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PSEUDOPARABOLIC REGULARIZATION OF FORWARD-BACKWARD PARABOLIC EQUATIONS: A LOGARITHMIC NONLINEARITY

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We study the initial-boundary value problem

$$\begin{cases} u_t = \Delta\varphi(u) + \varepsilon\Delta[\psi(u)]_t & \text{in } Q := \Omega\times(0,T], \\ \varphi(u) + \varepsilon[\psi(u)]_t = 0 & \text{in } \partial\Omega\times(0,T], \\ u = u_0 \ge 0 & \text{in } \Omega\times\{0\}, \end{cases}$$

with *measure-valued initial data*, assuming that the regularizing term ψ has logarithmic growth (the case of power-type ψ was dealt with in an earlier work). We prove that this case is intermediate between the case of power-type ψ and that of bounded ψ, to be addressed in a forthcoming paper. Specifically, the support of the singular part of the solution with respect to the Lebesgue measure remains constant in time (as in the case of power-type ψ), although the singular part itself need not be constant (as in the case of bounded ψ, where the support of the singular part can also increase). However, it turns out that the concentrated part of the solution with respect to the Newtonian capacity remains constant.

1. Introduction

In this paper we study the initial-boundary value problem

$$\begin{cases} u_t = \Delta\varphi(u) + \varepsilon\Delta[\psi(u)]_t & \text{in } Q := \Omega\times(0,T], \\ \varphi(u) + \varepsilon[\psi(u)]_t = 0 & \text{in } \partial\Omega\times(0,T], \\ u = u_0 \ge 0 & \text{in } \Omega\times\{0\}, \end{cases} \tag{1-1}$$

where ε and T are positive constants,

$$\psi(u) = \log(1+u) \quad \text{for } u \ge 0, \tag{1-2}$$

$\varphi : [0,\infty) \to [0,\infty)$ is nonmonotone, u_0 is a nonnegative Radon measure on Ω, and $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ is a bounded and connected domain, with smooth boundary $\partial\Omega$ if $N \ge 2$. More precisely, $\varphi \in C^\infty([0,\infty))$ is a Perona–Malik type nonlinearity which satisfies, for some $\alpha > 0$ and $q \in (1,\infty)$,

$$\varphi(0) = \varphi(\infty) = 0, \quad \varphi' > 0 \text{ in } [0,\alpha), \quad \varphi' < 0 \text{ in } (\alpha,\infty), \quad \varphi''(\alpha) \ne 0, \tag{1-3}$$

$$\varphi \in L^q((0,\infty)), \quad \varphi^{(j)} \in L^\infty((0,\infty)) \text{ for any } j \in \mathbb{N}, \tag{1-4}$$

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and, for some $C > 0$,

$$|\varphi'(u)| \le C\psi'(u) = \frac{C}{1+u} \quad \text{for } u \ge 0. \tag{1-5}$$

In particular, $0 < \varphi(u) \le \varphi(\alpha)$ holds for $u > 0$. A typical example is

$$\varphi(u) = \frac{u}{1+u^2}.$$

The partial differential equation in problem (1-1) can be regarded as the regularization of the forward-backward parabolic equation

$$u_t = \Delta\varphi(u),$$

which leads to ill-posed problems. The latter equation and its regularizations arise in several applications, such as edge detection in image processing [Perona and Malik 1990], aggregation models in population dynamics [Padrón 1998], and stratified turbulent shear flow [Barenblatt et al. 1993a].

This paper is the second of a series where we address problem (1-1) with *measure-valued initial data*; see [Bertsch et al. $\ge$ 2013]. It is natural to consider flows which allow measure-valued solutions, since it is known that initially smooth solutions may develop a singular part in finite time, if $N = 1$ and ψ is uniformly bounded [Barenblatt et al. 1993b]. On the other hand we have shown [Bertsch et al. $\ge$ 2013] that in the case of power-type nonlinearities,

$$\psi(u) = (1+u)^\theta - 1 \qquad (u \ge 0, \theta \in (0, 1]), \tag{1-6}$$

the singular part of the solutions does not evolve in time, and initially smooth functions remain smooth for each later time. Therefore, the qualitative behavior of measure-valued solutions turns out to depend critically on the behavior of the nonlinearity $\psi(u)$ as $u \to \infty$.

Our purpose is to make a detailed analysis of this dependence. Therefore we distinguish three cases in this series of papers: mild degeneracies (power-type ψ), strong degeneracies (bounded ψ), and the intermediate case of logarithmic ψ. Observe that if ψ' vanishes at infinity, the partial differential equation in problem (1-1) is of *degenerate pseudoparabolic type*. In the present paper we focus on the intermediate case of functions ψ with logarithmic growth, and we take (1-2) as a model case.

It turns out that the logarithmic ψ can be considered as a truly intermediate case, in the sense that

(i) as in the case of power-type ψ, singularities cannot appear spontaneously;

(ii) as in the case of bounded ψ, the singular part of u need not be constant with respect to t.

Specifically, in all three cases the singular part of the solution is nondecreasing in time: it is constant for a power-type ψ (see [Bertsch et al. $\ge$ 2013, Theorem 2.1]), whereas its support can expand (that is, new singularities can appear) in the case of bounded ψ. Instead, in the logarithmic case the support of the singular part is constant, yet the singular part can increase; see Theorem 3.5 and equalities (3-13)–(3-14).

To explain the above claims, let us discuss heuristically the behavior of solutions to problem (1-1) for a logarithmic ψ as in (1-2) or a power-type ψ as in (1-6); see [Bertsch et al. $\ge$ 2013]. By a suitable approximation procedure, which plays a key role in our approach (see Section 6), we prove in both cases that the *entropy solution* $u(\cdot\,,t)$ at time t of problem (1-1) and the corresponding value $v(\cdot\,,t)$ of the *chemical potential*

$$v := \varphi(u_r) + \varepsilon[\psi(u_r)]_t \tag{1-7}$$

satisfy a suitable elliptic problem. Here $u_r(\cdot\,,t)$ denotes the density of the absolutely continuous part of $u(\cdot\,,t)$; see after (2-5). When ψ is of power-type, (1-7) becomes

$$\begin{cases} -\varepsilon\Delta v(\cdot\,,t) + \dfrac{v(\cdot\,,t)}{\psi'(u_r(\cdot\,,t))} = \dfrac{\varphi(u_r(\cdot\,,t))}{\psi'(u_r(\cdot\,,t))} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{1-8}$$

for a.e. $t \in (0,T)$. Instead, for a logarithmic ψ the elliptic problem is

$$\begin{cases} -\varepsilon\Delta v(\cdot\,,t) + \dfrac{1}{\psi'([u(\cdot\,,t)]_{d,2})} v(\cdot\,,t) = \dfrac{\varphi(u_r(\cdot\,,t))}{\psi'(u_r(\cdot\,,t))} & \text{in } \Omega, \\ v(\cdot\,,t) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-9}$$

where $[u(\cdot\,,t)]_{d,2}$ denotes the diffuse part of $u(\cdot\,,t)$ with respect to the Newtonian C_2-capacity. Recalling that $1/\psi'(u) = 1+u$, the first equation of problem (1-9) is meant in the sense that

$$\begin{aligned} -\varepsilon\langle \Delta[v(\cdot\,,t)], \rho\rangle_\Omega + \big\langle \{1+u_r(\cdot\,,t) + [u_s(\cdot\,,t)]_{d,2}\}, v(\cdot\,,t)\rho\big\rangle_\Omega \\ = \int_\Omega [1+u_r(x,t)]\varphi(u_r(x,t))\rho(x)\,dx \end{aligned} \tag{1-10}$$

for any $\rho \in C_c(\Omega)$; here $u_s(\cdot\,,t)$ denotes the singular part of $u(\cdot\,,t)$ and, as we shall make precise in Section 2 (see (2-2) and Remark 2.1), $\langle\cdot\,,\cdot\rangle_\Omega$ denotes an extension of the duality map between the space $\mathcal{M}(\Omega)$ of finite Radon measures on Ω and the space $C_c(\Omega)$ of continuous functions with compact support. Notice that

$$0 \le (1+u_r)\varphi(u_r) \le \varphi(\alpha)(1+u_r) \in L^1(Q).$$

The presence of the singular term $\langle [u_s(\cdot\,,t)]_{d,2}, v(\cdot\,,t)\rho\rangle_\Omega$ in the left-hand side of (1-10), which does not appear in the power-type case (see (1-8)), depends on the weaker regularization properties of a logarithmic ψ with respect to a power-type ψ.

By the above definition of the chemical potential, the partial differential equation in (1-1) reads

$$u_t = \Delta v. \tag{1-11}$$

The coupling of the above evolutionary equation with the corresponding elliptic problem (either (1-8) or (1-9), depending on the choice of ψ) suggests that we could study the time evolution of $u_r(\cdot\,,t)$ and that of $u_s(\cdot\,,t)$ separately. For both choices of ψ our definition of the solution of problem (1-1) implies that $v \in L^1(Q)$; see Definition 3.1 and [Bertsch et al. $\ge$ 2013, Definition 2.1]. Then for a power-type ψ we obtain from (1-8) that $\Delta v \in L^1(Q)$, which, by (1-11), implies

$$u_s(\cdot\,,t) = u_{0s}, \quad [u_r]_t(\cdot\,,t) = u_t(\cdot\,,t) = \Delta v(\cdot\,,t), \tag{1-12}$$

namely, the singular part u_s does not evolve with time; see [Bertsch et al. $\ge$ 2013, Theorem 2.1].

Now consider a logarithmic ψ as in (1-2). By (1-11) and the arbitrariness of ρ, (1-10) gives

$$-\epsilon u_t(\cdot\,,t)+\{1+u_r(\cdot\,,t)+[u_s(\cdot\,,t)]_{d,2}\}v(\cdot\,,t)=[1+u_r(\cdot\,,t)]\varphi(u_r(\cdot\,,t)). \tag{1-13}$$

On the other hand, by definition of the chemical potential, we have

$$\epsilon[u_r]_t(\cdot\,,t)=[1+u_r(\cdot\,,t)][v(\cdot\,,t)-\varphi(u_r)(\cdot\,,t)], \tag{1-14}$$

which can be regarded as the equation governing the evolution of the regular part u_r, since $v\in L^1(Q)$. From (1-13)–(1-14) we obtain the following equation for the evolution of the singular part u_s:

$$\epsilon[u_s]_t(\cdot\,,t)=[u_s]_{d,2}(\cdot\,,t)v(\cdot\,,t), \tag{1-15}$$

namely,

$$\epsilon\langle[u_s]_t(\cdot\,,t),\rho\rangle_\Omega=\langle[u_s(\cdot\,,t)]_{d,2},v(\cdot\,,t)\rho\rangle_\Omega$$

for any $\rho\in C_c(\Omega)$. Since

$$u_s=u_{c,2}+[u_s]_{d,2} \tag{1-16}$$

(see (2-7)–(2-8)), from Equation (1-15) we obtain

$$u_{c,2}(\cdot\,,t)=[u_0]_{c,2}$$

(see Theorem 3.1 below) and

$$\langle[u_s]_{d,2}(\cdot\,,t),\rho\rangle_\Omega=\left\langle[u_{0s}]_{d,2},\exp\left\{\frac{1}{\epsilon}\int_0^t v(\cdot\,,s)\,ds\right\}\rho\right\rangle_\Omega, \tag{1-17}$$

which imply (see (3-1))

$$\langle u_s(\cdot\,,t),\rho\rangle_\Omega\le\exp\left\{\frac{\varphi(\alpha)t}{\varepsilon}\right\}\langle u_{0s},\rho\rangle_\Omega$$

for any $t\ge 0$ and $\rho\in C_c(\Omega)$.

If $N=1$, since every Radon measure is C_2-diffuse (see page 1725), problem (1-9) becomes

$$\begin{cases}-\varepsilon[v(\cdot\,,t)]_{xx}+\dfrac{1}{\psi'(u(\cdot\,,t))}v(\cdot\,,t)=\dfrac{\varphi(u_r(\cdot\,,t))}{\psi'(u_r(\cdot\,,t))} & \text{in } \Omega,\\ v(\cdot\,,t)=0 & \text{on } \partial\Omega.\end{cases} \tag{1-18}$$

Now the evolution of the singular part u_s is described by the equation

$$\epsilon[u_s]_t(\cdot\,,t)=u_s(\cdot\,,t)v(\cdot\,,t), \tag{1-19}$$

whence we obtain

$$\langle u_s(\cdot\,,t),\rho\rangle_\Omega=\left\langle u_{0s},\exp\left\{\frac{1}{\epsilon}\int_0^t v(\cdot\,,s)\,ds\right\}\rho\right\rangle_\Omega \tag{1-20}$$

for any $\rho\in C_c(\Omega)$.

In view of the above considerations, whether or not $u_s(\cdot, t)$ evolves in time clearly depends on the positivity of the chemical potential; see (1-17), (1-20). This point will be addressed by a generalized strong maximum principle (see Proposition 3.15). We shall also construct a solution of the form

$$u(\cdot, t) = u_r(\cdot, t) + A(t)\delta_{x_0}, \quad A(0) = 1,$$

δ_{x_0} denoting the Dirac mass centered at $x_0 \in \Omega$ (see Remark 3.20), to point out the importance of the elliptic problem (1-9) for ensuring uniqueness of the solutions of problem (1-1); see Theorem 3.11; a similar example was given in [Porzio et al. 2013, Remark 2.4]. Finally, in Theorem 3.17 we prove the existence of an *entropy solution* of (1-1) (see Definition 3.4), whereas in Theorem 3.18 we show that under suitable conditions this solution and the associated chemical potential satisfy problem (1-9).

2. Preliminaries

Nonnegative finite Radon measures. We denote by $\mathcal{M}(\Omega)$ the space of finite Radon measures on Ω, and by $\mathcal{M}^+(\Omega)$ the cone of positive (finite) Radon measures on Ω. By $\mathcal{M}^+_{ac}(\Omega)$ and $\mathcal{M}^+_s(\Omega)$ we denote the subsets of $\mathcal{M}^+(\Omega)$ whose elements are, respectively, absolutely continuous and singular with respect to the Lebesgue measure on Ω. We have $\mathcal{M}^+_{ac}(\Omega) \cap \mathcal{M}^+_s(\Omega) = \{0\}$, and for every $\mu \in \mathcal{M}^+(\Omega)$ there is a unique pair $(\mu_{ac} \in \mathcal{M}^+_{ac}(\Omega), \mu_s \in \mathcal{M}^+_s(\Omega))$ such that

$$\mu = \mu_{ac} + \mu_s. \tag{2-1}$$

For every $\mu \in \mathcal{M}^+(\Omega)$, we shall denote by μ_r the density of the absolutely continuous part μ_{ac} of μ; namely, according to the Radon–Nikodym Theorem, μ_r is the unique function in $L^1(\Omega)$ such that

$$\mu_{ac}(E) = \int_E \mu_r \, dx$$

for every Borel set $E \subseteq \Omega$.

Given $\mu \in \mathcal{M}(\Omega)$ and a Borel set $E \subseteq \Omega$, the restriction $\mu \llcorner E$ of μ to E is defined by

$$(\mu \llcorner E)(A) := \mu(E \cap A)$$

for every Borel set $A \subseteq \Omega$. We denote by $\langle \cdot, \cdot \rangle_\Omega$ the duality map between $\mathcal{M}(\Omega)$ and the space $C_c(\Omega)$ of continuous functions with compact support. For $\mu \in \mathcal{M}(\Omega)$ and $\rho \in L^1(\Omega, \mu)$ we set, by abuse of notation,

$$\langle \mu, \rho \rangle_\Omega := \int_\Omega \rho(x) \, d\mu(x) \quad \text{and} \quad \|\mu\|_{\mathcal{M}(\Omega)} := |\mu|(\overline{\Omega}). \tag{2-2}$$

Similar notations will be used for the space of Radon measures on $Q := \Omega \times (0, T)$. The Lebesgue measure of any Borel set $E \subseteq \Omega$ or $E \subseteq Q$, will be denoted by $|E|$. A Borel set E such that $|E| = 0$ is called a null set. By the expression “almost everywhere”, henceforth abbreviated a.e., we always mean “up to null sets”.

We denote by $L^\infty((0, T); \mathcal{M}^+(\Omega))$ the set of positive Radon measures $u \in \mathcal{M}^+(Q)$ such that for a.e. $t \in (0, T)$ there exists a measure $u(\cdot, t) \in \mathcal{M}^+(\Omega)$ satisfying the following conditions:

(i) For every $\zeta \in C(\bar{Q})$ the map $t \to \langle u(\cdot, t), \zeta(\cdot, t)\rangle_\Omega$ is Lebesgue measurable, and

$$\langle u, \zeta\rangle_Q = \int_0^T \langle u(\cdot, t), \zeta(\cdot, t)\rangle_\Omega \, dt. \tag{2-3}$$

(ii) $\operatorname{ess\,sup}_{t\in(0,T)} \|u(\cdot, t)\|_{\mathcal{M}(\Omega)} < \infty$.

