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THE J-FLOW ON KÄHLER SURFACES: A BOUNDARY CASE

HAO FANG, MIJIA LAI, JIAN SONG AND BEN WEINKOVE

We study the J-flow on Kähler surfaces when the Kähler class lies on the boundary of the open cone for which global smooth convergence holds and satisfies a nonnegativity condition. We obtain a C^0 estimate and show that the J-flow converges smoothly to a singular Kähler metric away from a finite number of curves of negative self-intersection on the surface. We discuss an application to the Mabuchi energy functional on Kähler surfaces with ample canonical bundle.

1. Introduction

The J-flow is a parabolic flow on Kähler manifolds with two Kähler classes. It was defined by Donaldson [1999] in the setting of moment maps and by Chen [2000] as the gradient flow of the $\mathcal{J}$-functional appearing in his formula for the Mabuchi energy [1986].

The J-flow is defined as follows. Let X be a compact Kähler manifold with two Kähler metrics ω and χ in different Kähler classes $[\omega]$ and $[\chi]$. Let $\mathcal{P}_\chi$ be the space of smooth χ-plurisubharmonic functions on X:

$$\mathcal{P}_\chi = \{\varphi \mid \chi_\varphi := \chi + dd^c\varphi > 0\}.$$

Then the J-flow is a flow defined in $\mathcal{P}_\chi$ by

$$\frac{\partial}{\partial t}\varphi = c - \frac{n\chi_\varphi^{n-1}\wedge\omega}{\chi_\varphi^n}, \qquad \varphi(0) = \varphi_0 \in \mathcal{P}_\chi, \tag{1-1}$$

where c is the topological constant given by

$$c = \frac{n[\chi]^{n-1}\cdot[\omega]}{[\chi]^n}.$$

A stationary point of (1-1) gives a critical Kähler metric $\tilde{\chi} \in [\chi]$ satisfying

$$c\tilde{\chi}^n = n\tilde{\chi}^{n-1}\wedge\omega. \tag{1-2}$$

Donaldson [1999] noted that a smooth critical metric exists only if the cohomological condition $[c\chi - \omega] > 0$ holds. In complex dimension 2, Chen [2000] showed that this necessary condition is sufficient for the existence of a smooth critical metric by observing that in this case, (1-2) is equivalent to the complex Monge–Ampère equation solved by Yau [1978] (see (2-2) below). Chen [2004] also

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established the long time existence for the J-flow (1-1) with any initial data. Weinkove [2004; 2006] showed that the J-flow converges to a critical metric if the cohomological condition $[c\chi-(n-1)\omega]>0$ holds. In particular, if X is a Kähler surface, a necessary and sufficient condition for convergence of the flow to a smooth critical metric is Donaldson's cohomological condition $[c\chi-\omega]>0$.

Song and Weinkove [2008] found a necessary and sufficient condition for the convergence of the J-flow in higher dimensions, which we now explain. Define

$$\mathscr{C}_\omega := \{[\chi]>0 \mid \text{there exists } \chi'\in[\chi] \text{ such that } c\chi'^{n-1}-(n-1)\chi'^{n-2}\wedge\omega>0\}. \tag{1-3}$$

Then the J-flow (1-1) converges smoothly to the critical metric solving (1-2) if and only if $[\chi]\in\mathscr{C}_\omega$.

In [Fang et al. 2011; Fang and Lai 2012b], the J-flow was generalized to the general inverse σ_k flow. An analogous necessary and sufficient condition is found to ensure the smooth convergence of the flow.

The behavior of the J-flow in the case when the condition $[\chi]\in\mathscr{C}_\omega$ does *not* hold is still largely open. However, recent progress was made by Fang and Lai [2012a] in the case of a family of Kähler manifolds satisfying the Calabi symmetry condition. It was shown (in the more general case of the inverse σ_k flow) that if the initial metric satisfies the Calabi symmetry, the flow converges to a Kähler current which is the sum of a Kähler metric with a conic singularity and a current of integration along a divisor.

We consider the case when X is a Kähler surface. As discussed above, a necessary and sufficient condition for convergence of the flow to a smooth critical metric is

$$[c\chi-\omega]>0. \tag{1-4}$$

Donaldson [1999] remarked that if this condition fails, then one might expect the J-flow to blow up over some curves of negative self-intersection. It was observed in [Song and Weinkove 2008, Proposition 4.5] that, applying the results of Buchdahl [1999] and Lamari [1999], there exist a finite number $N\geq 0$, say, of irreducible curves C_i with $C_i^2<0$ on X and positive real numbers a_i such that $[c\chi-\omega]-\sum_{i=1}^N a_i[C_i]$ is Kähler. It was shown in [Song and Weinkove 2008] that at least for some sequence of points approaching some C_i, the quantity $|\varphi|+|\Delta_\omega\varphi|$ blows up.

In this paper we describe the behavior of the J-flow for certain classes $[\chi]$ on the boundary of $\mathscr{C}_\omega$. First we introduce some notation: given a closed $(1,1)$-form α, write $[\alpha]\geq 0$ if there exists a smooth closed nonnegative $(1,1)$-form cohomologous to α. We consider any Kähler class $[\chi]$ satisfying

$$[c\chi-\omega]\geq 0. \tag{1-5}$$

All such classes $[\chi]$ lie in the closure of $\mathscr{C}_\omega$. The boundary of $\mathscr{C}_\omega$ consists of Kähler classes $[\chi]$ such that $[c\chi-\omega]$ is *nef*, which means that for every $\varepsilon>0$ there exists a representative of $[c\chi-\omega]$ which is bounded below by $-\varepsilon\omega$. Further, since

$$[c\chi-\omega]^2=[\omega]^2>0,$$

the class $[c\chi-\omega]$ is nef and big. Nevertheless, to our knowledge, this does not imply that it satisfies (1-5) — see Question 4.1 below. However, at least in many cases the condition (1-5) is equivalent to $[\chi]$ belonging to the closure of $\mathscr{C}_\omega$ in the Kähler cone. This holds for all Hirzebruch surfaces, for example, since explicit nonnegative (1, 1)-forms can be found representing all classes on the boundary of the Kähler cone (see the discussion in [Calabi 1982]).

