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STRICHARTZ ESTIMATES FOR SCHRÖDINGER EQUATIONS WITH VARIABLE COEFFICIENTS AND UNBOUNDED POTENTIALS

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This paper is concerned with Schrödinger equations with variable coefficients and unbounded electromagnetic potentials, where the kinetic energy part is a long-range perturbation of the flat Laplacian and the electric (respectively magnetic) potential can grow subquadratically (respectively sublinearly) at spatial infinity. We prove sharp (local-in-time) Strichartz estimates, outside a large compact ball centered at the origin, for any admissible pair including the endpoint. Under the nontrapping condition on the Hamilton flow generated by the kinetic energy, global-in-space estimates are also studied. Finally, under the nontrapping condition, we prove Strichartz estimates with an arbitrarily small derivative loss without asymptotic flatness on the coefficients.

1. Introduction

We study sharp (local-in-time) Strichartz estimates for Schrödinger equations with variable coefficients and unbounded electromagnetic potentials. More precisely, we consider the Schrödinger operator

$$H=\frac{1}{2}\sum_{j,k=1}^{d}(-i\partial_j-A_j(x))g^{jk}(x)(-i\partial_k-A_k(x))+V(x),\quad x\in\mathbb{R}^d,$$

where $d\ge 1$ is the spatial dimension. Throughout the paper we assume that g^{jk}, V, and A_j are smooth real-valued functions on $\mathbb{R}^d$ and that $(g^{jk}(x))_{j,k}$ is symmetric and positive definite:

$$\sum_{j,k=1}^{d}g^{jk}(x)\xi_j\xi_k\ge c|\xi|^2,\quad x,\xi\in\mathbb{R}^d,$$

with some $c>0$. Moreover, we suppose the following condition holds.

Assumption 1.1. There exists $\mu\ge 0$ such that for any $\alpha\in\mathbb{Z}_+^d$,

$$\begin{aligned}|\partial_x^\alpha(g^{jk}(x)-\delta_{jk})|&\le C_\alpha\langle x\rangle^{-\mu-|\alpha|},\\ |\partial_x^\alpha A_j(x)|&\le C_\alpha\langle x\rangle^{1-\mu-|\alpha|},\\ |\partial_x^\alpha V(x)|&\le C_\alpha\langle x\rangle^{2-\mu-|\alpha|},\quad x\in\mathbb{R}^d.\end{aligned}$$

Then it is well known that H admits a unique self-adjoint realization on $L^2(\mathbb{R}^d)$, which we denote by the

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same symbol H. By Stone's theorem, H generates a unique unitary propagator e^{-itH} on $L^2(\mathbb{R}^d)$ such that $u(t)=e^{-itH}\varphi$ is the solution to the Schrödinger equation

$$\begin{aligned} &i\partial_t u(t)=Hu(t), \quad t\in\mathbb{R},\\ &u|_{t=0}=\varphi\in L^2(\mathbb{R}^d). \end{aligned}$$

In order to explain the purpose of the paper, we recall some known results. Let us first recall well-known properties of the free propagator e^{-itH_0}, where $H_0=-\Delta/2$. The distribution kernel of e^{-itH_0} is given explicitly by $(2\pi it)^{-d/2}e^{i|x-y|^2/(2t)}$, and $e^{-itH_0}\varphi$ thus satisfies the dispersive estimate

$$\|e^{-itH_0}\varphi\|_{L^\infty(\mathbb{R}^d)}\le C|t|^{-d/2}\|\varphi\|_{L^1(\mathbb{R}^d)}, \quad t\neq 0.$$

Moreover, e^{-itH_0} enjoys the (global-in-time) Strichartz estimates

$$\|e^{-itH_0}\varphi\|_{L^p(\mathbb{R};L^q(\mathbb{R}^d))}\le C\|\varphi\|_{L^2(\mathbb{R}^d)},$$

where (p,q) satisfies the admissible condition

$$p\ge 2, \quad \frac{2}{p}=d\Big(\frac12-\frac1q\Big), \quad (d,p,q)\neq(2,2,\infty). \tag{1-1}$$

Strichartz estimates imply that, for any $\varphi\in L^2$, $e^{-itH_0}\varphi\in\bigcap_{q\in Q_d}L^q$ for a.e. $t\in\mathbb{R}$, where $Q_1=[2,\infty]$, $Q_2=[2,\infty)$ and $Q_d=[2,2d/(d-2)]$ for $d\ge 3$. These estimates can therefore be regarded as L^p-type smoothing properties of Schrödinger equations, and have been widely used in the study of nonlinear Schrödinger equations; see, for example, [Cazenave 2003]. Strichartz estimates for e^{-itH_0} were first proved in [Strichartz 1977] for a restricted pair of (p,q) with $p=q=2(d+2)/d$, and have been generalized for (p,q) satisfying (1-1) and $p\neq 2$ in [Ginibre and Velo 1985]. The endpoint estimate $(p,q)=(2,2d/(d-2))$ for $d\ge 3$ was obtained in [Keel and Tao 1998].

For Schrödinger operators with electromagnetic potentials, that is, $H=(1/2)(-i\partial_x-A)^2+V$, (short-time) dispersive and (local-in-time) Strichartz estimates have been extended with potentials decaying at infinity [Yajima 1987] or growing at infinity [Fujiwara 1980; Yajima 1991]. In particular, it was shown in the last two references that if $g^{jk}=\delta_{jk}$, V and A satisfy Assumption 1.1 with $\mu\ge 0$, and all derivatives of the magnetic field $B=dA$ of short-range type, then $e^{-itH}\varphi$ satisfies (short-time) dispersive estimate

$$\|e^{-itH}\varphi\|_{L^\infty(\mathbb{R}^d)}\le C|t|^{-d/2}\|\varphi\|_{L^1(\mathbb{R}^d)},$$

for sufficiently small $t\neq 0$. Local-in-time Strichartz estimates, which have the form

$$\|e^{-itH}\varphi\|_{L^p([-T,T];L^q(\mathbb{R}^d))}\le C_T\|\varphi\|_{L^2(\mathbb{R}^d)}, \quad T>0,$$

are immediate consequences of this estimate and the TT^*-argument in [Ginibre and Velo 1985] (see [Keel and Tao 1998] for the endpoint estimate). For the case with singular electric potentials or with supercritical electromagnetic potentials, we refer to [Yajima 1987; 1998; Yajima and Zhang 2004; D'Ancona and Fanelli 2009]. We mention that global-in-time dispersive and Strichartz estimates for scattering states have also been studied under suitable decaying conditions on potentials and assumptions for zero energy; see [Journé et al. 1991; Yajima 2005; Schlag 2007; Erdoğan et al. 2009; D'Ancona et al. 2010]. We also mention that there is no result on sharp global-in-time dispersive estimates for magnetic Schrödinger equations.

On the other hand, the influence of the geometry on the behavior of solutions to linear and nonlinear partial differential equations has been extensively studied. From this geometric viewpoint, sharp local-in-time Strichartz estimates for Schrödinger equations with variable coefficients (or, more generally, on manifolds) have recently been investigated by many authors under several conditions on the geometry; see, for example, [Staffilani and Tataru 2002; Burq et al. 2004; Robbiano and Zuily 2005; Hassell et al. 2006; Bouclet and Tzvetkov 2007; Bouclet 2011b; Burq et al. 2010; Mizutani 2012]. In [Staffilani and Tataru 2002; Robbiano and Zuily 2005; Bouclet and Tzvetkov 2007], the authors studied the case on the Euclidean space with nontrapping asymptotically flat metrics. The case on the nontrapping asymptotically conic manifold was studied in [Hassell et al. 2006; Mizutani 2012]. Bouclet [2011b] considered the case of a nontrapping asymptotically hyperbolic manifold. For the trapping case, it was shown in [Burq et al. 2004] that Strichartz estimates with a loss of derivative $1/p$ hold on any compact manifold without boundaries. They also proved that the loss $1/p$ is optimal in the case of $M = \mathbb{S}^d$. In [Bouclet and Tzvetkov 2007; Bouclet 2011b; Mizutani 2012], the authors proved sharp Strichartz estimates, outside a large compact set, without the nontrapping condition. It was shown in [Burq et al. 2010] that sharp Strichartz estimates still hold for the case with hyperbolic trapped trajectories of sufficiently small fractal dimension. We mention that there are also several works on global-in-time Strichartz estimates in the case of long-range perturbations of the flat Laplacian on $\mathbb{R}^d$ [Bouclet and Tzvetkov 2008; Tataru 2008; Marzuola et al. 2008].

While (local-in-time) Strichartz estimates are well studied for these two cases (at least under the nontrapping condition), the literature is sparser for the mixed case. In this paper we give a unified approach to a combination of these two kinds of results. More precisely, under Assumption 1.1 with $\mu > 0$, we prove

(1) sharp local-in-time Strichartz estimates, outside a large compact set centered at the origin, without the nontrapping condition, and

(2) global-in-space estimates with the nontrapping condition.

Under the nontrapping condition and Assumption 1.1 with $\mu \geq 0$, we also show local-in-time Strichartz estimates with an arbitrarily small derivative loss. We mention that all results include the endpoint estimates $(p, q) = (2, 2d/(d-2))$ for $d \geq 3$. This is a natural continuation of the author's previous work [Mizutani 2013], which was concerned with the nonendpoint estimates for the case with at most linearly growing potentials.

In the sequel, $F(*)$ denotes the characteristic function designated by $(*)$. We now state the main result.

Theorem 1.2 (Strichartz estimates near infinity). *Suppose that H satisfies Assumption 1.1 with $\mu > 0$. Then there exists $R_0 > 0$ such that for any $T > 0$, $p \geq 2$, $q < \infty$, $2/p = d(1/2 - 1/q)$, and $R \geq R_0$, we have*

$$\|F(|x| > R)e^{-itH}\varphi\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_T\|\varphi\|_{L^2(\mathbb{R}^d)}, \tag{1-2}$$

where $C_T > 0$ may be taken uniformly with respect to R.

To state the result on global-in-space estimates, we recall the *nontrapping condition*. We denote by

$$k(x,\xi)=\frac{1}{2}\sum_{j,k=1}^{d} g^{jk}(x)\xi_j\xi_k,$$

the classical kinetic energy, and by $(y_0(t,x,\xi),\eta_0(t,x,\xi))$ the Hamilton flow generated by $k(x,\xi)$:

$$\dot{y}_0(t)=\partial_\xi k(y_0(t),\eta_0(t)),\quad \dot{\eta}_0(t)=-\partial_x k(y_0(t),\eta_0(t)),\quad (y_0(0),\eta_0(0))=(x,\xi).$$

The Hamiltonian vector field $H_k=\partial_\xi k\cdot\partial_x-\partial_x k\cdot\partial_\xi$ generated by k is complete on $\mathbb{R}^{2d}$ since (g^{jk}) satisfies the uniform elliptic condition. Hence $(y_0(t,x,\xi),\eta_0(t,x,\xi))$ exists for all $t\in\mathbb{R}$.

Definition 1.3. We say that $k(x,\xi)$ satisfies the nontrapping condition if, for any $(x,\xi)\in\mathbb{R}^{2d}$ with $\xi\neq 0$,

$$|y_0(t,x,\xi)|\to+\infty \quad \text{as } t\to\pm\infty. \tag{1-3}$$

To control the asymptotic behavior of the flow, we also impose the following condition, which is the classical analogue of Mourre's inequality.

Assumption 1.4 (convexity near infinity). There exists $f\in C^\infty(\mathbb{R}^d)$ satisfying $f\geq 1$ and $f\to+\infty$ as $|x|\to+\infty$ such that $\partial^\alpha f\in L^\infty(\mathbb{R}^d)$ for any $|\alpha|\geq 2$ and

$$H_k(H_k f)(x,\xi)\geq ck(x,\xi)$$

on $\{(x,\xi)\in\mathbb{R}^{2d}: f(x)\geq R\}$ for some positive constants c, $R>0$.

Note that if $|\partial_x g^{jk}(x)|=o(|x|^{-1})$ as $|x|\to+\infty$, Assumption 1.4 holds with $f(x)=1+|x|^2$. In particular, Assumption 1.1 with $\mu>0$ implies Assumption 1.4. Moreover, if $g^{jk}(x)=(1+a_1\sin(a_2\log r))\delta_{jk}$ for $a_1\in\mathbb{R}, a_2>0$ with $a_1^2(1+a_2^2)<1$ and for $r=|x|\gg 1$, then Assumption 1.4 holds with $f(r)=(\int_0^r(1+a_1\sin(a_2\log t))^{-1}\,dt)^2$. For more examples, we refer to [Doi 2005, Section 2].

Theorem 1.5 (global-in-space Strichartz estimates). *Suppose that H satisfies Assumption 1.1 with $\mu\geq 0$. Let $T>0$, $p\geq 2$, $q<\infty$, and $2/p=d(1/2-1/q)$. Then, for any $r>0$, there exists $C_{T,r}>0$ such that*

$$\|F(|x|<r)e^{-itH}\varphi\|_{L^p([-T,T];L^q(\mathbb{R}^d))}\leq C_{T,r}\|\langle H\rangle^{1/(2p)}\varphi\|_{L^2(\mathbb{R}^d)}. \tag{1-4}$$

If we assume in addition that $k(x,\xi)$ satisfies the nontrapping condition (1-3) and Assumption 1.4,

$$\|F(|x|<r)e^{-itH}\varphi\|_{L^p([-T,T];L^q(\mathbb{R}^d))}\leq C_{T,r}\|\varphi\|_{L^2(\mathbb{R}^d)}. \tag{1-5}$$

In particular, combining with Theorem 1.2, we have the (global-in-space) Strichartz estimates

$$\|e^{-itH}\varphi\|_{L^p([-T,T];L^q(\mathbb{R}^d))}\leq C_T\|\varphi\|_{L^2(\mathbb{R}^d)},$$

under the nontrapping condition (1-3), provided that $\mu>0$.

For the general case we have the following partial result.

Theorem 1.6 (near sharp estimates without asymptotic flatness). *Suppose H satisfies Assumption 1.1 with $\mu \geq 0$ and $k(x,\xi)$ satisfies the nontrapping condition (1-3). Assume also Assumption 1.4. Let $T > 0$, $p \geq 2$, $q < \infty$, and $2/p = d(1/2 - 1/q)$. Then, for any $\varepsilon > 0$, there exists $C_{T,\varepsilon} > 0$ such that*

$$\|e^{-itH}\varphi\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_{T,\varepsilon}\|\langle H\rangle^{\varepsilon}\varphi\|_{L^2(\mathbb{R}^d)}.$$

Remark 1.7. (1) The estimates of forms (1-2), (1-4), and (1-5) have been proved [Staffilani and Tataru 2002; Bouclet and Tzvetkov 2007] when $A \equiv 0$ and V is of long-range type. Theorems 1.2 and 1.5 are therefore regarded as generalizations of their results for the case with growing electromagnetic potential perturbations.

(2) The only restriction for admissible pairs, in comparison to the flat case, is to exclude $(p,q) = (4,\infty)$ for $d = 1$, which is due to the use of the Littlewood–Paley decomposition.

(3) The missing derivative loss $\langle H\rangle^{\varepsilon}$ in Theorem 1.6 is due to the use of the following local smoothing effect, due to [Doi 2005]:

$$\|\langle x\rangle^{-1/2-\varepsilon}\langle D\rangle^{1/2}e^{-itH}\varphi\|_{L^2([-T,T];L^2(\mathbb{R}^d))} \leq C_{T,\varepsilon}\|\varphi\|_{L^2(\mathbb{R}^d)}.$$

It is well known that this estimate does not hold when $\varepsilon = 0$ even for $H = H_0$. We would expect that Theorem 1.2 still holds true for the case with critical electromagnetic potentials in the following sense:

$$\langle x\rangle^{-1}|\partial_x^{\alpha}A_j(x)| + \langle x\rangle^{-2}|\partial_x^{\alpha}V(x)| \leq C_{\alpha\beta}\langle x\rangle^{-|\alpha|},$$

(at least if g^{jk} satisfies the bounds in Assumption 1.1 with $\mu > 0$). However, this is beyond our techniques (see also Remark 4.2).

The rest of the paper is devoted to the proofs of Theorems 1.2, 1.5, and 1.6. Throughout the paper we use the following notations. $\langle x\rangle$ stands for $\sqrt{1+|x|^2}$. We write $L^q = L^q(\mathbb{R}^d)$ if there is no confusion. For Banach spaces X and Y, we denote by $\|\cdot\|_{X\to Y}$ the operator norm from X to Y. We write $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$ and denote the set of multi-indices by $\mathbb{Z}_+^d$. We denote by K the kinetic energy part of H and by H_0 the free Schrödinger operator:

$$K = -\frac{1}{2}\sum_{j,k=1}^{d}\partial_j g^{jk}(x)\partial_k, \quad H_0 = -\tfrac{1}{2}\Delta = -\frac{1}{2}\sum_{j=1}^{d}\partial_j^2.$$

We define the symbols $p(x,\xi)$ and $p_1(x,\xi)$ by

$$\begin{aligned} p(x,\xi) &= \frac{1}{2}\sum_{j,k=1}^{d} g^{jk}(x)(\xi_j - A_j(x))(\xi_k - A_k(x)) + V(x), \\ p_1(x,\xi) &= -\frac{i}{2}\sum_{j,k=1}^{d}\left(\frac{\partial g^{jk}}{\partial x_j}(x)(\xi_k - A_k(x)) - g^{jk}(x)\frac{\partial A_k}{\partial x_j}(x)\right). \end{aligned} \tag{1-6}$$

Assumption 1.1 implies

$$
\begin{aligned}
|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| &\le C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle \xi\rangle^{-|\beta|}(|\xi|^2+\langle x\rangle^{2-\mu}),\\
|\partial_x^\alpha \partial_\xi^\beta p_1(x,\xi)| &\le C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle \xi\rangle^{-|\beta|}(\langle x\rangle^{-1-\mu}|\xi|+\langle x\rangle^{-\mu}).
\end{aligned}
\tag{1-7}
$$

For $h \in (0,1]$ we consider $H^h := h^2 H$ as a semiclassical Schrödinger operator with h-dependent electromagnetic potentials $h^2 V$ and hA_j. The corresponding symbols p_h and $p_{1,h}$ are also defined by

$$
\begin{aligned}
p_h(x,\xi) &= \frac{1}{2}\sum_{j,k=1}^d g^{jk}(x)(\xi_j - hA_j(x))(\xi_k - hA_k(x)) + h^2 V(x),\\
p_{1,h}(x,\xi) &= -\frac{i}{2}\sum_{j,k=1}^d \biggl(\frac{\partial g^{jk}}{\partial x_j}(x)(\xi_k - hA_k(x)) - hg^{jk}(x)\frac{\partial A_k}{\partial x_j}(x)\biggr).
\end{aligned}
\tag{1-8}
$$

It is easy to see that $H = \mathrm{Op}(p) + \mathrm{Op}(p_1)$ and $H^h = \mathrm{Op}_h(p_h) + h\,\mathrm{Op}_h(p_{1,h})$.

Before starting the details of the proofs, we describe the main ideas. First we note that, since our Hamiltonian H is not bounded below, the Littlewood–Paley decomposition associated with H seems to be false for $p \ne 2$ in general. To overcome this difficulty, we consider the following partition of unity on the phase space $\mathbb{R}^{2d}$:

$$
\psi_\varepsilon(x,\xi) + \chi_\varepsilon(x,\xi) = 1,
$$

where ψ_ε is supported in $\{(x,\xi) : \langle x\rangle < \varepsilon|\xi|\}$ for some sufficiently small constant $\varepsilon > 0$. It is easy to see that the symbol $p(x,\xi)$ is elliptic on $\operatorname{supp}\psi_\varepsilon$:

$$
C^{-1}|\xi|^2 \le p(x,\xi) \le C|\xi|^2, \quad (x,\xi) \in \operatorname{supp}\psi_\varepsilon,
$$

and we can therefore prove a Littlewood–Paley type decomposition of the form

$$
\|\mathrm{Op}(\psi_\varepsilon)u\|_{L^q} \le C_q\|u\|_{L^2} + C_q\biggl(\sum_{\substack{h=2^{-j}\\ j\ge 0}} \|\mathrm{Op}_h(a_h) f(h^2 H)u\|_{L^q}^2\biggr)^{1/2},
$$

where $2 \le q < \infty$, the sequence $\{f(h^2\,\cdot\,) : h = 2^{-j},\ j \ge 0\}$ is a 4-adic partition of unity on $[1,\infty)$, a_h is an appropriate h-dependent symbol supported in $\{|x| < 1/h,\ |\xi| \in I\}$ for some open interval $I \Subset (0,\infty)$, and $\mathrm{Op}(\psi_\varepsilon)$, $\mathrm{Op}_h(a_h)$ denote the corresponding pseudodifferential and semiclassical pseudodifferential operators, respectively.

