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TWO-PHASE PROBLEMS WITH DISTRIBUTED SOURCES: REGULARITY OF THE FREE BOUNDARY

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We investigate the regularity of the free boundary for a general class of two-phase free boundary problems with nonzero right-hand side. We prove that Lipschitz or flat free boundaries are $C^{1,\gamma}$. In particular, viscosity solutions are indeed classical.

1. Introduction and main results

In this paper we consider two phase free boundary problems governed by uniformly elliptic equations with distributed sources. Our purpose is to investigate the regularity of the free boundary under additional hypotheses such as flatness or Lipschitz continuity. A model problem we have in mind is:

$$\begin{cases} \Delta u = f & \text{in } \Omega^+(u) \cup \Omega^-(u), \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 1 & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega. \end{cases} \tag{1-1}$$

Here, as usual for any bounded domain $\Omega \subset \mathbb{R}^n$,

$$\Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \quad \Omega^-(u) := \{x \in \Omega : u(x) \leq 0\}^\circ,$$

and u_ν^+ and u_ν^- denote the normal derivatives in the inward direction to $\Omega^+(u)$ and $\Omega^-(u)$.

Typical examples are the Prandtl–Batchelor model in fluid dynamics (see, e.g., [Batchelor 1956; Elcrat and Miller 1995]), where $f = \mathbf{1}_{\Omega^-(u)}$, the characteristic function of the negative phase, or the eigenvalue problem in magnetohydrodynamics (1,1) considered in [Friedman and Liu 1995], where $f = -\lambda u \mathbf{1}_{\Omega^-(u)}$. Other examples come from limits of singular perturbation problems with forcing term as in [Lederman and Wolanski 2006], where the authors analyze solutions to (1-1), arising in the study of flame propagation with nonlocal effects.

The homogeneous case $f \equiv 0$ was settled in the classical works of Caffarelli [1987; 1989]. A key step in these papers is the construction of a family of continuous sup-convolution deformations that act as comparison subsolutions.

The results in [Caffarelli 1987; 1989] have been widely generalized to different classes of homogeneous elliptic problems. See, for example, [Cerutti et al. 2004; Ferrari and Salsa 2007a; 2007b] for linear

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operators; [Argiolas and Ferrari 2009; Feldman 2001; 1997; Ferrari 2006; Wang 2000; 2002] for fully nonlinear operators; and [Lewis and Nyström 2010] for the p-Laplacian. All these papers follow the guidelines of [Caffarelli 1987; 1989].

De Silva [2011] introduced a new strategy to investigate inhomogeneous free boundary problems, motivated by a classical one phase problem in hydrodynamic. This method has been successfully applied in [De Silva and Roquejoffre 2012] to nonlocal one phase Bernoulli type problems, governed by the fractional Laplacian. For another application of the techniques in [De Silva 2011] see also [Leitão and Teixeira 2011].

Here we extend the method in [De Silva 2011] to two phase problems to prove that flat (see below) or Lipschitz free boundaries of (1-1) are $C^{1,\gamma}$.

In order to better emphasize the ideas involved, we first develop the regularity theory for free boundaries of viscosity solutions to problem (1-1) (see Section 2 for the relevant definitions), and then we extend our results to a more general class of free boundary problems. For simplicity, in order to avoid the machinery of L^p-viscosity solution, we assume that f is bounded in Ω and continuous in $\Omega^+(u) \cup \Omega^-(u)$. Our results may be extended to the case when f is merely bounded measurable.

We remark that in view of Theorem 4.5 in [Caffarelli et al. 2002], a viscosity solution to (1-1) is locally Lipschitz. In fact, as it can be easily checked, our viscosity solutions are also weak solutions in the sense of Definition 4.4 in that paper and both $\Delta u^\pm - f$ are nonnegative Radon measures.

We now state our first main results. Here constants depending only on n, $\|f\|_\infty$, and $\mathrm{Lip}(u)$ will be called universal.

Theorem 1.1 (flatness implies $C^{1,\gamma}$). *Let u be a (Lipschitz) viscosity solution to* (1-1) *in B_1. Assume that $f \in L^\infty(B_1)$ is continuous in $B_1^+(u) \cup B_1^-(u)$. There exists a universal constant $\bar{\delta} > 0$ such that, if*

$$\{x_n \le -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \le \delta\}, \tag{1-2}$$

with $0 \le \delta \le \bar{\delta}$, then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

Theorem 1.1 still holds when (1-2) is replaced by other common flatness conditions (see page 296).

Theorem 1.2 (Lipschitz implies $C^{1,\gamma}$). *Let u be a (Lipschitz) viscosity solution to* (1-1) *in B_1, with $0 \in F(u)$. Assume that $f \in L^\infty(B_1)$ is continuous in $B_1^+(u) \cup B_1^-(u)$. If $F(u)$ is a Lipschitz graph in a neighborhood of 0, then $F(u)$ is $C^{1,\gamma}$ in a (smaller) neighborhood of 0.*

The proof of Theorem 1.1 is based on an improvement of flatness, obtained via a compactness argument which linearizes the problem into a limiting one. The key tool is a geometric Harnack inequality that localizes the free boundary well, and allows the rigorous passage to the limit.

The main difficulty in the analysis comes from the case when u^- is degenerate, that is very close to zero without being identically zero. In this case the flatness assumption does not guarantee closeness of u to an "optimal" (two-plane) configuration. Thus one needs to work only with the positive phase u^+ to balance the situation in which u^+ highly predominates over u^- and the case in which u^- is not too small with respect to u^+.

Theorem 1.2 follows from Theorem 1.1 and the main result in [Caffarelli 1987], via a blow-up argument.

Sections 2–6 are devoted to the proof of the theorems above. In particular, in Section 2 we introduce the relevant definitions and some preliminary lemmas. In Section 3 we describe the linearized problem associated to (1-1). Section 4 is devoted to the proof of the Harnack inequality both in the nondegenerate and in the degenerate setting. In Section 5, we present the proof of the improvement of flatness lemmas. Section 6 contains the proof of the Theorem 1.1 and Theorem 1.2.

From Section 7 to Section 10 we deal with more general problems of the form

$$\begin{cases} \mathscr{L}u = f & \text{in } \Omega^+(u) \cup \Omega^-(u), \\ u_\nu^+ = G(u_\nu^-, x) & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega, \end{cases} \tag{1-3}$$

with f bounded on Ω and continuous in $\Omega^+(u) \cup \Omega^-(u)$, and u Lipschitz continuous with $\mathrm{Lip}(u) \le L$. Here

$$\mathscr{L} = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} + \boldsymbol{b} \cdot \nabla, \quad a_{ij} \in C^{0,\bar\gamma}(\Omega),\ \boldsymbol{b} \in C(\Omega) \cap L^\infty(\Omega),$$

is uniformly elliptic; that is, there exist $0 < \lambda \le \Lambda$ such that, for every $\xi \in \mathbb{R}^n$ and every $x \in \Omega$,

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2,$$

and

$$G(\eta, x) : [0,\infty) \times \Omega \to (0,\infty)$$

satisfies the following assumptions:

(H1) $G(\eta, \cdot) \in C^{0,\bar\gamma}(\Omega)$ uniformly in η; $G(\cdot, x) \in C^{1,\bar\gamma}([0, L])$ for every $x \in \Omega$.

(H2) $G'(\cdot, x) > 0$ with $G(0, x) \ge \gamma_0 > 0$ uniformly in x.

(H3) There exists $N > 0$ such that $\eta^{-N} G(\eta, x)$ is strictly decreasing in η, uniformly in x.

In this framework we prove the following main results. Here, a constant depending (possibly) on n, $\mathrm{Lip}(u)$, λ, Λ, $[a_{ij}]_{C^{0,\bar\gamma}}$, $\|\boldsymbol{b}\|_{L^\infty}$, $\|f\|_{L^\infty}$, $[G(\eta,\cdot)]_{C^{0,\bar\gamma}}$, γ_0 and N is called universal. The $C^{1,\bar\gamma}$ norm of $G(\cdot, x)$ may depend on x, and enters our proofs in a qualitative way only.

Theorem 1.3 (flatness implies $C^{1,\gamma}$). *Let u be a Lipschitz viscosity solution to* (1-3) *in B_1, with* $\mathrm{Lip}(u) \le L$. *Assume that f is continuous in $B_1^+(u) \cup B_1^-(u)$, $\|f\|_{L^\infty(B_1)} \le L$ and G satisfies assumptions* (H1)–(H3). *There exists a universal constant $\bar\delta > 0$ such that, if*

$$\{x_n \le -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \le \delta\},$$

with $0 \le \delta \le \bar\delta$, then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

Theorem 1.4 (Lipschitz implies $C^{1,\gamma}$). *Let u be a Lipschitz viscosity solution to* (1-3) *in B_1, with $0 \in F(u)$ and* $\mathrm{Lip}(u) \le L$. *Assume that f is continuous in $B_1^+(u) \cup B_1^-(u)$, $\|f\|_{L^\infty(B_1)} \le L$ and G satisfies assumptions* (H1)–(H3). *If $F(u)$ is a Lipschitz graph in a neighborhood of* 0, *then $F(u)$ is $C^{1,\gamma}$ in a* (*smaller*) *neighborhood of* 0.

Further extensions can be achieved with small extra effort: there is no problem in extending our results to the case when $\boldsymbol{b}$ and f are merely bounded measurable. However, as already said of the prototype problem, we wish to avoid too many technicalities.

In Theorems 1.3 and 1.4 we need to assume the Lipschitz continuity of our solution unless the operator can be put into divergence form. Indeed, in this case an almost monotonicity formula is available (see [Matevosyan and Petrosyan 2011]) and under the assumption $G(\eta, x) \to \infty$, as $\eta \to \infty$ one can reproduce the proof of Theorem 4.5 in [Caffarelli et al. 2002], to recover the Lipschitz continuity of a viscosity solution. Observe that then $f = f(x, u, \nabla u)$ is allowed, with $f(x, \cdot, \cdot)$ locally bounded.

2. Compactness and localization lemmas

In this section, we state basic definitions and we prove some elementary lemmas. First we need the following standard notion.

Definition 2.1. Given $u, \varphi \in C(\Omega)$, we say that φ touches u from below at $x_0 \in \Omega$ if $u(x_0) = \varphi(x_0)$ and

$$u(x) \geq \varphi(x) \quad \text{in a neighborhood } O \text{ of } x_0.$$

If this inequality is strict in $O \setminus \{x_0\}$, we say that φ touches u strictly from below. Touching (strictly) from above is defined similarly, replacing $\leq$ by $\geq$.

We retain the usual definition of C-viscosity sub/supersolutions and solutions of an elliptic PDE; see [Caffarelli and Cabré 1995], for example. Here is the definition of a viscosity solution to the problem (1-1):

Definition 2.2. Let u be a continuous function in Ω. We say that u is a *viscosity solution* to (1-1) in Ω if the following conditions are satisfied:

(i) $\Delta u = f$ in $\Omega^+(u) \cup \Omega^-(u)$ in the viscosity sense.

(ii) Let $x_0 \in F(u)$ and $v \in C^2(\overline{B^+(v)}) \cap C^2(\overline{B^-(v)})$ $(B = B_\delta(x_0))$ with $F(v) \in C^2$. If v touches u from below (resp. above) at $x_0 \in F(v)$, then

$$(v_\nu^+(x_0))^2 - (v_\nu^-(x_0))^2 \leq 1 \quad (\text{resp.} \geq 1).$$

For our arguments, it is convenient to introduce also the notion of comparison sub/supersolutions.

Definition 2.3. We say that $v \in C(\Omega)$ is a strict (comparison) subsolution (resp. supersolution) to (1-1) in Ω if $v \in C^2(\overline{\Omega^+(v)}) \cap C^2(\overline{\Omega^-(v)})$ and the following conditions are satisfied.

(i) $\Delta v > f$ (resp. $< f$) in $\Omega^+(v) \cup \Omega^-(v)$;

(ii) If $x_0 \in F(v)$, then

$$(v_\nu^+)^2 - (v_\nu^-)^2 > 1 \quad \big(\text{resp. } (v_\nu^+)^2 - (v_\nu^-)^2 < 1,\ v_\nu^+(x_0) \neq 0\big).$$

Notice that by the implicit function theorem, according to our definition the free boundary of a comparison sub/supersolution is C^2.

Remark 2.4. A strict comparison subsolution v cannot touch a viscosity solution u from below at any point in $F(u) \cap F(v)$. A strict comparison supersolution v cannot touch u from above at any point in $F(u) \cap F(v)$.

The next lemma shows that "δ-flat" viscosity solutions (in the sense of Theorem 1.1) enjoy non-degeneracy of the positive part δ-away from the free boundary:

Lemma 2.5. *Let u be a solution to* (1-1) *in B_2 with* $\mathrm{Lip}(u) \leq L$ *and* $\|f\|_{L^\infty} \leq L$. *If*

$$\{x_n \leq g(x') - \delta\} \subset \{u^+ = 0\} \subset \{x_n \leq g(x') + \delta\},$$

with g a Lipschitz function, $\mathrm{Lip}(g) \leq L$, $g(0) = 0$, *then*

$$u(x) \geq c_0(x_n - g(x')), \quad x \in \{x_n \geq g(x') + 2\delta\} \cap B_{\rho_0},$$

for some c_0, $\rho_0 > 0$ depending on n, L as long as $\delta \leq c_0$.

Proof. All constants in this proof will depend on n, L.

It suffices to show that our statement holds for $\{x_n \geq g(x') + C\delta\}$ for a possibly large constant C. Then one can apply the Harnack inequality to obtain the full statement.

We prove the statement above at $x = de_n$ (recall that $g(0) = 0$). Precisely, we want to show that

$$u(de_n) \geq c_0 d, \quad d \geq C\delta.$$

After rescaling, we reduce to proving that

$$u(e_n) \geq c_0$$

as long as $\delta \leq 1/C$, and $\|f\|_\infty$ is sufficiently small. Let $\gamma > 0$ and

$$w(x) = \frac{1}{2\gamma}(1 - |x|^{-\gamma})$$

be defined on the closure of the annulus $B_2 \setminus \bar{B}_1$ with $\|f\|_\infty$ small enough that

$$\Delta w < -\|f\| \quad \text{on } B_2 \setminus \bar{B}_1.$$

Extend $w = 0$ in B_1. Let

$$w_t(x) = w(x + te_n).$$

Notice that

$$((w_t)_\nu^+)^2 - ((w_t)_\nu^-)^2 < 1 \quad \text{on } F(w_t) = \partial B_1(-te_n). \tag{2-1}$$

From our flatness assumption for $t = C(L)$ sufficiently large (depending on the Lipschitz constant of g), w_t is above u. We decrease t continuously and let $\bar{t}$ be the smallest t such that w_t is above u. Notice that $\bar{t} > 0$.

Then, there is a touching point $z \in (\bar{B}_2 \setminus B_1) - \bar{t}e_n$. Since $w_{\bar{t}}$ is a strict supersolution to $\Delta u = f$ in $(B_2 \setminus \bar{B}_1) - \bar{t}e_n$ and (2-1) is satisfied, the touching point z can occur only on the $\eta := \frac{1}{2\gamma}(1 - 2^{-\gamma})$ level set in the positive phase of u. From the bounds on $\bar{t}$ it follows $|z| \leq C$ (C depending on L.)

Since u is Lipschitz continuous, we have $0 < u(z) = \eta \le Ld(z, F(u))$; that is, a full ball around z of radius η/L is contained in the positive phase of u. Thus, for $\bar{\delta}$ small depending on η, L, we have $B_{\eta/2L}(z) \subset \{x_n \ge g(x') + 2\bar{\delta}\}$. Since $x_n = g(x') + 2\bar{\delta}$ is Lipschitz we can connect e_n and z with a chain of intersecting balls included in the positive side of u with radii comparable to $\eta/2L$. The number of balls depends on L. Then we can apply the Harnack inequality and obtain

$$u(e_n) \ge cu(z) = c_0,$$

as desired. □

Next, we state a compactness lemma. For its proof, we refer the reader to Section 7 where the analogue of this result for a more general class of operators and free boundary conditions is stated and proved (see Lemma 7.3).

Lemma 2.6. *Let u_k be a sequence of viscosity solutions to* (1-1) *with right-hand side f_k satisfying $\|f_k\|_{L^\infty} \le L$. Assume $u_k \to u^*$ uniformly on compact sets, and $\{u_k^+ = 0\} \to \{(u^*)^+ = 0\}$ in the Hausdorff distance. Then*

$$-L \le \Delta u^* \le L \quad \text{in } \Omega^+(u^*) \cup \Omega^-(u^*)$$

in the viscosity sense and u^* satisfies the free boundary condition

$$(u_\nu^{*+})^2 - (u_\nu^{*-})^2 = 1 \quad \text{on } F(u^*)$$

in the viscosity sense of Definition 2.2.

We are now ready to reformulate our main Theorem 1.1 using the two lemmas above. First, we denote by U_β the following one-dimensional function,

$$U_\beta(t) = \alpha t^+ - \beta t^-, \quad \beta \ge 0, \quad \alpha = \sqrt{1+\beta^2},$$

where

$$t^+ = \max\{t, 0\}, \quad t^- = -\min\{t, 0\}.$$

Then $U_\beta(x) = U_\beta(x_n)$ is the so-called two-plane solution to (1-1) when $f \equiv 0$.

Lemma 2.7. *Let u be a solution to* (1-1) *in B_1 with* $\mathrm{Lip}(u) \le L$ *and $\|f\|_{L^\infty} \le L$. For any $\varepsilon > 0$ there exist $\bar{\delta}, \bar{r} > 0$ depending on ε, n, and L such that if*

$$\{x_n \le -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \le \delta\},$$

with $0 \le \delta \le \bar{\delta}$, then

$$\|u - U_\beta\|_{L^\infty(B_{\bar{r}})} \le \varepsilon \bar{r} \tag{2-2}$$

for some $0 \le \beta \le L$.

Proof. Given $\varepsilon > 0$ and $\bar{r}$ depending on ε to be specified later, assume by contradiction that there exist a sequence $\delta_k \to 0$ and a sequence of solutions u_k to the problem (1-1) with right-hand side f_k such that $\mathrm{Lip}(u_k)$, $\|f_k\| \le L$ and

$$\{x_n \le -\delta_k\} \subset B_1 \cap \{u_k^+(x) = 0\} \subset \{x_n \le \delta_k\}, \tag{2-3}$$

but the u_k do not satisfy the conclusion (2-2).

Then, up to a subsequence, the u_k converge uniformly on compacts to a function u^*. In view of (2-3) and the nondegeneracy of u_k^+ $2\delta_k$-away from the free boundary (Lemma 2.5), we can apply our compactness lemma and conclude that

$$-L \le \Delta u^* \le L \quad \text{in } B_{1/2} \cap \{x_n \neq 0\}$$

in the viscosity sense and also

$$(u_n^{*+})^2 - (u_n^{*-})^2 = 1 \quad \text{on } F(u^*) = B_{1/2} \cap \{x_n = 0\}, \tag{2-4}$$

with

$$u^* > 0 \quad \text{in } B_{\rho_0} \cap \{x_n > 0\}.$$

Thus,

$$u^* \in C^{1,\gamma}\big(B_{1/2} \cap \{x_n \ge 0\}\big) \cap C^{1,\gamma}\big(B_{1/2} \cap \{x_n \le 0\}\big)$$

for all γ and in view of (2-4) we have that (for any $\bar{r}$ small)

$$\|u^* - (\alpha x_n^+ - \beta x_n^-)\|_{L^\infty(B_{\bar{r}})} \le C(n, L)\bar{r}^{1+\gamma},$$

with $\alpha^2 = 1 + \beta^2$. If $\bar{r}$ is chosen depending on ε so that

$$C(n, L)\bar{r}^{1+\gamma} \le \frac{\varepsilon}{2}\bar{r},$$

since the u_k converge uniformly to u^* on $B_{1/2}$ we obtain that for all k large

$$\|u_k - (\alpha x_n^+ - \beta x_n^-)\|_{L^\infty(B_{\bar{r}})} \le \varepsilon\bar{r},$$

a contradiction. □

In view of Lemma 2.7, and after rescaling, our first main theorem (Theorem 1.1) follows from our second, which we now state:

Theorem 2.8. *Let u be a solution to* (1-1) *in B_1 with* $\mathrm{Lip}(u) \le L$ *and $\|f\|_{L^\infty} \le L$. There exists a universal constant $\bar{\varepsilon} > 0$ such that, if*

$$\|u - U_\beta\|_{L^\infty(B_1)} \le \bar{\varepsilon} \quad \textit{for some } 0 \le \beta \le L, \tag{2-5}$$

and

$$\{x_n \le -\bar{\varepsilon}\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \le \bar{\varepsilon}\} \quad \textit{and} \quad \|f\|_{L^\infty(B_1)} \le \bar{\varepsilon},$$

then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

The next lemma is elementary.

Lemma 2.9. *Let u be a continuous function. If, for $\eta > 0$ small, we have*

$$\|u - U_\beta\|_{L^\infty(B_2)} \le \eta \quad \textit{for } 0 \le \beta \le L,$$

and

$$\{x_n \leq -\eta\} \subset B_2 \cap \{u^+(x) = 0\} \subset \{x_n \leq \eta\},$$

then

- *if* $\beta \geq \eta^{1/3}$, *then* $U_\beta(x_n - \eta^{1/3}) \leq u(x) \leq U_\beta(x_n + \eta^{1/3})$ *in* B_1;
- *if* $\beta < \eta^{1/3}$, *then* $U_0(x_n - \eta^{1/3}) \leq u^+(x) \leq U_0(x_n + \eta^{1/3})$ *in* B_1.

3. The linearized problem

This section is devoted to the study of the linearized problem associated with our free boundary problem (1-1), that is, the following boundary value problem ($\tilde{\alpha} \neq 0$):

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_\rho \cap \{x_n \neq 0\}, \\ \tilde{\alpha}^2 (\tilde{u}_n)^+ - \tilde{\beta}^2 (\tilde{u}_n)^- = 0 & \text{on } B_\rho \cap \{x_n = 0\}. \end{cases} \tag{3-1}$$

Here $(\tilde{u}_n)^+$ (resp. $(\tilde{u}_n)^-$) denotes the derivative in the e_n direction of $\tilde{u}$ restricted to $\{x_n > 0\}$ (resp. $\{x_n < 0\}$).

We remark that Theorem 2.8 will follow, see Section 6, via a compactness argument from the regularity properties of viscosity solutions to (3-1).

Definition 3.1. A continuous function u is a viscosity solution to (3-1) if the following conditions are satisfied:

(i) $\Delta \tilde{u} = 0$ in $B_\rho \cap \{x_n \neq 0\}$, in the viscosity sense.

(ii) Let ϕ be a function of the form

$$\phi(x) = A + p x_n^+ - q x_n^- + B\,Q(x - y),$$

with

$$Q(x) = \tfrac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0),\ A \in \mathbb{R},\ B > 0$$

and

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0.$$

Then ϕ cannot touch u strictly from below at a point $x_0 = (x_0', 0) \in B_\rho$. Analogously, if

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q < 0,$$

then ϕ cannot touch u strictly from above at x_0.

We wish to prove the following regularity result for viscosity solutions to the linearized problem.

Theorem 3.2. *Let* $\tilde{u}$ *be a viscosity solution to* (3-1) *in* $B_{1/2}$ *such that* $\|\tilde{u}\|_\infty \leq 1$. *There exists a universal constant* $\bar{C}$ *such that*

$$\left|\tilde{u}(x) - \tilde{u}(0) - (\nabla_{x'}\tilde{u}(0) \cdot x' + \tilde{p} x_n^+ - \tilde{q} x_n^-)\right| \leq \bar{C} r^2 \quad \text{in } B_r, \tag{3-2}$$

for all $r \leq \frac{1}{4}$ *and with* $\tilde{\alpha}^2 \tilde{p} - \tilde{\beta}^2 \tilde{q} = 0$.

Before proving this, we first show that the problem (3-1) admits a classical solution:

Theorem 3.3. *Let h be a continuous function on ∂B_1. There exists a (unique) classical solution $\tilde v$ to* (3-1) *with $\tilde v = h$ on ∂B_1, that is, $\tilde v \in C^\infty(B_1 \cap \{x_n \ge 0\}) \cap C^\infty(B_1 \cap \{x_n \le 0\})$. In particular, there exists a universal constant $\widetilde C$ such that*

$$\left|\tilde v(x) - \tilde v(\bar x) - \left(\nabla_{x'}\tilde v(\bar x)\cdot(x'-\bar x') + \tilde p(\bar x)x_n^+ - \tilde q(\bar x)x_n^-\right)\right| \le \widetilde C \|\tilde v\|_{L^\infty} r^2 \quad \text{in } B_r(\bar x), \tag{3-3}$$

for all $r \le \frac14$, $\bar x = (\bar x', 0) \in B_{1/2}$ and with $\tilde\alpha^2 \tilde p(\bar x) - \tilde\beta^2 \tilde q(\bar x) = 0$.

Proof. Let w be the harmonic function in $B_1 \cap \{x_n > 0\}$ such that

$$\begin{aligned} w &= 0 && \text{on } B_1 \cap \{x_n = 0\},\\ w(x) &= h(x', x_n) - h(x', -x_n) && \text{on } \partial B_1 \cap \{x_n > 0\}. \end{aligned}$$

Then $w \in C^\infty(B_1 \cap \{x_n \ge 0\})$. Set

$$\phi(x') = w_n(x', 0), \qquad (x', 0) \in B_1.$$

Let

$$\tilde v_1(x) = w(x) + \tilde v_2(x', -x_n) \quad \text{in } \bar B_1 \cap \{x_n \ge 0\},$$

where $\tilde v_2$ is the solution to the problem

$$\begin{cases} \Delta \tilde v_2 = 0 & \text{in } B_1 \cap \{x_n < 0\},\\ \tilde v_2 = h & \text{on } \partial B_1 \cap \{x_n < 0\},\\ (\tilde v_2)_n = \tilde q\phi & \text{on } B_1 \cap \{x_n = 0\}, \end{cases}$$

with $\tilde q = \dfrac{\tilde\alpha^2}{\tilde\beta^2 + \tilde\alpha^2}$. Then it is easily verified that the function

$$\tilde v = \begin{cases} \tilde v_1 & \text{in } \bar B_1 \cap \{x_n \ge 0\},\\ \tilde v_2 & \text{in } \bar B_1 \cap \{x_n \le 0\} \end{cases}$$

is the unique classical solution to our problem and hence it satisfies the estimate (3-3) with

$$\tilde q(\bar x) = \tilde q\phi(\bar x), \qquad \tilde p(\bar x) = \tilde p\phi(\bar x), \qquad \tilde p = \frac{\tilde\beta^2}{\tilde\beta^2 + \tilde\alpha^2}. \qquad \square$$

Finally, to obtain our regularity result we only need to show the following fact.

Theorem 3.4. *Let $\tilde u$ be a viscosity solution to* (3-1) *in B_1 such that $\|\tilde u\|_\infty \le 1$ and let $\tilde v$ be the classical solution to* (3-1) *in $B_{1/2}$ with boundary data $\tilde u$. Then $\tilde u = \tilde v$.*

Proof. We prove that $\tilde v \le \tilde u$ in $B_{1/2}$. The opposite inequality is obtained in a similar way.

Let $\varepsilon > 0$, $t \in \mathbb{R}$ and set

$$\tilde v_{t,\varepsilon}(x) = \tilde v + \varepsilon|x_n| + \varepsilon x_n^2 - \varepsilon - t, \qquad x \in \bar B_{1/2}.$$

Since $\tilde u$ is bounded, for $t > 0$ large enough,

$$\tilde v_{t,\varepsilon} \le \tilde u. \tag{3-4}$$

Let $\bar{t}$ be the smallest t such that (3-4) holds and let $\bar{x}$ be the first touching point. We want to show that $\bar{t} < 0$. Assume $\bar{t} \geq 0$. Since

$$\tilde{v}_{\bar{t},\varepsilon} < \tilde{u} \quad \text{on } \partial B_{1/2},$$

such touching point must belong to $B_{1/2}$. However,

$$\begin{aligned} \Delta \tilde{v}_{\bar{t},\varepsilon}(x) > 0 \quad & \text{in } B_{1/2} \cap \{x_n \neq 0\}, \\ \Delta \tilde{u} = 0 \quad & \text{in } B_{1/2} \cap \{x_n \neq 0\}. \end{aligned}$$

Thus $\bar{x} \in B_{1/2} \cap \{x_n = 0\}$. We claim that there exists a function ϕ of the form

$$\phi(x) = A + p x_n^+ - q x_n^- + B Q(x - y)$$

with

$$Q(x) = \tfrac{1}{2}[(n-1)x_n^2 - |x'|^2], \qquad y = (y', 0),\ A \in \mathbb{R},\ B > 0$$

and

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0,$$

such that ϕ touches $\tilde{v}_{\bar{t},\varepsilon}(x)$ strictly from below at $\bar{x}$. This would contradict the definition of viscosity solutions, hence $\bar{t} < 0$. In particular,

$$\tilde{v} + \varepsilon |x_n| + \varepsilon x_n^2 - \varepsilon < \tilde{u} \quad \text{on } B_{1/2},$$

and for ε going to 0 we obtain as desired

$$\tilde{v} \leq \tilde{u} \quad \text{on } B_{1/2}.$$

We are left with the proof of the claim. Define

$$\nu' = \nabla_{x'} \tilde{v}(\bar{x}),$$

and set

$$y' = \bar{x}' + \frac{\nu'}{B}, \qquad A = \tilde{v}(\bar{x}) - \varepsilon - \bar{t} - B Q(\bar{x} - y),$$

with $B > 0$ to be chosen later. In view of the estimate (3-3), to verify that in a small neighborhood of $\bar{x}$

$$\phi(x) < \tilde{v}_{\bar{t},\varepsilon}(x), \qquad x \neq \bar{x},$$

we need to show that we can find $B > 0$, p, q such that for $|x - \bar{x}| \neq 0$ small enough ($\widetilde{C}$ universal),

$$\frac{B}{2}(n-1)x_n^2 - \frac{B}{2}|x' - \bar{x}'|^2 + p x_n^+ - q x_n^- < (\tilde{p} + \varepsilon) x_n^+ - (\tilde{q} - \varepsilon) x_n^- - \widetilde{C}|x - \bar{x}|^2$$

and

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0,$$

(for simplicity we dropped the dependence of $\tilde{p}, \tilde{q}$ on $\bar{x}$).

It is then enough to choose

$$B = 4\widetilde{C}, \qquad p = \tilde{p} + \frac{\varepsilon}{2}, \qquad q = \tilde{q} - \frac{\varepsilon}{2}. \qquad \square$$

4. The Harnack inequality

In this section we prove our main tool, a Harnack-type inequality for solutions to our free boundary problem. The results contained here will allow us to pass to the limit in the compactness argument for our improvement of flatness lemmas in Section 5.

Throughout this section we consider a Lipschitz solution u to (1-1) with $\mathrm{Lip}(u) \leq L$.

We need to distinguish two cases, which we call the nondegenerate and the degenerate case.

Nondegenerate case. In this case our solution u is trapped between two translations of a "true" two-plane solution U_β that is with $\beta \neq 0$.

Theorem 4.1 (Harnack inequality). *There exists a universal constant $\bar{\varepsilon}$ such that, if u satisfies at some point $x_0 \in B_2$*

$$U_\beta(x_n + a_0) \leq u(x) \leq U_\beta(x_n + b_0) \quad \textit{in } B_r(x_0) \subset B_2, \tag{4-1}$$

with

$$\|f\|_{L^\infty} \leq \varepsilon^2 \beta, \quad 0 < \beta \leq L,$$

and

$$b_0 - a_0 \leq \varepsilon r,$$

for some $\varepsilon \leq \bar{\varepsilon}$, then

$$U_\beta(x_n + a_1) \leq u(x) \leq U_\beta(x_n + b_1) \quad \textit{in } B_{r/20}(x_0),$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 \leq (1-c)\varepsilon r,$$

and $0 < c < 1$ universal.

Before giving the proof we deduce an important consequence.

If u satisfies (4-1) with, say $r = 1$, then we can apply the Harnack inequality repeatedly and obtain

$$U_\beta(x_n + a_m) \leq u(x) \leq U_\beta(x_n + b_m) \quad \text{in } B_{20^{-m}}(x_0),$$

with

$$b_m - a_m \leq (1-c)^m \varepsilon,$$

for all m such that

$$(1-c)^m 20^m \varepsilon \leq \bar{\varepsilon}.$$

This implies that for all such m, the oscillation of the function

$$\tilde{u}_\varepsilon(x) = \begin{cases} \dfrac{u(x) - \alpha x_n}{\alpha \varepsilon} & \text{in } B_2^+(u) \cup F(u), \\ \dfrac{u(x) - \beta x_n}{\beta \varepsilon} & \text{in } B_2^-(u), \end{cases}$$

in $B_r(x_0)$, $r = 20^{-m}$ is less than $(1-c)^m = 20^{-\gamma m} = r^\gamma$. Thus, the following corollary holds.

Corollary 4.2. *Let u be as in Theorem 4.1 satisfying* (4-1) *for $r = 1$. Then in $B_1(x_0)$ $\tilde{u}_\varepsilon$ has a Hölder modulus of continuity at x_0 outside the ball of radius $\varepsilon/\bar{\varepsilon}$; that is, for all $x \in B_1(x_0)$, with $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$,*

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C|x - x_0|^\gamma.$$

The proof of the Harnack inequality relies on the following lemma.

Lemma 4.3. *There exists a universal constant $\bar{\varepsilon} > 0$ such that if u satisfies*

$$u(x) \geq U_\beta(x) \quad \text{in } B_1,$$

with

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^2 \beta, \quad 0 < \beta \leq L, \tag{4-2}$$

then if at $\bar{x} = \frac{1}{5} e_n$,

$$u(\bar{x}) \geq U_\beta(\bar{x}_n + \varepsilon), \tag{4-3}$$

then

$$u(x) \geq U_\beta(x_n + c\varepsilon) \quad \text{in } \bar{B}_{1/2}, \tag{4-4}$$

for some universal c with $0 < c < 1$. Analogously, if $u(x) \leq U_\beta(x)$ in B_1 and $u(\bar{x}) \leq U_\beta(\bar{x}_n - \varepsilon)$, then

$$u(x) \leq U_\beta(x_n - c\varepsilon) \quad \text{in } \bar{B}_{1/2}.$$

Proof. We prove the first statement. For notational simplicity we drop the subindex β from U_β.

Let

$$w = c(|x - \bar{x}|^{-\gamma} - (3/4)^{-\gamma}) \tag{4-5}$$

be defined in the closure of the annulus

$$A := B_{3/4}(\bar{x}) \setminus \bar{B}_{1/20}(\bar{x}).$$

The constant c is such that w satisfies the boundary conditions

$$\begin{cases} w = 0 & \text{on } \partial B_{3/4}(\bar{x}), \\ w = 1 & \text{on } \partial B_{1/20}(\bar{x}). \end{cases}$$

Then, for a fixed $\gamma > n - 2$,

$$\Delta w \geq k(\gamma, n) = k(n) > 0, \quad 0 \leq w \leq 1 \text{ on } A.$$

Extend w to be equal to 1 on $B_{1/20}(\bar{x})$.

Notice that since $x_n > 0$ in $B_{1/10}(\bar{x})$ and $u \geq U$ in B_1, we get

$$B_{1/10}(\bar{x}) \subset B_1^+(u).$$

Thus $u - U \geq 0$ and solves $\Delta(u - U) = f$ in $B_{1/10}(\bar{x})$ and we can apply the Harnack inequality to obtain

$$u(x) - U(x) \geq c(u(\bar{x}) - U(\bar{x})) - C\|f\|_{L^\infty} \quad \text{in } \bar{B}_{1/20}(\bar{x}). \tag{4-6}$$

From the assumptions (4-2) and (4-3) we conclude that (for ε small enough)

$$u - U \geq \alpha c\varepsilon - C\alpha\varepsilon^2 \geq \alpha c_0 \varepsilon \quad \text{in } \bar{B}_{1/20}(\bar{x}). \tag{4-7}$$

Now set $\psi = 1 - w$ and

$$v(x) = U(x_n - \varepsilon c_0 \psi(x)), \quad x \in \bar{B}_{3/4}(\bar{x}),$$

and for $t \geq 0$,

$$v_t(x) = U(x_n - \varepsilon c_0 \psi(x) + t\varepsilon), \quad x \in \bar{B}_{3/4}(\bar{x}).$$

Then,

$$v_0(x) = U(x_n - \varepsilon c_0 \psi(x)) \leq U(x) \leq u(x), \quad x \in \bar{B}_{3/4}(\bar{x}).$$

Let $\bar{t}$ be the largest $t \geq 0$ such that

$$v_t(x) \leq u(x) \quad \text{in } \bar{B}_{3/4}(\bar{x}).$$

We want to show that $\bar{t} \geq c_0$. Then we get the desired statement. Indeed,

$$u(x) \geq v_{\bar{t}}(x) = U(x_n - \varepsilon c_0 \psi + \bar{t}\varepsilon) \geq U(x_n + c\varepsilon) \quad \text{in } B_{1/2} \Subset B_{3/4}(\bar{x}),$$

with c universal. In the last inequality we used that $\|\psi\|_{L^\infty(B_{1/2})} < 1$.

Suppose $\bar{t} < c_0$. Then at some $\tilde{x} \in \bar{B}_{3/4}(\bar{x})$ we have

$$v_{\bar{t}}(\tilde{x}) = u(\tilde{x}).$$

We show that such touching point can only occur on $\bar{B}_{1/20}(\bar{x})$. Indeed, since $w \equiv 0$ on $\partial B_{3/4}(\bar{x})$ from the definition of v_t we get that for $\bar{t} < c_0$,

$$v_{\bar{t}}(x) = U(x_n - \varepsilon c_0 \psi(x) + \bar{t}\varepsilon) < U(x) \leq u(x) \quad \text{on } \partial B_{3/4}(\bar{x}).$$

We now show that $\tilde{x}$ cannot belong to the annulus A. Indeed,

$$\Delta v_{\bar{t}} \geq \beta\varepsilon c_0 k(n) > \varepsilon^2 \beta \geq \|f\|_\infty \quad \text{in } A^+(v_{\bar{t}}) \cup A^-(v_{\bar{t}})$$

for ε small enough. Also,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 = 1 + \varepsilon^2 c_0^2 |\nabla\psi|^2 - 2\varepsilon c_0 \psi_n \quad \text{on } F(v_{\bar{t}}) \cap A.$$

Thus,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 > 1 \quad \text{on } F(v_{\bar{t}}) \cap A,$$

as long as

$$\psi_n < 0 \quad \text{on } F(v_{\bar{t}}) \cap A.$$

This can be easily verified from the formula for ψ (for ε small enough).

Thus, $v_{\bar{t}}$ is a strict subsolution to (1-1) in A which lies below u, hence by the definition of viscosity solution, $\tilde{x}$ cannot belong to A.

Therefore, $\tilde{x} \in \overline{B}_{1/20}(\bar{x})$ and

$$u(\tilde{x}) = v_{\bar{t}}(\tilde{x}) = U(\tilde{x}_n + \bar{t}\varepsilon) \le U(\tilde{x}) + \alpha\bar{t}\varepsilon < U(\tilde{x}) + \alpha c_0 \varepsilon,$$

contradicting (4-7).

The proof of the second statement follows from a similar argument. □

Proof of Theorem 4.1. Assume without loss of generality that $x_0 = 0$, $r = 1$. We distinguish three cases.

Case 1: $a_0 < -\frac{1}{5}$. In this case it follows from (4-1) that $B_{1/10} \subset \{u < 0\}$ and

$$0 \le v(x) := \frac{u(x) - \beta(x_n + a_0)}{\beta\varepsilon} \le 1,$$

with

$$|\Delta v| \le \varepsilon \quad \text{in } B_{1/10}.$$

The desired claim follows from the standard Harnack inequality applied to the function v.

Case 2: $a_0 > \frac{1}{5}$. In this case it follows from (4-1) that $B_{1/5} \subset \{u > 0\}$ and

$$0 \le v(x) := \frac{u(x) - \alpha(x_n + a_0)}{\alpha\varepsilon} \le 1,$$

with

$$|\Delta v| \le \varepsilon \quad \text{in } B_{1/5}.$$

Again, the desired claim follows from the standard Harnack inequality for v.

Case 3: $|a_0| \le 1/5$. Assumption (4-1) gives that

$$U_\beta(x_n + a_0) \le u(x) \le U_\beta(x_n + a_0 + \varepsilon) \quad \text{in } B_1.$$

Assume that (the other case is treated similarly)

$$u(\bar{x}) \ge U_\beta\big(\bar{x}_n + a_0 + \tfrac{1}{2}\varepsilon\big), \quad \bar{x} = \tfrac{1}{5}e_n. \tag{4-8}$$

Set

$$v(x) := u(x - a_0 e_n), \quad x \in B_{4/5}.$$

Then the inequality above reads

$$U_\beta(x_n) \le v(x) \le U_\beta(x_n + \varepsilon) \quad \text{in } B_{4/5}.$$

From (4-8), we have

$$v(\bar{x}) \ge U_\beta\big(\bar{x}_n + \tfrac{1}{2}\varepsilon\big).$$

Then, by Lemma 4.3,

$$v(x) \ge U_\beta(x_n + c\varepsilon) \quad \text{in } B_{2/5},$$

which gives the desired improvement

$$u(x) \ge U_\beta(x + a_0 + c\varepsilon) \quad \text{in } B_{3/5}.$$ □

Degenerate case. In this case, the negative part of u is negligible and the positive part is close to a one-plane solution (i.e., $\beta = 0$).

Theorem 4.4 (Harnack inequality). *There exists a universal constant $\bar{\varepsilon}$, such that if u satisfies at some point $x_0 \in B_2$*

$$U_0(x_n + a_0) \leq u^+(x) \leq U_0(x_n + b_0) \quad \textit{in } B_r(x_0) \subset B_2, \tag{4-9}$$

with

$$\|u^-\|_{L^\infty} \leq \varepsilon^2, \quad \|f\|_{L^\infty} \leq \varepsilon^4,$$

and

$$b_0 - a_0 \leq \varepsilon r,$$

for some $\varepsilon \leq \bar{\varepsilon}$, then

$$U_0(x_n + a_1) \leq u^+(x) \leq U_0(x_n + b_1) \quad \textit{in } B_{r/20}(x_0),$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 \leq (1-c)\varepsilon r,$$

and $0 < c < 1$ universal.

We can argue as in the nondegenerate case and get the following result.

Corollary 4.5. *Let u be as in Theorem 4.1 satisfying* (4-9) *for $r = 1$. Then in $B_1(x_0)$*

$$\tilde{u}_\varepsilon := \frac{u^+(x) - x_n}{\varepsilon}$$

has a Hölder modulus of continuity at x_0 outside the ball of radius $\varepsilon/\bar{\varepsilon}$; that is, for all $x \in B_1(x_0)$ with $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$,

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C|x - x_0|^\gamma.$$

The proof of the Harnack inequality can be deduced from the following lemma, as in the one-phase case [De Silva 2011].

Lemma 4.6. *There exists a universal constant $\bar{\varepsilon} > 0$ such that if u satisfies*

$$u^+(x) \geq U_0(x) \quad \textit{in } B_1,$$

with

$$\|u^-\|_{L^\infty} \leq \varepsilon^2, \quad \|f\|_{L^\infty} \leq \varepsilon^4, \tag{4-10}$$

then if at $\bar{x} = \frac{1}{5}e_n$

$$u^+(\bar{x}) \geq U_0(\bar{x}_n + \varepsilon), \tag{4-11}$$

then

$$u^+(x) \geq U_0(x_n + c\varepsilon) \quad \textit{in } \overline{B}_{1/2}, \tag{4-12}$$

for some universal c with $0 < c < 1$. Analogously, if $u^+(x) \leq U_0(x)$ in B_1 and $u^+(\bar{x}) \leq U_0(\bar{x}_n - \varepsilon)$, then

$$u^+(x) \leq U_0(x_n - c\varepsilon) \quad \textit{in } \overline{B}_{1/2}.$$

Proof. We prove the first statement. The proof follows the same line as in the nondegenerate case.

Since $x_n > 0$ in $B_{1/10}(\bar{x})$ and $u^+ \geq U_0$ in B_1 we get

$$B_{1/10}(\bar{x}) \subset B_1^+(u).$$

Thus $u - x_n \geq 0$ and solves $\Delta(u - x_n) = f$ in $B_{1/10}(\bar{x})$ and we can apply the Harnack inequality and the assumptions (4-10) and (4-11) to obtain that (for ε small enough)

$$u - x_n \geq c_0\varepsilon \quad \text{in } \bar{B}_{1/20}(\bar{x}). \tag{4-13}$$

Let w be as in the proof of Lemma 4.3 and $\psi = 1 - w$. Set

$$v(x) = (x_n - \varepsilon c_0\psi(x))^+ - \varepsilon^2 C_1(x_n - \varepsilon c_0\psi(x))^-, \quad x \in \bar{B}_{3/4}(\bar{x}),$$

and, for $t \geq 0$,

$$v_t(x) = (x_n - \varepsilon c_0\psi + t\varepsilon)^+ - \varepsilon^2 C_1(x_n - \varepsilon c_0\psi(x) + t\varepsilon)^-, \quad x \in \bar{B}_{3/4}(\bar{x}).$$

Here C_1 is a universal constant to be made precise later. We claim that

$$v_0(x) = v(x) \leq u(x), \quad x \in \bar{B}_{3/4}(\bar{x}).$$

This is readily verified in the set where u is nonnegative using that $u \geq x_n^+$. To prove our claim in the set where u is negative we wish to use the following fact:

$$u^- \leq C x_n^- \varepsilon^2 \quad \text{in } B_{19/20},\ C \text{ universal.} \tag{4-14}$$

This estimate is easily obtained using that $\{u < 0\} \subset \{x_n < 0\}$, $\|u^-\|_\infty < \varepsilon^2$ and the comparison principle with the function w satisfying

$$\Delta w = -\varepsilon^4 \quad \text{in } B_1 \cap \{x_n < 0\}, \qquad w = u^- \quad \text{on } \partial(B_1 \cap \{x_n < 0\}).$$

Thus our claim immediately follows from the fact that for $x_n < 0$ and $C_1 \geq C$,

$$\varepsilon^2 C_1(x_n - \varepsilon c_0\psi(x)) \leq C x_n \varepsilon^2.$$

Let $\bar{t}$ be the largest $t \geq 0$ such that

$$v_t(x) \leq u(x) \quad \text{in } \bar{B}_{3/4}(\bar{x}).$$

We want to show that $\bar{t} \geq c_0$. Then we get the desired statement. Indeed, it is easy to check that if

$$u(x) \geq v_{\bar{t}}(x) = (x_n - \varepsilon c_0\psi + \bar{t}\varepsilon)^+ - \varepsilon^2 C_1(x_n - \varepsilon c_0\psi(x) + \bar{t}\varepsilon)^- \quad \text{in } B_{3/4}(\bar{x}),$$

then

$$u^+(x) \geq U_0(x_n + c\varepsilon) \quad \text{in } B_{1/2} \Subset B_{3/4}(\bar{x}),$$

with c universal, $c < c_0 \inf_{B_{1/2}} w$.

Suppose $\bar{t} < c_0$. Then at some $\tilde{x} \in \bar{B}_{3/4}(\bar{x})$ we have

$$v_{\bar{t}}(\tilde{x}) = u(\tilde{x}).$$

We show that such a touching point can only occur on $\bar{B}_{1/20}(\bar{x})$. Indeed, since $w \equiv 0$ on $\partial B_{3/4}(\bar{x})$ from the definition of v_t we get that for $\bar{t} < c_0$

$$v_{\bar{t}}(x) = (x_n - \varepsilon c_0 + \bar{t}\varepsilon)^+ - \varepsilon^2 C_1 (x_n - \varepsilon c_0 + \bar{t}\varepsilon)^- < u(x) \quad \text{on } \partial B_{3/4}(\bar{x}).$$

In the set where $u \geq 0$, this can be seen using that $u \geq x_n^+$, while in the set where $u < 0$ again we can use the estimate (4-14).

We now show that $\tilde{x}$ cannot belong to the annulus A. Indeed,

$$\Delta v_{\bar{t}} \geq \varepsilon^3 c_0 k(n) > \varepsilon^4 \geq \|f\|_\infty \quad \text{in } A^+(v_{\bar{t}}) \cup A^-(v_{\bar{t}}),$$

for ε small enough.

Also,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 = (1 - \varepsilon^4 C_1^2)\big(1 + \varepsilon^2 c_0^2 |\nabla \psi|^2 - 2\varepsilon c_0 \psi_n\big) \quad \text{on } F(v_{\bar{t}}) \cap A.$$

Thus,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 > 1 \quad \text{on } F(v_{\bar{t}}) \cap A,$$

as long as ε is small enough (as in the nondegenerate case one can check that $\inf_{F(v_{\bar{t}}) \cap A}(-\psi_n) > c > 0$, with c universal.) Thus, $v_{\bar{t}}$ is a strict subsolution to (1-1) in A which lies below u, hence by definition $\tilde{x}$ cannot belong to A.

Therefore, $\tilde{x} \in \bar{B}_{1/20}(\bar{x})$ and

$$u(\tilde{x}) = v_{\bar{t}}(\tilde{x}) = (\tilde{x}_n + \bar{t}\varepsilon) < \tilde{x}_n + c_0 \varepsilon,$$

contradicting (4-13). □

5. Improvement of flatness

In this section we prove our key lemmas improving flatness. As in Section 4, we distinguish two cases.

Nondegenerate case. In this case our solution u is trapped between two translations of a two-plane solution U_β with $\beta \neq 0$. We plan to show that when we restrict to smaller balls, u is trapped between closer translations of another two-plane solution (in a different system of coordinates).

Lemma 5.1 (improvement of flatness). *Let u satisfy*

$$U_\beta(x_n - \varepsilon) \leq u(x) \leq U_\beta(x_n + \varepsilon) \quad \text{in } B_1,\ 0 \in F(u), \tag{5-1}$$

with $0 < \beta \leq L$ and

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^2 \beta.$$

If $0 < r \leq r_0$ for r_0 universal, and $0 < \varepsilon \leq \varepsilon_0$ for some ε_0 depending on r, then

$$U_{\beta'}\Big(x \cdot \nu_1 - r\frac{\varepsilon}{2}\Big) \leq u(x) \leq U_{\beta'}\Big(x \cdot \nu_1 + r\frac{\varepsilon}{2}\Big) \quad \text{in } B_r, \tag{5-2}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq \widetilde{C}\varepsilon$, and $|\beta - \beta'| \leq \widetilde{C}\beta\varepsilon$ for a universal constant $\widetilde{C}$.

Proof. We divide the proof of this lemma into three steps.

Step 1: compactness. Fix $r \le r_0$ with r_0 universal (the precise r_0 will be given in Step 3). Assume by contradiction that we can find a sequence $\varepsilon_k \to 0$ and a sequence u_k of solutions to (1-1) in B_1 with right-hand side f_k with L^∞ norm bounded by $\varepsilon_k^2 \beta_k$, such that

$$U_{\beta_k}(x_n - \varepsilon_k) \le u_k(x) \le U_{\beta_k}(x_n + \varepsilon_k) \quad \text{for } x \in B_1,\, 0 \in F(u_k), \tag{5-3}$$

with $L \ge \beta_k > 0$, but u_k does not satisfy the conclusion of the lemma, (5-2).

With $\alpha_k^2 = 1 + \beta_k^2$, set

$$\tilde{u}_k(x) = \begin{cases} \dfrac{u_k(x) - \alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(u_k) \cup F(u_k), \\ \dfrac{u_k(x) - \beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(u_k). \end{cases}$$

Then (5-3) gives

$$-1 \le \tilde{u}_k(x) \le 1 \quad \text{for } x \in B_1. \tag{5-4}$$

From Corollary 4.2, it follows that the function $\tilde{u}_k$ satisfies

$$|\tilde{u}_k(x) - \tilde{u}_k(y)| \le C|x - y|^\gamma, \tag{5-5}$$

for C universal, and

$$|x - y| \ge \varepsilon_k/\bar{\varepsilon}, \qquad x, y \in B_{1/2}.$$

From (5-3) it clearly follows that $F(u_k)$ converges to $B_1 \cap \{x_n = 0\}$ in the Hausdorff distance. This fact and (5-5) together with Ascoli–Arzelà give that as $\varepsilon_k \to 0$ the graphs of the $\tilde{u}_k$ converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function $\tilde{u}$ over $B_{1/2}$. Also, up to a subsequence we have

$$\beta_k \to \tilde{\beta} \ge 0,$$

and hence

$$\alpha_k \to \tilde{\alpha} = \sqrt{1 + \tilde{\beta}^2}.$$

Step 2: limiting solution. We now show that $\tilde{u}$ solves the following linearized problem (transmission problem):

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n \ne 0\}, \\ \tilde{\alpha}^2 (\tilde{u}_n)^+ - \tilde{\beta}^2 (\tilde{u}_n)^- = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases} \tag{5-6}$$

Since

$$|\Delta u_k| \le \varepsilon_k^2 \beta_k \quad \text{in } B_1^+(u_k) \cup B_1^-(u_k),$$

one easily deduces that $\tilde{u}$ is harmonic in $B_{1/2} \cap \{x_n \ne 0\}$.

Next, we prove that $\tilde{u}$ satisfies the boundary condition in (5-6) in the viscosity sense.

Let $\tilde{\phi}$ be a function of the form

$$\tilde{\phi}(x) = A + px_n^+ - qx_n^- + BQ(x-y),$$

with

$$Q(x) = \tfrac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0),\ A \in \mathbb{R},\ B > 0$$

and

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0.$$

Then we must show that $\tilde{\phi}$ cannot touch u strictly from below at a point $x_0 = (x_0', 0) \in B_{1/2}$ (the analogous statement from above follows with a similar argument).

Suppose that such a $\tilde{\phi}$ exists and let x_0 be the touching point.

Let

$$\Gamma(x) = \frac{1}{n-2}\big[(|x'|^2 + |x_n - 1|^2)^{\frac{2-n}{2}} - 1\big]$$

and

$$\Gamma_k(x) = \frac{1}{B\varepsilon_k}\Gamma(B\varepsilon_k(x-y) + AB\varepsilon_k^2 e_n). \tag{5-7}$$

Now, set

$$\phi_k(x) = a_k\Gamma_k^+(x) - b_k\Gamma_k^-(x) + \alpha_k(d_k^+(x))^2\varepsilon_k^{3/2} + \beta_k(d_k^-(x))^2\varepsilon_k^{3/2},$$

where

$$a_k = \alpha_k(1+\varepsilon_k p), \quad b_k = \beta_k(1+\varepsilon_k q),$$

and $d_k(x)$ is the signed distance from x to $\partial B_{1/(B\varepsilon_k)}\Big(y + e_n\Big(\dfrac{1}{B\varepsilon_k} - A\varepsilon_k\Big)\Big)$.

Finally, let

$$\tilde{\phi}_k(x) = \begin{cases} \dfrac{\phi_k(x) - \alpha_k x_n}{\alpha_k\varepsilon_k}, & x \in B_1^+(\phi_k) \cup F(\phi_k), \\[2ex] \dfrac{\phi_k(x) - \beta_k x_n}{\beta_k\varepsilon_k}, & x \in B_1^-(\phi_k). \end{cases}$$

By Taylor's theorem,

$$\Gamma(x) = x_n + Q(x) + O(|x|^3), \quad x \in B_1;$$

thus it is easy to verify that

$$\Gamma_k(x) = A\varepsilon_k + x_n + B\varepsilon_k Q(x-y) + O(\varepsilon_k^2), \quad x \in B_1,$$

with the constant in $O(\varepsilon_k^2)$ depending on A, B, and $|y|$ (later this constant will depend also on p, q).

It follows that in $B_1^+(\phi_k) \cup F(\phi_k)$ $(Q^y(x) = Q(x-y))$,

$$\tilde{\phi}_k(x) = A + BQ^y + px_n + A\varepsilon_k p + Bp\varepsilon_k Q^y + \varepsilon_k^{1/2} d_k^2 + O(\varepsilon_k),$$

and analogously in $B_1^-(\phi_k)$,

$$\tilde{\phi}_k(x) = A + BQ^y + qx_n + A\varepsilon_k p + Bq\varepsilon_k Q^y + \varepsilon_k^{1/2} d_k^2 + O(\varepsilon_k).$$

Hence, $\tilde{\phi}_k$ converges uniformly to $\tilde{\phi}$ on $B_{1/2}$. Since $\tilde{u}_k$ converges uniformly to $\tilde{u}$ and $\tilde{\phi}$ touches $\tilde{u}$ strictly from below at x_0, we conclude that there exist a sequence of constants $c_k \to 0$ and of points $x_k \to x_0$ such that the function

$$\psi_k(x) = \phi_k(x + \varepsilon_k c_k e_n)$$

touches u_k from below at x_k. We thus get a contradiction if we prove that ψ_k is a strict subsolution to our free boundary problem, that is,

$$\begin{cases} \Delta\psi_k > \varepsilon_k^2 \beta_k \geq \| f_k \|_\infty & \text{in } B_1^+(\psi_k) \cup B_1^-(\psi_k), \\ (\psi_k^+)_\nu^2 - (\psi_k^-)_\nu^2 > 1, & \text{on } F(\psi_k). \end{cases} \tag{5-8}$$

It is easily checked that, away from the free boundary,

$$\Delta\psi_k \geq \beta_k \varepsilon_k^{3/2} \Delta d_k^2(x + \varepsilon_k c_k e_n),$$

and the first condition in (5-8) is satisfied for k large enough.

Finally, since on the zero level set $|\nabla\Gamma_k| = 1$ and $|\nabla d_k^2| = 0$, the free boundary condition reduces to showing that

$$a_k^2 - b_k^2 > 1.$$

Using the definition of a_k, b_k we need to check that

$$(\alpha_k^2 p^2 - \beta_k^2 q^2)\varepsilon_k + 2(\alpha_k^2 p - \beta_k^2 q) > 0.$$

This inequality holds for k large in view of the fact that

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0.$$

Thus $\tilde{u}$ is a solution to the linearized problem.

Step 3: Contradiction. According to estimate (3-2), since $\tilde{u}(0) = 0$ we obtain that

$$|\tilde{u} - (x' \cdot \nu' + \tilde{p} x_n^+ - \tilde{q} x_n^-)| \leq Cr^2, \quad x \in B_r,$$

with

$$\tilde{\alpha}^2 \tilde{p} - \tilde{\beta}^2 \tilde{q} = 0, \quad |\nu'| = |\nabla_{x'} \tilde{u}(0)| \leq C.$$

Thus, since $\tilde{u}_k$ converges uniformly to $\tilde{u}$ (by slightly enlarging C) we get that

$$|\tilde{u}_k - (x' \cdot \nu' + \tilde{p} x_n^+ - \tilde{q} x_n^-)| \leq Cr^2, \quad x \in B_r. \tag{5-9}$$

Now set

$$\beta_k' = \beta_k(1 + \varepsilon_k \tilde{q}), \quad \nu_k = \frac{1}{\sqrt{1 + \varepsilon_k^2 |\nu'|^2}} (e_n + \varepsilon_k(\nu', 0)).$$

Then,

$$\alpha_k' = \sqrt{1 + \beta_k'^2} = \alpha_k(1 + \varepsilon_k \tilde{p}) + O(\varepsilon_k^2), \quad \nu_k = e_n + \varepsilon_k(\nu', 0) + \varepsilon_k^2 \tau, \quad |\tau| \leq C,$$

where to obtain the first equality we used that $\tilde{\alpha}^2\tilde{p} - \tilde{\beta}^2\tilde{q} = 0$ and hence

$$\frac{\beta_k^2}{\alpha_k^2}\tilde{q} = \tilde{p} + O(\varepsilon_k).$$

With these choices we can now show that (for k large and $r \le r_0$)

$$\widetilde{U}_{\beta'_k}\Big(x\cdot\nu_k - \varepsilon_k\frac{r}{2}\Big) \le \tilde{u}_k(x) \le \widetilde{U}_{\beta'_k}\Big(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\Big) \quad \text{in } B_r,$$

where again we are using the notation

$$\widetilde{U}_{\beta'_k}(x) = \begin{cases} \dfrac{\widetilde{U}_{\beta'_k}(x) - \alpha_k x_n}{\alpha_k\varepsilon_k}, & x \in B_1^+(\widetilde{U}_{\beta'_k}) \cup F(\widetilde{U}_{\beta'_k}), \\ \dfrac{\widetilde{U}_{\beta'_k}(x) - \beta_k x_n}{\beta_k\varepsilon_k}, & x \in B_1^-(\widetilde{U}_{\beta'_k}). \end{cases}$$

This will clearly imply that

$$U_{\beta'_k}\Big(x\cdot\nu_k - \varepsilon_k\frac{r}{2}\Big) \le u_k(x) \le U_{\beta'_k}\Big(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\Big) \quad \text{in } B_r$$

and hence will lead to a contradiction.

In view of (5-9), we need to show that in B_r,

$$\begin{aligned} \widetilde{U}_{\beta'_k}\Big(x\cdot\nu_k - \varepsilon_k\frac{r}{2}\Big) &\le (x'\cdot\nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-) - Cr^2, \\ \widetilde{U}_{\beta'_k}\Big(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\Big) &\ge (x'\cdot\nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-) + Cr^2. \end{aligned}$$

We show the second inequality. In the set where

$$x\cdot\nu_k + \varepsilon_k\frac{r}{2} < 0 \tag{5-10}$$

we have, by definition,

$$\widetilde{U}_{\beta'_k}\Big(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\Big) = \frac{1}{\beta_k\varepsilon_k}\Big(\beta'_k\Big(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\Big) - \beta_k x_n\Big),$$

which from the formula for β'_k, ν_k gives

$$\widetilde{U}_{\beta'_k}\Big(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\Big) \ge x'\cdot\nu' + \tilde{q}x_n + \frac{r}{2} - C_0\varepsilon_k.$$

Using (5-10) we then obtain

$$\widetilde{U}_{\beta'_k}\Big(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\Big) \ge x'\cdot\nu' + \tilde{p}x_n^+ - \tilde{q}x_n^- + \frac{r}{2} - C_1\varepsilon_k.$$

Thus to obtain the desired bound it suffices to fix $r_0 \le 1/(4C)$ and take k large enough.

The other case can be argued similarly. □

Degenerate case. In this case, the negative part of u is negligible and the positive part is close to a one-plane solution ($\beta = 0$). We prove below that in this setting only u^+ enjoys an improvement of flatness.

Lemma 5.2 (improvement of flatness). *Let u satisfy*

$$U_0(x_n - \varepsilon) \le u^+(x) \le U_0(x_n + \varepsilon) \quad \textit{in } B_1, 0 \in F(u), \tag{5-11}$$

with

$$\|f\|_{L^\infty(B_1)} \le \varepsilon^4 \quad \textit{and} \quad \|u^-\|_{L^\infty(B_1)} \le \varepsilon^2.$$

If $0 < r \le r_1$ for r_1 universal, and $0 < \varepsilon \le \varepsilon_1$ for some ε_1 depending on r, then

$$U_0\Big(x \cdot \nu_1 - r\frac{\varepsilon}{2}\Big) \le u^+(x) \le U_0\Big(x \cdot \nu_1 + r\frac{\varepsilon}{2}\Big) \quad \textit{in } B_r, \tag{5-12}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \le C\varepsilon$ for a universal constant C.

Proof. We argue similarly as in the nondegenerate case.

Step 1: compactness. Fix $r \le r_1$ with r_1 universal (made precise in Step 3). Assume for a contradiction that we can find a sequence $\varepsilon_k \to 0$ and a sequence u_k of solutions to (1-1) in B_1 with right-hand side f_k with L^∞ norm bounded by ε_k^4, such that

$$U_0(x_n - \varepsilon_k) \le u_k^+(x) \le U_0(x_n + \varepsilon_k) \quad \text{for } x \in B_1, 0 \in F(u_k), \tag{5-13}$$

with

$$\|u_k^-\|_\infty \le \varepsilon_k^2,$$

but u_k does not satisfy the conclusion (5-12) of the lemma. Set

$$\tilde{u}_k(x) = \frac{u_k(x) - x_n}{\varepsilon_k}, \quad x \in B_1^+(u_k) \cup F(u_k).$$

Then (5-13) gives

$$-1 \le \tilde{u}_k(x) \le 1 \quad \text{for } x \in B_1^+(u_k) \cup F(u_k). \tag{5-14}$$

As in the nondegenerate case, it follows from Corollary 4.5 that as $\varepsilon_k \to 0$ the graphs of the $\tilde{u}_k$ converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function $\tilde{u}$ over $B_{1/2} \cap \{x_n \ge 0\}$.

Step 2: limiting solution. We now show that $\tilde{u}$ solves the following Neumann problem:

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \tilde{u}_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases} \tag{5-15}$$

As before, the interior condition follows easily thus we focus on the boundary condition.

Let $\tilde{\phi}$ be a function of the form

$$\tilde{\phi}(x) = A + px_n + BQ(x - y),$$

with

$$Q(x) = \tfrac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0), \ A \in \mathbb{R}, B > 0$$

and $p > 0$. We must show that $\tilde{\phi}$ cannot touch u strictly from below at a point $x_0 = (x_0', 0) \in B_{1/2}$. Suppose that such a $\tilde{\phi}$ exists and let x_0 be the touching point.

Let Γ_k and d_k be as in the proof of the nondegenerate case (see (5-7) and subsequent lines). Set

$$\phi_k(x) = a_k \Gamma_k^+(x) + (d_k^+(x))^2 \varepsilon_k^2, \quad a_k = (1 + \varepsilon_k p).$$

Let

$$\tilde{\phi}_k(x) = \frac{\phi_k(x) - x_n}{\varepsilon_k}.$$

As in the previous case, it follows that in $B_1^+(\phi_k) \cup F(\phi_k)$ ($Q^y(x) = Q(x-y)$),

$$\tilde{\phi}_k(x) = A + BQ^y + px_n + A\varepsilon_k p + Bp\varepsilon_k Q^y + \varepsilon_k d_k^2 + O(\varepsilon_k).$$

Hence, $\tilde{\phi}_k$ converges uniformly to $\tilde{\phi}$ on $B_{1/2} \cap \{x_n \geq 0\}$. Since $\tilde{u}_k$ converges uniformly to $\tilde{u}$ and $\tilde{\phi}$ touches $\tilde{u}$ strictly from below at x_0, we conclude that there exist a sequence of constants $c_k \to 0$ and of points $x_k \to x_0$ such that the function

$$\psi_k(x) = \phi_k(x + \varepsilon_k c_k e_n)$$

touches u_k from below at $x_k \in B_1^+(u_k) \cup F(u_k)$. We claim that x_k cannot belong to $B_1^+(u_k)$. Otherwise, in a small neighborhood N of x_k we would have

$$\Delta\psi_k > \varepsilon_k^4 \geq \|f_k\|_\infty = \Delta u_k, \quad \psi_k < u_k \text{ in } N \setminus \{x_k\},\ \psi_k(x_k) = u_k(x_k),$$

a contradiction.

Thus $x_k \in F(u_k) \cap \partial B_{1/(B\varepsilon_k)}\big(y + e_n(\frac{1}{B\varepsilon_k} - A\varepsilon_k - \varepsilon_k c_k)\big)$. For simplicity we set

$$\mathcal{B} := B_{1/(B\varepsilon_k)}\left(y + e_n\left(\frac{1}{B\varepsilon_k} - A\varepsilon_k - \varepsilon_k c_k\right)\right).$$

Let N_ρ be a small neighborhood of x_k of size ρ. Since

$$\|u_k^-\|_\infty \leq \varepsilon_k^2, \quad u_k^+ \geq (x_n - \varepsilon_k)^+,$$

as in the proof of the Harnack inequality and using the fact that $x_k \in F(u_k) \cap \partial\mathcal{B}$, we can conclude by the comparison principle that

$$u_k^- \leq c\varepsilon_k^2 (d(x, \partial\mathcal{B}))^- \quad \text{in } N_{\frac{3}{4}\rho},$$

where d denotes again the signed distance from x to $\partial\mathcal{B}$.

Let

$$\Psi_k(x) = \begin{cases} \psi_k & \text{in } \mathcal{B}, \\ c\varepsilon_k^2(3d(x, \partial\mathcal{B}) + d^2(x, \partial\mathcal{B})) & \text{outside of } \mathcal{B}. \end{cases} \tag{5-16}$$

Then Ψ_k touches u_k strictly from below at $x_k \in F(u_k) \cap F(\Psi_k)$.

We will reach a contradiction if we show that

$$(\Psi_k^+)_\nu^2 - (\Psi_k^-)_\nu^2 > 1 \quad \text{on } F(\Psi_k).$$

This is equivalent to showing that

$$a_k^2 - c\varepsilon_k^4 > 1, \quad \text{or again} \quad (1+\varepsilon_k p)^2 - c\varepsilon_k^4 > 1.$$

This holds for k large enough, since $p > 0$. We finally reached a contradiction.

Step 3: contradiction. In this step we can argue as in the final step of the proof of Lemma 4.1 in [De Silva 2011]. □

6. Proof of the main theorems

In this section we exhibit the proofs of our main results, Theorems 1.1 and 1.2. As already pointed out, Theorem 1.2 will follow via a blow-up analysis from the flatness result. Thus, first we present the proof of Theorem 1.1 based on the improvement of flatness lemmas of the previous section.

Proof of Theorem 1.1. To complete the analysis of the degenerate case, we need to deal with the situation when u is close to a one-plane solution and yet the size of u^- is not negligible. More precisely:

Lemma 6.1. *Let u solve* (1-1) *in B_2 with*

$$\|f\|_{L^\infty(B_1)} \le \varepsilon^4,$$

and let it satisfy

$$U_0(x_n - \varepsilon) \le u^+(x) \le U_0(x_n + \varepsilon) \quad \text{in } B_1, \quad 0 \in F(u), \tag{6-1}$$

and

$$\|u^-\|_{L^\infty(B_2)} \le \bar{C}\varepsilon^2, \quad \|u^-\|_{L^\infty(B_1)} > \varepsilon^2,$$

for a universal constant $\bar{C}$. There is a universal $\varepsilon_2 > 0$ such that, if $\varepsilon \le \varepsilon_2$, the rescaling

$$u_\varepsilon(x) = \varepsilon^{-1/2} u(\varepsilon^{1/2} x)$$

satisfies in B_1

$$U_{\beta'}(x_n - C'\varepsilon^{1/2}) \le u_\varepsilon(x) \le U_{\beta'}(x_n + C'\varepsilon^{1/2}),$$

with $\beta' \sim \varepsilon^2$ and $C' > 0$ depending on $\bar{C}$.

Proof. For notational simplicity we set

$$v = \frac{u^-}{\varepsilon^2}.$$

From our assumptions we can deduce that

$$\begin{gathered} F(v) \subset \{-\varepsilon \le x_n \le \varepsilon\}, \\ v \ge 0 \quad \text{in } B_2 \cap \{x_n \le -\varepsilon\}, \quad v \equiv 0 \quad \text{in } B_2 \cap \{x_n > \varepsilon\}. \end{gathered} \tag{6-2}$$

Also,

$$|\Delta v| \le \varepsilon^2 \quad \text{in } B_2 \cap \{x_n < -\varepsilon\},$$

and

$$0 \le v \le \bar{C} \quad \text{on } \partial B_2, \tag{6-3}$$

$$v(\bar{x}) > 1 \quad \text{at some point } \bar{x} \text{ in } B_1. \tag{6-4}$$

Thus, using comparison with the function w such that

$$\begin{aligned} \Delta w &= -\varepsilon^2 \quad \text{in } D := B_2 \cap \{x_n < \varepsilon\}, \\ w &= v \quad \text{on } \partial D, \end{aligned}$$

we obtain that for some $k > 0$ universal

$$v \le k|x_n - \varepsilon| \quad \text{in } B_1. \tag{6-5}$$

This fact forces the point $\bar{x}$ in (6-4) to belong to $B_1 \cap \{x_n < -\varepsilon\}$ at a fixed distance δ from $x_n = -\varepsilon$.

Now, let w be the harmonic function in $B_1 \cap \{x_n < -\varepsilon\}$ such that

$$\begin{aligned} w &= 0 \quad \text{on } B_1 \cap \{x_n = -\varepsilon\}, \\ w &= v \quad \text{on } \partial B_1 \cap \{x_n \le -\varepsilon\}. \end{aligned}$$

By the maximum principle we conclude that

$$w + \varepsilon^2(|x|^2 - 3) \le v \quad \text{on } B_1 \cap \{x_n < -\varepsilon\}.$$

Also, for ε small, in view of (6-5) we obtain that

$$w - k\varepsilon(|x|^2 - 3) \ge v \quad \text{on } \partial(B_1 \cap \{x_n < -\varepsilon\}),$$

and hence also in the interior. Thus we conclude that

$$|w - v| \le c\varepsilon \quad \text{in } B_1 \cap \{x_n < -\varepsilon\}. \tag{6-6}$$

In particular this is true at $\bar{x}$, which forces

$$w(\bar{x}) \ge 1/2. \tag{6-7}$$

By expanding w around $(0, -\varepsilon)$ we then obtain, say, in $B_{1/2} \cap \{x_n \le -\varepsilon\}$,

$$\big|w - a|x_n + \varepsilon|\big| \le C|x|^2 + C\varepsilon.$$

This combined with (6-6) gives that

$$\big|v - a|x_n + \varepsilon|\big| \le C\varepsilon \quad \text{in } B_{\varepsilon^{1/2}} \cap \{x_n \le -\varepsilon\}.$$

Moreover, in view of (6-7) and the fact that $\bar{x}$ occurs at a fixed distance from $\{x_n = -\varepsilon\}$ we deduce from the Hopf lemma that

$$a \ge c > 0,$$

with c universal. In conclusion (see (6-5)),

$$\begin{aligned} \left|u^- - b\varepsilon^2|x_n+\varepsilon|\right| &\leq C\varepsilon^3 \quad \text{in } B_{\varepsilon^{1/2}} \cap \{x_n \leq -\varepsilon\},\\ u^- &\leq b\varepsilon^2|x_n-\varepsilon| \quad \text{in } B_1, \end{aligned}$$

with b comparable to a universal constant.

Combining the two inequalities above and the assumption (6-1) we conclude that in $B_{\varepsilon^{1/2}}$

$$(x_n-\varepsilon)^+ - b\varepsilon^2(x_n - C\varepsilon)^- \leq u(x) \leq (x_n+\varepsilon)^+ - b\varepsilon^2(x_n+C\varepsilon)^-,$$

with $C > 0$ universal and b larger than a universal constant. Rescaling, we obtain that in B_1

$$(x_n-\varepsilon^{1/2})^+ - \beta'(x_n - C\varepsilon^{1/2})^- \leq u_\varepsilon(x) \leq (x_n+\varepsilon^{1/2})^+ - \beta'(x_n+C\varepsilon^{1/2})^-,$$

with $\beta' \sim \varepsilon^2$. We finally need to check that this implies the desired conclusion in B_1

$$\alpha'(x_n-C\varepsilon^{1/2})^+ - \beta'(x_n - C\varepsilon^{1/2})^- \leq u_\varepsilon(x) \leq \alpha'(x_n+C\varepsilon^{1/2})^+ - \beta'(x_n+C\varepsilon^{1/2})^-,$$

with $\alpha'^2 = 1 + \beta'^2 \sim 1 + \varepsilon^4$. This clearly holds in B_1 for ε small, say by possibly enlarging C so that $C \geq 2$. $\square$

We are finally ready to exhibit the proof of Theorem 2.8, which as already observed immediately gives the result of Theorem 1.1.

Proof of Theorem 2.8. Let us fix a universal constant $\bar{r} > 0$ such that

$$\bar{r} \leq r_0, r_1, \tfrac{1}{16},$$

where r_0, r_1 are the universal constants in the improvement of flatness Lemmas 5.1 and 5.2. Also, let us fix a universal constant $\tilde{\varepsilon} > 0$ such that

$$2\tilde{\varepsilon} \leq 2\varepsilon_0(\bar{r}), \varepsilon_1(\bar{r}), \widetilde{C}^{-1}, \varepsilon_2,$$

where $\varepsilon_0, \varepsilon_1, \varepsilon_2, \widetilde{C}$ are the constants in Lemmas 5.1, 5.2 and 6.1. Now, let

$$\bar{\varepsilon} = \tilde{\varepsilon}^3.$$

We distinguish two cases. For notational simplicity we assume that u satisfies our assumptions in the ball B_2 and $0 \in F(u)$.

<u>Case 1</u>: $\beta \geq \tilde{\varepsilon}$. In this case, in view of Lemma 2.9 and our choice of $\tilde{\varepsilon}$, we obtain that u satisfies the assumptions of Lemma 5.1, namely

$$U_\beta(x_n - \tilde{\varepsilon}) \leq u(x) \leq U_\beta(x_n+\tilde{\varepsilon}) \quad \text{in } B_1, \quad 0 \in F(u), \tag{6-8}$$

with $0 < \beta \leq L$ and

$$\|f\|_{L^\infty(B_1)} \leq \tilde{\varepsilon}^3 \leq \tilde{\varepsilon}^2\beta.$$

Thus we can conclude that (with $\beta_1 = \beta'$)

$$U_{\beta_1}\Big(x\cdot\nu_1-\bar r\frac{\tilde\varepsilon}{2}\Big)\le u(x)\le U_{\beta_1}\Big(x\cdot\nu_1+\bar r\frac{\tilde\varepsilon}{2}\Big)\quad\text{in } B_{\bar r},$$

with $|\nu_1|=1$, $|\nu_1-e_n|\le\widetilde C\tilde\varepsilon$, and $|\beta-\beta_1|\le\widetilde C\beta\tilde\varepsilon$. In particular, by our choice of $\tilde\varepsilon$ we have

$$\beta_1\ge\tilde\varepsilon/2.$$

We can therefore rescale and iterate the argument above. Precisely, for $k=0,1,2,\dots$, set

$$\rho_k=\bar r^k,\quad \varepsilon_k=2^{-k}\tilde\varepsilon,\quad u_k(x)=\frac{1}{\rho_k}u(\rho_k x),\quad f_k(x)=\rho_k f(\rho_k x).$$

Also, let β_k be the constants generated at each k-iteration, hence satisfying (with $\beta_0=\beta$)

$$|\beta_k-\beta_{k+1}|\le\widetilde C\beta_k\varepsilon_k.$$

Then we obtain by induction that each u_k satisfies

$$U_{\beta_k}(x\cdot\nu_k-\varepsilon_k)\le u_k(x)\le U_{\beta_k}(x\cdot\nu_k+\varepsilon_k)\quad\text{in } B_1,\tag{6-9}$$

with $|\nu_k|=1$, $|\nu_k-\nu_{k+1}|\le\widetilde C\tilde\varepsilon_k$ ($\nu_0=e_n$).

Case 2: $\beta<\tilde\varepsilon$. In view of Lemma 2.9 we conclude that

$$U_0(x_n-\tilde\varepsilon)\le u^+(x)\le U_0(x_n+\tilde\varepsilon)\quad\text{in } B_1.$$

Moreover, from the assumption (2-5) and the fact that $\beta<\tilde\varepsilon$ we also obtain that

$$\|u^-\|_{L^\infty(B_1)}<2\tilde\varepsilon.$$

Let ε' be given by $\varepsilon'^2=2\tilde\varepsilon$. Then u satisfies the assumptions of Lemma 5.2 on improvement of flatness in the degenerate case:

$$U_0(x_n-\varepsilon')\le u^+(x)\le U_0(x_n+\varepsilon')\quad\text{in } B_1,$$

with

$$\|f\|_{L^\infty(B_1)}\le(\varepsilon')^4,\quad \|u^-\|_{L^\infty(B_1)}<\varepsilon'^2.$$

We conclude that

$$U_0\Big(x\cdot\nu_1-\bar r\frac{\varepsilon'}{2}\Big)\le u^+(x)\le U_0\Big(x\cdot\nu_1+\bar r\frac{\varepsilon'}{2}\Big)\quad\text{in } B_{\bar r},$$

with $|\nu_1|=1$, $|\nu_1-e_n|\le C\varepsilon'$ for a universal constant C. We now rescale as in the previous case and set, for $k=0,1,2,\dots$,

$$\rho_k=\bar r^k,\quad \varepsilon_k=2^{-k}\varepsilon',\quad u_k(x)=\frac{1}{\rho_k}u(\rho_k x),\quad f_k(x)=\rho_k f(\rho_k x).$$

We can iterate our argument and obtain that (with $|\nu_k|=1$, $|\nu_k-\nu_{k+1}|\le C\varepsilon_k$)

$$U_0(x\cdot\nu_k-\varepsilon_k)\le u_k^+(x)\le U_0(x\cdot\nu_k+\varepsilon_k)\quad\text{in } B_1,\tag{6-10}$$

as long as we can verify that

$$\|u_k^-\|_{L^\infty(B_1)}<\varepsilon_k^2.$$

Let $\bar{k}$ be the first integer $\bar{k} > 1$ for which this fails, that is,

$$\|u_{\bar{k}}^-\|_{L^\infty(B_1)} \geq \varepsilon_{\bar{k}}^2 \quad \text{and} \quad \|u_{\bar{k}-1}^-\|_{L^\infty(B_1)} < \varepsilon_{\bar{k}-1}^2.$$

Also,

$$U_0(x \cdot \nu_{\bar{k}-1} - \varepsilon_{\bar{k}-1}) \leq u_{\bar{k}-1}^+(x) \leq U_0(x \cdot \nu_{\bar{k}-1} + \varepsilon_{\bar{k}\_1}) \quad \text{in } B_1.$$

As argued several times (see for example (4-14)), we can then conclude from the comparison principle that

$$u_{\bar{k}-1}^- \leq M|x_n - \varepsilon_{\bar{k}-1}|\varepsilon_{\bar{k}-1}^2 \quad \text{in } B_{19/20},$$

for a universal constant $M > 0$. Thus, by rescaling we get that

$$\|u_{\bar{k}}^-\|_{L^\infty(B_2)} < \bar{C}\varepsilon_{\bar{k}}^2,$$

with $\bar{C}$ universal (depending on the fixed $\bar{r}$). We obtain that $u_{\bar{k}}$ satisfies all the assumptions of Lemma 6.1 and hence the rescaling

$$v(x) = \varepsilon_{\bar{k}}^{-1/2} u_{\bar{k}}(\varepsilon_{\bar{k}}^{1/2} x)$$

satisfies in B_1

$$U_{\beta'}(x_n - C'\varepsilon_{\bar{k}}^{1/2}) \leq v(x) \leq U_{\beta'}(x_n + C'\varepsilon_{\bar{k}}^{1/2}),$$

with $\beta' \sim \varepsilon_{\bar{k}}^2$. Set $\eta = \bar{C}\varepsilon_{\bar{k}}^{1/2}$. Then v satisfies our free boundary problem in B_1 with right-hand side

$$g(x) = \varepsilon_{\bar{k}}^{1/2} f_{\bar{k}}(\varepsilon_{\bar{k}}^{1/2} x)$$

and the flatness assumption

$$U_{\beta'}(x_n - \eta) \leq v(x) \leq U_{\beta'}(x_n + \eta).$$

Since $\beta' \sim \varepsilon_{\bar{k}}^2$ with a universal constant,

$$\|g\|_{L^\infty(B_1)} \leq \varepsilon_{\bar{k}}^{1/2} \varepsilon_{\bar{k}}^4 \leq \eta^2 \beta',$$

as long as $\tilde{\varepsilon} \leq C''$ depending on $\bar{C}$. In conclusion, choosing $\tilde{\varepsilon} \leq \varepsilon_0(\bar{r})^4/(2\bar{C}^4)$, v falls under the assumptions of Lemma 5.1 on improvement of flatness (nondegenerate) and we can use an iteration argument as in Case 1. □

Proof of Theorem 1.2. Although not strictly necessary, we use the following Liouville-type result for global viscosity solutions to a two-phase homogeneous free boundary problem, which could be of independent interest.

Lemma 6.2. *Let U be a global viscosity solution to*

$$\begin{cases} \Delta U = 0 & \text{in } \{U > 0\} \cup \{U \leq 0\}^0, \\ (U_\nu^+)^2 - (U_\nu^-)^2 = 1 & \text{on } F(U) := \partial\{U > 0\}. \end{cases} \tag{6-11}$$

Assume that $F(U) = \{x_n = g(x'),\ x' \in \mathbb{R}^{n-1}\}$ with $\operatorname{Lip}(g) \leq M$. Then g is linear and $U(x) = U_\beta(x)$ for some $\beta \geq 0$.

Proof. Assume for simplicity that $0 \in F(U)$. Also, balls (of radius ρ and centered at 0) in $\mathbb{R}^{n-1}$ are denoted by $\mathcal{B}_\rho$.

By the regularity theory in [Caffarelli 1987], since U is a solution in B_2, the free boundary $F(U)$ is $C^{1,\gamma}$ in B_1 with a bound depending only on n and on M. Thus,

$$|g(x') - g(0) - \nabla g(0) \cdot x'| \le C|x'|^{1+\alpha}, \quad x' \in \mathcal{B}_1,$$

with C depending only on n, M. Moreover, since U is a global solution, the rescaling

$$g_R(x') = \frac{1}{R} g(Rx'), \quad x' \in \mathcal{B}_2,$$

which preserves the same Lipschitz constant as g, satisfies the same inequality as above, that is,

$$|g_R(x') - g_R(0) - \nabla g_R(0) \cdot x'| \le C|x'|^{1+\alpha}, \quad x' \in \mathcal{B}_1.$$

This reads,

$$|g(Rx') - g(0) - \nabla g(0) \cdot Rx'| \le CR|x'|^{1+\alpha}, \quad x' \in \mathcal{B}_1.$$

Thus,

$$|g(y') - g(0) - \nabla g(0) \cdot y'| \le C\frac{1}{R^\alpha}|y'|^{1+\alpha}, \quad y' \in \mathcal{B}_R.$$

Passing to the limit as $R \to \infty$ we obtain the claim. □

Proof of Theorem 1.2. Let $\bar{\varepsilon}$ be the universal constant in Theorem 2.8. Consider the blow-up sequence

$$u_k(x) = \frac{u(\delta_k)}{\delta_k},$$

with $\delta_k \to 0$ as $k \to \infty$. Each u_k solves (1-1) with right-hand side

$$f_k(x) = \delta_k f(\delta_k x)$$

and we have

$$\|f_k(x)\| \le \delta_k \|f\|_{L^\infty} \le \bar{\varepsilon} \quad \text{for } k \text{ large enough.}$$

Standard arguments (see for example [Alt et al. 1984]) using the uniform Lipschitz continuity of the u_k and the nondegeneracy of their positive part u_k^+ (see Lemma 2.5) imply that (up to a subsequence)

$$u_k \to \tilde{u} \quad \text{uniformly on compacts}$$

and

$$\{u_k^+ = 0\} \to \{\tilde{u} = 0\} \quad \text{in the Hausdorff distance.}$$

The blow-up limit $\tilde{u}$ solves the global homogeneous two-phase free boundary problem

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } \{\tilde{u} > 0\} \cup \{\tilde{u} \le 0\}^0, \\ (\tilde{u}_\nu^+)^2 - (\tilde{u}_\nu^-)^2 = 1 & \text{on } F(\tilde{u}) := \partial\{\tilde{u} > 0\}. \end{cases} \tag{6-12}$$

Since $F(u)$ is a Lipschitz graph in a neighborhood of 0, it follows from Lemma 6.2 that $\tilde{u}$ is a two-plane solution, $\tilde{u} = U_\beta$ for some $\beta \geq 0$. Thus, for k large enough,

$$\|u_k - U_\beta\|_{L^\infty} \leq \bar{\varepsilon} \quad \text{and} \quad \{x_n \leq -\bar{\varepsilon}\} \subset B_1 \cap \{u_k^+(x) = 0\} \subset \{x_n \leq \bar{\varepsilon}\}.$$

Therefore, we can apply our flatness theorem (Theorem 2.8) and conclude that $F(u_k)$, and hence $F(u)$, are smooth. □

Flatness and ε-monotonicity. The flatness results present in the literature (see [Caffarelli 1989], for instance), are often stated in terms of "ε-monotonicity" along a large cone of directions $\Gamma(\theta_0, e)$ of axis e and opening θ_0. Precisely, a function u is said to be ε-monotone ($\varepsilon > 0$ small) along the direction τ in the cone $\Gamma(\theta_0, e)$ if for every $\varepsilon' \geq \varepsilon$,

$$u(x + \varepsilon'\tau) \leq u(x).$$

A variant of Theorem 1.1 states the following.

Theorem 6.3. *Let u be a solution to* (1-1) *in B_1, $0 \in F(u)$. Suppose that u^+ is nondegenerate. Then there exist $\theta_0 < \pi/2$ and $\varepsilon_0 > 0$ such that if u^+ is ε-monotone along every direction in $\Gamma(\theta_0, e_n)$ for some $\varepsilon \leq \varepsilon_0$, then u^+ is fully monotone in $B_{1/2}$ along any direction in $\Gamma(\theta_1, e_n)$ for some θ_1 depending on θ_0, ε_0. In particular $F(u)$ is the graph of a Lipschitz function.*

Geometrically, the ε-monotonicity of u^+ can be interpreted as ε-closeness of $F(u)$ to the graph of a Lipschitz function. Our flatness assumption requires ε-closeness of $F(u)$ to a hyperplane. While this looks like a somewhat stronger assumption, it is indeed a natural one since it is satisfied for example by rescaling of solutions around a "regular" point of the free boundary. Moreover, if $\|f\|_\infty$ is small enough, depending on ε, it is not hard to check that ε-flatness of $F(u)$ implies $c\varepsilon$-monotonicity of u^+ along the directions of a flat cone, for a c depending on its opening.

The proof of Theorem 6.3 follows immediately from the following elementary lemma:

Lemma 6.4. *Let u be a solution to* (1-1) *in B_1, with $0 \in F(u)$. Suppose that u^+ is Lipschitz and nondegenerate. Assume that u^+ is ε-monotone along every direction in $\Gamma(\theta_0, e_n)$ for some $\varepsilon \leq \varepsilon_0$. Then there exist a radius $r_0 > 0$ and $\delta_0 > 0$ depending on ε_0, θ_0 such that u^+ is δ_0-flat in B_{r_0}, that is,*

$$\{x_n \leq -\delta_0\} \subset B_{r_0} \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta_0\}.$$

7. More general operators and free boundary conditions

The setup. In this section we analyze the free boundary problem (1-3), that is,

$$\begin{cases} \mathscr{L}u = f & \text{in } \Omega^+(u) \cup \Omega^-(u), \\ u_\nu^+ = G(u_\nu^-, x) & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega, \end{cases} \tag{7-1}$$

where f is continuous in $\Omega^+(u) \cup \Omega^-(u)$ with $\|f\|_{L^\infty(\Omega)} \leq L$, and

$$\mathscr{L} = \sum_{i,j=1}^n a_{ij}(x) D_{ij} + \boldsymbol{b} \cdot \nabla, \quad a_{ij} \in C^{0,\bar{\gamma}}(\Omega), \boldsymbol{b} \in C(\Omega) \cap L^\infty(\Omega),$$

is uniformly elliptic with constants $0 < \lambda \leq \Lambda$.

We recall that our assumptions on G are:

(H1) $G(\eta, \cdot) \in C^{0,\bar\gamma}(\Omega)$ uniformly in η; $G(\cdot, x) \in C^{1,\bar\gamma}([0, L])$ for every $x \in \Omega$.

(H2) $G'(\cdot, x) > 0$ with $G(0, x) \geq \gamma_0 > 0$ uniformly in x.

(H3) There exists $N > 0$ such that $\eta^{-N} G(\eta, x)$ is strictly decreasing in η, uniformly in x.

We assume that $0 \in F(u)$ and that $a_{ij}(0) = \delta_{ij}$. Also, for notational convenience we set

$$G_0(\beta) = G(\beta, 0).$$

Let U_β be the two-plane solution to (7-1) when $\mathscr{L} = \Delta$, $f \equiv 0$ and $G = G_0$, that is,

$$U_\beta(x) = \alpha x_n^+ - \beta x_n^-, \quad \beta \geq 0, \quad \alpha = G_0(\beta).$$

The following definitions parallel those in Section 2.