Our main result is this:

Theorem 1.1. *Let X be a compact Kähler surface with Kähler metrics ω and χ such that*

$$[c\chi-\omega]\geq 0, \quad \textit{where } c=\frac{2[\chi]\cdot[\omega]}{[\chi]^2}.$$

Then there exist a finite number of curves C_i on X of negative self-intersection such that the solution $\varphi(t)$ of the J-flow (1-1) converges in $C^\infty_{\mathrm{loc}}\big(X\setminus\bigcup C_i\big)$ to a continuous function φ_∞, smooth on $X\setminus\bigcup C_i$, satisfying

$$c\chi^2_{\varphi_\infty}=2\chi_{\varphi_\infty}\wedge\omega, \quad \textit{for} \quad \chi_{\varphi_\infty}=\chi+dd^c\varphi_\infty\geq 0. \tag{1-6}$$

Moreover, φ_∞ is the unique continuous solution of (1-6) up to the addition of a constant.

Our result makes use of some recent works in the study of complex Monge–Ampère equations that appeared after the breakthrough of Kołodziej [1998]. Indeed, the existence of a unique weak solution to the critical equation (1-6) is a direct consequence of a result of Eyssidieux, Guedj, and Zeriahi [Eyssidieux et al. 2009] and Zhang [2006], who generalized Kołodziej's theorem to the degenerate complex Monge–Ampère equation. By comparing with this solution, we obtain our key uniform estimate for $\varphi(t)$ along the J-flow (Proposition 2.2 below). In addition, we use the viscosity methods introduced in [Eyssidieux et al. 2011] to give a second proof of our key estimate. The results of [Eyssidieux et al. 2011] allow us to conclude that the solution of (1-6) is continuous, and that (1-6) can be understood in both the pluripotential and the viscosity senses.

We have an application of our result to the *Mabuchi energy* [1986], a functional which is closely connected to the problem of algebraic stability and existence of constant scalar curvature Kähler (cscK) metrics [Yau 1993; Tian 1997; Donaldson 2002]. Given a Kähler surface (X,χ), the Mabuchi energy is the functional $\mathrm{Mab}:\mathcal{P}_\chi\to\mathbb{R}$ given by

$$\mathrm{Mab}(\varphi)=-\int_0^1\int_X\frac{\partial\varphi_t}{\partial t}(R_{\chi_{\varphi_t}}-\mu)\chi^n_{\varphi_t}\,dt,$$

where $\{\varphi_t\}_{0\leq t\leq 1}$ is a path in $\mathcal{P}_\chi$ between 0 and φ, $R_{\chi_{\varphi_t}}$ is the scalar curvature of the metric χ_{φ_t}, and μ is the average of the scalar curvature of χ. The value $\mathrm{Mab}(\varphi)$ is independent of the choice of path.

It was conjectured by Tian [1997], assuming X has no nontrivial holomorphic vector fields, that the existence of a cscK metric is equivalent to the *properness* of the Mabuchi energy, meaning that there exists an increasing function $f:[0,\infty)\to\mathbb{R}$ with $\lim_{x\to\infty}f(x)=\infty$ such that

$$\mathrm{Mab}(\varphi)\geq f(E(\varphi)), \quad \text{where} \quad E(\varphi)=\int_X\sqrt{-1}\,\partial\varphi\wedge\bar\partial\varphi\wedge(\chi_0+\chi_\varphi).$$

This conjecture holds whenever $[\chi]=-c_1(X)>0$ or if $[\chi]=c_1(X)>0$ and X has no nontrivial holomorphic vector fields [Tian 1997; 2000; Tian and Zhu 2000]. It also holds on all manifolds with $c_1(X)=0$, even in the presence of holomorphic vector fields [Tian 2000]. In fact in each case, the function f can be taken to be linear [Tian 2000; Phong et al. 2008]. Chen [2000] showed that on manifolds with

$c_1(X) < 0$, or equivalently, with ample canonical bundle K_X, the Mabuchi energy can be written as a sum of two terms: the first is the $\mathcal{J}$-functional with reference metric ω in $[K_X]$, and the second is a term which is bounded below. In fact, the second term is proper [Tian 2000] (see the discussion in [Song and Weinkove 2008]), and under the cohomological condition $[c\chi - \omega] \geq 0$, the $\mathcal{J}$-functional has a lower bound, as shown in Corollary 3.3 below. Hence we obtain:

Corollary 1.2. *Suppose that X is a compact Kähler surface with ample canonical bundle K_X. Then the Mabuchi energy is proper on the classes $[\chi]$ satisfying*

$$\left(\frac{2[\chi]\cdot[K_X]}{[\chi]^2}\right)[\chi]-[K_X]\geq 0. \tag{1-7}$$

Moreover, the function f in the definition of properness can be taken to be linear.