Then the idea of the proof of Theorem 1.2 is as follows. In view of the above Littlewood–Paley estimate, the proof is reduced to proving Strichartz estimates for $F(|x| > R)\,\mathrm{Op}_h(a_h)e^{-itH}$ and $\mathrm{Op}(\chi_\varepsilon)e^{-itH}$. In order to prove Strichartz estimates for $F(|x| > R)\,\mathrm{Op}_h(a_h)e^{-itH}$, we use semiclassical approximations of Isozaki–Kitada type. However, we note that, because of the unboundedness of potentials with respect to x, it is difficult to directly construct such approximations. To overcome this difficulty, we introduce a modified Hamiltonian $\widetilde{H}$ [Yajima and Zhang 2004] so that $\widetilde{H} = H$ for $|x| \le L/h$ and $\widetilde{H} = K$ for $|x| \ge 2L/h$ for some constant $L \ge 1$. Then $\widetilde{H}^h = h^2\widetilde{H}$ can be regarded as a "long-range perturbation" of the semiclassical free Schrödinger operator $H_0^h = h^2 H_0$. We also introduce the corresponding modified symbol $\tilde{p}_h(x,\xi)$ so that $\tilde{p}_h(x,\xi)=p_h(x,\xi)$ for $|x|\le L/h$ and $\tilde{p}_h(x,\xi)=k(x,\xi)$ for $|x|\ge 2L/h$. Let $a_h^\pm$ be supported in outgoing and incoming regions $\{R<|x|<1/h,\ |\xi|\in I,\ \pm\hat{x}\cdot\hat{\xi}>1/2\}$, respectively, so that $F(|x|>R)a_h=a_h^+ + a_h^-$, where $\hat{x}=x/|x|$. Rescaling $t\mapsto th$, we first construct the semiclassical approximations for $e^{-it\widetilde{H}^h/h}\operatorname{Op}_h(a_h^\pm)^*$ of the forms

$$e^{-it\widetilde{H}^h/h}\operatorname{Op}_h(a_h^\pm)^* = J_h(S_h^\pm, b_h^\pm)e^{-itH_0^h/h}J_h(S_h^\pm, c_h^\pm)^* + O(h^N),\quad 0\le\pm t\le 1/h,$$

respectively, where $S_h^\pm$ solves the eikonal equation associated to $\tilde{p}_h$ and $J_h(S_h^\pm, b_h^\pm)$ and $J_h(S_h^\pm, c_h^\pm)$ are the associated semiclassical Fourier integral operators (FIOs). The method of the construction is similar to that of [Robert 1994]. On the other hand, we will see that if $L\ge 1$ is large enough, the Hamilton flow generated by $\tilde{p}_h$ with initial conditions in $\operatorname{supp} a_h^\pm$ cannot escape from $\{|x|\le L/h\}$ for $0<\pm t\le 1/h$, respectively, that is,

$$\pi_x(\exp tH_{\tilde{p}_h}(\operatorname{supp} a_h^\pm))\subset\{|x|\le L/h\},\quad 0<\pm t\le 1/h.$$

Since $\tilde{p}_h=p_h$ for $|x|\le L/h$, we have

$$\exp tH_{\tilde{p}_h}(\operatorname{supp} a_h^\pm)=\exp tH_{p_h}(\operatorname{supp} a_h^\pm),\quad 0<\pm t\le 1/h.$$

We can thus expect (at least formally) that the corresponding two quantum evolutions are approximately equivalent modulo some smoothing operator. We will prove the following rigorous justification of this formal consideration:

$$\|(e^{-itH^h/h}-e^{-it\widetilde{H}^h/h})\operatorname{Op}_h(a_h^\pm)^*\|_{L^2\to L^2}\le C_M h^M,\quad 0\le\pm t\le 1/h,\ M\ge 0,$$

where $H^h=h^2H$. By using such approximations for $e^{-itH^h/h}\operatorname{Op}_h(a_h^\pm)^*$, we prove local-in-time dispersive estimates for $\operatorname{Op}_h(a_h^\pm)e^{-itH}\operatorname{Op}_h(a_h^\pm)^*$:

$$\|\operatorname{Op}_h(a_h^\pm)e^{-itH}\operatorname{Op}_h(a_h^\pm)^*\|_{L^1\to L^\infty}\le C|t|^{-d/2},\quad 0<h\ll 1,\ 0<|t|<1.$$

Strichartz estimates follow from these estimates and the abstract theorem due to Keel and Tao [1998].

Strichartz estimates for $\operatorname{Op}(\chi_\varepsilon)e^{-itH}$ follow from the short-time dispersive estimate

$$\|\operatorname{Op}(\chi_\varepsilon)e^{-itH}\operatorname{Op}(\chi_\varepsilon)^*\|_{L^1\to L^\infty}\le C_\varepsilon|t|^{-d/2},\quad 0<|t|<t_\varepsilon\ll 1.$$

To prove this, we first construct an approximation for $e^{-itH}\operatorname{Op}(\chi_\varepsilon)^*$ of the form

$$e^{-itH}\operatorname{Op}(\chi_\varepsilon)^*=J(\Psi,a)+O_{H^{-\gamma}\to H^\gamma}(1),\quad |t|<t_\varepsilon,\ \gamma>d/2,$$

where the phase function $\Psi=\Psi(t,x,\xi)$ is a solution to the time-dependent Hamilton–Jacobi equation associated to $p(x,\xi)$ and $J(\Psi,a)$ is the corresponding Fourier integral operator. In the construction, the fact that

$$|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)|\le C_{\alpha\beta},\quad (x,\xi)\in\operatorname{supp}\chi_\varepsilon,\ |\alpha+\beta|\ge 2,$$

plays an important role. We note that if $(g^{jk})_{jk}-\mathrm{Id}_d\ne 0$ depends on x, these bounds do not hold without such a restriction of the support. Using these bounds, we construct the phase function $\Psi(t,x,\xi)$ such that

$$|\partial_x^\alpha \partial_\xi^\beta (\Psi(t,x,\xi) - x\cdot\xi + p(x,\xi))| \le C_{\alpha\beta}|t|^2\langle x\rangle^{2-|\alpha+\beta|}.$$

Then we can follow a classical argument [Kitada and Kumano-go 1981] and construct the FIO $J(\Psi,a)$. By the composition formula, $\mathrm{Op}(\chi_\varepsilon)J(\Psi,a)$ is also an FIO and dispersive estimates for this operator follow from the standard stationary phase method. Finally, using an Egorov-type lemma, we prove that the remainder, $\mathrm{Op}(\chi_\varepsilon)(e^{-itH}\,\mathrm{Op}(\chi_\varepsilon)^* - J(\Psi,a))$, has a smooth kernel for sufficiently small t.

The proof of Theorem 1.5 is based on a standard idea [Staffilani and Tataru 2002]; see also [Burq et al. 2004; Bouclet and Tzvetkov 2007]. Strichartz estimates with loss of derivatives $\langle H\rangle^{1/(2p)}$ follow from semiclassical Strichartz estimates up to time scales of order h, which can be verified by the standard argument. Moreover, under the nontrapping condition, we will prove that the missing $1/p$ derivative loss can be recovered by using local smoothing effects [Doi 2005].

The proof of Theorem 1.6 is based on a slight modification of that of Theorem 1.5. By virtue of the Strichartz estimates for $\mathrm{Op}(\chi_\varepsilon)e^{-itH}$ and the Littlewood–Paley decomposition, it suffices to show

$$\|\mathrm{Op}_h(a_h)e^{-itH}\varphi\|_{L^p([-T,T];L^q)} \le C_T h^{-\varepsilon}\|\varphi\|_{L^2}, \quad 0<h\ll 1.$$

To prove this estimate, we first prove semiclassical Strichartz estimates for $e^{-itH}\,\mathrm{Op}_h(a_h)^*$ up to time scales of order hR, where $R=\inf|\pi_x(\operatorname{supp} a_h)|$. The proof is based on a refinement of the standard WKB approximation for the semiclassical propagator $e^{-itH^h/h}\,\mathrm{Op}_h(a_h)^*$. Combining semiclassical Strichartz estimates with a partition of unity argument with respect to x, we will obtain the following Strichartz estimate with an inhomogeneous error term:

$$\|\mathrm{Op}_h(a_h)e^{-itH}\varphi\|_{L^p([-T,T];L^q)} \le C_T\|\varphi\|_{L^2} + C\|\langle x\rangle^{-1/2-\varepsilon}h^{-1/2-\varepsilon}\,\mathrm{Op}_h(a_h)e^{-itH}\varphi\|_{L^2([-T,T];L^2)},$$

for any $\varepsilon>0$, which, combined with local smoothing effects, implies Theorem 1.6.

The paper is organized as follows. In Section 2 We record some known results on the semiclassical pseudodifferential calculus and prove the above Littlewood–Paley decomposition. Using dispersive estimates, which will be studied in Sections 4 and 5, we prove Theorem 1.2 in Section 3. We construct approximations of Isozaki–Kitada type and prove dispersive estimates for $\mathrm{Op}_h(a_h^\pm)e^{-itH}\,\mathrm{Op}_h(a_h^\pm)^*$ in Section 4. In Section 5 we discuss the dispersive estimates for $\mathrm{Op}(\chi_\varepsilon)e^{-itH}\,\mathrm{Op}(\chi_\varepsilon)^*$. The proofs of Theorems 1.5 and 1.6 are given in Sections 6 and 7, respectively.

2. Semiclassical functional calculus

Throughout this section we assume Assumption 1.1 with $\mu\ge 0$, that is,

$$|\partial_x^\alpha g^{jk}(x)| + \langle x\rangle^{-1}|\partial_x^\alpha A_j(x)| + \langle x\rangle^{-2}|\partial_x^\alpha V(x)| \le C_{\alpha\beta}\langle x\rangle^{-|\alpha|}. \tag{2-1}$$

The goal of this section is to prove a Littlewood–Paley type decomposition under a suitable restriction on the initial data. First we record (without proof) some known results on the pseudodifferential calculus which will be used throughout the paper. We refer to [Robert 1987; Martinez 2002] for the details of the proof.

Pseudodifferential calculus. For the metric $g = dx^2/\langle x\rangle^2 + d\xi^2/\langle\xi\rangle^2$ and a weight function $m(x,\xi)$ on the phase space $\mathbb{R}^{2d}$, we use Hörmander's symbol class notation $S(m,g)$, that is, $a \in S(m,g)$ if and only if $a \in C^\infty(\mathbb{R}^{2d})$ and

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} m(x,\xi)\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}, \quad \alpha,\beta \in \mathbb{Z}_+^d.$$

To a symbol $a \in C^\infty(\mathbb{R}^{2d})$ and $h \in (0,1]$, we associate the semiclassical pseudodifferential operator (h-PDO for short) $\mathrm{Op}_h(a)$ defined by

$$\mathrm{Op}_h(a)f(x) = \frac{1}{(2\pi h)^d}\int e^{i(x-y)\cdot\xi/h} a(x,\xi) f(y)\,dy\,d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

When $h = 1$ we write $\mathrm{Op}(a) = \mathrm{Op}_h(a)$ for simplicity. The Calderón–Vaillancourt theorem shows that for any symbol $a \in C^\infty(\mathbb{R}^{2d})$ satisfying $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}$, $\mathrm{Op}_h(a)$ is extended to a bounded operator on $L^2(\mathbb{R}^d)$ uniformly with respect to $h \in (0,1]$. Moreover, for any symbol a satisfying

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{-\gamma}, \quad \gamma > d,$$

$\mathrm{Op}_h(a)$ is extended to a bounded operator from $L^q(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ with the bounds

$$\|\mathrm{Op}_h(a)\|_{L^q\to L^r} \le C_{qr} h^{-d(1/q-1/r)}, \quad 1 \le q \le r \le \infty, \tag{2-2}$$

where $C_{qr} > 0$ is independent of $h \in (0,1]$. These bounds follow from the Schur lemma and an interpolation; see, for example, [Bouclet and Tzvetkov 2007, Proposition 2.4].

For two symbols $a \in S(m_1,g)$ and $b \in S(m_2,g)$, the composition $\mathrm{Op}_h(a)\,\mathrm{Op}_h(b)$ is also an h-PDO and is written in the form $\mathrm{Op}_h(c) = \mathrm{Op}_h(a)\,\mathrm{Op}_h(b)$ with a symbol $c \in S(m_1m_2,g)$ given by $c(x,\xi) = e^{ihD_\eta D_z} a(x,\eta) b(z,\xi)|_{z=x,\eta=\xi}$. Moreover, $c(x,\xi)$ has the expansion

$$c = \sum_{|\alpha|=0}^{N-1} \frac{h^{|\alpha|}}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha a \cdot \partial_x^\alpha b + h^N r_N \quad \text{with } r_N \in S(\langle x\rangle^{-N}\langle\xi\rangle^{-N} m_1 m_2, g). \tag{2-3}$$

The symbol of the adjoint $\mathrm{Op}_h(a)^*$ is given by $a^*(x,\xi) = e^{ihD_\eta D_z} a(z,\eta)|_{z=x,\eta=\xi} \in S(m_1,g)$ which has the expansion

$$a^* = \sum_{|\alpha|=0}^{N-1} \frac{h^{|\alpha|}}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha \partial_x^\alpha a + h^N r_N^* \quad \text{with } r_N^* \in S(\langle x\rangle^{-N}\langle\xi\rangle^{-N} m_1, g). \tag{2-4}$$

Littlewood–Paley decomposition. As we mentioned in the outline of the paper, H is not bounded below in general and hence we cannot expect that the Littlewood–Paley decomposition associated with H, which is of the form

$$\|u\|_{L^q} \le C_q\|u\|_{L^2} + C_q\bigg(\sum_{j=0}^\infty \|f(2^{-2j}H)u\|_{L^q}^2\bigg)^{1/2},$$

to hold if $q \ne 2$. The standard Littlewood–Paley decomposition associated with H_0 also does not work well in our case, since the commutator of H with the Littlewood–Paley projection $f(2^{-2j}H_0)$ can grow at spatial infinity. To overcome this difficulty, let us introduce an additional localization as follows. Given a parameter $\varepsilon > 0$ and a cut-off function $\varphi \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi \equiv 1$ on $[0, 1/2]$ and $\operatorname{supp}\varphi \subset [0, 1]$, we define $\psi_\varepsilon(x, \xi)$ by

$$\psi_\varepsilon(x, \xi) = \varphi\Big(\frac{\langle x\rangle}{\varepsilon|\xi|}\Big).$$

It is easy to see that, for each $\varepsilon > 0$, $\psi_\varepsilon \in S(1, g)$ and is supported in $\{(x, \xi) \in \mathbb{R}^{2d} : \langle x\rangle < \varepsilon|\xi|\}$. Moreover, for sufficiently small $\varepsilon > 0$, $p(x, \xi)$ is uniformly elliptic on the support of ψ_ε and thus $\mathrm{Op}(\psi_\varepsilon)H$ is essentially bounded below.

In this subsection we prove a Littlewood–Paley type decomposition on the range of $\mathrm{Op}(\psi_\varepsilon)$. We begin with the following proposition which tells us that, for any $f \in C_0^\infty(\mathbb{R})$ and $h \in (0, 1]$, $\mathrm{Op}(\psi_\varepsilon) f(h^2 H)$ is well approximated in terms of the h-PDO.

Proposition 2.1. *There exists $\varepsilon > 0$ such that, for any $f \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} f \Subset (0,\infty)$, we can construct bounded families $\{a_{h,j}\}_{h\in(0,1]} \subset \bigcap_{M\ge 0} S(\langle x\rangle^{-j}\langle\xi\rangle^{-M}, g)$, $j \ge 0$, such that*:

(1) *$a_{h,0}$ is given explicitly by $a_{h,0}(x, \xi) = \psi_\varepsilon(x, \xi/h) f(p_h(x, \xi))$. Moreover,*

$$\operatorname{supp} a_{h,j} \subset \operatorname{supp} \psi_\varepsilon(\,\cdot\,, \cdot/h) \cap \operatorname{supp} f(p_h) \subset \{(x, \xi) \in \mathbb{R}^{2d} : \langle x\rangle < 1/h,\ |\xi| \in I\},$$

for some relatively compact open interval $I \Subset (0,\infty)$. In particular, we have

$$\|\mathrm{Op}_h(a_{h,j})\|_{L^{q'}\to L^q} \le C_{jqq'} h^{-d(1/q'-1/q)}, \quad 1 \le q' \le q \le \infty,$$

uniformly in $h \in (0, 1]$.

(2) *For any integer $N > d + 2$, we set $a_h = \sum_{j=0}^{N-1} h^j a_{h,j}$. Then*

$$\|\mathrm{Op}(\psi_\varepsilon) f(h^2 H) - \mathrm{Op}_h(a_h)\|_{L^2\to L^q} \le C_{qN} h^2, \quad 2 \le q \le \infty,$$

uniformly in $h \in (0, 1]$.

The following is an immediate consequence of this proposition.

Corollary 2.2. *For any $2 \le q \le \infty$ and $h \in (0, 1]$, $\mathrm{Op}(\psi_\varepsilon) f(h^2 H)$ is bounded from $L^2(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ and satisfies*

$$\|\mathrm{Op}(\psi_\varepsilon) f(h^2 H)\|_{L^2\to L^q} \le C_q h^{-d(1/2-1/q)},$$

where $C_q > 0$ is independent of $h \in (0, 1]$.

For the low energy part we have the following.

Lemma 2.3. *For any $f_0 \in C_0^\infty(\mathbb{R})$ and $2 \le q \le \infty$, we have*

$$\|\mathrm{Op}(\psi_\varepsilon) f_0(H)\|_{L^2\to L^q} \le C_q.$$

Remark 2.4. If $V, A \equiv 0$, then Proposition 2.1, Corollary 2.2, and Lemma 2.3 hold without the additional term $\mathrm{Op}(\psi_\varepsilon)$. Moreover, in this case we see that the remainder satisfies

$$\| f(h^2 H) - \mathrm{Op}_h(a_h)\|_{L^2\to L^q} \le C_{qN} h^{N-d(1/2-1/q)}.$$

We refer to [Burq et al. 2004] (for the case on compact manifolds without boundary) and to [Bouclet and Tzvetkov 2007] (for the case with metric perturbations on $\mathbb{R}^d$). For more general cases with Laplace–Beltrami operators on noncompact manifolds with ends, we refer to [Bouclet 2010; 2011a]. Because of this result, we believe Proposition 2.1 is far from sharp. However, the bounds

$$\|\mathrm{Op}(\psi_\varepsilon) f(h^2H) - \mathrm{Op}_h(a_h)\|_{L^2\to L^q} \le C_{qN} h, \quad 2 \le q \le \infty,$$

are sufficient to obtain our Littlewood–Paley type decomposition (Proposition 2.5). For more details, we refer to Burq, Gérard, and Tvzetkov [2004, Corollary 2.3].

Proof of Proposition 2.1. We write

$$\mathrm{Op}(\psi_\varepsilon) = \mathrm{Op}_h(\psi_{\varepsilon/h}), \quad h \in (0, 1],$$

where $\psi_{\varepsilon/h}(x,\xi) = \psi_\varepsilon(x, \xi/h)$ satisfies $\operatorname{supp}\psi_{\varepsilon/h} \subset \{h\langle x\rangle < \varepsilon|\xi|\}$ and

$$|\partial_x^\alpha \partial_\xi^\beta \psi_{\varepsilon/h}(x,\xi)| \le C_{\alpha\beta\varepsilon} h^{-|\beta|}\langle x\rangle^{-|\alpha|}\langle \xi/h\rangle^{-|\beta|} \le C_{\alpha\beta\varepsilon}\langle x\rangle^{-|\alpha|}(h+|\xi|)^{-|\beta|}. \tag{2-5}$$

By using the Helffer–Sjöstrand formula [1989], we get

$$\mathrm{Op}_h(\psi_{\varepsilon/h}) f(h^2H) = -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial \tilde f}{\partial \bar z}(z)\, \mathrm{Op}_h(\psi_{\varepsilon/h})(h^2H - z)^{-1}\, dz \wedge d\bar z,$$

where $\tilde f(z)$ is an almost analytic extension of $f(\lambda)$. Since $f \in C_0^\infty(\mathbb{R})$, $\tilde f(z)$ is also compactly supported and satisfies

$$\partial_{\bar z}\tilde f(z) = O(|\mathrm{Im}\, z|^M)$$

for any $M > 0$. We may assume $|z| \le C$ on $\operatorname{supp}\tilde f$ with some $C > 0$. In order to use this formula, we shall construct a semiclassical approximation of $\mathrm{Op}_h(\psi_{\varepsilon/h})(h^2H - z)^{-1}$, in terms of the h-PDO, for $z \in \mathbb{C}\setminus[0,\infty)$ with $|z| \le C$. Although the method is based on the standard semiclassical parametrix construction (see, for example, [Robert 1987; Burq et al. 2004]), we give the details of the proof, since $\psi_{\varepsilon/h}$ is not uniformly bounded in $S(1, g)$ with respect to $h \in (0, 1]$.