Definition 7.1. Let u be a continuous function in Ω. We say that u is a viscosity solution to (1-3) in Ω, if the following conditions are satisfied:

(i) $\mathscr{L}u = f$ in $\Omega^+(u) \cup \Omega^-(u)$ in the viscosity sense.

(ii) Let $x_0 \in F(u)$ and $v \in C^2(\overline{B^+(v)}) \cap C^2(\overline{B^-(v)})$ $(B = B_\delta(x_0))$ with $F(v) \in C^2$. If v touches u from below (resp. above) at $x_0 \in F(v)$, then

$$v_\nu^+(x_0) \leq G(v_\nu^-(x_0), x_0) \quad \text{(resp. } \geq\text{)}.$$

Definition 7.2. We say that $v \in C(\Omega)$ is a C^2 strict (comparison) subsolution (resp. supersolution) to (7-1) in Ω, if $v \in C^2(\overline{\Omega^+(v)}) \cap C^2(\overline{\Omega^-(v)})$ and the following conditions are satisfied:

(i) $\mathscr{L}v > f$ (resp. $< f$) in $\Omega^+(v) \cup \Omega^-(v)$.

(ii) If $x_0 \in F(v)$, then

$$v_\nu^+(x_0) > G(v_\nu^-(x_0), x_0) \quad \big(\text{resp. } v_\nu^+(x_0) < G(v_\nu^-(x_0), x_0),\ v_\nu^+(x_0) \neq 0\big).$$

Observe that the free boundary of a strict comparison sub/supersolution is C^2.

From here after, most of the statements and proofs parallel those in Sections 2–6. Thus, we only point out the main differences as much as possible.

Compactness and localization. As for the problem (1-1), we prove some basic lemmas to reduce the statement of the flatness theorem to a proper normalized situation. We start with the compactness Lemma 2.6 which generalizes to operators of the form

$$\mathscr{L}_*^k = \sum a_{ij}^k D_{ij},$$

with $a_{ij}^k \in C^{0,\bar\gamma}$ uniformly elliptic with constants λ, Λ and free boundary conditions given by a G_k satisfying hypotheses (H1)–(H3).

Lemma 7.3. *Let u_k be a sequence of (Lipschitz) viscosity solutions to*

$$\begin{cases} |\mathcal{L}_*^k u_k| \le M & \text{in } \Omega^+(u_k) \cup \Omega^-(u_k), \\ (u_k^+)_\nu = G_k((u_k^-)_\nu, x) & \text{on } F(u_k). \end{cases} \tag{7-2}$$

Assume that

(i) $a_{ij}^k \to a_{ij}$, $u_k \to u^*$ *uniformly on compact sets,*

(ii) $G_k(\eta, \cdot) \to G(\eta, \cdot)$ *on compact sets, uniformly on* $0 \le \eta \le L = \mathrm{Lip}(u_k)$, *and*

(iii) $\{u_k^+ = 0\} \to \{(u^*)^+ = 0\}$ *in the Hausdorff distance.*

Then

$$\left|\sum a_{ij} D_{ij} u^*\right| \le M \quad \text{in } \Omega^+(u^*) \cup \Omega^-(u^*),$$

and u^* satisfies the free boundary condition

$$(u^*)_\nu^+ = G((u^*)_\nu^-, x) \quad \text{on } F(u^*),$$

both in the viscosity sense.

Proof. Set

$$\mathcal{L}_* := \sum a_{ij} D_{ij}.$$

The proof that

$$|\mathcal{L}_* u^*| \le M \quad \text{in } \Omega^+(u^*) \cup \Omega^-(u^*)$$

is standard. We show for example that

$$\mathcal{L}_* u^* + M \ge 0 \quad \text{in } \Omega^+(u^*).$$

Let $v \in C^2(\Omega^+(u^*))$ touch u^* from above at $\bar{x} \in \Omega^+(u^*)$ and assume by contradiction that

$$\mathcal{L}_* v(\bar{x}) + M < 0.$$

Without loss of generality we can assume that v touches u^* strictly from above; otherwise we replace v by

$$v + \frac{\eta}{2n\Lambda}|x - \bar{x}|^2,$$

with η small. Then, since $u_k \to u^*$ uniformly in compact sets and $\{u_k^+ = 0\} \to \{(u^*)^+ = 0\}$ in the Hausdorff distance, there exists $x_k \to \bar{x}$ and constants $c_k \to 0$ such that $v + c_k$ touches from above u_k at $x_k \in \Omega^+(u_k)$, for k large. Then, since $|\mathcal{L}_*^k u_k(x_k)| \le M$ we must have

$$\mathcal{L}_*^k v(x_k) + M \ge 0.$$

This implies, for $k \to \infty$,

$$\mathcal{L}_* v(\bar{x}) + M \ge 0,$$

which is a contradiction.

We now prove that the free boundary condition holds. Let $\bar{x} \in F(u^*)$ and $v \in C^2(\overline{B^+(v)}) \cap C^2(\overline{B^-(v)})$ with $F(v) \in C^2$ touch u^* from above at $\bar{x} \in F(v)$.

Assume
$$v_\nu^+(\bar{x}) < G(v_\nu^-(\bar{x}), \bar{x}), \quad v_\nu^+(\bar{x}) \neq 0.$$

We distinguish two cases. For notational simplicity let $\nu(\bar{x}) = e_n$. If $v_n^-(\bar{x}) \neq 0$, we can assume that the free boundaries $F(v)$ and $F(u^*)$ touch strictly and that
$$\mathscr{L}_* v + M < 0 \quad \text{in } \Omega^+(v) \cup \Omega^-(v) \tag{7-3}$$
holds up to $F(v)$. Otherwise, in a small neighborhood of $\bar{x}$ we replace v with
$$\bar{v}(x) = v(x + \eta |x' - \bar{x}'|^2 e_n) + \eta\, |\mathrm{dist}(x, F(v))| - C\, \mathrm{dist}(x, F(v))^2 \quad (\eta \text{ small}, C \text{ large}).$$

Then, for a suitable $c_k \to 0$, $v(x + c_k e_n)$ touches from above u_k at x_k with $x_k \to \bar{x}$. Then, either for every (large) k we have $x_k \in \Omega^+(u_k) \cup \Omega^-(u_k)$ or there exists a subsequence, which we still call $\{x_k\}$, such that $x_k \in F(u_k)$ for every large k. Thus, either
$$\sum a_{ij}^k(x_k) D_{ij} v(x_k + c_k e_n) + M \geq 0$$
or
$$\bar{v}_{\nu_k}^+(x_k + c_k e_n) \geq G_k(v_{\nu_k}^-(x_k + c_k e_n), x_k),$$
and we easily reach a contradiction for k large.

If $v_n^-(\bar{x}) = 0$, we replace v^- with zero and argue as above for v^+. □

Lemma 2.5 on the nondegeneracy of the positive part δ-away from the free boundary continues to hold unaltered; only choose
$$w(x) = \frac{G_0(0)}{2\gamma}(1 - |x|^{-\gamma}).$$

The analogue of Lemma 2.7 is the following:

Lemma 7.4. *Let u be a Lipschitz solution to* (1-3) *in B_1, with* $\mathrm{Lip}(u) \leq L$, $\|b\|_\infty$, $\|f\|_\infty \leq L$. *For any $\varepsilon > 0$ there exist $\bar{\delta}, \bar{r} > 0$ such that if*
$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\},$$
with $0 \leq \delta \leq \bar{\delta}$, *then*
$$\|u - U_\beta\|_{L^\infty(B_{\bar{r}})} \leq \varepsilon \bar{r}, \tag{7-4}$$
for some $0 \leq \beta \leq L$.

Proof. Given $\varepsilon > 0$ and $\bar{r}$ depending on ε to be specified later, assume by contradiction that there exist a sequence $\delta_k \to 0$ and a sequence of solutions u_k to the problem (7-2) with $M = L + L^2$, such that $\mathrm{Lip}(u_k) \leq L$ and
$$\{x_n \leq -\delta_k\} \subset B_1 \cap \{u_k^+(x) = 0\} \subset \{x_n \leq \delta_k\}, \tag{7-5}$$
but the u_k do not satisfy the conclusion (7-4).

Then, up to a subsequence, the u_k converge uniformly on compact set to a function u^*. In view of (7-5) and the nondegeneracy of u_k^+, δ_k-away from the free boundary (see remark above), we can apply our compactness Lemma 7.3 and conclude that, for some $\tilde{\mathscr{L}} := \sum \tilde{a}_{ij} D_{ij}$ and $\widetilde{G}$ in our class,

$$|\tilde{\mathscr{L}} u^*| \le M \quad \text{in } B_{1/2} \cap \{x_n \ne 0\}$$

and

$$(u^*)_n^+ = \widetilde{G}((u^*)_n^-, x) \quad \text{on } F(u^*) = B_{1/2} \cap \{x_n = 0\}, \tag{7-6}$$

in the viscosity sense, with

$$u^* > 0 \quad \text{in } B_{\rho_0} \cap \{x_n > 0\}.$$

Thus, by L^p Schauder estimates, we have

$$u^* \in C^{1,\tilde{\gamma}}\big(B_{1/2} \cap \{x_n \ge 0\}\big) \cap C^{1,\tilde{\gamma}}\big(B_{1/2} \cap \{x_n \le 0\}\big)$$

for all $\tilde{\gamma} < 1$ and (for any $\bar{r}$ small)

$$\big\|u^* - (\alpha x_n^+ - \beta x_n^-)\big\|_{L^\infty(B_{\bar{r}})} \le C(n, L)\bar{r}^{1+\tilde{\gamma}},$$

with $\beta = (u^*)_n^-(0)$ and $\alpha = (u^*)_n^+(0) > 0$. Thus, from (7-6), we have $\alpha = \widetilde{G}_0(\beta)$.

Then we reach a contradiction as in Lemma 2.7. □

In view of the lemma above, after proper rescaling, Theorem 1.3 follows from the following result.

Theorem 7.5. *Let u be a Lipschitz solution to* (1-3) *in B_1, with* $\mathrm{Lip}(u) \le L$. *There exists a universal constant $\bar{\varepsilon} > 0$ such that, if*

$$\begin{gathered} \|u - U_\beta\|_{L^\infty(B_1)} \le \bar{\varepsilon}, \quad \textit{for some } 0 \le \beta \le L, \\ \{x_n \le -\bar{\varepsilon}\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \le \bar{\varepsilon}\}, \end{gathered} \tag{7-7}$$

and

$$\begin{gathered} [a_{ij}]_{C^{0,\bar{\gamma}}(B_1)} \le \bar{\varepsilon}, \quad \|\boldsymbol{b}\|_{L^\infty(B_1)} \le \bar{\varepsilon}, \quad \|f\|_{L^\infty(B_1)} \le \bar{\varepsilon}, \\ [G(\eta, \cdot)]_{C^{0,\bar{\gamma}}(B_1)} \le \bar{\varepsilon} \quad \textit{for all } 0 \le \eta \le L, \end{gathered}$$

then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

Linearized problem. The linearized problem becomes ($\tilde{\alpha} > 0$)

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_\rho \cap \{x_n \ne 0\}, \\ \tilde{\alpha}(\tilde{u})_n^+ - \tilde{\beta} G_0'(\tilde{\beta})(\tilde{u})_n^- = 0 & \text{on } B_\rho \cap \{x_n = 0\}, \end{cases} \tag{7-8}$$

with $\tilde{\alpha} = G_0(\tilde{\beta})$.

Setting $\zeta^2 = \tilde{\alpha}$ and $\xi^2 = \tilde{\beta} G_0'(\tilde{\beta})$ we can write the free boundary condition as

$$\zeta^2 \tilde{u}_n^+ - \xi^2 \tilde{u}_n^- = 0.$$

Consequently, all the definitions and conclusions in Section 3 hold, in particular Theorems 3.2–3.4.

8. The nondegenerate case for general free boundary problems

In this section, we recover lemma on improvement of flatness in the nondegenerate case, that is, when the solution is trapped between parallel two-plane solutions U_β at ε distance, with $\beta > 0$. First we need the Harnack inequality.

The Harnack inequality. As in Section 4, the Harnack inequality follows from the following basic lemma.

Lemma 8.1. *Let u be a viscosity solution to* (7-1). *There exists a universal constant $\bar{\varepsilon} > 0$ such that, if u satisfies*

$$u(x) \geq U_\beta(x) \quad \textit{in } B_1,$$

with $0 < \beta \leq L$, and if furthermore we have

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^2 \min\{G_0(\beta), \beta\}, \quad \|\boldsymbol{b}\|_{L^\infty(B_1)} \leq \varepsilon^2, \tag{8-1}$$

$$\|G(\eta, x) - G_0(\eta)\|_{L^\infty(B_1)} \leq \varepsilon^2 \quad \textit{for all } 0 \leq \eta \leq L, \tag{8-2}$$

with $0 \leq \varepsilon \leq \bar{\varepsilon}$, then, if at $\bar{x} = \frac{1}{5} e_n$

$$u(\bar{x}) \geq U_\beta(\bar{x}_n + \varepsilon), \tag{8-3}$$

then

$$u(x) \geq U_\beta(x_n + c\varepsilon) \quad \textit{in } \bar{B}_{1/2}, \tag{8-4}$$

for some universal $0 < c < 1$. *Analogously, if $u(x) \leq U_\beta(x)$ in B_1 and $u(\bar{x}) \leq U_\beta(\bar{x}_n - \varepsilon)$, then*

$$u(x) \leq U_\beta(x_n - c\varepsilon) \quad \textit{in } \bar{B}_{1/2}.$$

Proof. We argue as in the proof of Lemma 4.3 and we only point out the main differences.

By our assumptions, in $B_{1/10}(\bar{x}) \subset B_1^+(u)$, $u - U_\beta \geq 0$ solves

$$\mathcal{L}(u - U_\beta) = f - \alpha b_n.$$

Recall that $\alpha = G_0(\beta)$. By the Harnack inequality, we obtain in $\bar{B}_{1/20}(\bar{x})$

$$\begin{aligned} u(x) - U_\beta(x) &\geq c(u(\bar{x}) - U_\beta(\bar{x})) - C\|f - \alpha b_n\|_{L^\infty} \\ &\geq c(u(\bar{x}) - U_\beta(\bar{x})) - C(\|f\|_{L^\infty} + \alpha\|b\|_{L^\infty}). \end{aligned}$$

From (8-1), (8-3) and the inequality above we conclude that for ε small enough,

$$u - U_\beta \geq \alpha c\varepsilon - \alpha C\varepsilon^2 \geq c_0\alpha\varepsilon \quad \text{in } \bar{B}_{1/20}(\bar{x}). \tag{8-5}$$

From (8-5) and the comparison principle it follows that for c_1 small universal

$$u - \alpha x_n \geq \alpha c_1 \varepsilon x_n, \quad x \in \{x_n > 0\} \cap \bar{B}_{19/20}. \tag{8-6}$$

To prove this claim, let ϕ solve

$$\mathcal{L}\phi = 0 \quad \text{in } R := (B_1 \cap \{x_n > 0\}) \setminus \bar{B}_{1/20}(\bar{x}),$$

with boundary data

$$\phi = 0 \quad \text{on } \partial(B_1 \cap \{x_n > 0\}),$$
$$\phi = 1 \quad \text{on } \partial B_{1/20}(\bar{x}).$$

Then, by boundary Harnack,

$$\phi \geq c x_n \quad \text{in } \bar{R} \cap B_{19/20}.$$

We now compare $u - \alpha x_n$ with $\frac{1}{2}\alpha c_0 \phi \varepsilon - 8\alpha\varepsilon^2 x_n + 4\alpha\varepsilon^2 x_n^2$ in the domain R to obtain the desired conclusion.

We now proceed similarly as in Lemma 4.3, with w the function defined in (4-5). We compute

$$\begin{aligned}\sum a_{ij} D_{ij} w(x) &= \gamma(\gamma+2)|x-\bar{x}|^{-\gamma-4}\operatorname{Tr}(A(x-\bar{x})\otimes(x-\bar{x})) - \gamma|x-\bar{x}|^{-\gamma-2}\operatorname{Tr}(A)\\ &\geq \gamma(\gamma+2)|x-\bar{x}|^{-\gamma-2} n\lambda - \gamma|x-\bar{x}|^{-\gamma-2} n\Lambda\\ &= \gamma|x-\bar{x}|^{-\gamma-2} n((\gamma+2)\lambda - \Lambda).\end{aligned}$$

Then

$$\begin{aligned}\mathcal{L}w &\geq \gamma|x-\bar{x}|^{-\gamma-2} n((\gamma+2)\lambda-\Lambda) - \gamma\|\boldsymbol{b}\|_{L^\infty}|x-\bar{x}|^{-\gamma-1}\\ &= \gamma|x-\bar{x}|^{-\gamma-2}\big(n((\gamma+2)\lambda-\Lambda) - \|\boldsymbol{b}\|_{L^\infty}|x-\bar{x}|\big)\\ &\geq \gamma|x-\bar{x}|^{-\gamma-2}\big(n((\gamma+2)\lambda-\Lambda) - \|\boldsymbol{b}\|_{L^\infty}\big) \equiv k_0(\gamma, c_0, n, \lambda, \Lambda) > 0,\end{aligned}$$

as long as γ satisfies

$$n((\gamma+2)\lambda-\Lambda) - \|\boldsymbol{b}\|_{L^\infty} > 0.$$

Now set $\psi = 1 - w$ and for $x \in \bar{B}_{3/4}(\bar{x})$ define

$$v_t(x) = \alpha(1+c_1\varepsilon)(x_n - \varepsilon c_0\delta\psi(x) + t\varepsilon)^+ - \beta(x_n - \varepsilon c_0\delta\psi(x) + t\varepsilon)^-,$$

with $\delta > 0$ small to be made precise later, and c_1 the constant in (8-6).

Then, for $t = -c_1$ one can easily verify that

$$v_{-c_1} \leq U_\beta \leq u, \quad x \in \bar{B}_{3/4}(\bar{x}).$$

Let $\bar{t}$ be the largest $t \geq -c_1$ such that

$$v_t(x) \leq u(x) \quad \text{in } \bar{B}_{3/4}(\bar{x}),$$

and let $\tilde{x}$ be the first touching point. To guarantee that $\tilde{x}$ cannot belong to $\partial B_{3/4}$ when $\bar{t} < c_0\delta$ we use (8-6). Indeed if $x \in \partial B_{3/4}$ and $v_{\bar{t}}(x) \geq 0$ then $x_n > 0$ and in view of (8-6)

$$v_{\bar{t}}(x) = \alpha(1+c_1\varepsilon)(x_n - \varepsilon c_0\delta + \bar{t}\varepsilon) < \alpha(1+c_1\varepsilon)x_n \leq u(x).$$

If $v_{\bar{t}}(x) < 0$ we use that $u \geq U_\beta$ to reach again the conclusion that $v_{\bar{t}}(x) < u(x)$. To proceed as in Lemma 4.3 we now need to show that for $\bar{t} < c_0\delta$, $v_{\bar{t}}$ is a strict subsolution in the annulus A.

Indeed, in $A^+(v_{\bar{t}})$ in view of the assumption (8-1) and the computation above for $\mathcal{L}w$, we have

$$\mathcal{L}v_{\bar{t}} \geq \alpha(\varepsilon c_0\delta k_0 + b_n) \geq \varepsilon^2\min\{\alpha, \beta\} \geq \|f\|_\infty.$$

A similar estimate holds in $A^-(v_{\bar t})$. Thus

$$\mathscr{L}v_{\bar t} \ge f \quad \text{in } A^+(v_{\bar t}) \cup A^-(v_{\bar t}),$$

for ε small enough.

Also, since $\psi_n < -c$ on $F(v_{\bar t}) \cap A$, for ε small, we have

$$\kappa \equiv |e_n - \varepsilon c_0 \nabla\psi| = \big(1 - 2\varepsilon c_0 \delta \psi_n + \varepsilon^2 c_0^2 \delta^2 |\nabla\psi|^2\big)^{1/2} = 1 + \tilde{k}\delta\varepsilon,$$

with $\tilde{k}$ between two universal constants.

Then, on $F(v_{\bar t}) \cap A$, using (8-2), we can write, as long as ε is sufficiently small,

$$\begin{aligned}(v_{\bar t}^+)_\nu - G((v_{\bar t}^-)_\nu, x) &= \alpha(1 + c_1\varepsilon)\kappa - G(\beta\kappa, x) \ge \alpha(1 + c_1\varepsilon)\kappa - G_0(\beta\kappa) - \epsilon^2 \\ &> (1 + c_1\varepsilon)G_0(\beta) - G_0(\beta)\kappa^N - \epsilon^2 \\ &\ge \varepsilon G_0(\beta)\Big(\frac{c_1}{2} - N\tilde{k}\delta\Big) > 0\end{aligned}$$

if $\delta < c_1/(2N\tilde{\kappa})$. We used that $G_0(\beta) \ge G_0(0) > 0$ and that $G_0(\beta\kappa) < G_0(\beta)\kappa^N$, since $\eta^{-N}G_0(\eta)$ is strictly decreasing.

Thus, $v_{\bar t}$ is a strict subsolution to (1-1) in A as desired. Hence $\bar t \ge c_0\delta$ and we conclude as in the Laplacian case. ☐

With Lemma 8.1 at hand, the Harnack inequality and its corollary follow as in Section 4. We only state the corollary, since it is indeed the tool used in the proof of the improvement of flatness lemma in the next subsection.

Corollary 8.2. *Let u satisfy at some point $x_0 \in B_2$*

$$U_\beta(x_n + a_0) \le u(x) \le U_\beta(x_n + b_0) \quad \text{in } B_1(x_0) \subset B_2, \tag{8-7}$$

for some $0 < \beta \le L$, with

$$b_0 - a_0 \le \varepsilon,$$

and let (8-1)–(8-2) hold, for $\varepsilon \le \bar\varepsilon$, $\bar\varepsilon$ universal. Then in $B_1(x_0)$ (with $\alpha = G_0(\beta)$) we have

$$\tilde{u}_\varepsilon(x) = \begin{cases} \dfrac{u(x) - \alpha x_n}{\alpha\varepsilon} & \text{in } B_2^+(u) \cup F(u), \\[2ex] \dfrac{u(x) - \beta x_n}{\beta\varepsilon} & \text{in } B_2^-(u), \end{cases}$$

has a Hölder modulus of continuity at x_0, outside the ball of radius $\varepsilon/\bar\varepsilon$, that is, for all $x \in B_1(x_0)$, with $|x - x_0| \ge \varepsilon/\bar\varepsilon$,

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \le C|x - x_0|^\gamma.$$

Improvement of flatness. We now extend the basic induction step towards $C^{1,\gamma}$ regularity at 0. We argue as in the proof of Lemma 5.1.

Lemma 8.3. *Let u be solution of* (1-3) *and suppose that*

$$U_\beta(x_n-\varepsilon) \le u(x) \le U_\beta(x_n+\varepsilon) \quad \text{in } B_1, \tag{8-8}$$

with $0<\beta\le L$,

$$\|a_{ij}-\delta_{ij}\|_{L^\infty(B_1)} \le \varepsilon, \quad \|f\|_{L^\infty(B_1)} \le \varepsilon^2 \min\{G_0(\beta),\beta)\}, \quad \|\boldsymbol{b}\|_{L^\infty(B_1)} \le \varepsilon^2,$$

and

$$\|G(\eta,\cdot)-G_0(\eta)\|_{L^\infty(B_1)} \le \varepsilon^2 \quad \text{for all } \ 0\le\eta\le L.$$

If $0<r\le r_0$ *for* r_0 *universal, and* $0<\varepsilon\le\varepsilon_0$ *for some* ε_0 *depending on* r, *then*

$$U_{\beta'}\left(x\cdot\nu_1 - r\frac{\varepsilon}{2}\right) \le u(x) \le U_{\beta'}\left(x\cdot\nu_1 + r\frac{\varepsilon}{2}\right) \quad \text{in } B_r, \tag{8-9}$$

with $|\nu_1|=1,\ |\nu_1-e_n|\le \widetilde{C}\varepsilon$, *and* $|\beta-\beta'|\le\widetilde{C}\beta\varepsilon$ *for a universal constant* $\widetilde{C}$.

Proof. We divide the proof into three steps.

Step 1: compactness. We keep the same notation of Lemma 5.1. In this case, the sequence u_k is a solution of problem (1-3) for operators

$$\mathscr{L}^k = \sum_{ij} a_{ij}{}^k D_{ij} + \boldsymbol{b}^k\cdot\nabla,$$

where (with $\alpha_k = G_k(\beta_k,0)$)

$$\|a^k_{ij}-\delta_{ij}\|_{L^\infty} \le \varepsilon_k, \quad \|f_k\|_{L^\infty} \le \varepsilon_k^2 \min\{\alpha_k,\beta_k\}, \quad \|\boldsymbol{b}^k\|_{L^\infty} \le \varepsilon_k^2,$$

and

$$\|G_k(\eta,\cdot)-G_k(\eta,0)\|_\infty \le \varepsilon_k^2 \quad \text{for all } \ 0\le\eta\le L. \tag{8-10}$$

The normalized functions $\tilde u_k$ are defined by the same formula. Up to a subsequence, $G_k(\cdot,0)$ converges, locally uniformly, to some C^1-function $\widetilde{G}_0$, while $\beta_k\to\tilde\beta$ so that $\alpha_k\to\tilde\alpha=\widetilde{G}_0(\tilde\beta)$. Moreover, by Corollary 8.2 the graphs of $\tilde u_k$ converge in the Hausdorff distance to a Hölder continuous $\tilde u$.

Step 2: limiting solution. We show that $\tilde u$ solves

$$\begin{cases} \Delta\tilde u = 0 & \text{in } B_{1/2}\cap\{x_n\ne 0\},\\ \tilde\alpha\tilde u_n^+ - \tilde\beta\widetilde{G}_0'(\tilde\beta)\tilde u_n^- = 0 & \text{on } B_{1/2}\cap\{x_n=0\}. \end{cases} \tag{8-11}$$

We can write in $\Omega^+(u^k)$ (in $\Omega^-(u^k)$ replace α_k with β_k)

$$\sum a^k_{ij} D_{ij}\tilde u_k = \frac{1}{\alpha_k\varepsilon_k}\sum a^k_{ij} D_{ij} u_k = \frac{1}{\alpha_k\varepsilon_k}(-\alpha_k\boldsymbol{b}^k\cdot\nabla u_k + f^k) \equiv F^k,$$

where $|F^k|\le C\varepsilon_k$.

Thus

$$\Delta \tilde{u}_k = \sum_{i,j=1}^{n} (\delta_{ij} - a_{ij}^k) D_{ij}\tilde{u}_k + F^k.$$

Hence recalling that $\|a_{ij}^k - \delta_{ij}\|_\infty \le \varepsilon_k$, and from interior L^p Schauder estimates for second derivatives, we conclude that, for instance, $\Delta \tilde{u}_k \to 0$ in L^p on every compact set contained in $\Omega^+(\tilde{u}^k)$ or in $\Omega^-(\tilde{u}^k)$. This shows that $\tilde{u}$ is harmonic in $B_{1/2} \cap \{x_n \neq 0\}$.

Next, we prove that $\tilde{u}$ satisfies the transmission condition in (8-11) in the viscosity sense.

Again we argue by contradiction. Let $\tilde{\phi}$ be a function of the form

$$\tilde{\phi}(x) = A + px_n^+ - qx_n^- + BQ(x-y),$$

with

$$Q(x) = \tfrac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0), \quad A, B > 0, \quad \tilde{\alpha}p - \tilde{\beta}\widetilde{G}_0'(\tilde{\beta})q > 0,$$

and assume that $\tilde{\phi}$ touches u strictly from below at a point $x_0 = (x_0', 0) \in B_{1/2}$. As in Lemma 5.1, let

$$\phi_k = a_k\Gamma_k^+(x) - b_k\Gamma_k^-(x) + \alpha_k(d_k^+(x))^2\varepsilon_k^{3/2} + \beta_k(d_k^-(x))^2\varepsilon_k^{3/2},$$

where, we recall,

$$a_k = \alpha_k(1+\varepsilon_k p), \quad b_k = \beta_k(1+\varepsilon_k q),$$

and $d_k(x)$ is the signed distance from x to $\partial B_{1/(B\varepsilon_k)}\Big(y + e_n\Big(\dfrac{1}{B\varepsilon_k} - A\varepsilon_k\Big)\Big)$. Moreover,

$$\psi_k(x) = \phi_k(x + \varepsilon_k c_k e_n)$$

touches u_k from below at x_k, with $c_k \to 0$, $x_k \to x_0$.

We get a contradiction if we prove that ψ_k is a strict subsolution to our free boundary problem, that is,

$$\begin{cases} \mathscr{L}^k\psi_k > f_k & \text{in } B_1^+(\psi_k) \cup B_1^-(\psi_k), \\ (\psi_k^+)_\nu - G_k((\psi_k^-)_\nu, x) > 0 & \text{on } F(\psi_k). \end{cases}$$

We have

$$|\nabla\Gamma_k| \le C, \quad |D_{ij}\Gamma_k| \le C\varepsilon_k, \quad |a_{ij} - \delta_{ij}| \le \varepsilon_k.$$

For k large enough, we can write, say in the positive phase of ψ_k,

$$\begin{aligned} \mathscr{L}_k\psi_k &= (\mathscr{L}^k - \Delta)\psi_k + \Delta\psi_k \ge -C\alpha_k\varepsilon_k^2 + \alpha_k\varepsilon_k^{3/2}\mathscr{L}^k d_k^2(x + \varepsilon c_k e_n) \\ &\ge c\min\{\alpha_k, \beta_k\}\varepsilon_k^{3/2} \ge \|f_k\|_{L^\infty}, \end{aligned}$$

and the first condition is satisfied. An analogous estimate holds in the negative phase.

Finally, since on the zero level set $|\nabla\Gamma_k| = 1$ and $|\nabla d_k^2| = 0$, the free boundary condition reduces to showing that

$$a_k - G_k(b_k, x) > 0.$$

Using the definition of a_k, b_k we need to check that

$$\alpha_k(1+\varepsilon_k p) - G_k(\beta_k(1+\varepsilon_k q), x) > 0.$$

From (8-10), it suffices to check that

$$\alpha_k(1+\varepsilon_k p) - G_k(\beta_k(1+\varepsilon_k q), 0) - \varepsilon_k^2 > 0.$$

This inequality holds for k large in view of the fact that

$$\tilde{\alpha} p - \tilde{\beta}\widetilde{G}_0'(\tilde{\beta})q > 0.$$

Thus $\tilde{u}$ is a viscosity solution to the linearized problem.

Step 3: contradiction. According to estimate (3-2), since $\tilde{u}(0)=0$ we obtain

$$|\tilde{u} - (x'\cdot\nu' + px_n^+ - qx_n^-)| \le Cr^2, \quad x \in B_r,$$

with

$$\tilde{\alpha} p - \tilde{\beta}\widetilde{G}_0'(\tilde{\beta})q = 0, \quad |\nu'| = |\nabla_{x'}\tilde{u}(0)| \le C.$$

Thus, since $\tilde{u}_k$ converges uniformly to $\tilde{u}$ (by slightly enlarging C) we get

$$|\tilde{u}_k - (x'\cdot\nu' + px_n^+ - qx_n^-)| \le Cr^2, \quad x \in B_r.$$

Now set

$$\beta_k' = \beta_k(1+\varepsilon_k q), \quad \nu_k = \frac{1}{\sqrt{1+\varepsilon_k^2|\nu'|^2}}(e_n + \varepsilon_k(\nu', 0)).$$

Then

$$\begin{aligned}\alpha_k' &= G_k(\beta_k(1+\varepsilon_k q), 0) = G_k(\beta_k, 0) + \beta_k G_k'(\beta_k, 0)\varepsilon_k q + O(\varepsilon_k^2)\\ &= \alpha_k\left(1+\beta_k\frac{G_k'(\beta_k,0)}{\alpha_k}q\varepsilon_k\right) + O(\varepsilon_k^2) = \alpha_k(1+\varepsilon_k p) + O(\varepsilon_k^2),\end{aligned}$$

since from the identity $\tilde{\alpha} p - \tilde{\beta}\tilde{G}_0'(\tilde{\beta})q = 0$ we derive that

$$\beta_k\frac{G_k'(\beta_k,0)}{\alpha_k}q = p + O(\varepsilon_k).$$

Moreover

$$\nu_k = e_n + \varepsilon_k(\nu', 0) + \varepsilon_k^2\tau, \quad |\tau| \le C.$$

With these choices, it follows as in Lemma 5.1 that (for k large and $r \le r_0$)

$$\widetilde{U}_{\beta_k'}\left(x\cdot\nu_k - \varepsilon_k\frac{r}{2}\right) \le \tilde{u}_k(x) \le \widetilde{U}_{\beta_k'}\left(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\right) \quad \text{in } B_r,$$

which leads to a contradiction. $\square$

9. The degenerate case for general free boundary problems

In this section, we recover the improvement of flatness lemma in the degenerate case, that is, when the negative part of u is negligible and the positive part is close to a one-plane solution ($\beta = 0$, $\alpha = G_0(0)$). First we need the Harnack inequality.

The Harnack inequality. As in Section 4, the Harnack inequality in the degenerate case is a consequence of the following basic lemma.

Lemma 9.1. *There exists a universal constant $\bar{\varepsilon} > 0$ such that if u satisfies*

$$u^+(x) \geq U_0(x) \quad \text{in } B_1,$$

with

$$\|u^-\|_{L^\infty} \leq \varepsilon^2, \quad \|\boldsymbol{b}\|_{L^\infty} \leq \varepsilon^2, \quad \|f\|_{L^\infty} \leq \varepsilon^4, \tag{9-1}$$

$$\|G(\eta, \cdot) - G_0(\eta)\| \leq \varepsilon^2, \quad 0 \leq \eta \leq C\varepsilon^2, \tag{9-2}$$

then if at $\bar{x} = \frac{1}{5} e_n$

$$u^+(\bar{x}) \geq U_0(\bar{x}_n + \varepsilon), \tag{9-3}$$

then

$$u^+(x) \geq U_0(x_n + c\varepsilon) \quad \text{in } \overline{B}_{1/2}, \tag{9-4}$$

for some universal c, with $0 < c < 1$. Analogously, if $u^+(x) \leq U_0(x)$ in B_1 and $u^+(\bar{x}) \leq U_0(\bar{x}_n - \varepsilon)$, then

$$u^+(x) \leq U_0(x_n - c\varepsilon) \quad \text{in } \overline{B}_{1/2}.$$

Proof. The proof is the same as for the model case in Lemma 4.6. To prove that

$$v_{\bar{t}}(x) = G_0(0)(x_n - \varepsilon c_0 \psi + \bar{t}\varepsilon)^+ - \varepsilon^2 C_1 (x_n - \varepsilon c_0 \psi(x) + \bar{t}\varepsilon)^-, \quad x \in \overline{B}_{3/4}(\bar{x})$$

is a subsolution in the annulus A, we use the following computation:

$$\mathscr{L} v_{\bar{t}} \geq c_0 C_1 \varepsilon^3 \mathscr{L} w - C_1 \varepsilon^2 |b_n| \geq \varepsilon^3 K(n, \lambda, \Lambda) > \varepsilon^4 \geq \|f\|_\infty \quad \text{in } A^+(v_{\bar{t}}) \cup A^-(v_{\bar{t}}),$$

for ε small enough. Here we have used as in Lemma 8.1 that $\mathscr{L} w \geq k_0 > 0$.

Moreover, on $F(v_{\bar{t}}) \cap A$ we have

$$(v_{\bar{t}}^+)_\nu - G((v_{\bar{t}}^-)_\nu) = G_0(0)|e_n - \varepsilon c_0 \nabla \psi| - G(\varepsilon^2 C_1 |e_n - \varepsilon c_0 \nabla \psi|, x) \geq C\varepsilon |\psi_n| + O(\varepsilon^2) > 0,$$

as long as ε is small enough. □

We state here the corollary that can be deduced by the degenerate Harnack inequality.

Corollary 9.2. *Let u satisfy at some point $x_0 \in B_2$*

$$U_0(x_n + a_0) \leq u(x) \leq U_0(x_n + b_0) \quad \text{in } B_1(x_0) \subset B_2, \tag{9-5}$$

with

$$b_0 - a_0 \leq \varepsilon,$$

and let (9-1)–(9-2) *hold with* $\varepsilon \leq \bar{\varepsilon}$, *where* $\bar{\varepsilon}$ *is universal. Then in* $B_1(x_0)$

$$\tilde{u}_\varepsilon := \frac{u^+(x) - G_0(0)x_n}{\varepsilon G_0(0)}$$

has a Hölder modulus of continuity at x_0, *outside the ball of radius* $\varepsilon/\bar{\varepsilon}$, *that is, for all* $x \in B_1(x_0)$ *with* $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$,

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C|x - x_0|^\gamma.$$

Improvement of flatness. We prove here the improvement of flatness in the degenerate setting. Recall that in this case one improves the flatness of u^+ only.

Lemma 9.3. *Let* u *satisfy*

$$U_0(x_n - \varepsilon) \leq u^+(x) \leq U_0(x_n + \varepsilon) \quad \text{in } B_1, 0 \in F(u), \tag{9-6}$$

with

$$\|a_{ij} - \delta_{ij}\| \leq \varepsilon, \quad \|f\|_{L^\infty(B_1)} \leq \varepsilon^4, \quad \|\boldsymbol{b}\|_{L^\infty(B_1)} \leq \varepsilon^2,$$
$$\|G(\eta, \cdot) - G_0(\eta)\|_{L^\infty} \leq \varepsilon^2, \quad 0 \leq \eta \leq C\varepsilon^2,$$

and

$$\|u^-\|_{L^\infty(B_1)} \leq \varepsilon^2.$$

If $0 < r \leq r_1$ *for* r_1 *universal, and* $0 < \varepsilon \leq \varepsilon_1$ *for some* ε_1 *depending on* r, *then*

$$U_0\Big(x \cdot \nu_1 - r\frac{\varepsilon}{2}\Big) \leq u^+(x) \leq U_0\Big(x \cdot \nu_1 + r\frac{\varepsilon}{2}\Big) \quad \text{in } B_r, \tag{9-7}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq C\varepsilon$ *for a universal constant* C.

Proof. Step 1: Compactness. As in Lemma 5.2, it follows from Corollary 9.2 that as $\varepsilon_k \to 0$ the graphs of the

$$\tilde{u}_k(x) = \frac{u_k(x) - G_k(0,0)x_n}{G_k(0,0)\varepsilon_k}, \quad x \in B_1^+(u_k) \cup F(u_k)$$

converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function $\tilde{u}$ over $B_{1/2} \cap \{x_n \geq 0\}$. Here the u_k solve our free boundary problem (1-3) with coefficients a_{ij}^k, $\boldsymbol{b}^k$, right-hand side f_k and free boundary condition G_k satisfying the assumptions of the lemma for a subsequence of ε_k going to 0.

Step 2: limiting solution. One shows that $\tilde{u}$ solves the following Neumann problem

$$\begin{cases} \Delta\tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \tilde{u}_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases} \tag{9-8}$$

We can easily adapt the proof of Lemma 5.2, choosing

$$\phi_k(x) = a_k\Gamma_k^+(x) + (d_k^+(x))^2\varepsilon_k^{3/2}, \quad a_k = G_k(0,0)(1 + \varepsilon_k p).$$

and

$$\Psi_k(x) = \begin{cases} \phi_k(x + c_k\varepsilon_k e_n) & \text{in } \mathcal{B}, \\ c\varepsilon_k^2(3d(x, \partial\mathcal{B}) + d^2(x, \partial\mathcal{B})) & \text{outside of } \mathcal{B}, \end{cases} \tag{9-9}$$

with

$$\mathcal{B} := B_{1/(B\varepsilon_k)}\left(y + e_n\left(\frac{1}{B\varepsilon_k} - A\varepsilon_k - \varepsilon_k c_k\right)\right).$$

To check the subsolution condition at the free boundary for the function $\Psi_k(x)$, we need that

$$(\Psi_k^+)_\nu > G_k((\Psi_k^-)_\nu, x) \quad \text{on } F(\Psi_k).$$

This is equivalent to showing that $G_k(0, 0)(1 + \varepsilon_k p) - G_k(c\varepsilon_k^2, x) > 0$ for k large. Since $p > 0$, this follows immediately from the assumptions on G_k.

Step 3: contradiction. In this step we can argue as in the final step of the proof of Lemma 4.1 in [De Silva 2011]. □

10. Proofs of the main theorems for general free boundary problems

The proof of Theorem 1.3 and Theorem 1.4 follow the same scheme of the model case. In particular, for Theorem 1.3, we take care of choosing $\bar{r}^{\bar{\gamma}} < \frac{1}{16}$, say, while the other assumptions on $\bar{r}$ remain the same. Also, $\tilde{\varepsilon}$ may have to be smaller, depending on γ_0. The dichotomy degenerate/nondegenerate is handled through Lemma 6.1 which extends to the variable coefficients case, with minor changes in the proof.

In the proof of Theorem 1.4, the blow-up limit $\tilde{u}$ solves the following global homogeneous two-phase free boundary problem

$$\begin{cases} \Delta\tilde{u} = 0, & \text{in } \{\tilde{u} > 0\} \cup \{\tilde{u} \le 0\}^0, \\ \tilde{u}_\nu^+ = G_0(\tilde{u}_\nu^-) & \text{on } F(\tilde{u}) := \partial\{\tilde{u} > 0\}. \end{cases} \tag{10-1}$$

Now, Lemma 6.2 holds with identical proof for the free boundary condition $U_\nu^+ = G_0(U_\nu^-)$, so that the proof of Theorem 1.4 does not present any further difficulty.

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MIURA MAPS AND INVERSE SCATTERING FOR THE NOVIKOV–VESELOV EQUATION

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We use the inverse scattering method to solve the zero-energy Novikov–Veselov (NV) equation for initial data of conductivity type, solving a problem posed by Lassas, Mueller, Siltanen, and Stahel. We exploit Bogdanov's Miura-type map which transforms solutions of the modified Novikov–Veselov (mNV) equation into solutions of the NV equation. We show that the Cauchy data of conductivity type considered by Lassas, Mueller, Siltanen, and Stahel lie in the range of Bogdanov's Miura-type map, so that it suffices to study the mNV equation. We solve the mNV equation using the scattering transform associated to the defocussing Davey–Stewartson II equation.

1. Introduction

In this paper we will use inverse scattering methods to solve the Novikov–Veselov (NV) equation, a completely integrable, dispersive nonlinear equation in two space and one time $(2+1)$ dimensions, for the class of *conductivity type* initial data that we define below. Our results solve a problem posed by Lassas, Mueller, Siltanen and Stahel [Lassas et al. 2012] in their analytical study of the inverse scattering method for the NV equation.

Denoting $z = x_1 + ix_2$, $\overline{\partial} = (1/2)(\partial_{x_1} + i\partial_{x_2})$, $\partial = (1/2)(\partial_{x_1} - i\partial_{x_2})$, the Cauchy problem for the NV equation is

$$\begin{aligned} &q_t + \partial^3 q + \overline{\partial}^3 q - \tfrac{3}{4}\partial(q\overline{\partial}^{-1}\partial q) - \tfrac{3}{4}\overline{\partial}(q\partial^{-1}\overline{\partial} q) = 0, \\ &q|_{t=0} = q_0. \end{aligned} \tag{1-1}$$

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where q_0 is a real-valued function that vanishes at infinity. The NV equation generalizes the celebrated KdV equation

$$q_t + q_{xxx} + 6qq_x = 0$$

in the sense that any solution of KdV (after rescaling) solves NV when regarded as a function of (x_1, x_2, t) with no x_2-dependence. As has recently been proved by Angelopoulos [2013], the Cauchy problem for the NV equation is locally well-posed in the Sobolev space $H^s(\mathbb{R}^2)$ for any $s > 1$. The inverse scattering method considered here yields solutions global in time, albeit for a more restrictive class of initial data.

The Novikov–Veselov equation is one of a hierarchy of dispersive nonlinear equations in $2+1$ dimensions discovered by Novikov and Veselov [1984; 1986]. Up to trivial scalings, our equation is the zero-energy ($E = 0$) case of the equation they studied, which reads

$$\begin{aligned} q_t &= 4\,\mathrm{Re}(4\partial^3 q + \partial(qw) - E\partial q), \\ \bar{\partial} w &= \partial q. \end{aligned} \tag{1-2}$$

In the papers cited, Novikov and Veselov constructed explicit solutions from the spectral data associated to a two-dimensional Schrödinger problem at a single energy. Novikov conjectured that the inverse problem for the two-dimensional Schrödinger operator at a fixed energy should be completely solvable (see the remarks in [Grinevich 2000]), and that inverse scattering for the Schrödinger equation at a fixed energy E could be used to solve the NV equation at the same energy E by inverse scattering. Subsequent studies [Grinevich 1986; Grinevich and Manakov 1986; Grinevich and Novikov 1985; 1986; 1988b; 1988a; 1995] further developed the inverse scattering method and constructed multisoliton solutions (see also [Kazeykina 2012a; 2012b; Kazeykina and Novikov 2011a; 2011b; 2011c] for further results). Independently, Boiti, Leon, Manna, and Pempinelli [Boiti et al. 1987] proposed an inverse scattering method to solve the NV equation at zero energy with data vanishing at infinity. We refer the reader to the recent survey [Croke et al. 2013] for further references and further information on the Novikov–Veselov equation. Recently, Angelopoulos [2013] has proved local well-posedness for the Novikov–Veselov equation in the space $H^s(\mathbb{R}^2)$ for $s > \frac{1}{2}$.

It has long been understood that the inverse Schrödinger scattering problem at zero energy poses special challenges (see, for example, the discussion in Part I of supplement 1 in [Grinevich and Novikov 1988a], and the comments in [Grinevich 2000, Section 7.3]). In particular, the scattering transform for the Schrödinger operator at zero energy is known to be well-behaved *only* for a special class of potentials, the potentials of "conductivity type", which may be thought of as follows.

Definition 1.1. A real-valued function $u \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ is called a *potential of conductivity type* if the equation $(-\Delta + q)\psi = 0$ admits a unique, strictly positive solution normalized so that $\psi(z) = 1$ in a neighborhood of infinity.

Remark 1.2. If q is a potential of conductivity type, it is not difficult to see that the corresponding Schrödinger operator has no eigenvalues (including no eigenvalues at zero energy), and that $q = \psi^{-1}(\Delta\psi)$ for a *unique* strictly positive function ψ with $\psi(z) = 1$ near infinity. See [Music et al. 2013] for further discussion.

The class of conductivity type potentials can also be defined for less regular q (see [Nachman 1996, Theorem 3]), but this definition will suffice for the present purpose. The terminology comes from the connection of the Schrödinger inverse problem at zero energy with Calderón's inverse conductivity problem [Calderón 1980] (see [Nachman 1996] for a solution for conductivities $\sigma \in W^{2,p}$ via the scattering transform, and see [Astala and Päivärinta 2006] for the solution to Calderón's inverse problem for general $\gamma \in L^\infty$, and for references to the literature). The problem is to reconstruct the conductivity γ of a conducting body $\Omega \subset \mathbb{R}^2$ from the Dirichlet to Neumann map, defined as follows. Let $f \in H^{1/2}(\partial\Omega)$ and let $u \in H^1(\Omega)$ solve the problem

$$\nabla \cdot (\gamma \nabla u) = 0, \quad u|_{\partial\Omega} = f.$$

This problem has a unique solution for conductivities $\gamma \in L^\infty(\Omega)$ with $\gamma(z) \geq c > 0$ for a.e. z. The Dirichlet to Neumann map is the mapping

$$\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad f \mapsto \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

Nachman [1996] exploited the fact that $v = \gamma^{1/2} u$ solves the Schrödinger equation at zero energy where

$$q = \gamma^{-1/2} \Delta(\gamma^{1/2}). \tag{1-3}$$

The Schrödinger problem also has a Dirichlet to Neumann map

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad f \mapsto \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega},$$

defined by the unique solution of

$$(-\Delta + q)v = 0, \quad v|_{\partial\Omega} = f.$$

The operator Λ_q determines and is determined by the scattering data for q of the form (1-3) at zero energy, and Λ_q determines Λ_γ. Note that q is of conductivity type if we take $\psi = \gamma^{1/2}$ and extend ψ to $\mathbb{R}^2 \backslash \Omega$ setting $\psi(z) = 1$. Nachman showed that the scattering transform at zero energy is well-defined *only* when q is of conductivity type (we give a precise statement below) and used the inverse scattering transform to reconstruct q from its scattering data.

The set of conductivity-type potentials is highly unstable, even under $C_0^\infty(\mathbb{R}^2)$ perturbations of arbitrarily small size. To explain this, we recall from [Murata 1986] (see also [Gesztesy and Zhao 1995] for more recent work and further references) that a Schrödinger operator is called

(i) *subcritical* if $-\Delta + q$ has a positive Green's function,

(ii) *critical* if $-\Delta + q$ does not have a positive Green's function, but the quadratic form

$$\mathfrak{q}(\varphi) = \int_{\mathbb{R}^2} \left(|(\nabla\varphi)(z)|^2 + q(z)|v(z)|^2 \right) dA(z)$$

on $C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$ is nonnegative, or

(iii) *supercritical* if the quadratic form $\mathfrak{q}$ is not nonnegative.

It follows from Theorem 3.1(iii) of [Murata 1986] that a conductivity-type potential is critical. From Theorem 2.4(i) of the same reference we may conclude that for any $w \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and any $\lambda > 0$, the potential $q_0 - \lambda w$ is subcritical and not of conductivity type. We refer the reader to Appendix B of [Music et al. 2013] for further details.

Thus, the set of conductivity-type potentials is nowhere dense in any reasonable function space! For this reason one expects the direct and inverse scattering maps for the Schrödinger operator at zero energy not to have good continuity properties as a function of the potential q.

Let us describe the direct scattering transform $\mathcal{T}$ and inverse scattering transform $\mathcal{Q}$ for the Schrödinger operator at zero energy in more detail (see [Nachman 1996] and [Lassas et al. 2012] for details and references). To define the direct scattering map $\mathcal{T}$ on potentials $q \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we seek complex geometric optics (CGO) solutions $\psi = \psi(z, k)$ of

$$(-\Delta + q)\psi = 0, \tag{1-4}$$

which satisfy the asymptotic condition

$$\lim_{|z|\to\infty} e^{-ikz}\psi(z, k) = 1 \tag{1-5}$$

for a fixed $k \in \mathbb{C}$. Let $m(z, k) = e^{-izk}\psi(z, k)$. Assuming that the problem (1-4)–(1-5) has a unique solution for all k, we define the scattering transform $\mathbf{t} = \mathcal{T}q$ via the formula

$$\mathbf{t}(k) = \int e^{i(\bar{k}\bar{z}+kz)} q(z) m(z, k)\, dA(z), \tag{1-6}$$

where $dA(z)$ is Lebesgue measure on $\mathbb{R}^2$. The surprising fact is that, if $\mathbf{t}$ is well-behaved, the solutions $\psi(z, k)$, and hence the potential q, may be recovered from $\mathbf{t}(k)$. This fact leads to an inverse scattering transform $q = \mathcal{Q}\mathbf{t}$ given by

$$q(z) = \frac{i}{\pi^2}\bar{\partial}_z\left(\int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} e^{-i(kz+\bar{k}\bar{z})}\overline{m(z, k)}\, dA(k)\right). \tag{1-7}$$

Boiti, Leon, Manna and Pempinelli [Boiti et al. 1987], proposed an inverse scattering solution to the Novikov–Veselov equation using these maps:

$$q(t) = \mathcal{Q}\big(e^{it((\diamond)^3+(\bar{\diamond})^3)}(\mathcal{T}q_0)(\diamond)\big), \tag{1-8}$$

and gave formal arguments to justify it. The maps were further studied in [Tsai 1993]. Lassas, Mueller, Siltanen, and Stahel [Lassas et al. 2012], building on [Lassas et al. 2007], showed that the scattering transforms are well-defined for certain potentials of conductivity type. For conductivity-type potentials, they proved that $\mathcal{T}$ and $\mathcal{Q}$ are inverses, and that (1-8) defines a continuous $L^p(\mathcal{R}^2)$-valued function of t for $p \in (1, 2)$. They conjectured that $q(t)$ is in fact a classical solution of (1-1) if q_0 is a smooth, decreasing, real-valued potential of conductivity type but were unable to prove that this was the case.

The fact, already mentioned, that conductivity-type potentials are a nowhere dense set in the space of potentials, suggests that studying the NV equation using the maps $\mathcal{T}$ and $\mathcal{Q}$ is likely to be technically challenging. The following result of Nachman makes the difficulty clearer. For given q, let $\mathcal{E}_q$ be the set of all k for which the problem (1-4)–(1-5) does *not* have a unique solution. Let $L^p_\rho(\mathbb{R}^2)$ denote the

Banach space of real-valued measurable functions q with

$$\|q\|_{L^p_\rho} := \left[\int (1+|z|)^{p\rho} |q(z)|^p \, dA(z) \right]^{1/p} < \infty.$$

Theorem 1.3 [Nachman 1996, Theorem 3]. *Suppose that $q \in L^p_\rho(\mathbb{R}^2)$ for some $p \in (1,2)$, and $\rho > 1$: The following are equivalent*:

(i) *The set $\mathcal{E}_q$ is empty and $|\mathbf{t}(k)| \leq C|k|^\varepsilon$ for some fixed $\varepsilon > 0$ and all sufficiently small k.*

(ii) *There is a real-valued function $\gamma \in L^\infty(\mathbb{R}^2)$ with $\gamma(z) \geq c > 0$ for a.e. z and a fixed constant c so that $q = \gamma^{-1/2}\Delta(\gamma^{1/2})$.*

One should think of γ as ψ^2 where ψ is the unique normalized positive solution of $(-\Delta + q)\psi = 0$ for a potential of conductivity type. Nachman's result suggests that non-conductivity type potentials will have singular scattering transforms: Music, Perry and Siltanen [Music et al. 2013] construct an explicit one-parameter deformation $\lambda \mapsto q_\lambda$ of a conductivity type potentials (q_0 is of conductivity type, but q_λ is not for $\lambda \neq 0$) for which the corresponding family $\lambda \mapsto \mathbf{t}_\lambda$ of scattering transforms has an essential singularity at $\lambda = 0$.

We will show that, nonetheless, the formula (1-8) does yield classical solutions of the NV equation for a much larger class of initial data than considered in [Lassas et al. 2012]. We achieve this result by circumventing the scattering maps studied in [Lassas et al. 2012]. Instead, we exploit Bogdanov's observation [1987] (see also [Dubrovsky and Gramolin 2008; 2009]) that the Miura-type map

$$\mathcal{M}(v) = 2\partial v + |v|^2 \tag{1-9}$$

takes solutions u of the modified Novikov–Veselov (mNV) equation

$$u_t + (\partial^3 + \bar\partial^3)u - NL(u) = 0, \tag{1-10}$$

where

$$NL(u) = \tfrac{3}{4}(\partial\bar u)\cdot(\bar\partial\partial^{-1}(|u|^2)) + \tfrac{3}{4}(\bar\partial u)\cdot(\bar\partial\partial^{-1}(|u|^2)) + \tfrac{3}{4}\bar u\bar\partial\partial^{-1}(\bar u\bar\partial u) + \tfrac{3}{4}u\partial^{-1}(\bar\partial(\bar u\bar\partial u)),$$

to solutions q of the NV equation. This map is an analogue of the celebrated Miura map $u \mapsto u_x + u^2$ which takes solutions of the modified Korteweg–de Vries equation to solutions of the Korteweg–de Vries equation [Miura 1968; Kappeler et al. 2005]. We remark that local well-posedness for the mNV equation in $H^s(\mathbb{R}^2)$ for any $s > 1$ was recently proved in [Angelopoulos 2013].

In (1-9), the domain of the Miura map is understood to be smooth functions v with $\partial v = \overline{\partial v}$. As we will show, the range of this Miura-type map consists exactly of initial data of conductivity type! In particular, we show that the range of $\mathcal{M}$ contains the conductivity-type potentials studied by in [Lassas et al. 2012].

Thus, to solve the NV equation for initial data of conductivity type, it suffices to solve the mNV equation and use the map $\mathcal{M}$ to obtain a solution of NV. The mNV equation is a member of the Davey–Stewartson II hierarchy, so the well-known scattering maps for the DS II hierarchy (see [Fokas and Ablowitz 1983; 1984; Beals and Coifman 1984; 1985; 1989; 1990; Brown 2001; Perry 2011; Sung 1994a; 1994b; 1994c]) can be used to solve the Cauchy problem for mNV. We denote by $\mathcal{R}$ and $\mathcal{I}$ respectively the scattering transform and inverse scattering transform associated to the defocusing DS II equation (see Section 3 for the definitions). We show in Appendix A that the function

$$u(t) = \mathcal{I}\big(\exp((\bar{\diamond}^3 - \diamond^3)t)(\mathcal{R}u_0)(\diamond)\big) \tag{1-11}$$

is a classical solution of the mNV equation (1-10) for initial data $u_0 \in \mathcal{S}(\mathbb{R}^2)$.

In order to obtain good mapping properties for the solution map $u_0 \mapsto u(t)$ defined by (1-11), we need local Lipschitz continuity of the maps $\mathcal{I}$ and $\mathcal{R}$ on spaces that are preserved under the flow (compare the treatment of the cubic NLS in one dimension in [Deift and Zhou 2003] and the Sobolev mapping properties for the scattering maps for NLS proven in [Zhou 1998]). In [Perry 2011] it was shown that $\mathcal{R}$ and $\mathcal{I}$ are mutually inverse mappings of $H^{1,1}(\mathbb{R}^2)$ into itself where

$$H^{m,n}(\mathbb{R}^2) = \big\{u \in L^2(\mathbb{R}^2) : (1-\Delta)^{m/2}u,\ (1+|\cdot|)^n u(\cdot) \in L^2(\mathbb{R}^2)\big\}.$$

In order to use (1-11), we need the following refined mapping property of $\mathcal{I}$ and $\mathcal{R}$.

Theorem 1.4. *The scattering maps $\mathcal{R}$ and $\mathcal{I}$ restrict to locally Lipschitz continuous maps*

$$\mathcal{R} : H^{2,1}(\mathbb{R}^2) \to H^{1,2}(\mathbb{R}^2), \quad \mathcal{I} : H^{1,2}(\mathbb{R}^2) \to H^{2,1}(\mathbb{R}^2).$$

This immediately implies that the solution formula (1-11) defines a continuous map

$$H^{2,1}(\mathbb{R}^2) \to C\big([0,T]; H^{2,1}(\mathbb{R}^2)\big), \quad t \mapsto u(t),$$

for any $T > 0$. We say that u is a weak solution of the mNV equation (see (5-1)) on $[0,T]$ if

$$(\varphi_t + \partial^3\varphi + \bar{\partial}^3\varphi, u) + (\varphi, NL(u)) = 0, \tag{1-12}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2 \times [0,T])$, where $(\,\cdot\,,\cdot\,)$ denotes the inner product on $L^2(\mathbb{R}^2 \times [0,T])$. We will show that (1-11) defines a weak solution in this sense and that, also, the flow (1-11) leaves the domain of $\mathcal{M}$ invariant. We will prove:

Theorem 1.5. *For $u_0 \in \mathcal{S}(\mathbb{R}^2)$, the solution formula* (1-11) *gives a classical solution of mNV. Moreover, if $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, $\partial u_0 = \overline{\partial u_0}$, and $\int u_0(z)\,dA(z) = 0$, then $u(t)$ is a weak solution of mNV and the relations $(\partial u)(\,\cdot\,,t) = \overline{(\partial u)(\,\cdot\,,t)}$ and $\int u(z,t)\,dA(z) = 0$ hold for all t.*

Now we can solve the NV equation using the solution map for mNV and the Miura map (1-9). We say that q is a weak solution of the NV equation on $[0,T]$ if

$$(\varphi_t + \partial^3\varphi + \bar{\partial}^3\varphi, q) + \tfrac{3}{4}(\partial\varphi, q\bar{\partial}^{-1}\partial q) + \tfrac{3}{4}(\bar{\partial}\varphi, q\partial^{-1}\bar{\partial} q) = 0, \tag{1-13}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2 \times (0,T))$. Using Theorem 1.5, we will prove:

Theorem 1.6. *Suppose that $q_0 = 2\partial u_0 + |u_0|^2$ where $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, $\partial u_0 = \overline{\partial u_0}$, and $\int u_0(z)\,dA(z) = 0$. Then*

$$q(t) = \mathcal{M}\big(\mathcal{I}\big(e^{2it((\diamond)^2 + (\bar{\diamond})^2)}(\mathcal{R}u_0)(\diamond)\big)\big) \tag{1-14}$$

is a weak solution the NV equation with initial data q_0. If $u_0 \in \mathcal{S}(\mathbb{R}^2)$, then $q(t)$ is a classical solution of the NV equation.

The class of initial data covered by Theorem 1.6 includes the conductivity-type potentials considered in [Lassas et al. 2012]. The connection between that work and ours is given in the following theorem.

Theorem 1.7. *Suppose that $u_0 \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with $\int u_0(z)\, dA(z) = 0$ and $\overline{\bar\partial u_0} = \partial u_0$, and let $q_0 = 2\partial u_0 + |u_0|^2$. Then, for any t,*

$$\mathcal{Q}\big(e^{it((\diamond)^3+(\bar\diamond)^3)}(\mathcal{T}q_0)(\diamond)\big) = \mathcal{MI}\big(e^{t((\bar\diamond)^3-(\diamond)^3)}(\mathcal{R}u_0)(\diamond)\big),$$

and their common value is a classical solution to the Novikov–Veselov equation.

It should be noted that the solution formula (1-14) provides a solution which exists *globally* in time. On the other hand, Taimanov and Tsaryov [2007; 2008a; 2008b; 2010]] have used Moutard transformations to construct explicit, nonsingular Cauchy data q_0 with rapid decay at infinity and having the following properties: (i) the Schrödinger operator $-\Delta + q_0$ has nonzero eigenvalues at zero energy (and so is not of conductivity type) and (ii) the solution of (1-1) with Cauchy data q_0 blows up in finite time.

To close this introduction, we comment on the seemingly restrictive hypothesis in Theorems 1.6 and 1.7. In both theorems, we assume that $\int u_0 = 0$. To understand what this assumption means, we recall that if $\phi_0 = \bar\partial^{-1} u_0$, then the unique, positive, normalized zero-energy solution of the Schrödinger equation (1-4) is given by $\psi_0 = \exp(\phi_0)$. For $u_0 \in \mathcal{S}(\mathbb{R}^2)$ say, we have from the integral expression for $\bar\partial^{-1}$ that

$$\phi_0(z) = -\frac{1}{\pi}\frac{\int u_0(\zeta)\, d\zeta}{z} + \mathcal{O}(|z|^{-2}),$$

so that, to leading order

$$\psi_0 - 1 = -\frac{1}{\pi}\frac{\int u_0(\zeta)\, d\zeta}{z} + \mathcal{O}(|z|^{-2}).$$

Recalling that $\gamma^{1/2}(z) = \psi_0(z)$ we see that the vanishing of $\int u_0(z)\, dA(z)$ implies that $\gamma(z) - 1 = \mathcal{O}(|z|^{-2})$ as $|z| \to \infty$. In particular, for conductivities with $\gamma = 1$ outside a compact set, $\int u_0(z)\, dA(z) = 0$.

Indeed, suppose that $q = \gamma^{-1/2}\Delta(\gamma^{1/2})$ in distribution sense, where $\gamma \in L^\infty(\mathbb{R}^2)$, $\gamma(z) \geq c > 0$, and suppose further that $\Delta(\nabla\gamma)$ and $\gamma - 1$ belong to $L^2(\mathbb{R}^2)$. It follows that $\varphi = \log\gamma \in H^{3,1}(\mathbb{R}^2)$ and the function

$$u = 2\bar\partial\varphi$$

belongs to $H^{2,1}$. We then compute that $q = 2\partial u + |u|^2$. If we have stronger decay of $\gamma(z)$ as $|z| \to \infty$, this will imply additional decay of $\varphi(z)$ that can be used to check $\int u(z)\, dA(z) = 0$ by Green's formula $\int_\Omega \bar\partial\varphi\, dA(z) = \frac{1}{2}\int_{\partial\Omega} \varphi(\nu_{x_1} + i\nu_{x_2})\, d\sigma$.

The structure of this paper is as follows. In Section 2 we review some important linear and multilinear estimates which will be used to study the scattering maps $\mathcal{R}$ and $\mathcal{I}$. In Section 3 we recall how the scattering maps $\mathcal{R}$ and $\mathcal{I}$ for the Davey–Stewartson system are defined, while in Section 4 we prove that $\mathcal{R}: H^{2,1}(\mathbb{R}^2) \to H^{1,2}(\mathbb{R}^2)$ and $\mathcal{I}: H^{1,2}(\mathbb{R}^2) \to H^{2,1}(\mathbb{R}^2)$ are locally Lipschitz continuous. In Section 5 we solve the mNV equation using the inverse scattering method and prove that, for initial data $u_0 \in H^{2,1}(\mathbb{R}^2)$ with $\partial u_0 = \overline{\partial u_0}$ and $\int_{\mathbb{R}^2} u_0(z)\, dA(z) = 0$, the condition $\partial u = \overline{\partial u}$ holds for all $t > 0$. In Section 6 we prove Theorem 1.6. In Section 7 we show that our class of potentials extends the class of conductivity type potentials considered in [Lassas et al. 2012], and that our solution coincides with theirs where the two constructions overlap. Appendix A sketches the solution of the mNV equation by scattering theory for initial data in the Schwarz class.

2. Preliminaries

Notation. In what follows, $\|\cdot\|_p$ denotes the usual L^p-norm and $p' = p/(p-1)$ denotes the conjugate exponent. If f is a function of (z,k), $f(z,\diamond)$ (resp. $f(\,\cdot\,,k)$) denotes f with a generic argument in the z (resp. k) variable. We will write L^p_z or L^p_k for L^p-spaces with respect to the z or k variable, and $L^p_z(L^q_k)$ for the mixed spaces with norm

$$\|f\|_{L^p_z(L^q_k)} = \left(\int \|f(z,\diamond)\|_q^p \, dA(z)\right)^{1/p}.$$

If f is a function of z and k, $\|f\|_\infty$ denotes $\|f\|_{L^\infty(\mathbb{R}^2_z \times \mathbb{R}^2_k)}$.

In what follows, $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the pairing

$$\langle f, g\rangle = \frac{1}{\pi}\int \overline{f(z)}\, g(z)\, dA(z).$$

We will call a mapping f from a Banach space X to a Banach space Y a *locally Lipschitz continuous map* (LLCM) if, for any bounded subset B of X, there is a positive constant $C = C(B)$ such that, for all $x_1, x_2 \in B$,

$$\|f(x_1) - f(x_2)\|_Y \le C(B)\|x_1 - x_2\|_X.$$

For example, if $M : X^m \to Y$ is a continuous multilinear map, then

$$f \mapsto M(f, f, \dots, f)$$

is an LLCM from X to Y.

Cauchy transforms. The integral operators

$$P\psi = \frac{1}{\pi}\int \frac{1}{z-\zeta} f(\zeta)\, dm(\zeta), \qquad \bar{P}\psi = \frac{1}{\pi}\int \frac{1}{\bar{z}-\bar{\zeta}} f(\zeta)\, dm(\zeta)$$

are formal inverses respectively of $\bar{\partial}$ and ∂. We denote by P_k and $\bar{P}_k$ the corresponding formal inverses of $\bar{\partial}_k$ and ∂_k. The following estimates are standard (see, for example, [Astala et al. 2009, Section 4.3] or [Vekua 1959]).

Lemma 2.1. (i) *For any $p \in (2,\infty)$ and $f \in L^{2p/(p+2)}$, $\|Pf\|_p \le C_p \|f\|_{2p/(p+2)}$.*

(ii) *For any p, q with $1 < q < 2 < p < \infty$ and any $f \in L^p \cap L^q$, $\|Pf\|_\infty \le C_{p,q}\|f\|_{L^p \cap L^q}$ and Pf is Hölder continuous of order $(p-2)/p$ with*

$$|(Pf)(z) - (Pf)(w)| \le C_p |z-w|^{(p-2)/p}\|f\|_p.$$

(iii) *For $2 < p, q$ and $u \in L^s$ for $q^{-1} + 1/2 = p^{-1} + s^{-1}$,*

$$\|P(u\psi)\|_q \le C_{p,q}\|u\|_s \|\psi\|_p.$$

Remark 2.2. If $p > 2$ and $u \in L^s$ for $s \in (1, \infty)$, then estimate (iii) holds true for any $q > 2$.

Beurling transform. The operator

$$(\mathcal{S} f)(z) = -\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{|z-w|>\varepsilon} \frac{1}{(z-w)^2} f(w)\, dw, \tag{2-1}$$

defined as a Calderón–Zygmund type singular integral, has the property that for $f \in C_0^\infty(\mathbb{R}^2)$ we have $\mathcal{S}(\bar{\partial} f) = \partial f$. The operator $\mathcal{S}$ is a bounded operator on L^p for $p \in (1, \infty)$ (see, for example, [Astala et al. 2009, Section 4.5.2]). This fact allows us to obtain L^p-estimates on ∂-derivatives of functions of interest from L^p-estimates on $\bar{\partial}$-derivatives.

We will also need the following trivial estimate on the Beurling transform of a smooth, rapidly decreasing function.

Lemma 2.3. *Suppose that $M > 2$ and $\sup_{|\alpha| \le 2} |D^\alpha g(z)| \le C(1+|z|)^{-M}$. For any β with $0 \le \beta < M-2$ and $\beta \le 2$, the estimate $|\mathcal{S}(g)| \le C(1+|z|)^{-\beta}$ holds.*

Proof. Compute

$$\int_{\varepsilon<|w|} \frac{1}{(z-w)^2} f(w)\, dw = \left(\int_{\varepsilon<|z-w|<1} + \int_{|z-w|\ge 1} \right) \frac{1}{(z-w)^2} f(w)\, dw.$$

In the first term we may Taylor-expand $f(w)$, note that $\int_{\varepsilon<|w|<1} (z-w)^{-2}\, dw = 0$, and conclude that the first term is estimated by

$$C \sup_{\substack{|\alpha|\le 2 \\ |z-w|\le 1}} |(D^\alpha f(w)|,$$

which is $\mathcal{O}(|z|^{-M})$ by hypothesis. The second term is estimated by a constant times

$$(1+|z|)^{-\beta} \int \frac{1}{(1+|z-w|)^{2-\beta}} \frac{1}{(1+|w|)^{M-\beta}}\, dw,$$

which gives the required decay. □

Brascamp–Lieb type estimates. A fundamental role is played by the following multilinear estimate due to Russell Brown [2001], who initiated their use in the analysis of the DS II scattering maps. See [Christ 2011] for a proof of these estimates using the methods of Bennett, Carbery, Christ and Tao [Bennett et al. 2008; 2010], and see [Nie and Brown 2011] for a different proof. Define

$$\Lambda_n(\rho, u_0, u_1, \dots, u_{2n}) = \int_{\mathbb{C}^{2n+1}} \frac{|\rho(\zeta)|\, |u_0(z_0)| \dots |u(z_{2n})|}{\prod_{j=1}^{2k} |z_{j-1} - z_j|}\, dA(z),$$

where $dA(z)$ is product measure on $\mathbb{C}^{2n+1}$, and set

$$\zeta = \sum_{j=0}^{2n} (-1)^j z_j. \tag{2-2}$$

Proposition 2.4 [Brown 2001]. *The estimate $\left|\Lambda_n(\rho, u_0, u_1, \dots, u_{2n})\right| \le C_n \|\rho\|_2 \prod_{j=0}^{2n} \|u_j\|_2$ holds.*

Remark 2.5. For $u_1, \dots, u_{2n} \in \mathcal{S}(\mathbb{R}^2)$, define operators W_j by $W_j \psi = P e_k u_j \overline{\psi}$. Proposition 2.4 implies that

$$F(k) = \langle e_k u_0, W_1 W_2 \dots W_{2n} 1 \rangle \tag{2-3}$$

is a multilinear $L^2_k(\mathbb{R}^2)$-valued function of $(u_0, \dots, u_{2n})$ with

$$\|F\|_2 \le C \prod_{j=0}^{2n} \|u_j\|_2.$$

Pseudodifferential operators. In Section 5 we will use pseudodifferential operators to prove key estimates on a third-order linear evolution equation. We recall that a function $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ belongs to the symbol class $S^m(\mathbb{R}^n)$ if for all multiindices α, β, the seminorms

$$\rho_{\alpha,\beta}(p) := \sup_{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n} \left| (1 + |\xi|)^{m-|\alpha|} p(x, \xi) \right|. \tag{2-4}$$

are finite. The corresponding pseudodifferential operator $P(x, D)$ is given by the Weyl quantization

$$(P(x, D) f)(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} p\left(\frac{x+y}{2}, \xi\right) e^{i(x-y)\cdot\xi} f(y)\, dy,$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, and we say that $P(x, D) \in \mathrm{OPS}^m(\mathbb{R}^n)$. We also write $\sigma(P)$ for p. For the Weyl quantization, if p is a real-valued symbol, then $p(x, D)$ is formally symmetric.

The celebrated Calderón–Vaillancourt theorem [1972] implies that if $p \in S^0(\mathbb{R}^n)$, then $p(x, D)$ extends to a bounded operator on $L^2(\mathbb{R}^n)$. If $\{p(x, \xi, t)\}_{t \in [0,T]}$ is a smooth family of symbols in $S^0(\mathbb{R}^n)$ with the seminorms (2-4) bounded uniformly in $t \in [0, T]$ for each fixed α, β, then $\|p(x, D, t)\|_{L^2}$ is bounded independently of $t \in [0, T]$.

We will also use a simple version of the sharp Gårding inequality: if $P \in \mathrm{OPS}^1(\mathbb{R}^n)$ and $p(x, \xi)$ is real-valued and nonnegative for $x \in \mathbb{R}^n$ and ξ outside a compact subset of $\mathbb{R}^n$, there is a constant C such that

$$(\varphi, P(x, D)\varphi) \ge -C\|\varphi\|^2 \tag{2-5}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. If $p(x, \xi, t)$ is a smooth family of symbols in $S^0(\mathbb{R}^n)$ such that

(i) the seminorms (2-4) are bounded uniformly in $t \in [0, T]$ for each fixed α, β, and

(ii) $p(x, \xi, t)$ is real-valued and nonnegative for $x \in \mathbb{R}^n$ and ξ outside a fixed compact subset of $\mathbb{R}^n$, independent of $t \in [0, T]$.

Then the lower bound (2-5) holds for a C independent of $t \in [0, T]$.

3. Scattering maps and an oscillatory $\bar\partial$-problem

First, we recall that the Davey–Stewartson scattering maps $\mathcal{R}$ and $\mathcal{I}$ are both defined by $\bar\partial$-problems; see [Perry 2011] for discussion. The inverse scattering method for the Davey–Stewartson II equation was developed by Ablowitz and Fokas [1983; 1984] and Beals and Coifman [1984; 1985; 1989; 1990]. Sung

[1994a; 1994b; 1994c] and Brown [2001] carried out detailed analytical studies of the direct and inverse scattering maps.

For a complex parameter k and for $z = x_1 + ix_2$, let

$$e_k = e^{\bar{k}\bar{z}-kz}.$$

Given $u \in H^{1,1}(\mathbb{R}^2)$ and $k \in \mathbb{C}$, there exists a unique bounded continuous solution of

$$\begin{aligned}\bar\partial\mu_1 &= \tfrac{1}{2}e_k u\overline{\mu_2},\\ \bar\partial\mu_2 &= \tfrac{1}{2}e_k u\overline{\mu_1},\\ \lim_{|z|\to\infty}(\mu_1(z,k),\,\mu_2(z,k)) &= (1,0).\end{aligned} \tag{3-1}$$

We then define $r = \mathcal{R}u$ by

$$r(k) = \frac{1}{\pi}\int e_k(z)u(z)\overline{\mu_1(zk)}\,dA(z). \tag{3-2}$$

On the other hand, it can be shown that

$$\nu_1 = \mu_1 \quad\text{and}\quad \nu_2 = e_k\overline{\mu_2} \tag{3-3}$$

solve a $\bar\partial$-problem in the k variable:

$$\begin{aligned}\bar\partial_k\nu_1 &= \tfrac{1}{2}e_k\overline{r\nu_2},\\ \bar\partial_k\nu_2 &= \tfrac{1}{2}e_k\overline{r\nu_1},\\ \lim_{|k|\to\infty}(\nu_1(z,k),\,\nu_2(z,k)) &= (1,0),\end{aligned} \tag{3-4}$$

and that this solution is unique within the space of bounded continuous functions. Given $r \in H^{1,1}(\mathbb{R}^2)$, we solve the $\bar\partial$-system (3-4) and define $u = \mathcal{I}r$ by

$$u(z) = \frac{1}{\pi}\int e_{-k}(z)r(k)\nu_1(z,k)\,dA(k). \tag{3-5}$$

Theorem 3.1 [Perry 2011]. *The maps $\mathcal{R}$ and $\mathcal{I}$, initially defined on $\mathcal{S}(\mathbb{R}^2)$, extend to LLCM's from $H^{1,1}(\mathbb{R}^2)$ to itself. Moreover $\mathcal{R}\circ\mathcal{I} = \mathcal{I}\circ\mathcal{R} = I$, where I denotes the identity map on $H^{1,1}(\mathbb{R}^2)$.*

In what follows, we will study the restriction of the maps $\mathcal{R}$ and $\mathcal{I}$ respectively to $H^{2,1}(\mathbb{R}^2)$ and $H^{1,2}(\mathbb{R}^2)$, and obtain refined continuity results. To do so, we first describe three basic tools used in [Perry 2011] to analyze the generic system

$$\begin{aligned}\bar\partial w_1 &= \tfrac{1}{2}e_k u\overline{w_2},\\ \partial w_2 &= \tfrac{1}{2}e_k u\overline{w_1},\\ \lim_{|z|\to\infty}(w_1(z,k),\,w_2(z,k)) &= (1,0),\end{aligned} \tag{3-6}$$

for unknown functions $w_1(z,k)$ and $w_2(z,k)$, where k is a complex parameter, and $u \in H^{1,1}(\mathbb{R}^2)$. We refer the reader to [Perry 2011] for the proofs. We don't state the obvious analogues of the facts below when the roles of k and z are reversed, but use them freely in what follows.

1. *Finite L^p-expansions.* In [Perry 2011] it is shown that the system (3-6) has a unique solution in L^∞_z. This result, and further analysis of the solution, is a consequence of the following facts, which we recall from Section 3 of the same reference. Let T be the antilinear operator

$$T\psi = \tfrac{1}{2} P e_k u \overline{\psi},$$

which is a bounded operator from L^p to itself for $p \in (2, \infty]$ if $u \in H^{1,1}$ by Lemma 2.1(i). The system (3-6) is equivalent to the integral equation

$$w_1 = 1 + T^2 w_1$$

and the auxiliary formula $w_2 = T w_1$. The operator $I - T^2$ has trivial kernel as a map from $L^p(\mathbb{R}^2)$ to itself for any $p \in (2, \infty]$, and the estimate

$$\|T^2\|_{L^p \to L^p} \le C_p \|u\|^2_{H^{1,1}} (1 + |k|)^{-1}$$

holds for any $p \in (2, \infty)$. For any $p \in (2, \infty)$, the resolvent $(I - T^2)^{-1}$ is bounded uniformly in $k \in \mathbb{C}$ and u in bounded subsets of $H^{1,1}$ as an operator from L^p to itself. Note that if $u \in H^{1,1}$, the expression $T1 = \frac{1}{2} P e_k u$ is a well-defined element of L^p for all $p \in (2, \infty]$. The unique solution of (3-6) is given by

$$w_1 - 1 = (I - T^2)^{-1} T^2 1, \quad w_2 = T w_1.$$

From these facts, one has (see [Perry 2011, Section 3]):

Lemma 3.2 (finite L^p-expansions). *For any positive integer N, the expansions*

$$w_1 - 1 = \sum_{j=1}^{N} T^{2j} 1 + R_{1,N} \quad \text{and} \quad w_2 = \sum_{j=1}^{N} T^{2j-1} 1 + R_{2,N}$$

hold, where the maps

$$u \mapsto (1 + |\diamond|)^N R_{1,N}(\,\cdot\,, \diamond), \quad u \mapsto (1 + |\diamond|)^N R_{2,N}(\,\cdot\,, \diamond)$$

are LLCMs from $H^{1,1}(\mathbb{R}^2)$ into $L^\infty_k(L^p_z)$.

2. *Multilinear estimates.* Substituting the expansions into the representation formulas (3-5) and (3-2) leads to expressions of the form

$$\langle e_* w, F_j \rangle,$$

where e_* denotes e_k or e_{-k}, w is a monomial in u and its derivatives, and F_j denotes $T^{2j} 1$ or $\overline{T^{2j} 1}$ for $j \ge 1$. We assume that w is bounded in L^2 norm by a power of $\|u\|_{H^{2,1}}$. The following fact is an immediate consequence of Remark 2.5.

Lemma 3.3. *The map $u \mapsto \langle e_* w, F_j \rangle$ is an LLCM from $H^{2,1}(\mathbb{R}^2)$ to $L^2_k(\mathbb{R}^2)$.*

3. *Large-parameter expansions.* Finally, the following large-z finite expansions for w_1 and w_2 will be useful. We omit the straightforward computational proof.

Lemma 3.4. *For $u \in H^{1,1}(\mathbb{R}^2)$,*

$$w_1(z,k)-1=\frac{1}{2\pi z}\int e_k(z')u(z')\overline{w_2(z',k)}\,dm(z')+\frac{1}{2\pi z}\int \frac{e_k(z')}{z-z'}z'u(z')\overline{w_2(z',k)}\,dm(z'),$$

and similarly

$$w_2(z,k)=\frac{1}{2\pi z}\int e_k(z')u(z')\overline{w_1(z',k)}\,dm(z')+\frac{1}{2\pi z}\int \frac{e_k(z')}{z-z'}z'u(z')\overline{w_1(z',k)}\,dm(z').$$

Analogous expansions hold for the $\bar\partial$-problem in the k variables.

4. Restrictions of scattering maps

In this section we prove Theorem 1.4. By virtue of Theorem 3.1, it suffices to show that the maps $H^{2,1} \ni u \mapsto |\diamond|^2 r(\diamond)$ and $H^{1,2} \ni r \mapsto \Delta u \in L^2$ are LLCMs. First, we prove:

Lemma 4.1. *The map $u \mapsto |\diamond|^2 r(\diamond)$ is an LLCM from $H^{2,1}(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.*

Proof. We carry out all computations on $u \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and extend by density to $H^{2,1}(\mathbb{R}^2)$. Note that $\|u\|_p \le C_p\|u\|_{H^{2,1}}$ for all $p \in (1,\infty)$ and $\|\partial u\|_p \le C_p\|u\|_{H^{2,1}}$ for $p \in [2,\infty)$. An integration by parts using (3-2) and the identity $\partial e_k = -k e_k$ shows that (up to trivial factors)

$$\begin{aligned}|k|^2 r(k) &= -\bar k\int e_k(\partial u)-\bar k\int e_k(\partial u)(\overline{\mu_1}-1)-\frac{\bar k}{2}\int |u|^2\mu_2\\ &= I_1+I_2+I_3,\end{aligned}$$

where in the last term we used

$$\bar\partial\mu_1=\tfrac12 e_k u\overline{\mu_2}. \tag{4-1}$$

I_1: This term is the Fourier transform of $\partial\bar\partial u$ and hence defines a linear map from $H^{2,1}$ to L^2_k.

I_2: An integration by parts using (3-2), the identity $\partial(e_k)=-ke_k$, and (4-1) again shows that

$$\begin{aligned}I_2 &= \frac{\bar k}{k}\left(\int e_k(\partial^2 u)(\overline{\mu_1}-1)+\frac12\int \bar u\,\partial u\,\mu_2\right)\\ &= I_{21}+I_{22}.\end{aligned}$$

In I_{21} we insert $1=\chi+(1-\chi)$, where $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^2)$ satisfies $0\le\chi(z)\le1$, $\chi(z)=1$ for $|z|\le1$, and $\chi(z)=0$ for $|z|\ge2$. Drop the unimodular factor $\bar k/k$ and write $I_{21}=I_{21}^{\text{in}}+I_{21}^{\text{out}}$ corresponding to this decomposition. Since $\chi\partial^2 u\in L^{p'}$ for any $p>2$, we may use Lemma 3.2 to get the expansion

$$I_{21}^{\text{in}}=\sum_{j=1}^{N}\int e_k(\partial^2 u)\chi(\overline{T^{2j}1})+\int e_k(\partial^2 u)\chi\overline{(I-T^2)^{-1}T^{2j+2}1}.$$

By Lemmas 3.2 and 3.3 and the fact that $\chi\partial^2 u\in L^{p'}$, each right-hand term defines an LLCM from $H^{2,1}$ to L^2_k, hence $u\mapsto I_{21}^{\text{rm}}$ is an LLCM. In I_{21}^{out}, we use Lemma 3.4 to write

$$\begin{aligned}&\int e_k(1-\chi)\partial^2 u(\overline{\mu_1}-1)\\&\qquad=-\frac{1}{2\pi}\left(\int e_k(1-\chi)(\partial^2 u)z^{-1}\right)\left(\int e_{-k}\bar{u}\mu_2\right)+\tfrac{1}{2}\big\langle e_{-k}(1-\chi)\overline{(\partial^2 u)z^{-1}},\,Pe_{-k}u_1(T\mu_1)\big\rangle.\end{aligned}\tag{4-2}$$

The first term on the second line of (4-2) is the product of the Fourier transform of the L^2-function $(1-\chi(z))(\partial^2 u)(z)z^{-1}$ and the function $\int e_{-k}\bar{u}\mu_2$. Since $u\in L^{p'}$ for all $p>2$ while $u\mapsto\mu_2$ is an LLCM from $H^{1,1}$ to $L^\infty_k(L^p_z)$, the map $u\mapsto\int e_{-k}\bar{u}\mu_2$ is an LLCM from $H^{2,1}$ to L^∞_k, so the first right-hand term in (4-2) defines an LLCM from $H^{2,1}$ to L^2_k. The second right-hand term in (4-2) may be controlled using Lemmas 3.2 and 3.3. This shows that $u\mapsto I_{21}^{\text{out}}$, and hence $u\mapsto I_{21}$, defines an LLCM from $H^{2,1}$ to L^2_k. Finally, to control I_{22}, we note that $\bar{u}\partial u\in L^{p'}$ for $p>2$. Hence, using Lemma 3.2 we obtain

$$I_{22}=\sum_{j=0}^{N}\int\bar{u}\partial u\,T^{2j+1}1+\int(\bar{u}\partial u)(I-T^2)T^{2j+1}1.\tag{4-3}$$

To control terms in the finite sum in (4-3), we write

$$\begin{aligned}\int\bar{u}\partial u\,T^{2j+1}1&=\big\langle u\partial\bar{u},\,P[e_k u(\overline{T^{2j}1})]\big\rangle\\&=-\big\langle e_{-k}\bar{u}\,\overline{P}(u\partial\bar{u}),\,\overline{T^{2j}1}\big\rangle.\end{aligned}$$

and apply Lemma 3.3 since $\|u\,\overline{P}(u\partial\bar{u})\|_2'\le C\|u\|_{H^{2,1}}$. The second right-hand term in (4-3) defines an LLCM from $H^{2,1}$ to L^2_k by Lemma 3.2. Hence, $u\mapsto I_2$ is a LLCM from $H^{2,1}$ to L^2_k.

I_3: Note that $|u|^2\in L^{p'}$ for all $p>2$ and use the expansion of μ_2 to write I_3 as

$$\sum_{j=1}^{N}-\frac{\bar{k}}{2}\int|u|^2T^{2j+1}1-\frac{\bar{k}}{2}\int|u|^2(I-T^2)^{-1}T^{2N+3}1.$$

The remainder is an LLCM from $H^{2,1}$ to L^2_k by Lemma 3.2. A given term in the finite sum is written (up to constant factors)

$$\begin{aligned}\bar{k}\big\langle|u|^2,\,P[e_k u(\overline{T^{2j}1})]\big\rangle&=\bar{k}\big\langle e_{-k}\bar{u}\,\overline{P}(|u|^2),\,\overline{T^{2j}1}\big\rangle\\&=-\big\langle\bar\partial(e_{-k}\bar{u}\,\overline{P}(|u|^2)),\,\overline{T^{2j}1}\big\rangle+\big\langle e_{-k}\bar\partial(\bar{u}\,\overline{P}(|u|^2)),\,\overline{T^{2j}1}\big\rangle,\end{aligned}\tag{4-4}$$

where we integrated by parts to remove the factor of $\bar{k}$. The first term on the second line of (4-4) is

$$\begin{aligned}\big\langle e_{-k}\bar{u}\,\overline{P}(|u|^2),\,\partial(\overline{T^{2j}1})\big\rangle&=\big\langle e_{-k}\bar{u}\,\overline{P}(|u|^2),\,e_{-k}\bar{u}P(e_k u\overline{T^{2j-2}1})\big\rangle\\&=\big\langle e_{-k}\bar{u}P(|u|^2P(|u|^2)),\,\overline{T^{2j-2}1}\big\rangle,\end{aligned}$$

which defines an LLCM from $H^{2,1}$ to L^2_k by Lemma 3.3 since $\bar{u}P(|u|^2P(|u|^2))\in L^2$. The second right-hand term is treated similarly. Hence $u\mapsto I_3$ is an LLCM from $H^{2,1}$ to L^2_k.

Collecting these results, we conclude that $u\mapsto|\diamond|^2r(\diamond)$ is an LLCM from $H^{2,1}$ to L^2_k. ☐

Lemma 4.2. *The map $r\mapsto\Delta u$ is an LLCM from $H^{2,1}(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.*

Proof. Since $r \in H^{1,2}$ we have $kr(k) \in L^p$ for all $p \in (1,2]$, $r \in L^p$ for all $p \in [1,\infty)$ and $\partial r \in L^p$ for all $p \in [2,\infty)$. A straightforward computation shows that

$$\begin{aligned}\partial\bar\partial u &= \int |k|^2 e_{-k} r + \int |k|^2 e_{-k} r(\nu_1 - 1) - \int \bar k e_{-k} r \partial\nu_1 + \int k e_{-k} r \bar\partial \nu_1 + \int e_{-k} r \partial\bar\partial\nu_1 \\ &= I_1 + I_2 + I_3 + I_4 + I_5,\end{aligned}$$

where all derivatives are taken with respect to z. We now show that each of I_1–I_5 defines a locally Lipschitz continuous map from $H^{2,1} \ni r$ into L^2_z.

I_1: This term is the Fourier transform of $\partial\bar\partial r$ and hence L^2.

