ANALYSIS & PDE

Volume 6 No. 8 2013

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ANALYSIS AND PDE
Vol. 6, No. 8, 2013
dx.doi.org/10.2140/apde.2013.6.1793

L^p AND SCHAUDER ESTIMATES FOR NONVARIATIONAL OPERATORS STRUCTURED ON HÖRMANDER VECTOR FIELDS WITH DRIFT

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Let

$$\mathscr{L} = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j + a_0(x) X_0,$$

where $X_0, X_1, \dots, X_q$ are real smooth vector fields satisfying Hörmander's condition in some bounded domain $\Omega \subset \mathbb{R}^n$ $(n > q+1)$, and the coefficients $a_{ij} = a_{ji}$, a_0 are real valued, bounded measurable functions defined in Ω, satisfying the uniform positivity conditions

$$\mu |\xi|^2 \le \sum_{i,j=1}^{q} a_{ij}(x)\xi_i\xi_j \le \mu^{-1}|\xi|^2, \quad \mu \le a_0(x) \le \mu^{-1},$$

for a.e. $x \in \Omega$, every $\xi \in \mathbb{R}^q$, and some constant $\mu > 0$.

We prove that if the coefficients a_{ij}, a_0 belong to the Hölder space $C_X^\alpha(\Omega)$ with respect to the distance induced by the vector fields, local Schauder estimates of the following kind hold:

$$\|X_i X_j u\|_{C_X^\alpha(\Omega')} + \|X_0 u\|_{C_X^\alpha(\Omega')} \le c\{\|Lu\|_{C_X^\alpha(\Omega)} + \|u\|_{L^\infty(\Omega)}\}$$

for any $\Omega' \Subset \Omega$.

If the coefficients a_{ij}, a_0 belong to the space $\mathrm{VMO}_{X,\mathrm{loc}}(\Omega)$ with respect to the distance induced by the vector fields, local L^p estimates of the following kind hold, for every $p \in (1, \infty)$:

$$\|X_i X_j u\|_{L^p(\Omega')} + \|X_0 u\|_{L^p(\Omega')} \le c\{\|Lu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\}.$$

This project was supported by the National Natural Science Foundation of China (grant numbers 11271299 and 11001221).
MSC2010: primary 35H20; secondary 42B20, 35B45, 53C17.
Keywords: Hörmander's vector fields, Schauder estimates, L^p estimates, drift.

1. Introduction

Let us consider a family of real smooth vector fields

$$X_i = \sum_{j=1}^{n} b_{ij}(x)\partial_{x_j}, \quad i = 0, 1, 2, \ldots, q$$

(here $q+1 < n$), defined in some bounded domain Ω of $\mathbb{R}^n$ and satisfying Hörmander's condition: the Lie algebra generated by the X_i at any point of Ω spans $\mathbb{R}^n$. Under these assumptions, Hörmander's operators

$$\mathcal{L} = \sum_{i=1}^{q} X_i^2 + X_0$$

have been studied since the late 1960s. Hörmander [1967] proved that $\mathcal{L}$ is hypoelliptic, while Rothschild and Stein [1976] proved that, for these operators, a priori estimates of L^p type for second order derivatives with respect to the vector fields hold, namely,

$$\sum_{i,j=1}^{q} \|X_i X_j u\|_{L^p(\Omega')} + \|X_0 u\|_{L^p(\Omega')} \le c\left\{ \|\mathcal{L}u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} + \sum_{i=1}^{q} \|X_i u\|_{L^p(\Omega)} \right\} \tag{1-1}$$

for any $p \in (1, \infty)$, $\Omega' \Subset \Omega$.

Note that the "drift" vector field X_0 has weight two, compared with the vector fields

$$X_i \quad \text{for } i = 1, 2, \ldots, q.$$

Many more results have been proved in the literature for operators without the drift term ("sum of squares" of Hörmander type) than for complete Hörmander's operators. On the other hand, complete operators owe their interest, for instance, to the class of Kolmogorov–Fokker–Planck operators, which arise naturally in many fields of physics, natural sciences, and finance as the transport-diffusion equations satisfied by the transition probability density of stochastic systems of ODEs which describe some real system governed by a basically deterministic law perturbed by some kind of white noise. The study of Kolmogorov–Fokker–Planck operators in the framework of Hörmander's operators received a strong impulse from [Lanconelli and Polidoro 1994], which started a lively line of research. We refer to [Lanconelli et al. 2002] for a good survey of this field, with further motivations for the study of these equations and related references.

Let us also note that the study of Hörmander's operators is considerably easier when $\mathcal{L}$ is left invariant with respect to a suitable Lie group of translations and homogeneous of degree two with respect to a suitable family of dilations (which are group automorphisms of the corresponding group of translations). In this case we say that $\mathcal{L}$ has an underlying structure of homogeneous group and, by a famous result due to Folland [1975], $\mathcal{L}$ possesses a homogeneous left invariant global fundamental solution, which turns out to be a precious tool in proving a priori estimates.

In the last ten years, more general classes of nonvariational operators structured on Hörmander's vector fields have been studied, namely,

$$\mathscr{L} = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j, \tag{1-2}$$

$$\mathscr{L} = \sum_{i,j=1}^{q} a_{ij}(x,t) X_i X_j - \partial_t, \tag{1-3}$$

$$\mathscr{L} = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j + a_0(x) X_0, \tag{1-4}$$

where the matrix $\{a_{ij}(\cdot)\}_{i,j=1}^{q}$ is symmetric positive definite and the coefficients are bounded (a_0 is bounded away from zero) and satisfy suitable mild regularity assumptions; for instance, they belong to Hölder or VMO spaces defined with respect to the distance induced by the vector fields. Since the a_{ij}'s are not C^∞, these operators are no longer hypoelliptic. Nevertheless, a priori estimates on second order derivatives with respect to the vector fields are a natural result which does not in principle require smoothness of the coefficients. Namely, a priori estimates in L^p (with coefficients a_{ij} in $\mathrm{VMO}_X \cap L^\infty$) have been proved for operators (1-2) [Bramanti and Brandolini 2000a] and for operators (1-4) [Bramanti and Brandolini 2000b] but in homogeneous groups; a priori estimates in C_X^α spaces (with coefficients a_{ij} in C_X^α) have been proved for operators (1-3) [Bramanti and Brandolini 2007] and for operators (1-4) [Gutiérrez and Lanconelli 2009] but in homogeneous groups. Here the Hölder space C_X^α and the VMO_X space are defined with respect to the distance induced by the vector fields (see Section 3D for precise definitions).

In the particular case of Kolmogorov–Fokker–Planck operators, which can be written as

$$\mathscr{L} = \sum_{i,j=1}^{q} a_{ij}(x) \partial^2_{x_i x_j} + X_0$$

for a suitable drift X_0, L^p estimates (when a_{ij} are VMO) have been proved [Bramanti et al. 1996] in homogeneous groups, while Schauder estimates (when a_{ij} are Hölder continuous) have been proved [Di Francesco and Polidoro 2006] under more general assumptions (namely, assuming the existence of translations but not necessarily dilations, adapted to the operator). We recall that the idea of proving L^p estimates for nonvariational operators with leading coefficients in $\mathrm{VMO} \cap L^\infty$ (instead of assuming their uniform continuity) appeared for the first time in [Chiarenza et al. 1991; Chiarenza et al. 1993] by Chiarenza, Frasca, and Longo, in the uniformly elliptic case.

The aim of the present paper is to prove both L^p and C^α local estimates for general operators (1-4) structured on Hörmander's vector fields "with drift", without assuming the existence of any group structure, under the appropriate assumptions on the coefficients a_{ij}, a_0. Namely, our basic estimates read as follows:

$$\|u\|_{S_X^{2,p}(\Omega')} \le c\{\|\mathscr{L}u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\} \tag{1-5}$$

for $p \in (1, \infty)$ and any $\Omega' \Subset \Omega$ if the coefficients are $\mathrm{VMO}_{X,\mathrm{loc}}(\Omega)$, and

$$\|u\|_{C_X^{2,\alpha}(\Omega')} \le c\{\|\mathscr{L}u\|_{C_X^{\alpha}(\Omega)} + \|u\|_{L^\infty(\Omega)}\} \tag{1-6}$$

for $\alpha \in (0, 1)$ and $\Omega' \Subset \Omega$ if the coefficients are $C_X^\alpha(\Omega)$. The related Sobolev and Hölder spaces $S_X^{2,p}$, $C_X^{2,\alpha}$ are those induced by the vector fields X_i, and will be precisely defined in Section 3D. Clearly, these estimates are more general than those contained in all the aforementioned papers.

At first sight, this kind of result could seem a straightforward generalization of existing theories. However, several difficulties exist, some hidden in subtle details. We are going to describe some of them. First of all, we have to remark that in [Rothschild and Stein 1976], although $S_X^{2,p}$ estimates are stated for both sum of squares and complete Hörmander's operators, proofs are given only in the first case. While some adaptations are quite straightforward, this is not always the case. Therefore, some results proved in the present paper can be seen also as a detailed proof of results stated in [Rothschild and Stein 1976], in the drift case. One of the new difficulties in the drift case is related to the proof of suitable representation formulas for second order derivatives $X_i X_j u$ of a test function, in terms of u and $\mathscr{L}u$, via singular integrals and commutators of singular integrals. In turn, the reason why these representation formulas are harder to prove in the presence of a drift relies on the fact that a technical result which allows us to exchange, in a suitable sense, the action of X_i-derivatives with that of suitable integral operators assumes a more involved form when the drift is present.

Once the suitable representation formulas are established, a real variable machinery similar to that used in [Bramanti and Brandolini 2000a; 2007] can be applied, and this is the reason why we have chosen to give in a single paper a unified treatment of L^p and C_X^α estimates. More specifically, one considers a bounded domain Ω endowed with the control distance induced by the vector fields X_i, which has been defined, in the drift case, by Nagel, Stein, and Wainger [Nagel et al. 1985], and the Lebesgue measure, which is locally doubling with respect to these metric balls, as proved in [Nagel et al. 1985]. However, a problem arises when trying to apply to this context known results about singular integrals in metric doubling spaces (or "spaces of homogeneous type", after [Coifman and Weiss 1971]). Namely, what we should know to apply this theory on some domain $\Omega' \Subset \Omega$ is a doubling property such as

$$\mu(B(x, 2r) \cap \Omega') \le c\mu(B(x, r) \cap \Omega') \quad \text{for any } x \in \Omega' \Subset \Omega,\ r > 0 \tag{1-7}$$

while what we actually know, in view of [Nagel et al. 1985], is

$$\mu(B(x, 2r)) \le c\mu(B(x, r)) \quad \text{for any } x \in \Omega' \Subset \Omega,\ 0 < r < r_0. \tag{1-8}$$

It has been known since [Franchi and Lanconelli 1983] that, when Ω' is for instance a metric ball, condition (1-7) follows from (1-8) as soon as the distance satisfies a kind of *segment property* which reads as follows: for any couple of points x_1, x_2 at distance r and for any number $\delta < r$ and $\varepsilon > 0$, there exists a point x_0 having distance $\le \delta$ from x_1 and $\le r - \delta + \varepsilon$ from x_2 (this fact explicitly appears, for instance, from the proof given in [Bramanti and Brandolini 2005, Lemma 4.2]). However, while when the drift term is lacking, the distance induced by the X_i is easily seen to satisfy this property, this is no longer the case when the field X_0 with weight two enters the definition of distance, and, as far as we know, a condition of kind (1-7) has never been proved in this context for a metric ball Ω', or for any other special kind of bounded domain Ω. Thus we are forced to apply a theory of singular integrals which does not require the full strength of the global doubling condition (1-7). A first possibility is to consider the context of *nondoubling spaces*, as studied by Tolsa, Nazarov, Treil, and Volberg, and other authors (see, for instance, [Tolsa 2001; Nazarov et al. 2003] and the references therein). Results of L^p and C^α continuity for singular integrals of this kind, applicable to our context, have been proved in [Bramanti 2010]. However, to prove our L^p estimates (1-5), we also need some *commutator estimates*, of the kind of the well-known result proved by [Coifman et al. 1976], which, as far as we know, are not presently available in the framework of general nondoubling quasimetric (or metric) spaces. For this reason, we have recently developed [Bramanti and Zhu 2012] a theory of *locally homogeneous spaces* which is quite a natural framework where all the results we need about singular integrals and their commutators with BMO functions can be proved. To give a unified treatment of both L^p and C^α estimates, here we have decided to prove both by exploiting the results in [Bramanti and Zhu 2012]. We note that our Schauder estimates could also be obtained by applying the results in [Bramanti 2010], while L^p estimates could not.

Once the basic estimates on second order derivatives are established, a natural, but nontrivial, extension consists in proving similar estimates for derivatives of (weighted) order $k+2$, in terms of k derivatives of $\mathscr{L}u$ (assuming, of course, that the coefficients of the operator possess the corresponding further regularity). In the presence of a drift, it is reasonable to restrict this study to the case of k even, as already appears from the analog result proved in homogeneous groups [Bramanti and Brandolini 2000b]. Even in this case, a proof of this extension seems to be a difficult task, and we have decided not to lengthen the paper to address this problem.

2. Assumptions and main results

We now state precisely our assumptions and main results. All the function spaces involved in the statements below will be precisely defined in Section 3D. Our basic assumption is as follows.

Assumption (H). Let

$$\mathscr{L} = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j + a_0(x) X_0,$$

where the $X_0, X_1, \dots, X_q$ are real smooth vector fields satisfying Hörmander's condition (see Section 3A) in some bounded domain $\Omega \subset \mathbb{R}^n$ and the coefficients $a_{ij} = a_{ji}$, a_0 are real valued, bounded measurable functions defined in Ω, satisfying the uniform positivity conditions

$$\mu |\xi|^2 \le \sum_{i,j=1}^{q} a_{ij}(x) \xi_i \xi_j \le \mu^{-1} |\xi|^2, \quad \mu \le a_0(x) \le \mu^{-1},$$

for a.e. $x \in \Omega$, every $\xi \in \mathbb{R}^q$, and some constant $\mu > 0$.

Our main results are contained in the next two theorems.

Theorem 2.1. *In addition to* (H), *assume that the coefficients* a_{ij}, a_0 *belong to* $C^{\alpha}_X(\Omega)$ *for some* $\alpha \in (0,1)$. *Then, for every domain* $\Omega' \Subset \Omega$, *there exists a constant* $c > 0$ *depending on* Ω', Ω, X_i, α, μ, $\|a_{ij}\|_{C^{\alpha}_X(\Omega)}$, *and* $\|a_0\|_{C^{\alpha}_X(\Omega)}$ *such that, for every* $u \in C^{2,\alpha}_X(\Omega)$, *one has*

$$\|u\|_{C^{2,\alpha}_X(\Omega')} \leq c\{\|\mathcal{L}u\|_{C^{\alpha}_X(\Omega)} + \|u\|_{L^{\infty}(\Omega)}\}.$$

Theorem 2.2. *In addition to* (H), *assume that the coefficients* a_{ij}, a_0 *belong to the space* $\mathrm{VMO}_{X,\mathrm{loc}}(\Omega)$. *Then, for every* $p \in (1,\infty)$, *any* $\Omega' \Subset \Omega$, *there exists a constant c depending on* X_i, n, q, p, μ, Ω', Ω, *and the* VMO *moduli of* a_{ij} *and* a_0 *such that, for every* $u \in S^{2,p}_X(\Omega)$,

$$\|u\|_{S^{2,p}_X(\Omega')} \leq c\{\|\mathcal{L}u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\}.$$

Remark 2.3. Under the assumptions of the previous theorems, it is not restrictive to assume $a_0(x)$ to be equal to 1, for we can always rewrite (1-4) in the form

$$\sum_{i,j=1}^{q} \frac{a_{ij}}{a_0} X_i X_j + X_0 = \frac{f}{a_0}$$

and apply the a priori estimates to this equation, controlling C^{α}_X or VMO moduli of the new coefficients a_{ij}/a_0 in terms of the analogous moduli of a_{ij}, a_0, and the constant μ. Therefore, throughout the following we will always take $a_0 \equiv 1$.

3. Known results and preparatory results from real analysis and geometry of vector fields

3A. ***Hörmander's vector fields, lifting, and approximation.*** Let $X_0, X_1, \dots, X_q$ be a system of real smooth vector fields

$$X_i = \sum_{j=1}^{n} b_{ij}(x)\partial_{x_j}, \quad i = 0, 1, 2, \dots, q$$

$(q+1 < n)$ defined in some bounded, open and connected subset Ω of $\mathbb{R}^n$. Let us assign to each X_i a weight p_i, saying that

$$p_0 = 2 \quad \text{and} \quad p_i = 1 \quad \text{for } i = 1, 2, \dots, q.$$

For any multiindex

$$I = (i_1, i_2, \dots, i_k), \quad 0 \leq i_j \leq q,$$

we define the weight of I as

$$|I| = \sum_{j=1}^{k} p_{i_j}$$

and we set

$$X_I = X_{i_1} X_{i_2} \cdots X_{i_k},$$
$$X_{[I]} = [X_{i_1}, [X_{i_2}, \dots [X_{i_{k-1}}, X_{i_k}] \dots]],$$

where $[X, Y] = XY - YX$ for any couple of vector fields X, Y.

We will say that $X_{[I]}$ is a *commutator of weight* $|I|$. As usual, $X_{[I]}$ can be seen either as a differential operator or as a vector field. We will write

$$X_{[I]} f$$

to denote the differential operator $X_{[I]}$ acting on a function f, and

$$(X_{[I]})_x$$

to denote the vector field $X_{[I]}$ evaluated at the point $x \in \Omega$.

We shall say that $X = \{X_0, X_1, \dots, X_q\}$ satisfies *Hörmander's condition of weight* s if these vector fields, together with their commutators of weight $\leq s$, span the tangent space at every point $x \in \Omega$.

Let ℓ be the free Lie algebra of weight s on $q+1$ generators, that is, the quotient of the free Lie algebra with $q+1$ generators by the ideal generated by the commutators of weight at least $s+1$. We say that the vector fields $X_0, \dots, X_q$, which satisfy Hörmander's condition of weight s at some point $x_0 \in \mathbb{R}^n$, are *free up to order* s *at* x_0 if $n = \dim \ell$, as a vector space (note that inequality $\leq$ always holds). The famous lifting theorem proved by Rothschild and Stein [1976, p. 272] reads as follows.

Theorem 3.1. *Let $X = (X_0, X_1, \dots, X_q)$ be C^∞ real vector fields on a domain $\Omega \subset \mathbb{R}^n$ satisfying Hörmander's condition of weight s in Ω. Then, for any $\bar{x} \in \Omega$, in terms of new variables, $h_{n+1}, \dots, h_N$, there exist smooth functions $\lambda_{il}(x, h)$ $(0 \leq i \leq q,\ n+1 \leq l \leq N)$ defined in a neighborhood $\widetilde{U}$ of $\bar{\xi} = (\bar{x}, 0) \in \mathbb{R}^N$ such that the vector fields $\widetilde{X}_i$ given by*

$$\widetilde{X}_i = X_i + \sum_{l=n+1}^{N} \lambda_{il}(x, h) \frac{\partial}{\partial h_l}, \quad i = 0, \dots, q,$$

satisfy Hörmander's condition of weight s and are free up to weight s at every point in $\widetilde{U}$.

Let $\widetilde{X} = (\widetilde{X}_0, \widetilde{X}_1, \dots, \widetilde{X}_q)$ be the lifted vector fields which are free up to weight s at some point $\xi \in \mathbb{R}^N$ and let ℓ be the free Lie algebra generated by $\widetilde{X}$. For each j, $1 \leq j \leq s$, we can select a family $\{\widetilde{X}_{j,k}\}_k$ of commutators of weight j, with $\widetilde{X}_{1,k} = \widetilde{X}_k$, $\widetilde{X}_{2,1} = \widetilde{X}_0$, $k = 1, 2, \dots, q$, such that $\{\widetilde{X}_{j,k}\}_{jk}$ is a basis of ℓ, that is to say, there exists a set A of double-indices α such that $\{\widetilde{X}_\alpha\}_{\alpha \in A}$ is a basis of ℓ. Note that Card $A = N$, which allows us to identify ℓ with $\mathbb{R}^N$.

Now, in $\mathbb{R}^N$ we can consider the group structure of $N(q+1, s)$, which is the simply connected Lie group associated to ℓ. We will write $\circ$ for the Lie group operation (which we think of as a *translation*) and assume that the group identity is the origin. It is also possible to assume that $u^{-1} = -u$ (the group inverse is the Euclidean opposite). We can naturally define *dilations* in $N(q+1, s)$ by

$$D(\lambda)((u_\alpha)_{\alpha \in A}) = (\lambda^{|\alpha|} u_\alpha)_{\alpha \in A} \tag{3-1}$$

with $|j, k| = j$. These are group automorphisms, hence $N(q+1, s)$ is a *homogeneous group*, in the sense of Stein [1993, pp. 618–622]. We will call this group $\mathbb{G}$, leaving the numbers q, s implicitly understood.

We can define in $\mathbb{G}$ a *homogeneous norm* $\|\cdot\|$ as follows. For any $u \in \mathbb{G}$, $u \neq 0$, set

$$\|u\| = r \Leftrightarrow \left| D\left(\frac{1}{r}\right) u \right| = 1,$$

where $|\cdot|$ denotes the Euclidean norm.

The function

$$d_{\mathbb{G}}(u,v)=\|v^{-1}\circ u\|$$

is a *quasidistance*, that is

$$\begin{aligned} &d_{\mathbb{G}}(u,v)\ge 0 \quad \text{and} \quad d_{\mathbb{G}}(u,v)=0 \quad \text{if and only if } u=v,\\ &d_{\mathbb{G}}(u,v)=d_{\mathbb{G}}(v,u),\\ &d_{\mathbb{G}}(u,v)\le c(d_{\mathbb{G}}(u,z)+d_{\mathbb{G}}(z,v)) \end{aligned} \tag{3-2}$$

for every $u,v,z\in\mathbb{G}$ and some positive constant $c(\mathbb{G})\ge 1$. We define the balls with respect to $d_{\mathbb{G}}$ as

$$B(u,r):=\{v\in\mathbb{R}^N : d_{\mathbb{G}}(u,v)<r\}.$$

It can be proved [Stein 1993, p. 619] that the Lebesgue measure in $\mathbb{R}^N$ is the Haar measure of $\mathbb{G}$. Therefore, by (3-1),

$$|B(u,r)|=|B(u,1)|r^Q$$

for every $u\in\mathbb{G}$ and $r>0$, where $Q=\sum_{\alpha\in A}|\alpha|$. We will call Q the *homogeneous dimension* of $\mathbb{G}$.

Let τ_u be the left translation operator acting on functions: $(\tau_u f)(v)=f(u\circ v)$. We say that a differential operator P on $\mathbb{G}$ is *left invariant* if $P(\tau_u f)=\tau_u(Pf)$ for every smooth function f.

We say that a differential operator P on $\mathbb{G}$ is *homogeneous of degree* $\delta>0$ if

$$P(f(D(\lambda)u))=\lambda^\delta(Pf)(D(\lambda)u)$$

for every test function f and every $\lambda>0$, $u\in\mathbb{G}$. We also say that a function f is *homogeneous of degree* $\delta\in\mathbb{R}$ if

$$f(D(\lambda)u)=\lambda^\delta f(u) \quad \text{for every } \lambda>0,\ u\in\mathbb{G}.$$

Clearly, if P is a differential operator homogeneous of degree δ_1 and f is a homogeneous function of degree δ_2, then Pf is a homogeneous function of degree $\delta_2-\delta_1$, while fP is a differential operator, homogeneous of degree $\delta_1-\delta_2$.

Let Y_α be the left invariant vector field which agrees with $\partial/(\partial u_\alpha)$ at 0 and set $Y_{1,k}=Y_k$, $k=1,\dots,q$, $Y_{2,1}=Y_0$. The differential operator $Y_{i,k}$ is homogeneous of degree i, and $\{Y_\alpha\}_{\alpha\in A}$ is a basis of the free Lie algebra ℓ.

A differential operator on $\mathbb{G}$ is said to have *local degree less than or equal to* λ if, after taking the Taylor expansion at 0 of its coefficients, each term obtained is a differential operator homogeneous of degree $\le\lambda$.

Also, a function on $\mathbb{G}$ is said to have *local degree greater than or equal to* λ if, after taking the Taylor expansion at 0 of its coefficients, each term obtained is a homogeneous function of degree $\ge\lambda$. For $\xi,\eta\in\widetilde{U}$, define the map

$$\Theta_\eta(\xi)=(u_\alpha)_{\alpha\in A}$$

with $\xi=\exp\bigl(\sum_{\alpha\in A}u_\alpha\widetilde{X}_\alpha\bigr)\eta$. We will also write $\Theta(\eta,\xi)=\Theta_\eta(\xi)$.

We can now state Rothschild and Stein's approximation theorem [1976, p. 273].

Theorem 3.2. *In the coordinates given by $\Theta(\eta,\cdot)$ we can write $\widetilde{X}_i = Y_i + R_i^\eta$ on an open neighborhood of* 0, *where R_i^η is a vector field of local degree at most* 0 *for $i = 1, \dots, q$ (and at most* 1 *for $i = 0$) depending smoothly on η. Explicitly, this means that, for every $f \in C_0^\infty(\mathbb{G})$,*

$$\widetilde{X}_i[f(\Theta(\eta,\cdot))](\xi) = (Y_i f + R_i^\eta f)(\Theta(\eta,\xi)). \tag{3-3}$$

More generally, for every double-index $(i,k) \in A$, we can write

$$\widetilde{X}_{i,k}[f(\Theta(\eta,\cdot))](\xi) = (Y_{i,k} f + R_{i,k}^\eta f)(\Theta(\eta,\xi)), \tag{3-4}$$

where $R_{i,k}^\eta$ is a vector field of local degree $\le i-1$ depending smoothly on η.

Some other important properties of the map Θ are stated in the next theorem (see [Rothschild and Stein 1976, pp. 284–287]).

Theorem 3.3. *Let $\bar{\xi} \in \mathbb{R}^N$ and $\widetilde{U}$ be a neighborhood of $\bar{\xi}$ such that for any $\eta \in \widetilde{U}$ the map $\Theta(\eta,\cdot)$ is well defined in $\widetilde{U}$. For $\xi, \eta \in \widetilde{U}$, define*

$$\rho(\eta,\xi) = \|\Theta(\eta,\xi)\|, \tag{3-5}$$

where $\|\cdot\|$ is the homogeneous norm defined above. Then

(a) *$\Theta(\eta,\xi) = \Theta(\xi,\eta)^{-1} = -\Theta(\xi,\eta)$ for every $\xi, \eta \in \widetilde{U}$;*

(b) *ρ is a quasidistance in $\widetilde{U}$ (that is satisfies the three properties* (3-2)*);*

(c) *under the change of coordinates $u = \Theta_\xi(\eta)$, the measure element becomes*

$$d\eta = c(\xi)\cdot(1+\omega(\xi,u))\,du, \tag{3-6}$$

where $c(\xi)$ is a smooth function, bounded and bounded away from zero in $\widetilde{U}$, $\omega(\xi,u)$ is a smooth function in both variables with

$$|\omega(\xi,u)| \le c\|u\|,$$

and an analogous statement is true for the change of coordinates $u = \Theta_\eta(\xi)$.

Remark 3.4. As we recalled in the introduction, in [Rothschild and Stein 1976] detailed proofs are given only when the drift term X_0 is lacking. A proof of the lifting and approximation results explicitly covering the drift case can be found in [Bramanti et al. 2010], where the theory is also extended to the case of nonsmooth Hörmander's vector fields. We refer to the introduction of [Bramanti et al. 2010] for further bibliographic remarks about existing alternative proofs of the lifting and approximation theorems.

3B. ***Metric induced by vector fields.*** Let us start by recalling the definition of control distance given by Nagel, Stein, Wainger [Nagel et al. 1985] for Hörmander's vector fields with drift.

Definition 3.5. For any $\delta > 0$, let $C(\delta)$ be the class of absolutely continuous mappings φ: $[0,1] \to \Omega$ which satisfy

$$\varphi'(t) = \sum_{|I|\le s} \lambda_I(t)(X_{[I]})_{\varphi(t)} \quad \text{for a.e. } t \in (0,1) \tag{3-7}$$

with $|\lambda_I(t)| \le \delta^{|I|}$. We define

$$d(x, y) = \inf\{\delta : \text{there exists } \varphi \in C(\delta) \text{ with } \varphi(0) = x, \varphi(1) = y\}.$$

The finiteness of d immediately follows by Hörmander's condition: since the vector fields $\{X_{[I]}\}_{|I|\le s}$ span $\mathbb{R}^n$, we can always join any two points x, y with a curve φ of the kind (3-7); moreover, d turns out to be a distance. Analogously to what Nagel, Stein, and Wainger [Nagel et al. 1985] do when X_0 is lacking, in [Bramanti et al. 2013] the following notion is introduced.

