies that

$$0 < T_1 < T_*(\kappa_0) < T_2 < 2T_*(\kappa_0) \quad \text{and} \quad 0 < \gamma = T_2 - T_1 \le \overline{T} \le T_0. \tag{11-26}$$

Then Theorem 10.7 allows us to extend solutions to the interval $[0, T_2]$, and it provides estimates on the extended interval $[T_1, T_2]$:

$$\sup_{T_1\le t\le T_2} \mathscr{E}_{2N}(t) + \sup_{T_1\le t\le T_2} \|\mathscr{I}_\lambda p(t)\|_0^2 + \int_{T_1}^{T_2} \mathscr{D}_{2N}(t)\,dt + \|\partial_t^{2N+1} u\|^2_{(\mathscr{X}_{(T_1,T_2)})^*} \\ \le C_2(\varepsilon + \|\mathscr{I}_\lambda u(T_1)\|_0^2 + \|\mathscr{I}_\lambda \eta(T_1)\|_0^2), \tag{11-27}$$

and

$$\sup_{T_1\le t\le T_2} \mathfrak{E}_{2N}(t) \le \varepsilon \quad \text{and} \quad \sup_{T_1\le t\le T_2} \mathscr{F}_{2N}(t) \le C_2 \mathscr{F}_{2N}(T_1) + \varepsilon. \tag{11-28}$$

Here, in (11-27), we understand that $\mathscr{X}_{(T_1,T_2)}$ is defined as in (1-11) except on the temporal interval (T_1, T_2) rather than $(0, T)$.

Having extended the existence interval, we will now show that $\mathscr{G}_{2N}(T_2) \le \delta$. Note that the constant δ, which comes from Theorem 9.8, is already smaller than the δ appearing in Lemma 2.6. Then the first estimate in (11-28) and the bound $\varepsilon \le \delta$ (a consequence of (11-19)) imply that $\sup_{T_1\le t\le T_2} \|\eta(t)\|_{5/2}^2$ is smaller than the δ in Lemma 2.6, which means we may use the second estimate in Proposition 11.1. We then combine the estimates (11-27)–(11-28) with (11-21)–(11-22) and the bound (11-2) of Proposition 11.1 to see that

$$\mathscr{G}_{2N}(T_2) < C_1 C_3 \kappa_0 + C_2(\varepsilon + C_1\kappa_0) + \frac{C_1 C_2 \kappa_0 (1+T_1) + \varepsilon}{(1+T_1)} + \varepsilon C_3 (T_2 - T_1)^2 (1+T_2)^{2+\lambda} \\ \le \kappa_0 C_1 (C_3 + 2C_2) + \varepsilon(1+C_2) + \varepsilon C_3 \gamma^2 (1+2T_*(\kappa_0))^{2+\lambda} \le \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta, \tag{11-29}$$

where the second inequality follows from (11-26) and the third follows from the choice of ε, κ_0, and γ given in (11-19), (11-20), and (11-25), respectively. Hence $\mathscr{G}_{2N}(T_2) \le \delta$, contradicting the definition of $T_*(\kappa_0)$. We then deduce that $T_*(\kappa_0) = \infty$, which completes the proof of the claim and the theorem. □

With this result in hand, it is a simple matter to prove Theorem 1.3.

Proof of Theorem 1.3. We set $N = 5$ in Theorem 11.2 to deduce all of the conclusions of Theorem 1.3 except the estimates (1-20)–(1-21). Proposition 3.9 implies that

$$\|u\|^2_{C^2(\Omega)} \le C(r)(\mathscr{E}_{10})^{r/(2+r)} (\mathscr{E}_{7,2})^{2/(2+r)} \tag{11-30}$$

for any $r \in (0, 1)$, where $C(r) > 0$ is a constant depending on r. Let $0 \le \rho < \lambda$ and then choose $r \in (0, 1)$ such that

$$0 < r \le 2\frac{2+\lambda}{2+\rho} - 2, \quad \text{or equivalently} \quad (2+\rho) \le (2+\lambda)\frac{2}{2+r}. \tag{11-31}$$

Then $C(r) = C(\rho)$ and the bound $\mathcal{G}_{10}(\infty) \le C_1(\mathcal{E}_{10}(0) + \mathcal{F}_{10}(0))$ implies that

$$\begin{aligned}\sup_{t\ge 0}(1+t)^{2+\rho}\|u(t)\|^2_{C^2(\Omega)} &\le C(\rho)C_1(\mathcal{E}_{10}(0)+\mathcal{F}_{10}(0))\sup_{t\ge 0}(1+t)^{2+\rho}\Big(\frac{1}{(1+t)^{2+\lambda}}\Big)^{2/(2+r)}\\ &\le C(\rho)C_1(\mathcal{E}_{10}(0)+\mathcal{F}_{10}(0)),\end{aligned} \tag{11-32}$$

which is (1-20). The estimate (1-21) follows similarly by using the interpolation estimates of Lemma 3.1 for the η terms and the interpolation estimates of Theorem 3.14 for $\|u\|_2^2$. In this case, though, no use of $r \in (0, 1)$ is necessary because it does not appear in the interpolations. □

Appendix: Analytic tools

Products in Sobolev spaces. We will need some estimates of the product of functions in Sobolev spaces.