I_2: Inserting $1 = \chi + (1-\chi)$ in I_2, where χ is as in the proof of Lemma 4.1 (except that, here, χ is a function of k, not z), we have $I_2 = I_{21} + I_{22}$, where

$$I_{21} = \int e_{-k} |k|^2 \chi r(\nu_1 - 1), \quad I_{22} = \int e_{-k}|k|^2 r(1-\chi)(\nu_1 - 1).$$

We will show that I_{21} and I_{22} are both LLCMs from $H^{1,2}$ to L^2_z. Since $|k|^2\chi r \in L^{p'}$ for any $p > 2$, we can use Lemma 3.2 for $\nu_1 - 1$ together with Lemma 3.3 to conclude that $r \mapsto I_{21}$ is an LLCM from $H^{1,2}$ to L^2_z. For I_{22} we use the one-step large-k expansion of $\nu_1 - 1$ (Lemma 3.4):

$$\nu_1(z,k) - 1 = -\frac{1}{2\pi k}\int e_{k'}(z)\overline{r(k')}\,\overline{\nu_2(z,k')}\,dm(k') - \frac{1}{2\pi k}\int \frac{e_{k'}(z)}{k-k'} k' r(\overline{k'})\,\overline{\nu_2(z,k')}\,dm(k').$$

We then have

$$I_{22} = \int e_{-k}\bar k r(1-\chi)(F_1 + F_2),$$

where

$$\begin{aligned} F_1(z) &= -\frac{1}{2\pi}\int e_{k'}\overline{r(k')}\,\overline{\nu_2(z,k')}\,dm(k'), \\ F_2(z,k) &= -\frac{1}{2\pi}\int \frac{e_{k'}(z)}{k-k'} k' r(\overline{k'})\,\overline{\nu_2(z,k')}\,dm(k'). \end{aligned}$$

It is easy to see that $\|F_1\|_{L^\infty_z} \le \|r\|_1 \|\nu_2\|_\infty$, so that $r \mapsto F_1$ is an LLCM from $H^{1,2}$ to L^∞_z. Moreover, $\int e_{-k}\bar k r(1-\chi)$ is the inverse Fourier transform of the L^2 function $(\bar\diamond) r(\diamond)(1-\chi(\diamond))$. Hence, the map $r \mapsto \int e_{-k}\bar k r(1-\chi)F_1$ is an LLCM from $H^{1,2}$ to L^2_z. Next, we may use Lemma 3.2 in F_2 to conclude that

$$F_2 = -\frac{1}{2}\sum_{j=1}^{N} P_k\big(e_k k\bar r\,\overline{T^{2j+1}1}\big) - \frac{1}{2} P_k\big(e_k k \bar r\,\overline{((I-T^2)^{-1}T^{2N+3}1)}\big). \tag{4-5}$$

The corresponding contributions to I_{22} from terms in the finite sum from (4-5) define LLCMs from $H^{1,2}$ to L^2_z by Lemma 3.3, while by the remainder estimate in Lemma 3.2, the mapping

$$r \mapsto P e_k k \bar r (I - T^2)^{-1} T^{2N+3} 1$$

is an LLCM from $H^{1,2}$ to $L^2_z(L^p_k)$ for $p > 2$. Using these estimates we may conclude that

$$r \mapsto \int e_{-k}\bar{k}r(1-\chi)F_2$$

is an LLCM from $H^{1,2}$ to L^2_z.

I_3: Since $\mu_1 = \nu_1$, we conclude from (4-1) and (3-3) that

$$\bar{\partial}_z \nu_1 = \tfrac{1}{2} e_k u \overline{\mu_2} = \tfrac{1}{2} u \nu_2, \tag{4-6}$$

so that

$$I_3 = -\int \bar{k} e_{-k} r (\partial \bar{\partial}^{-1})(\bar{\partial}\nu_1) = -\frac{1}{2}\int \bar{k} e_{-k} r (\partial\bar{\partial}^{-1})(u\nu_2).$$

Proceeding as in the analysis of I_{22} in Lemma 4.1, we use the one-step large-k expansion (Lemma 3.4) to obtain

$$\begin{aligned}\nu_2(z,k) &= -\frac{1}{2\pi k}\int e_{k'}(z)\overline{r(k')}\,\overline{\nu_2(z,k')}\,dm(k') - \frac{1}{2\pi k}\int \frac{e_{k'}(z)}{k-k'} k' r(\overline{k'})\,\overline{\nu_2(z,k')}\,dm(k') \\ &= F_1 + F_2.\end{aligned}$$

Hence, up to trivial factors,

$$I_3 = \int e_{-k} r (\partial\bar{\partial}^{-1})[u(F_1+F_2)].$$

By Minkowski's inequality,

$$\|I_3\|_{L^2_z} \le \frac{1}{2}\int |r| \left\| \partial\bar{\partial}^{-1}(u(F_1+F_2))\right\|_{L^2_z}.$$

Observe that $\|\partial\bar{\partial}^{-1}(uF_1)\|_{L^2_z} \le C\|uF_1\|_{L^2_z}$, while

$$\|\partial\bar{\partial}^{-1}(uF_2)\|_{L^p_k(L^2_z)} \le C_p\|u\|_2\|F_2\|_{L^p_k(L^\infty_z)} \le C_p\|u\|_2\|(\diamond)r(\diamond)\|_{2p/(p+2)}\|\nu_2\|_\infty$$

(where $\|\nu_2\|_\infty$ means $\|\nu_2\|_{L^\infty(\mathbb{R}^2_z\times\mathbb{R}^2_k)}$), so that altogether

$$\|I_3\|_{L^2_z} \le C\|u\|_2\|r\|_{H^{1,2}}(1+\|\nu_2\|_\infty).$$

Thus $I_3 \in L^2_z$. The local Lipschitz continuity of I_3 follows from that of $r \mapsto u$ and $r \mapsto \nu_2$.

I_4: Using (4-6) again, we compute

$$\int k e_{-k} r \bar{\partial}\nu_1 = \frac{u}{2}\int e_{-k} k r \nu_2,$$

so it suffices to show that $r \mapsto \int e_{-k} k r \nu_2$ is an LLCM from $H^{1,2}$ to L^∞_z. Since $kr \in L^{p'}$ for $p > 2$, and $r \mapsto \nu_2$ is an LLCM from $H^{1,1}$ to L^∞, the result follows.

I_5: Compute

$$I_5 = \int e_{-k} r \partial(u\nu_2) = \partial u \int e_{-k} r \nu_2 + u \int e_{-k} r(\partial \nu_2). \tag{4-7}$$

The first right-hand term in (4-7) defines an LLCM from $H^{1,2}$ to L^2_z since $r \mapsto \partial u$ has this property. Thus, to control the first right-hand term, it suffices to show that $r \mapsto \int e_{-k} r \nu_2$ defines an LLCM from $H^{1,2}$ to L^∞_z. To see this, note that $r \in L^{p'}$ for $p > 2$, and $r \mapsto \nu_2$ is an LLCM from $H^{1,1}$ to $L^\infty_z(L^p_z)$. To control the second right-hand term in (4-7), recall that $\nu_2 = e_k \overline{\mu_2}$, so that the second term is written

$$-u \int k r e_k \overline{\nu_2} + \frac{|u|^2}{2} \int e_{-k} r \nu_1. \tag{4-8}$$

Since u and $|u|^2$ belong to L^2 it is enough to show that the two integrals in (4-8) define LLCMs from $r \in H^{2,1}$ to L^∞_z. Since $kr \in L^{p'}$ for $p > 2$ and ν_2 is an LLCM from $H^{1,2}$ to $L^\infty_z(L^p_k)$, the first term in (4-8) clearly has this property. Since $r \in L^1$ and ν_1 is an LLCM from $r \in H^{2,1}$ to $L^\infty_z(L^\infty_k)$, we conclude that the second term also has this property. □

5. Solving the mNV equation

In this section we prove Theorem 1.5. Recall that the modified Novikov–Veselov (mNV) equation [Bogdanov 1987] is

$$u_t + (\partial^3 + \bar\partial^3)u - NL(u) = 0, \tag{5-1}$$

where

$$NL(u) = \tfrac{3}{4}(\partial \bar u) \cdot (\bar\partial \partial^{-1}(|u|^2)) + \tfrac{3}{4}(\bar\partial u) \cdot (\bar\partial \partial^{-1}(|u|^2)) + \tfrac{3}{4}\bar u \bar\partial \partial^{-1}(\bar u \bar\partial u) + \tfrac{3}{4} u \partial^{-1}(\bar\partial(\bar u \bar\partial u)).$$

By Theorem A, for $u_0 \in \mathcal{S}(\mathbb{R}^2)$, the formula

$$u(z,t) = \mathcal{I}\big(\exp((\bar\diamond^3 - \diamond^3)t)\mathcal{R}u_0(\diamond)\big)(z) \tag{5-2}$$

gives a classical solution of the mNV equation.

Proposition 5.1. *Suppose that $u_0 \in H^{2,1}(\mathbb{R}^2)$. Then* (5-2) *defines a weak solution of the mNV equation in the sense of* (1-12) *with* $\lim_{t\to 0} u(t) = u_0$ *in* $L^2(\mathbb{R}^2)$.

Proof. Let $r_0 = \mathcal{R}u_0$. By continuity of the maps $\mathcal{R}$, $r_0 \mapsto \exp((\bar\diamond^3 - \diamond^3)t) r_0(\diamond)$, and $\mathcal{I}$, the formula (5-2) extends to $u_0 \in H^{2,1}$, and exhibits the solution as a continuous curve in $H^{2,1}$ that depends continuously on the initial data. Since, for any $u_0 \in \mathcal{S}(\mathbb{R}^2)$, the function u given by (5-2) is a classical solution, it follows that u trivially satisfies (1-12). The same fact for $u(t)$ with $u_0 \in H^{2,1}$ follows from the density of $\mathcal{S}(\mathbb{R}^2)$ in $H^{2,1}$, the continuity of the map (5-2) in u_0, and an easy approximation argument. □

It remains to show:

Proposition 5.2. *Suppose that $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and that, also,*

$$\int u_0 \, dA(z) = 0, \qquad \partial u_0 = \overline{\partial \mu_0}. \tag{5-3}$$

Define $u(t)$ by (5-2). *Then*

$$\partial u = \overline{\partial u}, \tag{5-4}$$

for all t.

We will prove Proposition 5.2 by first showing that the relation (5-4) holds for initial data $u_0 \in S(\mathbb{R}^2)$ with the stated properties. We will then use Lipschitz continuity of the map $u_0 \to u(t)$ defined by (5-2) to extend to all $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ so that the conditions (5-3) hold.

First, we consider $u_0 \in \mathcal{S}(\mathbb{R}^2)$. It will be useful to consider the function

$$\varphi = \bar{\partial}^{-1} u,$$

which solves the Cauchy problem

$$\begin{aligned} \varphi_t &= -\partial^3\varphi - \bar{\partial}^3\varphi - \tfrac{1}{4}(\partial\varphi)^3 - \tfrac{1}{4}(\bar{\partial}\varphi)^3 + \tfrac{3}{4}\partial\varphi \cdot \bar{\partial}^{-1}\partial(|\partial\varphi|^2) + \tfrac{3}{4}\bar{\partial}\varphi \cdot \bar{\partial}^{-1}\partial(|\partial\varphi|^2), \qquad (5\text{-}5)\\ \varphi|_{t=0} &= \varphi_0. \end{aligned}$$

The condition $\partial u_0 = \overline{\partial \mu_0}$ implies that φ_0 is real. On the other hand, to show that $\partial u = \overline{\partial u}$, it suffices to show that φ is real for $t > 0$. To this end, we consider the function

$$w = \varphi - \bar{\varphi},$$

and derive a linear Cauchy problem satisfied by w. We will need to know that w is L^2 in the space variables.

Lemma 5.3. *Suppose that $u_0 \in \mathcal{S}(\mathbb{R}^2)$, that $u(t)$ solves the mNV equation, and $\varphi(z,t) = (\bar{\partial}^{-1}u)(t)$. Then for each t,*

$$\varphi(z,t) = \frac{c_0}{z} + \mathcal{O}_t(|z|^{-2}),$$

where $c_0 = \int u(z,t)\, dm(z)$ is independent of t. If $c_0 = 0$, then $\varphi(\,\cdot\,,t) \in L^2(\mathbb{R}^2)$ for all $t > 0$.

Proof. To see that φ has the stated form if $u_0 \in \mathcal{S}(\mathbb{R}^2)$, we note that $u(t) \in \mathcal{S}(\mathbb{R}^2)$ by the mapping properties of the scattering transform (see [Sung 1994a; 1994b; 1994c]) and that

$$\varphi(z,t) = -\frac{1}{\pi z}\int u(z,t)\, dt + \mathcal{O}_t(|z|^{-2})$$

differentiably in z, t. Let $c_0(t) = \int u(z,t)\, dm(z)$. Substituting in (5-5) we easily conclude that $c_0'(t) = 0$. It now follows that $\varphi(\diamond, t) \in L^2(\mathbb{R}^2)$ for each t as claimed. ☐

Next, we derive a linear Cauchy problem obeyed by w and show that, if $w|_{t=0} = 0$, then $w(t) = 0$ identically. If so, it follows that φ is real, and hence $\partial u = \overline{\partial u}$ for all $t > 0$.

Using (5-5) and its complex conjugate, we see that

$$w_t = Lw, \tag{5-6}$$

where

$$Lw = L_0 + A\partial w + \overline{A\partial w}$$

with

$$L_0 w = -\partial^3 w - \bar{\partial}^3 w$$

and

$$A = \tfrac{1}{4}[(\partial\varphi)^2 + (\partial\varphi)\cdot(\partial\bar{\varphi}) + (\partial\bar{\varphi})^2] + \tfrac{3}{4}\bar{\partial}^{-1}\partial(|\partial\varphi|^2). \tag{5-7}$$

We will need the following property of A. We say that $g(z)$ is *integrable along lines* if $\int_{-\infty}^{\infty} |g(\gamma(t))|\,dt$ is finite for any path $\gamma(t) = z_0 + z_1 t$. We say that g is *uniformly integrable along lines* if

$$\sup_{\substack{z_0 \in \mathbb{C} \\ |z_1|=1}} \int |g(\gamma(t))|\,dt < \infty.$$

Lemma 5.4. *Suppose that* $\varphi = \bar{\partial}^{-1} u$, *where* $u \in C([0, T]; \mathcal{S}(\mathbb{R}^2))$ *and*

$$\int u(z, t)\,dm(z) = 0$$

for all t. *Then, the function* $A(z, t)$ *is uniformly integrable along lines in* $\mathbb{R}^2$, *with estimates uniform in* $t \in [0, T]$.

Proof. Recall that if $f \in H^s(\mathbb{R}^2)$ then the restriction of f to a line belongs to $H^{s-1/2-\varepsilon}(\mathbb{R}^2)$ for any $\varepsilon > 0$. In particular, if $f \in H^1(\mathbb{R}^2)$, then f is square-integrable along lines. Note that $\partial\varphi = \partial\bar{\partial}^{-1}u$ and $\partial\bar{\varphi} = \bar{u}$ belong to $H^s(\mathbb{R}^2)$ for all $s > 0$ and each fixed $t \in [0, T]$ since $\partial\bar{\partial}^{-1}$ is a Fourier multiplier on H^s and $u \in H^s(\mathbb{R}^2)$ for all such s, uniformly in $t \in [0, T]$. In particular, $\partial\varphi$ and $\partial\bar{\varphi}$ restrict to square-integrable functions along lines in $\mathbb{R}^2$, so the first three terms in (5-7) are all integrable along lines with estimates bounded seminorms of u.

To handle the last term in (5-7), we note that $\partial\phi = \partial\bar{\partial}^{-1}u$. Hence, by Lemma 2.3 and the fact that differentiation commutes with the Beurling transform, we conclude that

$$\sup_{|\alpha| \le 2} |D^\alpha(|\partial\phi|^2)| \le C(1 + |z|)^{-4}.$$

It now follows from Lemma 2.3 that again $\bar{\partial}^{-1}\partial(|\partial\phi|)^2$ is $\mathcal{O}(|z|^{2-\varepsilon})$ for any $\varepsilon > 0$, and hence is integrable along lines with appropriate uniform estimates. □

We wish to prove an a priori estimate for the problem (5-6) that bounds $\|w(t)\|$ in terms of $\|w(0)\|$, proving uniqueness of the initial value problem. A formal computation of $\frac{d}{dt}\|w(t)\|^2$ leads to uncontrolled derivatives since the principal part of L is skew-adjoint. Instead, following the multiplier method of [Chihara 2004] (applied to third-order dispersive nonlinear equations; see [Doi 1994] for a similar pseudodifferential multiplier method applied to Schrödinger-type equations), we find a family of invertible pseudodifferential operators $K(t)$ such that

(1) $\|K(t)w(t)\|$ controls $\|w(t)\|$, and

(2) $\dfrac{d}{dt}\|K(t)w(t)\|^2$ is bounded above.

A formal computation shows that

$$\frac{d}{dt}\|K(t)w(t)\|^2 = \big(K(t)w(t), C(t)K(t)w(t)\big), \tag{5-8}$$

where

$$\begin{aligned} C(t) &= 2\,\mathrm{Re}\big\{K'(t)K(t)^{-1} + K(t)L(t)K(t)^{-1}\big\} \\ &= 2\,\mathrm{Re}\big\{K'(t)K(t)^{-1} + K(t)(A\partial + \overline{A\partial})K(t)^{-1} + [K(t), L_0]K(t)^{-1}\big\}. \end{aligned} \tag{5-9}$$

We will choose $K(t)$ so that $C(t)$ is the sum of a negative definite operator and a bounded operator.

The following lemma obtains the desired estimate. Note that Lemma 5.4 implies the existence of a function $\eta(z,t)$ satisfying the hypotheses of Lemma 5.5 if A is given by (5-7).

Lemma 5.5. *Suppose that $A(z,t)$ is a bounded smooth function on $\mathbb{R}^2 \times [0,T]$ and that $\eta(z,t)$ is a bounded smooth nonnegative function with $|A(z,t)| \le \eta(z,t)$ for $z \in \mathbb{C}$ and $t \in [0,T]$. Writing $\eta(z,t) = \eta(x_1,x_2,t)$, suppose that there is a constant c such that $\int |\eta(y,x_2,t)|\,dy \le c$ and $\int |\eta(x_1,y,t)|\,dy \le c$ uniformly in $(x_1,x_2) \in \mathbb{R}^2$ and $t \in [0,T]$. Finally, let w be a smooth solution of* (5-6) *with $w(\diamond,t) \in L^2(\mathbb{R}^2)$ for each $t > 0$. Then, there is a constant C such that*

$$\sup_{t\in[0,T]} \|w(t)\| \le e^{CT}\|w(0)\|.$$

Proof. Let η be a function with

$$2|A(z,t)| \le \eta(z,t),$$

and set

$$p_0(\xi) = \frac{i}{4}(\xi_1^3 - 3\xi_1\xi_2^2),$$

the symbol of the operator $-\partial^3 - \bar\partial^3$. With $z = x_1 + ix_2$ and $\lambda > 0$ to be chosen, let

$$\begin{aligned} b(t,x,\xi) = i\left(\int_{-\infty}^{x_1} \eta(y,x_2,t)\,dy\right) \times \frac{\partial p_0(\xi)}{\partial \xi_1}\frac{|\xi|}{|\nabla p_0(\xi)|^2}\chi\left(\frac{|\xi|}{\lambda}\right) \\ + i\left(\int_{-\infty}^{x_2} \eta(x_1,y,t)\,dy\right) \times \frac{\partial p_0(\xi)}{\partial \xi_2}\frac{|\xi|}{|\nabla p_0(\xi)|^2}\chi\left(\frac{|\xi|}{\lambda}\right), \end{aligned} \tag{5-10}$$

where $\chi \in \mathcal{C}_0^\infty([0,\infty))$ is a nonnegative function with $\chi(t) = 0$ for $0 \le t < 1/2$ and $\chi(t) = 1$ for $t \ge 1$. By the usual quantization, the pseudodifferential operator $b(t,x,D)$ belongs to the class $\mathrm{OPS}^{-1}(\mathbb{R}^n)$. It is easy to see that, also, the symbols

$$k(t,x,\xi) = e^{b(t,x,\xi)} \quad \text{and} \quad \tilde{k}(t,x,\xi) = e^{-b(t,x,\xi)}$$

define pseudodifferential operators $K(t) := K(t,x,D)$ and $\widetilde{K}(t) := \widetilde{K}(t,x,D)$ in $\mathrm{OPS}^0(\mathbb{R}^n)$ with

$$K(t)\widetilde{K}(t) - I \in \mathrm{OPS}^{-1}(\mathbb{R}^n) \quad \text{and} \quad \lim_{\lambda\to\infty}\sup_{t\in[0,T]} \|K(t)\widetilde{K}(t) - I\| = 0.$$

Thus, there is a $\lambda_0 > 0$ such that $K(t)$ is invertible for all $|\lambda| \ge \lambda_0$. We take $|\lambda| \ge \lambda_0$ from now on.

We claim that, if $w(t)$ is a solution of the evolution equation (5-6) belonging to $L^2(\mathbb{R}^2)$, the inequality

$$\|K(t)w(t)\| \le \|K(0)w(0)\|e^{CT} \tag{5-11}$$

holds for $t \in [0,T]$ and a constant C. Since $K(t)$ is invertible for λ sufficiently large and $t \in [0,T]$, this implies that $w(t) = 0$ for all t if $w(0) = 0$.

To prove the inequality (5-11), we use (5-8). We will show that

$$2\,\mathrm{Re}\left\{A\partial + \overline{A\partial} + [K(t),L_0]K(t)^{-1}\right\} = -Q_1(t) + Q_2(t), \tag{5-12}$$

where $Q_1(t) \in \mathrm{OPS}^{1,0}(\mathbb{R}^2)$ with $q_1(x,\xi) := \sigma(Q_1(t))$ nonnegative for $|\xi| \ge 2\lambda$, and $Q_2(t) \in \mathrm{OPS}^0(\mathbb{R}^2)$. If so, then by the Gårding inequality (2-5),

$$\mathrm{Re}(v, Q_1(t)v) \ge -C_1\|v\|^2, \tag{5-13}$$

with C_1 uniform in $t \in [0,T]$. Hence

$$\frac{d}{dt}\|K(t)w(t)\|^2 \le C_3\|K(t)w(t)\|^2,$$

where C_3 majorizes $C_1 + \sup_{t\in[0,T]}\big(\|Q_2(t)\| + \|K'(t)K^{-1}(t)\|\big)$. The desired result now follows from Gronwall's inequality.

Thus, to finish the proof of (5-11), we need only prove that (5-12) holds. From the computation

$$\sigma([K(t), L_0]) = -\frac{1}{i}\nabla_x(e^{\gamma(t,x,\zeta)})\cdot(\nabla_\xi p_0)(\xi),$$

it follows that the left-side of (5-12) has leading symbol $-q_1(x_1,x_2,\xi,t)$ where

$$q_1(x_1,x_2,\xi,t) = \frac{1}{i}\nabla_\xi p_0(\xi)\cdot\nabla_x\gamma(t,x_1,x_2,\xi) + \mathrm{Re}[A(x_1,x_2,t)(\xi_1 - i\xi_2)],$$

which is nonnegative for $|\xi| \ge 2\lambda$ since $|A(x_1,x_2,t)| \le \eta(x_1,x_2,t)$. This completes the proof. □

Proof of Proposition 5.2. First, suppose that $u_0 \in \mathcal{S}(\mathbb{R}^2)$, $\partial u_0 = \overline{\partial u_0}$, and $\int u_0(z)\,dm(z) = 0$. The function $\varphi_0 = \bar\partial^{-1}u_0$ is real-valued and if $u(t)$ solves the mNV equation with Cauchy data u_0, the function $\varphi(t) = (\bar\partial^{-1}u)(t)$ belongs to $L^2(\mathbb{R}^2)$ for all t. The same is true of $w(t) = \varphi(t) - \overline{\varphi(t)}$, and $w(0) = 0$. It now follows from Lemma 5.5 that $w(t) = 0$ and $\varphi(t)$ is real-valued for all t. This implies that $\partial u = \overline{\partial u}$ for all t.

To conclude that the proposition holds for $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, we first observe that there is a sequence $\{v_{n,0}\}$ from $\mathcal{S}(\mathbb{R}^2)$ with $v_{n,0} \to u_0$ in $H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Let f be a nonnegative C_0^∞ function with $\int f = 1$, and let $u_{n,0} = v_{n,0} - (\int u_{n,0})f$. It is easy to see that $\int v_{n,0} = 0$ and $v_{n,0} \to u_0$ in $H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Since

$$u_n(t) := \mathcal{I}\big(\exp((\bar\diamond^3 - \diamond^3)t)\mathcal{R}u_{0,n}(\diamond)\big)$$

converges to

$$u(t) = \mathcal{I}\big(\exp((\bar\diamond^3 - \diamond^3)t)\mathcal{R}u_{0,n}(\diamond)\big)$$

in $C([0,T], H^{2,1})$, it now follows that $\partial u = \overline{\partial u}$, as claimed. □

Proof of Theorem 1.5. An immediate consequence of Propositions 5.1 and 5.2. □

6. Solving the NV equation

In this section we prove Theorem 1.6. The key observation is due to Bogdanov [1987] and can be checked by straightforward computation. Recall the Miura map $\mathcal{M}$, defined in (1-9).

Lemma 6.1. *Suppose that $u(z,t)$ is a smooth classical solution of* (5-1) *with*

$$(\partial_z u)(z,t) = \overline{(\partial_z u)(z,t)},$$

and $\int u(z,t)\,dm(z) = 0$ *for all* t. *Then, the function*

$$q(z,t) = \mathcal{M}(u(\,\cdot\,,t))(z)$$

is a smooth classical solution of (1-1).

Remark 6.2. In [Bogdanov 1987], the mNV and NV are shown to be gauge-equivalent, and the Miura map is computed from the gauge equivalence. Note that our conventions differ slightly from those of Bogdanov in order to insure that the range of the Miura map consists of real-valued functions.

Proof of Theorem 1.6. Pick $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ so that $\partial u_0 = \overline{\partial u_0}$ and $\int u_0(z)\,dm(z) = 0$. Let $\{u_{0,n}\}$ be a sequence from $\mathcal{S}(\mathbb{R}^2)$ with $u_{n,0} \to u_0$ in $H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. By local Lipschitz continuity of the scattering maps, for any $T > 0$, the sequence $\{u_n\}$ from $C([0,T]; H^{2,1}(\mathbb{R}^2))$ given by

$$u_n(z,t) = \mathcal{I}\big(e^{t((\diamond)^3 - (\bar{\diamond})^3)}(\mathcal{R}u_{0,n})(\diamond)\big)(z)$$

converges in $C([0,T]; H^{2,1}(\mathbb{R}^2))$ to

$$u(z,t) := \mathcal{I}\big(e^{t((\diamond)^3 - (\bar{\diamond})^3)}(\mathcal{R}u_0)(\diamond)\big)(z).$$

This convergence implies that $q_n(z,t) := \mathcal{M}(u_n(\diamond,t))(z)$ converges in $L^2(\mathbb{R}^2)$.

Recall (1-13). Since $q_n \to q$ in $C([0,T]; L^2(\mathbb{R}^2))$ it follows from the L^2-boundedness of $\mathcal{S} = \partial\bar{\partial}^{-1}$ that the two nonlinear terms converge in L^1; i.e., $q_n\bar{\partial}^{-1}\partial q_n \to q\bar{\partial}^{-1}\partial q$ and $q_n\partial^{-1}\bar{\partial}q_n \to q\partial^{-1}\bar{\partial}q$ in $C([0,T], L^1(\mathbb{R}^2))$. We conclude that q is a weak solution of the NV equation. □

7. Conductivity-type potentials

In this section we show that our solution of NV coincides with that of [Lassas et al. 2012] in the cases they consider, proving Theorem 1.7.

We briefly recall some of the notation and results of [Lassas et al. 2007]. Assume first that $q \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and is of conductivity type. We denote by $\psi(x,\zeta)$ the unique solution of the problem

$$\begin{gathered}(-\Delta + q)\psi = 0,\\ \lim_{|z|\to\infty}(e^{-i(x\cdot\zeta)}\psi(x,\zeta) - 1) = 0,\end{gathered} \tag{7-1}$$

where $x = (x_1, x_2)$ and $\zeta \in \mathbb{C}^2$ satisfies $\zeta \cdot \zeta = 0$. Here $a \cdot b$ denotes the Euclidean inner product without complex conjugation. Henceforth, we set $\zeta = (k, ik)$ for $k \in \mathbb{C}$, which amounts to choosing a branch of the variety $\mathcal{V} = \{\zeta \in \mathbb{C}^2 : \zeta \cdot \zeta = 0\}$. Since q is of conductivity type, it follows from Theorem 3 in [Nachman 1996] that the problem (7-1) admits a unique solution for each $k \in \mathbb{C}$. We set $z = x_1 + ix_2$ and define

$$m(z,k) = e^{-ikz}\psi(x,\zeta), \tag{7-2}$$

for $\zeta = (k, ik)$.

The direct scattering map

$$\mathcal{T} : q \to \mathbf{t} \tag{7-3}$$

is defined by

$$\mathbf{t}(k) = \int e^{i(\bar{k}\bar{z}+kz)} q(z) m(z,k)\, dm(z). \tag{7-4}$$

The inverse map

$$\mathcal{Q} : \mathbf{t} \to q \tag{7-5}$$

is defined by

$$q(z) = \frac{i}{\pi^2} \bar{\partial}_z \left(\int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} e^{-i(kz+\bar{k}\bar{z})} \overline{m(z,k)}\, dm(k) \right), \tag{7-6}$$

where $m(z,k)$ is reconstructed from t via the $\bar{\partial}$-problem

$$\bar{\partial}_k m(x,k) = \frac{\mathbf{t}(k)}{4\pi k} e^{-i(kz+\bar{k}\bar{z})}(z) \overline{m(x,k)}. \tag{7-7}$$

Let

$$\mathbf{m}_t^n(k) = \exp(-i^n(k^n + \bar{k}^n)t),$$

for an odd positive integer n. Lassas, Mueller, Siltanen and Stahel proved:

Theorem 7.1 [Lassas et al. 2007, Theorem 1.1; 2012, Theorem 4.1]. *For $q_0 \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ radial and of conductivity type, $\mathcal{Q}\mathcal{T}(q_0) = q_0$. Moreover, if*

$$q(t) := \mathcal{Q}(\mathbf{m}_t^n \mathcal{T} q_0), \tag{7-8}$$

then $q(t)$ is a continuous, real-valued potential with $q(t) \in L^p(\mathbb{R}^2)$ for $p \in (1,2)$.

They conjecture that for $n = 3$, $q(t)$ given by (7-8) solves the NV equation, provided that q_0 obeys the hypotheses of Theorem 7.1. We will prove that this is the case (for a larger class of q_0) by proving Theorem 1.7.

We will prove Theorem 1.7 in two steps. First, we show that for $u_0 \in \mathcal{S}(\mathbb{R}^2)$ with $\partial u_0 = \overline{\partial u_0}$ and $\int u_0(z)\, dm(z) = 0$, the scattering data $r = \mathcal{R}u$ is related to the scattering transform $\mathbf{t} = \mathcal{T}q$ for $q = 2\partial u + |u|^2$ by the identity

$$\mathbf{t}(k) = -2\pi i \bar{k} \overline{r(ik)}.$$

Next, we show that for $\mathbf{t}$ of the above form with $r = \mathcal{R}u$, the identity

$$(\mathcal{Q}\mathbf{t})(z) = 2(\partial u)(z) + |u(z)|^2.$$

Theorem 1.7 is an easy consequence of these two identities.

The key to both computations is the following construction of complex geometric optics solutions for the potential $q = 2\partial u + |u|^2$ from the solutions $\mu = (\mu_1, \mu_2)^T$ of (3-1). First, suppose that $\Phi = (\Phi_1, \Phi_2)^T$ is a vector-valued solution of the linear system

$$\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \Phi = \frac{1}{2} \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} \Phi. \tag{7-9}$$

A straightforward calculation shows that the function

$$\widetilde{\psi} = \Phi_1 + \Phi_2$$

solves the zero-energy Schrödinger equation

$$(-\Delta + q)\widetilde{\psi} = 0 \tag{7-10}$$

for $q = 2\partial u + |u|^2$.

Recall that matrix-valued solutions of (7-9) are related to the solutions μ of (3-1) by

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \Phi_1 \\ \overline{\Phi_2} \end{pmatrix} e^{-kz},$$

so that

$$\Phi_1 + \Phi_2 = e^{kz}\mu_1(z,k) + e^{\bar{k}\bar{z}}\,\overline{\mu_2(z,k)} \tag{7-11}$$

solves (7-10). To compute its asymptotic behavior, using $(\mu_1, \mu_2) \to (1, 0)$ as $|z| \to \infty$ we conclude that $e^{-kz}\widetilde{\psi}(z,k) \to 1$ as $|z| \to \infty$. Hence, denoting by ψ the solution of the problem (7-10) with $\zeta = (k, ik)$ for $k \in \mathbb{C}$, we have

$$\psi(z,k) = \widetilde{\psi}(z, ik) = e^{ikz}\mu_1(z, ik) + e^{-i\bar{k}\bar{z}}\,\overline{\mu_2(z, ik)}, \tag{7-12}$$

so

$$m(z,k) = \mu_1(z,k) + e^{-i(kz+\bar{k}\bar{z})}\,\overline{\mu_2(z, ik)}.$$

Lemma 7.2. *Let $u \in C_0^\infty(\mathbb{R}^2)$ with $\partial u = \overline{\partial u}$, suppose $\int u(z)\,dm(z) = 0$, and let $q = 2\partial u + |u|^2$. Then*

$$(\mathcal{T}q)(k) = -2\pi i \bar{k}\,\overline{(\mathcal{R}u)(ik)}. \tag{7-13}$$

Proof. We compute

$$\begin{aligned}
(\mathcal{T}q)(k) &= \int q(z) e^{i\bar{k}\bar{z}} \psi(z,k)\,dm(z) \\
&= \int 2(\overline{\partial u})(z) e^{i(\bar{k}\bar{z}+kz)} \mu_1(z, ik)\,dm(z) \\
&\qquad + \int 2(\partial u)(z)\overline{\mu_2(z, ik)}\,dm(z) \\
&\qquad + \int |u(z)|^2 \bigl(e^{i(\bar{k}\bar{z}+kz)}\mu_1(z, ik) + \overline{\mu_2(z, ik)}\bigr)\,dm(z) \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where in the first right-hand term we used $\partial u = \overline{\partial u}$. We can integrate by parts in each of the first two right-hand terms and use (3-1) to obtain

$$\begin{aligned}
I_1 &= -2i\bar{k} \int \overline{u(z)}\, e^{i(kz+\bar{k}\bar{z})} \mu_1(z, ik)\,dm(z) - \int |u(z)|^2\, \overline{\mu_2(z, ik)}\,dm(z), \\
I_2 &= -\int |u(z)|^2 e^{i(kz+\bar{k}\bar{z})} \mu_1(z, ik)\,dm(z).
\end{aligned}$$

Using the relation (3-2), we recover (7-13). □

Next, we analyze the inverse scattering transform $\mathcal{Q}$ defined by (1-7).

Lemma 7.3. *Let $u \in \mathcal{S}(\mathbb{R}^2)$ with $\partial u = \overline{\partial u}$, and suppose that $\int u(z)\,dm(z) = 0$. Let $r = \mathcal{R}u$ and suppose that t is given by* (7-13). *Then*

$$(\mathcal{Q}t)(z) = 2(\partial u)(z) + |u(z)|^2.$$

Proof. We compute from (1-7), (7-13), and (7-12) that

$$\begin{aligned}(\mathcal{Q}t)(z) &= \frac{2}{\pi}\bar{\partial}_z\biggl(\int \overline{r(ik)}\,e^{-i(kz+\bar{k}\bar{z})}\,\overline{\mu_1(z,ik)}\,dm(k)\biggr) + \frac{2}{\pi}\bar{\partial}_z\biggl(\int \overline{r(ik)}\,\mu_2(z,ik)\,dm(k)\biggr)\\ &= T_1 + T_2.\end{aligned}$$

Changing variables to $\zeta = ik$ in T_1 we recover

$$T_1 = \frac{2}{\pi}\bar{\partial}_z\biggl(\int \overline{r(\zeta)}\,e^{\bar{\zeta}\bar{z}-\zeta z}\,\overline{\mu_1(z,\zeta)}\,dm(\zeta)\biggr) = 2(\overline{\partial u})(z)[5pt] = 2(\partial u)(z),$$

where we have used (3-5). Using (3-1) in T_2 we have

$$T_2 = \frac{1}{\pi}\int \overline{r(ik)}\,u(z)\,e^{-i(kz+\bar{k}\bar{z})}\,\overline{\mu_1(z,ik)}\,dm(k) = \frac{1}{\pi}u(z)\int r(\zeta)\,e^{\bar{\zeta}\bar{z}-\zeta z}\,\overline{\mu_1(z,\zeta)}\,dm(\zeta)[2pt] = |u(z)|^2.$$

Combining these computations gives the desired result. □

Proof of Theorem 1.7. For u_0 satisfying the hypotheses and $q = 2\partial u_0 + |u_0|^2$, we have by Lemma 7.2 that

$$(\mathcal{T}q_0)(k) = -2\pi i\,\bar{k}\,\overline{r(ik)},$$

where $r = \mathcal{R}(u_0)$, and hence

$$e^{-it(k^3+\bar{k}^3)}(\mathcal{T}q_0)(k) = -2\pi i\,\bar{k}\,\overline{\bigl(e^{t((\bar{\diamond})^3-(\diamond)^3)}r(\diamond)\bigr)(ik)}.$$

We can now apply Lemma 7.3 to conclude that

$$\mathcal{Q}\bigl(e^{-it((\diamond)^3+(\bar{\diamond})^3)}(\mathcal{T}q_0)(\diamond)\bigr) = \mathcal{M}\mathcal{I}\bigl(e^{t((\bar{\diamond})^3-(\diamond)^3)}r(\diamond)\bigr),$$

as claimed. □

Appendix: Schwarz class inverse scattering for the mNV equation

In this appendix we develop the Schwarz class inverse theory for the mNV equation, using freely the results and notation of [Perry 2011]. Our main result is this:

Theorem A. *Suppose that $u_0 \in \mathcal{S}(\mathbb{R}^2)$, and let $\mathcal{R}$ and $\mathcal{I}$ be the scattering maps defined respectively by* (3-2) *and* (3-5). *Finally, define*

$$u(t) = \mathcal{I}\bigl(e^{t((\diamond)^3-(\bar{\diamond})^3)}(\mathcal{R}u_0)(\diamond)\bigr).$$

Then $u(t)$ is a classical solution of the modified Novikov–Veselov equation (5-1).

The proof follows the method of [Beals and Coifman 1985; 1989; 1990; Sung 1994a; 1994b; 1994c] but necessitates some long computations.

A.1. ***Scattering solutions and tangent maps.*** First we recall the solutions ν and $\tilde{\nu}$ of the $\bar{\partial}$ problem with $\bar{\partial}$-data determined by the time-dependent coefficient r and the formulas from [Perry 2011] for the tangent maps.

We recall that $\nu = (\nu_1, \nu_2)^T$ is the unique solution of the $\bar{\partial}$ problem

$$\begin{aligned} \bar{\partial}_k \nu_1 &= \tfrac{1}{2} e_k \bar{r} \overline{\nu_2}, \\ \bar{\partial}_k \nu_2 &= \tfrac{1}{2} e_k \bar{r} \overline{\nu_1}, \\ \lim_{|k|\to\infty} \nu(z,k) &= (1,0), \end{aligned} \tag{A-1}$$

where $r = \mathcal{R}(u)$. Here

$$e_k(z) = e^{\bar{k}\bar{z} - kz}.$$

The function $\nu^\# = (\nu_1^\#, \nu_2^\#)$ solves the same problem but for $u^\#(\,\cdot\,) = -\bar{u}(-\,\cdot\,)$ and $r^\# = \mathcal{R}(u^\#) = -\bar{r}$ (see [Perry 2011, Lemma B.1]). Thus

$$\begin{aligned} \bar{\partial}_k \nu_1^\# &= -\tfrac{1}{2} e_k r \, \overline{\nu_2^\#}, \\ \bar{\partial}_k \nu_2^\# &= -\tfrac{1}{2} e_k \bar{r} \, \overline{\nu_1^\#}, \\ \lim_{|k|\to\infty} \nu^\#(z,k) &= (1,0). \end{aligned} \tag{A-2}$$

The tangent map formula gives an expression for u if $u = \mathcal{R}(r)$ where r is a C^1-curve in $\mathcal{S}(\mathbb{R}^2)$. Assuming the law of evolution

$$\dot{r} = (\bar{k}^3 - k^3) r,$$

and following the calculations in Appendix B of [Perry 2011], we find that

$$u = 2i(I_1 + \overline{I_2}), \tag{A-3}$$

where

$$I_1 = \frac{1}{\pi} \int k^3 \bar{\partial}_k [\nu_2^\#(-z,k) \nu_1(z,k)] \, dm(k), \tag{A-4}$$

$$I_2 = -\frac{1}{\pi} \int k^3 \bar{\partial}_k [\nu_1^\#(-z,k) \nu_2(z,k)] \, dm(k). \tag{A-5}$$

As in [Perry 2011, Appendix B], we evaluate these integrals using the following fact: if g is a $\mathcal{C}^\infty$ function with asymptotic expansion

$$g(k,\bar{k}) \sim 1 + \sum_{\ell \ge 0} \frac{g_\ell}{k^{\ell+1}}, \tag{A-6}$$

as $|k| \to \infty$ then

$$\lim_{R\to\infty} \left(-\frac{1}{\pi} \int_{|k|\le R} k^n (\overline{\partial_k} g)(k) \, dm(k) \right) = g_n. \tag{A-7}$$

Using (A-7) we get (noting the $-$ sign in (A-5))

$$I_1 = 2[\nu_1(z,\diamond) \nu_2^\#(-z,\diamond)]_3 \quad \text{and} \quad \overline{I_2} = 2[\nu_2(z,\diamond)\nu_1^\#(-z,\diamond)]_3,$$

so that

$$\dot{u} = 2\big\{[\nu_1(z,\diamond)\nu_2^{\#}(-z,\diamond)]_3 + \overline{[\nu_2(z,\diamond)\nu_1^{\#}(-z,\diamond)_3]}\big\} \tag{A-8}$$

Here $[\diamond]_n$ denotes the coefficient of k^{-n-1} in an asymptotic expansion of the form (A-6). The formulas

$$[\nu_1(z,\diamond)\nu_2^{\#}(-z,\diamond)]_n = (\nu_n^{\#})_{21} + \sum_{j=0}^{n-1} (\nu_{n-j-1}^{\#})_{21}(\nu_j)_{11},$$

$$[\nu_2(z,\diamond)\nu_1^{\#}(-z,\diamond)]_n = (\nu_n)_{12} + \sum_{j=0}^{n-1} (\nu_{n-1-j})_{12}(\nu_j^{\#})_{22}$$

will be used in concert with the residue formulae below to obtain the equation of motion.

A.2. ***Expansion coefficients for ν.*** Following the method of Appendix C in [Perry 2011], we can compute the additional coefficients in the asymptotic expansion

$$\nu \sim (1,0) + \sum_{\ell \geq 0} k^{-(\ell+1)}\nu^{(\ell)} \tag{A-9}$$

needed to compute $\dot{u}$ from the formula (A-8). Let us set $\nu^{(\ell)} = (\nu_{1,\ell}, \nu_{2,\ell})^T$. We recall from [Perry 2011] the "initial data"

$$\nu_{1,0} = \tfrac{1}{4}\bar{\partial}^{-1}(|u|^2), \qquad \nu_{2,0} = \tfrac{1}{2}\bar{u}, \tag{A-10}$$

and the recurrence relations

$$\nu_{2,\ell} = \tfrac{1}{2}\bar{u}\nu_{1,\ell-1} - \partial\nu_{2,\ell-1}, \qquad \nu_{1,\ell} = \tfrac{1}{2}P(u\nu_{2,\ell}).$$

The following formulas are a straightforward consequence.

$\ell = 0$:

$$\nu_{1,0} = \tfrac{1}{4}\bar{\partial}^{-1}(|u|^2), \tag{A-11}$$

$$\nu_{2,0} = \tfrac{1}{2}\bar{u}. \tag{A-12}$$

$\ell = 1$:

$$\nu_{1,1} = \tfrac{1}{16}\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2)) - \tfrac{1}{4}\bar{\partial}^{-1}(u\partial\bar{u}), \tag{A-13}$$

$$\nu_{2,1} = \tfrac{1}{8}\bar{u}\bar{\partial}^{-1}(|u|^2) - \tfrac{1}{2}\partial\bar{u}. \tag{A-14}$$

$\ell = 2$:

$$\nu_{1,2} = \tfrac{1}{64}\bar{\partial}^{-1}\big(|u|^2\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\big) \tag{A-15}$$
$$\qquad - \tfrac{1}{16}\big\{\bar{\partial}^{-1}\big(u\partial(\bar{u}\bar{\partial}^{-1}(|u|^2))\big) + \bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(u\partial\bar{u}))\big\} + \tfrac{1}{4}\bar{\partial}^{-1}(u\partial^2\bar{u}),$$

$$\nu_{2,2} = \tfrac{1}{32}\bar{u}\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2)) - \tfrac{1}{8}\big\{\partial(\bar{u}\bar{\partial}^{-1}(|u|^2)) + \bar{u}\bar{\partial}^{-1}(u\partial\bar{u})\big\} + \tfrac{1}{2}\partial^2\bar{u}. \tag{A-16}$$

$\ell = 3$:

$$\begin{aligned}\nu_{2,3} &= \tfrac{1}{128}\,\bar{u}\bar{\partial}^{-1}\big(|u|^2\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\big) \\ &\quad - \tfrac{1}{32}\left\{\bar{u}\bar{\partial}^{-1}\big(u\partial(\bar{u}\bar{\partial}^{-1}(|u|^2))\big) + \bar{u}\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(u\partial\bar{u})) + \partial\big(\bar{u}\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\big)\right\} \\ &\quad + \tfrac{1}{8}\left\{\bar{u}\bar{\partial}^{-1}(u\partial^2\bar{u}) + \partial^2(\bar{u}\bar{\partial}^{-1}(|u|^2)) + \partial(\bar{u}\bar{\partial}^{-1}(u\partial\bar{u}))\right\} \\ &\quad - \tfrac{1}{2}\,\partial^3\bar{u}.\end{aligned} \tag{A-17}$$

A.3. ***Expansion coefficients for*** $\boldsymbol{\nu^\#}$**.** The solution $\nu^\#$ corresponds to the potential $-\overline{u}(-z)$. To compute the corresponding residues for $\nu^\#(-z,k)$ we therefore make the following substitutions in the formulas above:

$$\begin{gathered}\bar{\partial}^{-1} \to -\bar{\partial}^{-1}, \quad \partial \to -\partial, \\ u \to -\lambda\bar{u}, \quad \bar{u} \to -\lambda u,\end{gathered}$$

Thus the overall sign change is $(-1)^{n_u+n_\partial}$ where n_u is the number of factors of u and $\bar{u}$, while n_∂ is the number of factors of ∂ and $\bar{\partial}^{-1}$. There is also an overall factor of $(\lambda)^{n_u}$, that is, λ if n_u is odd, or 1 if n_u is even. Applying these rules we obtain:

$\ell = 0$:

$$\nu^\#_{1,0} = -\tfrac{1}{4}\bar{\partial}^{-1}(|u|^2), \tag{A-18}$$

$$\nu^\#_{2,0} = -\tfrac{1}{2}u. \tag{A-19}$$

$\ell = 1$:

$$\nu^\#_{1,1} = \tfrac{1}{16}\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2)) - \tfrac{1}{4}\bar{\partial}^{-1}(\bar{u}\partial u), \tag{A-20}$$

$$\nu^\#_{2,1} = \tfrac{1}{8}u\bar{\partial}^{-1}(|u|^2) - \tfrac{1}{2}\partial u. \tag{A-21}$$

$\ell = 2$:

$$\begin{aligned}\nu^\#_{1,2} &= -\tfrac{1}{64}\bar{\partial}^{-1}\big(|u|^2\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\big) \\ &\qquad + \tfrac{1}{16}\left\{\bar{\partial}^{-1}\big(\bar{u}\partial(u\bar{\partial}^{-1}(|u|^2))\big) + \bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(\bar{u}\partial u))\right\} - \tfrac{1}{4}\bar{\partial}^{-1}(\bar{u}\partial^2 u),\end{aligned} \tag{A-22}$$

$$\nu^\#_{2,2} = -\tfrac{1}{32}u\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2)) + \tfrac{1}{8}\left\{\partial(u\bar{\partial}^{-1}(|u|^2)) + u\bar{\partial}^{-1}(\bar{u}\partial u)\right\} - \tfrac{1}{2}\partial^2 u. \tag{A-23}$$

$\ell = 3$:

$$\begin{aligned}\nu^\#_{2,3} &= \tfrac{1}{128}u\bar{\partial}^{-1}\big(|u|^2\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\big) \\ &\quad - \tfrac{1}{32}\left\{u\bar{\partial}^{-1}\big(\bar{u}\partial(u\bar{\partial}^{-1}(|u|^2))\big) + u\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(\bar{u}\partial u)) + \partial\big(u\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\big)\right\} \\ &\quad + \tfrac{1}{8}\left\{u\bar{\partial}^{-1}(\bar{u}\partial^2 u) + \partial^2(u\bar{\partial}^{-1}(|u|^2)) + \partial(u\bar{\partial}^{-1}(\bar{u}\partial u))\right\} \\ &\quad - \tfrac{1}{2}\partial^3 u.\end{aligned} \tag{A-24}$$

A.4. ***Inverse scattering method for mNV.*** We now compute the motion of the putative solution

$$u = \mathcal{I} r$$

if the reflection coefficient evolves according to the law

$$\dot{r} = -(k^3 - \overline{k}^3) r, \quad r|_{t=0} = \mathcal{R} u_0.$$

To use (A-8), we compute $[\nu_1(z,\diamond)\nu_2^\#(-z,\diamond)]_3$ and $[\nu_2(z,\diamond)\nu_1^\#(-z,\diamond)]_3$.

First, we have

$$[\nu_1(z,\diamond)\nu_2^\#(-z,\diamond)]_3 = \nu_{2,3}^\# + \nu_{2,2}^\# \nu_{1,0} + \nu_{2,1}^\# \nu_{1,1} + \nu_{2,0}^\# \nu_{1,2}. \tag{A-25}$$

From the formulas above we have

$$\begin{aligned} \nu_{2,2}^\# \nu_{1,0} = &-\tfrac{1}{128} u \bar\partial^{-1}(|u|^2 \bar\partial^{-1}(|u|^2)) \cdot (\bar\partial^{-1}|u|^2) \\ &+ \tfrac{1}{32}\big\{\partial(u\bar\partial^{-1}(|u|^2)) \cdot (\bar\partial^{-1}(|u|^2)) + u\bar\partial^{-1}(\bar{u}\partial u) \cdot (\bar\partial^{-1}(|u|^2))\big\} \\ &- \tfrac{1}{8}\partial^2 u \cdot \bar\partial^{-1}(|u|^2), \end{aligned} \tag{A-26}$$

$$\begin{aligned} \nu_{2,1}^\# \nu_{1,1} = &\tfrac{1}{128} u \bar\partial^{-1}(|u|^2) \cdot \bar\partial^{-1}(|u|^2 \bar\partial^{-1}(|u|^2)) \\ &- \tfrac{1}{32}\big\{u\bar\partial^{-1}(|u|^2) \cdot \bar\partial^{-1}(u\partial\bar{u}) + \partial u \cdot \big(\bar\partial^{-1}(|u|^2 \bar\partial^{-1}(|u|^2))\big)\big\} \\ &+ \tfrac{1}{8}\partial u \cdot \bar\partial^{-1}(u\partial\bar{u}), \end{aligned} \tag{A-27}$$

$$\begin{aligned} \nu_{2,0}^\# \nu_{1,2} = &-\tfrac{1}{128} u \bar\partial^{-1}\big(|u|^2 \bar\partial^{-1}(|u|^2 \bar\partial^{-1}(|u|^2))\big) \\ &+ \tfrac{1}{32}\big\{u\bar\partial^{-1}\big(u\partial(\bar{u}\bar\partial^{-1}(|u|^2))\big) + u\bar\partial^{-1}(|u|^2\bar\partial^{-1}(u\partial\bar{u}))\big\} \\ &- \tfrac{1}{8} u\bar\partial^{-1}(u\partial^2\bar{u}). \end{aligned} \tag{A-28}$$

Using (A-24) and (A-26)–(A-28) in (A-25) we see that seventh-order terms cancel, while fifth-order terms sum to zero, as may be shown using the identity

$$\bar\partial^{-1} f \cdot \bar\partial^{-1} g = \bar\partial^{-1}(f\bar\partial^{-1} g + g\bar\partial^{-1} f), \tag{A-29}$$

while third-order terms may be simplified using the same identity with $f = g$. The result is

$$\big[\nu_{11}(z,\diamond)\tilde\nu_{21}(-z,\diamond)\big]_3 = \tfrac{3}{8}\big[(\partial u) \cdot \big(\bar\partial^{-1}(\partial(|u|^2))\big)\big] + \tfrac{3}{8}\big[u\bar\partial^{-1}(\bar{u}\bar\partial u)\big] - \tfrac{1}{2}\partial^3 u. \tag{A-30}$$

Next, we compute

$$\big[\nu_2(z,\diamond)\nu_1^\#(-z,\diamond)\big]_3 = \nu_{2,3} + \nu_{2,2}\nu_{1,0}^\# + \nu_{2,1}\nu_{1,1}^\# + \nu_{2,0}\nu_{1,2}^\#. \tag{A-31}$$

From the formulas above we have

$$\begin{aligned}\nu_{2,2}\nu^{\#}_{1,0} &= -\tfrac{1}{128}\lambda\bar{u}\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\cdot\bar{\partial}^{-1}(|u|^2) \\ &\quad+\tfrac{1}{32}\bigl\{\partial(\bar{u}\bar{\partial}^{-1}(|u|^2))\cdot\bar{\partial}^{-1}(|u|^2)+\bar{u}\bar{\partial}^{-1}(u\partial\bar{u})\cdot(\bar{\partial}^{-1}(|u|^2))\bigr\} \\ &\quad-\tfrac{1}{8}\lambda\partial^2\bar{u}\cdot\bar{\partial}^{-1}(|u|^2),\end{aligned} \tag{A-32}$$

$$\begin{aligned}\nu_{2,1}\nu^{\#}_{1,1} &= \tfrac{1}{128}\bar{u}\bar{\partial}^{-1}(|u|^2)\cdot\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2)) \\ &\quad-\tfrac{1}{32}\bigl\{\bar{u}\bar{\partial}^{-1}(|u|^2)\cdot\bar{\partial}^{-1}(\bar{u}\partial u)+\partial\bar{u}\cdot\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\bigr\} \\ &\quad+\tfrac{1}{8}\partial\bar{u}\cdot\bar{\partial}^{-1}(\bar{u}\partial u),\end{aligned} \tag{A-33}$$

$$\begin{aligned}\nu_{2,0}\nu^{\#}_{1,2} &= -\tfrac{1}{128}\bar{u}\bar{\partial}^{-1}\bigl(|u|^2\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(|u|^2))\bigr) \\ &\quad+\tfrac{1}{32}\bigl\{\bar{u}\bar{\partial}^{-1}(\bar{u}\partial(u\bar{\partial}^{-1}(|u|^2)))+\bar{u}\bar{\partial}^{-1}(|u|^2\bar{\partial}^{-1}(\bar{u}\partial u))\bigr\} \\ &\quad-\tfrac{1}{8}\bar{u}\bar{\partial}^{-1}(\bar{u}\partial^2 u).\end{aligned} \tag{A-34}$$

Using (A-17) and (A-32)–(A-34) in (A-31), noting the cancellation of fifth-order terms, we obtain

$$[\nu_2(z,\diamond)\nu^{\#}_1(-z,\diamond)]_3 = \tfrac{3}{8}\bigl[\bar{u}\bar{\partial}^{-1}(\partial(u\partial\bar{u}))\bigr]+\tfrac{3}{8}(\partial\bar{u})\cdot\partial\bar{\partial}^{-1}(|u|^2)-\tfrac{1}{2}\partial^3\bar{u}, \tag{A-35}$$

or upon complex conjugation

$$[\overline{\nu_2(z,\diamond)\nu^{\#}_1(-z,\diamond)_3}] = \tfrac{3}{8}u\partial^{-1}(\bar{\partial}(\bar{u}\bar{\partial}u))+\tfrac{3}{8}(\bar{\partial}u)\cdot\partial^{-1}(\bar{\partial}(|u|^2))-\tfrac{1}{2}\bar{\partial}^3 u. \tag{A-36}$$

Using these equations in (A-8), we obtain the mNV equation:

$$\frac{\partial u}{\partial t} = -\partial^3 u-\bar{\partial}^3 u+\tfrac{3}{4}(\partial\bar{u})\cdot(\bar{\partial}\partial^{-1}(|u|^2))+\tfrac{3}{4}(\bar{\partial}u)\cdot(\bar{\partial}\partial^{-1}(|u|^2))+\tfrac{3}{4}\bar{u}\bar{\partial}\partial^{-1}(\bar{u}\bar{\partial}u)+\tfrac{3}{4}u\partial^{-1}(\bar{\partial}(\bar{u}\bar{\partial}u)).$$

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CONVEXITY OF AVERAGE OPERATORS FOR SUBSOLUTIONS TO SUBELLIPTIC EQUATIONS

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We study convexity properties of the average integral operators naturally associated with divergence-form second-order subelliptic operators $\mathscr{L}$ with nonnegative characteristic form. When $\mathscr{L}$ is the classical Laplace operator, these average operators are the usual average integrals over Euclidean spheres. In our subelliptic setting, the average operators are (weighted) integrals over the level sets

$$\partial\Omega_r(x) = \{y : \Gamma(x, y) = 1/r\}$$

of the fundamental solution $\Gamma(x, y)$ of $\mathscr{L}$. We shall obtain characterizations of the $\mathscr{L}$-subharmonic functions u (that is, the weak solutions to $-\mathscr{L}u \le 0$) in terms of the convexity (w.r.t. a power of r) of the average of u over $\partial\Omega_r(x)$, as a function of the radius r. Solid average operators will be considered as well. Our main tools are representation formulae of the (weak) derivatives of the average operators w.r.t. the radius. As applications, we shall obtain Poisson–Jensen and Bôcher type results for $\mathscr{L}$.

1. Introduction and main results

1A. ***Notation and definitions.*** Let u be a subharmonic function in an open set $\Omega \subseteq \mathbb{R}^N$, $N \ge 2$. Then, with fixed $x \in \Omega$, the map

$$m_r(u)(x) : (0, R(x)) \longrightarrow (-\infty, \infty),$$
$$r \mapsto m_r(u)(x) := \frac{1}{H^{N-1}(\partial B_r(x))} \int_{\partial B_r(x)} u(y)\, dH^{N-1}(y) \tag{1-1}$$

is convex with respect to $\log r$ if $N = 2$, and $1/r^{N-2}$ if $N \ge 3$. In (1-1), $B_r(x)$ denotes the Euclidean ball of radius r and center x; $R(x)$ stands for $\sup\{r > 0 : B_r(x) \subset \Omega\}$; H^{N-1} is the Hausdorff $(N-1)$-dimensional measure in $\mathbb{R}^N$. This quite well-known classical result has many important consequences and applications; see [Armitage and Gardiner 2001, Section 3.5; Hayman and Kennedy 1976, Section 2.7; Hörmander 1994, Section 3.2]. Of these applications, we only mention the Hadamard three-circles theorem, the Liouville-type theorem for bounded above subharmonic functions in $\mathbb{R}^2$, the applications to the theory of Hardy spaces, and the Bôcher theorem for harmonic functions in punctured balls (see [Armitage and Gardiner 2001, Chapter 3], for example).

The aim of the present paper is to study analogous properties for some weighted average operators acting on subsolutions to

$$-\mathscr{L}u = 0 \quad \text{in } \Omega \subseteq \mathbb{R}^N,\ N \ge 3,$$

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where $\mathscr{L}$ is a linear second order PDO with nonnegative characteristic form. Precisely, the operators we are dealing with are of the form

$$\mathscr{L} := \sum_{i,j=1}^{N} \partial_{x_i}(a_{i,j}(x)\partial_{x_j}) = \operatorname{div}(A(x)\nabla),$$

where $\nabla = (\partial_{x_1}, \dots, \partial_{x_N})^T$ and $A(x) = (a_{i,j}(x))_{i,j}$ is a symmetric matrix with smooth entries that is nonnegative definite at any point $x \in \mathbb{R}^N$. In Section 2 we will precisely fix our hypotheses on $\mathscr{L}$. Here we only need to mention the crucial ones: $\mathscr{L}$ is not totally degenerate, hypoelliptic, and endowed with a fundamental solution

$$\Gamma : \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y\} \longrightarrow (0, \infty),$$

with pole at any point of the diagonal $\{x = y\}$ and vanishing at infinity. For example, besides the classical Laplace operator on $\mathbb{R}^N$ ($N \geq 3$), any sub-Laplacian operator on a stratified Lie group (with homogeneous dimension ≥ 3) enjoys all these hypotheses; see, for example, [Bonfiglioli et al. 2007].

The main objects of our investigation are the average operators on the level sets of Γ, that is, on the sets

$$\partial\Omega_r(x) = \{y \in \mathbb{R}^N : \Gamma(x, y) = 1/r\}, \quad x \in \mathbb{R}^N, \ r > 0,$$

together with their solid counterparts, the average operators on the sets

$$\Omega_r(x) = \{y \in \mathbb{R}^N : \Gamma(x, y) > 1/r\}, \quad x \in \mathbb{R}^N, \ r > 0.$$

We call $\partial\Omega_r(x)$ and $\Omega_r(x)$, respectively, the $\mathscr{L}$-sphere and the $\mathscr{L}$-ball with radius r and center x. Owing to Sard's theorem, since Γ is smooth (in view of the hypoellipticity of $\mathscr{L}$), any $\mathscr{L}$-sphere is an $(N-1)$-dimensional manifold of class C^∞, for almost every radius. (For simplicity, we assume this to be true for every positive radius.)

If $\Omega \subseteq \mathbb{R}^N$ is open, given an upper semicontinuous (u.s.c.) function $u : \Omega \to [-\infty, \infty)$, for any $\mathscr{L}$-ball $\Omega_r(x)$ with closure contained in $\Omega_r(x)$, we set[1]

$$m_r(u)(x) := \int_{\partial\Omega_r(x)} u(y)k(x, y)\, dH^{N-1}(y),$$
$$M_r^\alpha(u)(x) := \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(x)} u(y)K_\alpha(x, y)\, dy$$

for any $\alpha > -1$. Set $\Gamma_x := \Gamma(x, \cdot)$. The weights k, K_α are defined on $\mathbb{R}^N \setminus \{x\}$ by

$$k(x, \cdot) := \frac{|\nabla_{\mathscr{L}}\Gamma_x|^2}{|\nabla\Gamma_x|}, \quad K_\alpha(x, \cdot) := \frac{|\nabla_{\mathscr{L}}\Gamma_x|^2}{\Gamma_x^{2+\alpha}}, \tag{1-2}$$

where $|\nabla_{\mathscr{L}}\Gamma_x(y)|^2 := \langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\rangle$. The average operators m_r and M_r^α can be used to characterize the solutions to $\mathscr{L}u = v$. Indeed, for every $u \in C^2(\Omega, \mathbb{R})$, the following representation formulae

[1] Obviously, in order to define $m_r(u)(x)$, we only need to require that Ω contains $\partial\Omega_r(x)$.

hold true [Bonfiglioli and Lanconelli 2013, Section 11]:

$$\begin{aligned} u(x) &= m_r(u)(x) - \int_{\Omega_r(x)} \Big(\Gamma(x,y) - \frac{1}{r}\Big) \mathscr{L}u(y)\,dy, \\ u(x) &= M_r^\alpha(u)(x) - \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \Big(\int_{\Omega_\rho(x)} \Big(\Gamma(x,y) - \frac{1}{\rho}\Big) \mathscr{L}u(y)\,dy \Big)\,d\rho \end{aligned} \tag{1-3}$$

for every $\mathscr{L}$-ball $\Omega_r(x)$ with closure contained in Ω. Thus, given $x \in \Omega$, the above formula is satisfied for any positive r such that $r < R(x)$, where

$$R(x) := \sup\{r > 0 : \Omega_r(x) \subset \Omega\}. \tag{1-4}$$

For $u \equiv 1$, these formulae give

$$1 = m_r(1)(x) = M_r^\alpha(1)(x) \quad \text{for every } x \in \mathbb{R}^N \text{ and } r > 0.$$

Therefore, since the kernels k and K_α are nonnegative (recall that $A(y) \geq 0$), $m_r(u)(x)$ and $M_r^\alpha(u)(x)$ are well-posed (possibly $-\infty$) for every u.s.c. function u. (Actually, as was recently proved in [Abbondanza and Bonfiglioli 2013], $k(x,\,\cdot)$ and $K_\alpha(x,\,\cdot)$ are positive on an open dense subset of $\mathbb{R}^N \setminus \{x\}$ for every $x \in \mathbb{R}^N$.)

It is also worth noticing that

$$M_r^\alpha(u)(x) = \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha m_\rho(u)(x)\,d\rho. \tag{1-5}$$

This can be proved by using Federer's co-area formula and suitable approximation arguments for u.s.c. functions.

In what follows, given a u.s.c. function u on an open set $\Omega \subseteq \mathbb{R}^N$, we say that

(1) u is m-continuous in Ω if $u(x) = \lim_{r\to 0+} m_r(u)(x)$ for every $x \in \Omega$;

(2) u is M^α-continuous in Ω if $u(x) = \lim_{r\to 0+} M_r^\alpha(u)(x)$ for every $x \in \Omega$.

A smooth function u will be called $\mathscr{L}$-harmonic in Ω if $\mathscr{L}u = 0$ in Ω. We call a u.s.c. function $u : \Omega \to [-\infty, \infty)$ $\mathscr{L}$-subharmonic in Ω if

(1) the set $\Omega(u) := \{x \in \Omega : u(x) > -\infty\}$ contains at least one point of every connected component of Ω;

(2) for every bounded open set $V \subset \overline{V} \subset \Omega$ and for every $\mathscr{L}$-harmonic function h in V, continuous up to ∂V, $u \leq h$ holds whenever $u \leq h$ on ∂V.

The family of the $\mathscr{L}$-subharmonic functions in Ω is a cone denoted by $\underline{\mathscr{S}}(\Omega)$.

In [Bonfiglioli and Lanconelli 2013, Section 8] it is proved that u is $\mathscr{L}$-subharmonic in Ω if and only if $u \in L^1_{\mathrm{loc}}(\Omega)$, $\mathscr{L}u \geq 0$ in the weak sense of distributions, and u is M^α-continuous in Ω. For this reason the $\mathscr{L}$-subharmonic functions are also said to be the subsolutions of $-\mathscr{L}$. As a consequence of the cited characterization, by the classical Riesz representation theorem, it follows that, given $u \in \underline{\mathscr{S}}(\Omega)$, there exists a nonnegative Radon measure μ_u on the Borel subsets of Ω (called the $\mathscr{L}$-Riesz measure of u) such that $\mathscr{L}u = \mu_u$ in Ω, in the weak sense of distributions.

Several other characterizations of the $\mathscr{L}$-subharmonicity have been provided in [Bonfiglioli and Lanconelli 2013] in terms of the average operators m_r and M_r^α. For our aim it is convenient to recall [Bonfiglioli and Lanconelli 2013, Theorem 4.2]; see also the notation in (1-4). Let $u : \Omega \to [-\infty, \infty)$ be a u.s.c. function such that $\Omega(u)$ contains at least one point of every connected component of Ω. Then $u \in \underline{\mathscr{S}}(\Omega)$ if and only if one of the following conditions is satisfied:

(A.1) $u(x) \le m_r(u)(x)$ for every $x \in \Omega$ and every $r \in (0, R(x))$;

(A.2) u is m-continuous in Ω and, for every $x \in \Omega$, $r \mapsto m_r(u)(x)$ is monotone nondecreasing on $(0, R(x))$.

One obtains further equivalent conditions by replacing, in (A.1) and (A.2), the surface average m_r with the solid average M_r^α, with $\alpha > -1$.

The following result will be used frequently in what follows.

Remark 1.1. By [Bonfiglioli and Lanconelli 2013, Proposition 6.10], if $u \in \underline{\mathscr{S}}(\Omega)$, the map $r \mapsto m_r(u)(x)$ is finite-valued and continuous on $(0, R(x))$ for every $x \in \Omega$. This follows from [Bonfiglioli and Lanconelli 2013, Theorem 6.4] and

$$m_r(\Gamma(\,\cdot\,, z))(x) = \min\{\Gamma(x, z), 1/r\} \tag{1-6}$$

jointly with a Riesz representation argument decomposing u, locally, as an $\mathscr{L}$-harmonic function plus the convolution of Γ with the Riesz measure of u. As a consequence, whenever $\alpha > 0$, the map $r \mapsto M_r^\alpha(u)(x)$ is finite-valued and continuous on $(0, R(x))$ for every $x \in \Omega$. This follows at once from (1-5) and (1-6), since $\rho^{\alpha-1}$ is integrable on $(0, r)$ for any positive α. The solid average $M_r^\alpha(u)(x)$ is finite-valued and continuous also when $-1 < \alpha \le 0$, provided that $x \in \Omega(u)$. To obtain this fact, it suffices to keep in mind identity (1-5) and the inequalities $-\infty < u(x) \le m_r(u)(x)$, valid for $x \in \Omega(u)$ and $0 < r < R(x)$.

In order to list the main results of this paper, we need a few more definitions. Let $I \subseteq \mathbb{R}$ be an interval and suppose that $\varphi : I \to \mathbb{R}$ is a strictly monotone continuous function. Following [Armitage and Gardiner 2001, Section 3.5], we say that $f : I \to \mathbb{R}$ is φ-convex if

$$f(r) \le \frac{\varphi(r_2) - \varphi(r)}{\varphi(r_2) - \varphi(r_1)} f(r_1) + \frac{\varphi(r) - \varphi(r_1)}{\varphi(r_2) - \varphi(r_1)} f(r_2) \tag{1-7}$$

for every $r_1, r, r_2 \in I$ such that $r_1 < r < r_2$. When $\varphi(r) = r$, (1-7) gives back the standard definition of a convex function. Moreover, clearly f is φ-convex if and only if $f \circ \varphi^{-1}$ is convex on the interval $\varphi(I)$, in the usual sense.

Finally, given a function $f : I \to \mathbb{R}$, we say that

(1) f is locally absolutely continuous (locally a.c.) if f is absolutely continuous on every compact subinterval of I;

(2) f is essentially monotone if there exists a monotone function $f^* : I \to \mathbb{R}$ such that $f = f^*$ almost everywhere in I.

1B. ***Main theorems.*** Our crucial results concern the derivative with respect to r of the average operators $m_r(u)(x)$ and $M_r^\alpha(u)(x)$, when u is $\mathscr{L}$-subharmonic. These are given in the following theorem.

Theorem 1.2 (derivatives of $m_r(u)$ and $M_r^\alpha(u)$). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let u be an $\mathscr{L}$-subharmonic function in Ω with $\mathscr{L}$-Riesz measure μ_u.*

(i) *For every $x \in \Omega$, the map $r \mapsto m_r(u)(x)$ is locally a.c. on $(0, R(x))$, and*

$$\frac{d}{dr} m_r(u)(x) = \frac{\mu_u(\Omega_r(x))}{r^2} \quad \text{for almost every } r \text{ in } (0, R(x)). \tag{1-8}$$

(ii) *For every $x \in \Omega$ and $\alpha > 0$, the map $r \mapsto M_r^\alpha(u)(x)$ is of class C^1 on $(0, R(x))$, and*

$$\frac{d}{dr} M_r^\alpha(u)(x) = \frac{\alpha+1}{r^{\alpha+2}} \int_{\Omega_r(x)} \left(f_\alpha(r) - f_\alpha\left(\frac{1}{\Gamma(x,y)}\right)\right) d\mu_u(y) \tag{1-9}$$

for every r in $(0, R(x))$, where f_α denotes an antiderivative of $r^{\alpha-1}$:

$$f_\alpha(r) := \begin{cases} \ln r, & \text{if } \alpha = 0, \\ r^\alpha/\alpha, & \text{if } \alpha \neq 0. \end{cases} \tag{1-10}$$

This also holds for $-1 < \alpha \le 0$ if $x \in \Omega(u)$.

A straightforward consequence of this theorem is the following corollary.

Corollary 1.3 (Poisson–Jensen type formula). *Let $u \in \underline{\mathcal{S}}(\Omega)$ and let μ_u be its $\mathscr{L}$-Riesz measure. The maps $r \mapsto m_r(u)(x)$ and $r \mapsto M_r^\alpha(u)(x)$ (for $\alpha > -1$) can be prolonged with continuity up to $r = 0$ if and only if $x \in \Omega(u)$.*

Furthermore, for every $x \in \Omega$ and $r \in (0, R(x))$, one has the following representation formulae (of Poisson–Jensen type):

$$u(x) = m_r(u)(x) - \int_0^r \frac{\mu_u(\Omega_\rho(x))}{\rho^2}\, d\rho = m_r(u)(x) - \int_{\Omega_r(x)} \left(\Gamma(x,y) - \frac{1}{r}\right) d\mu_u(y), \tag{1-11}$$

and, for $\alpha > 0$,

$$\begin{aligned} u(x) &= M_r^\alpha(u)(x) - \int_0^r \frac{\alpha+1}{\rho^{\alpha+2}} \left(\int_{\Omega_\rho(x)} \left(f_\alpha(\rho) - f_\alpha\left(\frac{1}{\Gamma(x,y)}\right)\right) d\mu_u(y)\right) d\rho \\ &= M_r^\alpha(u)(x) - \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left(\int_{\Omega_\rho(x)} \left(\Gamma(x,y) - \frac{1}{\rho}\right) d\mu_u(y)\right) d\rho. \end{aligned} \tag{1-12}$$

When $x \notin \Omega(u)$, all the sides of the previous formulae (1-11) and (1-12) are $-\infty$, and this happens if and only if $\mu_u(\{x\}) > 0$.

Formula (1-12) holds true also for $-1 < \alpha \le 0$, provided that $x \in \Omega(u)$.

Theorem 1.2, together with the following real analysis lemma, easily implies convexity properties of our average operators, and these will characterize the $\mathscr{L}$-subharmonic functions.

Lemma 1.4. *Let $I = (0, a)$ be an interval in $(0, \infty)$, and let $f : I \to \mathbb{R}$.*

(i) *If f is bounded from above and $r^{-\beta}$-convex for a real $\beta > 0$, then f is monotone nondecreasing.*

(ii) *Let f be locally a.c., and let $\beta \neq 0$. Then f is $r^{-\beta}$-convex if and only if $r \mapsto r^{\beta+1} f'(r)$ is essentially monotone nondecreasing.*

Here are our main results concerning convexity of the average operators.

Theorem 1.5 (subharmonicity and convexity of the average operators). *Suppose that $\Omega \subseteq \mathbb{R}^N$ is an open set, and let $u : \Omega \to [-\infty, \infty)$ be an u.s.c. function such that $\Omega(u)$ intersects every connected component of Ω.*

Then the following statements are equivalent.

(1) $u \in \underline{\mathcal{S}}(\Omega)$.

(2) *u is m-continuous and the map $r \mapsto m_r(u)(x)$ is $1/r$-convex on $(0, R(x))$ for every $x \in \Omega$.*

(3) *u is m-continuous and the map $r \mapsto m_r(u)(x)$ is $1/r$-convex on $(0, R(x))$ for every $x \in \Omega(u)$.*

(4) *u is M^α-continuous and for every $x \in \Omega$, the map $r \mapsto M_r^\alpha(u)(x)$ is $1/r^{\alpha+1}$-convex on $(0, R(x))$ for some (or for every) $\alpha > 0$.*

(5) *u is M^α-continuous and, for every $x \in \Omega(u)$, the map $r \mapsto M_r^\alpha(u)(x)$ is $1/r^{\alpha+1}$-convex on $(0, R(x))$ for some (or for every) $\alpha > -1$.*

We observe that, to the best of our knowledge, the implications (2), (3), (4), (5) $\Rightarrow$ (1) appear here for the first time, even when $\mathcal{L}$ is the classical Laplace operator.

Moreover, we shall prove that (in statements (2), (3), (4), (5) above) we can replace r^{-1}-convexity or $r^{-(\alpha+1)}$-convexity with $r^{-\gamma}$-convexity for infinitely many other values of $\gamma > 0$ (see Theorems 5.1 and 5.2 for the precise statements).

We observe that the convexity (w.r.t. suitable powers of r) of the maps $r \mapsto m_r(u)(x)$, $M_r^\alpha(u)(x)$ in Theorem 1.5 ensures that these functions have more regularity properties than those provided so far in Theorem 1.2: by Alexandrov's theorem, they are *twice differentiable* almost everywhere on $(0, R(x))$.

1C. *Ring-shaped domains, applications, and further developments.* Suitable versions of Theorems 1.2 and 1.5 hold true for $\mathcal{L}$-subharmonic functions in ring-shaped domains. Given a, b such that $0 \le a < b \le \infty$, and given $x_0 \in \mathbb{R}^N$, we define the Γ-annulus of center x_0 and radii a, b as follows:

$$A_{a,b}(x_0) := \left\{ x \in \mathbb{R}^N : a < \frac{1}{\Gamma(x_0, x)} < b \right\}. \tag{1-13}$$

The conventions $1/\infty = 0$ and $1/0 = \infty$ apply.

The following results (Corollary 1.7 and Theorems 1.8 and 1.9) improve [Bonfiglioli and Lanconelli 2007, Theorems 1.5, 1.8, and 1.9], proved in the case of sub-Laplacians $\mathcal{L}$ on stratified groups.

Theorem 1.6. *Let $u \in \underline{\mathcal{S}}(A_{a,b}(x_0))$ and let μ_u be its $\mathcal{L}$-Riesz measure. The map*

$$(a, b) \ni r \mapsto m_r(u)(x_0) \in \mathbb{R}$$

is locally a.c. and $1/r$*-convex. Moreover, for every fixed* α, β *such that* $a < \alpha < \beta < b$, *there exists a constant* $c \in \mathbb{R}$ (*depending on* $a, \alpha, \beta, b, u, x_0$) *such that*

$$r^2 \frac{d}{dr} m_r(u)(x_0) = \mu_u(A_{\alpha,r}(x_0)) + c \tag{1-14}$$

for almost every r *in* (α, β).

From this theorem we obtain the following result.

Corollary 1.7. *Suppose* u *is* $\mathscr{L}$*-harmonic in the* Γ*-annulus* $A_{a,b}(x_0)$. *Then*

$$m_r(u)(x_0) = \frac{c_1}{r} + c_2, \quad r \in (a, b),$$

for some real constants c_1, c_2.

As an application of the previous results on $\mathscr{L}$-subharmonic functions on ring-shaped domains, we will show a symmetry result, from which a Bôcher-type theorem for $\mathscr{L}$ will follow. The latter improves a result in [Bonfiglioli and Lanconelli 2007].

For our application we need (together with the structural assumptions (H1) and (H2) in Section 2) the following extra assumption on $\mathscr{L}$, a homogeneous Harnack inequality on Γ-spheres.

(HH) For every fixed $x_0 \in \mathbb{R}^N$ and every $0 < b < \infty$, there exist positive constants $C = C(x_0, b) > 1$ and $\theta = \theta(x_0, b) < 1$ such that

$$\sup_{\partial\Omega_r(x_0)} h \le C \inf_{\partial\Omega_r(x_0)} h$$

for every r such that $0 < r < \theta b$ and every $\mathscr{L}$-harmonic nonnegative function h in the Γ-annulus $A_{0,b}(x_0)$.

By standard arguments (see, for example, [Bony 1969]), this hypothesis is satisfied for the sum of squares of Hörmander vector fields $\mathscr{L} = \sum_{j=1}^m X_j^2$. Moreover, (HH) is fulfilled for $x_0 = 0$, when $\mathscr{L}$ is homogeneous of positive degree (in the sense recalled in Remark 7.1) w.r.t. a group of dilations; see [Bonfiglioli et al. 2007, Theorem 5.16.5, page 327].

Theorem 1.8. *Suppose* $\mathscr{L}$ *satisfies condition* (HH) *above.*

Let w *be nonnegative and* $\mathscr{L}$*-harmonic in the* Γ*-annulus* $A_{0,b}(x_0) = \Omega_b(x_0) \setminus \{x_0\}$ (*where* $b < \infty$) *and suppose that* w *is also continuous up to* $\partial\Omega_b(x_0)$ *and* $w \equiv 0$ *on* $\partial\Omega_b(x_0)$. *Then* w *is affine w.r.t.* Γ, *that is,*

$$w(x) = c(\Gamma(x_0, x) - 1/b), \quad x \in A_{0,b}(x_0),$$

for some positive constant c.

We prove this theorem as a consequence of Corollary 1.7, by following an idea exploited by Axler, Bourdon, and Ramey [Axler et al. 1992] in the classical case of the Laplace operator. From Theorem 1.8 one easily obtains the following Bôcher-type result.

Theorem 1.9 (Bôcher's theorem for $\mathscr{L}$). *Suppose $\mathscr{L}$ satisfies condition* (HH).

Let $\Omega \subseteq \mathbb{R}^N$ be open and $x_0 \in \Omega$. Let u be nonnegative and $\mathscr{L}$-harmonic in $\Omega \setminus \{x_0\}$. Then there exists an $\mathscr{L}$-harmonic function h on Ω and a constant $c \geq 0$ such that

$$u(x) = c\Gamma(x_0, x) + h(x) \quad \textit{for every } x \in \Omega \setminus \{x_0\}.$$

Further developments. We end the introduction by pointing out further applications of the results of this paper: they can be used for the investigations of *convex functions in Carnot groups* (as introduced in [Danielli et al. 2003; Lu et al. 2004]). Indeed (see [Juutinen et al. 2007] for the relevant results), since the so-called v-convex functions on Carnot groups are characterized in terms of their $\mathscr{L}$-subharmonicity w.r.t. the family of the sub-Laplacians $\{\mathscr{L}\}$ (a class of operators comprised in our present paper), by Theorem 1.5 it turns out that v-convexity can be characterized by the usual (Euclidean) convexity of the family of the real-variable functions $\{r \mapsto m_r(u)(x)\}$ (or of $\{r \mapsto M_r^\alpha(u)(x)\}$), as the average operators vary with $\{\mathscr{L}\}$. The characterization of v-convexity in [Bonfiglioli and Lanconelli 2012] can also be exploited to further simplify the investigation.

Finally, since our results apply to any Hörmander sum of squares of vector fields, we can use our characterization of v-convexity in order to obtain a new notion of convexity in more general frameworks than the Carnot setting (for instance, in the framework of Hörmander vector fields), as was done by Magnani and Scienza [2012]. We plan to develop this topic in a forthcoming study.

2. Main assumptions on $\mathscr{L}$ and recalls on $r^{-\beta}$-convexity

2A. ***Assumptions on*** $\mathscr{L}$**.** Throughout the paper, we let

$$\mathscr{L} := \sum_{i,j=1}^{N} \partial_{x_i}(a_{i,j}(x)\partial_{x_j}) \tag{2-1}$$

be a linear second order PDO in $\mathbb{R}^N$, in divergence form, with C^∞ coefficients, such that the matrix $A(x) := (a_{i,j}(x))_{i,j\leq N}$ is symmetric and nonnegative definite at every point $x \in \mathbb{R}^N$. The operator $\mathscr{L}$ is self-adjoint and it is (possibly) degenerate elliptic. However, we always assume without further comments that $\mathscr{L}$ is not totally degenerate, that is, there exists $i \in \{1, \ldots, N\}$ such that $a_{i,i}(x) > 0$ for every $x \in \mathbb{R}^N$. As is well-known, this ensures that $\mathscr{L}$ satisfies the weak maximum principle on every bounded open subset of $\mathbb{R}^N$.

Our main assumptions on $\mathscr{L}$ are as follows.

(H1) *$\mathscr{L}$ is a C^∞-hypoelliptic differential operator*, that is, for every open set $\Omega \subseteq \mathbb{R}^N$, and for every $f \in C^\infty(\Omega, \mathbb{R})$, if $u \in \mathscr{D}'(\Omega)$ is a solution of $\mathscr{L}u = f$ in the weak sense of distributions, u can be identified with a C^∞ function on Ω.

(H2) We assume that $\mathscr{L}$ is equipped with a *global fundamental solution*

$$\Gamma : D = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y\} \longrightarrow (0, \infty)$$

with the following properties:

(a) $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}^N) \cap C^\infty(D, \mathbb{R})$;

(b) for every fixed $x \in \mathbb{R}^N$, we have $\lim_{y \to x} \Gamma(x, y) = \infty$ and $\lim_{y \to \infty} \Gamma(x, y) = 0$;

(c) for every $\varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$ and every $x \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} \Gamma(x, y) \mathscr{L}\varphi(y)\, dy = -\varphi(x). \tag{2-2}$$

If $\Omega \subseteq \mathbb{R}^N$ is open, we say that u is $\mathscr{L}$-harmonic on Ω if $u \in C^\infty(\Omega, \mathbb{R})$ and $\mathscr{L}u = 0$ in Ω. A bounded open set $V \subset \mathbb{R}^N$ is said to be $\mathscr{L}$-regular if the following property is satisfied: for every $f \in C(\partial V, \mathbb{R})$, there exists a (unique) $\mathscr{L}$-harmonic function in V, denoted by H_f^V, satisfying $\lim_{y \to x} H_f^V(y) = f(x)$ for every $x \in \partial V$.

As described in [Bonfiglioli and Lanconelli 2013, Remark 2.2], $\mathscr{L}$ endows $\mathbb{R}^N$ with the structure of a $\mathfrak{S}^*$-harmonic space, in the sense of [Bonfiglioli et al. 2007, Definition 6.10.1]: this is a consequence of hypothesis (H1). As a very particular byproduct, we can use Bouligand's theorem to derive that the Γ-balls $\Omega_r(x)$ are $\mathscr{L}$-regular open sets (we shall use this last fact in the proof of Bôcher's Theorem 1.9).

2B. *Background results on $r^{-\beta}$-convexity.* Next we prove some results on φ-convexity, as introduced in Section 1. We begin by remarking that, obviously, given intervals $I, J \subseteq \mathbb{R}$ and given a function $\psi : J \to I$ which is monotone and continuous, a function $u : I \to \mathbb{R}$ is φ-convex on I if and only if $u \circ \psi$ is $(\varphi \circ \psi)$-convex on $\psi^{-1}(I)$. Another very simple lemma is in order.

Lemma 2.1. *Suppose $\beta \neq 0$. Let $I \subseteq (0, \infty)$ be an interval and let $u : I \to \mathbb{R}$. The following assertions are equivalent*:

(1) *$u(r)$ is $r^{-\beta}$-convex on I*;

(2) *$u(r^{-1/\beta})$ is convex on $\varphi(I)$, where $\varphi(r) = r^{-\beta}$*;

(3) *$r^\beta u(r)$ is r^β-convex on I.*

Proof. The equivalence of (1) and (2) follows from the remark preceding the lemma. Taking $\varphi(r) = 1/r^\beta$, a simple computation shows that (1-7) is equivalent to

$$r^\beta u(r) \le \frac{r_2^\beta - r^\beta}{r_2^\beta - r_1^\beta} r_1^\beta u(r_1) + \frac{r^\beta - r_1^\beta}{r_2^\beta - r_1^\beta} r_2^\beta u(r_2),$$

which is equivalent to the r^β-convexity of $r^\beta u(r)$. □

The following result will be crucial later.

Lemma 2.2. *Let $a > 0$. Suppose $f : (0, a) \to \mathbb{R}$ is bounded from above and $r^{-\beta}$-convex on $(0, a)$ for some $\beta > 0$. Then f is monotone nondecreasing.*

This lemma proves Lemma 1.4(i).

Proof. Let f be as in the assertion; by Lemma 2.1(2), $g(r) := f(r^{-1/\beta})$ is convex on $I := (a^{-\beta}, \infty)$. Since f is bounded from above on $(0, a)$, g is bounded from above on I. From elementary properties of convex functions, since I is unbounded, we infer that g is monotone nonincreasing on I; since $\beta > 0$, this means that f is monotone nondecreasing on $(0, a)$. □

We prove a condition for $r^{-\beta}$-convexity under a weak-differentiability assumption.

Lemma 2.3. *Suppose $\beta \neq 0$ and let $I \subseteq (0, \infty)$ be an open interval. Suppose that $u : I \to \mathbb{R}$ is a locally absolutely continuous function. Then u is $r^{-\beta}$-convex on I if and only if $r^{\beta+1}u'(r)$ is essentially monotone nondecreasing on I.*

This lemma proves Lemma 1.4(ii).

Proof. By Lemma 2.1(2), u is $r^{-\beta}$-convex if and only if $F(r) := u(r^{-1/\beta})$ is convex in its domain in the usual sense. On the other hand, since F is continuous, standard results (which we may omit) imply that F is convex if and only if F' is essentially nondecreasing. Summing up,

$$u \text{ is } r^{-\beta}\text{-convex if and only if } F' \text{ is essentially nondecreasing.} \tag{2-3}$$

In turn, F' is essentially nondecreasing if and only if the map $\rho \mapsto -\beta F'(\rho^{-\beta})$ is essentially nondecreasing on its domain. (Indeed, notice that if $\beta > 0$, then $-\beta < 0$ and $\rho^{-\beta}$ is decreasing; if $\beta < 0$, then $-\beta > 0$ and $\rho^{-\beta}$ is increasing.) Since

$$F'(r) = -\beta^{-1} r^{-(\beta+1)/\beta} u'(r^{-1/\beta}),$$

we get $-\beta F'(\rho^{-\beta}) = \rho^{\beta+1}u'(\rho)$. As a consequence, F' is essentially nondecreasing if and only this is true of $r^{\beta+1}u'(r)$, and this ends the proof, in view of (2-3). □

Convexity of a monotone C^2 function with respect to a power of r brings along convexity with respect to many other functions, as the following result shows.

Lemma 2.4. *Let $I \subseteq (0, \infty)$ be an open interval and suppose that $u : I \to \mathbb{R}$ is monotone nondecreasing and locally a.c. If u is $r^{-\gamma}$-convex on I, it is $r^{-\beta}$-convex of I for every $\beta \geq \gamma$.*

Proof. Suppose u is monotone nondecreasing, locally a.c., and $r^{-\gamma}$-convex on I. From Lemma 2.3, we know that $r^{\gamma+1}u'(r)$ is essentially nondecreasing on I. Since $u'(r) \geq 0$ almost everywhere on I, if $\beta \geq \gamma$, then $r^{\beta+1}u'(r) = r^{\beta-\gamma}(r^{\gamma+1}u'(r))$ is essentially nondecreasing as well. Again by Lemma 2.3, we deduce that u is $r^{-\beta}$-convex on I. □

We now investigate convexity properties of an average integral function.

Corollary 2.5. *Let $a > 0$ and $f : (0, a] \to \mathbb{R}$. Assume furthermore that $\alpha > -1$ and $r^\alpha f(r)$ is integrable on $(0, a)$. Let us consider the function*

$$F(r) = \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha f(\rho)\, d\rho, \qquad r \in (0, a].$$

(a) *If $\beta \neq 0$ and f is r^β-convex on $(0, a]$, the same is true of $F(r)$.*

(b) *Suppose that f is also continuous. Then $F(r)$ is $r^{-(\alpha+1)}$-convex on $(0, a]$ if and only if $f(r)$ is monotone nondecreasing.*

As $\alpha > -1$, note that the integrability of $r^\alpha f(r)$ is ensured, for example, whenever f is bounded on $(0, a)$ (for example, when f extends continuously on $[0, a]$).

Proof. We prove (a). Fix $r \in (0, a]$. The change of variable $\rho = rs$ gives $F(r) = (\alpha+1)\int_0^1 s^\alpha f(rs)\,ds$. Setting $r = t^{1/\beta}$, we have

$$F(t^{1/\beta}) = (\alpha+1)\int_0^1 s^\alpha f(t^{1/\beta}s)\,ds.$$

For every fixed $s \in [0, 1]$, the function $t \mapsto f(t^{1/\beta}s)$ is convex, since $f(t^{1/\beta}s) = f((ts^\beta)^{1/\beta})$, and since $r \mapsto f(r^{1/\beta})$ is convex by the assumption of r^β-convexity of f. This immediately gives the convexity of $F(t^{1/\beta})$, that is, the r^β-convexity of $F(r)$.