Definition 3.6. For any $\delta > 0$, let $C_1(\delta)$ be the class of absolutely continuous mappings $\varphi : [0, 1] \to \Omega$ which satisfy

$$\varphi'(t) = \sum_{i=0}^{q} \lambda_i(t)(X_i)_{\varphi(t)} \quad \text{for a.e. } t \in (0, 1)$$

with $|\lambda_0(t)| \le \delta^2$ and $|\lambda_j(t)| \le \delta$ for $j = 1, \dots, q$. We define

$$d_X(x, y) = \inf\{\delta : \text{there exists } \varphi \in C_1(\delta) \text{ with } \varphi(0) = x, \varphi(1) = y\}.$$

Note that the finiteness of $d_X(x, y)$ for any two points $x, y \in \Omega$ is not a trivial fact, but depends on a connectivity result ("Chow's theorem"); moreover, it can be proved that d and d_X are locally equivalent, and that d_X is still a distance (see [Bramanti et al. 2013], where these results are proved in the more general setting of nonsmooth vector fields). From now on we will always refer to d_X as the *control distance* induced by the system of Hörmander's vector fields X. It is well-known that this distance is topologically equivalent to the Euclidean one. For any $x \in \Omega$, we set

$$B(x, r) = \{y \in \Omega : d_X(x, y) < r\}.$$

The basic result about the measure of metric balls is the famous local doubling condition.

Theorem 3.7 [Nagel et al. 1985]. *For every $\Omega' \Subset \Omega$ there exist positive constants c, r_0 such that, for any $x \in \Omega'$, $r \le r_0$,*

$$|B(x, 2r)| \le c|B(x, r)|.$$

As already pointed out in the introduction, the distance d_X does *not* satisfy the segment property: given two points at distance r, it is generally impossible to find a third point at distance $r/2$ from both. A weaker property which this distance actually satisfies is contained in the next lemma, and will be useful when dealing with the properties of Hölder spaces C_X^α.

Lemma 3.8. *For any $x, y \in \Omega$, positive integer n, $\varepsilon > 0$, we can join x to y with a curve γ and find $n + 1$ points $p_0 = x, p_1, p_2, \dots, p_n = y$ on γ, such that*

$$d_X(p_j, p_{j+1}) \le \frac{1+\varepsilon}{\sqrt{n}} d_X(x, y) \quad \textit{for } j = 0, 2, \dots, n-1.$$

Proof. For any $x, y \in \Omega$ with $d_X(x, y) = R$, any $\varepsilon > 0$, by Definition 3.6 we can join x and y with a curve $\gamma(t)$ satisfying

$$\gamma(0) = x, \quad \gamma(1) = y, \quad \gamma'(t) = \sum_{i=0}^{q} \lambda_i(t)(X_i)_{\gamma(t)},$$

with $|\lambda_i(t)| \le R(1+\varepsilon)$, for $i = 1, \dots, q$ and $|\lambda_0(t)| \le (R(1+\varepsilon))^2$.

Let $\gamma_j(t) = \gamma((t+j)/n)$ for $j = 0, 1, 2, \dots, n-1$. Then $\gamma_j(t)$ satisfies

$$\gamma_j(0) = \gamma\left(\frac{j}{n}\right) =: p_j, \quad \gamma_j(1) = \gamma\left(\frac{j+1}{n}\right) = p_{j+1}.$$

In particular, $p_0 = x$ and $p_n = y$. Moreover,

$$\gamma_j'(t) = \frac{1}{n}\sum_{i=0}^{q} \lambda_i\left(\frac{t+j}{n}\right)(X_i)_{\gamma_j(t)}$$

with

$$\left|\frac{1}{n}\lambda_0\left(\frac{t+j}{n}\right)\right| \le \left(\frac{R(1+\varepsilon)}{\sqrt{n}}\right)^2, \quad \left|\frac{1}{n}\lambda_i\left(\frac{t+j}{n}\right)\right| < \frac{R(1+\varepsilon)}{\sqrt{n}}$$

for $i = 1, \dots, q$, $j = 0, 2, \dots, n-1$. Thus

$$d_X(p_j, p_{j+1}) \le \frac{R(1+\varepsilon)}{\sqrt{n}}$$

for $j = 0, 2, \dots, n-1$, so we are done. □

The free lifted vector fields $\widetilde{X}_i$ induce, in the neighborhood where they are defined, a control distance $d_{\widetilde{X}}$; we will denote by $\widetilde{B}(\xi, r)$ the corresponding metric balls. In this lifted setting we can also consider the quasidistance ρ defined in (3-5). The two functions turn out to be equivalent.

Lemma 3.9. *Let $\bar{\xi}, \widetilde{U}$ be as in Theorem 3.3. There exists $\widetilde{B}(\bar{\xi}, R) \subset \widetilde{U}$ such that the distance $d_{\widetilde{X}}$ is equivalent to the quasidistance ρ in* (3-5) *in $\widetilde{B}(\bar{\xi}, R)$, and both are greater than the Euclidean distance*; *namely*, *there exist positive constants c_1, c_2, c_3 such that*

$$c_1|\xi - \eta| \le c_2\rho(\eta, \xi) \le d_{\widetilde{X}}(\eta, \xi) \le c_3\rho(\eta, \xi) \quad \textit{for every } \xi, \eta \in \widetilde{B}(\bar{\xi}, R).$$

This fact is proved in [Nagel et al. 1985]; see also [Bramanti et al. 2010, Proposition 22].

3C. ***Locally homogeneous spaces.*** We are now going to recall the notion of *locally homogeneous space*, introduced in [Bramanti and Zhu 2012]. Roughly speaking, a locally homogeneous space is a set Ω endowed with a function d which is a quasidistance on any compact subset, and a measure μ which is locally doubling, in a sense which will be made precise below. In our concrete situation, our set is endowed with a function d which is a *distance* in Ω, and a locally doubling measure. We can therefore give the following definition, which is simpler than that given in [Bramanti and Zhu 2012].

Definition 3.10. Let (Ω, d) be a metric space, and let μ be a positive regular Borel measure in Ω.

Assume there exists an increasing sequence $\{\Omega_n\}_{n=1}^{\infty}$ of bounded measurable subsets of Ω such that

$$\bigcup_{n=1}^{\infty} \Omega_n = \Omega \tag{3-8}$$

and, for any $n = 1, 2, 3, \ldots,$

(i) the closure of Ω_n in Ω is compact,

(ii) there exists $\varepsilon_n > 0$ such that

$$\{x \in \Omega : d(x, y) < 2\varepsilon_n \quad \text{for some } y \in \Omega_n\} \subset \Omega_{n+1}, \tag{3-9}$$

(iii) there exists $C_n > 1$ such that, for any $x \in \Omega_n$, $0 < r \leq \varepsilon_n$, we have

$$0 < \mu(B(x, 2r)) \leq C_n \mu(B(x, r)) < \infty. \tag{3-10}$$

(Note that for $x \in \Omega_n$ and $r \leq \varepsilon_n$ we also have $B(x, 2r) \subset \Omega_{n+1}$.)

We say that $(\Omega, \{\Omega_n\}_{n=1}^{\infty}, d, \mu)$ is a *(metric) locally homogeneous space* if the above assumptions hold.

Any space satisfying the above definition a fortiori satisfies the definition of locally homogeneous space given in [Bramanti and Zhu 2012].

Next, we discuss some facts about local singular kernels. For fixed Ω_n, Ω_{n+1}, and a fixed ball $B(\bar{x}, R_0)$, with $\bar{x} \in \Omega_n$ and $R_0 < 2\varepsilon_n$ (hence $B(\bar{x}, R_0) \subset \Omega_{n+1}$), let $K(x, y)$ be a measurable function defined for $x, y \in B(\bar{x}, R_0)$, $x \neq y$. We now list a series of possible assumptions on the kernel K which are involved in the theorems that we will apply in the following.

(i) We say that K satisfies the *standard estimates* for some $\nu \in [0, 1)$ if the following hold:

$$|K(x, y)| \leq \frac{A d(x, y)^{\nu}}{\mu(B(x, d(x, y)))} \tag{3-11}$$

for $x, y \in B(\bar{x}, R_0)$ with $x \neq y$, and

$$|K(x_0, y) - K(x, y)| + |K(y, x_0) - K(y, x)| \leq \frac{B d(x_0, y)^{\nu}}{\mu(B(x_0, d(x_0, y)))} \left(\frac{d(x_0, x)}{d(x_0, y)}\right)^{\beta} \tag{3-12}$$

for any $x_0, x, y \in B(\bar{x}, R_0)$ with $d(x_0, y) > 2d(x_0, x)$, and some $\beta > 0$.

(ii) We say that K satisfies the *cancellation property* if the following holds: there exists $C > 0$ such that, for a.e. $x \in B(\bar{x}, R_0)$ and every $\varepsilon_1, \varepsilon_2$ such that $0 < \varepsilon_1 < \varepsilon_2$ and $B_\rho(x, \varepsilon_2) \subset \Omega_{n+1}$,

$$\left| \int_{\Omega_{n+1}, \varepsilon_1 < \rho(x,y) < \varepsilon_2} K(x, y)\, d\mu(y) \right| + \left| \int_{\Omega_{n+1}, \varepsilon_1 < \rho(x,z) < \varepsilon_2} K(z, x)\, d\mu(z) \right| \leq C, \tag{3-13}$$

where ρ is any *quasidistance* (see (3-2)) equivalent to d in Ω_{n+1} and B_ρ denotes ρ-balls.

(iii) We say that K satisfies the *convergence condition* if the following holds: for a.e. $x \in B(\bar{x}, R_0)$ such that $B_\rho(x, R) \subset \Omega_{n+1}$, there exists

$$h_R(x) \equiv \lim_{\varepsilon \to 0} \int_{\Omega_{n+1}, \varepsilon < \rho(x,y) < R} K(x, y)\, d\mu(y), \tag{3-14}$$

where ρ is any quasidistance equivalent to d in Ω_{n+1}.

Application of the abstract theory to our setting. Let's now explain how this abstract setting will be used to describe our concrete situation. The a priori estimates we will prove in Theorems 2.1 and 2.2 involve a fixed subdomain $\Omega' \Subset \Omega$. Let us fix this Ω' once and for all. For any $\bar{x} \in \Omega'$ we can perform in a suitable neighborhood of $\bar{x}$ the lifting and approximation procedure as explained in Section 3A. Let $\bar{\xi} = (\bar{x}, 0) \in \mathbb{R}^N$ and $\widetilde{B}(\bar{\xi}, R)$ be as in Lemma 3.9. Then we can choose

$$\widetilde{\Omega} = \widetilde{B}(\bar{\xi}, R);\ \widetilde{\Omega}_k = \widetilde{B}\left(\bar{\xi}, \frac{kR}{k+1}\right) \quad \text{for } k = 1, 2, 3, \dots.$$

By the properties of $d_{\widetilde{X}}$ that we have listed in Section 3B, and particularly Theorem 3.7, we see that

$$(\widetilde{\Omega}, \{\widetilde{\Omega}_k\}_{k=1}^{\infty}, d_{\widetilde{X}}, d\xi)$$

is a metric locally homogeneous space. The function $\rho(\xi, \eta) = \|\Theta(\eta, \xi)\|$ will play the role of the quasidistance appearing in conditions (3-13) and (3-14), in view of Lemma 3.9. This is the basic setting where we will apply several results about singular integrals in locally homogeneous spaces, which have been proved in [Bramanti and Zhu 2012]. Here we do not repeat the statements of all those theorems. Instead, we will give a precise reference to [Bramanti and Zhu 2012] for each one. We just note that, since in our situation we are dealing with a *metric* locally homogeneous space, the constants which are called B_n in [Bramanti and Zhu 2012], here are equal to 1.

In the space of the original variables (Ω, d_X, dx), instead, we will not apply singular integral estimates, but we will again use the local doubling condition when we establish some important properties of function spaces C^α and VMO (see Section 3D). Note that if Ω_k is an increasing sequence of domains with $\Omega_k \Subset \Omega_{k+1} \Subset \Omega$, we can say that

$$(\Omega, \{\Omega_k\}_k, d_X, dx)$$

is a metric locally homogeneous space.

3D. *Function spaces.* The aim of this section is twofold. First, we want to define the basic function spaces we will need and point out their main properties; second, we want to find a relation between function spaces defined over a ball $B(\bar{x}, r) \subset \Omega \subset \mathbb{R}^n$ and those over the corresponding lifted ball $\widetilde{B}(\bar{\xi}, r) \subset \mathbb{R}^N$. More precisely, we need to know that $f(x)$ belongs to some function space on B if and only if $\tilde{f}(x, h) = f(x)$ belongs to the analogous function space on $\widetilde{B}$. This last fact relies on the following known result; see [Nagel et al. 1985, Lemmas 3.1 and 3.2, p. 139].

Theorem 3.11. *Let us denote by B and $\widetilde{B}$ the balls defined with respect to d_X and $d_{\widetilde{X}}$, respectively. There exist constants $\delta_0 \in (0,1)$, $r_0, c_1, c_2 > 0$ such that*

$$c_1 \operatorname{vol}(\widetilde{B}_r(x,h)) \le \operatorname{vol}(B_r(x)) \cdot \operatorname{vol}\{h' \in \mathbb{R}^{N-n} : (z,h') \in \widetilde{B}_r(x,h)\} \le c_2 \operatorname{vol}(\widetilde{B}_r(x,h)) \tag{3-15}$$

for every $x \in \Omega$, $z \in B_{\delta_0 r}(x)$, and $r \le r_0$. (Here "vol" stands for the Lebesgue measure in the appropriate dimension, x denotes a point in $\mathbb{R}^n$, and h a point in $\mathbb{R}^{N-n}$). More precisely, the condition $z \in B_{\delta_0 r}(x)$ is needed only for the validity of the first inequality in (3-15)*. Moreover,*

$$d_{\widetilde{X}}((x,h),(x',h')) \ge d_X(x,x'). \tag{3-16}$$

Finally, the projection of the lifted ball $\widetilde{B}_r(x,h)$ on $\mathbb{R}^n$ is just the ball $B(x,r)$, and this projection is onto.

A consequence of the above theorem is the following.

Corollary 3.12. *For any positive function g defined in $B_r(x) \subset \Omega$, $r \le r_0$, one has*

$$\frac{c_1}{|B_{\delta_0 r}(x)|}\int_{B_{\delta_0 r}(x)} g(y)\,dy \le \frac{1}{|\widetilde{B}_r(x,h)|}\int_{\widetilde{B}_r(x,h)} g(y)\,dy\,dh' \le \frac{c_2}{|B_r(x)|}\int_{B_r(x)} g(y)\,dy, \tag{3-17}$$

where δ_0 is the constant in Theorem 3.11.

Proof. By (3-15) and the locally doubling condition, we have, for some fixed $\delta_0 < 1$ as in Theorem 3.11,

$$\begin{aligned}\frac{1}{|\widetilde{B}_r(x,h)|}\int_{\widetilde{B}_r(x,h)} g(y)\,dy\,dh' &= \frac{1}{|\widetilde{B}_r(x,h)|}\int_{B_r(x)} g(y)\,dy \int_{\{h'\in\mathbb{R}^{N-n}:(y,h')\in\widetilde{B}_r(x,h)\}} dh' \\ &\ge \frac{c_1}{|\widetilde{B}_r(x,h)|}\int_{B_{\delta_0 r}(x)} \frac{|\widetilde{B}_r(x,h)|}{|B_r(x)|} g(y)\,dy \ge \frac{c}{|B_{\delta_0 r}(x)|}\int_{B_{\delta_0 r}(x)} g(y)\,dy,\end{aligned}$$

where in the last inequality we exploited the doubling condition $|B_r(x)| \le c|B_{\delta_0 r}(x)|$, which holds because $B_r(x) \subset \Omega$ and $r \le r_0$. The proof of the second inequality in (3-17) is analogous but easier, since it involves the second inequality in (3-15), which does not require the condition $y \in B_{\delta_0 r}(x)$. □

3D.1. *Hölder spaces.*

Definition 3.13. For any $0 < \alpha < 1$, $u : \Omega \to \mathbb{R}$, let

$$\begin{aligned}|u|_{C_X^\alpha(\Omega)} &= \sup\left\{\frac{|u(x)-u(y)|}{d_X(x,y)^\alpha} : x, y \in \Omega, x \ne y\right\},\\ \|u\|_{C_X^\alpha(\Omega)} &= |u|_{C^\alpha(\Omega)} + \|u\|_{L^\infty(\Omega)},\\ C_X^\alpha(\Omega) &= \{u : \Omega \to \mathbb{R} : \|u\|_{C^\alpha(\Omega)} < \infty\}.\end{aligned}$$

Also, for any positive integer k, let

$$C_X^{k,\alpha}(\Omega) = \{u : \Omega \to \mathbb{R} : \|u\|_{C^{k,\alpha}(\Omega)} < \infty\},$$

with

$$\|u\|_{C_X^{k,\alpha}(\Omega)} = \sum_{|I|=1}^{k} \sum_{j_i=0}^{q} \|X_{j_1}\cdots X_{j_l} u\|_{C^\alpha(\Omega)} + \|u\|_{C^\alpha(\Omega)},$$

where $I = (j_1, j_2, \dots, j_l)$.

We will set $C^{\alpha}_{X,0}(\Omega)$ and $C^{k,\alpha}_{X,0}(\Omega)$ for the subspaces of $C^{\alpha}_{X}(\Omega)$ and $C^{k,\alpha}_{X}(\Omega)$ of functions which are compactly supported in Ω, and set $C^{\alpha}_{\widetilde{X}}(\widetilde{B})$, $C^{k,\alpha}_{\widetilde{X}}(\widetilde{B})$, $C^{\alpha}_{\widetilde{X},0}(\widetilde{B})$, and $C^{k,\alpha}_{\widetilde{X},0}(\widetilde{B})$ for the analogous function spaces over $\widetilde{B}$ defined by the $\widetilde{X}_i$.

We will also write $C^{k,0}_{X}(\Omega)$ to denote the space of functions with continuous X-derivatives up to weight k.

Let us note that we will sometimes also need to use the classical spaces of (possibly compactly supported) continuously differentiable functions, denoted as usual by C^1 (or C^1_0).

The next proposition, adapted from [Bramanti and Brandolini 2007, Proposition 4.2], collects some properties of C^{α} functions which will be useful later. We will apply these properties mainly in the context of lifted variables, that is, for the vector fields $\widetilde{X}_i$ on a ball $\widetilde{B}(\bar{\xi}, R)$.

Proposition 3.14. *Let $B(\bar{x}, 2R)$ be a fixed ball where the vector fields X_i and the control distance d are well defined.*

(i) *For any $\delta \in (0, 1)$ and any $f \in C^1(B(\bar{x}, (1+\delta)R))$, one has*

$$|f(x) - f(y)| \le \frac{c}{\delta} d_X(x, y)\left(\sum_{i=1}^{q} \sup_{B(\bar{x},(1+\delta)R)} |X_i f| + d_X(x, y) \sup_{B(\bar{x},(1+\delta)R)} |X_0 f|\right) \tag{3-18}$$

for any $x, y \in B(\bar{x}, R)$.

If $f \in C^1_0(B(\bar{x}, R))$, one can simply write, for any $x, y \in B(\bar{x}, R)$,

$$|f(x) - f(y)| \le c d_X(x, y)\left(\sum_{i=1}^{q} \sup_{B(\bar{x},R)} |X_i f| + d_X(x, y) \sup_{B(\bar{x},R)} |X_0 f|\right). \tag{3-19}$$

In particular, for $f \in C^1_0(B(\bar{x}, R))$,

$$|f|_{C^{\alpha}(B(\bar{x},R))} \le c R^{1-\alpha} \cdot \left(\sum_{i=1}^{q} \sup_{B(\bar{x},R)} |X_i f| + R \sup_{B(\bar{x},R)} |X_0 f|\right). \tag{3-20}$$

The assumption $f \in C^1$ (or C^1_0) can be replaced by $f \in C^2_X$ (or $C^2_{X,0}$, respectively).

(ii) *For any couple of functions $f, g \in C^{\alpha}_{X}(B(\bar{x}, R))$, one has*

$$|fg|_{C^{\alpha}_{X}(B(\bar{x},R))} \le |f|_{C^{\alpha}_{X}(B(\bar{x},R))} \|g\|_{L^{\infty}(B(\bar{x},R))} + |g|_{C^{\alpha}_{X}(B(\bar{x},R))} \|f\|_{L^{\infty}(B(\bar{x},R))}$$

and

$$\|fg\|_{C^{\alpha}_{X}(B(\bar{x},R))} \le 2\|f\|_{C^{\alpha}_{X}(B(\bar{x},R))} \|g\|_{C^{\alpha}_{X}(B(\bar{x},R))}. \tag{3-21}$$

Moreover, if both f and g vanish at least at a point of $B(\bar{x}, R)$, then

$$|fg|_{C^{\alpha}_{X}(B(\bar{x},R))} \le c R^{\alpha} |f|_{C^{\alpha}_{X}(B(\bar{x},R))} |g|_{C^{\alpha}_{X}(B(\bar{x},R))}. \tag{3-22}$$

(iii) *Let $B(x_i, r)$ $(i = 1, 2, \dots, k)$ be a finite family of balls of the same radius r such that $\bigcup_{i=1}^k B(x_i, 2r) \subset \Omega$. Then, for any $f \in C_X^\alpha(\Omega)$,*

$$\|f\|_{C_X^\alpha(\bigcup_{i=1}^k B(x_i,r))} \le c \sum_{i=1}^k \|f\|_{C_X^\alpha(B(x_i,2r))} \tag{3-23}$$

with c depending on the family of balls, but not on f.

(iv) *There exists $r_0 > 0$ such that, for any $f \in C_{X,0}^{2,\alpha}(B(\bar{x}, R))$ and $0 < r \le r_0$, we have the interpolation inequality*

$$\|X_0 f\|_{L^\infty(B(\bar{x},R))} \le r^{\alpha/2} |X_0 f|_{C_X^\alpha(B(\bar{x},R))} + \frac{2}{r} \|f\|_{L^\infty(B(\bar{x},R))}. \tag{3-24}$$

Proof. The proof of (ii)–(iii) is similar to that in [Bramanti and Brandolini 2007, Proposition 4.2], hence we will only prove (i) and (iv).

Throughout this proof we will write d for d_X. (Actually, we will apply this proposition both to d_X and to $d_{\widetilde{X}}$).

(i) Fix $\delta \in (0, 1)$ and let $R' = (1 + \delta)R$. Let us distinguish two cases.

Case 1: $d(x, y) < R' - \max(d(\bar{x}, x), d(\bar{x}, y))$. Let $\varepsilon > 0$ be such that

$$d(x, y) + \varepsilon < R' - \max(d(\bar{x}, x), d(\bar{x}, y)), \tag{3-25}$$

hence, by Definition 3.6, there exists a curve $\varphi(t)$ such that $\varphi(0) = x$, $\varphi(1) = y$, and

$$\varphi'(t) = \sum_{i=0}^q \lambda_i(t)(X_i)_{\varphi(t)}$$

with $|\lambda_i(t)| \le (d(x, y) + \varepsilon)$, $|\lambda_0(t)| \le (d(x, y) + \varepsilon)^2$ for $i = 1, \dots, q$. By (3-25),

$$B(x, d(x, y) + \varepsilon) \subset B(\bar{x}, R'),$$

hence every point $\gamma(t)$ for $t \in (0, 1)$ belongs to $B(\bar{x}, R')$. Then we can write

$$\begin{aligned} |f(x) - f(y)| &= \left| \int_0^1 \frac{d}{dt} f(\varphi(t))\, dt \right| = \left| \int_0^1 \sum_{i=0}^q \lambda_i(t)(X_i f)_{\varphi(t)}\, dt \right| \\ &\le (d(x, y) + \varepsilon) \sum_{i=1}^q \sup_{B(\bar{x},R')} |X_i f| + (d(x, y) + \varepsilon)^2 \sup_{B(\bar{x},R')} |X_0 f|, \end{aligned}$$

and since ε is arbitrary, this implies (3-19) and, in particular, (3-18). We note that the above argument relies on the differentiability of f along the curve φ, which holds under either the assumption $f \in C^1(B(\bar{x}, (1 + \delta)R))$ or $f \in C_X^2(B(\bar{x}, (1 + \delta)R))$ (since X_0 has weight two).

Case 2: $d(x, y) \ge R' - \max(d(\bar{x}, x), d(\bar{x}, y))$. Let us write

$$|f(x) - f(y)| \le |f(x) - f(\bar{x})| + |f(\bar{x}) - f(y)| = A + B.$$

Each of the terms A, B can be bounded by an argument similar to that in *Case* 1 (since both x and y can be joined to $\bar{x}$ by curves contained in $B(\bar{x}, R)$), giving

$$|f(x)-f(y)| \leq [d(x,\bar{x})+d(y,\bar{x})]\cdot\left\{\sum_{i=1}^{q}\sup_{B(\bar{x},R)}|X_i f| + [d(x,\bar{x})+d(y,\bar{x})]\sup_{B(\bar{x},R)}|X_0 f|\right\}.$$

Now it is enough to show that

$$d(x,\bar{x})+d(y,\bar{x}) \leq \frac{c}{\delta}d(x,y).$$

To show this, let $r := \max(d(\bar{x},x), d(\bar{x},y))$. Then

$$d(x,\bar{x})+d(y,\bar{x}) \leq 2r \leq \frac{2}{\delta}(R'-r) \leq \frac{2}{\delta}d(x,y),$$

where the second inequality holds since $r < R$ and $R' = (1+\delta)R$, and the last inequality is the assumption $d(x,y) \geq R' - \max(d(\bar{x},x), d(\bar{x},y))$. This completes the proof of (3-18), which immediately implies (3-19) and (3-20).

(iv) Let $f \in C^{2,\alpha}_{X,0}(B(\bar{x},R))$. For any $x \in B(\bar{x},R)$, let $\gamma(t)$ be the curve such that

$$\gamma'(t) = (X_0)_{\gamma(t)}, \quad \gamma(0) = x.$$

This $\gamma(t)$ will be defined at least for $t \in [0, r_0]$ where $r_0 > 0$ is a number only depending on $B(\bar{x},R)$ and X_0. Then, for any $r \in (0, r_0)$, we can write, for some $\theta \in (0,1)$,

$$f(\gamma(r)) - f(\gamma(0)) = r\frac{d}{dt}[f(\gamma(t))]_{t=\theta r} = r(X_0 f)(\gamma(\theta r)),$$

hence

$$(X_0 f)(x) = (X_0 f)(\gamma(0)) - (X_0 f)(\gamma(\theta r)) + \frac{1}{r}[f(\gamma(r)) - f(\gamma(0))]$$

and since, by definition of γ and d, $d(\gamma(0), \gamma(\theta r)) \leq (\theta r)^{1/2}$, we get

$$\begin{aligned}|(X_0 f)(x)| &\leq |(X_0 f)(\gamma(0)) - (X_0 f)(\gamma(\theta r))| + \frac{2}{r}\|f\|_{L^\infty} \\ &\leq (\theta r)^{\alpha/2}|X_0 f|_{C^\alpha_X(B(\bar{x},R))} + \frac{2}{r}\|f\|_{L^\infty(B(\bar{x},R))} \\ &\leq r^{\alpha/2}|X_0 f|_{C^\alpha_X(B(\bar{x},R))} + \frac{2}{r}\|f\|_{L^\infty(B(\bar{x},R))},\end{aligned}$$

so we are done. □

Next, we are going to study the relation between the spaces $C^\alpha_X(B_R)$ and $C^\alpha_{\widetilde{X}}(\widetilde{B}_R)$.

Proposition 3.15. *Let $\widetilde{B}(\bar{\xi}, R)$ be a lifted ball* (*see the end of Section 3C*), *with $\bar{\xi} = (\bar{x}, 0)$. If f is a function defined in $B(\bar{x}, R)$ and $\tilde{f}(x,h) = f(x)$ is regarded as a function defined on $\widetilde{B}_R(\bar{\xi}, R)$, the following inequalities hold* (*whenever the right-hand side is finite*):

$$\begin{aligned}&|\tilde{f}|_{C^\alpha_{\widetilde{X}}(\widetilde{B}(\bar{\xi},R))} \leq |f|_{C^\alpha_X(B(\bar{x},R))}, \\ &|f|_{C^\alpha_X(B(\bar{x},s))} \leq \frac{c}{(t-s)^2}|\tilde{f}|_{C^\alpha_{\widetilde{X}}(\widetilde{B}(\bar{\xi},t))} \quad \textit{for } 0 < s < t < R,\end{aligned} \tag{3-26}$$

where c also depends on R. Moreover,

$$|\widetilde{X}_{i_1}\widetilde{X}_{i_2}\cdots\widetilde{X}_{i_k}\tilde{f}|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},R))} \le |X_{i_1}X_{i_2}\cdots X_{i_k}f|_{C^{\alpha}_{X}(B(\bar{x},R))}, \tag{3-27}$$

$$|X_{i_1}X_{i_2}\cdots X_{i_k}f|_{C^{\alpha}_{X}(B(\bar{x},s))} \le \frac{c}{(t-s)^2}|\widetilde{X}_{i_1}\widetilde{X}_{i_2}\cdots\widetilde{X}_{i_k}\tilde{f}|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},t))} \tag{3-28}$$

for $0 < s < t < R$ *and* $i_j = 0, 1, 2, \dots, q$.

As already done in [Bramanti and Brandolini 2007, Proposition 8.3], to prove the above relation between Hölder spaces over B and $\widetilde{B}$ we have to exploit an equivalent integral characterization of Hölder continuous functions, analogous to the one established in the classical case by Campanato [1963]. However, to avoid integration over sets of the kind $\Omega \cap B(x,r)$ (with the related problem of assuring a suitable doubling condition), we need to apply the local version of this result which has been established in [Bramanti and Zhu 2012].