Thus, since the condition of K_X being ample implies that X has no nontrivial holomorphic vector fields, conjecturally, classes $[\chi]$ in the cone given by (1-7) should admit cscK metrics. The class $[K_X]$ is inside this cone and admits a cscK metric [Aubin 1978; Yau 1978]. The same is true for classes sufficiently close to $[K_X]$ (see [LeBrun and Simanca 1994]). On the other hand, Ross [2006] found Kähler classes on surfaces with K_X ample that do not admit cscK metrics. Corollary 1.2, together with the arguments of [LeBrun and Simanca 1994], suggests that the set of classes that admit cscK metrics is strictly larger than those lying in the cone (1-7).

An outline of the paper is as follows. In Section 2, we prove the key C^0 estimate. We provide two proofs: the first uses smooth maximum principle arguments and the second uses the notion of viscosity solutions from [Eyssidieux et al. 2011]. We complete the proof of the main theorem in Section 3, and in the last section we finish with some questions for further study.

2. The C^0 estimate

For convenience of notation, we assume from now on that $c = 1$. We may do this by considering $(1/c)[\chi]$ instead of $[\chi]$. In addition, we may assume, by modifying the initial data if necessary, that $\chi - \omega \geq 0$.

The key estimate we need is a uniform C^0 estimate for the solution $\varphi(t)$ of the J-flow. We need the following theorem on the degenerate complex Monge–Ampère equation (the C^0 estimate was proved independently in [Zhang 2006] under slightly less general hypotheses).

Theorem 2.1 [Eyssidieux et al. 2009; 2011]. *Let (M, ω) be a compact Kähler manifold of complex dimension n and let α be a semipositive $(1,1)$-form with $\int_M \alpha^n > 0$. For any nonnegative $f \in L^p(M, \omega^n)$, for $p > 1$, with $\int_M f\omega^n = \int_M \alpha^n$, there exists a unique continuous function φ on M with $\alpha + dd^c\varphi \geq 0$ and*

$$(\alpha + dd^c\varphi)^n = f\omega^n, \quad \sup_M \varphi = 0. \tag{2-1}$$

Moreover, $\|\varphi\|_{C^0(M)}$ is uniformly bounded by a constant depending only on p, M, ω, α and $\|f\|_{L^p(M)}$.

Given this, we immediately obtain a solution φ_∞ to (1-6), using the observation of Chen [2000] that the critical equation can be rewritten as a complex Monge–Ampère equation:

$$\chi_\varphi^2 = 2\chi_\varphi \wedge \omega \iff (\chi_\varphi - \omega)^2 = \omega^2. \tag{2-2}$$

Writing $\alpha := \chi - \omega \geq 0$ on the Kähler surface X, we can apply Theorem 2.1 to see that there exists a continuous function φ_∞ solving (1-6). Moverover, φ_∞ is unique up to the addition of a constant.

Next we use the uniform C^0 bound from Theorem 2.1 to obtain:

Proposition 2.2. *We assume that $\chi - \omega \geq 0$ as discussed above. Let $\varphi(t)$ be the solution of J-flow (1-1) on the compact Kähler surface X. Then there exists C depending only on the initial data such that for all $t \geq 0$,*

$$\|\varphi(t)\|_{C^0(X)} \leq C. \tag{2-3}$$

Proof. From the introduction, we know

$$[\chi - \omega] - \sum_{i=1}^{N} a_i [C_i] > 0, \tag{2-4}$$

for positive real numbers a_i and irreducible curves C_i of negative self-intersection. Since we are assuming $[\chi - \omega] \geq 0$, we may take the constants a_i to be arbitrarily small. However, we will not need to make use of this last fact.

It follows that there exist Hermitian metrics h_i on the line bundles $[C_i]$ associated to C_i such that

$$\chi - \omega - \sum_{i=1}^{N} a_i R_{h_i} > 0, \tag{2-5}$$

where $R_{h_i} = -dd^c \log h_i$ is the curvature of h_i. Let s_i be a holomorphic section of $[C_i]$ vanishing along C_i to order 1. Recall that we denote $\chi - \omega$ by α.

Next, we apply Theorem 2.1 and write ψ for the solution to the degenerate complex Monge–Ampère equation

$$(\alpha + dd^c \psi)^2 = \omega^2, \quad \alpha + dd^c \psi \geq 0, \tag{2-6}$$

subject to the condition $\sup_X \psi = 0$. We have $\|\psi\|_{C^0(X)} \leq C$.

It follows from a trick of Tsuji [1988], as used in [Eyssidieux et al. 2009], that ψ is smooth away from the curves C_i. Although the proof is the same, the precise statement we need does not seem to be quite contained in [Eyssidieux et al. 2009], so we briefly outline the idea here for the convenience of the reader. For $\delta > 0$, let ψ_δ be Yau's solution of the complex Monge–Ampère equation

$$(\alpha + \delta\omega + dd^c \psi_\delta)^2 = c_\delta \omega^2, \quad \alpha_\delta := \alpha + \delta\omega + dd^c \psi_\delta > 0, \tag{2-7}$$

for a constant c_δ chosen so that the integrals of both sides are equal. From Theorem 2.1, ψ_δ is uniformly bounded in C^0. To obtain a second-order estimate for ψ_δ, uniform in δ, we consider, for a constant $A > 0$,

$$Q_\delta = \log \operatorname{tr}_\omega \alpha_\delta - A\left(\psi_\delta - \sum_i a_i \log |s_i|^2_{h_i}\right), \tag{2-8}$$

which is well-defined on $X \setminus \bigcup C_i$ and tends to $-\infty$ on $\bigcup C_i$. Compute, at a point in $X \setminus \bigcup C_i$,

$$\Delta_{\alpha_\delta} Q_\delta \geq -C \operatorname{tr}_{\alpha_\delta} \omega - 2A + A \operatorname{tr}_{\alpha_\delta}\left(\alpha - \sum_i a_i R_{h_i}\right).$$