Lemma A.1. *Let U denote either Σ or Ω.*

(1) *Let $0 \le r \le s_1 \le s_2$ be such that $s_1 > n/2$. Let $f \in H^{s_1}(U)$, $g \in H^{s_2}(U)$. Then $fg \in H^r(U)$ and*

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}}\|g\|_{H^{s_2}}. \tag{A-1}$$

(2) *Let $0 \le r \le s_1 \le s_2$ be such that $s_2 > r + n/2$. Let $f \in H^{s_1}(U)$, $g \in H^{s_2}(U)$. Then $fg \in H^r(U)$ and*

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}}\|g\|_{H^{s_2}}. \tag{A-2}$$

(3) *Let $0 \le r \le s_1 \le s_2$ be such that $s_2 > r + n/2$. Let $f \in H^{-r}(\Sigma)$, $g \in H^{s_2}(\Sigma)$. Then $fg \in H^{-s_1}(\Sigma)$ and*

$$\|fg\|_{-s_1} \lesssim \|f\|_{-r}\|g\|_{s_2}. \tag{A-3}$$

Proof. The proofs of (A-1) and (A-2) are standard; the bounds are first proved in $\mathbb{R}^n$ with the Fourier transform, and then the bounds in sufficiently nice subsets of $\mathbb{R}^n$ are deduced by use of an extension operator. To prove (A-3) we argue by duality. For $\varphi \in H^{s_1}(\Sigma)$ we use (A-2) to bound

$$\int_\Sigma \varphi f g \lesssim \|\varphi g\|_r \|f\|_{-r} \lesssim \|\varphi\|_{s_1}\|g\|_{s_2}\|f\|_{-r}, \tag{A-4}$$

so that upon taking the supremum over φ with $\|\varphi\|_{s_1} \le 1$ we get (A-3). □

We will also need the following variant.

Lemma A.2. *Suppose that $f \in C^1(\Sigma)$ and $g \in H^{1/2}(\Sigma)$. Then*

$$\|fg\|_{1/2} \lesssim \|f\|_{C^1}\|g\|_{1/2}. \tag{A-5}$$

Proof. Consider the operator $F : H^k \to H^k$ given by $F(g) = fg$ for $k = 0, 1$. It is a bounded operator for $k = 0, 1$ since

$$\|fg\|_0 \le \|f\|_{C^1}\|g\|_0 \quad \text{and} \quad \|fg\|_1 \lesssim \|f\|_{C^1}\|g\|_1. \tag{A-6}$$

Then the theory of interpolation of operators implies that F is bounded from $H^{1/2}$ to itself, with operator norm less than a constant times $\sqrt{\|f\|_{C^1}}\sqrt{\|f\|_{C^1}} = \|f\|_{C^1}$, which is the desired result. □

Estimates of the Riesz potential $\mathcal{I}_\lambda$. Consider $\Omega = \mathbb{R}^2 \times (-b, 0)$ for $b > 0$. For a function f, defined on Ω, we define the Riesz potential $\mathcal{I}_\lambda f$ by

$$\mathcal{I}_\lambda f(x', x_3) = \int_{\mathbb{R}^2} \hat{f}(\xi, x_3)|\xi|^{-\lambda} e^{2\pi i x'\cdot\xi}\, d\xi, \tag{A-7}$$

where $\hat{}$ denotes the Fourier transform in (x_1, x_2). Similarly, for f defined on Σ, we set

$$\mathcal{I}_\lambda f(x') = \int_{\mathbb{R}^2} \hat{f}(\xi)|\xi|^{-\lambda} e^{2\pi i x'\cdot\xi}\, d\xi. \tag{A-8}$$

We have a product estimate that is a fractional analogue of the Leibniz rule.

Lemma A.3. *Let $\lambda \in (0, 1)$. If $f \in H^0(\Omega)$ and $g, Dg \in H^1(\Omega)$, then*

$$\|\mathcal{I}_\lambda(fg)\|_0 \lesssim \|f\|_0 \|g\|_1^\lambda \|Dg\|_1^{1-\lambda}. \tag{A-9}$$

If $f \in H^0(\Sigma)$ and $g \in H^1(\Sigma)$, then

$$\|\mathcal{I}_\lambda(fg)\|_{H^0(\Sigma)} \lesssim \|f\|_{H^0(\Sigma)} \|g\|_{H^0(\Sigma)}^\lambda \|Dg\|_{H^0(\Sigma)}^{1-\lambda}. \tag{A-10}$$

