ANALYSIS & PDE

Volume 7 No. 1 2014

HAO FANG, MIJIA LAI, JIAN SONG AND BEN WEINKOVE

THE J-FLOW ON KÄHLER SURFACES: A BOUNDARY CASE

msp

ANALYSIS AND PDE
Vol. 7, No. 1, 2014
dx.doi.org/10.2140/apde.2014.7.215

THE J-FLOW ON KÄHLER SURFACES: A BOUNDARY CASE

HAO FANG, MIJIA LAI, JIAN SONG AND BEN WEINKOVE

We study the J-flow on Kähler surfaces when the Kähler class lies on the boundary of the open cone for which global smooth convergence holds and satisfies a nonnegativity condition. We obtain a C^0 estimate and show that the J-flow converges smoothly to a singular Kähler metric away from a finite number of curves of negative self-intersection on the surface. We discuss an application to the Mabuchi energy functional on Kähler surfaces with ample canonical bundle.

1. Introduction

The J-flow is a parabolic flow on Kähler manifolds with two Kähler classes. It was defined by Donaldson [1999] in the setting of moment maps and by Chen [2000] as the gradient flow of the $\mathcal{J}$-functional appearing in his formula for the Mabuchi energy [1986].

The J-flow is defined as follows. Let X be a compact Kähler manifold with two Kähler metrics ω and χ in different Kähler classes $[\omega]$ and $[\chi]$. Let $\mathcal{P}_\chi$ be the space of smooth χ-plurisubharmonic functions on X:

$$\mathcal{P}_\chi = \{\varphi \mid \chi_\varphi := \chi + dd^c\varphi > 0\}.$$

Then the J-flow is a flow defined in $\mathcal{P}_\chi$ by

$$\frac{\partial}{\partial t}\varphi = c - \frac{n\chi_\varphi^{n-1}\wedge\omega}{\chi_\varphi^n}, \quad \varphi(0) = \varphi_0 \in \mathcal{P}_\chi, \tag{1-1}$$

where c is the topological constant given by

$$c = \frac{n[\chi]^{n-1}\cdot[\omega]}{[\chi]^n}.$$

A stationary point of (1-1) gives a critical Kähler metric $\tilde{\chi} \in [\chi]$ satisfying

$$c\tilde{\chi}^n = n\tilde{\chi}^{n-1}\wedge\omega. \tag{1-2}$$

Donaldson [1999] noted that a smooth critical metric exists only if the cohomological condition $[c\chi - \omega] > 0$ holds. In complex dimension 2, Chen [2000] showed that this necessary condition is sufficient for the existence of a smooth critical metric by observing that in this case, (1-2) is equivalent to the complex Monge–Ampère equation solved by Yau [1978] (see (2-2) below). Chen [2004] also

---

Research supported in part by NSF grants DMS-1008249, DMS-08047524 and DMS-1105373. This work was carried out while Weinkove was a member of the Mathematics Department at the University of California at San Diego.
MSC2010: 53C44, 53C55.
Keywords: Kähler, J-flow, complex Monge–Ampère.

established the long time existence for the J-flow (1-1) with any initial data. Weinkove [2004; 2006] showed that the J-flow converges to a critical metric if the cohomological condition $[c\chi - (n-1)\omega] > 0$ holds. In particular, if X is a Kähler surface, a necessary and sufficient condition for convergence of the flow to a smooth critical metric is Donaldson's cohomological condition $[c\chi - \omega] > 0$.

Song and Weinkove [2008] found a necessary and sufficient condition for the convergence of the J-flow in higher dimensions, which we now explain. Define

$$\mathscr{C}_\omega := \big\{[\chi] > 0 \mid \text{there exists } \chi' \in [\chi] \text{ such that } c{\chi'}^{n-1} - (n-1){\chi'}^{n-2} \wedge \omega > 0\big\}. \tag{1-3}$$

Then the J-flow (1-1) converges smoothly to the critical metric solving (1-2) if and only if $[\chi] \in \mathscr{C}_\omega$.

In [Fang et al. 2011; Fang and Lai 2012b], the J-flow was generalized to the general inverse σ_k flow. An analogous necessary and sufficient condition is found to ensure the smooth convergence of the flow.

The behavior of the J-flow in the case when the condition $[\chi] \in \mathscr{C}_\omega$ does *not* hold is still largely open. However, recent progress was made by Fang and Lai [2012a] in the case of a family of Kähler manifolds satisfying the Calabi symmetry condition. It was shown (in the more general case of the inverse σ_k flow) that if the initial metric satisfies the Calabi symmetry, the flow converges to a Kähler current which is the sum of a Kähler metric with a conic singularity and a current of integration along a divisor.

We consider the case when X is a Kähler surface. As discussed above, a necessary and sufficient condition for convergence of the flow to a smooth critical metric is

$$[c\chi - \omega] > 0. \tag{1-4}$$

Donaldson [1999] remarked that if this condition fails, then one might expect the J-flow to blow up over some curves of negative self-intersection. It was observed in [Song and Weinkove 2008, Proposition 4.5] that, applying the results of Buchdahl [1999] and Lamari [1999], there exist a finite number $N \geq 0$, say, of irreducible curves C_i with $C_i^2 < 0$ on X and positive real numbers a_i such that $[c\chi - \omega] - \sum_{i=1}^N a_i[C_i]$ is Kähler. It was shown in [Song and Weinkove 2008] that at least for some sequence of points approaching some C_i, the quantity $|\varphi| + |\Delta_\omega \varphi|$ blows up.