If $u \in L^\infty((0, T); \mathcal{M}^+(\Omega))$, both u_{ac} and u_s belong to $L^\infty((0, T); \mathcal{M}^+(\Omega))$. By (2-3), for all $\zeta \in C(\bar{Q})$,

$$\langle u_{ac}, \zeta\rangle_Q = \iint_Q u_r \zeta \, dx \, dt \quad \text{and} \quad \langle u_s, \zeta\rangle_Q = \int_0^T \langle u_s(\cdot, t), \zeta(\cdot, t)\rangle_\Omega \, dt.$$

It is easily checked that for a.e. $t \in (0, T)$ the measures $[u(\cdot, t)]_{ac}, [u(\cdot, t)]_s \in \mathcal{M}^+(\Omega)$ satisfy the equalities

$$u_{ac}(\cdot, t) = [u(\cdot, t)]_{ac}, \quad u_s(\cdot, t) = [u(\cdot, t)]_s. \tag{2-4}$$

Observe that the first equality above implies

$$u_r(\cdot, t) = [u(\cdot, t)]_r, \tag{2-5}$$

where $[u(\cdot, t)]_r$ denotes the density of the measure $[u(\cdot, t)]_{ac}$:

$$\langle [u(\cdot, t)]_{ac}, \zeta\rangle_\Omega = \int_\Omega u_r(\cdot, t)\zeta \, dx \quad \text{for } \zeta \in C(\bar{\Omega}) \text{ and a.e. t.}$$

C_p-capacity. Let $p \in [1, \infty)$. The C_p-capacity in Ω of a Borel set $E \subseteq \Omega$ is defined as

$$C_p(E) := \inf_{v \in \mathcal{U}_\Omega^E} \int_\Omega |\nabla v|^p \, dx,$$

where $\mathcal{U}_\Omega^E$ is the set of all functions $v \in H_0^{1,p}(\Omega)$ such that $0 \le v \le 1$ a.e. in Ω and $v = 1$ a.e. in a neighborhood of E (analogous definitions can be given in $\mathbb{R}^N$). If $\mathcal{U}_\Omega^E = \varnothing$ we adopt the usual convention that $\inf \varnothing = \infty$. We use the notation $C_p(E, \Omega)$ when we want to stress the dependence on Ω. If $K \subseteq \Omega$ is compact, then

$$C_p(K) := \inf_{v \in \mathcal{F}_\Omega^K} \int_\Omega |\nabla v|^p \, dx,$$

where $\mathcal{F}_\Omega^K$ is the set of all functions $v \in C_0^\infty(\Omega)$ such that $0 \le v \le 1$ in Ω and $v = 1$ in K. Moreover, if $p \in [1, \infty)$, for every Borel set $E \subseteq \Omega$,

$$C_p(E) = \inf\{C_p(U) \mid U \subseteq \Omega \text{ open, } E \subseteq U\},$$

and, if $1 < p < \infty$, for every open set $U \subseteq \Omega$,

$$C_p(U) = \sup\{C_p(K) \mid K \text{ compact, } K \subseteq U\}.$$

For any $p \in [1, \infty)$ define

$$\mathcal{M}_{d,p}^+(\Omega) := \{\mu \in \mathcal{M}^+(\Omega) \mid \mu(E) = 0 \text{ for every Borel set } E \subseteq \Omega, \ C_p(E) = 0\},$$

the set of finite (positive) Radon measures on Ω which are absolutely continuous with respect to the C_p-capacity. Analogously,

$$\mathcal{M}^+_{c,p}(\Omega) := \{\mu \in \mathcal{M}^+(\Omega) \mid \exists \text{ a Borel set } E \subseteq \Omega \text{ s.t. } C_p(E) = 0 \text{ and } \mu = \mu \llcorner E\}$$

is the set of finite (positive) Radon measures on Ω which are singular with respect to the C_p-capacity. Clearly, $\mathcal{M}^+_{c,p}(\Omega) \cap \mathcal{M}^+_{d,p}(\Omega) = \{0\}$. Observe that $\mathcal{M}^+_{d,p_1}(\Omega) \subseteq \mathcal{M}^+_{d,p_2}(\Omega)$ and $\mathcal{M}^+_{c,p_2}(\Omega) \subset \mathcal{M}^+_{c,p_1}(\Omega)$ if $p_1 < p_2$.

Recall that every subset $E \subseteq \Omega$ such that $C_p(E) = 0$ for $p \in [1,\infty)$ is Lebesgue measurable and satisfies $|E| = 0$. This plainly implies

$$\mathcal{M}^+_{c,p}(\Omega) \subseteq \mathcal{M}^+_s(\Omega), \quad \mathcal{M}^+_{ac}(\Omega) \subseteq \mathcal{M}^+_{d,p}(\Omega) \quad \text{for every } p \in [1,\infty). \tag{2-6}$$

In connection with the first inclusion in (2-6), observe that if $N = 1$, then $\mathcal{M}^+_{c,p}(\Omega) = \varnothing$ for any $p \in [1,\infty)$. In fact, for *singletons* $E = \{x\}$ ($x \in \Omega$), we have

$$C_p(\{x\},\Omega) > 0 \quad \text{if either } p > N \text{ or } p = N = 1.$$

Therefore, if $N = 1$, by monotonicity, we have $C_p(E) > 0$ ($p \in [1,\infty)$) for every nonempty Borel set $E \subseteq \Omega$. The claim follows.

For any $p \in (1,\infty)$ it is known that a measure $\mu \in \mathcal{M}^+(\Omega)$ belongs to $\mathcal{M}^+_{d,p}(\Omega)$ if and only if

$$\mu \in L^1(\Omega) + W^{-1,p'}(\Omega)$$

(where $W^{-1,p'}(\Omega)$ denotes the dual space of the Sobolev space $W^{1,p}_0(\Omega)$). Then the duality symbol $\langle \mu, \varphi\rangle_\Omega$ makes sense for any $\mu \in \mathcal{M}^+_{d,p}(\Omega)$ and $\varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$. Moreover, if $\mu \in \mathcal{M}^+_{d,p}(\Omega)$, every function $v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ also belongs to $L^\infty(\Omega,\mu)$; for example, see [Evans and Gariepy 1992].

For every $\mu \in \mathcal{M}^+(\Omega)$, $p \in [1,\infty)$, we define the *concentrated* and *diffuse* parts of μ with respect to C_p-capacity as the (unique, mutually singular) measures $\mu_{c,p} \in \mathcal{M}^+_{c,p}(\Omega)$ and $\mu_{d,p} \in \mathcal{M}^+_{d,p}(\Omega)$ such that

$$\mu = \mu_{c,p} + \mu_{d,p}. \tag{2-7}$$

Combining the decompositions in (2-1) and (2-7) and using (2-6) gives

$$\mu_{c,p} = [\mu_s]_{c,p}, \tag{2-8}$$

$$\mu_{d,p} = \mu_{ac} + [\mu_s]_{d,p}, \tag{2-9}$$

for every $\mu \in \mathcal{M}^+(\Omega)$. From (2-7)–(2-9) we obtain

$$\mu = \mu_{ac} + [\mu_s]_{d,p} + \mu_{c,p}, \tag{2-10}$$

which in the case $N = 1$ reduces to (2-1).

Finally, recall that a function $f : \Omega \to \mathbb{R}$ is *C_p-quasicontinuous in* Ω if for any $\epsilon > 0$ there exists a set $E \subseteq \Omega$, with $C_p(E) < \epsilon$, such that the restriction $f\big|_{\Omega\setminus E}$ is continuous in $\Omega \setminus E$ (it is not restrictive to assume that the set E is open). It can be proven (for example, see [Evans and Gariepy 1992]) that every function $u \in W^{1,p}(\Omega)$ has a C_p-quasicontinuous representative $\tilde{u}$; moreover, if $\bar{u}$ is another C_p-quasicontinuous representative of u, then the equality $\bar{u} = \tilde{u}$ holds C_p-almost everywhere in Ω. In the following, every function $u \in W^{1,p}(\Omega)$ will be identified with its unique C_p-quasicontinuous representative.

Remark 2.1. Recalling that $v(\cdot, t) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ for a.e. $t \in (0, T)$ (see Definition 3.1) and $[u_s(\cdot, t)]_{d,2} \in L^1(\Omega) + H^{-1}(\Omega)$ by the characterization of the diffuse measures, it is apparent that the singular term $\langle [u_s(\cdot, t)]_{d,2}, v(\cdot, t)\rho\rangle_\Omega$ in the left-hand side of (1-10) is well defined for any $\rho \in C_c^1(\Omega)$. Let us show that the same quantity is well defined for any $\rho \in C_c(\Omega)$.

In fact, let $\mu \in \mathcal{M}_{d,2}^+(\Omega)$, $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and let $\tilde{v}$ be its C_2-quasicontinuous representative. Let us show that $\tilde{v}\rho$ belongs to $L^1(\Omega, \mu)$, so that the quantity

$$\langle \mu, v\rho\rangle_\Omega = \int_\Omega \tilde{v}(x)\rho(x)\, d\mu(x)$$

is well defined.

Let $\{\rho_n\} \subseteq C_c^\infty(\Omega)$ be any sequence such that

$$\rho_n \to \rho \quad \text{in } C(\overline{\Omega}). \tag{2-11}$$

Since $\tilde{v}$ is defined C_2-almost everywhere in Ω and $\mu \in \mathcal{M}_{d,2}^+(\Omega)$,

$$\tilde{v}(x)\rho_n(x) \to \tilde{v}(x)\rho(x) \qquad \text{for } \mu\text{-a.e. } x \in \Omega. \tag{2-12}$$

Moreover, by (2-11) there exists $C > 0$ such that for every $n \in \mathbb{N}$ we have

$$|\tilde{v}\rho_n| \le C|\tilde{v}| \in L^1(\Omega, \mu).$$

Then by the dominated convergence theorem the claim follows.

3. Main results

Definitions.

Definition 3.1. Given $u_0 \in \mathcal{M}^+(\Omega)$, a measure $u \in L^\infty((0, T); \mathcal{M}^+(\Omega))$ is called a *solution* of problem (1-1) if the following holds:

(i) $[\psi(u_r)]_t \in L^\infty(Q)$, the chemical potential v defined by (1-7) belongs to $L^\infty((0, T); H_0^1(\Omega))$,

$$\Delta v \in L^\infty((0, T); \mathcal{M}(\Omega)),$$

and

$$0 \le v \le \varphi(\alpha) \quad \text{a.e. in } Q. \tag{3-1}$$

(ii) for every $\zeta \in C^1([0, T]; C_c(\Omega))$ with $\zeta(\cdot, T) = 0$ in Ω,

$$\int_0^T \langle u(\cdot, t), \zeta_t(\cdot, t)\rangle_\Omega\, dt + \int_0^T \langle \Delta v(\cdot, t), \zeta(\cdot, t)\rangle_\Omega\, dt = -\langle u_0, \zeta(\cdot, 0)\rangle_\Omega. \tag{3-2}$$

Observe that the assumption $\Delta v \in L^\infty((0, T); \mathcal{M}(\Omega))$ implies $u \in C([0, T]; \mathcal{M}^+(\Omega))$.

Remark 3.2. Since $0 \le \varphi(u) \le \varphi(\alpha)$ for $u \ge 0$ by (1-3), it follows from (1-7) and (3-1) that

$$|[\psi(u_r)]_t| \le \frac{\varphi(\alpha)}{\varepsilon} \quad \text{a.e. in } Q. \tag{3-3}$$

Remark 3.3. Since $v \in L^\infty((0,T); H_0^1(\Omega))$ and $\Delta v \in L^\infty((0,T); \mathcal{M}(\Omega))$, for a.e. $t \in (0,T)$ we have that $v(\cdot\,,t) \in H_0^1(\Omega)$ and $\Delta v(\cdot\,,t) := [\Delta v](\cdot\,,t) \in \mathcal{M}(\Omega)$. Observe that

$$\Delta v(\cdot\,,t) = \Delta[v(\cdot\,,t)] \in H^{-1}(\Omega) \tag{3-4}$$

for a.e. $t \in (0,T)$. In fact, let j_σ ($\sigma > 0$) be a standard mollifier. Then

$$\langle [\Delta v(\cdot\,,t)] * j_\sigma, \rho\rangle_\Omega = \langle \Delta[v(\cdot\,,t)] * j_\sigma], \rho\rangle_\Omega = \langle v(\cdot\,,t) * j_\sigma, \Delta\rho\rangle_\Omega$$

for any $\rho \in C_c^2(\Omega)$. Letting $\sigma \to 0$ we obtain

$$\langle \Delta v(\cdot\,,t), \rho\rangle_\Omega = \langle v(\cdot\,,t), \Delta\rho\rangle_\Omega,$$

which shows that $\Delta v(\cdot\,,t)$ is the distributional Laplacian of $v(\cdot\,,t) \in H_0^1(\Omega)$. Hence (3-4) follows.

Given $g \in C^1([0,\varphi(\alpha)])$, we set

$$G(z) := \int_0^z g(\varphi(u))\,du \quad \text{for } z \ge 0. \tag{3-5}$$

Definition 3.4. Let $u_0 \in \mathcal{M}^+(\Omega)$. A solution u of problem (1-1) is called an *entropy solution* if for all $g \in C^1([0,\varphi(\alpha)])$ such that $g' \ge 0$ and $g(0) = 0$, and for all $\zeta \in C^1([0,T]; C_c^1(\Omega))$ such that $\zeta \ge 0$, $\zeta(\cdot\,,T) = 0$ in Ω, the following *entropy inequality* holds:

$$\iint_Q \{G(u_r)\zeta_t - g(v)\nabla v \nabla\zeta - g'(v)|\nabla v|^2\zeta\}\,dx\,dt \ge -\int_\Omega G(u_{0r})\zeta(x,0)\,dx, \tag{3-6}$$

where G is defined by (3-5).

Inequality (3-6) is called the entropy inequality for problem (1-1) by analogy with the entropy inequality for viscous conservation laws; see [Evans 2004; Serre 1999]. Such an inequality is known to hold

(i) when $u_0 \in L^\infty(\Omega)$ and $\psi(u) = u$ (this is the so-called Sobolev regularization), both for a cubic-like φ and for a φ of Perona–Malik type (see [Novick-Cohen and Pego 1991; Smarrazzo 2008]);

(ii) for problem (1-1) if $N = 1$ and $\psi'(u) \to 0$ as $u \to \infty$ (see [Smarrazzo and Tesei 2012]).

In such cases, entropy inequalities play an important role both to describe the time evolution of solutions of (1-1) and to address the "vanishing viscosity limit" of the problem as $\epsilon \to 0$.

Persistence and monotonicity. Given any solution u of problem (1-1), we prove in Section 4 that the C_2-concentrated part $[u(\cdot\,,t)]_{c,2}$ does not evolve in time if $N \ge 2$ (recall that $\mathcal{M}_{c,2}^+(\Omega) = \varnothing$ if $N = 1$).

Theorem 3.5. *Let $N \ge 2$ and let u be a solution to problem* (1-1). *Then*

$$[u(\cdot\,,t)]_{c,2} = [u_0]_{c,2} \quad \textit{for a.e. } t \in (0,T). \tag{3-7}$$

Therefore, with respect to the case of a power-type ψ in which the first equality of (1-12) holds, in the present case it is only the concentrated part $[u(\cdot\,,t)]_{c,2} = [u_s(\cdot\,,t)]_{c,2}$ of the solution which remains constant.

Concerning the density of the absolutely continuous part of an entropy solution, the following holds. The proof is the same as that of [Bertsch et al. $\geq$ 2013, Proposition 2.5], thus we omit it.

Proposition 3.6. *Let u be an entropy solution of problem* (1-1). *Then there exists a null set $F^* \subset (0, T)$ such that, for any $t_0 \in (0, T) \setminus F^*$ and any Borel set $E \subseteq \Omega$,*

$$u_r(\cdot\,, t_0) \leq \alpha \text{ a.e. in } E \implies u_r(\cdot\,, t) \leq \alpha \text{ a.e. in } E \text{ for every } t \in (t_0, T) \setminus F^*.$$

The singular part of an entropy solution does not decrease if time evolves.