We first study the symbol of the resolvent $(h^2H - z)^{-1}$. Let p_h and $p_{1,h}$ be as in (1-8) so that $h^2H = \mathrm{Op}_h(p_h) + h\,\mathrm{Op}_h(p_{1,h})$. Since

$$h|A(x)| \lesssim |\xi|, \quad h^2|V(x)| \lesssim |\xi|^2,$$

on $\operatorname{supp}\psi_{\varepsilon/h}$, we obtain by (1-7) that

$$|\partial_x^\alpha \partial_\xi^\beta p_h(x,\xi)| \le C_{\alpha\beta}\langle x\rangle^{-|\alpha|}|\xi|^{2-|\beta|} \quad \text{if } |\beta| \le 2, \tag{2-6}$$

$$|\partial_x^\alpha \partial_\xi^\beta p_{1,h}(x,\xi)| \le C_{\alpha\beta}\langle x\rangle^{-1-|\alpha|}|\xi|^{1-|\beta|} \quad \text{if } |\beta| \le 1, \tag{2-7}$$

uniformly in $(x,\xi) \in \operatorname{supp}\psi_{\varepsilon/h}$ and $h \in (0,1]$. Moreover, if $\varepsilon > 0$ is sufficiently small, the uniform ellipticity of k implies that p_h is also uniformly elliptic on $\operatorname{supp}\psi_{\varepsilon/h}$:

$$C_1^{-2}|\xi|^2 \le p_h(x,\xi) \le C_1^2|\xi|^2 \quad \text{if } h\langle x\rangle < \varepsilon|\xi|,$$

with some $C_1 > 0$, which particularly implies

$$\frac{1}{|p_h(x,\xi)-z|} \lesssim \begin{cases} |\mathrm{Im}\, z|^{-1} & \text{if } |\xi| \le 2C_2, \\ \langle \xi \rangle^{-2} & \text{if } |\xi| \ge 2C_2 \end{cases} \tag{2-8}$$

for $(x,\xi) \in \operatorname{supp} \psi_{\varepsilon/h}$, $z \notin \mathbb{R}$, and $|z| \le C$, with some $C_2 > 0$.

Let us now consider a sequence of symbols $q_j^h = q_j^h(z,x,\xi)$ (depending holomorphically on $z \notin \mathbb{R}$) defined inductively by

$$\begin{aligned} q_0^h &= \frac{\psi_{\varepsilon/h}}{p_h - z}, \\ q_1^h &= -\frac{1}{p_h - z}\bigg(\sum_{|\alpha|=1} i^{-1}\partial_\xi^\alpha q_0^h \cdot \partial_x^\alpha p_h + q_0^h \cdot p_{1,h}\bigg), \\ q_j^h &= -\frac{1}{p_h - z}\bigg(\sum_{\substack{|\alpha|+k=j \\ |\alpha|\ge 1}} \frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha q_k^h \cdot \partial_x^\alpha p_h + \sum_{|\alpha|+k=j-1} \frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha q_k^h \cdot \partial_x^\alpha p_{1,h}\bigg), \quad j \ge 2. \end{aligned}$$

We then learn by (2-5), (2-6), and (2-8) that

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta q_0^h(z,x,\xi)| &\le C_{\alpha\beta\varepsilon} \begin{cases} \langle x \rangle^{-|\alpha|}(h+|\xi|)^{-|\beta|}|\mathrm{Im}\, z|^{-1-|\alpha+\beta|} & \text{if } |\xi| \le 2C_2, \\ \langle x \rangle^{-|\alpha|}\langle \xi \rangle^{-|\beta|-2} & \text{if } |\xi| \ge 2C_2, \end{cases} \\ &\le C_{\alpha\beta\varepsilon}\langle x \rangle^{-|\alpha|}(h+|\xi|)^{-|\beta|}|\mathrm{Im}\, z|^{-1-|\alpha+\beta|} \end{aligned} \tag{2-9}$$

for $z \notin \mathbb{R}$ with $|z| \le C$ and $h \in (0,1]$. Similarly, by using (2-6), (2-7), and (2-9), we obtain that if $h|\xi| \le 2C_2$,

$$\begin{aligned} &|\partial_x^\alpha \partial_\xi^\beta q_1^h(z,x,\xi)| \\ &\quad\le C_{\alpha\beta\varepsilon}\big(\langle x \rangle^{-1-|\alpha|}(h+|\xi|)^{-1-|\beta|}|\xi|^2|\mathrm{Im}\, z|^{-3-|\alpha+\beta|} + \langle x \rangle^{-1-|\alpha|}(h+|\xi|)^{-|\beta|}(h+|\xi|)|\mathrm{Im}\, z|^{-2-|\alpha+\beta|}\big) \\ &\quad\le C_{\alpha\beta\varepsilon}(h+|\xi|)^2\langle x \rangle^{-1-|\alpha|}(h+|\xi|)^{-1-|\beta|}|\mathrm{Im}\, z|^{-3-|\alpha+\beta|}, \end{aligned}$$

for $z \notin \mathbb{R}$ with $|z| \le C$ and $h \in (0,1]$. Here note that, in this case, $(h+|\xi|)^{-1}$ may have a singularity at $\xi = 0$ as $h \to +0$. In order to prove the remainder estimate, we will remove this singularity by using a rescaling $\xi \mapsto h\xi$ (see the estimates (2-12)). For $h|\xi| \ge 2C_2$, q_1^h does not have such a singularity and satisfies

$$|\partial_x^\alpha \partial_\xi^\beta q_1^h(z,x,\xi)| \le C_{\alpha\beta\varepsilon}\langle x \rangle^{-1-|\alpha|}\langle \xi \rangle^{-|\beta|-4}|\xi| \le C_{\alpha\beta\varepsilon}\langle x \rangle^{-1-|\alpha|}\langle \xi \rangle^{-|\beta|-3}$$

uniformly in $z \notin \mathbb{R}$ with $|z| \le C$ and $h \in (0,1]$. Since $1 \lesssim h+|\xi|$ if $h|\xi| \gtrsim 1$, summarizing these, we get

$$|\partial_x^\alpha \partial_\xi^\beta q_1^h(z,x,\xi)| \le C_{\alpha\beta\varepsilon}\langle x \rangle^{-1-|\alpha|}(h+|\xi|)^{1-|\beta|}|\mathrm{Im}\, z|^{-3-|\alpha+\beta|}, \quad z \notin \mathbb{R},\ |z| \le C,\ h \in (0,1].$$

The estimates (2-9) and a direct computation also show that q_1^h is of the form

$$q_1^h = q_{11}^h(p_h - z)^{-3} + q_{10}^h(p_h - z)^{-2},$$

where q_{1k}^h are supported in $\operatorname{supp} \psi_{\varepsilon/h}$, are independent of z, and satisfy

$$|\partial_x^\alpha \partial_\xi^\beta q_{1k}^h(x,\xi)| \le C_{\alpha\beta\varepsilon}\langle x \rangle^{-1-|\alpha|}(h+|\xi|)^{-|\beta|}\langle \xi \rangle^{N_1(k)}, \quad h \in (0,1],$$

with some positive integer $N_1(k) > 0$. For $j \ge 2$, an induction argument yields that

$$|\partial_x^\alpha \partial_\xi^\beta q_j^h(z,x,\xi)| \le C_{\alpha\beta\varepsilon} \langle x\rangle^{-j-|\alpha|} (h+|\xi|)^{2-j-|\beta|} |\mathrm{Im}\, z|^{-2j-1-|\alpha+\beta|}, \quad j \ge 2, \tag{2-10}$$

for $z \notin \mathbb{R}$ with $|z| \le C$ and $h \in (0,1]$. It also follows from an induction on j that there exists a sequence of z-independent symbols $(q_{jk}^h)_{k=0}^j$ supported in $\operatorname{supp} \psi_{\varepsilon/h}$ and satisfying

$$|\partial_x^\alpha \partial_\xi^\beta q_{jk}^h(x,\xi)| \le C_{\alpha\beta\varepsilon} \langle x\rangle^{-j-|\alpha|} (h+|\xi|)^{-|\beta|} \langle \xi\rangle^{N_j(k)} \tag{2-11}$$

with some $N_j(k) > 0$, such that q_j^h is of the form

$$q_j^h = \sum_{k=0}^{j} q_{jk}^h (p_h - z)^{-j-k-1}.$$

Rescaling $\xi \mapsto h\xi$, we learn by (2-9) and (2-10) that

$$q_0^h(z,x,h\xi) \in S(1,g), \quad h^j q_j^h(z,x,h\xi) \in S(h^2 \langle x\rangle^{-j} \langle \xi\rangle^{2-j}, g),$$

with uniform bounds in h and polynomially bounds in $|\mathrm{Im}\, z|^{-1}$. Then, by the construction of q_j^h, the standard symbolic calculus (not in the semiclassical regime), and the fact that

$$\mathrm{Op}(h^j q_j^h(z,x,h\xi)) = h^j \mathrm{Op}_h(q_j^h),$$

we obtain

$$\mathrm{Op}(\psi_\varepsilon) = \sum_{j=0}^{N-1} h^j \mathrm{Op}_h(q_j^h)(h^2 H - z) + h^2 \mathrm{Op}(r_{h,N,z}), \quad N \ge 1,$$

with some $r_{h,N,z} \in S(\langle x\rangle^{-N} \langle \xi\rangle^{2-N}, g)$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta r_{h,N,z}(x,\xi)| \le C_{\alpha\beta\varepsilon N} \langle x\rangle^{-N-|\alpha|} \langle \xi\rangle^{2-N-|\beta|} |\mathrm{Im}\, z|^{-2N-1-|\alpha+\beta|}, \tag{2-12}$$

where $C_{\alpha\beta\varepsilon N} > 0$ may be taken uniformly in $h \in (0,1]$, $z \in \mathbb{C} \setminus \mathbb{R}$ with $|z| \le C$ and $x, \xi \in \mathbb{R}^d$.

We now use the Helffer–Sjöstrand formula to obtain

$$\mathrm{Op}(\psi_\varepsilon) f(h^2 H) = \sum_{j=0}^{N-1} h^j \mathrm{Op}_h(a_{h,j}) + h^2 R(h,N),$$

where

$$a_{h,0}(x,\xi) = \psi_{\varepsilon/h}(x,\xi)(f \circ p_h)(x,\xi),$$

$$a_{h,j}(x,\xi) = \sum_{k=0}^{j} \frac{(-1)^{k+j}}{(k+j)!} q_{jk}^h(x,\xi)(f^{(j+k)} \circ p_h)(x,\xi), \;\; 1 \le j \le N-1,$$

$$R(h,N) = -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z) \mathrm{Op}_h(r_{h,N,z})(h^2 H - z)^{-1} \, dz \wedge d\bar{z}.$$

Since $\operatorname{supp} q_{jk} \subset \operatorname{supp} \psi_{\varepsilon/h} \subset \{h\langle x\rangle < \varepsilon|\xi|\}$ and p_h is uniformly elliptic (that is, $p_h \approx |\xi|^2$) on the latter region, taking $\varepsilon > 0$ smaller if necessary, we have

$$a_{h,j} \subset \operatorname{supp} \psi_{\varepsilon/h} \cap \operatorname{supp} f(p_h) \subset \{(x,\xi) : |x| < 1/h, \;\; C_0^{-1} \le |\xi| \le C_0\}$$

with some positive constant $C_0 > 0$, which, combined with (2-11), implies that $\{a_{h,j}\}_{h\in(0,1]}$ is bounded in $\bigcap_{M\ge 0} S(\langle x\rangle^{-j}\langle\xi\rangle^{-M}, g)$, since $h+|\xi| \gtrsim \langle\xi\rangle$ on $\operatorname{supp}\psi_{\varepsilon/h}\cap\operatorname{supp} f(p_h)$. By virtue of (2-2), we also obtain

$$\|\mathrm{Op}_h(a_{h,j})\|_{L^{q'}\to L^q} \le C_{jqq'} h^{-d(1/q'-1/q)}, \quad h\in(0,1],\ 1\le q'\le q\le\infty.$$

Finally, we prove the estimate on the remainder $R(h,N)$. If we choose $N > d+2$, then (2-12) and (2-2) (with $h=1$) imply

$$\|\mathrm{Op}(r_{h,N,z})\|_{L^2\to L^q} \le C_{qN}|\mathrm{Im}\, z|^{-n(N,q)}, \quad 2\le q\le\infty,$$

with some positive integer $n(N,q)\ge 2N+1$, where $C_{qN}>0$ is independent of h. Using the bounds $\|(h^2H-z)^{-1}\|_{L^2\to L^2}\le|\mathrm{Im}\, z|^{-1}$, $|\partial_{\bar z}\tilde f(z)|\le C_M|\mathrm{Im}\, z|^M$ for any $M\ge 0$ and the fact that $\tilde f$ is compactly supported, we conclude that

$$\begin{aligned}\|R(h,N)\|_{L^2\to L^q} &\le C_M\int_{\operatorname{supp}\tilde f}|\mathrm{Im}\, z|^M\|\mathrm{Op}(r_{h,N,z})\|_{L^2\to L^q}\|(h^2H-z)^{-1}\|_{L^2\to L^2}\, dz\wedge d\bar z\\ &\le C_{MNq}\int_{\operatorname{supp}\tilde f}|\mathrm{Im}\, z|^{M-n(N,q)-1}\, dz\wedge d\bar z\\ &\le C_{MNq},\end{aligned}$$

provided that M is large enough. This completes the proof. □

Proof of Lemma 2.3. By the same argument as above with $h=1$, we can see that

$$\mathrm{Op}(\psi_\varepsilon)f_0(H) = \sum_{j=0}^{N-1}\mathrm{Op}(a_j) + R(N)$$

where $a_j\in\bigcap_{M\ge0}S(\langle x\rangle^{-j}\langle\xi\rangle^{-M}, g)$ are supported in

$$\operatorname{supp}\psi_\varepsilon\cap\operatorname{supp} f_0(p)\subset\{(x,\xi)\in\mathbb{R}^{2d} : \langle x\rangle<\varepsilon|\xi|,\ |\xi|\lesssim 1\}$$

and $R(N)$ satisfies

$$\|R(h,N)\|_{L^2\to L^q}\le C_{Nq}, \quad 2\le q\le\infty,$$

if $N>d+2$. The assertion then follows from (2-2). □

Consider a 4-adic partition of unity

$$f_0(\lambda)+\sum_h f(h^2\lambda)=1, \quad \lambda\in\mathbb{R},$$

where $f_0, f\in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} f_0\subset[-1,1]$, $\operatorname{supp} f\subset[1/4,4]$ and $\sum_h$ means that, in the sum, h takes all negative powers of 2 as values, that is, $\sum_h=\sum_{h=2^{-j},j\ge0}$. Let $F\in C_0^\infty(\mathbb{R})$ be such that $\operatorname{supp} F\subset[1/8,8]$ and $F\equiv 1$ on $\operatorname{supp} f$. The spectral decomposition theorem implies

$$1=f_0(H)+\sum_h f(h^2H)=f_0(H)+\sum_h F(h^2H)f(h^2H).$$

Let $a_h \in S(1, g)$ be as in Proposition 2.1 with $f = F$. Using Proposition 2.1, we obtain a Littlewood–Paley type estimates on a range of $\mathrm{Op}(\psi_\varepsilon)$.

Proposition 2.5. *For any* $2 \le q < \infty$,

$$\|\mathrm{Op}(\psi_\varepsilon)u\|_{L^q(\mathbb{R}^d)} \le C_q\|u\|_{L^2(\mathbb{R}^d)} + C_q\Bigg(\sum_h \|\mathrm{Op}_h(a_h)f(h^2H)u\|_{L^q(\mathbb{R}^d)}\Bigg)^{1/2}.$$

Proof. The proof is the same as that of [Burq et al. 2004, Corollary 2.3] and we omit the details. □

Corollary 2.6. *Let* $\varepsilon > 0$ *and* ψ_ε *be as above and* $\chi_\varepsilon = 1 - \psi_\varepsilon$. *Let* $\rho \in C^\infty(\mathbb{R}^d)$ *be such that*

$$|\partial_x^\alpha \rho(x)| \le C_\alpha \langle x\rangle^{-|\alpha|}, \quad \alpha \in \mathbb{Z}_+^d.$$

Then, for any $T > 0$ *and any* (p, q) *satisfying* $p \ge 2$, $q < \infty$ *and* $2/p = d(1/2 - 1/q)$, *there exists* $C_T > 0$ *such that*

$$\begin{aligned}\|\rho e^{-itH}\varphi\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \le C_T\|\varphi\|_{L^2(\mathbb{R}^d)} + C\|\mathrm{Op}(\chi_\varepsilon)e^{-itH}\varphi\|_{L^p([-T,T];L^q(\mathbb{R}^d))}\\ + C\Bigg(\sum_h \|\mathrm{Op}_h(a_h)e^{-itH}f(h^2H)\varphi\|^2_{L^p([-T,T];L^q(\mathbb{R}^d))}\Bigg)^{1/2},\end{aligned}$$

where a_h *is given by Proposition 2.1 with* ψ_ε *replaced by* $\rho\psi_\varepsilon$. *In particular*, $a_h(x, \xi)$ *is supported in* $\operatorname{supp} \rho(x)\psi(x, \xi/h)F(p_h(x, \xi))$.

Proof. This proposition follows from the L^2-boundedness of e^{-itH}, Propositions 2.1 and 2.5 (with ψ_ε replaced by $\rho\psi_\varepsilon$), and the Minkowski inequality. □

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2 under Assumption 1.1 with $\mu > 0$. We first state two key estimates which we will prove in later sections. For $R > 0$, an open interval $I \Subset (0, \infty)$ and $\sigma \in (-1, 1)$, we define the outgoing and incoming regions $\Gamma^\pm(R, I, \sigma)$ by

$$\Gamma^\pm(R, I, \sigma) := \left\{(x, \xi) \in \mathbb{R}^{2d} : |x| > R,\ |\xi| \in I,\ \pm\frac{x\cdot\xi}{|x||\xi|} > -\sigma\right\},$$

respectively. We then have the following (local-in-time) dispersive estimates.

Proposition 3.1. *Suppose that* H *satisfies Assumption 1.1 with* $\mu > 0$. *Let* $I \Subset (0, \infty)$ *and* $\sigma \in (-1, 1)$. *Then, for sufficiently large* $R \ge 1$, *small* $h_0 > 0$, *and any symbols* $a_h^\pm \in S(1, g)$ *supported in* $\Gamma^\pm(R, I, \sigma) \cap \{x : |x| < 1/h\}$, *we have*

$$\|\mathrm{Op}_h(a_h^\pm)e^{-itH}\,\mathrm{Op}_h(a_h^\pm)^*\|_{L^1\to L^\infty} \le C|t|^{-d/2}, \quad 0 < |t| \le 1,$$

uniformly with respect to $h \in (0, h_0]$.

We prove this proposition in Section 4. In the region $\{|x| \gtrsim |\xi|\}$, we have the following (short-time) dispersive estimates.

Proposition 3.2. *Suppose that H satisfies Assumption 1.1 with $\mu \ge 0$. Let us fix arbitrarily $\varepsilon > 0$. Then there exists $t_\varepsilon > 0$ such that, for any symbol $\chi_\varepsilon \in S(1, g)$ supported in $\{(x, \xi) : \langle x\rangle \ge \varepsilon|\xi|\}$, we have*

$$\|\mathrm{Op}(\chi_\varepsilon)e^{-itH}\,\mathrm{Op}(\chi_\varepsilon)^*\|_{L^1\to L^\infty} \le C_\varepsilon|t|^{-d/2}, \quad 0 < |t| \le t_\varepsilon.$$

We prove this proposition in Section 5.