Then using (2-5), we may choose a uniform A sufficiently large that

$$A\Big(\alpha-\sum_i a_i R_{h_i}\Big)\geq (C+1)\omega.$$

The quantity Q_δ achieves a maximum at some point $x\in X\setminus\bigcup C_i$, and at this point we have $\Delta_{\alpha_\delta}Q_\delta\leq 0$. Hence, at x,

$$0\geq \operatorname{tr}_{\alpha_\delta}\omega-2A,$$

so $\operatorname{tr}_{\alpha_\delta}\omega$ is uniformly bounded from above. But by (2-7) we have at x

$$\operatorname{tr}_\omega \alpha_\delta=\left(\frac{\alpha_\delta^2}{\omega^2}\right)\operatorname{tr}_{\alpha_\delta}\omega=c_\delta \operatorname{tr}_{\alpha_\delta}\omega\leq C',$$

for some uniform C'. Since ψ_δ is uniformly bounded in C^0, we see that Q_δ is uniformly bounded from above at x, and hence everywhere.

This establishes a uniform upper bound for $\operatorname{tr}_\omega\alpha_\delta$ (and again by (2-7), also for $\operatorname{tr}_{\alpha_\delta}\omega$) on any compact subset of $X\setminus\bigcup C_i$. It follows that on such a fixed compact set, ω and α_δ are uniformly equivalent. Hence we have estimates, uniform in δ, for $dd^c\psi_\delta$ on compact subsets of $X\setminus\bigcup C_i$. The $C^\infty_{\text{loc}}(X\setminus\bigcup C_i)$ estimates for ψ_δ then follow from the usual Evans–Krylov local theory for the complex Monge–Ampère equation [Evans 1982; Krylov 1982]. Taking a limit as $\delta\to 0$ shows that ψ is smooth away from the C_i.

Fix $\varepsilon\in(0,1)$. We will apply the maximum principle to the quantity

$$\theta_\varepsilon=\varphi-(1+\varepsilon)\psi+\varepsilon\sum_{i=1}^N a_i\log|s_i|^2_{h_i}-A\varepsilon t,$$

where A is a constant to be determined. Observe that θ_ε is smooth on $X\setminus\bigcup C_i$ and tends to negative infinity along $\bigcup C_i$, and hence θ_ε achieves a maximum in the interior of $X\setminus\bigcup C_i$ for each time t.

We rewrite (1-1) as

$$\frac{\partial\varphi}{\partial t}=1-\frac{2\chi_\varphi\wedge\omega}{\chi_\varphi^2}=\frac{\chi_\varphi^2-2\chi_\varphi\wedge\omega}{\chi_\varphi^2}=\frac{(\chi_\varphi-\omega)^2-\omega^2}{\chi_\varphi^2}=\frac{\omega^2}{\chi_\varphi^2}\left(\frac{(\chi_\varphi-\omega)^2}{\omega^2}-1\right)=\frac{\omega^2}{\chi_\varphi^2}\left(\frac{\alpha_\varphi^2}{\alpha_\psi^2}-1\right). \tag{2-9}$$

Compute on $X\setminus\bigcup C_i$, using (2-9),

$$\begin{aligned}\frac{\partial}{\partial t}\theta_\varepsilon&=\frac{\omega^2}{\chi_\varphi^2}\left(\frac{(\alpha+dd^c\varphi)^2}{(\alpha+dd^c\psi)^2}-1\right)-A\varepsilon\\&=\frac{\omega^2}{\chi_\varphi^2}\left(\frac{\big((1+\varepsilon)\alpha+(1+\varepsilon)dd^c\psi-\varepsilon(\alpha-\sum a_iR_{h_i})+dd^c\theta_\varepsilon\big)^2}{(\alpha+dd^c\psi)^2}-1\right)-A\varepsilon.\end{aligned}$$

But $\alpha-\sum a_iR_{h_i}\geq 0$, and at the maximum of θ_ε, we have $dd^c\theta_\varepsilon\leq 0$. Hence at the maximum of θ_ε,

$$\frac{\partial}{\partial t}\theta_\varepsilon\leq\frac{\omega^2}{\chi_\varphi^2}\left((1+\varepsilon)^2\frac{(\alpha+dd^c\psi)^2}{(\alpha+dd^c\psi)^2}-1\right)-A\varepsilon<0, \tag{2-10}$$

if we choose

$$A = \sup_{X\times[0,\infty)} \frac{3\omega^2}{\chi_\varphi^2},$$

which is a uniform constant since χ_φ is always uniformly bounded from below away from zero along the J-flow. Indeed, this follows immediately from taking a time derivative of the J-flow equation and applying the maximum principle (see Lemma 4.1 in [Chen 2004]). Then (2-10) implies that θ_ε must achieve its maximum at time zero, and hence θ_ε is uniformly bounded from above by a constant independent of ε. Letting $\varepsilon \to 0$, we obtain the upper bound for φ.

The lower bound of φ is similar: just replace ε by $-\varepsilon$ and consider the minimum instead of the maximum. □

We provide a second proof. The proof is based on the equivalence of two notions of weak solution of (2-2): the pluripotential sense and the viscosity sense.

Second proof of Proposition 2.2. As in the first proof, write ψ for the solution to (2-6) with $\sup_X \psi = 0$. The function ψ is continuous on X and is smooth away from the curves C_i. We now apply Theorem 3.6 of [Eyssidieux et al. 2011], which states that ψ satisfies (2-6) in the viscosity sense as defined in that paper.