Proposition 3.7. *Let u be an entropy solution of problem* (1-1), *and let $\rho \in C_c(\Omega)$, $\rho \geq 0$. Then, for a.e. $0 \leq t_1 \leq t_2 \leq T$,*

$$\langle u_s(\cdot\,, t_1), \rho\rangle_\Omega \leq \langle u_s(\cdot\,, t_2), \rho\rangle_\Omega \tag{3-8}$$

and, for a.e. $t \in (0, T)$,

$$\langle u_{0s}, \rho\rangle_\Omega \leq \langle u_s(\cdot\,, t), \rho\rangle_\Omega. \tag{3-9}$$

Remark 3.8. If u is a solution of problem (1-1) satisfying (1-9), inequalities (3-8)–(3-9) immediately follow from (3-7) and (3-13) below. The relationship between entropy solutions and solutions satisfying (1-9) is addressed in Theorem 3.18.

Proposition 3.7 implies that a solution (satisfying estimate (3-10) below) with trivial absolutely continuous part is a steady state.

Corollary 3.9. *Let $u_0 \in \mathcal{M}^+(\Omega)$, let $\varphi \in C^\infty([0, \infty))$ satisfy* (1-3)–(1-5), *and let u be an entropy solution of problem* (1-1) *such that, for a.e. $t \in (0, T)$,*

$$\|u(\cdot\,, t)\|_{\mathcal{M}(\Omega)} \leq \|u_0\|_{\mathcal{M}(\Omega)}. \tag{3-10}$$

Then

$$u_{0r} = 0 \text{ a.e. in } \Omega \quad \implies \quad u_r(\cdot\,, t) = 0 \text{ a.e. in } \Omega,\ u_s(\cdot\,, t) = u_0 \text{ for a.e. } t \in (0, T).$$

Proposition 3.7 and Corollary 3.9 will be proved in Section 4.

Remark 3.10. By the considerations above,

$$u_r(\cdot\,, t) = 0 \text{ a.e. } t \in (0, T) \iff u_s(\cdot\,, t) = u_0 \text{ for a.e. } t \in (0, T).$$

In fact, if $u_r(\cdot\,, t) = 0$ for a.e. $t \in (0, T)$, by (1-7) we have $v = 0$ a.e. in Q, hence $u(\cdot\,, t) = u_s(\cdot\,, t) = u_0$ by equality (3-2). Conversely, if $u_s(\cdot\,, t) = u_0$ for a.e. $t \in (0, T)$, we have $u_0 = u_{0s}$, thus $u_{0r} = 0$ a.e. in Ω which implies $u_r(\cdot\,, t) = 0$ by (3-10).

Uniqueness. In this subsection we consider solutions u of problem (1-1) such that for a.e. $t \in (0, T)$ the trace $v(\cdot, t)$ of the chemical potential solves the elliptic problem (1-9). This means that for a.e. $t \in (0, T)$, $v(\cdot, t) \in H_0^1(\Omega)$, $\Delta[v(\cdot, t)] \in \mathcal{M}(\Omega)$, and equality (1-10) is satisfied for every $\rho \in C_c(\Omega)$. The results described in this subsection will be proved in Section 5.

Satisfying problem (1-9) guarantees uniqueness of solutions.

Theorem 3.11. *Let $\varphi \in C^\infty([0, \infty))$ satisfy (1-3)–(1-4). Let there exist $C > 0$ such that*

$$\left| \left(\frac{\varphi}{\psi'} \right)'(u) \right| \le C \quad \text{for } u \ge 0. \tag{3-11}$$

Then problem (1-1) has at most one solution satisfying (1-9).

Below we consider in more detail the qualitative properties of solutions of problem (1-1) which satisfy (1-9). In fact, it turns out that the logarithmic form of ψ makes it possible to give precise estimates of the time evolution both for u_r and for u_s.

Proposition 3.12. *Let $\varphi \in C^\infty([0, \infty))$ satisfy (1-3)–(1-4), and let u be a solution of problem (1-1) satisfying (1-9). Then, for a.e. $t \in (0, T)$ and for any $\rho \in C_c(\Omega)$, $\rho \ge 0$,*

$$\int_\Omega [1 + u_r(x, t)]\rho(x)\, dx \le \exp\left\{ \frac{\varphi(\alpha)t}{\varepsilon} \right\} \int_\Omega [1 + u_{0r}(x)]\rho(x)\, dx, \tag{3-12}$$

$$\langle [u_s]_{d,2}(\cdot, t), \rho \rangle_\Omega = \langle [u_{0s}]_{d,2}, \exp\left\{ \frac{1}{\epsilon} \int_0^t v(\cdot, s)\, ds \right\} \rho \rangle_\Omega, \tag{3-13}$$

$$\langle u_s(\cdot, t), \rho \rangle_\Omega \le \exp\left\{ \frac{\varphi(\alpha)t}{\varepsilon} \right\} \langle u_{0s}, \rho \rangle_\Omega. \tag{3-14}$$

In particular, $u_s(\cdot, t)$ is absolutely continuous with respect to u_{0s}, for a.e. $t \in (0, T)$.

The last statement above entails a regularity result: no singularity can arise at some positive time. Going into detail, we have the following remark.

Remark 3.13. By inequality (3-14), for any solution of problem (1-1) satisfying (1-9), we have:

(i) $u_0 \in L^1(\Omega)$, $u_0 \ge 0 \implies u \in L^1(Q)$, $u \ge 0$.

(ii) $u_{0s} \in \mathcal{M}^+_{c,p}(\Omega) \implies u_s(\cdot, t) \in \mathcal{M}^+_{c,p}(\Omega)$ for a.e. $t \in (0, T)$.

(iii) $u_0 \in \mathcal{M}^+_{d,p}(\Omega) \implies u(\cdot, t) \in \mathcal{M}^+_{d,p}(\Omega)$ for a.e. $t \in (0, T)$ $(p \in [1, \infty))$.

Remark 3.14. By the arbitrariness of ρ in (3-12)–(3-14), for every Borel set $E \subseteq \Omega$ and a.e. $t \in (0, T)$, we have

$$\int_E [1 + u_r(x, t)]\, dx \le \exp\left\{ \frac{\varphi(\alpha)t}{\varepsilon} \right\} \int_E [1 + u_{0r}(x)]\, dx,$$

$$u_s(\cdot, t)(E) \le \exp\left\{ \frac{\varphi(\alpha)t}{\varepsilon} \right\} u_{0s}(E).$$

Also observe that (3-12) and (3-14) imply

$$\langle [1 + u(\cdot, t)], \rho \rangle_\Omega \le \exp\left\{ \frac{\varphi(\alpha)t}{\varepsilon} \right\} \langle [1 + u_0], \rho \rangle_\Omega \tag{3-15}$$

for every $\rho \in C_c(\Omega)$, $\rho \geq 0$, thus

$$u(\,\cdot\,,t)(E) \leq \exp\left\{\frac{\varphi(\alpha)t}{\varepsilon}\right\}u_0(E) + \left(\exp\left\{\frac{\varphi(\alpha)t}{\varepsilon}\right\} - 1\right)|E|$$

for every Borel set $E \subseteq \Omega$.

Observe that by equalities (2-8) and (2-10)

$$u_s(\,\cdot\,,t) = [u_s(\,\cdot\,,t)]_{d,2} + [u(\,\cdot\,,t)]_{c,2}$$

for a.e. $t \in (0,T)$. Then from (3-7), (3-13) it is apparent that to describe the time evolution of $u_s(\,\cdot\,,t)$ it is important to know whether $v(\,\cdot\,,t)$ vanishes in Ω. In this sense the following maximum principle, which generalizes in a certain sense [Brezis and Ponce 2003, Theorem 1], is expedient.

Proposition 3.15. *Let $\mu \in \mathcal{M}^+(\Omega)$ be C_2-diffuse. Let $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfy*

$$-\Delta v + \mu v \geq 0 \quad \text{in } \Omega,$$

in the sense that

$$\int_\Omega \nabla v \cdot \nabla \rho \, dx + \langle \mu, v\rho \rangle_\Omega \geq 0 \quad \text{for any } \rho \in H_0^1(\Omega) \cap L^\infty(\Omega),\ \rho \geq 0. \tag{3-16}$$

Then $v \geq 0$ a.e. in Ω, and $v = 0$ a.e. in Ω if $v = 0$ a.e. on a subset $E \subseteq \Omega$ such that $C_2(E) > 0$.

If $N = 1$, we have the following.

Proposition 3.16. *Let $N = 1$, and let u be a solution of problem* (1-1) *satisfying* (1-18). *Then, for a.e. $t \in (0,T)$, either $v(\,\cdot\,,t) > 0$ in Ω or $v(\,\cdot\,,t) \equiv 0$ in Ω.*

Existence. Set

$$\psi_n(u) := \psi(u) + \frac{u}{n} = \log(1+u) + \frac{u}{n} \quad \text{for } u \geq 0. \tag{3-17}$$

Observe that $\psi_n \to \psi$ as $n \to \infty$ and $\psi_n' \geq 1/n > 0$, thus the nonlinearities ψ_n are nondegenerate. Consider the regularized problems

$$\begin{cases} u_{nt} = \Delta v_n & \text{in } Q, \\ v_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n = u_{0n} \geq 0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{P_n}$$

where

$$v_n := \varphi(u_n) + \varepsilon[\psi_n(u_n)]_t \tag{3-18}$$

and $\{u_{0n}\}$ is a sequence of smooth nonnegative functions with the properties stated in Lemma 6.1 (Section 6 is dedicated to the approximating problem P_n).

Theorem 3.17. *Let $u_0 \in \mathcal{M}^+(\Omega)$ and let $\varphi \in C^\infty([0,\infty))$ satisfy* (1-3)–(1-5). *Then problem* (1-1) *has an entropy solution u, which is a limiting point as $n \to \infty$ of the family of solutions of the approximating problems (P_n). Moreover*:

(i) *For a.e. $t \in (0,T)$, inequality* (3-10) *holds.*

(ii) *For a.e.* $t \in (0, T)$ *and for every Borel set* $E \subseteq \Omega$, *inequalities* (3-12) *and* (3-14) *hold. In particular*, $u_s(\,\cdot\,, t)$ *is absolutely continuous with respect to* u_{0s}.

In Theorem 3.18 below we show that the entropy solution given in Theorem 3.17 satisfies the elliptic problem (1-9) if $N = 1$; the same holds if $N \geq 2$ for a suitable class of initial data $u_0 \in \mathcal{M}^+(\Omega)$. In these cases claim (ii) of Theorem 3.17 follows directly from Proposition 3.12.

Theorem 3.18. *Let* $u_0 \in \mathcal{M}^+(\Omega)$, *and let* $\varphi \in C^\infty([0,\infty))$ *satisfy* (1-3)–(1-5). *Let* u *be the entropy solution of problem* (1-1) *given in Theorem 3.17 and let* v *be the chemical potential defined in* (1-7).

(a) *If* $N = 1$, *the pair* (u, v) *satisfies problem* (1-18).

(b) *Let* $N \geq 2$, *and let* u_0 *satisfy the following assumptions*:
   (i) $[u_0]_{c,2}$ *is concentrated on some compact* $K_0 \subset \Omega$ *such that* $C_2(K_0) = 0$;
   (ii) $[u_0]_{d,2} \in \mathcal{M}^+_{d,p}(\Omega)$ *for some* $p \in [1, 2)$.
   Then the pair (u, v) *satisfies problem* (1-9).

Theorems 3.17 and 3.18 will be proved in Sections 7 and 8, respectively.

For $N = 1$, from the above theorem we deduce that an entropy solution of problem (1-1) satisfying problem (1-9) (or equivalently (1-18)) can be obtained as a limiting point as $n \to \infty$ of the family of solutions to the approximating problems (P_n).

If $N \geq 2$, the same result holds for a suitable class of initial data u_0, subject to technical conditions involving both $[u_0]_{d,2}$ and $[u_0]_{c,2}$ (see Theorem 3.18-(b)). Assumption (ii) on $[u_0]_{d,2}$ is rather mild, yet the problem of removing it is open. On the other hand, the existence of an entropy solution of (1-1) satisfying (1-9) can also be proven without assumption (i). In fact, for every $u_0 \in \mathcal{M}^+(\Omega)$,

$$u_0 = [u_0]_{d,2} + [u_0]_{c,2},$$

with $[u_0]_{d,2} \in \mathcal{M}^+_{d,p}(\Omega)$ for some $p \in [1, 2)$, it suffices to consider the measure $u \in L^\infty((0, T); \mathcal{M}^+(\Omega))$ defined by setting

$$u(\,\cdot\,, t) := \tilde{u}(\,\cdot\,, t) + [u_0]_{c,2} \quad \text{for a.e. } t \in (0, T);$$

here $\tilde{u}$ denotes a solution of (1-1) with initial data $[u_0]_{d,2}$ which satisfies the elliptic problem (1-9) (the existence of such a solution is ensured by Theorem 3.18 above). Clearly, the solution u (whose uniqueness is ensured by Theorem 3.11, if (3-11) holds) need not be obtained by letting $n \to \infty$ in the associated problems (P_n).

Corollary 3.19. *Let* $u_0 \in \mathcal{M}^+(\Omega)$, *and let* $\varphi \in C^\infty([0,\infty))$ *satisfy* (1-3)–(1-5) *and* (3-11). *If either* $N = 1$, *or* $N \geq 2$ *and* $[u_0]_{d,2} \in \mathcal{M}^+_{d,p}(\Omega)$ *for some* $p \in [1, 2)$, *there is exactly one entropy solution of problem* (1-1) *satisfying problem* (1-9).

Remark 3.20. Problem (1-9) is essential to introduce a class of well-posedness for problem (1-1). In fact, it is easy to exhibit a weak solution to problem (1-1) which does not satisfy (1-9) and which, therefore, is different from the solution given by Theorem 3.17.

For this purpose, let $N = 1$ and $\Omega = (0, 1)$. Let $\hat{u}_0 \in C^\infty([0, 1])$ satisfy $0 < \hat{u}_0 < \alpha$ in $(0, 1)$, $\hat{u}_0(0) = \hat{u}_0(1) = 0$. Let $\hat{u}$ be the solution of problem (1-1) with Cauchy data $u_0 = u_{0r} = \hat{u}_0$ given by

Theorem 3.17 . Then $\hat{u} = \hat{u}_r \in C^\infty([0,1] \times [0,\infty))$, $0 < \hat{u} < \alpha$ in $[0,1] \times [0,\infty)$, and $\hat{u}_s \equiv 0$. By Theorem 3.18(i) the pair $(\hat{u}, \hat{v})$, where $\hat{v} := \varphi(\hat{u}) + \varepsilon[\psi(\hat{u})]_t$, satisfies the problem

$$\begin{cases} -\varepsilon \hat{v}_{xx} + (1+\hat{u})\hat{v} = (1+\hat{u})\varphi(\hat{u}) & \text{in } [0,1] \times [0,\infty), \\ \hat{v} = 0 & \text{in } \{0,1\} \times [0,\infty), \end{cases}$$

hence $0 < \hat{v} < \varphi(\alpha)$ in $(0,1) \times [0,\infty)$ by the maximum principle.

Let δ_{x_0} denote the Dirac mass centered at some point $x_0 \in \Omega$, and set

$$u_1 := \hat{u} + \delta_{x_0}.$$

On the other hand, let u_2 be the solution of problem (1-1) given by Theorem 3.17, with initial data $u_0 := \hat{u}_0 + \delta_{x_0}$. We claim that

$$u_1 \text{ is a solution of problem (1-1) different from } u_2.$$

It is easily seen that u_1 is a solution of (1-1). Clearly, $u_{1r} = \hat{u}$, so the corresponding potential $v_1 := \varphi(u_{1r}) + \varepsilon[\psi(u_{1r})]_t$ coincides with $\hat{v}$. Recalling that $\hat{u}_t = \hat{v}_{xx}$, we have

$$\int_0^T \langle u(\cdot,t), \zeta_t(\cdot,t)\rangle_\Omega \, dt = \int_0^T\int_0^1 \hat{u}\zeta_t \, dx\, dt - \zeta(x_0,0) = -\int_0^T\int_0^1 \hat{v}_{xx}\zeta \, dx\, dt = -\int_0^1 \hat{u}_0(x)\zeta(x,0)\, dx - \zeta(x_0,0),$$

namely, equality (3-2) for every $\zeta \in C^1([0,T]; C_c(\Omega))$ with $\zeta(\cdot\,,T) = 0$ in Ω.

On the other hand, by Theorem 3.18(i) the solution u_2 and the corresponding chemical potential satisfy the elliptic problem (1-18), whereas the pair $(u_1, v_1) = (u_1, \hat{v})$ does not. In fact, if it did, by equality (3-13) we would have

$$\langle u_{1s}(\cdot\,,t), \rho\rangle_\Omega = \exp\left\{\frac{1}{\epsilon}\int_0^t \hat{v}(x_0,s)\, ds\right\}\rho(x_0)$$

(since every Radon measure is C_2-diffuse if $N = 1$), whereas the very definition of u_1 implies that

$$\langle u_{1s}(\cdot\,,t), \rho\rangle_\Omega = \langle \delta_{x_0}, \rho\rangle_\Omega = \rho(x_0)$$

for every $t > 0$. Since $\hat{v} > 0$ in $(0,1) \times [0,\infty)$, this gives a contradiction if $\rho(x_0) \neq 0$. The claim follows.