Proof of Theorem 1.2. Taking $\rho \in C^\infty(\mathbb{R}^d)$ so that $0 \le \rho(x) \le 1$, $\rho(x) = 1$ for $|x| \ge 1$ and $\rho(x) = 0$ for $|x| \le 1/2$, we set $\rho_R(x) = \rho(x/R)$. In order to prove Theorem 1.2, it suffices to show

$$\|\rho_R e^{-itH}\varphi\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \le C_T\|\varphi\|_{L^2(\mathbb{R}^d)},$$

for sufficiently large $R \ge 1$. We may also assume without loss of generality that $T > 0$ is sufficiently small. Indeed, if the above estimate holds on $[-T_0, T_0]$ with some $T_0 > 0$, we obtain by the unitarity of e^{-itH} on L^2 that, for any $T > T_0$,

$$\begin{aligned}\|\rho_R e^{-itH}\varphi\|^p_{L^p([-T,T];L^q(\mathbb{R}^d))} &\lesssim \sum_{k=-[T/T_0]}^{[T/T_0]+1} \|\rho_R e^{-itH}e^{-i(k+1)H}\varphi\|^p_{L^p([-T_0,T_0];L^q(\mathbb{R}^d))}\\ &\lesssim (T/T_0)C^p_{T_0}\|\varphi\|^p_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Let a_h be as in Proposition 2.1. Replacing ψ_ε with $\rho_R\psi_\varepsilon$ and taking $\varepsilon > 0$ smaller if necessary, we may assume without loss of generality that $\operatorname{supp} a_h \subset \{(x, \xi) : R < |x| < 1/h, |\xi| \in I\}$ for some open interval $I \Subset (0, \infty)$. Choosing $\theta^\pm \in C^\infty([-1, 1])$ so that $\theta^+ + \theta^- = 1$, $\theta^+ = 1$ on $[1/2, 1]$ and $\theta^+ = 0$ on $[-1, -1/2]$, we set $a^\pm_h(x, \xi) = a_h(x, \xi)\theta^\pm(\hat{x}\cdot\hat{\xi})$, where $\hat{x} = x/|x|$. It is clear that $\{a^\pm_h\}_{h\in(0,1]}$ is bounded in $S(1, g)$ and $\operatorname{supp} a^\pm_h \subset \Gamma^\pm(R, I, 1/2)\cap\{x : |x| < 1/h\}$, and that $a_h = a^+_h + a^-_h$. We now apply Proposition 3.1 to $a^\pm_h$ and obtain the local-in-time dispersive estimate for $\mathrm{Op}_h(a^\pm_h)e^{-itH}\,\mathrm{Op}_h(a^\pm_h)^*$ (uniformly in $h \in (0, h_0]$), which, combined with the L^2-boundedness of $\mathrm{Op}_h(a^\pm_h)e^{-itH}$ and the abstract theorem [Keel and Tao 1998], implies the following Strichartz estimates for $\mathrm{Op}_h(a_h)e^{-itH}$:

$$\begin{aligned}\|\mathrm{Op}_h(a_h)e^{-itH}\varphi\|_{L^p([-1,1];L^q(\mathbb{R}^d))} &\le \sum_\pm \|\mathrm{Op}_h(a^\pm_h)e^{-itH}\varphi\|_{L^p([-1,1];L^q(\mathbb{R}^d))}\\ &\le C\|\varphi\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

uniformly with respect to $h \in (0, h_0]$. Since $\mathrm{Op}_h(a_h)$ is bounded from $L^2(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ with the bound of order $O(h^{-d(1/2-1/q)})$, for $h_0 < h \le 1$, we have

$$\sum_{h_0<h\le 1} \|\mathrm{Op}_h(a_h)e^{-itH}f(h^2H)\varphi\|^2_{L^p([-1,1];L^q(\mathbb{R}^d))} \le C(h_0)\|\varphi\|^2_{L^2(\mathbb{R}^d)}$$

with some $C(h_0) > 0$. Using these two bounds, we obtain

$$\begin{aligned}\sum_h \|\mathrm{Op}_h(a_h)e^{-itH}f(h^2H)\varphi\|^2_{L^p([-1,1];L^q(\mathbb{R}^d))} &\le C\sum_{0<h<h_0}\|f(h^2H)\varphi\|^2_{L^2(\mathbb{R}^d))} + C(h_0)\|\varphi\|^2_{L^2(\mathbb{R}^d)}\\ &\le C\|\varphi\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

On the other hand, Strichartz estimates for $\mathrm{Op}(\chi_\varepsilon)e^{-itH}$ are an immediate consequence of Proposition 3.2. Together with Corollary 2.6, this completes the proof. □

4. Semiclassical approximations for outgoing propagators

Throughout this section we assume Assumption 1.1 with $\mu > 0$. Here we study the behavior of

$$e^{-itH}\,\mathrm{Op}_h(a_h^\pm)^*,$$

where $a_h^\pm \in S(1, g)$ are supported in $\Gamma^\pm(R, I, \sigma) \cap \{|x| < 1/h\}$, respectively. The main goal of this section is to prove Proposition 3.1. For simplicity, we consider the outgoing propagator $e^{-itH}\,\mathrm{Op}_h(a_h^+)^*$ for $0 \le t \le 1$ only, and the proof for the incoming case is analogous.

In order to prove dispersive estimates, we construct a semiclassical approximation for the outgoing propagator $e^{-itH}\,\mathrm{Op}_h(a_h^+)^*$ by using the method of Isozaki–Kitada. Namely, rescaling $t \mapsto th$ and setting $H^h = h^2 H$, $H_0^h = -h^2\Delta/2$, we consider an approximation for the semiclassical propagator $e^{-itH^h/h}\,\mathrm{Op}_h(a_h^+)^*$ of the form

$$e^{-itH^h/h}\,\mathrm{Op}_h(a_h^+)^* = J_h(S_h^+, b_h^+)e^{-itH_0^h/h}J_h(S_h^+, c_h^+)^* + O(h^N), \quad 0 \le t \le h^{-1},$$

where S_h^+ solves a suitable eikonal equation in the outgoing region and $J(S_h^+, w)$ is the corresponding semiclassical Fourier integral operator (h-FIO for short):

$$J_h(S_h^+, w)f(x) = (2\pi h)^{-d}\int e^{i(S_h^+(x,\xi)-y\cdot\xi)/h}w(x,\xi)f(y)\,dy\,d\xi.$$

Such approximations (uniformly in time) have been studied for Schrödinger operators with long-range potentials [Robert and Tamura 1987] and for the case of long-range metric perturbations [Robert 1987; 1994; Bouclet and Tzvetkov 2007]. We also refer to the original paper by Isozaki and Kitada [1985], in which the existence and asymptotic completeness of modified wave operators (with time-independent modifiers) were established for the case of Schrödinger operators with long-range potentials. We note that, in these cases, we do not need the additional restriction of the initial data in $\{|x| < 1/h\}$. On the other hand, in [Mizutani 2013], we constructed such approximations (locally in time) for the case with long-range metric perturbations, combined with potentials growing subquadratically at infinity, under the additional restriction on the initial data into $\{|x| < 1/h\}$.

As we mentioned in the outline of the paper, we first construct an approximation for the modified propagator $e^{-it\widetilde{H}^h/h}$, where $\widetilde{H}^h$ is defined as follows. Taking arbitrarily a cut-off function $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \psi \le 1$, $\psi \equiv 1$ for $|x| \le 1/2$ and $\psi \equiv 0$ for $|x| \ge 1$, we define truncated electric and magnetic potentials, V_h and $A_h = (A_{h,j})_j$ by $V_h(x) := \psi(hx/L)V(x)$ and $A_{h,j}(x) = \psi(hx/L)A_j(x)$, respectively. It is easy to see that

$$V_h \equiv V,\ \ A_{h,j} \equiv A_j \text{ on } \{|x| \le L/(2h)\}, \quad \operatorname{supp} A_{h,j},\ \operatorname{supp} V_h \subset \{|x| \le L/h\},$$

and that, for any $\alpha \in \mathbb{Z}_+^d$, there exists $C_{L,\alpha} > 0$, independent of x, h, such that

$$h^2|\partial_x^\alpha V_h(x)| + h|\partial_x^\alpha A_h(x)| \le C_{\alpha,L}\langle x\rangle^{-\mu-|\alpha|}. \tag{4-1}$$

Let us define $\widetilde{H}^h$ by

$$\widetilde{H}^h = \frac{1}{2}\sum_{j,k=1}^{d}(-ih\partial_j - hA_{h,j}(x))g^{jk}(x)(-ih\partial_k - hA_{h,k}(x)) + h^2V_h(x).$$

We consider $\widetilde{H}^h$ as a "semiclassical" Schrödinger operator with h-dependent electromagnetic potentials h^2V_h and hA_h. By virtue of the estimates on g^{jk}, A_h, and V_h, $\widetilde{H}^h$ can be regarded as a long-range perturbation of the semiclassical free Schrödinger operator $H_0^h = -h^2\Delta/2$. Such a type modification has been used to prove Strichartz estimates and local smoothing effects (with loss of derivatives) for Schrödinger equations with superquadratic potentials; see [Yajima and Zhang 2004, Section 4]. Let us denote by $\tilde{p}_h$ the corresponding modified symbol

$$\tilde{p}_h(x,\xi) = \frac{1}{2}\sum_{j,k=1}^{d} g^{jk}(x)(\xi_j - hA_{h,j}(x))(\xi_k - hA_{h,k}(x)) + h^2V_h(x). \tag{4-2}$$

The following proposition provides the existence of the phase function of h-FIOs.

Proposition 4.1 [Robert 1994]. *Fix an open interval $I \Subset (0,\infty)$, $-1<\sigma<1$ and $L>0$. Then there exist $R_0, h_0 > 0$ and a family of smooth and real-valued functions*

$$\{S_h^+ : 0 < h \le h_0,\ R \ge R_0\} \subset C^\infty(\mathbb{R}^{2d} : \mathbb{R})$$

satisfying the eikonal equation associated to $\tilde{p}_h$:

$$\tilde{p}_h(x, \partial_x S_h^+(x,\xi)) = |\xi|^2/2, \quad (x,\xi) \in \Gamma^+(R, I, \sigma), \tag{4-3}$$

such that

$$|S_h^+(x,\xi) - x\cdot\xi| \le C\langle x\rangle^{1-\mu}, \quad x,\xi \in \mathbb{R}^d. \tag{4-4}$$

Moreover, for any $|\alpha+\beta| \ge 1$,

$$|\partial_x^\alpha \partial_\xi^\beta (S_h^+(x,\xi) - x\cdot\xi)| \le C_{\alpha\beta}\min\{R^{1-\mu-|\alpha|}, \langle x\rangle^{1-\mu-|\alpha|}\}, \quad x,\xi \in \mathbb{R}^d. \tag{4-5}$$

Here $C, C_{\alpha\beta} > 0$ are independent of x, ξ, R, and h.

Proof. Since h^2V_h and hA_h are of long-range type uniformly with respect to $h \in (0,1]$ (the constant $C_{L,\alpha}$ in (4-1) can be taken independently of h), the proof is the same as that of [Robert 1994, Proposition 4.1], and we omit it. For the R dependence, we refer to [Bouclet and Tzvetkov 2007, Proposition 3.1]. □

Remark 4.2. The crucial point to obtain the estimates (4-4) and (4-5) is the uniform bound (4-1), and we do not have to use the support properties of A_h and V_h. Suppose that A and V satisfy $\langle x\rangle^{-1}|\partial_x^\alpha A(x)| + \langle x\rangle^{-2}|\partial_x^\alpha V(x)| \le C_{\alpha\beta}\langle x\rangle^{-|\alpha|}$, and g^{jk} satisfies Assumption 1.1 with $\mu \ge 0$. Then there exists $L > 0$, independent of h, such that if $0 < L \le L_0$, we can still construct the solution S_h^+ to (4-3) by using the support properties of A_h and V_h. However, in this case, $S_h^+ - x\cdot\xi$ behaves like $\langle x\rangle^{1-\mu}h^{-1}$ as $h \to 0$, and we cannot obtain the uniform L^2-boundedness of the corresponding h-FIO. This is one of the reasons why we exclude the critical case $\mu = 0$.

To the phase S_h^+ and an amplitude $a \in S(1, g)$, we associate the h-FIO defined by

$$J_h(S_h^+, a) f(x) = (2\pi h)^{-d} \int e^{i(S_h^+(x,\xi) - y\cdot\xi)/h} a(x, \xi) f(y)\, dy\, d\xi.$$

Using (4-5), for sufficiently large $R > 0$, we have

$$|\partial_\xi \otimes \partial_x S_h^+(x, \xi) - \mathrm{Id}\,| \le C\langle R\rangle^{-\mu} < \tfrac{1}{2}, \quad |\partial_x^\alpha \partial_\xi^\beta S_h^+(x, \xi)| \le C_{\alpha\beta} \text{ for } |\alpha+\beta| \ge 2,$$

uniformly in $h \in (0, h_0]$. Therefore, the standard L^2-boundedness of FIOs implies that $J_h(S_h^+, a)$ is uniformly bounded on $L^2(\mathbb{R}^d)$ with respect to $h \in (0, h_0]$.

We now construct the outgoing approximation for $e^{-it\widetilde{H}^h/h}$.

Theorem 4.3. *Let us fix arbitrarily open intervals $I \Subset I_0 \Subset I_1 \Subset I_2 \Subset (0, \infty)$, $-1 < \sigma < \sigma_0 < \sigma_1 < \sigma_2 < 1$ and $L > 0$. Let R_0 and h_0 be as in Proposition 4.1 with I, σ replaced by I_2, σ_2, respectively. Then, for every integer $N \ge 0$, the following hold uniformly with respect to $R \ge R_0$ and $h \in (0, h_0]$.*

(1) *There exists a symbol*

$$b_h^+ = \sum_{j=0}^{N-1} h^j b_{h,j}^+ \quad \textit{with } b_{h,j}^+ \in S(\langle x\rangle^{-j}\langle\xi\rangle^{-j}, g),\ \operatorname{supp} b_{h,j}^+ \subset \Gamma^+(R^{1/3}, I_1, \sigma_1),$$

such that, for any $a^+ \in S(1, g)$ with $\operatorname{supp} a^+ \subset \Gamma^+(R, I, \sigma)$, we can find

$$c_h^+ = \sum_{j=0}^{N-1} h^j c_{h,j}^+ \quad \textit{with } c_{h,j}^+ \in S(\langle x\rangle^{-j}\langle\xi\rangle^{-j}, g),\ \operatorname{supp} c_{h,j}^+ \subset \Gamma^+(R^{1/2}, I_0, \sigma_0),$$

such that, for all $0 \le t \le h^{-1}$, $e^{-it\widetilde{H}^h/h} \operatorname{Op}_h(a^+)^*$ can be brought to the form

$$e^{-it\widetilde{H}^h/h} \operatorname{Op}_h(a^+)^* = J_h(S_h^+, b_h^+) e^{-itH_0^h/h} J_h(S_h^+, c_h^+)^* + Q_{\mathrm{IK}}^+(t, h, N),$$

where $J_h(S_h^+, w)$, $w = b_h^+, c_h^+$, are h-FIOs associated to the phase S_h^+ defined in Proposition 4.1 with R, I, and σ replaced by $R^{1/4}$, I_2, and σ_2, respectively. Moreover, for any integer $s \ge 0$ with $2s \le N-1$, the remainder $Q_{\mathrm{IK}}^+(t, h, N)$ satisfies

$$\|\langle D\rangle^s Q_{\mathrm{IK}}^+(t, h, N)\langle D\rangle^s\|_{L^2\to L^2} \le C_{Ns} h^{N-2s-1}, \tag{4-6}$$

uniformly with respect to $h \in (0, h_0]$ and $0 \le t \le h^{-1}$.

(2) *Let $K_{S_h^+}(t, x, y)$ be the distribution kernel of $J_h(S_h^+, b_h^+) e^{-itH_0^h/h} J_h(S_h^+, c_h^+)^*$. Then $K_{S_h^+}$ satisfies the dispersive estimate*

$$|K_{S_h^+}(t, x, y)| \le C|th|^{-d/2}, \tag{4-7}$$

uniformly with respect to $h \in (0, h_0]$, $x, y \in \mathbb{R}^d$ and $0 \le t \le h^{-1}$.

Proof. This theorem is basically known; hence we omit the proof. For the construction of the amplitudes b_h^+ and c_h^+, we refer to [Robert 1994, Section 4]; see also [Bouclet and Tzvetkov 2007, Section 3]. The remainder estimate (4-6) can be proved by the same argument as that in [Bouclet and Tzvetkov 2007, Proposition 3.3, Lemma 3.4] combined with the simple estimate

$$\|\langle D\rangle^s(\widetilde{H}^h + C_1)^{-s/2}\|_{L^2\to L^2} \le C_s h^{-s}, \quad s \ge 0.$$

where $C_1 > 0$ is a large constant. Note that this estimate follow from the obvious bounds

$$\|\langle D\rangle^s\langle hD\rangle^{-s}\|_{L^2\to L^2} \le C_s h^{-s}, \quad s \ge 0,$$

and the fact that $(\tilde{p}_h + h\tilde{p}_{1,h} + C_1)^{-s/2} \in S(\langle\xi\rangle^{-s}, g)$ since $\tilde{p}_h + h\tilde{p}_{1,h} + C_1$ is uniformly elliptic for sufficiently large $C_1 > 0$. The dispersive estimate (4-7) can be verified by the same argument as that in [Bouclet and Tzvetkov 2007, Lemma 4.4]. □

The following lemma, which has been essentially proved in [Mizutani 2013], tells us that we can still construct the semiclassical approximation for the original propagator $e^{-itH^h/h}$ if we restrict the support of initial data in the region $\Gamma^+(R, J, \sigma) \cap \{x : |x| < h^{-1}\}$.

Lemma 4.4. *Suppose that $\{a_h^+\}_{h\in(0,1]}$ is a bounded set in $S(1, g)$ with symbols supported in*

$$\Gamma^+(R, I, \sigma) \cap \{x : |x| < h^{-1}\}.$$

There exists $L > 1$ such that, for any $M, s \ge 0$, $h \in (0, h_0]$ and $0 \le t \le h^{-1}$, we have

$$\|(e^{-itH^h/h} - e^{-it\widetilde{H}^h/h}) \operatorname{Op}_h(a_h^+)^*\langle D\rangle^s\|_{L^2\to L^2} \le C_{M,s}h^{M-s},$$

where $C_{M,s} > 0$ is independent of h and t.

In order to prove this lemma, we need the following.

Lemma 4.5. *Let $f_h \in C^\infty(\mathbb{R}^d)$ be such that for any $\alpha \in \mathbb{Z}_+^d$,*

$$|\partial_x^\alpha f_h(x)| \le C_\alpha$$

uniformly with respect to $h \in (0, h_0]$ and such that $\operatorname{supp} f_h \subset \{|x| \ge L/(2h)\}$. *Let $L > 1$ be large enough. Then, under the conditions in Lemma 4.4, we have*

$$\|f_h(x)\langle D\rangle^\gamma e^{-it\widetilde{H}^h/h} \operatorname{Op}_h(a_h^+)^*\langle D\rangle^s\|_{L^2\to L^2} \le C_{M,s,\gamma}h^{M-s-\gamma},$$

for any $s, \gamma \ge 0$ and $M \ge 0$, uniformly with respect $h \in (0, h_0]$ and $0 \le t \le 1/h$.

Proof. We apply Theorem 4.3 to $e^{-it\widetilde{H}^h/h} \operatorname{Op}_h(a_h^+)^*$ and obtain

$$e^{-it\widetilde{H}^h/h} \operatorname{Op}_h(a_h^+)^* = J_h(S_h^+, b_h^+)e^{-itH_0^h/h}J_h(S_h^+, c_h^+)^* + Q_{\mathrm{IK}}^+(t, h, N).$$

By virtue of (4-6), the remainder $f_h(x)\langle D\rangle^\gamma Q_{\mathrm{IK}}^+(t, h, N)\langle D\rangle^s$ is bounded on $L^2(\mathbb{R}^d)$ with the norm dominated by $C_{Ns\gamma}h^{N-\gamma-s-1}$, uniformly with respect $h \in (0, h_0]$ and $t \in [0, 1/h]$. On the other hand, by virtue of (4-5), the phase of $K_{S_h^+}(t, x, y)$, which is given by

$$\Phi_h^+(t, x, y, \xi) = S_h^+(x, \xi) - \tfrac{1}{2}t|\xi|^2 - S_h^+(y, \xi),$$

satisfies $\partial_\xi\Phi_h^+(t, x, y, \xi) = (x - y)(\mathrm{Id} + O(R^{-\mu/4})) - t\xi$. Here we recall that

$$\operatorname{supp} c_h^+ \subset \{(y,\xi)\in\mathbb{R}^{2d} : a_h^+(y,\partial_\xi S_h^+(y,\xi))\neq 0\};$$

see [Mizutani 2013, Lemma 3.2] and its proof. In particular, $c_h^+(y,\xi)$ vanishes in the region $\{y : |y|\ge 1/h\}$. We now set $L=4\sqrt{\sup I_2}+2$, where I_2 is given in Theorem 4.3. Since $|x|\ge L/(2h)$, $|y|<1/h$, and $|\xi|^2\in I_2$ on the support of the amplitude $f_h(x)b_h^+(x,\xi)\overline{c_h^+(y,\xi)}$, we obtain

$$|\partial_\xi \Phi_h^+(t,x,y,\xi)| > c(1+|x|+|y|+|\xi|+t+h^{-1}),\quad 0\le t\le h^{-1},$$

for some universal constant $c>0$. The assertion now follows from an integration by parts and the L^2-boundedness of h-FIOs. ☐

Proof of Lemma 4.4. The Duhamel formula yields

$$\begin{aligned}
&(e^{-itH^h/h}-e^{-it\widetilde{H}^h/h})\\
&\quad=-\frac{i}{h}\int_0^t e^{-i(t-s)H^h/h}W_0^h e^{-is\widetilde{H}^h/h}\,ds\\
&\quad=-\frac{i}{h}\int_0^t e^{-i(t-s)H^h/h}e^{-is\widetilde{H}^h/h}W_0^h\,ds+\frac{1}{h^2}\int_0^t e^{-i(t-s)H^h/h}\int_0^s e^{-i(s-\tau)\widetilde{H}^h/h}[\widetilde{H}^h,W_0^h]e^{-i\tau\widetilde{H}^h/h}\,d\tau\,ds,
\end{aligned}$$

where $W_0^h := H^h-\widetilde{H}^h$ consists of two parts,

$$\frac{ih^2}{2}\sum_{j,k}(\partial_j g^{jk}(1-\psi(hx/L))A_k+(1-\psi(hx/L))A_j g^{jk}\partial_k)$$

and

$$\frac{h^2}{2}\sum_{j,k}(1-\psi(hx/L))^2 g^{jk}A_jA_k+h^2(1-\psi(hx/L))V.$$

In particular, W_0^h is a first order differential operator of the form

$$h^2\sum_{|\alpha|=1} f_\alpha^h(x)\partial_x^\alpha + h^2 f_0^h(x),$$

where f_α^h, f_0^h are supported in $\{|x|\ge L/(2h)\}$ and satisfy

$$|\partial_x^\beta f_\alpha^h(x)|\le C_{\alpha\beta}\langle x\rangle^{1-\mu-|\beta|},\quad |\partial_x^\beta f_0^h(x)|\le C_{\alpha\beta}\langle x\rangle^{2-\mu-|\beta|}. \tag{4-8}$$

Since $\{|x|\ge L/(2h)\}\cap\pi_x(\operatorname{supp} a_h^+)=\varnothing$ if $L>1$, we have

$$\|W_0^h \operatorname{Op}_h(a_h^+)^*\langle D\rangle^s\|_{L^2\to L^2}\le C_{M,s}h^{M-s},\quad M\ge 0,\ s\in\mathbb{R}.$$

Therefore, the first term of the right-hand side of the above Duhamel formula satisfies the desired estimates since $e^{-itH^h/h}$ and $e^{-it\widehat{H}^h/h}$ are unitary on L^2.