Definition 3.16. For $\bar{x} \in \Omega'$, $B(\bar{x},R) \subset \Omega$, $f \in L^1(B(\bar{x},R))$, $\alpha \in (0,1)$, and $0 < s < t \le 1$, let

$$M_{\alpha,B_{sR},B_{tR}}(f) = \sup_{x\in B(\bar{x},sR),\, r\le (t-s)R}\ \inf_{c\in\mathbb{R}} \frac{1}{r^{\alpha}|B_r(x)|}\int_{B_r(x)} |f(y)-c|\,dy.$$

If $f \in C^{\alpha}_X(B(\bar{x},R))$, then

$$M_{\alpha,B_{sR},B_{tR}}(f) \le |f|_{C^{\alpha}(B_R(x_0))}.$$

Moreover, we get the following.

Lemma 3.17. *For* $\bar{x} \in \Omega'$, $B(\bar{x},2R_0) \subset \Omega$, $R < R_0$, $\alpha \in (0,1)$, *and* $0 < s < t \le 1$, *if* $f \in L^1(B(\bar{x},tR))$ *is a function such that* $M_{\alpha,B_{sR},B_{tR}}(f) < \infty$, *then there exists a function* f^*, *a.e. equal to* f, *such that* $f^* \in C^{\alpha}_X(B(\bar{x},sR))$ *and*

$$|f^*|_{C^{\alpha}_X(B(\bar{x},sR))} \le \frac{c}{(t-s)^2} M_{\alpha,B_{sR},B_{tR}}(f)$$

for some c independent of f, s, t.

Proof. We can apply [Bramanti and Zhu 2012, Theorem 9.2] choosing $\Omega_k = B(\bar{x},sR)$, $\Omega_{k+1} = B(\bar{x},tR)$, $\varepsilon_n = R(t-s)$. The locally doubling constant can be chosen independently of R, since $B(\bar{x},2R_0) \subset \Omega$, $R < R_0$. We conclude that there exists a function f^*, a.e. equal to f, such that

$$|f^*(x) - f^*(y)| \le c M_{\alpha,B_{sR},B_{tR}}(f) d_X(x,y)^{\alpha}$$

for any $x, y \in B(\bar{x},sR)$ with $d_X(x,y) \le R(t-s)/2$.

Now if x, y are any two points in $B_{sR}(x_0)$, and $r = d_X(x,y)$, by Lemma 3.8 we can find $n+1$ points $x_0 = x, x_1, x_2, \dots, x_n = y$ in $B_{sR}(x_0)$ such that

$$d_X(x_i, x_{i-1}) \le \frac{2r}{\sqrt{n}}.$$

Let n be the least integer such that $2r/\sqrt{n} \le R(t-s)/2$. Then

$$|f^*(x) - f^*(y)| \le \sum_{i=1}^{n} |f^*(x_i) - f^*(x_{i-1})| \le \sum_{i=1}^{n} c M_{\alpha, B_{sR}, B_{tR}}(f) d_X(x_i, x_{i-1})^{\alpha}$$
$$\le ncM_{\alpha, B_{sR}, B_{tR}}(f) d_X(x, y)^{\alpha}.$$

Let us find an upper bound on n. We know that

$$\sqrt{n} \le c \frac{d_X(x, y)}{R(t-s)} \le \frac{c}{t-s},$$

since $d_X(x, y) \le 2R$ for $x, y \in B_{tR}(x_0)$. Hence $n \le c/(t-s)^2$ and the lemma is proved. □

Proof of Proposition 3.15. The first inequality immediately follows by (3-16). To prove the second one, let $0 < s < t < 1$ and $x \in B(\bar{x}, \delta_0 sR)$, where δ_0 is the number in Theorem 3.11, $r \le R(t-s)$, $\bar{\xi} = (\bar{x}, 0)$. Since the projection $\pi : \widetilde{B}((x, s), \delta) \to B(x, \delta)$ is onto (see Theorem 3.11), there exists $h \in \mathbb{R}^{N-n}$ such that $\xi = (x, h) \in \widetilde{B}(\bar{\xi}, \delta_0 sR)$. Then, by Corollary 3.12, we have

$$\frac{1}{r^{\alpha}} \frac{c}{|B_{\delta_0 r}(x)|} \int_{B_{\delta_0 r}(x)} |f(y) - k| \, dy \le \frac{c}{r^{\alpha}} \frac{1}{|\widetilde{B}(\xi, r)|} \int_{\widetilde{B}(\xi, r)} |\tilde{f}(\eta) - k| \, d\eta; \tag{3-29}$$

choosing $k = f(x) = \tilde{f}(\xi)$, the latter quantity is

$$\le \frac{c}{r^{\alpha}} |\tilde{f}|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\xi, r))} r^{\alpha} = c |\tilde{f}|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\xi, r))}.$$

Since $r \le R(t-s)$ and $d(\xi, \bar{\xi}) < \delta_0 sR$, we have the inclusion

$$\widetilde{B}(\xi, r) \subset \widetilde{B}(\bar{\xi}, \delta_0 sR + R(t-s)) =: \widetilde{B}(\bar{\xi}, R')$$

so that (3-29) implies

$$M_{\alpha, B(\bar{x}, \delta_0 sR), B(\bar{x}, \delta_0 tR)}(f) \le c |\tilde{f}|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, R'))},$$

and, by Lemma 3.17, we conclude

$$|f^*|_{C^{\alpha}_X(B(\bar{x}, \delta_0 sR))} \le \frac{c}{(t-s)^2} |\tilde{f}|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, R'))}.$$

Note that $R' - \delta_0 sR = R(t-s)$, hence, changing our notation to

$$\delta_0 sR = s', \quad R' = t',$$

we get

$$|f^*|_{C^{\alpha}_X(B(\bar{x}, s'))} \le \frac{c}{(t'-s')^2} |\tilde{f}|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, t'))}$$

for $0 < s' < t' < R$, with c also depending on R. This is (3-26).

Now inequalities (3-27) and (3-28) also follow, because $\widetilde{X}_i \tilde{f} = \widetilde{X_i f}$, hence the same reasoning can be iterated to higher order derivatives. □

3D.2. *Sobolev spaces.*

Definition 3.18. If $X = (X_0, X_1, \dots, X_q)$ is any system of smooth vector fields satisfying Hörmander's condition in a domain $\Omega \subset \mathbb{R}^n$, the Sobolev space $S_X^{2,p}(\Omega)$ $(1 < p < \infty)$ consists of L^p-functions with 2 (weighted) derivatives with respect to the vector fields X_i, in L^p. Explicitly,

$$\|u\|_{S_X^{2,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{2} \|D^i u\|_{L^p(\Omega)},$$

$$\text{where } \|D^1 u\|_{L^p(\Omega)} = \sum_{i=1}^{q} \|X_i u\|_{L^p(\Omega)}; \ \|D^2 u\|_{L^p(\Omega)} = \|X_0 u\|_{L^p(\Omega)} + \sum_{i,j=1}^{q} \|X_i X_j u\|_{L^p(\Omega)}.$$

Also, we can define the spaces of functions vanishing at the boundary saying that $u \in S_{0,X}^{2,p}(\Omega)$ if there exists a sequence $\{u_k\}$ of $C_0^\infty(\Omega)$ functions converging to u in $S_X^{2,p}(\Omega)$. Similarly, we can define the Sobolev spaces $S_{\widetilde{X}}^{2,p}(\widetilde{B})$, $S_{\widetilde{X},0}^{2,p}(\widetilde{B})$ over a lifted ball $\widetilde{B}$, induced by the $\widetilde{X}$.

The following has been proved [Bramanti and Brandolini 2000a, Proposition 3.5].

Proposition 3.19. *If $u \in S_X^{2,p}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$, then $u\varphi \in S_{0,X}^{2,p}(\Omega)$, and an analogous property holds for the space $S_{0,\widetilde{X}}^{2,p}(\widetilde{B})$.*

Moreover, we have the following.

Theorem 3.20. *Let $f \in L^p(B(x,r)$, $\tilde{f}(x,h) = f(x)$, and $\widetilde{B}(\xi, r)$ be the lifted ball of $B(x,r)$, with $\xi = (x, 0) \in \mathbb{R}^N$. Then*

$$c_1 \|f\|_{L^p(B(x,\delta_0 r))} \le \|\tilde{f}\|_{L^p(\widetilde{B}(\xi,r))} \le c_2 \|f\|_{L^p(B(x,r))},$$
$$c_1 \|f\|_{S_X^{2,p}(B(x,\delta_0 r))} \le \|\tilde{f}\|_{S_{\widetilde{X}}^{2,p}(\widetilde{B}(\xi,r))} \le c_2 \|f\|_{S_X^{2,p}(B(x,r))},$$

where $\delta_0 < 1$ is the number appearing in Theorem 3.11.

Proof. The first inequality follows by Theorem 3.11; the second follows by the first, since

$$\widetilde{X}_i \tilde{f} = X_i \tilde{f} = \widetilde{(X_i f)}. \qquad \square$$

3D.3. *Vanishing mean oscillation.* Let us recall the following abstract definition.

Definition 3.21 [Bramanti and Zhu 2012, Definition 6.1]. Let $(\Omega, \{\Omega_n\}_{n=1}^\infty, d, \mu)$ be a metric locally homogeneous space (see Section 3C). For any function $u \in L^1(\Omega_{n+1})$ and $r > 0$ with $r \le \varepsilon_n$, set

$$\eta^*_{u,\Omega_n,\Omega_{n+1}}(r) = \sup_{t \le r} \sup_{x_0 \in \Omega_n} \frac{1}{\mu(B(x_0,t))} \int_{B(x_0,t)} |u(x) - u_B| \, d\mu(x),$$

where $u_B = \mu(B(x_0,t))^{-1} \int_{B(x_0,t)} u$. We say that $u \in \mathrm{BMO}_{\mathrm{loc}}(\Omega_n, \Omega_{n+1})$ if

$$\|u\|_{\mathrm{BMO}_{\mathrm{loc}}(\Omega_n,\Omega_{n+1})} = \sup_{r \le \varepsilon_n} \eta^*_{u,\Omega_n,\Omega_{n+1}}(r) < \infty.$$

We say that $u \in \mathrm{VMO}_{\mathrm{loc}}(\Omega_n, \Omega_{n+1})$ if $u \in \mathrm{BMO}_{\mathrm{loc}}(\Omega_n, \Omega_{n+1})$ and

$$\eta^*_{u,\Omega_n,\Omega_{n+1}}(r) \to 0 \quad \text{as } r \to 0.$$

The function $\eta^*_{u,\Omega_n,\Omega_{n+1}}$ will be called the VMO local modulus of u in (Ω_n, Ω_{n+1}).

We need to specialize this definition to our concrete situation. First, let us endow our domain Ω with the structure

$$(\Omega, \{\Omega_k\}_k, d_X, dx)$$

of locally homogeneous space described at the end of Section 3C. Then:

Definition 3.22 (local VMO). We say that $a \in \mathrm{VMO}_{X,\mathrm{loc}}(\Omega)$ if

$$a \in \mathrm{VMO}_{\mathrm{loc}}(\Omega_k, \Omega_{k+1}) \quad \text{for every } k.$$

More explicitly, this means that, for any fixed $\Omega' \Subset \Omega$, the function

$$\eta^*_{u,\Omega',\Omega}(r) = \sup_{t \le r} \sup_{x_0 \in \Omega'} \frac{1}{|B_t(x_0)|} \int_{B_t(x_0)} |u(x) - u_{B_t(x_0)}| \, dx,$$

is finite for $r \le r_0$ and vanishes for $r \to 0$, where r_0 is the number such that the local doubling condition of Theorem 3.7 holds:

$$|B(x, 2r)| \le c|B(x, r)| \quad \text{for any } x \in \Omega',\ r \le r_0.$$

As for Hölder continuous and Sobolev functions, we need a comparison result for VMO functions in the original variables and the lifted ones. By Corollary 3.12 we immediately have the following.

Proposition 3.23. *Let $a \in \mathrm{VMO}_{X,\mathrm{loc}}(\Omega)$. Then, for any $\Omega' \Subset \Omega$, $x_0 \in \Omega'$, $B(x_0, R)$, and $\widetilde{\Omega}_k = \widetilde{B}(\xi_0, kR/(k+1))$ as before, we have that $\tilde{a}(x, h) = a(x)$ belongs to the class $\mathrm{VMO}_{\mathrm{loc}}(\widetilde{\Omega}_k, \widetilde{\Omega}_k)$ for every k, with*

$$\eta^*_{\tilde{a},\widetilde{\Omega}_k,\widetilde{\Omega}_{k+1}}(r) \le c\eta^*_{a,\Omega',\Omega}(r).$$

In other words, the $\mathrm{VMO}_{\mathrm{loc}}$ modulus of the original function a controls the $\mathrm{VMO}_{\mathrm{loc}}$ modulus of its lifted version.

4. Operators of type λ and representation formulas

4A. ***Differential operators and fundamental solutions.*** We now define various differential operators that we will handle in the following. Our main interest is to study the operator

$$\mathscr{L} = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j + X_0,$$

under the assumption (H) in Section 2. Recall that, in view of Remark 2.3, we have set $a_0(x) \equiv 1$.

For any $\bar{x} \in \Omega$, we can apply the "lifting theorem" to the vector fields X_i (see Section 3A for the statement and notation), obtaining new vector fields $\widetilde{X}_i$ which are free up to weight s and satisfy

Hörmander's condition of weight s in a neighborhood of $\bar{\xi} = (\bar{x}, 0) \in \mathbb{R}^N$. For $\xi = (x, t) \in \widetilde{B}(\bar{\xi}, R)$, with $\widetilde{B}(\bar{\xi}, R)$ as in Lemma 3.9, set

$$\tilde{a}_{ij}(x, t) = a_{ij}(x),$$

and let

$$\widetilde{\mathscr{L}} = \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi)\widetilde{X}_i\widetilde{X}_j + \widetilde{X}_0 \tag{4-1}$$

be the lifted operator, defined in $\widetilde{B}(\bar{\xi}, R)$. Next, we freeze $\widetilde{\mathscr{L}}$ at some point $\xi_0 \in \widetilde{B}(\bar{\xi}, R)$, and consider the frozen lifted operator

$$\widetilde{\mathscr{L}}_0 = \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi_0)\widetilde{X}_i\widetilde{X}_j + \widetilde{X}_0. \tag{4-2}$$

To study $\widetilde{\mathscr{L}}_0$, in view of the "approximation theorem" (Theorem 3.2), we will consider the approximating operator, defined on the homogeneous group $\mathbb{G}$,

$$\mathscr{L}_0^* = \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi_0)Y_iY_j + Y_0,$$

and its transpose,

$$\mathscr{L}_0^{*T} = \sum_{i,j=1}^{q} \tilde{a}_{ij}(\xi_0)Y_iY_j - Y_0,$$

where $\{Y_i\}$ are the left invariant vector fields on the group $\mathbb{G}$ defined in Section 3A.

We will apply to $\mathscr{L}_0^*$ and $\mathscr{L}_0^{*T}$ several results proved in [Bramanti and Brandolini 2000b], which in turn are based on [Folland 1975, Theorem 2.1 and Corollary 2.8; Folland and Stein 1974, Proposition 8.5]. They are collected in the following theorem.

Theorem 4.1. *Assume that the homogeneous dimension of* $\mathbb{G}$ *is* $Q \geq 3$. *For every* $\xi_0 \in \widetilde{B}(\bar{\xi}, R)$, *the operator* $\mathscr{L}_0^*$ *has a unique fundamental solution* $\Gamma(\xi_0; \cdot)$ *such that*

(a) $\Gamma(\xi_0; \cdot) \in C^\infty(\mathbb{R}^N \setminus \{0\})$;

(b) $\Gamma(\xi_0; \cdot)$ *is homogeneous of degree* $(2 - Q)$;

(c) *for every test function* f *and every* $v \in \mathbb{R}^N$,

$$f(v) = \int_{\mathbb{R}^N} \Gamma(\xi_0; u^{-1} \circ v)\mathscr{L}_0^* f(u)\, du;$$

moreover, for every $i, j = 1, \dots, q$, *there exist constants* $\alpha_{ij}(\xi_0)$ *such that*

$$Y_iY_j f(v) = \mathrm{PV}\int_{\mathbb{R}^N} Y_iY_j\Gamma(\xi_0; u^{-1} \circ v)\mathscr{L}_0^* f(u)\, du + \alpha_{ij}(\xi_0) \cdot \mathscr{L}_0^* f(v); \tag{4-3}$$

(d) $Y_iY_j\Gamma(\xi_0; \cdot)$ *is homogeneous of degree* $-Q$;

(e) *for every* $R > r > 0$,

$$\int_{r<\|u\|<R} Y_i Y_j \Gamma(\xi_0; u)\, du = \int_{\|u\|=1} Y_i Y_j \Gamma(\xi_0; u)\, d\sigma(u) = 0.$$

In (4-3) the notation PV $\int_{\mathbb{R}^N} \cdots du$ stands for $\lim_{\varepsilon\to 0} \int_{\|u^{-1}\circ v\|>\varepsilon} \cdots du$.

Remark 4.2. By [Folland 1975, remark on p. 174], we know that the fundamental solution of the transposed operator $\mathscr{L}_0^{*T}$ is

$$\Gamma^T(\xi_0; u) = \Gamma(\xi_0; u^{-1}) = \Gamma(\xi_0; -u).$$

(However, beware that $Y_i \Gamma^T(\xi_0; u) \neq \pm Y_i \Gamma(\xi_0; -u)$.)

Throughout the following, we will set, for $i, j = 1, \dots, q$,

$$\begin{aligned} \Gamma_{ij}(\xi_0; u) &= Y_i Y_j [\Gamma(\xi_0; \cdot)](u), \\ \Gamma_{ij}^T(\xi_0; u) &= Y_i Y_j [\Gamma^T(\xi_0; \cdot)](u). \end{aligned}$$

A second fundamental result we need contains a bound on the derivatives of Γ, uniform with respect to ξ_0.

Theorem 4.3 [Bramanti and Brandolini 2000b, Theorem 12]. *For every multi-index β, there exists a constant $c = c(\beta, \mathbb{G}, \mu)$ such that, for any $i, j = 1, \dots, q$,*

$$\sup_{\substack{\xi\in\widetilde{B}(\bar{\xi},R)\\ \|u\|=1}} \left| \left(\frac{\partial}{\partial u}\right)^{\beta} \Gamma_{ij}(\xi; u) \right| \leq c;$$

moreover, for the α_{ij} appearing in (4-3), *the uniform bound*

$$\sup_{\xi\in\widetilde{B}(\bar{\xi},R)} |\alpha_{ij}(\xi)| \leq c_2$$

holds for some constant $c_2 = c_2(\mathbb{G}, \mu)$.

Remark 4.4. Theorems 4.1 and 4.3 still hold replacing Γ by Γ^T and Γ_{ij} by Γ_{ij}^T.

4B. ***Operators of type λ.*** As in [Rothschild and Stein 1976; Bramanti and Brandolini 2000a], we are going to build a parametrix for $\widetilde{\mathscr{L}}$ shaped on the homogeneous fundamental solution of $\mathscr{L}_0^*$. More generally, we need to define a class of integral operators with different degrees of singularity. The next definition is adapted from [Bramanti and Brandolini 2000a], the difference being the necessity, in the present case, to consider integral kernels shaped on the fundamental solutions of both $\mathscr{L}_0^*$ and $\mathscr{L}_0^{*T}$.

Definition 4.5. For any $\xi_0 \in \widetilde{B}(\bar{\xi}, R)$, we say that $k(\xi_0; \xi, \eta)$ is a *frozen kernel of type* λ (over the ball $\widetilde{B}(\bar{\xi}, R)$) for some nonnegative integer λ (we will use $\lambda = 0, 1, 2$) if, for every positive integer m, we can write, for $\xi, \eta \in \widetilde{B}(\bar{\xi}, R)$,

$$
\begin{aligned}
k(\xi_0; \xi, \eta) &= k'(\xi_0; \xi, \eta) + k''(\xi_0; \xi, \eta) \\
&= \left\{ \sum_{i=1}^{H_m} a_i(\xi) b_i(\eta) D_i \Gamma(\xi_0; \cdot) + a_0(\xi) b_0(\eta) D_0 \Gamma(\xi_0; \cdot) \right\} (\Theta(\eta, \xi)) \\
&\qquad + \left\{ \sum_{i=1}^{H_m} a_i'(\xi) b_i'(\eta) D_i' \Gamma^T(\xi_0; \cdot) + a_0'(\xi) b_0'(\eta) D_0' \Gamma^T(\xi_0; \cdot) \right\} (\Theta(\eta, \xi)),
\end{aligned}
$$

where $a_i, b_i, a_i', b_i' \in C_0^\infty(\widetilde{B}(\bar{\xi}, R))$ $(i = 0, 1, \dots, H_m)$, and D_i and D_i' are differential operators such that, for $i = 1, \dots, H_m$, D_i and D_i' are homogeneous of degree $\leq 2 - \lambda$ (so that $D_i \Gamma(\xi_0; \cdot)$, and $D_i' \Gamma^T(\xi_0; \cdot)$ are homogeneous functions of degree $\geq \lambda - Q$); D_0 and D_0' are differential operators such that $D_0 \Gamma(\xi_0; \cdot)$ and $D_0' \Gamma^T(\xi_0; \cdot)$ have m (weighted) derivatives with respect to the vector fields Y_i $(i = 0, 1, \dots, q)$. Moreover, the coefficients of the differential operators D_i, D_i' for $i = 0, 1, \dots, H_m$ possibly depend also on the variables ξ, η, in such a way that the joint dependence on (ξ, η, u) is smooth.

In order to simplify notation, we will not always express explicitly this dependence of the coefficients of D_i on ξ, η. Only if necessary will we write, for instance, $a_i(\xi) b_i(\eta) D_i^{\xi,\eta} \Gamma(\xi_0; \Theta(\eta, \xi))$ to recall this dependence.

Remark 4.6. Note that if a smooth function $c(\xi, \eta, u)$ is $D(\lambda)$-homogeneous of some degree β with respect to u, any ξ or η derivative of c has the same homogeneity with respect to u, since

$$
c(\xi, \eta, D(\lambda) u) = \lambda^\beta c(\xi, \eta, u) \quad \text{implies } \frac{\partial c}{\partial \xi_i}(\xi, \eta, D(\lambda) u) = \lambda^\beta \frac{\partial c}{\partial \xi_i}(\xi, \eta, u).
$$

Hence any derivative

$$
\left(\frac{\partial}{\partial \xi_i} D_i^{\xi,\eta} \right) \Gamma(\xi_0; \cdot), \qquad \left(\frac{\partial}{\partial \eta_i} D_i^{\xi,\eta} \right) \Gamma(\xi_0; \cdot)
$$

has the same homogeneity as

$$
D_i^{\xi,\eta} \Gamma(\xi_0; \cdot).
$$

Here and in the following, the symbol $((\partial / \partial \xi_i) D_i^{\xi,\eta}) f$ means that we have taken the ξ_i-derivative of the coefficients of the differential operator $D_i^{\xi,\eta}$, which acts on the u variables but contains ξ, η as parameters; the resulting differential operator acts on the function $f(u)$.

Definition 4.7. For any $\xi_0 \in \widetilde{B}(\bar{\xi}, R)$, we say that $T(\xi_0)$ is a *frozen operator of type* $\lambda \geq 1$ (over the ball $\widetilde{B}(\bar{\xi}, R)$) if $k(\xi_0; \xi, \eta)$ is a frozen kernel of type λ and

$$
T(\xi_0) f(\xi) = \int_{\widetilde{B}} k(\xi_0; \xi, \eta) f(\eta) \, d\eta
$$

for $f \in C_0^\infty(\widetilde{B}(\bar{\xi}, R))$. We say that $T(\xi_0)$ is a *frozen operator of type* 0 if $k(\xi_0; \xi, \eta)$ is a frozen kernel of type 0 and

$$
T(\xi_0) f(\xi) = \mathrm{PV} \int_{\widetilde{B}} k(\xi_0; \xi, \eta) f(\eta) \, d\eta + \alpha(\xi_0, \xi) f(\xi),
$$

where α is a bounded measurable function, smooth in ξ, and the principal value integral exists. Explicitly, this principal value is defined by

$$\mathrm{PV}\int_{\widetilde{B}} k(\xi_0;\xi,\eta) f(\eta), d\eta = \lim_{\varepsilon\to 0}\int_{\|\Theta(\eta,\xi)\|>\varepsilon} k(\xi_0;\xi,\eta) f(\eta)\, d\eta.$$

Definition 4.8. If $k(\xi_0;\xi,\eta)$ is a frozen kernel of type $\lambda \geq 0$, we say that $k(\xi;\xi,\eta)$ is a *variable kernel of type* λ (over the ball $\widetilde{B}(\bar{\xi},R)$), and

$$Tf(\xi) = \int_{\widetilde{B}} k(\xi;\xi,\eta) f(\eta)\, d\eta$$

is a *variable operator of type* λ. If $\lambda = 0$, the integral must be taken in principal value sense and a term $\alpha(\xi,\xi) f(\xi)$ must be added.

In reference to Definition 4.5, we will call the k' and k'' parts of k "the frozen kernels of type λ modeled on Γ and Γ^T", respectively. Analogously we will sometimes speak of frozen operators of type λ modeled on Γ or Γ^T, to denote that the kernel has this special form.

A common operation on frozen operators is *transposition*.

Definition 4.9. If $T(\xi_0)$ is a frozen operator of type $\lambda \geq 0$ over $\widetilde{B}(\bar{\xi},R)$, we will denote by $T(\xi_0)^T$ the transposed operator, formally defined by

$$\int_{\widetilde{B}} f(\xi) T(\xi_0)^T g(\xi)\, d\xi = \int_{\widetilde{B}} g(\xi) T(\xi_0) f(\xi)\, d\xi$$

for any $f, g \in C_0^\infty(\widetilde{B}(\bar{\xi},R))$.

Clearly, if $k(\xi_0,\xi,\eta)$ is the kernel of $T(\xi_0)$, then $k(\xi_0,\eta,\xi)$ is the kernel of $T(\xi_0)^T$. It is useful to note the following.

Proposition 4.10. *If $T(\xi_0)$ is a frozen operator of type $\lambda \geq 0$ over $\widetilde{B}(\bar{\xi},R)$, modeled on Γ or Γ^T, then $T(\xi_0)^T$ is a frozen operator of type λ, modeled on Γ^T or Γ, respectively. In particular, the transpose of a frozen operator of type λ is still a frozen operator of type λ.*

Proof. Let D be any differential operator on the group $\mathbb{G}$. For any $f \in C_0^\infty(\widetilde{B}(\bar{\xi},R))$, let $f'(u) = f(-u)$. Let D' be the differential operator defined by the identity

$$D'f = (D(f'))'.$$

Clearly, if D is homogeneous of some degree β, the same is true for D'; if $D\Gamma(\xi_0;\cdot)$ or $D\Gamma^T(\xi_0;\cdot)$ has m (weighted) derivatives with respect to the vector fields Y_i $(i = 0, 1, \ldots, q)$, the same is true for $D'\Gamma(\xi_0;\cdot)$ or $D'\Gamma^T(\xi_0;\cdot)$. Also, recalling that $\Gamma^T(\xi_0;u) = \Gamma(\xi_0;-u)$, we have

$$(D'\Gamma)(u) = (D\Gamma^T)(-u) \quad \text{and} \quad (D'\Gamma^T)(u) = (D\Gamma)(-u).$$

Moreover, these identities can be iterated, for instance,

$$(D_1D_2\Gamma)(-u) = (D_1(D_2\Gamma))(-u) = (D_1'(D_2\Gamma)')(u) = (D_1'D_2'\Gamma^T)(u).$$

Therefore, if

$$k'(\xi_0,\xi,\eta)=\left\{\sum_{i=1}^{H_m}a_i(\xi)b_i(\eta)D_i\Gamma(\xi_0;\,\cdot\,)+a_0(\xi)b_0(\eta)D_0\Gamma(\xi_0;\,\cdot\,)\right\}(\Theta(\eta,\xi))$$

is a frozen kernel of type λ modeled on Γ, then

$$\begin{aligned}k'(\xi_0,\eta,\xi)&=\left\{\sum_{i=1}^{H_m}a_i(\eta)b_i(\xi)D_i\Gamma(\xi_0;\,\cdot\,)+a_0(\xi)b_0(\eta)D_0\Gamma(\xi_0;\,\cdot\,)\right\}(-\Theta(\eta,\xi))\\&=\left\{\sum_{i=1}^{H_m}a_i(\eta)b_i(\xi)D_i'\Gamma^T(\xi_0;\,\cdot\,)+a_0(\xi)b_0(\eta)D_0'\Gamma^T(\xi_0;\,\cdot\,)\right\}(\Theta(\eta,\xi))\end{aligned}$$

is a frozen kernel of type λ modeled on Γ^T. Analogously one can prove the converse. □

We now have to deal with the relations between operators of type λ and the differential operators represented by the vector fields $\widetilde{X}_i$. This is a study which was carried out in [Rothschild and Stein 1976, Section 14] and adapted to nonvariational operators in [Bramanti and Brandolini 2000a]. We are interested in two main results. Roughly speaking, the first says that the composition, in any order, of an operator of type λ with the $\widetilde{X}_i$ or $\widetilde{X}_0$ derivative is an operator of type $\lambda-1$ or $\lambda-2$, respectively. The second says that the $\widetilde{X}_i$ derivative of an operator of type λ can be rewritten as the sum of other operators of type λ, each acting on a different $\widetilde{X}_j$ derivative, plus a suitable remainder. In [Rothschild and Stein 1976] these results are proved only for a system of Hörmander vector fields of weight one (that is, without the drift), and several arguments are very condensed. Hence we need to extend and modify some arguments in [Rothschild and Stein 1976, Section 14] to cover the present situation. Moreover, as in [Bramanti and Brandolini 2000a], we need to keep under careful control the dependence of any quantity on the frozen point ξ_0 appearing in $\Gamma(\xi_0,\cdot)$. For these and other technical reasons, we prefer to write complete proofs of these properties. The first result is the following.