Proof. The Hardy–Littlewood–Sobolev inequality (see, for example, Theorem 4.3 of [Lieb and Loss 2001]) implies that $\mathcal{I}_\lambda : L^{2/(1+\lambda)}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a bounded linear operator for $\lambda \in (0, 1)$. We may then employ Fubini's theorem and apply this result to each slice $\{x_3 = z\}$ for $z \in (-b, 0)$ to estimate

$$\begin{aligned}\int_\Omega |\mathcal{I}_\lambda(fg)|^2 &= \int_{-b}^0 \int_{\mathbb{R}^2} |\mathcal{I}_\lambda(fg)|^2\, dx' dx_3 \lesssim \int_{-b}^0 \left(\int_{\mathbb{R}^2} |fg|^{2/(1+\lambda)}\, dx' \right)^{1+\lambda} dx_3 \\ &\le \int_{-b}^0 \left(\int_{\mathbb{R}^2} |f|^2\, dx' \right) \left(\int_{\mathbb{R}^2} |g|^{2/\lambda} dx' \right)^{\lambda} dx_3 \le \sup_{-b \le x_3 \le 0} \|g(\cdot, x_3)\|^2_{L^{2/\lambda}(\mathbb{R}^2)} \int_\Omega |f|^2,\end{aligned} \tag{A-11}$$

where, in the second inequality, we have applied Hölder's inequality. By the Gagliardo–Nirenberg interpolation inequality on $\mathbb{R}^2$ we may bound

$$\|g(\cdot, x_3)\|_{L^{2/\lambda}(\mathbb{R}^2)} \lesssim \|g(\cdot, x_3)\|^\lambda_{L^2(\mathbb{R}^2)} \|Dg(\cdot, x_3)\|^{1-\lambda}_{L^2(\mathbb{R}^2)}, \tag{A-12}$$

but, by trace theory, we also have

$$\|g(\cdot, x_3)\|_{L^2(\mathbb{R}^2)} \lesssim \|g\|_1 \quad \text{and} \quad \|Dg(\cdot, x_3)\|_{L^2(\mathbb{R}^2)} \lesssim \|Dg\|_1, \tag{A-13}$$

so that

$$\sup_{-b \le x_3 \le 0} \|g(\cdot, x_3)\|^2_{L^{2/\lambda}(\mathbb{R}^2)} \lesssim \|g\|_1^\lambda \|Dg\|_1^{1-\lambda}. \tag{A-14}$$

Chaining together (A-11) and (A-14) then yields the estimate (A-9). A similar argument, not employing Fubini's theorem or trace theory, provides the estimate (A-10). □

Our next result shows how $\mathcal{I}_\lambda$ interacts with horizontal derivatives in Ω.

Lemma A.4. *Let $\lambda \in (0, 1)$. If $f \in H^k(\Omega)$ for $k \ge 1$ an integer, then*

$$\|\mathcal{I}_\lambda D^k f\|_0 \lesssim \|D^{k-1} f\|_0^\lambda \|D^k f\|_0^{1-\lambda}. \tag{A-15}$$

Proof. On a fixed horizontal slice $\{x_3 = z\}$ for $z \in (-b, 0)$, Parseval's theorem implies that

$$\begin{aligned}\int_{\mathbb{R}^2} |\mathcal{I}_\lambda D^k f(x', x_3)|^2 \, dx' &\lesssim \int_{\mathbb{R}^2} |\xi|^{2(k-\lambda)} |\hat{f}(\xi, x_3)|^2 \, d\xi \\ &= \int_{\mathbb{R}^2} (|\xi|^{2(k-1)} |\hat{f}(\xi, x_3)|^2)^\lambda (|\xi|^{2k} |\hat{f}(\xi, x_3)|^2)^{1-\lambda} \, d\xi \\ &\lesssim \left(\int_{\mathbb{R}^2} |D^{k-1} f(x', x_3)|^2 \, dx' \right)^\lambda \left(\int_{\mathbb{R}^2} |D^k f(x', x_3)|^2 \, dx' \right)^{1-\lambda}. \end{aligned} \tag{A-16}$$

Here, in the second inequality, we have used Hölder and Parseval. Integrating both sides of this inequality with respect to $x_3 \in (-b, 0)$ and again applying Hölder's inequality yields the estimate (A-15). □

Poisson integral. For a function f defined on $\Sigma = \mathbb{R}^2$, the Poisson integral in $\mathbb{R}^2 \times (-\infty, 0)$ is defined by

$$\mathcal{P} f(x', x_3) = \int_{\mathbb{R}^2} \hat{f}(\xi) e^{2\pi|\xi|x_3} e^{2\pi i x' \cdot \xi} \, d\xi. \tag{A-17}$$

Although $\mathcal{P} f$ is defined in all of $\mathbb{R}^2 \times (-\infty, 0)$, we will only need bounds on its norm in the restricted domain $\Omega = \mathbb{R}^2 \times (-b, 0)$. This yields a couple improvements of the usual estimates of $\mathcal{P} f$ on the set $\mathbb{R}^2 \times (-\infty, 0)$. Recall that we use the conventions for sums of derivatives described on page 1443, which in particular means that ∇^q involves x_3 derivatives.

