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GREGORY ESKIN AND JAMES RALSTON

In memory of Hans Duistermaat

We show that in the absence of a magnetic field the spectrum of the magnetic Schrödinger operator in an annulus depends on the cosine of the flux associated with the magnetic potential. This result follows from an analysis of a singularity in the "wave trace" for this Schrödinger operator, and hence shows that even in the absence of a magnetic field the magnetic potential can change the asymptotics of the Schrödinger spectrum; that is, the Aharonov–Bohm effect takes place. We also study the Aharonov–Bohm effect for the magnetic Schrödinger operator on a torus.

1. Introduction

Let Ω be the exterior of a bounded region in $\mathbb{R}^2$ with smooth boundary, and let

$$H_{A,V} = \tfrac{1}{2}(i\,\partial_{x_1} + A_1(x))^2 + \tfrac{1}{2}(i\,\partial_{x_2} + A_2(x))^2 - V(x).$$

This is the Schrödinger operator for a particle of mass 1 and charge -1 moving in Ω under the influence of the magnetic potential $A = (A_1, A_2)$ and the electric potential V. We assume that

$$\partial_{x_2} A_1 - \partial_{x_1} A_2 = 0 \quad \text{in } \Omega, \tag{1-1}$$

that is, the magnetic field vanishes in Ω. Given a simple, closed curve γ in Ω encircling the complement of Ω, we define the magnetic flux by

$$\alpha_\gamma = \int_\gamma A(x) \cdot dx.$$

In view of (1-1) α_γ only depends on the orientation of γ.

In a seminal paper, Aharonov and Bohm [1959] showed that if $\alpha_\gamma \neq 0 \bmod 2\pi$, then one can detect the cosine of the magnetic flux in the scattering of particles in this quantum system, that is, the magnetic potential has a physical impact even when the magnetic field is zero in Ω. This is called the Aharonov–Bohm effect. Aharonov and Bohm found this by computing the scattering cross-section explicitly for $\Omega = \mathbb{R}^2\setminus\{0\}$, when $A(x) = (-x_2/|x|^2, x_1/|x|^2)$ and $V(x) = 0$. They also proposed an experiment to demonstrate this effect. However, the first generally accepted experimental verification of the Aharonov–Bohm (AB) effect was done many years later in [Tonomura et al. 1986]. For further mathematical work

on the AB effect, see [Nicoleau 2000; Weder 2002; Roux and Yafaev 2002; Eskin 2013; Eskin et al. 2010].

Helffer [1988] showed that $A(x)$ can influence the spectrum of $H_{A,V}$ when the magnetic field is zero in Ω. In the semiclassical setting with $V(x) \to \infty$, as $|x| \to \infty$, and $\Omega = \{|x| > 1\}$ he showed that the lowest Dirichlet eigenvalue depended on the cosine of the magnetic flux. Earlier related results on magnetic Schrödinger operators are due to Lavine and O'Carroll [1977].

In this paper we study the Schrödinger operator in the domain $\Omega_R = \Omega \cap \{|x| < R\}$ with Dirichlet boundary conditions on $|x| = R$ and $\partial\Omega$. We compute the singularity at $t = 3R\sqrt{3}$ of the distribution trace of the fundamental solution of the initial-boundary value problem

$$
\begin{aligned}
u_{tt} + H_{A,V}u &= 0 & &\text{in } \Omega_R \times \mathbb{R}_t, \\
u(x, 0) &= f(x) \text{ and } u_t(x, 0) = 0 & &\text{in } \Omega_R, \\
u(x, t) &= 0 & &\text{when } x \in \partial\Omega_R.
\end{aligned}
\tag{1-2}
$$

This distribution trace is known as the "wave trace" for this problem, and it is given by

$$
\sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j}),
$$

where $\{\lambda_j\}_{j=1}^{\infty}$ are the Dirichlet eigenvalues of $H_{A,V}$ in Ω_R. Hence its singularities are determined by the behavior of the λ_j as $j \to \infty$. These singularities are well-known to appear only at the lengths of periodic broken ray paths in Ω_R. The singularity at $t = 3R\sqrt{3}$ comes from equilateral triangles in Ω_R with vertices on $|x| = R$. To compute this singularity we need to know that $3R\sqrt{3}$ is isolated in the set of lengths of broken periodic rays. To ensure that we assume that the complement of Ω, Ω^c is strictly convex and contained in $\{|x| < 1\}$ and $R \geq 8$ (see Remark 1.1), but any assumption that makes the length of the inscribed equilateral triangles isolated in the lengths of periodic reflected ray paths will suffice. The geometry that we have chosen makes the singularity unchanged when one changes the sign of α_γ. Hence we cannot recover more than the cosine of α_γ from it (see Remark 1.2).

A definitive computation of leading singularities in wave traces was given by Duistermaat and Guillemin [1975] for manifolds without boundary. For manifolds with boundary the analogous computation has not been done in that generality. To carry it out in here we have taken this opportunity to present a different method of computation that replaces Fourier integral operators with superpositions of Gaussian beams (see [Combescure et al. 1999] and Chapter 5 of [Combescure and Robert 2012]). In Section 5 we briefly discuss the computation of wave trace singularities using the global theory of Fourier integral operators (see [Hörmander 2003; 2005; 2007; 2009; Duistermaat 1974; Maslov and Fedoriuk 1976; Eskin 2011]). Both approaches lead to the following:

Theorem 1.1. *The distribution*

$$
\sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j})
$$

has an isolated singularity at $t = L = 3R\sqrt{3}$. The leading term in that singularity is the distribution

$$-2^{-5/2}3^{1/4}R^{3/2}\cos\left(\int_\gamma A(x)\cdot dx\right)(t-L)_+^{-3/2}. \tag{1-3}$$

Hence the wave trace determines the cosine of the magnetic flux.

In the final section of this paper we consider $H_{A,V}$ on (flat) 2-torus and obtain essentially the same result: under a nondegeneracy assumption on the torus the singularities in the wave trace at times equal to the lengths of curves in a homology basis determine the cosines of magnetic fluxes around those curves (see Theorem 6.1).

Remark 1.1. The only fact from geometry needed here — we only need it for circles — is: a ray and its reflections inside an ellipse are all tangent to an ellipse confocal with the boundary ellipse. So rays in $|x| \le R$ tangent to a circle $|x| = r > 1$ will never enter $|x| < 1$ after reflection in $|x| = R$, while rays that enter $|x| < 1$ will always reenter $|x| < 1$ after reflection in $|x| = R$. Since the boundary curve C is convex, rays entering $|x| < 1$ will leave $|x| < 1$ after at most one reflection. This gives the following bounds on the length L of periodic ray paths that hit C. For rays that close after entering $|x| < 1$ k times

$$2kR - 2k < L < 2kR + 2k.$$

So periodic rays that enter $|x| < 1$ more than three times have lengths are greater than $8R - 8$, and the equilateral triangles are the (isolated) shortest periodic rays that never enter $|x| < 1$ (assuming $R > 2$). So we need $4R + 4 < 3R\sqrt{3} < 6R - 6$. That happens as soon as $R \ge 8$ (picking the first whole number that works).

Remark 1.2. If $\Omega = \{|x| > 1\}$ and $V \equiv 0$, the mapping $u(x) \to u(-x)$ sends eigenfunctions of $H_{A,0}$ to eigenfunctions of $H_{-A,0}$ bijectively. Thus the wave traces of these operators must be identical. The leading singularity in the wave trace at $t = 3\sqrt{3}R$ does not depend on the boundary of Ω or $V(x)$, hence it will be unchanged when A is replaced by $-A$ in these cases, too. Therefore, one cannot distinguish α_γ and $-\alpha_\gamma$ using the leading singularity. The same ambiguity arises in the results in [Aharonov and Bohm 1959; Helffer 1988].

2. Singularities of the wave trace

Let $E(x, y, t)$ denote the fundamental solution for the initial-boundary value problem (1-2). The wave front set of the distribution kernel of E is contained in the canonical relation for the bicharacteristic flow (see [Melrose and Sjöstrand 1978; 1982]). For this problem the canonical relation is defined as follows: Let $\nu(x)$ denote the outer unit normal to $\partial\Omega_R$ at x. Given (y_0, η_0) with $y_0 \in \Omega_R$ and $|\eta_0| = 1$, define $(x(s, y_0, \eta_0), \xi(s, y_0, \eta_0)) = (y_0 + s\eta_0, \eta_0)$ until, at $s = s_1$, $y_1 = x(s_1, \eta_0, y_0) \in \partial\Omega_R$. Then, if $\eta_0 \cdot \nu(y_1) \ne 0$, continue $(x(s, y_0, \eta_0), \xi(s, y_0, \eta_0))$ for $s > s_1$ as $(y_1 + s\eta_1, \eta_1)$, where $\eta_1 = \eta_0 - 2(\nu(y_1) \cdot \eta_0)\nu(y_1)$. Continue the bicharacteristic this way, reflecting when $x(s, y_0, \eta_0)$ hits $\partial\Omega_R$, as long as $x(s, y_0, \eta_0)$ does not intersect $\partial\Omega_R$ tangentially. At points of tangential intersection one has to distinguish grazing and gliding points. However, since we assume that the boundary of Ω^c is strictly convex, points of tangential

intersection with $\partial\Omega$ are grazing points and bicharacteristics continue unaffected by these intersections. When y_0 is in the interior of Ω_R, a bicharacteristic with initial data (y_0, η_0) will never intersect $|x| = R$ tangentially. Hence, the wave front set of the kernel of $E(\cdot, \cdot, t)$ is the union over $y_0 \in \Omega_R$ and $\eta_0 \in \mathbb{S}^1$ of the points

$$\big(x(t, y_0, \eta_0), \xi(t, y_0, \eta_0), y_0, -\eta_0\big),$$

where $\big(x(t, y_0, \eta_0), \xi(t, y_0, \eta_0)\big)$ are the reflected bicharacteristics described above. Strictly speaking, the wave front set is the closure of that set and includes a "boundary wave front set" over $|x| = R$ (see [Melrose and Sjöstrand 1978; 1982] for details).