We finally prove (b). By Lemma 2.1(3) (with $\beta=\alpha+1$), $F(r)$ is $r^{-(\alpha+1)}$-convex if and only if $r^{\alpha+1}F(r)$ is $r^{\alpha+1}$-convex. In turn, this last condition is equivalent to the fact that the function $G(r) := r^{-\alpha}(r^{\alpha+1}F(r))'$ is nondecreasing, this time by applying Lemma 2.3 to $u(r) = r^{\alpha+1}F(r)$ and $\beta = -\alpha-1$. Now, the fundamental theorem of integral calculus ensures that $G(r) = (\alpha+1)f(r)$, and this function is monotone nondecreasing if and only if the same is true of $f(r)$. □

3. Derivatives of the average operators in the C^2 case

In order to prove Theorem 1.2, we first need the derivatives of $r \mapsto m_r(u)(x)$ and $M_r^\alpha(u)(x)$ for u of class C^2. An approximation argument will eventually yield the weak derivatives in the $\mathscr{L}$-subharmonic case (see Section 4).

Proposition 3.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in C^2(\Omega, \mathbb{R})$. For every fixed $x \in \Omega$, the functions*

$$(0, R(x)) \ni r \mapsto m_r(u)(x),\ M_r^\alpha(u)(x)$$

are differentiable and their derivatives are given by

$$\frac{d}{dr}m_r(u)(x) = \frac{1}{r^2}\int_{\Omega_r(x)} \mathscr{L}u(y)\,dy, \tag{3-1}$$

$$\frac{d}{dr}M_r^\alpha(u)(x) = \frac{\alpha+1}{r^{\alpha+2}}\int_{\Omega_r(x)}\left(f_\alpha(r) - f_\alpha\left(\frac{1}{\Gamma(x,y)}\right)\right)\mathscr{L}u(y)\,dy, \tag{3-2}$$

where f_α is an antiderivative of $r^{\alpha-1}$ on $(0,\infty)$ (see (1-10)).

Proof. We fix the notation in the statement of the proposition. From the first mean-value formula for $\mathscr{L}$ in (1-3), we get

$$\begin{aligned}
\frac{d}{dr}m_r(u)(x) &= \frac{d}{dr}\left(u(x) + \int_{\Omega_r(x)}\left(\Gamma_x - \frac{1}{r}\right)\mathscr{L}u\right) \qquad \text{(by the co-area formula)}\\
&= \frac{d}{dr}\int_0^r\left(\int_{t=1/\Gamma_x}\left(\Gamma_x - \frac{1}{r}\right)\mathscr{L}u\frac{dH^{N-1}}{|\nabla(1/\Gamma_x)|}\right)dt\\
&= \int_{r=1/\Gamma_x}\left(\Gamma_x - \frac{1}{r}\right)\mathscr{L}u\frac{dH^{N-1}}{|\nabla(1/\Gamma_x)|} + \int_0^r\left(\int_{t=1/\Gamma_x}\frac{1}{r^2}\mathscr{L}u\frac{dH^{N-1}}{|\nabla(1/\Gamma_x)|}\right)dt\\
&= \frac{1}{r^2}\int_{\Omega_r(x)}\mathscr{L}u
\end{aligned}$$

(the first integral is 0; we use the co-area formula again in the second one). We next prove (3-2). From the second mean-value formula for $\mathscr{L}$ (1-3), we get

$$\frac{d}{dr}M_r^\alpha(u)(x) = -\frac{(\alpha+1)^2}{r^{\alpha+2}}\int_0^r \rho^\alpha\Big(\int_{\Omega_\rho(x)}\Big(\Gamma_x-\frac{1}{\rho}\Big)\mathscr{L}u\Big)\,d\rho + \frac{\alpha+1}{r}\int_{\Omega_r(x)}\Big(\Gamma_x-\frac{1}{r}\Big)\mathscr{L}u =: -\mathrm{I}+\mathrm{II}.$$

By applying Fubini's theorem to the summand I we get

$$\mathrm{I} = \frac{(\alpha+1)^2}{r^{\alpha+2}}\int_{\Omega_r(x)}\mathscr{L}u(y)\Big(\int_{1/\Gamma(x,y)}^r\Big(\rho^\alpha\Gamma(x,y)-\rho^{\alpha-1}\Big)\,d\rho\Big)\,dy.$$

By recalling (1-10), since the inner integral in ρ is equal to

$$f_\alpha\Big(\frac{1}{\Gamma(x,y)}\Big) - f_\alpha(r) + \frac{r^{\alpha+1}}{\alpha+1}\Big(\Gamma(x,y) - \frac{1}{r^{\alpha+1}\Gamma^\alpha(x,y)}\Big),$$

we derive for $-\mathrm{I}+\mathrm{II}$ the expression

$$\begin{aligned}
\frac{(\alpha+1)^2}{r^{\alpha+2}}\int_{\Omega_r(x)}\mathscr{L}u\Big(f_\alpha(r)-f_\alpha\Big(\frac{1}{\Gamma_x}\Big)\Big) - \frac{\alpha+1}{r}\int_{\Omega_r(x)}\mathscr{L}u\Big(\Gamma_x-\frac{1}{r^{\alpha+1}\Gamma_x^\alpha}\Big) + \frac{\alpha+1}{r}\int_{\Omega_r(x)}\Big(\Gamma_x-\frac{1}{r}\Big)\mathscr{L}u \\
= \frac{(\alpha+1)^2}{r^{\alpha+2}}\int_{\Omega_r(x)}\mathscr{L}u\Big(f_\alpha(r)-f_\alpha\Big(\frac{1}{\Gamma_x}\Big)\Big) + \frac{\alpha+1}{r}\int_{\Omega_r(x)}\mathscr{L}u\Big(\frac{1}{r^{\alpha+1}\Gamma_x^\alpha}-\frac{1}{r}\Big) \\
= \frac{\alpha+1}{r^{\alpha+2}}\int_{\Omega_r(x)}\mathscr{L}u\Big((\alpha+1)f_\alpha(r)-(\alpha+1)f_\alpha\Big(\frac{1}{\Gamma_x}\Big)+\frac{1}{\Gamma_x^\alpha}-r^\alpha\Big).
\end{aligned}$$

Now, the inner term in parentheses is equal to

$$\begin{cases} f_\alpha(r)-f_\alpha\Big(\dfrac{1}{\Gamma_x}\Big), & \text{if } \alpha=0,\\ (\alpha+1)f_\alpha(r)-(\alpha+1)f_\alpha\Big(\dfrac{1}{\Gamma_x}\Big)+\alpha f_\alpha\Big(\dfrac{1}{\Gamma_x}\Big)-\alpha f_\alpha(r), & \text{if } \alpha\neq 0,\end{cases}$$

and, in turn, this equals $f_\alpha(r) - f_\alpha(1/\Gamma_x)$ after a cancelation in the formula for $\alpha \neq 0$. Because $(d/dr)M_r^\alpha(u)(x) = -\mathrm{I}+\mathrm{II}$, the proof is complete. □

Proposition 3.1 allows us to prove the needed characterization of the $\mathscr{L}$-subharmonicity in the C^2 case.

Proposition 3.2. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let $u \in C^2(\Omega, \mathbb{R})$. Then the following conditions are equivalent (here $\alpha > -1$).*

(1) *u is $\mathscr{L}$-subharmonic on Ω.*

(2) *$\mathscr{L}u \geq 0$ on Ω.*

(3) *For every $x \in \Omega$, the function $r \mapsto m_r(u)(x)$ is $1/r$-convex on $(0, R(x))$.*

(4) *For every $x \in \Omega$, the function $r \mapsto M_r^\alpha(u)(x)$ is $1/r^{\alpha+1}$-convex on $(0, R(x))$.*

The interval $(0, R(x))$ in (3) and (4) above can be replaced with $(0, \varepsilon(x))$ (for some $\varepsilon(x) > 0$), that is, two other characterizations hold true:

(5) *for every $x \in \Omega$, there exists $0 < \varepsilon(x) \leq R(x)$ such that the function $r \mapsto m_r(u)(x)$ is $1/r$-convex on $(0, \varepsilon(x))$;*

(6) *for every* $x \in \Omega$, *there exists* $0 < \varepsilon(x) \le R(x)$ *such that the function* $r \mapsto M_r^\alpha(u)(x)$ *is* $1/r^{\alpha+1}$*-convex on* $(0, \varepsilon(x))$.

Proof. Owing to the submean characterizations of the $\mathcal{L}$-subharmonicity recalled in Section 1, $u \in \underline{\mathcal{S}}(\Omega)$ if and only if $u(x) \le m_r(u)(x)$ for every $x \in \Omega$ and every $r \in (0, R(x))$. When u is C^2, due to the representation formula (1-3), this is clearly equivalent to $\mathcal{L}u \ge 0$ on Ω (recall that $\Gamma(x, y) - 1/r$ is positive on $\Omega_r(x)$). This proves the equivalence of conditions (1) and (2) above.

We now prove the equivalence of conditions (2) and (3). Since $m_r(u)(x)$ is differentiable w.r.t. r (see Proposition 3.1), by Lemma 2.3 we obtain that condition (3) holds true if and only if the function

$$F(r) := r^2 \frac{d}{dr} m_r(u)(x)$$

is monotone nondecreasing on $(0, R(x))$. By (3-1), we have $F(r) = \int_{\Omega_r(x)} \mathcal{L}u$, and this function is nondecreasing if and only if $\mathcal{L}u \ge 0$ (indeed, recall that $\Omega_r(x)$ shrinks to $\{x\}$ as $r \to 0$). This shows the equivalence of (2) and (3).

The equivalence of (2) and (4) can be proved analogously, by showing that

$$F_\alpha(r) := r^{\alpha+2} \frac{d}{dr} M_r^\alpha(u)(x)$$

is monotone nondecreasing on $(0, R(x))$, this time by using (3-2) (and the fact that f_α is strictly increasing *for every* α; see (1-10)).

Obviously, condition (3) implies condition (5), and (4) implies (6).

Finally, we prove that conditions (5) and (6) imply condition (2). Suppose by contradiction that $\mathcal{L}u(x) < 0$ at some point $x \in \Omega$, and hence on some neighborhood $U \subset \Omega$ of x. Due to our hypothesis (H2)(b) on the fundamental solution Γ, we can choose $r_2 > 0$ so small that $r_2 < \varepsilon(x)$ and such that $\overline{\Omega_{r_2}(x)} \subset U$. If r_1 is any positive number less than r_2, we derive that $F(r_2) < F(r_1)$ and $F_\alpha(r_2) < F_\alpha(r_1)$, with the notations above for F and F_α. This shows that conditions (5) and (6) cannot be true, since they are equivalent to the nondecreasing monotonicity on $(0, \varepsilon(x))$ of F and F_α, respectively (by Lemma 2.3). This ends the proof. □

Remark 3.3. We observe that the equivalence "(2) ⇔ (4)" may also be proved as follows, without the aid of formula (3-2). By (3-1), condition (2) holds true if and only if $m_r(u)(x)$ is nondecreasing w.r.t. r on $(0, R(x))$; now we can apply Corollary 2.5(b), which ensures that this last condition is satisfied if and only if

$$r \mapsto \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha m_\rho(u)(x)\, d\rho \text{ is } \frac{1}{r^{\alpha+1}}\text{-convex on } (0, R(x)).$$

Owing to (1-5), this last assertion is nothing but condition (4).

4. Weak derivatives of the average operators of $u \in \underline{\mathcal{S}}(\Omega)$

Our next task is to prove analogues of (3-1) and (3-2) (in the sense of weak derivatives) *for arbitrary* $\mathcal{L}$*-subharmonic functions*. To this end, we need to recall that the $\mathcal{L}$-Riesz measure μ_u of u is characterized by the identity

$$\int_{\Omega} u(x)\mathscr{L}\varphi(x)\,dx = \int_{\Omega} \varphi(x)\,d\mu_u(x) \quad \text{for every } \varphi \in C_0^\infty(\Omega, \mathbb{R}). \tag{4-1}$$

We notice that, fixing a positive r, the average operators $m_r(u)(x)$ and $M_r^\alpha(u)(x)$ are well posed, as functions of the center x, for any $x \in \Omega^r$, where

$$\Omega^r := \{x \in \Omega : \overline{\Omega_r(x)} \subset \Omega\}, \tag{4-2}$$

if this set is nonempty. By our hypothesis (H2)(b) on the fundamental solution Γ, it is easy to see that $\Omega^\varepsilon \uparrow \Omega$ as $\varepsilon \downarrow 0$. Moreover, it is not difficult to prove that

$$\text{for every compact set } K \subset \Omega, \text{ there exists } \varepsilon > 0 \text{ such that } K \subset \Omega^\varepsilon. \tag{4-3}$$

We are ready to give the following keystone result, whose proof is quite delicate.

Theorem 4.1 (derivatives of $m_r(u)$ and $M_r^\alpha(u)$). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in \underline{\mathcal{S}}(\Omega)$ with $\mathscr{L}$-Riesz measure μ_u on Ω. Finally let $x \in \Omega$ be fixed.*

(i) *The function $r \mapsto m_r(u)(x)$ is locally absolutely continuous, hence it is almost everywhere differentiable and its weak derivative (coinciding with its derivative at the points where the latter exists) is given by*

$$\frac{d}{dr} m_r(u)(x) = \frac{\mu_u(\Omega_r(x))}{r^2}. \tag{4-4}$$

Moreover, $m_r(u)(x)$ can be prolonged with continuity at $r = 0$ if and only if $x \in \Omega(u)$, and in this case one has, for every $r \in [0, R(x))$,

$$\begin{aligned} m_r(u)(x) &= u(x) + \int_0^r \frac{\mu_u(\Omega_\rho(x))}{\rho^2}\,d\rho \\ &= u(x) + \int_{\Omega_r(x)} \left(\Gamma(x, y) - \frac{1}{r}\right) d\mu_u(y). \end{aligned} \tag{4-5}$$

(ii) *Let $\alpha > 0$. The function $r \mapsto M_r^\alpha(u)(x)$ is of class C^1 on $(0, R(x))$; its derivative is*

$$\frac{d}{dr} M_r^\alpha(u)(x) = \frac{\alpha+1}{r^{\alpha+2}} \int_{\Omega_r(x)} \left(f_\alpha(r) - f_\alpha\left(\frac{1}{\Gamma(x, y)}\right)\right) d\mu_u(y), \tag{4-6}$$

where f_α is as in (1-10). Moreover, $M_r^\alpha(u)(x)$ can be prolonged with continuity at $r = 0$ if and only if $x \in \Omega(u)$, and in this case one has, for $r \in [0, R(x))$,

$$\begin{aligned} M_r^\alpha(u)(x) &= u(x) + \int_0^r \frac{\alpha+1}{\rho^{\alpha+2}} \left(\int_{\Omega_\rho(x)} \left(f_\alpha(\rho) - f_\alpha\left(\frac{1}{\Gamma(x, y)}\right)\right) d\mu_u(y)\right) d\rho \\ &= u(x) + \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left(\int_{\Omega_\rho(x)} \left(\Gamma(x, y) - \frac{1}{\rho}\right) d\mu_u(y)\right) d\rho. \end{aligned} \tag{4-7}$$

(iii) *The same result as in* (ii) *holds true also for $-1 < \alpha \le 0$, provided that $x \in \Omega(u)$ (in which case* (4-7) *is also satisfied).*

We observe that Theorem 4.1 proves Theorem 1.2.

Remark 4.2. We remark that, for $\alpha > 0$ and $x \in \Omega$ (and for $-1 < \alpha \le 0$, provided that $x \in \Omega(u)$), (4-5) and (4-7) produce the representation formulae

$$u(x) = m_r(u)(x) - \int_{\Omega_r(x)} \left(\Gamma(x, y) - \frac{1}{r}\right) d\mu_u(y),$$

$$u(x) = M_r^\alpha(u)(x) - \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left(\int_{\Omega_\rho(x)} \left(\Gamma(x, y) - \frac{1}{\rho}\right) d\mu_u(y)\right) d\rho.$$

This demonstrates Corollary 1.3. The above formulae are the analogues, of *Poisson–Jensen* type, of the representation formulae (1-3).

Proof of Theorem 4.1. Let us fix $\varepsilon > 0$. Given $u \in \underline{\mathscr{S}}(\Omega)$, by the smoothing result in [Bonfiglioli and Lanconelli 2013, Theorem 7.1] (requiring the C^∞-hypoellipticity of $\mathscr{L}$), there exists a nonincreasing sequence u_n of smooth $\mathscr{L}$-subharmonic functions on the set Ω^ε (see (4-2)) converging point-wise to u on Ω^ε. Given $x \in \Omega^\varepsilon$, if we set

$$R^\varepsilon(x) := \sup\{r > 0 : \Omega_r(x) \subseteq \Omega^\varepsilon\},$$

then $\lim_{\varepsilon\to 0^+} R^\varepsilon(x) = R(x)$ holds. This is a direct consequence of (4-3).

Hence, the theorem is proved if we show that, for any given $x \in \Omega^\varepsilon$, the functions of $r \in [0, R^\varepsilon(x))$ given by $m_r(u)(x)$ and $M_r^\alpha(u)(x)$ are locally a.c. on $(0, R^\varepsilon(x))$, and that their weak derivatives are given by (4-4) and (4-6).

Since $u_n \in C^\infty(\Omega^\varepsilon, \mathbb{R})$, from (3-1) we have

$$\frac{d}{dr} m_r(u_n)(x) = \frac{1}{r^2} \int_{\Omega_r(x)} \mathscr{L} u_n$$

for every $x \in \Omega^\varepsilon$ and every $r \in (0, R^\varepsilon(x))$. Let $\psi(r)$ be a smooth function compactly supported in $(0, R^\varepsilon(x))$; we multiply both sides of the above equality by $\psi(r)$, we integrate with respect to $r \in (0, R^\varepsilon(x))$, and we use integration by parts in the left-hand side, thus getting

$$\int \psi'(r) m_r(u_n)(x)\, dr = \int \psi(r) \left(\frac{1}{r^2} \int_{\Omega_r(x)} \mathscr{L} u_n\right) dr. \tag{4-8}$$

We aim to let $n \to \infty$ in this identity. To begin with, we claim that

$$\lim_{n\to\infty} \int \psi'(r) m_r(u_n)(x)\, dr = \int \psi'(r) m_r(u)(x)\, dr. \tag{4-9}$$

To prove this claim, we observe that, by arguing as in the proof of (5-2), we have

$$\lim_{n\to\infty} m_r(u_n)(x) = m_r(u)(x) \quad \text{for all } x \in \Omega^\varepsilon, r \in (0, R^\varepsilon(x)). \tag{4-10}$$

As a consequence of (4-10), (4-9) holds true if we prove that, in the left-hand side of (4-9), it is possible to apply the dominated convergence theorem. This is indeed possible as a direct consequence of the bounds

$$u \le u_n \le u_1 \Longrightarrow -\infty < m_r(u)(x) \le m_r(u_n)(x) \le m_r(u_1)(x) < \infty.$$

We next investigate the right-hand side of (4-8). If we denote by $[a, b]$ the support of ψ (recall that $0 < a < b < R^\varepsilon(x)$), by Fubini's theorem we have

$$\begin{aligned}\int \psi(r)\Big(\frac{1}{r^2}\int_{\Omega_r(x)} \mathscr{L}u_n\Big)\,dr &= \int \psi(r)\Big(\frac{1}{r^2}\int_{\Gamma(x,y)>1/r} \mathscr{L}u_n(y)\,dy\Big)\,dr \\ &= \int_{\Omega_b(x)} \mathscr{L}u_n(y)\Big(\int_{\max\{1/\Gamma(x,y),a\}}^{b} \psi(r)\frac{dr}{r^2}\Big)\,dy =: \int_{\Omega_b(x)} \mathscr{L}u_n(y)\Psi(y)\,dy.\end{aligned}$$

Now the function Ψ is supported in $\Omega_b(x)$, it is identically equal to the constant function $\int_a^b \psi(r)\,dr/r^2$ on $\Omega_a(x)$, and it is smooth because $\Gamma(x,\,\cdot)$ is smooth outside x. Hence we can integrate by parts two times to derive

$$\int \psi(r)\Big(\frac{1}{r^2}\int_{\Omega_r(x)} \mathscr{L}u_n\Big)\,dr = \int u_n(y)\mathscr{L}\Psi(y)\,dy.$$

From $u \le u_n \le u_1$ we get $|u_n| \le \max\{|u|, |u_1|\}$; hence, by recalling that $\mathscr{L}$-subharmonic functions are locally integrable [Negrini and Scornazzani 1987], and by observing that $\mathscr{L}\Psi \in C_0^\infty(\Omega^\varepsilon)$, a dominated convergence argument finally proves that

$$\lim_{n\to\infty} \int \psi(r)\Big(\frac{1}{r^2}\int_{\Omega_r(x)} \mathscr{L}u_n\Big)\,dr = \int u(y)\mathscr{L}\Psi(y)\,dy \overset{\text{(4-1)}}{=} \int \Psi(y)\,d\mu_u(y).$$

On the other hand, again by Fubini's theorem, we infer that

$$\begin{aligned}\int \Psi(y)\,d\mu_u(y) &= \int_{\Omega_b(x)} \Big(\int_{\max\{1/\Gamma(x,y),a\}}^{b} \psi(r)\frac{dr}{r^2}\Big)\,d\mu_u(y) \\ &= \int \psi(r)\Big(\frac{1}{r^2}\int_{\Omega_r(x)} d\mu_u(y)\Big)\,dr = \int \psi(r)\frac{\mu_u(\Omega_r(x))}{r^2}\,dr.\end{aligned}$$

Summing up, we have proved that

$$\lim_{n\to\infty} \int \psi(r)\Big(\frac{1}{r^2}\int_{\Omega_r(x)} \mathscr{L}u_n\Big)\,dr = \int \psi(r)\frac{\mu_u(\Omega_r(x))}{r^2}\,dr. \tag{4-11}$$

Gathering together (4-9) and (4-11), from (4-8) we derive

$$\int \psi'(r)m_r(u)(x)\,dr = \int \psi(r)\frac{\mu_u(\Omega_r(x))}{r^2}\,dr.$$

From the arbitrariness of $\psi \in C_0^\infty((0, R^\varepsilon(x)), \mathbb{R})$, this shows that $m_r(u)(x)$ possesses a weak derivative on $(0, R^\varepsilon(x))$, and this is equal to $\mu_u(\Omega_r(x))/r^2$. From the arbitrariness of $\varepsilon > 0$, we infer that $m_r(u)(x)$ is weakly differentiable on (α, β) for every α, β such that $0 < \alpha < \beta < R(x)$, and its weak derivative is $\mu_u(\Omega_r(x))/r^2$. Note that this function is integrable on (α, β), since

$$\int_\alpha^\beta \frac{\mu_u(\Omega_r(x))}{r^2}\,dr \le \frac{\mu_u(\overline{\Omega_\beta(x)})}{\alpha^2}(\beta - \alpha) < \infty,$$

the last inequality following from the finiteness of μ_u on the compact subsets of Ω.

This proves that $m_r(u)(x)$ is equal almost everywhere to a continuous function on $(0, R(x))$, say $m(r)$, and $m(r)$ is locally a.c. on $(0, R(x))$, with weak derivative given by $\mu_u(\Omega_r(x))/r^2$; since absolutely continuous functions are almost everywhere differentiable, we also get $m'(r) = \mu_u(\Omega_r(x))/r^2$ for almost every $r \in (0, R(x))$. Moreover, since absolutely continuous functions satisfy the fundamental theorem of calculus, we also have

$$m(r) = m(r_1) + \int_r^{r_2} \frac{1}{\rho^2} \mu_u(\Omega_\rho(x))\, d\rho,$$

whenever $0 < r_1 < r < R(x)$.

As $m_r(u)(x)$ is monotone (see (A.2) in Section 1), it can be equal almost everywhere to $m(r)$ (which is a continuous function) only if $m_r(u)(x) = m(r)$ for every $r \in (0, R(x))$. Thus $m_r(u)(x)$ inherits all the above properties of $m(r)$. In particular, whenever $0 < r_1 < r < R(x)$, we get

$$m_r(u)(x) - m_{r_1}(u)(x) = \int_{r_1}^{r} \frac{1}{\rho^2} \mu_u(\Omega_\rho(x))\, d\rho.$$

Letting $r_1 \to 0^+$, by Beppo Levi's theorem and by exploiting the m-continuity of $\mathscr{L}$-subharmonic functions (see property (A.2)), we obtain

$$m_r(u)(x) - u(x) = \int_0^r \frac{1}{\rho^2} \mu_u(\Omega_\rho(x))\, d\rho,$$

where both sides are $+\infty$ if and only if $u(x) = -\infty$ (recall that $m_r(u)(x)$ is always finite). Otherwise, when $x \in \Omega(u)$ both sides are finite, and we get the first formula in (4-5). In this latter case, we derive that $\mu_u(\Omega_\rho(x))/\rho^2$ is integrable on every compact subinterval of $[0, R(x))$, so that the function $r \mapsto m_r(u)(x)$ (defined as $u(x)$ when $r = 0$) is locally a.c. on $[0, R(x))$. The second formula in (4-5) can be obtained by Tonelli's theorem, since

$$\begin{aligned}\int_0^r \frac{1}{\rho^2} \mu_u(\Omega_\rho(x))\, d\rho &= \int_0^r \frac{1}{\rho^2} \left(\int_{1/\Gamma(x,y)<\rho} d\mu_u(y) \right) d\rho \\ &= \int_{1/\Gamma(x,y)<r} \left(\int_{1/\Gamma(x,y)}^{r} \frac{1}{\rho^2}\, d\rho \right) d\mu_u(y) = \int_{\Omega_r(x)} \left(\Gamma(x, y) - \frac{1}{r} \right) d\mu_u(y).\end{aligned}$$

This completes the proof of the theorem where surface average operators are concerned. The case of solid average operators can be proved analogously, this time starting from (3-2), and by recalling that $M_r^\alpha(u)(x)$ is always finite if $\alpha > 0$, and it is finite for $-1 < \alpha \le 0$ if $x \in \Omega(u)$.

Note that the fact that $M_r^\alpha(u)(x)$ is of class C^1 is a consequence of identity (1-5), together with the continuity of $m_r(u)(x)$ up to $r = 0$ (when $x \in \Omega(u)$). The fact that the two formulae in (4-7) are equivalent to one another can be proved by direct computations, by taking into account that

$$\int_0^r \left(\int_{\Omega_\rho(x)} g(\rho, y)\, d\mu_u(y) \right) d\rho = \int_{\Omega_r(x)} \left(\int_{1/\Gamma(x,y)}^{r} g(\rho, y)\, d\rho \right) d\mu_u(y)$$

for every integrable function $g(\rho, y)$. $\square$

5. Subharmonicity and convexity of the average operators

We are ready to give the proof of Theorem 1.5. We highlight the fact that, over the course of this section, we shall provide finer versions of Theorem 1.5, namely, Theorems 5.1 and 5.2 below.

Proof of Theorem 1.5. We split the proof into six short parts.

(1) $\Rightarrow$ (2). If $u \in \underline{\mathcal{S}}(\Omega)$ and $x \in \Omega$, by Theorem 4.1, $r \mapsto m(r) := m_r(u)(x)$ is locally a.c. on $(0, R(x))$ and, due to identity (4-4), one has (for almost every $r \in (0, R(x))$)

$$r^2 m'(r) = \mu_u(\Omega_r(x)),$$

the latter being a nondecreasing function of r. This shows that $r^2 m'(r)$ is essentially monotone nondecreasing on $(0, R(x))$. By Lemma 2.3 (for $\beta = 1$) we see that $m(r)$ is r^{-1}-convex. Finally, the m-continuity of u is contained in (A.2). This proves statement (2) of the theorem.

(2) $\Rightarrow$ (3). This is obvious.

(3) $\Rightarrow$ (1). Let $x \in \Omega(u)$. By the assumption (3), the map $r \mapsto m(r) := m_r(u)(x)$ is r^{-1}-convex on $(0, R(x))$. On the other hand, for $0 < r \le a < R(x)$, one has

$$m(r) \le \sup\{u(y) : y \in \overline{\Omega_a(x)}\} < \infty,$$

due to $m_r(1)(x) = 1$, the upper semicontinuity of u, and the compactness of $\overline{\Omega_a(x)}$. Thus $m(r)$ is bounded from above on $(0, a)$ for every positive $a < R(x)$. An application of Lemma 2.2 (for $\beta = 1$) shows that $m(r)$ is monotone nondecreasing on $(0, R(x))$. Since u is m-continuous by assumption (3), this gives

$$u(x) = \lim_{r \to 0^+} m_r(u)(x) \le m_r(u)(x) \quad \text{for all } x \in \Omega(u),\, r \in (0, R(x)).$$

On the other hand, the inequality $u(x) \le m_r(u)(x)$ is trivially satisfied when $x \notin \Omega(u)$ (because this means that $u(x) = -\infty$). Therefore, one has $u(x) \le m_r(u)(x)$ for every $r \in (0, R(x))$ and every $x \in \Omega$. By the characterization (A.1) of the $\mathcal{L}$-subharmonicity, we deduce that $u \in \underline{\mathcal{S}}(\Omega)$.

(1) $\Rightarrow$ (4). Let $\alpha > 0$. If $u \in \underline{\mathcal{S}}(\Omega)$ and $x \in \Omega$, by Theorem 4.1, the function $r \mapsto M(r) := M_r^\alpha(u)(x)$ is C^1 on $(0, R(x))$ and, due to identity (4-6), one has

$$r^{\alpha+2} M'(r) = (\alpha + 1) \int_{\Omega_r(x)} \left(f_\alpha(r) - f_\alpha\left(\frac{1}{\Gamma(x, y)}\right)\right) d\mu_u(y),$$

where f_α is as in (1-10). Note that the function in the right-hand side is nondecreasing w.r.t. r, because this is true of f_α (and $r > 1/\Gamma(x, y)$ on $\Omega_r(x)$). This shows that $r^{\alpha+2} M'(r)$ is monotone nondecreasing on $(0, R(x))$. An application of Lemma 2.3 (for $\beta = \alpha + 1$) proves that $M(r)$ is $r^{-(\alpha+1)}$-convex. Finally, the M^α-continuity of u is contained in (A.2) (with m_r replaced with M_r^α). This proves statement (4) of the theorem.

(4) $\Rightarrow$ (1). Suppose there exists $\alpha > 0$ such that $r \mapsto M_r^\alpha(u)(x)$ is $r^{-(\alpha+1)}$-convex on $(0, R(x))$ for every $x \in \Omega$. By arguing as in the above proof of “(3) $\Rightarrow$ (1)”, an application of Lemma 2.2 (for $\beta = \alpha + 1$) shows that $M_r^\alpha(u)(x)$ is monotone nondecreasing on $(0, R(x))$. Since u is M^α-continuous by assumption

(4), we get (see the above argument) $u(x) \le M_r^\alpha(u)(x)$ for every $x \in \Omega$ and $r \in (0, R(x))$. By the characterization (A.1) of the $\mathscr{L}$-subharmonicity (with m_r replaced with M_r^α), we deduce that $u \in \underline{\mathcal{S}}(\Omega)$.

(1) $\Leftrightarrow$ (5). This can be proved by using similar arguments as above (this time invoking identity (4-6) for $\alpha \in (-1, 0]$ and $x \in \Omega(u)$; note that f_α is increasing also for nonpositive values of α; see (1-10)). □

We next turn to proving a more refined versions of the implications (1) $\Rightarrow$ (2), (3), (4), (5) of Theorem 1.5.

Theorem 5.1 (subharmonicity implies convexity of the average operators). *Suppose that $\Omega \subseteq \mathbb{R}^N$ is an open set, and let $u \in \underline{\mathcal{S}}(\Omega)$. Then we have the following.*

(1) *For every $x \in \Omega$ the average operator $m_r(u)(x)$ is $1/r$-convex on $(0, R(x))$; furthermore, it is $1/r^\beta$-convex also for $\beta \ge 1$.*

(2) *When $\alpha > 0$, for every $x \in \Omega$, the average operator $M_r^\alpha(u)(x)$ is $1/r^{\alpha+1}$-convex on $(0, R(x))$; furthermore, it is $1/r^\beta$-convex also for $\beta \ge 1$.*

(3) *When $-1 < \alpha \le 0$, for every $x \in \Omega(u)$, the average operator $M_r^\alpha(u)(x)$ is $1/r^{\alpha+1}$-convex on $(0, R(x))$; furthermore, it is $1/r^\beta$-convex also for $\beta \ge \alpha + 1$.*

Proof. Let us fix $\varepsilon > 0$. Given $u \in \underline{\mathcal{S}}(\Omega)$, by the smoothing result in [Bonfiglioli and Lanconelli 2013, Theorem 7.1] (recall that we assumed $\mathscr{L}$ to be C^∞-hypoelliptic), there exists a nonincreasing sequence u_n of smooth $\mathscr{L}$-subharmonic functions on the set Ω^ε (see (4-2)) converging point-wise to u on Ω^ε. Given $x \in \Omega^\varepsilon$, if we set

$$R^\varepsilon(x) := \sup\{r > 0 : \Omega_r(x) \subseteq \Omega^\varepsilon\},$$

then $\lim_{\varepsilon\to 0^+} R^\varepsilon(x) = R(x)$ holds. (This is a direct consequence of (4-3).)

Hence, the theorem is proved if we show that, for any given $x \in \Omega^\varepsilon$, the functions of $r \in (0, R^\varepsilon(x))$ given by $m_r(u)(x)$ and $M_r^\alpha(u)(x)$ are $r^{-\beta}$-convex, respectively, for $\beta \ge 1$ and for $\beta \ge \min\{1, \alpha+1\}$.

To this end, let us observe that, since $u_n \in \underline{\mathcal{S}}(\Omega^\varepsilon) \cap C^\infty(\Omega^\varepsilon, \mathbb{R})$, from Proposition 3.2 we know the following.

- $m_r(u_n)(x)$ is r^{-1}-convex on $(0, R^\varepsilon(x))$; since this function is smooth w.r.t. r and monotone nondecreasing (see (3-1) and recall that $\mathscr{L}u_n \ge 0$), by Lemma 2.4 we infer that it is also $r^{-\beta}$-convex for every $\beta \ge 1$.
- $M_r^\alpha(u_n)(x)$ is $r^{-(\alpha+1)}$-convex on $(0, R^\varepsilon(x))$; from the r^{-1}-convexity of the surface mean $m_r(u_n)(x)$ we derive that $M_r^\alpha(u_n)(x)$ is also r^{-1}-convex, owing to Corollary 2.5(a); since $M_r^\alpha(u_n)(x)$ is smooth w.r.t. r and monotone nondecreasing (see (3-2)), by Lemma 2.4 we infer that it is also $r^{-\beta}$-convex for every $\beta \ge \min\{1, \alpha+1\}$.

We now show that the above properties are inherited by $m_r(u)(x)$ and $M_r^\alpha(u)(x)$, by passing to the limit as $n \to \infty$. We prove it for solid average operators, the argument for surface average operators being completely analogous. Let $\beta \ge \min\{1, \alpha+1\}$. We know that (setting $\varphi(r) = r^{-\beta}$)

$$M_r^\alpha(u_n)(x) \le \frac{\varphi(r_2)-\varphi(r)}{\varphi(r_2)-\varphi(r_1)} M_{r_1}^\alpha(u_n)(x) + \frac{\varphi(r)-\varphi(r_1)}{\varphi(r_2)-\varphi(r_1)} M_{r_2}^\alpha(u_n)(x) \tag{5-1}$$

for every $r_1, r, r_2 \in (0, R^\varepsilon(x))$ such that $r_1 < r < r_2$. We claim that

$$\lim_{n\to\infty} M_r^\alpha(u_n)(x) = M_r^\alpha(u)(x) \quad \text{for all } x \in \Omega^\varepsilon,\ r \in (0, R^\varepsilon(x)). \tag{5-2}$$

Once this claim is proved, letting $n \to \infty$ in (5-1), we get

$$M_r^\alpha(u)(x) \le \frac{\varphi(r_2)-\varphi(r)}{\varphi(r_2)-\varphi(r_1)} M_{r_1}^\alpha(u)(x) + \frac{\varphi(r)-\varphi(r_1)}{\varphi(r_2)-\varphi(r_1)} M_{r_2}^\alpha(u)(x)$$

for every $r_1, r, r_2 \in (0, R^\varepsilon(x))$ such that $r_1 < r < r_2$. This is precisely what we aim to prove, that is, $M_r^\alpha(u)(x)$ is an $r^{-\beta}$-convex function of r on $(0, R^\varepsilon(x))$.

We finally turn to prove the claimed (5-2). We fix any $x \in \Omega^\varepsilon$ and any $r \in (0, R^\varepsilon(x))$. Let us consider the sequence v_n defined by

$$v_n(x) := u_1(x) - u_n(x), \quad x \in \Omega^\varepsilon.$$

Since $\{u_n\}_n$ is monotone nonincreasing, we infer that $\{v_n\}_n$ is monotone nondecreasing and nonnegative. Moreover, by construction of u_n, we have $v_n \to u_1 - u$, as $n \to \infty$, point-wise on Ω^ε. As $K_\alpha \ge 0$ (see (1-2)), we are therefore entitled to apply the monotone convergence theorem to derive that

$$\lim_{n\to\infty} \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(x)} v_n(y) K_\alpha(x,y)\,dy = \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(x)} (u_1(y) - u(y)) K_\alpha(x,y)\,dy.$$

Recalling that $M_r^\alpha(u)(x)$ is finite valued (see Remark 1.1) for any $\alpha > 0$, and also for $-1 < \alpha \le 0$ provided that $x \in \Omega(u)$, we obtain the following identity from the above one (whenever $M_r^\alpha(u)(x) > -\infty$):

$$\lim_{n\to\infty} (M_r^\alpha(u_1)(x) - M_r^\alpha(u_n)(x)) = M_r^\alpha(u_1)(x) - M_r^\alpha(u)(x).$$

By canceling out $M_r^\alpha(u_1)(x)$ (when it is finite), we get (5-2) and the proof of statements (2) and (3) of the theorem is complete. The proof of (1) is analogous, taking into account that $m_r(u)(x)$ is always finite (see Remark 1.1). $\square$

The next result provides the reverse implication of Theorem 5.1. Also, it proves refined versions of the implications (2), (3), (4), (5) $\Rightarrow$ (1) of Theorem 1.5.

Theorem 5.2 (convexity of the average operators implies subharmonicity). *Suppose that $\Omega \subseteq \mathbb{R}^N$ is an open set and $\alpha > -1$. Let $u : \Omega \to [-\infty, \infty)$ be an u.s.c. function such that $\Omega(u)$ intersects every connected component of Ω.*

Then, any of the following conditions implies that u is $\mathcal{L}$-subharmonic in Ω:

(1) *u is m-continuous in Ω and, for every fixed $x \in \Omega(u)$, the average operator $m_r(u)(x)$ is $1/r^\gamma$-convex on $(0, R(x))$ for some $\gamma > 0$.*

(2) *u is M^α-continuous in Ω and, for every fixed $x \in \Omega(u)$, the average operator $M_r^\alpha(u)(x)$ is $1/r^\gamma$-convex on $(0, R(x))$ for some $\gamma > 0$.*

We explicitly point out that this result holds true for every sub-Laplacian $\mathcal{L}$ on any Carnot group of homogeneous dimension $Q > 2$, since $\mathcal{L}$ satisfies all the properties in Section 2 (see, for example, [Bonfiglioli et al. 2007]); hence, as a very special case, *Theorem 5.2 holds true for the classical Laplace operator* Δ *on* $\mathbb{R}^N$, *with* $N \geq 3$. This result seems to be new in the literature.

Proof. Since u is u.s.c., u is locally bounded from above. This ensures that, for every fixed $x \in \Omega$, $m_r(u)(x)$ and $M_r^\alpha(u)(x)$ are bounded from above on $(0, a]$ for every positive $a < R(x)$. If condition (1) of Theorem 5.2 holds true (respectively condition (2)), we can apply Lemma 2.2 to derive that, for every $x \in \Omega(u)$, the average operator $m_r(u)(x)$ (respectively $M_r^\alpha(u)(x)$) is monotone nondecreasing on $(0, a]$ for every $a < R(x)$. Hence it is nondecreasing on the whole of $(0, R(x))$. Since u is supposed to be m-continuous (respectively M^α-continuous), we infer that, for every $x \in \Omega(u)$, one has

$$u(x) = \lim_{r\to 0^+} m_r(u)(x) \leq m_r(u)(x) \quad (\text{respectively } u(x) = \lim_{r\to 0^+} M_r^\alpha(u)(x) \leq M_r^\alpha(u)(x))$$

for every $r \in (0, R(x))$. On the other hand, the inequality $u(x) \leq m_r(u)(x)$ (respectively $u(x) \leq M_r^\alpha(u)(x)$) is trivially satisfied when $x \notin \Omega(u)$ (since this means that $u(x) = -\infty$). Therefore, one has $u(x) \leq m_r(u)(x)$ (respectively $u(x) \leq M_r^\alpha(u)(x)$) for every $r \in (0, R(x))$ and every $x \in \Omega$. By the characterization (A.1) of the $\mathscr{L}$-subharmonicity (respectively the analogue of (A.1) with m_r replaced with M_r^α), we deduce that u is $\mathscr{L}$-subharmonic in Ω. □

6. The case of Γ-annuli

In this section we use the following notation: given a, b such that $0 \leq a < b \leq \infty$, and given $x_0 \in \mathbb{R}^N$, we set

$$A_{a,b}(x_0) := \left\{ x \in \mathbb{R}^N : \frac{1}{b} < \Gamma(x_0, x) < \frac{1}{a} \right\} \tag{6-1}$$

(with the convention that $1/\infty = 0$ and $1/0 = \infty$), and we say that $A_{a,b}(x_0)$ is the Γ-*annulus* of center x_0 and radii a, b. The notation $A_{a,b}$ will apply instead of $A_{a,b}(x_0)$ whenever x_0 is understood. Our main task is to prove the following result, from which applications will be derived in Section 7.

Theorem 6.1. *Let* $0 \leq a < b \leq \infty$ *and* $x_0 \in \mathbb{R}^N$ *be fixed. Suppose* u *is* $\mathscr{L}$*-subharmonic on* $A_{a,b}(x_0)$. *Then the function*

$$(a, b) \ni r \mapsto m_r(u)(x_0)$$

is r^{-1}*-convex and locally absolutely continuous on* (a, b). *For every* α, β *such that* $a < \alpha < \beta < b$, *there exists* c (*depending on* $a, \alpha, \beta, b, u, x_0$) *such that the* (*weak*) *derivative of* $m_r(u)(x_0)$ *on* (α, β) *is given by*

$$\frac{d}{dr} m_r(u)(x_0) = \frac{1}{r^2}(\mu_u(A_{\alpha,r}(x_0)) + c) \tag{6-2}$$

for almost every $r \in (\alpha, \beta)$. *As usual,* μ_u *is the* $\mathscr{L}$*-Riesz measure of* u *on* $A_{a,b}(x_0)$.

This proves Theorem 1.6.

Remark 6.2. We cannot expect that analogues of Theorems 5.1 and 5.2 will hold true in the case of Γ-annuli, since, in the case of a Γ-annulus

- $\mathscr{L}$-subharmonicity does not necessarily imply $r^{-\beta}$-convexity, when $\beta > 1$;

- solid α-means are not well-posed;
- $m_r(u)(x_0)$ is not necessarily monotone nondecreasing.

See Remark 6.5 at the end of the section for related results (and a converse of Theorem 6.1 for C^2 functions which are "radial" with respect to Γ).

In order to prove Theorem 6.1, we need a substitute for identity (3-1). This is given in the next result.

Lemma 6.3. *Let $x_0 \in \mathbb{R}^N$ and $0 \le R_1 < R_2 \le \infty$ be fixed, and suppose that $u \in C^2(A_{R_1,R_2}(x_0), \mathbb{R})$. Given any $R \in (R_1, R_2)$, one has*

$$\frac{d}{dr}\Big|_{r=R} m_r(u)(x_0) = \frac{1}{R^2}\left(\int_{\partial\Omega_\rho(x_0)} \langle A\nabla u, \nu\rangle \, dH^{N-1} + \int_{A_{\rho,R}(x_0)} \mathscr{L}u\right) \tag{6-3}$$

for every $\rho \in (R_1, R)$. In particular, we have

$$r_2^2 \frac{d}{dr}\Big|_{r=r_2} m_r(u)(x_0) - r_1^2 \frac{d}{dr}\Big|_{r=r_1} m_r(u)(x_0) = \int_{A_{r_1,r_2}(x_0)} \mathscr{L}u \tag{6-4}$$

for every r_1, r_2 such that $R_1 < r_1 < r_2 < R_2$.

Proof. Let $\Omega \subset \mathbb{R}^N$ be any bounded open set whose boundary is regular enough to support the divergence theorem. The divergence form (2-1) of $\mathscr{L} = \operatorname{div}(A\nabla)$ gives

$$\int_\Omega (u\mathscr{L}v - v\mathscr{L}u) = \int_{\partial\Omega} (u\langle A\nabla v, \nu_{\text{est}}\rangle - v\langle A\nabla u, \nu_{\text{est}}\rangle) \, dH^{N-1} \tag{6-5}$$

for every $u, v \in C^2(\overline{\Omega}, \mathbb{R})$. Here ν_{est} denotes the exterior normal unit vector on $\partial\Omega$. Let $u \in C^2(A_{R_1,R_2}(x_0))$ and let us take any ρ, r such that $R_1 < \rho < r < R_2$. Choosing $\Omega = A_{\rho,r}(x_0)$ and $v = -\Gamma_{x_0}$, and since

$$\nu_{\text{est}}(x) = \begin{cases} -\nu(x) := +\nabla\Gamma_{x_0}(x)/|\nabla\Gamma_{x_0}(x)|, & \text{if } x \in \partial\Omega_\rho(x_0), \\ +\nu(x) := -\nabla\Gamma_{x_0}(x)/|\nabla\Gamma_{x_0}(x)|, & \text{if } x \in \partial\Omega_r(x_0), \end{cases} \tag{6-6}$$

from (6-5) we derive (recalling that Γ_{x_0} is $\mathscr{L}$-harmonic on $\mathbb{R}^N \setminus \{x_0\}$)

$$\int_{A_{\rho,r}(x_0)} \Gamma_{x_0}\mathscr{L}u = m_r(u)(x_0) - m_\rho(u)(x_0) + \frac{1}{r} J_r(u)(x_0) - \frac{1}{\rho} J_\rho(u)(x_0), \tag{6-7}$$

where m_r is the usual surface average operator, while

$$J_R(u)(x_0) := \int_{\partial\Omega_R(x_0)} \langle A\nabla u, \nu\rangle \, dH^{N-1} \quad \text{for } R = r \text{ and } R = \rho, \tag{6-8}$$

and ν is as in (6-6) (note that ν is the normal unit vector on $\partial\Omega_R(x_0)$ which is exterior to the set $\Omega_R(x_0)$). If in (6-5) we take $v \equiv -1$ and $\Omega = A_{\rho,r}(x_0)$, we get

$$\int_{A_{\rho,r}(x_0)} \mathscr{L}u = J_r(u)(x_0) - J_\rho(u)(x_0). \tag{6-9}$$

We set $f(r) := m_r(u)(x_0)$ for brevity and we differentiate both sides of (6-7) w.r.t. r:

$$\frac{d}{dr}\int_{A_{\rho,r}(x_0)} \Gamma_{x_0}\mathscr{L}u = f'(r) - \frac{1}{r^2} J_r(u)(x_0) + \frac{1}{r}\frac{d}{dr} J_r(u)(x_0). \tag{6-10}$$

On the one hand, owing to the co-area formula, we have

$$\begin{aligned}
&\frac{d}{dr}\int_{A_{\rho,r}(x_0)} \Gamma_{x_0}\mathscr{L}u \\
&\qquad= \frac{d}{dr}\int_{A_{\rho,r}(x_0)} \Big(\frac{1}{r}+\Gamma_{x_0}-\frac{1}{r}\Big)\mathscr{L}u \\
&\qquad= -\frac{1}{r^2}\int_{A_{\rho,r}(x_0)} \mathscr{L}u + \frac{1}{r}\frac{d}{dr}\int_{A_{\rho,r}(x_0)} \mathscr{L}u + \frac{d}{dr}\int_\rho^r \Big(\int_{1/\Gamma_{x_0}=t}\Big(\Gamma_{x_0}-\frac{1}{r}\Big)\mathscr{L}u \frac{dH^{N-1}}{|\nabla(1/\Gamma_{x_0})|}\Big)\,dt \\
&\qquad= -\frac{1}{r^2}\int_{A_{\rho,r}(x_0)} \mathscr{L}u + \frac{1}{r}\frac{d}{dr}\int_{A_{\rho,r}(x_0)} \mathscr{L}u \\
&\qquad\qquad\qquad + \int_{1/\Gamma_{x_0}=r}\Big(\Gamma_{x_0}-\frac{1}{r}\Big)\mathscr{L}u \frac{dH^{N-1}}{|\nabla(1/\Gamma_{x_0})|} + \int_\rho^r \Big(\int_{1/\Gamma_{x_0}=t}\frac{1}{r^2}\mathscr{L}u \frac{dH^{N-1}}{|\nabla(1/\Gamma_{x_0})|}\Big)\,dt.
\end{aligned}$$

The third summand is 0, while the fourth is the opposite of the first one. Thus

$$\frac{d}{dr}\int_{A_{\rho,r}(x_0)} \Gamma_{x_0}\mathscr{L}u = \frac{1}{r}\frac{d}{dr}\int_{A_{\rho,r}(x_0)} \mathscr{L}u \overset{(6\text{-}9)}{=} \frac{1}{r}\frac{d}{dr}J_r(u)(x_0).$$

This shows that the left-hand side of (6-10) and the last summand of its right-hand side are equal. Thus (6-10) is equivalent to

$$f'(r) = \frac{1}{r^2}J_r(u)(x_0).$$

Taking into consideration (6-9) again, we get

$$f'(r) = \frac{1}{r^2}\Big(J_\rho(u)(x_0) + \int_{A_{\rho,r}(x_0)} \mathscr{L}u\Big), \quad R_1 < \rho < r < R_2. \tag{6-11}$$

This proves (6-3). Equivalently, we also obtain that

$$r^2 f'(r) = J_\rho(u)(x_0) + \int_{A_{\rho,r}(x_0)} \mathscr{L}u, \quad R_1 < \rho < r < R_2. \tag{6-12}$$

If r_1, r_2 are such that $R_1 < r_1 < r_2 < R_2$, we can choose any ρ satisfying $R_1 < \rho < r_1$. Taking $r = r_2$ in (6-12) and subtracting side by side what we get by taking $r = r_1$ in (6-12), we finally obtain

$$r_2^2 f'(r_2) - r_1^2 f'(r_1) = \int_{A_{\rho,r_2}(x_0)} \mathscr{L}u - \int_{A_{\rho,r_1}(x_0)} \mathscr{L}u = \int_{A_{r_1,r_2}(x_0)} \mathscr{L}u,$$

which is (6-4). □

We remark that, if $u \in C^2(\Omega_{R_2}(x_0), \mathbb{R})$, letting $\rho \to 0^+$ in (6-3), one gets back formula (3-1). Indeed,

$$\lim_{\rho\to 0^+}\int_{\partial\Omega_\rho(x_0)} \langle A\nabla u, \nu\rangle\, dH^{N-1} = 0,$$

as it follows from the identity $\int_{\partial\Omega_\rho(x_0)}\langle A\nabla u, \nu\rangle\, dH^{N-1} = \int_{\partial\Omega_\rho(x_0)} \mathscr{L}u$ (a consequence of (6-5) taking $v \equiv -1$ and $\Omega = \Omega_\rho(x_0)$).

Proof of Theorem 6.1. First we observe that Theorem 6.1 holds true if, together with the other assumptions, u is of class C^2. Indeed, if u is C^2 and $\mathscr{L}$-subharmonic, we have $\mathscr{L}u \geq 0$ on $A_{a,b}$; thus (6-4) proves that $r^2(d/dr)(m_r(u)(x_0))$ is monotone nondecreasing on (a, b). Lemma 2.1(3) ensures that $m_r(u)(x_0)$ is r^{-1}-convex on (a, b) and that formula (6-4) holds true.

The general case of $u \in \underline{\mathscr{S}}(A_{a,b})$ can be proved by the very same approximation technique as in the proofs of Theorems 4.1 and 5.1. □

Remark 6.4. Another example of a convex function naturally associated to an $\mathscr{L}$-subharmonic function is

$$B(r) := \sup_{\partial\Omega_r(x_0)} u.$$

Indeed, let us prove that, *if $u \in \underline{\mathscr{S}}(A_{a,b}(x_0))$, then $B(r)$ is an r^{-1}-convex function of $r \in (a, b)$.* Fix any r_1, r_2 such that $a < r_1 < r_2 < b$. We need to prove that $B(r) \leq I(r)$ for every $r \in (r_1, r_2)$, where

$$I(r) = \frac{1/r_2 - 1/r}{1/r_2 - 1/r_1} B(r_1) + \frac{1/r - 1/r_1}{1/r_2 - 1/r_1} B(r_2).$$

We remark that $I(r_i) = B(r_i)$ for $i = 1, 2$ and

$$I(r) = \frac{1}{r}a + b, \quad \text{where } a = \frac{B(r_2) - B(r_1)}{1/r_2 - 1/r_1},\ b = \frac{B(r_1)/r_2 - B(r_2)/r_1}{1/r_2 - 1/r_1}.$$

With these same notations, we set $v(x) := I(1/\Gamma(x_0, x)) = a\Gamma(x_0, x) + b$. Clearly v is $\mathscr{L}$-harmonic in $\mathbb{R}^N \setminus \{x_0\}$; moreover, for every $x \in \partial\Omega_{r_i}(x_0)$, one has

$$v(x) = I(r_i) = B(r_i) = \sup_{\partial\Omega_{r_i}(x_0)} u \geq u(x).$$

By the weak maximum principle for the $\mathscr{L}$-subharmonic function $u - v$ on the bounded open set $A_{r_1,r_2}(x_0)$, we infer that $u(x) \leq v(x)$ for every $x \in A_{r_1,r_2}(x_0)$. In particular, if we take $x \in \partial\Omega_r(x_0)$, we get $u(x) \leq v(x) = I(r)$; taking the supremum over $\partial\Omega_r(x_0)$, we get exactly the needed inequality $B(r) \leq I(r)$.

Remark 6.5. (a) *Surface average operators of $\mathscr{L}$-subharmonic functions on a Γ-annulus need not be monotone nondecreasing.* Indeed, if for example $\mathscr{L} = \Delta$ is the classical Laplace operator on $\mathbb{R}^3$, the function

$$u(x) = (\|x\| - 2)^2, \quad \text{where } \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

is subharmonic on the annulus $\{4/3 < \|x\| < 3\}$, but $m_r(u)(0) = (r - 2)^2$ is not monotone on $(4/3, 3)$.

(b) *A converse of Theorem 6.1 holds true for C^2 functions which are "radial" with respect to* Γ. More precisely, suppose u has the form

$$u(x) = f(\Gamma(x_0, x)), \quad x \in A_{a,b}(x_0),$$

for some $f \in C^2((1/b, 1/a), \mathbb{R})$. A direct computation based on (2-1) and on the $\mathscr{L}$-harmonicity of $\Gamma(x_0, \cdot)$ on $\mathbb{R}^N \setminus \{x_0\}$, proves that

$$\mathscr{L}u = f'(\Gamma_{x_0})\mathscr{L}\Gamma_{x_0} + f''(\Gamma_{x_0}) \sum_{i,j} a_{i,j} \partial_i \Gamma_{x_0} \partial_j \Gamma_{x_0} = f''(\Gamma_{x_0}) \langle A\nabla\Gamma_{x_0}, \nabla\Gamma_{x_0}\rangle.$$

Thus u is $\mathscr{L}$-subharmonic on $A_{a,b}(x_0)$ (that is, $\mathscr{L}u \geq 0$) if and only if (recall that A is positive semidefinite) $f'' \geq 0$ on $(1/b, 1/a)$. On the other hand, if $r \in (a, b)$,

$$m_r(u)(x_0) = \int_{\Gamma(x_0,x)=1/r} f(\Gamma(x_0, x))k(x_0, x)\, dH^{N-1}(x) = f(1/r).$$

Thus, $m_r(u)(x_0)$ is r^{-1}-convex on (a, b) if and only if $f(r)$ is convex on $(1/b, 1/a)$. This proves that u is $\mathscr{L}$-subharmonic on $A_{a,b}(x_0)$ if and only if $m_r(u)(x_0)$ is an r^{-1}-convex function on (a, b).

(c) If u is as in part (b), then $m_r(u)(x_0)$ is $r^{-\beta}$-convex on (a, b) if and only if (see Lemma 2.1) $f(r^{1/\beta})$ is convex on $(b^{-\beta}, a^{-\beta})$; this last condition holds if and only if

$$f''(\rho) - (\beta - 1)\frac{f'(\rho)}{\rho} \geq 0 \quad \text{for all } \rho \in (b^{-1}, a^{-1}).$$

Now, when $\beta > 1$, it is very simple to produce a function f satisfying this last condition on some open interval (b^{-1}, a^{-1}), but violating $f'' \geq 0$ on the same interval (recall that this last condition is equivalent to u being $\mathscr{L}$-subharmonic on $A_{a,b}$): for instance, $f(\rho) = -\rho^\beta$ does the job. With this choice of f, the associated function $u(x) = -(\Gamma(x_0, x))^\beta$ is *not* $\mathscr{L}$-subharmonic on any annulus $A_{a,b}(x_0)$, but $m_r(u)(x_0)$ is $r^{-\beta}$-convex on every subinterval (b^{-1}, a^{-1}) of $(0, \infty)$.

7. Applications

We are ready to give the following proofs.

Proof of Corollary 1.7. From (6-4) we derive that $r^2(d/dr)m_r(u)(x_0)$ is constant on (a, b), that is, there exists $c_1 \in \mathbb{R}$ such that

$$\frac{d}{dr}(m_r(u)(x_0)) = -\frac{c_1}{r^2} = \frac{d}{dr}\left(\frac{c_1}{r}\right)$$

for every r in the interval (a, b). □

We now prove the Γ-symmetry result in Theorem 1.8. Hypothesis (HH) in Section 1 is assumed.

Remark 7.1. Thanks to our hypoellipticity assumption (H1), by the strong maximum principle for $\mathscr{L}$ (proved in [Abbondanza and Bonfiglioli 2013, Theorem 3.4]) we infer that the harmonic sheaf associated with $\mathscr{L}$ is elliptic (in the sense of [Constantinescu and Cornea 1972]). By standard techniques, hypothesis (HH) is then fulfilled, for example, in the following cases:

(1) if $\mathscr{L}$ can be put in the form $\mathscr{L} = \sum_{j=1}^m X_j^2$, where $X_1, \dots, X_m$ are smooth vector fields satisfying Hörmander's rank condition on $\mathbb{R}^N$;

(2) for $x_0 = 0$, if $\mathscr{L}$ is homogeneous w.r.t. some group of dilations on $\mathbb{R}^N$ (this is true, for example, if $\mathscr{L}$ is a sub-Laplacian on a Carnot group).

Here we agree to say that a family of maps $\{\delta_\lambda\}_{\lambda>0}$ is a group of dilations if

$$\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_\lambda(x_1, \dots, x_N) = (\lambda^{\sigma_1}x_1, \dots, \lambda^{\sigma_N}x_N),$$

where the exponents σ_j are strictly positive real numbers. Moreover, we say that $\mathscr{L}$ is δ_λ-homogeneous of positive degree if there exists $\sigma > 0$ such that $\mathscr{L}(u \circ \delta_\lambda) = \lambda^\sigma (\mathscr{L}u) \circ \delta_\lambda$ for every $u \in C^2(\mathbb{R}^N, \mathbb{R})$.

Proof of Theorem 1.8. We follow an idea [Axler et al. 1992] exploited in the classical case of the Laplace operator. The proof is split into three steps.

(I) We set $A := A_{0,b}(x_0)$. Given $u \in C(A, \mathbb{R})$, we introduce the operator

$$S(u)(x) := m_{1/\Gamma(x_0,x)}(u)(x_0), \quad x \in A.$$

Clearly, one has $S(1) \equiv 1$ and, moreover, Γ_{x_0} is another fixed function for S, since

$$\begin{aligned} S(\Gamma_{x_0})(x) = m_{1/\Gamma(x_0,x)}(\Gamma_{x_0})(x_0) &= \int_{\Gamma(x_0,y)=\Gamma(x_0,x)} \Gamma(x_0, y)k(x_0, y)\, dH^{N-1}(y) \\ &= \Gamma(x_0, x)S(1)(x) = \Gamma_{x_0}(x). \end{aligned}$$

We observe that if u is $\mathscr{L}$-harmonic in A, then (by Corollary 1.7)

$$S(u)(x) = c\Gamma(x_0, x) + c_2, \quad x \in A, \tag{7-1}$$

for some constants c, c_2. In particular, if u is $\mathscr{L}$-harmonic in A, then $S(u)$ is $\mathscr{L}$-harmonic in A (actually $S(u)$ extends to an $\mathscr{L}$-harmonic function in $\mathbb{R}^N \setminus \{x_0\}$). Furthermore, by the above results ensuring that 1 and Γ_{x_0} are fixed functions for S, we infer that

$$\textit{if } u \textit{ is } \mathscr{L}\textit{-harmonic in } A\textit{, then } S(S(u)) = S(u). \tag{7-2}$$

Next we see how S behaves on a function w enjoying the hypotheses of the theorem. First notice that, since w vanishes on $\partial\Omega_b(x_0)$ with continuity, the same is true of $S(w)$. Moreover $S(w)$ is $\mathscr{L}$-harmonic in A (since this is true of w) and

$$S(w)(x) = c(\Gamma(x_0, x) - 1/b). \tag{7-3}$$

Here we used (7-1), observing that $c_2 = -c/b$ is the only choice for c_2 which ensures the vanishing of $S(w)$ on $\partial\Omega_b(x_0)$.

Comparing (7-3) with the thesis of the theorem, we recognize that *the theorem is proved if we are able to show that w is fixed by S*, that is, $S(w) = w$ on A.

(II) We let $\boldsymbol{c} := C^{-1}$, where C is the constant in hypothesis (HH). Note that $0 < \boldsymbol{c} < 1$. We claim that the following property holds true:

If h is $\mathscr{L}$-harmonic in A and continuous up to $\partial\Omega_b(x_0)$

$$\text{with } h \equiv 0 \text{ on } \partial\Omega_b(x_0) \text{ and } h \geq 0 \text{ on } A\text{, then } h \geq \boldsymbol{c}S(h) \text{ on } A. \tag{7-4}$$

With this result in hand, the proof of Theorem 1.8 follows. Indeed, suppose w enjoys the hypothesis of the theorem; let us prove by induction that, setting

$$\boldsymbol{c}_n := 1 - (1 - \boldsymbol{c})^n, \quad n \in \mathbb{N} \cup \{0\}, \tag{7-5}$$

we have

$$w \geq \boldsymbol{c}_n S(w) \text{ on } A \quad \text{for any } n \in \mathbb{N} \cup \{0\}. \tag{7-6}$$

The case $n = 0$ follows from the nonnegativity of w on A and $c_0 = 0$.

Suppose (7-6) holds true, and let us prove it for $n+1$ replacing n. The function $h := w - \boldsymbol{c}_n S(w)$ satisfies the hypothesis of statement (7-4): indeed, from the last remarks of part I above, it follows that h is $\mathscr{L}$-harmonic in A, continuous up to $\partial\Omega_b(x_0)$ and vanishing there. Finally $h \geq 0$ on A is the inductive assumption.

Consequently, from the claimed result in (7-4), we have on A

$$\begin{aligned} 0 \leq h - \boldsymbol{c}S(h) &= w - \boldsymbol{c}_n S(w) - \boldsymbol{c}S(w - \boldsymbol{c}_n S(w)) \\ &= w - \boldsymbol{c}_n S(w) - \boldsymbol{c}S(w) + \boldsymbol{c}\boldsymbol{c}_n S(w) \\ &= w - \boldsymbol{c}_{n+1} S(w). \end{aligned}$$

Here we used (7-2) together with $\boldsymbol{c}_n + \boldsymbol{c} - \boldsymbol{c}\boldsymbol{c}_n = \boldsymbol{c}_{n+1}$ (see the very definition (7-5) of $\boldsymbol{c}_n$). Thus (7-6) is proved by induction.

Letting $n \to \infty$ in it, we infer $w \geq S(w)$ on A, since $\boldsymbol{c}_n \to 1$, as $0 < 1 - \boldsymbol{c} < 1$. Recalling what we proved in part I, we are done if we can also prove the reverse inequality $w \leq S(w)$. Suppose by contradiction that $w(\overline{x}) > S(w)(\overline{x})$ for some $\overline{x} \in A$; by (7-2), this gives $S(w)(\overline{x}) > S\big(S(w)\big)(\overline{x}) = S(w)(\overline{x})$, a contradiction. Note that the above inequality is a consequence of $S(1) = 1$ and of the fact that S is a nondecreasing operator (that is, if $u \leq v$ on A, then $S(u) \leq S(v)$ on A).

(III) We are thus left with the proof of the claimed (7-4). Notice that (HH) can be restated as follows:

$$\boldsymbol{c}h(z) \leq h(x), \quad \text{whenever } (\theta b)^{-1} < \Gamma(x_0, z) = \Gamma(x_0, x) < \infty \text{ and } h \geq 0 \text{ is } \mathscr{L}\text{-harmonic in } A. \tag{7-7}$$

Let h be as in (7-4). Arguing as in part I of the proof, we infer that $H := h - \boldsymbol{c}S(h)$ is $\mathscr{L}$-harmonic in A, continuous up to $\partial\Omega_b(x_0)$, and $H = 0$ on $\partial\Omega_b(x_0)$. Let us fix any arbitrary $r \in (0, \theta b)$. We take $x, z \in \partial\Omega_r(x_0)$; recall that this means

$$\Gamma(x_0, x) = \Gamma(x_0, z) = 1/r.$$

Let us consider the inequality in the left-hand side of (7-7), which is fulfilled since $(\theta b)^{-1} < 1/r < \infty$; by multiplication by $k(x_0, z)$ (see the notation in (1-2)), and by integration w.r.t. $z \in \partial\Omega_r(x_0)$, we get $\boldsymbol{c}m_r(h)(x_0) \leq h(x)$. Recalling that $r = 1/\Gamma(x_0, x)$, we infer

$$\boldsymbol{c}m_{1/\Gamma(x_0,x)}(h)(x_0) \leq h(x), \quad \text{that is,} \quad \boldsymbol{c}S(h)(x) \leq h(x).$$

The arbitrariness of $x \in \partial\Omega_r(x_0)$ implies that $H(x) \geq 0$ on $\partial\Omega_r(x_0)$. By the weak minimum principle applied to the $\mathscr{L}$-harmonic function H and to the bounded open set $A_{r,b}(x_0)$, we derive $H \geq 0$ on $A_{r,b}(x_0)$. Since $r \in (0, \theta b)$ is arbitrary, this yields $H \geq 0$ on $A_{0,b}(x_0) = A$, that is, $h \geq \boldsymbol{c}S(h)$ on A. This proves (7-4). □

We end the paper by giving the following proof.

Proof of Theorem 1.9. Let $\varepsilon > 0$ be so small that $\overline{\Omega_\varepsilon(x_0)} \subset \Omega$. Since $V := \Omega_\varepsilon(x_0)$ is an $\mathscr{L}$-regular open set, setting $f := u|_{\partial\Omega_\varepsilon(x_0)}$, we can consider H_f^V, the unique $\mathscr{L}$-harmonic function in V, continuous up to

∂V, coinciding with u on ∂V. Let

$$w(x) := u(x) - H_f^V(x) + \Gamma(x_0, x) - 1/\varepsilon, \quad x \in O := \Omega_\varepsilon(x_0) \setminus \{x_0\}.$$

The function w is $\mathscr{L}$-harmonic in O and continuous up to $\partial\Omega_\varepsilon(x_0)$, where it vanishes; moreover, $\liminf_{x\to x_0} w(x) \geq -H_f^V(x_0) - 1/\varepsilon + \lim_{x\to x_0}\Gamma(x_0, x) = \infty$, the inequality following from the hypothesis $u \geq 0$. The weak minimum principle for w and for the bounded open set O proves that $w \geq 0$ on O. Note that O is the Γ-annulus $A_{0,\varepsilon}(x_0)$. We are therefore entitled to apply Theorem 1.8 and derive that $w = c_1(\Gamma_{x_0} - 1/\varepsilon)$ on O, for some constant c_1. As a consequence, we get $u = c\Gamma_{x_0} + H$ on O, where $c = c_1 - 1$ and $H = H_f^V - c/\varepsilon$. From $u = c\Gamma_{x_0} + H$, the finiteness of $H(x_0)$ and the hypothesis $u \geq 0$, we get $c \geq 0$. This proves that the function h defined on $\Omega \setminus \{x_0\}$ by $h(x) := u(x) - c\Gamma(x_0, x)$ is not only $\mathscr{L}$-harmonic, but (as it coincides with H on O) it extends $\mathscr{L}$-harmonically through x_0. □

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GLOBAL UNIQUENESS FOR AN IBVP FOR THE TIME-HARMONIC MAXWELL EQUATIONS

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In this paper we prove uniqueness for an inverse boundary value problem (IBVP) arising in electrodynamics. We assume that the electromagnetic properties of the medium, namely the magnetic permeability, the electric permittivity, and the conductivity, are described by continuously differentiable functions.

1. Introduction

Let Ω be a bounded nonempty open subset of $\mathbb{R}^3$ with boundary denoted by $\partial\Omega$. Consider functions $\mu, \varepsilon, \sigma \in L^\infty(\Omega)$, representing magnetic permeability, electric permittivity, and conductivity, respectively, such that $\mu(x) \geq \mu_0$, $\varepsilon(x) \geq \varepsilon_0$, and $\sigma(x) \geq 0$ almost everywhere in Ω for positive constants μ_0 and ε_0. At frequency $\omega > 0$, for each medium characterized by $(\mu, \varepsilon, \sigma)$, we have access to all available data of the boundary tangential components of electric and magnetic fields. More specifically, we have access to the *Cauchy data set* $C(\mu, \varepsilon, \sigma; \omega)$ consisting of all boundary graded forms $f^1 + f^2 \in TH^\delta(\partial\Omega; \Lambda^1\mathbb{R}^3) \oplus TH^d(\partial\Omega; \Lambda^2\mathbb{R}^3)$ (see the Appendix for the definitions of these spaces and results related to l-forms) such that there exists $u^1 + u^2 \in H^d(\Omega; \Lambda^1\mathbb{R}^3) \oplus H^\delta(\Omega; \Lambda^2\mathbb{R}^3)$ satisfying

$$\delta u^2 + i\omega\varepsilon u^1 - du^1 + i\omega\mu u^2 = \sigma u^1 \tag{1-1}$$

almost everywhere in Ω and

$$\delta \operatorname{tr} u^2 + d \operatorname{tr} u^1 = f^1 + f^2 \tag{1-2}$$

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in the sense of $TH^{\delta}\big(\partial\Omega;\, \Lambda^1\mathbb{R}^3\big) \oplus TH^{d}\big(\partial\Omega;\, \Lambda^2\mathbb{R}^3\big)$. Here u^1 is the 1-form representation of the electric field and u^2 is the 2-form representation of the magnetic field. It is worth pointing out that the graded equations (1-1) and (1-2) are equivalent to the following systems of time-harmonic Maxwell equations:

$$\begin{cases} \delta u^2 + i\omega\varepsilon u^1 = \sigma u^1, \\ du^1 - i\omega\mu u^2 = 0 \end{cases}$$

almost everywhere in Ω and

$$\begin{cases} \delta \operatorname{tr} u^2 = f^1, \\ d \operatorname{tr} u^1 = f^2 \end{cases}$$

in the sense of the space $TH^{\delta}\big(\partial\Omega;\, \Lambda^1\mathbb{R}^3\big)$ for the 1-form equation and in the sense of $TH^{d}\big(\partial\Omega;\, \Lambda^2\mathbb{R}^3\big)$ for the 2-form equation. Throughout this paper, for convenience, we follow the graded form notation rather than the l-form system.

We are interested in the inverse boundary value problem (IBVP) of recovering $\mu, \varepsilon, \sigma \in L^\infty(\Omega)$ from the knowledge of $C(\mu, \varepsilon, \sigma; \omega)$. This problem is just a reformulation in differential forms of the usual IBVP for the time-harmonic Maxwell equations proposed in [Somersalo et al. 1992], where $\partial\Omega$ was smooth enough, the electromagnetic fields $(\mathbf{E}, \mathbf{H})$ satisfied

$$\begin{cases} \nabla \times \mathbf{E} - i\omega\mu\mathbf{H} = 0, \\ \nabla \times \mathbf{H} + i\omega(\varepsilon + i\sigma/\omega)\mathbf{E} = 0 \end{cases}$$

almost everywhere in Ω, and the Cauchy set $C(\mu, \varepsilon, \sigma; \omega)$ consisted of pairs

$$(\nu \times \mathbf{E}|_{\partial\Omega},\, \nu \times \mathbf{H}|_{\partial\Omega}) \in TH^{1/2}_{\mathrm{Div}}(\partial\Omega) \times TH^{1/2}_{\mathrm{Div}}(\partial\Omega)$$

(see [Somersalo et al. 1992] for precise definitions) with ν denoting the unit outer normal vector to $\partial\Omega$. The uniqueness question associated to this problem is as follows. Given a frequency $\omega > 0$ and two sets of parameters $\{\mu_j, \varepsilon_j, \sigma_j\} \subset L^\infty(\Omega)$ with $j \in \{1, 2\}$ such that $\mu_j(x) \geq \mu_0$, $\varepsilon_j(x) \geq \varepsilon_0$, and $\sigma_j(x) \geq 0$ almost everywhere in Ω, does $C(\mu_1, \varepsilon_1, \sigma_1; \omega) = C(\mu_2, \varepsilon_2, \sigma_2; \omega)$ imply $\mu_1 = \mu_2$, $\varepsilon_1 = \varepsilon_2$, and $\sigma_1 = \sigma_2$?

In this paper we provide the answer to this question in the case where Ω is locally described by the graph of a Lipschitz function and μ, ε, and σ are continuously differentiable in Ω. This is stated in our main theorem as follows.

Theorem 1.1. *Let Ω be a bounded nonempty open subset of $\mathbb{R}^3$. Assume that $\partial\Omega$ is locally described by the graph of a Lipschitz function. Let μ_j, ε_j, and σ_j with $j \in \{1, 2\}$ belong to $C^1(\bar{\Omega})$. At frequency $\omega > 0$, suppose $\partial^\alpha\mu_1(x) = \partial^\alpha\mu_2(x)$, $\partial^\alpha\varepsilon_1(x) = \partial^\alpha\varepsilon_2(x)$, and $\partial^\alpha\sigma_1(x) = \partial^\alpha\sigma_2(x)$ for $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq 1$ and all $x \in \partial\Omega$. Then*

$$C(\mu_1, \varepsilon_1, \sigma_1, \omega) = C(\mu_2, \varepsilon_2, \sigma_2, \omega) \Longrightarrow \mu_1 = \mu_2,\ \varepsilon_1 = \varepsilon_2 \text{ and } \sigma_1 = \sigma_2.$$

A precise definition of the space denoted by $C^1(\bar{\Omega})$ is given at the beginning of Section 3. Our result assumes the coefficients to be equal up to order one on the boundary. This is required to extend them identically outside the domain. As far as we know, the only available results about uniqueness on the boundary in this context are due to Joshi and McDowall [McDowall 1997; Joshi and McDowall 2000], where $\partial\Omega$ is assumed to be locally described by a smooth function and the Cauchy data sets are given by the graph of a bounded map.

The IBVP considered in this paper was first proposed by Somersalo, Isaacson, and Cheney [Somersalo et al. 1992]. Lassas [1997] found a relation between this IBVP and the inverse conductivity problem proposed by Calderón [2006]. In general terms, the latter problem can be seen as the low-frequency limit of the former. Calderón's problem in electrical impedance tomography consists of reconstructing the conductivity of a domain by measuring electric voltages and currents on the boundary. The uniqueness question arising in this problem is whether the conductivity σ ($\sigma \in L^\infty(\Omega)$ and $\sigma(x) \geq \sigma_0 > 0$ for almost every $x \in \Omega$), in a divergence type equation $\nabla \cdot (\sigma \nabla u) = 0$ in Ω, can be determined uniquely by the boundary Dirichlet-to-Neumann map $\Lambda_\sigma : H^1(\Omega)/H^1_0(\Omega) \longrightarrow (H^1(\Omega)/H^1_0(\Omega))^*$ defined as

$$\langle \Lambda_\sigma f \mid g \rangle = \int_\Omega \sigma \nabla u \cdot \nabla v \, dx$$

for any $f, g \in H^1(\Omega)/H^1_0(\Omega)$, where $u \in H^1(\Omega)$ is the weak solution of the conductivity equation $\nabla \cdot (\sigma \nabla u) = 0$ in Ω with $u|_{\partial\Omega} = f$ and $v \in H^1(\Omega)$ with $v|_{\partial\Omega} = g$. A significant number of works have been devoted to answering not only the question of uniqueness but also the questions of reconstruction and stability. The most successful approach to treat this problem was introduced by Sylvester and Uhlmann [1987] and it is based on the construction of complex geometrical optics (CGO) solutions. In dimension 2, the problem is rather well understood and some important results can be found in [Astala and Päivärinta 2006; Clop et al. 2010; Nachman 1996]. In dimension greater than 2, there are still many open questions about the sharp smoothness to ensure uniqueness, stability, and reconstruction. Some important results can be found in [Haberman and Tataru 2013; Sylvester and Uhlmann 1987; Nachman 1988; Alessandrini 1988]. Some recent results are [Caro et al. 2013; García and Zhang 2012]. For a more complete list of papers on this problem, we refer to the survey papers [Uhlmann 2009; 2008].

The literature for the IBVP in electrodynamics under consideration is not as extensive as for Calderón's problem. [Somersalo et al. 1992] contains the first partial results for the linearization of the problem at constant electromagnetic parameters, and Sun and Uhlmann [1992] provided a local uniqueness theorem. The first global uniqueness result is due to Ola, Päivärinta, and Somersalo [Ola et al. 1993], where the authors assume that the electromagnetic coefficients are C^3-functions and $\partial\Omega$ is of class $C^{1,1}$. They also provided a reconstruction algorithm to recover the coefficients. The arguments in [Ola et al. 1993] are rather complicated, since the method developed by Sylvester and Uhlmann [1987] does not immediately apply. The lack of ellipticity of Maxwell's equations makes the problem more complicated than Calderón's. Ola and Somersalo [1996] simplified the proof in [Ola et al. 1993] by establishing a relation between Maxwell's equations and a matrix Helmholtz equation with a potential. This relation helps to deal with the lack of ellipticity, allowing them to produce exponentially growing solutions for Maxwell's equations from the CGOs for the matrix Helmholtz equation. This idea has been extensively used in proving many other results and it will be used in this paper as well. There are other results related to the IBVP under consideration in the literature. Kenig, Salo, and Uhlmann [Kenig et al. 2011] proved uniqueness for the corresponding IBVP in some noneuclidean geometries. With certain types of partial boundary data, the uniqueness was addressed by Caro, Ola, and Salo [Caro et al. 2009]; see also [Caro 2011]. The question of stability has been studied in [Caro 2010] assuming full data and in [Caro 2011] assuming partial data. Zhou [2010] used the enclosure method to reconstruct electromagnetic obstacles.

In our paper, Theorem 1.1 lowers significantly the regularity of the coefficients and the smoothness of the boundary of Ω assumed in previous results (despite the fact that domains with Lipschitz boundaries were already considered in [Caro 2010]) and it matches the regularity assumptions made in [Haberman and Tataru 2013] for Calderón's problem.

The general line of our paper follows the argument in [Ola and Somersalo 1996], relating (1-1) with an equation given by a compactly supported zeroth order perturbation of the graded Hodge–Helmholtz operator, namely

$$(\delta d + d\delta - \omega^2 \mu_0 \varepsilon_0) w_j + Q_j w_j = 0, \tag{1-3}$$

where $Q_j = Q(\varepsilon_j + i\sigma_j/\omega, \mu_j, \omega)$ with $j \in \{1, 2\}$ has to be thought of as a weak potential containing second partial derivatives of μ_j, ε_j, and σ_j. Using this relation, we are able to prove the integral formula

$$\langle (Q_2 - Q_1) w_1 \mid v_2 \rangle = 0, \tag{1-4}$$

where w_1 is a solution to (1-3) that produces a solution to (1-1) and v_2 is a solution to a first order elliptic equation (see Section 3 for more details). This integral formula, with CGOs w_1 and v_2 as inputs, will be the starting point of our proof.

To lower the regularity of the electromagnetic parameters, we adopt a recent improvement of Sylvester and Uhlmann's method that Haberman and Tataru developed [2013] to prove uniqueness of the Calderón problem with continuously differentiable conductivities. For such regularity, solving a conductivity equation can be reduced to solving a Schrödinger equation, $-\Delta v + m_q v = 0$, where m_q denotes the multiplication operator by the compactly supported weak potential $q = \Delta\sqrt{\sigma}/\sqrt{\sigma}$. Note that this reduction was first used by Sylvester and Uhlmann [1987] for smooth conductivities and later by Brown [1996] for less regular conductivities, all followed by the construction of CGOs in proper function spaces. Haberman and Tataru [2013] proved the existence of CGO solutions $v(x) = e^{x\cdot\zeta}(1 + \psi_\zeta(x))$ with $\zeta \in \mathbb{C}^n$ and $\zeta \cdot \zeta = 0$ to the Schrödinger equation. Roughly speaking, the construction is based on solving the equation $-(\Delta + 2\zeta \cdot \nabla)\psi_\zeta + m_q \psi_\zeta = 0$ in a Bourgain-type space $\dot{X}^b_\zeta$ whose norm includes the potential $|p_\zeta(\xi)|^{2b} = ||\xi|^2 - 2i\zeta \cdot \xi|^{2b}$ as a weight. In this way, the ζ-dependence is transferred into the space norms and it is shown [Haberman and Tataru 2013] that

$$\|(\Delta + 2\zeta \cdot \nabla)^{-1}\|_{\dot{X}^{-1/2}_\zeta \to \dot{X}^{1/2}_\zeta} = 1, \quad \|m_q\|_{\dot{X}^{1/2}_\zeta \to \dot{X}^{-1/2}_\zeta} < 1,$$

which guarantee the convergence of the Neumann series for ψ_ζ. Furthermore, Haberman and Tataru obtained an average decaying property for $\|\psi_\zeta\|_{\dot{X}^{1/2}_\zeta}$, from which they deduced the existence of a sequence $\{\zeta^m\}$ such that $\{\psi_{\zeta^m}\}$ vanishes as m grows.

In this paper, we adopt the idea and several of the estimates in [Haberman and Tataru 2013] to construct the CGOs w_1 and v_2 with desired properties. Nevertheless, we avoid the argument of extracting the sequence of $\{\zeta^m\}$, and directly use the decay in average. This has been previously done by Caro, García, and Reyes [Caro et al. 2013] to prove stability of the Calderón problem for $C^{1,\epsilon}$-conductivities. When plugging in the CGOs w_1 and v_2, the output of (1-4) will be certain nonlinear relations of $\varepsilon_1 + i\sigma_1/\omega$, μ_1, $\varepsilon_2 + i\sigma_2/\omega$, and μ_2 involving second weak partial derivatives of the coefficients. Thus, to conclude the proof of our theorem we will need a unique continuation property for a system of the form

$$\begin{aligned} -\Delta f + V f + a f + b g &= 0, \\ -\Delta g + W g + c g + d f &= 0, \end{aligned}$$

where a, b, c, and d are compactly supported and belong to $L^\infty(\mathbb{R}^3)$, while V and W are again weak potentials. We will again apply the argument with Bourgain-type spaces to prove the required unique continuation property, which seems not to be available in the literature.

The paper is organized as follows. In Section 2 we show the relation between (1-1) and (1-3). The proof of the integral formula (1-4) is given in Section 3. The CGO solutions are constructed in Section 4, where we will directly refer several times to the estimates proven in [Haberman and Tataru 2013] rather than listing them in the paper. In Section 5, we complete our proof by plugging the CGOs into (1-4) and using the unique continuation principle that we will derive. An appendix is provided at the end of the paper, gathering basic facts and notations in the framework of differential forms, as well as including some technical computations for the electromagnetic IBVP.

2. An auxiliary graded equation

In this section we establish a relation between

$$\delta u^2 + i\omega\varepsilon u^1 - du^1 + i\omega\mu u^2 = \sigma u^1$$

and an auxiliary graded Hodge–Helmholtz equation with zeroth order perturbation (following the idea in [Ola and Somersalo 1996]), which allows the construction of CGOs. For our purposes, it would be enough to have solutions in Ω, but for convenience we will conduct our analysis in the whole $\mathbb{R}^3$. This gives us certain freedoms in extending the coefficients outside Ω. Thus, set $B = \{x \in \mathbb{R}^3 : |x| < R\}$ with $R > 0$ such that $\bar{\Omega} \subset B$. Let ω, μ_0, and ε_0 be three positive constants. At this point, we consider μ, ε, and σ in $W^{1,\infty}(\mathbb{R}^3)$, the space of measurable functions modulo those vanishing almost everywhere such that they and their first weak partial derivatives are essentially bounded in $\mathbb{R}^3$. Furthermore, we assume that μ, ε, and σ are real-valued,

$$\operatorname{supp}(\mu - \mu_0) \subset B, \quad \operatorname{supp}(\varepsilon - \varepsilon_0) \subset B, \quad \operatorname{supp}(\sigma) \subset B,$$

and $\mu(x) \geq \mu_0$, $\varepsilon(x) \geq \varepsilon_0$, and $\sigma(x) \geq 0$ for almost every x in $\mathbb{R}^3$. For simplicity, write $\gamma = \varepsilon + i\sigma/\omega$. It is sufficient for us to produce weak solutions to

$$\delta u^2 + i\omega\gamma u^1 - du^1 + i\omega\mu u^2 = 0 \tag{2-1}$$

in $\mathbb{R}^3$, namely, forms $u^1 + u^2$ with $u^l \in L^2_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$ satisfying

$$\langle \delta u^2 + i\omega\gamma u^1 - du^1 + i\omega\mu u^2 \mid \varphi^1 + \varphi^2 \rangle = 0$$

for all $\varphi^1+\varphi^2$ with $\varphi^l \in C_0^\infty(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)$. Here $\langle \cdot \mid \cdot \rangle$ denotes the duality bracket for distributions.

In order to derive the auxiliary equation, we augment (2-1) by adding

$$-\gamma^{-1}\delta(\gamma u^1)+\mu^{-1}d(\mu u^2)=0,$$

which is derived directly from (2-1).

Next, we consider an equation of the graded form $\sum_{l=0}^3 u^l$ where $u^l \in L^2_{\text{loc}}(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)$:

$$-\gamma^{-1}\delta(\gamma u^1)+i\omega\mu u^0+\mu^{-1}d(\mu u^0)+\delta u^2+i\omega\gamma u^1-\gamma^{-1}\delta(\gamma u^3)-du^1+i\omega\mu u^2+\mu^{-1}d(\mu u^2)+i\omega\gamma u^3=0.$$

Multiplying 0, 2-forms by $\gamma^{1/2}$ and 1, 3-forms by $\mu^{1/2}$, we obtain

$$\begin{aligned}-\gamma^{-1/2}\delta(\gamma u^1)+i\omega\gamma^{1/2}\mu u^0+\mu^{-1/2}d(\mu u^0)+\mu^{1/2}\delta u^2+i\omega\gamma\mu^{1/2}u^1&\\ -\gamma^{-1/2}\delta(\gamma u^3)-\gamma^{1/2}du^1+i\omega\gamma^{1/2}\mu u^2+\mu^{-1/2}d(\mu u^2)+i\omega\gamma\mu^{1/2}u^3&=0.\end{aligned}$$

Throughout this paper $(\,\cdot\,)^{1/2}$ will denote the principal branch of the square root, and the same convention will apply to log. If we now set

$$v=\sum_{l=0}^3 v^l=\mu^{1/2}u^0+\gamma^{1/2}u^1+\mu^{1/2}u^2+\gamma^{1/2}u^3,$$

we end up with the equation

$$P(d+\delta;\gamma,\mu,\omega)v=0, \tag{2-2}$$

where

$$\begin{aligned}&P(d+\delta;\gamma,\mu,\omega)v\\ &\quad=(d+\delta)\sum_{l=0}^3(-1)^l v^l+da\wedge v^1+da\vee(v^1+v^3)+db\wedge(v^0+v^2)-db\vee v^2+i\omega\gamma^{1/2}\mu^{1/2}v,\end{aligned}$$

$a=\frac12\log\gamma$ and $b=\frac12\log\mu$. The key point of this derivation to take note of is that $v=\sum_0^3 v^l$ with $v^l\in L^2_{\text{loc}}(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)$ is a weak solution of (2-2) in $\mathbb{R}^3$ (that is, for every $\varphi=\sum_0^3\varphi^l$ with $\varphi^l\in C_0^\infty(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)$, $\langle P(d+\delta;\gamma,\mu,\omega)v \mid \varphi\rangle=0$ with $\langle\cdot\mid\cdot\rangle$ denoting the duality bracket for distributions) and $v^0+v^3=0$ if and only if $u^1+u^2=\gamma^{-1/2}v^1+\mu^{-1/2}v^2$ with $u^l\in L^2_{\text{loc}}(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)$ is a weak solution of (2-1) in $\mathbb{R}^3$. For convenience, let us define an operator

$$\begin{aligned}&P(d+\delta;\gamma,\mu,\omega)^t w\\ &:=(d+\delta)\sum_{l=0}^3(-1)^{l+1}w^l+db\wedge w^1+db\vee(w^1+w^3)+da\wedge(w^0+w^2)-da\vee w^2+i\omega\gamma^{1/2}\mu^{1/2}w\end{aligned} \tag{2-3}$$

for $w=\sum_0^3 w^l$ with $w^l\in H^\delta_{\text{loc}}(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)\cap H^d_{\text{loc}}(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)$. Note that $P(d+\delta;\gamma,\mu,\omega)^t$ is the formal transpose of $P(d+\delta;\gamma,\mu,\omega)$.

