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GLOBAL WELL-POSEDNESS FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH DERIVATIVE IN ENERGY SPACE

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In this paper, we prove that there exists some small $\varepsilon_* > 0$ such that the derivative nonlinear Schrödinger equation (DNLS) is globally well-posed in the energy space, provided that the initial data $u_0 \in H^1(\mathbb{R})$ satisfies $\|u_0\|_{L^2} < \sqrt{2\pi} + \varepsilon_*$. This result shows us that there are no blow-up solutions whose masses slightly exceed 2π, even if their energies are negative. This phenomenon is much different from the behavior of the nonlinear Schrödinger equation with critical nonlinearity. The technique used is a variational argument together with the momentum conservation law. Further, for the DNLS on the half-line $\mathbb{R}^+$, we show the blow-up for the solution with negative energy.

1. Introduction

We study the following Cauchy problem of the nonlinear Schrödinger equation with derivative (DNLS):

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\lambda\partial_x(|u|^2 u), & t \in \mathbb{R},\ x \in \mathbb{R}, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \tag{1-1}$$

where $\lambda \in \mathbb{R}$. It arises from studying the propagation of circularly polarized Alfvén waves in magnetized plasma with a constant magnetic field; see [Mio et al. 1976; Mjolhus 1976; Sulem and Sulem 1999] and the references therein.

This equation is L^2-critical in the sense that both the equation and the L^2-norm are invariant under the scaling transform

$$u_\alpha(t, x) = \alpha^{1/2} u(\alpha^2 t, \alpha x), \quad \alpha > 0.$$

It has the same scaling invariance as the quintic nonlinear Schrödinger equation,

$$i\partial_t u + \partial_x^2 u + \mu|u|^4 u = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R},$$

and the quintic generalized Korteweg–de Vries equation,

$$\partial_t u + \partial_x^3 u + \mu\partial_x(u^5) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}.$$

One may always take $\lambda = 1$ in (1-1), since the general case can be reduced to this case by the following two transforms. First, we apply the transform

$$u(t, x) \mapsto \bar{u}(-t, x),$$

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then reduce the equation to the case of $\lambda > 0$. Then we take the rescaling transform

$$u(t,x) \mapsto \frac{1}{\sqrt{\lambda}} u(t,x)$$

and reduce it to the case of $\lambda = 1$. So in this sense, (1-1) can always be regarded as the focusing equation. From now on, we always assume that $\lambda = 1$ in (1-1).

The H^1-solution of (1-1) obeys three conservation laws. The first is the conservation of the mass

$$M(u(t)) := \int_{\mathbb{R}} |u(t)|^2 \, dx = M(u_0); \tag{1-2}$$

the second is the conservation of energy

$$E_D(u(t)) := \int_{\mathbb{R}} \bigl(|u_x(t)|^2 + \tfrac{3}{2} \operatorname{Im} |u(t)|^2 u(t) \overline{u_x(t)} + \tfrac{1}{2} |u(t)|^6 \bigr) \, dx = E_D(u_0); \tag{1-3}$$

and the third is the conservation of momentum (see (3-4) below),

$$P_D(u(t)) := \operatorname{Im} \int_{\mathbb{R}} \bar{u}(t) u_x(t) \, dx - \frac{1}{2} \int_{\mathbb{R}} |u(t)|^4 \, dx = P_D(u_0). \tag{1-4}$$

Local well-posedness for the Cauchy problem (1-1) is well understood. It was proved for the energy space $H^1(\mathbb{R})$ in [Hayashi 1993; Hayashi and Ozawa 1992; 1994]; see also [Guo and Tan 1991] for an earlier result in smooth spaces. For rough data below the energy space, Takaoka [1999] proved local well-posedness in $H^s(\mathbb{R})$ for $s \geq \frac{1}{2}$. This result was shown to be sharp in the sense that the flow map fails to be uniformly C^0 for $s < \frac{1}{2}$; see [Biagioni and Linares 2001; Takaoka 2001].

The global well-posedness for (1-1) has also been widely studied. By using mass and energy conservation laws, and by developing the gauge transformations, Hayashi and Ozawa [Hayashi and Ozawa 1994; Ozawa 1996] proved that the problem (1-1) is globally well-posed in energy space $H^1(\mathbb{R})$ under the condition

$$\|u_0\|_{L^2} < \sqrt{2\pi}. \tag{1-5}$$

Further, for initial data of regularity below the energy space, Colliander et al. [2001; 2002] proved the global well-posedness for (1-1) in $H^s(\mathbb{R})$ for $s > \frac{1}{2}$, under the condition (1-5). Recently, Miao, Wu, and Xu [Miao et al. 2011] proved that (1-1) is globally well-posed in the critical space $H^{1/2}(\mathbb{R})$, also under the condition (1-5). For other work on the DNLS in the periodic case, see for example [Grünrock and Herr 2008; Herr 2006; Nahmod et al. 2012; Win 2010].

As mentioned above, all the results on global existence for initial data were obtained under the assumption (1-5). Since $\sqrt{2\pi}$ is just the mass of the ground state of the corresponding elliptic problem, the condition (1-5) was naturally used to keep the energy positive; see [Colliander et al. 2001; Miao et al. 2011] for examples. Now one may wonder what happens to the well-posedness for the solution when (1-5) is not fulfilled. Our first main result in this paper is to improve the assumption (1-5) and obtain the global well-posedness as follows.