Since $E(x, y, t)$ is a distribution in t depending smoothly on $(x, y) \in \Omega_R \times \Omega_R$, $\int_{\Omega_R} E(x, x, t) \, dx$ is well-defined, and we have the relation

$$T =_{\text{def}} \sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j}) = \int_{\Omega_R} E(x, x, t) \, dx.$$

The singular support of T is contained in the set of t such that $(y_0, \eta_0, y_0, -\eta_0) \in WF(E(x, y, t))$ for some $y_0 \in \overline{\Omega}_R$, (see [Guillemin and Melrose 1979]). The choice of Ω and R here implies that, for t in a sufficiently small neighborhood of $3R\sqrt{3}$, $(y_0, \eta_0, y_0, -\eta_0) \in WF(E(x, y, t))$ only if the ray $x(s, y_0, \eta_0)$ traces an inscribed equilateral triangle.

To compute the singularities in the wave trace we need a parametrix for the initial-boundary value problem (1-2). Since this parametrix will differ from $E(x, y, t)$ by an integral operator with a smooth kernel, we can use it to compute singularities. Since we are only interested in singularities arising from inscribed equilateral triangles, we only need a parametrix which captures the singularities of $\int_{\Omega_R} E(x, y, t) f(y) \, dy$ when $WF(f) \subset \{y, \eta) : y \in \Omega_R, |y \cdot \eta^\perp| = R/2\}$, where $(\eta_1, \eta_2)^\perp = (\eta_2, -\eta_1)$. These singularities hit $\partial\Omega_R$ nontangentially, and hence this parametrix construction can be done with reflection at the boundary. This observation applies equally well to constructions with Fourier integral operators and the Gaussian beam superpositions used here.

3. The Gaussian beam construction

Here we will outline the construction of a parametrix for (1-3), for initial data with wave fronts projecting onto the inscribed equilateral triangles. We will continue to let η have length one. The Gaussian beam method allows one to do the following (see [Ralston 1982] for more details):

(i) For any ray, $(x(t), t) = (z + t\eta, t)$, in space-time, one can construct a function $\phi(x, t; z, \eta)$ satisfying:

  (a) For any given integer N, $(\phi_t)^2 - |\phi_x|^2$ vanishes to order N on $(x(t), t)$ and $\text{Im}\{\phi_{xx}\}$ is positive definite on $(x(t), t)$.

  (b) $\phi(x, 0; z, \eta) = x \cdot \eta + \frac{i}{2}|x - z|^2$ on $|x - z| < \delta$.

  (c) $\phi_t(x, 0; z, \eta) = -1$.

Moreover, if Γ is a curve with unit normal ν at $x(t_0)$ and η is not tangent to Γ, then one can construct $\phi^r = \phi$ on Γ, satisfying (a) for the reflected ray $(x(t_0) + (t - t_0)\eta^r, t)$, where $\eta^r = \omega - 2(\nu \cdot \eta)\nu$. Reflection of beams is discussed in [Ralston 1982, Section 2.2].

(ii) Once ϕ has been constructed, for any given integer N, one can solve the transport equations

$$2\phi_t(a_0)_t - 2\phi_x \cdot (a_0)_x + \big(2i A(x) \cdot \phi_x + \phi_{tt} - \Delta\phi\big)a_0 = 0,$$
$$2\phi_t(a_j)_t - 2\phi_x \cdot (a_j)_x + \big(2i A(x) \cdot \phi_x + \phi_{tt} - \Delta\phi\big)a_j = -\big(\partial_t^2 - (\partial_x + i A(x))^2\big)a_{j-1}, \quad j > 0 \tag{3-1}$$

to order N on $(x(t), t)$, and impose the initial conditions $a_0(0, x; z, \eta) = 1$ and $a_j(0, x; z, \eta) = 0$ for $j > 0$ on $|x - z| < \delta$.

For the singularity computation we need to know the leading amplitude a_0 on the ray beginning at z in direction η.

We define $a(x, t; z, \eta, r)$ to be the formal sum

$$a(x, t; z, \eta, r) = \sum_{j \geq 0} a_j(x, t; z, \eta)r^{-j}. \tag{3-2}$$

As before one can reflect in a plane curve Γ which is transverse to the ray, and we impose $a^r = -a$ on Γ to satisfy Dirichlet boundary conditions.

Using the preceding constructions we can construct the operator

$$[V(t)f](x) = \tfrac{1}{2}\big([V_+(t)f](x) + [V_-(t)f](x)\big),$$

where

$$[V_\pm(t)f](x) = \sum_{k \geq 0} \frac{1}{(2\pi)^3} \int_{\mathbb{R}_+ \times S^1 \times \{|z| < R + \delta\}} e^{ir\phi^k(x, \pm t; z, \eta)} \cdot a^k(x, \pm t; z, \eta, r)\hat{f}(r\eta)r^2 \, dr \, d\eta \, dz. \tag{3-3}$$

Here, ϕ^0 is the phase function with $\phi^0(x, 0; z, \eta) = x \cdot \eta + i/2|x - z|^2$, and for $k > 0$,

$$e^{ir\phi^k(x, t; z, \eta)}a^k(x, t; z, \eta, r)$$

is the (Dirichlet) reflection of $e^{ir\phi^{k-1}(x, t; z, \eta)}a^{k-1}(x, t; z, \eta, r)$ in the circle $|x| = R$. Since Gaussian beams can be constructed to for any finite ray segment, we can assume that each term in (3-3) is defined on $\{|x| \leq 2R\}$ when necessary. Note that in this notation the variables (z, η) in ϕ^k remain the initial data **at t=0** for the ray where $\text{Im}\{\phi^k\} = 0$. Note also that the integration in r in (3-3) is in the sense of distributions.

For the parametrix construction we need $V(0)f = f + Kf$ where K is an operator with a smooth kernel. From (3-3) we have

$$[V(0)f](x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}_+ \times S^1 \times \{|z| < 2R\}} e^{irx \cdot \eta - r|x - z|^2/2}\hat{f}(r\eta)r^2 \, dr \, d\eta \, dz.$$

Since

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}_+ \times S^1 \times \mathbb{R}_z^2} e^{irx \cdot \eta - r|x - z|^2/2}\hat{f}(r\eta)r^2 \, dr \, d\eta \, dz = f(x)$$

and f is supported in $\{|x| < R\}$, it follows that omitting the contribution from $\{|z| > R + \delta\}$ in (3-3) only adds an operator with a smooth kernel.

To compute singularities of the wave trace we need to make the kernels of the operators $V_{\pm}(t)$ explicit. The distribution kernels of these operators are sums of terms of the form

$$S(t) = \int_{\mathbb{R}_+ \times S^1 \times \mathbb{R}_z^2} e^{ir\phi(x,t;z,\eta) - ir\eta \cdot y} a(x,t;z,\eta,r) r^2 \, dr \, d\eta \, dz. \tag{3-4}$$

As was stated earlier, these operators are smooth in (x, y), and we can compute their traces by integrating these kernels over the diagonal $y = x$. Thus the (distribution) trace of $V(t)$ is a sum of terms of the form

$$\mathrm{Tr}(\phi, a) = \int_{D \times \mathbb{R}_+ \times S^1 \times \mathbb{R}_z^2} e^{ir\phi(x,t;z,\omega) - ir\eta \cdot x} a(x,t;z,\eta,r) r^2 \, dr \, d\eta \, dz \, dx. \tag{3-5}$$

We want to compute the singularity in t of this trace at $t = L = 3R\sqrt{3}$, and we only need to consider t in $|t - L| < \delta$, where δ is small enough that $\{t : |t - L| < \delta\}$ contains no other lengths of periodic rays in the disk $|x| < R$.

4. Calculation of the singularity at $t = L = 3\sqrt{3}R$

For $\eta = (\eta_1, \eta_2)$ with $|\eta| = 1$ define $\eta^{\perp} = (\eta_2, -\eta_1)$, the "right hand" normal. To compute the singularity at $t = L$ we only need the parametrix restricted to $R/2 - \epsilon < |z \cdot \eta^{\perp}| < R/2 + \epsilon$ for any fixed positive ϵ. Since the broken ray $x(t, z, \eta)$ is initially of the form $x = z + t\eta$, $\eta^{\perp} \cdot z > 0$ corresponds to rays going counterclockwise around $z = 0$, and $\eta^{\perp} \cdot z < 0$ corresponds to rays going clockwise around $z = 0$.

In the preceding section we concluded that the singularity in the wave trace at $t = L$ could be calculated from a sum of integrals of the form

$$\frac{1}{2} \sum_{\pm} \int_0^{\infty} r^2 dr \int_{\mathscr{S}^1} d\eta \left(\int a_0(x, \pm t, z, \eta) e^{ir(\phi(x, \pm t, z, \eta) - x \cdot \eta)} \, dx \, dz \right). \tag{4-1}$$

The integral in r is to be taken in distribution sense. Until the end of this section we will consider (4-1) in the case that the phase ϕ is the beam phase resulting from reflecting the bicharacteristic with initial data $(x, \xi) = (z, \eta)$ three times in $|x| = R$. The amplitudes $a_0(x, t, z, \eta)$ are determined by the transport Equation (3-1). The contributions to the singularity from the $+$ and $-$ terms in (4-1) are complex conjugates of each other, and from here one we only consider the "$+$" term.

We assume that a_0 vanishes when $|z \cdot \eta^{\perp}|$ is not close to $R/2$. Note that we can assume that $\phi(x, t, z, \eta)$ is defined for all (x, z, t) when $|z \cdot \eta^{\perp}|$ is sufficiently close to $R/2$.