We refer to [Eyssidieux et al. 2011] for the precise definition of a viscosity solution to (2-6) and state two consequences of this definition which are sufficient for our purposes:

(i) If x_0 is any point on X and q is any smooth function defined in a neighborhood of x_0 such that

$$\psi - q \text{ has a local maximum at } x_0,$$

then $(\alpha + dd^c q)^2 \geq \omega^2$ at x_0.

(ii) If x_0 is any point on X and q is any smooth function defined in a neighborhood of x_0 such that

$$\psi - q \text{ has a local minimum at } x_0,$$

then $(\alpha + dd^c q)^2 \leq \omega^2$ at x_0.

Indeed, (i) follows from the definition of a viscosity subsolution, and (ii) from the definition of a viscosity supersolution (see Section 2 in [Eyssidieux et al. 2011]).

We first find an upper bound for φ. Let $\varepsilon > 0$ and define $H_\varepsilon = \varphi - \psi - \varepsilon t$. We wish to show that H_ε attains its maximum value at $t = 0$. Note that H_ε satisfies the equation

$$\frac{\partial H_\varepsilon}{\partial t} = 1 - \frac{2\chi_\varphi \wedge \omega}{\chi_\varphi^2} - \varepsilon.$$

Suppose that H_ε attains a maximum at a point (x_0, t_0) on $X \times [0, T]$ for some finite $T > 0$, and assume for a contradiction that $t_0 > 0$. Then $\partial H_\varepsilon/\partial t\,(x_0, t_0) \geq 0$. Define a smooth function q on X by $q(x) = \varphi(x, t_0) - H_\varepsilon(x_0, t_0) - \varepsilon t_0$. The function

$$x \mapsto (\psi - q)(x) = -H_\varepsilon(x, t_0) + H_\varepsilon(x_0, t_0)$$

achieves its minimum at x_0. Then we can apply (ii) to see that $(\alpha + dd^c q)^2 \le \omega^2$ at x_0, or in other words

$$(\chi - \omega + dd^c \varphi)^2 \le \omega^2, \quad \text{at } (x_0, t_0),$$

which is equivalent to

$$\chi_\varphi^2 \le 2\chi_\varphi \wedge \omega \quad \text{at } (x_0, t_0).$$

It follows that

$$\frac{\partial H_\varepsilon}{\partial t}(x_0, t_0) = 1 - \frac{2\chi_\varphi \wedge \omega}{\chi_\varphi^2} - \varepsilon < 0,$$

contradicting the fact that $\partial H_\varepsilon / \partial t\,(x_0, t_0) \ge 0$. Hence H_ε attains its maximum value at $t = 0$ and is uniformly bounded from above independent of ε. Letting $\varepsilon \to 0$ gives the desired upper bound for φ.

Applying a similar argument, using (i) instead of (ii), gives a uniform lower bound for φ. □

We can now apply Theorem 1.3 of [Song and Weinkove 2008] together with the standard local theory for (1-1) to obtain higher-order estimates.

Proposition 2.3. *As above, assume that $\chi - \omega \ge 0$ on the compact Kähler surface X and let $\varphi(t)$ be the solution of the J-flow (1-1). For any compact subset $K \subset X \setminus \bigcup C_i$ and any $k \ge 0$, there exists a constant $C_{k,K}$ such that for all t,*

$$\|\varphi(t)\|_{C^k(K)} \le C_{k,K}.$$

Here, the C_i are the irreducible curves of negative self-intersection chosen to satisfy (2-4).

3. Proof of the main theorem

Again we assume in this section that $[\chi]$ is scaled so that $c = 1$. Before proving the main theorem we first discuss the $\mathcal{J}$ and $\mathcal{I}$-functionals. Define $\mathcal{J}_{\omega,\chi}$ and $\mathcal{I}_{\omega,\chi}$ by

$$\mathcal{J}_{\omega,\chi}(\varphi) := \int_0^1 \int_X \dot{\varphi}_t \big(2\chi_{\varphi_t} \wedge \omega - \chi_{\varphi_t}^2\big)\, dt,$$
$$\mathcal{I}_{\omega,\chi}(\varphi) := \int_0^1 \int_X \dot{\varphi}_t \chi_{\varphi_t}^2 \, dt,$$

where φ_t is a smooth path in $\mathcal{P}_\chi$ connecting 0 and φ. For simplicity, we will omit the subscripts.

If $\varphi(t)$ is the solution of the J-flow, then

$$\frac{d}{dt}\mathcal{J}(\varphi(t)) = -\int_X \dot{\varphi}(t)^2 \chi_{\varphi(t)}^2, \qquad \frac{d}{dt}\mathcal{I}(\varphi(t)) = 0. \tag{3-1}$$

In particular, the J-flow is the gradient flow of $\mathcal{J}$.

One can write explicit formulae for $\mathcal{J}$, $\mathcal{I}$ as follows:

$$\mathcal{J}(\varphi) = \int_X \varphi\big(\chi_\varphi \wedge \omega + \chi \wedge \omega\big) - \frac{1}{3}\int_X \varphi\big(\chi_\varphi^2 + \chi_\varphi \wedge \chi + \chi^2\big), \tag{3-2}$$

$$\mathcal{I}(\varphi) = \frac{1}{3}\int_X \varphi\big(\chi_\varphi^2 + \chi_\varphi \wedge \chi + \chi^2\big). \tag{3-3}$$

Thus an immediate corollary of Proposition 2.2 is:

Proposition 3.1. *There exists a uniform constant C such that, for $\varphi(t)$ the solution of the J-flow, we have*

$$\mathcal{J}(\varphi(t)) \geq -C$$

for all $t \geq 0$.