Theorem 1.1. *There exists a small $\varepsilon_* > 0$ such that, for any $u_0 \in H^1(\mathbb{R})$ with*

$$\int_{\mathbb{R}} |u_0(x)|^2 \, dx < 2\pi + \varepsilon_*, \tag{1-6}$$

the Cauchy problem (1-1) ($\lambda = 1$) *is globally well-posed in $H^1(\mathbb{R})$ and the solution u satisfies*

$$\|u\|_{L^\infty_t H^1_x} \le C(\varepsilon_*, \|u_0\|_{H^1}).$$

The technique used to prove Theorem 1.1 is a variational argument together with the momentum and energy conservation laws. The key ingredient is the momentum conservation law, rather than the energy conservation law, upon which many (subcritical) problems rely when studying the global existence. We argue by contradiction. Suppose that the solution of (1-1) blows up at finite/infinite time T and t_n is a time sequence tending to T such that $u(t_n)$ tends to infinity in $H^1(\mathbb{R})$ norm. Then, thanks to the energy conservation law and a variational lemma from Merle [2001], $u(t_n)$ is close to the ground state Q (see below for its definition) up to a spatial transformation, a phase rotation, and a scaling transformation. On the one hand, since $u(t_n)$ blows up at T, the scaling parameter λ_n decays to zero; on the other hand, the conservation of momentum prevents λ_n from tending to zero. This leads to a contradiction.

As mentioned above, Theorem 1.1 improves the smallness of the L^2-norm of the initial data of the previous works on global existence [Hayashi and Ozawa 1994; Ozawa 1996]. More importantly, it reveals some special features of the derivative nonlinear Schrödinger equation. As discussed before, the smallness condition (1-5) in the previous works is imposed to guarantee the positivity of the energy $E_D(u(t))$. Indeed, by using a variant gauge transformation

$$v(t,x) := e^{-(3/4)i \int_{-\infty}^{x} |u(t,y)|^2 \, dy} u(t,x), \tag{1-7}$$

the energy is deduced to be

$$E_D(u(t)) = \|v_x(t)\|^2_{L^2_x} - \tfrac{1}{16}\|v(t)\|^6_{L^6_x} := E(v(t)), \tag{1-8}$$

and then the positivity of $E(v)$ is followed by the sharp Gagliardo–Nirenberg inequality (see [Weinstein 1982/83])

$$\|f\|^6_{L^6} \le \frac{4}{\pi^2}\|f\|^4_{L^2}\|f_x\|^2_{L^2}. \tag{1-9}$$

Once the mass is greater than 2π, the positive energy can not be maintained. To see this, we first make use of the gauge transformation (1-7), and rewrite (1-1) as

$$i\partial_t v + \partial_x^2 v = \frac{i}{2}|v|^2 v_x - \frac{i}{2} v^2 \bar{v}_x - \tfrac{3}{16}|v|^4 v. \tag{1-10}$$

Then there exists a standing wave $e^{it}Q$ of (1-10), where Q is the unique (up to some symmetries) positive solution of the elliptic equation

$$-Q_{xx} + Q - \tfrac{3}{16}Q^5 = 0.$$

This leads to the standing wave solution corresponding to (1-1),

$$R(t,x) := e^{it + (3/4)i \int_{-\infty}^{x} Q^2 \, dy} Q(x).$$

So on the one hand, as a byproduct, our result implies the stability of the standing wave solution, which has been proved by Colin and Ohta [2006]. On the other hand,

$$\|Q\|_{L^2} = \sqrt{2\pi}, \quad E(Q) = 0,$$

and the Fréchet derivation of the functional $E(v)$ at Q satisfies $\delta E(Q) \cdot Q = -2\pi < 0$. These relations imply that there exists a u_0 such that u_0 obeys (1-6) and $E_D(u_0) < 0$. Therefore, there indeed exist global solutions with negative energy, as stated in Theorem 1.1. Obviously this is much different from the focusing, quintic nonlinear Schrödinger equation (3-1) and focusing, quintic generalized Korteweg–de Vries equation (3-2). For (3-1), Ogawa and Tsutsumi [1991] proved that the solutions with the initial data belonging to $H^1(\mathbb{R})$ and negative energy must blow up in finite time; for (3-2), Martel and Merle [Martel and Merle 2002; Merle 2001] proved that the solutions with the initial data belonging to $H^1(\mathbb{R})$, negative energy, and obeying some further decay conditions blow up in finite time. In Section 3 below we will discuss some differences among these three equations, in particular from the viewpoint of the virial arguments.

Moreover, the situation of the Cauchy problem and the initial boundary value problem of (1-1) are much different. We consider the following Cauchy–Dirichlet problem of the nonlinear Schrödinger equation with derivative on the half-line $\mathbb{R}^+$:

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x(|u|^2 u), & t \in \mathbb{R},\ x \in (0, +\infty), \\ u(0, x) = u_0(x), \\ u(t, 0) = 0. \end{cases} \tag{1-11}$$

We show that under some assumptions, the solution must blow up in finite time if its energy is negative.