Due to the rescaling by $\gamma^{1/2}$ and $\mu^{1/2}$ that we chose, it can be verified that $P(d+\delta;\gamma,\mu,\omega)\circ P(d+\delta;\gamma,\mu,\omega)^t$ is a zeroth order perturbation of the graded Hodge–Helmholtz operator. For any graded forms

$w = \sum_0^3 w^l$ and $\varphi = \sum_0^3 \varphi^l$ with $w^l, \varphi^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$, set

$$
\begin{aligned}
\langle Q(\gamma,\mu,\omega)w \mid \varphi\rangle = -\int_{\mathbb{R}^3} &\omega^2(\gamma\mu - \varepsilon_0\mu_0)\langle w, \varphi\rangle \, dx \\
&+ \int_{\mathbb{R}^3} \langle i2\omega d(\gamma^{1/2}\mu^{1/2}) \vee (w^1 + w^3) + i2\omega d(\gamma^{1/2}\mu^{1/2}) \wedge (w^0 + w^2), \varphi\rangle \, dx \\
&+ \int_{\mathbb{R}^3} \langle da, da\rangle \langle w^0 + w^2, \varphi^0 + \varphi^2\rangle + \langle db, db\rangle \langle w^1 + w^3, \varphi^1 + \varphi^3\rangle \, dx \\
&+ \int_{\mathbb{R}^3} \langle da, d\langle -w^0 + w^2, \varphi^0 + \varphi^2\rangle\rangle + \langle db, d\langle w^1 - w^3, \varphi^1 + \varphi^3\rangle\rangle \, dx \\
&+ \int_{\mathbb{R}^3} \langle db, D^*(w^1 \odot \varphi^1)\rangle \, dx + \int_{\mathbb{R}^3} \langle da, D^*(*w^2 \odot *\varphi^2)\rangle \, dx. \qquad (2\text{-}4)
\end{aligned}
$$

Proposition 2.1. *Let $w = \sum_0^3 w^l$ be a graded form with $w^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$ and assume that*

$$
\int_{\mathbb{R}^3} \langle \delta w, \delta\varphi\rangle + \langle dw, d\varphi\rangle - \omega^2 \varepsilon_0\mu_0 \langle w, \varphi\rangle \, dx + \langle Q(\gamma,\mu,\omega)w \mid \varphi\rangle = 0 \qquad (2\text{-}5)
$$

for all $\varphi = \sum_0^3 \varphi^l$ with $\varphi^l \in C_0^\infty(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$. Then $v = \sum_0^3 v^l$ defined by

$$
v = P(d+\delta; \gamma, \mu, \omega)^t w \qquad (2\text{-}6)
$$

is a weak solution to (2-2) *in $\mathbb{R}^3$ and $v^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$.*

Proof. We first prove that v is a weak solution to (2-2). Since $v^l \in L^2_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$, it is enough to show that

$$
\begin{aligned}
\int_{\mathbb{R}^3} &\langle P(d+\delta;\gamma,\mu,\omega)^t w, P(d+\delta;\gamma,\mu,\omega)^t \varphi\rangle \, dx \\
&= \int_{\mathbb{R}^3} \langle \delta w, \delta\varphi\rangle + \langle dw, d\varphi\rangle - \omega^2\varepsilon_0\mu_0\langle w, \varphi\rangle \, dx + \langle Q(\gamma,\mu,\omega)w \mid \varphi\rangle. \qquad (2\text{-}7)
\end{aligned}
$$

To check this, by direct computation, the first four terms on the left-hand side are

$$
\int_{\mathbb{R}^3} \Big\langle (d+\delta)\sum_{l=0}^3 (-1)^{l+1} w^l, (d+\delta)\sum_{l=0}^3 (-1)^{l+1}\varphi^l \Big\rangle dx = \int_{\mathbb{R}^3} \langle \delta w, \delta\varphi\rangle + \langle dw, d\varphi\rangle \, dx, \qquad (2\text{-}8)
$$

$$
\int_{\mathbb{R}^3} \langle i\omega\gamma^{1/2}\mu^{1/2} w, i\omega\gamma^{1/2}\mu^{1/2}\varphi\rangle \, dx = -\int_{\mathbb{R}^3} \omega^2\gamma\mu\langle w, \varphi\rangle \, dx, \qquad (2\text{-}9)
$$

$$
\begin{aligned}
\int_{\mathbb{R}^3} &\Big\langle (d+\delta)\sum_{l=0}^3 (-1)^{l+1} w^l, i\omega\gamma^{1/2}\mu^{1/2}\varphi \Big\rangle dx + \int_{\mathbb{R}^3} \Big\langle i\omega\gamma^{1/2}\mu^{1/2} w, (d+\delta)\sum_{l=0}^3 (-1)^{l+1}\varphi^l \Big\rangle dx \\
&= \int_{\mathbb{R}^3} \langle i\omega d(\gamma^{1/2}\mu^{1/2}) \vee (w^1 + w^3) + i\omega d(\gamma^{1/2}\mu^{1/2}) \wedge (w^0 + w^2), \varphi\rangle \, dx \\
&\quad + \int_{\mathbb{R}^3} \langle i\omega d(\gamma^{1/2}\mu^{1/2}) \vee w^2 - i\omega d(\gamma^{1/2}\mu^{1/2}) \wedge w^1, \varphi\rangle \, dx, \qquad (2\text{-}10)
\end{aligned}
$$

and

$$\int_{\mathbb{R}^3} \langle db\wedge w^1 + db\vee (w^1+w^3) + da\wedge (w^0+w^2) - da\vee w^2, i\omega\gamma^{1/2}\mu^{1/2}\varphi\rangle \\ + \langle i\omega\gamma^{1/2}\mu^{1/2} w, db\wedge\varphi^1 + db\vee(\varphi^1+\varphi^3) + da\wedge(\varphi^0+\varphi^2) - da\vee\varphi^2\rangle\, dx \\ = \int_{\mathbb{R}^3} \langle i\omega d(\gamma^{1/2}\mu^{1/2})\vee (w^1+w^3) + i\omega d(\gamma^{1/2}\mu^{1/2})\wedge (w^0+w^2), \varphi\rangle\, dx \\ - \int_{\mathbb{R}^3} \langle i\omega d(\gamma^{1/2}\mu^{1/2})\vee w^2 - i\omega d(\gamma^{1/2}\mu^{1/2})\wedge w^1, \varphi\rangle\, dx. \quad (2\text{-}11)$$

By Corollary A.2, the fifth term gives

$$\int_{\mathbb{R}^3} \langle db\wedge w^1 + db\vee (w^1+w^3) + da\wedge (w^0+w^2) \\ - da\vee w^2, db\wedge\varphi^1 + db\vee(\varphi^1+\varphi^3) + da\wedge(\varphi^0+\varphi^2) - da\vee\varphi^2\rangle\, dx \\ = \int_{\mathbb{R}^3} \langle da, da\rangle\langle w^0+w^2, \varphi^0+\varphi^2\rangle + \langle db, db\rangle\langle w^1+w^3, \varphi^1+\varphi^3\rangle\, dx. \quad (2\text{-}12)$$

By Proposition A.6, the last term yields

$$\int_{\mathbb{R}^3} \Big\langle db\wedge w^1 + db\vee (w^1+w^3) + da\wedge (w^0+w^2) - da\vee w^2, (d+\delta)\sum_{l=0}^{3}(-1)^{l+1}\varphi^l \Big\rangle \\ + \Big\langle (d+\delta)\sum_{l=0}^{3}(-1)^{l+1} w^l, db\wedge\varphi^1 + db\vee(\varphi^1+\varphi^3) + da\wedge(\varphi^0+\varphi^2) - da\vee\varphi^2 \Big\rangle\, dx \\ = \int_{\mathbb{R}^3} \langle da, d\langle -w^0+w^2, \varphi^0+\varphi^2\rangle\rangle + \langle db, d\langle w^1-w^3, \varphi^1+\varphi^3\rangle\rangle\, dx + \int_{\mathbb{R}^3} \langle db, D^*(w^1\odot\varphi^1)\rangle\, dx \\ + \int_{\mathbb{R}^3} \langle da, D^*(*w^2\odot *\varphi^2)\rangle\, dx = 0. \quad (2\text{-}13)$$

Summing up identities (2-8) through (2-13) gives identity (2-7).

It remains to prove that $v^l\in H^1_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$. Since $v^l\in L^2_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$, we have

$$(d+\delta)\sum_{0}^{3}(-1)^l v^l \in \bigoplus_{0}^{3} L^2_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$$

by (2-2). Therefore, Lemma A.7 allows us to conclude the proof. □

Remark 2.2. Identity (2-7) holds even for $\varphi=\sum_0^3\varphi^l$ with $\varphi^l\in H^1_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$.

Similar calculations verify that the same property holds for $P(d+\delta;\gamma,\mu,\omega)^t\circ P(d+\delta;\gamma,\mu,\omega)$ as stated in Proposition 2.3. Define

$$\begin{aligned}\langle \widetilde{Q}(\gamma,\mu,\omega)w \mid \varphi\rangle = -\int_{\mathbb{R}^3} \omega^2(\gamma\mu-\varepsilon_0\mu_0)\langle w,\varphi\rangle\,dx \\ + \int_{\mathbb{R}^3} \langle -i2\omega d(\gamma^{1/2}\mu^{1/2})\vee w^2 + i2\omega d(\gamma^{1/2}\mu^{1/2})\wedge w^1, \varphi\rangle\,dx \\ + \int_{\mathbb{R}^3} \langle db,db\rangle\langle w^0+w^2,\varphi^0+\varphi^2\rangle + \langle da,da\rangle\langle w^1+w^3,\varphi^1+\varphi^3\rangle\,dx \\ + \int_{\mathbb{R}^3} \langle db, d\langle w^0-w^2,\varphi^0+\varphi^2\rangle\rangle + \langle da, d\langle -w^1+w^3,\varphi^1+\varphi^3\rangle\rangle\,dx \\ - \int_{\mathbb{R}^3}\langle da, D^*(w^1\odot\varphi^1)\rangle\,dx - \int_{\mathbb{R}^3}\langle db, D^*(*w^2\odot *\varphi^2)\rangle\,dx \end{aligned} \tag{2-14}$$

for $w=\sum_0^3 w^l$ and $\varphi=\sum_0^3\varphi^l$ with $w^l,\varphi^l\in H^1_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$.

Proposition 2.3. *Let $w=\sum_0^3 w^l$ be a graded form with $w^l\in H^1_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$ and assume that*

$$\int_{\mathbb{R}^3}\langle\delta w,\delta\varphi\rangle+\langle dw,d\varphi\rangle-\omega^2\varepsilon_0\mu_0\langle w,\varphi\rangle\,dx+\langle\widetilde{Q}(\gamma,\mu,\omega)w\mid\varphi\rangle=0 \tag{2-15}$$

for all $\varphi=\sum_0^3\varphi^l$ with $\varphi^l\in C_0^\infty(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$. Then $v=\sum_0^3 v^l$ defined by

$$v=P(d+\delta;\gamma,\mu,\omega)w$$

is a weak solution of

$$P(d+\delta;\gamma,\mu,\omega)^t v=0$$

in $\mathbb{R}^3$ and $v^l\in H^1_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$.

Recall that $v=\sum_0^3 v^l$ with $v^l\in L^2_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$ is a weak solution to (2-2) and satisfies $v^0+v^3=0$ in $\mathbb{R}^3$ if and only if $u^1+u^2=\gamma^{-1/2}v^1+\mu^{-1/2}v^2$ with $u^l\in L^2_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$ is a weak solution of (2-1) in $\mathbb{R}^3$. We finish this section by singling out the equation of v^0+v^3 from (2-15), which is used later to show that the CGOs we will construct in Section 4 satisfy $v^0+v^3=0$.

Proposition 2.4. *Let $v=\sum_0^3 v^l$ with $v^l\in H^1_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$ satisfy*

$$P(d+\delta;\gamma,\mu,\omega)v=0$$

in any bounded open subset of $\mathbb{R}^3$. For any $\varphi=\varphi^0+\varphi^3$ with φ^l belonging to $C_0^\infty(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3}\langle\delta(v^0+v^3),\delta\varphi\rangle+\langle d(v^0+v^3),d\varphi\rangle-\omega^2\varepsilon_0\mu_0\langle v^0+v^3,\varphi\rangle\,dx+\langle\tilde q(\gamma,\mu,\omega)(v^0+v^3)\mid\varphi\rangle=0, \tag{2-16}$$

where

$$\begin{aligned}\langle\tilde q(\gamma,\mu,\omega)(v^0+v^3)\mid\varphi\rangle=-\int_{\mathbb{R}^3}\omega^2(\gamma\mu-\varepsilon_0\mu_0)\langle v^0+v^3,\varphi\rangle\,dx+\int_{\mathbb{R}^3}\langle db,db\rangle\langle v^0,\varphi^0\rangle+\langle da,da\rangle\langle v^3,\varphi^3\rangle \\ +\langle db,d\langle v^0,\varphi^0\rangle\rangle+\langle da,d\langle v^3,\varphi^3\rangle\rangle\,dx.\end{aligned}$$

Proof. This is immediate from the proof of Proposition 2.3 and the fact that $\widetilde{Q}(\gamma,\mu,\omega)$ decouples for v^0+v^3. □

3. An integral formula

In this section we provide an integral formula that serves as the starting point to prove uniqueness of the IBVP. To do this, we exploit the computations which allow us to produce solutions for (2-1) from solutions of (2-5) (see Proposition 2.1).

Let Ω be a bounded nonempty open subset in $\mathbb{R}^3$ whose boundary $\partial\Omega$ can be locally described by the graph of a Lipschitz function. Throughout the rest of the paper, we assume that μ_j, ε_j, and σ_j belong to $C^1(\bar{\Omega})$ with $j \in \{1, 2\}$ such that $\mu_j(x) \geq \mu_0$, $\varepsilon_j(x) \geq \varepsilon_0$, and $\sigma_j(x) \geq 0$ everywhere in Ω. Here we say that f is in $C^1(\bar{\Omega})$ if $f : \Omega \longrightarrow \mathbb{C}$ is continuously differentiable in Ω, its partial derivatives $\partial^\alpha f$ are uniformly continuous in Ω for $\alpha \in \mathbb{N}^3$ and $|\alpha| = 1$, and

$$|\partial^\alpha f(x)| \leq C \quad \text{for all } x \in \Omega,\ |\alpha| \leq 1, \tag{3-1}$$

for a certain positive constant C. The norm on $C^1(\bar{\Omega})$, defined as the smallest constant C for which (3-1) holds, makes $C^1(\bar{\Omega})$ a Banach space. Since $\partial\Omega$ is of Lipschitz class, f defined as above is uniformly continuous and, consequently, $\partial^\alpha f$ possesses a unique bounded continuous extension to $\bar{\Omega}$ for any $|\alpha| \leq 1$. This extension will still be denoted by f.

Consider $C_j = C(\mu_j, \varepsilon_j, \sigma_j; \omega)$, the Cauchy data set associated to μ_j, ε_j, and σ_j at frequency $\omega > 0$. Write $\gamma_j = \varepsilon_j + i\sigma_j/\omega$ and assume $\partial^\alpha \gamma_1(x) = \partial^\alpha \gamma_2(x)$ and $\partial^\alpha \mu_1(x) = \partial^\alpha \mu_2(x)$ for all $x \in \partial\Omega$ and $|\alpha| \leq 1$. We can extend[1] γ_j and μ_j to continuously differentiable functions in $\mathbb{R}^3$, still denoted by γ_j and μ_j, such that $|\partial^\alpha \gamma_j(x)| + |\partial^\alpha \mu_j(x)| \leq C$, $\mu_j(x) \geq \mu_0$, $\varepsilon_j(x) \geq \varepsilon_0$, and $\sigma_j(x) \geq 0$ for all $x \in \mathbb{R}^3$, $|\alpha| \leq 1$ and a certain constant $C > 0$,

$$\operatorname{supp}(\mu_j - \mu_0) \subset B, \quad \operatorname{supp}(\gamma_j - \varepsilon_0) \subset B,$$

where $B = \{x \in \mathbb{R}^3 : |x| < R\} \supset \bar{\Omega}$, and $\gamma_1(x) = \gamma_2(x)$ and $\mu_1(x) = \mu_2(x)$ for all $x \in \mathbb{R}^3 \setminus \Omega$. For convenience, we write $a_j = \frac{1}{2}\log\gamma_j$ and $b_j = \frac{1}{2}\log\mu_j$.

Proposition 3.1. *Let $w_1 = \sum_0^3 w_1^l$ be a graded form with $w_1^l \in H^1_{\text{loc}}(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)$ satisfying*

$$\int_{\mathbb{R}^3} \langle \delta w_1, \delta\varphi\rangle + \langle dw_1, d\varphi\rangle - \omega^2\varepsilon_0\mu_0\langle w_1, \varphi\rangle\, dx + \langle Q(\gamma_1, \mu_1, \omega)w_1 \mid \varphi\rangle = 0 \tag{3-2}$$

for all $\varphi = \sum_0^3 \varphi^l$ with $\varphi^l \in C_0^\infty(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)$. Assume that $v_1 = \sum_0^3 v_1^l$, defined by

$$v_1 = P(d+\delta; \gamma_1, \mu_1, \omega)^t w_1, \tag{3-3}$$

satisfies $v_1^0 + v_1^3 = 0$. Let $v_2 = \sum_0^3 v_2^l$ with $v_2^l \in H^1_{\text{loc}}(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)$ satisfy

$$P(d+\delta; \gamma_2, \mu_2, \omega)^t v_2 = 0 \tag{3-4}$$

[1] The extensions we want to perform here are of Whitney type. These kinds of extensions hold for functions defined on any closed subset of $\mathbb{R}^n$ whenever the functions can be approximated by certain polynomials. In order to ensure the existence of such polynomials, we use that $\partial\Omega$ is of Lipschitz class. The argument to prove the existence of such polynomials is similar to the one carried out in Section 2 of [Caro et al. 2013] for $C^{1,\varepsilon}(\bar{\Omega})$ functions with the only difference being that, where the authors referred to Chapter VI, Section 2 of [Stein 1970], we refer to Chapter VI, Section 4.7 of [Stein 1970].

in any bounded open subset of $\mathbb{R}^3$. *Then* $C_1 = C_2$ *implies*

$$\langle (Q(\gamma_2, \mu_2, \omega) - Q(\gamma_1, \mu_1, \omega)) w_1 \mid v_2 \rangle = 0.$$

Proof. By Remark 2.2 and because $\gamma_1(x) = \gamma_2(x)$ and $\mu_1(x) = \mu_2(x)$ for all $x \in \mathbb{R}^3 \setminus \Omega$, we know that

$$\begin{aligned}\langle (Q(\gamma_2, \mu_2, \omega) - Q(\gamma_1, \mu_1, \omega)) w_1 \mid v_2 \rangle &= \int_\Omega \langle P(d+\delta; \gamma_2, \mu_2, \omega)^t w_1, P(d+\delta; \gamma_2, \mu_2, \omega)^t v_2 \rangle \, dx \\ &\quad - \int_\Omega \langle P(d+\delta; \gamma_1, \mu_1, \omega)^t w_1, P(d+\delta; \gamma_1, \mu_1, \omega)^t v_2 \rangle \, dx \\ &= -\int_\Omega \langle v_1, P(d+\delta; \gamma_1, \mu_1, \omega)^t v_2 \rangle \, dx.\end{aligned}$$

The last equality follows from (3-4) and (3-3).

Since $v_1^0 + v_1^3 = 0$, we have that $u_1^1 + u_1^2 = \gamma_1^{-1/2} v_1^1 + \mu_1^{-1/2} v_1^2$ satisfies

$$\delta u_1^2 + i\omega\gamma_1 u_1^1 - d u_1^1 + i\omega\mu_1 u_1^2 = 0 \tag{3-5}$$

almost everywhere in Ω (see Section 2). The definitions of boundary traces $\delta\,\mathrm{tr}$ and $d\,\mathrm{tr}$ (see Section A3) give

$$\begin{aligned}&-\int_\Omega \langle v_1, P(d+\delta; \gamma_1, \mu_1, \omega)^t v_2 \rangle \, dx \\ &\quad = \langle \delta\,\mathrm{tr}(\gamma_1 u_1^1) \mid \gamma_1^{-1/2} v_2^0 \rangle + \langle \delta\,\mathrm{tr}\, u_1^2 \mid \mu_1^{1/2} v_2^1 \rangle - \langle d\,\mathrm{tr}\, u_1^1 \mid \gamma_1^{1/2} v_2^2 \rangle + \langle d\,\mathrm{tr}(\mu_1 u_1^2) \mid \mu_1^{-1/2} v_2^3 \rangle.\end{aligned} \tag{3-6}$$

Suppose $f = f^1 + f^2$ with $f^l \in L^2_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$ is a weak solution to

$$\delta f^2 + i\omega\gamma_2 f^1 - d f^1 + i\omega\mu_2 f^2 = 0 \tag{3-7}$$

in $\mathbb{R}^3$. Note that then $f^1 \in H^d(\Omega; \Lambda^1\mathbb{R}^3)$, $f^2 \in H^\delta(\Omega; \Lambda^2\mathbb{R}^3)$. Set $g = g^1 + g^2 = \gamma_2^{1/2} f^1 + \mu_2^{1/2} f^2$. By (3-4), we obviously have

$$\int_\Omega \langle g, P(d+\delta; \gamma_2, \mu_2, \omega)^t v_2 \rangle \, dx = 0.$$

Once more by the definitions of $\delta\,\mathrm{tr}$ and $d\,\mathrm{tr}$, we have

$$\begin{aligned}0 &= \int_\Omega \langle g, P(d+\delta; \gamma_2, \mu_2, \omega)^t v_2 \rangle \, dx \\ &= -\langle \delta\,\mathrm{tr}(\gamma_2 f^1) \mid \gamma_2^{-1/2} v_2^0 \rangle - \langle \delta\,\mathrm{tr}\, f^2 \mid \mu_2^{1/2} v_2^1 \rangle + \langle d\,\mathrm{tr}\, f^1 \mid \gamma_2^{1/2} v_2^2 \rangle - \langle d\,\mathrm{tr}(\mu_2 f^2) \mid \mu_2^{-1/2} v_2^3 \rangle.\end{aligned} \tag{3-8}$$

Since $\delta\,\mathrm{tr}\, u_1^2 + d\,\mathrm{tr}\, u_1^1 \in C_1 = C_2$ by assumption, there exists $u_2 = u_2^1 + u_2^2$ with $u_2^1 \in H^d(\Omega; \Lambda^1\mathbb{R}^3)$ and $u_2^2 \in H^\delta(\Omega; \Lambda^2\mathbb{R}^3)$ a solution to (3-7) in Ω such that

$$\delta\,\mathrm{tr}\, u_1^2 + d\,\mathrm{tr}\, u_1^1 = \delta\,\mathrm{tr}\, u_2^2 + d\,\mathrm{tr}\, u_2^1.$$

Define[2] $f(x)=u_2(x)$ for almost every $x\in\Omega$ and $f(x)=u_1(x)$ for almost every $x\in\mathbb{R}^3\setminus\Omega$. Using (3-6) and (3-8) and noting that $\gamma_1(x)=\gamma_2(x)$ and $\mu_1(x)=\mu_2(x)$ for all $x\in\partial\Omega$, we can conclude

$$\begin{aligned}&\langle (Q(\gamma_2,\mu_2,\omega)-Q(\gamma_1,\mu_1,\omega))w_1 \mid v_2\rangle\\&=-\frac{1}{i\omega}\langle\delta\operatorname{tr}(\delta u_1^2)\mid\gamma_2^{-1/2}v_2^0\rangle+\frac{1}{i\omega}\langle d\operatorname{tr}(du_1^1)\mid\mu_2^{-1/2}v_2^3\rangle+\frac{1}{i\omega}\langle\delta\operatorname{tr}(\delta u_2^2)\mid\gamma_2^{-1/2}v_2^0\rangle-\frac{1}{i\omega}\langle d\operatorname{tr}(du_2^1)\mid\mu_2^{-1/2}v_2^3\rangle.\end{aligned}$$

The result follows by Lemma A.5. □

4. The construction of CGO solutions

In this section we construct the CGO solutions that will be plugged into the integral formula in Proposition 3.1. To deal with less regular electromagnetic coefficients than those in [Ola and Somersalo 1996], we adopt Bourgain-type spaces introduced by Haberman and Tataru [2013].

Let $\zeta=\sum_1^3\zeta_j\,dx^j$ be a constant 1-differential form in $\mathbb{R}^3$ and let p_ζ denote the polynomial

$$p_\zeta(\xi)=|\xi|^2-2i\langle\zeta,\xi\rangle.$$

For any $b\in\mathbb{R}$, let $\dot X_\zeta^b$ denote the space of graded forms $w=\sum_0^3 w^l$ such that $w^l\in\mathscr{S}'(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$ and its Fourier transform

$$\widehat{w^l}\in L^2(\mathbb{R}^3,|p_\zeta|^{2b}d\xi;\Lambda^l\mathbb{R}^3).$$

The functional

$$w\in\dot X_\zeta^b\longmapsto\|w\|_{\dot X_\zeta^b}=\Bigl(\sum_{l=0}^3\||p_\zeta|^b\widehat{w^l}\|^2_{L^2(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)}\Bigr)^{1/2}$$

makes $\dot X_\zeta^b$ a normed space. Moreover, if $b<1$, then $\dot X_\zeta^b$ is a Hilbert space. As in [Haberman and Tataru 2013], we will only use the cases where $b\in\{1/2,-1/2\}$. Note that $\dot X_\zeta^{-1/2}$ can be identified as the dual space of $\dot X_\zeta^{1/2}$. The simplest feature of these spaces is that the operator $(\Delta_\zeta+\langle\zeta,\zeta\rangle)^{-1}$ (defined by the symbol $(p_\zeta)^{-1}$) is a bounded linear operator from $\dot X_\zeta^{-1/2}$ to $\dot X_\zeta^{1/2}$ with norm

$$\|(\Delta_\zeta+\langle\zeta,\zeta\rangle)^{-1}\|_{\dot X_\zeta^{-1/2}\to\dot X_\zeta^{1/2}}=1. \tag{4-1}$$

Let Δ_ζ denote the conjugate operator $\Delta_\zeta=e_{-\zeta}(d\delta+\delta d)\circ e_\zeta$ where $e_\zeta(x)=e^{\zeta\cdot x}$ and $\zeta\cdot x=\sum_1^3\zeta_j x^j$.

Remark 4.1. Given $f\in\dot X_\zeta^{-1/2}$, it is an obvious consequence of the definition of $\dot X_\zeta^{1/2}$ that there exists a unique $u\in\dot X_\zeta^{1/2}$ satisfying

$$\Delta_\zeta u+\langle\zeta,\zeta\rangle u=f.$$

Remark 4.2. If $u\in\dot X_\zeta^{1/2}$ with $u=\sum_0^3 u^l$, then $u^l\in H^1_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$. This is a simple consequence of (5) and (6) in Lemma 2.2 of [Haberman and Tataru 2013] and the finite band property (sometimes called Bernstein's inequality).

[2]This definition satisfies the appropriate conditions, since $\gamma_1(x)=\gamma_2(x)$ and $\mu_1(x)=\mu_2(x)$ for all $x\in\mathbb{R}^3\setminus\Omega$.

4A. ***The construction of w_1.*** Let ζ_1 be a complex-valued constant 1-form in $\mathbb{R}^3$ satisfying $\langle \zeta_1, \zeta_1 \rangle = -k^2$ where $k = \omega^{1/2}\mu_0\epsilon_0$. We are looking for $w_1 = \sum_0^3 w_1^l$ with $w_1^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$, the solution to (3-2) of the form

$$w_1 = e_{\zeta_1}(A_{\zeta_1} + R_{\zeta_1}) \tag{4-2}$$

with A_{ζ_1} a constant graded differential form in $\mathbb{R}^3$ and $R_{\zeta_1} \in \dot{X}^{1/2}_{\zeta_1}$. Moreover, we want R_{ζ_1} to bear a certain sense of smallness. Note that this is equivalent to finding R_{ζ_1}, which solves

$$(\Delta_{\zeta_1} - k^2)R_{\zeta_1} + Q(\gamma_1, \mu_1, \omega)R_{\zeta_1} = -Q(\gamma_1, \mu_1, \omega)A_{\zeta_1} \tag{4-3}$$

in $\dot{X}^{1/2}_{\zeta_1}$. Note that $Q(\gamma_1, \mu_1, \omega)A_{\zeta_1} \in \dot{X}^{-1/2}_{\zeta_1}$. In the scalar case, this was done in [Haberman and Tataru 2013] for such Bourgain-type spaces. In the original case of smooth coefficients, such equations were solved in weighted L^2 spaces in [Sylvester and Uhlmann 1987] for the scalar case and in [Ola and Somersalo 1996] for systems.

Lemma 4.3. *Let ζ_1 and A_{ζ_1} be as above. For $|\zeta_1|$ large enough, there exists a solution $R_{\zeta_1} \in \dot{X}^{1/2}_{\zeta_1}$ to* (4-3) *such that*

$$\|R_{\zeta_1}\|_{\dot{X}^{1/2}_{\zeta_1}} \lesssim \|Q(\gamma_1, \mu_1, \omega)A_{\zeta_1}\|_{\dot{X}^{-1/2}_{\zeta_1}}, \tag{4-4}$$

where the implicit constant (*incorporated in the symbol* $\lesssim$) *is independent of* ζ_1.

Proof. By using a Neumann series argument (see [Sylvester and Uhlmann 1987]), we can show the existence of $R_{\zeta_1} \in \dot{X}^{1/2}_{\zeta_1}$ satisfying

$$\|R_{\zeta_1}\|_{\dot{X}^{1/2}_{\zeta_1}} \leq \|(I + (\Delta_{\zeta_1} - k^2)^{-1} Q(\gamma_1, \mu_1, \omega))^{-1}\|_{\dot{X}^{1/2}_{\zeta_1} \to \dot{X}^{1/2}_{\zeta_1}} \|Q(\gamma_1, \mu_1, \omega)A_{\zeta_1}\|_{\dot{X}^{-1/2}_{\zeta_1}}$$

for $|\zeta_1|$ large enough, as a simple consequence of (4-1) and

$$\|Q(\gamma_1, \mu_1, \omega)\|_{\dot{X}^{1/2}_{\zeta_1} \to \dot{X}^{-1/2}_{\zeta_1}} = o(\mathbf{1}(|\zeta_1|)). \tag{4-5}$$

Here $\mathbf{1}(t) = 1$ for any $t \in \mathbb{R}$.

To prove (4-5), let u and v belong to $\dot{X}^{1/2}_{\zeta_1}$. By a slight modification of Corollary 2.1 in [Haberman and Tataru 2013], we have that

$$\begin{aligned}
|\langle Q(\gamma_1, \mu_1, \omega)u \mid v\rangle| \lesssim{}& |\zeta_1|^{-1} \|u\|_{\dot{X}^{1/2}_{\zeta_1}} \|v\|_{\dot{X}^{1/2}_{\zeta_1}} \\
&+ \left| \int_{\mathbb{R}^3} \langle \alpha_h, d\langle -u^0 + u^2, v^0 + v^2 \rangle\rangle + \langle \beta_h, d\langle u^1 - u^3, \varphi^1 + v^3 \rangle\rangle \, dx \right| \\
&+ \left| \int_{\mathbb{R}^3} \langle \beta_h, D^*(u^1 \odot v^1)\rangle \, dx \right| + \left| \int_{\mathbb{R}^3} \langle \alpha_h, D^*(*u^2 \odot *v^2)\rangle \, dx \right| \\
&+ \left| \int_{\mathbb{R}^3} \langle da_1 - \alpha_h, d\langle -u^0 + u^2, v^0 + v^2 \rangle\rangle + \langle db_1 - \beta_h, d\langle u^1 - u^3, \varphi^1 + v^3 \rangle\rangle \, dx \right| \\
&+ \left| \int_{\mathbb{R}^3} \langle db_1 - \beta_h, D^*(u^1 \odot v^1)\rangle \, dx \right| + \left| \int_{\mathbb{R}^3} \langle da_1 - \alpha_h, D^*(*u^2 \odot *v^2)\rangle \, dx \right|
\end{aligned}$$

where α_h and β_h are 1-forms in $\mathbb{R}^3$ defined by

$$\alpha_h = \varphi_h * da_1, \quad \beta_h = \varphi_h * db_1$$

(here $*$ denotes convolution) with $0 < h \le 1$, $\varphi_h(x) = h^{-3}\varphi(x/h)$, $\varphi \in C_0^\infty(\mathbb{R}^3)$, $0 \le \varphi(x) \le 1$ for all $x \in \mathbb{R}^3$ and $\int_{\mathbb{R}^3} \varphi\, dx = 1$. Note that the implicit constant depends on ε_0, μ_0, Ω, and the C^1-norms of γ_1 and μ_1. A further modification of Lemma 2.3 in [Haberman and Tataru 2013] gives

$$\begin{aligned}
|\langle Q(\gamma_1,\mu_1,\omega)u \mid v\rangle| &\lesssim |\zeta_1|^{-1}\|u\|_{\dot X^{1/2}_{\zeta_1}}\|v\|_{\dot X^{1/2}_{\zeta_1}} + |\zeta_1|^{-1}(\|\delta\alpha_h\|_{L^\infty(\mathbb{R}^3)} + \|\delta\beta_h\|_{L^\infty(\mathbb{R}^3)})\|u\|_{\dot X^{1/2}_{\zeta_1}}\|v\|_{\dot X^{1/2}_{\zeta_1}} \\
&\qquad + (\|da_1 - \alpha_h\|_{L^\infty(\mathbb{R}^3)} + \|db_1 - \beta_h\|_{L^\infty(\mathbb{R}^3)})\|u\|_{\dot X^{1/2}_{\zeta_1}}\|v\|_{\dot X^{1/2}_{\zeta_1}} \\
&\lesssim (|\zeta_1|^{-1}h^{-1} + o(\mathbf{1}(h)))\|u\|_{\dot X^{1/2}_{\zeta_1}}\|v\|_{\dot X^{1/2}_{\zeta_1}}
\end{aligned}$$

as h vanishes. Choosing $h = |\zeta_1|^{-1/2}$, this implies (4-1) and the lemma is proven. □

Up to this point, nothing has been said about the smallness of R_{ζ_1}. We will see in the next lemma that estimate (4-4) yields such smallness in an average sense. This idea is one of the key points in [Haberman and Tataru 2013].

Lemma 4.4. *Let $s \in \mathbb{R}$ satisfy $s \ge 1$. Given a real-valued constant 1-form ρ in $\mathbb{R}^3$, choose η_1 and η_2 also real-valued constant 1-forms such that $\langle \eta_1, \eta_2\rangle = 0$, $\langle \eta_j, \rho\rangle = 0$, and $|\eta_j| = 1$ for $j \in \{1, 2\}$. Set*

$$\zeta_1 = -\sqrt{s^2 + \frac{|\rho|^2}{4}}\,\eta_1 + i\left(\frac{\rho}{2} - \sqrt{s^2+k^2}\,\eta_2\right),$$

and assume $|A_{\zeta_1}|$ is bounded as a function of s, η_1. Then the R_{ζ_1} obtained in Lemma 4.3 satisfies

$$\frac{1}{\lambda}\int_{S^1}\int_\lambda^{2\lambda} \|R_{\zeta_1}\|^2_{\dot X^{1/2}_{\zeta_1}}\, ds\, d\eta_1 = o(1(\lambda)) \tag{4-6}$$

as λ becomes large. Here S^1 denotes the intersection between the unit sphere in $\mathbb{R}^3$ and the plane defined by η_1 and η_2.

Proof. By the definition of $Q(\gamma_1,\mu_1,\omega)$, the identity (2-13), the fact that A_{ζ_1} is constant, and the fact that $Q(\gamma_1,\mu_1,\omega)$ is compactly supported, we have

$$|\langle Q(\gamma_1,\mu_1,\omega)A_{\zeta_1} \mid v\rangle| \lesssim \sum_{l=0}^{3}\|\chi v^l\|_{L^2(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)} + \|\chi(d+\delta)f_{\zeta_1}\|_{\dot X^{-1/2}_{\zeta_1}}\|v\|_{\dot X^{1/2}_{\zeta_1}},$$

where $v = \sum_0^3 v^l$, $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\chi(x) = 1$ for all $x \in \operatorname{supp} d\gamma_1 \cup \operatorname{supp} d\mu_1$ and

$$f_{\zeta_1} = db_1 \wedge A^1_{\zeta_1} + db_1 \vee (A^1_{\zeta_1} + A^3_{\zeta_1}) + da_1 \wedge (A^0_{\zeta_1} + A^2_{\zeta_1}) - da_1 \vee A^2_{\zeta_1}$$

with $A_{\zeta_1} = \sum_0^3 A^l_{\zeta_1}$. By (5) in Lemma 2.2 of [Haberman and Tataru 2013], this gives

$$\|Q(\gamma_1,\mu_1,\omega)A_{\zeta_1}\|_{\dot X^{-1/2}_{\zeta_1}} \lesssim s^{-1/2} + \|\chi(d+\delta)f_{\zeta_1}\|_{\dot X^{-1/2}_{\zeta_1}}.$$

Now an immediate modification of Lemma 3.1 in [Haberman and Tataru 2013] allows us to check that

$$\frac{1}{\lambda}\int_{S^1}\int_{\lambda}^{2\lambda} \|\chi(d+\delta) f_{\zeta_1}\|^2_{\dot{X}^{-1/2}_{\zeta_1}}\, ds\, d\eta_1 = o(\mathbf{1}(\lambda)),$$

which implies

$$\frac{1}{\lambda}\int_{S^1}\int_{\lambda}^{2\lambda} \|Q(\gamma_1,\mu_1,\omega)A_{\zeta_1}\|^2_{\dot{X}^{-1/2}_{\zeta_1}}\, ds\, d\eta_1 = o(\mathbf{1}(\lambda)) \tag{4-7}$$

as λ becomes large. By (4-4), we obtain (4-6). □

From the construction of $R_{\zeta_1} \in \dot{X}^{1/2}_{\zeta_1}$ solving (4-3), the existence of w_1 of the form (4-2) that solves (3-2) is immediate. However, it turns out that for such a w_1 to satisfy the condition in Proposition 3.1, the constant 1-form A_{ζ_1} has to be chosen carefully.

Lemma 4.5. *Let $w_1 = \sum_{l=0}^{3} w_1^l$ as in* (4-2) *with ζ_1, A_{ζ_1}, and R_{ζ_1} as in Lemma 4.3. Then*

$$w_1^l \in H^1_{\mathrm{loc}}\big(\mathbb{R}^3;\Lambda^l\mathbb{R}^3\big)$$

and w_1 is a solution of (3-2). *Moreover, if A_{ζ_1} satisfies the relation*

$$-\zeta_1 \vee A^1_{\zeta_1} + ikA^0_{\zeta_1} - \zeta_1\wedge A^2_{\zeta_1} + ikA^3_{\zeta_1} = 0, \tag{4-8}$$

then $v_1 = \sum_0^3 v_1^l$ defined as in (3-3) *satisfies $v_1^0 + v_1^3 = 0$ for $|\zeta_1|$ large enough.*

Proof. We can ensure w_1^l is in $H^1_{\mathrm{loc}}\big(\mathbb{R}^3;\Lambda^l\mathbb{R}^3\big)$ since $R_{\zeta_1}\in \dot{X}^{1/2}_{\zeta_1}$ (See Remark 4.2). Additionally, w_1 is a solution of (3-2) since $R_{\zeta_1}\in\dot{X}^{1/2}_{\zeta_1}$ solves[3] (4-3).

In order to prove the second part of this lemma, note that $v_1^l \in H^1_{\mathrm{loc}}\big(\mathbb{R}^3;\Lambda^l\mathbb{R}^3\big)$ and

$$P(d+\delta;\gamma_1,\mu_1,\omega)v_1 = 0$$

in any bounded open subset of $\mathbb{R}^3$ by Proposition 2.1. Then by Proposition 2.4 we know that $v_1^0+v_1^3$ is a weak solution to

$$(\delta d + d\delta - k^2)(v_1^0+v_1^3) + \tilde{q}(\gamma_1,\mu_1,\omega)(v_1^0+v_1^3) = 0$$

in $\mathbb{R}^3$. By (3-3), we can write $v_1^l = e_{\zeta_1}(B^l_{\zeta_1}+S^l_{\zeta_1})$ with $l\in\{0,3\}$, where

$$\begin{aligned}
B^0_{\zeta_1} &= -\zeta_1\vee A^1_{\zeta_1} + ikA^0_{\zeta_1},\\
S^0_{\zeta_1} &= -\zeta_1\vee R^1_{\zeta_1} + \delta R^1_{\zeta_1} + db\vee(A^1_{\zeta_1}+R^1_{\zeta_1}) + i\omega\gamma_1^{1/2}\mu_1^{1/2}R^0_{\zeta_1} + i(\omega\gamma_1^{1/2}\mu_1^{1/2}-k)A^0_{\zeta_1}, &\text{(4-9)}\\
B^3_{\zeta_1} &= -\zeta_1\wedge A^2_{\zeta_1} + ikA^3_{\zeta_1},\\
S^3_{\zeta_1} &= -\zeta_1\wedge R^2_{\zeta_1} - dR^2_{\zeta_1} + da\wedge(A^2_{\zeta_1}+R^2_{\zeta_1}) + i\omega\gamma_1^{1/2}\mu_1^{1/2}R^3_{\zeta_1} + i(\omega\gamma_1^{1/2}\mu_1^{1/2}-k)A^3_{\zeta_1}. &\text{(4-10)}
\end{aligned}$$

Then relation (4-8) implies $B^0_{\zeta_1}+B^3_{\zeta_1}=0$, and hence that $v_1^0+v_1^3 = e_{\zeta_1}(S^0_{\zeta_1}+S^3_{\zeta_1})$ is a weak solution of

$$(\Delta_{\zeta_1}-k^2)(S^0_{\zeta_1}+S^3_{\zeta_1}) + \tilde{q}(\gamma_1,\mu_1,\omega)(S^0_{\zeta_1}+S^3_{\zeta_1}) = 0 \tag{4-11}$$

in $\mathbb{R}^3$.

[3] See also (A-20).

To complete the proof, it is sufficient to show that (4-11) is uniquely solvable in $\dot{X}^{1/2}_{\zeta_1}$ for $|\zeta_1|$ large enough and $S^0_{\zeta_1} + S^3_{\zeta_1}$ belongs to $\dot{X}^{1/2}_{\zeta_1}$.

Using the same argument as in proving (4-5), we see that $\tilde{q}(\gamma_1, \mu_1, \omega)$ is a bounded linear operator from $\dot{X}^{1/2}_{\zeta_1}$ to $\dot{X}^{-1/2}_{\zeta_1}$ and its operator norm is $o(\mathbf{1}(|\zeta_1|))$. Then, by Remark 4.1, identity (4-1), and the Banach fixed-point theorem, (4-11) is uniquely solvable in $\dot{X}^{1/2}_{\zeta_1}$ for $|\zeta_1|$ large enough.

Since $e_{\zeta_1} S^l_{\zeta_1} = v^l_1 \in H^1_{\text{loc}}\big(\mathbb{R}^3; \Lambda^l \mathbb{R}^3\big)$ for $l \in \{0, 3\}$, we know that $\chi(S^0_{\zeta_1} + S^3_{\zeta_1}) \in \dot{X}^{1/2}_{\zeta_1}$ for $\chi \in C^\infty_0(\mathbb{R}^3)$ such that $\chi(x) = 1$ for all $x \in (\operatorname{supp} d\gamma_1 \cup \operatorname{supp} d\mu_1)$. Therefore, the right-hand side of

$$(\Delta_{\zeta_1} - k^2)(S^0_{\zeta_1} + S^3_{\zeta_1}) = -\tilde{q}(\gamma_1, \mu_1, \omega)\chi(S^0_{\zeta_1} + S^3_{\zeta_1})$$

is in $\dot{X}^{-1/2}_{\zeta_1}$. Further, it is not hard to see from (4-9) and (4-10) that $\widehat{S^l_{\zeta_1}}$ belongs to $L^2_{\text{loc}}\big(\mathbb{R}^3; \Lambda^l \mathbb{R}^3\big)$ with $l \in \{0, 3\}$. The last two facts imply that $S^0_{\zeta_1} + S^3_{\zeta_1} \in \dot{X}^{1/2}_{\zeta_1}$. □

Remark 4.6. The condition given by (4-8) is necessary in our proof since $B^0_{\zeta_1} + B^3_{\zeta_1}$ does not belong to $\dot{X}^{1/2}_{\zeta_1}$.

As a conclusion of these lemmas, we can state the constructions of w_1 in the following theorem.

Theorem 4.7. *Let $s \in \mathbb{R}$ satisfy $s \geq 1$. Given a real-valued constant 1-form ρ in $\mathbb{R}^3$, choose η_1 and η_2 also real-valued constant 1-forms in $\mathbb{R}^3$ such that $\langle \eta_1, \eta_2 \rangle = 0$, $\langle \eta_j, \rho \rangle = 0$, and $|\eta_j| = 1$ for $j \in \{1, 2\}$. Set*

$$\zeta_1 = -\sqrt{s^2 + \frac{|\rho|^2}{4}}\,\eta_1 + i\left(\frac{\rho}{2} - \sqrt{s^2 + k^2}\,\eta_2\right)$$

and

$$A_{\zeta_1} = \frac{\sqrt{2}}{|\zeta_1|}(\zeta_1 \vee \alpha + ik\alpha + ik\beta + \zeta_1 \wedge \beta),$$

where either $\alpha = \eta_1$ and $\beta = 0$ or $\alpha = 0$ and $\beta = |\rho|^{-1}\eta_2 \wedge \rho$. Then, for $|\zeta_1|$ large enough, there exists $w_1 = \sum_0^3 w^l_1$ with $w^l_1 \in H^1_{\text{loc}}\big(\mathbb{R}^3; \Lambda^l \mathbb{R}^3\big)$ of the form

$$w_1 = e_{\zeta_1}(A_{\zeta_1} + R_{\zeta_1}),$$

which is a weak solution to

$$(d\delta + \delta d - k^2)w_1 + Q(\gamma_1, \mu_1, \omega)w_1 = 0$$

in $\mathbb{R}^3$. Moreover, we have $R_{\zeta_1} \in \dot{X}^{1/2}_{\zeta_1}$ satisfies

$$\frac{1}{\lambda}\int_{S^1}\int_{\lambda}^{2\lambda} \|R_{\zeta_1}\|^2_{\dot{X}^{1/2}_{\zeta_1}}\, ds\, d\eta_1 = o(1(\lambda))$$

as λ becomes large. Here S^1 denotes the intersection between the unit sphere in $\mathbb{R}^3$ and the plane defined by η_1 and η_2. Furthermore, $v_1 = \sum_0^3 v^l_1$ defined by

$$v_1 = P(d + \delta; \gamma_1, \mu_1, \omega)^t w_1$$

satisfies $v^0_1 + v^3_1 = 0$ for $|\zeta_1|$ large enough.

4B. ***The construction of v_2.*** Let ζ_2 be a complex-valued constant 1-form in $\mathbb{R}^3$ satisfying $\langle \zeta_2, \zeta_2\rangle = -k^2$. We are looking for the solution $v_2 = \sum_0^3 v_2^l$ with $v_2^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$ to (3-4) in any bounded subset of $\mathbb{R}^3$ of the form

$$v_2 = e_{\zeta_2}(B_{\zeta_2} + S_{\zeta_2}), \tag{4-12}$$

where B_{ζ_2} is a constant graded differential form in $\mathbb{R}^3$ and $S_{\zeta_2} \in \dot{X}^{1/2}_{\zeta_2}$. In addition, we want S_{ζ_2} to be small in the sense of (4-6). To construct such a v_2, by Proposition 2.3, we start with the construction of a solution w_2 to

$$\int_{\mathbb{R}^3} \langle \delta w_2, \delta\varphi\rangle + \langle d w_2, d\varphi\rangle - \omega^2 \varepsilon_0 \mu_0 \langle w_2, \varphi\rangle \, dx + \langle \widetilde{Q}(\gamma_2, \mu_2, \omega) w_2 \mid \varphi\rangle = 0 \tag{4-13}$$

for all $\varphi = \sum_0^3 \varphi^l$, with $\varphi^l \in C_0^\infty(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$.

Lemma 4.8. *Let $A_{\zeta_2} = A^1_{\zeta_2} + A^2_{\zeta_2}$ be a constant graded differential form in $\mathbb{R}^3$. For $|\zeta_2|$ large enough, there exists $R_{\zeta_2} = R^1_{\zeta_2} + R^2_{\zeta_2} \in \dot{X}^{1/2}_{\zeta_2}$ such that $w_2 = w_2^1 + w_2^2$ with*

$$w_2^l = e_{\zeta_2}(A^l_{\zeta_2} + R^l_{\zeta_2})$$

and $w_2^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$, is a solution of (4-13) *in* $\mathbb{R}^3$.

Proof. Analogous to the proof of Lemma 4.3, the existence of a general $R_{\zeta_2} = \sum_0^3 R^l_{\zeta_2}$ for a given constant $A_{\zeta_2} = \sum_0^3 A^l_{\zeta_2}$ is immediate by

$$\|\widetilde{Q}(\gamma_2, \mu_2, \omega)\|_{\dot{X}^{1/2}_{\zeta_2} \to \dot{X}^{-1/2}_{\zeta_2}} = o(\mathbf{1}(|\zeta_2|))$$

as $|\zeta_2|$ becomes large. Since $\widetilde{Q}(\gamma_2, \mu_2, \omega)$ decouples for 1 and 2 forms, we can ensure that $R_{\zeta_2} = R^1_{\zeta_2} + R^2_{\zeta_2}$ for $A_{\zeta_2} = A^1_{\zeta_2} + A^2_{\zeta_2}$. □

Now Proposition 2.3 states that $v_2 = P(d+\delta; \gamma_2, \mu_2, \omega) w_2$ is a solution to (3-4). Moreover, we can write v_2 as in (4-12). However, we still need to show the smallness of S_{ζ_2}.

Theorem 4.9. *Let $s \in \mathbb{R}$ satisfy $s \geq 1$. Given a real-valued constant 1-form ρ in $\mathbb{R}^3$, we choose η_1 and η_2 two other real-valued constant 1-forms in $\mathbb{R}^3$ such that $\langle \eta_1, \eta_2\rangle = 0$, $\langle \eta_j, \rho\rangle = 0$, and $|\eta_j| = 1$ for $j \in \{1, 2\}$. Set*

$$\zeta_2 = \sqrt{s^2 + \frac{|\rho|^2}{4}}\, \eta_1 + i\left(\frac{\rho}{2} + \sqrt{s^2 + k^2}\, \eta_2\right)$$

and let α and β be as in Theorem 4.7. If $|\zeta_2|$ is large enough, there exists $v_2 = \sum_0^3 v_2^l$ with

$$v_2^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$$

of the form

$$v_2 = e_{\zeta_2}(B_{\zeta_2} + S_{\zeta_2}),$$

where

$$B_{\zeta_2} = -\frac{\sqrt{2}}{|\zeta_2|}(\zeta_2 \vee (\alpha+\beta) + \zeta_2 \wedge (-\alpha+\beta) + ik(\alpha+\beta)) \tag{4-14}$$

and $S_{\zeta_2} \in \dot{X}^{1/2}_{\zeta_2}$, *which solves*

$$P(d+\delta;\gamma_2,\mu_2,\omega)^t v_2 = 0$$

in any bounded open subset of $\mathbb{R}^3$ *and satisfies*

$$\frac{1}{\lambda}\int_{S^1}\int_{\lambda}^{2\lambda} \|S_{\zeta_2}\|^2_{\dot{X}^{1/2}_{\zeta_2}}\,ds\,d\eta_1 = o(1(\lambda)) \tag{4-15}$$

as λ *becomes large. Here* S^1 *denotes the intersection between the unit sphere in* $\mathbb{R}^3$ *and the plane defined by* η_1 *and* η_2.

Proof. Let w_2 be as in Lemma 4.8 with $A_{\zeta_2} = A^1_{\zeta_2} + A^2_{\zeta_2} = -\sqrt{2}(\alpha+\beta)$. By Proposition 2.3, we know that $v_2 = \sum_0^3 v_2^l$ defined by

$$v_2 = P(d+\delta;\gamma_2,\mu_2,\omega)w_2$$

satisfies that $v_2^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$ and solves

$$P(d+\delta;\gamma_2,\mu_2,\omega)^t v_2 = 0 \tag{4-16}$$

in any bounded open subset of $\mathbb{R}^3$. One can easily write

$$v_2 = e_{\zeta_2}(B_{\zeta_2} + S_{\zeta_2})$$

and check that B_{ζ_2} is given by (4-14) and

$$\begin{aligned} S_{\zeta_2} = \frac{1}{|\zeta_2|}\big(&\zeta_2 \vee (R^1_{\zeta_2}+R^2_{\zeta_2}) + \zeta_2 \wedge (-R^1_{\zeta_2}+R^2_{\zeta_2}) + (d+\delta)(-R^1_{\zeta_2}+R^2_{\zeta_2}) \\ &+ da_2 \wedge (A^1_{\zeta_2}+R^1_{\zeta_2}) + da_2 \vee (A^1_{\zeta_2}+R^1_{\zeta_2}) + db_2 \wedge (A^2_{\zeta_2}+R^2_{\zeta_2}) - db_2 \vee (A^2_{\zeta_2}+R^2_{\zeta_2}) \\ &+ i\omega\gamma_2^{1/2}\mu_2^{1/2}(R^1_{\zeta_2}+R^2_{\zeta_2}) + i(\omega\gamma_2^{1/2}\mu_2^{1/2}-k)(A^1_{\zeta_2}+A^2_{\zeta_2})\big). \end{aligned}$$

Moreover, by (4-16) and (2-7), we know that S_{ζ_2} satisfies the familiar equation

$$(\Delta_{\zeta_2} - k^2)S_{\zeta_2} + Q(\gamma_2,\mu_2,\omega)S_{\zeta_2} = -Q(\gamma_2,\mu_2,\omega)B_{\zeta_2}. \tag{4-17}$$

Since $Q(\gamma_2,\mu_2,\omega)B_{\zeta_2} \in \dot{X}^{-1/2}_{\zeta_2}$, (4-17) is uniquely solvable in $\dot{X}^{1/2}_{\zeta_2}$. Therefore, since $S_{\zeta_2} \in \dot{X}^{1/2}_{\zeta_2}$ and $|B_{\zeta_2}| = \mathcal{O}(\mathbf{1}(|\zeta_2|))$, S_{ζ_2} satisfies (4-15). □

5. Proof of uniqueness

To complete the proof of Theorem 1.1, the final step is to plug into the integral formula given in Proposition 3.1 the w_1 and v_2 obtained in Theorem 4.7 and Theorem 4.9 and to let λ go to ∞. The output turns out to be certain nonlinear relations of γ_1, μ_1, γ_2, μ_2, and their weak partial derivatives up to the second order. Then a unique continuation principle argument can be used to conclude the uniqueness.

Throughout this section we let Q_j denote $Q(\gamma_j,\mu_j,\omega)$ with $j \in \{1,2\}$. If

$$\begin{aligned} A_1 &= -(\eta_1 + i\eta_2)\vee\alpha - (\eta_1+i\eta_2)\wedge\beta, \\ B_2 &= -(\eta_1+i\eta_2)\vee(\alpha+\beta) - (\eta_1+i\eta_2)\wedge(-\alpha+\beta) \end{aligned}$$

with α and β as in Theorem 4.7, we see that, for any ρ, $|A_{\zeta_1} - A_1| + |B_{\zeta_2} - B_2| = \mathbb{O}(s^{-1})$ for s large enough and all $\eta_1, \eta_2 \in S^1$. The implicit constant (incorporated in the symbol $\mathbb{O}$) here depends on ρ. On the other hand, plugging w_1 and v_2 into Proposition 3.1, as in Theorem 4.7 and Theorem 4.9, we get

$$\langle (Q_2-Q_1)e_{i\rho}A_1 \mid B_2\rangle = \langle (Q_1-Q_2)(A_{\zeta_1}+R_{\zeta_1}) \mid e_{i\rho}(B_{\zeta_2}-B_2+S_{\zeta_2})\rangle + \langle (Q_1-Q_2)B_2 \mid e_{i\rho}(A_{\zeta_1}-A_1+R_{\zeta_1})\rangle.$$

We know that, for each ρ, Q_j is bounded from $\dot{X}^{1/2}_{\zeta_j}$ to $\dot{X}^{-1/2}_{\zeta_j}$ and its norm is $o(\mathbf{1}(s))$ for s large enough and all η_1 (see (4-5) and the same applies to Q_2). The same is true for $Q_1 - Q_2$ from $\dot{X}^{1/2}_{\zeta_1}$ to $\dot{X}^{-1/2}_{\zeta_2}$ as an immediate consequence of the proof of Lemma 2.3 in [Haberman and Tataru 2013]. Thus, for each ρ, we have

$$\begin{aligned} |\langle (Q_2 - Q_1)e_{i\rho}A_1 \mid B_2\rangle| \lesssim{}& \|(Q_1 - Q_2)B_2\|_{\dot{X}^{-1/2}_{\zeta_1}} [\|\chi(A_{\zeta_1} - A_1)\|_{\dot{X}^{1/2}_{\zeta_1}} + \|R_{\zeta_1}\|_{\dot{X}^{1/2}_{\zeta_1}}] \\ &+ [\|(Q_1 - Q_2)A_{\zeta_1}\|_{\dot{X}^{-1/2}_{\zeta_2}} + \|R_{\zeta_1}\|_{\dot{X}^{1/2}_{\zeta_1}}][\|\chi(B_{\zeta_2} - B_2)\|_{\dot{X}^{1/2}_{\zeta_2}} + \|S_{\zeta_2}\|_{\dot{X}^{1/2}_{\zeta_2}}], \end{aligned} \tag{5-1}$$

where $\chi \in \mathbb{C}^\infty_0(\mathbb{R}^3)$ such that $\chi(x) = 1$ for all $x \in \operatorname{supp} d\gamma_2 \cup \operatorname{supp} d\mu_2$. Here the implicit constant might depend on ρ.

If $\alpha = \eta_1$ and $\beta = 0$, then $A_1 = -1$, $B_2 = -1 + i\eta_2 \wedge \eta_1$, and the left-hand side of (5-1) gives

$$\begin{aligned} &\langle (Q_2 - Q_1)e_{i\rho}A_1 \mid B_2\rangle \\ &= \int_{\mathbb{R}^3} \langle d(a_1 - a_2), de_{i\rho}\rangle\, dx + \int_{\mathbb{R}^3} \langle d(a_1 + a_2), d(a_2 - a_1)\rangle e_{i\rho}\, dx + \int_{\mathbb{R}^3} \omega^2(\gamma_1\mu_1 - \gamma_2\mu_2)e_{i\rho}\, dx. \end{aligned} \tag{5-2}$$

If $\alpha = 0$ and $\beta = |\rho|^{-1}\eta_2 \wedge \rho$, then

$$A_1 = -|\rho|^{-1}\eta_1 \wedge \eta_2 \wedge \rho, \quad B_2 = -|\rho|^{-1}(\eta_1 + i\eta_2) \vee (\eta_2 \wedge \rho) - |\rho|^{-1}\eta_1 \wedge \eta_2 \wedge \rho,$$

and we have

$$\begin{aligned} &\langle (Q_2 - Q_1)e_{i\rho}A_1 \mid B_2\rangle \\ &= \int_{\mathbb{R}^3} \langle d(b_1 - b_2), de_{i\rho}\rangle\, dx + \int_{\mathbb{R}^3} \langle d(b_1 + b_2), d(b_2 - b_1)\rangle e_{i\rho}\, dx + \int_{\mathbb{R}^3} \omega^2(\gamma_1\mu_1 - \gamma_2\mu_2)e_{i\rho}\, dx. \end{aligned} \tag{5-3}$$

Meanwhile, by the choice of A_1 and B_2 above, we have

$$\begin{aligned} \Big(\frac{1}{\lambda}\int_{S^1}\int_\lambda^{2\lambda} \|\chi(A_{\zeta_1} - A_1)\|^2_{\dot{X}^{1/2}_{\zeta_1}}\, ds\, d\eta_1\Big)^{1/2} &= \mathbb{O}(\mathbf{1}(\lambda)), \\ \Big(\frac{1}{\lambda}\int_{S^1}\int_\lambda^{2\lambda} \|\chi(B_{\zeta_2} - B_2)\|^2_{\dot{X}^{1/2}_{\zeta_2}}\, ds\, d\eta_1\Big)^{1/2} &= \mathbb{O}(\mathbf{1}(\lambda)). \end{aligned}$$

Then, after averaging (5-1) on $(s, \eta_1) \in [\lambda, 2\lambda] \times S^1$ and using the Cauchy–Schwartz inequality, we get

$$\begin{aligned} |\langle (Q_2 - Q_1)e_{i\rho}A_1 \mid B_2\rangle| \lesssim{}& [\mathbb{O}(\mathbf{1}(\lambda)) + o(\mathbf{1}(\lambda))]\Big(\frac{1}{\lambda}\int_{S^1}\int_\lambda^{2\lambda} \|(Q_1 - Q_2)B_2\|^2_{\dot{X}^{-1/2}_{\zeta_1}}\, ds\, d\eta_1\Big)^{1/2} \\ &+ [\mathbb{O}(\mathbf{1}(\lambda)) + o(\mathbf{1}(\lambda))]\Big[\Big(\frac{1}{\lambda}\int_{S^1}\int_\lambda^{2\lambda} \|(Q_1 - Q_2)A_{\zeta_1}\|^2_{\dot{X}^{-1/2}_{\zeta_2}}\, ds\, d\eta_1\Big)^{1/2} + o(\mathbf{1}(\lambda))\Big], \end{aligned}$$

where Theorems 4.7 and 4.9 are used. It is not hard to see this converges to zero as λ goes to ∞ by the same argument we used in proving (4-7) and by noticing that the left-hand side is independent of λ. Thus, by (5-2) and (5-3), we arrive at

$$\int_{\mathbb{R}^3} \langle d(a_2-a_1), de_{i\rho}\rangle\,dx - \int_{\mathbb{R}^3} \langle d(a_1+a_2), d(a_2-a_1)\rangle e_{i\rho}\,dx + \int_{\mathbb{R}^3} \omega^2(\gamma_2\mu_2-\gamma_1\mu_1)e_{i\rho}\,dx = 0 \quad (5\text{-}4)$$

and

$$\int_{\mathbb{R}^3} \langle d(b_2-b_1), de_{i\rho}\rangle\,dx - \int_{\mathbb{R}^3} \langle d(b_1+b_2), d(b_2-b_1)\rangle e_{i\rho}\,dx + \int_{\mathbb{R}^3} \omega^2(\gamma_2\mu_2-\gamma_1\mu_1)e_{i\rho}\,dx = 0 \quad (5\text{-}5)$$

for any ρ. So far, this shows that

$$\begin{cases} \delta d(a_2-a_1) - \langle d(a_1+a_2), d(a_2-a_1)\rangle + \omega^2(\gamma_2\mu_2-\gamma_1\mu_1) = 0, \\ \delta d(b_2-b_1) - \langle d(b_1+b_2), d(b_2-b_1)\rangle + \omega^2(\gamma_2\mu_2-\gamma_1\mu_1) = 0, \end{cases}$$

a system that has to be understood in the weak sense. Finally, some simple computations yield a system of second order equations of the form

$$\begin{cases} -\Delta(\gamma_2^{1/2}-\gamma_1^{1/2}) + V(\gamma_2^{1/2}-\gamma_1^{1/2}) + a(\gamma_2^{1/2}-\gamma_1^{1/2}) + b(\mu_2^{1/2}-\mu_1^{1/2}) = 0, \\ -\Delta(\mu_2^{1/2}-\mu_1^{1/2}) + W(\mu_2^{1/2}-\mu_1^{1/2}) + c(\mu_2^{1/2}-\mu_1^{1/2}) + d(\gamma_2^{1/2}-\gamma_1^{1/2}) = 0, \end{cases}$$

again in the weak sense with

$$V = -\frac{\delta d(\gamma_1^{1/2}+\gamma_2^{1/2})}{\gamma_1^{1/2}+\gamma_2^{1/2}}, \quad W = -\frac{\delta d(\mu_1^{1/2}+\mu_2^{1/2})}{\mu_1^{1/2}+\mu_2^{1/2}}$$

and

$$a = \mathbf{1}_\Omega \omega^2 \gamma_1^{1/2}\gamma_2^{1/2}(\mu_1+\mu_2), \quad b = -\mathbf{1}_\Omega \omega^2 \gamma_1^{1/2}\gamma_2^{1/2}(\gamma_1+\gamma_2)\frac{\mu_1^{1/2}+\mu_2^{1/2}}{\gamma_1^{1/2}+\gamma_2^{1/2}},$$

$$c = \mathbf{1}_\Omega \omega^2 \mu_1^{1/2}\mu_2^{1/2}(\gamma_1+\gamma_2), \quad d = -\mathbf{1}_\Omega \omega^2 \mu_1^{1/2}\mu_2^{1/2}(\mu_1+\mu_2)\frac{\gamma_1^{1/2}+\gamma_2^{1/2}}{\mu_1^{1/2}+\mu_2^{1/2}},$$

where $\mathbf{1}_\Omega$ is the characteristic function of Ω. Note that $\gamma_2^{1/2}-\gamma_1^{1/2}$ and $\mu_2^{1/2}-\mu_1^{1/2}$ belong to $H^1(\mathbb{R}^3)$ and they are compactly supported. Thus the next unique continuation result implies that $\gamma_2=\gamma_1$ and $\mu_2=\mu_1$.

Lemma 5.1. *Let f and g belong to $H^1(\mathbb{R}^3)$ and assume that they are compactly supported. Then f and g vanish if and only if they satisfy*

$$\begin{cases} -\Delta f + Vf + af + bg = 0, \\ -\Delta g + Wg + cg + df = 0. \end{cases} \quad (5\text{-}6)$$

Proof. Let $\zeta \in \mathbb{C}^n$ satisfies $\zeta\cdot\zeta = 0$. Set $u(x) = e^{\zeta\cdot x} f(x)$ and $v(x) = e^{\zeta\cdot x} g(x)$. Since f and g belong to $H^1(\mathbb{R}^3)$ and they are compactly supported, u and v also belong to $H^1(\mathbb{R}^3)$ and, consequently, to $\dot{X}_\zeta^{1/2}$. Moreover, u and v solve

$$\begin{cases} -(\Delta + 2\zeta\cdot\nabla)u + Vu + au + bv = 0, \\ -(\Delta + 2\zeta\cdot\nabla)v + Wv + cv + du = 0. \end{cases} \quad (5\text{-}7)$$

Let $w = w^0 + w^3$ be the graded form given by $w^0 = u$ and $w^3 = *v$ and define

$$\langle Qw \mid \varphi\rangle = -\int_{\mathbb{R}^3}\Big\langle d(\gamma_1^{1/2}+\gamma_2^{1/2}),\, d\frac{\langle w^0,\varphi^0\rangle}{\gamma_1^{1/2}+\gamma_2^{1/2}}\Big\rangle\,dx + \int_{\mathbb{R}^3}\langle aw^0+bw^3,\varphi^0\rangle\,dx \\ -\int_{\mathbb{R}^3}\Big\langle d(\mu_1^{1/2}+\mu_2^{1/2}),\, d\frac{\langle w^3,\varphi^3\rangle}{\mu_1^{1/2}+\mu_2^{1/2}}\Big\rangle\,dx + \int_{\mathbb{R}^3}\langle dw^0+cw^3,\varphi^3\rangle\,dx$$

for any $\varphi = \varphi^0 + \varphi^3$ with $\varphi^l \in H^1(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$. Then $w \in \dot{X}_\zeta^{1/2}$ and (5-7) reads

$$\Delta_\zeta w + Qw = 0. \tag{5-8}$$

Here we have identified ζ with a 1-form also denoted by ζ. Following the same argument as in Lemma 4.3, we can prove

$$\|Q\|_{\dot{X}_\zeta^{1/2}\to\dot{X}_\zeta^{-1/2}} = o(\mathbf{1}(|\zeta|)) \tag{5-9}$$

as $|\zeta|$ becomes large. Then Remark 4.1, identity (4-1), (5-9), and the Banach fixed-point theorem imply that (5-8) has a unique solution belonging to $\dot{X}_\zeta^{1/2}$. Therefore, $w = 0$, which in turn implies $f = g = 0$. □

Appendix: The framework of differential forms

Since the tools used in this paper are scattered throughout the literature, to make the paper more self-contained, we summarized them in this appendix. We start with collecting several basics required in the framework of differential forms (see [Taylor 1996] and [Federer 1969] for some details of differential forms and Grassman graded algebra), and the basic functional spaces and properties for the current discussion of PDEs. Then we show a useful identity used in the paper, and end our discussion with recalling basic facts about the Fourier transform of graded forms.

A1. ***Tools of multivariable calculus.*** For $x \in \mathbb{R}^n$ and $n \in \mathbb{N}\backslash\{0\}$, let $T_x\mathbb{R}^n$ denote the complex vector space of distributions X of order one in $\mathbb{R}^n$ satisfying $\operatorname{supp} X = \{x\}$ and $\langle X \mid c\rangle = 0$ for any constant function c (See Theorem 2.3.4 in [Hörmander 1983] for the justification of this definition). Such X can be uniquely extended to a linear form on $C^1(\mathbb{R}^n)$, the space of continuously differentiable functions in $\mathbb{R}^n$. Let $\partial_{x^j}|_x$ denote the distribution given by

$$\langle \partial_{x^j}|_x \mid \phi\rangle = \partial_{x^j}\phi(x)$$

for any $\phi \in C^1(\mathbb{R}^n)$. The set $\{\partial_{x^1}|_x, \dots, \partial_{x^n}|_x\}$ is a base of $T_x\mathbb{R}^n$. Let $T_x^*\mathbb{R}^n$ denote the dual vector space of $T_x\mathbb{R}^n$ with $\{dx^1|_x, \dots, dx^n|_x\}$ being the dual base. We define on $T_x^*\mathbb{R}^n$ the inner product $\langle\cdot,\cdot\rangle$ given by the bilinear extension of $\langle dx^j|_x, dx^k|_x\rangle = \delta_{jk}$ (Kronecker delta). Note that it is not a Hermitian product.

A1.1. *Differential forms.* Let $\Lambda^l\mathbb{R}^n$ with $l \in \{0, 1, \dots, n\}$ and $n \geq 2$ denote the smooth complex vector bundle over $\mathbb{R}^n$ whose fiber at $x \in \mathbb{R}^n$ consists of $\Lambda^l T_x^*\mathbb{R}^n$, the l-fold exterior product of $T_x^*\mathbb{R}^n$. By convention, a 0-fold is just a complex number and a 1-fold is an element of $T_x^*\mathbb{R}^n$. Let E be a nonempty subset of $\mathbb{R}^n$; an l-form on E is a section u of $\Lambda^l\mathbb{R}^n$ over E, so $u(x) = u|_x \in \Lambda^l T_x^*\mathbb{R}^n$ for any $x \in E$. Any

l-form on E with $l \in \{1, \dots, n\}$ can be written as

$$u = \sum_{\alpha \in S^l} u_\alpha \, dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_l}$$

with $S^l = \{(\alpha_1, \dots, \alpha_l) \in \{1, \dots, n\}^l : \alpha_1 < \cdots < \alpha_l\}$ and $u_\alpha : E \longrightarrow \mathbb{C}$. It is convenient to call u_α with $\alpha \in S^l$ the component functions of u.

The exterior product of an l-form u and an m-form v, both on E, is denoted by $(u \wedge v)(x) = u|_x \wedge v|_x$ for any $x \in E$. Recall that the exterior product is bilinear, associative and anticommutative:

$$u \wedge v = (-1)^{lm} v \wedge u. \tag{A-1}$$

Since a 0-form v on E is nothing but a map from E to $\mathbb{C}$, it holds that $u \wedge v = v \wedge u = vu$ for any l-form u on E.

The inner product of two l-forms on E with $l \in \{2, \dots, n\}$ can be defined at each point $x \in E$ as the bilinear extension of

$$\langle (dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_l})|_x, (dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_l})|_x \rangle = \det \langle dx^{\alpha_j}|_x, dx^{\beta_k}|_x \rangle,$$

where the right-hand side stands for the determinant of the matrix

$$(\langle dx^{\alpha_j}|_x, dx^{\beta_k}|_x \rangle)_{jk}.$$

The inner product of two 0-forms is just the usual product of functions. The inner product on l-forms can be immediately extended to graded forms $u(x) = \sum_0^n u^l(x)$ and $v(x) = \sum_0^n v^l(x)$ on E, with u^l and v^l l-forms on E, as follows:

$$\langle u, v \rangle(x) = \sum_{l=0}^n \langle u^l|_x, v^l|_x \rangle.$$

Associated to this inner product, we consider the norm satisfying $|u|^2 = \langle u, \bar{u} \rangle$.

Now let $T_x^* \mathbb{R}^n$ be endowed with an orientation. The Hodge star operator of an l-form on E with $l \in \{1, \dots, n-1\}$ is defined at each point $x \in E$ as the linear extension of

$$*(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_l})|_x = (dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_{n-l}})|_x,$$

where $(\beta_1, \dots, \beta_{n-l}) \in \{1, \dots, n\}^{n-l}$ is chosen such that

$$\{dx^{\alpha_1}, \dots dx^{\alpha_l}, dx^{\beta_1}, \dots, dx^{\beta_{n-l}}\}$$

is a positive base of $T_x^* \mathbb{R}^n$. The case of 0-forms and n-forms follows from

$$*1|_x = (dx^1 \wedge \cdots \wedge dx^n)|_x, \quad *(dx^1 \wedge \cdots \wedge dx^n)|_x = 1|_x,$$

where 1 denotes the constant function taking the value 1 at any point. Now, if u and v are l-forms on E,

$$**u(x) = (-1)^{l(n-l)}u(x), \tag{A-2}$$

$$\langle u, v\rangle(x) = *(u|_x \wedge *v|_x) = *(v|_x \wedge *u|_x), \tag{A-3}$$

$$\langle u, v\rangle = \langle *u, *v\rangle. \tag{A-4}$$

Let u be an l-form on E and let v be an m-form on E. The vee product of v and u at each point $x \in E$ is defined as

$$(v \vee u)(x) = (-1)^{(n+m-l)(l-m)} *(v|_x \wedge *u|_x). \tag{A-5}$$

Note that whenever $m > l$, $(v \vee u)(x) = 0$ for all $x \in E$. The vee product is bilinear, but it is neither associative nor commutative. The product satisfies

$$\langle w \wedge v, u\rangle = \langle w, v \vee u\rangle \tag{A-6}$$

for any k-form w on E.

Proposition A.1. *If u and v are 1-forms and w is an l-form with $l \in \{0, \dots, n\}$, then*

$$u \vee (v \wedge w) - v \wedge (u \vee w) = (-1)^l \langle u, v\rangle w. \tag{A-7}$$

Corollary A.2. *If u^1 and v^1 are 1-forms and u^l and v^l are l-forms with $l \in \{0, \dots, n\}$, then*

$$\langle u^1 \vee u^l, v^1 \vee v^l\rangle + \langle v^1 \wedge u^l, u^1 \wedge v^l\rangle = \langle u^1, v^1\rangle\langle u^l, v^l\rangle.$$

Proof. Since

$$\langle u^1 \vee u^l, v^1 \vee v^l\rangle + \langle v^1 \wedge u^l, u^1 \wedge v^l\rangle = (-1)^l \langle u^1 \vee (v^1 \wedge u^l) - v^1 \wedge (u^1 \vee u^l), v^l\rangle,$$

the identity follows from (A-7). □

Let G be a nonempty open subset of $\mathbb{R}^n$ and k a positive integer. An l-form u on G with $l \in \{1, \dots, n\}$ is said to be k-times continuously differentiable if its component functions are k-times continuously differentiable in G. We write $u \in C^k(G; \Lambda^l\mathbb{R}^n)$. If $u \in C^k(G; \Lambda^l\mathbb{R}^n)$ for any positive integer k, we say that u is smooth and we write $u \in C^\infty(G; \Lambda^l\mathbb{R}^n)$. Furthermore, $u \in C^k(G; \Lambda^l\mathbb{R}^n)$ (respectively $u \in C^\infty(G; \Lambda^l\mathbb{R}^n)$) is said to be compactly supported if its component functions are compactly supported in G, in which case we write $u \in C_0^\infty(G; \Lambda^l\mathbb{R}^n)$ (respectively $u \in C_0^\infty(G; \Lambda^l\mathbb{R}^n)$). These definitions are naturally generalized to 0-forms, where the conventional function space notations are also used.

The exterior derivative of $u \in C^1(G; \Lambda^0\mathbb{R}^n)$ is a 1-form defined by

$$du|_x(X) = \langle X \mid \chi_x u\rangle$$

for each $x \in G$ and $X \in T_x\mathbb{R}^n$. Here $\chi_x \in C_0^\infty(G)$ with $\chi_x(x) = 1$ on G, and $\chi_x u$ is understood as the extension of u by zero outside G. The exterior derivative of $u \in C^1(G; \Lambda^l\mathbb{R}^n)$ with $l \in \{1, \dots, n\}$ is defined by

$$du = \sum_{\alpha \in S^l} du_\alpha \wedge dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_l}.$$

Recall that $d(du) = 0$ for any $u \in C^2\big(G;\, \Lambda^l\mathbb{R}^3\big)$ and

$$d(u \wedge v) = du \wedge v + (-1)^l u \wedge dv, \tag{A-8}$$

for any $u \in C^1\big(G;\, \Lambda^l\mathbb{R}^3\big)$ and $v \in C^1\big(G;\, \Lambda^m\mathbb{R}^3\big)$.

A1.2. *Symmetric tensors.* Let $\Sigma^l\mathbb{R}^n$ with $l \in \mathbb{N}$ and $n \geq 2$ denote the smooth complex vector bundle over $\mathbb{R}^n$ whose fiber at $x \in \mathbb{R}^n$ consists of $\Sigma^l T_x^*\mathbb{R}^n$, the l-fold symmetric tensor product of $T_x^*\mathbb{R}^n$. By convention, a 0-fold is just a complex number and a 1-fold is an element of $T_x^*\mathbb{R}^n$. Let E be a nonempty subset of $\mathbb{R}^n$; an l-symmetric tensor on E is a section u of $\Sigma^l\mathbb{R}^n$ over E, so $u(x) = u|_x \in \Sigma^l T_x^*\mathbb{R}^n$ for any $x \in E$. Any l-symmetric tensor on E with $l \in \{1, \dots, n\}$ can be written as

$$u = \sum_{\alpha \in T^l} u_\alpha\, dx^{\alpha_1} \odot \cdots \odot dx^{\alpha_l}$$

with $T^l = \{(\alpha_1, \dots, \alpha_l) \in \{1, \dots, n\}^l : \alpha_1 \leq \cdots \leq \alpha_l\}$ and $u_\alpha : E \longrightarrow \mathbb{C}$. It is convenient to call u_α with $\alpha \in T^l$ the component functions of u and to point out that $\Sigma^l T_x^*\mathbb{R}^n = \Lambda^l T_x^*\mathbb{R}^n$ for $l \in \{0, 1\}$, which in turn implies $\Sigma^l\mathbb{R}^n = \Lambda^l\mathbb{R}^n$ for $l \in \{0, 1\}$.

The symmetric tensor product of an l-symmetric tensor u and an m-symmetric tensor v, both on E, is denoted by $(u \odot v)(x) = u|_x \odot v|_x$ for any $x \in E$. Recall that the symmetric tensor product is bilinear, associative, and commutative. Moreover, if u and v are 1-symmetric tensors,

$$u \odot v = \tfrac{1}{2}(u \otimes v + v \otimes u).$$

The inner product of two l-symmetric tensors on E with $l \in \mathbb{N} \setminus \{0, 1\}$ can be defined at each point $x \in E$ as the bilinear extension of

$$\langle (dx^{\alpha_1} \odot \cdots \odot dx^{\alpha_l})|_x, (dx^{\beta_1} \odot \cdots \odot dx^{\beta_l})|_x \rangle = |\det\langle dx^{\alpha_j}|_x, dx^{\beta_k}|_x\rangle|.$$

Let G be a nonempty open subset of $\mathbb{R}^n$. An l-symmetric tensor u on G with $l \in \mathbb{N}$ is said to be k-times continuously differentiable if its component functions are k-times continuously differentiable in G, and we write $u \in C^k(G;\, \Sigma^l\mathbb{R}^n)$. Furthermore, $u \in C^k(G;\, \Sigma^l\mathbb{R}^n)$ with $l \in \mathbb{N}$ is said to be compactly supported if its component functions are compactly supported in G, and we write $u \in C_0^k(G;\, \Sigma^l\mathbb{R}^n)$. These definitions extend naturally to 0-symmetric tensors on G.

The symmetric derivative of a smooth l-symmetric tensor u on G with $l \in \mathbb{N} \setminus \{0\}$ is defined by

$$i\,Du = \sum_{\alpha \in T^l} du_\alpha \odot dx^{\alpha_1} \odot \cdots \odot dx^{\alpha_l}.$$

A2. ***Functional spaces.*** Let $L^1_{\mathrm{loc}}\big(E;\, \Lambda^l\mathbb{R}^n\big)$ denote the space of locally integrable l-forms (whose component functions are in $L^1_{\mathrm{loc}}(E)$) modulo those which vanish almost everywhere (a.e.) in E. The space $L^p\big(E;\, \Lambda^l\mathbb{R}^n\big)$, with $p \in [1, +\infty)$, consists of all $u \in L^1_{\mathrm{loc}}\big(E;\, \Lambda^l\mathbb{R}^n\big)$ such that

$$\int_E \langle u, \bar{u}\rangle^{p/2}\, dx < +\infty.$$

Endowed with the norm

$$\|u\|_{L^p(E;\Lambda^l\mathbb{R}^n)} = \left(\int_E \langle u, \bar{u}\rangle^{p/2}\,dx\right)^{1/p},$$

$L^p\big(E;\Lambda^l\mathbb{R}^n\big)$ is a Banach space. Moreover, $L^2\big(E;\Lambda^l\mathbb{R}^n\big)$ is a Hilbert space.

Let $u \in L^1_{\mathrm{loc}}\big(G;\Lambda^l\mathbb{R}^n\big)$ with $l \in \{1,\dots,n\}$. We say that $v \in L^1_{\mathrm{loc}}\big(G;\Lambda^{l-1}\mathbb{R}^n\big)$ is the formal adjoint derivative of u, denoted by $v = \delta u$, if

$$\int_G \langle v, w\rangle\,dx = \int_G \langle u, dw\rangle\,dx$$

for any $w \in C^1_0\big(G;\Lambda^{l-1}\mathbb{R}^n\big)$. If $u \in L^1_{\mathrm{loc}}\big(G;\Lambda^0\mathbb{R}^n\big)$, we define $\delta u = 0$. For all $u \in L^1_{\mathrm{loc}}\big(G;\Lambda^l\mathbb{R}^n\big)$ with $l \in \{0,\dots,n\}$ such that $\delta u \in L^1_{\mathrm{loc}}\big(G;\Lambda^{l-1}\mathbb{R}^n\big)$, one has $\delta(\delta u) = 0$. Moreover, if $u \in C^1\big(G;\Lambda^l\mathbb{R}^n\big)$, then

$$\delta u = (-1)^{n(l+1)+1} {*d*u}. \tag{A-9}$$

Proposition A.3. *Consider $u \in L^1_{\mathrm{loc}}\big(G;\Lambda^l\mathbb{R}^n\big)$ and $v \in C^1\big(G;\Lambda^m\mathbb{R}^n\big)$. If $\delta u \in L^1_{\mathrm{loc}}\big(G;\Lambda^{l-1}\mathbb{R}^n\big)$, then $\delta(v \vee u) \in L^1_{\mathrm{loc}}\big(G;\Lambda^{l-m-1}\mathbb{R}^n\big)$ and*

$$\delta(v \vee u) = (-1)^{l-m} dv \vee u + v \vee \delta u. \tag{A-10}$$

Let $u \in L^1_{\mathrm{loc}}\big(G;\Lambda^l\mathbb{R}^n\big)$ with $l \in \{0,\dots,(n-1)\}$. We say that $v \in L^1_{\mathrm{loc}}\big(G;\Lambda^{l+1}\mathbb{R}^n\big)$ is the (weak) exterior derivative of u, denoted by $v = du$, if

$$\int_G \langle v, w\rangle\,dx = \int_G \langle u, \delta w\rangle\,dx$$

for any $w \in C^1_0\big(G;\Lambda^{l+1}\mathbb{R}^n\big)$. If $u \in L^1_{\mathrm{loc}}\big(G;\Lambda^n\mathbb{R}^n\big)$, we define $du = 0$. For all $u \in L^1_{\mathrm{loc}}\big(G;\Lambda^l\mathbb{R}^n\big)$ with $l \in \{0,\dots,n\}$ such that $du \in L^1_{\mathrm{loc}}\big(G;\Lambda^{l+1}\mathbb{R}^n\big)$, one has $d(du) = 0$.

Proposition A.4. *Let $u \in L^1_{\mathrm{loc}}\big(G;\Lambda^l\mathbb{R}^n\big)$ such that $\delta u \in L^1_{\mathrm{loc}}\big(G;\Lambda^{l-1}\mathbb{R}^n\big)$. Then ${*d*u} \in L^1_{\mathrm{loc}}\big(G;\Lambda^{l-1}\mathbb{R}^n\big)$ and*

$$\delta u = (-1)^{n(l+1)+1} {*d*u}. \tag{A-11}$$

We now present certain Sobolev spaces of forms, in which our PDEs are discussed. Let $H^d\big(G;\Lambda^l\mathbb{R}^n\big)$ (respectively $H^\delta\big(G;\Lambda^l\mathbb{R}^n\big)$) denote the space of $u \in L^2\big(G;\Lambda^l\mathbb{R}^n\big)$ such that $du \in L^2\big(G;\Lambda^{l+1}\mathbb{R}^n\big)$ (respectively $\delta u \in L^2\big(G;\Lambda^{l-1}\mathbb{R}^n\big)$), endowed with the norm

$$\|u\|_{H^d(G;\Lambda^l\mathbb{R}^n)} = (\|u\|^2_{L^2(G;\Lambda^l\mathbb{R}^n)} + \|du\|^2_{L^2(G;\Lambda^{l+1}\mathbb{R}^n)})^{1/2}$$
$$\big(\text{respectively } \|u\|_{H^\delta(G;\Lambda^l\mathbb{R}^n)} = (\|u\|^2_{L^2(G;\Lambda^l\mathbb{R}^n)} + \|\delta u\|^2_{L^2(G;\Lambda^{l-1}\mathbb{R}^n)})^{1/2}\big).$$

It is observed that $H^d\big(G;\Lambda^l\mathbb{R}^n\big)$ and $H^\delta\big(G;\Lambda^l\mathbb{R}^n\big)$ are Hilbert spaces and $C^1_0\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big)$ is dense in them. Let $H^d_{\mathrm{loc}}\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big)$ and $H^\delta_{\mathrm{loc}}\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big)$ denote the spaces of $u \in L^1_{\mathrm{loc}}\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big)$ such that $u|_U \in H^d\big(U;\Lambda^l\mathbb{R}^n\big)$ and $u|_U \in H^\delta\big(U;\Lambda^l\mathbb{R}^n\big)$, respectively, for any bounded nonempty open subset U in $\mathbb{R}^n$.

Finally, by a density argument, we have

$$\int_{\mathbb{R}^n} \langle du, v\rangle \, dx = \int_{\mathbb{R}^n} \langle u, \delta v\rangle \, dx \tag{A-12}$$

for all $u \in H^d\big(\mathbb{R}^n; \Lambda^{l-1}\mathbb{R}^n\big)$ and $v \in H^\delta\big(\mathbb{R}^n; \Lambda^l\mathbb{R}^n\big)$ with $l \in \{1, \dots, n\}$.

A3. ***Traces.***[4]

Let U be a nonempty bounded open subset of $\mathbb{R}^n$, and let $H^1\big(U; \Lambda^l\mathbb{R}^n\big)$ denote the space of all $u \in L^2\big(U; \Lambda^l\mathbb{R}^n\big)$ whose component functions u_α satisfy $du_\alpha \in L^2\big(U; \Lambda^1\mathbb{R}^n\big)$ for all $\alpha \in S^l$, endowed with the norm

$$\|u\|_{H^1(U;\Lambda^l\mathbb{R}^n)} = \bigg(\|u\|^2_{L^2(U;\Lambda^l\mathbb{R}^n)} + \sum_{\alpha\in S^l} \|du_\alpha\|^2_{L^2(U;\Lambda^1\mathbb{R}^n)}\bigg)^{1/2}. \tag{A-13}$$

Given G, a nonempty open subset of $\mathbb{R}^n$, by $H^1_{\text{loc}}\big(G; \Lambda^l\mathbb{R}^n\big)$ we denote the space of $u \in L^1_{\text{loc}}\big(G; \Lambda^l\mathbb{R}^n\big)$ such that $u|_U \in H^1\big(U; \Lambda^l\mathbb{R}^n\big)$ for any bounded nonempty open subset U of G.

It is a consequence of (A-11) that, for any $u \in H^1\big(U; \Lambda^l\mathbb{R}^n\big)$, one has

$$\|u\|_{H^\delta(U;\Lambda^l\mathbb{R}^n)} \le \|u\|_{H^1(U;\Lambda^l\mathbb{R}^n)}. \tag{A-14}$$

Let $H^1_0\big(U; \Lambda^l\mathbb{R}^n\big)$ denote the closure in $H^1\big(U; \Lambda^l\mathbb{R}^n\big)$ of $C^\infty_0\big(U; \Lambda^l\mathbb{R}^n\big)$ modulo those vanishing a.e. in U. We then define the space

$$TH^1\big(\partial U; \Lambda^l\mathbb{R}^n\big) = H^1\big(U; \Lambda^l\mathbb{R}^n\big)/H^1_0\big(U; \Lambda^l\mathbb{R}^n\big).$$

If $f \in TH^1\big(\partial U; \Lambda^l\mathbb{R}^n\big)$, let $u_f \in H^1\big(U; \Lambda^l\mathbb{R}^n\big)$ denote a representative of f. This space can be endowed with the norm

$$\|f\|_{TH^1(\partial U;\Lambda^l\mathbb{R}^n)} = \inf\{\|u\|_{H^1(U;\Lambda^l\mathbb{R}^n)} : u - u_f \in H^1_0\big(U; \Lambda^l\mathbb{R}^n\big)\}.$$

Let $TH^1\big(\partial U; \Lambda^l\mathbb{R}^n\big)^*$ denote the dual space of $TH^1\big(\partial U; \Lambda^l\mathbb{R}^n\big)$ with the functional $\|\cdot\|_{TH^1(\partial U;\Lambda^l\mathbb{R}^n)^*}$ standing for the dual norm.

The latter spaces will be used as auxiliary spaces to define certain traces on $H^d\big(U; \Lambda^l\mathbb{R}^n\big)$ and $H^\delta\big(U; \Lambda^l\mathbb{R}^n\big)$. Firstly, define the d-trace of $v \in H^d\big(U; \Lambda^l\mathbb{R}^n\big)$ with $l \in \{0, \dots, n-1\}$ as

$$\langle d \operatorname{tr} v \mid f\rangle = \int_U \langle dv, u\rangle \, dx - \int_U \langle v, \delta u\rangle \, dx$$

for any $f \in TH^1\big(\partial U; \Lambda^{l+1}\mathbb{R}^n\big)$ where $u \in H^1\big(U; \Lambda^{l+1}\mathbb{R}^n\big)$ such that $u - u_f \in H^1_0\big(U; \Lambda^{l+1}\mathbb{R}^n\big)$. Since (A-14) holds, we have

$$\langle d \operatorname{tr} v \mid f\rangle \le \|v\|_{H^d(U;\Lambda^l\mathbb{R}^n)} \|u\|_{H^1(U;\Lambda^{l+1}\mathbb{R}^n)}$$

for all $u \in H^1\big(U; \Lambda^{l+1}\mathbb{R}^n\big)$ such that $u - u_f \in H^1_0\big(U; \Lambda^{l+1}\mathbb{R}^n\big)$. Hence $d \operatorname{tr} v \in TH^1\big(\partial U; \Lambda^{l+1}\mathbb{R}^n\big)^*$ and

$$\|d \operatorname{tr} v\|_{TH^1(\partial U;\Lambda^{l+1}\mathbb{R}^n)^*} \le \|v\|_{H^d(U;\Lambda^l\mathbb{R}^n)}.$$

[4]For more details on traces see [Mitrea 2004; Schwarz 1995].

This motivates the definition of $TH^d(\partial U;\, \Lambda^{l+1}\mathbb{R}^n)$ to be the space of all $g \in TH^1(\partial U;\, \Lambda^{l+1}\mathbb{R}^n)^*$ such that $d\,\mathrm{tr}\, v = g$ for some $v \in H^d(U;\, \Lambda^l\mathbb{R}^n)$. The endowed norm is then given by

$$\|g\|_{TH^d(\partial U;\Lambda^{l+1}\mathbb{R}^n)} = \inf\{\|v\|_{H^d(U;\Lambda^l\mathbb{R}^n)} : d\,\mathrm{tr}\, v = g\}.$$

Finally, we define the δ-trace of $v \in H^\delta(U;\, \Lambda^l\mathbb{R}^n)$ with $l \in \{1, \dots, n\}$ as

$$\langle \delta\,\mathrm{tr}\, v \mid f\rangle = (-1)^l \int_U \langle \delta v, u\rangle\, dx - (-1)^l \int_U \langle v, du\rangle\, dx$$

for any $f \in TH^1(\partial U;\, \Lambda^{l-1}\mathbb{R}^n)$ where $u \in H^1(U;\, \Lambda^{l-1}\mathbb{R}^n)$ such that $u - u_f \in H^1_0(U;\, \Lambda^{l-1}\mathbb{R}^n)$. Similarly we would have $\delta\,\mathrm{tr}\, v \in TH^1(\partial U;\, \Lambda^{l-1}\mathbb{R}^n)^*$ and

$$\|\delta\,\mathrm{tr}\, v\|_{TH^1(\partial U;\Lambda^{l-1}\mathbb{R}^n)^*} \le \|v\|_{H^\delta(U;\Lambda^l\mathbb{R}^n)}.$$

Moreover, we define $TH^\delta(\partial U;\, \Lambda^{l-1}\mathbb{R}^n)$, the space consisting of all g belonging to $TH^1(\partial U;\, \Lambda^{l-1}\mathbb{R}^n)^*$, such that there exists $v \in H^\delta(U;\, \Lambda^l\mathbb{R}^n)$ with $\delta\,\mathrm{tr}\, v = g$ with norm

$$\|g\|_{TH^\delta(\partial U;\Lambda^{l-1}\mathbb{R}^n)} = \inf\{\|v\|_{H^\delta(U;\Lambda^l\mathbb{R}^n)} : \delta\,\mathrm{tr}\, v = g\}.$$

Then we will need the following lemma about these spaces.