4. Proofs of persistence and monotonicity results

The proof of the following lemma is almost identical to that of [Bertsch et al. $\geq$ 2013, Lemma 3.1]; thus we omit it.

Lemma 4.1. *Let u be a solution of problem* (1-1). *Then there exists a null set $F^* \subseteq (0,T)$ such that, for every $t \in (0,T) \setminus F^*$ and $\rho \in C_c(\Omega)$,*

$$\langle u(\cdot\,,t), \rho\rangle_\Omega - \langle u_0, \rho\rangle_\Omega = \int_0^t \langle \Delta v(\cdot\,,s), \rho\rangle_\Omega \, ds, \tag{4-1}$$

$$\lim_{n\to\infty} \frac{n}{2}\int_{t-1/n}^{t+1/n} |\langle u_s(\cdot\,,s), \rho\rangle_\Omega - \langle u_s(\cdot\,,t), \rho\rangle_\Omega| \, ds = 0. \tag{4-2}$$

Proof of Theorem 3.5. Let $F^* \subseteq (0,T)$ be the null set given by Lemma 4.1. For every $t \in (0,T) \setminus F^*$ consider the map

$$F_t : C_c(\Omega) \to \mathbb{R}, \quad \rho \to \int_0^t \langle \Delta v(\,\cdot\,, s), \rho \rangle_\Omega \, ds.$$

By (4-1) we have $F_t \in \mathcal{M}(\Omega)$. Moreover, $F_t \in H^{-1}(\Omega)$ by Remark 3.3; thus $F_t \in \mathcal{M}_{d,2}(\Omega)$. Then (4-1) becomes

$$\langle [u(\,\cdot\,,t)]_{c,2}, \rho \rangle_\Omega - \langle [u_0]_{c,2}, \rho \rangle_\Omega = \langle F_t, \rho \rangle_\Omega - \langle [u(\,\cdot\,,t)]_{d,2} - [u_0]_{d,2}, \rho \rangle_\Omega. \tag{4-3}$$

By equality (4-3) the difference $[u(\,\cdot\,,t)]_{c,2} - [u_0]_{c,2}$ is both C_2-diffuse and C_2-concentrated; thus

$$[u(\,\cdot\,,t)]_{c,2} - [u_0]_{c,2} = 0. \qquad \square$$

Proof of Proposition 3.7. Let $\{g_n\} \subseteq \mathrm{Lip}([0, \varphi(\alpha)])$ be defined by

$$g_n(s) := \begin{cases} ns & \text{if } 0 \le s \le \dfrac{1}{n}, \\ 1 & \text{if } \dfrac{1}{n} < s \le \varphi(\alpha), \end{cases}$$

and let G_n be the function (3-5) with $g = g_n$. By standard approximation arguments, inequality (3-6) is still valid with $G = G_n$. Therefore,

$$\iint_Q \{G_n(u_r)\zeta_t - g_n(v)\nabla v \nabla \zeta\} \, dx \, dt \ge -\int_\Omega G_n(u_{0r}(x))\zeta(x,0)\, dx \tag{4-4}$$

for $\zeta \in C^1([0,T]; C^1_c(\Omega))$, $\zeta \ge 0$, $\zeta(\,\cdot\,,T) = 0$ in Ω.

Since $0 \le G_n(u_r) \le u_r$ a.e. in Q, $0 \le G_n(u_{0r}) \le u_{0r}$ a.e. in Ω, and $g_n(s) \to 1$ for any $s \in (0, \varphi(\alpha)]$, as $n \to \infty$, by the dominated convergence theorem, we have

$$G_n(u_r) \to u_r \text{ in } L^1(Q), \quad G_n(u_{0r}) \to u_{0r} \text{ in } L^1(\Omega). \tag{4-5}$$

Moreover,

$$g_n(v)\nabla v = \nabla\left(\int_0^v g_n(s)\, ds\right) \quad \text{a.e. in } Q, \tag{4-6}$$

and

$$\| g_n(v) |\nabla v| \|_{L^2(Q)} \le \| |\nabla v| \|_{L^2(Q)}.$$

Therefore the sequence $\{g_n(v)\nabla v\}$ is weakly relatively compact in $[L^2(Q)]^N$. By (4-6), since

$$\int_0^{v(x,t)} g_n(s)\, ds \to v(x,t) \quad \text{as } n \to \infty \quad \text{for a.e. } (x,t) \in Q,$$

we obtain

$$g_n(v)\nabla v \rightharpoonup \nabla v \quad \text{in } [L^2(Q)]^N. \tag{4-7}$$

By (4-5) and (4-7), letting $n \to \infty$ in inequality (4-4), we have

$$\iint_\Omega \{u_r \zeta_t - \nabla v \nabla \zeta\} \, dx \, dt \ge -\int_\Omega u_{0r}(x)\zeta(x,0)\, dx, \tag{4-8}$$

whence, by (3-2),

$$-\int_0^T \langle u_s(\cdot\,,t), \zeta_t(\cdot\,,t)\rangle_\Omega\,dt \ge \langle u_{0s}, \zeta(\cdot\,,0)\rangle_\Omega \tag{4-9}$$

for any ζ as above.

To prove inequality (3-8), let $t_1, t_2 \in (0,T)\setminus F^*$, where $F^* \subseteq (0,T)$ is the null set defined by Lemma 4.1, and set

$$h_1(t) := \begin{cases} 0 & \text{if } t < t_1 - \dfrac{1}{n}, \\ n\Big(t - t_1 + \dfrac{1}{n}\Big) & \text{if } t_1 - \dfrac{1}{n} \le t \le t_1, \\ 1 & \text{if } t_1 < t < t_2, \\ -n\Big(t - t_2 - \dfrac{1}{n}\Big) & \text{if } t_2 \le t \le t_2 + \dfrac{1}{n}, \\ 0 & \text{if } t \ge t_2 + \dfrac{1}{n}. \end{cases}$$

Choosing $\zeta(x,t) = \rho(x)h_1(t)$ in (4-9), with any $\rho \in C_c^1(\Omega)$, $\rho \ge 0$, we obtain

$$n\int_{t_2}^{t_2+1/n} \langle u_s(\cdot\,,t), \rho\rangle_\Omega\,dt \ge n\int_{t_1-1/n}^{t_1} \langle u_s(\cdot\,,t), \rho\rangle_\Omega\,dt.$$

Letting $n \to \infty$ in the above inequality and using (4-2), we obtain (3-8).

The proof of inequality (3-9) is similar. For any $\tau \in (0,T)\setminus F^*$ define

$$h_2(t) := \begin{cases} 1 & \text{if } t \le \tau, \\ -n\Big(t - \tau - \dfrac{1}{n}\Big) & \text{if } \tau < t < \tau + \dfrac{1}{n}, \\ 0 & \text{if } t \ge \tau + \dfrac{1}{n}. \end{cases}$$

Substitution of $\zeta(x,t) = \rho(x)h_2(t)$ in (4-9) gives

$$n\int_\tau^{\tau+1/n} \langle u_s(\cdot\,,t), \rho\rangle_\Omega\,dt \ge \langle u_{0s}, \rho\rangle_\Omega,$$

whence we obtain (3-9) as $n \to \infty$. This completes the proof. □

Proof of Corollary 3.9. Since by assumption $u_0 = u_{0s}$, by inequality (3-10) we have

$$\|u_s(\cdot\,,t)\|_{\mathcal{M}(\Omega)} \le \|u(\cdot\,,t)\|_{\mathcal{M}(\Omega)} \le \|u_{0s}\|_{\mathcal{M}(\Omega)}$$

for a.e. $t \in (0,T)$. On the other hand, by inequality (3-9)

$$\|u_{0s}\|_{\mathcal{M}(\Omega)} = \sup_{\rho\in C_c(\Omega),|\rho|\le 1} \langle u_{0s}, \rho\rangle_\Omega \le \sup_{\rho\in C_c(\Omega),|\rho|\le 1} \langle u_s(\cdot\,,t), \rho\rangle_\Omega = \|u_s(\cdot\,,t)\|_{\mathcal{M}(\Omega)}.$$

The above inequalities imply

$$\|u_s(\cdot\,,t)\|_{\mathcal{M}(\Omega)} = \|u(\cdot\,,t)\|_{\mathcal{M}(\Omega)} = \|u_{0s}\|_{\mathcal{M}(\Omega)} = \|u_0\|_{\mathcal{M}(\Omega)}, \tag{4-10}$$

whence $\|u_r(\cdot\,,t)\|_{L^1(\Omega)} = 0$ for a.e. $t \in (0,T)$.

It remains to prove that $u_s(\,\cdot\,,t)=u_0$ for a.e. $t\in(0,T)$. By inequality (3-9) and the arbitrariness of ρ, for every Borel set $E\subseteq\Omega$ and for a.e. $t\in(0,T)$,

$$u_s(\,\cdot\,,t)(E)\ge u_{0s}(E)=u_0(E). \tag{4-11}$$

So, arguing by contradiction, we suppose that there exists a Borel set $\widetilde{E}\subseteq\Omega$ such that

$$u_s(\,\cdot\,,t)(\widetilde{E})>u_0(\widetilde{E}). \tag{4-12}$$

By (4-10)–(4-12) and the identities

$$\|u_0\|_{\mathcal{M}(\Omega)}=u_0(\Omega),\qquad \|u_s(\,\cdot\,,t)\|_{\mathcal{M}(\Omega)}=u_s(\,\cdot\,,t)(\Omega),$$

we obtain

$$u_0(\Omega\setminus\widetilde{E})\le u_s(\,\cdot\,,t)(\Omega\setminus\widetilde{E})=u_s(\,\cdot\,,t)(\Omega)-u_s(\,\cdot\,,t)(\widetilde{E})<u_0(\Omega)-u_0(\widetilde{E})=u_0(\Omega\setminus\widetilde{E}),$$

a contradiction. Hence the conclusion follows. □

5. Proof of uniqueness

Proof of Theorem 3.11. Let u_1, u_2 be two solutions of problem (1-1) satisfying (1-9), and let v_1, v_2 be the corresponding potentials defined by (1-7). By Theorem 3.5 it is sufficient to prove that

$$[u_1(\,\cdot\,,t)]_{d,2}=[u_2(\,\cdot\,,t)]_{d,2}\quad\text{for a.e. } t\in(0,T).$$

By (3-2), for each $\rho\in C_c(\Omega)$ and for a.e. $t\in(0,T)$,

$$\langle u_1(\cdot,t)-u_2(\cdot,t),\rho\rangle_\Omega=\int_0^t\langle\Delta[v_1(\cdot,s)-v(\cdot,s)],\rho\rangle_\Omega\,ds\le\|\rho\|_{C(\bar{\Omega})}\int_0^t\|\Delta[v_1(\cdot,s)-v_2(\cdot,s)]\|_{\mathcal{M}(\Omega)}\,ds,$$

thus

$$\|u_1(\cdot,t)-u_2(\cdot,t)\|_{\mathcal{M}(\Omega)}=\sup_{\rho\in C_c(\Omega),|\rho|\le1}\langle u_1(\cdot,t)-u_2(\cdot,t),\rho\rangle_\Omega\le\int_0^t\|\Delta[v_1(\cdot,s)-v_2(\cdot,s)]\|_{\mathcal{M}(\Omega)}\,ds. \tag{5-1}$$

Let

$$w(x,t):=v_1(x,t)-v_2(x,t)\qquad((x,t)\in Q).$$

By (1-9), $w\in L^\infty((0,T);H_0^1(\Omega)\cap L^\infty(\Omega))$, $\Delta w\in L^\infty((0,T);\mathcal{M}(\Omega))$, and w solves the elliptic equation

$$\begin{aligned}-\varepsilon\Delta w(\,\cdot\,,t)&+[u_1(\,\cdot\,,t)]_{d,2}w(\,\cdot\,,t)+w(\,\cdot\,,t)\\&=-([u_1(\,\cdot\,,t)]_{d,2}-[u_2(\,\cdot\,,t)]_{d,2})v_2(\,\cdot\,,t)+\left[\frac{\varphi(u_{1r})}{\psi'(u_{1r})}-\frac{\varphi(u_{2r})}{\psi'(u_{2r})}\right](\,\cdot\,,t)\quad\text{in }\mathcal{M}(\Omega)\end{aligned} \tag{5-2}$$

for a.e. $t\in(0,T)$.

Let $\{f_j\}\subseteq C^\infty(\mathbb{R})$ satisfy

$$\begin{cases}f_j(0)=0,\quad \|f_j\|_\infty\le1,\quad f_j'\ge0\text{ in }\mathbb{R},\\ |f_j'(s)s|\le1\text{ for every }s\in\mathbb{R},\quad f_j(s)\to\dfrac{s}{|s|}\text{ for every }s\ne0.\end{cases} \tag{5-3}$$

Since $f_j(w) \in L^\infty((0,T); H_0^1(\Omega) \cap L^\infty(\Omega))$ for every $j \in \mathbb{N}$, it makes sense to use $[f_j(w)](\cdot\,,t)$ as test function for equality (5-2). Using inequalities (3-1) and (3-11), this gives

$$\begin{aligned}
\varepsilon \int_\Omega f_j'(w)(x,t)|\nabla w|^2(x,t)\,dx + \langle [u_1(\cdot\,,t)]_{d,2}, [f_j(w)w](\cdot\,,t)\rangle_\Omega + \int_\Omega [f_j(w)w](x,t)\,dx \\
\le \varphi(\alpha)\|[u_1(\cdot\,,t)]_{d,2} - [u_2(\cdot\,,t)]_{d,2}\|_{\mathcal{M}(\Omega)} + \int_\Omega \left| \frac{\varphi(u_{1r})}{\psi'(u_{1r})} - \frac{\varphi(u_{2r})}{\psi'(u_{2r})} \right| (x,t) f_j(w)(x,t)\,dx \\
\le \varphi(\alpha)\|[u_1(\cdot\,,t)]_{d,2} - [u_2(\cdot\,,t)]_{d,2}\|_{\mathcal{M}(\Omega)} + C\|u_{1r}(\cdot\,,t) - u_{2r}(\cdot\,,t)\|_{L^1(\Omega)} \\
\le L\|[u_1(\cdot\,,t)]_{d,2} - [u_2(\cdot\,,t)]_{d,2}\|_{\mathcal{M}(\Omega)}
\end{aligned} \tag{5-4}$$

for a.e. $t \in (0,T)$ with some constant $L > 0$. By the properties of $\{f_j\}$ (see (5-3)) we have

$$\|\,|\nabla[f_j(w)w]|\,\|_{L^2(Q)} \le 2\|\,|\nabla w|\,\|_{L^2(Q)} \tag{5-5}$$

for every $j \in \mathbb{N}$; hence the sequence $\{\nabla[f_j(w)w]\}$ is weakly relatively compact in $[L^2(Q)]^N$. Since

$$[f_j(w)w](\cdot\,,t)) \to |w(\cdot\,,t)| \text{ a.e. in } \Omega$$

and $\|w\|_{L^\infty(Q)} \le \varphi(\alpha)$ by inequality (3-1), by the dominated convergence theorem we have

$$[f_j(w)w](\cdot\,,t)) \to |w(\cdot\,,t)| \text{ in } L^1(\Omega), \quad [f_j(w)w](\cdot\,,t)) \overset{*}{\rightharpoonup} |w(\cdot\,,t)| \text{ in } L^\infty(\Omega).$$

Moreover, by (5-5)

$$[f_j(w)w](\cdot\,,t)) \rightharpoonup |w(\cdot\,,t)| \quad \text{in } H_0^1(\Omega).$$

Then, letting $n \to \infty$ in (5-4) and recalling that $f_j' \ge 0$, we get

$$\langle [u_1(\cdot\,,t)]_{d,2}, |w(\cdot\,,t)|\rangle_\Omega + \int_\Omega |w(x,t)|\,dx \le L\|[u_1(\cdot\,,t)]_{d,2} - [u_2(\cdot\,,t)]_{d,2}\|_{\mathcal{M}(\Omega)}.$$

On the other hand, since $u_1(\cdot\,,t)$ is a nonnegative Radon measure, for any $\rho \in C_c(\Omega)$ we have

$$\begin{aligned}
\langle [u_1(\cdot,t)]_{d,2}, |w(\cdot,t)|\rho\rangle_\Omega + \int_\Omega |w(x,t)|\rho(x)\,dx \le \|\rho\|_{C(\bar\Omega)} \left\{ \langle [u_1(\cdot,t)]_{d,2}, |w(\cdot,t)|\rangle_\Omega + \int_\Omega |w(x,t)|\,dx \right\} \\
\le L\|\rho\|_{C(\bar\Omega)}\|[u_1(\cdot\,,t)]_{d,2} - [u_2(\cdot\,,t)]_{d,2}\|_{\mathcal{M}(\Omega)}.
\end{aligned}$$