We next study the second term. Again by the Duhamel formula, we have

$$[\widetilde{H}^h,W_0^h]e^{-i\tau\widetilde{H}^h/h}=e^{-i\tau\widetilde{H}^h/h}[\widetilde{H}^h,W_0^h]+\frac{i}{h}\int_0^\tau e^{-i(\tau-u)\widetilde{H}^h/h}[\widetilde{H}^h,[\widetilde{H}^h,W_0^h]]e^{-iu\widetilde{H}^h/h}\,du.$$

Since the coefficients of the commutator $[\widetilde{H}^h, W_0^h]$ are supported in $\{|x| \geq L/(2h)\}$, the support property of a_h^+ again implies that $[\widetilde{H}^h, W_0^h]\operatorname{Op}_h(a_h^+)^*\langle D\rangle^s = O_{L^2\to L^2}(h^{M-s})$ for any $M \geq 0$ and $s \in \mathbb{R}$. Furthermore, by virtue of (4-1), (4-8), and the symbolic calculus, the coefficients of $[\widetilde{H}^h, [\widetilde{H}^h, W_0^h]]$ are uniformly bounded in x and supported in $\{|x| \geq L/(2h)\}$. We now apply Lemma 4.5 to

$$[\widetilde{H}^h, [\widetilde{H}^h, W_0^h]]e^{-iu\widetilde{H}^h/h}\operatorname{Op}_h(a_h^+)^*$$

and obtain the assertion. □

Proof of Proposition 3.1. Rescaling $t \to th$, it suffices to show

$$\|\operatorname{Op}_h(a_h^+)e^{-itH^h/h}\operatorname{Op}_h(a_h^+)^*\|_{L^1\to L^\infty} \leq C_\varepsilon |th|^{-d/2}, \quad 0 < |t| \leq h^{-1},$$

where $H^h = h^2 H$. Let $A_h(x, y)$ be the distribution kernel of $\operatorname{Op}_h(a_h^+)$:

$$A_h(x, y) = (2\pi h)^{-d}\int e^{i(x-y)\cdot\xi/h}a_h^+(x, \xi)\,d\xi.$$

Since $a_h^+ \in S(1, g)$ is compactly supported in I with respect to ξ, we easily see that

$$\sup_x \int |A_h(x, y)|\,dy + \sup_y \int |A_h(x, y)|\,dx \leq C, \quad h \in (0, 1].$$

Moreover, since $\langle\xi\rangle^s a_h^+\langle\xi\rangle^\gamma \in S(1, g)$ for any s, γ, we have

$$\|\langle D\rangle^s \operatorname{Op}_h(a_h^+)\langle D\rangle^\gamma\|_{L^2\to L^2} \leq C_s h^{-s-\gamma}. \tag{4-9}$$

Combining these two estimates with Theorem 4.3 and Lemma 4.4, we can write

$$\operatorname{Op}_h(a_h^+)e^{-itH^h/h}\operatorname{Op}_h(a_h^+)^* = K_1(t, h, N) + K_2(t, h, N),$$

where

$$K_1(t, h, N) = \operatorname{Op}_h(a_h^+)J_h(S_h^+, b_h^+)e^{-itH_0^h/h}J_h(S_h^+, c_h^+)^*,$$
$$K_2(t, h, N) = \operatorname{Op}_h(a_h^+)Q_{\mathrm{IK}}^+(t, h, N) + \operatorname{Op}_h(a_h^+)(e^{-itH^h/h} - e^{-it\widetilde{H}^h/h})\operatorname{Op}_h(a_h^+)^*.$$

By (4-7), the distribution kernel of $K_1(t, h, N)$, which we denote by $K_1(t, x, y)$, satisfies

$$|K_1(t, x, y)| \leq \int |A_h(x, z)||K_{S_h^+}(t, z, y)|\,dz \leq C_N |th|^{-d/2}, \quad 0 < t \leq h^{-1},$$

uniformly in $h \in (0, h_0]$. On the other hand, (4-6), Lemma 4.4, and (4-9) imply

$$\|\langle D\rangle^s K_2(t, h, N)\langle D\rangle^s\|_{L^2\to L^2} \leq C_{N,s}h^{N-2s-1}.$$

If we choose $N \geq d+2$ and $s > d/2$, it follows from the Sobolev embedding that the distribution kernel of $K_2(t, h, N)$ is uniformly bounded in $\mathbb{R}^{2d}$ with respect to $h \in (0, h_0]$ and $0 < t \leq h^{-1}$. Therefore, $\operatorname{Op}_h(a_h^+)e^{-itH^h/h}\operatorname{Op}_h(a_h^+)^*$ has the distribution kernel $K(t, x, y)$ satisfying dispersive estimates for $0 < t \leq h^{-1}$:

$$|K(t, x, y)| \leq C_N |th|^{-d/2}, \quad x, y \in \mathbb{R}^d. \tag{4-10}$$

Finally, using the relation

$$\operatorname{Op}_h(a_h^+)e^{-itH^h/h}\operatorname{Op}_h(a_h^+)^* = (\operatorname{Op}_h(a_h^+)e^{itH^h/h}\operatorname{Op}_h(a_h^+)^*)^*,$$

we learn that $K(t,x,y) = \overline{K(-t,y,x)}$ and (4-10) also holds for $0 < -t \le h^{-1}$. For the incoming case, the proof is analogous and we omit it. □

5. Fourier integral operators with the time dependent phase

Throughout this section we assume Assumption 1.1 with $\mu \ge 0$. Consider a symbol $\chi_\varepsilon \in S(1,g)$ supported in a region

$$\Omega(\varepsilon) := \{(x,\xi) \in \mathbb{R}^{2d} : \langle x\rangle > \varepsilon|\xi|/2\},$$

where $\varepsilon > 0$ is an arbitrarily small fixed constant. In this section we prove the dispersive estimate

$$\|\operatorname{Op}(\chi_\varepsilon)e^{-itH}\operatorname{Op}(\chi_\varepsilon)^*\|_{L^1\to L^\infty} \le C_\varepsilon|t|^{-d/2}, \quad 0 < |t| \le t_\varepsilon,$$

where $t_\varepsilon > 0$ is a small constant depending on ε. This estimate, combined with the L^2-boundedness of $\operatorname{Op}(\chi_\varepsilon)$ and e^{-itH}, implies the Strichartz estimates for $\operatorname{Op}(\chi_\varepsilon)e^{-itH}$.

Let us give a short summary of the steps of the proof. Choose $\chi_\varepsilon^* \in S(1,g)$ so that $\operatorname{supp}\chi_\varepsilon^* = \operatorname{supp}\chi_\varepsilon$ and $\operatorname{Op}(\chi_\varepsilon)^* = \operatorname{Op}(\chi_\varepsilon^*) + \operatorname{Op}(r_N)$ with some $r_N \in S(\langle x\rangle^{-N}\langle\xi\rangle^{-N}, g)$ for sufficiently large $N > d/2$. We first construct an approximation for $e^{-itH}\operatorname{Op}(\chi_\varepsilon^*)$ in terms of the FIO with a time dependent phase

$$J(\Psi, b^N)f(x) = \frac{1}{(2\pi)^d}\int e^{i(\Psi(t,x,\xi)-y\cdot\xi)}b(t,x,\xi)f(y)\,dy\,d\xi,$$

where Ψ is a generating function of the Hamilton flow associated to $p(x,\xi)$ and $(\partial_\xi\Psi,\xi) \mapsto (x,\partial_x\Psi)$ is the corresponding canonical map, and the amplitude

$$b = b_0 + b_2 + \cdots + b_{N-1}$$

solves the corresponding transport equations. Although such parametrix constructions are well known as WKB approximations (at least if χ_ε^* is compactly supported in ξ and the time scale depends on the size of frequency), we give the details of the proof since, in the present case, $\operatorname{supp}\chi_\varepsilon^*$ is not compact with respect to ξ and t_ε is independent of the size of frequency. The crucial point is that $p(x,\xi)$ is of *quadratic type* on $\Omega(\varepsilon)$:

$$|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)| \le C_{\alpha\beta}, \quad (x,\xi) \in \Omega(\varepsilon),\ |\alpha+\beta| \ge 2,$$

which allows us to follow a classical argument (see, for example, [Kitada and Kumano-go 1981]) and construct the approximation for $|t| < t_\varepsilon$ if $t_\varepsilon > 0$ is small enough. The composition $\operatorname{Op}(\chi_\varepsilon)J(\Psi,b)$ is also an FIO with the same phase, and a standard stationary phase method can be used to prove dispersive estimates for $0 < |t| < t_\varepsilon$. It remains to obtain the $L^1 \to L^\infty$ bounds of the remainders $\operatorname{Op}(\chi_\varepsilon)e^{-itH}\operatorname{Op}(r_N)$ and $\operatorname{Op}(\chi_\varepsilon)e^{-itH}(\operatorname{Op}(\chi_\varepsilon^*) - J(\Psi,b^N))$. If e^{-itH} maps from the Sobolev space $H^{d/2}(\mathbb{R}^d)$ to itself, then $L^1 \to L^\infty$ bounds are direct consequences of the Sobolev embedding and L^2-boundedness of PDOs. However, our Hamiltonian H is not bounded below (on $\{|x| \gtrsim |\xi|\}$) and such a property does not hold in general. To overcome this difficulty, we use an Egorov-type lemma as follows. By the Sobolev embedding and the Littlewood–Paley decomposition, the proof is reduced to that of the estimate

$$\sum_{j\ge 0} \|2^{j\gamma} S_j(D)\operatorname{Op}(\chi_\varepsilon)e^{-itH}\operatorname{Op}(r_N)\langle D\rangle^\gamma f\|_{L^2}^2 \le C\|f\|_{L^2}^2, \tag{5-1}$$

where $\gamma > d/2$ and S_j is a dyadic partition of unity. Then we will prove that there exists $\eta_j(t,\cdot,\cdot)\in S(1,g)$ such that

$$2^j \le C(1+|x|+|\xi|) \quad \text{on } \operatorname{supp}\eta_j(t),$$

and that

$$S_j(D)\operatorname{Op}(\chi_\varepsilon)e^{-itH} = e^{-itH}\operatorname{Op}(\eta_j(t)) + O_{L^2\to L^2}(2^{-jN}), \quad |t| < t_\varepsilon \ll 1.$$

Choosing $\delta > 0$ with $\gamma+\delta \le N/2$, we learn that $2^{j(\gamma+\delta)}\eta_j(t)r_N\langle\xi\rangle^\gamma \in S(1,g)$, and hence (5-1) holds. $\operatorname{Op}(\chi_\varepsilon)e^{-itH}(\operatorname{Op}(\chi_\varepsilon^*) - J(\Psi,b))$ can be controlled similarly.

Short-time behavior of the Hamilton flow. We now discuss the classical mechanics generated by $p(x,\xi)$. We denote by $(X(t),\Xi(t)) = (X(t,x,\xi),\Xi(t,x,\xi))$ the solution to the Hamilton equations

$$\begin{cases} \dot X_j = \dfrac{\partial p}{\partial \xi_j}(X,\Xi) = \displaystyle\sum_k g^{jk}(X)(\Xi_k - A_k(X)), \\ \dot\Xi_j = -\dfrac{\partial p}{\partial x_j}(X,\Xi) \\ \quad = -\dfrac{1}{2}\displaystyle\sum_{k,l}\frac{\partial g^{kl}}{\partial x_j}(X)(\Xi_k - A_k(X))(\Xi_l - A_l(X)) + \sum_{k,l} g^{kl}(X)\frac{\partial A_k}{\partial x_j}(X)(\Xi_l - A_l(X)) - \frac{\partial V}{\partial x_j}(X) \end{cases}$$

with the initial condition $(X(0),\Xi(0)) = (x,\xi)$, where $\dot f = \partial_t f$. We first observe that the flow conserves the energy:

$$p(x,\xi) = p(X(t),\Xi(t)),$$

which, combined with the uniform ellipticity of g^{jk}, implies

$$\begin{aligned} |\Xi(t) - A(X(t))|^2 &\lesssim p(X(t),\Xi(t)) - V(X(t)) \\ &= p(x,\xi) - V(X(t)) \\ &\lesssim |\xi - A(x)|^2 + |V(x)| + |V(X(t))|, \end{aligned}$$

and hence $|\Xi(t)| \lesssim |\xi| + \langle x\rangle + \langle X(t)\rangle$. By the Hamilton equation, we then have

$$|\dot X(t)| + |\dot\Xi(t)| \le C(1+|\xi|+|x|+|X(t)|+|\Xi(t)|).$$

Applying Gronwall's inequality to this estimate, we obtain an a priori bound:

$$|X(t)-x| + |\Xi(t)-\xi| \le C_T|t|(1+|x|+|\xi|), \quad |t|\le T,\ x,\xi\in\mathbb{R}^d.$$

Using this estimate, we obtain more precise behavior of the flow with initial conditions in $\Omega(\varepsilon)$.

Lemma 5.1. *Let $\varepsilon > 0$. Then, for sufficiently small $t_\varepsilon > 0$ and all $\alpha, \beta \in \mathbb{Z}_+^d$,*

$$|\partial_x^\alpha \partial_\xi^\beta (X(t, x, \xi) - x)| + |\partial_x^\alpha \partial_\xi^\beta (\Xi(t, x, \xi) - \xi| \le C_{\alpha\beta\varepsilon} |t| \langle x \rangle^{1-|\alpha+\beta|},$$

uniformly with respect to $(t, x, \xi) \in (-t_\varepsilon, t_\varepsilon) \times \Omega(\varepsilon)$.

Proof. We only consider the case with $t \ge 0$, the proof for the opposite case is similar. Let $(x, \xi) \in \Omega(\varepsilon)$. First we remark that, for sufficiently small $t_\varepsilon > 0$,

$$|x|/2 \le |X(t, x, \xi)| \le 2\langle x \rangle, \quad |t| \le t_\varepsilon. \tag{5-2}$$

For $|\alpha + \beta| = 0$, the assertion is obvious. We let $|\alpha + \beta| = 1$ and differentiate the Hamilton equations with respect to $\partial_x^\alpha \partial_\xi^\beta$:

$$\frac{d}{dt}\begin{pmatrix} \partial_x^\alpha \partial_\xi^\beta X \\ \partial_x^\alpha \partial_\xi^\beta \Xi \end{pmatrix} = \begin{pmatrix} \partial_x \partial_\xi p(X, \Xi) & \partial_\xi^2 p(X, \Xi) \\ -\partial_x^2 p(X, \Xi) & -\partial_\xi \partial_x p(X, \Xi) \end{pmatrix} \begin{pmatrix} \partial_x^\alpha \partial_\xi^\beta X \\ \partial_x^\alpha \partial_\xi^\beta \Xi \end{pmatrix}. \tag{5-3}$$

Using (5-2), we learn that $p(X(t), \Xi(t))$ is of quadratic type in $\Omega(\varepsilon)$:

$$|(\partial_x^\alpha \partial_\xi^\beta p)(X(t), \Xi(t))| \le C_{\alpha\beta\varepsilon} \langle x \rangle^{2-|\alpha+\beta|}, \quad (t, x, \xi) \in (-t_\varepsilon, t_\varepsilon) \times \Omega(\varepsilon).$$

Hence all entries of the above matrix are uniformly bounded in $(t, x, \xi) \in (-t_\varepsilon, t_\varepsilon) \times \Omega(\varepsilon)$. Taking $t_\varepsilon > 0$ smaller if necessary, integrating (5-3) with respect to t, and applying Gronwall's inequality, we have the assertion with $|\alpha + \beta| = 1$. For $|\alpha + \beta| \ge 2$, we prove the estimate for $\partial_{\xi_1}^2 X(t)$ and $\partial_{\xi_1}^2 \Xi(t)$ only, where $\xi = (\xi_1, \xi_2, \dots, \xi_d)$. Proofs for other cases are similar, and proofs for higher derivatives follow from an induction on $|\alpha + \beta|$. By the Hamilton equation, we learn

$$\frac{d}{dt}\partial_{\xi_1}^2 X(t) = \partial_x \partial_\xi p(X(t), \Xi(t)) \partial_{\xi_1}^2 X(t) + \partial_\xi^2 p(X(t), \Xi(t)) \partial_{\xi_1}^2 \Xi(t) + Q(X(t), \Xi(t)),$$

where $Q(X(t), \Xi(t))$ satisfies

$$\begin{aligned} |Q(X(t), \Xi(t))| &\le C_\varepsilon \sum_{|\alpha+\beta|=3, |\beta| \ge 1} |(\partial_x^\alpha \partial_\xi^\beta p)(X(t), \Xi(t))| |\partial_{\xi_1} X(t)|^{|\alpha|} |\partial_{\xi_1} \Xi(t)|^{|\beta|} \\ &\le C_\varepsilon \langle x \rangle^{-1}. \end{aligned}$$

We similarly obtain

$$\frac{d}{dt}\partial_{\xi_1}^2 \Xi(t) = -\partial_x^2 p(X(t), \Xi(t)) \partial_{\xi_1}^2 X(t) - \partial_\xi \partial_x p(X(t), \Xi(t)) \partial_{\xi_1}^2 \Xi(t) + O(\langle x \rangle^{-1}).$$

Applying Gronwall's inequality, we have the desired estimates. □

Lemma 5.2. (1) *Let $t_\varepsilon > 0$ be small enough. Then, for any $|t| < t_\varepsilon$, the map*

$$g(t) : (x, \xi) \mapsto (X(t, x, \xi), \xi)$$

is a diffeomorphism from $\Omega(\varepsilon/2)$ onto its range, and satisfies

$$\Omega(\varepsilon) \subset g(t, \Omega(\varepsilon/2)) \quad \textit{for all } |t| < t_\varepsilon.$$

(2) *Let* $\Omega(\varepsilon) \ni (x,\xi) \mapsto (Y(t,x,\xi),\xi) \in \Omega(\varepsilon/2)$ *be the inverse map of* $g(t)$. *Then* $Y(t,x,\xi)$ *and* $\Xi(t, Y(t,x,\xi),\xi)$ *satisfy the same estimates as those for* $X(t,x,\xi)$ *and* $\Xi(t,x,\xi)$ *of* Lemma 5.1, *respectively*:

$$|\partial_x^\alpha \partial_\xi^\beta (Y(t,x,\xi)-x)| + |\partial_x^\alpha \partial_\xi^\beta (\Xi(t, Y(t,x,\xi),\xi)-\xi| \le C_{\alpha\beta\varepsilon}|t|\langle x\rangle^{1-|\alpha+\beta|},$$

uniformly with respect to $(t,x,\xi) \in (-t_\varepsilon, t_\varepsilon) \times \Omega(\varepsilon)$.

Proof. Choosing a cutoff function $\rho \in S(1,g)$ such that $0 \le \rho \le 1$, $\operatorname{supp}\rho \subset \Omega(\varepsilon/3)$, and $\rho \equiv 1$ on $\Omega(\varepsilon/2)$, we modify $g(t)$ as follows:

$$g_\rho(t,x,\xi) = (X_\rho(t,x,\xi),\xi), \quad X_\rho(t,x,\xi) = (1-\rho(x,\xi))x + \rho(x,\xi)X(t,x,\xi).$$

It is easy to see that, for $(t,x,\xi) \in (-t_\varepsilon, t_\varepsilon) \times \Omega(\varepsilon/2)$, $g_\rho(t,x,\xi)$ is smooth and Lemma 5.1 implies

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta g_\rho(t,x,\xi)| &\le C_{\alpha\beta\varepsilon}, \quad |\alpha+\beta| \ge 1, \\ |J(g_\rho)(t,x,\xi) - \mathrm{Id}\,| &\le C_\varepsilon t_\varepsilon, \end{aligned}$$

where $J(g_\rho)$ is the Jacobi matrix with respect to (x,ξ) and the constant $C_\varepsilon > 0$ is independent of t, x, and ξ. Choosing $t_\varepsilon > 0$ so small that $C_\varepsilon t_\varepsilon < 1/2$, and applying the Hadamard global inverse mapping theorem, we see that, for any fixed $|t| < t_\varepsilon$, $g_\rho(t)$ is a diffeomorphism from $\mathbb{R}^{2d}$ onto itself. By definition, $g(t)$ is diffeomorphic from $\Omega(\varepsilon/2)$ onto its range. Since $g_\rho(t)$ is bijective, it remains to check that

$$\Omega(\varepsilon)^c \supset g_\rho(t, \Omega(\varepsilon/2)^c), \quad |t| < t_\varepsilon.$$

Suppose that $(x,\xi) \in \Omega(\varepsilon/2)^c$. If $(x,\xi) \in \Omega(\varepsilon/3)^c$, the assertion is obvious since $g_\rho(t) \equiv \mathrm{Id}$ outside $\Omega(\varepsilon/3)$. If $(x,\xi) \in \Omega(\varepsilon/3) \setminus \Omega(\varepsilon/2)$, then, by Lemma 5.1 and the support property of ρ, we have

$$|X_\rho(t,x,\xi)| \le |x| + \rho(x,\xi)|(X(t,x,\xi)-x)| \le (\varepsilon/2 + C_0 t_\varepsilon)\langle\xi\rangle$$

for some $C_0 > 0$ independent of x, ξ, and t_ε. Choosing $t_\varepsilon < \varepsilon/(2C_0)$, we obtain the assertion.