In what follows, we will need to make use of a simple continuity-type result for the $\mathcal{I}$ and $\mathcal{J}$ functionals.

Lemma 3.2. *Let $\varphi_j \in \mathcal{P}_\chi$ and let φ be a continuous function on X satisfying $\chi + dd^c\varphi \geq 0$. Let Y be a proper subvariety of X. Suppose that*

(a) *there exists C such that $\|\varphi_j\|_{C^0(X)} \leq C$;*

(b) *$\varphi_j \to \varphi$ in $C^\infty_{\mathrm{loc}}(X \setminus Y)$ as $j \to \infty$.*

Then

$$\mathcal{J}(\varphi_j) \to \mathcal{J}(\varphi) \quad \text{and} \quad \mathcal{I}(\varphi_j) \to \mathcal{I}(\varphi) \quad \text{as } j \to \infty.$$

Proof. The proof is a simple exercise in pluripotential theory (we refer the reader to [Kołodziej 2005] for an introduction to this theory). For the convenience of the reader, we sketch the proof here. For φ continuous with $\chi + dd^c\varphi \geq 0$, the quantities χ_φ^2, $\chi \wedge \chi_\varphi$ and $\chi_\varphi \wedge \omega$ define finite measures on X and hence by (3-2) and (3-3), the functionals $\mathcal{I}(\varphi)$ and $\mathcal{J}(\varphi)$ are well-defined.

We may choose a sequence of open tubular neighborhoods Y_k of Y such that $Y_k \downarrow Y$ as $k \to \infty$. Since Y is pluripolar, the capacity $\mathrm{Cap}_\chi(Y)$ of Y with respect to χ (in the sense of [Kołodziej 1998]) is zero. By the properties of this capacity (see [Guedj and Zeriahi 2005], for example) we have

$$\lim_{k\to\infty} \mathrm{Cap}_\chi(Y_k) = \mathrm{Cap}_\chi(Y) = 0.$$

Since the φ_j are uniformly bounded, it follows that $\int_{Y_k} \varphi_j \beta \wedge \gamma \to 0$ as $k \to \infty$, uniformly in j, where β, γ are each one of ω, χ or χ_{φ_j}. The same holds if we replace φ_j by φ. The result then follows from the expressions (3-2) and (3-3) together with condition (b). □

Proof of Theorem 1.1. Since $\mathcal{J}$ is decreasing and bounded from below, there exists a constant C such that

$$\int_0^\infty \int_X \dot{\varphi}(t)^2 \chi^2_{\varphi(t)}\, dt < C. \tag{3-4}$$

We claim that for each fixed point $p \in X \setminus \bigcup C_i$, we have $\dot{\varphi}(p,t) \to 0$ as $t \to \infty$. Suppose not. Then there exists $\varepsilon > 0$ and a sequence of times $t_i \to \infty$ such that $|\dot{\varphi}(t_i)| > \varepsilon$ for all i. But since we have bounds for $\dot{\varphi}$ and all its time and space derivatives in a fixed neighborhood U, say, of p with $U \subset X \setminus \bigcup C_i$, it follows that $|\dot{\varphi}(t)| > \varepsilon/2$ for $t \in [t_i, t_i + \delta]$ for a uniform $\delta > 0$. This contradicts (3-4) and establishes the claim.

Since we have $C^\infty_{\mathrm{loc}}\big(X \setminus \bigcup C_i\big)$ bounds for $\dot{\varphi}$, the uniqueness of limits implies that $\dot{\varphi}$ converges to zero in $C^\infty_{\mathrm{loc}}\big(X \setminus \bigcup C_i\big)$.

We have uniform C^∞ bounds for $\varphi(t)$ on compact subsets of $X \setminus \bigcup C_i$, and hence we can apply the Arzelà–Ascoli theorem to see that for a sequence of times $t_i \to \infty$, we have $\varphi(t_i) \to \varphi_\infty$ for a smooth (bounded) function φ_∞ on $X \setminus \bigcup C_i$. Since $\dot{\varphi} \to 0$, φ_∞ satisfies the equation $\chi^2_{\varphi_\infty} = 2\chi_{\varphi_\infty} \wedge \omega$ as in the statement of the theorem.

We also have $\mathcal{I}(\varphi_\infty) = \lim_{t\to\infty} \mathcal{I}(\varphi(t)) = \mathcal{I}(\varphi_0)$, using Lemma 3.2 and the fact that $\mathcal{I}$ is constant along the flow. Applying Theorem 2.1, we know that (1-6) has a unique solution up to the addition of a constant. Thus φ_∞ is the unique solution of (1-6) subject to the condition $\mathcal{I}(\varphi_\infty) = \mathcal{I}(\varphi_0)$.