In this paper we describe the behavior of the J-flow for certain classes $[\chi]$ on the boundary of $\mathscr{C}_\omega$. First we introduce some notation: given a closed $(1,1)$-form α, write $[\alpha] \geq 0$ if there exists a smooth closed nonnegative $(1,1)$-form cohomologous to α. We consider any Kähler class $[\chi]$ satisfying

$$[c\chi - \omega] \geq 0. \tag{1-5}$$

All such classes $[\chi]$ lie in the closure of $\mathscr{C}_\omega$. The boundary of $\mathscr{C}_\omega$ consists of Kähler classes $[\chi]$ such that $[c\chi - \omega]$ is *nef*, which means that for every $\varepsilon > 0$ there exists a representative of $[c\chi - \omega]$ which is bounded below by $-\varepsilon\omega$. Further, since

$$[c\chi - \omega]^2 = [\omega]^2 > 0,$$

the class $[c\chi - \omega]$ is nef and big. Nevertheless, to our knowledge, this does not imply that it satisfies (1-5) — see Question 4.1 below. However, at least in many cases the condition (1-5) is equivalent to $[\chi]$ belonging to the closure of $\mathscr{C}_\omega$ in the Kähler cone. This holds for all Hirzebruch surfaces, for example, since explicit nonnegative (1, 1)-forms can be found representing all classes on the boundary of the Kähler cone (see the discussion in [Calabi 1982]).

Our main result is this:

Theorem 1.1. *Let X be a compact Kähler surface with Kähler metrics ω and χ such that*

$$[c\chi - \omega] \geq 0, \quad \textit{where } c = \frac{2[\chi]\cdot[\omega]}{[\chi]^2}.$$

Then there exist a finite number of curves C_i on X of negative self-intersection such that the solution $\varphi(t)$ of the J-flow (1-1) *converges in $C^\infty_{\mathrm{loc}}(X \setminus \bigcup C_i)$ to a continuous function φ_∞, smooth on $X \setminus \bigcup C_i$, satisfying*

$$c\chi^2_{\varphi_\infty} = 2\chi_{\varphi_\infty} \wedge \omega, \quad \textit{for} \quad \chi_{\varphi_\infty} = \chi + dd^c\varphi_\infty \geq 0. \tag{1-6}$$

Moreover, φ_∞ is the unique continuous solution of (1-6) *up to the addition of a constant.*

Our result makes use of some recent works in the study of complex Monge–Ampère equations that appeared after the breakthrough of Kołodziej [1998]. Indeed, the existence of a unique weak solution to the critical equation (1-6) is a direct consequence of a result of Eyssidieux, Guedj, and Zeriahi [Eyssidieux et al. 2009] and Zhang [2006], who generalized Kołodziej's theorem to the degenerate complex Monge–Ampère equation. By comparing with this solution, we obtain our key uniform estimate for $\varphi(t)$ along the J-flow (Proposition 2.2 below). In addition, we use the viscosity methods introduced in [Eyssidieux et al. 2011] to give a second proof of our key estimate. The results of [Eyssidieux et al. 2011] allow us to conclude that the solution of (1-6) is continuous, and that (1-6) can be understood in both the pluripotential and the viscosity senses.

We have an application of our result to the *Mabuchi energy* [1986], a functional which is closely connected to the problem of algebraic stability and existence of constant scalar curvature Kähler (cscK) metrics [Yau 1993; Tian 1997; Donaldson 2002]. Given a Kähler surface (X, χ), the Mabuchi energy is the functional $\mathrm{Mab} : \mathcal{P}_\chi \to \mathbb{R}$ given by

$$\mathrm{Mab}(\varphi) = -\int_0^1 \int_X \frac{\partial \varphi_t}{\partial t} (R_{\chi_{\varphi_t}} - \mu)\chi^n_{\varphi_t}\, dt,$$

where $\{\varphi_t\}_{0\leq t\leq 1}$ is a path in $\mathcal{P}_\chi$ between 0 and φ, $R_{\chi_{\varphi_t}}$ is the scalar curvature of the metric χ_{φ_t}, and μ is the average of the scalar curvature of χ. The value $\mathrm{Mab}(\varphi)$ is independent of the choice of path.

It was conjectured by Tian [1997], assuming X has no nontrivial holomorphic vector fields, that the existence of a cscK metric is equivalent to the *properness* of the Mabuchi energy, meaning that there exists an increasing function $f : [0, \infty) \to \mathbb{R}$ with $\lim_{x\to\infty} f(x) = \infty$ such that

$$\mathrm{Mab}(\varphi) \geq f(E(\varphi)), \quad \text{where} \quad E(\varphi) = \int_X \sqrt{-1}\, \partial\varphi \wedge \overline{\partial}\varphi \wedge (\chi_0 + \chi_\varphi).$$

This conjecture holds whenever $[\chi] = -c_1(X) > 0$ or if $[\chi] = c_1(X) > 0$ and X has no nontrivial holomorphic vector fields [Tian 1997; 2000; Tian and Zhu 2000]. It also holds on all manifolds with $c_1(X) = 0$, even in the presence of holomorphic vector fields [Tian 2000]. In fact in each case, the function f can be taken to be linear [Tian 2000; Phong et al. 2008]. Chen [2000] showed that on manifolds with

$c_1(X) < 0$, or equivalently, with ample canonical bundle K_X, the Mabuchi energy can be written as a sum of two terms: the first is the $\mathcal{J}$-functional with reference metric ω in $[K_X]$, and the second is a term which is bounded below. In fact, the second term is proper [Tian 2000] (see the discussion in [Song and Weinkove 2008]), and under the cohomological condition $[c\chi - \omega] \geq 0$, the $\mathcal{J}$-functional has a lower bound, as shown in Corollary 3.3 below. Hence we obtain:

Corollary 1.2. *Suppose that X is a compact Kähler surface with ample canonical bundle K_X. Then the Mabuchi energy is proper on the classes $[\chi]$ satisfying*

$$\left(\frac{2[\chi]\cdot[K_X]}{[\chi]^2}\right)[\chi] - [K_X] \geq 0. \tag{1-7}$$

Moreover, the function f in the definition of properness can be taken to be linear.