Theorem 4.11 [Rothschild and Stein 1976, Theorem 8]. *Suppose $T(\xi_0)$ is a frozen operator of type $\lambda\geq 1$. Then $\widetilde{X}_kT(\xi_0)$ and $T(\xi_0)\widetilde{X}_k$ $(k=1,2,\dots,q)$ are operators of type $\lambda-1$. If $\lambda\geq 2$, then $\widetilde{X}_0T(\xi_0)$ and $T(\xi_0)\widetilde{X}_0$ are operators of type $\lambda-2$.*

To prove this, we begin by stating the following two lemmas.

Lemma 4.12. *If $k(\xi_0;\xi,\eta)$ is a frozen kernel of type $\lambda\geq 1$ over $\widetilde{B}(\bar{\xi},R)$, then $(\widetilde{X}_jk)(\xi_0;\cdot,\eta)(\xi)$ $(j=1,2,\dots,q)$ is a frozen kernel of type $\lambda-1$. If $\lambda\geq 2$, then $(\widetilde{X}_0k)(\xi_0;\cdot,\eta)(\xi)$ is a frozen kernel of type $\lambda-2$.*

Proof. This basically follows by the definition of kernel of type λ and Theorem 3.2. When the $\widetilde{X}_j$ derivative acts on the ξ variable of a kernel $D_i^{\xi}\Gamma(\xi_0,\cdot)$, one also has to take into account Remark 4.6.

Here we just want to point out the following fact. The prototype of a frozen kernel of type 2 is the function

$$a(\xi)\Gamma(\xi_0;\Theta(\eta,\xi))b(\eta).$$

Note that the computation

$$\widetilde{X}_i[a(\cdot)\Gamma(\xi_0;\Theta(\eta,\cdot))b(\eta)](\xi)=a(\xi)[(Y_i+R_i^\eta)\Gamma(\xi_0;\cdot)](\Theta(\eta,\xi))b(\eta)+(\widetilde{X}_i a)(\xi)\Gamma(\xi_0;\Theta(\eta,\xi))b(\eta)$$

in particular generates the term

$$a(\xi)(R_i^\eta\Gamma)(\xi_0;\cdot)(\Theta(\eta,\xi))b(\eta),$$

where the differential operator R_i^η has coefficients depending on η. In the proof of Theorem 4.11 we will see another basic computation on frozen kernels which generates differential operators with coefficients also depending on ξ. This is the reason why Definition 4.5 allows for this kind of dependence. □

Lemma 4.13. *If $T(\xi_0)$ is a frozen operator of type $\lambda\geq 1$ over $\widetilde{B}(\bar{\xi},R)$, then $\widetilde{X}_i T(\xi_0)$ $(i=1,2,\dots,q)$ is a frozen operator of type $\lambda-1$. If $\lambda\geq 2$, then $\widetilde{X}_0 T(\xi_0)$ is a frozen operator of type $\lambda-2$.*

Proof. With reference to Definition 4.5, it is enough to consider the part k' of the kernel of T, the proof for k'' being completely analogous. So, let us consider the operator

$$\widetilde{X}_i T(\xi_0)\quad (i=1,2,\dots,q),$$

where $T(\xi_0)$ has kernel k'.

If $\lambda>1$, the result immediately follows by the previous lemma. If $\lambda=1$, then

$$T(\xi_0)f(\xi)=\int_{\widetilde{B}(\bar{\xi},R)} a(\xi)b(\eta)D_1\Gamma(\xi_0;\Theta(\eta,\xi))f(\eta)\,d\eta+T'(\xi_0)f(\xi),$$

where $T'(\xi_0)$ is a frozen operator of type 2 and D_1 is a 1-homogeneous differential operator. We already know that $\widetilde{X}_i T'(\xi_0)$ is a frozen operator of type 1, so it remains to show that

$$\widetilde{X}_i\int_{\widetilde{B}(\bar{\xi},R)} a(\xi)b(\eta)D_1\Gamma(\xi_0;(\Theta(\eta,\xi)))f(\eta)\,d\eta$$

is a frozen operator of type 0. To do this, we have to apply a distributional argument, which will be used several times in the following. Let us compute, for any $\omega\in C_0^\infty(\widetilde{B}(\bar{\xi},R))$,

$$\begin{aligned}\int_{\widetilde{B}(\bar{\xi},R)}\widetilde{X}_i^T\omega(\xi)\int_{\widetilde{B}(\bar{\xi},R)} a(\xi)b(\eta)D_1^\xi\Gamma(\xi_0;(\Theta(\eta,\xi)))f(\eta)\,d\eta\,d\xi\\ =\lim_{\varepsilon\to 0}\int_{\widetilde{B}(\bar{\xi},R)}\widetilde{X}_i^T\omega(\xi)\int_{\widetilde{B}(\bar{\xi},R)} a(\xi)b(\eta)\varphi_\varepsilon(\Theta(\eta,\xi))D_1^\xi\Gamma(\xi_0;(\Theta(\eta,\xi)))f(\eta)\,d\eta\,d\xi,\end{aligned}$$

where $\varphi_\varepsilon(u)=\varphi(D(\varepsilon^{-1})u)$ and $\varphi\in C_0^\infty(\mathbb{R}^N)$, $\varphi(u)=0$ for $\|u\|<1$, $\varphi(u)=1$ for $\|u\|>2$. Here we have written D_1^ξ to recall that the coefficients of the differential operator D_1 also depend (smoothly) on ξ

as a parameter. By Theorem 3.2,

$$\begin{aligned}
&\int_{\widetilde{B}(\bar{\xi},R)} \widetilde{X}_i^T \omega(\xi) \int_{\widetilde{B}(\bar{\xi},R)} a(\xi)b(\eta)\varphi_\varepsilon(\Theta(\eta,\xi))D_1^\xi \Gamma(\xi_0;(\Theta(\eta,\xi)))f(\eta)\,d\eta\,d\xi \\
&\qquad= \int_{\widetilde{B}(\bar{\xi},R)} b(\eta)f(\eta)\int_{\widetilde{B}(\bar{\xi},R)} (\widetilde{X}_i^T\omega)(\xi)a(\xi)\varphi_\varepsilon(\Theta(\eta,\xi))D_1^\xi\Gamma(\xi_0;(\Theta(\eta,\xi)))\,d\xi\,d\eta \\
&\qquad= \int_{\widetilde{B}(\bar{\xi},R)} b(\eta)f(\eta)\int_{\widetilde{B}(\bar{\xi},R)} \omega(\xi)(\widetilde{X}_i a)(\xi)\varphi_\varepsilon(\Theta(\eta,\xi))D_1^\xi\Gamma(\xi_0;(\Theta(\eta,\xi)))\,d\xi\,d\eta \\
&\qquad\quad+ \int_{\widetilde{B}(\bar{\xi},R)} b(\eta)f(\eta)\int_{\widetilde{B}(\bar{\xi},R)} \omega(\xi)a(\xi)\varphi_\varepsilon(\Theta(\eta,\xi))(\widetilde{X}_i D_1^\xi)\Gamma(\xi_0;(\Theta(\eta,\xi)))\,d\xi\,d\eta \\
&\qquad\quad+ \int_{\widetilde{B}(\bar{\xi},R)} b(\eta)f(\eta)\int_{\widetilde{B}(\bar{\xi},R)} \omega(\xi)a(\xi)[(Y_i+R_i^\eta)(\varphi_\varepsilon D_1^\xi\Gamma(\xi_0;\cdot))](\Theta(\eta,\xi))\,d\xi\,d\eta \\
&\qquad=: A_\varepsilon + B_\varepsilon + C_\varepsilon. \qquad (4\text{-}4)
\end{aligned}$$

(For the meaning of the symbol $\widetilde{X}_i D_1^\xi$ appearing in the term B_ε, see Remark 4.6.) Now, for $\varepsilon \to 0$,

$$\begin{aligned}
A_\varepsilon \to &\int_{\widetilde{B}(\bar{\xi},R)} b(\eta)f(\eta)\int_{\widetilde{B}(\bar{\xi},R)} \omega(\xi)(\widetilde{X}_i a)(\xi)D_1\Gamma(\xi_0;(\Theta(\eta,\xi)))\,d\xi\,d\eta \\
&= \int_{\widetilde{B}(\bar{\xi},R)} f(\eta)S_1(\xi_0)\omega(\eta)\,d\eta = \int_{\widetilde{B}(\bar{\xi},R)} \omega(\eta)S_1(\xi_0)^T f(\eta)\,d\eta, \qquad (4\text{-}5)
\end{aligned}$$

where $S_1(\xi_0)$ is a frozen operator of type 1, and $S_1(\xi_0)^T$ is still a frozen operator of type 1, by Proposition 4.10. Next,

$$\begin{aligned}
B_\varepsilon \to &\int_{\widetilde{B}(\bar{\xi},R)} b(\eta)f(\eta)\int_{\widetilde{B}(\bar{\xi},R)} \omega(\xi)a(\xi)(\widetilde{X}_i D_1^\xi)\Gamma(\xi_0;(\Theta(\eta,\xi)))\,d\xi\,d\eta \\
&= \int_{\widetilde{B}(\bar{\xi},R)} f(\eta)S_1'(\xi_0)\omega(\eta)\,d\eta = \int_{\widetilde{B}(\bar{\xi},R)} \omega(\eta)S_1'(\xi_0)^T f(\eta)\,d\eta, \qquad (4\text{-}6)
\end{aligned}$$

where, by Remark 4.6, $S_1'(\xi_0)$ is a frozen operator of type 1, and the same is still true for $S_1'(\xi_0)^T$. Finally,

$$\begin{aligned}
C_\varepsilon = &\int_{\widetilde{B}(\bar{\xi},R)} b(\eta)f(\eta)\int_{\widetilde{B}(\bar{\xi},R)} \omega(\xi)a(\xi)[\varphi_\varepsilon Y_i D_1\Gamma(\xi_0;\cdot)](\Theta(\eta,\xi))\,d\xi\,d\eta \\
&+ \int_{\widetilde{B}(\bar{\xi},R)} b(\eta)f(\eta)\int_{\widetilde{B}(\bar{\xi},R)} \omega(\xi)a(\xi)[\varphi_\varepsilon R_i^\eta D_1\Gamma(\xi_0;\cdot)](\Theta(\eta,\xi))\,d\xi\,d\eta \\
&+ \int_{\widetilde{B}(\bar{\xi},R)} b(\eta)f(\eta)\int_{\widetilde{B}(\bar{\xi},R)} \omega(\xi)a(\xi)[(Y_i+R_i^\eta)\varphi_\varepsilon D_1\Gamma(\xi_0;\cdot)](\Theta(\eta,\xi))\,d\xi\,d\eta \\
=:\,& C_\varepsilon^1 + C_\varepsilon^2 + C_\varepsilon^3. \qquad (4\text{-}7)
\end{aligned}$$

Now

$$C_\varepsilon^1 \to \int_{\widetilde{B}(\bar{\xi},R)} \omega(\xi)\left\{\mathrm{PV}\int_{\widetilde{B}(\bar{\xi},R)} a(\xi)Y_i D_1\Gamma(\xi_0;\Theta(\eta,\xi))b(\eta)f(\eta)\,d\eta\right\}d\xi = \int_{\widetilde{B}(\bar{\xi},R)} \omega(\xi)T(\xi_0)f(\xi)\,d\xi, \quad (4\text{-}8)$$

where $T(\xi_0)$ is a frozen operator of type 0. Note that the principal value exists because the kernel $Y_i D_1\Gamma(\xi_0;u)$ has a vanishing integral over spherical shells $\{u\in\mathbb{G}: r_1<\|u\|<r_2\}$ (see Theorem 4.1).

$$C_\varepsilon^2 \to \int_{\widetilde{B}(\bar{\xi},R)} \omega(\xi)\left\{\int_{\widetilde{B}(\bar{\xi},R)} a(\xi)R_i^\eta D_1\Gamma(\xi_0;\Theta(\eta,\xi))b(\eta)f(\eta)\,d\eta\right\}d\xi = \int_{\widetilde{B}(\bar{\xi},R)} \omega(\xi)S(\xi_0)f(\xi)\,d\xi, \tag{4-9}$$

where $S(\xi_0)$ is a frozen operator of type 1. To handle C_ε^3, let us perform the change of variables $u=\Theta(\eta,\xi)$, which, by Theorem 3.3, gives

$$C_\varepsilon^3 = \int_{\widetilde{B}(\bar{\xi},R)} (bf)(\eta)\int_{\|u\|<R}(\omega a)(\Theta(\eta,\cdot)^{-1}(u))[(Y_i+R_i^\eta)\varphi_\varepsilon D_1\Gamma(\xi_0;\cdot)](u)\cdot c(\eta)(1+O(\|u\|))\,du\,d\eta.$$

On the other hand, $Y_i\varphi_\varepsilon(u)=(1/\varepsilon)Y_i\varphi(D(1/\varepsilon)u)$, while $R_i^\eta\varphi_\varepsilon(u)$ is uniformly bounded in ε. Hence the change of variables $D(1/\varepsilon)u=v$ gives

$$\begin{aligned}
C_\varepsilon^3 &= \int_{\widetilde{B}(\bar{\xi},R)} (bf)(\eta)\int_{\|v\|<R/\varepsilon}(\omega a)(\Theta(\eta,\cdot)^{-1}(D(\varepsilon)v))\left[\frac{1}{\varepsilon}Y_i\varphi(v)+O(1)\right]\\
&\qquad\qquad\qquad\qquad\cdot c(\eta)\varepsilon^{1-Q}D_1^\eta\Gamma(\xi_0;v)(1+O(\varepsilon\|v\|))\varepsilon^Q\,dv\,d\eta\\
&\to \int_{\widetilde{B}(\bar{\xi},R)} (bcf)(\eta)\int_{\|v\|\le 2}(\omega a)(\Theta(\eta,\cdot)^{-1}(0))Y_i\varphi(v)D_1^\eta\Gamma(\xi_0;v)\,dv\,d\eta\\
&= \int_{\widetilde{B}(\bar{\xi},R)} (\omega abcf)(\eta)\int_{\|v\|\le 2}Y_i\varphi(v)D_1^\eta\Gamma(\xi_0;v)\,dv\,d\eta\\
&= \int_{\widetilde{B}(\bar{\xi},R)} (\omega abcf)(\eta)\alpha(\xi_0,\eta)\,d\eta,
\end{aligned}\tag{4-10}$$

which is the integral of ω times the multiplicative part of a frozen operator of type 0. It is worthwhile (although not logically necessary to prove the theorem) to realize that the quantity $\alpha(\xi_0,\eta)$ appearing in (4-10) actually does not depend on the function φ. Namely, recalling that $Y_i\varphi(v)$ is supported in the spherical shell $1\le\|v\|\le 2$ with $\varphi(u)=1$ for $\|u\|=2$ and $\varphi(u)=0$ for $\|u\|=1$, an integration by parts gives

$$\int_{1\le\|v\|\le 2} Y_i\varphi(v)D_1^\eta\Gamma(\xi_0;v)\,dv = -\int_{1\le\|v\|\le 2}\varphi(v)Y_iD_1^\eta\Gamma(\xi_0;v)\,dv+\int_{\|v\|=2}D_1^\eta\Gamma(\xi_0;v)n_i\,d\sigma(v)$$

with $n_i=\sum_{j=1}^N b_{ij}(u)\nu_j$, where $Y_i=\sum_{j=1}^N b_{ij}(u)\partial_{u_j}$ and ν is the outer normal on $\|v\|=2$. The vanishing property of the kernel $Y_iD_1^\xi\Gamma(\xi_0;\cdot)$ implies that if φ is a radial function, the first integral vanishes. Therefore,

$$\alpha(\xi_0,\eta)=\int_{\|v\|=2}D_1^\eta\Gamma(\xi_0;v)n_i\,d\sigma(v),$$

which also shows that $\alpha(\xi_0,\eta)$ smoothly depends on η and is bounded in ξ_0 (by Theorem 4.3). By (4-4)–(4-6) and (4-8)–(4-10) we have therefore proved that

$$\widetilde{X}_iT(\xi_0)f(\xi)=S_1(\xi_0)^Tf(\xi)+S_1'(\xi_0)^Tf(\xi)+T(\xi_0)f(\xi)+\alpha(\xi_0,\xi)(abcf)(\xi),$$

which is a frozen operator of type 0. This completes the proof of the first statement. The proof of the fact that if $\lambda \geq 2$, then $\widetilde{X}_0 T(\xi_0)$ is a frozen operator of type $\lambda - 2$ is completely analogous. $\square$

The above two lemmas imply the assertion on $\widetilde{X}_k T(\xi_0)$ and $\widetilde{X}_0 T(\xi_0)$ in Theorem 4.11. To prove the assertions about $T(\xi_0)\widetilde{X}_k$ and $T(\xi_0)\widetilde{X}_0$ we need a way to express ξ-derivatives of the integral kernel in terms of η-derivatives of the kernel, in order to integrate by parts. This will involve the use of *right invariant vector fields* on the group $\mathbb{G}$: throughout the following, we will denote by

$$Y_{i,k}^R$$

the right invariant vector field on $\mathbb{G}$ satisfying $Y_{i,k}^R f(0) = Y_{i,k} f(0)$.

Lemma 4.14. *For any $f \in C_0^\infty(\mathbb{G})$ and η, ξ in a neighborhood of ξ_0, we can write, for any $i = 1, 2, \ldots, s$, $k = 1, 2, \ldots, k_i$ (recall s is the step of the Lie algebra),*

$$\widetilde{X}_{i,k}[f(\Theta(\cdot\,,\xi))](\eta) = -(Y_{i,k}^R f)(\Theta(\eta,\xi)) + ((R_{i,k}^\xi)' f)(\Theta(\eta,\xi)), \tag{4-11}$$

where $(R_{i,k}^\xi)'$ is a vector field of local degree $\leq i-1$ smoothly depending on ξ.

Proof. We start with the following.

Claim. For any function f defined on $\mathbb{G}$, let

$$f'(u) = f(-u)$$

(recall that $-u = u^{-1}$); then the following identities hold:

$$Y_{i,k}(f') = -(Y_{i,k}^R f)'. \tag{4-12}$$

Proof. Let us define the vector fields $\widehat{Y}_{i,k}$ by

$$Y_{i,k}(f') = -(\widehat{Y}_{i,k} f)'. \tag{4-13}$$

Then, for any $a \in \mathbb{G}$, denoting by L_a, R_a the corresponding operators of left and right translation, respectively (acting on functions), we have

$$(\widehat{Y}_{i,k} R_a f)' = -Y_{i,k}((R_a f)') = -Y_{i,k}(L_{-a} f') = -L_{-a} Y_{i,k} f' = L_{-a}(-Y_{i,k} f') = L_{-a}(\widehat{Y}_{i,k} f)' = (R_a \widehat{Y}_{i,k} f)',$$

hence $\widehat{Y}_{i,k}$ are right invariant vector fields. Also, note that, for any vector field $Y = \sum a_j(u)\partial_{u_j}$, we have

$$Y(f')(0) = -(Yf)(0),$$

because

$$Y(f')(u) = \sum a_j(u)\partial_{u_j}[f(-u)] = -\sum a_j(u)(\partial_{u_j} f)(-u) \text{ implies}$$
$$Y(f')(0) = -\sum a_j(0)(\partial_{u_j} f)(0) = -(Yf)(0).$$

Hence, by (4-13), we know that $\widehat{Y}_k f(0) = Y_k f(0)$. Therefore $\widehat{Y}_k$ is the right invariant vector field which coincides with Y_k at the origin, that is, $\widehat{Y}_k = Y_k^R$. $\square$

By (3-4) and (4-12),

$$\begin{aligned}\widetilde{X}_{i,k}[f(\Theta(\,\cdot\,,\xi))](\eta)\\ &= \widetilde{X}_{i,k}[f'(\Theta(\xi,\,\cdot\,))](\eta) = (Y_{i,k}f' + R^{\xi}_{i,k}f')(\Theta(\xi,\eta))\\ &= -(Y^R_{i,k}f)'(\Theta(\xi,\eta)) + R^{\xi}_{i,k}f'(\Theta(\xi,\eta)) = -(Y^R_{i,k}f)(\Theta(\eta,\xi)) + ((R^{\xi}_{i,k})'f)(\Theta(\eta,\xi)), \qquad (4\text{-}14)\end{aligned}$$

where

$$((R^{\xi}_{i,k})'f)(u) = (R^{\xi}_{i,k}f')(-u)$$

is a differential operator of degree $\le i-1$. This proves (4-11). □

Proof of Theorem 4.11. As we noted after Lemma 4.13, we are left to prove the assertion about $T(\xi_0)\widetilde{X}_i$ and $T(\xi_0)\widetilde{X}_0$. We only give the proof for the case $\lambda \ge 1$, $i = 1, \ldots, q$. The proof for $\lambda \ge 2$, $i = 0$ being very similar. Like in the proof of Lemma 4.13, it is enough to consider the part k' of the kernel of T, the proof for k'' being completely analogous (see Definition 4.5). Let us expand

$$k'(\xi_0;\xi,\eta) = \left\{\sum_{j=1}^{H_m} a_j(\xi)b_j(\eta)D_j\Gamma(\xi_0;\,\cdot\,) + a_0(\xi)b_0(\eta)D_0\Gamma(\xi_0;\,\cdot\,)\right\}(\Theta(\eta,\xi)),$$

where $D_0\Gamma(\xi_0;\,\cdot\,)$ has bounded Y_i-derivatives ($i = 1, 2, \ldots, q$). We can consider each of the terms

$$T_j(\xi_0)\widetilde{X}_i f(\xi) \equiv \int a_j(\xi)b_j(\eta)D^{\eta}_j\Gamma(\xi_0;\Theta(\eta,\xi))\widetilde{X}_i f(\eta)\,d\eta$$

(this time it is important to recall the η-dependence of the coefficients of D_j) and distinguish 2 cases:

(i) $D_j\Gamma$ is homogeneous of degree $\ge 2-Q$ or it is regular (that is, $D_j\Gamma$ has bounded Y_i-derivatives);

(ii) $T_j(\xi_0)$ is a frozen operator of type 1 and $D_j\Gamma$ is homogeneous of degree $1-Q$.

Case (i). We can integrate by parts, recalling that the transpose of $\widetilde{X}_i$ is

$$(\widetilde{X}_i)^T g(\eta) = -\widetilde{X}_i g(\eta) + c_i(\eta)g(\eta)$$

with c_i smooth functions:

$$\begin{aligned}&T_j(\xi_0)\widetilde{X}_i f(\xi)\\ &= \int c_i(\eta)a_j(\xi)b_j(\eta)D^{\eta}_j\Gamma(\xi_0;\Theta(\eta,\xi))f(\eta)\,d\eta - \int a_j(\xi)(\widetilde{X}_i b_j)(\eta)D^{\eta}_j\Gamma(\xi_0;\Theta(\eta,\xi))f(\eta)\,d\eta\\ &\quad - \int a_j(\xi)b_j(\eta)\widetilde{X}_i[D^{\eta}_j\Gamma(\xi_0;\Theta(\,\cdot\,,\xi))](\eta)f(\eta)\,d\eta - \int a_j(\xi)b_j(\eta)(\widetilde{X}^{\eta}_i D^{\eta}_j)\Gamma(\xi_0;\Theta(\eta,\xi))f(\eta)\,d\eta\\ &= A(\xi)+B(\xi)+C(\xi)+D(\xi).\end{aligned}$$

Now, $A(\xi)+B(\xi)$ is still an operator of type λ, applied to f; in particular, it can be seen as operator of type $\lambda-1$; the same is true for $D(\xi)$, by Remark 4.6. To study $C(\xi)$, we apply Lemma 4.14, which gives

$$\widetilde{X}_i[D^{\eta}_j\Gamma(\xi_0;\Theta(\,\cdot\,,\xi))](\eta) = -(Y^R_i D^{\eta}_j\Gamma)(\xi_0,\Theta(\eta,\xi)) + ((R^{\xi}_i)'D^{\eta}_j\Gamma)(\xi_0,\Theta(\eta,\xi)).$$

Since Y_i^R is homogeneous of degree 1, $a_j(\xi)b_j(\eta)Y_i^R D_j^\eta \Gamma(\xi_0, \Theta(\eta,\xi))$ is a kernel of type $\lambda - 1$. Since $(R_i^\xi)'$ is a differential operator of degree ≤ 0, the kernel $a_j(\xi)b_j(\eta)((R_i^\xi)' D_j^\eta \Gamma)(\xi_0, \Theta(\eta,\xi))$ is of type λ.

Note that, even when the coefficients of the differential operator D_j (in the expression $D_j\Gamma(\xi_0; \Theta(\eta,\xi))$) do not depend on ξ and η, this procedure introduces, with the operator $(R_i^\xi)'$, a new ξ-dependence of the coefficients. Compare this with our remark in the proof of Lemma 4.12.

Case (ii). In this case the kernel $(Y_i^R D_j \Gamma)$ is singular, so that the computation must be handled with more care. We can write

$$T_j(\xi_0)\widetilde{X}_i f(\xi) = \lim_{\varepsilon\to 0} \int a_j(\xi)b_j(\eta)\varphi_\varepsilon(\Theta(\xi,\eta))D_j\Gamma(\xi_0; \Theta(\eta,\xi))\widetilde{X}_i f(\eta)\, d\eta \equiv \lim_{\varepsilon\to 0} T_\varepsilon(\xi)$$

with φ_ε as in the proof of Lemma 4.13. Note that, choosing a radial φ, we have $\varphi_\varepsilon(\Theta(\xi,\eta)) = \varphi_\varepsilon(\Theta(\eta,\xi))$. Then

$$\begin{aligned} T_\varepsilon(\xi) = &\int c_i(\eta)a_j(\xi)b_j(\eta)\varphi_\varepsilon(\Theta(\xi,\eta))D_j\Gamma(\xi_0; \Theta(\eta,\xi)) f(\eta)\, d\eta \\ &- \int a_j(\xi)(\widetilde{X}_i b_j)(\eta)\varphi_\varepsilon(\Theta(\xi,\eta))D_j\Gamma(\xi_0; \Theta(\eta,\xi)) f(\eta)\, d\eta \\ &- \int a_j(\xi)b_j(\eta)\widetilde{X}_i[\varphi_\varepsilon(\Theta(\,\cdot\,,\xi))D_j\Gamma(\xi_0; \Theta(\,\cdot\,,\xi))](\eta) f(\eta)\, d\eta \\ &- \int a_j(\xi)b_j(\eta)\varphi_\varepsilon(\Theta(\xi,\eta))(\widetilde{X}_i^\eta D_j^\eta)\Gamma(\xi_0; \Theta(\eta,\xi)) f(\eta)\, d\eta \\ =:&\ A_\varepsilon(\xi) + B_\varepsilon(\xi) + C_\varepsilon(\xi) + D_\varepsilon(\xi). \end{aligned}$$

Now $A_\varepsilon(\xi) + B_\varepsilon(\xi) + D_\varepsilon(\xi)$ converge to an operator of type λ, as $A(\xi)$, $B(\xi)$, $D(\xi)$ are in Case (i), while, by Theorem 3.2 and Lemma 4.14,

$$\begin{aligned} C_\varepsilon(\xi) = &- \int a_j(\xi)b_j(\eta) f(\eta)(Y_i\varphi_\varepsilon)(\Theta(\eta,\xi))D_j\Gamma(\xi_0; \Theta(\eta,\xi))\, d\eta \\ &- \int a_j(\xi)b_j(\eta) f(\eta)(R_i^\xi\varphi_\varepsilon)(\Theta(\eta,\xi))D_j\Gamma(\xi_0; \Theta(\eta,\xi))\, d\eta \\ &+ \int a_j(\xi)b_j(\eta) f(\eta)\varphi_\varepsilon(\Theta(\eta,\xi))(Y_i^R D_j\Gamma)(\xi_0, \Theta(\eta,\xi))\, d\eta \\ &- \int a_j(\xi)b_j(\eta) f(\eta)\varphi_\varepsilon(\Theta(\eta,\xi))((R_i^\xi)' D_j\Gamma)(\xi_0, \Theta(\eta,\xi))\, d\eta \\ =:&\ E_\varepsilon(\xi) + F_\varepsilon(\xi) + G_\varepsilon(\xi) + H_\varepsilon(\xi). \end{aligned}$$

Now $H_\varepsilon(\xi)$ tends to an operator of type 1 and $G_\varepsilon(\xi)$ tends to

$$\mathrm{PV}\int a_j(\xi)b_j(\eta) f(\eta)(Y_i^R D_j\Gamma)(\xi_0, \Theta(\eta,\xi))\, d\eta,$$

which is an operator of type 0. As to $E_\varepsilon(\xi)$, the same computation as in the proof of Lemma 4.13 gives

$$E_\varepsilon(\xi) \to \alpha(\xi_0,\xi)(abcf)(\xi)$$

with

$$\alpha(\xi_0, \xi) = \int Y_i \varphi(v) D_1^{\xi} \Gamma(\xi_0; v)\, dv,$$

which is the multiplicative part of an operator of type 0. A similar computation shows that $F_\varepsilon(\xi) \to 0$. □

Let us come to the second main result of this section. In [Rothschild and Stein 1976, corollary on p. 296], the following fact is proved for a family of Hörmander's vector fields without the drift $\widetilde{X}_0$: for any frozen operator $T(\xi_0)$ of type 1, $i = 1, 2, \ldots, q$, there exist operators $T_{ij}(\xi_0)$, $T_i(\xi_0)$ of type 1 such that

$$\widetilde{X}_i T(\xi_0) = \sum_{j=1}^{q} T_{ij}(\xi_0) \widetilde{X}_j + T_i(\xi_0).$$

This possibility of exchanging the order of integral and differential operators will be crucial in the proof of representation formulas. However, such an identity cannot be proved in this form when the drift $\widetilde{X}_0$ is present. Instead, we are going to prove the following, which will be enough for our purposes.