Theorem 1.2. *Let $u_0 \in H^2(\mathbb{R}^+)$ and $xu_0 \in L^2(\mathbb{R}^+)$, and let u be the corresponding solution of* (1-11) *which exists on the* (*right*) *maximal lifetime* $[0, T_*)$. *If $E_D(u_0) < 0$, then $T_* < \infty$. Moreover, there exists a constant $C = C(u_0) > 0$ such that*

$$\|u_x(t, x)\|_{L^2(\mathbb{R}^+)} \geq \frac{C}{\sqrt{T_* - t}} \to \infty \quad \text{as } t \nearrow T^*.$$

For related results on the blow-up solution to the DNLS equation on bounded domain with the Dirichlet condition, see [Tan 2004].

Lastly, we remark that it remains open for the DNLS equation (1-1) whether there exists an $H^1(\mathbb{R})$ initial data of much larger L^2-norm such that the corresponding solution blows up in finite time. Moreover, it may be interesting to study the existence of global rough solutions when the condition (1-5) on initial data is relaxed.

This paper is organized as follows. In Section 2, we present the gauge transformation and prove the virial identities of DNLS. In Section 3, we discuss the differences among the DNLS, the quintic NLS, and the quintic gKdV equations. In Section 4, we study the initial boundary value problem of the DNLS on the half-line and give the proof of Theorem 1.2. In Section 5, we prove Theorem 1.1.

2. Gauge transformations, virial identities

Gauge transformations. The gauge transformation is an important and very nice tool to study the nonlinear Schrödinger equation with derivative [Hayashi 1993; Hayashi and Ozawa 1992; 1994]. It gives some improvement of the nonlinearity. In this subsection, we present the various gauge transformations and their properties. See [Colliander et al. 2001; Ozawa 1996] for more details. We define

$$\mathcal{G}_a u(t,x) = e^{ia\int_{-\infty}^{x}|u(t,y)|^2\,dy}u(t,x).$$

Then $\mathcal{G}_a\mathcal{G}_{-a} = \mathrm{Id}$, the identity transform. For any function f,

$$\partial_x \mathcal{G}_a f = e^{ia\int_{-\infty}^{x}|f(t,y)|^2\,dy}(ia|f|^2 f + f_x). \tag{2-1}$$

Further, we have the following.

Lemma 2.1. *If u is the solution of* (1-1) (*where* $\lambda = 1$), $v = \mathcal{G}_a u$ *is the solution of the equation*

$$i\partial_t v + \partial_x^2 v - i2(a+1)|v|^2 v_x - i(2a+1)v^2\bar{v}_x + \tfrac{1}{2}a(2a+1)|v|^4 v = 0.$$

Moreover,

$$E_D(u) = \|\partial_x\mathcal{G}_a u\|_2^2 + (2a+\tfrac{3}{2})\,\mathrm{Im}\int_{\mathbb{R}}|\mathcal{G}_a u|^2\mathcal{G}_a u\cdot\partial_x\overline{\mathcal{G}_a u}\,dx + (a^2+\tfrac{3}{2}a+\tfrac{1}{2})\int_{\mathbb{R}}|\mathcal{G}_a u|^6\,dx.$$

The proof of this lemma follows from a direct computation and is omitted.

To understand how the gauge transform improves the nonlinearity in the present form (1-1), we introduce the following two transforms used in [Hayashi and Ozawa 1994; Ozawa 1996]. Let

$$\phi = \mathcal{G}_{-1}u, \quad \psi = \mathcal{G}_{1/2}\partial_x\mathcal{G}_{-1/2}u.$$

Then (ϕ, ψ) solves the following system of nonlinear Schrödinger equations:

$$\begin{cases} i\partial_t\phi + \partial_x^2\phi = -i\phi^2\bar{\psi}, \\ i\partial_t\psi + \partial_x^2\psi = \psi^2\bar{\phi}. \end{cases} \tag{2-2}$$

Compared with the original equation (1-1), the system above has no loss of derivatives. Thus it is much more convenient to get the local solvability of (1-1) for suitable smooth data by considering the system (2-2) instead.

As mentioned above, it is convenient to consider $v = \mathcal{G}_{-3/4}u$. Then, by Lemma 2.1, the equation (1-1) of u reduces to (1-10), that is,

$$i\partial_t v + \partial_x^2 v = \tfrac{1}{2}i|v|^2 v_x - \tfrac{1}{2}2v^2\bar{v}_x - \tfrac{3}{16}|v|^4 v.$$

Moreover, the energy $E_D(u)$ in (1-3) is changed into $E(v)$ in (1-8). In the sequel we shall consider (1-10) and the energy (1-8) of v instead.