Thus, since the condition of K_X being ample implies that X has no nontrivial holomorphic vector fields, conjecturally, classes $[\chi]$ in the cone given by (1-7) should admit cscK metrics. The class $[K_X]$ is inside this cone and admits a cscK metric [Aubin 1978; Yau 1978]. The same is true for classes sufficiently close to $[K_X]$ (see [LeBrun and Simanca 1994]). On the other hand, Ross [2006] found Kähler classes on surfaces with K_X ample that do not admit cscK metrics. Corollary 1.2, together with the arguments of [LeBrun and Simanca 1994], suggests that the set of classes that admit cscK metrics is strictly larger than those lying in the cone (1-7).

An outline of the paper is as follows. In Section 2, we prove the key C^0 estimate. We provide two proofs: the first uses smooth maximum principle arguments and the second uses the notion of viscosity solutions from [Eyssidieux et al. 2011]. We complete the proof of the main theorem in Section 3, and in the last section we finish with some questions for further study.

2. The C^0 estimate

For convenience of notation, we assume from now on that $c = 1$. We may do this by considering $(1/c)[\chi]$ instead of $[\chi]$. In addition, we may assume, by modifying the initial data if necessary, that $\chi - \omega \geq 0$.

The key estimate we need is a uniform C^0 estimate for the solution $\varphi(t)$ of the J-flow. We need the following theorem on the degenerate complex Monge–Ampère equation (the C^0 estimate was proved independently in [Zhang 2006] under slightly less general hypotheses).

Theorem 2.1 [Eyssidieux et al. 2009; 2011]. *Let (M, ω) be a compact Kähler manifold of complex dimension n and let α be a semipositive $(1, 1)$-form with $\int_M \alpha^n > 0$. For any nonnegative $f \in L^p(M, \omega^n)$, for $p > 1$, with $\int_M f\omega^n = \int_M \alpha^n$, there exists a unique continuous function φ on M with $\alpha + dd^c\varphi \geq 0$ and*

$$(\alpha + dd^c\varphi)^n = f\omega^n, \quad \sup_M \varphi = 0. \tag{2-1}$$

Moreover, $\|\varphi\|_{C^0(M)}$ is uniformly bounded by a constant depending only on p, M, ω, α and $\|f\|_{L^p(M)}$.

Given this, we immediately obtain a solution φ_∞ to (1-6), using the observation of Chen [2000] that the critical equation can be rewritten as a complex Monge–Ampère equation:

$$\chi_\varphi^2 = 2\chi_\varphi \wedge \omega \iff (\chi_\varphi - \omega)^2 = \omega^2. \tag{2-2}$$

Writing $\alpha := \chi - \omega \geq 0$ on the Kähler surface X, we can apply Theorem 2.1 to see that there exists a continuous function φ_∞ solving (1-6). Moverover, φ_∞ is unique up to the addition of a constant.

Next we use the uniform C^0 bound from Theorem 2.1 to obtain:

Proposition 2.2. *We assume that $\chi - \omega \geq 0$ as discussed above. Let $\varphi(t)$ be the solution of J-flow* (1-1) *on the compact Kähler surface X. Then there exists C depending only on the initial data such that for all $t \geq 0$,*

$$\|\varphi(t)\|_{C^0(X)} \leq C. \tag{2-3}$$

Proof. From the introduction, we know

$$[\chi - \omega] - \sum_{i=1}^{N} a_i [C_i] > 0, \tag{2-4}$$

for positive real numbers a_i and irreducible curves C_i of negative self-intersection. Since we are assuming $[\chi - \omega] \geq 0$, we may take the constants a_i to be arbitrarily small. However, we will not need to make use of this last fact.

It follows that there exist Hermitian metrics h_i on the line bundles $[C_i]$ associated to C_i such that

$$\chi - \omega - \sum_{i=1}^{N} a_i R_{h_i} > 0, \tag{2-5}$$

where $R_{h_i} = -dd^c \log h_i$ is the curvature of h_i. Let s_i be a holomorphic section of $[C_i]$ vanishing along C_i to order 1. Recall that we denote $\chi - \omega$ by α.

Next, we apply Theorem 2.1 and write ψ for the solution to the degenerate complex Monge–Ampère equation

$$(\alpha + dd^c\psi)^2 = \omega^2, \quad \alpha + dd^c\psi \geq 0, \tag{2-6}$$

subject to the condition $\sup_X \psi = 0$. We have $\|\psi\|_{C^0(X)} \leq C$.

It follows from a trick of Tsuji [1988], as used in [Eyssidieux et al. 2009], that ψ is smooth away from the curves C_i. Although the proof is the same, the precise statement we need does not seem to be quite contained in [Eyssidieux et al. 2009], so we briefly outline the idea here for the convenience of the reader. For $\delta > 0$, let ψ_δ be Yau's solution of the complex Monge–Ampère equation

$$(\alpha + \delta\omega + dd^c\psi_\delta)^2 = c_\delta \omega^2, \quad \alpha_\delta := \alpha + \delta\omega + dd^c\psi_\delta > 0, \tag{2-7}$$

for a constant c_δ chosen so that the integrals of both sides are equal. From Theorem 2.1, ψ_δ is uniformly bounded in C^0. To obtain a second-order estimate for ψ_δ, uniform in δ, we consider, for a constant $A > 0$,

$$Q_\delta = \log \operatorname{tr}_\omega \alpha_\delta - A\left(\psi_\delta - \sum_i a_i \log |s_i|^2_{h_i}\right), \tag{2-8}$$

which is well-defined on $X \setminus \bigcup C_i$ and tends to $-\infty$ on $\bigcup C_i$. Compute, at a point in $X \setminus \bigcup C_i$,

$$\Delta_{\alpha_\delta} Q_\delta \geq -C \operatorname{tr}_{\alpha_\delta} \omega - 2A + A \operatorname{tr}_{\alpha_\delta}\left(\alpha - \sum_i a_i R_{h_i}\right).$$