Theorem 4.15. *If $T(\xi_0)$ is a frozen operator of type $\lambda \geq 1$, $i = 1, 2, \ldots, q$, then*

$$\widetilde{X}_i T(\xi_0) = \sum_{k=1}^{q} T_k^i(\xi_0) \widetilde{X}_k + \sum_{h,j=1}^{q} \tilde{a}_{hj}(\xi_0) T^{hi}(\xi_0) \widetilde{X}_j + T_0^i(\xi_0) + T^i(\xi_0) \widetilde{\mathscr{L}}_0, \tag{4-15}$$

where $T_k^i(\xi_0)$ $(k = 0, 1, \ldots, q)$ and $T^{hi}(\xi_0)$ are frozen operators of type λ, $T^i(\xi_0)$ are frozen operators of type $\lambda + 1$, and $\tilde{a}_{hj}(\xi_0)$ are the frozen coefficients of $\widetilde{\mathscr{L}}_0$.

If $T(\xi_0)$ is a frozen operator of type $\lambda \geq 2$, then

$$\widetilde{X}_0 T(\xi_0) = \sum_{k=1}^{q} T_k(\xi_0) \widetilde{X}_k + \sum_{h,j=1}^{q} \tilde{a}_{hj}(\xi_0) T^h(\xi_0) \widetilde{X}_j + T_0(\xi_0) + T(\xi_0) \widetilde{\mathscr{L}}_0, \tag{4-16}$$

where $T_k(\xi_0)$ $(k = 0, 1, \ldots, q)$ and $T^h(\xi_0)$ are frozen operators of type $\lambda - 1$, $T(\xi_0)$ is a frozen operator of type λ.

We start with the following lemma, similar to that proved in [Rothschild and Stein 1976, p. 296].

Lemma 4.16. *For any vector field $\widetilde{X}_{j_0,k_0}$ $(j_0 = 1, 2, \ldots, s$, $k_0 = 1, 2, \ldots, k_{j_0})$, there exist smooth functions*

$$\{a_{jk}^{j_0 k_0 \eta}\}_{\substack{j=1,2,\ldots,s \\ k=1,2,\ldots,h_j}}$$

having local degree $\geq \max\{j - j_0, 0\}$ and smoothly depending on η, such that, for any $f \in C_0^\infty(\mathbb{G})$, one can write

$$\widetilde{X}_{j_0,k_0}[f(\Theta(\eta, \cdot))](\xi) = \sum_{\substack{j=1,2,\ldots,s \\ k=1,2,\ldots,k_j}} a_{jk}^{j_0 k_0 \eta}(\Theta(\eta, \xi)) \widetilde{X}_{j,k}[f(\Theta(\cdot, \xi))](\eta) + (R_{j_0,k_0}^{\xi,\eta} f)(\Theta(\eta, \xi)), \tag{4-17}$$

where $R_{j_0}^{\xi,\eta}$ is a vector field of local degree $\leq j_0 - 1$, smoothly depending on ξ, η.

Proof. By Theorem 3.2 we know that

$$\widetilde{X}_{j_0,k_0}[f(\Theta(\eta,\cdot))](\xi)=(Y_{j_0,k_0}f+R^{\eta}_{j_0,k_0}f)(\Theta(\eta,\xi))\equiv(Z^{\eta}_{j_0,k_0}f)(\Theta(\eta,\xi)), \tag{4-18}$$

where $Z^{\eta}_{j_0,k_0}$ is a vector field of local degree $\le j_0$, smoothly depending on η. To rewrite $(Z^{\eta}_{j_0,k_0}f)$ in a suitable form, we start from the following identities:

$$Y_{i,k}=\frac{\partial}{\partial u_{ik}}+\sum_{i<l\le s}\sum_{r=1}^{k_l}g^{ik}_{lr}(u)\frac{\partial}{\partial u_{lr}} \tag{4-19}$$

for any $i=1,2,\dots,s$ and $k=1,2,\dots,k_i$;

$$Y_{i,k}=\sum g^{ik}_{lr}(u)Y^{R}_{l,r}, \tag{4-20}$$

where $g^{ik}_{lr}(u)$ are homogeneous of degree $l-i$; see [Rothschild and Stein 1976, p. 295]. Hence we can write

$$Z^{\eta}_{j_0,k_0}=\sum a^{\eta}_{jk}(u)\frac{\partial}{\partial u_{jk}},$$

where a_{jk} has local degree $\ge j-j_0$ and smoothly depends on η. By inverting (for any i,k) the triangular system (4-19), we obtain

$$\frac{\partial}{\partial u_{jk}}=Y_{j,k}+\sum_{j<l\le s}\sum_{r=1}^{k_l}f^{jk}_{lr}(u)Y_{l,r},$$

where each $f^{jk}_{lr}(u)$ is homogeneous of degree $l-j$. Also using (4-20), we have

$$(Z^{\eta}_{j_0,k_0}f)(u)=\sum a^{\eta}_{jk}(u)[(Y_{j,k}f)(u)+\sum_{j<l\le s}f^{jk}_{lr}(u)(Y_{l,r}f)(u)]=\sum b^{\eta}_{lr}(u)(Y^{R}_{l,r}f)(u), \tag{4-21}$$

where

$$b^{\eta}_{lr}\text{ has local degree}\ge\max\{l-j_0,0\} \tag{4-22}$$

and smoothly depends on η. Then, by Lemma 4.14,

$$(Z^{\eta}_{j_0,k_0}f)(\Theta(\eta,\xi))=\sum_{l,r}-b^{\eta}_{lr}(\Theta(\eta,\xi))\widetilde{X}_{l,r}[f(\Theta(\cdot\,,\xi))](\eta)+\sum_{l,r}(b^{\eta}_{lr}(R^{\xi}_{l,r})'f)(\Theta(\eta,\xi)), \tag{4-23}$$

where $(R^{\xi}_{l,r})'$ is a differential operator of local degree $\le l-1$, hence the differential operator on $\mathbb{G}$

$$R^{\xi,\eta}_{j_0,k_0}\equiv\sum_{l,r}b^{\eta}_{lr}(R^{\xi}_{l,r})'\text{ has local degree}\le j_0-1 \tag{4-24}$$

and depends smoothly on ξ,η. Collecting (4-18), (4-22), (4-23), (4-24), the lemma is proved, with $a^{j_0k_0\eta}_{jk}=-b^{\eta}_{jk}$. $\square$

Thanks to this lemma, we can prove the following, which is similar to [Rothschild and Stein 1976, Theorem 9].

Theorem 4.17. (i) *Suppose $T(\xi_0)$ is a frozen operator of type $\lambda \geq 1$. Given a vector field $\widetilde{X}_i$ for $i = 1, 2, \dots, q$, there exist frozen operators $T^i(\xi_0)$ of type λ, and $T^i_{jk}(\xi_0)$, frozen operators of type $\lambda + j - 1$, such that*

$$\widetilde{X}_i T(\xi_0) = \sum_{j,k} T^i_{jk}(\xi_0) \widetilde{X}_{j,k} + T^i(\xi_0). \tag{4-25}$$

(ii) *Suppose $T(\xi_0)$ is a frozen operator of type $\lambda \geq 2$. There exist $T^0(\xi_0)$ and $T^0_{jk}(\xi_0)$, frozen operators of type $\lambda - 1$ and $\lambda + \max\{j-2, 0\}$, respectively, such that*

$$\widetilde{X}_0 T(\xi_0) = \sum_{j,k} T^0_{jk}(\xi_0) \widetilde{X}_{j,k} + T^0(\xi_0). \tag{4-26}$$

Proof. First of all, it is enough to consider the part k' of the kernel of $T(\xi_0)$, the proof for k'' being completely analogous (see Definition 4.5).

(i) If $T(\xi_0)$ is a frozen operator of type $\lambda \geq 1$ with kernel k', we can write it as

$$T(\xi_0) f(\xi) = \int a(\xi) D\Gamma(\xi_0; \Theta(\eta, \xi)) b(\eta) f(\eta)\, d\eta + T'(\xi_0) f(\xi),$$

where $D\Gamma(\xi_0, \cdot)$ is homogeneous of degree $\lambda - Q$ and $T'(\xi_0)$ is a frozen operator of degree $\lambda + 1$. Since $\widetilde{X}_i T'(\xi_0)$ is a frozen operator of type λ, it already has the form $T^i(\xi_0)$ required by the theorem, hence it is enough to prove that

$$\widetilde{X}_i \int a(\xi) D\Gamma(\xi_0; \Theta(\eta, \xi)) b(\eta) f(\eta)\, d\eta$$

can be rewritten in the form

$$\sum_{j,k} T^i_{jk}(\xi_0) \widetilde{X}_{j,k} f(\xi) + T^i(\xi_0) f(\xi)$$

with $T^i_{jk}(\xi_0)$ and $T^i(\xi_0)$ frozen operators of type $\lambda + j - 1$ and λ, respectively. Next, we have to distinguish two cases.

Case 1: $\lambda \geq 2$. In this case the $\widetilde{X}_i$ derivative can be taken under the integral sign, writing

$$\begin{aligned}
&\widetilde{X}_i \int a(\xi) D\Gamma(\xi_0; \Theta(\eta, \xi)) b(\eta) f(\eta)\, d\eta \\
&\qquad = \int (\widetilde{X}_i a)(\xi) D\Gamma(\xi_0; \Theta(\eta, \xi)) b(\eta) f(\eta)\, d\eta + \int a(\xi) \widetilde{X}_i [D\Gamma(\Theta(\eta, \cdot))](\xi) b(\eta) f(\eta)\, d\eta \\
&\qquad =: A(\xi) + B(\xi).
\end{aligned}$$

Now $A(\xi)$ is a frozen operator of type λ, while applying Lemma 4.16 with $j_0 = 1$ we get

$$\begin{aligned}
B(\xi) = &\int a(\xi) \sum_{l,r} a^i_{lr}(\Theta(\eta, \xi)) \widetilde{X}_{l,r} [D\Gamma(\xi_0; \Theta(\cdot\,, \xi))](\eta) b(\eta) f(\eta)\, d\eta \\
&+ \int a(\xi) (R^\xi_i D\Gamma)(\xi_0; \Theta(\eta, \xi)) b(\eta) f(\eta)\, d\eta \\
=:&\ C(\xi) + D(\xi),
\end{aligned}$$

where R_i^ξ are differential operators of local degree ≤ 0, and the a_{lr}^i have local degree $\geq l-1$. Hence D is a frozen operator of type λ, while, since the transposed vector field of $\widetilde{X}_{l,r}$ is

$$\widetilde{X}_{l,r}^T = -\widetilde{X}_{l,r} + c_{l,r}$$

with $c_{l,r}$ smooth functions,

$$\begin{aligned} C(\xi) = & -a(\xi) \sum_{l,r} \int \widetilde{X}_{l,r}[a_{lr}^i(\Theta(\,\cdot\,,\xi))b(\,\cdot\,)](\eta) D\Gamma(\xi_0;\Theta(\eta,\xi)) f(\eta)\, d\eta \\ & + a(\xi) \sum_{l,r} \int a_{lr}^i(\Theta(\eta,\xi)) D\Gamma(\xi_0;\Theta(\eta,\xi)) c_{l,r}(\eta) b(\eta) f(\eta)\, d\eta \\ & - a(\xi) \sum_{l,r} \int a_{lr}^i(\Theta(\eta,\xi)) D\Gamma(\xi_0;\Theta(\eta,\xi)) b(\eta) \widetilde{X}_{l,r} f(\eta)\, d\eta. \end{aligned}$$

The first two terms in the last expression are still frozen operators of type λ applied to f, while the third is a sum of operators of type $\lambda+l-1$ applied to $\widetilde{X}_{l,r} f$, as required by the theorem.

Case 2: $\lambda = 1$. In this case we have to compute the derivative of the integral in a distributional sense, as was already done in the proof of Lemma 4.13. With the same meaning of φ_ε, let us compute

$$\lim_{\varepsilon\to 0} \widetilde{X}_i \int a(\xi)\varphi_\varepsilon(\Theta(\eta,\xi)) D\Gamma(\xi_0;\Theta(\eta,\xi)) b(\eta) f(\eta)\, d\eta.$$

Actually, this gives exactly the same result as in case 1:

$$\begin{aligned} &\widetilde{X}_i \int a(\xi)\varphi_\varepsilon(\Theta(\eta,\xi)) D\Gamma(\xi_0;\Theta(\eta,\xi)) b(\eta) f(\eta)\, d\eta \\ &= \int (\widetilde{X}_i a)(\xi)\varphi_\varepsilon(\Theta(\eta,\xi)) D\Gamma(\xi_0;\Theta(\eta,\xi)) b(\eta) f(\eta)\, d\eta + \int a(\xi)\widetilde{X}_i[(\varphi_\varepsilon D\Gamma)(\Theta(\eta,\cdot))](\xi) b(\eta) f(\eta)\, d\eta \\ &= A_\varepsilon(\xi) + B_\varepsilon(\xi), \end{aligned}$$

where $A_\varepsilon(\xi) \to \int (\widetilde{X}_i a)(\xi) D\Gamma(\xi_0;\Theta(\eta,\xi)) b(\eta) f(\eta)\, d\eta$ and

$$\begin{aligned} B_\varepsilon(\xi) = & \int a(\xi) \sum_{l,r} a_{lr}^i(\Theta(\eta,\xi)) \widetilde{X}_{l,r}[\varphi_\varepsilon(\Theta(\,\cdot\,,\xi)) D\Gamma(\xi_0;\Theta(\,\cdot\,,\xi))](\eta) b(\eta) f(\eta)\, d\eta \\ & + \int a(\xi)(R_i^\xi(\varphi_\varepsilon D\Gamma))(\xi_0;\Theta(\eta,\xi)) b(\eta) f(\eta)\, d\eta \\ =: & \; C_\varepsilon(\xi) + D_\varepsilon(\xi), \end{aligned}$$

where $C_\varepsilon(\xi)$ converges to the expression called $C(\xi)$ in the computation of case 1; as for $D_\varepsilon(\xi)$,

$$R_i^\xi(\varphi_\varepsilon D\Gamma) = (R_i^\xi \varphi_\varepsilon) D\Gamma + \varphi_\varepsilon R_i^\xi D\Gamma.$$

Now, $\varphi_\varepsilon R_i^\xi D\Gamma \to R_i^\xi D\Gamma$ while $(R_i^\xi \varphi_\varepsilon) D\Gamma \to 0$, R_i^ξ being a vector field of local degree ≤ 0. Hence $D_\varepsilon(\xi)$ also converges to the expression called $D(\xi)$ in the computation of case 1, and we are done.

(ii) Now let $T(\xi_0)$ be a frozen operator of type $\lambda \geq 2$ with kernel k'. As in (i), it is enough to prove that

$$\widetilde{X}_0 \int a(\xi) D\Gamma(\xi_0; \Theta(\eta, \xi)) b(\eta) f(\eta)\, d\eta,$$

where $D\Gamma$ is homogeneous of degree $\lambda - Q$ can be rewritten in the form

$$\sum_{j,k} T^0_{jk}(\xi_0) \widetilde{X}_{j,k} f(\xi) + T^0(\xi_0) f(\xi)$$

with $T^0_{jk}(\xi_0)$ and $T^0(\xi_0)$ frozen operators of type $\lambda + j - 2$ and $\lambda - 1$, respectively. Let us consider only the case $\lambda \geq 3$, the case $\lambda = 2$ being handled with the modification seen in (i), Case 2. By Lemma 4.16,

$$\begin{aligned}\widetilde{X}_0 \int a(\xi) D\Gamma(\xi_0; \Theta(\eta, \xi)) b(\eta) f(\eta)\, d\eta \\ &= \int (\widetilde{X}_0 a)(\xi) D\Gamma(\xi_0; \Theta(\eta, \xi)) b(\eta) f(\eta)\, d\eta \\ &\quad + \int a(\xi) \sum_{l,r} a^0_{lr}(\Theta(\eta, \xi)) \widetilde{X}_{l,r}[D\Gamma(\xi_0; \Theta(\,\cdot\,, \xi))](\eta) b(\eta) f(\eta)\, d\eta \\ &\quad + \int a(\xi) (R^\xi_0 D\Gamma)(\xi_0; \Theta(\eta, \xi)) b(\eta) f(\eta)\, d\eta \\ &=: A(\xi) + C(\xi) + D(\xi),\end{aligned}$$

where R^ξ_0 are now differential operators of local degree ≤ 1, and the a^0_{lr} have local degree $\geq \max\{j-2, 0\}$. Then $A(\xi)$ is a frozen operator of type λ, applied to f; $D(\xi)$ is a frozen operator of type $\lambda - 1$, applied to f. Moreover,

$$\begin{aligned} C(\xi) = &- a(\xi) \sum_{l,r} \int \widetilde{X}_{l,r}[a^0_{lr}(\Theta(\,\cdot\,, \xi)) b(\,\cdot\,)](\eta) D\Gamma(\xi_0; \Theta(\eta, \xi)) f(\eta)\, d\eta \\ &+ a(\xi) \sum_{l,r} \int a^0_{lr}(\Theta(\eta, \xi)) D\Gamma(\xi_0; \Theta(\eta, \xi)) c_{l,r}(\eta) b(\eta) f(\eta)\, d\eta \\ &- a(\xi) \sum_{l,r} \int a^0_{lr}(\Theta(\eta, \xi)) D\Gamma(\xi_0; \Theta(\eta, \xi)) b(\eta) \widetilde{X}_{l,r} f(\eta)\, d\eta,\end{aligned}$$

where the first two terms are still frozen operators of type λ, applied to f, while the third is the sum of frozen operators of type $\lambda + \max\{j-2, 0\}$ applied to $\widetilde{X}_{l,r} f$. □

Proof of Theorem 4.15. It suffices to prove (4-15), since the proof of (4-16) is similar. So, if $\widetilde{X}_i T(\xi_0)$ is like in (4-15), let us apply Theorem 4.17 and rewrite $\widetilde{X}_i T(\xi_0)$ like in (4-25). Now, let us consider one of the terms $T^i_{jk}(\xi_0) \widetilde{X}_{j,k}$ appearing in (4-25).

If $j = 1$, the term is already in the form required by the theorem we are proving.

If $j = 2$, then $\widetilde{X}_{2,k}$ can be written as a combination of commutators of the vector fields $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_q$, plus (possibly) the field $\widetilde{X}_0$. Then $T^i_{2k}(\xi_0) \widetilde{X}_{2,k}$ contains terms $T^i_{2k}(\xi_0) \widetilde{X}_h \widetilde{X}_j$ and possibly a term $T^i_{2k}(\xi_0) \widetilde{X}_0$.

By Theorem 4.17, we know T^i_{2k} is a frozen operator of type $\lambda+1$. Now

$$T^i_{2k}(\xi_0)\widetilde{X}_h\widetilde{X}_j = (T^i_{2k}(\xi_0)\widetilde{X}_h)\widetilde{X}_j = T^i_k(\xi_0)\widetilde{X}_j,$$

where, by Theorem 4.11, $T^i_k(\xi_0)$ is a frozen operator of type λ; on the other hand, by (4-2),

$$\begin{aligned} T^i_{2k}(\xi_0)\widetilde{X}_0 &= T^i_{2k}(\xi_0)\Bigg(\widetilde{\mathcal{L}}_0 - \sum_{h,j=1}^q \tilde{a}_{hj}(\xi_0)\widetilde{X}_h\widetilde{X}_j\Bigg) \\ &= T^i_{2k}(\xi_0)\widetilde{\mathcal{L}}_0 - \sum_{h,j=1}^q \tilde{a}_{hj}(\xi_0)(T^i_{2k}(\xi_0)\widetilde{X}_h)\widetilde{X}_j \;=\; T^i_{2k}(\xi_0)\widetilde{\mathcal{L}}_0 - \sum_{h,j=1}^q \tilde{a}_{hj}(\xi_0)T^i_{h,k}(\xi_0)\widetilde{X}_j, \end{aligned}$$

with $T^i_{2k}(\xi_0)$ and $T^i_{h,k}(\xi_0)$ frozen operators of type $\lambda+1$ and λ, respectively, which is in the form allowed by the thesis of the theorem we are proving.

Finally, if $j>2$, it is enough to look at the final part of the differential operator $\widetilde{X}_{j,k}$. It is always possible to rewrite $\widetilde{X}_{j,k}$ either as $\widetilde{X}_{j-1,k}\widetilde{X}_{1,k}$ or as $\widetilde{X}_{j-2,k}\widetilde{X}_{2,k}$. In the first case, we have

$$T^i_{jk}(\xi_0)\widetilde{X}_{j,k} = (T^i_{jk}(\xi_0)\widetilde{X}_{j-1,k})\widetilde{X}_{1,k} = T'^i_{jk}(\xi_0)\widetilde{X}_{1,k},$$

with $T'^i_{jk}(\xi_0)$ frozen operator of type λ, which is already in the proper form; in the second case, we have

$$T^i_{jk}(\xi_0)\widetilde{X}_{j,k} = (T^i_{jk}(\xi_0)\widetilde{X}_{j-2,k})\widetilde{X}_{2,k} = T'^i_j(\xi_0)\widetilde{X}_{2,k}$$

with $T'^i_{jk}(\xi_0)$ frozen operator of type $\lambda+1$, and then we can proceed as in the case $j=2$. □

4C. *Parametrix and representation formulas.* Throughout this subsection we will make extensive use of computations on frozen operators of type λ. To make our formulas more readable, we will use the symbols

$$T(\xi_0), \quad S(\xi_0), \quad P(\xi_0)$$

(possibly with some indices) to denote frozen operators of type 0, 1, 2, respectively.

In order to prove representation formulas for second order derivatives, we start with the following parametrix identities, analogous to [Rothschild and Stein 1976, Theorem 10; Bramanti and Brandolini 2000a, Theorem 3.1].

Theorem 4.18. *Given $a\in C_0^\infty(\widetilde{B}(\bar{\xi},R))$, there exist $S_{ij}(\xi_0)$, $S_0(\xi_0)$, $S^*_{ij}(\xi_0)$, $S^*_0(\xi_0)$, frozen operators of type* 1 *and $P(\xi_0)$, $P^*(\xi_0)$, frozen operators of type* 2 (*over the ball $\widetilde{B}(\bar{\xi},R)$*) *such that*

$$aI = \widetilde{\mathcal{L}}_0^T P^*(\xi_0) + \sum_{i,j=1}^q \tilde{a}_{ij}(\xi_0)S^*_{ij}(\xi_0) + S^*_0(\xi_0), \tag{4-27}$$

$$aI = P(\xi_0)\widetilde{\mathcal{L}}_0 + \sum_{i,j=1}^q \tilde{a}_{ij}(\xi_0)S_{ij}(\xi_0) + S_0(\xi_0), \tag{4-28}$$

where I denotes the identity. Moreover, $S_{ij}^*(\xi_0)$, $S_0^*(\xi_0)$, $P^*(\xi_0)$ are modeled on Γ^T, while $S_{ij}(\xi_0)$, $S_0(\xi_0)$, $P(\xi_0)$ are modeled on Γ. Explicitly,

$$P^*(\xi_0)f(\xi) = -\frac{a(\xi)}{c(\xi)}\int_{\widetilde{B}}\Gamma^T(\xi_0;\Theta(\eta,\xi))\,b(\eta)\,f(\eta)\,d\eta,$$

$$P(\xi_0)f(\xi) = -b(\xi)\int_{\widetilde{B}}\frac{a(\eta)}{c(\eta)}\Gamma(\xi_0;\Theta(\eta,\xi))\,f(\eta)\,d\eta,$$

where c is the function appearing in Theorem 3.3(c).

Sketch of the proof. Let us define

$$P^*(\xi_0)f(\xi) = -\frac{a(\xi)}{c(\xi)}\int_{\widetilde{B}}\Gamma^T(\xi_0;\Theta(\eta,\xi))b(\eta)f(\eta)\,d\eta,$$

where $a, b \in C_0^\infty(\widetilde{B}(\bar{\xi}, R))$ such that $ab = a$ and $c(\xi)$ is the function appearing in the formula of change of variables (3-6). Let us compute $\widetilde{\mathscr{L}}_0^T P^*(\xi_0)f$ for $f \in C_0^\infty(\widetilde{B}(\bar{\xi}, R))$. We can apply a distributional argument like in the proof of Lemma 4.13. For $\omega \in C_0^\infty(\widetilde{B}(\bar{\xi}, R))$, let us evaluate

$$\int_{\widetilde{B}}\widetilde{\mathscr{L}}_0\omega(\xi)P^*(\xi_0)f(\xi)\,d\xi = \lim_{\varepsilon\to 0}\int_{\widetilde{B}}\widetilde{\mathscr{L}}_0\omega(\xi)P_\varepsilon^*(\xi_0)f(\xi)\,d\xi,$$

where

$$P_\varepsilon^*(\xi_0)f(\xi) = -\frac{a(\xi)}{c(\xi)}\int_{\widetilde{B}}\varphi_\varepsilon(\Theta(\eta,\xi))\Gamma^T(\xi_0;\Theta(\eta,\xi))b(\eta)f(\eta)\,d\eta$$

with φ_ε as in the proof of Lemma 4.13. Now, computing the integral

$$\int_{\widetilde{B}}\widetilde{\mathscr{L}}_0\omega(\xi)P_\varepsilon^*(\xi_0)f(\xi)\,d\xi$$

and taking its limit for $\varepsilon \to 0$, by the same techniques used in Section 4B, we can prove (4-27). Transposing this identity, one finds (4-28). □

Now, starting from (4-28) and reasoning as in the proof of [Bramanti and Brandolini 2000a, Theorem 3.2], applying Theorem 4.11 and Theorem 4.15, one can easily prove the next two theorems.

Theorem 4.19 (representation of $\widetilde{X}_m\widetilde{X}_l u$ by frozen operators). *Let $a \in C_0^\infty(\widetilde{B}(\bar{\xi}, R))$, $\xi_0 \in \widetilde{B}(\bar{\xi}, R)$. Then, for any $m, l = 1, 2, \dots, q$, there exist frozen operators over the ball $\widetilde{B}(\bar{\xi}, R)$ such that, for any $u \in C_0^\infty(\widetilde{B}(\bar{\xi}, R))$,*

$$\begin{aligned}\widetilde{X}_m\widetilde{X}_l(au) &= T_{lm}(\xi_0)\widetilde{\mathscr{L}}_0 u + \sum_{k=1}^q T_{lm,k}(\xi_0)\widetilde{X}_k u + T_{lm}^0(\xi_0)u \\ &\quad + \sum_{i,j=1}^q \tilde{a}_{ij}(\xi_0)\left\{\sum_{k=1}^q T_{lm,k}^{ij}(\xi_0)\widetilde{X}_k u + \sum_{h,k=1}^q \tilde{a}_{hk}(\xi_0)T_{lm,h}^{\prime ij}(\xi_0)\widetilde{X}_k u + S_{lm}^{ij}(\xi_0)\widetilde{\mathscr{L}}_0 u + T_{lm}^{ij}(\xi_0)u\right\}.\end{aligned} \quad (4\text{-}29)$$

(All the $T_{\dots}(\xi_0)$ are frozen operators of type 0 *and $S^{ij}_{lm}(\xi_0)$ are of type* 1.*) Also,*

$$
\begin{aligned}
\widetilde{X}_m\widetilde{X}_l(au) = T_{lm}(\xi_0)\widetilde{\mathscr{L}}u + T_{lm}(\xi_0)\biggl(\sum_{i,j=1}^{q}[\tilde{a}_{ij}(\xi_0)-\tilde{a}_{ij}(\cdot)]\widetilde{X}_i\widetilde{X}_j u\biggr) + \sum_{k=1}^{q} T_{lm,k}(\xi_0)\widetilde{X}_k u + T^0_{lm}(\xi_0)u \\
+ \sum_{i,j=1}^{q}\tilde{a}_{ij}(\xi_0)\biggl\{\sum_{k=1}^{q} T^{ij}_{lm,k}(\xi_0)\widetilde{X}_k u + \sum_{h,k=1}^{q}\tilde{a}_{hk}(\xi_0)T'^{ij}_{lm,h}(\xi_0)\widetilde{X}_k u + S^{ij}_{lm}(\xi_0)\widetilde{\mathscr{L}}u \\
+ S^{ij}_{lm}(\xi_0)\biggl(\sum_{i,j=1}^{q}[\tilde{a}_{ij}(\xi_0)-\tilde{a}_{ij}(\cdot)]\widetilde{X}_i\widetilde{X}_j u\biggr) + T^{ij}_{lm}(\xi_0)u\biggr\}. \quad (4\text{-}30)
\end{aligned}
$$

Remark 4.20. The representation formulas of the above theorem have a cumbersome aspect, due to the presence of the coefficients $\tilde{a}_{ij}(\xi_0)$ which appear several times as multiplicative factors. Anyway, if we agree to leave implicitly understood in the symbol of frozen operators the possible multiplication by the coefficients $\tilde{a}_{ij}$, our formulas assume the following more compact form

$$\widetilde{X}_m\widetilde{X}_l(au) = T_{lm}(\xi_0)\widetilde{\mathscr{L}}_0 u + \sum_{k=1}^{q} T^{lm}_k(\xi_0)\widetilde{X}_k u + T^0_{lm}(\xi_0)u$$

and

$$\widetilde{X}_m\widetilde{X}_l(au) = T_{lm}(\xi_0)\widetilde{\mathscr{L}}u + T_{lm}(\xi_0)\biggl(\sum_{i,j=1}^{q}[\tilde{a}_{ij}(\xi_0)-\tilde{a}_{ij}(\cdot)]\widetilde{X}_i\widetilde{X}_j u\biggr) + \sum_{k=1}^{q} T^{lm}_k(\xi_0)\widetilde{X}_k u + T^0_{lm}(\xi_0)u.$$

In the proof of a priori estimates, when we take $C^{\alpha}_{\widetilde{X}}$ or L^p norms of both sides of these identities, the multiplicative factors $\tilde{a}_{hj}$ will be simply bounded by taking, respectively, the $C^{\alpha}_{\widetilde{X}}$ or the L^{∞} norms of the $\tilde{a}_{hj}$; hence leaving these factors implicitly understood is harmless.