Virial identities. In this subsection, we discuss some virial identities for the nonlinear Schrödinger equation with derivative. Formally, one may find that the virial quantity of v is similar to that of the mass-critical nonlinear Schrödinger equation. However, it is in fact the difference that gives the different conclusions of these two equations. Let $\psi = \psi(x)$ be a smooth real function. Define

$$I(t) = \int_{\mathbb{R}} \psi |v(t)|^2 \, dx, \tag{2-3}$$

$$J(t) = 2 \operatorname{Im} \int_{\mathbb{R}} \psi \bar{v}(t) v_x(t) \, dx + \frac{1}{2} \int \psi |v(t)|^4 \, dx. \tag{2-4}$$

Lemma 2.2. *Let v be the solution of* (1-10) *with $v(0) = v_0 \in H^1(\mathbb{R})$, and let $\psi \in C^3$. Then*

$$I'(t) = 2 \operatorname{Im} \int_{\mathbb{R}} \psi' \bar{v}(t) v_x(t) \, dx, \tag{2-5}$$

$$J'(t) = 4 \int_{\mathbb{R}} \psi' \big(|v_x(t)|^2 - \tfrac{1}{16} |v(t)|^6\big) \, dx - \int_{\mathbb{R}} \psi''' |v(t)|^2 \, dx. \tag{2-6}$$

Proof. Employing the gauge transform

$$w(t,x) := \mathcal{G}_{-1/2} u(t,x) = \mathcal{G}_{1/4} v(t,x),$$

by Lemma 2.1, w obeys the equation

$$i w_t + w_{xx} = i |w|^2 w_x.$$

Moreover, since $v(t,x) = \mathcal{G}_{-1/4} w(t,x)$, by (2-1),

$$\partial_x v(t,x) = e^{-i(1/4)\int_{-\infty}^x |w(t,y)|^2 \, dy} \big(-\tfrac{1}{4} i |w|^2 w + w_x\big).$$

Thus we have

$$I(t) = \int_{\mathbb{R}} \psi |w(t)|^2 \, dx \quad \text{and} \quad J(t) = 2 \operatorname{Im} \int_{\mathbb{R}} \psi \bar{w}(t) w_x(t) \, dx.$$

Now, by a direct computation, we get

$$\begin{aligned} I'(t) &= 2 \operatorname{Re} \int_{\mathbb{R}} \psi \bar{w}(t,x) \partial_t w(t,x) \, dx = 2 \operatorname{Re} \int_{\mathbb{R}} \psi \bar{w} (i w_{xx} + |w|^2 w_x) \, dx \\ &= 2 \operatorname{Im} \int_{\mathbb{R}} \psi' \bar{w} w_x \, dx - \frac{1}{2} \int_{\mathbb{R}} \psi' |w|^4 \, dx. \end{aligned} \tag{2-7}$$

Applying (2-1) again,

$$\partial_x w(t,x) = e^{(1/4) i \int_{-\infty}^x |v(t,y)|^2 \, dy} \big(\tfrac{1}{4} i |v|^2 v + v_x\big). \tag{2-8}$$

This together with (2-7) gives (2-5). Now we turn to (2-6). For this, we get

$$\begin{aligned} J'(t) &= 2 \operatorname{Im} \int_{\mathbb{R}} \psi \bar{w}_t(t,x) w_x(t,x) \, dx + 2 \operatorname{Im} \int_{\mathbb{R}} \psi \bar{w}(t,x) w_{xt}(t,x) \, dx \\ &= -4 \operatorname{Im} \int_{\mathbb{R}} \psi w_t \bar{w}_x \, dx - 2 \operatorname{Im} \int_{\mathbb{R}} \psi' \bar{w} w_t \, dx \end{aligned}$$

$$
\begin{aligned}
&= -4\operatorname{Im}\int_{\mathbb{R}} \psi \bar{w}_x (i w_{xx} + |w|^2 w_x)\,dx - 2\operatorname{Im}\int_{\mathbb{R}} \psi' \bar{w}(i w_{xx} + |w|^2 w_x)\,dx \\
&= -4\operatorname{Re}\int_{\mathbb{R}} \psi \bar{w}_x w_{xx}\,dx - 2\operatorname{Re}\int_{\mathbb{R}} \psi' \bar{w} w_{xx}\,dx - 2\operatorname{Im}\int_{\mathbb{R}} \psi' |w|^2 \bar{w} w_x\,dx \\
&= 4\int_{\mathbb{R}} \psi' |w_x|^2\,dx + 2\operatorname{Re}\int_{\mathbb{R}} \psi'' \bar{w} w_x\,dx - 2\operatorname{Im}\int_{\mathbb{R}} \psi' |w|^2 \bar{w} w_x\,dx \\
&= 4\int_{\mathbb{R}} \psi' |w_x|^2\,dx - \int_{\mathbb{R}} \psi''' |w|^2\,dx - 2\operatorname{Im}\int_{\mathbb{R}} \psi' |w|^2 \bar{w} w_x\,dx.
\end{aligned} \tag{2-9}
$$

Now, using (2-8), we have

$$|w_x|^2 = |v_x|^2 + \tfrac{1}{2}\operatorname{Im}(|v|^2 \bar{v} v_x) + \tfrac{1}{16}|v|^6$$

and

$$|w|^2 = |v|^2, \quad \operatorname{Im}(|w|^2 \bar{w} w_x) = \operatorname{Im}(|v|^2 \bar{v} v_x) + \tfrac{1}{4}|v|^6.$$

These insert into (2-9) and we obtain (2-6). □

3. A comparison between DNLS, NLS-5, and gKdV-5

In this section, we discuss the nonlinear Schrödinger equation with derivative (1-10), the focusing, quintic nonlinear Schrödinger equation (NLS-5), which reads

$$i\partial_t u + \partial_x^2 u + \tfrac{3}{16}|u|^4 u = 0, \tag{3-1}$$

and the focusing, quintic generalized Korteweg–de Vries equation (gKdV-5),

$$\partial_t u + \partial_x^3 u + \tfrac{3}{16}\partial_x(u^5) = 0. \tag{3-2}$$

The first two equations have the same standing wave solutions as $e^{it}Q$, and the last one has a traveling wave solution $Q(x-t)$. These three equations have the same energies in the form of (1-8). So by the sharp Gagliardo–Nirenberg inequality, all of them are globally well-posed in $H^1(\mathbb{R})$ when the initial data $\|u_0\|_{L^2} < \|Q\|_{L^2} = \sqrt{2\pi}$.