Then using (2-5), we may choose a uniform A sufficiently large that

$$A\biggl(\alpha - \sum_i a_i R_{h_i}\biggr) \geq (C+1)\omega.$$

The quantity Q_δ achieves a maximum at some point $x \in X \setminus \bigcup C_i$, and at this point we have $\Delta_{\alpha_\delta} Q_\delta \leq 0$. Hence, at x,

$$0 \geq \operatorname{tr}_{\alpha_\delta} \omega - 2A,$$

so $\operatorname{tr}_{\alpha_\delta} \omega$ is uniformly bounded from above. But by (2-7) we have at x

$$\operatorname{tr}_\omega \alpha_\delta = \biggl(\frac{\alpha_\delta^2}{\omega^2}\biggr) \operatorname{tr}_{\alpha_\delta} \omega = c_\delta \operatorname{tr}_{\alpha_\delta} \omega \leq C',$$

for some uniform C'. Since ψ_δ is uniformly bounded in C^0, we see that Q_δ is uniformly bounded from above at x, and hence everywhere.

This establishes a uniform upper bound for $\operatorname{tr}_\omega \alpha_\delta$ (and again by (2-7), also for $\operatorname{tr}_{\alpha_\delta} \omega$) on any compact subset of $X \setminus \bigcup C_i$. It follows that on such a fixed compact set, ω and α_δ are uniformly equivalent. Hence we have estimates, uniform in δ, for $dd^c \psi_\delta$ on compact subsets of $X \setminus \bigcup C_i$. The $C^\infty_{\mathrm{loc}}\bigl(X \setminus \bigcup C_i\bigr)$ estimates for ψ_δ then follow from the usual Evans–Krylov local theory for the complex Monge–Ampère equation [Evans 1982; Krylov 1982]. Taking a limit as $\delta \to 0$ shows that ψ is smooth away from the C_i.

Fix $\varepsilon \in (0, 1)$. We will apply the maximum principle to the quantity

$$\theta_\varepsilon = \varphi - (1+\varepsilon)\psi + \varepsilon \sum_{i=1}^N a_i \log |s_i|^2_{h_i} - A\varepsilon t,$$

where A is a constant to be determined. Observe that θ_ε is smooth on $X \setminus \bigcup C_i$ and tends to negative infinity along $\bigcup C_i$, and hence θ_ε achieves a maximum in the interior of $X \setminus \bigcup C_i$ for each time t.

We rewrite (1-1) as

$$\frac{\partial \varphi}{\partial t} = 1 - \frac{2\chi_\varphi \wedge \omega}{\chi_\varphi^2} = \frac{\chi_\varphi^2 - 2\chi_\varphi \wedge \omega}{\chi_\varphi^2} = \frac{(\chi_\varphi - \omega)^2 - \omega^2}{\chi_\varphi^2} = \frac{\omega^2}{\chi_\varphi^2}\biggl(\frac{(\chi_\varphi - \omega)^2}{\omega^2} - 1\biggr) = \frac{\omega^2}{\chi_\varphi^2}\biggl(\frac{\alpha_\varphi^2}{\alpha_\psi^2} - 1\biggr). \tag{2-9}$$

Compute on $X \setminus \bigcup C_i$, using (2-9),

$$\begin{aligned}\frac{\partial}{\partial t}\theta_\varepsilon &= \frac{\omega^2}{\chi_\varphi^2}\biggl(\frac{(\alpha + dd^c\varphi)^2}{(\alpha + dd^c\psi)^2} - 1\biggr) - A\varepsilon \\ &= \frac{\omega^2}{\chi_\varphi^2}\biggl(\frac{\bigl((1+\varepsilon)\alpha + (1+\varepsilon)dd^c\psi - \varepsilon(\alpha - \sum a_i R_{h_i}) + dd^c\theta_\varepsilon\bigr)^2}{(\alpha + dd^c\psi)^2} - 1\biggr) - A\varepsilon.\end{aligned}$$

But $\alpha - \sum a_i R_{h_i} \geq 0$, and at the maximum of θ_ε, we have $dd^c\theta_\varepsilon \leq 0$. Hence at the maximum of θ_ε,

$$\frac{\partial}{\partial t}\theta_\varepsilon \leq \frac{\omega^2}{\chi_\varphi^2}\biggl((1+\varepsilon)^2 \frac{(\alpha + dd^c\psi)^2}{(\alpha + dd^c\psi)^2} - 1\biggr) - A\varepsilon < 0, \tag{2-10}$$

if we choose

$$A = \sup_{X\times[0,\infty)} \frac{3\omega^2}{\chi_\varphi^2},$$

which is a uniform constant since χ_φ is always uniformly bounded from below away from zero along the J-flow. Indeed, this follows immediately from taking a time derivative of the J-flow equation and applying the maximum principle (see Lemma 4.1 in [Chen 2004]). Then (2-10) implies that θ_ε must achieve its maximum at time zero, and hence θ_ε is uniformly bounded from above by a constant independent of ε. Letting $\varepsilon \to 0$, we obtain the upper bound for φ.

The lower bound of φ is similar: just replace ε by $-\varepsilon$ and consider the minimum instead of the maximum. □

We provide a second proof. The proof is based on the equivalence of two notions of weak solution of (2-2): the pluripotential sense and the viscosity sense.