The above theorem is suited to the proof of $C^{\alpha}_{\widetilde{X}}$ estimates for $\widetilde{X}_i\widetilde{X}_j u$. In order to prove L^p estimate for $\widetilde{X}_i\widetilde{X}_j u$ we need the following variation.

Theorem 4.21 (representation of $\widetilde{X}_m\widetilde{X}_l u$ by variable operators). *Given $a \in C^{\infty}_0(\widetilde{B}(\bar{\xi}, R))$, for any $m, l = 1, 2, \dots, q$, there exist variable operators over the ball $\widetilde{B}(\bar{\xi}, R)$ such that, for any $u \in C^{\infty}_0(\widetilde{B}(\bar{\xi}, R))$,*

$$
\begin{aligned}
&\widetilde{X}_m\widetilde{X}_l(au) \\
&\quad = T_{lm}\widetilde{\mathscr{L}}u + \sum_{i,j=1}^{q}[\tilde{a}_{ij}, T_{lm}]\widetilde{X}_i\widetilde{X}_j u + \sum_{k=1}^{q} T_{lm,k}\widetilde{X}_k u + T^0_{lm}u \\
&\qquad + \sum_{i,j=1}^{q}\tilde{a}_{ij}\biggl\{\sum_{k=1}^{q} T^{ij}_{lm,k}\widetilde{X}_k u + \sum_{h,k=1}^{q}\tilde{a}_{hk}T'^{ij}_{lm,h}\widetilde{X}_k u + S^{ij}_{lm}\widetilde{\mathscr{L}}u + \sum_{i,j=1}^{q}[\tilde{a}_{ij}, S^{ij}_{lm}]\widetilde{X}_i\widetilde{X}_j u + T^{ij}_{lm}u\biggr\}. \quad (4\text{-}31)
\end{aligned}
$$

Here all the $T_{\dots}$ are variable operators of type 0, *S^{ij}_{lm} is of type* 1, *$[a, T]$ denotes the commutator of the multiplication for a with the operator T, and $\tilde{a}_{ij}$ are the coefficients of the operator $\widetilde{\mathscr{L}}$ (which are no longer frozen at ξ_0).*

Remark 4.22. The above representation formula can be written in a shorter way as

$$\widetilde{X}_m \widetilde{X}_l(au) = T_{lm}\widetilde{\mathcal{L}}u + \sum_{i,j=1}^{q} [\tilde{a}_{ij}, T_{lm}]\, \widetilde{X}_i \widetilde{X}_j u + \sum_{k=1}^{q} T_{lm,k}\widetilde{X}_k u + T^0_{lm}u$$

if we leave understood in the symbol of variable operators the possible multiplication by the coefficients $\tilde{a}_{ij}$; see the previous remark.

5. Singular integral estimates for operators of type zero

The proof of a priori estimates on the derivatives $\widetilde{X}_i \widetilde{X}_j u$ will follow, as will be explained in Section 6 and Section 7, combining the representation formulas proved in Section 4C with suitable C^α or L^p estimates for "operators of type zero". To be more precise, the results we need are the $C^\alpha_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, R))$ continuity of a *frozen operator of type zero* and the $L^p(\widetilde{B}(\bar{\xi}, R))$ continuity of a *variable operator of type zero*, together with the $L^p(\widetilde{B}(\bar{\xi}, r))$ estimate for the commutator of a variable operator of type zero with the multiplication with a VMO function, implying that the $L^p(\widetilde{B}(\bar{\xi}, r))$ norm of the commutator vanishes as $r \to 0$. All these results will be derived in the present section, as an application of abstract results proved in [Bramanti and Zhu 2012] in the context of locally homogeneous spaces (see Section 3C). To apply them, we need to check that our kernels of type zero satisfy suitable properties. Moreover, to study *variable* operators of type zero, we also have to resort to the classical technique of expansion in series of spherical harmonics, dating back to Calderón and Zygmund [1957], and already applied in the framework of vector fields in [Bramanti and Brandolini 2000b; 2000a]. This study will be split into two subsections, the first devoted to frozen operators on C^α and the second to variable operators on L^p.

5A. $C^\alpha_{\widetilde{X}}$ *continuity of frozen operators of type* 0.

The goal of this section is the proof of the following.

Theorem 5.1. *Let $\widetilde{B}(\bar{\xi}, R)$ be as before, $\xi_0 \in \widetilde{B}(\bar{\xi}, R)$, and let $T(\xi_0)$ be a frozen operator of type $\lambda \geq 0$ over $\widetilde{B}(\bar{\xi}, R)$. Then there exists $c > 0$ depending on R, $\{\widetilde{X}_i\}$, α, and μ, such that, for any $r < R$ and $u \in C^\alpha_{\widetilde{X},0}(\widetilde{B}(\bar{\xi}, r))$,*

$$\|T(\xi_0)u\|_{C^\alpha_{\widetilde{X}}(\widetilde{B}(\bar{\xi},r))} \leq c\|u\|_{C^\alpha_{\widetilde{X}}(\widetilde{B}(\bar{\xi},r))}. \tag{5-1}$$

To prove this, we will apply theorems proved in [Bramanti and Zhu 2012] about the C^α continuity of singular and fractional integrals in spaces of locally homogeneous type, taking

$$\Omega_k = \widetilde{B}\left(\bar{\xi}, \frac{kR}{k+1}\right) \quad \text{for } k = 1, 2, 3, \ldots. \tag{5-2}$$

Following notation and assumptions in Definition 4.5, our frozen kernel of type zero can be written as

$$k(\xi_0; \xi, \eta) = k'(\xi_0; \xi, \eta) + k''(\xi_0; \xi, \eta).$$

We will prove Theorem 5.1 for the operator with kernel k', the proof for k'' being completely analogous. Let us split k' as

$$k'(\xi_0;\xi,\eta)=a_1(\xi)b_1(\eta)D_1\Gamma(\xi_0;\Theta(\eta,\xi))+\left\{\sum_{i=2}^{H_m}a_i(\xi)b_i(\eta)D_i\Gamma(\xi_0;\cdot)+a_0(\xi)b_0(\eta)D_0\Gamma(\xi_0;\cdot)\right\}(\Theta(\eta,\xi))$$
$$=: k_S(\xi,\eta)+k_F(\xi,\eta),$$

where $D_1\Gamma(\xi_0;u)$ is homogeneous of degree $-Q$ while all the kernels $D_i\Gamma(\xi_0;u)$ are homogeneous of some degree $\geq 1-Q$ and $D_0\Gamma(\xi_0;u)$ is regular. Recall that all these kernels may also have an explicit (smooth) dependence on ξ,η; we will write $D_i^{\xi,\eta}\Gamma(\xi_0;\Theta(\eta,\xi))$ to point out this fact when it is important.

We want to apply [Bramanti and Zhu 2012, Theorem 5.4] (about singular integrals) to the kernel k_S and [Bramanti and Zhu 2012, Theorem 5.8] (about fractional integrals) to each term of the kernel k_F.

We start with the following result, very similar to [Bramanti and Brandolini 2000a, Proposition 2.17].

Proposition 5.2. *Let $W^{\xi,\eta}(\cdot)$ be a function defined on the homogeneous group $\mathbb{G}$, smooth outside the origin and homogeneous of degree $\ell-Q$ for some nonnegative integer ℓ, smoothly depending on the parameters $\xi,\eta\in\widetilde{B}(\bar{\xi},R)$, and let*

$$K(\xi,\eta)=W^{\xi,\eta}(\Theta(\eta,\xi))$$

be defined for $\xi,\eta\in\widetilde{B}(\bar{\xi},R)$. Then K satisfies the following.

(i) *The growth condition: there exists a constant c such that*

$$|K(\xi,\eta)|\leq c\cdot\sup_{\|u\|=1}|W^{\xi,\eta}(u)|\cdot d_{\widetilde{X}}(\xi,\eta)^{\ell-Q}.$$

(ii) *The mean value inequality: there exists a constant $c>0$ such that, for every ξ_0,ξ,η with $d_{\widetilde{X}}(\xi_0,\eta)\geq 2d_{\widetilde{X}}(\xi_0,\xi)$,*

$$|K(\xi_0,\eta)-K(\xi,\eta)|+|K(\eta,\xi_0)-K(\eta,\xi)|\leq C\frac{d_{\widetilde{X}}(\xi_0,\xi)}{d_{\widetilde{X}}(\xi_0,\eta)^{Q+1-\ell}},\tag{5-3}$$

where the constant C has the form

$$c\sup_{\substack{\|u\|=1\\ \xi,\eta\in\widetilde{B}(\bar{\xi},R)}}\{|\nabla_u W^{\xi,\eta}(u)|+|\nabla_\xi W^{\xi,\eta}(u)|+|\nabla_\eta W^{\xi,\eta}(u)|\}.$$

(iii) *The cancellation property: if $\ell=0$ and W satisfies the vanishing property*

$$\int_{r<\|u\|<R}W^{\xi,\eta}(u)\,du=0\quad\text{for every } R>r>0 \text{ and } \xi,\eta\in\widetilde{B}(\bar{\xi},R),\tag{5-4}$$

then, for any positive integer k, for every $\varepsilon_2>\varepsilon_1>0$ and $\xi\in\Omega_k$ (see (5-2)) such that $\widetilde{B}(\xi,\varepsilon_2)\subset\Omega_{k+1}$,

$$\left|\int_{\Omega_{k+1},\varepsilon_1<\rho(\xi,\eta)<\varepsilon_2}K(\xi,\eta)\,d\eta\right|+\left|\int_{\Omega_{k+1},\varepsilon_1<\rho(\xi,\eta)<\varepsilon_2}K(\eta,\xi)\,d\eta\right|\leq C\cdot(\varepsilon_2-\varepsilon_1),\tag{5-5}$$

where the constant C has the form

$$c \sup_{\substack{\|u\|=1 \\ \xi,\eta\in\widetilde{B}(\bar{\xi},R)}} \{|W^{\xi,\eta}(u)| + |\nabla_\xi W^{\xi,\eta}(u)| + |\nabla_\eta W^{\xi,\eta}(u)|\}.$$

Proof. (i) is trivial, by the homogeneity of W, and the equivalence between $d_{\widetilde{X}}$ and ρ (see Lemma 3.9).

In order to prove (ii), fix ξ_0, η and let $r = \frac{1}{2}\rho(\eta,\xi_0)$. Condition $\rho(\eta,\xi_0) > 2\rho(\xi,\xi_0)$ means that ξ is a point ranging in $\widetilde{B}_r(\xi_0)$. Applying (3-18) to the function

$$f(\xi) = K(\xi,\eta),$$

we can write

$$|f(\xi)-f(\xi_0)| \le c d_{\widetilde{X}}(\xi,\xi_0)\biggl(\sum_{i=1}^{q} \sup_{\zeta\in\widetilde{B}(\xi_0,\frac{3}{4}d_{\widetilde{X}}(\xi_0,\eta))} |\widetilde{X}_i f(\zeta)| + d_{\widetilde{X}}(\xi,\xi_0) \sup_{\zeta\in\widetilde{B}(\xi_0,\frac{3}{4}d_{\widetilde{X}}(\xi_0,\eta))} |\widetilde{X}_0 f(\zeta)|\biggr).$$

Noting that, for $\zeta \in \widetilde{B}(\xi_0, \frac{3}{4}d_{\widetilde{X}}(\xi_0,\eta))$,

$$\begin{aligned} |\widetilde{X}_i K(\,\cdot\,,\eta)(\zeta)| &= |\widetilde{X}_i (W^{\zeta,\eta}(\Theta(\,\cdot\,,\eta)))(\zeta) + (\widetilde{X}_i W^{\cdot,\eta}(\Theta(\zeta,\eta)))(\zeta)| \\ &\le |(Y_i W + R_i^{\eta} W)(\Theta(\eta,\zeta))| + |(\widetilde{X}_i W^{\cdot,\eta}(\Theta(\zeta,\eta)))(\zeta)| \end{aligned}$$

and recalling that, by Remark 4.6, $\nabla_\zeta W^{\zeta,\eta}(u)$ has the same u homogeneity as $W^{\zeta,\eta}(u)$, we get

$$\begin{aligned} |\widetilde{X}_i K(\,\cdot\,,\eta)(\zeta)| &\le \sup_{\substack{\|u\|=1 \\ \zeta,\eta\in\widetilde{B}(\bar{\xi},R)}} |\nabla_u W^{\xi,\eta}(u)| \frac{c}{\rho(\zeta,\eta)^{Q-\ell+1}} + \sup_{\substack{\|u\|=1 \\ \zeta,\eta\in\widetilde{B}(\bar{\xi},R)}} |\nabla_\zeta W^{\zeta,\eta}(u)| \frac{c}{\rho(\zeta,\eta)^{Q-\ell}} \\ &\le \sup_{\substack{\|u\|=1 \\ \zeta,\eta\in\widetilde{B}(\bar{\xi},R)}} \{|\nabla_u W^{\zeta,\eta}(u)| + |\nabla_\zeta W^{\zeta,\eta}(u)|\} \frac{c}{d_{\widetilde{X}}(\xi_0,\eta)^{Q-\ell+1}}. \end{aligned}$$

Analogously,

$$|\widetilde{X}_0 K(\,\cdot\,,\eta)(\zeta)| \le \sup_{\substack{\|u\|=1 \\ \zeta,\eta\in\widetilde{B}(\bar{\xi},R)}} \{|\nabla_u W^{\zeta,\eta}(u)| + |\nabla_\zeta W^{\zeta,\eta}(u)|\} \frac{c}{d_{\widetilde{X}}(\xi_0,\eta)^{Q-\ell+2}},$$

hence

$$|K(\xi,\eta) - K(\xi_0,\eta)| \le C \frac{d_{\widetilde{X}}(\xi,\xi_0)}{d_{\widetilde{X}}(\xi_0,\eta)^{Q-\ell+1}}$$

with

$$C = c \sup_{\substack{\|u\|=1 \\ \zeta,\eta\in\widetilde{B}(\bar{\xi},R)}} \{|\nabla_u W^{\zeta,\eta}(u)| + |\nabla_\zeta W^{\zeta,\eta}(u)|\}.$$

To get the analogous bound for $|K(\eta,\xi_0) - K(\eta,\xi)|$, it is enough to apply the previous estimate to the function

$$\widetilde{K}(\xi,\eta) = \widetilde{W}^{\xi,\eta}(\Theta(\eta,\xi)) \quad \text{with } \widetilde{W}^{\xi,\eta}(u) = W^{\eta,\xi}(u^{-1}).$$

This completes the proof of (ii).

To prove (iii), we first ignore the dependence on the parameters ξ, η, and then we will show how to modify our argument to take them into account. By the change of variables $u = \Theta(\eta, \xi)$, Theorem 3.3(c) gives

$$\int_{\Omega_{k+1}, \varepsilon_1 < \rho(\xi,\eta) < \varepsilon_2} W(\Theta(\eta, \xi))\, d\eta = c(\xi) \int_{\varepsilon_1 < \|u\| < \varepsilon_2} W(u)(1 + \omega(\xi, u)))\, du,$$

which, by the vanishing property of W, equals

$$c(\xi) \int_{\varepsilon_1 < \|u\| < \varepsilon_2} W(u)\omega(\xi, u)\, du.$$

Then

$$\begin{aligned}\left| \int_{\Omega_{k+1}, \varepsilon_1 < \rho(\xi,\eta) < \varepsilon_2} W(\Theta(\eta, \xi))\, d\eta \right| &\le c \cdot \int_{\varepsilon_1 < \|u\| < \varepsilon_2} |W(u)|\, \|u\|\, du \\ &\le c \cdot \sup_{\|u\|=1} |W| \cdot \int_{\varepsilon_1 < \|u\| < \varepsilon_2} \|u\|^{1-Q}\, du \le c \cdot \sup_{\|u\|=1} |W| \cdot (\varepsilon_2 - \varepsilon_1).\end{aligned}$$

Analogously, one can prove the bound on $W(\Theta(\xi, \eta))$. Now, to keep track of the possible dependence of W on the parameters ξ, η, let us write

$$\begin{aligned}&\int_{\Omega_{k+1}, \varepsilon_1 < \rho(\xi,\eta) < \varepsilon_2} W^{\xi,\eta}(\Theta(\eta, \xi))\, d\eta \\ &\quad = \int_{\Omega_{k+1}, \varepsilon_1 < \rho(\xi,\eta) < \varepsilon_2} W^{\xi,\xi}(\Theta(\eta, \xi))\, d\eta + \int_{\Omega_{k+1}, \varepsilon_1 < \rho(\xi,\eta) < \varepsilon_2} [W^{\xi,\eta}(\Theta(\eta, \xi)) - W^{\xi,\xi}(\Theta(\eta, \xi))\,]\, d\eta \\ &\quad =: I + II.\end{aligned}$$

The term I can be bounded as above, while

$$|W^{\xi,\eta}(u) - W^{\xi,\xi}(u)| \le |\xi - \eta| |\nabla_\eta W^{\xi,\eta'}(u)|$$

for some point η' near ξ and η. Recalling again that the function $\nabla_\eta W^{\xi,\eta'}(\,\cdot\,)$ has the same homogeneity as $W^{\xi,\eta'}(\,\cdot\,)$, while

$$|\xi - \eta| \le c d_{\widetilde{X}}(\xi, \eta) \le c\rho(\xi, \eta),$$

we have

$$|II| \le c \sup_{\substack{\|u\|=1 \\ \xi,\eta \in \widetilde{B}(\bar{\xi}, R)}} |\nabla_\eta W^{\xi,\eta}(u)| \int_{\Omega_{k+1}, \varepsilon_1 < \|u\| < \varepsilon_2} \|u\|^{1-Q}\, du$$

and the same reasoning as above applies. This proves the bound on $|\int K(\xi, \eta)\, d\eta|$ in (5-5). The proof of the bound on $|\int K(\eta, \xi)\, d\eta|$ is analogous, since the vanishing property (5-4) also implies the same bound for $\int_{r < \|u\| < R} W^{\xi,\eta}(u^{-1})\, du$. □

Proposition 5.2 implies that $D_1\Gamma(\xi_0; \Theta(\eta, \xi))$ satisfies the standard estimates, cancellation property, and convergence condition stated in Section 3C. Note that (5-5) implies both the cancellation property and the convergence condition.

In order to apply [Bramanti and Zhu 2012, Theorem 5.4] to the kernel $k_S(\xi,\eta)$, we still need to prove that the singular integral T with kernel $k_S(\xi,\eta)$ satisfies a condition $T(1)\in C^{\gamma}_{\widetilde{X}}$. This result is more delicate than the previous conditions, and is contained in the following.

Proposition 5.3. *Let*

$$\widetilde{h}(\xi) := \lim_{\varepsilon\to 0}\int_{\rho(\xi,\eta)>\varepsilon}\widetilde{K}(\xi,\eta)\,d\eta$$

with

$$\widetilde{K}(\xi,\eta) = a_1(\xi)b_1(\eta)D_1^{\xi,\eta}\Gamma(\xi_0;\Theta(\eta,\xi)),$$

$D_1^{\xi,\eta}\Gamma(\xi_0;\cdot)$ *homogeneous of degree $-Q$ and satisfying the vanishing property*

$$\int_{r<\|u\|<R} D_1^{\xi,\eta}\Gamma(\xi_0;u)\,du = 0 \quad \text{for every } R>r>0,\ \text{any } \xi,\eta\in\widetilde{B}(\bar{\xi},R).$$

Then

$$\widetilde{h}\in C^{\gamma}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},R)) \quad \text{for any } \gamma\in(0,1). \tag{5-6}$$

Proof. Since a_1, b_1 are compactly supported in $\widetilde{B}(\bar{\xi},R)$, we can choose a radial cutoff function

$$\phi(\xi,\eta) = f(\rho(\xi,\eta))$$

with

$$f(\|u\|) = 1 \quad \text{for } \|u\|\le R, \qquad f(\|u\|) = 0 \quad \text{for } \|u\|\ge 2R,$$

so that $\widetilde{K}(\xi,\eta) = \widetilde{K}(\xi,\eta)\phi(\xi,\eta)$. To begin with, let us prove the assertion without taking into consideration the dependence of $D_1^{\xi,\eta}\Gamma(\xi_0;u)$ on ξ,η. Then

$$\begin{aligned}\widetilde{h}(\xi) = a_1(\xi)b_1(\xi)\lim_{\varepsilon\to 0}\int_{\rho(\xi,\eta)>\varepsilon}&\phi(\xi,\eta)D_1\Gamma(\xi_0;\Theta(\eta,\xi))\,d\eta \\ &+ a_1(\xi)\int\phi(\xi,\eta)D_1\Gamma(\xi_0;\Theta(\eta,\xi))[b_1(\eta)-b_1(\xi)]\,d\eta \\ =: I(\xi)+II(\xi).&\end{aligned}$$

Now,

$$\begin{aligned} I(\xi) &= a_1(\xi)b_1(\xi)c(\xi)\lim_{\varepsilon\to 0}\int_{\|u\|>\varepsilon} f(\|u\|)D_1\Gamma(\xi_0;u)(1+\omega(\xi,u))\,du \\ &= a_1(\xi)b_1(\xi)c(\xi)\int f(\|u\|)D_1\Gamma(\xi_0;u)\omega(\xi,u)\,du,\end{aligned}$$

by the vanishing property, with ω smoothly depending on ξ and uniformly bounded by $c\|u\|$. Hence $I(\xi)$ is Lipschitz continuous and, in particular, Hölder continuous of any exponent $\gamma\in(0,1)$. Moreover,

$$II(\xi) = a_1(\xi)\int_{\widetilde{B}(\bar{\xi},R)}\kappa(\xi,\eta)\,d\eta \quad \text{with } \kappa(\xi,\eta) = \phi(\xi,\eta)D_1\Gamma(\xi_0;\Theta(\eta,\xi))[b_1(\eta)-b_1(\xi)].$$

It is not difficult to check that the kernel $\kappa(\xi,\eta)$ satisfies the standard estimates of *fractional* integrals (3-11) and (3-12) for any $\nu\in(0,1)$ (actually, for $\nu=1$). Hence, by [Bramanti and Zhu 2012, Theorem 5.8], the operator with kernel κ is continuous on $C^\gamma(\widetilde{B}(\bar{\xi}, R))$; in particular, it maps the function 1 into $C^\gamma(\widetilde{B}(\bar{\xi}, R))$, which proves that $II(\xi)$ is Hölder continuous.

To conclude the proof, we have to show how to take into account the possible dependence of $D_1^{\xi,\eta}\Gamma(\xi_0; u)$ on ξ, η. Let us start with the η dependence.

$$\begin{aligned}\widetilde{h}(\xi) = a_1(\xi)b_1(\xi) \lim_{\varepsilon\to 0}\int_{\rho(\xi,\eta)>\varepsilon} &\phi(\xi, \eta) D_1^\eta\Gamma(\xi_0; \Theta(\eta, \xi))\, d\eta \\ &+ a_1(\xi)\int \phi(\xi, \eta) D_1^\eta\Gamma(\xi_0; \Theta(\eta, \xi))[b_1(\eta) - b_1(\xi)]\, d\eta \\ =: I'(\xi) + II'(\xi).&\end{aligned}$$

The term $II'(\xi)$ can be handled as the term $II(\xi)$ above. As to $I'(\xi)$,

$$\begin{aligned}I'(\xi) = a_1(\xi)b_1(\xi)c(\xi) \lim_{\varepsilon\to 0}\int_{\|u\|>\varepsilon} &f(\|u\|) D_1^{\Theta(\cdot,\xi)^{-1}(u)}\Gamma(\xi_0; u)\, du \\ &+ a_1(\xi)b_1(\xi)c(\xi)\int f(\|u\|) D_1^{\Theta(\cdot,\xi)^{-1}(u)}\Gamma(\xi_0; u)\omega(\xi, u)\, du.\end{aligned}$$

The second term can be handled as above, while the first one requires some care. By the vanishing property of $D_1^\zeta\Gamma(\xi_0; u)$ for any fixed ζ, we can write

$$\lim_{\varepsilon\to 0}\int_{\|u\|>\varepsilon} f(\|u\|) D_1^{\Theta(\cdot,\xi)^{-1}(u)}\Gamma(\xi_0; u)\, du = \lim_{\varepsilon\to 0}\int_{\|u\|>\varepsilon} f(\|u\|)[D_1^{\Theta(\cdot,\xi)^{-1}(u)}\Gamma(\xi_0; u) - D_1^\xi\Gamma(\xi_0; u)]\, du.$$

On the other hand,

$$D_1^{\Theta(\cdot,\xi)^{-1}(u)}\Gamma(\xi_0; u) = D_1^\xi\Gamma(\xi_0; u) + D_0^\xi\Gamma(\xi_0; u),$$

where D_0^ξ is a vector field of local weight ≤ 0, smoothly depending on ξ. Hence

$$\lim_{\varepsilon\to 0}\int_{\|u\|>\varepsilon} f(\|u\|) D_1^{\Theta(\cdot,\xi)^{-1}(u)}\Gamma(\xi_0; u)\, du = \int f(\|u\|) D_0^\xi\Gamma(\xi_0; u)\, du,$$

which can be handled as the term $I(\xi)$ above.

Dependence on the variable ξ can be taken into account as follows. If

$$\begin{aligned}\widetilde{h}(\xi) &= a_1(\xi)b_1(\xi) \lim_{\varepsilon\to 0}\int_{\rho(\xi,\eta)>\varepsilon} \phi(\xi, \eta) D_1^{\xi,\eta}\Gamma(\xi_0; \Theta(\eta, \xi))\, d\eta \\ &\equiv \lim_{\varepsilon\to 0}\int F_\varepsilon(\xi, \xi, \eta) \quad \text{with } F_\varepsilon(\zeta, \xi, \eta) = a_1(\xi)b_1(\xi)\chi_{\rho(\xi,\eta)>\varepsilon}(\eta)\phi(\xi, \eta) D_1^{\zeta,\eta}\Gamma(\xi_0; \Theta(\eta, \xi))\, d\eta,\end{aligned}$$

then

$$\begin{aligned}\widetilde{h}(\xi_1) - \widetilde{h}(\xi_2) &= \lim_{\varepsilon\to 0}\int [F_\varepsilon(\xi_1, \xi_1, \eta) - F_\varepsilon(\xi_2, \xi_1, \eta)]\, d\eta + \lim_{\varepsilon\to 0}\int [F_\varepsilon(\xi_2, \xi_1, \eta) - F_\varepsilon(\xi_2, \xi_2, \eta)]\, d\eta \\ &=: A(\xi_1, \xi_2) + B(\xi_1, \xi_2).\end{aligned}$$

Now,

$$|A(\xi_1, \xi_2)| \leq c\rho(\xi_1, \xi_2)$$

by the smoothness of $\xi \mapsto D_1^{\xi,\eta}\Gamma(\xi_0; u)$. As to $B(\xi_1, \xi_2)$, it is enough to apply the previous reasoning to $D_1^{\zeta,\eta}\Gamma(\xi_0; \Theta(\eta, \xi))$, for any fixed ζ, to conclude that

$$\left|\lim_{\varepsilon\to 0}\int [F(\zeta, \xi_1, \eta) - F(\zeta, \xi_2, \eta)]\, d\eta\right| \le c\rho(\xi_1, \xi_2)^\gamma$$

for some constant uniformly bounded in ζ, and then apply this inequality taking $\zeta = \xi_2$. □

Conclusion of the proof of Theorem 5.1. Recall that a frozen operator of type zero is written as

$$T(\xi_0) f(\xi) = \mathrm{PV}\int_{\widetilde{B}} k(\xi_0; \xi, \eta)\, f(\eta)\, d\eta + \alpha(\xi_0, \xi) f(\xi),$$

where α is a bounded measurable function, smooth in ξ. The multiplicative part

$$f(\xi) \longmapsto \alpha(\xi_0, \xi) f(\xi)$$

clearly maps $C^\alpha_{\widetilde{X}}$ in $C^\alpha_{\widetilde{X}}$, since $\alpha(\xi_0, \cdot)$ is smooth, with operator norm bounded by some constant depending on the vector fields and the ellipticity constant μ, by Theorem 4.3.