Then from (5-2), arguing as in the proof of (5-4), we obtain plainly

$$\varepsilon\langle \Delta w(\cdot\,,t), \rho\rangle_\Omega \le \widetilde{L}\|\rho\|_{C(\bar\Omega)}\|[u_1(\cdot\,,t)]_{d,2} - [u_2(\cdot\,,t)]_{d,2}\|_{\mathcal{M}(\Omega)}$$

for some constant $\widetilde{L} > 0$ and any $\rho \in C_c(\Omega)$, whence

$$\varepsilon\|\Delta[v_1(\cdot\,,t) - v_2(\cdot\,,t)]\|_{\mathcal{M}(\Omega)} = \varepsilon\|\Delta w(\cdot\,,t)\|_{\mathcal{M}(\Omega)} \le \widetilde{L}\|[u_1(\cdot\,,t)]_{d,2} - [u_2(\cdot\,,t)]_{d,2}\|_{\mathcal{M}(\Omega)}$$

for a.e. $t \in (0,T)$. Combined with equality (5-1) this yields

$$\varepsilon\|u_1(\cdot\,,t) - u_2(\cdot\,,t)\|_{\mathcal{M}(\Omega)} \le \widetilde{L}\int_0^t \|u_1(\cdot\,,s) - u_2(\cdot\,,s)\|_{\mathcal{M}(\Omega)}\,ds,$$

and since $u_1(\cdot\,,0)=u_2(\cdot\,,0)=u_0$, it follows from Gronwall's inequality that

$$\|u_1(\cdot\,,t)-u_2(\cdot\,,t)\|_{\mathcal{M}(\Omega)}=0 \quad \text{for a.e. } t\in(0,T). \qquad \square$$

Proof of Proposition 3.12. (i) Since $[\psi(u_r)]_t\in L^\infty(Q)$ (see Remark 3.2), the map $t\to\psi(u_r)(x,t)$ is Lipschitz continuous, and hence differentiable a.e. in $(0,T)$ for a.e. $x\in\Omega$. Differentiating the identity $u_r(\cdot\,,t)=\psi^{-1}[\psi(u_r)](\cdot\,,t)$, we obtain that the derivative u_{rt} exists a.e. in $(0,T)$ and the equality $[\psi(u_r)]_t=\psi'(u_r)u_{rt}$ holds, whence, by (1-7),

$$\varepsilon u_{rt}=(1+u_r)[v-\varphi(u_r)]\in L^1(Q). \tag{5-6}$$

Integrating the above equality in $(0,t)$, we obtain

$$\varepsilon u_r(x,t)-\varepsilon u_{0r}(x)=\int_0^t\{(1+u_r)[v-\varphi(u_r)]\}(x,s)\,ds \tag{5-7}$$

for a.e. $x\in\Omega$, whence, by inequality (3-1),

$$\varepsilon u_r(x,t)-\varepsilon u_{0r}(x)\le\varphi(\alpha)\int_0^t(1+u_r)(x,s)\,ds.$$

Then by Gronwall's inequality

$$1+u_r(x,t)\le[1+u_{0r}(x)]\exp\left\{\frac{\varphi(\alpha)t}{\varepsilon}\right\}\qquad(t\in(0,T))$$

for a.e. $x\in\Omega$, which implies (3-12).

(ii) By (4-1) and (1-10) we have

$$\begin{aligned}\varepsilon\int_\Omega&[u_r(x,t)-u_{0r}(x)]\rho(x)\,dx+\varepsilon\langle[u_s(\cdot\,,t)-u_{0s}],\rho\rangle_\Omega\\&=\int_0^t\int_\Omega\rho(x)\{(1+u_r)[v-\varphi(u_r)]\}(x,s)\,dx\,ds+\int_0^t\langle[u_s(\cdot\,,s)]_{d,2},v(\cdot\,,s)\rho\rangle_\Omega\,ds\end{aligned} \tag{5-8}$$

for any $\rho\in C_c(\Omega)$. Then by (5-7)–(5-8) we get

$$\varepsilon\langle[u_s(\cdot\,,t)-u_{0s}],\rho\rangle_\Omega=\int_0^t\langle[u_s(\cdot\,,s)]_{d,2},v(\cdot\,,s)\rho\rangle_\Omega\,ds.$$

It follows that the map

$$g:(0,T)\to\mathcal{M}^+_{d,2}(\Omega),\quad g(t):=[u_s(\cdot\,,t)]_{d,2}\qquad(t\in(0,T))$$

satisfies the problem

$$\begin{cases}\varepsilon\dfrac{d}{dt}\langle f(t),\rho\rangle_\Omega=\langle f(t),v(\cdot\,,t)\rho\rangle_\Omega & \text{in } (0,T),\\ \langle f(0),\rho\rangle_\Omega=\langle[u_{0s}]_{d,2},\rho\rangle_\Omega\end{cases} \tag{5-9}$$

for any $\rho\in C_c(\Omega)$.

Claim. *The unique solution of problem* (5-9) *is*

$$f:(0,T)\to\mathcal{M}^+_{d,2}(\Omega),\quad f(t):=[u_{0s}]_{d,2}\exp\left\{\frac{1}{\epsilon}\int_0^t v(\cdot\,,s)\,ds\right\}\qquad(t\in(0,T)).$$

This implies that

$$[u_s(\cdot\,,t)]_{d,2}=[u_{0s}]_{d,2}\exp\left\{\frac{1}{\epsilon}\int_0^t v(\cdot\,,s)\,ds\right\}\ \text{in}\ \mathcal{M}^+_{d,2}(\Omega)\ \text{for any}\ t\in(0,T),\tag{5-10}$$

whence equality (3-13) follows. Then inequality (3-14) follows by (3-7) and (3-13), which completes the proof.

To prove the claim, observe preliminarily that

$$\exp\left\{\frac{1}{\epsilon}\int_0^t v(\cdot\,,s)\,ds\right\}\in H^1(\Omega)\cap L^\infty(\Omega),$$

thus

$$\langle f(t),\rho\rangle_\Omega:=\left\langle [u_{0s}]_{d,2},\exp\left\{\frac{1}{\epsilon}\int_0^t v(\cdot\,,s)\,ds\right\}\rho\right\rangle_\Omega$$

is well defined for any $\rho\in C_c(\Omega)$. Then for any $t_0,t_0+h\in(0,T)$ we have

$$\left\langle f(t_0+h)-f(t_0)-\frac{h}{\varepsilon}[u_{0s}]_{d,2}\exp\left\{\frac{1}{\epsilon}\int_0^{t_0} v(\cdot\,,s)\,ds\right\}v(\cdot\,,t_0),\rho\right\rangle_\Omega$$
$$=\frac{|h|^2}{\epsilon^2}\left\langle [u_{0s}]_{d,2},\exp\left\{\frac{1}{\epsilon}\int_0^{t_0+\theta h} v(\cdot\,,s)\,ds\right\}v^2(\cdot\,,t_0),\rho\right\rangle_\Omega$$

for some $\theta\in(0,1)$ and any $\rho\in C_c(\Omega)$. Hence there exists $C>0$, only depending on the norm of v in $L^\infty((0,T);H^1_0(\Omega)\cap L^\infty(\Omega))$, such that

$$\left\| f(t_0+h)-f(t_0)-\frac{h}{\varepsilon}[u_{0s}]_{d,2}\exp\left\{\frac{1}{\epsilon}\int_0^{t_0} v(\cdot\,,s)\,ds\right\}v(\cdot\,,t_0)\right\|_{\mathcal{M}(\Omega)}\le\frac{C}{\epsilon^2}\|u_0\|_{\mathcal{M}(\Omega)}|h|^2.$$

This proves that f is differentiable and satisfies the first equation of problem (5-9). Since $f(0)=[u_{0s}]_{d,2}$, f is a solution of the problem.

Let us show that no other solutions exist, so that equality (5-10) holds. In fact, if f_1 and f_2 both solve problem (5-9), plainly we obtain

$$\|f_1(t)-f_2(t)\|_{\mathcal{M}(\Omega)}\le\frac{\varphi(\alpha)}{\epsilon}\int_0^t\|f_1(s)-f_2(s)\|_{\mathcal{M}(\Omega)}\,ds\quad\text{for any}\ t\in(0,T),$$

whence $f_1=f_2$ in $(0,T)$ by Gronwall's inequality. This proves the claim, and Proposition 3.12 follows. □

Proof of Proposition 3.15. Writing $v=v_+-v_-$ and choosing $\rho=v_-$ in (3-16), we get

$$-\int_\Omega|\nabla v_-|^2\,dx-\langle\mu,v_-^2\rangle_\Omega\ge 0,$$

whence $v=v_+\ge 0$ a.e. in Ω. Therefore the function $1/(v+\delta)$ belongs to $H^1(\Omega)\cap L^\infty(\Omega)$ and we can choose in (3-16) $\rho=\chi^2/(v+\delta)$ for any $\chi\in C_c^\infty(\Omega)$ and $\delta>0$, thus obtaining

$$-\int_\Omega\nabla v\cdot\nabla\left(\frac{\chi^2}{v+\delta}\right)dx\le\left\langle\mu,\frac{v}{v+\delta}\chi^2\right\rangle_\Omega.\tag{5-11}$$

Integrating by parts, we plainly get

$$\begin{aligned}\int_\Omega \nabla v\cdot\nabla\Big(\frac{\chi^2}{v+\delta}\Big)\,dx &= -\int_\Omega \frac{|\nabla v|^2}{(v+\delta)^2}\chi^2\,dx + 2\int_\Omega \frac{\chi\nabla\chi\cdot\nabla v}{v+\delta}\,dx\\ &\le -\frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{(v+\delta)^2}\chi^2\,dx + 2\int_\Omega |\nabla\chi|^2\,dx.\end{aligned} \tag{5-12}$$

Since

$$\frac{\nabla v}{v+\delta} = \nabla\Big[\log\Big(1+\frac{v}{\delta}\Big)\Big],$$

by (5-11)–(5-12) we have

$$\tfrac{1}{2}\int_\Omega \Big|\nabla\Big[\log\Big(1+\frac{v}{\delta}\Big)\Big]\Big|^2\chi^2\,dx \le \langle\mu,\chi^2\rangle_\Omega + 2\int_\Omega|\nabla\chi|^2\,dx.$$

Then, arguing as in the proof of [Brezis and Ponce 2003, Theorem 1], the conclusion follows. □

Proof of Proposition 3.16. Since $N=1$, for a.e. $t\in(0,T)$ $v(\cdot,t)\in C(\overline{\Omega})$ and every singleton $E=\{x_0\}$ ($x_0\in\Omega$) has positive C_2-capacity. The conclusion follows by Proposition 3.15. □

6. The approximating problems

Lemma 6.1. *Let $u_0\in\mathcal{M}^+(\Omega)$,*

$$u_0 = u_{0ac} + [u_{0s}]_{d,2} + [u_0]_{c,2} = u_{0ac} + u_{0s},$$

and let u_{0r} denote the density of the absolutely continuous part u_{0ac}. Then there exist sequences $\{u_{0rn}\}$, $\{([u_{0s}]_{d,2})_n\}$ $\{([u_0]_{c,2})_n\}\subseteq C_c^\infty(\Omega)$ of nonnegative functions such that

$$\|u_{0rn}\|_{L^1(\Omega)} \le \|u_{0r}\|_{L^1(\Omega)}; \tag{6-1}$$

$$\|([u_{0s}]_{d,2})_n\|_{L^1(\Omega)} \le \|[u_{0s}]_{d,2}\|_{\mathcal{M}(\Omega)}, \quad \|([u_0]_{c,2})_n\|_{L^1(\Omega)} \le \|[u_0]_{c,2}\|_{\mathcal{M}(\Omega)}; \tag{6-2}$$

$$u_{0rn}\to u_{0r} \text{ in } L^1(\Omega); \tag{6-3}$$

$$([u_{0s}]_{d,2})_n \overset{*}{\rightharpoonup} [u_{0s}]_{d,2}, \quad ([u_0]_{c,2})_n \overset{*}{\rightharpoonup} [u_0]_{c,2}, \quad u_{0sn} \overset{*}{\rightharpoonup} u_{0s} \text{ in } \mathcal{M}(\Omega), \tag{6-4}$$

$$u_{0n}\to u_{0r} \text{ a.e. in } \Omega, \quad u_{0n} \overset{*}{\rightharpoonup} u_0 \text{ in } \mathcal{M}(\Omega), \tag{6-5}$$

where $u_{0sn} := ([u_{0s}]_{d,2})_n + ([u_0]_{c,2})_n$, $u_{0n} := u_{0rn} + u_{0sn}$. In addition, there exists $C>0$ such that

$$\|u_{0n}\|_{L^\infty(\Omega)} \le C\sqrt{n} \quad \text{for all } n. \tag{6-6}$$

Proof. Define $\tilde u_0\in\mathcal{M}^+(\mathbb{R}^N)$ by setting $\tilde u_0 := \tilde u_{0r} + \tilde u_{0s}$, where

$$\tilde u_{0r}(x) := \begin{cases} u_{0r}(x) & \text{if } x\in\Omega,\\ 0 & \text{otherwise}\end{cases}$$

and

$$[\tilde u_{0s}]_{d,2}(E) := [u_{0s}]_{d,2}(\Omega\cap E), \quad [\tilde u_0]_{c,2}(E) := [u_0]_{c,2}(\Omega\cap E), \quad \tilde u_{0s}(E) := [\tilde u_{0s}]_{d,2}(E) + [\tilde u_0]_{c,2}(E)$$

for every Borel set $E \subseteq \mathbb{R}^N$. Observe that by definition

$$\tilde{u}_0 = \tilde{u}_0 \llcorner \Omega, \quad \tilde{u}_0(E) = u_0(E) \quad \text{for every Borel set } E \subseteq \Omega.$$

Hence, if $\rho \in C_c(\Omega)$ and $\tilde{\rho} \in C_c(\mathbb{R}^N)$ denotes its trivial extension to $\mathbb{R}^N$, we get

$$\langle \tilde{u}_0, \tilde{\rho} \rangle_{\mathbb{R}^N} = \langle u_0, \rho \rangle_\Omega.$$

Consider the sequence $\{\tilde{u}_{0n}\} \subset C_c^\infty(\mathbb{R}^N)$ where

$$\tilde{u}_{0n} := \tilde{u}_0 * j_n,$$

$\{j_n\} \subseteq C_c^\infty(\mathbb{R}^N)$ being a regularizing sequence. We also define

$$\tilde{u}_{0rn} := \tilde{u}_{0r} * j_n, \quad ([\tilde{u}_{0s}]_{d,2})_n := ([\tilde{u}_{0s}]_{d,2}) * j_n, \quad ([\tilde{u}_0]_{c,2})_n := ([\tilde{u}_0]_{c,2}) * j_n, \quad \tilde{u}_{0sn} := \tilde{u}_{0s} * j_n$$

with j_n as above. To be specific, we choose

$$j_n(x) = \frac{n^N}{\int_{\mathbb{R}^N} j(x)\,dx} \zeta(nx) \qquad (x \in \mathbb{R}^N),$$

where $j \in C_c^\infty(\mathbb{R}^N)$, $j(x) = j(|x|)$ is a standard mollifier.

Next, choose any sequence $\{\eta_n\} \subseteq C_c^\infty(\mathbb{R}^N)$ such that $\eta_n \in C_c^\infty(\Omega_{n+1})$, $0 \le \eta_n \le 1$, $\eta_n = 1$ in $\overline{\Omega}_n$; here Ω_n is open, $\overline{\Omega}_n \subset \Omega_{n+1} \subset \Omega$ for every $n \in \mathbb{N}$ and $\bigcup_{n=1}^\infty \Omega_n = \Omega$. Finally, set

$$u_{0rn} := \tilde{u}_{0rn}\eta_n, \quad ([u_{0s}]_{d,2})_n := ([\tilde{u}_{0s}]_{d,2})_n\eta_n, \quad ([u_0]_{c,2})_n := ([\tilde{u}_0]_{c,2})_n\eta_n, \quad u_{0sn} := \tilde{u}_{0sn}\eta_n.$$

It is easily checked that the sequences $\{u_{0rn}\}$, $\{([u_{0s}]_{d,2})_n\}$ $\{([u_0]_{c,2})_n\}$, $\{u_{0sn}\}$, and $\{u_{0n}\}$ have the asserted properties. □

Definition 6.2. A nonnegative function $u_n \in C^1([0, T]; C(\overline{\Omega}))$ is called a *solution* of problem (P_n) if the function v_n defined by (3-17) belongs to $C([0, T]; C_0(\overline{\Omega}) \cap H^{2,p}(\Omega))$ for all $p \in [1, \infty)$, $\Delta v_n \in C(\overline{Q})$, and the pair (u_n, v_n) satisfies (P_n) in the strong sense.