The main step in isolating the singularity is an application of the method of stationary phase to (4-1). For that we introduce the change of coordinates

$$x = u + v\eta + w\eta^{\perp}, \quad z = v\eta + w\eta^{\perp}, \quad u \in \mathbb{R}^2, \; v, w \in \mathbb{R}.$$

Our objective is the elimination of the integral in (u, w) by stationary phase. To see when the phase is real and stationary in these variables, note that

(i) the phase is real only when $x = x(t, z, \eta)$;

(ii) the derivative of the phase with respect to u at $x = x(t, z, \eta)$ is

$$\phi_x - \eta = \xi(t, z, \eta) - \eta,$$

which vanishes precisely when three reflections have made ξ return to its initial value. That implies $|z \cdot \eta^\perp| = R/2$. Since the reflected ray will return to z when $t = L$ and it is propagating in the direction η, $x(t, z, \eta) = z + (t - L)\eta$. Hence $u = (t - L)\eta$ and $|w| = R/2$ on the stationary set in u. The derivative of the phase with respect to w at $x = x(t, z, \eta)$ is

$$\eta^\perp \cdot \phi_x + \eta^\perp \cdot \phi_z - \eta \cdot \eta^\perp,$$

which vanishes, since $\phi_z(x(t, z, \eta), t, z, \eta) = \phi_z(x(0, z, \eta), 0, z, \eta) = \partial_z(x \cdot \eta + i|x - z|^2/2)|_{x=z} = 0$. Thus we will need to do the stationary phase computation at $(u, w) = ((t - L)\eta, \pm R/2)$.

Calculation of asymptotics by stationary phase requires the computation of the determinant of the Hessian of the phase, and here this computation is rather long. We have found it useful to consider the phase and the bicharacteristics defined for all $\eta \neq 0$ by homogeneity. That makes the Jacobian matrix

$$F(t) = \begin{pmatrix} \partial x/\partial z(t, z, \eta) & \partial x/\partial \eta(t, z, \eta) \\ \partial \xi/\partial z(t, z, \eta) & \partial \xi/\partial \eta(t, z, \eta) \end{pmatrix} =_{\text{def}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

symplectic. Using $\phi_x(x(t, z, \eta), t, z, \eta) = \xi(t, z, \eta)$ and $\phi_z(x(t, z, \eta), t, z, \eta) = 0$, and setting $M = \phi_{xx}(x(t, z, \eta), t, z, \eta)$, one computes directly that at $x = x(t, z, \eta)$:

$$H =_{\text{def}} \begin{pmatrix} \phi_{xx} & \phi_{xz} \\ \phi_{zx} & \phi_{zz} \end{pmatrix} = \begin{pmatrix} M & c - Ma \\ c^t - a^t M & a^t Ma - a^t c \end{pmatrix}.$$

Letting O_η be the matrix with columns η and $\eta^\perp$, one sees that the Hessian of the phase in (4-1) with respect to the variables (u, v, w) is $B^t H B$ where

$$B = \begin{pmatrix} I & O_\eta \\ 0 & O_\eta \end{pmatrix}.$$

However, we need the Hessian with respect to (u, w). We will see that $\binom{\eta}{\eta}$ is a null vector for H, and we have

$$B \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \eta \\ \eta \end{pmatrix}.$$

Moreover, letting P_η denote the orthogonal projection of $\mathbb{R}^2$ onto $\langle \eta \rangle$, one computes

$$B^t \begin{pmatrix} 0 & 0 \\ 0 & P_\eta \end{pmatrix} B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$\det \begin{pmatrix} \phi_{u_1 u_1} & \phi_{u_1 u_2} & 0 & \phi_{u_1 w} \\ \phi_{u_2 u_1} & \phi_{u_2 u_2} & 0 & \phi_{u_2 w} \\ 0 & 0 & 1 & 0 \\ \phi_{w u_1} & \phi_{w u_2} & 0 & \phi_{w w} \end{pmatrix} = \det \begin{pmatrix} M & c - Ma \\ c^t - a^t M & a^t M a - a^t c + P_\eta \end{pmatrix}. \tag{4-2}$$

To proceed with this computation we need to know $F(t)$. The computation begins with the formulas for $x(t, z, \eta)$ and $\xi(t, z, \eta)$ after three reflections:

$$x(t, z, \eta) = w \frac{\xi^\perp}{|\xi|} + \left(t + \frac{z \cdot \eta}{|\eta|} - 6\sqrt{R^2 - w^2} \right) \frac{\xi}{|\xi|}.$$

Setting $\eta = |\eta|(\cos\theta, \sin\theta)$, we get

$$\xi(t, z, \eta) = |\eta| \left(\cos\left(\theta + \pi - 6 \sin^{-1} \frac{w}{R} \right), \sin\left(\theta + \pi - 6 \sin^{-1} \frac{w}{R} \right) \right).$$

One checks that $\partial_z w = \eta^\perp / |\eta|$ and $\partial_\eta w = -(z \cdot \eta)(\eta^\perp / |\eta|^3)$, and this implies that the Jacobian $\partial\xi/\partial z$ at $w = \pm R/2$ is $(4\sqrt{3}/R)|\eta| P_{\eta^\perp}$. So $c = (4\sqrt{3}/R)|\eta| P_{\eta^\perp}$. Using $\partial_\eta \theta = -\eta^\perp / |\eta|^2$, one finds that at $w = \pm R/2$

$$\frac{\partial\xi}{\partial\eta} = P_\eta + P_\eta^\perp - \frac{4\sqrt{3}}{R} \frac{z \cdot \eta}{\eta} P_{\eta^\perp} = I - \frac{4\sqrt{3}}{R} \frac{z \cdot \eta}{|\eta|} P_{\eta^\perp}.$$

So $d = I - (4\sqrt{3}/R)v P_{\eta^\perp}$.

The computations of the derivatives of $x(t, z, \eta)$ are longer, but they are simplified by the observation that $|\xi(t, z, \eta)| = |\eta|$. At $w = \pm R/2$ one has

$$\frac{\partial x}{\partial z} = P_{\eta^\perp} \mp 2\sqrt{3} \frac{\eta}{|\eta|} \left\langle \frac{\eta^\perp}{|\eta|}, \cdot \right\rangle + \frac{\eta}{|\eta|} \left\langle \frac{\eta}{|\eta|} \pm 2\sqrt{3} \frac{\eta^\perp}{|\eta|}, \cdot \right\rangle + \left(t - L + \frac{z \cdot \eta}{|\eta|} \right) \frac{4\sqrt{3}}{R} P_{\eta^\perp}$$

$$= I + \left(t - L + \frac{z \cdot \eta}{|\eta|} \right) \frac{4\sqrt{3}}{R} P_{\eta^\perp}.$$

So $a = I + (t - L + v)(4\sqrt{3}/R) P_{\eta^\perp}$.

To compute $\partial x/\partial\eta$ at $w = \pm R/2$ one uses

$$\left(\frac{\xi}{|\xi|} \right)_\eta = \frac{1}{|\eta|} \left(1 - \frac{4\sqrt{3}}{R} \frac{z \cdot \eta}{|\eta|} \right) P_{\eta^\perp}$$

at $w = \pm R/2$, and the less obvious result that

$$\left(\frac{\xi^\perp}{|\xi|} \right)_\eta = \left(-1 + \frac{4\sqrt{3}}{R} \frac{z \cdot \eta}{|\eta|} \right) \frac{\eta}{|\eta|^2} \left\langle \frac{\eta^\perp}{|\eta|}, \cdot \right\rangle.$$

Combining those with $\partial_\eta v = (z \cdot \eta^\perp)(\eta^\perp/|\eta|^3) = \pm(R/2|\eta|^2)\eta^\perp$, one has

$$\frac{\partial x}{\partial \eta} = \frac{\eta^\perp}{|\eta|}\left\langle -(z \cdot \eta)\frac{\eta^\perp}{|\eta|^3}, \cdot \right\rangle \pm \frac{R}{2}\left(-1 + \frac{4\sqrt{3}}{R}\frac{z \cdot \eta}{|\eta|}\right)\frac{\eta}{|\eta|^2}\left\langle \frac{\eta^\perp}{|\eta|}, \cdot \right\rangle$$

$$+ \frac{\eta}{|\eta|}\left\langle \pm\frac{R}{2|\eta|^2}\eta^\perp \mp 2\sqrt{3}(z \cdot \eta)\frac{\eta^\perp}{|\eta|^3}, \cdot \right\rangle + \frac{1}{|\eta|}\left(t - L + \frac{z \cdot \eta}{|\eta|}\right)\left(1 - \frac{4\sqrt{3}}{R}\frac{z \cdot \eta}{|\eta|}\right)P_{\eta^\perp}$$

$$= \frac{\left(t - L + \frac{z \cdot \eta}{|\eta|}\right)}{|\eta|}\left(1 - \frac{4\sqrt{3}}{R}\frac{z \cdot \eta}{|\eta|}\right)P_{\eta^\perp} - \frac{(z \cdot \eta)}{|\eta|^2}P_{\eta^\perp}.$$

Thus, when $(5\sqrt{3})/2 - v < t < (7\sqrt{3})/2 - v$,

$$F(t) = \begin{pmatrix} I + (t - L + v)\frac{4\sqrt{3}}{R}P_{\eta^\perp} & \frac{(t-L)}{|\eta|}\left(1 - \frac{4\sqrt{3}}{R}v\right)P_{\eta^\perp} - \frac{4\sqrt{3}}{R}\frac{v^2}{|\eta|}P_{\eta^\perp} \\ \frac{4\sqrt{3}}{R}|\eta|P_{\eta^\perp} & I - \frac{4\sqrt{3}}{R}vP_{\eta^\perp} \end{pmatrix}. \tag{4-3}$$

From this point onward we will assume that $|\eta| = 1$, that is, $\eta = (\cos\theta, \sin\theta)$. Note that this implies $|\xi(t, z, \eta)| \equiv 1$.