Let us now consider the integral part. With the notation introduced at the beginning of this section, let us consider first

$$k_S(\xi, \eta) = a_1(\xi) b_1(\eta) D_1^{\xi,\eta}\Gamma(\xi_0; \Theta(\eta, \xi)),$$

where $D_1^{\xi,\eta}\Gamma(\xi_0; u)$ is homogeneous of degree $-Q$ and satisfies the vanishing property (5-4). By Proposition 5.2, $k_S(\xi, \eta)$ satisfies conditions (i), (ii), and (iii) in Section 3C, with constants bounded by

$$c \sup_{\|u\|=1} \{|D^2\Gamma(\xi_0, u)| + |D^3\Gamma(\xi_0, u)|\}, \tag{5-7}$$

where the symbols D^2, D^3 denote standard derivatives of orders 2, 3, respectively, with respect to u, and the constant c depends on the vector fields but not on the point ξ_0. By Proposition 5.3, condition (5-6) is also satisfied by $k_S(\xi, \eta)$, with the $C^\gamma_{\widetilde{X}}$ norm bounded by a quantity of the kind (5-7). Hence, by [Bramanti and Zhu 2012, Theorem 5.4], the operator with kernel $k_S(\xi, \eta)$ satisfies the assertion of the theorem we are proving, with a constant bounded by a quantity like (5-7). In turn, by Theorem 4.3, this quantity can be bounded by a constant depending on the vector fields and the ellipticity constant μ of the matrix $a_{ij}(x)$.

Let us now come to the kernel

$$k_F(\xi, \eta) = \left\{\sum_{i=2}^{H} a_i(\xi) b_i(\eta) D_i^{\xi,\eta}\Gamma(\xi_0; \cdot) + a_0(\xi) b_0(\eta) D_0^{\xi,\eta}\Gamma(\xi_0; \cdot)\right\}(\Theta(\eta, \xi)),$$

where each function $D_i^{\xi,\eta}\Gamma(\xi_0; u)$ $(i = 2, 3, \ldots, H)$ is homogeneous of some degree $\ge 1 - Q$, while $D_0^{\xi,\eta}\Gamma(\xi_0; u)$ is bounded and smooth. By Proposition 5.2, each kernel

$$a_i(\xi) b_i(\eta) D_i^{\xi,\eta}\Gamma(\xi_0; \Theta(\eta, \xi))$$

satisfies the standard estimates (i) in Section 3C for some $\nu > 0$, hence we can apply [Bramanti and Zhu 2012, Theorem 5.8] to the integral operators defined by these kernels, and conclude as above. Finally, the integral operator with regular kernel is clearly C^γ continuous. □

5B. *L^p continuity of variable operators of type 0 and their commutators.* In this subsection we are going to prove the following.

Theorem 5.4. *Let T be a variable operator of type 0 (see Section 4B) over the ball $\widetilde{B}(\bar{\xi}, R)$, and $p \in (1, \infty)$. Then*

(i) *there exists $c > 0$, depending on p, R, $\{\widetilde{X}_i\}_{i=0}^q$, and μ such that*

$$\|Tu\|_{L^p(\widetilde{B}(\bar{\xi},r))} \leq c\|u\|_{L^p(\widetilde{B}(\bar{\xi},r))}$$

for every $u \in L^p(\widetilde{B}(\bar{\xi}, r))$ and $r \leq R$;

(ii) *for every $a \in \mathrm{VMO}_{X,\mathrm{loc}}(\Omega)$, any $\varepsilon > 0$, there exists $r \leq R$ such that, for every $u \in L^p(\widetilde{B}(\bar{\xi}, r))$,*

$$\|T(\tilde{a}u) - \tilde{a} \cdot Tu\|_{L^p(\widetilde{B}(\bar{\xi},r))} \leq \varepsilon\|u\|_{L^p(\widetilde{B}(\bar{\xi},r))}, \tag{5-8}$$

where $\tilde{a}(x, h) = a(x)$. The number r depends on p, R, $\{\widetilde{X}_i\}_{i=0}^q$, μ, $\eta^*_{a,\Omega',\Omega}$, and ε (see Section 3D.3 for the definition of $\mathrm{VMO}_{X,\mathrm{loc}}(\Omega)$ and $\eta^*_{a,\Omega',\Omega}$).

A basic difference between the context here and that of the previous section is that here we are considering *variable* kernels and operators of type zero. To reduce the study of these operators to that of constant operators of type zero we will make use of the classical technique of expansion in series of spherical harmonics, as already done in [Bramanti and Brandolini 2000a].

Proof. This proof is similar to that of [Bramanti and Brandolini 2000a, Theorem 2.11]. Recall that a variable operator of type zero is written as

$$Tf(\xi) = \mathrm{PV}\int_{\widetilde{B}} k(\xi; \xi, \eta) f(\eta)\, d\eta + \alpha(\xi, \xi) f(\xi),$$

where $\alpha(\xi_0, \xi)$ is a bounded measurable function in ξ_0, smooth in ξ. The multiplicative part

$$f(\xi) \longmapsto \alpha(\xi, \xi) f(\xi)$$

clearly maps L^p into L^p, with operator norm bounded by some constant depending on the vector fields and the ellipticity constant μ, by Theorem 4.3. Moreover, this part does not affect the commutator of T.

As to the integral part of T, let us split the variable kernel as

$$k(\xi; \xi, \eta) = k'(\xi; \xi, \eta) + k''(\xi; \xi, \eta).$$

Like in the previous section, it is enough to prove our result for the kernel k'. Let us expand it as

$$\begin{aligned} k'(\xi; \xi, \eta) &= \sum_{i=1}^{H} a_i(\xi) b_i(\eta) D_i^{\xi,\eta}\Gamma(\xi; \Theta(\eta, \xi)) + a_0(\xi) b_0(\eta) D_0^{\xi,\eta}\Gamma(\xi; \Theta(\eta, \xi)) \\ &=: k_U(\xi; \xi, \eta) + k_B(\xi; \xi, \eta), \end{aligned}$$

where the kernels $D_i^{\xi,\eta}\Gamma(\xi;u)$ (for $i=1,2,3,\dots,H$) are homogeneous of some degree $\geq -Q$, $D_i^{\xi,\eta}\Gamma(\xi;u)$ satisfies the cancellation property, and $D_0^{\xi,\eta}\Gamma(\xi;u)$ is bounded in u and smooth in ξ,η. The kernels k_U and k_B are "unbounded" and "bounded", respectively.

The operator with kernel k_B is obviously L^p continuous. Moreover, it satisfies the commutator estimate (5-8) by [Bramanti and Zhu 2012, Theorem 7.3], since

$$|k_B(\xi;\xi,\eta)| \leq c a_0(\xi) b_0(\eta)$$

and the constant function 1 obviously satisfies the standard estimates (3-11), (3-12) with $\nu = 1$.

To handle the kernel k_U, we expand each of its terms in series of spherical harmonics, exactly like in [Bramanti and Brandolini 2000a, Section 2.4]:

$$D_i^{\xi,\eta}\Gamma(\xi;u) = \sum_{m=0}^{\infty}\sum_{k=1}^{g_m} c_{i,km}^{\xi,\eta}(\xi) K_{i,km}(u),$$

where $K_{i,km}(u)$ are homogeneous kernels which, on the sphere $\|u\|=1$, coincide with the spherical harmonics, and $c_{i,km}^{\xi,\eta}(\cdot)$ are the corresponding Fourier coefficients.

Let us first prove the assertion without taking into account the dependence of the coefficients $c_{i,km}^{\xi,\eta}(\xi)$ on η. Then the operator with kernel k_U can be written as

$$Sf(\xi) = \sum_{m=0}^{\infty}\sum_{k=1}^{g_m} c_{i,km}^{\xi}(\xi) S_{i,km} f(\xi) \tag{5-9}$$

with

$$S_{i,km} f(\xi) = a_i(\xi)\int_{\widetilde{B}} b_i(\eta) K_{i,km}(\Theta(\eta,\xi)) f(\eta)\, d\eta.$$

The number g_m in (5-9) is the dimension of the space of spherical harmonics of degree m in $\mathbb{R}^N$; it is known that

$$g_m \leq c(N)\cdot m^{N-2} \quad \text{for every } m = 1,2,\dots. \tag{5-10}$$

For every $p\in(1,\infty)$ we can write

$$\|Sf\|_{L^p(\widetilde{B}(\bar{\xi},r))} \leq \sum_{m=0}^{\infty}\sum_{k=1}^{g_m} \|c_{i,km}^{\cdot}(\cdot)\|_{L^\infty(\widetilde{B}(\bar{\xi},r))} \|S_{i,km} f\|_{L^p(\widetilde{B}(\bar{\xi},r))}$$

and

$$\|S(\tilde{a}f) - \tilde{a}\cdot Sf\|_{L^p(\widetilde{B}(\bar{\xi},r))} \leq \sum_{m=0}^{\infty}\sum_{k=1}^{g_m} \|c_{i,km}^{\cdot}(\cdot)\|_{L^\infty(\widetilde{B}(\bar{\xi},r))} \|S_{i,km}(\widetilde{a}f) - \tilde{a}\cdot S_{i,km} f\|_{L^p(\widetilde{B}(\bar{\xi},r))}.$$

Now each $S_{i,km}$ is a frozen operator of type $\lambda\geq 0$, and the same arguments as in the previous section show that the kernel of $S_{i,km}$ satisfies the assumptions (i), (ii), and (iii) in Section 3C with constants bounded by

$$c\cdot \sup_{\|u\|=1} |\nabla_u K_{km}(u)|,$$

(with c depending on the vector fields); in turn, by known properties of spherical harmonics, we have

$$\sup_{\|u\|=1} |\nabla_u K_{km}(u)| \le c(N) m^{N/2},$$

so that, by [Bramanti and Zhu 2012, Theorems 5.3 and 5.7], we conclude as in [Bramanti and Brandolini 2000a, p. 807] that

$$\|S_{i,km} f\|_{L^p(\widetilde{B}(\bar{\xi},r))} \le c \cdot m^{N/2} \|f\|_{L^p(\widetilde{B}(\bar{\xi},r))} \quad \text{for } i = 1, 2, \ldots, H,$$

where we have also taken into account Remark 5.5 below.

Analogously, applying [Bramanti and Zhu 2012, Theorems 7.1 and 7.2], we have the commutator estimate

$$\|S_{i,km}(\tilde{a} f) - \tilde{a} \cdot S_{i,km} f\|_{L^p(\widetilde{B}(\bar{\xi},r))} \le \varepsilon \cdot m^{N/2} \|f\|_{L^p(\widetilde{B}(\bar{\xi},r))} \quad \text{for } i = 1, 2, \ldots, H,$$

for any $\varepsilon > 0$, provided r is small enough, depending on ε and $\eta^*_{\tilde{a},\Omega_{k+2},\Omega_{k+3}}$ (see (5-2) and Definition 3.21 for the meaning of symbols). Then, by Proposition 3.23, the constant r depends on the function a only through the local VMO modulus $\eta^*_{a,\Omega',\Omega}$.

Next, again by known properties of spherical harmonics, we can say that, for any positive integer h, there exists c_h such that

$$|c^{\zeta}_{i,km}(\xi)| \le c_h \cdot m^{-2h} \sup_{\|u\|=1, |\beta|=2h} \left| \left(\frac{\partial}{\partial u}\right)^{\beta} D^{\zeta}_i \Gamma(\xi; u) \right|.$$

By the uniform estimates contained in Theorem 4.3, the last expression is bounded by Cm^{-2h}, for some constant C depending on h, the vector fields, and the ellipticity constant μ. Also taking into account (5-10) and choosing h large enough, we conclude

$$\|Sf\|_{L^p(\widetilde{B}(\bar{\xi},r))} \le \sum_{m=0}^{\infty} C g_m m^{-2h} m^{N/2} \|f\|_{L^p(\widetilde{B}(\bar{\xi},r))} = c\|f\|_{L^p(\widetilde{B}(\bar{\xi},r))}$$

and

$$\|S(\tilde{a} f) - \tilde{a} \cdot Sf\|_{L^p(\widetilde{B}(\bar{\xi},r))} \le c\varepsilon \|f\|_{L^p(\widetilde{B}(\bar{\xi},r))}$$

for any $\varepsilon > 0$, provided r is small enough.

We are left to show how the previous argument needs to be modified to take into account the possible dependence of $D^{\xi,\eta}_i \Gamma(\xi; u)$ (and then of $c^{\xi,\eta}_{i,km}(\xi)$) on η. Let us expand

$$D^{\zeta,\Theta(\cdot,\zeta)^{-1}(u)}_i \Gamma(\xi; u) = \sum_{m=0}^{\infty} \sum_{k=1}^{g_m} c^{\zeta}_{i,km}(\xi) K_{i,km}(u)$$

so that

$$D^{\zeta,\eta}_i \Gamma(\xi; \Theta(\eta,\zeta)) = \sum_{m=0}^{\infty} \sum_{k=1}^{g_m} c^{\zeta}_{i,km}(\xi) K_{i,km}(\Theta(\eta,\zeta)).$$

The kernels $K_{i,km}$ are the same as above, hence the estimates on the operators $S_{i,km}$ and their commutators remain unchanged. As to the coefficients $c^{\zeta}_{i,km}(\xi)$, we now have to write, for any positive integer h and some constant c_h,

$$|c^{\zeta}_{i,km}(\xi)| \le c_h \cdot m^{-2h} \sup_{\|u\|=1,|\beta|=2h} \left| \left(\frac{\partial}{\partial u}\right)^{\beta} (D_i^{\zeta,\Theta(\cdot,\zeta)^{-1}(u)} \Gamma(\xi;u)) \right|.$$

Now, from the identity

$$\frac{\partial}{\partial u_j}(D_i^{\zeta,\Theta(\cdot,\zeta)^{-1}(u)}\Gamma(\xi;u)) = \frac{\partial}{\partial u_j}(D_i^{\zeta,\eta}\Gamma(\xi;u))_{/\eta=\Theta(\cdot,\zeta)^{-1}(u)} + \sum_l \frac{\partial}{\partial \eta_l}(D_i^{\zeta,\eta}\Gamma(\xi;u)) \frac{\partial}{\partial u_j}(\Theta(\cdot,\zeta)^{-1}(u))_l,$$

it is easy to see that we can still get a uniform bound of the kind

$$|c^{\zeta}_{i,km}(\xi)| \le C \cdot m^{-2h}$$

with C depending on h, the vector fields, and the ellipticity constant μ. □

Remark 5.5. In the statements of all the theorems about singular integrals proved in [Bramanti and Zhu 2012], the constant depends on the kernel only through the constants involved in the assumptions. Actually, we need some additional information about this dependence. A standard sublinearity argument allows us to say that if, for example, our assumptions on the kernel are (3-11), (3-12), and (3-13), the constant in our upper bound will have the form

$$c \cdot (A + B + C),$$

where A, B, and C are the constants appearing in (3-11), (3-12), and (3-13), and c does not depend on the kernel. This fact has been used in the above proof and will be used again.

6. Schauder estimates

We are now in position to apply all the machinery presented in the previous sections to prove our main results, that is, C^{α}_X and L^p estimates on X_iX_ju in terms of u and $\mathcal{L}u$. We will prove C^{α}_X estimates (Theorem 2.1) in this section, and L^p estimates (Theorem 2.2) in Section 7.

Let us recall the setting described at the end of Section 3C. For a fixed subdomain $\Omega' \Subset \Omega \subset \mathbb{R}^n$ and a fixed point $\bar{x} \in \Omega'$, let us consider a lifted ball $\widetilde{B}(\bar{\xi}, R) \subset \mathbb{R}^N$ (with $\bar{\xi} = (\bar{x}, 0)$) where the lifted vector fields $\widetilde{X}_i$ are defined and satisfy Hörmander's condition and the map Θ is defined and satisfies the properties stated in Section 3A.

According to the procedure followed in [Bramanti and Brandolini 2007, Section 5], the proof of C^{α}_X a priori estimates for second order derivatives will proceed in three steps: first, in the space of lifted variables, for test functions supported in a ball $\widetilde{B}(\bar{\xi}, r)$ with r small enough; then for any function in $C^{2,\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, r))$ (not necessarily vanishing at the boundary); then for any function in $C^{2,\alpha}_X(B(\bar{x}, r))$, that is in the original space.

The first step in the proof of Schauder estimates is contained in the following.

Theorem 6.1. *Let $\widetilde{B}(\bar{\xi}, R)$ be as before. Then there exist $R_0 < R$ and $c > 0$ such that, for every $u \in C^{2,\alpha}_{\widetilde{X},0}(\widetilde{B}(\bar{\xi}, R_0))$,*

$$\|u\|_{C^{2,\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},R_0))} \le c\{\|\widetilde{\mathscr{L}}u\|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},R_0))} + \|u\|_{L^\infty(\widetilde{B}(\bar{\xi},R_0))}\},$$

where c and R_0 depend on R, $\{\widetilde{X}_i\}$, α, μ, and $\|\tilde{a}_{ij}\|_{C^\alpha(\widetilde{B}(\bar{\xi},R))}$.

The proof is quite similar to that of [Bramanti and Brandolini 2007, Theorem 5.2] and will be omitted. We just point out the facts which it relies upon:

- the representation formula proved in Theorem 4.19;
- Theorem 5.1 about singular integrals on $C^{\alpha}_{\widetilde{X}}$;
- several properties of $C^{2,\alpha}_{\widetilde{X}}$ functions, collected in Proposition 3.14.

The second step in the proof of Schauder estimates consists in establishing a priori estimates for functions not necessarily compactly supported.

Theorem 6.2. *There exist $r_0 < R_0$ and $c, \beta > 0$ (with R_0 as in Theorem 6.1) such that, for every $u \in C^{2,\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, r_0))$, $0 < t < s < r_0$,*

$$\|u\|_{C^{2,\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},t))} \le \frac{c}{(s-t)^\beta}\{\|\widetilde{\mathscr{L}}u\|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},s))} + \|u\|_{L^\infty(\widetilde{B}(\bar{\xi},s))}\},$$

where r_0, c depend on R, $\{\widetilde{X}_i\}_{i=1}^q$, α, μ, and $\|\tilde{a}_{ij}\|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},R))}$ and β depends on $\{\widetilde{X}_i\}_{i=0}^q$ and α.

As in [Bramanti and Brandolini 2007], this result relies on interpolation inequalities for $C^{k,\alpha}_{\widetilde{X}}$ norms and the use of suitable cutoff function. The following result can be proved as [Bramanti and Brandolini 2007, Lemma 6.2] by the results in Proposition 3.14.

Lemma 6.3 (cutoff functions). *For any $0 < \rho < r$ and $\xi \in \widetilde{B}(\bar{\xi}, R)$, there exists $\varphi \in C_0^\infty(\mathbb{R}^N)$ with the following properties.*

(i) $0 \leqslant \varphi \leqslant 1$, $\varphi \equiv 1$ *on* $\widetilde{B}(\xi, \rho)$, *and* $\operatorname{sprt}\varphi \subseteq \widetilde{B}(\xi, r)$.

(ii) *For* $i, j = 1, 2, \dots, q$,

$$|\widetilde{X}_i\varphi| \leqslant \frac{c}{r-\rho};\ |\widetilde{X}_0\varphi|, |\widetilde{X}_i\widetilde{X}_j\varphi| \leqslant \frac{c}{(r-\rho)^2}. \tag{6-1}$$

(iii) *For any $f \in C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, R))$ and $r - \rho$ small enough,*

$$\begin{aligned} \|f\,\widetilde{X}_i\varphi\|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},R))} &\leqslant \frac{c}{(r-\rho)^2}\|f\|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},R))}, \\ \|f\,\widetilde{X}_0\varphi\|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},R))}, \|f\widetilde{X}_i\widetilde{X}_j\varphi\|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},R))} &\leqslant \frac{c}{(r-\rho)^3}\|f\|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},R))}. \end{aligned} \tag{6-2}$$

We will write

$$\widetilde{B}_\rho(\xi) \prec \varphi \prec \widetilde{B}_r(\xi)$$

to indicate that φ satisfies all the previous properties.

Proposition 6.4 (interpolation inequality for test functions). *Let*

$$H = \sum_{i=1}^{q} \widetilde{X}_i^2 + \widetilde{X}_0$$

and let $\widetilde{B}(\bar{\xi}, R)$ be as before. Then, for every $\alpha \in (0, 1)$, there exist constants $\gamma \geq 1$ and $c > 0$, depending on α, R and $\{\widetilde{X}_i\}$, such that, for every $\varepsilon \in (0, 1)$ and every $f \in C_0^\infty(\widetilde{B}(\bar{\xi}, R/2))$,

$$\|\widetilde{X}_l f\|_{C^\alpha_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, R/2))} \leq \varepsilon \|Hf\|_{C^\alpha_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, R/2))} + \frac{c}{\varepsilon^\gamma} \|f\|_{L^\infty(\widetilde{B}(\bar{\xi}, R/2))} \tag{6-3}$$

for $l = 1, 2, \dots, q$; moreover, we have

$$\|Df\|_{C^\alpha_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, R/2))} \leq \varepsilon \|\widetilde{\mathscr{L}} f\|_{C^\alpha_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, R/2))} + \frac{c}{\varepsilon^\gamma} \|f\|_{L^\infty(\widetilde{B}(\bar{\xi}, R/2))}, \tag{6-4}$$

where D is any vector field of local degree ≤ 1.

To prove Proposition 6.4, we need the following.

Lemma 6.5. *Let $P(\xi_0)$ be a frozen operator of type $\lambda \geq 1$ over $\widetilde{B}(\bar{\xi}, R)$ and $\alpha \in (0, 1)$. Then there exist positive constants $\gamma > 1$ and c, depending on α, μ, and $\{\widetilde{X}_i\}$, such that, for every $f \in C_0^\infty(\widetilde{B}(\bar{\xi}, R))$ and $\varepsilon \in (0, 1)$,*

$$\|PHf\|_{C^\alpha_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, R))} \leq \varepsilon \|Hf\|_{C^\alpha_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, R))} + \frac{c}{\varepsilon^\gamma} \|f\|_{L^\infty(\widetilde{B}(\bar{\xi}, R))}. \tag{6-5}$$

Moreover, (6-5) still holds if H is replaced by any differential operator of weight two, like $\widetilde{X}_i \widetilde{X}_j$ or $\widetilde{X}_0$.

The proof of this lemma is very similar to that of [Bramanti and Brandolini 2007, Lemma 7.2]. It exploits the properties of cutoff functions (Lemma 6.3), inequality (3-19), and fractional integral estimates, relying on [Bramanti and Zhu 2012, Theorem 5.7] and Remark 5.5.

Proof of Proposition 6.4. By Theorem 4.18, we can write

$$af = PHf(\xi) + Sf,$$

where P and S are frozen operators of type 2 and 1, respectively, over $\widetilde{B}(\bar{\xi}, R)$. More precisely, they should be called "constant kernels of type 2 and 1", since they satisfy the definition of frozen kernels with the matrix $\{\tilde{a}_{ij}(\xi_0)\}$ replaced by the identity matrix.

If we assume $a = 1$ on $\widetilde{B}(\bar{\xi}, R/2)$, then, for $f \in C_0^\infty(\widetilde{B}(\bar{\xi}, R/2))$, we obtain

$$f = PHf(\xi) + Sf, \tag{6-6}$$

and therefore, by Theorem 4.11,

$$\widetilde{X}_i f = S_1 Hf(\xi) + Tf, \tag{6-7}$$

where S_1 and T are frozen operators of type 1 and 0, respectively. Substituting (6-6) in (6-7) yields

$$\widetilde{X}_i f = S_1 Hf(\xi) + TPHf + TSf,$$

and therefore, by Theorem 5.1 and Lemma 6.5,

$$\|\widetilde{X}_i f\|_\alpha \le \|S_1 Hf\|_\alpha + c\{\|PHf\|_\alpha + \|Sf\|_\alpha\} \le c\{\varepsilon\|Hf\|_\alpha + \varepsilon^{-\gamma}\|f\|_\infty + \|Sf\|_\alpha\}, \tag{6-8}$$

where all the norms are taken over $\widetilde{B}(\bar{\xi}, R/2)$. We end the proof by showing that, for an operator S of type 1,

$$\|Sf\|_\alpha \le c\|f\|_{L^\infty},$$

which by (6-8) will complete the proof of the first inequality in the proposition. Indeed, if

$$Sf(\xi) = \int_{\widetilde{B}_R} k(\xi, \eta) f(\eta)\, d\eta,$$

we have

$$|Sf(\xi_1) - Sf(\xi_2)| \le \|f\|_{L^\infty(\widetilde{B}_R)} \int_{\widetilde{B}(\bar{\xi}, R)} |k(\xi_1, \eta) - k(\xi_2, \eta)|\, d\eta. \tag{6-9}$$

Moreover,

$$\begin{aligned}\int_{\widetilde{B}_R} |k(\xi_1, \eta) - k(\xi_2, \eta)|\, d\eta &= \int_{\widetilde{B}(\bar{\xi}, R), \rho(\xi_1, \eta) > M\rho(\xi_1, \xi_2)} |k(\xi_1, \eta) - k(\xi_2, \eta)|\, d\eta \\ &\qquad + \int_{\widetilde{B}(\bar{\xi}, R), \rho(\xi_1, \eta) \le M\rho(\xi_1, \xi_2)} |k(\xi_1, \eta) - k(\xi_2, \eta)|\, d\eta \\ &=: I + II.\end{aligned}$$

Then

$$\begin{aligned} I &\le \int_{\rho(\xi_1, \eta) > M\rho(\xi_1, \xi_2)} \frac{c}{\rho(\xi_1, \eta)^{Q-1}} \frac{\rho(\xi_1, \xi_2)}{\rho(\xi_1, \eta)}\, d\eta \\ &= \rho(\xi_1, \xi_2)^\alpha \int_{\rho(\xi_1, \eta) > M\rho(\xi_1, \xi_2)} \frac{\rho(\xi_1, \eta)^{1-\alpha}}{\rho(\xi_1, \eta)^Q} \frac{\rho(\xi_1, \xi_2)^{1-\alpha}}{\rho(\xi_1, \eta)^{1-\alpha}}\, d\eta \\ &\le c\rho(\xi_1, \xi_2)^\alpha \int_{\widetilde{B}_R} \frac{\rho(\xi_1, \eta)^{1-\alpha}}{\rho(\xi_1, \eta)^Q}\, d\eta \le c\rho(\xi_1, \xi_2)^\alpha R^{1-\alpha}, \end{aligned}$$

where in the last inequality we have used the following standard computation (which will be useful again):

$$\int_{\widetilde{B}(\bar{\xi}, R), \rho(\xi_1, \eta) < r} \frac{d\eta}{\rho(\xi_1, \eta)^{Q-\beta}} \le cr^\beta \quad \text{for any } \xi_1 \in \widetilde{B}(\bar{\xi}, R). \tag{6-10}$$

As to II,

$$II \le \int_{\rho(\xi_1, \eta) \le M\rho(\xi_1, \xi_2)} |k(\xi_1, \eta)|\, d\eta + \int_{\rho(\xi_1, \eta) \le M\rho(\xi_1, \xi_2)} |k(\xi_2, \eta)|\, d\eta.$$

Since there exists $M_1 > 0$ such that if $\rho(\xi_1, \eta) \le M\rho(\xi_1, \xi_2)$, then $\rho(\xi_2, \eta) \le M_1\rho(\xi_1, \xi_2)$,

$$II \le c\left\{\int_{\rho(\xi_1, \eta) \le M\rho(\xi_1, \xi_2)} \frac{1}{\rho(\xi_1, \eta)^{Q-1}}\, d\eta + \int_{\rho(\xi_2, \eta) \le M_1\rho(\xi_1, \xi_2)} \frac{1}{\rho(\xi_2, \eta)^{Q-1}}\, d\eta\right\},$$

which, again by (6-10), is

$$\le c\rho(\xi_1, \xi_2) \le c\rho(\xi_1, \xi_2)^\alpha R^{1-\alpha}.$$

Hence, for every $\alpha \in (0,1)$,

$$\int_{\widetilde{B}_R} |k(\xi_1,\eta)-k(\xi_2,\eta)|\,d\eta \le c_\alpha \rho(\xi_1,\xi_2)^\alpha R^{1-\alpha},$$

and, by (6-9),

$$|Sf|_\alpha \le c\|f\|_{L^\infty}.$$

Moreover,

$$|Sf(\xi)| \le \int_{\widetilde{B}_R} |k(\xi,\eta)f(\eta)|\,d\eta \le \|f\|_{L^\infty}\int_{\rho(\xi,\eta)\le cR} \frac{c}{\rho(\xi,\eta)^{Q-1}}\,d\eta \le cR\|f\|_{L^\infty},$$

hence

$$\|Sf\|_\alpha \le c\|f\|_{L^\infty}.$$

This completes the proof of (6-3). A similar argument gives (6-4). □

Theorem 6.6 (interpolation inequality). *There exist positive constants c, γ and $r_1 < R$ such that, for any $u \in C^{2,\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},r_1))$, $0<\rho<r_1$, $0<\delta<1/3$,*

$$\|\widetilde{D}u\|_{C^\alpha_{\widetilde{X}}(\widetilde{B}(\bar{\xi},\rho))} \le \delta\sum_{i=1}^{q}\|\widetilde{D}^2u\|_{C^\alpha_{\widetilde{X}}(\widetilde{B}(\bar{\xi},r_1))} + \frac{c}{\delta^\gamma(r_1-\rho)^{2\gamma}}\|u\|_{L^\infty(\widetilde{B}(\bar{\xi},r_1))},$$

where

$$\|\widetilde{D}u\| \equiv \sum_{i=1}^{q}\|\widetilde{X}_iu\| \quad \textit{and} \quad \|\widetilde{D}^2u\| \equiv \sum_{i,j=1}^{q}\|\widetilde{X}_i\widetilde{X}_iu\| + \|\widetilde{X}_0u\|.$$

The constants c, r_1, γ depend on α, $\{\widetilde{X}_i\}$; γ is as in Proposition 6.4.