Second proof of Proposition 2.2. As in the first proof, write ψ for the solution to (2-6) with $\sup_X \psi = 0$. The function ψ is continuous on X and is smooth away from the curves C_i. We now apply Theorem 3.6 of [Eyssidieux et al. 2011], which states that ψ satisfies (2-6) in the viscosity sense as defined in that paper.

We refer to [Eyssidieux et al. 2011] for the precise definition of a viscosity solution to (2-6) and state two consequences of this definition which are sufficient for our purposes:

(i) If x_0 is any point on X and q is any smooth function defined in a neighborhood of x_0 such that

$$\psi - q \text{ has a local maximum at } x_0,$$

then $(\alpha + dd^c q)^2 \geq \omega^2$ at x_0.

(ii) If x_0 is any point on X and q is any smooth function defined in a neighborhood of x_0 such that

$$\psi - q \text{ has a local minimum at } x_0,$$

then $(\alpha + dd^c q)^2 \leq \omega^2$ at x_0.

Indeed, (i) follows from the definition of a viscosity subsolution, and (ii) from the definition of a viscosity supersolution (see Section 2 in [Eyssidieux et al. 2011]).

We first find an upper bound for φ. Let $\varepsilon > 0$ and define $H_\varepsilon = \varphi - \psi - \varepsilon t$. We wish to show that H_ε attains its maximum value at $t = 0$. Note that H_ε satisfies the equation

$$\frac{\partial H_\varepsilon}{\partial t} = 1 - \frac{2\chi_\varphi \wedge \omega}{\chi_\varphi^2} - \varepsilon.$$

Suppose that H_ε attains a maximum at a point (x_0, t_0) on $X \times [0, T]$ for some finite $T > 0$, and assume for a contradiction that $t_0 > 0$. Then $\partial H_\varepsilon/\partial t\,(x_0, t_0) \geq 0$. Define a smooth function q on X by $q(x) = \varphi(x, t_0) - H_\varepsilon(x_0, t_0) - \varepsilon t_0$. The function

$$x \mapsto (\psi - q)(x) = -H_\varepsilon(x, t_0) + H_\varepsilon(x_0, t_0)$$

achieves its minimum at x_0. Then we can apply (ii) to see that $(\alpha + dd^c q)^2 \le \omega^2$ at x_0, or in other words

$$(\chi - \omega + dd^c\varphi)^2 \le \omega^2, \quad \text{at } (x_0, t_0),$$

which is equivalent to

$$\chi_\varphi^2 \le 2\chi_\varphi \wedge \omega \quad \text{at } (x_0, t_0).$$

It follows that

$$\frac{\partial H_\varepsilon}{\partial t}(x_0, t_0) = 1 - \frac{2\chi_\varphi \wedge \omega}{\chi_\varphi^2} - \varepsilon < 0,$$

contradicting the fact that $\partial H_\varepsilon / \partial t\,(x_0, t_0) \ge 0$. Hence H_ε attains its maximum value at $t = 0$ and is uniformly bounded from above independent of ε. Letting $\varepsilon \to 0$ gives the desired upper bound for φ.

Applying a similar argument, using (i) instead of (ii), gives a uniform lower bound for φ. □

We can now apply Theorem 1.3 of [Song and Weinkove 2008] together with the standard local theory for (1-1) to obtain higher-order estimates.

Proposition 2.3. *As above, assume that $\chi - \omega \ge 0$ on the compact Kähler surface X and let $\varphi(t)$ be the solution of the J-flow* (1-1). *For any compact subset $K \subset X \setminus \bigcup C_i$ and any $k \ge 0$, there exists a constant $C_{k,K}$ such that for all t,*

$$\|\varphi(t)\|_{C^k(K)} \le C_{k,K}.$$

Here, the C_i are the irreducible curves of negative self-intersection chosen to satisfy (2-4).

3. Proof of the main theorem

Again we assume in this section that $[\chi]$ is scaled so that $c = 1$. Before proving the main theorem we first discuss the $\mathcal{J}$ and $\mathcal{I}$-functionals. Define $\mathcal{J}_{\omega,\chi}$ and $\mathcal{I}_{\omega,\chi}$ by

$$\mathcal{J}_{\omega,\chi}(\varphi) := \int_0^1 \int_X \dot{\varphi}_t \big(2\chi_{\varphi_t} \wedge \omega - \chi_{\varphi_t}^2\big)\, dt,$$

$$\mathcal{I}_{\omega,\chi}(\varphi) := \int_0^1 \int_X \dot{\varphi}_t \chi_{\varphi_t}^2\, dt,$$

where φ_t is a smooth path in $\mathcal{P}_\chi$ connecting 0 and φ. For simplicity, we will omit the subscripts.

If $\varphi(t)$ is the solution of the J-flow, then

$$\frac{d}{dt}\mathcal{J}(\varphi(t)) = -\int_X \dot{\varphi}(t)^2 \chi_{\varphi(t)}^2, \quad \frac{d}{dt}\mathcal{I}(\varphi(t)) = 0. \tag{3-1}$$

In particular, the J-flow is the gradient flow of $\mathcal{J}$.

One can write explicit formulae for $\mathcal{J}$, $\mathcal{I}$ as follows:

$$\mathcal{J}(\varphi) = \int_X \varphi\big(\chi_\varphi \wedge \omega + \chi \wedge \omega\big) - \frac{1}{3}\int_X \varphi\big(\chi_\varphi^2 + \chi_\varphi \wedge \chi + \chi^2\big), \tag{3-2}$$

$$\mathcal{I}(\varphi) = \frac{1}{3}\int_X \varphi\big(\chi_\varphi^2 + \chi_\varphi \wedge \chi + \chi^2\big). \tag{3-3}$$

Thus an immediate corollary of Proposition 2.2 is:

Proposition 3.1. *There exists a uniform constant C such that, for $\varphi(t)$ the solution of the J-flow, we have*

$$\mathcal{J}(\varphi(t)) \geq -C$$

for all $t \geq 0$.