Now we can resume the computation of the Hessian. First we compute the determinant of the Hessian. For this the only facts that we need from the computation of the symplectic matrix $F(t)$ — it is a good check on the computation to verify that it *is* symplectic — are that a, b, c and d commute with P_η with $aP_\eta = dP_\eta = P_\eta$ and $bP_\eta = bP_\eta = 0$. We will also eventually use the exact form of c. Note that since $F(t)$ is symplectic $a^t c$ and $d^t b$ are symmetric and $a^t d - c^t b = I$.

Returning to (4-2) we have

$$\begin{pmatrix} M & c - Ma \\ c^t - a^t M & a^t Ma - a^t c + P_\eta \end{pmatrix}\begin{pmatrix} I & a \\ 0 & I \end{pmatrix} = \begin{pmatrix} M & c \\ c^t - a^t M & P_\eta \end{pmatrix},$$

$$\begin{pmatrix} I & 0 \\ a^t & I \end{pmatrix}\begin{pmatrix} M & c \\ c^t - a^t M & P_\eta \end{pmatrix} = \begin{pmatrix} M & c \\ c^t & a^t c + P_\eta \end{pmatrix}.$$

Since $M = (c + id)(a + ib)^{-1}$ (see [Combescure et al. 1999]),

$$\begin{pmatrix} M & c \\ c^t & a^t c + P_\eta \end{pmatrix}\begin{pmatrix} a + ib & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} c + id & c \\ c^t a + ic^t b & a^t c + P_\eta \end{pmatrix}$$

$$\begin{pmatrix} -a^t & I \\ I & 0 \end{pmatrix}\begin{pmatrix} c + id & c \\ c^t a + ic^t b & a^t c + P_\eta \end{pmatrix} = \begin{pmatrix} i(c^t b - a^t d) & P_\eta \\ c + id & c \end{pmatrix} = \begin{pmatrix} -iI & P_\eta \\ c + id & c \end{pmatrix}.$$

Finally

$$\begin{pmatrix} -ic + d & I \\ I & 0 \end{pmatrix}\begin{pmatrix} -iI & P_\eta \\ c + id & c \end{pmatrix} = \begin{pmatrix} 0 & P_\eta + c \\ -iI & P_\eta \end{pmatrix}.$$

From the preceding, using the exact form of c, one can read off the determinant of the Hessian of the phase (at $u = (t - L)\eta$, $w = \pm R/2$). It is

$$(-1)\left(\frac{4\sqrt{3}}{R}\right)\det((a + ib)^{-1}). \tag{4-4}$$

At this point it is convenient to calculate the amplitude a_0. Note that

$$\phi_t(a_0)_t - \phi_x \cdot (a_0)_x = -(d/dt)a_0(x(t,z,\eta),t,z,\eta).$$

Hence (3-1) implies that, after three reflections,

$$a_0(x(t,z,\eta),t,z,\eta) = (-1)^3 e^{i\int_0^t A(x(s))\dot{x}(s)\,ds} e^{(\int_0^t [\phi_{tt} - \Delta\phi](x(s),s)\,ds)/2}. \qquad (4\text{-}5)$$

Note that $|\phi_x| + \phi_t$ vanishes to second order when $x = x(t,z,\eta)$ and thus $\phi_{tt} + \phi_{tx} \cdot \dot{x} = 0$ and $\phi_x = \xi(t,z,\eta)$ when $x = x(t,z,\eta)$. Differentiating $|\phi_x| + \phi_t = 0$ with respect to x and using $\phi_{tt} = -\phi_{tx} \cdot \dot{x}$, we have $\phi_{tt} - \Delta\phi = \xi \cdot M\xi - \text{trace}(M)$, when $x = x(t,z,\eta)$.

Differentiating $\dot{x} = \xi/|\xi|$ with respect to z and η and restricting to $|\eta| = 1$ one sees that $\dot{a} + i\dot{b} = (I - P_\xi)(c + id)$. Hence, using $M = (c + id)(a + ib)^{-1}$, we see that, when $x(t,z,\eta)$ is not a reflection point,

$$(d/dt)(\log\det(a+ib)) = \text{trace}((\dot{a}+i\dot{b})(a+ib)^{-1}) = \text{trace}((I-P_\xi)M) = \Delta\phi - \phi_{tt}. \qquad (4\text{-}6)$$

At reflection points $a + ib$ jumps to $(1 - 2P_\nu)(a + ib)$, where ν is normal to the boundary. Thus $\det(a+ib)$ is multiplied by -1. Note that, since the imaginary part of M is positive definite and the trace of $(I - P_\xi)M$ equals the trace of $(I - P_\xi)M(I - P_\xi)$, (4-6) shows that the argument of $\det(a + ib)$ is strictly increasing away from reflection points. Thus we can make the argument of $(\det(a + ib))^{1/2}$ increasing by defining it to be 1 when $t = 0$, to be multiplied by i at each reflection point, and to be continuous between reflection points. With this definition of $(\det(a + ib))^{1/2}$, we can conclude that after three reflections

$$a_0(x(t,z,\eta),t,z,\eta) = i(\det(a+ib))^{-1/2} e^{i\int_0^t A(x(s))\dot{x}(s)\,ds}. \qquad (4\text{-}7)$$

We have $\int_0^L A(x(s))\dot{x}(s)\,ds = \alpha_\gamma$, where γ is the equilateral triangle traced by $x(s,z,\eta)$ with $z = v\eta + (R/2)\eta^\perp$ or $z = v\eta - (R/2)\eta^\perp$. Since the magnetic field vanishes in Ω, α_γ is independent of v and η, and its value when $z = v\eta + (R/2)\eta^\perp$ is the negative of its value when $z = v\eta - (R/2)\eta^\perp$.

Now we can evaluate the integral in (u, w) asymptotically by the method of stationary phase. The standard form of the stationary phase lemma [Hörmander 2003, Theorem 7.7.5], gives the following: if $f(y)$ is a smooth function such that $\text{Im}\{f\} \geq 0$, $f_y(y_0) = 0$ and the Hessian $f_{yy}(y_0)$ is nonsingular, then for a smooth with support in a sufficiently small neighborhood of y_0, one has the asymptotic expansion

$$\int_{\mathbb{R}^n} e^{irf(y)} a(y)\,dy = \left(\frac{2\pi}{r}\right)^{n/2} \sum_{j=0}^{\infty} c_j r^{-j},$$

and the leading coefficient is given by

$$c_0 = e^{irf(y_0)} a(y_0)(\det(-if_{yy}(y_0)))^{-1/2}. \qquad (4\text{-}8)$$

Here the square root of the determinant in $(\det(-if_{yy}(y_0)))^{-1/2}$ is the analytic continuation to symmetric matrices with nonnegative real part of the positive square root for positive definite matrices (see [Hörmander 2003, Theorem 7.7.5]).

In our case we will use stationary phase to eliminate the integrations in u and w in (4-1) — recall that $z = v\eta + w\eta^\perp$ and $x = u + v\eta + w\eta^\perp$. The stationary point y_0 in (4-5) is either $(u, w) = ((t - L)\eta, R/2)$ or $(u, w) = ((t - L)\eta, -R/2)$. Since

$$\phi(x(t, z, \eta), t, z, \eta) = \phi(x(0, z, \eta), 0, z, \eta) = z \cdot \eta,$$

and we have

$$f(y_0) = \phi(x(t, z, \eta), t, z, \eta) - x(t, z, \eta) \cdot \eta$$

evaluated at $(u, w) = ((t - L)\eta, R/2)$ or $(u, w) = ((t - L)\eta, -R/2)$, it follows that $f(y_0) = -(t - L)$. The domain of integration in (u, v, w, η) is

$$\left\{ (u, v, w, \eta) : |\eta| = 1, \ |u + v\eta + w\eta^\perp| \le R \text{ and } \sqrt{w^2 + v^2} < R + \delta \right\}. \tag{4-9}$$

We consider (4-1) as an iterated integral with the integrations in (u, w) done first. After we use the stationary phase lemma in those integrations, the resulting integrand is evaluated at $(u, w) = ((t - L)\eta, \pm R/2)$, and, since we can assume that $|t - L|$ is smaller than δ, the domain of integration in (v, η) becomes

$$D =_{\text{def}} \left[-\frac{\sqrt{3}}{2} R - (t - L), \frac{\sqrt{3}}{2} R - (t - L) \right] \times S^1.$$

The stationary phase argument needs to be modified when v is near $\pm\sqrt{3}R/2$. There, since the integration in (u, w) should not cross $|x| = R$, the stationary phase lemma does not apply. However, there is a simple remedy for this. Let $\rho = |u + v\eta + w\eta^\perp|$. On the sphere $\rho = R$ we can introduce coordinates $(\theta_1, \theta_2, \theta_3)$, functions of (u, w) depending on v as a parameter, near the points $(u, v, w) = ((t-L)\eta, \pm\sqrt{3}R/2, \pm R/2)$. Next using smooth cutoffs one can write the trace integral as the sum of an integral over a region when $\rho < R - \delta$, where the stationary phase argument applies as given earlier, and a region where $R - 2\delta < \rho < R$. In the second region, near the points where the phase is stationary, one writes the integral in the variables $(\theta_1, \theta_2, \theta_3, v, \eta)$, and applies stationary phase in $(\theta_1, \theta_2, \theta_3)$. The stationary set will be the image in these coordinates of $(u, w) = ((t - L)\eta, \pm R/2)$ and it will depend on v. Likewise, letting Q denote the Hessian in (u, w) of the phase at the stationary points, the Hessian at the stationary points will now be $J^t Q J$, where J is the Jacobian matrix of (u, w) with respect to $(\theta_1, \theta_2, \theta_3)$. Since the θ variables are tangential, one can use the stationary phase expansion *uniformly* in v. The leading term will be an integral over the stationary set. On that set $(\det Q)^{-1/2}$ will be replaced by $(\det J^t Q J)^{-1/2} = |\det J|^{-1}(\det Q)^{-1/2}$. However, the new factor $|\det J|^{-1}$ is canceled by the Jacobian in the volume form (we have $du\,dw = |\det J|d\theta_1 d\theta_2 d\theta_3$). Hence, the stationary phase expansion holds uniformly up to $v = \pm\sqrt{3}R/2$. The result is that (4-4), (4-7) and (4-8) give, uniformly for $(v, \eta) \in D$,