Now we continue to discuss the difference between the first equation (DNLS) and the last two (NLS-5, gKdV-5).

First of all, we give some products from Lemma 2.2. We always assume that v is smooth enough. Taking $\psi = x$ and $\psi = x^2$, by (2-5), we have

$$\frac{d}{dt}\int_{\mathbb{R}} x|v(t)|^2\,dx = 2\operatorname{Im}\int_{\mathbb{R}} \bar{v}(t) v_x(t)\,dx$$

and

$$\frac{d}{dt}\int_{\mathbb{R}} x^2|v(t)|^2\,dx = 4\operatorname{Im}\int_{\mathbb{R}} x\bar{v}(t) v_x(t)\,dx, \tag{3-3}$$

respectively. Note that these two identities resemble the corresponding identity of the mass-critical nonlinear Schrödinger equation (3-1).

Now we take $\psi = 1$ in (2-6), which gives the momentum conservation law,

$$P(v(t)) := \operatorname{Im} \int_{\mathbb{R}} \bar{v}(t) v_x(t)\,dx + \frac{1}{4} \int_{\mathbb{R}} |v(t)|^4\,dx = P(v_0). \tag{3-4}$$

Then, taking $\psi = x$, we have

$$\frac{d}{dt}\left(2 \operatorname{Im} \int_{\mathbb{R}} x \bar{v}(t) v_x(t)\,dx + \frac{1}{2} \int_{\mathbb{R}} x |v(t)|^4\,dx\right) = 4E(v_0). \tag{3-5}$$

This equality is different from the situation of the mass-critical nonlinear Schrödinger equation (3-1). More precisely, for the solution u of (3-1) with the initial data u_0, we have

$$\frac{d}{dt}\left(2 \operatorname{Im} \int_{\mathbb{R}} x \bar{u}(t) u_x(t)\,dx\right) = 4E(u_0). \tag{3-6}$$

Compared with the identity (3-6), there is an additional term $\frac{1}{2}\int x|v(t)|^4\,dx$ in (3-5). Indeed, for the solution of (3-1), combining with the same identity as in (3-3), one has

$$\frac{d^2}{dt^2} \int_{\mathbb{R}} x^2 |u(t)|^2\,dx = 8E(u_0). \tag{3-7}$$

But this does not hold for the solution of (1-10). The "surplus" term $\frac{1}{2}\int x|v(t)|^4\,dx$ in (3-5) breaks the convexity of the variance. It is precisely this difference that leads to the distinct phenomena of the solutions of these two equations, at least at the technical level.

Using the virial identity (3-7), Glassey [1977] proved that the solution u of the mass-critical nonlinear Schrödinger equation

$$\partial_t u + \Delta u + |u|^{4/N} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

blows up in finite time when $u_0 \in H^1(\mathbb{R}^N)$, $xu_0 \in L^2(\mathbb{R}^N)$, and $E(u_0) < 0$. Further, in the 1D case, Ogawa and Tsutsumi [1991] proved that the solutions of (3-1) blow up in finite time when $u_0 \in H^1(\mathbb{R})$ and $E(u_0) < 0$. See also [Du et al. 2013; Holmer and Roudenko 2010; Glangetas and Merle 1995; Nawa 1999], where all the solutions of the nonlinear Schrödinger equations with power nonlinearity blow up in finite time or infinite time if their energies are negative. However, Theorem 1.1 depicts a different scene, where there exist global and uniformly bounded solutions even if $E(v_0) < 0$.

The situation is also different from the mass-critical generalized KdV equation (3-2). The latter also has virial identity

$$\frac{d}{dt} \int_{\mathbb{R}} (x+t)|u(t)|^2\,dx = \int_{\mathbb{R}} u^2\,dx - 3 \int_{\mathbb{R}} |u_x|^2\,dx - \frac{1}{3} \int_{\mathbb{R}} |u|^6\,dx.$$

The blow-up of the solutions to (3-2) also occurs when the initial data u_0 satisfies $E(u_0) < 0$, (1-6), and some decay conditions; see [Martel and Merle 2002; Merle 2001].