In what follows, we will need to make use of a simple continuity-type result for the $\mathcal{I}$ and $\mathcal{J}$ functionals.

Lemma 3.2. *Let $\varphi_j \in \mathcal{P}_\chi$ and let φ be a continuous function on X satisfying $\chi + dd^c\varphi \geq 0$. Let Y be a proper subvariety of X. Suppose that*

(a) *there exists C such that $\|\varphi_j\|_{C^0(X)} \leq C$;*

(b) *$\varphi_j \to \varphi$ in $C^\infty_{\mathrm{loc}}(X \setminus Y)$ as $j \to \infty$.*

Then

$$\mathcal{J}(\varphi_j) \to \mathcal{J}(\varphi) \quad \textit{and} \quad \mathcal{I}(\varphi_j) \to \mathcal{I}(\varphi) \quad \textit{as } j \to \infty.$$

Proof. The proof is a simple exercise in pluripotential theory (we refer the reader to [Kołodziej 2005] for an introduction to this theory). For the convenience of the reader, we sketch the proof here. For φ continuous with $\chi + dd^c\varphi \geq 0$, the quantities χ_φ^2, $\chi \wedge \chi_\varphi$ and $\chi_\varphi \wedge \omega$ define finite measures on X and hence by (3-2) and (3-3), the functionals $\mathcal{I}(\varphi)$ and $\mathcal{J}(\varphi)$ are well-defined.

We may choose a sequence of open tubular neighborhoods Y_k of Y such that $Y_k \downarrow Y$ as $k \to \infty$. Since Y is pluripolar, the capacity $\mathrm{Cap}_\chi(Y)$ of Y with respect to χ (in the sense of [Kołodziej 1998]) is zero. By the properties of this capacity (see [Guedj and Zeriahi 2005], for example) we have

$$\lim_{k\to\infty} \mathrm{Cap}_\chi(Y_k) = \mathrm{Cap}_\chi(Y) = 0.$$

Since the φ_j are uniformly bounded, it follows that $\int_{Y_k} \varphi_j \beta \wedge \gamma \to 0$ as $k \to \infty$, uniformly in j, where β, γ are each one of ω, χ or χ_{φ_j}. The same holds if we replace φ_j by φ. The result then follows from the expressions (3-2) and (3-3) together with condition (b). ☐

Proof of Theorem 1.1. Since $\mathcal{J}$ is decreasing and bounded from below, there exists a constant C such that

$$\int_0^\infty \int_X \dot{\varphi}(t)^2 \chi_{\varphi(t)}^2 \, dt < C. \tag{3-4}$$

We claim that for each fixed point $p \in X \setminus \bigcup C_i$, we have $\dot{\varphi}(p, t) \to 0$ as $t \to \infty$. Suppose not. Then there exists $\varepsilon > 0$ and a sequence of times $t_i \to \infty$ such that $|\dot{\varphi}(t_i)| > \varepsilon$ for all i. But since we have bounds for $\dot{\varphi}$ and all its time and space derivatives in a fixed neighborhood U, say, of p with $U \subset X \setminus \bigcup C_i$, it follows that $|\dot{\varphi}(t)| > \varepsilon/2$ for $t \in [t_i, t_i + \delta]$ for a uniform $\delta > 0$. This contradicts (3-4) and establishes the claim.

Since we have $C^\infty_{\mathrm{loc}}\big(X \setminus \bigcup C_i\big)$ bounds for $\dot{\varphi}$, the uniqueness of limits implies that $\dot{\varphi}$ converges to zero in $C^\infty_{\mathrm{loc}}\big(X \setminus \bigcup C_i\big)$.

We have uniform C^∞ bounds for $\varphi(t)$ on compact subsets of $X \setminus \bigcup C_i$, and hence we can apply the Arzelà–Ascoli theorem to see that for a sequence of times $t_i \to \infty$, we have $\varphi(t_i) \to \varphi_\infty$ for a smooth (bounded) function φ_∞ on $X \setminus \bigcup C_i$. Since $\dot{\varphi} \to 0$, φ_∞ satisfies the equation $\chi_{\varphi_\infty}^2 = 2\chi_{\varphi_\infty} \wedge \omega$ as in the statement of the theorem.

We also have $\mathcal{I}(\varphi_\infty) = \lim_{t\to\infty} \mathcal{I}(\varphi(t)) = \mathcal{I}(\varphi_0)$, using Lemma 3.2 and the fact that $\mathcal{I}$ is constant along the flow. Applying Theorem 2.1, we know that (1-6) has a unique solution up to the addition of a constant. Thus φ_∞ is the unique solution of (1-6) subject to the condition $\mathcal{I}(\varphi_\infty) = \mathcal{I}(\varphi_0)$.

Finally we claim that $\varphi(t)$ converges in $C^\infty_{\mathrm{loc}}\big(X \setminus \bigcup C_i\big)$ to φ_∞. Suppose not. Then there exist $\varepsilon > 0$ and a sequence of times $t_i \to \infty$ such that $\|\varphi(t_i) - \varphi_\infty\|_{C^k(K)} > \varepsilon$ for all i, for some integer k and compact $K \subset X \setminus \bigcup C_i$. Since we have uniform C^∞ bounds for $\varphi(t)$ on K, we can pass to a subsequence and assume that $\varphi(t_i)$ converges to a function $\varphi'_\infty \neq \varphi_\infty$. But φ'_∞ will also satisfy the equations $\chi^2_{\varphi'_\infty} = 2\chi_{\varphi'_\infty} \wedge \omega$ and $\mathcal{I}(\varphi'_\infty) = \mathcal{I}(\varphi_0)$, contradicting the uniqueness. □