We next prove the estimates on $Y(t)$. Since $(Y(t,x,\xi),\xi) \in \Omega(\varepsilon/2)$, we learn

$$\begin{aligned} |Y(t,x,\xi) - x| &= |X(0, Y(t,x,\xi),\xi) - X(t, Y(t,x,\xi),\xi)| \\ &\le \sup_{(x,\xi)\in\Omega(\varepsilon/2)} |X(t,x,\xi) - x| \\ &\le C_\varepsilon |t|\langle x\rangle. \end{aligned}$$

For $\alpha, \beta \in \mathbb{Z}_+^d$ with $|\alpha+\beta| = 1$, apply $\partial_x^\alpha \partial_\xi^\beta$ to the equality $x = X(t, Y(t,x,\xi),\xi)$. We then have the equality

$$A(t, Z(t,x,\xi))\partial_x^\alpha \partial_\xi^\beta (Y(t,x,\xi)-x) = \partial_y^\alpha \partial_\eta^\beta (y - X(t,y,\eta))|_{(y,\eta)=Z(t,x,\xi)},$$

where $Z(t,x,\xi) = (Y(t,x,\xi),\xi)$ and $A(t,Z) = (\partial_x X)(t,Z)$ is a $d \times d$ matrix. By Lemma 5.1 and a similar argument to that in the proof of Lemma 5.2(1), we learn that $A(t, Z(t,x,\xi))$ is invertible if $t_\varepsilon > 0$ is small enough, and that $A(t, Z(t,x,\xi))$ and $A(t, Z(t,x,\xi))^{-1}$ are bounded uniformly in

$(t, x, \xi) \in (-t_\varepsilon, t_\varepsilon) \times \Omega(\varepsilon/2)$. Therefore,

$$|\partial_x^\alpha \partial_\xi^\beta (Y(t, x, \xi) - x)| \le C_{\alpha\beta} \sup_{(x,\xi)\in\Omega(\varepsilon/2)} |\partial_x^\alpha \partial_\xi^\beta (x - X(t, x, \xi))|$$
$$\le C_{\alpha\beta} |t| \langle x\rangle^{1-|\alpha+\beta|}.$$

Proofs for higher derivatives are obtained by induction in $|\alpha + \beta|$ and proofs for $\Xi(t, Y(t, x, \xi), \xi)$ are similar. □

***The parametrix for* $\mathrm{Op}(\chi_\varepsilon)e^{-itH}\,\mathrm{Op}(\chi_\varepsilon)^*$.** Before starting the construction of parametrix, we prepare two lemmas. The following Egorov-type theorem will be used to control the remainder term. We write $\exp tH_p(x, \xi) = (X(t, x, \xi), \Xi(t, x, \xi))$.

Lemma 5.3. *For $h \in (0, 1]$, consider a h-dependent symbol $\eta_h \in S(1, g)$ such that $\operatorname{supp} \eta_h \subset \Omega(\varepsilon) \cap \{1/(2h) < |\xi| < 2/h\}$. Then, for sufficiently small $t_\varepsilon > 0$, independent of h, and any integer $N \ge 0$, there exists a bounded family of symbols*

$$\{\eta_h^N(t, \cdot, \cdot) : |t| < t_\varepsilon,\ 0 < h \le 1\} \subset S(1, g)$$

such that

$$\operatorname{supp} \eta_h^N(t, \cdot, \cdot) \subset \exp(-t)H_p(\operatorname{supp} \eta_h)$$

and

$$\|e^{itH}\,\mathrm{Op}(\eta_h)e^{-itH} - \mathrm{Op}(\eta_h^N(t))\|_{L^2\to L^2} \le C_{N\varepsilon} h^N,$$

uniformly with respect to $0 < h \le 1$ and $|t| < t_\varepsilon$.

Proof. Let $\eta_h^0(t, x, \xi) = \eta_h(\exp tH_p(x, \xi)) = \eta_h(X(t, x, \xi), \Xi(t, x, \xi))$. It is easy to see that

$$\operatorname{supp} \eta_h^0 \subset \exp(-t)H_p(\operatorname{supp} \eta_h).$$

Moreover, Lemma 5.1 implies that $\{\eta_h^0 : |t| < t_\varepsilon, 0 < h \le 1\}$ is a bounded subset of $S(1, g)$. By a direct computation, η_h^0 solves

$$\partial_t \eta_h^0 = \{p, \eta_h^0\}, \quad \eta_h^0|_{t=0} = \eta_h,$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. Then, by standard pseudodifferential calculus, there exists a bounded set $\{r_h^0(t, \cdot, \cdot) : 0 \le t < t_\varepsilon, 0 < h \le 1\} \subset S(1, g)$ with $\operatorname{supp} r_h^0 \subset \exp(-t)H_p(\operatorname{supp} \eta_h)$ such that

$$\frac{d}{dt}\mathrm{Op}(\eta_h^0) = i[H, \mathrm{Op}(\eta_h^0)] + h\,\mathrm{Op}(r_h^0).$$

We next set

$$\eta_h^1(t, x, \xi) = \int_0^t r_h^0(s, X(t-s, x, \xi), \Xi(t-s, x, \xi))\,ds.$$

Again, we learn that $\{\eta_h^1(t, \cdot, \cdot) : |t| < t_\varepsilon, 0 < h \le 1\} \subset S(1, g)$ is also bounded and that

$$\operatorname{supp} \eta_h^1 \subset \exp(-t)H_p(\operatorname{supp} \eta_h)$$

for all $|t| < t_\varepsilon$ and $0 < h \le 1$. Moreover, η_h^1 solves

$$\partial_t \eta_h^1 = \{p, \eta_h^1\} + r_h^0, \quad \eta_h^1|_{t=0} = 0,$$

which implies

$$\frac{d}{dt} \operatorname{Op}(\eta_h^0 + h\eta_h^1) = i[H, \operatorname{Op}(\eta_h^0 + h\eta_h^1)] + h^2 \operatorname{Op}(r_h^1)$$

with some $\{r_h^1 : 0 \le t < t_\varepsilon, 0 < h \le 1\} \subset S(1, g)$ and $\operatorname{supp} r_h^1 \subset \exp(-t)H_p(\operatorname{supp} \eta_h)$. Iterating this procedure and putting $\eta_h^N = \sum_{j=0}^{N-1} h^j \eta_h^j$, we obtain the assertion. □

Using this lemma, we have the following.

Lemma 5.4. *Let $\varepsilon > 0$. Then, for any symbol $\chi_\varepsilon \in S(1, g)$ with* $\operatorname{supp} \chi_\varepsilon \subset \Omega(\varepsilon)$ *and any integer $N \ge 1$, there exists $\chi_\varepsilon^* \in S(1, g)$ with* $\operatorname{supp} \chi_\varepsilon^* \subset \Omega(\varepsilon)$ *such that for any $\gamma < N/2$,*

$$\sup_{|t|<t_\varepsilon} \|\operatorname{Op}(\chi_\varepsilon) e^{-itH} \operatorname{Op}(\chi_\varepsilon)^* - \operatorname{Op}(\chi_\varepsilon) e^{-itH} \operatorname{Op}(\chi_\varepsilon^*)\|_{H^{-\gamma}(\mathbb{R}^d) \to H^\gamma(\mathbb{R}^d)} \le C_{N\gamma\varepsilon}.$$

Proof. By the expansion formula (2-4), there exists $\chi_\varepsilon^* \in S(1, g)$ with $\operatorname{supp} \chi_\varepsilon^* \subset \Omega(\varepsilon)$ such that

$$\operatorname{Op}(\chi_\varepsilon)^* = \operatorname{Op}(\chi_\varepsilon^*) + \operatorname{Op}(r_0(N))$$

with some $r_0(N) \in S(\langle x\rangle^{-N} \langle \xi\rangle^{-N}, g)$. For $\delta > 0$ with $2\gamma + \delta \le N$, we split

$$\langle D\rangle^\gamma \operatorname{Op}(\chi_\varepsilon) e^{-itH} \operatorname{Op}(r_0(N)) \langle D\rangle^\gamma = \langle D\rangle^\gamma \operatorname{Op}(\chi_\varepsilon) e^{-itH} \langle D\rangle^{-\gamma-\delta} \langle x\rangle^{-\gamma-\delta} \cdot \langle x\rangle^{\gamma+\delta} \langle D\rangle^{\gamma+\delta} \operatorname{Op}(r_0(N)) \langle D\rangle^\gamma.$$

Since $\langle x\rangle^{\gamma+\delta} \langle \xi\rangle^{\gamma+\delta} r_0(N) \langle \xi\rangle^\gamma \in S(1, g)$, $\langle x\rangle^{\gamma+\delta} \langle D\rangle^{\gamma+\delta} \operatorname{Op}(r_0(N)) \langle D\rangle^\gamma$ is bounded on L^2. In order to prove the L^2-boundedness of the first term of the right hand side, we use the standard Littlewood–Paley decomposition and Lemma 5.3 as follows. Consider a dyadic partition of unity with respect to the frequency:

$$\sum_{j=0}^{\infty} S_j(D) = 1,$$

where $S_j(\xi) = S(2^{-j}\xi)$, $j \ge 1$, with some $S \in C_0^\infty(\mathbb{R}^d)$ supported in $\{1/2 < |\xi| < 2\}$ and $S_0 \in C_0^\infty(\mathbb{R}^d)$ supported in $\{|\xi| < 1\}$. Then

$$\|\langle D\rangle^\gamma \operatorname{Op}(\chi_\varepsilon) e^{-itH} \langle D\rangle^{-\gamma-\delta} \langle x\rangle^{-\gamma-\delta} f\|_{L^2} \le C \biggl(\sum_{j=0}^{\infty} \|2^{j\gamma} S_j(D) \operatorname{Op}(\chi_\varepsilon) e^{-itH} \langle D\rangle^{-\gamma-\delta} \langle x\rangle^{-\gamma-\delta} f\|_{L^2}^2 \biggr)^{1/2}.$$

By the expansion formula (2-3), there exists a sequence of symbols $\eta_j \in S(1, g)$ supported in

$$\Omega(\varepsilon) \cap \{2^{j-1} < |\xi| < 2^{j+1}\}$$

such that

$$S_j(D) \operatorname{Op}(\chi_\varepsilon) = \operatorname{Op}(\eta_j) + Q_1(j, N), \quad \|Q_1(j, N)\|_{L^2 \to L^2} = O(2^{-jN}).$$

We then learn from Lemma 5.3 with $h = 2^{-j}$ that there exists $\{\eta_j^N(t) : |t| < t_\varepsilon\} \subset S(1, g)$ such that

$$\operatorname{Op}(\eta_j)e^{-itH} = e^{-itH}\operatorname{Op}(\eta_j^N(t)) + Q_2(t, j, N), \quad \sup_{|t|<t_\varepsilon} \|Q_2(t, j, N)\|_{L^2\to L^2} = O(2^{-jN}).$$

Since $N \geq \gamma + \delta$, the remainder satisfies

$$\sup_{|t|<t_\varepsilon} \|2^{j\gamma}(Q_1(j, N)e^{-itH} + Q_2(t, j, N))\langle D\rangle^{-\gamma-\delta}\langle x\rangle^{-\gamma-\delta} f\|_{L^2}^2 \leq C2^{-2j\delta}\|f\|_{L^2}^2.$$

Suppose that $(x, \xi) \in \operatorname{supp}\eta_j^N(t)$. Since $\operatorname{supp}\eta_j^N(t) \subset \exp(-t)H_p(\operatorname{supp}\eta_j)$, we have

$$|X(t, x, \xi)| > \varepsilon\langle\Xi(t, x, \xi)\rangle, \quad 2^{j-1} < |\Xi(t, x, \xi)| < 2^{j+1}.$$

Using Lemma 5.1 with the initial data $(X(t, x, \xi), \Xi(t, x, \xi))$, we learn

$$|x - X(t, x, \xi)| + |\xi - \Xi(t, x, \xi)| \leq Ct_\varepsilon\langle X(t, x, \xi)\rangle, \quad |t| < t_\varepsilon.$$

Combining these two estimates, we see that

$$2^j \leq C(1 + |x| + |\xi|), \quad (x, \xi) \in \operatorname{supp}\eta_j^N(t), \ |t| < t_\varepsilon,$$

where the constant $C > 0$ is independent of x, ξ, and t, provided that $t_\varepsilon > 0$ is small enough. Therefore, $2^{j(\gamma+\delta)}\eta_j^N(t)\langle\xi\rangle^{-\gamma-\delta}\langle x\rangle^{-\gamma-\delta} \in S(1, g)$ and the corresponding PDO is bounded on L^2. Finally, we obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty} \|2^{j\gamma}\operatorname{Op}(\eta_j)e^{-itH}\langle D\rangle^{-\gamma-\delta}\langle x\rangle^{-\gamma-\delta} f\|_{L^2}^2 \\
&\qquad \leq C\sum_{j=0}^{\infty}\big(\|2^{-j\delta}2^{j(\gamma+\delta)}\operatorname{Op}(\eta_j^N(t))\langle D\rangle^{-\gamma-\delta}\langle x\rangle^{-\gamma-\delta} f\|_{L^2}^2 + 2^{-2j\delta}\|f\|_{L^2}^2\big) \\
&\qquad \leq C\sum_{j=0}^{\infty} 2^{-2j\delta}\|f\|_{L^2}^2 \\
&\qquad \leq C\|f\|_{L^2}^2, \qquad \square
\end{aligned}$$

We now consider a parametrix construction of $\operatorname{Op}(\chi_\varepsilon)e^{-itH}\operatorname{Op}(\chi_\varepsilon^*)$. Let us first make the following ansatz:

$$v(t, x) = \frac{1}{(2\pi)^d}\int e^{i(\Psi(t,x,\xi)-y\cdot\xi)} b^N(t, x, \xi) f(y)\, dy\, d\xi,$$

where $b^N = \sum_{j=0}^{N-1} b_j$. In order to approximately solve the Schrödinger equation

$$i\partial_t v(t) = Hv(t), \quad v|_{t=0} = \operatorname{Op}(\chi_\varepsilon^*)\varphi,$$

the phase function Ψ and the amplitude b^N should satisfy respectively the Hamilton–Jacobi equation

$$\partial_t\Psi + p(x, \partial_x\Psi) = 0, \quad \Psi|_{t=0} = x\cdot\xi \tag{5-4}$$

and the transport equations

$$\begin{cases} \partial_t b_0 + \mathscr{X} \cdot \partial_x b_0 + \mathscr{Y} b_0 = 0, & b_0|_{t=0} = \chi_\varepsilon, \\ \partial_t b_j + \mathscr{X} \cdot \partial_x b_j + \mathscr{Y} b_j + iKb_{j-1} = 0, & b_j|_{t=0} = 0, \ 1 \le j \le N-1, \end{cases} \tag{5-5}$$

where K is the kinetic part of H, and the vector field $\mathscr{X}$ and function $\mathscr{Y}$ are defined by

$$\mathscr{X}_j(t, x, \xi) := (\partial_{\xi_j} p)(x, \partial_x \Psi(t, x, \xi)), \quad j = 1, \dots, d,$$
$$\mathscr{Y}(t, x, \xi) := [k(x, \partial_x)\Psi + p_1(x, \partial_x \Psi)](t, x, \xi).$$

Here p, p_1 are given by (1-6). We first construct the phase function Ψ.

Proposition 5.5. *Let us fix $\varepsilon > 0$ arbitrarily. Then, for sufficiently small $t_\varepsilon > 0$, we can construct a smooth and real-valued function $\Psi \in C^\infty((-t_\varepsilon, t_\varepsilon) \times \mathbb{R}^{2d}; \mathbb{R})$ which solves the Hamilton–Jacobi equation (5-4) for $(x, \xi) \in \Omega(\varepsilon)$ and $|t| \le t_\varepsilon$. Moreover, for all $\alpha, \beta \in \mathbb{Z}_+^d$, $x, \xi \in \mathbb{R}^d$ and $|t| \le t_\varepsilon$,*

$$|\partial_x^\alpha \partial_\xi^\beta (\Psi(t, x, \xi) - x \cdot \xi + tp(x, \xi)| \le C_{\alpha\beta\varepsilon} |t|^2 \langle x \rangle^{2-|\alpha+\beta|}, \tag{5-6}$$

where $C_{\alpha\beta\varepsilon} > 0$ is independent of x, ξ and t.

Proof. We consider the case when $t \ge 0$, and the proof for $t \le 0$ is similar. We first define the action integral $\widetilde{\Psi}(t, x, \xi)$ on $[0, t_\varepsilon) \times \Omega(\varepsilon/2)$ by

$$\widetilde{\Psi}(t, x, \xi) := x \cdot \xi + \int_0^t L(X(s, Y(t, x, \xi), \xi), \Xi(s, Y(t, x, \xi), \xi))\, ds,$$

where $L(x, \xi) = \xi \cdot \partial_\xi p(x, \xi) - p(x, \xi)$ is the Lagrangian associated to $p(x, \xi)$, and X, Ξ, and Y are given by Lemma 5.2(2) with ε replaced by $\varepsilon/2$. The smoothness of $\widetilde{\Psi}(t, x, \xi)$ follows from corresponding properties of $X(t)$, $\Xi(t)$, and $Y(t)$. It is well known that $\widetilde{\Psi}(t, x, \xi)$ solves the Hamilton–Jacobi equation

$$\partial_t \widetilde{\Psi}(t, x, \xi) + p(x, \partial_x \widetilde{\Psi}(t, x, \xi)) = 0, \quad \Psi|_{t=0} = x \cdot \xi,$$

for $(x, \xi) \in \Omega(\varepsilon/2)$, and satisfies

$$\partial_x \widetilde{\Psi}(t, x, \xi) = \Xi(t, Y(t, x, \xi), \xi), \quad \partial_\xi \widetilde{\Psi}(t, x, \xi) = Y(t, x, \xi).$$

Lemma 5.2(2) shows that $p(Y(t, x, \xi), \xi)$ is of quadratic type:

$$|\partial_x^\alpha \partial_\xi^\beta p(Y(t, x, \xi), \xi)| \le C_{\alpha\beta\varepsilon} \langle x \rangle^{2-|\alpha+\beta|}, \quad (t, x, \xi) \in [0, t_\varepsilon) \times \Omega(\varepsilon/2),$$

which, combined with the energy conservation

$$p(x, \partial_x \widetilde{\Psi}(t, x, \xi)) = p(Y(t, x, \xi), \xi),$$

implies

$$|\partial_x^\alpha \partial_\xi^\beta (\widetilde{\Psi}(t, x, \xi) - x \cdot \xi)| \le C_{\alpha\beta\varepsilon} |t| \langle x \rangle^{2-|\alpha+\beta|}, \quad (t, x, \xi) \in [0, t_\varepsilon) \times \Omega(\varepsilon/2).$$

We similarly obtain, for $(t, x, \xi) \in [0, t_\varepsilon) \times \Omega(\varepsilon/2)$,

$$\begin{aligned}|p(x, \partial_x \widetilde{\Psi}(t, x, \xi)) - p(x, \xi)| &= \left|(\partial_x \widetilde{\Psi}(t, x, \xi) - \xi) \cdot \int_0^1 (\partial_\xi p)(x, \theta \partial_x \widetilde{\Psi}(t, x, \xi) + (1-\theta)\xi)\, d\theta\right| \\ &\le C_\varepsilon |t| \langle x\rangle^2,\end{aligned}$$

and, more generally,

$$|\partial_x^\alpha \partial_\xi^\beta \big(p(x, \partial_x \widetilde{\Psi}(t, x, \xi)) - p(x, \xi)\big)| \le C_{\alpha\beta\varepsilon} |t| \langle x\rangle^{2-|\alpha+\beta|}.$$

Therefore, integrating the Hamilton–Jacobi equation with respect to t, we have

$$|\partial_x^\alpha \partial_\xi^\beta (\widetilde{\Psi}(t, x, \xi) - x \cdot \xi + tp(x, \xi))| \le C_{\alpha\beta\varepsilon} |t|^2 \langle x\rangle^{2-|\alpha+\beta|}.$$

Finally, choosing a cutoff function $\rho \in S(1, g)$ so that $0 \le \rho \le 1$, $\rho \equiv 1$ on $\Omega(\varepsilon)$, and $\operatorname{supp} \rho \subset \Omega(\varepsilon/2)$, we define

$$\Psi(t, x, \xi) := x \cdot \xi - tp(x, \xi) + \rho(x, \xi)(\widetilde{\Psi}(t, x, \xi) - x \cdot \xi + tp(x, \xi)).$$

$\Psi(t, x, \xi)$ clearly satisfies the statement of Proposition 5.5. □

Using the phase function constructed in Proposition 5.5, we can define the FIO $J(\Psi, a) : \mathscr{S} \to \mathscr{S}'$ by

$$J(\Psi, a) f(x) = \frac{1}{(2\pi)^d} \int e^{i(\Psi(t,x,\xi) - y\cdot\xi)} a(x, \xi) f(y)\, dy\, d\xi, \qquad f \in \mathscr{S}(\mathbb{R}^d),$$

where $a \in S(1, g)$. Moreover, we have the following.