Finally we claim that $\varphi(t)$ converges in $C^\infty_{\mathrm{loc}}\big(X \setminus \bigcup C_i\big)$ to φ_∞. Suppose not. Then there exist $\varepsilon > 0$ and a sequence of times $t_i \to \infty$ such that $\|\varphi(t_i) - \varphi_\infty\|_{C^k(K)} > \varepsilon$ for all i, for some integer k and compact $K \subset X \setminus \bigcup C_i$. Since we have uniform C^∞ bounds for $\varphi(t)$ on K, we can pass to a subsequence and assume that $\varphi(t_i)$ converges to a function $\varphi'_\infty \neq \varphi_\infty$. But φ'_∞ will also satisfy the equations $\chi^2_{\varphi'_\infty} = 2\chi_{\varphi'_\infty} \wedge \omega$ and $\mathcal{I}(\varphi'_\infty) = \mathcal{I}(\varphi_0)$, contradicting the uniqueness. □

As a consequence:

Corollary 3.3. *The $\mathcal{J}$-functional is bounded from below on $\mathcal{P}_\chi$.*

Proof. Take any $\varphi_0 \in \mathcal{P}_\chi$. Then running the J-flow from φ_0, which by Theorem 1.1 converges to φ_∞, we obtain (applying Lemma 3.2)

$$\mathcal{J}(\varphi_0) \geq \lim_{t\to\infty} \mathcal{J}(\varphi(t)) = \mathcal{J}(\varphi_\infty),$$

since $\mathcal{J}$ is decreasing along the flow. □

Proof of Corollary 1.2. Combine Corollary 3.3 and Lemma 4.1 of [Song and Weinkove 2008]. □

4. Further questions

Question 4.1. In general, it does not appear to be known whether a nef and big class on a Kähler surface can always be represented by a smooth nonnegative $(1, 1)$-form (for a counterexample in higher dimensions, see Example 5.4 in [Boucksom et al. 2010]). However, an example of Zariski shows that a nef and big class is not necessarily semiample (see Section 2.3A of [Lazarsfeld 2004]). Also, the nef condition alone is not sufficient for the existence of a nonnegative representative (see Example 1.7 of [Demailly et al. 1994]). What can be proved if we assume only that $[\chi - \omega]$ is nef and big? In this case, by [Boucksom et al. 2010], we know that we can produce a solution ψ of (2-2) with very mild singularities along C_i (less than any log pole). Can it be translated into an estimate for the solution $\varphi(t)$ of the J-flow? Does it imply that the J-functional is bounded from below?

Question 4.2. The results of [Fang and Lai 2012a] indicate a possible picture when $[\chi]$ is outside of $\mathcal{C}_\omega$. But they assume both ω and χ are of Calabi ansatz. Can one prove a general result on Kähler surfaces? In this case, presumably the $\mathcal{J}$-functional is not bounded from below.

Question 4.3. For general n, it would be interesting to investigate the weak solution of the critical equation (1-2) when $[\chi]$ does not lie in $\mathcal{C}_\omega$.

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References

[Aubin 1978] T. Aubin, "Équations du type Monge–Ampère sur les variétés Kählériennes compactes", *Bull. Sci. Math.* (2) **102**:1 (1978), 63–95. MR 81d:53047 Zbl 0374.53022

[Boucksom et al. 2010] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi, "Monge–Ampère equations in big cohomology classes", *Acta Math.* **205**:2 (2010), 199–262. MR 2011k:32049 Zbl 1213.32025

[Buchdahl 1999] N. Buchdahl, "On compact Kähler surfaces", *Ann. Inst. Fourier* (*Grenoble*) **49**:1 (1999), 287–302. MR 2000f:32029 Zbl 0926.32025

[Calabi 1982] E. Calabi, "Extremal Kähler metrics", pp. 259–290 in *Seminar on Differential Geometry*, edited by S.-T. Yau, Ann. of Math. Stud. **102**, Princeton University Press, 1982. MR 83i:53088 Zbl 0487.53057

[Chen 2000] X. Chen, "On the lower bound of the Mabuchi energy and its application", *Int. Math. Res. Not.* **2000**:12 (2000), 607–623. MR 2001f:32042 Zbl 0980.58007

[Chen 2004] X. Chen, "A new parabolic flow in Kähler manifolds", *Comm. Anal. Geom.* **12**:4 (2004), 837–852. MR 2005h:53116 Zbl 1073.53089

[Demailly et al. 1994] J.-P. Demailly, T. Peternell, and M. Schneider, "Compact complex manifolds with numerically effective tangent bundles", *J. Algebraic Geom.* **3**:2 (1994), 295–345. MR 95f:32037 Zbl 0827.14027

[Donaldson 1999] S. K. Donaldson, "Moment maps and diffeomorphisms", *Asian J. Math.* **3**:1 (1999), 1–15. MR 2001a:53122 Zbl 0999.53053

[Donaldson 2002] S. K. Donaldson, "Scalar curvature and stability of toric varieties", *J. Differential Geom.* **62**:2 (2002), 289–349. MR 2005c:32028 Zbl 1074.53059

[Evans 1982] L. C. Evans, "Classical solutions of fully nonlinear, convex, second-order elliptic equations", *Comm. Pure Appl. Math.* **35**:3 (1982), 333–363. MR 83g:35038 Zbl 0469.35022

[Eyssidieux et al. 2009] P. Eyssidieux, V. Guedj, and A. Zeriahi, "Singular Kähler–Einstein metrics", *J. Amer. Math. Soc.* **22**:3 (2009), 607–639. MR 2010k:32031 Zbl 1215.32017

[Eyssidieux et al. 2011] P. Eyssidieux, V. Guedj, and A. Zeriahi, "Viscosity solutions to degenerate complex Monge–Ampère equations", *Comm. Pure Appl. Math.* **64**:8 (2011), 1059–1094. MR 2012e:32062 Zbl 1227.32042

[Fang and Lai 2012a] H. Fang and M. Lai, "Convergence of general inverse σ_k-flow on Kähler manifolds with Calabi ansatz", preprint, 2012. arXiv 1203.5253

[Fang and Lai 2012b] H. Fang and M. Lai, "On the geometric flows solving Kählerian inverse σ_k equations", *Pacific J. Math.* **258**:2 (2012), 291–304. MR 2981955 Zbl 1267.53067 arXiv 1203.2571