As a consequence:

Corollary 3.3. *The $\mathcal{J}$-functional is bounded from below on $\mathcal{P}_\chi$.*

Proof. Take any $\varphi_0 \in \mathcal{P}_\chi$. Then running the J-flow from φ_0, which by Theorem 1.1 converges to φ_∞, we obtain (applying Lemma 3.2)

$$\mathcal{J}(\varphi_0) \geq \lim_{t\to\infty} \mathcal{J}(\varphi(t)) = \mathcal{J}(\varphi_\infty),$$

since $\mathcal{J}$ is decreasing along the flow. □

Proof of Corollary 1.2. Combine Corollary 3.3 and Lemma 4.1 of [Song and Weinkove 2008]. □

4. Further questions

Question 4.1. In general, it does not appear to be known whether a nef and big class on a Kähler surface can always be represented by a smooth nonnegative (1, 1)-form (for a counterexample in higher dimensions, see Example 5.4 in [Boucksom et al. 2010]). However, an example of Zariski shows that a nef and big class is not necessarily semiample (see Section 2.3A of [Lazarsfeld 2004]). Also, the nef condition alone is not sufficient for the existence of a nonnegative representative (see Example 1.7 of [Demailly et al. 1994]). What can be proved if we assume only that $[\chi - \omega]$ is nef and big? In this case, by [Boucksom et al. 2010], we know that we can produce a solution ψ of (2-2) with very mild singularities along C_i (less than any log pole). Can it be translated into an estimate for the solution $\varphi(t)$ of the J-flow? Does it imply that the J-functional is bounded from below?

Question 4.2. The results of [Fang and Lai 2012a] indicate a possible picture when $[\chi]$ is outside of $\mathcal{C}_\omega$. But they assume both ω and χ are of Calabi ansatz. Can one prove a general result on Kähler surfaces? In this case, presumably the $\mathcal{J}$-functional is not bounded from below.

Question 4.3. For general n, it would be interesting to investigate the weak solution of the critical equation (1-2) when $[\chi]$ does not lie in $\mathcal{C}_\omega$.

Acknowledgements

We thank Valentino Tosatti for some helpful suggestions and discussions. We also thank the referees for their comments, which helped to improve the exposition, and for pointing out some inaccuracies in a previous version of this paper.

References

[Aubin 1978] T. Aubin, "Équations du type Monge–Ampère sur les variétés Kählériennes compactes", *Bull. Sci. Math.* (2) **102**:1 (1978), 63–95. MR 81d:53047 Zbl 0374.53022

[Boucksom et al. 2010] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi, "Monge–Ampère equations in big cohomology classes", *Acta Math.* **205**:2 (2010), 199–262. MR 2011k:32049 Zbl 1213.32025

[Buchdahl 1999] N. Buchdahl, "On compact Kähler surfaces", *Ann. Inst. Fourier* (*Grenoble*) **49**:1 (1999), 287–302. MR 2000f:32029 Zbl 0926.32025

[Calabi 1982] E. Calabi, "Extremal Kähler metrics", pp. 259–290 in *Seminar on Differential Geometry*, edited by S.-T. Yau, Ann. of Math. Stud. **102**, Princeton University Press, 1982. MR 83i:53088 Zbl 0487.53057

[Chen 2000] X. Chen, "On the lower bound of the Mabuchi energy and its application", *Int. Math. Res. Not.* **2000**:12 (2000), 607–623. MR 2001f:32042 Zbl 0980.58007

[Chen 2004] X. Chen, "A new parabolic flow in Kähler manifolds", *Comm. Anal. Geom.* **12**:4 (2004), 837–852. MR 2005h:53116 Zbl 1073.53089

[Demailly et al. 1994] J.-P. Demailly, T. Peternell, and M. Schneider, "Compact complex manifolds with numerically effective tangent bundles", *J. Algebraic Geom.* **3**:2 (1994), 295–345. MR 95f:32037 Zbl 0827.14027

[Donaldson 1999] S. K. Donaldson, "Moment maps and diffeomorphisms", *Asian J. Math.* **3**:1 (1999), 1–15. MR 2001a:53122 Zbl 0999.53053

[Donaldson 2002] S. K. Donaldson, "Scalar curvature and stability of toric varieties", *J. Differential Geom.* **62**:2 (2002), 289–349. MR 2005c:32028 Zbl 1074.53059

[Evans 1982] L. C. Evans, "Classical solutions of fully nonlinear, convex, second-order elliptic equations", *Comm. Pure Appl. Math.* **35**:3 (1982), 333–363. MR 83g:35038 Zbl 0469.35022

[Eyssidieux et al. 2009] P. Eyssidieux, V. Guedj, and A. Zeriahi, "Singular Kähler–Einstein metrics", *J. Amer. Math. Soc.* **22**:3 (2009), 607–639. MR 2010k:32031 Zbl 1215.32017

[Eyssidieux et al. 2011] P. Eyssidieux, V. Guedj, and A. Zeriahi, "Viscosity solutions to degenerate complex Monge–Ampère equations", *Comm. Pure Appl. Math.* **64**:8 (2011), 1059–1094. MR 2012e:32062 Zbl 1227.32042

[Fang and Lai 2012a] H. Fang and M. Lai, "Convergence of general inverse σ_k-flow on Kähler manifolds with Calabi ansatz", preprint, 2012. arXiv 1203.5253

[Fang and Lai 2012b] H. Fang and M. Lai, "On the geometric flows solving Kählerian inverse σ_k equations", *Pacific J. Math.* **258**:2 (2012), 291–304. MR 2981955 Zbl 1267.53067 arXiv 1203.2571