Proof. The proof can be carried out exactly as in [Bramanti and Brandolini 2007, Proposition 7.4], exploiting the properties of cutoff functions (Lemma 6.3), the interpolation inequality for test functions (Proposition 6.4), and (3-20) in Proposition 3.14. □

We are now ready to complete the second step in the proof of Schauder estimates.

Proof of Theorem 6.2. This proof can now be carried out exactly like in [Bramanti and Brandolini 2007, Theorem 5.3], exploiting Schauder estimates for functions with small support (Theorem 6.1), the properties of Hölder continuous functions contained in (3-20), (3-21), and (3-24), the properties of cutoff functions (Lemma 6.3), and the interpolation inequalities contained in Theorem 6.6 and (6-4). □

Proof of Theorem 2.1. We finally come back to our original context, which we are going to recall. We have a bounded domain Ω where our vector fields and coefficients are defined, and a fixed subdomain $\Omega' \Subset \Omega$. Fix $\bar{x} \in \Omega'$ and R such that in $B(\bar{x},R) \subset \Omega$ all the construction of the previous two subsections (lifting to $\widetilde{B}(\bar{\xi},R)$ and so on) can be performed. Let r_0 be as in Theorem 6.2. To begin with, we want to prove Schauder estimates for functions $u \in C^{2,\alpha}_X(B(\bar{x},r_0))$. By Proposition 3.15 we know that the function $\widetilde{u}(x,h)=u(x)$ belongs to $C^{2,\alpha}_{\widetilde{X}}(B(\bar{\xi},r_0))$, so we can apply to $\widetilde{u}$ Schauder estimates contained in Theorem 6.2. Combining this fact with the two estimates in Proposition 3.15 and choosing t, s such that

$$r_0 > t > s > 0 \quad \text{and} \quad t-s = r_0-t,$$

we get, for some exponent $\omega > 2$,

$$\begin{aligned} \|u\|_{C^{2,\alpha}_X(B(\bar{x},s))} &\le \frac{c}{(t-s)^2}\|\widetilde{u}\|_{C^{2,\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},t))} \\ &\le \frac{c}{(r_0-t)^\omega}(\|\widetilde{\mathscr{L}}\widetilde{u}\|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},r_0))} + \|\widetilde{u}\|_{L^\infty(\widetilde{B}(\bar{\xi},r_0))}) \\ &\le \frac{c}{(r_0-s)^\omega}(\|\mathscr{L}u\|_{C^{\alpha}_X(B(\bar{x},r_0))} + \|u\|_{L^\infty(B(\bar{x},r_0))}), \end{aligned} \tag{6-11}$$

since $\widetilde{\mathscr{L}}\widetilde{u} = \widetilde{(\mathscr{L}u)}$. Next, let us choose a family of balls $B(x_i, r_0)$ in Ω such that

$$\Omega' \subset \bigcup_{i=1}^{k} B(x_i, r_0/2) \subset \bigcup_{i=1}^{k} B(x_i, r_0) \subset \Omega.$$

Then, by Proposition 3.14(v) and (6-11), with $s = r_0/2$,

$$\begin{aligned} \|u\|_{C^{2,\alpha}_X(\Omega')} \le \|u\|_{C^{2,\alpha}_X(\cup B(x_i,r_0/2))} &\le c\sum_{i=1}^{k}\|u\|_{C^{2,\alpha}_X(B(x_i,r_0))} \\ &\le c\sum_{i=1}^{k}\{\|\mathscr{L}u\|_{C^{\alpha}_X(B(x_i,r_0))} + \|u\|_{L^\infty(B(x_i,r_0))}\} \\ &\le c\{\|\mathscr{L}u\|_{C^{\alpha}_X(\Omega)} + \|u\|_{L^\infty(\Omega)}\} \end{aligned}$$

with c also depending on r_0. Finally, let us note that the constant c depends on the coefficients a_{ij} through the norms

$$\|\tilde{a}_{ij}\|_{C^{\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},R_0))},$$

which in turn are bounded by the norms

$$\|a_{ij}\|_{C^{\alpha}_X(B(\bar{x},R_0))}$$

(by Proposition 3.15), and hence by $\|a_{ij}\|_{C^{\alpha}_X(\Omega)}$ (or more precisely, by $\|a_{ij}\|_{C^{\alpha}_X(\Omega'')}$ for some Ω'' such that $\Omega' \Subset \Omega'' \Subset \Omega$). □

7. L^p estimates

The logical structure of this section, as well as the general setting, is very similar to that of the previous one, following as closely as possible the strategy of [Bramanti and Brandolini 2000a]. The basic difference with the setting of Schauder estimates is the fact that here we start with representation formulas where the "frozen" point has finally been unfrozen; therefore, singular integrals with *variable* kernels are now involved, together with their commutators with VMO functions. This makes the singular integral part of the theory more involved.

The first step is contained in the following.

Theorem 7.1. *Let $\widetilde{B}(\bar{\xi}, R)$ be as in the previous section, and $p \in (1, \infty)$. There exists $R_0 < R$ such that, for every $u \in C_0^\infty(\widetilde{B}(\bar{\xi}, R_0))$,*

$$\|u\|_{S^{2,p}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},R_0))} \le c\{\|\widetilde{\mathscr{L}}u\|_{L^p(\widetilde{B}(\bar{\xi},R_0))} + \|u\|_{L^p(\widetilde{B}(\bar{\xi},R_0))}\} \tag{7-1}$$

for some constant c depending on $\{\widetilde{X}_i\}_{i=0}^q$, p, μ, and R; the number R_0 also depends on the local VMO moduli $\eta^*_{a_{ij},\Omega',\Omega}$.

The proof can be carried out exactly like in [Bramanti and Brandolini 2000a, Theorem 3.2], exploiting the representation formula proved in Theorem 4.21 and the results about singular integrals and commutators contained in Theorem 5.4.

Next, we have to remove the restriction to compactly supported functions.

Theorem 7.2. *Let $\widetilde{B}(\bar{\xi}, R)$ be as before. There exists $r_0 < R$ and, for any $r \le r_0$, there exists $c > 0$ such that, for any $u \in S^{2,p}_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, r))$, we have*

$$\|u\|_{S^{2,p}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},r/2))} \le c\{\|\widetilde{\mathscr{L}}u\|_{L^p(\widetilde{B}(\bar{\xi},r))} + \|u\|_{L^p(\widetilde{B}(\bar{\xi},r))}\}.$$

The constants c, r_0 depend on $\{\widetilde{X}_i\}_{i=0}^q$, p, μ, R, and $\eta^*_{a_{ij},\Omega',\Omega}$; c also depends on r.

Analogously to what we have seen in Theorem 6.2, the proof of the above theorem relies on interpolation inequalities for Sobolev norms and the use of cutoff functions. Regarding cutoff functions, we need the following statement.

Lemma 7.3 (radial cutoff functions). *For any $\sigma \in (\frac{1}{2}, 1)$, $r > 0$ and $\xi \in \widetilde{B}(\bar{\xi}, r)$, there exists $\varphi \in C_0^\infty(\mathbb{R}^N)$ with the following properties.*

(i) *$\widetilde{B}_{\sigma r}(\xi) \prec \varphi \prec \widetilde{B}_{\sigma' r}(\xi)$ with $\sigma' = (1+\sigma)/2$ (this means that $\varphi = 1$ in $\widetilde{B}_{\sigma r}(\xi)$ and it is supported in $\widetilde{B}_{\sigma' r}(\xi)$).*

(ii) *For $i, j = 1, \dots, q$, we have*

$$|\widetilde{X}_i\varphi| \le \frac{c}{(1-\sigma)r}, \quad |\widetilde{X}_0\varphi|, |\widetilde{X}_i\widetilde{X}_j\varphi| \le \frac{c}{(1-\sigma)^2r^2}. \tag{7-2}$$

The above lemma, very similar to [Bramanti and Brandolini 2000a, Lemma 3.3], is actually contained in Lemma 6.3, but we prefer to state it explicitly because it is formulated in a slightly different notation, suitable to our application to L^p estimates.

Theorem 7.4 (interpolation inequality for Sobolev norms). *Let $\widetilde{B}(\bar{\xi}, R)$ be as before. For every $p \in (1, \infty)$, there exists $c > 0$ and $r_1 < R$ such that, for every $0 < \varepsilon \le 4r_1$, $u \in C_0^\infty(\widetilde{B}(\bar{\xi}, r_1))$,*

$$\|\widetilde{X}_iu\|_{L^p(\widetilde{B}(\bar{\xi},r_1))} \le \varepsilon\|Hu\|_{L^p(\widetilde{B}(\bar{\xi},r_1))} + \frac{c}{\varepsilon}\|u\|_{L^p(\widetilde{B}(\bar{\xi},r_1))} \tag{7-3}$$

for every $i = 1, \dots, q$, where $H := \sum_{i=1}^q \widetilde{X}_i^2 + \widetilde{X}_0$.

Proof. The proof of this proposition is adapted from [Bramanti and Brandolini 2000a, Theorem 3.6], but also improves that result, which is stated with a generic constant $c(\varepsilon)$ instead of c/ε.

Let r_1 be a small number to be fixed later. Like in the proof of Proposition 6.4, we can write, for any $u \in C_0^\infty(\widetilde{B}(\bar{\xi}, r_1))$ and $\xi \in \widetilde{B}(\bar{\xi}, r_1)$,

$$\widetilde{X}_i u(\xi) = SHu(\xi) + Tu(\xi),$$

where S and T are constant operators of type 1 and 0, respectively, over $\widetilde{B}(\bar{\xi}, 2r_1)$, provided $2r_1 < R$. (See the proof of Proposition 6.4 for the explanation of the term "constant operators of type λ".) Since

$$\|Tu\|_{L^p(\widetilde{B}(\bar{\xi},r_1))} \le c\|u\|_{L^p(\widetilde{B}(\bar{\xi},r_1))},$$

the result will follow if we prove that

$$\|SHu\|_{L^p(\widetilde{B}(\bar{\xi},r_1))} \le \varepsilon\|Hu\|_{L^p(\widetilde{B}(\bar{\xi},r_1))} + \frac{c}{\varepsilon}\|u\|_{L^p(\widetilde{B}(\bar{\xi},r_1))}. \tag{7-4}$$

Let $k(\xi, \eta)$ be the kernel of S, and, for any fixed $\xi \in \widetilde{B}(\bar{\xi}, r_1)$, let φ_ε be a cutoff function (as in Lemma 7.3) with

$$\widetilde{B}_{\varepsilon/2}(\xi) \prec \varphi_\varepsilon \prec \widetilde{B}_\varepsilon(\xi).$$

Let us split $SHu(\xi)$ as

$$\begin{aligned} SHu(\xi) &= \int_{\widetilde{B}(\bar{\xi},r_1),\rho(\xi,\eta)>\varepsilon/2} k(\xi,\eta)[1-\varphi_\varepsilon(\eta)]Hu(\eta)\,d\eta + \int_{\widetilde{B}(\bar{\xi},r_1),\rho(\xi,\eta)\le\varepsilon} k(\xi,\eta)Hu(\eta)\varphi_\varepsilon(\eta)\,d\eta \\ &=: I(\xi) + II(\xi). \end{aligned}$$

Then

$$\begin{aligned} |I(\xi)| &= \left| \int_{\widetilde{B}(\bar{\xi},r_1),\rho(\xi,\eta)>\varepsilon/2} H^T(k(\xi,\cdot)[1-\varphi_\varepsilon(\cdot)])(\eta)u(\eta)\,d\eta \right| \\ &\le \int_{\widetilde{B}(\bar{\xi},r_1),\rho(\xi,\eta)>\varepsilon/2} \Big\{ \big|[1-\varphi_\varepsilon]H^T k(\xi,\cdot)\big| + c\sum \big|\widetilde{X}_i[1-\varphi_\varepsilon]\cdot \widetilde{X}_j k(\xi,\cdot)\big| \\ &\qquad\qquad + \big|k(\xi,\cdot)H^T[1-\varphi_\varepsilon]|(\eta)|u(\eta)\big| \Big\}\,d\eta \\ &=: A(\xi) + B(\xi) + C(\xi). \end{aligned}$$

Recall that, for $i, j = 1, 2, \dots, q$,

$$\begin{aligned} |k(\xi,\eta)| &\le \frac{c}{d(\xi,\eta)^{Q-1}}, \\ |\widetilde{X}_i k(\xi,\eta)| &\le \frac{c}{d(\xi,\eta)^{Q}}, \\ |H^T k(\xi,\cdot)(\eta)| &\le \frac{c}{d(\xi,\eta)^{Q+1}}, \\ |\widetilde{X}_i(1-\varphi_\varepsilon)(\eta)| &\le \frac{c}{\varepsilon}, \\ |H^T(1-\varphi_\varepsilon)(\eta)| &\le \frac{c}{\varepsilon^2}, \end{aligned}$$

and the derivatives of $(1-\varphi_\varepsilon)$ are supported in the annulus $\varepsilon/2 \le d(\xi,\eta) \le \varepsilon$. Since $\xi,\eta \in \widetilde{B}(\bar{\xi},r_1)$, we have $d(\xi,\eta) < 2r_1$. Hence, letting k_0 be the integer such that $2^{k_0-1}\varepsilon < 2r_1 \le 2^{k_0}\varepsilon$, we have

$$
\begin{aligned}
|A(\xi)| &\le c \sum_{k=0}^{k_0} \int_{2^{k-1}\varepsilon<\rho(\xi,\eta)\le 2^k\varepsilon} \frac{c}{d(\xi,\eta)^{Q+1}} |u(\eta)|\, d\eta \\
&\le c \sum_{k=0}^{k_0} \frac{1}{2^{k-1}\varepsilon} \frac{1}{(\varepsilon 2^{k-1})^Q} \int_{\rho(\xi,\eta)\le 2^k\varepsilon} |u(\eta)|\, d\eta \\
&\le \frac{c}{\varepsilon} \cdot \sup_{r\le 4r_1} \frac{1}{|\widetilde{B}(\xi,r)|} \int_{\widetilde{B}(\xi,r)} |u(\eta)|\, d\eta. \qquad (7\text{-}5)
\end{aligned}
$$

We now have to recall the definition of a *local maximal function* $\mathcal{M}$ in a (metric) locally homogeneous space $(\Omega,\{\Omega_n\},d,d\mu)$, given in [Bramanti and Zhu 2012]. Fix Ω_n, Ω_{n+1} (see Section 3C for the notation) and, for any $f \in L^1(\Omega_{n+1})$, define

$$
\mathcal{M}_{\Omega_n,\Omega_{n+1}} f(x) = \sup_{r\le r_n} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y) \quad \text{for } x \in \Omega_n,
$$

where $r_n = \frac{2}{5}\varepsilon_n$. Applying this definition to our situation where $4r_1 = r_n = \frac{2}{5}\varepsilon_n$, we get $\varepsilon_n = 10r_1$ and, for $\xi \in \widetilde{B}(\bar{\xi},r_1)$, we have $\widetilde{B}(\xi,\varepsilon_n) \subset \widetilde{B}(\bar{\xi},11r_1)$. Therefore, by (7-5), we can write

$$
|A(\xi)| \le \frac{c}{\varepsilon} \cdot \mathcal{M}_{\widetilde{B}(\bar{\xi},r_1),\widetilde{B}(\bar{\xi},11r_1)} u(\xi),
$$

and, by [Bramanti and Zhu 2012, Theorem 8.3], we have

$$
\|A\|_{L^p(\widetilde{B}(\bar{\xi},r_1))} \le \frac{c}{\varepsilon} \|u\|_{L^p(\widetilde{B}(\bar{\xi},11r_1))} = \frac{c}{\varepsilon} \|u\|_{L^p(\widetilde{B}(\bar{\xi},r_1))},
$$

since $u \in C_0^\infty(\widetilde{B}(\bar{\xi},r_1))$, provided $11r_1 < R$. Also,

$$
\begin{aligned}
|B(\xi)| &\le c \int_{\frac{\varepsilon}{2}<\rho(\xi,\eta)\le\varepsilon,} \frac{1}{\varepsilon} \cdot \frac{1}{d(\xi,\eta)^Q} |u(\eta)|\, d\eta \le \frac{c}{\varepsilon^{Q+1}} \int_{\rho(\xi,\eta)\le\varepsilon} |u(\eta)|\, d\eta \\
&\le \frac{c}{\varepsilon} \cdot \sup_{r\le\varepsilon} \frac{1}{|\widetilde{B}(\xi,r)|} \int_{\widetilde{B}(\xi,r)} |u(\eta)|\, d\eta \le \frac{c}{\varepsilon} \cdot \mathcal{M}_{\widetilde{B}(\bar{\xi},r_1),\widetilde{B}(\bar{\xi},11r_1)} u(\xi)
\end{aligned}
$$

provided $\varepsilon < 4r_1$. As before, we have

$$
\|B\|_{L^p(\widetilde{B}(\bar{\xi},r_1))} \le \frac{c}{\varepsilon} \|u\|_{L^p(\widetilde{B}(\bar{\xi},r_1))}.
$$

Finally,

$$
|C(\xi)| \le c \int_{\varepsilon/2<\rho(\xi,\eta)\le\varepsilon} \frac{1}{\varepsilon^2} \cdot \frac{1}{d(\xi,\eta)^{Q-1}} |u(\eta)| \eta\, dy \le \frac{c}{\varepsilon^{Q+1}} \int_{\rho(\xi,\eta)\le\varepsilon} |u(\eta)|\, d\eta.
$$

Therefore, as for the term $B(\xi)$,

$$
\|I\|_{L^p(\widetilde{B}(\bar{\xi},r_1))} \le \frac{c}{\varepsilon} \|u\|_{L^p(\widetilde{B}(\bar{\xi},r_1))}.
$$

Let us bound II:

$$|II(\xi)| \le c \int_{\rho(\xi,\eta)\le\varepsilon} \frac{|Hu(\eta)|}{\rho(\xi,\eta)^{Q-1}}\, d\eta.$$

Then a computation similar to that of $C(\xi)$ gives

$$|II(\xi)| \le c\varepsilon \mathcal{M}_{\widetilde{B}(\bar{\xi},r_1),\widetilde{B}(\bar{\xi},11r_1)} u(\xi) \quad \text{and} \quad \|II\|_{L^p(\widetilde{B}(\bar{\xi},r_1))} \le c\varepsilon \|u\|_{L^p(\widetilde{B}(\bar{\xi},r_1))},$$

provided $\varepsilon < 4r_1$. □

Theorem 7.5. *For any $u \in S^{2,p}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},r))$, $p \in [1,\infty)$, $0 < r < r_1$ (where r_1 is the number in Theorem 7.4), define the following quantities*:

$$\Phi_k = \sup_{1/2<\sigma<1} ((1-\sigma)^k r^k \|\widetilde{D}^k u\|_{L^p(\widetilde{B}_{r\sigma})}) \quad \textit{for } k = 0, 1, 2.$$

Then, for any $\delta > 0$ (small enough),

$$\Phi_1 \le \delta\, \Phi_2 + \frac{c}{\delta} \Phi_0.$$

Proof. This result follows exactly as in [Bramanti and Brandolini 2000b, Theorem 21], exploiting the interpolation result for compactly supported functions (Theorem 7.4), cutoff functions (Lemma 7.3), and Proposition 3.19. □

Proof of Theorem 7.2. This proof is similar to that of theorem [Bramanti and Brandolini 2000b, Theorem 3]. Due to the different context, we include a complete proof for the convenience of the reader.

Pick $r_0 = \min(R_0, r_1)$ where R_0 and r_1 are the numbers appearing in Theorems 7.1 and 7.4, respectively. For $r \le r_0$, let $u \in S^{2,p}_{\widetilde{X}}(\widetilde{B}(\bar{\xi},r))$. Let φ be a cutoff function as in Lemma 7.3:

$$\widetilde{B}(\bar{\xi},\sigma r) \prec \varphi \prec \widetilde{B}(\bar{\xi},\sigma' r).$$

By Theorem 7.1, $\varphi u \in S^{2,p}_{\widetilde{X},0}(\widetilde{B}(\bar{\xi},r))$; then, by density, we can apply Theorem 7.1 to φu:

$$\|\varphi u\|_{S^{2,p}(\widetilde{B}(\bar{\xi},r))} \le c\{\|\widetilde{\mathcal{L}}(\varphi u)\|_{L^p(\widetilde{B}(\bar{\xi},r))} + \|\varphi u\|_{L^p(\widetilde{B}(\bar{\xi},r))}\}.$$

For $1 \le i, j \le q$, from the above inequality we get

$$\|\widetilde{X}_i \widetilde{X}_j u\|_{L^p(\widetilde{B}_{\sigma r})} \le c\{\|\widetilde{\mathcal{L}}u\|_{L^p(\widetilde{B}_{\sigma' r})} + \frac{1}{(1-\sigma)r}\|\widetilde{D}u\|_{L^p(\widetilde{B}_{\sigma' r})} + \frac{1}{(1-\sigma)^2 r^2}\|u\|_{L^p(\widetilde{B}_{\sigma' r})}\}.$$

Multiplying both sides by $(1-\sigma)^2 r^2$, we get

$$(1-\sigma)^2 r^2 \parallel \widetilde{X}_i \widetilde{X}_j u \parallel_{L^p(\widetilde{B}_{\sigma r})} \le c\{(1-\sigma)^2 r^2 \|\widetilde{\mathcal{L}}u\|_{L^p(\widetilde{B}_{\sigma' r})} + (1-\sigma) r (\|\widetilde{D}u\|_{L^p(\widetilde{B}_{\sigma' r})}) + \|u\|_{L^p(\widetilde{B}_{\sigma' r})}\}. \tag{7-6}$$

Next, we compute $(1-\sigma)^2 r^2 \|\widetilde{X}_0 u\|_{L^p(\widetilde{B}_{\sigma r})}$:

$$\begin{aligned}(1-\sigma)^2 r^2 \|\widetilde{X}_0 u\|_{L^p(\widetilde{B}_{\sigma r})} &= (1-\sigma)^2 r^2 \|\widetilde{\mathcal{L}}u - \sum_{i,j=1}^{q} \tilde{a}_{ij} \widetilde{X}_i \widetilde{X}_j u\|_{L^p(\widetilde{B}_{\sigma r})} \\ &\le c(1-\sigma)^2 r^2 (\|\widetilde{\mathcal{L}}u\|_{L^p(\widetilde{B}_{\sigma r})} + \|\widetilde{X}_i \widetilde{X}_j u\|_{L^p(\widetilde{B}_{\sigma r})}).\end{aligned} \tag{7-7}$$

Combining (7-6) and (7-7), we have

$$(1-\sigma)^2 r^2 \|\widetilde{D}^2 u\|_{L^p(\widetilde{B}_{\sigma r})} \le c\{(1-\sigma)^2 r^2 \|\widetilde{\mathscr{L}} u\|_{L^p(\widetilde{B}_{\sigma' r})} + (1-\sigma) r \|\widetilde{D} u\|_{L^p(\widetilde{B}_{\sigma' r})} + \|u\|_{L^p(\widetilde{B}_{\sigma' r})}\}. \tag{7-8}$$

Adding $(1-\sigma) r \|\widetilde{D} u\|_{L^p(\widetilde{B}_{\sigma r})}$ to both sides of (7-8),

$$\begin{aligned}(1-\sigma) r \|\widetilde{D} u\|_{L^p(\widetilde{B}_{\sigma r})} + (1-\sigma)^2 r^2 \|\widetilde{D}^2 u\|_{L^p(\widetilde{B}_{\sigma r})} \\ \le c\{(1-\sigma)^2 r^2 \|\widetilde{\mathscr{L}} u\|_{L^p(\widetilde{B}_{\sigma' r})} + (1-\sigma) r \|\widetilde{D} u\|_{L^p(\widetilde{B}_{\sigma' r})} + \|u\|_{L^p(\widetilde{B}_{\sigma' r})}\},\end{aligned} \tag{7-9}$$

which, by Theorem 7.5, is

$$\le c\{(1-\sigma)^2 r^2 \|\widetilde{\mathscr{L}} u\|_{L^p(\widetilde{B}_{\sigma' r})} + (\delta \Phi_2 + \frac{c}{\delta} \Phi_0) + \|u\|_{L^p(\widetilde{B}_{\sigma' r})}\}.$$

Choosing δ small enough, we have

$$\Phi_2 + \Phi_1 \le c\{r^2 \|\widetilde{\mathscr{L}} u\|_{L^p(\widetilde{B}_r)} + \|u\|_{L^p(\widetilde{B}_r)}\}.$$

Then

$$r^2 \|\widetilde{D}^2 u\|_{L^p(\widetilde{B}(\bar{\xi}, r/2))} + r \|\widetilde{D} u\|_{L^p(\widetilde{B}(\bar{\xi}, r/2))} \le c\{r^2 \|\widetilde{\mathscr{L}} u\|_{L^p(\widetilde{B}(\bar{\xi}, r))} + \|u\|_{L^p(\widetilde{B}(\bar{\xi}, r))}\},$$

hence

$$\|u\|_{S^{2,p}_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, r/2))} \le c\{\|\widetilde{\mathscr{L}} u\|_{L^p(\widetilde{B}(\bar{\xi}, r))} + \|u\|_{L^p(\widetilde{B}(\bar{\xi}, r))}\},$$

which is the desired result. □

Proof of Theorem 2.2. This follows from Theorem 7.2 in a way which is analogous to that followed in Section 6 to prove Schauder estimates. Namely, fix $\bar{x} \in \Omega' \Subset \Omega$ and R such that in $B(\bar{x}, R) \subset \Omega$ all the construction of the previous two subsections (lifting to $\widetilde{B}(\bar{\xi}, R)$ and so on) can be performed. Let $r_0 < R$ as in Theorem 7.2, and let $u \in S^{2,p}_X(B(\bar{x}, r_0))$. By Theorem 3.20 we know that the function $\widetilde{u}(x, h) = u(x)$ belongs to $S^{2,p}_{\widetilde{X}}(B(\bar{\xi}, r_0))$, so we can apply to $\widetilde{u}$ the L^p estimates contained in Theorem 7.2. Combining this fact with the two estimates in Theorem 3.20, we get

$$\begin{aligned}\|u\|_{S^{2,\alpha}_X(B(\bar{x}, \delta_0 r_0/2))} &\le c \|\widetilde{u}\|_{S^{2,\alpha}_{\widetilde{X}}(\widetilde{B}(\bar{\xi}, r_0/2))} \\ &\le c(\|\widetilde{\mathscr{L}} \widetilde{u}\|_{L^p(\widetilde{B}(\bar{\xi}, r_0))} + \|\widetilde{u}\|_{L^p(\widetilde{B}(\bar{\xi}, r_0))}) \\ &\le c(\|\mathscr{L} u\|_{L^p(B(\bar{x}, r_0))} + \|u\|_{L^p(B(\bar{x}, r_0))}),\end{aligned}$$

since $\widetilde{\mathscr{L}} \widetilde{u} = \widetilde{(\mathscr{L} u)}$. Next, let us choose a family of balls $B(x_i, r_0)$ in Ω such that

$$\Omega' \subset \bigcup_{i=1}^{k} B(x_i, \delta_0 r_0/2) \subset \bigcup_{i=1}^{k} B(x_i, r_0) \subset \Omega.$$

Therefore,

$$\begin{aligned}\|u\|_{S_X^{2,p}(\Omega')} \le \|u\|_{S_X^{2,p}(\cup B(x_i,\delta_0 r_0/2))} &\le \sum_{i=1}^{k} \|u\|_{S_X^{2,p}(B(x_i,\delta_0 r_0/2))}\\ &\le c\sum_{i=1}^{k}\{\|\mathscr{L}u\|_{L^p(B(x_i,r_0))} + \|u\|_{L^p(B(x_i,r_0))}\}\\ &\le c\{\|\mathscr{L}u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\}\end{aligned}$$

with c also depending on r_0. □

Acknowledgements

This research was mainly carried out while Maochun Zhu was visiting the Department of Mathematics of Politecnico di Milano, which we wish to thank for the hospitality.

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Received 24 Dec 2011. Revised 5 Apr 2013. Accepted 13 Jun 2013.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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ANALYSIS & PDE

Volume 6 No. 8 2013

2157-5045(2013)6:8;1-A