Lemma 5.6. *Let $t_\varepsilon > 0$ be small enough. Then, for any bounded family of symbols*

$$\{a(t) : |t| < t_\varepsilon\} \subset S(1, g),$$

$J(\Psi, a)$ is bounded on $L^2(\mathbb{R}^d)$ uniformly with respect to $|t| < t_\varepsilon$:

$$\sup_{|t| \le t_\varepsilon} \|J(\Psi, a)\|_{L^2 \to L^2} \le C_\varepsilon.$$

Proof. For sufficiently small $t_\varepsilon > 0$, the estimates (5-6) imply

$$|(\partial_\xi \otimes \partial_x \Psi)(t, x, \xi) - \mathrm{Id}\,| \le C_\varepsilon t_\varepsilon < \tfrac{1}{2}, \qquad |\partial_x^\alpha \partial_\xi^\beta \Psi(t, x, \xi)| \le C_{\alpha\beta\varepsilon} \quad \text{for } |\alpha + \beta| \ge 2,$$

uniformly with respect to $(t, x, \xi) \in (-t_\varepsilon, t_\varepsilon) \times \mathbb{R}^{2d}$. Therefore, the assertion is a consequence of the standard L^2-boundedness of FIOs, or, equivalently, Kuranishi's trick and the L^2-boundedness of PDOs; see, for example, [Robert 1987; Mizutani 2013, Lemma 4.2]. □

We next construct the amplitude.

Proposition 5.7. *Let $\Psi(t, x, \xi)$ be as in Proposition 5.5 with ε replaced by $\varepsilon/3$. Then, for any integer $N \ge 0$, there exist families of symbols $\{b_j(t, \cdot, \cdot) : |t| < t_\varepsilon\} \subset S(\langle x\rangle^{-j} \langle \xi\rangle^{-j}, g)$, $j = 0, 1, 2, \dots, N-1$, such that* $\operatorname{supp} b_j(t, \cdot, \cdot) \subset \Omega(\varepsilon/2)$ *and b_j solve the transport equations (5-5).*

Proof. We consider the case $t \geq 0$ only. Symbols b_j can be constructed by a standard method of characteristics along the flow generated by $\mathscr{X}(t,x,\xi)$ as follows. First note that Assumption 1.1 and (5-6) imply that

$$|\partial_x^\alpha \partial_\xi^\beta \mathscr{X}(t,x,\xi)| \leq C_{\alpha\beta\varepsilon} \langle x\rangle^{1-|\alpha+\beta|}, \tag{5-7}$$

$$|\partial_x^\alpha \partial_\xi^\beta \mathscr{Y}(t,x,\xi)| \leq C_{\alpha\beta\varepsilon} \langle x\rangle^{-|\alpha+\beta|}, \tag{5-8}$$

uniformly with respect to $0 \leq t \leq t_\varepsilon$ and $(x,\xi) \in \Omega(\varepsilon/3)$. For all $0 \leq s, t \leq t_\varepsilon$, we consider the solution to the ODE

$$\partial_t z(t,s,x,\xi) = \mathscr{X}(t, z(t,s,x,\xi), \xi), \quad z(s,s) = x.$$

We learn from (5-7) and an argument as in the proof of Lemma 5.1 that $z(t,s)$ is well defined for $0 \leq s, t \leq t_\varepsilon$ and $(x,\xi) \in \Omega(\varepsilon/3)$, and that

$$|\partial_x^\alpha \partial_\xi^\beta (z(t,s,x,\xi) - x)| \leq C_{\alpha\beta\varepsilon} t_\varepsilon \langle x\rangle^{1-|\alpha+\beta|}, \quad (x,\xi) \in \Omega(\varepsilon/3). \tag{5-9}$$

Then $b_j(t)$ are defined inductively by

$$b_0(t,x,\xi) = \chi_\varepsilon^*(z(0,t,x,\xi),\xi) \exp\biggl(\int_0^t \mathscr{Y}(s, z(s,t,x,\xi),\xi)\,ds\biggr),$$

$$b_j(t,x,\xi) = -\int_0^t (iKb_{j-1})(s, z(s,t,x,\xi),\xi) \exp\biggl(\int_u^t \mathscr{Y}(u, z(u,t,x,\xi),\xi)\,du\biggr)\,ds.$$

Since $\operatorname{supp}\chi_\varepsilon^* \subset \Omega(\varepsilon)$, by (5-9) and an argument as in the proof of Lemma 5.2(1), we see that $b_j(t,x,\xi)$ is smooth with respect to (x,ξ) and that $\partial_x^\alpha \partial_\xi^\beta b_j(t,x,\xi)$ are supported in $\Omega(\varepsilon/2)$ for all $0 \leq t \leq t_\varepsilon$. Thus, if we extend b_j on $\mathbb{R}^{2d}$ so that $b_j(t,x,\xi) = 0$ outside $\Omega(\varepsilon/2)$, then b_j is still smooth in (x,ξ). Furthermore, we learn by (5-8) and (5-9) that $\{b_j(t,\cdot,\cdot) : t \in [0,t_\varepsilon],\ 0 \leq j \leq N-1\}$ is a bounded set in $S(\langle x\rangle^{-j}\langle\xi\rangle^{-j}, g)$. Finally, a standard Hamilton–Jacobi theory shows that $b_j(t)$ solve the transport equations (5-5). □

We now state the main result in this section.

Theorem 5.8. *Fix $\varepsilon > 0$ arbitrarily. Then, for any sufficiently small $t_\varepsilon > 0$, any nonnegative integer $N \geq 0$ and any symbol $\chi_\varepsilon \in S(1,g)$ supported in $\Omega(\varepsilon)$, we can find a bounded family of symbols $\{a^N(t,\cdot,\cdot) : |t| < t_\varepsilon\} \subset S(1,g)$ such that $\operatorname{Op}(\chi_\varepsilon)e^{-itH}\operatorname{Op}(\chi_\varepsilon)^*$ can be brought to the form*

$$\operatorname{Op}(\chi_\varepsilon)e^{-itH}\operatorname{Op}(\chi_\varepsilon)^* = J(\Psi, a^N) + Q(t,N),$$

where $J(\Psi,a^N)$ is the FIO with the phase $\Psi(t,x,\xi)$ constructed in Proposition 5.5 with ε replaced by $\varepsilon/3$. The distribution kernel of $J(\Psi,a^N)$, which we denote by $K_{\Psi,a^N}(t,x,y)$, satisfies the dispersive estimate

$$|K_{\Psi,a^N}(t,x,y)| \leq C_{N,\varepsilon}|t|^{-d/2}, \quad 0 < |t| < t_\varepsilon,\ x,\xi \in \mathbb{R}^d.$$

Moreover, for any $\gamma \geq 0$ with $N > 2\gamma$, the remainder $Q(t,N)$ satisfies

$$\|\langle D\rangle^\gamma Q(t,N)\langle D\rangle^\gamma\|_{L^2\to L^2} \leq C_{N\gamma\varepsilon}|t|, \quad |t| < t_\varepsilon. \tag{5-10}$$

In particular, if we choose $N \geq d+1$, the distribution kernel of $Q(t,N)$ is uniformly bounded in $\mathbb{R}^{2d}$ with respect to $|t|<t_\varepsilon$. Hence

$$\|\mathrm{Op}(\chi_\varepsilon)e^{-itH}\,\mathrm{Op}(\chi_\varepsilon)^*\|_{L^1\to L^\infty} \leq C_\varepsilon |t|^{-d/2}, \quad 0<|t|<t_\varepsilon.$$

Proof. We consider the case when $t \geq 0$ and the proof for the opposite case is similar. By virtue of Lemma 5.4, we may replace $\mathrm{Op}(\chi_\varepsilon)^*$ by $\mathrm{Op}(\chi_\varepsilon^*)$ for some $\chi_\varepsilon^* \in S(1,g)$ supported in $\Omega(\varepsilon)$, without loss of generality. Let $b^N=\sum_{j=0}^{N-1} b_j$ with b_j constructed in Proposition 5.7. Since $J(\Psi,b^N)|_{t=0}=\mathrm{Op}(\chi_\varepsilon^*)$, we have the Duhamel formula

$$\mathrm{Op}(\chi_\varepsilon)e^{-itH}\,\mathrm{Op}(\chi_\varepsilon^*) = \mathrm{Op}(\chi_\varepsilon)J(\Psi,b^N) - i\int_0^t \mathrm{Op}(\chi_\varepsilon)e^{-i(t-s)H}(D_t+H)J(\Psi,b^N)|_{t=s}\,ds.$$

Estimates on the remainder. It suffices to show that

$$\sup_{|t|<t_\varepsilon} \|\langle D\rangle^\gamma\,\mathrm{Op}(\chi_\varepsilon)e^{-itH}(D_t+H)J(\Psi,b^N)\langle D\rangle^\gamma\|_{L^2\to L^2} \leq C_{N\gamma\varepsilon}.$$

Since Ψ, b_j solve the Hamilton–Jacobi equation (5-4) and transport equations (5-5), respectively, a direct computation yields

$$e^{-i\Psi(t,x,\xi)}(D_t+H)\biggl(e^{i\Psi(t,x,\xi)}\sum_{j=0}^{N-1} b_j(t,x,\xi)\biggr) = r_N(t,x,\xi),$$

with some $\{r_N(t,\cdot,\cdot) : 0 \leq t \leq t_\varepsilon\} \subset S(\langle x\rangle^{-N}\langle\xi\rangle^{-N},g)$. In particular,

$$(D_t+H)J(\Psi,b^N) = J(\Psi,r_N).$$

A standard L^2-boundedness of FIOs then implies

$$\sup_{|t|<t_\varepsilon} \|\langle x\rangle^{\gamma+\delta}\langle D\rangle^{\gamma+\delta}J(\Psi,r_N)\langle D\rangle^\gamma\|_{L^2\to L^2} \leq C_{N\gamma\delta},$$

for any $\gamma,\delta \geq 0$ with $2\gamma+\delta \leq N$. Since, in the proof of Lemma 5.4, we already proved that

$$\sup_{|t|\leq t_\varepsilon} \|\langle D\rangle^\gamma\,\mathrm{Op}(\chi_\varepsilon)e^{-itH}\langle D\rangle^{-\gamma-\delta}\langle x\rangle^{-\gamma-\delta}\|_{L^2\to L^2} \leq C_{\gamma\delta},$$

we obtain the desired estimate.

Dispersive estimates. By the composition formula of PDOs and FIOs (cf. [Robert 1987]),

$$\mathrm{Op}(\chi_\varepsilon)J(\Psi,b^N)$$

is also an FIO with the same phase Ψ and the amplitude

$$a^N(t,x,\xi) = \frac{1}{(2\pi)^d}\int e^{iy\cdot\eta}\chi_\varepsilon(x,\eta+\widetilde{\Xi}(t,x,y,\xi))b^N(t,x+y,\xi)\,dy\,d\eta,$$

where $\widetilde{\Xi}(t,x,y,\xi)=\int_0^1(\partial_x\Psi)(t,y+\lambda(x-y),\xi)\,d\lambda$. By virtue of (5-6), $\widetilde{\Xi}$ satisfies

$$|\partial_x^\alpha\partial_y^{\alpha'}\partial_\xi^\beta(\widetilde{\Xi}(t,x,y,\xi)-\xi)| \leq C_{\alpha\alpha'\beta}|t|, \quad |\alpha+\alpha'+\beta| \geq 1.$$

Combining this with the relations $\chi_\varepsilon, b^N \in S(1, g)$, $\operatorname{supp} \chi_\varepsilon \subset \Omega(\varepsilon)$, and $\operatorname{supp} b^N(t, \cdot, \cdot) \subset \Omega(\varepsilon/2)$, we see that $\{a^N : 0 \le t < t_\varepsilon\}$ is bounded in $S(1, g)$. The distribution kernel of $J(\Psi, a^N)$ is given by

$$K_{\Psi,a^N}(t, x, y) = \frac{1}{(2\pi)^d} \int e^{i(\Psi(t,x,\xi)-y\cdot\xi)} a^N(t, x, \xi)\, d\xi.$$

By virtue of Proposition 5.5, we have

$$\begin{aligned} &\sup_{|t|\le t_\varepsilon} |\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma (\Psi(t, x, \xi) - y\cdot\xi)| \le C_{\alpha\beta\gamma}, \quad && |\alpha+\beta+\gamma| \ge 2, \\ &\partial_\xi^2 \Psi(t, x, \xi) = -t(g^{jk}(x))_{j,k} + O(t^2), && |t| \to 0. \end{aligned}$$

As a consequence, since $g^{jk}(x)$ is uniformly elliptic, the phase function $\Psi(t, x, \xi) - y\cdot\xi$ has a unique nondegenerate critical point for all $|t| < t_\varepsilon$ and we can apply the stationary phase method to $K_{\Psi,a^N}(t, x, y)$, provided that $t_\varepsilon > 0$ is small enough. Therefore,

$$|K_{\Psi,a^N}(t, x, y)| \le C|t|^{-d/2}, \quad 0 < |t| \le t_\varepsilon,\ x, \xi \in \mathbb{R}^d. \qquad \square$$

6. Proof of Theorem 1.5

We now give the proof of Theorem 1.5. Suppose that H satisfies Assumption 1.1 with $\mu \ge 0$. In view of Corollary 2.6, (1-4) is a consequence of the following proposition.

Proposition 6.1. *For any symbol $a \in C_0^\infty(\mathbb{R}^{2d})$ and $T > 0$,*

$$\|\mathrm{Op}_h(a) e^{-itH} \varphi\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \le C_T h^{-1/p} \|\varphi\|_{L^2(\mathbb{R}^d)},$$

uniformly with respect to $h \in (0, 1]$, provided that (p, q) satisfies (1-1).

Proof. This proposition follows from the standard WKB approximation for $e^{-itH} \mathrm{Op}_h(a)$ up to time scales of order $1/h$. The proof is essentially the same as that in the case for the Laplace–Beltrami operator on compact manifolds without boundaries [Burq et al. 2004, Section 2]. We omit the details. $\square$

Using this proposition, we have the semiclassical Strichartz estimates with inhomogeneous error terms.

Proposition 6.2. *Let $a \in C_0^\infty(\mathbb{R}^{2d})$. Then, for any $T > 0$ and any (p, q) satisfying the admissible condition (1-1),*

$$\begin{aligned} &\|\mathrm{Op}_h(a) e^{-itH} \varphi\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \\ &\quad \le C_T \|\mathrm{Op}_h(a)\varphi\|_{L^2(\mathbb{R}^d)} + C_T h \|\varphi\|_{L^2(\mathbb{R}^d)} + C h^{-1/2} \|\mathrm{Op}_h(a) e^{-itH}\varphi\|_{L^2([-T,T];L^2(\mathbb{R}^d))} \\ &\qquad + C h^{1/2} \|[\mathrm{Op}_h(a), H] e^{-itH} \varphi\|_{L^2([-T,T];L^2(\mathbb{R}^d))}, \end{aligned}$$

uniformly with respect to $h \in (0, 1]$.

This proposition has been proved by [Bouclet and Tzvetkov 2007] for the case with $V, A \equiv 0$. We give a refinement of this proposition with its proof in Section 7.

Next, we shall prove that if $k(x,\xi)$ satisfies the nontrapping condition (1-3), the missing $1/p$ derivative can be recovered. We first recall the local smoothing effects for Schrödinger operators proved by Doi [2005]. For any $s\in\mathbb{R}$, we set $\mathscr{B}^s:=\{f\in L^2(\mathbb{R}^d):\langle x\rangle^s f,\langle D\rangle^s f\in L^2(\mathbb{R}^d)\}$. Define a symbol $e_s(x,\xi)$ by

$$e_s(x,\xi):=(k(x,\xi)+|x|^2+L(s))^{s/2}\in S((1+|x|+|\xi|)^s,g),$$

where $L(s)>1$ is a large constant depending on s. We denote by E_s its Weyl quantization,

$$E_s f(x)=\mathrm{Op}^w(e_s)f(x)=\frac{1}{(2\pi)^d}\int e^{i(x-y)\cdot\xi}e_s\Big(\frac{x+y}{2},\xi\Big)f(y)\,dy\,d\xi.$$

Then, for any $s\in\mathbb{R}$, there exists $L(s)>0$ such that E_s is a homeomorphism from $\mathscr{B}^{r+s}$ to $\mathscr{B}^r$ for all $r\in\mathbb{R}$, and $(E_s)^{-1}$ is still a Weyl quantization of a symbol in $S((1+|x|+|\xi|)^{-s},g)$; see, [Doi 2005, Lemma 4.1].

Proposition 6.3 (the local smoothing effects [Doi 2005]). *Suppose that $k(x,\xi)$ satisfies the nontrapping condition (1-3) and Assumption 1.4. Then, for any $T>0$ and $\sigma>0$, there exists $C_{T,\sigma}>0$ such that*

$$\|\langle x\rangle^{-1/2-\sigma}E_{1/2}e^{-itH}\varphi\|_{L^2([-T,T];L^2(\mathbb{R}^d))}\le C_{T,\sigma}\|\varphi\|_{L^2(\mathbb{R}^d)}. \tag{6-1}$$

Remark 6.4. (6-1) implies a standard local smoothing effect,

$$\|\langle x\rangle^{-1/2-\sigma}\langle D\rangle^{1/2}e^{-itH}\varphi\|_{L^2([-T,T];L^2(\mathbb{R}^d))}\le C_{T,\sigma}\|\varphi\|_{L^2(\mathbb{R}^d)}. \tag{6-2}$$

Indeed, we compute

$$\begin{aligned}\langle x\rangle^{-1/2-\sigma}\langle D\rangle^{1/2}&=\langle D\rangle^{1/2}\langle x\rangle^{-1/2-\sigma}+[\langle D\rangle^{1/2},\langle x\rangle^{-1/2-\sigma}]\\&=\langle D\rangle^{1/2}(E_{1/2})^{-1}E_{1/2}\langle x\rangle^{-1/2-\sigma}+[\langle D\rangle^{1/2},\langle x\rangle^{-1/2-\sigma}]\\&=\langle D\rangle^{1/2}(E_{1/2})^{-1}(\langle x\rangle^{-1/2-\sigma}E_{1/2}+[E_{1/2},\langle x\rangle^{-1/2-\sigma}])+[\langle D\rangle^{1/2},\langle x\rangle^{-1/2-\sigma}].\end{aligned}$$

It is easy to see that $\langle D\rangle^{1/2}(E_{1/2})^{-1}$, $[E_{1/2},\langle x\rangle^{-1/2-\sigma}]$, and $[\langle D\rangle^{1/2},\langle x\rangle^{-1/2-\sigma}]$ are bounded on $L^2(\mathbb{R}^d)$ since their symbols belong to $S(1,g)$. Therefore, (6-1) implies (6-2).

Proof of (1-5) of Theorem 1.5. It is clear that (1-5) follows from Proposition 6.2, (6-2), and Corollary 2.6, since a is compactly supported with respect to x and $\{a,p\}\in S(\langle\xi\rangle,g)$, where $p=p(x,\xi)$. □

7. Near sharp Strichartz estimates without asymptotic flatness

This section is devoted to proving Theorem 1.6. We may assume $\mu=0$ without loss of generality.

Proposition 7.1. *Let $I\Subset(0,\infty)$ be a relatively compact open interval and $C_0>1$. Then there exist $\delta_0,h_0>0$ such that for any $0<\delta\le\delta_0$, $0<h\le h_0$, $1\le R\le 1/h$, and any symbol $a_h\in S(1,g)$ supported in $\{(x,\xi):R<|x|<C_0/h,\ |\xi|\in I\}$, we have*

$$\|\mathrm{Op}_h(a_h)e^{-itH}\mathrm{Op}_h(a_h)^*\|_{L^1\to L^\infty}\le C_\delta|t|^{-d/2},\quad 0<|t|<\delta hR, \tag{7-1}$$

where $C_\delta>0$ may be taken uniformly with respect to h and R.

Remark 7.2. When $|t| > 0$ in (7-1) is small and independent of R, (7-1) is well known and the proof is given by the standard method of the short-time WKB approximation for $e^{-itH^h/h}\operatorname{Op}_h(a_h)^*$; see, for example, [Burq et al. 2004].