[Fang et al. 2011] H. Fang, M. Lai, and X. Ma, "On a class of fully nonlinear flows in Kähler geometry", *J. Reine Angew. Math.* **653** (2011), 189–220. MR 2012g:53134 Zbl 1222.53070

[Guedj and Zeriahi 2005] V. Guedj and A. Zeriahi, "Intrinsic capacities on compact Kähler manifolds", *J. Geom. Anal.* **15**:4 (2005), 607–639. MR 2006j:32041 Zbl 1087.32020

[Kołodziej 1998] S. Kołodziej, "The complex Monge–Ampère equation", *Acta Math.* **180**:1 (1998), 69–117. MR 99h:32017 Zbl 0913.35043

[Kołodziej 2005] S. Kołodziej, "The complex Monge–Ampère equation and pluripotential theory", *Mem. Amer. Math. Soc.* **178**:840 (2005). MR 2006d:32054 Zbl 1084.32027

[Krylov 1982] N. V. Krylov, "Ограниченно неоднородные эллиптические и параболические уравнения", *Izv. Akad. Nauk SSSR Ser. Mat.* **46**:3 (1982), 487–523. Translated as "Boundedly nonhomogeneous elliptic and parabolic equations" in *Math. USSR Izv.* **20**:3 (1983), 459–492. MR 84a:35091 Zbl 0529.35026

[Lamari 1999] A. Lamari, "Le cône Kählérien d'une surface", *J. Math. Pures Appl.* (9) **78**:3 (1999), 249–263. MR 2000e:32029 Zbl 0941.32007

[Lazarsfeld 2004] R. Lazarsfeld, *Positivity in algebraic geometry, I: Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **48**, Springer, Berlin, 2004. MR 2005k:14001a Zbl 1093.14501

[LeBrun and Simanca 1994] C. LeBrun and S. R. Simanca, "Extremal Kähler metrics and complex deformation theory", *Geom. Funct. Anal.* **4**:3 (1994), 298–336. MR 95k:58041 Zbl 0801.53050

[Mabuchi 1986] T. Mabuchi, “K-energy maps integrating Futaki invariants”, *Tohoku Math. J.* (2) **38**:4 (1986), 575–593. MR 88b:53060 Zbl 0619.53040

[Phong et al. 2008] D. H. Phong, J. Song, J. Sturm, and B. Weinkove, “The Moser–Trudinger inequality on Kähler–Einstein manifolds”, *Amer. J. Math.* **130**:4 (2008), 1067–1085. MR 2009e:32027 Zbl 1158.58005

[Ross 2006] J. Ross, “Unstable products of smooth curves”, *Invent. Math.* **165**:1 (2006), 153–162. MR 2007c:32027 Zbl 1107.14011

[Song and Weinkove 2008] J. Song and B. Weinkove, “On the convergence and singularities of the J-flow with applications to the Mabuchi energy”, *Comm. Pure Appl. Math.* **61**:2 (2008), 210–229. MR 2009a:32038 Zbl 1135.53047

[Tian 1997] G. Tian, “Kähler–Einstein metrics with positive scalar curvature”, *Invent. Math.* **130**:1 (1997), 1–37. MR 99e:53065 Zbl 0892.53027

[Tian 2000] G. Tian, *Canonical metrics in Kähler geometry*, Birkhäuser, Basel, 2000. MR 2001j:32024 Zbl 0978.53002

[Tian and Zhu 2000] G. Tian and X. Zhu, “A nonlinear inequality of Moser–Trudinger type”, *Calc. Var. Partial Differential Equations* **10**:4 (2000), 349–354. MR 2001f:32044 Zbl 0987.32017

[Tsuji 1988] H. Tsuji, “Existence and degeneration of Kähler–Einstein metrics on minimal algebraic varieties of general type”, *Math. Ann.* **281**:1 (1988), 123–133. MR 89e:53075 Zbl 0631.53051

[Weinkove 2004] B. Weinkove, “Convergence of the J-flow on Kähler surfaces”, *Comm. Anal. Geom.* **12**:4 (2004), 949–965. MR 2005g:32027 Zbl 1073.53092

[Weinkove 2006] B. Weinkove, “On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy”, *J. Differential Geom.* **73**:2 (2006), 351–358. MR 2007a:32026 Zbl 1107.53048

[Yau 1978] S.-T. Yau, “On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I”, *Comm. Pure Appl. Math.* **31**:3 (1978), 339–411. MR 81d:53045 Zbl 0369.53059

[Yau 1993] S.-T. Yau, “Open problems in geometry”, pp. 1–28 in *Differential geometry, 1: Partial differential equations on manifolds* (Los Angeles, 1990), edited by R. E. Greene and S.-T. Yau, Proc. Sympos. Pure Math. **54**, American Mathematical Society, Providence, RI, 1993. MR 94k:53001 Zbl 0801.53001

[Zhang 2006] Z. Zhang, “On degenerate Monge–Ampère equations over closed Kähler manifolds”, *Int. Math. Res. Not.* **2006** (2006), Art. ID #63640. MR 2007b:32058 Zbl 1112.32021

Received 9 Jan 2013. Accepted 22 Aug 2013.

HAO FANG: hao-fang@uiowa.edu
Department of Mathematics, University of Iowa, Iowa City, IA 52242, United States

MIJIA LAI: lai@math.rochester.edu
Department of Mathematics, University of Rochester, Rochester, NY 14642, United States

JIAN SONG: jiansong@math.rutgers.edu
Department of Mathematics, Rutgers University, Piscataway, NJ 08854, United States

BEN WEINKOVE: weinkove@math.northwestern.edu
Mathematics Department, Northwestern University, Evanston, IL 60208, United States

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2014 is US $180/year for the electronic version, and $355/year (+$50, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 7 No. 1 2014

2157-5045(2014)7:1;1-F