Remark 6.3. If u is a solution of problem (P_n), then $v \in C(\overline{Q})$ and $v_{x_i} \in C(\overline{Q})$ for $i \in \{1, \dots, N\}$. Moreover, v admits second order weak derivatives $v_{x_i x_j} \in L^p(Q)$ for all $p \in [1, \infty)$, and for every $t \in [0, T]$

$$v_{x_i x_j}(\cdot, t) = [v(\cdot, t)]_{x_i x_j} \quad \text{a.e. in } \Omega.$$

We omit the proof of the following result, as it is almost identical to those of [Bertsch et al. ≥ 2013, Theorems 4.1–4.2].

Theorem 6.4. *Let $\varphi \in C^\infty([0, \infty))$ satisfy* (1-3)–(1-4). *Then, for any $n \in \mathbb{N}$, problem (P_n) has a unique solution $u_n \ge 0$, and*

$$u_n = [\psi_n(u_n)]_t = 0 \quad \text{on } \partial\Omega \times [0, T].$$

The function $v_n(\cdot\,,t)$ *defined by* (3-18) *satisfies, for a.e.* $t \in (0,T)$,

$$\begin{cases} --\epsilon\Delta[v_n(\cdot\,,t)] + \dfrac{v_n(\cdot\,,t)}{\psi_n'(u_n(\cdot\,,t))} = \dfrac{\varphi(u_n(\cdot\,,t))}{\psi_n'(u_n(\cdot\,,t))} & \textit{in } \Omega, \\ v_n(\cdot\,,t) = 0 & \textit{on } \partial\Omega, \end{cases} \tag{6-7}$$

$$0 \le v_n(\cdot\,,t) \le \varphi(\alpha) \quad \textit{in } \Omega, \tag{6-8}$$

$$\frac{\partial v_n}{\partial \nu}(\cdot\,,t) \le 0 \quad \textit{on } \partial\Omega, \tag{6-9}$$

where $\partial/(\partial\nu)$ *denotes the outer derivative at* $\partial\Omega$.

In addition, $v_n \in C^1(\bar{Q}_T)$, $v_{nt} \in C([0,T]; C_0(\bar{\Omega}) \cap H^{2,p}(\Omega))$ *for* $p \in [1,\infty)$ *and, for a.e* $t \in (0,T)$, $v_{nt}(\cdot\,,t)$ *satisfies*

$$\begin{cases} -\varepsilon\Delta[v_{nt}(\cdot\,,t)] + \dfrac{v_{nt}(\cdot\,,t)}{\psi_n'(u_n(\cdot\,,t))} = \left[\dfrac{\varphi'(u_n)u_{nt} + \varepsilon\psi_n''(u_n)u_{nt}^2}{\psi_n'(u_n)}\right](\cdot\,,t) & \textit{in } \Omega, \\ v_{nt}(\cdot\,,t) = 0 & \textit{on } \partial\Omega. \end{cases}$$

The following result is analogous to [Bertsch et al. $\geq$ 2013, Proposition 4.3]. The proof is omitted.

Proposition 6.5. *Let* u_n *be the solution of problem* (P_n), *let* $g \in C^1([0,\varphi(\alpha)])$ *with* $g' \ge 0$, *and let* G *be defined by* (3-5). *Then, for any* $\zeta \in C^1([0,T]; C_c^1(\Omega))$, $\zeta \ge 0$ *and for any* $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned} \int_\Omega G(u_n(x,t_2))\zeta(x,t_2)\,dx - \int_\Omega G(u_n(x,t_1))\zeta(x,t_1)\,dx \\ \le \int_{t_1}^{t_2}\int_\Omega \{G(u_n)\zeta_t - g(v_n)\nabla v_n\nabla\zeta - g'(v_n)|\nabla v_n|^2\zeta\}\,dx\,dt. \end{aligned} \tag{6-10}$$

Next, the following a priori estimates hold.

Proposition 6.6. *Let* $\varphi \in C^\infty([0,\infty))$ *satisfy* (1-3)–(1-5). *Let* u_n *be the solution of problem* (P_n). *Then*

$$\|u_n\|_{L^\infty((0,T);L^1(\Omega))} \le \|u_0\|_{\mathcal{M}(\Omega)}, \tag{6-11}$$

$$\|[\psi_n(u_n)]_t\|_{L^\infty(Q)} \le \frac{\varphi(\alpha)}{\varepsilon}. \tag{6-12}$$

Moreover, there exists $C > 0$ *such that, for any* $n \in \mathbb{N}$,

$$\|v_n\|_{L^\infty((0,T);H_0^1(\Omega))} \le C, \tag{6-13}$$

$$\|v_{nt}\|_{L^\infty((0,T);L^1(\Omega))} \le C, \tag{6-14}$$

$$\|\Delta v_n\|_{L^\infty((0,T);L^1(\Omega))} \le C. \tag{6-15}$$

For the proofs of inequalities (6-11)–(6-14) we refer the reader to those of the analogous statements in [Bertsch et al. $\geq$ 2013, Proposition 5.1]. Let us only mention that in the proof of (6-13)–(6-14) we use the inequalities

$$\frac{\varphi(u_n)v_n}{\psi_n'(u_n)} \le [\varphi(\alpha)]^2(1+u_n)$$

and

$$\frac{|\psi_n''(u)|}{[\psi'(u)]^3} \le (1+u) \quad \text{for any } u \ge 0,$$

respectively.

Concerning inequality (6-15), observe that by, (6-7)–(6-8), we have

$$\varepsilon \int_\Omega |\Delta v_n|\, dx \le \int_\Omega \frac{|v_n - \varphi(u_n)|}{\psi_n'(u_n)}\, dx \le \varphi(\alpha) \int_\Omega [1+u_n]\, dx$$

for all $t \in (0, T)$. Then (6-15) follows from (6-11).

Finally, let us show that, for every $t \in (0, T)$, the sequence $\{1 + u_n(\cdot\,, t)\}$ satisfies an inequality analogous to (3-12).

Proposition 6.7. *Let $\varphi \in C^\infty([0,\infty))$ satisfy* (1-3)–(1-4). *Let u_n be the solution of problem (P_n). Then, for any $t \in (0, T)$ and $\rho \in C_c(\Omega)$, $\rho \ge 0$,*

$$\int_\Omega [1+u_n(x,t)]\rho(x)\, dx \le \exp\left\{\frac{\varphi(\alpha)t}{\varepsilon}\right\} \int_\Omega [1+u_{0n}(x)]\rho(x)\, dx. \tag{6-16}$$

Proof. From (3-18) we obtain

$$\varepsilon u_{nt} = \frac{v_n - \varphi(u_n)}{\psi_n'(u_n)}.$$

Integrating the above equality in $(0, t)$ and using inequality (6-8), we obtain, for every $x \in \Omega$,

$$\varepsilon[1+u_n(x,t)] - \varepsilon[1+u_{0n}(x)] \le \varphi(\alpha) \int_0^t [1+u_n(x,s)]\, ds.$$

Then, by Gronwall's inequality,

$$1+u_n(x,t) \le [1+u_{0n}(x)] \exp\left\{\frac{\varphi(\alpha)t}{\varepsilon}\right\} \qquad (t \in (0,T)) \tag{6-17}$$

for every $x \in \Omega$, which implies (6-16). □

7. Proof of existence results

To prove Theorem 3.17 we need some preliminary results concerning convergence of solutions of the sequences $\{u_n\}$, $\{v_n\}$. From the estimates in Proposition 6.6 we obtain the following.

Proposition 7.1. *Let $\varphi \in C^\infty([0,\infty))$ satisfy* (1-3)–(1-5). *Let u_n be the solution of problem (P_n) and let v_n be defined by* (3-18). *Then there exist $u \in L^\infty((0,T); \mathcal{M}^+(\Omega))$, $v \in L^\infty((0,T); H_0^1(\Omega)) \cap BV(Q)$*

with $\Delta v \in L^\infty((0,T);\mathcal{M}(\Omega))$, *and subsequences* $\{u_{n_k}\}$, $\{v_{n_k}\}$ *such that*

$$u_{n_k}(\cdot,t) \overset{*}{\rightharpoonup} u(\cdot,t) \quad \text{in } \mathcal{M}(\Omega), \tag{7-1}$$

$$v_{n_k} \to v \quad \text{a.e. in } Q, \tag{7-2}$$

$$\Delta v_{n_k} \overset{*}{\rightharpoonup} \Delta v \quad \text{in } \mathcal{M}(Q), \tag{7-3}$$

$$v_{n_k} \rightharpoonup v \quad \text{in } L^p((0,T);H_0^1(\Omega)) \qquad (p \in [1,\infty)), \tag{7-4}$$

$$v_n(\cdot,t) \rightharpoonup v(\cdot,t) \quad \text{in } H_0^1(\Omega) \tag{7-5}$$

for a.e. $t \in (0,T)$. *In addition*,

$$\|u\|_{L^\infty((0,T);\mathcal{M}(\Omega))} \le \|u_0\|_{\mathcal{M}(\Omega)} \tag{7-6}$$

and v *satisfies inequality* (3-1).

Proof. The convergence in (7-1) and inequality (7-6) are proven as in [Bertsch et al. ≥ 2013, Proposition 5.3]. The convergence in (7-2)–(7-4) and inequality (3-1) follow from (6-13)–(6-15) and (6-8).

To prove the convergence in (7-5), observe that, by (7-2),

$$v_{n_k}(\cdot,t) \to v(\cdot,t) \quad \text{a.e. in } \Omega$$

for a.e. $t \in (0,T)$. Hence, by inequality (6-8) and the dominated convergence theorem,

$$v_{n_k}(\cdot,t) \to v(\cdot,t) \quad \text{in } L^1(\Omega),$$

On the other hand, by inequality (6-13), the sequence $\{v_n(\cdot,t)\}$ is contained in a weakly compact subset of $H_0^1(\Omega)$ for a.e. $t \in (0,T)$; hence the conclusion follows. □

The sequence $\{u_{n_k}\}$ converges a.e. in Q to the density u_r of u_{ac}.

Proposition 7.2. *Let* $\varphi \in C^\infty([0,\infty))$ *satisfy* (1-3)–(1-5). *Let* $\{u_{n_k}\}$, u, *and* v *be as in Proposition 7.1, and let* $u_r \in L^1(Q)$ *be the density of the absolutely continuous part of* u. *Then*

$$u_{n_k} \to u_r \quad \text{a.e. in } Q, \tag{7-7}$$

$$[\psi(u_r)]_t \in L^\infty(Q), \quad u_{rt} \in L^1(Q), \tag{7-8}$$

$$[\psi_{n_k}(u_{n_k})]_t \overset{*}{\rightharpoonup} [\psi(u_r)]_t \quad \text{in } L^\infty(Q). \tag{7-9}$$

Moreover,

(i) *we have*

$$v = \varphi(u_r) + \varepsilon[\psi(u_r)]_t \quad \text{a.e. in } Q, \tag{7-10}$$

$$\|[\psi(u_r)]_t\|_{L^\infty(Q)} \le \frac{\varphi(\alpha)}{\varepsilon}; \tag{7-11}$$

(ii) $u_r(\cdot,t)$, $u_s(\cdot,t)$, $u(\cdot,t)$ *satisfy inequalities* (3-12), (3-14), (3-15), *respectively, for a.e.* $t \in (0,T)$ *and for any* $\rho \in C_c(\Omega)$, $\rho \ge 0$.

Proof. Arguing as in [Bertsch et al. $\geq$ 2013, Proposition 5.4], it can be proven that $u_{n_k} \to z$ a.e. in Q for some $z \in L^1(Q)$, $z \geq 0$. Let us show that

$$z = u_r \quad \text{a.e. in } Q. \tag{7-12}$$

For a.e. $t \in (0, T)$, we can assume without loss of generality that

$$u_{n_k}(\,\cdot\,, t) \to z(\,\cdot\,, t) \quad \text{a.e. in } \Omega \tag{7-13}$$

and the convergence in (7-1) holds. As in the proof of [Bertsch et al. $\geq$ 2013, Proposition 5.5], there exist a subsequence $\{u_{n_{k_j}}(\,\cdot\,, t)\}$ (possibly depending on t) and a sequence of subsets $\{A_j\}$, with $A_{j+1} \subseteq A_j \subseteq \Omega$ for any j and $|A_j| \to 0$, such that the family $\{u_{n_{k_j}}(\,\cdot\,, t)\chi_{\Omega\setminus A_j}\}$ is uniformly integrable in Ω and

$$u_{n_{k_j}}(\,\cdot\,, t)\chi_{\Omega\setminus A_j} \rightharpoonup z(\,\cdot\,, t) \quad \text{in } L^1(\Omega).$$

For example, see [Valadier 1994]. Then, by (7-1), we have

$$u_{n_{k_j}}(\,\cdot\,, t)\chi_{A_j} \overset{*}{\rightharpoonup} u(\,\cdot\,, t) - z(\,\cdot\,, t) =: \mu(\,\cdot\,, t) \quad \text{in } \mathcal{M}(\Omega). \tag{7-14}$$

Since $u_{n_{k_j}}(\,\cdot\,, t)\chi_{A_j} \geq 0$ in Ω for every j, the measure $\mu(\,\cdot\,, t)$ is nonnegative.

By (6-16), for every $\rho \in C_c(\Omega)$, $\rho \geq 0$, we get

$$\begin{aligned}\int_{A_j} u_{n_{k_j}}(x, t)\rho(x)\,dx &\leq \int_{A_j} [1 + u_{n_{k_j}}(x, t)]\rho(x)\,dx \leq \exp\left\{\frac{\varphi(\alpha)t}{\varepsilon}\right\} \int_{A_j} [1 + u_{0n_{k_j}}(x)]\rho(x)\,dx \\ &\leq \exp\left\{\frac{\varphi(\alpha)t}{\varepsilon}\right\}\left\{\int_{A_j} [1 + u_{0rn_{k_j}}(x)]\rho(x)\,dx + \int_\Omega u_{0sn_{k_j}}(x)\rho(x)\,dx\right\}.\end{aligned} \tag{7-15}$$

Since $u_{0rn_{k_j}} \to u_{0r}$ in $L^1(\Omega)$, $|A_j| \to 0$, and $u_{0sn_{k_j}} \overset{*}{\rightharpoonup} u_{0s}$ in $\mathcal{M}(\Omega)$ as $j \to \infty$,

$$\lim_{j\to\infty}\left\{\int_{A_j} [1 + u_{0rn_{k_j}}(x)]\rho(x)\,dx + \int_\Omega u_{0sn_{k_j}}(x)\rho(x)\,dx\right\} = \langle u_{0s}, \rho\rangle_\Omega.$$

Then, letting $j \to \infty$ in (7-15) and using (7-14), we have

$$\langle \mu(\,\cdot\,, t), \rho\rangle_\Omega \leq \exp\left\{\frac{\varphi(\alpha)t}{\varepsilon}\right\}\langle u_{0s}, \rho\rangle_\Omega \tag{7-16}$$

for every ρ, as above.

Since $\mu(\,\cdot\,, t)$ is nonnegative, by (7-16) it is absolutely continuous with respect to u_{0s}, thus singular with respect to the Lebesgue measure over Ω. Therefore, since $z(\,\cdot\,, t) \in L^1(\Omega)$ and $u(\,\cdot\,, t) = z(\,\cdot\,, t) + \mu(\,\cdot\,, t)$ by definition, the uniqueness of the Lebesgue decomposition of $u(\,\cdot\,, t)$ ensures that

$$z(\,\cdot\,, t) = [u(\,\cdot\,, t)]_r = [u_r(\,\cdot\,, t)], \quad \mu(\,\cdot\,, t) = [u(\,\cdot\,, t)]_s = [u_s(\,\cdot\,, t)], \tag{7-17}$$

(see (2-4)–(2-5)). This proves (7-12), whence (7-7) follows. By the same token, inequality (7-16) and the second equality in (7-17) show that $u_s(\,\cdot\,, t)$ satisfies inequality (3-14).

Let us prove the remaining claims. By inequality (6-11) and the convergence in (7-7), we have

$$\psi_{n_k}(u_{n_k}) \to \psi(u_r) \quad \text{in } L^1(Q). \tag{7-18}$$

Then $[\psi(u_r)]_t \in L^\infty(Q)$, by (7-18) and inequality (6-12). The convergence in (7-9) follows. Inequality (7-11) follows by (6-12), (7-9), and the lower semicontinuity of the norm. By the continuity of φ, from (7-7) and the results in Proposition 7.1, we obtain equality (7-10). On the other hand, the fact that $u_{rt} \in L^1(Q)$ follows as in the proof of Proposition 3.12.