[Fang et al. 2011] H. Fang, M. Lai, and X. Ma, "On a class of fully nonlinear flows in Kähler geometry", *J. Reine Angew. Math.* **653** (2011), 189–220. MR 2012g:53134 Zbl 1222.53070

[Guedj and Zeriahi 2005] V. Guedj and A. Zeriahi, "Intrinsic capacities on compact Kähler manifolds", *J. Geom. Anal.* **15**:4 (2005), 607–639. MR 2006j:32041 Zbl 1087.32020

[Kołodziej 1998] S. Kołodziej, "The complex Monge–Ampère equation", *Acta Math.* **180**:1 (1998), 69–117. MR 99h:32017 Zbl 0913.35043

[Kołodziej 2005] S. Kołodziej, "The complex Monge–Ampère equation and pluripotential theory", *Mem. Amer. Math. Soc.* **178**:840 (2005). MR 2006d:32054 Zbl 1084.32027

[Krylov 1982] N. V. Krylov, "Ограниченно неоднородные эллиптические и параболические уравнения", *Izv. Akad. Nauk SSSR Ser. Mat.* **46**:3 (1982), 487–523. Translated as "Boundedly nonhomogeneous elliptic and parabolic equations" in *Math. USSR Izv.* **20**:3 (1983), 459–492. MR 84a:35091 Zbl 0529.35026

[Lamari 1999] A. Lamari, "Le cône Kählérien d'une surface", *J. Math. Pures Appl.* (9) **78**:3 (1999), 249–263. MR 2000e:32029 Zbl 0941.32007

[Lazarsfeld 2004] R. Lazarsfeld, *Positivity in algebraic geometry, I: Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **48**, Springer, Berlin, 2004. MR 2005k:14001a Zbl 1093.14501

[LeBrun and Simanca 1994] C. LeBrun and S. R. Simanca, "Extremal Kähler metrics and complex deformation theory", *Geom. Funct. Anal.* **4**:3 (1994), 298–336. MR 95k:58041 Zbl 0801.53050

[Mabuchi 1986] T. Mabuchi, "K-energy maps integrating Futaki invariants", *Tohoku Math. J.* (2) **38**:4 (1986), 575–593. MR 88b:53060 Zbl 0619.53040

[Phong et al. 2008] D. H. Phong, J. Song, J. Sturm, and B. Weinkove, "The Moser–Trudinger inequality on Kähler–Einstein manifolds", *Amer. J. Math.* **130**:4 (2008), 1067–1085. MR 2009e:32027 Zbl 1158.58005

[Ross 2006] J. Ross, "Unstable products of smooth curves", *Invent. Math.* **165**:1 (2006), 153–162. MR 2007c:32027 Zbl 1107.14011

[Song and Weinkove 2008] J. Song and B. Weinkove, "On the convergence and singularities of the J-flow with applications to the Mabuchi energy", *Comm. Pure Appl. Math.* **61**:2 (2008), 210–229. MR 2009a:32038 Zbl 1135.53047

[Tian 1997] G. Tian, "Kähler–Einstein metrics with positive scalar curvature", *Invent. Math.* **130**:1 (1997), 1–37. MR 99e:53065 Zbl 0892.53027

[Tian 2000] G. Tian, *Canonical metrics in Kähler geometry*, Birkhäuser, Basel, 2000. MR 2001j:32024 Zbl 0978.53002

[Tian and Zhu 2000] G. Tian and X. Zhu, "A nonlinear inequality of Moser–Trudinger type", *Calc. Var. Partial Differential Equations* **10**:4 (2000), 349–354. MR 2001f:32044 Zbl 0987.32017

[Tsuji 1988] H. Tsuji, "Existence and degeneration of Kähler–Einstein metrics on minimal algebraic varieties of general type", *Math. Ann.* **281**:1 (1988), 123–133. MR 89e:53075 Zbl 0631.53051

[Weinkove 2004] B. Weinkove, "Convergence of the J-flow on Kähler surfaces", *Comm. Anal. Geom.* **12**:4 (2004), 949–965. MR 2005g:32027 Zbl 1073.53092

[Weinkove 2006] B. Weinkove, "On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy", *J. Differential Geom.* **73**:2 (2006), 351–358. MR 2007a:32026 Zbl 1107.53048

[Yau 1978] S.-T. Yau, "On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I", *Comm. Pure Appl. Math.* **31**:3 (1978), 339–411. MR 81d:53045 Zbl 0369.53059

[Yau 1993] S.-T. Yau, "Open problems in geometry", pp. 1–28 in *Differential geometry, 1: Partial differential equations on manifolds* (Los Angeles, 1990), edited by R. E. Greene and S.-T. Yau, Proc. Sympos. Pure Math. **54**, American Mathematical Society, Providence, RI, 1993. MR 94k:53001 Zbl 0801.53001

[Zhang 2006] Z. Zhang, "On degenerate Monge–Ampère equations over closed Kähler manifolds", *Int. Math. Res. Not.* **2006** (2006), Art. ID #63640. MR 2007b:32058 Zbl 1112.32021

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HAO FANG: hao-fang@uiowa.edu
Department of Mathematics, University of Iowa, Iowa City, IA 52242, United States

MIJIA LAI: lai@math.rochester.edu
Department of Mathematics, University of Rochester, Rochester, NY 14642, United States

JIAN SONG: jiansong@math.rutgers.edu
Department of Mathematics, Rutgers University, Piscataway, NJ 08854, United States

BEN WEINKOVE: weinkove@math.northwestern.edu
Mathematics Department, Northwestern University, Evanston, IL 60208, United States

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