Lemma A.5. *Given the definitions above,*

(a) *if $u \in H^d(U;\, \Lambda^l\mathbb{R}^n)$ with $l \in \{0, \dots, n-2\}$ and $d\,\mathrm{tr}\, u = 0$, then $d\,\mathrm{tr}(du) = 0$;*

(b) *if $u \in H^\delta(U;\, \Lambda^l\mathbb{R}^n)$ with $l \in \{2, \dots, n\}$ and $\delta\,\mathrm{tr}\, u = 0$, then $\delta\,\mathrm{tr}(\delta u) = 0$.*

Proof. In order to prove (a), let us consider the bounded linear operator

$$(\nu \vee \cdot\,) : TH^d(\partial U;\, \Lambda^l\mathbb{R}^n) \longrightarrow TH^\delta(\partial U;\, \Lambda^{l-1}\mathbb{R}^n)^*$$

given by

$$\langle \nu \vee f \mid g\rangle = \int_U \langle du, v\rangle\, dx - \int_U \langle u, \delta v\rangle\, dx,$$

where $u \in H^d(U;\, \Lambda^{l-1}\mathbb{R}^n)$, $v \in H^\delta(U;\, \Lambda^l\mathbb{R}^n)$, $d\,\mathrm{tr}\, u = f$, and $\delta\,\mathrm{tr}\, v = g$. Here $TH^\delta(\partial U;\, \Lambda^{l-1}\mathbb{R}^n)^*$ denotes the dual of $TH^\delta(\partial U;\, \Lambda^{l-1}\mathbb{R}^n)$. Let u be as in (a) and $g \in TH^1(U;\, \Lambda^{l+2}\mathbb{R}^n)$. Then

$$\langle d\,\mathrm{tr}(du) \mid g\rangle = -\langle \nu \vee d\,\mathrm{tr}\, u \mid \delta\,\mathrm{tr}(\delta v_g)\rangle,$$

where $v_g \in H^1(U;\, \Lambda^{l+2}\mathbb{R}^n)$ denotes a representative of g. Therefore (a) holds.

A similar proof applies to (b) by considering the operator

$$(\nu \vee \cdot\,) : TH^\delta(\partial U;\, \Lambda^l\mathbb{R}^n) \longrightarrow TH^d(\partial U;\, \Lambda^{l+1}\mathbb{R}^n)^*$$

defined by

$$\langle \nu \wedge f \mid g\rangle = (-1)^{l+1} \int_U \langle \delta u, v\rangle\, dx - (-1)^{l+1} \int_U \langle u, dv\rangle\, dx,$$

where $u \in H^\delta(U;\, \Lambda^{l+1}\mathbb{R}^n)$, $v \in H^d(U;\, \Lambda^l\mathbb{R}^n)$, $\delta\,\mathrm{tr}\, u = f$, and $d\,\mathrm{tr}\, v = g$. We leave the proof to the readers. $\square$

A4. ***A useful identity.*** Given G, a nonempty open subset of $\mathbb{R}^n$, let $L^1_{\mathrm{loc}}(G;\Sigma^l\mathbb{R}^n)$ denote the space of locally integrable l-symmetric tensors (whose component functions are in $L^1_{\mathrm{loc}}(E)$) modulo those which vanish a.e. in E.

For $u\in L^1_{\mathrm{loc}}(G;\Sigma^l\mathbb{R}^n)$ with $l\in\mathbb{N}\setminus\{1,2\}$, we say that $v\in L^1_{\mathrm{loc}}(G;\Sigma^{l-1}\mathbb{R}^n)$ is the formal adjoint (symmetric) derivative of u, denoted by $v=D^*u$, if

$$\int_G\langle v,w\rangle\,dx=\int_G\langle u,Dw\rangle\,dx$$

for any $w\in C^1_0(G;\Sigma^{l-1}\mathbb{R}^n)$.

Note that if

$$u=\sum_{j=1}^n u_j\,dx^j\quad\text{and}\quad v=\sum_{j=1}^n v_j\,dx^j$$

are such that $u\odot v\in L^1_{\mathrm{loc}}(G;\Sigma^2\mathbb{R}^n)$ and $D^*(u\odot v)\in L^1_{\mathrm{loc}}(G;\Sigma^1\mathbb{R}^n)$, then

$$D^*(u\odot v)=-\sum_{k=1}^n\biggl(\sum_{j=1}^n\partial_{x^j}(u_jv_k+u_kv_j)\biggr)dx^k. \tag{A-15}$$

Proposition A.6. *Given u and v in $H^1_{\mathrm{loc}}(G;\Lambda^1\mathbb{R}^n)$, we have that $d\langle u,v\rangle$ and $D^*(u\odot v)$ belong to $L^1_{\mathrm{loc}}(G;\Lambda^1\mathbb{R}^n)$ and the following identity holds*:

$$u\vee dv+v\vee du+\delta u\vee v+\delta v\vee u=d\langle u,v\rangle+D^*(u\odot v).$$

A5. ***Local regularity.*** Here we prove a local regularity lemma for the operator $(d+\delta)\sum_0^n(-1)^l$.

Lemma A.7. *Let $v=\sum_0^n v^l$ be such that $v^l\in L^2_{\mathrm{loc}}(\mathbb{R}^n;\Lambda^l\mathbb{R}^n)$ and*

$$(d+\delta)\sum_{l=0}^n(-1)^lv^l\in\bigoplus_{l=0}^n L^2_{\mathrm{loc}}(\mathbb{R}^n;\Lambda^l\mathbb{R}^n).$$

Then $v^l\in H^1_{\mathrm{loc}}(\mathbb{R}^n;\Lambda^l\mathbb{R}^n)$ for $l\in\{0,\dots,n\}$.

Proof. By using Corollary A.2 and the identity

$$\langle\xi\wedge\widehat{\phi^{l-1}}(\xi),\overline{\xi\vee\widehat{\phi^{l+1}}(\xi)}\rangle=0,$$

we can check that

$$\|\phi\|^2_{L^2}=\|\phi\|^2_{H^{-1}}+\biggl\|(d+\delta)\sum_{l=0}^n(-1)^l\phi^l\biggr\|^2_{H^{-1}} \tag{A-16}$$

for all $\phi=\sum_0^n\phi^l$ such that $\phi^l\in L^2(\mathbb{R}^n;\Lambda^l\mathbb{R}^n)$. Here we are using the notation $\|\varphi\|^2_Y=\sum_0^n\|\varphi^l\|^2_{Y(\mathbb{R}^n;\Lambda^l\mathbb{R}^n)}$ for $\varphi=\sum_0^n\varphi^l$ with $\varphi^l\in Y(\mathbb{R}^n;\Lambda^l\mathbb{R}^n)$, where Y denotes either L^2 or H^{-1}. Recall that $\|\varphi^l\|^2_{H^{-1}(\mathbb{R}^n;\Lambda^l\mathbb{R}^n)}=\int_{\mathbb{R}^n}(1+|\xi|^2)^{-1}|\widehat{\varphi^l}(\xi)|^2\,d\xi$.

Let ψ be a compactly supported smooth function in $\mathbb{R}^n$ and let $\Delta_h^j\phi$ be defined as

$$\Delta_h^j\phi(x) = \frac{1}{h}(\phi(x+he_j) - \phi(x))$$

with ϕ as in (A-16), h a positive parameter, and e_j the j-th element of the orthonormal basis of $\mathbb{R}^n$. By (A-16) and the commutativity between Δ_h^j and $(d+\delta)\sum_0^n(-1)^l$, we have

$$\|\Delta_h^j(\psi v)\|_{L^2}^2 = \|\Delta_h^j(\psi v)\|_{H^{-1}}^2 + \|\Delta_h^j(d+\delta)\sum_{l=0}^n(-1)^l(\psi v^l)\|_{H^{-1}}^2. \tag{A-17}$$

Since

$$(d+\delta)\sum_{l=0}^n(-1)^l(\psi v^l) = \psi(d+\delta)\sum_{l=0}^n(-1)^l v^l + \sum_{l=0}^n(-1)^l d\psi \wedge v^l + d\psi \vee v^l$$

and v and $(d+\delta)\sum_0^n(-1)^l v^l$ belong to $\bigoplus_0^n L^2_{\mathrm{loc}}\big(\mathbb{R}^n; \Lambda^l\mathbb{R}^n\big)$, the statement of the result follows by making the parameter h go to zero in the identity (A-17)[5]. □

A6. *Fourier transform of forms and operator* $\boldsymbol{\Delta_\zeta}$. An l-form u with $l \in \{0, \dots, n\}$ is said to belong to the Schwartz space $\mathcal{S}\big(\mathbb{R}^n; \Lambda^l\mathbb{R}^n\big)$ if its component functions u_α ($\alpha \in S^l$) are in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. We can define the space $\mathcal{S}'\big(\mathbb{R}^n; \Lambda^l\mathbb{R}^n\big)$ of l-form-valued tempered distributions similarly. The Fourier Transform of $u \in \mathcal{S}\big(\mathbb{R}^n; \Lambda^l\mathbb{R}^n\big)$ is then defined by

$$\hat{u} = \sum_{\alpha\in S^l} \widehat{u_\alpha}\, d\xi^{\alpha_1} \wedge \cdots \wedge d\xi^{\alpha_l} \in \mathcal{S}\big(\mathbb{R}^n; \Lambda^l\mathbb{R}^n\big).$$

The Fourier Transform $\hat{u}$ for $u \in \mathcal{S}'\big(\mathbb{R}^n; \Lambda^l\mathbb{R}^n\big)$ can be defined by duality. One can easily verify the following identities for $u \in \mathcal{S}\big(\mathbb{R}^n; \Lambda^l\mathbb{R}^n\big)$;

$$\widehat{du}(\xi) = i\xi \wedge \hat{u}(\xi), \quad \widehat{\delta u}(\xi) = i(-1)^l \xi \vee \hat{u}(\xi), \tag{A-18}$$

where $\xi \in \mathbb{R}^3\backslash\{0\}$ can be viewed as a 1-form. For $u, v \in L^2\big(\mathbb{R}^n; \Lambda^l\mathbb{R}^n\big)$, we have

$$\int_{\mathbb{R}^n} \langle u, \bar{v}\rangle\, dx = \int_{\mathbb{R}^n} \langle \hat{u}, \bar{\hat{v}}\rangle\, dx, \tag{A-19}$$

making the Fourier transform a unitary map on $L^2\big(\mathbb{R}^n; \Lambda^l\mathbb{R}^n\big)$.

Given $\zeta = \sum_1^n \zeta_j\, dx^j$, a constant 1-differential form in $\mathbb{R}^n$, consider the conjugated Hodge–Laplacian operator $\Delta_\zeta = e_{-\zeta}(d\delta + \delta d) \circ e_\zeta$, where $e_\zeta(x) = e^{\zeta\cdot x}$ and $\zeta \cdot x = \sum_1^n \zeta_j x^j$. When acting on an l-form $u \in H^d\big(\mathbb{R}^n; \Lambda^l\mathbb{R}^n\big) \cap H^\delta\big(\mathbb{R}^n; \Lambda^l\mathbb{R}^n\big)$, it reads

$$\Delta_\zeta u = (d\delta + \delta d)u + (-1)^l d(\zeta \vee u) + \zeta \wedge \delta u + \delta(\zeta \wedge u) + (-1)^{l+1}\zeta \vee du - \langle\zeta, \zeta\rangle u, \tag{A-20}$$

(understood in the weak sense). Moreover, it is easy to verify that the symbol of Δ_ζ is $|\xi|^2 - 2i\langle\zeta, \xi\rangle - \langle\zeta, \zeta\rangle$ by (A-18).

[5] See Theorem (6.19) of [Folland 1995] for more details.

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CONVEXITY ESTIMATES FOR HYPERSURFACES MOVING BY CONVEX CURVATURE FUNCTIONS

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We consider the evolution of compact hypersurfaces by fully nonlinear, parabolic curvature flows for which the normal speed is given by a smooth, convex, degree-one homogeneous function of the principal curvatures. We prove that solution hypersurfaces on which the speed is initially positive become weakly convex at a singularity of the flow. The result extends the convexity estimate of Huisken and Sinestrari [*Acta Math.* **183**:1 (1999), 45–70] for the mean curvature flow to a large class of speeds, and leads to an analogous description of "type-II" singularities. We remark that many of the speeds considered are positive on larger cones than the positive mean half-space, so that the result in those cases also applies to non-mean-convex initial data.

1. Introduction

Given a smooth, compact immersion $X_0 : M^n \to \mathbb{R}^{n+1}$, $n > 1$, we consider smooth families $X : M \times [0, T) \to \mathbb{R}^{n+1}$ of smooth immersions $X(\cdot, t)$ solving the curvature flow

$$\frac{\partial X}{\partial t}(x, t) = -s(x, t)\nu(x, t), \quad X(\cdot, 0) = X_0, \tag{1-1}$$

where ν is the outer unit normal field of the solution, and the speed s is determined by a function of the principal curvatures κ_i (with respect to ν). That is,

$$s(x, t) = f(\kappa_1(x, t), \dots, \kappa_n(x, t)). \tag{1-2}$$

We require that the speed function f satisfies the following conditions:

Conditions.

(i) *$f \in C^\infty(\Gamma)$ for some connected, open, symmetric cone $\Gamma \subset \mathbb{R}^n$.*

(ii) *f is monotone increasing in each argument.*

(iii) *f is homogeneous of degree one.*

(iv) $f > 0$.

(v) *Γ is preserved by the flow* (1-1).

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Condition (v) is intended as follows: Let X be a solution of (1-1)–(1-2) such that the initial hypersurface satisfies $(\kappa_1(x,0),\dots,\kappa_n(x,0)) \in \Gamma$ for all $x \in M$. Then there is a connected, open, symmetric subcone Γ_0 of Γ satisfying $\overline{\Gamma}_0 \setminus \{0\} \subset \Gamma$ such that the principal curvatures of the solution satisfy $(\kappa_1(x,t),\dots,\kappa_n(x,t)) \in \Gamma_0$ for all $(x,t) \in M \times [0,T)$. We refer to Γ_0 as a *preserved cone* of the flow. This is discussed further below.

Observe that, since the normal points outwards and f is homogeneous, we lose no generality in assuming further that $(1,\dots,1) \in \Gamma$, and that f is normalised such that $f(1,\dots,1) = 1$. Furthermore, since f is symmetric, we may at each point reorder the principal curvatures such that $\kappa_n \geq \cdots \geq \kappa_1$.

For most of the paper, we will also require that f satisfies the following two conditions, which are somewhat distinct from conditions (i)–(v):

Conditions.

(vi) *f is locally convex.*

(vii) $(\partial f/\partial z_p - \partial f/\partial z_q)\big|_z \geq 0$ *whenever $z \in \Gamma$ is such that $z_p \geq z_q$.*

We will say that s is an *admissible speed* for the flow (1-1) if s is given by (1-2) such that f satisfies conditions (i)–(vii).

Some discussion of conditions (i)–(vii) is in order: The symmetry of f is a geometric condition — it allows us to write s as a smooth function of the Weingarten map of the solution, which ensures geometric invariance of the flow. The monotonicity of f then ensures that the flow is parabolic, which guarantees short time existence of a solution if the principal curvatures of the initial immersion lie in Γ. Condition (v) is then a requirement that the principal curvatures do not "move out of" Γ during the flow. In general, some such condition is necessary (see [Andrews et al. 2013b, Theorem 3]), although, in particular, it automatically holds in each of the following situations (Lemma 2.4):

Ancillary conditions.

(viii) *Conditions* (i)–(iv) *and* (vi) *hold, and Γ is convex.*

(ix) *Conditions* (i)–(iv) *and* (vi) *hold, and* $f\big|_{\partial\Gamma} = 0$.

(x) *Conditions* (i)–(iv) *hold, and* $n = 2$.

For the purposes of Theorem 1.1, however, we need only assume that the weaker condition (v) holds. We remark that ancillary condition (ix) makes sense because any function satisfying conditions (i)–(iv) has a continuous extension to $\partial\Gamma$. This is proved for $\Gamma = \Gamma_+$ in [Andrews et al. 2013b], but the proof is easily modified for the present situation.

In the presence of condition (i), conditions (vi)–(vii) are equivalent to requiring that the speed is a smooth, convex function of the Weingarten map (Lemma 2.1). We note that condition (vii) is automatically true in each of the following situations:

Ancillary conditions.

(xi) *Conditions* (i)–(iii) *and* (vi) *hold, and Γ is convex.*

(xii) *Conditions* (i)–(iii) *and* (vi) *hold, and f extends as a convex function to $\mathbb{R}^n$* (*for example, if* $f\big|_{\partial\Gamma} = 0$).

(xiii) *Conditions* (i)–(iv) *and* (vi) *hold, and* $n = 2$.

The above assertions are discussed in greater detail in Section 2.

We now list some examples of admissible speeds.

Examples 1.1. The following functions define admissible speeds for the flow (1-1):

(1) *The arithmetic mean*: $f(z_1, \dots, z_n) = z_1 + \dots + z_n$ on the half-space $\Gamma = \{z \in \mathbb{R}^n : z_1 + \dots + z_n > 0\}$. The corresponding flow is the (mean convex) mean curvature flow.

(2) *The power means*: $f_p(z_1, \dots, z_n) = (z_1^p + \dots + z_n^p)^{1/p}$, for $p \geq 1$, on the positive cone $\Gamma_+^n = \{z \in \mathbb{R}^n : z_i > 0 \text{ for all } i\}$. The case $p = 2$ corresponds to the flow by the norm of the Weingarten map.

(3) *Positive linear combinations*: If $f_1, \dots, f_k$ are admissible on Γ, then, for all $(s_1, \dots, s_k) \in \Gamma_+^k$, the function $f = s_1 f_1 + \dots + s_k f_k$ is admissible on Γ. For example, the function

$$f(z_1, \dots, z_n) = z_1 + \dots + z_n + \sqrt{z_1^2 + \dots + z_n^2}$$

on the cone Γ_+ defines an admissible speed. In fact, the functions

$$f_\alpha(z_1, \dots, z_n) = z_1 + \dots + z_n + \alpha\sqrt{z_1^2 + \dots + z_n^2}$$

for $\alpha \in [0, 1]$ on the larger cones $\Gamma_\alpha = \{z \in \mathbb{R}^n : z_1 + \dots + z_n + \alpha\sqrt{z_1^2 + \dots + z_n^2} > 0\}$ define admissible speeds. We remark that the cones Γ_α contain the half-space $\{z \in \mathbb{R}^n : z_1 + \dots + z_n > 0\}$ when $\alpha > 0$.

(4) *Concave functions*: If $g \in C^\infty(\Gamma)$ is symmetric, homogeneous degree one and concave, then an admissible speed is defined by the function $f = H - \varepsilon g$ on the subcone of Γ for which $H > \varepsilon g$ and $\dot{g}^i < 1/\varepsilon$ for all i. The class of concave functions discussed in [Andrews 2007] then provide an interesting class of admissible speeds.

(5) *Convex homogeneous combinations*: Let ϕ satisfy conditions (i)–(iv) and (vi)–(vii) on a cone $\widetilde{\Gamma} \subset \mathbb{R}^k$, and suppose that the functions $f_1, \dots, f_k$ define admissible speeds on a cone $\Gamma_k \subset \mathbb{R}^n$. Then the function

$$f(z_1, \dots, z_n) := \phi\big(f_1(z_1, \dots, z_n), \dots, f_k(z_1, \dots, z_n)\big)$$

on the cone $\{z \in \Gamma : (f_1(z), \dots, f_k(z)) \in \widetilde{\Gamma}\}$ defines an admissible speed. For example, the function $f_\varepsilon(z_1, \dots, z_n) = H_p(z_1 + \varepsilon H, \dots, z_n + \varepsilon H)$ on the cone $\Gamma_\varepsilon := \{z \in \mathbb{R}^n : z_i + \varepsilon H > 0 \text{ for all } i\}$ defines an admissible speed.

Curvature problems of the form (1-1)–(1-2) have been studied extensively, although mostly under the assumption that the initial hypersurface is locally convex, that is, having Weingarten map everywhere positive definite. The most well-known result in this case is Huisken's theorem [1984], which states that, when the speed is given by the mean curvature, uniformly locally convex initial hypersurfaces remain uniformly locally convex and shrink to round points, "round" meaning that the solution approaches total umbilicity at the final point. Chow showed that this behaviour is true also for the flows by the n-th root of the Gauss curvature [1985], and, if an initial curvature pinching condition is assumed, the square root of the scalar curvature [1987]. Each of these flows satisfy conditions (i)–(iv) on the positive cone $\Gamma = \Gamma_+ := \{x \in \mathbb{R}^n : x_i > 0,\ i = 1, \dots, n\}$. More general degree-one homogeneous speeds were treated in [Andrews 1994a; 2007; 2010], where it was shown that uniformly convex hypersurfaces will contract to round points under the flow (1-1)–(1-2), so long as the speed satisfies conditions (i)–(iv) and, in addition, either

(1) $n = 2$, or

(2) f is convex, or

(3) f is concave, and *inverse concave*, that is, the function

$$f_*(z_1, \dots, z_n) = f(z_1^{-1}, \dots, z_n^{-1})^{-1}$$

is concave.

These conditions were weakened in [Andrews et al. 2013b], and their necessity demonstrated by the construction, in dimensions $n > 2$, of concave speed functions satisfying conditions (i)–(iv) for which convex initial hypersurfaces do not remain convex under the corresponding flow [ibid., Theorem 3].

In the case of nonconvex initial hypersurfaces, much less is known about the behaviour of solutions of (1-1), although in many cases the analogy with the mean curvature flow continues. For example, a simple calculation shows that spheres shrink to points in finite time under flows (1-1)–(1-2) satisfying conditions (i)–(iv). The avoidance principle (see[1] [Andrews et al. 2013a, Theorem 5]) then implies that any compact solution of (1-1) must become singular in finite time. If, in addition, the flow admits second derivative Hölder estimates (for example, if the speed function is a concave or convex function of the principal curvatures [Evans 1982; Krylov 1982], or if $n = 2$ [Andrews 2004]), one can deduce, by standard methods, that a singularity is characterised by a curvature blow-up [Andrews et al. 2012].

For the mean curvature flow, a crucial part of the current understanding of singularities is the asymptotic convexity estimate of Huisken and Sinestrari [1999a], which states that any mean convex initial hypersurface flowing by mean curvature becomes weakly convex at a singularity. This, together with the monotonicity formula of Huisken [1990] and the Harnack inequality of Hamilton [1995b] allows a rather complete description of singularities in the positive mean curvature case. We note that asymptotic convexity is necessary for the application of the Harnack inequality to deduce that "fast-forming" or "type-II" singularities are asymptotic to convex translation solutions of the flow.

For other flows, the understanding of singularities is far less developed, except in some specific settings such as axial symmetry (see [McCoy et al. 2014], for example). There are several reasons for this: First, there is no analogue available for the monotonicity formula, which is used to show that "slowly forming" or "type-I" singularities of the mean curvature flow are asymptotically self-similar. Second, there is in general no Harnack inequality available sufficient to classify type-II singularities, although the latter is known for quite a wide subclass of flows [Andrews 1994b]. And finally, there is so far no analogue of the Huisken–Sinestrari asymptotic convexity estimate for most other flows, with the notable exception of the recent result of Alessandroni and Sinestrari, which applies to a class of flows by functions of the mean

[1] We remark that the avoidance principle proved in [Andrews et al. 2013a, Theorem 5] is not in general true when the cone of definition of the speed is nonconvex. However, a slight modification reveals that it is still possible to compare compact solutions with spheres.

curvature having a certain asymptotic behaviour [Alessandroni and Sinestrari 2010]. In a companion paper [Andrews et al. 2012], we were able to exploit the simplified structure of the evolution equation for the second fundamental form in two dimensions (see also [Schulze 2006; Andrews 2007; McCoy 2011]) to prove that an asymptotic convexity estimate holds in surprising generality for flows of surfaces, namely for any surface flow (1-1)–(1-2) satisfying conditions (i)–(iv). On the other hand, one would expect this result should fail in higher dimensions in such generality, due to the aforementioned examples of "nice" speeds which fail to preserve local convexity of initial data. In this paper, we show that an asymptotic convexity estimate is possible in higher dimensions in the presence of the additional convexity conditions (vi)-(vii).

Theorem 1.1. *Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be a solution of* (1-1) *with s an admissible speed. Then for all $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that*

$$-\kappa_1(x, t) \leq \varepsilon s(x, t) + C_\varepsilon$$

for all $(x, t) \in M \times [0, T)$.

The proof of Theorem 1.1 utilises a Stampacchia–De Giorgi iteration procedure analogous to those of [Huisken 1984; Huisken and Sinestrari 1999b; 1999a; Chow 1985; 1987] (see also [Andrews et al. 2012]), in contrast to the result of [Alessandroni and Sinestrari 2010] (see also [Schulze 2006]), which is proved using the maximum principle. We remark that, by carefully constructing our curvature pinching function, we are able to avoid the rather technical induction on the elementary symmetric functions of curvature that is necessary in [Huisken and Sinestrari 1999a].

Combining Theorem 1.1 with the Harnack estimate of [Andrews 1994b] (see also [Hamilton 1995b]) as in [Huisken and Sinestrari 1999b; 1999a], we are led to the following classification of type-II blow-up limits about type-II singularities.

Corollary 1.2. *If s is an admissible speed, then any type-II blow-up limit of a solution of the corresponding flow* (1-1) *about a type-II singularity decomposes as a product $X : \Sigma^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n+1}$, such that $X\big|_{\Sigma^k} : \Sigma^k \times \mathbb{R} \to \mathbb{R}^{k+1} \subset \mathbb{R}^{n+1}$ is a strictly convex (k-dimensional) translation solution of the flow* (1-1).

Corollary 1.2 is proved in Section 6.

2. Notation and preliminary results

We now describe some important background results necessary for the subsequent sections. We begin with flow-independent results to do with symmetric functions, and prove, in Lemma 2.2, that each of the ancillary conditions (xi)–(xiii) implies condition (vii). We then discuss flow-dependent results, and prove, in Lemma 2.4, that each of the ancillary conditions (viii)–(x) implies condition (v). We follow the conventions of [Andrews et al. 2013b; Andrews 2007; 2010; McCoy 2005], where proofs or references for much of this section may be found. Many of the results can also be found in the book [Gerhardt 2006].

The curvature function f is a smooth, symmetric function defined on an open, convex, symmetric cone Γ. Denote by $\mathcal{S}_\Gamma$ the cone of symmetric $n \times n$ matrices with n-tuple of eigenvalues, $\lambda := (\lambda_1, \dots, \lambda_n)$, lying in Γ. A result of Glaeser [1963] implies that there is a smooth, $\mathrm{GL}(n)$-invariant function $F : \mathcal{S}_\Gamma \to \mathbb{R}$

such that $f(\lambda(A)) = F(A)$. The invariance of F under similarity transformations implies that the speed $s(x,t) = f(\kappa_1(x,t), \dots, \kappa_n(x,t))$ is a well-defined, smooth function of the Weingarten map $\mathcal{W}$, that is, $s(x,t) = F(\mathcal{W}(x,t)) := F(W(x,t))$, where $W(x,t)$ is the component matrix of $\mathcal{W}(x,t)$ with respect to some basis for $T_x^*M \otimes T_xM$. If we restrict attention to orthonormal bases, then $W_i{}^j = h_{ij}$, where the h_{ij} are the components of the second fundamental form.

We shall use dots to indicate derivatives of f and F as follows:

$$\begin{aligned} \dot f^i(\lambda)v_i &:= \frac{d}{ds}\bigg|_{s=0} f(\lambda+sv), & \ddot f^{ij}(\lambda)v_iv_j &:= \frac{d^2}{ds^2}\bigg|_{s=0} f(\lambda+sv),\\ \dot F^{ij}(A)B_{ij} &:= \frac{d}{ds}\bigg|_{s=0} F(A+sB), & \ddot F^{pq,rs}(A)B_{pq}B_{rs} &:= \frac{d^2}{ds^2}\bigg|_{s=0} F(A+sB). \end{aligned} \tag{2-1}$$

The derivatives of f and F are related in the following way:

Lemma 2.1 [Gerhardt 1990; Andrews 1994a; 2007]. *Suppose that the function f satisfies condition* (i). *Define the function $F : \mathcal{S}_\Gamma :\to \mathbb{R}$ by $F(A) := f(\lambda(A))$ as above. Then for any diagonal $A \in \mathcal{S}_\Gamma$ we have*

$$\dot F^{kl}(A) = \dot f^k(\lambda(A))\delta^{kl}, \tag{2-2}$$

and for any diagonal $A \in \mathcal{S}_\Gamma$ and symmetric $B \in \mathrm{GL}(n)$, we have

$$\ddot F^{pq,rs}(A)B_{pq}B_{rs} = \ddot f^{pq}(\lambda(A))B_{pp}B_{qq} + 2\sum_{p>q} \frac{\dot f^p(\lambda(A)) - \dot f^q(\lambda(A))}{\lambda_p(A)-\lambda_q(A)}(B_{pq})^2. \tag{2-3}$$

Note that (2-3) *holds* (*as a limit*) *even if A has eigenvalues of multiplicity greater than one.*

In particular, in an orthonormal frame of eigenvectors of $\mathcal{W}$, we have

$$\begin{aligned} \dot F^{kl}(\mathcal{W}) &= \dot f^k(\kappa)\delta^{kl}\\ \ddot F^{pq,rs}(\mathcal{W})B_{pq}B_{rs} &= \ddot f^{pq}(\kappa)B_{pp}B_{qq} + 2\sum_{p>q}\frac{\dot f^p(\kappa)-\dot f^q(\kappa)}{\kappa_p-\kappa_q}(B_{pq})^2. \end{aligned}$$

Observe that, by (2-2), conditions (i)–(ii) imply that (1-1)–(1-2) is parabolic. The methods of [Gerhardt 2006, Section 2.5] (see also [Giga and Goto 1992] and [Baker 2010]) then imply short time existence of solutions, so long as the principal curvatures of the initial immersion lie in Γ.

It follows from (2-3) that the function F is convex if and only if the function f is convex and satisfies $(\dot f^p - \dot f^q)(z_p - z_q) \ge 0$. We now show that in most cases of interest the second condition is automatic.

Lemma 2.2. *Suppose that f satisfies one of the ancillary conditions* (xi), (xii) *or* (xiii). *Then f satisfies condition* (vii).

Proof. Suppose first that condition (xi) is satisfied, so that Γ is convex. If $\Gamma = \Gamma_+$ then the claim is proved in [Andrews 1994a, Lemma 2.2] (see also [Ecker and Huisken 1989]). However, the proof applies to any convex cone: Consider an arbitrary point $z \in \Gamma$. Since f is smooth and convex, for any $v \in \mathbb{R}^n$ and any

$s \in \mathbb{R}$ such that $z + sv \in \Gamma$ we have

$$0 \leq \frac{d^2}{ds^2} f(z+sv) = \frac{d}{ds} \dot{f}^i(z+sv)v_i.$$

Therefore, if $s > 0$,

$$\dot{f}^i(z+sv)v_i \geq \dot{f}^i(z)v_i.$$

Setting $v = -(e_p - e_q)$, where e_i is the basis vector in the direction of z_i, we obtain

$$(\dot{f}^p - \dot{f}^q)\big|_z \geq (\dot{f}^p - \dot{f}^q)\big|_{z-s(e_p-e_q)}.$$

If $z_p \geq z_q$ then there is some $s_0 > 0$ such that $(z - s_0(e_p - e_q))_p = (z - s_0(e_p - e_q))_q$. By the symmetry and convexity of Γ, this point is in Γ. Since f is symmetric, $\dot{f}^p = \dot{f}^q$ at this point and the claim follows.

Now suppose that (xii) is satisfied, so that f extends to a convex, symmetric function on $\mathbb{R}^n$. If the extension is smooth, then the claim follows as above. If not, then we need to be more careful; we make use of the fact that the difference quotient $\big(f(\gamma(s)) - f(\gamma(t))\big)/(s-t)$ is nondecreasing in both s and t along all lines $\gamma(s) = z + sv$.

Consider a point $z \in \Gamma$ and a direction $v \in \mathbb{R}^n$. Then, for any $s \in \mathbb{R}$ and any $s_0 > 0$, we have

$$\frac{f(z+sv) - f(z+s_0v)}{s - s_0} \geq \frac{f(z+sv) - f(z)}{s} \geq \lim_{s\searrow 0} \frac{f(z+sv) - f(z)}{s} = \dot{f}^i\big|_z v_i.$$

Setting $v = -(e_p - e_q)$, it follows that

$$-(\dot{f}^p - \dot{f}^q)\big|_z = \dot{f}^i\big|_z v_i \leq \frac{f(z+sv) - f(z+s_0v)}{s - s_0} \leq \lim_{s\nearrow s_0} \frac{f(z+sv) - f(z+s_0v)}{s - s_0} = \psi'_-(0),$$

where we have defined $\psi(\sigma) := f(z + (\sigma + s_0)v)$. We note that the left derivative $\psi'_-(0)$ exists, and is no greater than the right derivative ψ'_+, by convexity of ψ. Supposing without loss of generality that $z_p \geq z_q$, we may choose s_0 such that $z_p - s_0 = z_q + s_0$. With this choice, it is easily checked that ψ is an even function. Since ψ is convex, we have

$$\begin{aligned}\psi'_-(0) \leq \psi'_+(0) &= \lim_{s\searrow 0} \frac{\psi(s) - \psi(0)}{s} \\ &= -\lim_{s\nearrow 0} \frac{\psi(-s) - \psi(0)}{s} = -\lim_{s\nearrow 0} \frac{\psi(s) - \psi(0)}{s} = -\psi'_-(0).\end{aligned}$$

It follows that $\psi'_-(0) \leq 0$ and we obtain $(\dot{f}^p - \dot{f}^q)\big|_z \geq 0$ as required.

Finally, suppose that (xiii) is satisfied, so that $\Gamma \subset \mathbb{R}^2$. Consider some point $z \in \Gamma$ and suppose $p \neq q$ are such that $z_p \geq z_q$. Since f is homogeneous of degree one, we have $f = \dot{f}^1 z_1 + \dot{f}^2 z_2$. Then, since f, $\dot{f}^1$ and $\dot{f}^2$ are positive on Γ, we must have $z_p > 0$. Now,

$$2f = 2(\dot{f}^p z_p + \dot{f}^q z_q) = (\dot{f}^p - \dot{f}^q)(z_p - z_q) + (\dot{f}^p + \dot{f}^q)(z_p + z_q),$$

so that

$$(\dot{f}^p - \dot{f}^q)(z_p - z_q) = 2f - (\dot{f}^p + \dot{f}^q)(z_p + z_q).$$

If $z_p + z_q \le 0$, then we are done (since f, $\dot f^1$ and $\dot f^2$ are positive). Otherwise, z lies in the open, symmetric, convex cone $\{z \in \mathbb{R}^2 : z_1 + z_2 > 0\}$. But we have just proved that the claim already holds in this case. This completes the proof. □

In the following, we are interested in the behaviour of solutions of the flow equation (1-1)–(1-2). We consider speeds $s = f(\kappa)$ such that f satisfies condition (i), and denote the corresponding function of $\mathcal{W}$ by F. We will use the following convention in order to simplify notation: If g satisfies condition (i), and $G(A) = g(\lambda(A))$ is the corresponding function on $\mathcal{S}_\Gamma$, then we write $g(x,t) \equiv g(\kappa(x,t))$ and $G(x,t) \equiv G(\mathcal{W}(x,t))$. Similarly, $\dot G(x,t) \equiv \dot G(\mathcal{W}(x,t))$ and $\ddot G(x,t) \equiv \ddot G(\mathcal{W}(x,t))$. This convention makes the notation s for the speed unnecessary, and from here on the speed will be denoted by F.

We recall the following evolution equations:

Lemma 2.3 [Andrews 1994a; 2007; Andrews et al. 2013b; Gerhardt 2006; McCoy 2005]. *Let*

$$X : M \times [0,T) \to \mathbb{R}^{n+1}$$

be a solution of the flow (1-1)–(1-2) *such that f satisfies conditions* (i)–(iii). *Then the following evolution equations hold along X*:

(1) $(\partial_t - \mathcal{L})h_i{}^j = (\nabla_i dF)^j + F h_i{}^k h_k{}^j = \ddot F^{pq,rs}\nabla_i h_{pq}\nabla^j h_{rs} + \dot F^{kl} h^2_{kl} h_i{}^j$.

(2) $(\partial_t - \mathcal{L})F = F \dot F^{kl} h^2_{kl}$.

(3) $\partial_t\, d\mu = -HF\, d\mu$.

(4) $(\partial_t - \mathcal{L})G = (\dot G^{kl}\ddot F^{pq,rs} - \dot F^{kl}\ddot G^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} + \dot G^{pq} h_{pq}\dot F^{kl} h^2_{kl}$.

Here $\mathcal{L}$ is the elliptic operator $\dot F^{ij}\nabla_i\nabla_j$, $h^2_{ij} = h_i{}^k h_{kj}$, $\mu(t)$ is the measure induced on M by the immersion $X(\,\cdot\,,t)$, and G is any function given by $G(x,t) := g(\kappa_1(x,t), \dots, \kappa_n(x,t))$ for some smooth, *symmetric $g : \Gamma \to \mathbb{R}$.*

Applying the maximum principle to Lemma 2.3(2), we see that F remains positive for all $t \in [0,T)$ whenever it is initially positive. It then follows from Euler's theorem and the monotonicity of f that the largest principal curvature also remains positive.

In the case that g is homogeneous of degree one, Euler's theorem simplifies Lemma 2.3(4) to

$$(\partial_t - \mathcal{L})G = (\dot G^{kl}\ddot F^{pq,rs} - \dot F^{kl}\ddot G^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} + \dot F^{kl} h^2_{kl} G. \tag{2-4}$$

It follows that

$$(\partial_t - \mathcal{L})\Big(\frac{G}{F}\Big) = \frac{1}{F}(\dot G^{kl}\ddot F^{pq,rs} - \dot F^{kl}\ddot G^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} - \frac{2}{F}\dot F^{kl}\nabla_k F\,\nabla_l\Big(\frac{G}{F}\Big). \tag{2-5}$$

Therefore $\max_{M\times\{t\}}(G/F)$ will be nonincreasing in t whenever G satisfies the condition

$$(\dot G^{kl}\ddot F^{pq,rs} - \dot F^{kl}\ddot G^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} \le 0. \tag{2-6}$$

These observations help us to find preserved cones for the flow: Suppose that f satisfies conditions (i)–(iii). If there is a smooth, nonnegative, symmetric, homogeneous degree-one function $g : \Gamma \to \mathbb{R}$ such that

$$(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})T_{kpq}T_{lrs} \le 0$$

for any totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$, where G is the corresponding function on $\mathcal{S}_\Gamma$, then any solution of the corresponding flow admits a preserved cone. Namely, the cone

$$\Gamma_0 := \left\{ z \in \mathbb{R}^n : g(z) < \max_{M\times\{0\}}\left(\frac{G}{F}\right) f(z) \right\}$$

is preserved.

In general, finding such a function g will be highly specific to the choice of flow speed f, however, in many cases we can be sure preserved cones exists:

Lemma 2.4. *Suppose f satisfies one of the ancillary conditions* (viii), (ix), *or* (x). *Then f satisfies condition* (v).

Proof. Suppose that condition (viii) holds, so that the cone Γ is convex. It follows from Lemma 2.2 that condition (vii) holds, so that $\ddot{F} \ge 0$ by Lemma 2.1. Let X be a solution of (1-1)–(1-2). Then the Weingarten map of X satisfies

$$(\partial_t - \mathcal{L})h_i{}^j \ge \dot{F}^{kl}h^2_{kl}h_i{}^j. \tag{2-7}$$

Let Γ_0 be the interior of the symmetrised convex conic hull in $\mathbb{R}^n$ of the principal curvatures of X_0. Then $\overline{\Gamma}_0 \setminus \{0\} \subset \Gamma$. The preservation of Γ_0 by the flow follows by applying a slight modification of Hamilton's tensor maximum principle [1986, Section 3] to (2-7) (for details, see [Andrews 2007, Theorem 3.2] and [Andrews and Hopper 2011, Chapter 6]).

Now suppose that (ix) is satisfied, so that f vanishes on $\partial\Gamma$. If $X : M \times [0, T) \to \mathbb{R}^{n+1}$ is a solution of the corresponding flow, then F is initially positive, and the maximum principle implies that it remains so. Then we may consider the function $G_1(x,t) := g_1(\kappa_1(x,t), \dots, \kappa_n(x,t))$, where g_1 is the function defined by (3-1) of the following section. Observe that f extends to a convex function on $\mathbb{R}^n$ by setting $f = 0$ outside Γ, so that, by Lemma 2.2, condition (vii) holds. Then we may proceed as in Lemma 3.2 to obtain

$$Z := (\dot{G}_1^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}_1^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} \le 0, \tag{2-8}$$

and it follows that $G_1/F \le c_0 := \max_{M\times\{0\}} G_1/F$. So consider $\Gamma_0 := \{z \in \mathbb{R}^n : g_1(z) < c_0 f(z)\}$. Since $g_1(z) = 0$ if and only if $z \in \overline{\Gamma}_+ \cap \Gamma$ and, by convexity of the extension of f, $\{z \in \mathbb{R}^n : z_1 + \cdots + z_n > 0\} \subset \Gamma$, we have $(\partial\Gamma \cap \partial\Gamma_0) \setminus \{0\} = \varnothing$. It follows that Γ_0 is a preserved cone.

Finally, consider the case that condition (x) holds, so that $\Gamma \subset \mathbb{R}^2$. Observe that, in this case, it is sufficient to obtain an estimate on the pinching ratio of the solution (which in this case follows from an estimate on G_1/F), since any open, connected, symmetric cone Γ in $\mathbb{R}^2$ that contains the positive ray is of the form $\{z \in \mathbb{R}^2 : z_{\min} > \varepsilon z_{\max}\}$. However, we can no longer use any convexity properties of f to control G_1/F, and the above proof that $Z \le 0$ no longer applies. On the other hand, by carefully analysing each of the terms in the expression for Z, it is possible to write the terms involving second derivatives of the speed as gradient terms, and the remaining terms turn out to be automatically favourable for obtaining the desired estimate on Z. We refer the reader to the papers [Andrews 2007; Andrews et al. 2012] for the proof of this assertion. □

The existence of a preserved cone ensures that the flow is uniformly parabolic:

Lemma 2.5. *Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be a solution of* (1-1), *with an admissible speed F. Then there is a constant $c_1 > 0$ such that for all $(x, t) \in M \times [0, T)$ it holds that*

$$\frac{1}{c_1}|v|^2 \leq \dot{F}^{kl}(x,t)v_k v_l \leq c_1|v|^2$$

for all $v \in T_x M$, where $|\cdot|$ is the norm induced on TM by the immersion $X(\cdot, t)$.

Proof. In an orthonormal frame of eigenvectors of the Weingarten map, we have, by (2-2), that $\dot{F}^{kl} = \dot{f}^k \delta^{kl}$. Let Γ_0 be a preserved cone for the flow. Since $\overline{\Gamma}_0 \setminus \{0\} \subset \Gamma$, and $\dot{f}^k > 0$ on Γ for all k, we see that the derivatives $\dot{f}^k$ are bounded by positive constants on the compact set $K := \{z \in \overline{\Gamma}_{c_0} : |z| = 1\}$. Since the derivatives $\dot{f}^k$ are homogeneous of degree zero, these bounds extend to the cone $\overline{\Gamma}_{c_0} \setminus \{0\}$, which completes the proof. □

The following long time existence result then follows using standard methods.

Proposition 2.6 [Andrews et al. 2012]. *Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be a maximally extended solution of* (1-1), *with an admissible speed. Then $T < \infty$, and $\max_{M \times \{t\}} |\mathcal{W}| \to \infty$ as $t \to T$.*

We now focus on the proof of Theorem 1.1 and Corollary 1.2, so for the rest of the paper we will assume that f defines an admissible speed, and $X : M \times [0, T) \to \mathbb{R}^{n+1}$ is a maximally extended solution of the corresponding flow (1-1).

3. The pinching function

In this section, we carefully construct an appropriate curvature pinching function to be used in the proof of Theorem 1.1. That is, we construct a smooth, symmetric, homogeneous (degree-one, say) function $G(x, t) = g(\kappa_1(x, t), \ldots, \kappa_n(x, t))$ of the principal curvatures that vanishes only if the hypersurface is weakly convex. Our goal is to show that the ratio G/F vanishes asymptotically along the flow. In particular, this ratio should be nonincreasing. In view of (2-5) we would therefore like G to satisfy

$$(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq} \nabla_l h_{rs} \leq 0.$$

In fact, as we shall see, the following two estimates will be essential.

Properties. (1) *For all $\varepsilon > 0$, there is a constant $c_\varepsilon > 0$ such that*

$$(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq} \nabla_l h_{rs} \leq -c_\varepsilon \frac{|\nabla \mathcal{W}|^2}{F}$$

whenever $G > \varepsilon F$.

(2) *For all $\varepsilon > 0$, there is a constant $\gamma_\varepsilon > 0$ such that*

$$(F\dot{G}^{kl} - G\dot{F}^{kl})h_{kl}^2 \leq -\gamma_\varepsilon F |\mathcal{W}|^2$$

whenever $G > \varepsilon F$.

These estimates are needed to show that the positive part of the function $G_{\varepsilon,\sigma} := (G/F - \varepsilon)F^{\sigma}$ is bounded in $L^p(M \times [0, T))$ for any $\varepsilon > 0$, so long as σ is sufficiently small. This is done in Section 4. The proof of Theorem 1.1 then follows from standard arguments, which we recall in Section 5. But first, we construct our pinching function. We first try a smoothed out version of the natural choice, $\max\{-\kappa_1, 0\}$. The function we obtain possesses the second of the above properties, but the first property only weakly (that is, with $c_\varepsilon = 0$). By making this function slightly more convex (namely, strictly convex in nonradial directions) we are able to obtain a function satisfying both estimates uniformly (without harming the other properties).

We begin with a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ which is strictly convex and positive, except on $\mathbb{R}_+$, where it vanishes identically. Such a function is easily constructed; for example, we could use

$$\phi(r) = \begin{cases} r^4 e^{-1/r^2} & \text{if } r < 0, \\ 0 & \text{if } r \geq 0. \end{cases}$$

Now consider the following function, defined on Γ:

$$g_1(z) := f(z) \sum_{i=1}^{n} \phi\left(\frac{z_i}{f(z)}\right). \tag{3-1}$$

Observe that g_1 is nonnegative and vanishes on (and only on) $\overline{\Gamma}_+ \cap \Gamma$. Furthermore, g_1 is clearly smooth, symmetric, and homogeneous of degree one. We now calculate

$$\begin{aligned} \dot{g}_1^k &= \dot{f}^k \sum_{i=1}^{n} \phi\left(\frac{z_i}{f}\right) + \sum_{i=1}^{n} \dot{\phi}\left(\frac{z_i}{f}\right)\left(\delta_i^k - \frac{z_i}{f}\dot{f}^k\right) \\ &= \dot{\phi}\left(\frac{z_k}{f}\right) + \dot{f}^k \sum_{i=1}^{n}\left[\phi\left(\frac{z_i}{f}\right) - \frac{z_i}{f}\dot{\phi}\left(\frac{z_i}{f}\right)\right]. \end{aligned}$$

It follows easily from the convexity of ϕ that $\phi(r) - r\dot{\phi}(r) \leq \phi(0) = 0$. Since ϕ is positive and $\dot{\phi}$ vanishes on $\mathbb{R}_+$, we must also have $\dot{\phi}(r) \leq 0$ for all $r \in \mathbb{R}$. Moreover, equality holds in the above inequalities only if $r \geq 0$. Therefore $\dot{g}_1^k(z) \leq 0$ for each k, with equality if and only if $z \in \overline{\Gamma}_+ \cap \Gamma$.

Now compute

$$\ddot{g}_1^{pq} = \ddot{f}^{pq} \sum_{i=1}^{n}\left[\phi\left(\frac{z_i}{f}\right) - \frac{z_i}{f}\dot{\phi}\left(\frac{z_i}{f}\right)\right] + \frac{1}{f}\sum_{i=1}^{n} \ddot{\phi}\left(\frac{z_i}{f}\right)\left(\delta_i{}^p - \frac{z_i}{f}\dot{f}^p\right)\left(\delta_i{}^q - \frac{z_i}{f}\dot{f}^q\right).$$

and

$$\dot{g}_1^k \ddot{f}^{pq} - \dot{f}^k \ddot{g}_1^{pq} = \dot{\phi}\left(\frac{z_k}{f}\right)\ddot{f}^{pq} - \frac{\dot{f}^k}{f}\sum_{i=1}^{n} \ddot{\phi}\left(\frac{z_i}{f}\right)\left(\delta_i{}^p - \frac{z_i}{f}\dot{f}^p\right)\left(\delta_i{}^q - \frac{z_i}{f}\dot{f}^q\right). \tag{3-2}$$

This forms a nonpositive definite matrix for each k. Finally, consider

$$\dot{g}_1^k \frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^k \frac{\dot{g}_1^p - \dot{g}_1^q}{z_p - z_q} = \dot{\phi}\left(\frac{z_k}{f}\right)\frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^k \frac{\dot{\phi}(z_p/f) - \dot{\phi}(z_q/f)}{z_p - z_q}. \tag{3-3}$$

This is also nonpositive for each k, since convexity of ϕ implies $\frac{\dot\phi(r)-\dot\phi(s)}{r-s}\geq 0$. Putting (3-2) and (3-3) together using Lemma 2.1, we see that

$$(\dot G_1^{kl}\ddot F^{pq,rs}-\dot F^{kl}\ddot G_1^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs}\leq 0.$$

To obtain the uniform estimate, we modify the function g_1 to introduce a slightly stronger convexity property. We use the good convexity properties of the Euclidean norm: Consider the function g defined by

$$g:=K(g_1,g_2):=\frac{g_1^2}{g_2}, \tag{3-4}$$

where g_2 is a positive, monotone, degree-one homogeneous function of the principal curvatures which is *strictly convex in nonradial directions*. The function defined by

$$g_2(z):=Rf(z)+\sum_{i=1}^n z_i-|z|$$

has the properties we require, so long as the constant $R>0$ may be chosen such that $g_2>0$ (at least along the flow). Let's first show that such a choice is possible.

Lemma 3.1. *There exists a constant $R>0$ such that*

$$RF(x,t)+H(x,t)-|\mathcal{W}(x,t)|>0$$

for all $(x,t)\in M\times[0,T)$.

Proof. Define $G_2(x,t):=g_2(\kappa_1(x,t),\dots,\kappa_n(x,t))$. Since $F(\,\cdot\,,0)>0$ and M is compact, we may choose $R>0$ such that $G_2(\,\cdot\,,0)>0$. By (2-4), it suffices to show that

$$(\dot G_2^{kl}\ddot F^{pq,rs}-\dot F^{kl}\ddot G_2^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs}\geq 0.$$

First calculate

$$\dot g_2^k=R\dot f^k+1-\frac{z_k}{|z|}$$

and

$$\ddot g_2^{pq}=R\ddot f^{pq}-\frac{1}{|z|^3}(|z|^2\delta_{pq}-z_pz_q).$$

It follows that

$$\dot g_2^k\ddot f^{pq}-\dot f^k\ddot g_2^{pq}=\left(1-\frac{z_k}{|z|}\right)\ddot f^{pq}+\frac{\dot f^k}{|z|^3}(|z|^2\delta_{pq}-z_pz_q), \tag{3-5}$$

which, by the Cauchy–Schwarz inequality, is nonnegative definite for each k.

Finally,

$$\dot g_2^k\frac{\dot f^p-\dot f^q}{z_p-z_q}-\dot f^k\frac{\dot g_2^p-\dot g_2^q}{z_p-z_q}=\left(1-\frac{z_k}{|z|}\right)\frac{\dot f^p-\dot f^q}{z_p-z_q}+\frac{1}{|z|}\dot f^k,$$

which is also nonnegative definite for each k. It now follows from (2-2) and (2-3) that

$$(\dot G_2^{kl}\ddot F^{pq,rs}-\dot F^{kl}\ddot G_2^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs}\geq 0$$

as required. □

So the function G is well defined. We show that it also satisfies property (1) (page 416) weakly:

Lemma 3.2. *There is a constant $c_0 > 0$ such that*

$$G(x,t) \le c_0 F(x,t)$$

for all $(x,t) \in M \times [0,T)$.

Proof. By a straightforward calculation, we find

$$(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs}) = \dot{K}^1(\dot{G}_1^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}_2^{pq,rs}) + \dot{K}^2(\dot{G}_2^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}_2^{pq,rs}) - \dot{F}^{kl}\ddot{K}^{\alpha\beta}\dot{g}_\alpha^p\dot{g}_\beta^q$$

at any diagonal matrix. Noting that $\dot{K}^1(x,y) > 0$, $\dot{K}^2(x,y) < 0$ and $\ddot{K}(x,y) \ge 0$ whenever x and y are positive, we see that

$$(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} \le 0. \tag{3-6}$$

In view of (2-5), the claim now follows from the maximum principle. □

We now show that G satisfies the required properties (1) and (2) (page 416) uniformly:

Lemma 3.3. *For all $\varepsilon > 0$ there exist constants $c_2 > 0$ and $\gamma > 0$ such that*

$$-c_2\frac{|\nabla\mathscr{W}|^2}{F} \le (\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} \le -\frac{1}{c_2}\frac{|\nabla\mathscr{W}|^2}{F} \tag{3-7}$$

and

$$(F\dot{G}^{kl} - G\dot{F}^{kl})h_{kl}^2 \le -\gamma F|\mathscr{W}|^2 \tag{3-8}$$

whenever $G > \varepsilon F$.

Proof. Let $A \in \mathrm{GL}(n)$ be a diagonal matrix and $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$ be a totally symmetric tensor. Define

$$Q(A,T) := -(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\big|_A T_{kpq}T_{lrs} \ge 0. \tag{3-9}$$

Recalling the application of the Cauchy–Schwarz inequality to (3-5) reveals that equality occurs in (3-9) only if T is radial, that is, if for each k we have $T_{kpq} = \mu_k A_{pq}$ for some constant μ_k.

Define the set $\Gamma_\varepsilon := \{x \in \Gamma : \varepsilon f(z) \le g(z) \le c_0 f(z)\}$. Then, to prove (3-7), we need to demonstrate uniform positive bounds for $FQ(A,T)$ whenever A has eigenvalues in Γ_ε and $|T| \ne 0$. Since Q is homogeneous of degree two with respect to T, we may assume without loss of generality that $|T| = 1$. Moreover, since Q is homogeneous of degree -1 with respect to A, it suffices to obtain the required bounds on the compact slice $K := \{A \in \mathscr{S}_\Gamma : \varepsilon F(A) \le G(A) \le c_0 F(A), |A| = 1\}$. The upper bound now follows immediately from the continuity of Q.

To prove the lower bound, it suffices to show that $Q(A,T) = 0$ for $A \in K$ only if $|T| = 0$. We have seen that $Q(A,T)$ can only vanish if T is radial. Then, since A is diagonal, it follows that T is also diagonal: $T_{klm} \ne 0$ only if $k = l = m$. Since $A \ne 0$, there is some p for which $\lambda_p(A) \ne 0$. But, since $T_{klm} = \mu_k A_{lm} = \mu_k \lambda_l(A)\delta_{lm}$, we have for any k

$$T_{kkk} = \frac{\lambda_k(A)}{\lambda_p(A)} T_{kpp}.$$

But T_{kpp} vanishes unless $k = p$. Thus T has at most one nonzero component: T_{ppp}. It follows that A has at most one nonzero eigenvalue: If instead we had $\lambda_q > 0$ for some $q \neq p$, then we could obtain the contradiction $T_{ppp} = (\lambda_p/\lambda_q)T_{qpp} = 0$. Since $A \in \mathcal{S}_{\Gamma_\varepsilon} \subset \mathcal{S}_\Gamma$, we must have $\lambda_p(A) > 0$. But this implies that $G(A) = 0$, so that $A \notin K$, a contradiction. Therefore Q can only vanish if T vanishes. This completes the proof of (3-7).

For the second estimate, we observe that, in an orthonormal basis of eigenvectors of $\mathcal{W}$,

$$(F\dot{G}^{kl} - G\dot{F}^{kl}) \leq F\dot{G}^{kl} = F\dot{g}^k\delta^{kl} \leq 2F\frac{g_1}{g_2}\dot{g}_1^{kl}\delta^{kl}.$$

Now g_1/g_2 is positive on Γ_ε and therefore has a strictly positive lower bound on the compact slice $\Gamma_\varepsilon \cap \{|z| = 1\}$. Similarly, $\dot{g}_1^k < 0$ on Γ_ε, and therefore has a strictly negative upper bound on $\Gamma_\varepsilon \cap \{|z| = 1\}$. Since both terms are homogeneous of degree zero, these bounds extend unharmed to Γ_ε, and the claim follows. □

Now consider, for some positive constants ε and σ, the function

$$G_{\varepsilon,\sigma} := \left(\frac{G}{F} - \varepsilon\right)F^\sigma.$$

Observe that the upper bound $G/F < c_0$ implies

$$G_{\varepsilon,\sigma} < c_0F^\sigma. \tag{3-10}$$

Lemma 3.4. *The function $G_{\varepsilon,\sigma}$ satisfies the evolution equation*

$$\begin{aligned}(\partial_t - \mathcal{L})G_{\varepsilon,\sigma} = F^{\sigma-1}&(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} \\ &+ \frac{2(1-\sigma)}{F}\langle\nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F - \frac{\sigma(1-\sigma)}{F^2}|\nabla F|_F^2 + \sigma G_{\varepsilon,\sigma}|\mathcal{W}|_F^2,\end{aligned} \tag{3-11}$$

where we have introduced the notation $\langle u, v\rangle_F := \dot{F}^{kl}u_k u_l$ and $|\mathcal{W}|_F^2 := \dot{F}^{kl}h_{kl}^2$.

Proof. We first compute

$$\nabla G_{\varepsilon,\sigma} = F^{\sigma-1}\left(\nabla G - \frac{G}{F}\nabla F\right) + \frac{\sigma}{F}G_{\varepsilon,\sigma}\nabla F.$$

It follows that

$$\mathcal{L}G_{\varepsilon,\sigma} = F^{\sigma-1}\left(\mathcal{L}G - \frac{G}{F}\mathcal{L}F\right) + \frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F - 2\frac{\sigma-1}{F}\langle\nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F - \frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2. \tag{3-12}$$

Therefore,

$$\begin{aligned}(\partial_t - \mathcal{L})G_{\varepsilon,\sigma} &= F^{\sigma-1}\left((\partial_t - \mathcal{L})G - \frac{G}{F}(\partial_t - \mathcal{L})F\right) + \frac{\sigma}{F}G_{\varepsilon,\sigma}(\partial_t - \mathcal{L})F \\ &\qquad + 2\,\frac{1-\sigma}{F}\langle\nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F - \frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2 \\ &= F^{\sigma-1}(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} + \sigma G_{\varepsilon,\sigma}|h|_F^2 \\ &\qquad + 2\,\frac{1-\sigma}{F}\langle\nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F + \frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2\end{aligned}$$

as required. □

Just as for the mean curvature flow, it is the final two terms of the evolution equation (3-11) that obstruct the application of the maximum principle. We will proceed by the Stampacchia–De Giorgi iteration method as applied in [Huisken 1984; Huisken and Sinestrari 1999b]. The first step is to show that the spatial L^p norms of the positive part, $(G_{\varepsilon,\sigma})_+ := \max\{G_{\varepsilon,\sigma}, 0\}$, of $G_{\varepsilon,\sigma}$ are nonincreasing in t, so long as σ is sufficiently small. As in [Huisken 1984; Huisken and Sinestrari 1999b; 1999a], this leads to a uniform upper bound on $G_{\varepsilon,\sigma}$ for small, nonzero σ.

4. The integral estimates

Proposition 4.1. *For all $\varepsilon > 0$ there exist constants $\ell, L > 0$ such that for all $p > L$ and $0 < \sigma < \ell p^{-\frac{1}{2}}$, the $L^p(M, \mu(t))$ norm of $(G_{\varepsilon,\sigma})_+$ is nonincreasing in t.*

To simplify notation somewhat, we fix $\varepsilon > 0$ and denote $E := (G_{\varepsilon,\sigma})_+$. Then E^p is C^1 in t for $p > 1$, with $\partial_t E^p = pE^{p-1}\partial_t G_{\varepsilon,\sigma}$. The evolution equation (3-11) for $G_{\varepsilon,\sigma}$ then implies

$$\begin{aligned}\frac{d}{dt}\int E^p\,d\mu = p\int E^{p-1}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu - p\int E^{p-1}F^{\sigma-1}Q\,d\mu \\ + 2(1-\sigma)p\int E^{p-1}\frac{\langle\nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F}{F}\,d\mu - \sigma(1-\sigma)p\int E^p\frac{|\nabla F|_F^2}{F^2}\,d\mu \\ + \sigma p\int E^p|\mathcal{W}|_F^2\,d\mu - \int E^p HF\,d\mu, \end{aligned} \tag{4-1}$$

where we have defined $Q = (\dot{F}^{kl}\ddot{G}^{pq,rs} - \dot{G}^{kl}\ddot{F}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs}$. It will be useful to estimate $|\nabla F|_F$ in terms of $|\nabla\mathcal{W}|$:

Lemma 4.2. *There is a constant $c_3 > 0$ for which $|\nabla F|_F^2 \le c_3|\nabla\mathcal{W}|^2$.*

Proof. Since $\nabla_k F = \dot{f}^p\nabla_k h_{pp}$ in an orthonormal basis of eigenvectors of $\mathcal{W}$, the claim follows from the uniform positive bounds on $\dot{f}^i$ along the flow. □

For $p > 2$, we can integrate the first term of (4-1) by parts:

$$\int E^{p-1}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu = -(p-1)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu - \int E^{p-1}\ddot{F}^{kl,rs}\nabla_k h_{rs}\nabla_l G_{\varepsilon,\sigma}\,d\mu.$$

The first term on the right will be useful. We estimate the second term (when $G_{\varepsilon,\sigma} > 0$) using Young's inequality as follows:

$$\begin{aligned}-\ddot{F}^{kl,rs}\nabla_k h_{rs}\nabla_l G_{\varepsilon,\sigma} &\le \frac{2c_4}{F}\sum_{k,l,r,s}|\nabla_k h_{rs}\nabla_l G_{\varepsilon,\sigma}| \\ &\le c_4 E\sum_{k,l,r,s}\left(\frac{(\nabla_k h_{rs})^2}{p^{\frac{1}{2}}F^2} + \frac{p^{\frac{1}{2}}(\nabla_l G_{\varepsilon,\sigma})^2}{E^2}\right) \\ &= c_4 E\left(p^{-\frac{1}{2}}\frac{|\nabla\mathcal{W}|^2}{F^2} + p^{\frac{1}{2}}\frac{|\nabla G_{\varepsilon,\sigma}|^2}{E^2}\right),\end{aligned} \tag{4-2}$$

where we have estimated each of the homogeneous terms $\ddot{F}^{kl,rs}$ above by $2c_4/F$.

A useful term is obtained from the second term of (4-1) using the first estimate of Lemma 3.3. We estimate the third term using Young's inequality as follows:

$$\int E^p \left\langle \frac{\nabla G_{\varepsilon,\sigma}}{E}, \frac{\nabla F}{F} \right\rangle_F d\mu \le \frac{p^{\frac{1}{2}}}{2} \int E^{p-2} |\nabla G_{\varepsilon,\sigma}|_F^2 \, d\mu + \frac{p^{-\frac{1}{2}}}{2} \int E^p \frac{|\nabla F|_F^2}{F} \, d\mu. \tag{4-3}$$

Putting this back together, we obtain:

Lemma 4.3. *For all* $\sigma \in (0, 1)$ *it holds that*

$$\frac{d}{dt} \int E^p \, d\mu \le \left((c_1 + c_4) p^{\frac{3}{2}} - c_1^{-1} p(p-1) \right) \int E^{p-2} |\nabla G_{\varepsilon,\sigma}|^2 \, d\mu \\ + \left((c_3 + c_4) p^{\frac{1}{2}} - \frac{1}{c_0 c_2} p \right) \int E^p \frac{|\nabla \mathcal{W}|^2}{F^2} \, d\mu + c_5 (\sigma p + 1) \int E^p |\mathcal{W}|^2 \, d\mu. \tag{4-4}$$

Proof. Since $-HF/|\mathcal{W}|^2$ is homogeneous of degree zero in the principal curvatures, it may be estimated above by some constant, which allows us to estimate the final term in (4-1). Now apply the estimates of Lemmata 2.5, 4.2 and 3.3, and the inequalities (3-10), (4-2) and (4-3) to the remaining terms. □

Notice that, for any fixed large p, the first two terms of (4-4) become nonpositive for sufficiently small σ (of order $p^{-\frac{1}{2}}$). We now estimate the final term in a similar fashion.

Proposition 4.4. *There are positive constants* A_1, A_2, A_3, B_1, B_2, *independent of p and* σ, *such that*

$$\int E^p |\mathcal{W}|^2 \le (A_1 p^{\frac{3}{2}} + A_2 p^{\frac{1}{2}} + A_3) \int E^{p-2} |\nabla G_{\varepsilon,\sigma}|^2 \, d\mu + (B_1 p^{\frac{1}{2}} + B_2) \int E^p \frac{|\nabla \mathcal{W}|^2}{F^2} \, d\mu. \tag{4-5}$$

Proof. We begin with the commutation formula (see [Andrews and Baker 2010, Proposition 5])

$$\nabla_k \nabla_l h_{pq} = \nabla_p \nabla_q h_{kl} + h_{kl} h_{pq}^2 - h_{pq} h_{kl}^2 + h_{kq} h_{pl}^2 - h_{pl} h_{kq}^2,$$

which holds on a general hypersurface of $\mathbb{R}^{n+1}$. This contracts to the Simons-type identity

$$\mathcal{L} h_{pq} = \dot{F}^{kl} \nabla_p \nabla_q h_{kl} + F h_{pq}^2 - \dot{F}^{kl} h_{pq} h_{kl}^2 + \dot{F}^{kl} h_{kq} h_{pl}^2 - \dot{F}^{kl} h_{pl} h_{kq}^2.$$

Contracting further with $\dot{G}$ yields

$$\dot{G}^{pq} \mathcal{L} h_{pq} = \dot{G}^{pq} \dot{F}^{kl} \nabla_p \nabla_q h_{kl} + (F \dot{G}^{kl} - G \dot{F}^{kl}) h_{kl}^2.$$

On the other hand, we have

$$\dot{F}^{kl} \nabla_p \nabla_q h_{kl} = \nabla_p \nabla_q F - \ddot{F}^{kl,rs} \nabla_p h_{rs} \nabla_q h_{kl},$$

so that

$$\dot{G}^{pq} \mathcal{L} h_{pq} = \dot{G}^{pq} \nabla_p \nabla_q F - \dot{G}^{pq} \ddot{F}^{kl,rs} \nabla_p h_{rs} \nabla_q h_{kl} + (F \dot{G}^{kl} - G \dot{F}^{kl}) h_{kl}^2. \tag{4-6}$$

We now recall (3-12):

$$\begin{aligned}\mathcal{L}G_{\varepsilon,\sigma} &= F^{\sigma-1}\Big(\mathcal{L}G-\frac{G}{F}\mathcal{L}F\Big)+\frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F-2\,\frac{1-\sigma}{F}\langle\nabla G_{\varepsilon,\sigma},\nabla F\rangle_F+\frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2\\ &= F^{\sigma-1}\Big(\dot F^{kl}\dot G^{pq}\nabla_k\nabla_l h_{pq}+\dot F^{kl}\ddot G^{pq,rs}\nabla_k h_{pq}\nabla_l h_{rs}-\frac{G}{F}\mathcal{L}F\Big)\\ &\qquad +\frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F-2\,\frac{1-\sigma}{F}\langle\nabla G_{\varepsilon,\sigma},\nabla F\rangle_F+\frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2. \end{aligned}\tag{4-7}$$

Putting (4-6) and (4-7) together, we obtain

$$\begin{aligned}\mathcal{L}G_{\varepsilon,\sigma} &= F^{\sigma-1}\big(\dot F^{kl}\ddot G^{pq,rs}-\dot G^{kl}\ddot F^{pq,rs}\big)\nabla_k h_{pq}\nabla_l h_{rs}\\ &\qquad + F^{\sigma-2}(F\dot G^{kl}-G\dot F^{kl})\nabla_k\nabla_l F+F^{\sigma-1}\big(F\dot G^{kl}-G\dot F^{kl}\big)h^2_{kl}\\ &\qquad +\frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F-2\,\frac{(1-\sigma)}{F}\langle\nabla F,\nabla G_{\varepsilon,\sigma}\rangle_F+\frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2. \end{aligned}\tag{4-8}$$

The first and third terms on the right may be estimated from below using Lemma 3.3.

Applying Young's inequality to the term involving the inner product, we obtain

$$-2\,\frac{(1-\sigma)}{F}\langle\nabla F,\nabla G_{\varepsilon,\sigma}\rangle_F\le(1-\sigma)E\bigg(\frac{|\nabla F|_F^2}{F^2}+\frac{|\nabla G_{\varepsilon,\sigma}|_F^2}{E^2}\bigg)$$

wherever $G_{\varepsilon,\sigma}>0$. Recalling the estimates of Lemmata 2.5, 3.3 and 4.2, and Equation (3-10), we obtain

$$\begin{aligned}\mathcal{L}G_{\varepsilon,\sigma} &\le (c_0c_1+c_2+c_0c_3)F^\sigma\frac{|\nabla\mathcal{W}|^2}{F^2}+F^{\sigma-2}(F\dot G^{kl}-G\dot F^{kl})\nabla_k\nabla_l F\\ &\qquad -\gamma F^\sigma|\mathcal{W}|^2+\frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F+c_0c_1F^\sigma\frac{|\nabla G_{\varepsilon,\sigma}|^2}{E^2}.\end{aligned}$$

Now put the $\gamma F^\sigma|\mathcal{W}|^2$ term on the left, multiply the equation by $E^pF^{-\sigma}$, and integrate over M to obtain

$$\begin{aligned}\gamma\int E^p|\mathcal{W}|^2\,d\mu &\le -\int E^pF^{-\sigma}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu+(c_0c_1+c_2+c_0c_3)\int E^p\frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu\\ &\qquad +\int E^pF^{-2}(F\dot G^{kl}-G\dot F^{kl})\nabla_k\nabla_l F\,d\mu\\ &\qquad +\sigma\int E^{p+1}F^{-1-\sigma}\mathcal{L}F\,d\mu+c_0c_1\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,d\mu.\end{aligned}\tag{4-9}$$

Integrating the first term on the right by parts, we obtain the following estimate:

Lemma 4.5. *If $\sigma\in(0,1)$ and $p>2$, there are constants $C_1, C_2, D_1>0$, independent of σ and p, such that*

$$-\int E^pF^{-\sigma}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu\le(C_1p+C_2)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,d\mu+D_1\int E^p\frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu.$$

Proof. Integrating by parts, we find

$$\begin{aligned}&-\int E^pF^{-\sigma}\mathcal{L}G_{\varepsilon,\sigma}\,d\mu\\ &\quad =p\int E^{p-1}F^{-\sigma}|\nabla G_{\varepsilon,\sigma}|_F^2\,d\mu-\sigma\int E^pF^{-\sigma-1}\langle\nabla G_{\varepsilon,\sigma},\nabla F\rangle_F\,d\mu+\int E^pF^{-\sigma}\ddot F^{kl,rs}\nabla_k h_{rs}\nabla_l G_{\varepsilon,\sigma}\,d\mu.\end{aligned}$$

Estimating each of the coefficients of $\ddot{F}$ above by $2c_4/F$ and applying Young's inequality to the second and third terms, we obtain

$$-\int E^p F^{-\sigma} \mathscr{L} G_{\varepsilon,\sigma}\, d\mu \leq c_0 p \int E^{p-2} |\nabla G_{\varepsilon,\sigma}|_F^2 \, d\mu + \frac{c_0\sigma}{2} \int E^p \left(\frac{|\nabla G_{\varepsilon,\sigma}|_F^2}{E^2} + \frac{|\nabla F|_F^2}{F^2} \right) d\mu \\ + \frac{c_0 c_4}{2} \int E^p \left(\frac{|\nabla \mathscr{W}|^2}{F^2} + \frac{|\nabla G_{\varepsilon,\sigma}|^2}{E^2} \right) d\mu.$$

Therefore,

$$-\int E^p F^{-\sigma} \mathscr{L} G_{\varepsilon,\sigma}\, d\mu \\ \leq \left(c_0 c_1 p + \frac{c_0 c_1 \sigma}{2} + \frac{c_0 c_4}{2} \right) \int E^{p-2} |\nabla G_{\varepsilon,\sigma}|^2 \, d\mu + \left(\frac{c_0 c_1 c_2 \sigma}{2} + \frac{c_0 c_4}{2} \right) \int E^p \frac{|\nabla \mathscr{W}|^2}{F^2} \, d\mu. \quad \square$$

In the same way, we obtain the following estimate on the third term of (4-9):

Lemma 4.6. *There are constants* $C_3, C_4, D_3, D_4 > 0$, *independent of* $p > 2$ *and* $\sigma \in (0, 1)$, *such that*

$$\int E^p F^{-2} (F \dot{G}^{kl} - G \dot{F}^{kl}) \nabla_k \nabla_l F \, d\mu \\ \leq (C_3 p^{\frac{3}{2}} + C_4) \int E^{p-2} |\nabla G_{\varepsilon,\sigma}|^2 \, d\mu + (D_3 p^{\frac{1}{2}} + D_4) \int E^p \frac{|\nabla \mathscr{W}|^2}{F^2} \, d\mu.$$

And the fourth term:

Lemma 4.7. *There are constants* $C_5, C_6, D_5, D_6 > 0$, *independent of* p *and* σ, *such that*

$$\int E^{p+1} F^{-1-\sigma} \mathscr{L} F \, d\mu \leq (C_5 p^{\frac{3}{2}} + C_6) \int E^{p-2} |\nabla G_{\varepsilon,\sigma}|^2 \, d\mu + (D_5 p^{\frac{1}{2}} + D_6) \int E^p \frac{|\nabla \mathscr{W}|^2}{F^2} \, d\mu.$$

This completes the proof of Proposition 4.4.

Combining Proposition 4.4 with Lemma 4.3, we obtain

$$\frac{d}{dt} \int E^p \, d\mu \leq -(c_1 p^2 - \alpha_1 \sigma p^{\frac{5}{2}} - \alpha_2 \sigma p^2 - \alpha_3 p^{\frac{3}{2}} - \alpha_4 p) \int E^{p-2} |G_{\varepsilon,\sigma}|^2 \, d\mu \\ - (\beta_1 p - \beta_2 \sigma p - \beta_3 p^{\frac{1}{2}} - \beta_4) \int E^p \frac{|\nabla \mathscr{W}|^2}{F^2} \, d\mu.$$

for some constants $\alpha_i, \beta_i > 0$, which are independent of σ and p. Proposition 4.1 follows easily.

5. Proof of Theorem 1.1

We are now able to proceed just as in [Huisken 1984, Section 5] and [Huisken and Sinestrari 1999b, Section 3], using Proposition 4.1 and the following lemma to derive the desired bound on $G_{\varepsilon,\sigma}$.

Lemma 5.1 [Stampacchia 1966]. *Let* $\varphi : [k_0, \infty) \to \mathbb{R}$ *be a nonnegative, nonincreasing function satisfying*

$$\varphi(h) \leq \frac{C}{(h-k)^\alpha} \varphi(k)^\beta, \quad h > k > k_0, \tag{5-1}$$

for some constants $C > 0$, $\alpha > 0$ and $\beta > 1$. Then

$$\varphi(k_0 + d) = 0,$$

where $d^\alpha = C\varphi(k_0)^{\beta-1} 2^{\alpha\beta/(\beta-1)}$.

Now, given any $k \geq k_0$, where $k_0 := \sup_{\sigma\in(0,1)} \sup_M G_{\varepsilon,\sigma}(\,\cdot\,, 0)$, set

$$v_k(x,t) := \big(G_{\varepsilon,\sigma}(x,t) - k\big)_+^{p/2} \quad \text{and} \quad A_k(t) := \{x \in M : v_k(x,t) > 0\}.$$

We will show that $\varphi(k) = |A_k| := \int_0^T \int_{A_k(t)} d\mu(\,\cdot\,, t)\,dt$ satisfies the conditions of Stampacchia's lemma for some $k_1 \geq k_0$. This provides us with a constant d for which $|A_{k_1+d}|$ vanishes. Theorem 1.1 then follows. Observe that $|A_k|$ is nonnegative and nonincreasing with respect to k. Then we only need to demonstrate that an inequality of the form (5-1) holds.

Lemma 5.2. *There are constants $L_1 \geq L$ and $c_6 > 0$ such that for all $p > L_1$ we have*

$$\frac{d}{dt}\int v_k^2\,d\mu + c_1^{-1}\int |\nabla v_k|^2\,d\mu \leq c_6(\sigma p + 1)\int_{A_k} F^2 G_{\varepsilon,\sigma}^p\,d\mu. \tag{5-2}$$

Proof. Observe that

$$\frac{d}{dt}\int v_k^2\,d\mu = \int_{A_k} p(G_{\varepsilon,\sigma} - k)_+^{p-1}\,\partial_t G_{\varepsilon,\sigma}\,d\mu - \int v_k^2 H F\,d\mu.$$

The result is then obtained by proceeding as in Lemma 4.3, applying

$$|\nabla v_k|^2 = \frac{p^2}{4}(G_{\varepsilon,\sigma} - k)_+^{p-2} |\nabla G_{\varepsilon,\sigma}|^2,$$

and estimating $|\mathcal{W}|^2 \leq CF^2$ using the degree-zero homogeneity of $|\mathcal{W}|^2/F^2$. □

Now set $\sigma' = \sigma + n/p$. Then

$$\int_{A_k} F^n\,d\mu \leq \int_{A_k} F^n \frac{(G_{\varepsilon,\sigma})_+^p}{k^p}\,d\mu = k^{-p}\int_{A_k} (G_{\varepsilon,\sigma'})_+^p\,d\mu \leq k^{-p}\int (G_{\varepsilon,\sigma'})_+^p\,d\mu. \tag{5-3}$$

If $p \geq \max\{L_1, 4n^2/\ell^2\}$ and $\sigma \leq (\ell/2)p^{-\frac{1}{2}}$, then $p \geq L_1$ and $\sigma' \leq \ell p^{-\frac{1}{2}}$, so that, by Proposition 4.1,

$$\int_{A_k} F^n\,d\mu \leq k^{-p}\int \big(G_{\varepsilon,\sigma'}(\,\cdot\,,0)\big)_+^p\,d\mu_0 \leq \mu_0(M)\Big(\frac{k_0}{k}\Big)^p. \tag{5-4}$$

Choosing k sufficiently large, the right hand side of this inequality can be made arbitrarily small. We will use this fact in conjunction with the following Sobolev inequality to exploit the good gradient term in (5-2).

Lemma 5.3 [Huisken 1984]. *There is a constant c_S (independent of σ, p, and ε) such that*

$$\Big(\int v_k^{2q}\,d\mu\Big)^{\frac{1}{q}} \leq c_S\bigg(\int |\nabla v_k|^2\,d\mu + \Big(\int_{A_k} F^n\,d\mu\Big)^{\frac{2}{n}}\Big(\int v_k^{2q}\,d\mu\Big)^{\frac{1}{q}}\bigg), \tag{5-5}$$

where q is equal to $n/(n-2)$ if $n > 2$, or any positive number if $n = 2$.

Proof. Since we have the estimate $H^2 < CF^2$ (by degree-zero homogeneity of the quantity H^2/F^2) this follows from the Michael–Simon–Sobolev inequality [1973] just as in [Huisken 1984]. □

It follows from (5-4) and (5-5) that there is some $k_1 > k_0$ such that for all $k > k_1$ we have

$$\left(\int v_k^{2q}\,d\mu\right)^{\frac{1}{q}} \le 2c_S \int |\nabla v_k|^2\,d\mu.$$

Therefore, from (5-2), we have for all $k > k_1$

$$\frac{d}{dt}\int v_k^2\,d\mu + \frac{1}{2c_1c_S}\left(\int v^{2q}\,d\mu\right)^{\frac{1}{q}} \le c_6(\sigma p+1)\int_{A_k} F^2 G^p_{\varepsilon,\sigma}\,d\mu.$$

Integrating this over time, and noting that $A_k(0)=\varnothing$, we find (since we may assume $2c_1c_S \ge 1$) that

$$\sup_{[0,T)}\left(\int_{A_k} v_k^2\,d\mu\right) + \int_0^T\left(\int v^{2q}\,d\mu\right)^{\frac{1}{q}} dt \le 4c_1c_Sc_6(\sigma p+1)\int_0^T\int_{A_k} F^2G^p_{\varepsilon,\sigma}\,d\mu\,dt. \tag{5-6}$$

We now exploit the interpolation inequality for L^p spaces:

$$|f|_{q_0} \le |f|_r^{1-\theta}|f|_q^{\theta}, \tag{5-7}$$

where $\theta\in(0,1)$ and $1/q_0=\theta/q+(1-\theta)/r$. Setting $r=1$ and $\theta=1/q_0$, we may assume $1<q_0<q$. Then applying (5-7) we find

$$\int_{A_k} v_k^{2q_0}\,d\mu \le \left(\int_{A_k} v_k^2\,d\mu\right)^{q_0-1}\left(\int_{A_k} v^{2q}\,d\mu\right)^{\frac{1}{q}}.$$

Now, applying the Hölder inequality, we find

$$\left(\int_0^T\int_{A_k} v_k^{2q_0}\,d\mu\,dt\right)^{\frac{1}{q_0}} \le \left(\sup_{[0,T)}\int_{A_k} v_k^2\,d\mu\right)^{\frac{q_0-1}{q_0}}\left(\int_0^T\left(\int_{A_k} v^{2q}\,d\mu\right)^{\frac{1}{q}}dt\right)^{\frac{1}{q_0}}.$$

Using Young's inequality, $ab \le (1-1/q_0)a^{q_0/(q_0-1)} + (1/q_0)b^{q_0}$, on the right hand side, we obtain

$$\begin{aligned}\left(\int_0^T\int_{A_k} v_k^{2q_0}\,d\mu\,dt\right)^{\frac{1}{q_0}} &\le \left(1-\frac{1}{q_0}\right)\sup_{[0,T)}\int_{A_k} v_k^2\,d\mu + \frac{1}{q_0}\int_0^T\left(\int_{A_k} v^{2q}\,d\mu\right)^{\frac{1}{q}}dt\\ &\le \sup_{[0,T)}\int_{A_k} v_k^2\,d\mu + \int_0^T\left(\int_{A_k} v^{2q}\,d\mu\right)^{\frac{1}{q}}dt.\end{aligned}$$

Recalling (5-6), we arrive at

$$\left(\int_0^T\int_{A_k} v_k^{2q_0}\,d\mu\,dt\right)^{\frac{1}{q_0}} \le 4c_1c_Sc_6(\sigma p+1)\int_0^T\int_{A_k} F^2G^p_{\varepsilon,\sigma}\,d\mu\,dt. \tag{5-8}$$

Application of the Hölder inequality yields the inequalities

$$\int_0^T\int_{A_k} F^2G^p_{\varepsilon,\sigma}\,d\mu\,dt \le |A_k|^{1-\frac{1}{r}}\left(\int_0^T\int_{A_k} F^{2r}G^{pr}_{\varepsilon,\sigma}\,d\mu\,dt\right)^{\frac{1}{r}} \le c_7|A_k|^{1-\frac{1}{r}} \tag{5-9}$$

and

$$\int_0^T \int_{A_k} v_k^2 \, d\mu \, dt \le |A_k|^{1-\frac{1}{q_0}} \left(\int_0^T \int_{A_k} v_k^{2q_0} \, d\mu \, dt \right)^{\frac{1}{q_0}}, \tag{5-10}$$

where the integral on the right hand side of (5-9) was estimated in a similar manner to (5-4), with $c_7 := k_0^2 (T\mu_0(M))^{1/r}$ (so long as $\sigma \le (l/4)p^{-1/2}$, and $2r > L_2 := \max\{L_1, 4n^2/l^2, 64/l^2\}$, say). Finally, for $h > k \ge k_1$ we may estimate

$$|A_h| := \int_0^T \int_{A_h} d\mu \, dt = \int_0^T \int_{A_h} \frac{(G_{\varepsilon,\sigma} - k)_+^p}{(G_{\varepsilon,\sigma} - k)_+^p} \, d\mu \, dt \le \int_0^T \int_{A_h} \frac{(G_{\varepsilon,\sigma} - k)_+^p}{(h-k)^p} \, d\mu \, dt.$$

Since $A_h(t) \subset A_k(t)$ for all $t \in [0.T)$, and $v_k^2 := (G_{\varepsilon,\sigma} - k)_+^p$, we obtain

$$(h-k)^p |A_h| \le \int_0^T \int_{A_k} v_k^2 \, d\mu \, dt. \tag{5-11}$$

Putting together estimates (5-8), (5-9), (5-10) and (5-11), we arrive at

$$|A_h| \le \frac{4c_1 c_S c_6 c_7 (\sigma p + 1)}{(h-k)^p} |A_k|^\gamma$$

for all $h > k \ge k_1$, where $\gamma := 2 - 1/q_0 - 1/r$. Now fix $p := 2L_2$ and choose $\sigma < (\ell/4)p^{-\frac{1}{2}}$ sufficiently small that $\sigma p < 1$. Then, choosing $r > \max\{q_0/(q_0-1), L_2\}$, so that $\gamma > 1$, we may apply Stampacchia's lemma. We conclude that $|A_k| = 0$ for all $k > k_1 + d$, where $d^p = c_1 c_S c_6 c_7 2^{3+\gamma p/(\gamma-1)} |A_{k_1}|^{\gamma-1}$. We note that d is finite, since T is finite and

$$\int_{A_{k_1}} d\mu \le \int_{A_{k_1}} \frac{(G_{\varepsilon,\sigma})_+^p}{k_1^p} \, d\mu \le k_1^{-p} \int (G_{\varepsilon,\sigma})_+^p \, d\mu \le k_1^{-p} \int (G_{\varepsilon,\sigma}(\,\cdot\,, 0))_+^p \, d\mu_0,$$

where the final estimate follows from Proposition 4.1.

It follows that

$$G \le \varepsilon F + (k_1 + d) F^{1-\sigma} \le 2\varepsilon F + C_\varepsilon$$

for some suitably large constant $C_\varepsilon > 0$. Theorem 1.1 follows.

6. Rescaling about type-II singularities

We now analyse the structure of fast forming singularities. Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be a smooth, compact solution of (1-1) satisfying the following ansatz: For all $C > 0$ there is a time $t_C \in [0, T)$ such that

$$\max_{x \in M} |\mathcal{W}(x, t)|^2 \ge \frac{C}{T - t} \tag{6-1}$$

for all $t \in [t_C, T)$. We say that the flow undergoes a *type-II singularity*. To analyse the shape of type-II singularities, we consider, following Hamilton [1995a] and Huisken and Sinestrari [1999b], the following sequence of parabolic rescalings: For each $k \in \mathbb{N}$, choose a sequence (t_k) of times $t_k \in [0, T - 1/k]$ and a sequence (x_k) of points $x_k \in M$ such that

$$|\mathcal{W}(x_k, t_k)|^2\Big(T - \frac{1}{k} - t_k\Big) = \max_{(x,t)\in M\times[0,T-1/k]} |\mathcal{W}(x,t)|^2\Big(T - \frac{1}{k} - t\Big).$$

Now set

$$L_k := |\mathcal{W}(x_k, t_k)|^2, \quad \alpha_k := -L_k t_k, \quad \sigma_k := L_k\Big(T - \frac{1}{k} - t_k\Big).$$

Lemma 6.1. *As* $k \to \infty$, *we have*

$$t_k \to T, \quad L_k \to \infty, \quad \alpha_k \to -\infty, \quad \sigma_k \to \infty.$$

Proof. By the ansatz (6-1), for all $R > 0$ there exists $t_R \in [0, T)$ and $x_R \in M$ such that

$$|\mathcal{W}(x_R, t_R)|^2(T - t_R) > 2R.$$

On the other hand, there is some sufficiently large $k_R \in \mathbb{N}$ such that

$$t_R < T - \frac{1}{k}, \quad |\mathcal{W}(x_R, t_R)|^2\Big(T - \frac{1}{k} - t_R\Big) > R$$

for all $k > k_R$. Therefore, by definition,

$$\sigma_k \geq |\mathcal{W}(x_R, t_R)|^2\Big(T - \frac{1}{k} - t_R\Big) > R$$

for all $k > k_R$. Since R was arbitrary, we find $\sigma_k \to \infty$ as $k \to \infty$.

Since $(T - 1/k - t_k)$ is bounded, it follows from the definition of σ_k that $L_k \to \infty$ as $k \to \infty$. Therefore, since $|\mathcal{W}|$ remains bounded whilst $t < T$, we must have $t_k \to T$. It follows that $\alpha_k \to -\infty$. □

Now consider the rescalings

$$X_k(x,t) = \sqrt{L_k}\Big(X\Big(x, \frac{t}{L_k} + t_k\Big) - X(x_k, t_k)\Big) \quad \text{for } t \in [\alpha_k, \sigma_k].$$

It is straightforward to compute

$$\begin{aligned}
\frac{\partial X_k}{\partial t}(x,t) &= -L_k^{-\frac{1}{2}} F\Big(x, \frac{t}{L_k} + t_k\Big)\nu\Big(x, \frac{t}{L_k} + t_k\Big);\\
\frac{\partial X_k}{\partial x^i}(x,t) &= \sqrt{L_k}\frac{\partial X}{\partial x^i}\Big(x, \frac{t}{L_k} + t_k\Big) \quad \Rightarrow \quad (g_k)_{ij}(x,t) = L_k g_{ij}\Big(x, \frac{t}{L_k} + t_k\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \Rightarrow \quad (g_k)^{ij}(x,t) = \frac{1}{L_k} g^{ij}\Big(x, \frac{t}{L_k} + t_k\Big);
\end{aligned}$$

and

$$\begin{aligned}
\nu_k(x,t) = \nu\Big(x, \frac{t}{L_k} + t_k\Big) \quad &\Rightarrow \quad {}^kD_i \nu_k(x,t) = {}^kD_i \nu\Big(x, \frac{t}{L_k} + t_k\Big)\\
&\Rightarrow \quad \mathcal{W}_k(x,t) = L_k^{-\frac{1}{2}} \mathcal{W}\Big(x, \frac{t}{L_k} + t_k\Big)\\
&\Rightarrow \quad F_k(x,t) = L_k^{-\frac{1}{2}} F\Big(x, \frac{t}{L_k} + t_k\Big),
\end{aligned}$$

where we used the script k to distinguish quantities related to the rescaling X_k (in particular, ${}^kD$ is the pullback of the Euclidean connection along X_k). We refer to the sequence (X_k) as a *blow-up sequence*.