Finally, arguing as in the proof of Proposition 3.12, from equality (5-6), we obtain that $u_r(\cdot, t)$ satisfies inequality (3-12). As a consequence of (3-12) and (3-14), $u(\cdot, t)$ satisfies (3-15). This completes the proof. □

The proof of the following result is the same as that of [Bertsch et al. ≥ 2013, Proposition 5.6], hence we omit it.

Proposition 7.3. *Let $\varphi \in C^\infty([0, \infty))$ satisfy* (1-3)–(1-5). *The pair (u, v) defined by Proposition 7.1 satisfies the entropy inequality* (3-6).

Proof of Theorem 3.17. Let u and v be defined by Proposition 7.1. Then $u \in L^\infty((0, T); \mathcal{M}^+(\Omega))$, $v \in L^\infty((0, T); H^1_0(\Omega))$, and $\Delta v \in L^\infty((0, T); \mathcal{M}(\Omega))$. Moreover, $[\psi(u_r)]_t \in L^\infty(Q)$ by (7-11), equality (7-10) holds, and inequality (3-1) is satisfied.

By (6-5), (6-11), (7-1), (7-3), and the dominated convergence theorem, letting $n \to \infty$ in the weak formulation of (P_n) shows that the limiting measure u satisfies equality (3-2) for any $\zeta \in C^1([0, T]; C_c(\Omega))$. The other claims follow by Propositions 7.1–7.2. This completes the proof. □

8. Proof of Theorem 3.18

Let us first prove Theorem 3.18 when $N = 1$. This is the content of the following proposition.

Proposition 8.1. *Let $N = 1$, $u_0 \in \mathcal{M}^+(\Omega)$, and let $\varphi \in C^\infty([0, \infty))$ satisfy* (1-3)–(1-5). *Let u be the entropy solution of problem* (1-1) *given in Theorem 3.17 and v the chemical potential defined in* (1-7). *Then the pair (u, v) satisfies problem* (1-18).

Proof. Fix any $t \in (0, T)$ such that

$$\begin{aligned} u_{n_k}(\cdot, t) &\overset{*}{\rightharpoonup} u(\cdot, t) && \text{in } \mathcal{M}(\Omega),\\ u_{n_k}(\cdot, t) &\to u_r(\cdot, t) && \text{a.e. in } \Omega,\\ v_{n_k}(\cdot, t) &\rightharpoonup v(\cdot, t) && \text{in } H^1_0(\Omega)\end{aligned}$$

(see (7-1), (7-5), and (7-12)–(7-13)). By inequality (6-13) we can also assume

$$v_{n_k}(\cdot, t) \to v(\cdot, t) \quad \text{in } C(\overline{\Omega}).$$

Given $\rho \in C_c^1(\Omega)$, let us study the limit as $k \to \infty$ of the weak formulation of (6-7) with $n = n_k$, namely,

$$\varepsilon \int_\Omega v_{n_k x}(x,t)\rho_x(x)\,dx + \int_\Omega \frac{v_{n_k}(x,t)}{\psi'_{n_k}(u_{n_k}(x,t))}\rho(x)\,dx = \int_\Omega \frac{\varphi(u_{n_k}(x,t))}{\psi'_{n_k}(u_{n_k}(x,t))}\rho(x)\,dx. \tag{8-1}$$

(i) Since $\varphi \in L^q([\alpha,\infty))$ (see (1-4)) and

$$\{(1+u)[\varphi(u)]^q\}' = [\varphi(u)]^q + q[(1+u)[\varphi(u)]^{q-1}]\varphi'(u) \le [\varphi(u)]^q \quad \text{for any } u \ge \alpha,$$

we have

$$(1+u)[\varphi(u)]^q \le (1+\alpha)[\varphi(\alpha)]^q + \int_\alpha^u [\varphi(u)]^q\,ds = (1+\alpha)[\varphi(\alpha)]^q + \|\varphi\|^q_{L^q(\mathbb{R}^+)} \quad \text{for any } u \ge \alpha,$$

whence we get

$$[\varphi(u)] \le C(1+u)^{-1/q} \quad \text{for any } u \ge 0,$$

for some constant $C > 0$. It follows that

$$\frac{\varphi(u_{n_k})}{\psi'_{n_k}(u_{n_k})} \le (1+u_{n_k})\varphi(u_{n_k}) \le C(1+u_{n_k})^{1-1/q} \quad \text{a.e. in } Q. \tag{8-2}$$

Then, for every Borel set $E \subseteq \Omega$ and for a.e. $t \in (0,T)$,

$$\int_E \frac{\varphi(u_{n_k}(x,t))}{\psi'_{n_k}(u_{n_k}(x,t))}\,dx \le C\int_E [1+u_{n_k}(x,t)]^{1-1/q}\,dx \le |E|^{1/q}\left(\int_E [1+u_{n_k}(x,t)]\,dx\right)^{1-1/q}. \tag{8-3}$$

Inequalities (6-11) and (8-3) imply that the sequence

$$\left\{\frac{\varphi(u_{n_k}(\cdot\,,t))}{\psi'_{n_k}(u_{n_k}(\cdot\,,t))}\right\}$$

is bounded in $L^1(\Omega)$ and uniformly integrable in Ω. As a consequence, there exists a subsequence, for simplicity, denoted again by

$$\left\{\frac{\varphi(u_{n_k}(\cdot\,,t))}{\psi'_{n_k}(u_{n_k}(\cdot\,,t))}\right\},$$

such that

$$\frac{\varphi(u_{n_k}(\cdot\,,t))}{\psi'_{n_k}(u_{n_k}(\cdot\,,t))} \rightharpoonup \frac{\varphi(u_r(\cdot\,,t))}{\psi'(u_r(\cdot\,,t))} \quad \text{in } L^1(\Omega). \tag{8-4}$$

(ii) By inequalities (6-6) and (6-17),

$$1+u_{n_k} \le \exp\left\{\frac{\varphi(\alpha)T}{\varepsilon}\right\}(1+\sqrt{n_k}) \quad \text{a.e. in } Q. \tag{8-5}$$

Observe that

$$\left|\frac{1}{\psi'_{n_k}(u)} - \frac{1}{\psi'(u)}\right| = \frac{1}{n_k}\left(\frac{1+u}{1/(1+u)+1/n_k}\right) \le \frac{(1+u)^2}{n_k}. \tag{8-6}$$

Then, by (6-11) and (8-5)–(8-6),

$$\left\| \frac{1}{\psi'_{n_k}(u_{n_k}(\cdot,t))} - \frac{1}{\psi'(u_{n_k}(\cdot,t))} \right\|_{L^1(\Omega)} \le \frac{2}{\sqrt{n_k}} \exp\left\{\frac{\varphi(\alpha)T}{\varepsilon}\right\} \int_\Omega [1+u_{n_k}(x,t)]\,dx$$
$$\le \frac{2}{\sqrt{n_k}} \exp\left\{\frac{\varphi(\alpha)T}{\varepsilon}\right\} [|\Omega| + \|u_0\|_{\mathcal{M}(\Omega)}] \to 0 \quad \text{as } k\to\infty. \tag{8-7}$$

Since $v_{n_k}(\cdot\,,t) \to v(\cdot\,,t)$ in $C(\overline{\Omega})$ and

$$\frac{1}{\psi'(u_{n_k}(\cdot\,,t))} = 1 + u_{n_k}(\cdot\,,t)) \overset{*}{\rightharpoonup} 1 + u(\cdot\,,t) \quad \text{in } \mathcal{M}(\Omega),$$

we have

$$\int_\Omega \frac{v_{n_k}(x,t)}{\psi'_{n_k}(u_{n_k}(x,t))} \rho(x)\,dx \to \langle [1+u(\cdot\,,t)],\, v(\cdot\,,t)\rho\rangle_\Omega. \tag{8-8}$$

Now let $k\to\infty$ in equality (8-1). By (7-5), (8-4), and (8-8), we obtain

$$\varepsilon \int_\Omega v_x(x,t)\rho_x(x)\,dx + \langle [1+u(\cdot\,,t)],\, \rho v(\cdot\,,t)\rangle_\Omega = \int_\Omega \frac{\varphi(u_r(x,t))}{\psi'(u_r(x,t))}\rho(x)\,dx.$$

Since by Definition 3.1, $v_{xx} \in L^\infty((0,T);\mathcal{M}(\Omega))$, this implies

$$-\varepsilon \langle v_{xx}(\cdot\,,t), \rho\rangle_\Omega + \langle [1+u(\cdot\,,t)],\, \rho v(\cdot\,,t)\rangle_\Omega = \int_\Omega \frac{\varphi(u_r(x,t))}{\psi'(u_r(x,t))}\rho(x)\,dx$$

for a.e. $t\in(0,T)$ and any $\rho \in C_c(\Omega)$. Hence the result follows. □

To complete the proof of Theorem 3.18, let us prove the following result.

Proposition 8.2. *Let $u_0 \in \mathcal{M}^+(\Omega)$, and let $\varphi \in C^\infty([0,\infty))$ satisfy* (1-3)–(1-5). *Let u be the entropy solution of problem* (1-1) *given in Theorem 3.17 and v the chemical potential defined in* (1-7). *Let $N \ge 2$, and let u_0 satisfy the following assumptions*:

(i) $[u_0]_{c,2}$ *is concentrated on some compact $K_0 \subset \Omega$ such that $C_2(K_0)=0$;*

(ii) $[u_0]_{d,2} \in \mathcal{M}^+_{d,p}(\Omega)$ *for some $p\in[1,2)$.*

Then the pair (u,v) satisfies problem (1-9).

The main step in the proof of Proposition 8.2 is given by the following lemma.

Lemma 8.3. *Let $\varphi \in C^\infty([0,\infty))$ satisfy* (1-3)–(1-5). *Let $\{u_{n_k}\}$, $\{v_{n_k}\}$ be the subsequences given by Proposition 7.1. Then, for every $\rho\in C^1_c(\Omega)$,*

$$\lim_{k\to\infty} \int_\Omega [1+u_{n_k}(x,t)]v_{n_k}(x,t)\rho(x)\,dx = \langle [1+u(\cdot\,,t)]_{d,2},\, v(\cdot\,,t)\rho\rangle_\Omega. \tag{8-9}$$

Proof of Proposition 8.2. Fix any $t\in(0,T)$ such that the convergence in (7-1) and (7-5) hold, namely,

$$\begin{aligned} u_{n_k}(\cdot\,,t) &\overset{*}{\rightharpoonup} u(\cdot\,,t) && \text{in } \mathcal{M}(\Omega),\\ v_{n_k}(\cdot\,,t) &\rightharpoonup v(\cdot\,,t) && \text{in } H^1_0(\Omega),\\ u_{n_k}(\cdot\,,t) &\to u_r(\cdot\,,t) && \text{a.e. in } \Omega \end{aligned}$$

(see (7-12)–(7-13)). Consider the weak formulation of (6-7) with $n = n_k$, namely,

$$\varepsilon \int_\Omega \nabla v_{n_k}(x,t) \cdot \nabla \rho(x)\,dx + \int_\Omega \frac{v_{n_k}(x,t)}{\psi'_{n_k}(u_{n_k}(x,t))}\rho(x)\,dx = \int_\Omega \frac{\varphi(u_{n_k}(x,t))}{\psi'_{n_k}(u_{n_k}(x,t))}\rho(x)\,dx \tag{8-10}$$

where $\rho \in C^1_c(\Omega)$. Arguing as in the proof of Proposition 8.1, it is easily seen that

$$\lim_{k\to\infty} \int_\Omega \nabla v_{n_k}(x,t) \cdot \nabla \rho(x)\,dx = \int_\Omega \nabla v(x,t) \cdot \nabla \rho(x)\,dx;$$
$$\lim_{k\to\infty} \int_\Omega \frac{\varphi(u_{n_k}(x,t))}{\psi'_{n_k}(u_{n_k}(x,t))}\rho(x)\,dx = \int_\Omega \frac{\varphi(u_r(x,t))}{\psi'(u_r(x,t))}\rho(x)\,dx;$$
$$\lim_{k\to\infty} \left\| \frac{1}{\psi'_{n_k}(u_{n_k}(\,\cdot\,,t))} - \frac{1}{\psi'(u_{n_k}(\,\cdot\,,t))} \right\|_{L^1(\Omega)} = 0.$$

thus

$$\lim_{k\to\infty} \int_\Omega \frac{v_{n_k}(x,t)}{\psi'_{n_k}(u_{n_k}(x,t))}\rho(x)\,dx = \lim_{k\to\infty} \int_\Omega \frac{v_{n_k}(x,t)}{\psi'(u_{n_k}(x,t))}\rho(x)\,dx$$

(here we use (6-8)). Then, by Lemma 8.3, the conclusion follows. □

The proof of Lemma 8.3, which was used in the proof of Proposition 8.2, requires a few intermediate steps. Let $K_0 \subset \Omega$, $C_2(K_0) = 0$, be a compact set where $[u_0]_{c,2}$ is concentrated. Then for every $\delta > 0$ there exists an open set $\Omega^c_\delta \subseteq \Omega$ such that

$$K_0 \subset \Omega^c_\delta, \quad C_2(\Omega^c_\delta) < \delta. \tag{8-11}$$

Set

$$\Omega^d_\delta := \Omega \setminus \Omega^c_\delta. \tag{8-12}$$

Moreover, observe that the convergence in (7-5) guarantees the existence of a compact set $E_\delta \subseteq \Omega^d_\delta$ such that

$$C_p(E^c_\delta) < \delta, \qquad \text{where } E^c_\delta := \Omega^d_\delta \setminus E_\delta \tag{8-13}$$

and $p \in [1, 2)$ is chosen so that $[u_0]_{d,2} \in \mathcal{M}^+_{d,p}(\Omega)$, and

$$v_{n_k}(\,\cdot\,,t) \to v(\,\cdot\,,t) \quad \text{uniformly in } E_\delta. \tag{8-14}$$

By (8-12) and the definition in (8-13), we have the disjoint union

$$\Omega = \Omega^c_\delta \cup E^c_\delta \cup E_\delta.$$

Therefore

$$\int_\Omega [1+u_{n_k}(x,t)]v_{n_k}(x,t)\rho(x)\,dx = \int_{\Omega^c_\delta} [1+u_{n_k}(x,t)]v_{n_k}(x,t)\rho(x)\,dx + \int_{E^c_\delta} [1+u_{n_k}(x,t)]v_{n_k}(x,t)\rho(x)\,dx$$
$$+ \int_{E_\delta} [1+u_{n_k}(x,t)]v_{n_k}(x,t)\rho(x)\,dx. \tag{8-15}$$

Concerning the first two integrals in the right-hand side of (8-15), we have the following two lemmata, whose proofs will be given at the end of this section.

Lemma 8.4. *Let* $\Omega_\delta^c \subseteq \Omega$ *be the set in* (8-11), *and* $\rho \in C_c^1(\Omega)$. *Then there exists a function*

$$f_1 = f_1(\delta) \geq 0$$

with $f_1(\delta) \to 0$ *as* $\delta \to 0$, *such that*

$$\limsup_{k\to\infty} \int_{\Omega_\delta^c} [1 + u_{n_k}(x,t)] v_{n_k}(x,t) |\rho(x)|\, dx \leq f_1(\delta). \tag{8-16}$$

Lemma 8.5. *Let* E_δ^c *be the set in* (8-13), *and* $\rho \in C_c^1(\Omega)$. *Then there exists a function* $f_2 = f_2(\delta) \geq 0$, $f_2(\delta) \to 0$ *as* $\delta \to 0$, *such that*

$$\limsup_{k\to\infty} \int_{E_\delta^c} [1 + u_{n_k}(x,t)] v_{n_k}(x,t) |\rho(x)|\, dx \leq f_2(\delta). \tag{8-17}$$

We also prove the following result.

Lemma 8.6. *Let* $\rho \in C_c^1(\Omega)$ *and let* $\phi_\delta \in C_c^\infty(\Omega)$ *such that*

$$\begin{cases} 0 \leq \phi_\delta \leq 1 & \text{a.e. in } \Omega, \\ \phi_\delta = 1 & \text{a.e. in } E_\delta, \\ \operatorname{dist}(K_0, \operatorname{supp} \phi_\delta) > 0. \end{cases} \tag{8-18}$$

Then there exists a function $f_3 = f_3(\delta) \geq 0$, $f_3(\delta) \to 0$ *as* $\delta \to 0$, *such that*

$$\limsup_{k\to\infty} \int_{\Omega_\delta^c \cup E_\delta^c} [1 + u_{n_k}(x,t)] v(x,t) \phi_\delta(x) |\rho(x)|\, dx \leq f_3(\delta), \tag{8-19}$$

Relying on the above results we can prove Lemma 8.3.