$$\int_{D(v, \eta)} a_0(x, t, z, \eta) e^{ir(\phi(x, t, z, \eta) - x \cdot \eta)} \, du\,dw = \pm \frac{c(R)}{r^{3/2}} K(t) e^{-ir(t - L)} + O\left(\frac{1}{r^{5/2}}\right), \tag{4-10}$$

where $D(v, \eta) = \left\{ (u, w) : |u + v\eta + w\eta^\perp| \le R \right\}$, and $c(R) = (2\pi)^{3/2}(R/4\sqrt{3})^{1/2} e^{3\pi i/4}$. The choice of sign $\pm$ is determined by (4-7) and (4-8): it is $+1$ when the square roots of $\det(a + ib)$ implicit in (4-7)

and (4-8) agree and -1 when they do not. The factor

$$K(t) = \exp\left(i \int_0^t A(x^+(s)) \cdot \dot{x}^+(s)\,ds\right) + \exp\left(i \int_0^t A(x^-(s)) \cdot \dot{x}^-(s)\,ds\right)$$

arises from adding the contributions from stationary points with $w = -R/2$ and $w = R/2$. The path $x^-(s)$ with $w = -R/2$ goes clockwise around the origin, and the path $x^+(s)$ with $w = R/2$ is counterclockwise. Hence $K(L) = 2\cos(\int_\gamma A(x) \cdot dx)$.

To compute the singularity we need the distribution calculation

$$\int_0^\infty e^{-i(t-L)r} r^{1/2}\,dr = \frac{e^{-3\pi i/4}\Gamma(3/2)}{(t-L-i0)^{3/2}} = e^{-3\pi i/4}\Gamma(3/2)(t-L)_+^{-3/2} + e^{3\pi i/4}\Gamma(3/2)(t-L)_-^{-3/2}, \quad (4\text{-}11)$$

where the homogeneous distributions $(s)_\pm^{-3/2}$ are defined by integration by parts and vanish on functions supported in $\mp s > 0$. Note that the contribution to the trace from $V_-(t)$ is the complex conjugate of the contribution from $V_+(t)$. Hence, integrating over (v, η, r), and adding the contributions from $V_-(t)$ and $V_+(t)$ gives the leading singularity in the trace at $t = L$ as

$$\pm 2^{-5/2} R^{3/2} 3^{1/4} \cos\left(\int_\gamma A(x) \cdot dx\right)(t-L)_+^{-3/2}. \tag{4-12}$$

The computation up to this point has not determined the choice of sign $(\pm)$ in (4-12). That will be done in Remark 4.1, and there is an alternative derivation in Section 5. However, since the choice of sign in (4-12) does not depend on A, (4-12) is sufficient to conclude that the trace determines the cosine of the magnetic flux.

The final step in this argument is showing that (4-12) really is the leading term in the singularity. We have not discussed the contributions of the beams with phases ϕ^j in (3-3) for $j \neq 3$. However, those phases are never stationary near the periodic orbits, and give smooth contributions to the trace by the "nonstationary phase" argument. Note that we can apply that argument up to $|x| = R$ by using the coordinates $(\theta_1, \theta_2, \theta_3)$ as before.

Remark 4.1. The sign "$\pm$" in the leading singularity is actually "$-$". To verify that we need to determine the signs of $(\det(a+ib))^{1/2}$ in both the stationary phase computation and the amplitude computation.

We begin with the stationary phase calculation. The matrix on the right in (4-2) can be rewritten as

$$\tilde{H} = \begin{pmatrix} M & c - Ma \\ c^t - a^t M & a^t Ma - a^t c + P_\eta \end{pmatrix} = \begin{pmatrix} (c+id)(a+ib)^{-1} & -i(a+ib)^{-1} \\ -i(a^t+ib^t)^{-1} & i(a+ib)^{-1}a + P_\eta \end{pmatrix}.$$

This is a consequence of $F(t)$ being a symplectic matrix. Then, using (4-3) with $t = L$, one sees that $\tilde{H}$ has the invariant subspaces $V_1 = \langle (\eta, \eta), (\eta, -\eta) \rangle$ and $V_2 = \langle (\eta^\perp, \eta^\perp), (\eta^\perp, -\eta^\perp) \rangle$. The product of the eigenvalues of $\tilde{H}$ from eigenvectors in V_1 is i (the eigenvalues are $1/2 + (1 \pm \sqrt{3}/2)i$) and the product of the eigenvalues from eigenvectors in V_2 is $iC(A+iB)^{-1}$ where $A = \eta^\perp \cdot a\eta^\perp$, $B = \eta^\perp \cdot b\eta^\perp$ and $C = \eta^\perp \cdot c\eta^\perp$. Since all the eigenvalues have nonnegative imaginary parts, this makes

$$(\det(-i\tilde{H}))^{-1/2} = \frac{\sqrt{A+iB}}{\sqrt{C}} e^{i\pi/4} = \tfrac{1}{2} R^{1/2} 3^{-1/4} e^{i\pi/4} \sqrt{A+iB},$$

in the stationary phase formula, where $\sqrt{A+iB}$ is in the lower half-plane. That $\sqrt{A+iB}$ here is in $\text{Im}\{z\} < 0$ is the point of the calculation, note that $A + iB = \det(a + ib)$ at $t = L$.

To calculate $(det(a + ib))^{-1/2}$ in the amplitude we need to consider the entire ray path tracing an equilateral triangle beginning at $z = (z \cdot \eta)\eta \pm (R/2)\eta^{\perp}$ when $t = 0$ and returning to that point when $t = L$. Without loss of generality we will assume that $z = (z \cdot \eta)\eta + (R/2)\eta^{\perp}$. Recall that $a(t) + ib(t) = (\partial x / \partial z)(t, z, \eta) + i(\partial x / \partial \eta)(t, z, \eta)$. As we observed in the calculation of the amplitude a_0, $\det(a + ib))$ is multiplied by -1 at each reflection. Geometric optics, following the reflection rule in Remark 1.1, shows that, after the first reflection at $(x, t) = ((\sqrt{3}R/2)\eta + (R/2)\eta^{\perp}, \sqrt{3}R/2 - z \cdot \eta)$, there is exactly one "focal point" where $\det(\partial x / \partial z) = 0$ on each side of the triangle. Moreover, the homogeneity of $x(t, z, \eta)$ in η of degree zero, implies that $(\partial x / \partial \eta)\eta \equiv 0$. That implies that the real part of $\det(a(t) + ib(t))$ changes sign from negative to positive at the points where $\det(\partial x / \partial z) = 0$. Since the argument of $\det(a(t) + ib(t))$ is increasing, this makes it possible to track the its change as t goes from 0 to L: the total change when the path reaches the third focal point is $2\pi + 2\pi + 3\pi/2$. Since the argument of $\big(\det(a(0) + ib(0))\big)^{1/2}$ was chosen to be zero, this means that at the third focal point, its argument will be $3\pi/4$ and $\big(\det(a(L) + ib(L))\big)^{1/2}$ will be in the upper half plane. Thus, the choices of $(\det(a(L) + ib(L))^{1/2}$ in the stationary phase computation and the amplitude computations have opposite signs, and the sign of the leading singularity in (4-10) is "$-$".

Remark 4.2. We used triangular periodic orbits here because it was easy to give conditions that would make their lengths isolated in the set of lengths of periodic orbits (Remark 4.2). However, it is easy to extend the trace formulas for periodic orbits that are regular N-gons. These would give the same results when one can show that their lengths are isolated in the lengths of periodic orbits.

For a regular inscribed N-gon the length of a side is $h_N = 2R \sin \pi/N$, and its total length is $L_N = Nh_N$. For the N-gon the entries in the first column of the Jacobian from (4-3) become

$$\frac{\partial x}{\partial z}(t, z, \eta) = I + (t + v - L_N)\frac{4N}{h_N}P_{\eta^{\perp}} \quad \text{and} \quad \frac{\partial \xi}{\partial z}(z, \eta) = \frac{4N}{h_N}P_{\eta^{\perp}}.$$

One can use either the analysis in Remark 4.1 or the Fourier integral approach in Section 5 to show that the only changes this makes in the leading singularity are the following. The factor of $(\sqrt{3}R)(R/4\sqrt{3})^{1/2}$, which arose from integration in v and $(\det(\partial \xi/\partial z))^{-1/2}$ from the stationary phase, is replaced by $(h_N)(h_N/4N)^{1/2}$. The initial ± 1 in (4-12) — note that this is -1 by Remark 4.1 — is replaced by $(i)^{N-1}$, since there is one focal point on each side. If one combines that with (4-11) and (4-12), the result is that the leading singularity in the trace is

$$(-1)^{(N-1)/2}C(N, \alpha_\gamma)(t - L_N)_+^{-3/2} \quad \text{for } N \text{ odd} \tag{4-13a}$$

and

$$(-1)^{N/2-1}C(N, \alpha_\gamma)(t - L)_-^{-3/2} \quad \text{for } N \text{ even,} \tag{4-13b}$$

where

$$C(N, \alpha_\gamma) = 2^{-5/2}h_N^{3/2}N^{-1/2}\cos(\alpha_\gamma) = \tfrac{1}{2}N^{-1/2}(R\sin(\pi/N))^{3/2}\cos(\alpha_\gamma).$$

5. A Fourier integral operator approach

This problem provides an opportunity for direct comparison of Gaussian beam superpositions and Fourier integral operators. In this section we describe the computation of the singularities in the wave trace using global Fourier integral operators as in [Hörmander 2003; 2005; 2007; 2009; Duistermaat 1974; Maslov and Fedoriuk 1976; Eskin 2011]. This method requires a detailed description of the singularities in the projection of bicharacteristics to x-space, but in a simple situations like ours one can arrive at the formula for the leading singularity quickly. There are analytical arguments needed to justify that computation, and we will sketch them. Both methods make essential use of the computations of $\partial x/\partial z$ and $\partial \xi/\partial z$ in (4-3).