Observe that the rescalings satisfy the flow equation (1-1). We also note the following properties (compare [Huisken and Sinestrari 1999b, Lemma 4.4]):

Lemma 6.2. (1) *For each $k \in \mathbb{N}$, $X_k(x_k, 0) = 0$ and $|\mathcal{W}(x_k, 0)| = 1$.*

(2) *For any $\varepsilon > 0$ and $\Sigma > 0$ there exists $k_0 \in \mathbb{N}$ such that $\sigma_k > \Sigma$ and*

$$\max_{M\times[\alpha_{k_0},\Sigma]} |\mathcal{W}_k|^2 \le 1+\varepsilon \tag{6-2}$$

for all $k \ge k_0$.

(3) *For any $\varepsilon > 0$ there exists C_ε such that*

$$-\kappa_1^{(k)}(x,t) \le \varepsilon F_k(x,t) + \frac{C_\varepsilon}{\sqrt{L_k}} \tag{6-3}$$

for all $(x,t) \in M \times [\alpha_k, \sigma_k]$, where $\kappa_1^{(k)}$ is the smallest principal curvature of X_k.

Proof. Part (1) is immediate from the definitions and our calculation of $\mathcal{W}_k$.

To prove part (2), first note that

$$|\mathcal{W}_k(x,t)|^2 = L_k^{-1}|\mathcal{W}(x, L_k^{-1}t + t_k)|^2.$$

By the definition of L_k and the choice of (x_k, t_k) we also have

$$|\mathcal{W}(x, L_k^{-1}t+t_k)|^2\Big(T - \frac{1}{k} - (L_k^{-1}t + t_k)\Big) \le L_k\Big(T - \frac{1}{k} - t_k\Big).$$

Therefore

$$|\mathcal{W}_k(x,t)|^2 \le \frac{T-\frac{1}{k}-t_k}{T-\frac{1}{k}-t_k-L_k^{-1}t} = \frac{\sigma_k}{\sigma_k - t} = 1 + \frac{t}{\sigma_k - t}.$$

Since $\sigma_k \to \infty$, the claim follows.

For part (3), we have

$$\kappa_1^k(x,t) = \frac{1}{\sqrt{L_k}}\kappa_1(x, L_k^{-1}t + t_k).$$

Therefore, by Theorem 1.1, for all $\varepsilon > 0$ there exists C_ε such that

$$-\kappa_1^k(x,t) \le \frac{1}{\sqrt{L_k}}\big(\varepsilon F(x, L_k^{-1}t + t_k) + C_\varepsilon\big) = \varepsilon F_k(x,t) + \frac{C_\varepsilon}{\sqrt{L_k}}$$

for all $(x,t) \in M \times [-\alpha_k, \sigma_k]$. □

Proof of Corollary 1.2. Since the flow speed is a convex function of the Weingarten map, the flow admits second derivative Hölder estimates, and we may proceed as in [Baker 2011, Section 3], using Lemma 6.2, to obtain a sublimit $X_\infty : M_\infty \times I_\infty \to \mathbb{R}^{n+1}$ of the blow-up sequence. Since for each k the rescaled immersion X_k is a solution of the flow on the time interval $[\alpha_k, \sigma_k]$, we deduce from Lemma 6.1 that X_∞ is an eternal solution of the flow (1-1) (that is, $I_\infty = \mathbb{R}$). Part (3) of Lemma 6.2 implies that X_∞

is weakly convex. Applying the strong tensor maximum principle [Hamilton 1982] (see also [Andrews 2007, Theorem 3.1]) to the evolution equation for the Weingarten map

$$\partial_t h_i{}^j = \mathscr{L} h_i{}^j + \ddot{F}^{pq,rs} \nabla_i h_{pq} \nabla^j h_{rs} + \dot{F}^{kl} h^2_{kl} h_i{}^j,$$

we deduce, just as in [Huisken and Sinestrari 1999a, Theorem 4.1], that the rank of $\mathscr{W}$ is constant and its null-space is invariant under parallel transport. The same use of Frobenius' theorem as in [Huisken 1993, Theorem 5.1] (compare [Hamilton 1986]) then implies that M_∞ splits isometrically as a product $\mathbb{R}^{n-k} \times \Sigma^k_\infty$ for some $1 \le k \le n$, where Σ^k_∞ is strictly convex. Moreover, $X_\infty\big|_{\Sigma^k_\infty}$ solves the flow (1-1) in $\mathbb{R}^{k+1}$.

Now observe that, by Lemma 6.2(i) and (ii), the maximum value of $|\mathscr{W}_\infty|$ is 1, and occurs at $(x_\infty, 0)$; it follows that the maximum value of F is also attained here. We complete the proof by applying the differential Harnack inequality of [Andrews 1994b] to deduce that $X_\infty\big|_{\Sigma^k_\infty}(\Sigma^k_\infty)$ moves by translation (compare [Hamilton 1995b]).

Proposition 6.3. *Let $X : \Sigma^k \times \mathbb{R} \to \mathbb{R}^{k+1}$ be a strictly convex, eternal solution of* (1-1) *with admissible speed F such that* $\sup_{\Sigma\times\mathbb{R}} F$ *is attained. Then X moves by translation.*

Proof. Consider the function $\Phi(A) = -F(A^{-1})$, where $F : \mathscr{S}_+ \to \mathbb{R}$ gives the flow speed as a function of the Weingarten map (here, $\mathscr{S}_+$ is the cone of symmetric, positive definite matrices). For any $A \in \mathscr{S}_+$, $B \in \mathrm{GL}(n)$, we have

$$\dot{\Phi}\big|_A(B) = \frac{d}{ds}\bigg|_{s=0} \Phi(A + sB) = -\frac{d}{ds}\bigg|_{s=0} F([A + sB]^{-1}) = \dot{F}\big|_A(A^{-1}BA^{-1})$$

and

$$\ddot{\Phi}\big|_A(B, B) = \frac{d^2}{ds^2}\bigg|_{s=0} \Phi(A + sB) = -\ddot{F}\big|_A(A^{-1}BA^{-1}, A^{-1}BA^{-1}) - 2\dot{F}\big|_A(A^{-1}BA^{-1}BA^{-1}).$$

Since $\ddot{F} \ge 0$, $\dot{F} > 0$, and $F > 0$, it follows that

$$\ddot{\Phi} + \frac{1-\alpha}{\alpha} \frac{\dot{\Phi} \otimes \dot{\Phi}}{\Phi} \le 0$$

for all $\alpha \in (0, 1)$. That is, Φ is α-concave for all $\alpha \in (0, 1)$. Thus Corollary 5.11 of [Andrews 1994b] may be applied. We deduce that any strictly convex solution of (1-1) satisfies

$$\partial_t F - g(\mathscr{W}^{-1}(\operatorname{grad} F), \operatorname{grad} F) + \frac{(\alpha - 1)F}{\alpha(t - t_0)} \ge 0 \tag{6-4}$$

for all $t > t_0$, where t_0 is the initial time, and grad is the gradient operator on M. It follows that any strictly convex, eternal solution of (1-1) satisfies

$$P := \partial_t F - g(\mathscr{W}^{-1}(\operatorname{grad} F),\ \operatorname{grad} F) \ge 0.$$

Moreover, (6-4) is deduced from the maximum principle applied to the time evolution of P, such that equality is attained at a space-time point only if equality holds identically. Since by assumption $\sup_{\Sigma\times\mathbb{R}} F$ is attained, P vanishes identically.

We now recall from [Andrews 1994b, Equation 5.2] that, in the Gauss map parametrisation, the Harnack quantity P satisfies:

$$(\partial_t - \overline{\mathscr{L}})P = \dot{\Phi}(\mathrm{Id})P + \ddot{\Phi}(\overline{Q}, \overline{Q}),$$

where $\overline{Q}$ is the time derivative of the inverse of the Weingarten map in the Gauss map parametrisation, and $\overline{\mathscr{L}}$ is the contraction of the covariant Hessian on S^n by $\dot{\Phi}$. Since P is identically zero, this simply says $\ddot{\Phi}(\overline{Q}, \overline{Q}) = 0$. Recalling the equation for $\ddot{\Phi}$, positive definiteness of $\dot{F}$ and strict convexity of Σ imply that $\overline{Q}$ must vanish. Returning to the standard parametrisation (for example, using [Andrews 1994b, Lemma 3.10]), we find $0 = Q = -\mathscr{W}^{-1} \circ (\partial_t \mathscr{W} + \nabla_V \mathscr{W}) \circ \mathscr{W}^{-1}$, where we have defined the vector field $V := -\mathscr{W}^{-1}(\operatorname{grad} F)$. Substituting $\partial_t \mathscr{W} = \nabla \operatorname{grad} F + F\mathscr{W}^2$, we have, for all $u \in T\Sigma$,

$$\begin{aligned} 0 &= \nabla_u \operatorname{grad} F + F\mathscr{W}^2(u) + \nabla_u \mathscr{W}(V) \\ &= \nabla_u(\operatorname{grad} F + \mathscr{W}(V)) + \mathscr{W}(F\mathscr{W}(u) - \nabla_u V). \end{aligned}$$

It follows that $\nabla V - F\mathscr{W} = 0$.

Now define the vector field $T := X_* V - F\nu$. Then, for any $u \in T\Sigma$,

$${}^X D_u T = (\nabla_u V - F\mathscr{W}(u)) - g(\mathscr{W}(V) + \operatorname{grad} F, u)\nu = 0.$$

Furthermore,

$${}^X D_t T = {}^X D_t X_* V - \partial_t F \nu - F \operatorname{grad} F,$$

where ${}^X D$ is the pullback of the Euclidean connection D by X. Since $P \equiv 0$, this becomes

$${}^X D_t T = {}^X D_t X_* V - g(\mathscr{W}^{-1}(\operatorname{grad} F), \operatorname{grad} F)\nu - F \operatorname{grad} F = {}^X D_t X_* V + g(V, \operatorname{grad} F)\nu - F \operatorname{grad} F.$$

Since V is tangential, we have

$$\left\langle {}^X D_t X_* V, \nu \right\rangle = -\left\langle X_* V, {}^X D_t \nu \right\rangle = -g(V, \operatorname{grad} F).$$

Thus the normal component of ${}^X D_t T$ is zero. The tangential part of ${}^X D_t X_* V$ is $({}^X D_t X_* V)^\top = -F\mathscr{W}(V) = F \operatorname{grad} F$; so the tangential component of ${}^X D_t T$ also vanishes. We have proved that T is parallel. Now set $\widetilde{X}(x,t) := X(\phi(x,t), t)$, where ϕ is the solution of $d\phi^i/dt = V^i$ with initial condition $\phi(x, 0) = x$. Then

$$\frac{\partial \widetilde{X}}{\partial t} = \frac{\partial X}{\partial x^i}\frac{d\phi^i}{dt} + \frac{\partial X}{\partial t} = T. \qquad \square$$

This completes the proof of Corollary 1.2. $\square$

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SPECTRAL ESTIMATES ON THE SPHERE

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In this article we establish optimal estimates for the first eigenvalue of Schrödinger operators on the d-dimensional unit sphere. These estimates depend on L^p norms of the potential, or of its inverse, and are equivalent to interpolation inequalities on the sphere. We also characterize a semiclassical asymptotic regime and discuss how our estimates on the sphere differ from those on the Euclidean space.

1. Introduction

Let Δ be the Laplace–Beltrami operator on the unit d-dimensional sphere $\mathbb{S}^d$. Our first result is concerned with the sharp estimate of the first negative eigenvalue $\lambda_1 = \lambda_1(-\Delta - V)$ of the Schrödinger operator $-\Delta - V$ on $\mathbb{S}^d$ (with potential $-V$) in terms of L^p-norms of V.

The literature on spectral estimates for the negative eigenvalues of Schrödinger operators on manifolds is limited. P. Federbusch [1969] and O. S. Rothaus [1981] established a link between logarithmic Sobolev inequalities and the ground state energy of Schrödinger operators. The Rozenbljum–Lieb–Cwikel inequality (case $\gamma = 0$ with standard notations: see below) on manifolds has been studied in [Levin and Solomyak 1997, Section 5]; we may also refer to [Lieb 1976] for the semiclassical regime, and to [Levin 2006; Ouhabaz and Poupaud 2010] for more recent results in this direction. A. Ilyin, in two articles [1993; 2012] on Lieb–Thirring type inequalities (see also [Levin 2006; Ouhabaz and Poupaud 2010] for other results on manifolds), considers Schrödinger operators on unit spheres restricted to the space of functions orthogonal to constants and uses the original method of E. Lieb and W. Thirring [1976]. The exclusion of the zero mode of the Laplace–Beltrami operator results in *semiclassical* estimates similar to those for negative eigenvalues of Schrödinger operators in Euclidean spaces.

The results in this paper are somewhat complementary. We show that if the L^p-norm of V is smaller than an explicit value, the first eigenvalue $\lambda_1(-\Delta - V)$ cannot satisfy the semiclassical inequality and thus it is impossible to obtain standard Lieb–Thirring type inequalities for the whole negative spectrum. However, we show that if the L^p-norm of the potential is large, the first eigenvalue behaves semiclassically and the best constant in the inequality asymptotically coincides with the best constants $\mathrm{L}^1_{\gamma,d}$ of the corresponding inequality in the Euclidean space of the same dimension (see below). In this regime the first eigenfunction is concentrated around some point on $\mathbb{S}^d$ and can be identified with an eigenfunction of the Schrödinger operator on the tangent space, up to a small error. In Appendix A, we illustrate the

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transition between the small L^p-norm regime and the asymptotic, semiclassical regime by numerically computing the optimal estimates for the eigenvalue $\lambda_1(-\Delta - V)$ in terms of the norms $\|V\|_{\mathrm{L}^p(\mathbb{S}^d)}$.

In order to formulate our first theorem, let us introduce the measure $d\omega$ induced by the Lebesgue measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ and the uniform probability measure $d\sigma = d\omega/|\mathbb{S}^d|$ with $|\mathbb{S}^d| = \omega(\mathbb{S}^d)$. We shall denote by $\|\cdot\|_{\mathrm{L}^q(\mathbb{S}^d)}$ the quantity $\|u\|_{\mathrm{L}^q(\mathbb{S}^d)} = (\int_{\mathbb{S}^d} |u|^q \, d\sigma)^{1/q}$ for any $q > 0$ (including the case $q \in (0, 1)$, for which $\|\cdot\|_{\mathrm{L}^q(\mathbb{S}^d)}$ is no longer a norm, but is only a quasinorm). Because of the normalization of $d\sigma$, when making comparisons with corresponding results in the Euclidean space, we will need the constant

$$\kappa_{q,d} := |\mathbb{S}^d|^{1-2/q}.$$

The well-known optimal constant $\mathrm{L}^1_{\gamma,d}$ in the one bound state Keller–Lieb–Thirring inequality is defined as follows: for any function ϕ on $\mathbb{R}^d$, if $\lambda_1(-\Delta - \phi)$ denotes the lowest negative eigenvalue of the Schrödinger operator $-\Delta - \phi$ (with potential $-\phi$) when it exists, and 0 otherwise, we have

$$|\lambda_1(-\Delta - \phi)|^\gamma \le \mathrm{L}^1_{\gamma,d} \int_{\mathbb{R}^d} \phi_+^{\gamma + d/2} \, dx, \tag{1}$$

provided $\gamma \ge 0$ if $d \ge 3$, $\gamma > 0$ if $d = 2$, and $\gamma \ge 1/2$ if $d = 1$. Notice that only the positive part ϕ_+ of ϕ is involved in the right-hand side of the above inequality. Assuming that $\gamma > 1 - d/2$ if $d = 1$ or 2, we shall consider the exponents

$$q = 2\frac{2\gamma + d}{2\gamma + d - 2} \quad \text{and} \quad p = \frac{q}{q-2} = \gamma + \frac{d}{2},$$

which are therefore such that

$$2 < q = \frac{2p}{p-1} \le 2^*$$

with $2^* := 2d/(d-2)$ if $d \ge 3$, and $q = 2p/(p-1) \in (2, +\infty)$ if $d = 1$ or 2. To simplify notation, we adopt the convention $2^* := \infty$ if $d = 1$ or 2. It is also convenient to introduce the notation

$$\alpha_* := \frac{1}{4} d(d-2).$$

In Section 2 we shall prove the following result.

Theorem 1. *Let $d \ge 1$ and $p \in (\max\{1, d/2\}, +\infty)$. Then there exists a convex increasing function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ with $\alpha(\mu) = \mu$ for any $\mu \in [0, (d/2)(p-1)]$ and $\alpha(\mu) > \mu$ for any $\mu \in ((d/2)(p-1), +\infty)$, such that*

$$|\lambda_1(-\Delta - V)| \le \alpha(\|V\|_{\mathrm{L}^p(\mathbb{S}^d)}) \tag{2}$$

for any nonnegative $V \in \mathrm{L}^p(\mathbb{S}^d)$. Moreover, for large values of μ, we have

$$\alpha(\mu)^{p-d/2} = \mathrm{L}^1_{p-d/2,d} (\kappa_{q,d}\,\mu)^p (1 + o(1)).$$

The estimate (2) is optimal in the sense that there exists a nonnegative function V such that $\mu = \|V\|_{\mathrm{L}^p(\mathbb{S}^d)}$ and $|\lambda_1(-\Delta - V)| = \alpha(\mu)$ for any $\mu \in ((d/2)(p-1), +\infty)$. If $\mu \le (d/2)(p-1)$, equality in (2) is achieved by constant potentials.

If $p = d/2$ and $d \geq 3$, then (2) *is satisfied with $\alpha(\mu) = \mu$ only for $\mu \in [0, \alpha_*]$. If $d = p = 1$, then* (2) *is also satisfied for some nonnegative, convex function α on $\mathbb{R}^+$ such that $\mu \leq \alpha(\mu) \leq \mu + \pi^2\mu^2$ for any $\mu \in (0, +\infty)$, equality in* (2) *is achieved and $\alpha(\mu) = \pi^2\mu^2(1 + o(1))$ as $\mu \to +\infty$.*

Since $\lambda_1(-\Delta - V)$ is nonpositive for any nonnegative, nontrivial V, inequality (2) is a lower estimate. We have indeed found that

$$0 \geq \lambda_1(-\Delta - V) \geq -\alpha(\|V\|_{\mathrm{L}^p(\mathbb{S}^d)}).$$

If V changes sign, the above inequality still holds if V is replaced by the positive part V_+ of V, provided the lowest eigenvalue is negative. We can then write

$$|\lambda_1(-\Delta - V)| \leq \alpha(\|V_+\|_{\mathrm{L}^p(\mathbb{S}^d)}) \quad \text{for all } V \in \mathrm{L}^p(\mathbb{S}^d). \tag{3}$$

The expression of $\mathrm{L}^1_{\gamma,d}$ is not explicit (except in the case $d = 1$: see [Lieb and Thirring 1976, page 290]), but can be given in terms of an optimal constant in some Gagliardo–Nirenberg–Sobolev inequality (see [Lieb and Thirring 1976] and (9)–(10) in Section 2.1). In case $d = p = 1$, notice that $\mathrm{L}^1_{1/2,1} = \frac{1}{2}$ (see Section B.2 in Appendix B) and $\kappa_{\infty,1} = 2\pi$ so that our formula in the asymptotic regime $\mu \to +\infty$ is consistent with the other cases.

The reader is invited to check that Theorem 1 can be reformulated in a more standard language of spectral theory as follows. We recall that $\gamma = p - d/2$ and that $d\omega$ is the standard measure induced on the unit sphere $\mathbb{S}^d$ by the Lebesgue measure on $\mathbb{R}^{d+1}$.

Corollary 2. *Let $d \geq 1$ and consider a nonnegative function V. For $\mu = \|V\|_{\mathrm{L}^{\gamma+d/2}(\mathbb{S}^d)}$ large, we have*

$$|\lambda_1(-\Delta - V)|^\gamma \lesssim \mathrm{L}^1_{\gamma,d} \int_{\mathbb{S}^d} V^{\gamma+d/2}\, d\omega \tag{4}$$

if either $\gamma > \max\{0, 1 - d/2\}$ or $\gamma = 1/2$ and $d = 1$. However, if $\mu = \|V\|_{\mathrm{L}^{\gamma+d/2}(\mathbb{S}^d)} \leq \frac{1}{4}d(2\gamma + d - 2)$, we have

$$|\lambda_1(-\Delta - V)|^{\gamma+d/2} \leq \int_{\mathbb{S}^d} V^{\gamma+d/2}\, d\omega \tag{5}$$

for any $\gamma \geq \max\{0, 1 - d/2\}$ and this estimate is optimal.

Here the notation $f \lesssim g$ as $\mu \to +\infty$ means that $f \leq c(\mu)g$ with $\lim_{\mu\to\infty} c(\mu) = 1$. The limit case $\gamma = \max\{0, 1 - d/2\}$ in (5) is covered by approximations. We may also notice that optimality in (5) is achieved by constant potentials. Let us give some details.

If we consider a sequence of constant functions $(V_n)_{n\in\mathbb{N}}$ uniformly converging towards 0, for instance $V_n = 1/n$, we get that

$$\lim_{n\to\infty} \frac{|\lambda_1(-\Delta - V_n)|^\gamma}{\int_{\mathbb{S}^d} V_n^{\gamma+d/2}\, d\omega} = +\infty,$$

which clearly forbids the possibility of an inequality of the same type as (4) for small values of $\int_{\mathbb{S}^d} V^{\gamma+d/2}\, d\omega$. This is however compatible with the results of Ilyin in dimension $d = 2$. In [Ilyin 2012, Theorem 2.1], the author states that if P is the orthogonal projection defined by $Pu := u - \int_{\mathbb{S}^2} u\, d\omega$, the negative eigenvalues $\lambda_k(P(-\Delta - V)P)$ satisfy the semiclassical inequality

$$\sum_k |\lambda_k(P(-\Delta - V)P)| \le \frac{3}{8}\int_{\mathbb{S}^2} V^2\, d\omega.$$

Another way of seeing that inequalities like (4) are incompatible with small potentials is based on the following observation. Inequality (5) shows that

$$|\lambda_1(-\Delta - V)| \le \left(\int_{\mathbb{S}^2} V^2\, d\omega\right)^{1/2}$$

if the L^2-norm of V is smaller than 1. Since such an inequality is sharp, the semiclassical Lieb–Thirring inequalities for the Schrödinger operator on the sphere $\mathbb{S}^2$ are therefore impossible for small potentials and can be achieved only in a semiclassical asymptotic regime, that is, when the norm $\|V\|_{\mathrm{L}^2(\mathbb{S}^2)}$ is large.

Our second main result is concerned with the estimates from below for the first eigenvalue of Schrödinger operators with positive potentials. In this case, by analogy with (1), it is convenient to introduce the constant $\mathrm{L}^1_{-\gamma,d}$ with $\gamma > d/2$, which is the optimal constant in the inequality

$$\lambda_1(-\Delta + \phi)^{-\gamma} \le \mathrm{L}^1_{-\gamma,d}\int_{\mathbb{R}^d} \phi^{d/2-\gamma}\, dx, \tag{6}$$

where ϕ is any positive potential on $\mathbb{R}^d$ and $\lambda_1(-\Delta+\phi)$ denotes the lowest positive eigenvalue if it exists, or $+\infty$ otherwise. Inequality (6) is less standard than (1); we refer to [Dolbeault et al. 2006, Theorem 12] for a statement and a proof. As in Theorem 1, we shall also introduce exponents p and q such that

$$q = 2\frac{2\gamma - d}{2\gamma - d + 2} \quad \text{and} \quad p = \frac{q}{2-q} = \gamma - \frac{d}{2},$$

so that p (respectively $q = 2p/p+1$) takes arbitrary values in $(0, +\infty)$ (respectively $(0, 2)$). With these notations, we have the counterpart of Theorem 1 in the case of positive potentials.

Theorem 3. *Let $d \ge 1$, $p \in (0, +\infty)$. There exists a concave increasing function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ with $\nu(\beta) = \beta$ for any $\beta \in [0, (d/2)(p+1)]$ if $p > 1$, $\nu(\beta) \le \beta$ for any $\beta > 0$ and $\nu(\beta) < \beta$ for any $\beta \in ((d/2)(p+1), +\infty)$, such that*

$$\lambda_1(-\Delta + W) \ge \nu(\beta) \quad \text{with } \beta = \|W^{-1}\|^{-1}_{\mathrm{L}^p(\mathbb{S}^d)}, \tag{7}$$

for any positive potential W such that $W^{-1} \in \mathrm{L}^p(\mathbb{S}^d)$. Moreover, for large values of β, we have

$$\nu(\beta)^{-(p+d/2)} \lesssim \mathrm{L}^1_{-(p+d/2),d}(\kappa_{q,d}\beta)^{-p}.$$

The estimate (7) is optimal in the sense that there exists a nonnegative potential W such that $\beta^{-1} = \|W^{-1}\|_{\mathrm{L}^p(\mathbb{S}^d)}$ and $\lambda_1(-\Delta + W) = \nu(\beta)$ for any positive β and p. If $\beta \le (d/2)(p+1)$ and $p > 1$, equality in (7) is achieved by constant potentials.

Again the expression of $\mathrm{L}^1_{-\gamma,d}$ is not explicit when $d \ge 2$ but can be given in terms of an optimal constant in some Gagliardo–Nirenberg–Sobolev inequality; see [Dolbeault et al. 2006] and (17)–(18) in Section 4.

We can rewrite Theorem 3 in terms of $\gamma = p + d/2$ and explicit integrals involving W.

Corollary 4. *Let $d \ge 1$ and $\gamma > d/2$. For $\beta = \|W^{-1}\|^{-1}_{\mathrm{L}^{\gamma-d/2}(\mathbb{S}^d)}$ large, we have*

$$(\lambda_1(-\Delta + W))^{-\gamma} \lesssim \mathrm{L}^1_{-\gamma,d} \int_{\mathbb{S}^d} W^{d/2-\gamma}\, d\omega.$$

However, if $\gamma \ge d/2 + 1$ and if $\beta = \|W^{-1}\|^{-1}_{\mathrm{L}^{\gamma-d/2}(\mathbb{S}^d)} \le \frac{1}{4} d(2\gamma - d + 2)$, we have

$$(\lambda_1(-\Delta + W))^{d/2-\gamma} \le \int_{\mathbb{S}^d} W^{d/2-\gamma}\, d\omega,$$

and this estimate is optimal.

This paper is organized as follows. Section 2 contains various results on interpolation inequalities; the most important one for our purpose is stated in Lemma 5. Theorem 1, Corollary 2 and, various spectral estimates for Schrödinger operators with negative potentials are established in Section 3. Section 4 deals with the case of positive potentials and contains the proofs of Theorem 3 and Corollary 4. Section 5 is devoted to the threshold case ($q = 2$, that is, $p, \gamma \to +\infty$) of exponential estimates for eigenvalues, or, in terms of interpolation inequalities, to logarithmic Sobolev inequalities. Finally, numerical and technical results have been collected in two appendices.

2. Interpolation inequalities and consequences for negative potentials

2.1. ***Inequalities in the Euclidean space.*** Let us start with some considerations on inequalities in the Euclidean space, which play a crucial role in the semiclassical regime.

We recall that we denote by 2^* the Sobolev critical exponent $2d/(d-2)$ if $d \ge 3$ and consider Sobolev's inequality on $\mathbb{R}^d$, $d \ge 3$,

$$\|v\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \le \mathsf{S}_d \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}^d) \tag{8}$$

where S_d is the optimal constant and $\mathcal{D}^{1,2}(\mathbb{R}^d)$ is the Beppo Levi space obtained by completion of smooth compactly supported functions with respect to the norm $v \mapsto \|\nabla v\|_{\mathrm{L}^2(\mathbb{R}^d)}$. See Section B.4 for details and comments on the expression of S_d.

Assume now that $d \ge 1$ and recall that $2^* = +\infty$ if $d = 1$ or 2. In the subcritical case, that is, $q \in (2, 2^*)$, let

$$\mathsf{K}_{q,d} := \inf_{v \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}}$$

be the optimal constant in the Gagliardo–Nirenberg–Sobolev inequality

$$\mathsf{K}_{q,d} \|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)} \le \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \text{for all } v \in \mathrm{H}^1(\mathbb{R}^d). \tag{9}$$

The optimal constant $\mathrm{L}^1_{\gamma,d}$ in the one bound state Keller–Lieb–Thirring inequality is such that

$$\mathrm{L}^1_{\gamma,d} := (\mathsf{K}_{q,d})^{-p} \quad \text{with } p = \gamma + \frac{d}{2},\ q = 2\frac{2\gamma + d}{2\gamma + d - 2}. \tag{10}$$

See Section B.5 for a proof and references and [Lieb and Thirring 1976] for a detailed discussion. Also see [Barnes 1976] for numerical values of $\mathsf{K}_{q,d}$.

We shall also define the exponent

$$\vartheta := d\frac{q-2}{2q}$$

which plays an important role in the scale invariant form of the Gagliardo–Nirenberg–Sobolev interpolation inequalities associated to $\mathsf{K}_{q,d}$: see Section B.1 for details.

2.2. ***Interpolation inequalities on the sphere.*** Using the inverse stereographic projection (see Section B.3), it is possible to relate interpolation inequalities on $\mathbb{R}^d$ with interpolation inequalities on $\mathbb{S}^d$. In this section we consider the case of the sphere. Notice that $\alpha_* = d/(q-2)$ when $q = 2^* = 2d/(d-2)$, $d \geq 3$.

Lemma 5. *Let $q \in (2, 2^*)$. There exists a concave increasing function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ with the properties*

$$\begin{aligned} &\mu(\alpha) = \alpha && \text{for all } \alpha \in \Big[0, \frac{d}{q-2}\Big], \\ &\mu(\alpha) < \alpha && \text{for all } \alpha \in \Big(\frac{d}{q-2}, +\infty\Big), \\ &\mu(\alpha) = \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}} \alpha^{1-\vartheta}(1+o(1)) && \text{as } \alpha \to +\infty, \end{aligned}$$

and such that

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \alpha \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \mu(\alpha) \|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \quad \text{for all } u \in \mathrm{H}^1(\mathbb{S}^d). \tag{11}$$

If $d \geq 3$ and $q = 2^*$, the inequality also holds for any $\alpha > 0$ with $\mu(\alpha) = \min\{\alpha, \alpha_*\}$.

The remainder of this section is mostly devoted to the proof of Lemma 5. A fundamental tool is a *rigidity* result proved by M.-F. Bidaut-Véron and L. Véron [1991, Theorem 6.1] for $q > 2$, which goes as follows. Any positive solution of

$$-\Delta f + \alpha f = f^{q-1} \tag{12}$$

has a unique solution $f \equiv \alpha^{1/(q-2)}$ for any $0 < \alpha \leq d/(q-2)$. A straightforward consequence of this rigidity result is the following interpolation inequality [Bidaut-Véron and Véron 1991, Corollary 6.2]:

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\sigma \geq \frac{d}{q-2}\bigg[\Big(\int_{\mathbb{S}^d} |u|^q \, d\sigma\Big)^{2/q} - \int_{\mathbb{S}^d} |u|^2 \, d\sigma\bigg] \quad \text{for all } u \in \mathrm{H}^1(\mathbb{S}^d, d\sigma). \tag{13}$$

Inequality (13) holds for any $q \in [1, 2) \cup (2, 2^*]$ if $d \geq 3$ and for any $q \in [1, 2) \cup (2, \infty)$ if $d = 1$ or 2. An alternative proof of (13) has been established in [Beckner 1993] for $q > 2$ using previous results by Lieb [1983] and the Funk–Hecke formula [Funk 1915; Hecke 1917]. The whole range $p \in [1, 2) \cup (2, 2^*)$ was covered in the case of the ultraspherical operator [Bentaleb and Fahlaoui 2009; 2010]. Also see [Bakry and Ledoux 1996; Ledoux 2000] for the *carré du champ* method, and [Dolbeault et al. 2013] for an elementary proof. Inequality (13) is *tight* as defined by D. Bakry [2006, Section 2], in the sense that equality is achieved only by constants.

Remark 6. Inequality (13) is equivalent to

$$\inf_{u\in \mathrm{H}^1(\mathbb{S}^d)\setminus\{0\}} \frac{(q-2)\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}=d.$$

Although we will not make use of them in this paper, we may notice that the following properties hold true:

(i) If $q<2^*$, the above infimum is not achieved in $\mathrm{H}^1(\mathbb{S}^d)\setminus\{0\}$, but

$$\lim_{\varepsilon\to 0_+} \frac{(q-2)\|\nabla u_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u_\varepsilon\|^2_{\mathrm{L}^q(\mathbb{S}^d)}-\|u_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}=d$$

if $u_\varepsilon := 1+\varepsilon\varphi$, where φ is a nontrivial eigenfunction of the Laplace–Beltrami operator corresponding to the first nonzero eigenvalue (see Section 2.3).

(ii) If $q=2^*$, $d\geq 3$, there are nontrivial optimal functions for (13), due to the conformal invariance. Alternatively, these solutions can be constructed from the family of Aubin–Talenti optimal functions for Sobolev's inequality, using the inverse stereographic projection.

(iii) If $\alpha>\alpha_*$ and $q=2^*$, $d\geq 3$, there are no optimal functions for (11), since otherwise $\alpha\mapsto\mu(\alpha)$ would not be constant on (α_*,α): see Proposition 7 below.

2.3. ***Properties of the function $\alpha\mapsto\mu(\alpha)$ in the subcritical case.*** Assume that $q\in(2,2^*)$. For any $\alpha>0$, consider

$$\mathcal{Q}_\alpha[u] := \frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\alpha\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)}} \quad \text{for all } u\in \mathrm{H}^1(\mathbb{S}^d,d\sigma).$$

It is a standard result of the calculus of variations that

$$\inf_{\substack{u\in \mathrm{H}^1(\mathbb{S}^d,d\sigma)\\ \int_{\mathbb{S}^d}|u|^q\,d\sigma=1}} \mathcal{Q}_\alpha[u] := \mu(\alpha)$$

is achieved by a minimizer $u\in \mathrm{H}^1(\mathbb{S}^d,d\sigma)$ which solves the Euler–Lagrange equations

$$-\Delta u+\alpha u-\mu(\alpha)u^{q-1}=0. \tag{14}$$

Indeed, we know that there is a Lagrange multiplier associated to the constraint $\int_{\mathbb{S}^d}|u|^q\,d\sigma=1$, and multiplying (14) by u and integrating on $\mathbb{S}^d$, we can identify it with $\mu(\alpha)$. As a corollary, we have shown that (11) holds. The fact that the Lagrange multiplier can be identified so easily is a consequence of the fact that all terms in (11) are two-homogeneous.

We can now list some basic properties of the function $\alpha\mapsto\mu(\alpha)$.

(1) For any $\alpha>0$, $\mu(\alpha)$ is positive, since the infimum is achieved by a nonnegative function u and $u=0$ is incompatible with the constraint $\int_{\mathbb{S}^d}|u|^q\,d\sigma=1$. By taking a constant test function, we see that $\mu(\alpha)\leq\alpha$ for all $\alpha>0$. The function $\alpha\mapsto\mu(\alpha)$ is monotone nondecreasing since for a given $u\in \mathrm{H}^1(\mathbb{S}^d,d\sigma)\setminus\{0\}$, the function $\alpha\mapsto\mathcal{Q}_\alpha[u]$ is monotone increasing. It is actually strictly monotone.

Indeed, if $\mu(\alpha_1) = \mu(\alpha_2)$ with $\alpha_1 < \alpha_2$, one can notice that $\mathcal{Q}_{\alpha_1}[u_2] < \mu(\alpha_1)$ if u_2 is a minimizer of $\mathcal{Q}_{\alpha_2}$ satisfying the constraint $\int_{\mathbb{S}^d} |u_2|^q \, d\sigma = 1$, which provides an obvious contradiction.

(2) We have

$$\mu(\alpha) = \alpha \quad \text{for all } \alpha \in \left(0, \frac{d}{q-2}\right].$$

Indeed, if u is a solution of (14), $f = \mu(\alpha)^{1/(q-2)} u$ solves (12) and is therefore a constant function if $\alpha \le d/(q-2)$ according to [Bidaut-Véron and Véron 1991, Theorem 6.1], and so is u as well. Because of the normalization constraint $\|u\|_{\mathrm{L}^q(\mathbb{S}^d)} = 1$, we get that $u = 1$, which proves the statement.

On the contrary, we have

$$\mu(\alpha) < \alpha \quad \text{for all } \alpha > \frac{d}{q-2}.$$

Let us prove this. Let φ be a nontrivial eigenfunction of the Laplace–Beltrami operator corresponding to the first nonzero eigenvalue:

$$-\Delta\varphi = d\varphi.$$

If $x = (x_1, x_2, \ldots, x_d, z)$ are cartesian coordinates of $x \in \mathbb{R}^{d+1}$ so that $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ is characterized by the condition $\sum_{i=1}^d x_i^2 + z^2 = 1$, a simple choice of such a function φ is $\varphi(x) = z$. By orthogonality with respect to the constants, we know that $\int_{\mathbb{S}^d} \varphi \, d\sigma = 0$. We may now Taylor expand $\mathcal{Q}_\alpha$ around $u = 1$ by considering $u = 1 + \varepsilon\varphi$ as $\varepsilon \to 0$ and obtain that

$$\mu(\alpha) \le \mathcal{Q}_\alpha[1+\varepsilon\varphi] = \frac{(d+\alpha)\varepsilon^2 \int_{\mathbb{S}^d} |\varphi|^2 \, d\sigma + \alpha}{\left(\int_{\mathbb{S}^d} |1+\varepsilon\varphi|^q \, d\sigma\right)^{2/q}} = \alpha + [d + \alpha(2-q)]\varepsilon^2 \int_{\mathbb{S}^d} |\varphi|^2 \, d\sigma + o(\varepsilon^2).$$

By taking ε small enough, we get $\mu(\alpha) < \alpha$ for all $\alpha > d/(q-2)$. Optimizing on the value of $\varepsilon > 0$ (not necessarily small) provides an interesting test function: see Section A.1.

(3) The function $\alpha \mapsto \mu(\alpha)$ is concave, because it is the minimum of a family of affine functions.

2.4. ***More estimates on the function $\boldsymbol{\alpha \mapsto \mu(\alpha)}$.*** We first consider the critical case $q = 2^*$, $d \ge 3$. As in the subcritical case $q < 2^*$, we have $\mu(\alpha) = \alpha$ for $\alpha \le \alpha^*$. For $\alpha > \alpha^*$, the function $\alpha \mapsto \mu(\alpha)$ is constant.

Proposition 7. *With the notations of Lemma 5, if $d \ge 3$ and $q = 2^*$, then*

$$\mu(\alpha) = \alpha_* \quad \textit{for all } \alpha > \alpha_* = \frac{d}{q-2} = \tfrac{1}{4} d(d-2).$$

Proof. Consider the Aubin–Talenti optimal functions for Sobolev's inequality and, more specifically, let us choose the functions

$$v_\varepsilon(x) := \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{d-2}{2}} \quad \text{for all } x \in \mathbb{R}^d \text{ and all } \varepsilon > 0,$$

which are such that $\|v_\varepsilon\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = \|v_1\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$ is independent of ε. With standard notations (see Section B.3), let $\mathrm{N} \in \mathbb{S}^d$ be the north pole. Using the stereographic projection Σ, that is, for the functions defined for any $y \in \mathbb{S}^d \setminus \{\mathrm{N}\}$ by

$$u_\varepsilon(y) = \left(\frac{|x|^2+1}{2}\right)^{\frac{d-2}{2}} v_\varepsilon(x) \quad \text{with } x = \Sigma(y),$$

we find that $\|u_\varepsilon\|_{\mathrm{L}^{2^*}(\mathbb{S}^d)} = \|v_1\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$ for any $\varepsilon > 0$, so that

$$\mu(\alpha) \le \mathcal{Q}_\alpha[u_\varepsilon] = \frac{\|\nabla v_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + (\alpha-\alpha_*)\int_{\mathbb{R}^d} |v_\varepsilon|^2 (2/(1+|x|^2))^2\,dx}{\kappa_{2^*,d}\|v_\varepsilon\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}} = \alpha_* + 4|\mathbb{S}^d|^{1-2/d}(\alpha-\alpha_*)\frac{\delta(d,\varepsilon)}{\|v_1\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}},$$

where we have used the fact that $\kappa_{2^*,d}\mathsf{S}_d = 1/\alpha_*$ (see Section B.4) and

$$\delta(d,\varepsilon) := \int_0^\infty \left(\frac{\varepsilon}{\varepsilon^2+r^2}\right)^{d-2} \frac{r^{d-1}}{(1+r^2)^2}\,dr = \varepsilon^2 \int_0^\infty \left(\frac{1}{1+s^2}\right)^{d-2} \frac{s^{d-1}}{(1+\varepsilon^2 s^2)^2}\,ds.$$

One can check that $\lim_{\varepsilon\to 0_+} \delta(d,\varepsilon) = 0$ since

$$\delta(d,\varepsilon) \le \varepsilon^2 \int_0^\infty \frac{s^{d-1}}{(1+s^2)^{d-2}}\,ds \quad \text{if } d \ge 5 \quad \text{and} \quad \delta(d,\varepsilon) \le \varepsilon c_d \int_0^{+\infty} \frac{ds}{(1+s^2)^2} \quad \text{if } d = 3 \text{ or } 4,$$

with $c_3 = 1$ and $c_4 = 3\sqrt{3}/16$. □

The next step is devoted to a lower estimate for the function $\alpha \mapsto \mu(\alpha)$ in the subcritical case, which shows that $\lim_{\alpha\to+\infty} \mu(\alpha) = +\infty$ in contrast with the critical case.

Proposition 8. *With the notations of Lemma 5, if $d \ge 3$ and $q \in (2, 2^*)$, then, for any $\alpha > \alpha_*$, we have*

$$\alpha > \mu(\alpha) \ge \alpha_*^{\vartheta} \alpha^{1-\vartheta}$$

with $\vartheta = d(q-2)/2q$. For every $s \in (2, 2^*)$, if $d \ge 3$, or every $s \in (2, +\infty)$ if $d = 1$ or 2, such that $s > q$, we also have that

$$\alpha > \mu(\alpha) \ge \left(\frac{d}{s-2}\right)^{\theta} \alpha^{1-\theta}$$

for any $\alpha > d/(s-2)$ and $\theta = \theta(s,q,d) := s(q-2)/(q(s-2))$.

Proof. The first case can be seen as a limit case of the second one as $s \to 2^*$ and $\vartheta = \theta(2^*, q, d)$. Using Hölder's inequality, we can estimate $\|u\|_{\mathrm{L}^q(\mathbb{S}^d)}$ by

$$\|u\|_{\mathrm{L}^q(\mathbb{S}^d)} \le \|u\|^{\theta}_{\mathrm{L}^s(\mathbb{S}^d)} \|u\|^{1-\theta}_{\mathrm{L}^2(\mathbb{S}^d)}$$

and get the result using

$$\mathcal{Q}_\alpha[u] \ge \left(\frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \alpha\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^s(\mathbb{S}^d)}}\right)^{\theta} \left(\frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \alpha\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)^{1-\theta} \ge \left(\frac{d}{s-2}\right)^{\theta} \alpha^{1-\theta}. \quad □$$

Proposition 9. *With the notations of Lemma 5, for every $q \in (2, 2^*)$, we have*

$$\limsup_{\alpha\to+\infty} \alpha^{\vartheta-1}\mu(\alpha) \le \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}.$$

Proof. Let v be an optimal function for $\mathsf{K}_{q,d}$ and define for any $x \in \mathbb{R}^d$ the function

$$v_\alpha(x) := v\big(2\sqrt{\alpha-\alpha_*}\,x\big)$$

with $\alpha_* = \frac{1}{4}d(d-2)$ and $\alpha > \alpha_*$, so that

$$\int_{\mathbb{R}^d} |\nabla v_\alpha|^2\,dx = 2^{2-d}(\alpha-\alpha_*)^{1-d/2}\int_{\mathbb{R}^d}|\nabla v|^2\,dx,$$

$$\int_{\mathbb{R}^d} |v_\alpha|^q \left(\frac{2}{1+|x|^2}\right)^{d-(d-2)q/1} dx = 2^{-(d-2)q/2}(\alpha-\alpha_*)^{-d/2}\int_{\mathbb{R}^d}|v|^q\left(1+\frac{|x|^2}{4(\alpha-\alpha_*)}\right)^{-d+(d-2)q/2} dx.$$

Now we observe that the function $u_\alpha(y) := ((|x|^2+1)/2)^{(d-2)/2}v_\alpha(x)$, where $y = \Sigma^{-1}(x)$ and Σ is the stereographic projection (see Section B.3), is such that

$$\mathcal{Q}_\alpha[u_\alpha] = \frac{1}{\kappa_{q,d}}\frac{\int_{\mathbb{R}^d}|\nabla v_\alpha|^2\,dx + (\alpha-\alpha_*)\int_{\mathbb{R}^d}|v_\alpha|^2(2/(1+|x|^2))^2\,dx}{[\int_{\mathbb{R}^d}|v_\alpha|^q(2/(1+|x|^2))^{d-(d-2)q/2}\,dx]^{2/q}}.$$

Passing to the limit as $\alpha \to +\infty$, we get

$$\lim_{\alpha\to+\infty}\int_{\mathbb{R}^d}|v|^q\left(1+\frac{|x|^2}{4(\alpha-\alpha_*)}\right)^{-d+(d-2)q/2} dx = \int_{\mathbb{R}^d}|v|^q\,dx$$

by Lebesgue's theorem of dominated convergence. The limit also holds with q replaced by 2. This proves that

$$\mathcal{Q}_\alpha[u_\alpha] = (\alpha-\alpha_*)^{1-d/2+d/q}\left(\frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}+o(1)\right)\quad \text{as } \alpha\to+\infty,$$

which concludes the proof because $\vartheta = d(q-2)/(2q)$. □

2.5. *The semiclassical regime: behavior of the function $\alpha \mapsto \mu(\alpha)$ as $\alpha \to +\infty$.* Assume $q \in (2, 2^*)$. If we combine the results of Propositions 8 and 9, we know that $\mu(\alpha) \sim \alpha^{1-\vartheta}$ as $\alpha \to +\infty$ if $d \geq 3$. If $d = 1$ or 2, we know that $\lim_{\alpha\to+\infty}\mu(\alpha) = +\infty$ with a growth at least equivalent to $\alpha^{2/q-\varepsilon}$ with $\varepsilon > 0$, arbitrarily small, according to Proposition 8, and at most equivalent to $\alpha^{1-\vartheta}$ by Proposition 9. To complete the proof of Lemma 5, it remains to determine the precise behavior of $\mu(\alpha)$ as $\alpha \to +\infty$.

Proposition 10. *With the notations of Lemma 5, for every $q \in (2, 2^*)$, with $\vartheta = d(q-2)/(2q)$ we have*

$$\mu(\alpha) = \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\alpha^{1-\vartheta}(1+o(1))\quad \textit{as } \alpha\to+\infty.$$

Proof. Suppose by contradiction that there is a positive constant η and a sequence $(\alpha_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty}\alpha_n = +\infty$ and

$$\lim_{n\to+\infty}\alpha_n^{\vartheta-1}\mu(\alpha_n) \leq \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}} - \eta. \tag{15}$$

Consider a sequence $(u_n)_{n\in\mathbb{N}}$ of functions in $\mathrm{H}^1(\mathbb{S}^d)$ such that $\mathcal{Q}_{\alpha_n}[u_n] = \mu(\alpha_n)$ and $\|u_n\|_{\mathrm{L}^q(\mathbb{S}^d)} = 1$ for any $n \in \mathbb{N}$. From (15), we know that

$$\alpha_n\|u_n\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \leq \mathcal{Q}_{\alpha_n}[u_n] = \mu(\alpha_n) \leq \alpha_n^{1-\vartheta}\left(\frac{\mathsf{K}_{q,d}}{\kappa_{q,d}} - \eta\right)(1+o(1))\quad \text{as } n\to+\infty,$$

that is,

$$\limsup_{n\to+\infty} \alpha_n^{\vartheta}\,\|u_n\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}} - \eta.$$

The normalization $\|u_n\|_{\mathrm{L}^q(\mathbb{S}^d)} = 1$ for any $n \in \mathbb{N}$ and the limit $\lim_{n\to+\infty}\|u_n\|_{\mathrm{L}^2(\mathbb{S}^d)} = 0$ mean that the sequence $(u_n)_{n\in\mathbb{N}}$ concentrates: there exists a sequence $(y_i)_{i\in\mathbb{N}}$ of points in $\mathbb{S}^d$ (eventually finite) and two sequences of positive numbers $(\zeta_i)_{i\in\mathbb{N}}$ and $(r_{i,n})_{i,n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} r_{i,n} = 0$, $\sum_{i\in\mathbb{N}} \zeta_i = 1$, and $\int_{\mathbb{S}^d\cap B(y_i,r_{i,n})} |u_{i,n}|^q\,d\sigma = \zeta_i + o(1)$, where $u_{i,n} \in \mathrm{H}^1(\mathbb{S}^d)$, $u_{i,n} = u_n$ on $\mathbb{S}^d \cap B(y_i, r_{i,n})$, and

$$\operatorname{supp} u_{i,n} \subset \mathbb{S}^d \cap B(y_i, 2r_{i,n}).$$

Here $o(1)$ means that uniformly with respect to i, the remainder term converges towards 0 as $n \to +\infty$. Using a computation similar to those of the proof of Proposition 9, we can *blow up* each function $u_{i,n}$ and prove

$$(\alpha_n - \alpha_*)^{\vartheta-1}\int_{\mathbb{S}^d} (|\nabla u_{i,n}|^2 + \alpha_n |u_{i,n}|^2)\,d\sigma \ge \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\zeta_i^{2/q} + o(1) \quad \text{for all } i.$$

Let us choose an integer N such that $\big(\sum_{i=1}^N \zeta_i\big)^{2/q} > 1 - \kappa_{q,d}\,\eta/(2\mathsf{K}_{q,d})$. Then we find that

$$\begin{aligned}(\alpha_n - \alpha_*)^{\vartheta-1}\int_{\mathbb{S}^d} (|\nabla u_n|^2 + \alpha_n |u_n|^2)\,d\sigma &\ge \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\sum_1^N \zeta_i^{2/q} + o(1) \ge \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\Big(\sum_1^N \zeta_i\Big)^{2/q} + o(1)\\ &\ge \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}} - \frac{\eta}{2} + o(1),\end{aligned}$$

a contradiction with (15). □

For details on the behavior of $\mathsf{K}_{q,d}$ as q varies, see Proposition 15. Collecting all results of this section completes the proof of Lemma 5.

3. Spectral estimates for the Schrödinger operator on the sphere

This section is devoted to the proof of Theorem 1. As a consequence of the results of Lemma 5, the function $\alpha \mapsto \mu(\alpha)$ is invertible, of inverse $\mu \mapsto \alpha(\mu)$, if $d = 1, 2$ or $d \ge 3$ and $q < 2^*$, and we have the inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2\,d\sigma - \mu\bigg(\int_{\mathbb{S}^d} |u|^q\,d\sigma\bigg)^{\frac{2}{q}} \ge -\alpha(\mu)\int_{\mathbb{S}^d} |u|^2\,d\sigma \quad \text{for all } u \in \mathrm{H}^1(\mathbb{S}^d, d\sigma) \text{ and all } \mu > 0. \tag{16}$$

Moreover, the function $\mu \mapsto \alpha(\mu)$ is monotone increasing, convex, and satisfies $\alpha(\mu) = \mu$ for any $\mu \in (0, d/(q-2)]$ and $\alpha(\mu) > \mu$ for any $\mu > d/(q-2)$.

Consider the Schrödinger operator $-\Delta - V$ for some function $V \in \mathrm{L}^p(\mathbb{S}^d)$ and the corresponding energy functional

$$\mathscr{E}[u] := \int_{\mathbb{S}^d} |\nabla u|^2\,d\sigma - \int_{\mathbb{S}^d} V|u|^2\,d\sigma.$$

Let

$$\lambda_1(-\Delta - V) \quad := \inf_{\substack{u\in\mathrm{H}^1(\mathbb{S}^d,d\sigma)\\ \int_{\mathbb{S}^d}|u|^2\,d\sigma = 1}} \mathscr{E}[u].$$

By Hölder's inequality, we have

$$\mathscr{E}[u] \geq \int_{\mathbb{S}^d} |\nabla u|^2 \, d\sigma - \|V_+\|_{\mathrm{L}^p(\mathbb{S}^d)} \|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)}$$

with $1/p + 2/q = 1$. From Section 2, with $\mu = \|V_+\|_{\mathrm{L}^p(\mathbb{S}^d)}$, we deduce

$$\mathscr{E}[u] \geq -\alpha(\mu) \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \text{for all } u \in \mathrm{H}^1(\mathbb{S}^d, d\sigma) \text{ and all } V \in \mathrm{L}^p(\mathbb{S}^d),$$

which amounts to a Keller–Lieb–Thirring inequality on the sphere (3), or equivalently,

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\sigma - \int_{\mathbb{S}^d} V |u|^2 \, d\sigma + \alpha(\|V_+\|_{\mathrm{L}^p(\mathbb{S}^d)}) \int_{\mathbb{S}^d} |u|^2 \, d\sigma \geq 0 \quad \text{for all } u \in \mathrm{H}^1(\mathbb{S}^d, d\sigma) \text{ and all } V \in \mathrm{L}^p(\mathbb{S}^d).$$

Notice that this inequality simultaneously contains (3) and (16), by optimizing either on u or on V.

Optimality in (3) still needs to be proved. This can be done by taking an arbitrary $\mu \in (0, \infty)$ and considering an optimal function for (16), for which we have

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\sigma - \mu \left(\int_{\mathbb{S}^d} |u|^q \, d\sigma \right)^{\frac{2}{q}} = \alpha(\mu) \int_{\mathbb{S}^d} |u|^2 \, d\sigma.$$

Because the above expression is homogeneous of degree two, there is no restriction to assume that $\int_{\mathbb{S}^d} |u|^q \, d\sigma = 1$, and since the solution is optimal, it solves the Euler–Lagrange equation

$$-\Delta u - V u = \alpha(\mu) u$$

with $V = \mu u^{q-2}$, such that

$$\|V_+\|_{\mathrm{L}^p(\mathbb{S}^d)} = \mu \|u\|^{q/p}_{\mathrm{L}^q(\mathbb{S}^d)} = \mu.$$

Hence such a function V realizes the equality in (3).

Taking into account Lemma 5 and (10), this completes the proof of Theorem 1 in the general case. The case $d = 1$ and $\gamma = 1/2$ has to be treated specifically. Using $u \equiv 1$ as a test function, we know that $|\lambda_1(-\Delta - V)| \leq \mu = \int_{\mathbb{S}^1} V \, dx$. On the other hand, consider $u \in \mathrm{H}^1(\mathbb{S}^1)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{S}^1)} = 1$. Since $\mathrm{H}^1(\mathbb{S}^1)$ is embedded into $C^{0,1/2}(\mathbb{S}^1)$, there exists $x_0 \in \mathbb{S}^1 \approx [0, 2\pi)$ such that $u(x_0) = 1$ and

$$|u(x)|^2 - 1 = 2 \int_{x_0}^{x} u(y) u'(y) \, dy = 2 \int_{x_0 + 2\pi}^{x} u(y) u'(y) \, dy$$

can be estimated by

$$\begin{aligned} \big| |u(x)|^2 - 1 \big| &\leq 2 \int_{x_0}^{x} |u(y)| \, |u'(y)| \, dy = 2 \int_{x_0 + 2\pi}^{x} |u(y)| \, |u'(y)| \, dy \\ &\leq \int_0^{2\pi} |u(y)| \, |u'(y)| \, dy \leq \left(\int_0^{2\pi} |u(y)|^2 \, dy \int_0^{2\pi} |u'(y)|^2 \, dy \right)^{1/2} \end{aligned}$$

using the Cauchy–Schwarz inequality, that is,

$$\big| |u(x)|^2 - 1 \big| \leq 2\pi \, \|u'\|_{\mathrm{L}^2(\mathbb{S}^1)},$$

since $\|u'\|^2_{L^2(\mathbb{S}^1)} = (1/(2\pi)) \int_0^{2\pi} |u'(y)|^2\,dy$ and $\|u\|^2_{L^2(\mathbb{S}^1)} = (1/(2\pi)) \int_0^{2\pi} |u(y)|^2\,dy = 1$ (recall that $d\sigma$ is a probability measure). Thus we get

$$|u(x)|^2 \le 1 + 2\pi \|u'\|_{L^2(\mathbb{S}^1)},$$

from which it follows that

$$\lambda_1(-\Delta - V) \ge \|u'\|^2_{L^2(\mathbb{S}^1)} - \mu(1 + 2\pi \|u'\|_{L^2(\mathbb{S}^1)}) \ge -\mu - \pi^2\mu^2.$$

This shows that $\mu \le \alpha(\mu) \le \mu + \pi^2\mu^2$. By the Arzelà–Ascoli theorem, the embedding of $H^1(\mathbb{S}^1)$ into $C^{0,1/2}(\mathbb{S}^1)$ is compact. When $d = 1$ and $\gamma = 1/2$, the proof of the asymptotic behavior of $\alpha(\mu)$ as $\mu \to +\infty$ can then be completed as in the other cases.

4. Spectral inequalities in the case of positive potentials

In this section we address the case of Schrödinger operators $-\Delta + W$ where W is a positive potential on $\mathbb{S}^d$ and we derive estimates from below for the first eigenvalue of such operators. In order to do so, we first study interpolation inequalities in the Euclidean space $\mathbb{R}^d$, like those studied in Section 2 (for $q > 2$).

For this purpose, let us define for $q \in (0, 2)$ the constant

$$\mathsf{K}^*_{q,d} := \inf_{v \in H^1(\mathbb{R}^d)\setminus\{0\}} \frac{\|\nabla v\|^2_{L^2(\mathbb{R}^d)} + \|v\|^2_{L^q(\mathbb{R}^d)}}{\|v\|^2_{L^2(\mathbb{R}^d)}},$$

that is, the optimal constant in the Gagliardo–Nirenberg–Sobolev inequality

$$\mathsf{K}^*_{q,d} \|v\|^2_{L^2(\mathbb{R}^d)} \le \|\nabla v\|^2_{L^2(\mathbb{R}^d)} + \|v\|^2_{L^q(\mathbb{R}^d)} \quad \text{for all } v \in H^1(\mathbb{R}^d) \tag{17}$$

(with the convention that the right-hand side is infinite if $|v|^q$ is not integrable).

The optimal constant $L^1_{-\gamma,d}$ in (6) is such that

$$L^1_{-\gamma,d} := (\mathsf{K}^*_{q,d})^{-\gamma} \quad \text{with } q = 2\frac{2\gamma - d}{2\gamma - d + 2}. \tag{18}$$

See Section B.6 for a proof. Let us define the exponent

$$\delta := \frac{2q}{2d - q(d-2)}.$$

Lemma 11. *Let $q \in (0, 2)$ and $d \ge 1$. Then there exists a concave increasing function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ with the properties*

$$\nu(\beta) \le \beta \quad \textit{for all } \beta > 0 \quad \textit{and} \quad \nu(\beta) < \beta \quad \textit{for all } \beta \in \left(\frac{d}{2-q}, +\infty\right),$$

$$\nu(\beta) = \beta \quad \textit{for all } \beta \in \left[0, \frac{d}{2-q}\right] \quad \textit{if } q \in [1, 2) \quad \textit{and} \quad \lim_{\beta \to 0_+} \frac{\nu(\beta)}{\beta} = 1 \quad \textit{if } q \in (0, 1),$$

$$\nu(\beta) = \mathsf{K}^*_{q,d} (\kappa_{q,d}\beta)^\delta (1 + o(1)) \quad \textit{as } \beta \to +\infty,$$

such that

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \beta\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \geq \nu(\beta)\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \textit{for all } u \in \mathrm{H}^1(\mathbb{S}^d). \tag{19}$$

Proof. Inequality (19) is obtained by minimizing the left-hand side the constraint $\|u\|_{\mathrm{L}^2(\mathbb{S}^d)} = 1$: there is a minimizer which satisfies

$$-\Delta u + \beta u^{q-1} - \nu(\beta)u = 0.$$

Case $q \in (1, 2)$. The proof is very similar to that of Lemma 5, so we leave it to the reader. Written for the optimal value of $\nu(\beta)$, inequality (19) is optimal in the following sense:

(i) If $0 < \beta \leq d/(2-q)$, equality is achieved by constants. See [Dolbeault et al. 2013] for rigidity results on $\mathbb{S}^d$.

(ii) If $\beta = d/(2-q)$, the sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n := 1 + (1/n)\varphi$, where φ is an eigenfunction of the Laplace–Beltrami operator, is a minimizing sequence of the quotient to the left-hand side of (19) divided by the right-hand side which converges to the optimal value of $\nu(\beta) = \beta = d/(2-q)$, that is,

$$\lim_{n\to\infty} \frac{\|\nabla u_n\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u_n\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u_n\|^2_{\mathrm{L}^q(\mathbb{S}^d)}} = \frac{d}{2-q}.$$

(iii) If $\beta > d/(2-q)$, there exists a nonconstant positive function $u \in \mathrm{H}^1(\mathbb{S}^d) \setminus \{0\}$ such that equality holds in (19).

Case $q \in (0, 1]$. In this case, since $\mathbb{S}^d$ is compact, the case $q \leq 1$ does not differ from the case $q \in (1, 2)$ as far as the existence of $\nu(\beta)$ is concerned. The only difference is that there is no known rigidity result for $q < 1$. However, we can prove that

$$\lim_{\beta\to 0_+} \frac{\nu(\beta)}{\beta} = 1.$$

Indeed, let us notice that $\nu(\beta) \leq \beta$ (use constants as test functions). On the other hand, let $u_\beta = c_\beta + v_\beta$ be a minimizer for $\nu(\beta)$ such that $c_\beta = \int_{\mathbb{S}^d} u_\beta \, d\sigma$ and, as a consequence, $\int_{\mathbb{S}^d} v_\beta \, d\sigma = 0$. Without loss of generality we can set $\int_{\mathbb{S}^d} |c_\beta + v_\beta|^2 \, d\sigma = c_\beta^2 + \int_{\mathbb{S}^d} |v_\beta|^2 \, d\sigma = 1$. Using the Poincaré inequality, we know that $\|\nabla v_\beta\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq d\|v_\beta\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$, and hence

$$d\|v_\beta\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \beta\|c_\beta + v_\beta\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \leq \|\nabla v_\beta\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \beta\|c_\beta + v_\beta\|^2_{\mathrm{L}^q(\mathbb{S}^d)} = \nu(\beta) \leq \beta,$$

which shows that $\lim_{\beta\to 0_+} \|v_\beta\|_{\mathrm{L}^2(\mathbb{S}^d)} = 0$ and $\lim_{\beta\to 0_+} c_\beta = 1$. As a consequence, $\|c_\beta + v_\beta\|^2_{\mathrm{L}^q(\mathbb{S}^d)} = c_\beta^2(1 + o(1))$ as $\beta \to 0_+$ and we obtain that

$$\beta(1 + o(1)) = \beta c_\beta^2(1 + o(1)) \leq \nu(\beta),$$

which concludes the proof.

Asymptotic behavior of $\nu(\beta)$. Finally, the asymptotic behavior of $\nu(\beta)$ when β is large can be investigated using concentration-compactness methods similar to those used in the proofs of Propositions 8, 9, and 10. Details are left to the reader. □

Proof of Theorem 3. By Hölder's inequality we have

$$\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} = \left(\int_{\mathbb{S}^d} W^{-q/2}(W|u|^2)^{q/2}\,d\sigma\right)^{2/q} \le \|W^{-1}\|_{\mathrm{L}^{q/(2-q)}(\mathbb{S}^d)} \int_{\mathbb{S}^d} W|u|^2\,d\sigma.$$

Using (19), we get

$$\int_{\mathbb{S}^d} |\nabla u|^2\,d\sigma + \int_{\mathbb{S}^d} W|u|^2\,d\sigma \ge \int_{\mathbb{S}^d} |\nabla u|^2\,d\sigma + \|W^{-1}\|^{-1}_{\mathrm{L}^p(\mathbb{S}^d)}\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \ge \nu(\|W^{-1}\|^{-1}_{\mathrm{L}^p(\mathbb{S}^d)}) \int_{\mathbb{S}^d} |u|^2\,d\sigma$$

with $p = q/(2-q)$, which proves (7). Then Theorem 3 is an easy consequence of Lemma 11. □

5. The threshold case: $q = 2$

The limiting case $q = 2$ in the interpolation inequality (13) corresponds to the logarithmic Sobolev inequality

$$\int_{\mathbb{S}^d} |u|^2 \log \frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\,d\sigma \le \frac{2}{d}\int_{\mathbb{S}^d} |\nabla u|^2\,d\sigma \quad \text{for all } u \in \mathrm{H}^1(\mathbb{S}^d, d\sigma),$$

which has been studied, for example, in [Beckner 1993; Brouttelande 2003b; 2003a]. For earlier results on the sphere, see [Federbush 1969; Rothaus 1981; Mueller and Weissler 1982] and the references therein (in particular for the circle). Now, if we consider inequality (11), in the limiting case $q = 2$ we obtain the following interpolation inequality.

Lemma 12. *For any $p > \max\{1, d/2\}$, there exists a concave nondecreasing function $\xi : (0, +\infty) \to \mathbb{R}$ with the properties*

$$\xi(\alpha) = \alpha \quad \textit{for all } \alpha \in (0, \alpha_0) \quad \textit{and} \quad \xi(\alpha) < \alpha \quad \textit{for all } \alpha > \alpha_0$$

for some $\alpha_0 \in [(d/2)(p-1), (d/2)p]$, and

$$\xi(\alpha) \sim \alpha^{1-d/(2p)} \quad \textit{as } \alpha \to +\infty$$

such that

$$\int_{\mathbb{S}^d} |u|^2 \log \frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\,d\sigma + p\log\frac{\xi(\alpha)}{\alpha}\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le p\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \log\left(1 + \frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\alpha\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)$$
$$\textit{for all } u \in \mathrm{H}^1(\mathbb{S}^d). \quad (20)$$

Proof. Consider Hölder's inequality: $\|u\|_{\mathrm{L}^r(\mathbb{S}^d)} \le \|u\|^\theta_{\mathrm{L}^2(\mathbb{S}^d)}\|u\|^{1-\theta}_{\mathrm{L}^q(\mathbb{S}^d)}$, with $2 \le r < q$ and $\theta = \frac{2}{r}\frac{q-r}{q-2}$. To emphasize the dependence of θ in r, we shall write $\theta = \theta(r)$. By taking the logarithm of both sides of the inequality, we find that

$$\frac{1}{r}\log\int_{\mathbb{S}^d} |u|^r\,d\sigma \le \frac{\theta(r)}{2}\log\int_{\mathbb{S}^d} |u|^2\,d\sigma + \frac{1-\theta(r)}{q}\log\int_{\mathbb{S}^d} |u|^q\,d\sigma.$$

The inequality becomes an equality when $r = 2$, so that we may differentiate at $r = 2$ and get, with $q = 2p/(p-1) < 2^*$, that is, $p = q/(q-2)$, the *logarithmic Hölder* inequality

$$\int_{\mathbb{S}^d} |u|^2 \log \frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \, d\sigma \le p \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \log \frac{\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \quad \text{for all } u \in \mathrm{H}^1(\mathbb{S}^d).$$

We may now use inequality (11) to estimate

$$\frac{\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \le \frac{\alpha}{\mu(\alpha)} \left(1 + \frac{1}{\alpha} \frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right),$$

where $\mu = \mu(\alpha)$ is the constant which appears in Lemma 5. Thus we get

$$\int_{\mathbb{S}^d} |u|^2 \log \frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \, d\sigma + p \log \frac{\mu(\alpha)}{\alpha} \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le p \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \log \left(1 + \frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\alpha \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right),$$

which proves that the inequality

$$\int_{\mathbb{S}^d} |u|^2 \log \frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \, d\sigma + p \log \xi(\alpha) \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le p \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \log \left(\alpha + \frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right)$$

holds for some optimal constant $\xi(\alpha) \ge \mu(\alpha)$, which is therefore concave, and such that $\lim_{\alpha \to +\infty} \xi(\alpha) = +\infty$. This establishes (20). The fact that equality is achieved for every $\alpha > 0$ follows from the method of [Dolbeault and Esteban 2012, Proposition 3.3].

Testing (20) with constant functions, we find that $\xi(\alpha) \le \alpha$ for any $\alpha > 0$. On the other hand, $\xi(\alpha) \ge \mu(\alpha) = \alpha$ for any $\alpha \le d/(q-2) = (d/2)(p-1)$. Testing (20) with $u = 1 + \varepsilon\varphi$, we find that $\xi(\alpha) < \alpha$ if $\alpha > (d/2)p$.

By Proposition 10, we know that $\xi(\alpha) \ge \mu(\alpha) \sim \alpha^{1-\vartheta}$ with $\vartheta = d(q-2)/(2q) = d/(2p)$ as $\alpha \to +\infty$. As in the proof of Propositions 9 and 10, let us consider an optimal function u_α for (20). Then we have

$$\begin{aligned} p \log \frac{\xi(\alpha)}{\alpha} \\ &= p \log \left(1 + \frac{1}{\alpha} \|\nabla u_\alpha\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right) - \int_{\mathbb{S}^d} |u_\alpha|^2 \log |u_\alpha|^2 \, d\sigma \sim \frac{p}{\alpha} \|\nabla u_\alpha\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \int_{\mathbb{S}^d} |u_\alpha|^2 \log |u_\alpha|^2 \, d\sigma \end{aligned}$$

as $\alpha \to +\infty$ and u_α concentrates at a single point like in the case $q > 2$ so that, after a stereographic projection which transforms u_α into v_α, the function v_α is, up to higher order terms, optimal for the Euclidean logarithmic Sobolev inequality

$$\int_{\mathbb{R}^d} |v|^2 \log \frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} \, dx + \frac{d}{2} \log(\pi \varepsilon e^2) \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \varepsilon \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)},$$

which holds for any $\varepsilon > 0$ and any $v \in \mathrm{H}^1(\mathbb{R}^d)$. Here we have of course $\varepsilon = p/\alpha$ and we find that

$$p \log \frac{\xi(\alpha)}{\alpha} = \frac{d}{2} \log \left(\pi \frac{p}{\alpha} e^2 \right) (1 + o(1)) \quad \text{as } \alpha \to +\infty. \qquad \square$$

Corollary 13. *With the notations of Lemma 12, for any* $\alpha > 0$, *we have*

$$\frac{\alpha}{p}\int_{\mathbb{S}^d}|u|^2\log\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\,d\sigma+\alpha\log\frac{\xi(\alpha)}{\alpha}\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\le\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\quad\textit{for all } u\in\mathrm{H}^1(\mathbb{S}^d).$$

Proof. This is a straightforward consequence of Lemma 12 using the fact that $\log(1+x)\le x$ for any $x>0$. ☐

As in the case $q\ne 2$, Corollary 13 provides some spectral estimates. Let $u\in\mathrm{H}^1(\mathbb{S}^d)$ be such that $\|u\|_{\mathrm{L}^2(\mathbb{S}^d)}=1$. A straightforward optimization with respect to an arbitrary function W shows that

$$\inf_W\left[\int_{\mathbb{S}^d}W|u|^2\,d\sigma+\mu\log\int_{\mathbb{S}^d}e^{-W/\mu}\,d\sigma\right]=-\mu\int_{\mathbb{S}^d}|u|^2\log|u|^2\,d\sigma,$$

with the optimality case achieved by W such that

$$|u|^2=\frac{e^{-W/\mu}}{\int_{\mathbb{S}^d}e^{-W/\mu}\,d\sigma}.$$

Notice that, up to the addition of a constant, we can always assume that $\int_{\mathbb{S}^d}e^{-W/\mu}\,d\sigma=1$, which uniquely determines the optimal W. Now, by Corollary 13 applied with $\mu=\alpha/p$, we find that

$$\int_{\mathbb{S}^d}|\nabla u|^2\,d\sigma+\int_{\mathbb{S}^d}W|u|^2\,d\sigma\ge\alpha\log\frac{\xi(\alpha)}{\alpha}-\frac{\alpha}{p}\log\int_{\mathbb{S}^d}e^{-pW/\alpha}\,d\sigma.$$

This leads us to the following statement.

Corollary 14. *Let* $d\ge 1$. *With the notations of Lemma 12, we have the estimate*

$$e^{-\lambda_1(-\Delta-W)/\alpha}\le\frac{\alpha}{\xi(\alpha)}\left(\int_{\mathbb{S}^d}e^{-pW/\alpha}\,d\sigma\right)^{1/p}$$

for any function W *such that* $e^{-pW/\alpha}$ *is integrable. This estimate is optimal in the sense that there exists a nonnegative function* W *for which the inequality becomes an equality.*

Appendix A. Further estimates and numerical results

A.1. ***A refined upper estimate.*** Let $q\in(2,2^*)$. For $\alpha>d/(q-2)$, we can give an upper estimate of the optimal constant $\mu(\alpha)$ in inequality (11) of Lemma 5. Consider functions which depend only on z, with the notations of Section 2.3. Then (11) is equivalent to an inequality that can be written as

$$\mathsf{F}_\alpha[f]:=\frac{\int_{-1}^1|f'|^2\nu\,d\nu_d+\alpha\int_{-1}^1|f|^2\,d\nu_d}{\left(\int_{-1}^1|f|^q\,d\nu_d\right)^{2/q}}\ge\mu(\alpha),$$

where $d\nu_d$ is the probability measure defined by

$$\nu_d(z)\,dz=d\nu_d(z):=Z_d^{-1}\nu^{d/2-1}\,dz\quad\text{with }\nu(z):=1-z^2,\ Z_d:=\sqrt{\pi}\,\frac{\Gamma(d/2)}{\Gamma((d+1)/2)}.$$

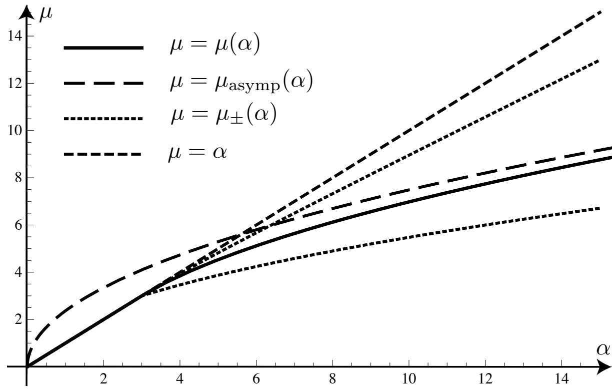

Figure 1. *In the case $q > 2$, the optimal constant is given by $\mu = \alpha$ for $\alpha \le d/(q-2)$ and the curve $\mu = \mu(\alpha)$ for $\alpha > d/(q-2)$. An upper estimate is given by the curve $\mu = \mu_+(\alpha)$ obtained by optimizing the function $h_\alpha(\varepsilon)$ in terms of $\varepsilon \in (0,1)$, while a lower estimate, namely $\mu = \mu_-(\alpha) = \alpha_*^{\vartheta}\alpha^{1-\vartheta}$, has been established in Proposition 8. The asymptotic regime is governed by $\mu(\alpha) \sim \mu_{\text{asymp}}(\alpha) = \mathsf{K}_{q,d}\kappa_{q,d}^{-1}\alpha^{1-\vartheta}$ as $\alpha \to +\infty$ according to Lemma 5. The above plot shows the various curves in the special case $d = 3$ and $q = 3$.*

See [Dolbeault et al. 2013] for details. To get an estimate, it is enough to take a well chosen test function. Consider $f_\varepsilon(z) := 1 + \varepsilon\varphi(z)$ and as in Section 2.3 we can choose $\varphi(z) = z$. Then one can optimize $h_\alpha(\varepsilon) = \mathsf{F}_\alpha[f_\varepsilon]$ with respect to $\varepsilon \in (0,1)$, and observe that $\int_{-1}^1 |f_\varepsilon'|^2 \nu \, d\nu_d = d\varepsilon^2 \int_{-1}^1 z^2 \, d\nu_d$, so that $h_\alpha(\varepsilon)$ can be written as

$$h_\alpha(\varepsilon) = \frac{\alpha + (d+\alpha)\varepsilon^2 \int_{-1}^1 z^2 \, d\nu_d}{\left(\int_{-1}^1 |1+\varepsilon z|^q \, d\nu_d\right)^{2/q}} \ge \mu(\alpha).$$

When $\varepsilon \to 0_+$, we recover that $h_\alpha(\varepsilon) - \alpha \sim [d - \alpha(q-2)]\varepsilon^2 \int_{-1}^1 z^2 \, d\nu_d < 0$ if $\alpha > d/(q-2)$, but a better estimate can be achieved simply by considering $\mu_+(\alpha) := \inf_{\varepsilon \in (0,1)} h_\alpha(\varepsilon)$ so that $\mu(\alpha) \le \mu_+(\alpha) < \alpha$. The function $\alpha \mapsto \mu_+(\alpha)$ can be computed explicitly (using hypergeometric functions) and is shown in Figure 1.

A.2. ***Numerical results.*** In this section, we illustrate the various estimates obtained in this paper by numerical computations done in the special case $d = 3$ and $q = 3$. See Figure 1 for the computation of the curve $\alpha \mapsto \mu(\alpha)$ and how it behaves compared to the theoretical estimates obtained in this paper. We emphasize that our upper and lower estimates $\alpha \mapsto \mu_\pm(\alpha)$ bifurcate from the line $\mu = \alpha$ precisely at $\alpha = d/(q-2)$ if $q \in (2, 2^*)$ (and at $\alpha = d/(2-q)$ if $q \in (1,2)$). The curve corresponding to the asymptotic regime is also plotted, but gives relevant information only as $\alpha \to \infty$.

The convergence towards the asymptotic regime is illustrated in Figure 2 which shows the convergence of $\mu(\alpha)/\mu_{\text{asymp}}(\alpha)$ towards 1 as $\alpha \to +\infty$ in the special case $d = 3$ and $q = 3$. In terms of spectral properties, for large potentials, eigenvalues of the Schrödinger operator can be estimated according to

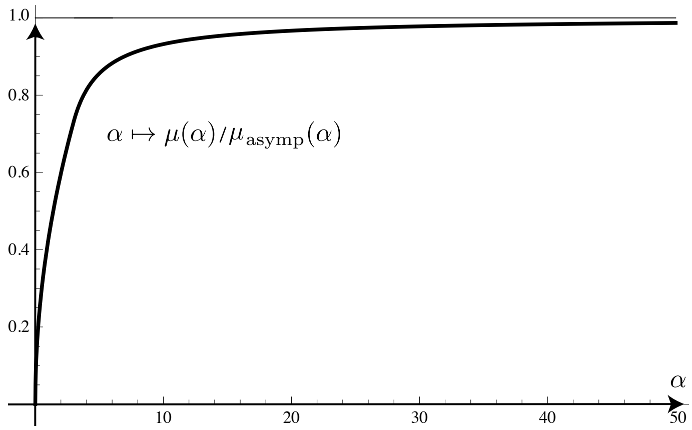

Figure 2. *The asymptotic regime corresponding to $\alpha \to +\infty$ has the interesting feature that, up to a dependence in $\alpha^{1-\vartheta}$ and a normalization factor proportional to $\kappa_{q,d}$, the optimal constant $\mu(\alpha)$ behaves like the optimal constant in the Euclidean space, as has been established in Proposition 10.*

Theorem 1 by the Euclidean Keller–Lieb–Thirring constant that has been numerically computed for instance in [Barnes 1976].

Appendix B. Constants on the Euclidean space

B.1. ***Scaling of the Gagliardo–Nirenberg–Sobolev inequality.*** Let $q > 2$ and denote by $\mathsf{K}_{\mathrm{GN}}(q)$ the optimal constant in the Gagliardo–Nirenberg–Sobolev inequality, given by

$$\mathsf{K}_{\mathrm{GN}}(q) := \inf_{u \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \frac{\|\nabla u\|^{2\vartheta}_{\mathrm{L}^2(\mathbb{R}^d)} \|u\|^{2(1-\vartheta)}_{\mathrm{L}^2(\mathbb{R}^d)}}{\|u\|^2_{\mathrm{L}^q(\mathbb{R}^d)}} \quad \text{with } \vartheta = \vartheta(q,d) = d\,\frac{q-2}{2q}.$$

An optimization of the quotient in the definition of $\mathsf{K}_{q,d}$, which has been defined in Section 2, allows us to relate this constant with $\mathsf{K}_{\mathrm{GN}}(q)$. Indeed, if we optimize $\mathcal{N}[u] := \int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} |u|^2\,dx$ under the scaling $\lambda \mapsto u_\lambda(x) := \lambda^{d/q} u(\lambda x)$, we find that

$$\mathcal{N}[u_\lambda] = \lambda^{2(1-\vartheta)} \int_{\mathbb{R}^d} |\nabla u|^2\,dx + \lambda^{-2\vartheta} \int_{\mathbb{R}^d} |u|^2\,dx$$

achieves its minimum at

$$\lambda_\star = \sqrt{\frac{\vartheta}{1-\vartheta}\,\frac{\|u\|_{\mathrm{L}^2(\mathbb{R}^d)}}{\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}}},$$

so that

$$\mathcal{N}[u_{\lambda_\star}] = \vartheta^{-\vartheta}(1-\vartheta)^{-(1-\vartheta)} \|\nabla u\|^{2\vartheta}_{\mathrm{L}^2(\mathbb{R}^d)} \|u\|^{2(1-\vartheta)}_{\mathrm{L}^2(\mathbb{R}^d)},$$

thus proving that $\mathsf{K}_{q,d}$ can be computed in terms of $\mathsf{K}_{\mathrm{GN}}(q)$ as

$$\mathsf{K}_{q,d} = \vartheta^{-\vartheta}(1-\vartheta)^{-(1-\vartheta)} \mathsf{K}_{\mathrm{GN}}(q).$$

B.2. ***Asymptotic regimes in Gagliardo–Nirenberg–Sobolev inequalities.*** Let $q > 2$ and consider the constant $\mathsf{K}_{q,d}$ as above. To handle the case of dimension $d = 1$, we may observe that, for any smooth compactly supported function u on $\mathbb{R}$, we can write either

$$|u(x)|^2 = 2\left|\int_{-\infty}^{x} u(y)u'(y)\,dy\right| \le \|u\|^2_{\mathrm{L}^2(-\infty,x)} + \|u'\|^2_{\mathrm{L}^2(-\infty,x)} \quad \text{for all } x \in \mathbb{R}$$

or

$$|u(x)|^2 = 2\left|\int_{x}^{+\infty} u(y)u'(y)\,dy\right| \le \|u\|^2_{\mathrm{L}^2(x,+\infty)} + \|u'\|^2_{\mathrm{L}^2(x,+\infty)} \quad \text{for all } x \in \mathbb{R},$$

thus proving that

$$|u(x)|^2 \le \tfrac{1}{2}(\|u\|^2_{\mathrm{L}^2(\mathbb{R})} + \|u'\|^2_{\mathrm{L}^2(\mathbb{R})}) \quad \text{for all } x \in \mathbb{R},$$

that is, the Agmon inequality

$$\frac{\|u\|^2_{\mathrm{L}^2(\mathbb{R})} + \|u'\|^2_{\mathrm{L}^2(\mathbb{R})}}{\|u\|^2_{\mathrm{L}^\infty(\mathbb{R})}} \ge 2,$$

and hence $\mathsf{K}_{\infty,1} \ge 2$. Equality is achieved by the function $u(x) = e^{-|x|}$, $x \in \mathbb{R}$, and we have shown that

$$\mathsf{K}_{\infty,1} = 2.$$

Proposition 15. *Assume that $q > 2$. For all $d \ge 1$,*

$$\lim_{q\to 2_+} \mathsf{K}_{q,d} = 1$$

and, for all $d \ge 3$,

$$\lim_{q\to 2^*} \mathsf{K}_{q,d} = \mathsf{S}_d,$$

where S_d is the best constant in inequality (8). *If $d = 1$, then* $\lim_{q\to+\infty} \mathsf{K}_{q,1} = \mathsf{K}_{\infty,1}$.

Proof. For any $v \in \mathrm{H}^1(\mathbb{R}^d)$ and $d \ge 3$, we have

$$\lim_{q\to 2^*} \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}} \ge \lim_{q\to 2^*} \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}} = \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}} \ge \mathsf{S}_d,$$

thus proving that $\lim_{q\to 2^*} \mathsf{K}_{q,d} \ge \mathsf{S}_d$. On the other hand, we may use the Aubin–Talenti function

$$\bar{u}(x) = (1 + |x|^2)^{-(d-2)/2} \quad \text{for all } x \in \mathbb{R}^d \tag{21}$$

as a test function for $\mathsf{K}_{q,d}$ if $d \ge 5$, that is,

$$\mathsf{K}_{q,d} \le \vartheta^{-\vartheta}(1-\vartheta)^{-(1-\vartheta)} \frac{\|\nabla \bar{u}\|^{2\vartheta}_{\mathrm{L}^2(\mathbb{R}^d)} \|\bar{u}\|^{2(1-\vartheta)}_{\mathrm{L}^2(\mathbb{R}^d)}}{\|\bar{u}\|^2_{\mathrm{L}^q(\mathbb{R}^d)}}$$

and observe that the right-hand side converges to S_d, since $\lim_{q\to 2^*} \vartheta(q, d) = 1$. If $d = 3$ or 4, standard additional truncations are needed. The case corresponding to $q \to \infty$, $d = 1$ is dealt with as above.

Now we investigate the limit as $q \to 2_+$. For any $v \in \mathrm{H}^1(\mathbb{R}^d)$, we have

$$\lim_{q\to 2_+} \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}} \ge \lim_{q\to 2_+} \frac{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}} = 1,$$

thus proving that $\lim_{q\to 2_+} \mathsf{K}_{q,d} \ge 1$, and for any $v \in \mathrm{H}^1(\mathbb{R}^d)$, the right-hand side in

$$\mathsf{K}_{q,d} \le \vartheta^{-\vartheta}(1-\vartheta)^{-(1-\vartheta)} \frac{\|\nabla v\|^{2\vartheta}_{\mathrm{L}^2(\mathbb{R}^d)} \|v\|^{2(1-\vartheta)}_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}}$$

converges to 1 as $q \to 2_+$. This completes the proof. □

B.3. ***Stereographic projection.*** On $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, we can introduce the coordinates $y = (\rho\phi, z) \in \mathbb{R}^d \times \mathbb{R}$ such that $\rho^2 + z^2 = 1$, $z \in [-1, 1]$, $\rho \ge 0$, and $\phi \in \mathbb{S}^{d-1}$, and consider the stereographic projection

$$\Sigma : \mathbb{S}^d \setminus \{\mathrm{N}\} \to \mathbb{R}^d$$

defined by $\Sigma(y) = x$, where, using the above notations, $x = r\phi$ with $r = \sqrt{(1+z)/(1-z)}$ for any $z \in [-1, 1)$. In this setting, the *north pole* N corresponds to $z = 1$ (and is formally sent at infinity) while the *equator* (corresponding to $z = 0$) is sent onto the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. Hence $x \in \mathbb{R}^d$ is such that $r = |x|$, $\phi = x/|x|$, and we have the useful formulae

$$z = \frac{r^2-1}{r^2+1} = 1 - \frac{2}{r^2+1}, \quad \rho = \frac{2r}{r^2+1}.$$

With these notations in hand, we can transform any function u on $\mathbb{S}^d$ into a function v on $\mathbb{R}^d$ using

$$u(y) = \left(\frac{r}{\rho}\right)^{\frac{d-2}{2}} v(x) = \left(\frac{r^2+1}{2}\right)^{\frac{d-2}{2}} v(x) = (1-z)^{-(d-2)/2} v(x),$$

and a painful but straightforward computation shows that, with $\alpha_* = \frac{1}{4}d(d-2)$,

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\omega + \alpha_* \int_{\mathbb{S}^d} |u|^2 \, d\omega = \int_{\mathbb{R}^d} |\nabla v|^2 \, dx \quad \text{and} \quad \int_{\mathbb{S}^d} |u|^q \, d\omega = \int_{\mathbb{R}^d} |v|^q \left(\frac{2}{1+|x|^2}\right)^{d-(d-2)q/2} dx.$$

As a consequence, Inequalities (11) and (19) are transformed, respectively, into

$$\int_{\mathbb{R}^d} |\nabla v|^2 \, dx + 4(\alpha - \alpha_*) \int_{\mathbb{R}^d} |v|^2 \frac{dx}{(1+|x|^2)^2} \ge \mu(\alpha)\kappa_{q,d} \left[\int_{\mathbb{R}^d} |v|^q \left(\frac{2}{1+|x|^2}\right)^{d-(d-2)q/2} dx\right]^{2/q} \quad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

if $q \in (2, 2^*)$ and $\alpha \ge \alpha_*$, and

$$\int_{\mathbb{R}^d} |\nabla v|^2 \, dx + \beta\kappa_{q,d} \left[\int_{\mathbb{R}^d} |v|^q \left(\frac{2}{1+|x|^2}\right)^{d-(d-2)q/2} dx\right]^{2/q} \ge 4(\nu(\beta) + \alpha_*) \int_{\mathbb{R}^d} |v|^2 \frac{dx}{(1+|x|^2)^2} \quad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

if $q \in (1, 2)$ and $\beta > 0$.

B.4. ***Sobolev's inequality: expression of the constant and references.*** The proof that Sobolev's inequality (8) becomes an equality if and only if $u=\bar{u}$ given by (21) up to a multiplication by a constant, a translation, and a scaling is due to T. Aubin [1976] and G. Talenti [1976]. However, G. Rosen [1971] showed (by linearization) that the function given by (21) is a local minimum when $d=3$ and computed the critical value.

Much earlier, G. Bliss [1930] (also see [Hardy and Littlewood 1930]) established that, among radial functions, the inequality

$$\left(\int_{\mathbb{R}^d} |f|^p |x|^{r+1-d-p}\,dx\right)^{\frac{2}{p}} \le \mathsf{C}_{\mathrm{Bliss}} \int_{\mathbb{R}^d} |\nabla f|^2 |x|^{1-d}\,dx$$

holds when $r=p/2-1$. With the change of variables $f(x)=v(|x|^{-1/(d-2)}x/|x|)$, the inequality is changed into

$$\left(\int_{\mathbb{R}^d} |v|^{2d/(d-2)}\,dx\right)^{\frac{d-2}{d}} \le \frac{\mathsf{C}_{\mathrm{Bliss}}}{(d-2)^{2(d-1)/d}} \int_{\mathbb{R}^d} |\nabla v|^2\,dx$$

if $p=2^*$, and it is a straightforward consequence of [Bliss 1930] that the equality is achieved with $v=\bar{u}$.

According to the duplication formula (see, for instance, [Abramowitz and Stegun 1964]) for the Γ function, we know that

$$\Gamma(x)\Gamma(x+\tfrac{1}{2}) = 2^{1-2x}\sqrt{\pi}\,\Gamma(2x).$$

As a consequence, the best constant in Sobolev's inequality (8) can be written either as

$$\mathsf{S}_d = \frac{4}{d(d-2)|\mathbb{S}^d|^{2/d}},$$

where the surface of the d-dimensional unit sphere is given by $|\mathbb{S}^d| = 2\pi^{(d+1)/2}/\,\Gamma\big(\frac{d+1}{2}\big)$ (see, for instance, [Beckner 1993]), or as

$$\mathsf{S}_d = \frac{1}{\pi d(d-2)}\left(\frac{\Gamma(d)}{\Gamma(d/2)}\right)^{\frac{2}{d}}$$

according to [Aubin 1976; Bliss 1930; Rosen 1971; Talenti 1976]. This last expression can easily be recovered using the fact that optimality in (8) is achieved by $\bar{u}$ defined in (21), while the first one, namely $1/\mathsf{S}_d = \frac{1}{4}d(d-2)\kappa_{2^*,d}$, is an easy consequence of the stereographic projection and the computations of Section B.3 with $\alpha=\alpha_*$ and $q=2^*$.