For $h \in (0, 1]$, $R \ge 1$, an open interval $I \Subset (0, \infty)$, and $C_0 > 1$, we set

$$\Gamma(R, h, I) := \{(x, \xi) \in \mathbb{R}^{2d} : R < |x| < C_0/h,\ |\xi| \in I\}.$$

Equation (7-1) is a consequence of the same argument as in the proof of Proposition 3.1 and the following proposition.

Proposition 7.3. *Let $I \Subset I_1 \Subset (0, \infty)$ and $C_0 > 1$. Then there exist $\delta_0, h_0 > 0$ such that the following hold for any $0 < \delta \le \delta_0$, $0 < h \le h_0$, and $1 \le R \le C_0/h$.*

(1) *There exists $\Phi_h(t, x, \xi) \in C^\infty((-\delta R, \delta R) \times \mathbb{R}^{2d})$ such that Φ_h solves the Hamilton–Jacobi equation*

$$\begin{cases} \partial_t \Phi_h(t, x, \xi) = -p_h(x, \partial_x \Phi_h(t, x, \xi)), & |t| < \delta R,\ (x, \xi) \in \Gamma(R/2, h/2, I_1), \\ \Phi_h(0, x, \xi) = x \cdot \xi, & (x, \xi) \in \Gamma(R/2, h/2, I_1). \end{cases} \tag{7-2}$$

Furthermore, we have

$$|\partial_x^\alpha \partial_\xi^\beta (\Phi_h(t, x, \xi) - x \cdot \xi + t p_h(x, \xi))| \le C_{\alpha\beta} R^{-|\alpha|} h |t|^2, \quad \alpha, \beta \in \mathbb{Z}_+^d, \tag{7-3}$$

uniformly with respect to $x, \xi \in \mathbb{R}^d$, $h \in (0, h_0]$, $0 \le R \le C_0/h$, and $|t| < \delta R$.

(2) *For any $a_h \in S(1, g)$ with $\operatorname{supp} a_h \subset \Gamma(R, h, I)$ and any integer $N \ge 0$, we can find $b_h^N(t, \cdot, \cdot) \in S(1, g)$ such that*

$$e^{-it\widetilde{H}^h/h}\operatorname{Op}_h(a_h)^* = J_h(\Phi_h, b_h^N) + Q_{\mathrm{WKB}}(t, h, N),$$

where $J_h(\Phi_h, b_h^N)$ is the h-FIO with phase function Φ_h and amplitude b_h^N, and its distribution kernel satisfies

$$|K_{\mathrm{WKB}}(t, h, x, y)| \le C|th|^{-d/2}, \quad h \in (0, h_0],\ 0 < |t| \le \delta R,\ x, \xi \in \mathbb{R}^d. \tag{7-4}$$

Moreover the remainder $Q_{\mathrm{WKB}}(t, h, N)$ satisfies

$$\|\langle D\rangle^s Q_{\mathrm{WKB}}(t, h, N)\langle D\rangle^s\|_{L^2 \to L^2} \le C_{N,s} h^{N-2s}|t|, \quad h \in (0, h_0],\ |t| \le \delta R.$$

Sketch of proof. The proof is similar to that of Theorem 5.8; in particular, the proof of the second claim is completely the same. Thus, we just outline the construction of Φ_h. We may assume $C_0 = 1$ without loss of generality. Denote by (X_h, Ξ_h) the Hamilton flow generated by p_h. To construct the phase function, the most important step is to study the inverse map of $(x, \xi) \mapsto (X_h(t, x, \xi), \xi)$. Choose an open interval $\tilde{I}_1$ so that $I_1 \Subset \tilde{I}_1 \Subset (0, \infty)$. The following bound was proved in [Mizutani 2013]:

$$|\partial_x^\alpha \partial_\xi^\beta (X_h(t, x, \xi) - x)| + \langle x\rangle |\partial_x^\alpha \partial_\xi^\beta (\Xi_h(t, x, \xi) - \xi)| \le C_{\alpha\beta}\langle x\rangle^{-|\alpha|}|t|$$

for $(x, \xi) \in \Gamma(R/3, h/3, \tilde{I}_1)$ and $|t| \le \delta R$. For sufficiently small $\delta > 0$ and for any fixed $|t| \le \delta R$, this implies

$$|\partial_x X_h(t) - \mathrm{Id}| \le C R^{-1}|t| \le C\delta < \tfrac{1}{2}.$$

By the same argument as that in the proof of Lemma 5.2, the map $(x,\xi)\mapsto(X_h(t,x,\xi),\xi)$ is a diffeomorphism from $\Gamma(R/3,h/3,\tilde I_1)$ onto its range and the corresponding inverse $(x,\xi)\mapsto(Y_h(t,x,\xi),\xi)$ is well-defined for $|t|<\delta R$ and $(x,\xi)\in\Gamma(R/2,h/2,I_1)$. Moreover, $Y_h(t)$ satisfies an estimate like the one for $X_h(t)$:

$$|\partial_x^\alpha\partial_\xi^\beta(Y_h(t,x,\xi)-x)|\le C_{\alpha\beta}\langle x\rangle^{-|\alpha|}|t|,\quad |t|<\delta R,\ (x,\xi)\in\Gamma(R/2,h/2,I_1).$$

We now define Φ_h by

$$\Phi_h(t,x,\xi):=x\cdot\xi+\int_0^t L_h\big(X_h(s,Y(t,x,\xi),\xi),\,\Xi(s,y(t,x,\xi),\xi)\big)\,ds,$$

where $L_h=\xi\cdot\partial_\xi p_h-p_h$. By the standard Hamilton–Jacobi theory, Φ_h solves (7-2). Moreover, using the energy conservation $p_h(x,\partial_x\Phi_h(t))=p_h(Y_h(t),\xi)$ and the above estimates on X_h, Ξ_h, and Y_h, we see that

$$\begin{aligned}|p_h(x,\partial_x\Phi_h(t))-p_h(x,\xi)|&=|p_h(Y_h(t),\xi)-p_h(x,\xi)|\\&\le|Y_h(t)-x|\left|\int_0^\lambda(\partial_x p_h)(\lambda Y_h(t)-(1-\lambda)x,\xi)\,d\lambda\right|\\&\le C|y(t)-x|(h+h^2\langle x\rangle^2)\\&\le Ch|t|\end{aligned}$$

and that

$$|\partial_x^\alpha\partial_\xi^\beta(p_h(x,\partial_x\Phi_h)-p_h(x,\xi))|\le C_{\alpha\beta}\langle x\rangle^{-|\alpha|}h|t|.$$

Using these estimates, we can check that Φ_h satisfies (7-3). Finally, we extend Φ_h to the whole space so that $\Phi_h(t,x,\xi)=x\cdot\xi-tp_h(x,\xi)$ outside $\Gamma(R/3,h/3,\tilde I_1)$. □

Using Proposition 7.1, we obtain a refinement of Proposition 6.2.

Proposition 7.4. *Let $0<R\le 1/h$ and let $a_h\in S(1,g)$ be supported in $\{(x,\xi):R<|x|<1/h,\ |\xi|\in I\}$. Then, for any $T>0$ and (p,q) satisfying the admissible condition (1-1),*

$$\begin{aligned}&\|\mathrm{Op}_h(a_h)e^{-itH}\varphi\|_{L^p([-T,T];L^q(\mathbb{R}^d))}\\&\quad\le C_T\|\mathrm{Op}_h(a_h)\varphi\|_{L^2(\mathbb{R}^d)}+C_Th\|\varphi\|_{L^2(\mathbb{R}^d)}+C_T(hR)^{-1/2}\|\mathrm{Op}_h(a_h)e^{-itH}\varphi\|_{L^2([-T,T];L^2(\mathbb{R}^d))}\\&\qquad+C_T(hR)^{1/2}\|[H,\mathrm{Op}_h(a_h)]e^{-itH}\varphi\|_{L^2([-T,T];L^2(\mathbb{R}^d))},\end{aligned}$$

uniformly with respect to $h\in(0,h_0]$.

Proof. The proof is similar to that of [Bouclet and Tzvetkov 2007, Proposition 5.4]. By time reversal invariance we can restrict our considerations to the interval $[0,T]$. We may assume $T\ge hR$ without loss of generality and split $[0,T]$ as follows: $[0,T]=J_0\cup J_1\cup\cdots\cup J_N$, where $J_j=[jhR,(j+1)hR]$, $0\le j\le N-1$, and $J_N=[T-\delta hR,T]$. For $j=0$, we have the Duhamel formula

$$\mathrm{Op}_h(a_h)e^{-itH}=e^{-itH}\,\mathrm{Op}_h(a_h)-i\int_0^t e^{-i(t-s)H}[\mathrm{Op}_h(a_h),H]e^{-isH}\,ds,\quad t\in J_0.$$

Here we choose $b_h\in S(1,g)$ so that $b_h\equiv 1$ on $\operatorname{supp}a$ and b_h is supported in a sufficiently small neighborhood of $\operatorname{supp} a_h$. By Proposition 7.1, $\mathrm{Op}_h(b_h)e^{-i(t-s)H}\,\mathrm{Op}_h(b_h)^*$ satisfies dispersive estimates (7-1) for $0<|t-s|<\delta hR$ with some $\delta>0$ small enough. Using the Keel–Tao theorem [1998] and the unitarity of e^{-itH}, we then learn that for any interval J_R of size $|J_R|\le 2hR$, the following homogeneous and inhomogeneous Strichartz estimates hold uniformly with respect to $h\in(0,h_0]$:

$$\|\mathrm{Op}_h(b_h)e^{-itH}\varphi\|_{L^p(J_R;L^q(\mathbb{R}^d))}\le C\|\varphi\|_{L^2(\mathbb{R}^d)},\tag{7-5}$$

$$\left\|\int_0^t F(s\in J_R)\,\mathrm{Op}_h(b_h)e^{-i(t-s)H}\,\mathrm{Op}_h(b_h)^*g(s)\,ds\right\|_{L^p(J_R;L^q(\mathbb{R}^d))}\le C\|g\|_{L^1(J_R;L^2(\mathbb{R}^d))},\tag{7-6}$$

where $F(s\in J_R)$ is the characteristic function of J_R and (p,q) satisfies the admissible condition (1-1). On the other hand, using the expansions (2-3) and (2-4), we see that for any $M\ge 0$,

$$\begin{aligned}\mathrm{Op}_h(a_h)&=\mathrm{Op}_h(b_h)\,\mathrm{Op}_h(a_h)+h^M\,\mathrm{Op}_h(r_{1,h})\\&=\mathrm{Op}_h(b_h)^*\,\mathrm{Op}_h(a_h)+h^M\,\mathrm{Op}_h(r_{2,h}),\\ [\mathrm{Op}_h(a_h),H]&=\mathrm{Op}_h(b_h)^*[\mathrm{Op}_h(a_h),H]+h^M\,\mathrm{Op}_h(r_{3,h}),\end{aligned}$$

with some $\{r_{l,h}\}_{h\in(0,1]}$, $l=1,2,3$, which are bounded in $S(\langle x\rangle^{-M}\langle\xi\rangle^{-M},g)$. Therefore, we can write

$$\begin{aligned}&\mathrm{Op}_h(a_h)e^{-itH}\\&\quad=\mathrm{Op}_h(b_h)e^{-itH}\,\mathrm{Op}_h(a_h)-i\int_0^t\mathrm{Op}_h(b_h)e^{-i(t-s)H}\,\mathrm{Op}_h(b_h)^*[\mathrm{Op}_h(a_h),H]e^{-isH}\,ds+Q(t,h,M),\end{aligned}$$

where the remainder $Q(t,h,M)$ satisfies

$$\|Q(t,h,M)\|_{L^2\to L^q}\le C_Mh^{M-1-d(1/2-1/q)},\quad 2\le q\le\infty,$$

uniformly in $h\in(0,1]$. Combining this estimate with (7-5) and (7-6), we obtain

$$\begin{aligned}\|\mathrm{Op}_h(a_h)e^{-itH}\varphi\|_{L^p(J_0;L^q)}&\le C\|\mathrm{Op}_h(a_h)\varphi\|_{L^2}+Ch\|\varphi\|_{L^2}+C\|[\mathrm{Op}_h(a_h),H]e^{-itH}\varphi\|_{L^1(J_0;L^2)}\\&\le C\|\mathrm{Op}_h(a_h)\varphi\|_{L^2}+Ch\|\varphi\|_{L^2}+C(hR)^{1/2}\|[\mathrm{Op}_h(a_h),H]e^{-itH}\varphi\|_{L^2(J_0;L^2)}.\end{aligned}$$

We similarly obtain the same bound for $j=N$:

$$\|\mathrm{Op}_h(a_h)e^{-itH}\varphi\|_{L^p(J_N;L^q)}\le C\|\mathrm{Op}_h(a_h)\varphi\|_{L^2}+Ch\|\varphi\|_{L^2}+C(hR)^{1/2}\|[\mathrm{Op}_h(a_h),H]e^{-itH}\varphi\|_{L^2(J_N;L^2)}.$$

For $j=1,2,\dots,N-1$, taking $\theta\in C_0^\infty(\mathbb{R})$ so that $\theta\equiv 1$ on $[-1/2,1/2]$ and $\operatorname{supp}\theta\subset[-1,1]$, we set $\theta_j(t)=\theta(t/(hR)-j-1/2))$. It is easy to see that $\theta_j\equiv 1$ on J_j and $\operatorname{supp}\theta_j\subset\widetilde{J}_j=J_j+[-hR/2,hR/2]$. We consider $v_j=\theta_j(t)\,\mathrm{Op}_h(a_h)e^{-itH}\varphi$, which solves

$$i\partial_tv_j=Hv_j+\theta_j'\,\mathrm{Op}_h(a_h)e^{-itH}\varphi+\theta_j[\mathrm{Op}_h(a_h),H]e^{-itH}\varphi,\quad v_j|_{t=0}=0.$$

An argument as above and the Duhamel formula then imply that, for any $t\in\widetilde{J}_j$ and $M\ge 0$, v_j satisfies

$$v_j=-i\int_0^t\mathrm{Op}_h(b_h)e^{-i(t-s)H}\,\mathrm{Op}_h(b_h)^*\big(\theta_j'(s)\,\mathrm{Op}_h(a_h)+\theta_j(s)[\mathrm{Op}_h(a_h),H]\big)e^{-isH}\varphi\,ds+\widetilde{Q}(t,h,M),$$

where the remainder $\widetilde{Q}(t,h,M)$ satisfies

$$\|\widetilde{Q}(t,h,M)\|_{L^2\to L^q}\le C_Mh^{M-1-d(1/2-1/q)},\quad 2\le q\le\infty,$$

uniformly in $h \in (0,1]$ and $t \in \widetilde{J}_j$. Taking $M \geq 0$ large enough, we learn

$$\begin{aligned}\|v_j\|_{L^p(J_j;L^q)} &\leq Ch^2\|\varphi\|_{L^2} + C(hR)^{-1}\|\mathrm{Op}_h(a_h)e^{-itH}\varphi\|_{L^1(\widetilde{J}_j;L^2)} + C\|[\mathrm{Op}_h(a_h), H]e^{-itH}\varphi\|_{L^1(\widetilde{J}_j;L^2)}\\ &\leq Ch^2\|\varphi\|_{L^2} + C(hR)^{-1/2}\|\mathrm{Op}_h(a_h)e^{-itH}\varphi\|_{L^2(\widetilde{J}_j;L^2)} + C(hR)^{1/2}\|[\mathrm{Op}_h(a_h), H]e^{-itH}\varphi\|_{L^2(\widetilde{J}_j;L^2)}.\end{aligned}$$

Since $N \leq T/h$ and $p \geq 2$, summing over $j = 0, 1, \dots, N$, we have the assertion by Minkowski's inequality. □

Proof of Theorem 1.6. In view of Corollary 2.6, Theorem 1.5, and Proposition 3.2, it suffices to show that, for any $a_h \in S(1, g)$ with

$$\operatorname{supp} a_h \in \{(x,\xi) : 2 \leq |x| \leq 1/h,\ |\xi| \in I\}$$

and any $\varepsilon > 0$,

$$\sum_h \|\mathrm{Op}_h(a_h)e^{-itH}f(h^2H)\varphi\|^2_{L^p([-T,T];L^q)} \leq C_{T,\varepsilon}\|\langle H\rangle^{\varepsilon}\varphi\|^2_{L^2}.$$

Let us consider a dyadic partition of unity:

$$\sum_{1\leq j\leq j_h} \chi(2^{-j}x) = 1, \quad 2 \leq |x| \leq 1/h,$$

where $\chi \in C_0^\infty(\mathbb{R}^d)$ with

$$\operatorname{supp}\chi \subset \{1/2 < |x| < 2\}$$

and $j_h \leq [\log(1/h)] + 1$. We set

$$\chi_j(x) = \chi(2^{-j}x).$$

Proposition 7.4 then implies

$$\begin{aligned}\|\chi_j \mathrm{Op}_h(a_h)e^{-itH}\varphi\|_{L^p([-T,T];L^q)} &\leq C_T\|\chi_j \mathrm{Op}_h(a_h)\varphi\|_{L^2} + C_T h\|\varphi\|_{L^2} + C_T(h2^j)^{-1/2}\|\chi_j \mathrm{Op}_h(a_h)e^{-itH}\varphi\|_{L^2([-T,T];L^2)}\\ &\quad + C_T(h2^j)^{1/2}\|[\chi_j \mathrm{Op}_h(a_h), H]e^{-itH}\varphi\|_{L^2([-T,T];L^2)}.\end{aligned}$$

Since $2^{j-1} \leq |x| \leq 2^{j+1}$ and $|x| \leq 1/h$ on $\operatorname{supp}\chi_j a_h$, we have, for any $\varepsilon \geq 0$,

$$(h2^j)^{-1/2}\|\chi_j \mathrm{Op}_h(a_h)e^{-itH}\varphi\|_{L^2([-T,T];L^2)} \leq C\|\chi_j\langle x\rangle^{-1/2-\varepsilon}h^{-1/2-\varepsilon}\mathrm{Op}_h(a_h)e^{-itH}\varphi\|_{L^2([-T,T];L^2)}.$$

Since $\{\chi_j a_h, p\} \in S(\langle x\rangle^{-1}\langle\xi\rangle, g)$, we similarly obtain

$$\begin{aligned}(h2^j)^{1/2}\|\chi_j[\mathrm{Op}_h(a_h), H]e^{-itH}\varphi\|_{L^2([-T,T];L^2)}\\ \leq \|\widetilde{\chi}_j\langle x\rangle^{-1/2-\varepsilon}h^{-1/2-\varepsilon}\mathrm{Op}_h(b_h)e^{-itH}\varphi\|_{L^2([-T,T];L^2)} + C_T h\|\varphi\|_{L^2},\end{aligned}$$

where $\widetilde{\chi}_j(x) = \widetilde{\chi}(2^{-j}x)$ for some $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^d)$ satisfying $\widetilde{\chi} \equiv 1$ on $[1/2, 2]$ and $\operatorname{supp}\widetilde{\chi} \subset [1/4, 4]$, and $b_h \in S(1, g)$ is supported in a neighborhood of $\operatorname{supp} a_h$ so that $b_h \equiv 1$ on $\operatorname{supp} a_h$. Summing over

$1 \le j \le j_h$ and using the local smoothing effect (6-2), since $p, q \ge 2$, we obtain

$$\begin{aligned}
&\|\mathrm{Op}_h(a_h)e^{-itH}\varphi\|^2_{L^p([-T,T];L^q)} \\
&\qquad \le \sum_{1\le j\le j_h} \|\chi_j \,\mathrm{Op}_h(a_h)e^{-itH}\varphi\|^2_{L^p([-T,T];L^q)} \\
&\qquad \le C_T \sum_{1\le j\le j_h} (\|\chi_j \,\mathrm{Op}_h(a_h)\varphi\|^2_{L^2} + h\|\varphi\|^2_{L^2}) \\
&\qquad\qquad + C \sum_{1\le j\le j_h} \|\widetilde{\chi}_j \langle x\rangle^{-1/2-\varepsilon} h^{-1/2-\varepsilon}\,\mathrm{Op}_h(a_h+b_h)e^{-itH}\varphi\|^2_{L^2([-T,T];L^2)} \\
&\qquad \le C_T\|\varphi\|^2_{L^2} + C\|\langle x\rangle^{-1/2-\varepsilon}h^{-1/2-\varepsilon}\,\mathrm{Op}_h(a_h+b_h)e^{-itH}\varphi\|^2_{L^2([-T,T];L^2)} \\
&\qquad \le C_{T,\varepsilon}h^{-2\varepsilon}\|\varphi\|^2_{L^2},
\end{aligned}$$

which implies

$$\sum_h \|\mathrm{Op}_h(a_h)e^{-itH}f(h^2H)\varphi\|^2_{L^p([-T,T];L^q)} \le C_{T,\varepsilon}\sum_h h^{-2\varepsilon}\|f(h^2H)\varphi\|^2_{L^2} \le C_{T,\varepsilon}\|\langle H\rangle^{\varepsilon/2}\varphi\|^2_{L^2}.$$

This completes the proof. □

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