4. Blow-up for the DNLS on the half line

In this section, we use the virial identities obtained in Lemma 2.2 to study the blow-up solutions for the nonlinear Schrödinger equation with derivative on the half line. Consider the problem (1-11), and set

$$v(t,x)=\exp\biggl(-\tfrac{3}{4}i\int_0^x |u(t,y)|^2\,dy\biggr)u(t,x),$$

Using the gauge transformation, we see that v is the solution of

$$\begin{cases} i\partial_t v+\partial_x^2 v=\frac{1}{2}i|v|^2v_x-\frac{1}{2}iv^2\bar{v}_x-\frac{3}{16}|v|^4v, & t\in\mathbb{R},\ x\in(0,+\infty),\\ v(0,x)=v_0(x),\\ v(t,0)=0. \end{cases} \tag{4-1}$$

Note that after replacing the integral domain $\mathbb{R}$ by $\mathbb{R}^+$, the energy conservation law and all of the virial identities obtained in Section 2 also hold true for v.

Now using the virial identities and Glassey's argument [1977], we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let v be the solution to (4-1). Define

$$I(t)=\int_0^\infty x^2|v(t,x)|^2\,dx.$$

Then, by the identity analogous to (3-3), we have

$$I'(t)=4\operatorname{Im}\int_0^\infty x\bar{v}(t)v_x(t)\,dx=2\biggl(2\operatorname{Im}\int_0^\infty x\bar{v}(t)v_x(t)\,dx+\frac{1}{2}\int_0^\infty x|v(t)|^4\,dx\biggr)-\int_0^\infty x|v(t)|^4\,dx.$$

Now, by the identity analogous to (3-5), we get

$$\frac{d}{dt}\biggl(2\operatorname{Im}\int_0^\infty x\bar{v}(t)v_x(t)\,dx+\frac{1}{2}\int_0^\infty x|v(t)|^4\,dx\biggr)=4E(v_0).$$

Therefore, using these two identities, we obtain

$$I''(t)=8E(v_0)-\frac{d}{dt}\int_0^\infty x|v(t)|^4\,dx.$$

Integrating in time twice, we have

$$\begin{aligned} I(t)&=I(0)+I'(0)t+\int_0^t\int_0^s I''(\tau)\,d\tau\,ds\\ &=I(0)+I'(0)t+\int_0^t\int_0^s\biggl(8E(v_0)-\frac{d}{d\tau}\int_0^\infty x|v(\tau)|^4\,dx\biggr)d\tau\,ds\\ &=4E(v_0)t^2+\biggl(I'(0)+\int_0^\infty x|v_0|^4\,dx\biggr)t+I(0)-\int_0^t\int_0^\infty x|v(s)|^4\,dx\,ds\\ &\le 4E(v_0)t^2+\biggl(I'(0)+\int_0^\infty x|v_0|^4\,dx\biggr)t+I(0). \end{aligned} \tag{4-2}$$

Since $E(v_0)=E_D(u_0)<0$, there exists a finite time $T_*>0$ such that $I(T_*)=0$,

$$I(t)>0 \quad \text{for } 0<t<T_*,$$

and

$$I(t)=O(T_*-t) \quad \text{as } t\nearrow T_*.$$

Note that

$$\int_0^\infty |v_0(x)|^2\,dx = \int_0^\infty |v(t,x)|^2\,dx = -2\,\mathrm{Re}\int_0^\infty x v(t,x)\overline{v_x}(t,x)\,dx$$
$$\le 2\|xv(t,x)\|_{L^2_x(\mathbb{R}^+)}\|v_x(t,x)\|_{L^2_x(\mathbb{R}^+)} = 2\sqrt{I(t)}\|v_x(t,\cdot)\|_{L^2(\mathbb{R}^+)}.$$

Then there is a constant $C = C(v_0) > 0$ such that

$$\|v_x(t,\cdot)\|_{L^2(\mathbb{R}^+)} \ge \frac{\int_0^\infty |v_0(x)|^2\,dx}{2\sqrt{I(t)}} \ge \frac{C}{\sqrt{T_* - t}}, \tag{4-3}$$

and the right-hand side goes to ∞ as $t \nearrow T^*$. Therefore, $v(t)$ blows up at time $T_* < +\infty$. Since

$$v_x = \exp\left(-\tfrac{3}{4}i\int_0^x |u(t,y)|^2\,dy\right)(-i\tfrac{3}{4}|u|^2u + u_x),$$

by the Gagliardo–Nirenberg inequality and the mass conservation law, there exists $C = C(u_0)$ such that

$$\|v_x(t,\cdot)\|_{L^2(\mathbb{R}^+)} \le \|u_x(t,\cdot)\|_{L^2(\mathbb{R}^+)} + \tfrac{3}{4}\|u(t,\cdot)\|^3_{L^6(\mathbb{R}^+)} \le C\|u_x(t,\cdot)\|_{L^2(\mathbb{R}^+)}.$$

Thus, by (4-3), this gives the analogous estimate on u. □

One may note from the proof that the key ingredient to obtain the blow-up result of the initial boundary value problem on the half-line case is the positivity of the "surplus" term $\int_0^\infty x|v(t)|^4\,dx$. This is not true for the Cauchy problem.