Lemma A.5. *Let $\mathcal{P} f$ be the Poisson integral of a function f that is either in $\dot{H}^q(\Sigma)$ or $\dot{H}^{q-1/2}(\Sigma)$ for $q \in \mathbb{N}$ (here $\dot{H}^s$ is the usual homogeneous Sobolev space of order s). Then*

$$\|\nabla^q \mathcal{P} f\|_0^2 \lesssim \int_{\mathbb{R}^2} |\xi|^{2q} |\hat{f}(\xi)|^2 \left(\frac{1 - e^{-4\pi b|\xi|}}{|\xi|} \right) d\xi, \tag{A-18}$$

and in particular

$$\|\nabla^q \mathcal{P} f\|_0^2 \lesssim \|f\|_{\dot{H}^{q-1/2}(\Sigma)}^2 \quad \text{and} \quad \|\nabla^q \mathcal{P} f\|_0^2 \lesssim \|f\|_{\dot{H}^q(\Sigma)}^2. \tag{A-19}$$

Proof. Employing Fubini, the horizontal Fourier transform, and Parseval, we may bound

$$\begin{aligned}\|\nabla^q \mathcal{P} f\|_0^2 \lesssim \int_{\mathbb{R}^2} \int_{-b}^0 |\xi|^{2q} |\hat{f}(\xi)|^2 e^{4\pi|\xi|x_3} \, dx_3 d\xi &\le \int_{\mathbb{R}^2} |\xi|^{2q} |\hat{f}(\xi)|^2 \left(\int_{-b}^0 e^{4\pi|\xi|x_3} \, dx_3 \right) d\xi \\ &\lesssim \int_{\mathbb{R}^2} |\xi|^{2q} |\hat{f}(\xi)|^2 \left(\frac{1 - e^{-4\pi b|\xi|}}{|\xi|} \right) d\xi. \end{aligned} \tag{A-20}$$

This is (A-18). To deduce (A-19) from (A-18), we simply note that

$$\frac{1 - e^{-4\pi b|\xi|}}{|\xi|} \le \min\left\{ 4\pi b, \frac{1}{|\xi|} \right\}, \tag{A-21}$$

which means we are free to bound the right side of (A-20) by either $\|f\|_{\dot{H}^{q-1/2}(\Sigma)}^2$ or $\|f\|_{\dot{H}^q(\Sigma)}^2$. □

Interpolation estimates. Assume that $\Sigma = \mathbb{R}^2$ and $\Omega = \Sigma \times (-b, 0)$. We begin with an interpolation result for Poisson integrals, as defined by (A-17).

Lemma A.6. *Let $\mathcal{P} f$ be the Poisson integral of f, defined on Σ. Let $\lambda \geq 0$, $q \in \mathbb{N}$, $s \geq 0$, and $r \geq 0$.*

(1) *Let*

$$\theta = \frac{s}{q+s+\lambda} \quad \textit{and} \quad 1-\theta = \frac{q+\lambda}{q+s+\lambda}. \tag{A-22}$$

Then

$$\|\nabla^q \mathcal{P} f\|_0^2 \lesssim (\|\mathcal{I}_\lambda f\|_0^2)^\theta (\|D^{q+s} f\|_0^2)^{1-\theta}. \tag{A-23}$$

(2) *Let $r+s>1$,*

$$\theta = \frac{r+s-1}{q+s+r+\lambda}, \quad \textit{and } 1-\theta = \frac{q+\lambda+1}{q+s+r+\lambda}. \tag{A-24}$$

Then

$$\|\nabla^q \mathcal{P} f\|_{L^\infty}^2 \lesssim (\|\mathcal{I}_\lambda f\|_0^2)^\theta (\|D^{q+s} f\|_r^2)^{1-\theta}. \tag{A-25}$$

(3) *Let $s>1$. Then*

$$\|\nabla^q \mathcal{P} f\|_{L^\infty}^2 \lesssim \|D^q f\|_s^2. \tag{A-26}$$

Proof. Employing Fubini, the horizontal Fourier transform, and Parseval, we may bound

$$\|\nabla^q \mathcal{P} f\|_0^2 \lesssim \int_{\mathbb{R}^2} \int_{-b}^0 |\xi|^{2q} |\hat{f}(\xi)|^2 e^{4\pi|\xi| x_3}\, dx_3\, d\xi \lesssim \int_{\mathbb{R}^2} |\xi|^{2q} |\hat{f}(\xi)|^2\, d\xi.$$
$$= \int_{\mathbb{R}^2} (|\xi|^{2(q+s)} |\hat{f}(\xi)|^2)^{1-\theta} (|\xi|^{-2\lambda} |\hat{f}(\xi)|^2)^\theta\, d\xi \tag{A-27}$$

for θ and $1-\theta$ defined by (A-22). An application of Hölder's inequality and a second application of Parseval's theorem then provides the estimate (A-23).