Let $E(t)$ be the fundamental solution for the boundary value problem (1-3). We will construct a parametrix for $E(t)$, microlocalized near the periodic rays, as a global Fourier integral operator. For f supported in Ω_R let

$$[W(t)f](x) = [W_+(t)f](x) + [W_-(t)f](x)$$
$$= \frac{1}{2(2\pi)^2} \int_{\mathbb{R}^2} (W_+(x, t, \eta) + W_-(x, t, \eta)) \hat{f}(\eta) d\eta,$$

where

$$[W_\pm(0)f](x) = \frac{1}{2(2\pi)^2} \int_{\mathbb{R}^2} e^{ix \cdot \eta} \hat{f}(\eta) d\eta = \frac{1}{2} f(x).$$

Since the analysis of $W_+(t)$ and $W_-(t)$ is the same, we will work with $W_+(t)$ from here on.

The kernel $W_+(x, t, \eta)$ is given by $\exp(-it|\eta| + ix \cdot \eta)$ plus terms arising from reflection in $|x| = R$, Of course, the phase and amplitude develop singularities, and in a neighborhood of those the form of $W_+(t)$ is more complicated, involving integrals over auxiliary variables. The Schwartz kernel of $W_+(t)$ is given by

$$\int_{\mathbb{R}^2} W_+(x, t, \eta) e^{-iy \cdot \eta} d\eta.$$

This is a distribution in t depending smoothly on (x, y). Hence, the distribution trace of $W_+(t)$ is given by

$$\int_{\Omega_R} \left(\int_{\mathbb{R}^2} e^{-ix \cdot \eta} W_+(x, t, \eta) d\eta \right) dx. \tag{5-1}$$

Denote the reflected bicharacteristics with initial data $(x(0), \xi(0)) = (z, \eta)$ by $(x(t, z, \eta), \xi(t, z, \eta))$ as in Section 2. We will write $\eta = |\eta|\hat{\eta}$ with $\hat{\eta} = (\cos\theta, \sin\theta)$ and $\hat{\eta}^\perp = (\sin\theta, -\cos\theta)$. Note that, since $x(t, z, \eta)$ is homogeneous of degree zero in η, we have $x(t, z, \eta) = x(t, z, \hat{\eta})$. In what follows $\hat{\eta}$ will be treated as a parameter; all estimates will be uniform in $\hat{\eta} \in \mathbb{S}^1$. We will use the coordinates (v, w) in x-space, where $x = v\hat{\eta} + w\hat{\eta}^\perp$, and the coordinates $(\tilde{v}, \tilde{w})$ in z-space, where $z = \tilde{v}\hat{\eta} + \tilde{w}\hat{\eta}^\perp$. Since only periodic ray paths contribute to the singularities of the wave trace, we only need to consider $(\tilde{v}, \tilde{w})$ with $|\tilde{w} - R/2| < \delta$ or $|\tilde{w} + R/2| < \delta$. Since the analysis is identical in both cases, we will only consider $|\tilde{w} - R/2| < \delta$. We are only interested in t close to L. For convenience of notation we will use $(x(\tilde{v}, \tilde{w}), \xi(\tilde{v}, \tilde{w})) =_{\text{def}} (x(L, \tilde{v}\hat{\eta} + \tilde{w}\hat{\eta}^\perp, \hat{\eta}), \xi(L, \tilde{v}\hat{\eta} + \tilde{w}\hat{\eta}^\perp, \hat{\eta}))$.

We will use the formulas for bicharacteristics after three reflections that were used to derive (4-3). From those formulas one sees that when $t = L$ the Jacobian $\partial(v, w)/\partial(\tilde{v}, \tilde{w})$ vanishes on the set $\tilde{\Sigma}$ where $\tilde{v} = (35/6)\sqrt{R^2 - \tilde{w}^2} - L$. We define Σ to be the image under the mapping $x = x(\tilde{v}, \tilde{w})$ of the intersection of $\tilde{\Sigma}$ with $|\tilde{w} - R/2| < \delta$. The set Σ is usually called the "caustic set" for the bicharacteristics.

Let $\chi_0(z, \hat{\eta})$, $\chi_\pm(z, \hat{\eta})$ be C^∞ functions in $\tilde{U} = \{|\tilde{w} - (R/2)| < \delta, |\tilde{v}| < \sqrt{R^2 - \tilde{w}^2}\}$ equal to zero near $|\tilde{w} - (R/2)| = \delta$ and such that $\chi_0(z, \hat{\eta}) = 0$ for $|\tilde{v} - \tilde{v}(\tilde{w})| > 2\epsilon$, $\chi_+(z, \hat{\eta}) = 0$ for $\tilde{v} - \tilde{v}(\tilde{w}) < \epsilon$, and $\chi_-(z, \hat{\eta}) = 0$ for $\tilde{v} - \tilde{v}(\tilde{w}) > -\epsilon$, where $\tilde{v} = \tilde{v}(\tilde{w})$ is the equation of $\tilde{\Sigma}$, and ϵ is fixed. We assume also that $\chi_0 + \chi_+ + \chi_- = 1$ for $|\tilde{w} - (R/2)| < \delta/2$. Denote by $\tilde{G}_\pm$ the supports of χ_0, $\chi_\pm$, respectively, and let $G_\pm$ be the images of $\tilde{G}_\pm$ under the mapping $x = x(\tilde{v}, \tilde{w})$. Denote by $V_0(x, t, \eta)e^{-iz\eta}$, $V_\pm(x, t, \eta)e^{-iz\eta}$ the distribution kernels corresponding to the initial conditions

$$\frac{1}{2(2\pi)^2} \chi_0(z, \hat{\eta})e^{i(x-z)\cdot\eta} \quad \text{and} \quad \frac{1}{2(2\pi)^2} \chi_\pm(z, \hat{\eta})e^{i(x-z)\cdot\eta},$$

respectively. Note that the difference $W_+(x, t, \eta) - \big(V_0(x, t, \eta) + V_+(x, t, \eta) + V_-(x, t, \eta)\big)$ does not contribute to the singularity near $t = L$.

It follows from [Maslov and Fedoriuk 1976] and [Eskin 2011, Section 66], that $V_\pm(x, t, \eta)$ has the following form on $G_\pm$: $V_\pm(x, t, \eta) = V_\pm^0(x, t, \eta)(1 + R^\pm(x, t, \eta))$, where

$$V_\pm^0(x, t, \eta) = \frac{(-1)^3}{8\pi^2} \chi_\pm(z^\pm(x, t, \hat{\eta}), \hat{\eta})\Big|\det\frac{\partial x^\pm}{\partial z}\Big|^{-1/2} \cdot \exp\Big(i\Big[\frac{\pi}{4}\sigma^\pm + \alpha(t) + \phi^\pm(x, t, \eta)\Big]\Big), \quad (5\text{-}2)$$

and $R^\pm \approx \sum_{k\geq 1} r_k^\pm(x, t, \hat{\eta})|\eta|^{-k}$ is an asymptotic series in $|\eta|$. Here $\phi^\pm(x, t, \eta) = z^\pm(x, t, \hat{\eta}) \cdot \hat{\eta}$, where $z = z^\pm(x, t, \eta)$ is the inverse function to $x = x(t, z, \eta)$ in $\tilde{G}_\pm$, and $\partial x^\pm/\partial z = (\partial x/\partial z)(t, z^\pm(x, t, \hat{\eta}), \hat{\eta})$. The piecewise constant function $\sigma^\pm$ in (5-2) is the sum of the "phase shifts" at the focal points on the ray paths used to define $\phi^\pm$. The sum of these phase shifts along the curve $x(t, z, \eta)$, $0 \leq t \leq L$ is called "Maslov index" of this curve (see [Maslov and Fedoriuk 1976, Section 1.7] or [Eskin 2011, Section 66]). The computation of the phase shifts at the focal points here can be done as in [Eskin 2011, Section 66.46–66.48], and the result is that the contribution to σ is -2 for each focal point that $x(t, \tilde{v}\eta + R/2\eta^\perp, \eta)$ has passed through up to time t. This makes $\sigma^+ = \sigma^- - 2$. The function $\alpha(t) = \int_0^t A(x(s, z, \eta)) \cdot \dot{x}(s, z, \eta)\,ds$, and the factor $(-1)^3$ comes from the three reflections of a ray on $0 \leq t \leq L$. Note that $V_\pm$ decay rapidly in $|\eta|$ outside $G_\pm$, respectively.

We denote the leading term of $\int_{\Omega_R}(V_1 + V_2)e^{-ix\cdot\eta}\,dx$ by $I(t, \eta) = I_+ + I_-$, where

$$I_\pm(t, \eta) = e^{-i|\eta|(t-L)} \int_{G_\pm} V_\pm^0(x, L, \eta)e^{-ix\cdot\eta}\,dx.$$

The phase in $I_\pm(t, \eta)$ is $\Phi_\pm(x, L, \eta) = \phi_\pm(x, L, \eta) - x \cdot \eta$. The phase functions $\phi^\pm(x, t, \eta)$ satisfy $\phi_t^\pm + |\phi_x|^2 = 0$, and we have

$$\phi_x^\pm(x(t, z, \eta), t, \eta) = \xi(t, z, \eta), \quad \phi_\eta^\pm(x(t, z, \eta), t, \eta) = z. \quad (5\text{-}3)$$

Since $|\phi_x^\pm| = |\eta|$ we have $\phi_t^\pm = -|\eta|$. Therefore $\phi^\pm(x, t, \eta) = \phi^\pm(x, L, \eta) - |\eta|(t - L)$. The critical points of $\Phi^\pm(x, L, \eta)$ are solutions of $\phi_x^\pm(x, L, \eta) - \eta = 0$, $\phi_\eta^\pm(x, L, \eta) - x = 0$. It follows from (5-3)

that $\xi(L, z, \eta) = \eta$ and $z = x(L, z, \eta)$. In the geometry here this means that the periodic orbit is an equilateral triangle inscribed in $|x| \leq R$, and $L = 3R\sqrt{3}$. Since any point of this triangle is a critical point, we need to use the stationary phase expansion in the transversal variable w.