Proof of Lemma 8.3. For every $k \in \mathbb{N}$ we have

$$\begin{aligned} \biggl| \int_\Omega & [1 + u_{n_k}(x,t)] v_{n_k}(x,t) \rho(x)\, dx - \langle [1 + u(\,\cdot\,,t)]_{d,2}, v(\,\cdot\,,t)\rho \rangle_\Omega \biggr| \\ &\leq \biggl| \int_{E_\delta} [1 + u_{n_k}(x,t)] v_{n_k}(x,t) \rho(x)\, dx - \langle [1 + u(\,\cdot\,,t)]_{d,2}, v(\,\cdot\,,t)\rho \rangle_\Omega \biggr| \\ &\qquad\qquad + \int_{\Omega_\delta^c \cup E_\delta^c} [1 + u_{n_k}(x,t)] v_{n_k}(x,t) |\rho(x)|\, dx \\ &\leq \int_{E_\delta} [1 + u_{n_k}(x,t)] |v_{n_k}(x,t) - v(x,t)| |\rho(x)|\, dx \\ &\qquad + \biggl| \int_\Omega [1 + u_{n_k}(x,t)] v(x,t) \phi_\delta(x) \rho(x)\, dx - \langle [1 + u(\,\cdot\,,t)]_{d,2}, v(\,\cdot\,,t)\phi_\delta \rho \rangle_\Omega \biggr| \\ &\qquad\qquad + \int_{\Omega_\delta^c \cup E_\delta^c} [1 + u_{n_k}(x,t)] [v_{n_k}(x,t) + v(x,t)\phi_\delta(x)] |\rho(x)|\, dx \\ &\qquad\qquad + |\langle [1 + u(\,\cdot\,,t)]_{d,2}, (1 - \phi_\delta) v(\,\cdot\,,t) |\rho| \rangle_\Omega|; \end{aligned} \tag{8-20}$$

here we have used the equality (recall that $\phi_\delta = 1$ a.e. in E_δ)

$$\int_{E_\delta}[1+u_{n_k}(x,t)]v_{n_k}(x,t)\rho(x)\,dx=\int_{E_\delta}[1+u_{n_k}(x,t)]v_{n_k}(x,t)\phi_\delta(x)\rho(x)\,dx$$
$$=\int_\Omega[1+u_{n_k}(x,t)]v_{n_k}(x,t)\phi_\delta(x)\rho(x)\,dx-\int_{\Omega_\delta^c\cup E_\delta^c}[1+u_{n_k}(x,t)]v_{n_k}(x,t)\phi_\delta(x)\rho(x)\,dx.$$

By (6-11) and (8-14), we have

$$\lim_{k\to\infty}\int_{E_\delta}[1+u_{n_k}(x,t)]|v_{n_k}(x,t)-v(x,t)|\rho(x)\,dx=0;$$

while by (8-16)–(8-19),

$$\limsup_{k\to\infty}\int_{\Omega_\delta^c\cup E_\delta^c}[1+u_{n_k}(x,t)][v_{n_k}(x,t)+v(x,t)\phi_\delta(x)]|\rho(x)|\,dx\le f_1(\delta)+f_2(\delta)+f_3(\delta).$$

Moreover, observe that, by (8-11) and (8-13),

$$C_p(\Omega_c^\delta\cup E_\delta^c)\le C_p(\Omega_c^\delta)+C_p(E_\delta^c)\le AC_2(\Omega_\delta^c)+C_p(E_\delta^c)<(A+1)\delta \tag{8-21}$$

for some constant $A>0$ (here we used the condition $p<2$). Since the support of the function $(1-\phi_\delta)$ is contained in the set $\Omega_c^\delta\cup E_\delta^c$, by (8-21) and the assumption $[u_0]_{d,2}\in\mathcal{M}^+_{d,p}(\Omega)$, there exists a function $f_4=f_4(\delta)\ge 0$, $f_4(\delta)\to 0$ as $\delta\to 0$, such that

$$\bigl|\bigl\langle[1+u(\cdot,t)]_{d,2},(1-\phi_\delta)v(\cdot,t)|\rho|\bigr\rangle_\Omega\bigr|\le f_4(\delta). \tag{8-22}$$

In addition, we prove that

$$\lim_{k\to\infty}\int_\Omega[1+u_{n_k}(x,t)]v(x,t)\phi_\delta(x)\rho(x)\,dx=\langle[1+u(\cdot,t)]_{d,2},v(\cdot,t)\phi_\delta\rho\rangle_\Omega. \tag{8-23}$$

Then, from (8-20), we obtain

$$\limsup_{k\to\infty}\left|\int_\Omega[1+u_{n_k}(x,t)]v_{n_k}(x,t)\rho(x)\,dx-\langle[1+u(\cdot,t)]_{d,2},v(\cdot,t)\rho\rangle_\Omega\right|$$
$$\le f_1(\delta)+f_2(\delta)+f_3(\delta)+f_4(\delta)\qquad\text{for any }\delta>0. \tag{8-24}$$

By the arbitrariness of δ the conclusion follows.

It remains to prove equality (8-23). By the weak formulation of (P_n), we have

$$\int_\Omega u_{n_k}(x,t)v(x,t)\phi_\delta(x)\rho(x)\,dx$$
$$=-\int_0^t\int_\Omega\nabla v_{n_k}(x,s)\cdot\nabla[v(x,t)\phi_\delta(x)\rho(x)]\,dx\,ds+\int_\Omega u_{0n_k}(x)v(x,t)\phi_\delta(x)\rho(x)\,dx, \tag{8-25}$$

where

$$\int_\Omega u_{0n_k}v(x,t)\phi_\delta(x)\rho(x)\,dx=\int_\Omega([u_0]_{d,2})_{n_k}v(x,t)\phi_\delta(x)\rho(x)\,dx \tag{8-26}$$

for every k large enough, since $\mathrm{dist}(K_0,\mathrm{supp}\phi_\delta)>0$ and K_0 is the set where $[u_0]_{c,2}$ is concentrated. Therefore, by (7-4), letting $k\to\infty$ in equality (8-25), we have

$$\lim_{k\to\infty}\int_\Omega u_{n_k}(x,t)v(x,t)\phi_\delta(x)\rho(x)\,dx \\ = -\int_0^t\int_\Omega \nabla v(x,s)\cdot\nabla[v(x,t)\phi_\delta(x)\rho(x)]\,dx\,ds + \langle [u_0]_{d,2}, v(\,\cdot\,,t)\phi_\delta\rho\rangle_\Omega. \quad (8\text{-}27)$$

On the other hand, in view of (3-7), equality (4-1) gives

$$\langle [u(\,\cdot\,,t)]_{d,2},\rho\rangle_\Omega - \langle [u_0]_{d,2},\rho\rangle_\Omega = \int_0^t \langle \Delta v(\,\cdot\,,s),\rho\rangle_\Omega\,ds,$$

which makes sense for any $\rho\in H_0^1(\Omega)\cap L^\infty(\Omega)$. Therefore we can choose $v(\,\cdot\,,t)\phi_\delta\rho$ as a test function, obtaining

$$\langle [u(\,\cdot\,,t)]_{d,2}, v(\,\cdot\,,t)\phi_\delta\rho\rangle_\Omega - \langle [u_0]_{d,2}, v(\,\cdot\,,t)\phi_\delta\rho\rangle_\Omega = -\int_0^t\int_\Omega \nabla v(x,s)\cdot\nabla[v(x,t)\phi_\delta(x)\rho(x)]\,dx\,ds.$$

Comparing this equality with (8-27), we obtain (8-23). This completes the proof. □

Finally, let us prove Lemmata 8.4–8.6.

Proof of Lemma 8.4. Since $C_2(\Omega_c^\delta)<\delta$, there exists $\eta_\delta\in H_0^1(\Omega)$ such that

$$\begin{cases} \|\eta_\delta\|_{H_0^1(\Omega)}\le 2\delta, & \\ 0\le\eta_\delta\le 1 & \text{a.e. in } \Omega,\\ \eta_\delta = 1 & \text{a.e. in } \Omega_c^\delta. \end{cases}$$

By (8-5)–(8-6), we have

$$\int_{\Omega_\delta^c}[1+u_{n_k}(x,t)]v_{n_k}(x,t)|\rho(x)|\,dx \\ \le \int_\Omega\left|\frac{1}{\psi'(u_{n_k})}-\frac{1}{\psi'_{n_k}(u_{n_k})}\right|(x,t)v_{n_k}(x,t)|\rho(x)|\eta_\delta(x)\,dx + \int_\Omega \frac{v_{n_k}}{\psi'_{n_k}(u_{n_k})}(x,t)|\rho(x)|\eta_\delta(x)\,dx \\ \le C\int_\Omega \eta_\delta\,dx + \int_\Omega \frac{v_{n_k}}{\psi'_{n_k}(u_{n_k})}(x,t)|\rho(x)|\eta_\delta(x)\,dx.$$

Since $|\rho|\eta_\delta\in H_0^1(\Omega)$, by (6-7) we get

$$\int_{\Omega_\delta^c}[1+u_{n_k}(x,t)]v_{n_k}(x,t)|\rho(x)|\,dx \\ \le \epsilon\int_\Omega |\nabla v_{n_k}(x,t)|\,|\nabla(|\rho|\eta_\delta)|\,dx + \int_\Omega \frac{\varphi(u_{n_k})}{\psi'_{n_k}(u_{n_k})}(x,t)|\rho(x)|\eta_\delta(x)\,dx + C\int_\Omega \eta_\delta(x)\,dx,$$

whence we get

$$\int_{\Omega_\delta^c}[1+u_{n_k}(x,t)]v_{n_k}(x,t)|\rho(x)|\,dx \le C_1\|\,|\rho|\eta_\delta\|_{H_0^1(\Omega)} + C_2\int_\Omega u_{n_k}^{1-1/q}(x,t)\eta_\delta(x)\,dx + C\int_\Omega \eta_\delta(x)\,dx \\ \le \widetilde{C}\left[\|\,|\rho|\eta_\delta\|_{H_0^1(\Omega)} + \left(\int_\Omega \eta_\delta^q(x)\,dx\right)^{1/q} + \int_\Omega \eta_\delta(x)\,dx\right]$$

(here we used (6-11), (6-13), and (8-2)). Setting

$$f_1(\delta) := \widetilde{C}\left[\| |\rho| \eta_\delta \|_{H_0^1(\Omega)} + \left(\int_\Omega \eta_\delta^q(x)\,dx\right)^{1/q} + \int_\Omega \eta_\delta(x)\,dx\right],$$

the conclusion follows. □

Proof of Lemma 8.5. By (6-16) (see also Remark 3.14) we obtain

$$\begin{aligned}\int_{E_\delta^c} [1+u_{n_k}(x,t)]v_{n_k}(x,t)|\rho(x)|\,dx &\le C_1 \int_{E_\delta^c} [1+u_{0n_k}(x)]v_{n_k}(x,t)|\rho(x)|\,dx \\ &\le C_1 \int_{E_\delta^c} u_{0n_k}(x)v_{n_k}(x,t)|\rho(x)|\,dx + C_2|E_\delta^c|. \qquad (8\text{-}28)\end{aligned}$$

Moreover, by the definition of the sequence $\{u_{0n}\}$ in Lemma 6.1, we have

$$u_{0n_k} = ([u_0]_{c,2})_{n_k} + ([u_0]_{d,2})_{n_k},$$

where

$$([u_0]_{d,2})_{n_k} := u_{0rn_k} + ([u_{0s}]_{d,2})_{n_k},$$

and

$$\int_{E_\delta^c} ([u_0]_{c,2})_{n_k}(x)\,dx = 0 \qquad (8\text{-}29)$$

holds for every k large enough. In fact, recall that the sequence $([u_0]_{c,2})_n$ is defined by convolution, $[u_0]_{c,2}$ is concentrated on the compact set $K_0 \subset \Omega_\delta^c$, the set Ω_δ^c is open, and $E_\delta^c \subseteq \Omega \setminus \Omega_\delta^c$. Combining (8-28) with (8-29) gives

$$\int_{E_\delta^c} [1+u_{n_k}(x,t)]v_{n_k}(x,t)|\rho(x)|\,dx \le C_1 \int_{E_\delta^c} ([u_0]_{d,2})_{n_k}(x)v_{n_k}(x,t)|\rho(x)|\,dx + C_2|E_\delta^c| \qquad (8\text{-}30)$$

for every k sufficiently large. Moreover, since $C_p(E_\delta^c) < \delta$ (see (8-13)) there exists $\rho_\delta \in H_0^{1,p}(\Omega)$ such that

$$\begin{cases} \|\rho_\delta\|_{H_0^{1,p}(\Omega)} \le 2\delta, & \\ 0 \le \rho_\delta \le 1 & \text{a.e. in } \Omega, \\ \rho_\delta = 1 & \text{a.e. in } E_\delta^c. \end{cases}$$

By the above remarks, using inequality (6-8), we obtain

$$\int_{E_\delta^c} ([u_0]_{d,2})_{n_k}(x)v_{n_k}(x,t)|\rho(x)|\,dx \le C_3 \int_\Omega ([u_0]_{d,2})_{n_k}(x)\rho_\delta(x)|\rho(x)|\,dx. \qquad (8\text{-}31)$$

Since, by assumption, $[u_0]_{d,2} \in \mathcal{M}_{d,p}^+(\Omega)$, by the first convergence in (6-4) we have

$$\lim_{k\to\infty} \int_\Omega ([u_0]_{d,2})_{n_k}(x)\rho_\delta(x)|\rho(x)|\,dx = \langle [u_0]_{d,2}, \rho_\delta|\rho| \rangle_\Omega.$$

Hence, by (8-30) and (8-31), we obtain

$$\limsup_{k\to\infty}\int_{E_\delta^c}[1+u_{n_k}(x,t)]v_{n_k}(x,t)|\rho(x)|\,dx \le C_2|E_\delta^c|+C_3\langle [u_0]_{d,2},\rho_\delta|\rho|\rangle_\Omega. \tag{8-32}$$

Then, setting

$$f_2(\delta):=C_2|E_\delta^c|+C_3\langle [u_0]_{d,2},\rho_\delta|\rho|\rangle_\Omega,$$

by (8-32) and the assumption $[u_0]_{d,2}\in\mathcal{M}^+_{d,p}(\Omega)$, the conclusion follows. □

Proof of Lemma 8.6. By (6-16) (see also Remark 3.14), for every k sufficiently large we have

$$\begin{aligned}\int_{\Omega_\delta^c\cup E_\delta^c}[1+u_{n_k}(x,t)]v(x,t)\phi_\delta(x)|\rho(x)|\,dx &\le C\int_{\Omega_\delta^c\cup E_\delta^c}u_{0n_k}(x)v(x,t)\phi_\delta(x)|\rho(x)|\,dx\\ &= C\int_{\Omega_\delta^c\cup E_\delta^c}([u_0]_{d,2})_{n_k}(x)v(x,t)\phi_\delta(x)|\rho(x)|\,dx.\end{aligned} \tag{8-33}$$

In fact, for k sufficiently large

$$\int_{\Omega_\delta^c\cup E_\delta^c}([u_0]_{c,2})_{n_k}(x)v(x,t)\phi_\delta(x)|\rho(x)|\,dx=0,$$

since $\operatorname{dist}(K_0,\operatorname{supp}\phi_\delta)>0$ and $[u_0]_{c,2}$ is concentrated on K_0.

Let $g_\delta\in H_0^{1,p}(\Omega)$ be any function such that

$$\begin{cases}\|g_\delta\|_{H_0^{1,p}(\Omega)}\le 4\delta,\\ 0\le g_\delta\le 1 & \text{a.e. in } \Omega,\\ g_\delta=1 & \text{a.e. in } \Omega\setminus E_\delta.\end{cases}$$

In view of (8-21), since $[u_0]_{d,2}\in\mathcal{M}^+_{d,p}(\Omega)$, we have

$$\begin{aligned}\limsup_{k\to\infty}\int_{\Omega_\delta^c\cup E_\delta^c}([u_0]_{d,2})_{n_k}(x)v(x,t)\phi_\delta(x)|\rho(x)|\,dx &\le C\lim_{k\to\infty}\int_\Omega([u_0]_{d,2})_{n_k}(x)g_\delta(x)\,dx\\ &= C\langle [u_0]_{d,2},g_\delta\rangle_\Omega.\end{aligned} \tag{8-34}$$

Since

$$f_3(\delta):=C\langle [u_0]_{d,2},g_\delta\rangle_\Omega\to 0 \quad \text{as } \delta\to 0,$$

by (8-33)–(8-34), the conclusion follows. □

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