For the L^∞ estimate (A-25), we use the definition of $\mathcal{P} f$ in conjunction with the trivial estimate $\exp(2\pi|\xi| x_3) \leq 1$ in Ω to bound

$$\|\nabla^q \mathcal{P} f\|_{L^\infty} \lesssim \int_{\mathbb{R}^2} |\xi|^q |\hat{f}(\xi)|\, d\xi. \tag{A-28}$$

We write B_R for the open ball of radius R, B_R^c for its complement, and $\langle \xi \rangle = \sqrt{1+|\xi|^2}$. For $R>0$ we split into high and low frequencies to see that

$$\begin{aligned}
\int_{\mathbb{R}^2} |\xi|^q |\hat{f}(\xi)|\, d\xi &= \int_{B_R} |\xi|^{q+\lambda} |\xi|^{-\lambda} |\hat{f}(\xi)|\, d\xi + \int_{B_R^c} |\xi|^{q+s} \langle \xi \rangle^r \langle \xi \rangle^{-r} |\xi|^{-s} |\hat{f}(\xi)|\, d\xi \\
&\lesssim \left(\int_{B_R} |\xi|^{2(q+\lambda)}\, d\xi \right)^{1/2} \|\mathcal{I}_\lambda f\|_0 + \left(\int_{B_R^c} |\xi|^{-2s} \langle \xi \rangle^{-2r}\, d\xi \right)^{1/2} \|D^{q+s} f\|_r \\
&\lesssim R^{q+\lambda+1} \|\mathcal{I}_\lambda f\|_0 + R^{-(r+s-1)} \|D^{q+s} f\|_r.
\end{aligned} \tag{A-29}$$

The condition $r+s>1$ guarantees that the integral over B_R^c is finite. Minimizing the right side with respect to $R \in (0,\infty)$ then yields (A-25).

The estimate (A-26) follows from the easy bound

$$\int_{\mathbb{R}^2} |\xi|^q |\hat{f}(\xi)|\, d\xi \lesssim \|D^q f\|_s \left(\int_{\mathbb{R}^2} \langle \xi \rangle^{-2s}\, d\xi \right)^{1/2} \lesssim \|D^q f\|_s, \tag{A-30}$$

which holds when $s > 1$. □

The next result is a similar interpolation result for functions defined only on Σ.

Lemma A.7. *Let f be defined on Σ. Let $\lambda \geq 0$.*

(1) *Let $q, s \in [0, \infty)$ and*

$$\theta = \frac{s}{q+s+\lambda} \quad \text{and} \quad 1-\theta = \frac{q+\lambda}{q+s+\lambda}. \tag{A-31}$$

Then

$$\|D^q f\|_0^2 \lesssim (\|\mathcal{I}_\lambda f\|_0^2)^\theta (\|D^{q+s} f\|_0^2)^{1-\theta}. \tag{A-32}$$

(2) *Let $q, s \in \mathbb{N}$, $r \geq 0$, $r+s > 1$,*

$$\theta = \frac{r+s-1}{q+s+r+\lambda}, \quad \text{and } 1-\theta = \frac{q+\lambda+1}{q+s+r+\lambda}. \tag{A-33}$$

Then

$$\|D^q f\|_{L^\infty}^2 \lesssim (\|\mathcal{I}_\lambda f\|_0^2)^\theta (\|D^{q+s} f\|_r^2)^{1-\theta}. \tag{A-34}$$

Proof. For the H^0 estimate we use

$$\|D^q f\|_0^2 \lesssim \int_{\mathbb{R}^2} |\xi|^{2q} |\hat{f}(\xi)|^2\, d\xi \tag{A-35}$$

and argue as in Lemma A.6. For the L^∞ estimate we bound

$$\|D^q f\|_{L^\infty} \lesssim \int_{\mathbb{R}^2} |\xi|^q |\hat{f}(\xi)|\, d\xi \tag{A-36}$$

and again argue as in Lemma A.6. □

Now we record a similar result for functions defined on Ω that are not Poisson integrals. The result follows from estimates on fixed horizontal slices.

Lemma A.8. *Let f be a function on Ω. Let $\lambda \geq 0$, $q, s \in \mathbb{N}$, and $r \geq 0$.*

(1) *Let*

$$\theta = \frac{s}{q+s+\lambda} \quad \text{and} \quad 1-\theta = \frac{q+\lambda}{q+s+\lambda}. \tag{A-37}$$

Then

$$\|D^q f\|_0^2 \lesssim (\|\mathcal{I}_\lambda f\|_0^2)^\theta (\|D^{q+s} f\|_0^2)^{1-\theta}. \tag{A-38}$$

(2) *Let $r+s > 1$,*

$$\theta = \frac{r+s-1}{q+s+r+\lambda}, \quad \text{and} \quad 1-\theta = \frac{q+\lambda+1}{q+s+r+\lambda}. \tag{A-39}$$

Then

$$\|D^q f\|_{L^\infty}^2 \lesssim (\|\mathcal{I}_\lambda f\|_1^2)^\theta \left(\|D^{q+s} f\|_{r+1}^2\right)^{1-\theta} \tag{A-40}$$

and

$$\|D^q f\|_{L^\infty(\Sigma)}^2 \lesssim (\|\mathcal{I}_\lambda f\|_1^2)^\theta (\|D^{q+s} f\|_{r+1}^2)^{1-\theta}. \tag{A-41}$$

Proof. We employ the horizontal Fourier transform and Parseval in conjunction with Fubini to bound

$$\|D^q f\|_0^2 \lesssim \int_{-b}^0 \int_{\mathbb{R}^2} |\xi|^{2q} |\hat{f}(\xi, x_3)|^2 \, d\xi dx_3. \tag{A-42}$$