Note that $z^\pm(x, L, \eta) = x = v\hat{\eta} + (R/2)\hat{\eta}^\perp$, $x \in G_\pm$. Hence

$$\Phi^\pm(v\hat{\eta} + (R/2)\hat{\eta}^\perp, L, \eta) = \phi^\pm(v\hat{\eta} + (R/2)\hat{\eta}^\perp, L, \eta) - x \cdot \eta = 0.$$

Also

$$\Phi_w^\pm(v\hat{\eta} + (R/2)\hat{\eta}^\perp, L, \eta) = \phi_x^\pm(v\hat{\eta} + (R/2)\hat{\eta}^\perp, L, \eta) \cdot \hat{\eta}^\perp = 0,$$

since $\phi_x^\pm - \eta = 0$ and $\eta \cdot \hat{\eta}^\perp = 0$. Compute now

$$\Phi_{ww}^\pm(v\hat{\eta} + (R/2)\hat{\eta}^\perp, L, \eta) = \hat{\eta}^\perp \cdot \phi_{xx}^\pm(v\hat{\eta} + (R/2)\hat{\eta}^\perp, L, \eta)\hat{\eta}^\perp.$$

Differentiating $\phi_x^\pm(x, L, \eta) = \xi(L, z^\pm(x, L, \eta), \eta)$ in x we get

$$\phi_{xx}^\pm = \frac{\partial \xi}{\partial z}\left(\frac{\partial x}{\partial z}\right)^{-1} \quad \text{at } x = v\hat{\eta} + (R/2)\hat{\eta}^\perp, \ x \in G_\pm.$$

It follows from (4-3) that

$$\Phi_{ww}^\pm\left(v\hat{\eta} + \frac{R}{2}\hat{\eta}^\perp, L, \eta\right) = \frac{4\sqrt{3}}{R}\left(1 + v\frac{4\sqrt{3}}{R}\right)^{-1}.$$

Note that $\Phi_{ww}^\pm > 0$ when $v > -R/4\sqrt{3}$ and $\Phi_{ww}^\pm < 0$ when $v < -R/4\sqrt{3}$.

At this point we have the data needed in the stationary phase formula, but we need to consider the behavior of the amplitude that comes from (5-2). Since $\det(\partial x/\partial z)(L, z, \eta) = 1 + v(4\sqrt{3}/R)$, the factor $|\det(\partial x/\partial z)|^{-1/2}$ in the amplitude is canceled by part of the factor $|\Phi_{ww}^\pm|^{-1/2}$ in the stationary phase formula. Hence the stationary phase expansion in w has the leading terms

$$\frac{(-1)^3}{8\pi^2}\left(\frac{2\pi}{|\eta|}\right)^{1/2}\left(\frac{R}{4\sqrt{3}}\right)^{1/2}\chi_-\left(v, \frac{R}{2}, \hat{\eta}\right)\exp\left(i\left[(L-t)|\eta| + \frac{\pi}{4}\sigma^- + \alpha(L) - \pi/4\right]\right), \quad \text{for } v < -R/(4\sqrt{3}),$$

$$\frac{(-1)^3}{8\pi^2}\left(\frac{2\pi}{|\eta|}\right)^{1/2}\left(\frac{R}{4\sqrt{3}}\right)^{1/2}\chi_+\left(v, \frac{R}{2}, \hat{\eta}\right)\exp\left(i[(L-t)|\eta| + \frac{\pi}{4}\sigma^+ + \alpha(L) + \pi/4]\right), \quad \text{for } v > -R/(4\sqrt{3}),$$

where σ^- and σ^+ are the values of σ before and after crossing the focal point at $v = -R/(4\sqrt{3})$. Since $\sigma^- = -4$ and $\sigma_+ = -6$, the two formulas above can be combined to give the leading term in the integrand in (5-2) after integration in w

$$\frac{2(\chi_+ + \chi_-)}{8\pi^2}\cos(\alpha(L))\left(\frac{2\pi}{|\eta|}\right)^{1/2}\left(\frac{R}{4\sqrt{3}}\right)^{1/2}\exp\left(i[(L-t)|\eta| - \pi/4]\right). \tag{5-4}$$

Here we have included the contributions from both $w = R/2$ and $w = -R/2$ which have $\alpha(L)$ with opposite signs.

Now we will find the contribution of $\int_{\Omega_R} V_0(x, t, \eta)e^{-ix\cdot\eta}dx$. The caustic set Σ is a fold-type singularity (see [Duistermaat 1974] and [Eskin 2011, Example 66.1]). Therefore $V_0(x, t, \eta)$ is given by an integral

representation (see [Eskin 2011, Section 66.53] and also [Ludwig 1966]):

$$V_0(x, t, \eta) = \frac{|\eta|^{1/2} e^{i(L-t)|\eta|}}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} a(v, \xi_2, |\eta|) e^{i|\eta|(S(v,\xi_2,L)+w\xi_2)} d\xi_2. \tag{5-5}$$

Computing the stationary points in (5-5) for $x \in G_- \cap \{d(x, \Sigma) < \epsilon\}$ we see that the stationary points are given by $S_{\xi_2}(v, p^-(v, w), L) + w = 0$ and the phase is $S(v, p^-, L) + wp^- = \phi^-(x, t, \eta)$, where $\phi^-(x, t, \eta)$ is the same as in (5-2). The amplitude $a(v, \xi_2, |\eta|)$ in (5-5) is an asymptotic series $\sum_{k \geq 0} a_k(v, \xi_2)|\eta|^{-k}$, where

$$a_0(v, \xi_2) = \frac{(-1)^3}{8\pi^2} \chi_0(z(v, \xi_1), \hat{\eta}) e^{i[\alpha(L)+(\pi/4)\sigma_- -(\pi/4)]} \left| \det \frac{\partial(v, \xi_2)}{\partial z} \right|^{-1/2}. \tag{5-6}$$

Note that the factor $e^{-i(\pi/4)}$ arises because

$$S_{\xi_2^2}(v, p^-(v, w), L) > 0;$$

see [Eskin 2011, Section 66.44].

   To evaluate the contribution of $\int_{\Omega_R} V_0 e^{-x \cdot \eta} dv \, dw$ we apply the stationary phase method to the double integral in ξ_2 and w. The phase function is $S(v, \xi_2, t) + w\xi_2 - v$. The equations for the stationary points are

$$S_{\xi_2}(v, \xi_2, t) + w = 0, \quad \xi_2 = 0.$$

Note that $t = L$. We will show that $w = -S_{\xi_2}(v, 0, L) = R/2$: Let $\xi_2 - \alpha(v) = 0$ be the equation of the caustic set, that is,

$$S_{\xi_2^2}(v, \alpha(v), L) = 0.$$

In our situation

$$S_{\xi_2^3}(v, \alpha(v), L) \neq 0.$$

Expand $S_{\xi_2}(v, \xi_2, L)$ by the Taylor's formula with a remainder at $\xi_2 = \alpha(v)$. When $\xi_2 = 0$, that gives

$$S_{\xi_2}(v, 0, L) = S_{\xi_2}(v, \alpha(v), L) + c(v)(0 - \alpha(v))^2.$$

Therefore

$$S_{\xi_2}(v, \alpha(v), L) = S_{\xi_2}(v, 0, L) - c(v)\alpha^2(v).$$

The equation of the caustic set in (v, w) coordinates is

$$w = -S_{\xi_2}(v, \alpha(v), L) = -S_{\xi_2}(v, 0, L) + c(v)\alpha^2(v).$$

On the other hand, using the mapping $x(\tilde{v}, \tilde{w})$, one sees that near $(v, w) = (v_0, R/2)$ with $v_0 = -R/(4\sqrt{3})$, the caustic set Σ is given by

$$w = (R/2) - c_1(v)(v - v_0)^2.$$

Comparing these two expressions for the caustic set we get

$$-S_{\xi_2}(v, 0, L) = R/2 \quad \text{and} \quad \alpha(v) = c_2(v)(v - v_0)^2.$$

Note that the determinant of the Hessian at the critical point $(0, R/2)$ is -1. Therefore the standard stationary phase lemma in (ξ_2, w) gives the asymptotic expansion $\sum_{i \geq 0} r_k^0(v)|\xi|^{-(1/2)-k}$, where

$$r_0^0 = \frac{(-1)^3}{8\pi^2}\left(\frac{2\pi}{|\eta|}\right)^{1/2}\chi_0\left(v\hat{\eta} + \frac{R}{2}\hat{\eta}^\perp, \eta\right)e^{i(\alpha(L)+(\pi/4)\sigma_- - \pi/4)}\left(\frac{4\sqrt{3}}{R}\right)^{-1/2}. \tag{5-7}$$

In (5-7) we substituted the value of the Jacobian in (5-6). By (4-3) that is equal to $4\sqrt{3}/R$ at $\xi_2 = 0$, $w = R/2$.

Combining the contributions of (5-7) for $w = R/2$ and $w = -R/2$ with the contribution of (5-4) and then integrating in (v, θ) we get the leading terms of the contribution of $W_+(t)$ to the trace:

$$\left(\frac{1}{(2\pi)^2}(R\sqrt{3})(2\pi)\right)(2\pi)^{1/2}\left(\frac{R}{4\sqrt{3}}\right)^{1/2}\int_0^\infty \cos(\alpha(L))e^{i[(L-t)|\eta|-\pi/4]}|\eta|^{1/2}d|\eta|. \tag{5-8}$$

This is consistent with (4-10), and therefore the final form of the singularity is again the one given in (1-3).