5. Proof of Theorem 1.1

Proof. Let $(-T_-(u_0), T_+(u_0))$ be the maximal lifespan of the solution u of (1-1). To prove Theorem 1.1, it is sufficient to obtain the (indeed uniformly) *a priori* estimate of the solutions on H^1-norm, that is,

$$\sup_{t\in(-T_-(u_0),T_+(u_0))} \|v_x(t)\|_{L^2} < +\infty.$$

Now we argue by contradiction and suppose that there exists a sequence $\{t_n\}$ with

$$t_n \to -T_-(u_0) \quad \text{or} \quad T_+(u_0)$$

such that

$$\|v_x(t_n)\|_{L^2} \to +\infty, \quad \text{as } n \to \infty. \tag{5-1}$$

Let

$$\lambda_n = \|Q_x\|_{L^2}/\|v_x(t_n)\|_{L^2} \tag{5-2}$$

and

$$w_n(x) = \lambda_n^{1/2} v(t_n, \lambda_n x). \tag{5-3}$$

Then, by (5-1),

$$\|\partial_x w_n\|_{L^2} = \|Q_x\|_{L^2} \quad \text{and} \quad \lambda_n \to 0,\ \text{as } n \to \infty.$$

First we have the following lemma.

Lemma 5.1. *For any $\varepsilon > 0$, there exists a small $\varepsilon_* = \varepsilon_*(\varepsilon) > 0$ such that if the function $f \in H^1(\mathbb{R})$ satisfies*

$$\int_{\mathbb{R}} |f(x)|^2\,dx < 2\pi + \varepsilon_*, \quad \|\partial_x f\|_{L^2} = \|\partial_x Q\|_{L^2}, \quad E(f) < \varepsilon_*,$$

then there exist $\gamma_0, x_0 \in \mathbb{R}$ such that

$$\|f - e^{-i\gamma_0} Q(\cdot - x_0)\|_{H^1} \le \varepsilon.$$

We put the proof of Lemma 5.1 at the end of this section and apply it to prove Theorem 1.1. Let $\varepsilon_0 > 0$ be a fixed small constant which will be chosen later, and let $\varepsilon_* = \varepsilon_*(\varepsilon_0) > 0$ be the number defined in Lemma 5.1. By (1-6), (5-3), and a simple computation,

$$\int_{\mathbb{R}} |w_n(x)|^2\,dx = \int_{\mathbb{R}} |v_0(x)|^2\,dx < 2\pi + \varepsilon_*,$$

and

$$\|\partial_x w_n\|_{L^2} = \|Q_x\|_{L^2}, \quad E(w_n) = \lambda_n^2 E(v_0) \to 0.$$

Then, by Lemma 5.1, we may inductively construct the sequences $\{\gamma_n\}$, $\{x_n\}$ which satisfy

$$\|w_n - e^{-i\gamma_n} Q(\cdot - x_n)\|_{H^1} \le \varepsilon_0 \quad \text{for any } n \ge n_0, \tag{5-4}$$

where $n_0 = n_0(\varepsilon_0)$ is a positive large number. Let

$$\varepsilon(t_n, x) = e^{i\gamma_n} w_n(x + x_n) - Q.$$

Then

$$w_n(x) = e^{-i\gamma_n} Q(x - x_n) + e^{-i\gamma_n} \varepsilon(t_n, x - x_n). \tag{5-5}$$

Therefore, by (5-3), (5-5), and (5-4), we have

$$v(t_n, x) = e^{-i\gamma_n} \lambda_n^{-1/2} (\varepsilon + Q)(t_n, \lambda_n^{-1} x - x_n), \quad \|\varepsilon(t_n)\|_{H^1} \le \varepsilon_0. \tag{5-6}$$

By the momentum and (5-6), one has

$$\begin{aligned}
P(v(t_n)) &= \operatorname{Im} \int_{\mathbb{R}} \bar{v}(t_n) v_x(t_n)\,dx + \frac{1}{4} \int_{\mathbb{R}} |v(t_n)|^4\,dx \\
&= \lambda_n^{-2} \operatorname{Im} \int_{\mathbb{R}} (\bar{\varepsilon} + Q)(t_n, \lambda_n^{-1} x - x_n) \cdot (\varepsilon_x + Q_x)(t_n, \lambda_n^{-1} x - x_n)\,dx \\
&\qquad\qquad + \tfrac{1}{4} \lambda_n^{-2} \int_{\mathbb{R}} |(\varepsilon + Q)(t_n, \lambda_n^{-1} x - x_n)|^4\,dx \\
&= \lambda_n^{-1} \operatorname{Im} \int_{\mathbb{R}} (\bar{\varepsilon}(t_n) + Q)(\varepsilon_x(t_n) + Q_x)\,dx + \tfrac{1}{4} \lambda_n^{-1} \int_{\mathbb{R}} |\varepsilon(t_n) + Q|^4\,dx \\
&= \lambda_n^{-1} \left(\tfrac{1}{4} \|Q\|_{L^4}^4 + \operatorname{Im} \int_{\mathbb{R}} (Q_x \varepsilon(t_n) + Q \varepsilon_x(t_n) + \bar{\varepsilon} \varepsilon_x(t_n))\,dx + \frac{1}{4} \int_{\mathbb{R}} (|\varepsilon(t_n) + Q|^4 - Q^4)\,dx \right) \\
&= \lambda_n^{-1} \big(\tfrac{1}{4} \|Q\|_{L^4}^4 + O(\|\varepsilon(t_n)\|_{H^1}) \big) \ge \lambda_n^{-1} \big(\tfrac{1}{4} \|Q\|_{L^4}^4 - C\varepsilon_0 \big).
\end{aligned} \tag{5-7}$$