For a fixed x_3 we may argue as in Lemma A.6 to show that

$$\int_{\mathbb{R}^2} |\xi|^{2q} |\hat{f}(\xi, x_3)|^2 \, d\xi \leq (\|\mathcal{I}_\lambda f(\cdot, x_3)\|_0^2)^\theta (\|D^{q+s} f(\cdot, x_3)\|_0^2)^{1-\theta} \tag{A-43}$$

for θ and $1-\theta$ given by (A-37). Combining these two inequalities with Hölder's inequality then shows that

$$\|D^q f\|_0^2 \lesssim \int_{-b}^0 (\|\mathcal{I}_\lambda f(\cdot, x_3)\|_0^2)^\theta (\|D^{q+s} f(\cdot, x_3)\|_0^2)^{1-\theta} \, dx_3 \leq (\|\mathcal{I}_\lambda f\|_0^2)^\theta (\|D^{q+s} f\|_0^2)^{1-\theta}, \tag{A-44}$$

which is (A-38).

Now, for the L^∞ estimate, we first work on a horizontal slice $\{x_3 = z\}$ for some $z \in [-b, 0]$. Indeed, using the horizontal Fourier transform on the slice, we have

$$\|D^q f(\cdot, x_3)\|_{L^\infty} \lesssim \int_{\mathbb{R}^2} |\xi|^q |\hat{f}(\xi, x_3)| \, d\xi. \tag{A-45}$$

We may then argue as in Lemma A.6 to show that

$$\int_{\mathbb{R}^2} |\xi|^q |\hat{f}(\xi, x_3)| \, d\xi \lesssim (\|\mathcal{I}_\lambda f(\cdot, x_3)\|_0)^\theta (\|D^{q+s} f(\cdot, x_3)\|_r)^{1-\theta} \tag{A-46}$$

for θ and $1-\theta$ given by (A-39). By the usual trace theory

$$\|\mathcal{I}_\lambda f(\cdot, x_3)\|_0 \lesssim \|\mathcal{I}_\lambda f\|_1 \quad \text{and} \quad \|D^{q+s} f(\cdot, x_3)\|_r \lesssim \|D^{q+s} f\|_{r+1}. \tag{A-47}$$

Combining (A-45)–(A-47) and taking the supremum over $x_3 \in [-b, 0]$ then gives (A-40). A similar argument yields (A-41). □

Transport estimate. Consider the equation

$$\begin{cases} \partial_t \eta + u \cdot D\eta = g & \text{in } \Sigma \times (0, T), \\ \eta(t=0) = \eta_0 \end{cases} \tag{A-48}$$

with $T \in (0, \infty]$ and $\Sigma = \mathbb{R}^2$. We have the following estimate of the transport of regularity for solutions to (A-48), which is a particular case of a more general result proved in [Danchin 2005].

Lemma A.9 [Danchin 2005, Proposition 2.1]. *Let η be a solution to (A-48). Then there is a universal constant $C > 0$ such that, for any $0 \leq s < 2$,*

$$\sup_{0\leq r\leq t} \|\eta(r)\|_{H^s} \leq \exp\left(C\int_0^t \|Du(r)\|_{H^{3/2}}\,dr\right)\left(\|\eta_0\|_{H^s} + \int_0^t \|g(r)\|_{H^s}\,dr\right). \tag{A-49}$$

Proof. Use $p = p_2 = 2$, $N = 2$, and $\sigma = s$ in Proposition 2.1 of [Danchin 2005] along with the embedding $H^{3/2} \hookrightarrow B^1_{2,\infty} \cap L^\infty$. □

Poincaré-type inequalities. Let Σ and Ω be as before.

Lemma A.10. *We have*

$$\|f\|^2_{L^2(\Omega)} \lesssim \|f\|^2_{L^2(\Sigma)} + \|\partial_3 f\|^2_{L^2(\Omega)} \tag{A-50}$$

for all $f \in H^1(\Omega)$. Also, if $f \in W^{1,\infty}(\Omega)$, then

$$\|f\|^2_{L^\infty(\Omega)} \lesssim \|f\|^2_{L^\infty(\Sigma)} + \|\partial_3 f\|^2_{L^\infty(\Omega)}\,. \tag{A-51}$$

Proof. By density we may assume that f is smooth. Writing $x = (x', x_3)$ for $x' \in \Sigma$ and $x_3 \in (-b, 0)$, we have

$$|f(x',x_3)|^2 = |f(x',0)|^2 - 2\int_{x_3}^0 f(x',z)\partial_3 f(x',z)\,dz \leq |f(x',0)|^2 + 2\int_{-b}^0 |f(x',z)||\partial_3 f(x',z)|\,dz. \tag{A-52}$$

We may integrate this with respect to $x_3 \in (-b, 0)$ to get

$$\int_{-b}^0 |f(x',x_3)|^2\,dx_3 \lesssim |f(x',0)|^2 + 2\int_{-b}^0 |f(x',z)||\partial_3 f(x',z)|\,dz. \tag{A-53}$$

Now we integrate over $x' \in \Sigma$ to find

$$\int_\Omega |f(x)|^2\,dx \leq C\|f\|^2_{L^2(\Sigma)} + 2C\int_\Omega |f(x)||\partial_3 f(x)|\,dx \leq C\|f\|^2_{L^2(\Sigma)} + \varepsilon\|f\|^2_{L^2(\Omega)} + \frac{C}{\varepsilon}\|\partial_3 f\|^2_{L^2(\Omega)} \tag{A-54}$$

for any $\varepsilon > 0$. Choosing $\varepsilon > 0$ sufficiently small then yields (A-50). The estimate (A-51) follows similarly, taking suprema rather than integrating. □

A simple modification of the proof of Lemma A.10 yields the following estimates.