Note that contributions from neighborhoods on reflection points can be treated by introduction of the natural angular coordinate place of w as in the final part of Section 4.

6. The Aharonov–Bohm effect on a torus

The Aharonov–Bohm effect only arises when the underlying domain is not simply connected. In the previous sections the domain was an annulus. Here we consider the Schrödinger operator on a torus. Let $L = \{m_1 e_1 + m_2 e_2 : m \in \mathbb{Z}^2\}$, where $\{e_1, e_2\}$ is a basis for $\mathbb{R}^2$. We assume that the lattice L has the property: For $d, d' \in L$, if $|d'| = |d|$, then $d' = \pm d$. This is a generic condition that implies that the group of isometries of L consists of lattice translations and the inversion $d \to -d$. Associated to L one has the dual lattice $L^* = \{\delta \in \mathbb{R}^2 : \delta \cdot d \in \mathbb{Z} \text{ for all } d \in L\}$.

We consider the Schrödinger operator,

$$H_{A,V} = \tfrac{1}{2}(i\partial_{x_1} + A_1(x))^2 + \tfrac{1}{2}(i\partial_{x_2} + A_2(x))^2 - V(x),$$

acting on functions on $\mathbb{T}^2 = \mathbb{R}^2/L$. The functions $A = (A_1, A_2)$ and V are assumed to be smooth on $\mathbb{T}^2$ and hence they have smooth extensions to $\mathbb{R}^2$ satisfying $A(x+d) = A(x)$ and $V(x+d) = V(x)$ for all $d \in L$. As before we assume that the magnetic field vanishes

$$\partial_{x_2}A_1 - \partial_{x_1}A_2 = 0 \quad \text{on } \mathbb{T}^2. \tag{6-1}$$

Thus for any closed curve γ on $\mathbb{T}^2$ the flux

$$\alpha_\gamma = \int_\gamma A(x) \cdot dx,$$

is determined by the homology class of γ. We let γ_1 and γ_2 be a basis for the homology group, for instance

$$\gamma_j = \{te_j, t \in [0, 1)\}, \quad j = 1, 2, \tag{6-2}$$

and denote the corresponding fluxes by α_1 and α_2.

Let $g(x) \in C^\infty(\mathbb{T}^2)$ be such that $|g(x)| = 1$. The conjugation of $H_{A,V}$ by the unitary operator of multiplication by $g(x)$ transforms $H_{A,V}$ to $H_{\tilde{A},V}$, where $\tilde{A} = A + ig^{-1}\nabla g$. The condition $|g(x)| = 1$ on $\mathbb{T}^2$ implies that $g(x) = \exp(2\pi i \delta \cdot x + \varphi(x))$, where $\delta \in L^*$ and $\varphi(x)$ is periodic. Hence

$$\alpha_1(\tilde{A}) = \alpha_1(A) - 2\pi \delta \cdot e_1, \, \alpha_2(\tilde{A}) = \alpha_2(A) - 2\pi \delta \cdot e_2.$$

Therefore if A and $\tilde{A}$ are gauge equivalent we have

$$\alpha_j(\tilde{A}) = \alpha_j(A) \mod 2\pi, \, j = 1, 2. \tag{6-3}$$

Expanding $A(x)$ in a Fourier series we have

$$A(x) = A_0 + \sum_{\delta \in L^* \setminus \{0\}} A_\delta e^{2\pi i \delta \cdot x},$$

where $A_0 = |\mathbb{T}^2|^{-1} \int_{\mathbb{T}^2} A(x)\,dx$, $|\mathbb{T}^2|$ denotes the area of $\{se_1 + te_2; 0 \le s, t \le 1\}$. Since $\partial_{x_2} A_1 = \partial_{x_1} A_1$ we have $A(x) = A_0 + \nabla\varphi(x)$, where

$$\varphi(x) = \sum_{\delta \in L^* \setminus \{O\}} \frac{\delta \cdot A_\delta}{2\pi i \delta \cdot \delta} e^{2\pi i \delta \cdot x}.$$

Therefore when (6-1) holds $A(x)$ is gauge equivalent to the constant potential A_0. Two constant magnetic potentials A_0 and $\tilde{A}_0$ are not gauge equivalent if (6-3) does not hold. When $\tilde{A}_0$ is not gauge equivalent to either A_0 or $-A_0$ the potentials A_0 and $\tilde{A}_0$ have a different physical impact, in particular, the spectra of $H_{A_0,V}$ and $H_{\tilde{A}_0,V}$ are not the same.

The last assertion is a consequence of the following theorem.

Theorem 6.1. *Suppose* (6-1) *holds. The spectrum of $H_{A,V}$ as a self-adjoint operator on $L^2(\mathbb{T}^2)$ determines* $\cos\alpha_1$ *and* $\cos\alpha_2$, *where* $\alpha_j = \int_{\gamma_j} A(x) \cdot dx, \, j = 1, 2.$

Theorem 6.1 complements the results of [Guillemin 1990; Eskin and Ralston 2009; Eskin 1989]. In particular it shows that, if A and $\tilde{A}$ give rise to zero magnetic fields on $\mathbb{T}^2$ but different values for $\cos\alpha_1$ and $\cos\alpha_2$, the Schrödinger operators, $H_{A,V}$ and $H_{\tilde{A},V}$ will have different spectra. This proves the Aharonov–Bohm effect on the torus.

Proof of Theorem 6.1. As in the preceding sections we start with the wave trace formula

$$\sum_{j=1}^\infty \cos(t\sqrt{\lambda_j}) = \int_{\mathbb{T}^2} E_{\mathbb{T}^2}(x, x, t)\,dx,$$

where $\{\lambda_j\}_{j=1}^\infty$ is the spectrum of $H_{A,V}$ on $\mathbb{T}^2$ and $E_{\mathbb{T}^2}(x, y, t)$ is the solution to $E_{tt} + H_{A,V}E = 0$ on $\mathbb{T}^2 \times \mathbb{R}$ satisfying $E(x, y, 0) = \delta(x - y)$ and $E_t(x, y, 0) = 0$. Note that

$$E_{\mathbb{T}^2}(x, y, t) = \sum_{d \in L} E_{\mathbb{R}^2}(x + d, y, t),$$

where $E_{\mathbb{R}^2}$ is the solution to $E_{tt} + H_{A,V} E = 0$ on $\mathbb{R}^2 \times \mathbb{R}$ satisfying $E(x, y, 0) = \delta(x - y)$ and $E_t(x, y, 0) = 0$ once $H_{A,V}$ is extended to $\mathbb{R}^2$ by making its coefficients periodic: $A(x + d) = A(x)$ and $V(x + d) = V(x)$ for all $d \in L$. Hence

$$\int_{\mathbb{T}^2} E_{\mathbb{T}^2}(x, x, t) \, dx = \sum_{d \in L} \int_{\mathbb{T}^2} E_{\mathbb{R}^2}(x + d, x, t) \, dx.$$

Since $E_{\mathbb{R}^2}$ is smooth off the cone $|x - y|^2 = t^2$, and our assumption on L implies that only two lattice vectors can have $|d|^2 = t^2$ for a fixed value of t, the singularity in the wave trace at $t = |d|$, must come from (compare [Eskin et al. 1984a; 1984b; Eskin and Ralston 2007])

$$\int_{\mathbb{T}^2} E_{\mathbb{R}^2}(x + d, x, t) \, dx + \int_{\mathbb{T}^2} E_{\mathbb{R}^2}(x - d, x, t) \, dx.$$

To compute the leading singularities in this trace we will use the Hadamard–Hörmander parametrix [Hörmander 2003; 2005; 2007; 2009]. We have

$$E_{\mathbb{R}^2}(x, y, t) = \partial_t \big(E_+(x, y, t) - E_+(x, y, -t) \big),$$

where E_+ is the forward fundamental solution.

The Hadamard–Hörmander parametrix construction for E_+ writes E_+ as an asymptotic sum of terms with increasing regularity. The first term is $a_0(x, y) e_0(|x - y|, t)$, where

$$e_0 = \frac{1}{2\sqrt{\pi}} (t^2 - |x - y|^2)_+^{-1/2} \quad \text{when } t > 0 \text{ and } e_0 = 0 \text{ when } t < 0,$$

and

$$a_0(x, y) = \exp\left(i \int_0^1 (x - y) \cdot A(y + s(x - y)) \, ds \right).$$

Therefore, by [Eskin and Ralston 2009], the singularity of the trace at $t = |d|$ determines $I(d) + I(-d)$ where

$$I(d) = \int_{\mathbb{T}^2} \exp\left(i \int_0^1 d \cdot A(x + sd) \, ds \right) dx.$$

Since $A(x) = A_0 + \nabla \varphi(x)$, where $\varphi(x)$ is periodic, we have

$$\int_0^1 d \cdot A(x + sd) \, ds = d \cdot A_0 \quad \text{since} \quad \int_0^1 d \cdot \nabla \varphi(x + sd) \, ds = 0.$$

Therefore $I(d) = e^{id \cdot A_0} |\mathbb{T}^2|$ and hence the singularity of the wave trace at $t = |d|$ determines $\cos(A_0 \cdot d)$ for all $d \in L$. In particular, when $d = e_j$ and $\gamma_j = \{te_j, t \in [0, 1)\}$, $j = 1, 2$, we get

$$\alpha_j = \int_{\gamma_j} A(x) \cdot dx = e_j \cdot A_0.$$

Thus the singularities of the wave trace when $t = |e_j|$ determine $\cos \alpha_j$ for $j = 1, 2$. When $V(x) = V(-x)$, then $H_{A_0,V}$ and $H_{-A_0,V}$ are isospectral and one can only recover $\cos \alpha_j$, $j = 1, 2$, from the spectrum. When V is not even, the question of whether one could recover $\exp(i\alpha_j)$, $j = 1, 2$, is open. $\square$

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