Thus, by choosing ε_0 small enough such that $C\varepsilon_0 \le \frac{1}{8}\|Q\|_{L^4}^4$, one has $P(v(t_n)) \ge \lambda_n^{-1}\cdot\frac{1}{8}\|Q\|_{L^4}^4$. By the momentum conservation law, this proves that $P(v_0)\lambda_n \ge \frac{1}{8}\|Q\|_{L^4}^4$. That is, by (5-2),

$$\|v_x(t_n)\|_{L^2} \le 8P(v_0)\|Q_x\|_{L^2}/\|Q\|_{L^4}^4. \tag{5-8}$$

This violates (5-1). Therefore, we prove that there exists $C_0 = C_0(\varepsilon_*, \|v_0\|_{H^1})$, such that

$$\sup_{t\in\mathbb{R}} \|v_x(t)\|_{L^2} \le C_0.$$

Now, for the solution u of (1-1) (with $\lambda = 1$), we have $u = \mathcal{G}_{3/4}v$. Thus, by (2-1), we have

$$u_x = e^{i(3/4)\int_{-\infty}^x |v(t,y)|^2\,dy}\big(i\tfrac{3}{4}|v|^2 v + v_x\big).$$

Therefore, by (1-9) and the mass conservation law, for any $t\in\mathbb{R}$,

$$\|u_x(t)\|_{L^2} \le \|v_x(t)\|_{L^2} + \tfrac{3}{4}\|v(t)\|_{L^6}^3 \le \|v_x(t)\|_{L^2} + \frac{3}{2\pi}\|v(t)\|_{L^2}^2\|v_x(t)\|_{L^2} \le C_0\Big(1+\frac{3}{2\pi}\|u_0\|_{L^2}^2\Big). \qquad \square$$

Proof of Lemma 5.1. The proof follows from the standard variational argument; see [Merle 2001; Weinstein 1986] for examples; see also [Banica 2004; Hmidi and Keraani 2005] for its applications. Here we prove it by using the profile decomposition (see [Gérard 1998] for example) for the sake of the completeness. Let $\{f_n\}\subset H^1(\mathbb{R})$ be any sequence satisfying

$$\|f_n\|_{L^2} \to \|Q\|_{L^2}, \quad \|\partial_x f_n\|_{L^2} = \|Q_x\|_{L^2}, \quad E(f_n)\to 0.$$

Then, by the profile decomposition, there exist $\{V^j\}$, $\{x_n^j\}$ such that, up to a subsequence,

$$f_n = \sum_{j=1}^{L} V^j(\cdot - x_n^j) + R_n^L,$$

where, for $j\ne k$, we have $|x_n^j - x_n^k|\to\infty$ as $n\to\infty$, and

$$\lim_{L\to\infty}\lim_{n\to\infty}\|R_n^L\|_{L^6} = 0. \tag{5-9}$$

Moreover,

$$\begin{aligned} \|f_n\|_{H^s}^2 &= \sum_{j=1}^{L}\|V^j\|_{H^s}^2 + \|R_n^L\|_{H^s}^2 + o_n(1) \quad \text{for } s = 0, 1,\\ E(f_n) &= \sum_{j=1}^{L} E(V^j) + E(R_n^L) + o_n(1). \end{aligned} \tag{5-10}$$

Since $\|f_n\|_{L^2}\to\|Q\|_{L^2}$, one has, by (5-10),

$$\|V^j\|_{L^2} \le \|Q\|_{L^2} \quad \text{for any } j\ge 1. \tag{5-11}$$

This implies, by the sharp Gagliardo–Nirenberg inequality (1-9), that $E(V^j)\ge 0$ for any $j\ge 1$. Further, by (5-9), one has

$$\lim_{L\to\infty}\lim_{n\to\infty} E(R_n^L) \ge 0.$$

Since $E(f_n) \to 0$, we have $E(V^j) = 0$ for any $j \geq 1$. Combining with (5-11) and (1-9), this again yields

$$\|V^j\|_{L^2} = \|Q\|_{L^2} \quad \text{or} \quad V^j = 0.$$

Since $\|f_n\|_{L^2} \to \|Q\|_{L^2}$, there exists exactly one j, say $j = 1$, such that

$$\|V^1\|_{L^2} = \|Q\|_{L^2}, \quad V^j = 0 \text{ for any } j \geq 2.$$

Moreover, by (5-10) and (1-9), when $n \to \infty$, we have $R_n^L \to 0$ in $L^2(\mathbb{R})$, and then further in $H^1(\mathbb{R})$. Therefore,

$$\|\partial_x V^1\|_{L^2} = \|Q_x\|_{L^2}, \quad E(V^1) = 0,$$

and $f_n \to V^1$ in $H^1(\mathbb{R})$ as $n \to \infty$. Now we note that V^1 attains the sharp Gagliardo–Nirenberg inequality (1-9). Thus, by the uniqueness of the minimizer of the Gagliardo–Nirenberg inequality [Weinstein 1982/83], we have $V^1 = e^{-i\gamma_0} Q(\cdot - x_0)$ for some $\gamma_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$. This proves the lemma. □

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