Lemma A.11. *We have $\|f\|_{H^0(\Sigma)} \lesssim \|\partial_3 f\|_{H^0(\Omega)}$ for $f \in H^1(\Omega)$ such that $f = 0$ on Σ_b. Moreover, $\|f\|_{L^\infty(\Sigma)} \lesssim \|\partial_3 f\|_{L^\infty(\Omega)}$ for $f \in W^{1,\infty}(\Omega)$ such that $f = 0$ on Σ_b.*

We will need a version of Korn's inequality, which is proved, for instance, in Lemma 2.7 of [Beale 1981].

Lemma A.12. *We have $\|u\|_1 \lesssim \|\mathbb{D}u\|_0$ for all $u \in H^1(\Omega;\mathbb{R}^3)$ such that $u = 0$ on Σ_b.*

We also record the standard Poincaré inequality, which applies for functions taking either vector or scalar values.

Lemma A.13. *We have $\|f\|_0 \lesssim \|f\|_1 \lesssim \|\nabla f\|_0$ for all $f \in H^1(\Omega)$ such that $f = 0$ on Σ_b. Also, $\|f\|_{L^\infty(\Omega)} \lesssim \|f\|_{W^{1,\infty}(\Omega)} \lesssim \|\nabla f\|_{L^\infty(\Omega)}$ for all $f \in W^{1,\infty}(\Omega)$ such that $f = 0$ on Σ_b.*

An elliptic estimate. The proof of the following estimate may be found in [Beale 1981].

Lemma A.14. *Suppose* (u, p) *solve*

$$\begin{cases} -\Delta u + \nabla p = \phi \in H^{r-2}(\Omega), \\ \operatorname{div} u = \psi \in H^{r-1}(\Omega), \\ (pI - \mathbb{D}(u))e_3 = \alpha \in H^{r-3/2}(\Sigma), \\ u|_{\Sigma_b} = 0. \end{cases} \tag{A-55}$$

Then, for $r \geq 2$,

$$\|u\|_{H^r}^2 + \|p\|_{H^{r-1}}^2 \lesssim \|\phi\|_{H^{r-2}}^2 + \|\psi\|_{H^{r-1}}^2 + \|\alpha\|_{H^{r-3/2}}^2. \tag{A-56}$$

Acknowledgements

Y. Guo thanks Hyung Ju Hwang for several stimulating discussions.

References

[Adams 1975] R. A. Adams, *Sobolev spaces*, Pure and Applied Mathematics **65**, Academic Press, New York, 1975. MR 56 #9247 Zbl 0314.46030

[Bae 2011] H. Bae, "Solvability of the free boundary value problem of the Navier–Stokes equations", *Discrete Contin. Dyn. Syst.* **29**:3 (2011), 769–801. MR 2012f:35587 Zbl 1222.35215

[Beale 1981] J. T. Beale, "The initial value problem for the Navier–Stokes equations with a free surface", *Comm. Pure Appl. Math.* **34**:3 (1981), 359–392. MR 83d:35134 Zbl 0464.76028

[Beale 1984] J. T. Beale, "Large-time regularity of viscous surface waves", *Arch. Rational Mech. Anal.* **84**:4 (1984), 307–352. MR 85c:35074 Zbl 0545.76029

[Beale and Nishida 1985] J. T. Beale and T. Nishida, "Large-time behavior of viscous surface waves", pp. 1–14 in *Recent topics in nonlinear PDE* (Sendai, 1984), vol. 2, edited by K. Masuda and M. Mimura, North-Holland Math. Stud. **128**, North-Holland, Amsterdam, 1985. MR 88f:35121 Zbl 0642.76048

[Castro et al. 2011] A. Castro, D. Córdoba, C. Fefferman, F. Gancedo, and J. Gómez-Serrano, "Finite time singularities for the free boundary incompressible Euler equations", preprint, 2011. arXiv math/1112.2170

[Castro et al. 2012] A. Castro, D. Córdoba, C. Fefferman, F. Gancedo, and M. López-Fernández, "Rayleigh–Taylor breakdown for the Muskat problem with applications to water waves", *Ann. Math.* (2) **175**:2 (2012), 909–948. Zbl 06025004

[Danchin 2005] R. Danchin, "Estimates in Besov spaces for transport and transport-diffusion equations with almost Lipschitz coefficients", *Rev. Mat. Iberoam.* **21**:3 (2005), 863–888. MR 2007j:35101 Zbl 1098.35038

[Germain et al. 2009] P. Germain, N. Masmoudi, and J. Shatah, "Global solutions for the gravity water waves equation in dimension 3", *C. R. Math. Acad. Sci. Paris* **347**:15-16 (2009), 897–902. MR 2010h:35312 Zbl 1177.35168

[Guo and Tice 2013a] Y. Guo and I. Tice, "Almost exponential decay of periodic viscous surface waves without surface tension", *Arch. Ration. Mech. Anal.* **207**:2 (2013), 459–531. MR 3005322 Zbl 06148932

[Guo and Tice 2013b] Y. Guo and I. Tice, "Local well-posedness of the viscous surface wave problem without surface tension", *Anal. PDE* **6**:2 (2013), 287–369. MR 3071393 Zbl 06212804

[Hataya 2009] Y. Hataya, "Decaying solution of a Navier–Stokes flow without surface tension", *J. Math. Kyoto Univ.* **49**:4 (2009), 691–717. MR 2011c:35404 Zbl 05696853

[Hataya and Kawashima 2009] Y. Hataya and S. Kawashima, "Decaying solution of the Navier–Stokes flow of infinite volume without surface tension", *Nonlinear Anal.* **71**:12 (2009), e2535–e2539. MR 2011d:35373 Zbl 1239.35111

[Lannes 2005] D. Lannes, "Well-posedness of the water-waves equations", *J. Amer. Math. Soc.* **18**:3 (2005), 605–654. MR 2005m:76021 Zbl 1069.35056

[Lieb and Loss 2001] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., Graduate Studies in Mathematics **14**, American Mathematical Society, Providence, RI, 2001. MR 2001i:00001 Zbl 0966.26002

[Nishida et al. 2004] T. Nishida, Y. Teramoto, and H. Yoshihara, "Global in time behavior of viscous surface waves: horizontally periodic motion", *J. Math. Kyoto Univ.* **44**:2 (2004), 271–323. MR 2005f:35246 Zbl 1095.35028

[Solonnikov 1977] V. A. Solonnikov, "Solvability of the problem of the motion of a viscous incompressible fluid that is bounded by a free surface", *Math. USSR-Izv.* **11**:6 (1977), 1323–1358. MR 58 #14297 Zbl 0398.76024

[Strain and Guo 2006] R. M. Strain and Y. Guo, "Almost exponential decay near Maxwellian", *Comm. Partial Differential Equations* **31**:1-3 (2006), 417–429. MR 2006m:82042 Zbl 1096.82010

[Sylvester 1990] D. L. G. Sylvester, "Large time existence of small viscous surface waves without surface tension", *Comm. Partial Differential Equations* **15**:6 (1990), 823–903. MR 91f:35216 Zbl 0731.35081

[Tani and Tanaka 1995] A. Tani and N. Tanaka, "Large-time existence of surface waves in incompressible viscous fluids with or without surface tension", *Arch. Rational Mech. Anal.* **130**:4 (1995), 303–314. MR 96i:35103 Zbl 0844.76025

[Wu 1997] S. Wu, "Well-posedness in Sobolev spaces of the full water wave problem in 2-D", *Invent. Math.* **130**:1 (1997), 39–72. MR 98m:35167 Zbl 0892.76009

[Wu 1999] S. Wu, "Well-posedness in Sobolev spaces of the full water wave problem in 3-D", *J. Amer. Math. Soc.* **12**:2 (1999), 445–495. MR 2001m:76019 Zbl 0921.76017

[Wu 2009] S. Wu, "Almost global wellposedness of the 2-D full water wave problem", *Invent. Math.* **177**:1 (2009), 45–135. MR 2010i:35303 Zbl 1181.35205

[Wu 2011] S. Wu, "Global wellposedness of the 3-D full water wave problem", *Invent. Math.* **184** (2011), 125–220. Zbl 1221.35304

Received 15 Oct 2012. Accepted 15 Nov 2012.

YAN GUO: guoy@dam.brown.edu
Division of Applied Mathematics, Brown University, 182 George Street, Providence, Rhode Island 02912, USA

IAN TICE: iantice@andrew.cmu.edu
Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA 15213, USA

Guidelines for Authors

Authors may submit manuscripts in PDF format on-line at the Submission page at msp.berkeley.edu/apde.

Originality. Submission of a manuscript acknowledges that the manuscript is original and and is not, in whole or in part, published or under consideration for publication elsewhere. It is understood also that the manuscript will not be submitted elsewhere while under consideration for publication in this journal.

Language. Articles in APDE are usually in English, but articles written in other languages are welcome.

Required items. A brief abstract of about 150 words or less must be included. It should be self-contained and not make any reference to the bibliography. If the article is not in English, two versions of the abstract must be included, one in the language of the article and one in English. Also required are keywords and subject classifications for the article, and, for each author, postal address, affiliation (if appropriate), and email address.

Format. Authors are encouraged to use LaTeX but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages (Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc.) allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@msp.org with details about how your graphics were generated.

White space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Proofs. Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

ANALYSIS & PDE

Volume 6 No. 6 2013