B.5. ***A proof of* (10).** Assume that $q>2$ and let us relate the optimal constant $\mathrm{L}^1_{\gamma,d}$ in the one bound state Keller–Lieb–Thirring inequality (1) with the optimal constant $\mathsf{K}_{q,d}$ in the Gagliardo–Nirenberg–Sobolev inequality (9). In this case, recall that $p=q/(q-2)=\gamma+d/2$. For any nonnegative function ϕ defined on $\mathbb{R}^d$ such that $\|\phi\|_{\mathrm{L}^p(\mathbb{R}^d)} = \mathsf{K}_{q,d}$, using Hölder's inequality, we can write that

$$\int_{\mathbb{R}^d} (|\nabla v|^2 - \phi|v|^2)\,dx \ge \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|\phi\|_{\mathrm{L}^p(\mathbb{R}^d)}\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}$$

for any $v \in \mathrm{H}^1(\mathbb{R}^d)$. Using (9), namely

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{K}_{q,d}\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)} \ge -\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)},$$

this proves that

$$|\lambda_1(-\Delta - \phi)| \le 1 \quad \text{for all } \phi \in \mathrm{L}^p(\mathbb{R}^d) \quad \text{such that} \quad \|\phi\|_{\mathrm{L}^p(\mathbb{R}^d)} = \mathsf{K}_{q,d}. \tag{22}$$

Next one can observe that inequality (1) can be rephrased as

$$\mathrm{L}^1_{\gamma,d} = \sup_{\phi\in\mathrm{L}^p(\mathbb{S}^d)} \sup_{v\in\mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} (\mathcal{R}[v,\phi])^\gamma \quad \text{with } \mathcal{R}[v,\phi] := \frac{\int_{\mathbb{R}^d} (\phi|v|^2 - |\nabla v|^2)\,dx}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\|\phi\|^{2p/(2p-d)}_{\mathrm{L}^p(\mathbb{R}^d)}},$$

where $p = \gamma + d/2$ so that the exponent $2p/(2p-d)$ is precisely the one for which we get the scaling invariance of $\mathcal{R}$. Indeed, with $v_\lambda(x) := v(\lambda x)$ and $\phi_\lambda(x) := \phi(\lambda x)$, we get that $\mathcal{R}[v_\lambda, \lambda^2\phi_\lambda] = \mathcal{R}[v,\phi]$ for any $\lambda > 0$. Hence we find that

$$\sup_{v\in\mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \mathcal{R}[v,\phi] = \frac{|\lambda_1(-\Delta-\phi)|}{\|\phi\|^{2p/(2p-d)}_{\mathrm{L}^p(\mathbb{R}^d)}} = \sup_{v\in\mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \mathcal{R}[v_\lambda, \lambda^2\phi_\lambda] = \frac{|\lambda_1(-\Delta-\lambda^2\phi_\lambda)|}{\|\lambda^2\phi_\lambda\|^{2p/(2p-d)}_{\mathrm{L}^p(\mathbb{R}^d)}},$$

and if we choose λ such that

$$\lambda^{(2p-d)/p}\|\phi\|_{\mathrm{L}^p(\mathbb{R}^d)} = \|\lambda^2\phi_\lambda\|_{\mathrm{L}^p(\mathbb{R}^d)} = \mathsf{K}_{q,d},$$

we obtain

$$\frac{|\lambda_1(-\Delta-\phi)|}{\|\phi\|^{2p/(2p-d)}_{\mathrm{L}^p(\mathbb{R}^d)}} \le \frac{1}{\mathsf{K}^{2p/(2p-d)}_{q,d}}$$

using (22), which proves that $\mathrm{L}^1_{\gamma,d} \le (\mathsf{K}_{q,d})^{-p}$ with $p = \gamma + d/2$. Since optimality can be preserved at each step, this actually proves (10). See [Keller 1961; Lieb and Thirring 1976; Veling 2002; 2003; Benguria and Loss 2004; Dolbeault et al. 2006] for further details.

In the Euclidean case, notice that the equivalence can be extended to the case of systems on the one hand and to Lieb–Thirring inequalities on the other hand: see [Lieb and Thirring 1976; Lieb 1984; Dolbeault et al. 2006].

B.6. ***A proof of* (18).** As in [Dolbeault et al. 2006], we can also relate $\mathrm{L}^1_{-\gamma,d}$ and $\mathsf{K}^*_{q,d}$ when $q = 2(2\gamma-d)/(2\gamma-d+2)$ takes values in $(0,2)$. The method is similar to that of Section B.5. For any function $v \in \mathrm{H}^1(\mathbb{R}^d)$ such that v^q is integrable and any positive potential ϕ such that ϕ^{-1} is in $\mathrm{L}^p(\mathbb{R}^d)$ with $p = q/(2-q)$, we can use Hölder's inequality as in the proof of Theorem 3 and get

$$\int_{\mathbb{R}^d} (|\nabla v|^2 + \phi|v|^2)\,dx \ge \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \frac{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}}{\|\phi^{-1}\|_{\mathrm{L}^p(\mathbb{R}^d)}}.$$

Using (17), namely $\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)} \ge \mathsf{K}^*_{q,d}\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, this proves that

$$\lambda_1(-\Delta+\phi) \ge \mathsf{K}^*_{q,d} \quad \text{for all } \phi \in \mathrm{L}^p(\mathbb{R}^d) \quad \text{such that} \quad \|\phi^{-1}\|_{\mathrm{L}^p(\mathbb{R}^d)} = 1.$$

Inequality (6) can be rephrased as

$$\mathrm{L}^1_{-\gamma,d} = \sup_{\phi\in \mathrm{L}^p(\mathbb{S}^d)} \sup_{v\in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} (\mathcal{R}[v,\phi])^{-\gamma} \quad \text{with } \mathcal{R}[v,\phi] := \frac{\int_{\mathbb{R}^d} (|\nabla v|^2+\phi|v|^2)\,dx}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} \|\phi^{-1}\|^{p/\gamma}_{\mathrm{L}^p(\mathbb{R}^d)}$$

with $\gamma = p + d/2$. The same scaling as in Section B.5 applies: with $v_\lambda(x) := v(\lambda x)$ and $\phi_\lambda(x) := \phi(\lambda x)$, we get that $\mathcal{R}[v_\lambda, \lambda^2\phi_\lambda] = \mathcal{R}[v,\phi]$ for any $\lambda > 0$, and hence

$$\mathrm{L}^1_{-\gamma,d} = (\mathsf{K}^*_{q,d})^{-\gamma},$$

which completes the proof of (18).

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NONDISPERSIVE DECAY FOR THE CUBIC WAVE EQUATION

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We consider the hyperboloidal initial value problem for the cubic focusing wave equation

$$(-\partial_t^2+\Delta_x)v(t,x)+v(t,x)^3=0, \quad x\in\mathbb{R}^3.$$

Without symmetry assumptions, we prove the existence of a codimension-4 Lipschitz manifold of initial data that lead to global solutions in forward time which do not scatter to free waves. More precisely, for any $\delta\in(0,1)$, we construct solutions with the asymptotic behavior

$$\|v-v_0\|_{L^4(t,2t)L^4(B_{(1-\delta)t})}\lesssim t^{-\frac{1}{2}+}$$

as $t\to\infty$, where $v_0(t,x)=\sqrt{2}/t$ and $B_{(1-\delta)t}:=\{x\in\mathbb{R}^3:|x|<(1-\delta)t\}$.

1. Introduction

We consider the cubic focusing wave equation

$$(-\partial_t^2+\Delta_x)v(t,x)+v(t,x)^3=0 \tag{1-1}$$

in three spatial dimensions. Equation (1-1) admits the conserved energy

$$E(v(t,\cdot),v_t(t,\cdot))=\tfrac{1}{2}\|(v(t,\cdot),v_t(t,\cdot))\|^2_{\dot H^1\times L^2(\mathbb{R}^3)}-\tfrac{1}{4}\|v(t,\cdot)\|^4_{L^4(\mathbb{R}^3)},$$

and it is well-known that solutions with small $\dot H^1\times L^2(\mathbb{R}^3)$-norm exist globally and scatter to zero [Strauss 1981; Mochizuki and Motai 1985; 1987; Pecher 1988], whereas solutions with negative energy blow up in finite time [Glassey 1973; Levine 1974]. There exists an explicit blowup solution $\tilde v_T(t,x)=\sqrt{2}/(T-t)$, which describes a stable blowup regime [Donninger and Schörkhuber 2012b] and the blowup speed (but not the profile) of any blowup solution [Merle and Zaag 2005]; see also [Bizoń et al. 2004] for numerical work. By the time translation and reflection symmetries of (1-1) we obtain from $\tilde v_T$ the explicit solution $v_0(t,x)=\sqrt{2}/t$, which is now global for $t\geq 1$ and decays in a nondispersive manner. However, in the context of the standard Cauchy problem, where one prescribes data at $t=t_0$ for some t_0 and considers the evolution for $t\geq t_0$, the role of v_0 for the study of global solutions is unclear because v_0 has infinite energy. In the present paper we argue that this is not a defect of the solution v_0 but rather a problem of the usual viewpoint concerning the Cauchy problem. Consequently, we study a different type of initial

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value problem for (1-1) where we prescribe data on a spacelike hyperboloid. In this formulation there exists a different "energy" which is finite for v_0.

Hyperboloidal initial value formulations have many advantages over the standard Cauchy problem and are well-known in numerical and mathematical relativity [Eardley and Smarr 1979; Friedrich 1983; Frauendiener 2004; Zenginoğlu 2008]. However, in the mathematical literature on wave equations in flat spacetime, hyperboloidal initial value formulations are less common (with notable exceptions such as [Christodoulou 1986]). We provide a thorough discussion of hyperboloidal methods in Section 2, where we argue that the hyperboloidal initial value problem is natural for hyperbolic equations in view of the underlying Minkowski geometry.

To state our main result, we consider a foliation of the future of the forward null cone emanating from the origin by spacelike hyperboloids

$$\Sigma_T := \left\{(t,x) \in \mathbb{R}\times\mathbb{R}^3 : t = -\frac{1}{2T} + \sqrt{\frac{1}{4T^2}+|x|^2}\right\},$$

where $T \in (-\infty, 0)$. Each Σ_T is parametrized by

$$\Phi_T : B_{|T|} \subset \mathbb{R}^3 \to \mathbb{R}^4, \quad \Phi_T(X) = \left(-\frac{T}{T^2-|X|^2}, \frac{X}{T^2-|X|^2}\right),$$

where $B_R := \{X \in \mathbb{R}^3 : |X| < R\}$ for $R > 0$. The ball $B_{|T|}$ shrinks in time as $T \to 0-$, but its image under Φ_T is an unbounded spacelike hypersurface in Minkowski space. The transformation $(T, X) \mapsto \Phi_T(X)$ has also been used by Christodoulou [1986] to study semilinear wave equations and is known as the Kelvin inversion [Tao 2008]. Note that in four-dimensional notation it can be written as $X^\mu \mapsto -X^\mu/(X^\nu X_\nu)$ (up to a sign in the zero component). To illustrate the resulting initial value problem, we plot the spacelike hyperboloids Σ_T for various values of $T \in (-\infty, 0)$ in a spacetime diagram (left panel) and in a Penrose diagram (right panel) in Figure 1 along with a null surface emanating from the origin. In our formulation of the initial value problem we prescribe data on the hypersurface Σ_{-1} and consider the future development. We refer the reader to Section 2 for a discussion on hyperboloidal foliations and their relation to wave equations.

We define a differential operator ∇_n by

$$\frac{(\nabla_n v)\circ\Phi_T(X)}{T^2-|X|^2} = \partial_T \frac{(v\circ\Phi_T)(X)}{T^2-|X|^2},$$

which one should think of as the normal derivative to the surface Σ_T (although this is not quite correct due to the additional factor $1/(T^2-|X|^2)$). Explicitly, we have

$$\nabla_n v(t,x) = (t^2+|x|^2)\,\partial_t v(t,x) + 2tx^j\,\partial_j v(t,x) + 2tv(t,x).$$

On each leaf Σ_T we define the norms

$$\|v\|^2_{L^2(\Sigma_T)} := \int_{B_{|T|}} \left|\frac{v\circ\Phi_T(X)}{T^2-|X|^2}\right|^2 dX, \quad \|v\|^2_{\dot H^1(\Sigma_T)} := \int_{B_{|T|}} \left|\nabla_X \frac{v\circ\Phi_T(X)}{T^2-|X|^2}\right|^2 dX, \tag{1-2}$$

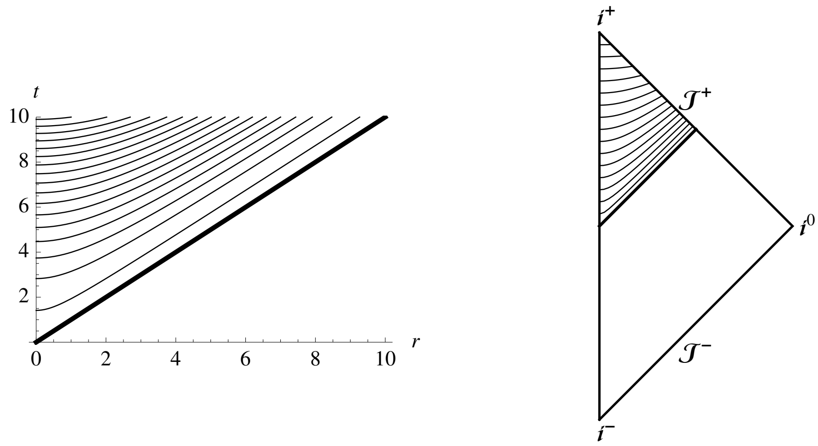

Figure 1. The spacelike hyperboloids Σ_T in a spacetime diagram (left panel) and a Penrose diagram (right panel) together with the null surface emanating from the origin (thick line with 45 degrees to the horizontal). Compare Figure 2.

and we write $\|\cdot\|^2_{H^1(\Sigma_T)} = \|\cdot\|^2_{\dot H^1(\Sigma_T)} + |T|^{-2}\|\cdot\|^2_{L^2(\Sigma_T)}$. We emphasize that

$$v_0 \circ \Phi_T(X) = \sqrt{2}\,\frac{T^2 - |X|^2}{(-T)},$$

and thus, $\|v_0\|_{H^1(\Sigma_T)} + \|\nabla_n v_0\|_{L^2(\Sigma_T)} \simeq |T|^{-\frac{1}{2}}$. Finally, for any subset $A \subset \mathbb{R}^4$ we denote its future domain of dependence by $D^+(A)$. With this notation at hand, we state our main result.

Theorem 1.1. *There exists a codimension-*4 *Lipschitz manifold* $\mathcal{M}$ *of functions in* $H^1(\Sigma_{-1}) \times L^2(\Sigma_{-1})$ *with* $(0,0) \in \mathcal{M}$ *such that the following holds. For data* $(f, g) \in \mathcal{M}$ *the hyperboloidal initial value problem*

$$\begin{cases} (-\partial_t^2 + \Delta_x)v(t,x) + v(t,x)^3 = 0, \\ v\big|_{\Sigma_{-1}} = v_0\big|_{\Sigma_{-1}} + f, \\ \nabla_n v\big|_{\Sigma_{-1}} = \nabla_n v_0\big|_{\Sigma_{-1}} + g \end{cases}$$

has a unique solution v *defined on* $D^+(\Sigma_{-1})$ *such that*

$$|T|^{\frac{1}{2}}\big(\|v - v_0\|_{H^1(\Sigma_T)} + \|\nabla_n v - \nabla_n v_0\|_{L^2(\Sigma_T)}\big) \lesssim |T|^{\frac{1}{2}-}$$

for all $T \in [-1, 0)$. *As a consequence, for any* $\delta \in (0,1)$, *we have*

$$\|v - v_0\|_{L^4(t,2t)L^4(B_{(1-\delta)t})} \lesssim t^{-\frac{1}{2}+}$$

as $t \to \infty$, *i.e.*, v *converges to* v_0 *in a localized Strichartz sense.*

Some remarks are in order.

- As usual, by a "solution" we mean a function which solves the equation in an appropriate weak sense, not necessarily in the sense of classical derivatives.

- The manifold $\mathcal{M}$ can be represented as a graph of a Lipschitz function. More precisely, let $\mathcal{H} := H^1(\Sigma_{-1}) \times L^2(\Sigma_{-1})$ and denote by $\mathcal{B}_R(0)$ the open ball of radius $R > 0$ around 0 in $\mathcal{H}$. We prove that there exists a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $\dim \mathcal{H}_2 = 4$ and a function $F : \mathcal{H}_1 \cap \mathcal{B}_\delta(0) \to \mathcal{H}_2$ such that $\mathcal{M} = \{\vec{u} + F(\vec{u}) : \vec{u} \in \mathcal{H}_1 \cap \mathcal{B}_\delta(0)\}$ provided $\delta > 0$ is chosen sufficiently small. Furthermore, F satisfies

$$\|F(\vec{u}) - F(\vec{v})\|_{\mathcal{H}} \lesssim \delta^{\frac{1}{2}} \|\vec{u} - \vec{v}\|_{\mathcal{H}}$$

  for all $\vec{u}, \vec{v} \in \mathcal{H}_1 \cap \mathcal{B}_\delta(0)$ and $F(\vec{0}) = \vec{0}$.

- The reason for the codimension-4 instability of the attractor v_0 is the invariance of (1-1) under time translations and Lorentz transforms (combined with the Kelvin inversion). The Lorentz boosts do not destroy the nondispersive character of the solution v_0 whereas the time translation does — see the beginning of Section 4 below for a more detailed discussion. In this sense, one may say that there exists a codimension-*one* manifold of data that lead to nondispersive solutions. However, if one fixes v_0, as we have done in our formulation, there are 4 unstable directions.

There was tremendous recent progress in the understanding of universal properties of global solutions to nonlinear wave equations, in particular in the energy critical case; see, for example, [Duyckaerts et al. 2012; 2013; Cote et al. 2012; Kenig et al. 2013]. A guiding principle for all these studies is the soliton resolution conjecture, that is, the idea that global solutions to nonlinear dispersive equations decouple into solitons plus radiation as time tends to infinity. It is known that, in such a strict sense, soliton resolution does not hold in most cases. One possible obstacle is the existence of global solutions which do not scatter. Recently, the first author and Krieger constructed nonscattering solutions for the energy critical focusing wave equation [Donninger and Krieger 2013]; see also [Ortoleva and Perelman 2013] for similar results in the context of the nonlinear Schrödinger equation. These solutions are obtained by considering a rescaled ground state soliton, the existence of which is typical for critical dispersive equations. The cubic wave equation under consideration is energy subcritical and does not admit solitons. Consequently, our result is of a completely different nature. Instead of considering moving solitons, we obtain the nonscattering solutions by perturbing the *self-similar* solution $v_0(t, x) = \sqrt{2}/t$. This can only be done in the framework of a hyperboloidal initial value formulation because the standard energy for the self-similar solution v_0 is infinite.

Another novel feature of our result is a precise description of the data which lead to solutions that converge to v_0: They lie on a Lipschitz manifold of codimension 4. In this respect we believe that our result is also interesting from the perspective of infinite-dimensional dynamical systems theory for wave equations, which is currently a very active field; see, for example, [Krieger et al. 2013a; 2013b; 2012].

Finally, we mention that the present work is motivated by numerical investigations undertaken by Bizoń and the second author [Bizoń and Zenginoğlu 2009]. In particular, the conformal symmetry for the cubic wave equation has been used in [Bizoń and Zenginoğlu 2009] to translate the (linear) stability analysis for blowup to asymptotic results for decay. We exploit this idea in a similar way: If v solves

(1-1) then u, defined by

$$u(T,X)=\frac{1}{T^2-|X|^2}v\left(-\frac{T}{T^2-|X|^2},\frac{X}{T^2-|X|^2}\right)=\frac{v\circ\Phi_T(X)}{T^2-|X|^2},$$

solves $(-\partial_T^2+\Delta_X)u(T,X)+u(T,X)^3=0$. The point is that the coordinate transformation $(t,x)\mapsto(T,X)$ with

$$T=-\frac{t}{t^2-|x|^2},\quad X=\frac{x}{t^2-|x|^2}$$

maps the forward light cone $\{(t,x):|x|<t,\,t>0\}$ to the backward light cone $\{(T,X):|X|<-T,\,T<0\}$ and $t\to\infty$ translates into $T\to 0-$ (see Figure 1). Moreover,

$$\frac{1}{T^2-|X|^2}v_0\left(-\frac{T}{T^2-|X|^2},\frac{X}{T^2-|X|^2}\right)=\frac{\sqrt{2}}{(-T)}=:u_0(T,X)$$

and thus, we are led to the study of the stability of the self-similar blowup solution u_0 in the backward light cone of the origin. In the context of radial symmetry, this problem was recently addressed by Donninger and Schörkhuber [2012b]; see also [Donninger 2011; 2012; Donninger and Schörkhuber 2012a] for similar results in the context of wave maps, Yang–Mills equations, and supercritical wave equations. However, in the present paper we do not assume any symmetry of the data and hence, we develop a stability theory similar to [Donninger and Schörkhuber 2012b] but beyond the radial context. Furthermore, the instabilities of u_0 have a different interpretation in the current setting and lead to the codimension-4 condition in Theorem 1.1 whereas the blowup studied in [Donninger and Schörkhuber 2012b] is stable. The conformal symmetry, although convenient, does not seem crucial for our argument. It appears that one can employ similar techniques to study nondispersive solutions for semilinear wave equations $(-\partial_t^2+\Delta_x)v(t,x)+v(t,x)|v(t,x)|^{p-1}=0$ with more general $p>3$.

Notation. The arguments for functions defined on Minkowski space are numbered by 0, 1, 2, 3 and we write ∂_μ, $\mu\in\{0,1,2,3\}$, for the respective derivatives. Our sign convention for the Minkowski metric η is $(-,+,+,+)$. We use the notation ∂_y for the derivative with respect to the variable y. We employ Einstein's summation convention throughout with Latin indices running from 1 to 3 and Greek indices running from 0 to 3, unless otherwise stated. We denote by $\mathbb{R}_0^+$ the set of positive real numbers including 0.

The letter C (possibly with indices to indicate dependencies) denotes a generic positive constant which may have a different value at each occurrence. The symbol $a\lesssim b$ means $a\leq Cb$ and we abbreviate $a\lesssim b\lesssim a$ by $a\simeq b$. We write $f(x)\sim g(x)$ for $x\to a$ if $\lim_{x\to a}f(x)/g(x)=1$.

For a closed linear operator $\boldsymbol{L}$ on a Banach space we denote its domain by $\mathcal{D}(\boldsymbol{L})$, its spectrum by $\sigma(\boldsymbol{L})$, and its point spectrum by $\sigma_p(\boldsymbol{L})$. We write $\boldsymbol{R_L}(z):=(z-\boldsymbol{L})^{-1}$ for $z\in\rho(\boldsymbol{L})=\mathbb{C}\backslash\sigma(\boldsymbol{L})$. The space of bounded operators on a Banach space $\mathcal{X}$ is denoted by $\mathcal{B}(\mathcal{X})$.

2. Wave equations and geometry

In this section, we present the motivation for using hyperboloidal coordinates in our analysis and provide some background. We discuss the main arguments and tools in a pedagogical manner to emphasize the relation between spacetime geometry and wave equations for readers not familiar with relativistic terminology.

Geometric preliminaries. A spacetime $(\mathcal{M}, g)$ is a four-dimensional paracompact Hausdorff manifold $\mathcal{M}$ with a time-oriented Lorentzian metric g. The cubic wave equation (1-1) is posed on the Minkowski spacetime $(\mathbb{R}^4, \eta)$. In standard time t and Cartesian coordinates (x, y, z) the Minkowski metric reads

$$\eta = -dt^2 + dx^2 + dy^2 + dz^2, \qquad (t, x, y, z) \in \mathbb{R}^4.$$

Minkowski spacetime is *spherically symmetric*, i.e., the group SO(3) acts nontrivially by isometry on $(\mathbb{R}^4, \eta)$. We introduce the quotient space $\mathcal{Q} = \mathbb{R}^4/\,\mathrm{SO}(3)$ and the area radius $r : \mathcal{Q} \to \mathbb{R}$ such that the group orbits of points $p \in \mathcal{Q}$ have area $4\pi r^2(p)$. The area radius can be written as $r = \sqrt{x^2 + y^2 + z^2}$ with respect to Cartesian coordinates. The flat metric can then be written as $\eta = q + r^2\, d\sigma^2$, where q is a rank-2 Lorentzian metric and $d\sigma^2$ is the standard metric on S^2. Choosing the usual angular variables for $d\sigma^2$, we obtain the familiar form of the flat spacetime metric in spherical coordinates

$$\eta = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad (t, r, \theta, \phi) \in \mathbb{R} \times \mathbb{R}_+ \times [0, \pi] \times [0, 2\pi).$$

A codimension-one submanifold is called a *hypersurface* and a *foliation* is a one-parameter family of nonintersecting spacelike hypersurfaces. A foliation can also be defined by a time function from $\mathcal{M}$ to the real line $\mathbb{R}$, whose level sets are the hypersurfaces of the foliation.

We can restrict our discussion of the interaction between hyperbolic equations and spacetime geometry to spherical symmetry without loss of generality because the radial direction is sufficient for exploiting the Lorentzian structure. Working in the two-dimensional quotient spacetime $(\mathcal{Q}, q)$ also allows us to illustrate the geometric definitions in two-dimensional plots.

Compactification and Penrose diagrams. It is useful to introduce Penrose diagrams to depict global features of time foliations in spherically symmetric spacetimes. Penrose presented the construction of the diagrams in his study of the asymptotic behavior of gravitational fields in 1963 [Penrose 2011]. A beautiful exposition of Penrose diagrams has been given in [Dafermos and Rodnianski 2005]. As we are working in Minkowski spacetime only, the main features of Penrose diagrams of interest to us are the compactification and the preservation of the causal structure. See, for example, [Christodoulou 1986; Keel and Tao 1998] for the application of Penrose compactification to study wave equations in flat spacetime.

The image of the Penrose diagram is a two-dimensional Minkowski spacetime with a *bounded* global null coordinate system. Causal concepts extend through the boundary of the map. Consider the rank-2 Minkowski metric q on the quotient manifold $\mathcal{Q}$

$$q = -dt^2 + dr^2, \quad (t, r) \in \mathbb{R} \times \mathbb{R}_0^+. \tag{2-1}$$

To map this metric to a global, bounded, null coordinate system, define $u = t - r$ and $v = t + r$ for $v \geq u$, and compactify by $U = \arctan u$ and $V = \arctan v$. The quotient metric becomes

$$q = -\frac{1}{\cos^2 V \cos^2 U}\, dV\, dU \quad (-\pi/2 < U \leq V < \pi/2).$$

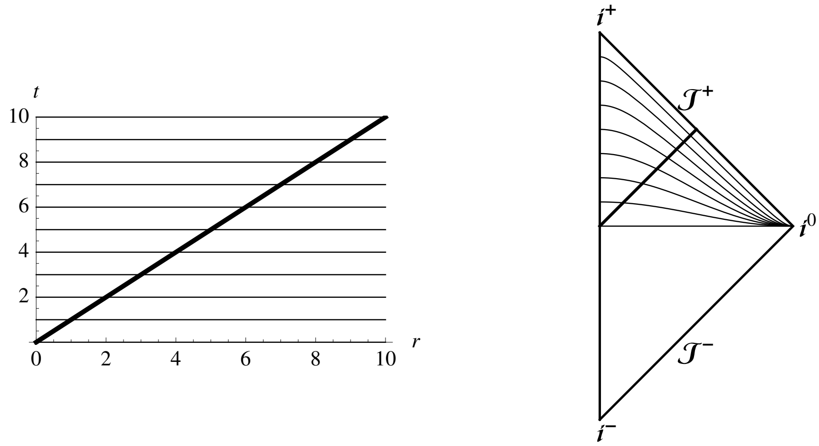

Figure 2. The level sets of the standard time t depicted in a spacetime diagram (left panel) and a Penrose diagram (right panel) together with a characteristic line from the origin. The boundary of the Penrose diagram includes the spatial origin and various notions of infinity. Past and future timelike infinity are depicted by points i^- and i^+. The vertical line connecting i^- and i^+ is the spatial origin $r = 0$. Spatial infinity is denoted by the point i^0. Null curves reach past and future null infinity, denoted by $\mathscr{I}^-$ and $\mathscr{I}^+$, for infinite values of their affine parameter.

Points at infinity with respect to the original coordinates have finite values with respect to the compactifying coordinates. The singular behavior of the metric in compactifying coordinates at the boundary can be compensated by a conformal rescaling with the conformal factor $\Omega = \cos V \cos U$, so that the rescaled metric

$$\bar{q} = \Omega^2 q = -dU\,dV$$

is well defined on the domain $(-\pi/2 \le U \le V \le \pi/2)$ including points that are at infinity with respect to q. We say that q can be conformally extended beyond infinity.

The Penrose diagram is then drawn using time and space coordinates $T = (V+U)/2$ and $R = (V-U)/2$ (see Figure 2). The resulting metric $\bar{q} = -dT^2 + dR^2$ is flat. The combined Penrose map is given by

$$t \mapsto \frac{1}{2}\left(\tan\frac{T+R}{2} + \tan\frac{T-R}{2}\right), \qquad r \mapsto \frac{1}{2}\left(\tan\frac{T+R}{2} - \tan\frac{T-R}{2}\right).$$

The boundary $\partial\bar{\mathscr{D}} = \{T = \pm(\pi - R),\ R \in [0, \pi]\}$ corresponds to points at infinity with respect to the original Minkowski metric. Asymptotic behavior of fields on $\mathscr{D}$ can be studied using local differential geometry near this boundary where the conformal factor $\Omega = \cos T + \cos R$ vanishes. The part of the boundary without the points at $R = 0, \pi$ is denoted by $\mathscr{I} = \{T = \pm(\pi - R),\ R \in (0, \pi)\}$. This part is referred to as null infinity because null geodesics reach it for an infinite value of their affine parameter. The differential of the conformal factor is nonvanishing at $\mathscr{I}$, $d\Omega|_{\mathscr{I}} \neq 0$, and $\mathscr{I}$ consists of two parts $\mathscr{I}^-$ and $\mathscr{I}^+$ referred to as past and future null infinity.

Hyperboloidal coordinates and wave equations. Equipped with the tools above we now turn to the interplay between wave equations and spacetime geometry. Consider the free wave equation

$$u_{tt} - \Delta u = -\eta^{\mu\nu}\, \partial_\mu \partial_\nu u = 0. \tag{2-2}$$

Radial solutions for the rescaled field $v := ru$ obey the two-dimensional free wave equation

$$v_{tt} - v_{rr} = 0, \tag{2-3}$$

on $(t, r) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ with vanishing boundary condition at the origin. Initial data are specified on the $t = 0$ hypersurface. The general solution to this system is such that the data propagate to infinity and leave nothing behind due to the validity of Huygens' principle. Intuitively, this behavior seems to contradict two well-known properties of the free wave equation: *conservation of energy* and *time reversibility*.

The conserved energy for the free wave equation (2-3) reads

$$E(v) = \int_0^\infty \tfrac{1}{2}\big(v_t(t, r)^2 + v_r(t, r)^2\big)\, dr.$$

The conservation of energy is counterintuitive because the waves propagate to infinity leaving nothing behind. One would expect a natural energy norm to decrease rapidly to zero with a nonpositive energy flux at infinity. The conservation of energy, however, implies that at very late times the solution is in some sense similar to the initial state [Tao 2008].

Another counterintuitive property of the free wave equation is its time reversibility, meaning that if $u(t, r)$ solves the equation, so does $u(-t, r)$. Data on a Cauchy hypersurface determine the solution at all future *and past* times in contrast to parabolic (dissipative) equations which are solvable only forward in time due to loss of energy to the future.

Both of these counterintuitive properties depend on our description of the problem. We can choose coordinates in which energy conservation and time reversibility are violated. Of course, it is always possible to find coordinates which break symmetries or hide features of an equation. We argue below that the hyperboloidal coordinates we employ emphasize the intuitive properties of the equation rather than blur them.

The reason behind the conservation of energy integrated along level sets of t can be seen in the Penrose diagram Figure 2. The outgoing characteristic line along which the wave propagates to infinity intersects all leaves of the t-foliation. When the energy expression is integrated globally, the energy of the initial wave will therefore still contribute to the result. The hyperboloidal T-foliation depicted in Figure 1, however, allows for outgoing null rays to leave the leaves of the foliation. Therefore one would expect that the energy flux through infinity is negative when integrated along the leaves of the hyperboloidal foliation.

The wave equation (2-3) has the same form in hyperboloidal coordinates:

$$w_{TT} - w_{RR} = 0,$$

where

$$w(T, R) = v\left(-\frac{T}{T^2 - R^2}, \frac{R}{T^2 - R^2}\right).$$

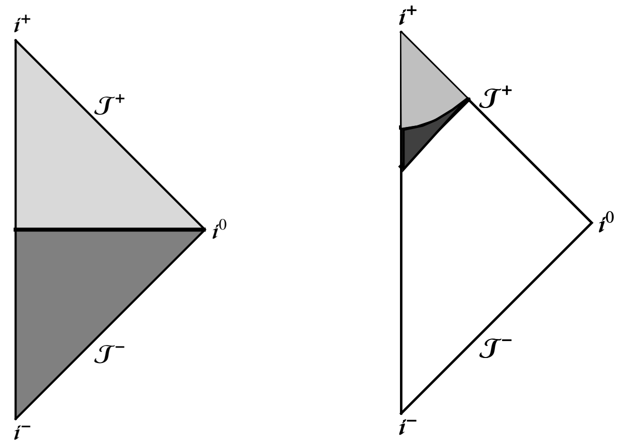

Figure 3. Comparison of the future (light gray) and past (dark gray) domains of dependence for the Cauchy surface $t = 0$ (left) and the hyperboloidal surface $T = -1$ (right).

Energy conservation and time reversibility seem valid for this equation as well, but here we have the shrinking, bounded spatial domain $R \in [0, -T)$ where $T \to 0-$. The energy integrated along the leaves of this domain

$$E(w) = \int_0^{-T} \tfrac{1}{2}\big(w_T(T, R)^2 + w_R(T, R)^2\big)\, dR$$

decays in time. The energy flux reads

$$\frac{\partial E}{\partial T} = -\tfrac{1}{2}\big(w_T(T, -T) - w_R(T, -T)\big)^2 \leq 0.$$

The energy flux through infinity vanishes only if the solution is constant or is propagating along future null infinity. When the solution has an outgoing component through future null infinity, the energy decays in time. This behavior is in accordance with physical intuition.

Consider the time reversibility. The equation in the new coordinates is time-reversible, but the hyperboloidal initial value problem is not. Formally, this is again a consequence of the time dependence of the spatial domain given by $R < -T$. Geometrically, we see in Figure 3 that the union of the past and future domain of dependence of the hyperboloidal surface $T = -1$ covers only a portion of Minkowski spacetime whereas for the Cauchy surface $t = 0$ such a union gives the global spacetime.

In summary, the hyperboloidal foliation given by the Kelvin inversion captures quantitatively the propagation of energy to infinity and leads to a time-irreversible wave propagation problem. Further, the transformation translates asymptotic analysis for $t \to \infty$ to local analysis for $T \to 0-$.

3. Derivation of the equations and preliminaries

First-order formulation and similarity coordinates. We start from $(-\partial_T^2 + \Delta_X)u(T, X) + u(T, X)^3 = 0$ in the hyperboloidal coordinates $T = -t/(t^2 - |x|^2)$, $X = x/(t^2 - |x|^2)$ for the rescaled unknown

$$u(T,X)=\frac{1}{T^2-|X|^2}v\left(-\frac{T}{T^2-|X|^2},\frac{X}{T^2-|X|^2}\right).$$

As discussed in the introduction, the domain we are interested in is $T\in[-1,0)$ and $|X|<|T|$. Our intention is to study the stability of the self-similar solution $u_0(T)=\sqrt{2}/(-T)$. Thus, it is natural to introduce the similarity coordinates

$$\tau=-\log(-T),\quad \xi=\frac{X}{-T} \tag{3-1}$$

with domain $\tau\geq 0$ and $|\xi|<1$. The derivatives transform according to

$$\partial_T=e^\tau(\partial_\tau+\xi^j\partial_{\xi^j}),\quad \partial_{X^j}=e^\tau\partial_{\xi^j}.$$

This implies

$$\partial_T^2=e^{2\tau}(\partial_\tau^2+\partial_\tau+2\xi^j\partial_{\xi^j}\partial_\tau+\xi^j\xi^k\partial_{\xi^j}\partial_{\xi^k}+2\xi^j\partial_{\xi^j})$$

and $\partial_{X_j}\partial_{X^j}=e^{2\tau}\partial_{\xi_j}\partial_{\xi^j}$. Consequently, for the function

$$U(\tau,\xi):=u(-e^{-\tau},e^{-\tau}\xi)$$

we obtain from $(-\partial_T^2+\partial_{X_j}\partial_{X^j})u(T,X)+u(T,X)^3=0$ the equation

$$[\partial_\tau^2+\partial_\tau+2\xi^j\partial_{\xi^j}\partial_\tau-(\delta^{jk}-\xi^j\xi^k)\partial_{\xi^j}\partial_{\xi^k}+2\xi^j\partial_{\xi^j}]U(\tau,\xi)=e^{-2\tau}U(\tau,\xi)^3.$$

To get rid of the time-dependent prefactor on the right-hand side, we rescale and set $U(\tau,\xi)=e^\tau\psi(\tau,\xi)$, which yields

$$[\partial_\tau^2+3\partial_\tau+2\xi^j\partial_{\xi^j}\partial_\tau-(\delta^{jk}-\xi^j\xi^k)\partial_{\xi^j}\partial_{\xi^k}+4\xi^j\partial_{\xi^j}+2]\psi(\tau,\xi)=\psi(\tau,\xi)^3. \tag{3-2}$$

The fundamental self-similar solution is given by

$$\psi_0(\tau,\xi):=e^{-\tau}u_0(-e^{-\tau},e^{-\tau}\xi)=\sqrt{2}.$$

Writing $\psi=\sqrt{2}+\phi$ we find the equation

$$[\partial_\tau^2+3\partial_\tau+2\xi^j\partial_{\xi^j}\partial_\tau-(\delta^{jk}-\xi^j\xi^k)\partial_{\xi^j}\partial_{\xi^k}+4\xi^j\partial_{\xi^j}+2]\phi(\tau,\xi)$$
$$=6\phi(\tau,\xi)+3\sqrt{2}\phi(\tau,\xi)^2+\phi(\tau,\xi)^3. \tag{3-3}$$

In summary, we have applied the coordinate transformation

$$\tau=-\log\frac{t}{t^2-|x|^2},\quad \xi=\frac{x}{t}$$

with inverse

$$t=\frac{e^\tau}{1-|\xi|^2},\quad x=\frac{e^\tau\xi}{1-|\xi|^2}$$

and $\phi(\tau,\xi)$ solves (3-3) for $\tau>0$ and $|\xi|<1$ if and only if

$$v(t,x)=\frac{\sqrt{2}}{t}+\frac{1}{t}\phi\left(-\log\frac{t}{t^2-|x|^2},\frac{x}{t}\right) \tag{3-4}$$

solves $(-\partial_t^2+\Delta_x)v(t,x)+v(t,x)^3=0$ for $(t,x)\in D^+(\Sigma_{-1})$.

We have $\partial_T u(T,X)=e^{2\tau}(\partial_\tau+\xi^j\partial_{\xi^j}+1)\psi(\tau,\xi)$ and thus, it is natural to use the variables $\phi_1=\phi$, $\phi_2=\partial_0\phi+\xi^j\partial_j\phi+\phi$ in a first-order formulation. We obtain

$$\begin{aligned}\partial_0\phi_1&=-\xi^j\partial_j\phi_1-\phi_1+\phi_2,\\ \partial_0\phi_2&=\partial_j\partial^j\phi_1-\xi^j\partial_j\phi_2-2\phi_2+6\phi_1+3\sqrt{2}\phi_1^2+\phi_1^3.\end{aligned}\tag{3-5}$$

For later reference we also note that (3-4) implies

$$t^2\partial_t v(t,x)=-\sqrt{2}-\frac{2t^2}{t^2-|x|^2}\left(\frac{x^j}{t}\partial_j\phi_1+\phi_1\right)+\frac{t^2+|x|^2}{t^2-|x|^2}\phi_2,\tag{3-6}$$

where it is understood, of course, that $\phi_1(\tau,\xi)$ and $\phi_2(\tau,\xi)$ are evaluated at $\tau=-\log\frac{t}{t^2-|x|^2}$ and $\xi=x/t$.

Norms. Since our approach is perturbative in nature, the function space in which we study (3-5) should be determined by the *free* version of (3-5), i.e.,

$$\begin{aligned}\partial_0\phi_1&=-\xi^j\partial_j\phi_1-\phi_1+\phi_2,\\ \partial_0\phi_2&=\partial_j\partial^j\phi_1-\xi^j\partial_j\phi_2-2\phi_2.\end{aligned}$$

The natural choice for a norm is derived from the standard energy $\dot H^1\times L^2$ of the free wave equation. In the present formulation this translates into

$$\|\phi_1(\tau,\cdot)\|_{\dot H^1(B)}+\|\phi_2(\tau,\cdot)\|_{L^2(B)},$$

where $B=\{\xi\in\mathbb{R}^3:|\xi|<1\}$. However, there is a slight technical problem since this is only a seminorm (the point is that we are working on the bounded domain B). In order to go around this difficulty, let us for the moment return to the radial context and consider the free wave equation in $\mathbb{R}^{1+3}$

$$u_{tt}-u_{rr}-\frac{2}{r}u_r=0,$$

in the standard coordinates t and $r=|x|$. Now we make the following observation. The conserved energy is given by

$$E(u)=\frac{1}{2}\int_0^\infty[u_t^2+u_r^2]r^2\,dr.$$

On the other hand, by setting $v=ru$, we obtain

$$v_{tt}-v_{rr}=0$$

with conserved energy $\frac{1}{2}\int_0^\infty[v_t^2+v_r^2]\,dr$, or, in terms of u,

$$E'(u)=\frac{1}{2}\int_0^\infty[r^2u_t^2+(ru_r+u)^2]\,dr.$$

The obvious question now is: how are E and E' related? An integration by parts shows that E and E' are equivalent, up to a boundary term $\lim_{r\to\infty}ru(r)^2$ which may be ignored by assuming some decay at spatial infinity. However, if we consider the *local* energy contained in a ball of radius R, the boundary term can no longer be ignored and one has the identity

$$E'_R(u) := \frac{1}{2}\int_0^R [r^2u_t^2 + (ru_r+u)^2]\,dr = \frac{1}{2}Ru(R)^2 + \frac{1}{2}\int_0^R [u_t^2+u_r^2]r^2\,dr.$$

The expression on the right-hand side is the standard energy with the term $\frac{1}{2}Ru(R)^2$ added. This small modification has important consequences because unlike the standard energy, this now defines a norm. Furthermore, $E'_R(u)$ is bounded along the wave flow since it is the local version of a positive definite conserved quantity.

In the nonradial context the above discussion suggests to take

$$\|\phi_1(\tau,\cdot)\|_{\dot H^1(B)} + \|\phi_1(\tau,\cdot)\|_{L^2(\partial B)} + \|\phi_2(\tau,\cdot)\|_{L^2(B)}.$$

This norm is not very handy, but fortunately we have equivalence to $H^1\times L^2(B)$ as the following result shows.

Lemma 3.1. *We have*[1]

$$\|f\|_{H^1(B)} \simeq \|f\|_{\dot H^1(B)} + \|f\|_{L^2(\partial B)}.$$

Proof. For $x\in\mathbb{R}^3$ we write $r=|x|$ and $\omega = x/|x|$. With this notation we have $f(x)=f(r\omega)$ and

$$\|f\|^2_{L^2(B)} = \int_0^1\int_{\partial B}|f(r\omega)|^2\,d\sigma(\omega)r^2\,dr,$$

where $d\sigma$ denotes the surface measure on the sphere. First, we prove $\|f\|_{L^2(B)} \lesssim \|f\|_{\dot H^1(B)} + \|f\|_{L^2(\partial B)}$. By density it suffices to consider $f\in C^\infty(\bar B)$. The fundamental theorem of calculus and Cauchy–Schwarz imply

$$rf(r\omega) = \int_0^r \partial_s[sf(s\omega)]\,ds \le \left(\int_0^1 |\partial_r[rf(r\omega)]|^2\,dr\right)^{1/2}.$$

Expanding the square and integrating by parts yields

$$\begin{aligned}\int_0^1 |\partial_r[rf(r\omega)]|^2\,dr &= \int_0^1 |\partial_r f(r\omega)|^2r^2\,dr + \int_0^1 r\,\partial_r|f(r\omega)|^2\,dr + \int_0^1 |f(r\omega)|^2\,dr\\ &= |f(\omega)|^2 + \int_0^1 |\partial_r f(r\omega)|^2 r^2\,dr\end{aligned}$$

and thus,

$$r^2|f(r\omega)|^2 \le |f(\omega)|^2 + \int_0^1 |\omega^j\partial_j f(r\omega)|^2r^2\,dr.$$

Integrating this inequality over the ball B yields the desired estimate. In order to finish the proof, it suffices to show that $\|f\|_{L^2(\partial B)} \lesssim \|f\|_{H^1(B)}$, but this is just the trace theorem (see, e.g., [Evans 1998, p. 258, Theorem 1]). $\square$

[1]As usual, $\|f\|_{L^2(\partial B)}$ has to be understood in the trace sense.

4. Linear perturbation theory

The goal of this section is to develop a functional analytic framework for studying the Cauchy problem for the *linearized* equation

$$\begin{aligned}\partial_0\phi_1 &= -\xi^j\partial_j\phi_1 - \phi_1 + \phi_2,\\ \partial_0\phi_2 &= \partial_j\partial^j\phi_1 - \xi^j\partial_j\phi_2 - 2\phi_2 + 6\phi_1.\end{aligned} \tag{4-1}$$

The main difficulty lies with the fact that the differential operators involved are not self-adjoint. It is thus natural to apply semigroup theory for studying (4-1). Before doing so, however, we commence with a heuristic discussion on instabilities. The equation $(-\partial_T^2 + \Delta_X)u(T,X) + u(T,X)^3 = 0$ is invariant under time translations $T \mapsto T - a$ and the three Lorentz boosts for each direction X^j

$$\begin{cases} T & \mapsto & T\cosh a - X^j \sinh a,\\ X^j & \mapsto & -T\sinh a + X^j\cosh a,\\ X^k & \mapsto & X^k \quad (k \neq j),\end{cases}$$

where $a \in \mathbb{R}$ is a parameter (the rapidity in case of the Lorentz boost). In general, if u_a is a one-parameter family of solutions to a nonlinear equation $F(u_a) = 0$, one obtains (at least formally)

$$0 = \partial_a F(u_a) = DF(u_a)\partial_a u_a$$

and thus, $\partial_a u_a$ is a solution of the linearization of $F(u) = 0$ at $u = u_a$. In our case we linearize around the solution $u_0(T,X) = \sqrt{2}/(-T)$. The time translation symmetry yields the one-parameter family $u_a(T,X) := \sqrt{2}/(a-T)$ and we have $\partial_a u_a(T,X)|_{a=0} = -\sqrt{2}/T^2$. Taking into account the above transformations that led from u to ϕ_1, ϕ_2, we obtain (after a suitable normalization) the functions

$$\phi_1(\tau,\xi) = e^\tau, \quad \phi_2(\tau,\xi) = 2e^\tau, \tag{4-2}$$

and a simple calculation shows that (4-2) indeed solve (4-1). Thus, there exists a growing solution of (4-1). Similarly, for the Lorentz boosts we consider

$$u_{a,j}(T,X) = \frac{\sqrt{2}}{X^j\sinh a - T\cosh a}$$

and thus, $\partial_a u_{a,j}(T,X)|_{a=0} = -(\sqrt{2}/T^2)X^j$. By recalling that $X^j/(-T) = \xi^j$, this yields the functions

$$\phi_1(\tau,\xi) = \xi^j, \quad \phi_2(\tau,\xi) = 2\xi^j, \tag{4-3}$$

and it is straightforward to check that (4-3) indeed solve (4-1). This time the solution (4-3) is not growing in τ but it is not decaying either. It is important to emphasize that in our context, the time translation symmetry leads to a *real* instability. The reason is that $u_a(T,X) = \sqrt{2}/(a-T)$ yields the solution

$$v_a(t,x) = \frac{1}{t^2-|x|^2}\,\frac{\sqrt{2}}{a + \frac{t}{t^2-|x|^2}} = \frac{\sqrt{2}}{t + a(t^2-|x|^2)}$$

of the original problem. This solution is part of a two-parameter family conjectured to describe generic radial solutions of the focusing cubic wave equation [Bizoń and Zenginoğlu 2009]. If $a \neq 0$, $v_a(t,x)$ decays like t^{-2} as $t \to \infty$ for each fixed $x \in \mathbb{R}^3$. This is the generic (dispersive) decay. On the other hand, the Lorentz transforms lead to apparent instabilities since the function $u_{a,j}$ yields the solution $v_{a,j}(t,x) = \sqrt{2}/(t \cosh a + x^j \sinh a)$ of the original problem which still displays the nondispersive decay. Consequently, we expect a codimension-one manifold of initial data that lead to nondispersive decay, as mentioned in the introduction. Since we are working with a fixed u_0, however, there is a four-dimensional unstable subspace of the linearized operator (to be defined below). This observation eventually leads to the codimension-4 statement in our Theorem 1.1. Note that other symmetries of the equation such as scaling, space translations, and space rotations do not play a role in this context as the solution u_0 is invariant under these.

A semigroup formulation for the free evolution. We start the rigorous treatment by considering the free wave equation in similarity coordinates given by the system

$$\begin{aligned} \partial_0\phi_1 &= -\xi^j\partial_j\phi_1 - \phi_1 + \phi_2, \\ \partial_0\phi_2 &= \partial_j\partial^j\phi_1 - \xi^j\partial_j\phi_2 - 2\phi_2. \end{aligned} \tag{4-4}$$

From (4-4) we read off the generator

$$\tilde{\boldsymbol{L}}_0\boldsymbol{u}(\xi) = \begin{pmatrix} -\xi^j\partial_j u_1(\xi) - u_1(\xi) + u_2(\xi) \\ \partial_j\partial^j u_1(\xi) - \xi^j\partial_j u_2(\xi) - 2u_2(\xi) \end{pmatrix},$$

acting on functions in $\mathfrak{D}(\tilde{\boldsymbol{L}}_0) := H^2(B) \cap C^2(\bar{B}\backslash\{0\}) \times H^1(B) \cap C^1(\bar{B}\backslash\{0\})$. With this notation we rewrite (4-4) as an ODE

$$\frac{d}{d\tau}\Phi(\tau) = \tilde{\boldsymbol{L}}_0\Phi(\tau).$$

The appropriate framework for studying such a problem is provided by semigroup theory, i.e., our goal is to find a suitable Hilbert space $\mathcal{H}$ such that there exists a map $\boldsymbol{S}_0 : [0,\infty) \to \mathcal{B}(\mathcal{H})$ satisfying

- $\boldsymbol{S}_0(0) = \mathrm{id}_{\mathcal{H}}$,
- $\boldsymbol{S}_0(\tau)\boldsymbol{S}_0(\sigma) = \boldsymbol{S}_0(\tau+\sigma)$ for all $\tau, \sigma \geq 0$,
- $\lim_{\tau\to 0+} \boldsymbol{S}_0(\tau)\boldsymbol{u} = \boldsymbol{u}$ for all $\boldsymbol{u} \in \mathcal{H}$,
- $\lim_{\tau\to 0+}(1/\tau)[\boldsymbol{S}_0(\tau)\boldsymbol{u} - \boldsymbol{u}] = \boldsymbol{L}_0\boldsymbol{u}$ for all $\boldsymbol{u} \in \mathfrak{D}(\boldsymbol{L}_0)$, where $\boldsymbol{L}_0$ is the closure of $\tilde{\boldsymbol{L}}_0$.

Given such an $\boldsymbol{S}_0$, the function $\Phi(\tau) = \boldsymbol{S}_0(\tau)\Phi(0)$ solves $d\Phi(\tau)/d\tau = \boldsymbol{L}_0\Phi(\tau)$.

Motivated by the above discussion we define a sesquilinear form on $\tilde{\mathcal{H}} := H^1(B) \cap C^1(B) \times L^2(B) \cap C(B)$ by

$$(\boldsymbol{u} \mid \boldsymbol{v}) := \int_B \partial_j u_1(\xi)\overline{\partial^j v_1(\xi)}\, d\xi + \int_{\partial B} u_1(\omega)\overline{v_1(\omega)}\, d\sigma(\omega) + \int_B u_2(\xi)\overline{v_2(\xi)}\, d\xi.$$

Lemma 3.1 implies that $(\cdot \mid \cdot)$ is an inner product on $\tilde{\mathcal{H}}$, and as usual we denote the induced norm by $\|\cdot\|$. Furthermore, we write $\mathcal{H}$ for the completion of $\tilde{\mathcal{H}}$ with respect to $\|\cdot\|$. We remark that $\mathcal{H}$ is equivalent to $H^1(B) \times L^2(B)$ as a Banach space by Lemma 3.1.

Proposition 4.1. *The operator $\tilde{\boldsymbol{L}}_0 : \mathcal{D}(\tilde{\boldsymbol{L}}_0) \subset \mathcal{H} \to \mathcal{H}$ is closable and its closure, denoted by $\boldsymbol{L}_0$, generates a strongly continuous semigroup $\boldsymbol{S}_0 : [0,\infty) \to \mathcal{B}(\mathcal{H})$ satisfying $\|\boldsymbol{S}_0(\tau)\| \le e^{-\frac{1}{2}\tau}$ for all $\tau \ge 0$. In particular, we have $\sigma(\boldsymbol{L}_0) \subset \{z \in \mathbb{C} : \operatorname{Re} z \le -\frac{1}{2}\}$.*

The proof of Proposition 4.1 requires the following technical lemma.

Lemma 4.2. *Let $f \in L^2(B)$ and $\varepsilon > 0$ be arbitrary. Then there exists a function $u \in H^2(B) \cap C^2(\bar{B}\backslash\{0\})$ such that $g \in L^2(B) \cap C(\bar{B}\backslash\{0\})$, defined by*

$$g(\xi) := -(\delta^{jk} - \xi^j \xi^k)\partial_j \partial_k u(\xi) + 5\xi^j \partial_j u(\xi) + \tfrac{15}{4} u(\xi), \tag{4-5}$$

satisfies $\|f - g\|_{L^2(B)} < \varepsilon$.

Proof. Since $C^\infty(\bar{B}) \subset L^2(B)$ is dense, we can find a $\tilde{g} \in C^\infty(\bar{B})$ such that $\|f - \tilde{g}\|_{L^2(B)} < \varepsilon/2$. We consider the equation

$$-(\delta^{jk} - \xi^j \xi^k)\partial_j \partial_k u(\xi) + 5\xi^j \partial_j u(\xi) + \tfrac{15}{4} u(\xi) = \tilde{g}(\xi). \tag{4-6}$$

In order to solve (4-6) we define $\rho(\xi) = |\xi|$, $\omega(\xi) = \xi/|\xi|$ and note that

$$\partial_j \rho(\xi) = \omega_j(\xi), \quad \partial_j \omega^k(\xi) = \frac{\delta_j{}^k - \omega_j(\xi)\omega^k(\xi)}{\rho(\xi)}.$$

Thus, interpreting ρ and ω as new coordinates, we obtain

$$\begin{aligned} \xi^j \partial_j u(\xi) &= \rho \partial_\rho u(\rho\omega), \\ \xi^j \xi^k \partial_j \partial_k u(\xi) &= \xi^j \partial_{\xi^j}[\xi^k \partial_{\xi^k} u(\xi)] - \xi^j \partial_j u(\xi) = \rho^2 \partial_\rho^2 u(\rho\omega) \end{aligned}$$

as well as

$$\partial^j \partial_j u(\rho\omega) = \left[\partial_\rho^2 + \frac{d-1}{\rho}\partial_\rho + \frac{\delta^{jk} - \omega^j \omega^k}{\rho^2}\partial_{\omega^j}\partial_{\omega^k} - \frac{d-1}{\rho^2}\omega^j \partial_{\omega^j}\right] u(\rho\omega),$$

where $d = 3$ is the spatial dimension. Consequently, (4-6) can be written as

$$\left[-(1-\rho^2)\partial_\rho^2 - \frac{2}{\rho}\partial_\rho + 5\rho\partial_\rho + \frac{15}{4} - \frac{1}{\rho^2}\Delta_{S^2}\right] u(\rho\omega) = \tilde{g}(\rho\omega), \tag{4-7}$$

where $-\Delta_{S^2}$ is the Laplace–Beltrami operator on S^2. The operator $-\Delta_{S^2}$ is self-adjoint on $L^2(S^2)$ and we have $\sigma(-\Delta_{S^2}) = \sigma_p(-\Delta_{S^2}) = \{\ell(\ell+1) : \ell \in \mathbb{N}_0\}$. The eigenspace to the eigenvalue $\ell(\ell+1)$ is $(2\ell+1)$-dimensional and spanned by the spherical harmonics $\{Y_{\ell,m} : m \in \mathbb{Z}, -\ell \le m \le \ell\}$ which are obtained by restricting harmonic homogeneous polynomials in $\mathbb{R}^3$ to the two-sphere S^2; see, for example, [Atkinson and Han 2012] for an up-to-date account of this classical subject. We may expand $\tilde{g}$ according to

$$\tilde{g}(\rho\omega) = \sum_{\ell,m}^{\infty} g_{\ell,m}(\rho) Y_{\ell,m}(\omega),$$

where $\sum_{\ell,m}^{\infty}$ is shorthand for $\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}$ and for any fixed $\rho\in[0,1]$, the sum converges in $L^2(S^2)$; see [Atkinson and Han 2012, p. 66, Theorem 2.34]. The expansion coefficient $g_{\ell,m}(\rho)$ is given by

$$g_{\ell,m}(\rho)=(\tilde g(\rho\,\cdot\,)\mid Y_{\ell,m})_{L^2(S^2)}:=\int_{S^2}\tilde g(\rho\omega)\overline{Y_{\ell,m}(\omega)}\,d\sigma(\omega)$$

and by dominated convergence it follows that $g_{\ell,m}\in C^\infty[0,1]$. Furthermore, by using the identity $Y_{\ell,m}=[\ell(\ell+1)]^{-1}(-\Delta_{S^2})Y_{\ell,m}$ and the self-adjointness of $-\Delta_{S^2}$ on $L^2(S^2)$, we obtain

$$\begin{aligned}g_{\ell,m}(\rho)&=\frac{1}{\ell(\ell+1)}\big(\tilde g(\rho\,\cdot\,)\,\big|\,(-\Delta_{S^2})Y_{\ell,m}\big)_{L^2(S^2)}\\&=\frac{1}{\ell(\ell+1)}\big((-\Delta_{S^2})\tilde g(\rho\,\cdot\,)\,\big|\,Y_{\ell,m}\big)_{L^2(S^2)}.\end{aligned}$$

Consequently, by iterating this argument we see that the smoothness of $\tilde g$ implies the pointwise decay $\|g_{\ell,m}\|_{L^\infty(0,1)}\le C_M\ell^{-M}$ for any $M\in\mathbb{N}$ and all $\ell\in\mathbb{N}$. Now we set

$$g_N(\xi):=\sum_{\ell,m}^{N}g_{\ell,m}(|\xi|)Y_{\ell,m}\Big(\frac{\xi}{|\xi|}\Big)$$

and note that $\|g_N(\rho\,\cdot\,)-\tilde g(\rho\,\cdot\,)\|_{L^2(S^2)}\to 0$ as $N\to\infty$. Furthermore, by $\|g_{\ell,m}\|_{L^\infty(0,1)}\lesssim\ell^{-2}$ for all $\ell\in\mathbb{N}$ we infer

$$\sup_{\rho\in(0,1)}\|g_N(\rho\,\cdot\,)-\tilde g(\rho\,\cdot\,)\|_{L^2(S^2)}\lesssim 1$$

for all $N\in\mathbb{N}$ and dominated convergence yields

$$\|g_N-\tilde g\|^2_{L^2(B)}=\int_0^1\|g_N(\rho\,\cdot\,)-\tilde g(\rho\,\cdot\,)\|^2_{L^2(S^2)}\rho^2\,d\rho\to 0$$

as $N\to\infty$. Thus, we may choose N so large that $\|g_N-\tilde g\|_{L^2(B)}<\varepsilon/2$.

By making the ansatz $u(\rho\omega)=\sum_{\ell,m}^{N}u_{\ell,m}(\rho)Y_{\ell,m}(\omega)$ we derive from (4-7) the (decoupled) system

$$\Big[-(1-\rho^2)\partial_\rho^2-\frac{2}{\rho}\partial_\rho+5\rho\partial_\rho+\frac{15}{4}+\frac{\ell(\ell+1)}{\rho^2}\Big]u_{\ell,m}(\rho)=g_{\ell,m}(\rho)\tag{4-8}$$

for $\ell\in\mathbb{N}_0$, $\ell\le N$, and $-\ell\le m\le\ell$. Equation (4-8) has regular singular points at $\rho=0$ and $\rho=1$ with Frobenius indices $\{\ell,-\ell-1\}$ and $\{0,-\frac12\}$, respectively. In fact, solutions to (4-8) can be given in terms of hypergeometric functions. In order to see this, define a new variable $v_{\ell,m}$ by $u_{\ell,m}(\rho)=\rho^\ell v_{\ell,m}(\rho^2)$. Then, (4-8) with $g_{\ell,m}=0$ is equivalent to

$$z(1-z)v''_{\ell,m}(z)+[c-(a+b+1)z]v'_{\ell,m}(z)-abv_{\ell,m}(z)=0\tag{4-9}$$

with $a=\frac12\big(\frac32+\ell\big)$, $b=a+\frac12$, $c=\frac32+\ell$, and $z=\rho^2$. We immediately obtain the two solutions

$$\phi_{0,\ell}(z)={}_2F_1\Big(\frac{3+2\ell}{4},\frac{5+2\ell}{4},\frac{3+2\ell}{2};z\Big),\quad \phi_{1,\ell}(z)={}_2F_1\Big(\frac{3+2\ell}{4},\frac{5+2\ell}{4},\frac{3}{2};1-z\Big),$$

where ${}_2F_1$ is the standard hypergeometric function; see [Olver et al. 2010; Kristensson 2010]. For later reference we also state a third solution, $\tilde{\phi}_{1,\ell}$, given by

$$\tilde{\phi}_{1,\ell}(z) = (1-z)^{-\frac{1}{2}} {}_2F_1\Big(\frac{3+2\ell}{4}, \frac{1+2\ell}{4}, \frac{1}{2}; 1-z\Big). \tag{4-10}$$

Note that $\phi_{0,\ell}$ is analytic around $z=0$ whereas $\phi_{1,\ell}$ is analytic around $z=1$. As a matter of fact, $\phi_{1,\ell}$ can be represented in terms of elementary functions and we have

$$\phi_{1,\ell}(z) = \frac{1}{(2\ell+1)\sqrt{1-z}}\big[(1-\sqrt{1-z})^{-\ell-\frac{1}{2}} - (1+\sqrt{1-z})^{-\ell-\frac{1}{2}}\big]; \tag{4-11}$$

see [Olver et al. 2010]. This immediately shows that $|\phi_{1,\ell}(z)| \to \infty$ as $z \to 0+$ which implies that $\phi_{0,\ell}$ and $\phi_{1,\ell}$ are linearly independent. Transforming back, we obtain the two solutions $\psi_{j,\ell}(\rho) = \rho^\ell \phi_{j,\ell}(\rho^2)$, $j=0, 1$, of (4-8) with $g_{\ell,m}=0$. By differentiating the Wronskian $W(\psi_{0,\ell}, \psi_{1,\ell}) = \psi_{0,\ell}\psi_{1,\ell}' - \psi_{0,\ell}'\psi_{1,\ell}$ and inserting the equation, we infer

$$W(\psi_{0,\ell}, \psi_{1,\ell})'(\rho) = \Big(\frac{3\rho}{1-\rho^2} - \frac{2}{\rho}\Big) W(\psi_{0,\ell}, \psi_{1,\ell})(\rho),$$

which implies

$$W(\psi_{0,\ell}, \psi_{1,\ell})(\rho) = \frac{c_\ell}{\rho^2(1-\rho^2)^{\frac{3}{2}}} \tag{4-12}$$

for some constant c_ℓ. In order to determine the precise value of c_ℓ, we first note that

$$\psi_{j,\ell}'(\rho) = 2\rho^{\ell+1}\phi_{j,\ell}'(\rho^2) + \ell\rho^{\ell-1}\phi_{j,\ell}(\rho^2).$$

For the following we recall the differentiation formula [Olver et al. 2010]

$$\frac{d}{dz}{}_2F_1(a, b, c; z) = \frac{ab}{c}{}_2F_1(a+1, b+1, c+1; z), \tag{4-13}$$

which is a direct consequence of the series representation of the hypergeometric function. Furthermore, by the formula [Olver et al. 2010]

$$\lim_{z\to 1-}[(1-z)^{a+b-c}{}_2F_1(a, b, c; z)] = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \tag{4-14}$$

valid for $\operatorname{Re}(a+b-c) > 0$, we obtain

$$\lim_{z\to 1-}[(1-z)^{\frac{1}{2}}\phi_{0,\ell}(z)] = \frac{\Gamma\big(\frac{3+2\ell}{2}\big)\Gamma\big(\frac{1}{2}\big)}{\Gamma\big(\frac{3+2\ell}{4}\big)\Gamma\big(\frac{5+2\ell}{4}\big)} = 2^{\ell+\frac{1}{2}} \tag{4-15}$$

as well as

$$\lim_{z\to 1-}[(1-z)^{\frac{3}{2}}\phi_{0,\ell}'(z)] = \frac{\Gamma\big(\frac{3+2\ell}{2}\big)\Gamma\big(\frac{3}{2}\big)}{\Gamma\big(\frac{3+2\ell}{4}\big)\Gamma\big(\frac{5+2\ell}{4}\big)} = 2^{\ell-\frac{1}{2}},$$

where we used the identity $\Gamma(x)\Gamma(x+\frac{1}{2})=\pi^{\frac{1}{2}}2^{1-2x}\Gamma(2x)$. This yields

$$\begin{aligned}c_\ell &= \rho^2(1-\rho^2)^{\frac{3}{2}}W(\psi_{0,\ell},\psi_{1,\ell})(\rho)\\ &= \rho^2(1-\rho^2)^{\frac{3}{2}}\rho^\ell\phi_{0,\ell}(\rho^2)[2\rho^{\ell+1}\phi'_{1,\ell}(\rho^2)+\ell\rho^{\ell-1}\phi_{1,\ell}(\rho^2)]\\ &\qquad\qquad -\rho^2(1-\rho^2)^{\frac{3}{2}}\rho^\ell\phi_{1,\ell}(\rho^2)[2\rho^{\ell+1}\phi'_{0,\ell}(\rho^2)+\ell\rho^{\ell-1}\phi_{0,\ell}(\rho^2)]\\ &= -2\lim_{\rho\to 1-}(1-\rho^2)^{\frac{3}{2}}\phi'_{0,\ell}(\rho^2)\\ &= -2^{\ell+\frac{1}{2}}.\end{aligned}$$

By the variation of constants formula, a solution to (4-8) is given by

$$u_{\ell,m}(\rho)=-\psi_{0,\ell}(\rho)\int_\rho^1\frac{\psi_{1,\ell}(s)}{W(\psi_{0,\ell},\psi_{1,\ell})(s)}\frac{g_{\ell,m}(s)}{1-s^2}\,ds-\psi_{1,\ell}(\rho)\int_0^\rho\frac{\psi_{0,\ell}(s)}{W(\psi_{0,\ell},\psi_{1,\ell})(s)}\frac{g_{\ell,m}(s)}{1-s^2}\,ds. \quad (4\text{-}16)$$

We claim that $u_{\ell,m}\in C^2(0,1]$. By formally differentiating (4-16) we find

$$u''_{\ell,m}(\rho)=-\frac{g_{\ell,m}(\rho)}{1-\rho^2}-\psi''_{0,\ell}(\rho)I_{1,\ell}(\rho)-\psi''_{1,\ell}(\rho)I_{0,\ell}(\rho),$$

where $I_{j,\ell}$, $j=0,1$, denote the respective integrals in (4-16). This implies $u_{\ell,m}\in C^2(0,1)$ but $u''_{\ell,m}(\rho)$ has an apparent singularity at $\rho=1$. We have the asymptotics $\psi''_{0,\ell}(\rho)I_{1,\ell}(\rho)\simeq(1-\rho)^{-1}$ and $\psi''_{1,\ell}(\rho)I_{0,\ell}(\rho)\simeq 1$ as $\rho\to 1-$. Thus, a necessary condition for $\lim_{\rho\to 1-}u''_{\ell,m}(\rho)$ to exist is

$$a_{\ell,m}:=\lim_{\rho\to 1-}[(1-\rho^2)\psi''_{0,\ell}(\rho)I_{1,\ell}(\rho)]=-g_{\ell,m}(1).$$

This limit can be computed by l'Hôpital's rule, i.e., we write

$$\begin{aligned}a_{\ell,m} &= \lim_{\rho\to 1-}\frac{I_{1,\ell}(\rho)}{[(1-\rho^2)\psi''_{0,\ell}(\rho)]^{-1}}\\ &= \lim_{\rho\to 1-}\frac{I'_{1,\ell}(\rho)}{-[(1-\rho^2)\psi''_{0,\ell}(\rho)]^{-2}[(1-\rho^2)\psi^{(3)}_{0,\ell}(\rho)-2\rho\psi''_{0,\ell}(\rho)]}.\end{aligned}$$

We have

$$\lim_{\rho\to 1-}[(1-\rho^2)^{-\frac{1}{2}}I'_{1,\ell}(\rho)]=-\frac{1}{c_\ell}\lim_{\rho\to 1-}[\rho^2\psi_{1,\ell}(\rho)g_{\ell,m}(\rho)]=-\frac{g_{\ell,m}(1)}{c_\ell},$$

and thus it suffices to show that

$$-\frac{1}{c_\ell}=\lim_{\rho\to 1-}\frac{(1-\rho^2)\psi^{(3)}_{0,\ell}(\rho)-2\rho\psi''_{0,\ell}(\rho)}{(1-\rho^2)^{\frac{1}{2}}[(1-\rho^2)\psi''_{0,\ell}(\rho)]^2}=\lim_{\rho\to 1-}\frac{(1-\rho^2)^{\frac{7}{2}}\psi^{(3)}_{0,\ell}(\rho)-2\rho(1-\rho^2)^{\frac{5}{2}}\psi''_{0,\ell}(\rho)}{[(1-\rho^2)^{\frac{5}{2}}\psi''_{0,\ell}(\rho)]^2}. \quad (4\text{-}17)$$

Note that

$$\begin{aligned}\psi''_{0,\ell}(\rho)&=4\rho^{\ell+2}\phi''_{0,\ell}(\rho^2)+\text{lower order derivatives},\\ \psi^{(3)}_{0,\ell}(\rho)&=8\rho^{\ell+3}\phi^{(3)}_{0,\ell}(\rho^2)+\text{lower order derivatives}.\end{aligned}$$

Consequently, from the definition of $\phi_{0,\ell}$ and equations (4-13) and (4-14), we infer

$$\lim_{\rho\to 1-}[(1-\rho^2)^{\frac{5}{2}}\psi''_{0,\ell}(\rho)] = 4\lim_{z\to 1-}[(1-z)^{\frac{5}{2}}\phi''_{0,\ell}(z)] = -3c_\ell,$$
$$\lim_{\rho\to 1-}[(1-\rho^2)^{\frac{7}{2}}\psi^{(3)}_{0,\ell}(\rho)] = 8\lim_{z\to 1-}[(1-z)^{\frac{7}{2}}\phi^{(3)}_{0,\ell}(z)] = -15c_\ell,$$

which proves (4-17). We have $g_{\ell,m}\in C^\infty[0,1]$ and thus, in order to prove the claim $u_{\ell,m}\in C^2(0,1]$, it suffices to show that $\rho\mapsto(1-\rho^2)\psi''_{0,\ell}(\rho)I_{1,\ell}(\rho)$ belongs to $C^1(0,1]$. We write the integrand in $I_{1,\ell}$ as

$$\frac{\psi_{1,\ell}(s)}{W(\psi_{0,\ell},\psi_{1,\ell})(s)}\frac{g_{\ell,m}(s)}{1-s^2} = (1-s)^{\frac{1}{2}}O(1),$$

where in the following, $O(1)$ stands for a suitable function in $C^\infty(0,1]$. Consequently, we infer $I_{1,\ell}(\rho)=(1-\rho)^{\frac{3}{2}}O(1)$. We have $\psi_{0,\ell}=a_\ell\psi_{1,\ell}+\tilde a_\ell\tilde\psi_{1,\ell}$ where $\tilde\psi_{1,\ell}(\rho):=\rho^\ell\tilde\phi_{1,\ell}(\rho^2)$ — see (4-10) — and $a_\ell,\tilde a_\ell\in\mathbb{C}$ are suitable constants. This yields

$$\psi''_{0,\ell}(\rho)=(1-\rho)^{-\frac{5}{2}}O(1)+O(1)$$

and thus, $(1-\rho^2)\psi''_{0,\ell}(\rho)I_{1,\ell}(\rho)=O(1)+(1-\rho)^{\frac{5}{2}}O(1)$. Consequently, $\rho\mapsto(1-\rho^2)\psi''_{0,\ell}(\rho)I_{1,\ell}(\rho)$ belongs to $C^1(0,1]$ and by l'Hôpital's rule we infer $u_{\ell,m}\in C^2(0,1]$ as claimed.

Next, we turn to the endpoint $\rho=0$. The integrand of $I_{1,\ell}$ is bounded by $C_\ell\rho^{-\ell+1}$ and thus, we obtain

$$\begin{aligned}
&|I_{1,\ell}(\rho)|\lesssim 1 && \text{for } \ell\in\{0,1\},\\
&|I_{1,2}(\rho)|\lesssim|\log\rho|,\\
&|I_{1,\ell}(\rho)|\lesssim\rho^{-\ell+2} && \text{for } \ell\in\mathbb{N},\ 3\le\ell\le N,
\end{aligned}$$

for all $\rho\in(0,1]$. The integrand of $I_{0,\ell}$ is bounded by $C_\ell\rho^{\ell+2}$ and this implies $|I_{0,\ell}(\rho)|\lesssim\rho^{\ell+3}$ for all $\rho\in[0,1]$ and $\ell\in\mathbb{N}_0$, $\ell\le N$. Thus, we obtain for all $\rho\in(0,1]$ and $k\in\{0,1,2\}$ the estimates

$$\begin{aligned}
&|u^{(k)}_{0,m}(\rho)|\lesssim 1,\\
&|u^{(k)}_{1,m}(\rho)|\lesssim\rho^{\max\{1-k,0\}},\\
&|u^{(k)}_{2,m}(\rho)|\lesssim\rho^{2-k}|\log\rho|+\rho^{2-k},\\
&|u^{(k)}_{\ell,m}(\rho)|\lesssim\rho^{2-k}\quad\text{for } \ell\in\mathbb{N},\ 3\le\ell\le N.
\end{aligned}\tag{4-18}$$

Now we define the function $u:\bar B\backslash\{0\}\to\mathbb{C}$ by

$$u(\xi):=\sum_{\ell,m}^{N}u_{\ell,m}(|\xi|)Y_{\ell,m}\left(\frac{\xi}{|\xi|}\right).\tag{4-19}$$

From the bounds (4-18) we obtain[2] $|\partial_j\partial_k u(\xi)|\lesssim|\xi|^{-1}$ which implies $u\in H^2(B)\cap C^2(\bar B\backslash\{0\})$ and by construction, u satisfies

$$-(\delta^{jk}-\xi^j\xi^k)\partial_j\partial_k u(\xi)+5\xi^j\partial_j u(\xi)+\tfrac{15}{4}u(\xi)=g_N(\xi),$$

[2] Note that $Y_{0,0}(\omega)=1/\sqrt{4\pi}$.

where $\|f-g_N\|_{L^2(B)} \leq \|f-\tilde{g}\|_{L^2(B)} + \|\tilde{g}-g_N\|_{L^2(B)} < \varepsilon$. $\square$

Proof of Proposition 4.1. First note that $\tilde{\boldsymbol{L}}_0$ is densely defined. Furthermore, we claim that

$$\operatorname{Re}\,(\tilde{\boldsymbol{L}}_0\boldsymbol{u} \mid \boldsymbol{u}) \leq -\tfrac{1}{2}\|\boldsymbol{u}\|^2 \tag{4-20}$$

for all $\boldsymbol{u} \in \mathfrak{D}(\tilde{\boldsymbol{L}}_0)$. We write $[\tilde{\boldsymbol{L}}_0\boldsymbol{u}]_A$ for the A-th component of $\tilde{\boldsymbol{L}}_0\boldsymbol{u}$, where $A \in \{1,2\}$. Then we have

$$\partial_k[\tilde{\boldsymbol{L}}_0\boldsymbol{u}]_1(\xi) = -\xi^j\partial_j\partial_k u_1(\xi) - 2\partial_k u_1(\xi) + \partial_k u_2(\xi).$$

By noting that

$$\begin{aligned}\operatorname{Re}\,[\partial_j\partial_k u_1\overline{\partial^k u_1}] &= \tfrac{1}{2}\partial_j[\partial_k u_1\overline{\partial^k u_1}],\\ \xi^j\partial_j f(\xi) &= \partial_{\xi^j}[\xi^j f(\xi)] - 3f(\xi),\end{aligned}$$

we infer

$$\operatorname{Re}\,[\xi^j\partial_j\partial_k u_1(\xi)\overline{\partial^k u_1(\xi)}] = \tfrac{1}{2}\partial_{\xi^j}[\xi^j\partial_k u_1(\xi)\overline{\partial^k u_1(\xi)}] - \tfrac{3}{2}\partial_k u_1(\xi)\overline{\partial^k u_1(\xi)},$$

and the divergence theorem implies

$$\begin{aligned}&\operatorname{Re}\int_B \partial_k[\tilde{\boldsymbol{L}}_0\boldsymbol{u}]_1(\xi)\,\overline{\partial^k u_1(\xi)}\,d\xi\\ &\qquad= -\frac{1}{2}\int_{\partial B}\partial_k u_1(\omega)\,\overline{\partial^k u_1(\omega)}\,d\sigma(\omega) - \frac{1}{2}\int_B \partial_k u_1(\xi)\,\overline{\partial^k u_1(\xi)}\,d\xi + \operatorname{Re}\int_B \partial_k u_2(\xi)\,\overline{\partial^k u_1(\xi)}\,d\xi.\end{aligned}$$

Furthermore, we have

$$\int_B \partial_j\partial^j u_1(\xi)\overline{u_2(\xi)}\,d\xi = \int_{\partial B}\omega^j\partial_j u_1(\omega)\overline{u_2(\omega)}\,d\sigma(\omega) - \int_B \partial_j u_1(\xi)\overline{\partial^j u_2(\xi)}\,d\xi$$

and

$$\operatorname{Re}\int_B \xi^j\partial_j u_2(\xi)\overline{u_2(\xi)}\,d\xi = \frac{1}{2}\int_{\partial B}|u_2(\omega)|^2\,d\sigma(\omega) - \frac{3}{2}\int_B |u_2(\xi)|^2\,d\xi,$$

which yields

$$\begin{aligned}&\operatorname{Re}\int_B [\tilde{\boldsymbol{L}}_0\boldsymbol{u}]_2(\xi)\overline{u_2(\xi)}\,d\xi\\ &\qquad= \operatorname{Re}\int_{\partial B}\omega^j\partial_j u_1(\omega)\overline{u_2(\omega)}\,d\sigma(\omega) - \tfrac{1}{2}\|u_2\|^2_{L^2(\partial B)} - \operatorname{Re}\int_B \partial_j u_1(\xi)\overline{\partial^j u_2(\xi)}\,d\xi - \tfrac{1}{2}\|u_2\|^2_{L^2(B)}.\end{aligned}$$

In summary, we infer

$$\operatorname{Re}\,(\tilde{\boldsymbol{L}}_0\boldsymbol{u} \mid \boldsymbol{u}) = -\tfrac{1}{2}\|u_1\|^2_{\dot{H}^1(B)} - \tfrac{1}{2}\|u_2\|^2_{L^2(B)} + \int_{\partial B} A(\omega)\,d\sigma(\omega)$$

with

$$\begin{aligned}A(\omega) &= -\tfrac{1}{2}|u_1(\omega)|^2 - \tfrac{1}{2}|u_1(\omega)|^2 - \tfrac{1}{2}|\nabla u_1(\omega)|^2 - \tfrac{1}{2}|u_2(\omega)|^2\\ &\qquad - \operatorname{Re}\,[\omega^j\partial_j u_1(\omega)\overline{u_1(\omega)}] + \operatorname{Re}\,[\omega^j\partial_j u_1(\omega)\overline{u_2(\omega)}] + \operatorname{Re}\,[u_2(\omega)\overline{u_1(\omega)}]\\ &\leq -\tfrac{1}{2}|u_1(\omega)|^2,\end{aligned}$$

where we have used the inequality

$$\operatorname{Re}(\bar{a}b) + \operatorname{Re}(\bar{a}c) - \operatorname{Re}(\bar{b}c) \le \tfrac{1}{2}(|a|^2 + |b|^2 + |c|^2), \quad a, b, c \in \mathbb{C}$$

which follows from $0 \le |a - b - c|^2$. This proves (4-20).

The estimate (4-20) implies

$$\begin{aligned}\left\|[\lambda - (\tilde{\boldsymbol{L}}_0 + \tfrac{1}{2})]\boldsymbol{u}\right\|^2 &= \lambda^2\|\boldsymbol{u}\|^2 - 2\lambda \operatorname{Re}\big((\tilde{\boldsymbol{L}}_0 + \tfrac{1}{2})\boldsymbol{u} \,\big|\, \boldsymbol{u}\big) + \left\|(\tilde{\boldsymbol{L}}_0 + \tfrac{1}{2})\boldsymbol{u}\right\|^2 \\ &\ge \lambda^2\|\boldsymbol{u}\|^2\end{aligned}$$

for all $\lambda > 0$ and $\boldsymbol{u} \in \mathfrak{D}(\tilde{\boldsymbol{L}}_0)$. Thus, in view of the Lumer–Phillips theorem [Engel and Nagel 2000, p. 83, Theorem 3.15] it suffices to prove density of the range of $\lambda - \tilde{\boldsymbol{L}}_0$ for some $\lambda > -\frac{1}{2}$. Let $\boldsymbol{f} \in \mathcal{H}$ and $\varepsilon > 0$ be arbitrary. We consider the equation $(\lambda - \tilde{\boldsymbol{L}}_0)\boldsymbol{u} = \boldsymbol{f}$. From the first component we infer $u_2 = \xi^j\partial_j u_1 + (\lambda+1)u_1 - f_1$ and inserting this in the second component we arrive at the degenerate elliptic problem

$$-(\delta^{jk} - \xi^j\xi^k)\partial_j\partial_k u(\xi) + 2(\lambda+2)\xi^j\partial_j u(\xi) + (\lambda+1)(\lambda+2)u(\xi) = f(\xi) \tag{4-21}$$

for $u = u_1$ and $f(\xi) := \xi^j\partial_j f_1(\xi) + (\lambda+2)f_1(\xi) + f_2(\xi)$. Note that by assumption we have $f \in L^2(B)$. Setting $\lambda = \frac{1}{2}$ we infer from Lemma 4.2 the existence of functions $u \in H^2(B) \cap C^2(\bar{B}\backslash\{0\})$ and $g \in L^2(B)$ such that

$$-(\delta^{jk} - \xi^j\xi^k)\partial_j\partial_k u(\xi) + 5\xi^j\partial_j u(\xi) + \tfrac{15}{4}u(\xi) = g(\xi)$$

and $\|f - g\|_{L^2(B)} < \varepsilon$. We set $u_1 := u$, $u_2(\xi) := \xi^j\partial_j u(\xi) + \frac{3}{2}u - f_1$, $g_1 := f_1$, and $g_2(\xi) := g(\xi) - \xi^j\partial_j f_1(\xi) - \frac{5}{2}f_1(\xi)$. Then we have $\boldsymbol{u} \in \mathfrak{D}(\tilde{\boldsymbol{L}}_0)$, $\boldsymbol{g} \in \mathcal{H}$,

$$\|\boldsymbol{f} - \boldsymbol{g}\| = \|f_2 - g_2\|_{L^2(B)} = \|f - g\|_{L^2(B)} < \varepsilon$$

and by construction, $\big(\frac{1}{2} - \tilde{\boldsymbol{L}}_0\big)\boldsymbol{u} = \boldsymbol{g}$. Since $\boldsymbol{f} \in \mathcal{H}$ and $\varepsilon > 0$ were arbitrary, this shows that $\operatorname{rg}\big(\frac{1}{2} - \tilde{\boldsymbol{L}}_0\big)$ is dense in $\mathcal{H}$, which finishes the proof. □

Well-posedness for the linearized problem. Next, we include the potential term and consider the system

$$\begin{aligned}\partial_0\phi_1 &= -\xi^j\partial_j\phi_1 - \phi_1 + \phi_2, \\ \partial_0\phi_2 &= \partial_j\partial^j\phi_1 - \xi^j\partial_j\phi_2 - 2\phi_2 + 6\phi_1.\end{aligned} \tag{4-22}$$

We define an operator $\boldsymbol{L}'$, acting on $\mathcal{H}$, by

$$\boldsymbol{L}'\boldsymbol{u}(\xi) := \begin{pmatrix} 0 \\ 6u_1 \end{pmatrix}.$$

Then we may rewrite (4-22) as an ODE

$$\frac{d}{d\tau}\Phi(\tau) = (\boldsymbol{L}_0 + \boldsymbol{L}')\Phi(\tau)$$

for a function $\Phi : [0, \infty) \to \mathcal{H}$.

Lemma 4.3. *The operator $\boldsymbol{L} := \boldsymbol{L}_0 + \boldsymbol{L}' : \mathcal{D}(\boldsymbol{L}_0) \subset \mathcal{H} \to \mathcal{H}$ generates a strongly continuous one-parameter semigroup $\boldsymbol{S} : [0,\infty) \to \mathcal{B}(\mathcal{H})$ satisfying $\|\boldsymbol{S}(\tau)\| \leq e^{(-\frac{1}{2}+\|\boldsymbol{L}'\|)\tau}$. Furthermore, for the spectrum of the generator we have $\sigma(\boldsymbol{L}) \backslash \sigma(\boldsymbol{L}_0) = \sigma_p(\boldsymbol{L})$.*

Proof. The first assertion is an immediate consequence of the bounded perturbation theorem of semigroup theory; see [Engel and Nagel 2000, p. 158, Theorem 1.3]. In order to prove the claim about the spectrum, we note that the operator $\boldsymbol{L}' : \mathcal{H} \to \mathcal{H}$ is compact by the compactness of the embedding $H^1(B) \hookrightarrow L^2(B)$ (Rellich–Kondrachov) and Lemma 3.1. Assume that $\lambda \in \sigma(\boldsymbol{L})$ and $\lambda \notin \sigma(\boldsymbol{L}_0)$. Then we may write $\lambda - \boldsymbol{L} = [1 - \boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)](\lambda - \boldsymbol{L}_0)$. Observe that the operator $\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)$ is compact. Furthermore, $1 \in \sigma(\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda))$ since otherwise we would have $\lambda \in \rho(\boldsymbol{L})$, a contradiction to our assumption. By the spectral theorem for compact operators we infer $1 \in \sigma_p(\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda))$ which shows that there exists a nontrivial $\boldsymbol{f} \in \mathcal{H}$ such that $[1 - \boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)]\boldsymbol{f} = \boldsymbol{0}$. Thus, by setting $\boldsymbol{u} := \boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}$, we infer $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}_0)$, $\boldsymbol{u} \neq \boldsymbol{0}$, and $(\lambda - \boldsymbol{L})\boldsymbol{u} = \boldsymbol{0}$ which implies $\lambda \in \sigma_p(\boldsymbol{L})$. □

Spectral analysis of the generator. In order to improve the rough growth bound for $\boldsymbol{S}$ given in Lemma 4.3, we need more information on the spectrum of $\boldsymbol{L}$. Thanks to Lemma 4.3 we are only concerned with point spectrum. To begin with, we need the following result concerning $\mathcal{D}(\boldsymbol{L}_0)$.

Lemma 4.4. *Let $\delta \in (0,1)$ and $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}_0)$. Then*

$$\boldsymbol{u}|_{B_{1-\delta}} \in H^2(B_{1-\delta}) \times H^1(B_{1-\delta}),$$

where $B_{1-\delta} := \{\xi \in \mathbb{R}^3 : |\xi| < 1-\delta\}$.

Proof. Let $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}_0)$. By definition of the closure there exists a sequence $(\boldsymbol{u}_n) \subset \mathcal{D}(\tilde{\boldsymbol{L}}_0)$ such that $\boldsymbol{u}_n \to \boldsymbol{u}$ and $\tilde{\boldsymbol{L}}_0\boldsymbol{u}_n \to \boldsymbol{L}_0\boldsymbol{u}$ in $\mathcal{H}$ as $n \to \infty$. We set $\boldsymbol{f}_n := \tilde{\boldsymbol{L}}_0\boldsymbol{u}_n$ and note that $\boldsymbol{f}_n \in H^1(B) \cap C^1(B) \times L^2(B) \cap C(B)$ for all $n \in \mathbb{N}$ by the definition of $\mathcal{D}(\tilde{\boldsymbol{L}}_0)$. We obtain $u_{2n}(\xi) = \xi^j \partial_j u_{1n}(\xi) + u_{1n}(\xi) + f_{1n}(\xi)$ and

$$-(\delta^{jk} - \xi^j \xi^k)\partial_j \partial_k u_{1n}(\xi) + 4\xi^j \partial_j u_{1n}(\xi) + 2u_{1n}(\xi) = f_n(\xi), \tag{4-23}$$

where $f_n(\xi) := -\xi^j \partial_j f_{1n}(\xi) - 2f_{1n}(\xi) - f_{2n}(\xi)$; compare (4-21). By assumption we have $f_n \to f$ in $L^2(B)$ for some $f \in L^2(B)$. Since

$$(\delta^{jk} - \xi^j \xi^k)\eta_j \eta_k \geq |\eta|^2 - |\xi|^2|\eta|^2 \geq \frac{\delta}{2}|\eta|^2$$

for all $\xi \in B_{1-\delta/2}$ and all $\eta \in \mathbb{R}^3$, we see that the differential operator in (4-23) is uniformly elliptic on $B_{1-\delta/2}$. Thus, by standard elliptic regularity theory (see [Evans 1998, p. 309, Theorem 1]) we obtain the estimate

$$\|u_{1n}\|_{H^2(B_{1-\delta})} \leq C_\delta(\|u_{1n}\|_{L^2(B_{1-\delta/2})} + \|f_n\|_{L^2(B_{1-\delta/2})}) \tag{4-24}$$

and since $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\mathcal{H}$ implies $u_{1n} \to u_1$ in $L^2(B)$, we infer $u_1|_{B_{1-\delta}} \in H^2(B_{1-\delta})$. Finally, from $u_{2n}(\xi) = \xi^j \partial_j u_{1n}(\xi) + u_{1n}(\xi) + f_{1n}(\xi)$ we conclude $u_2|_{B_{1-\delta}} \in H^1(B_{1-\delta})$. □

The next result allows us to obtain information on the spectrum of $\boldsymbol{L}$ by studying an ODE. For the following we define the space $H^1_{\mathrm{rad}}(a,b)$ by

$$\|f\|^2_{H^1_{\mathrm{rad}}(a,b)} := \int_a^b |f'(\rho)|^2\rho^2\,d\rho + \int_a^b |f(\rho)|^2\rho^2\,d\rho.$$

Lemma 4.5. *Let $\lambda \in \sigma_p(\boldsymbol{L})$. Then there exists an $\ell \in \mathbb{N}_0$ and a nonzero function $u \in C^\infty(0,1) \cap H^1_{\mathrm{rad}}(0,1)$ such that*

$$-(1-\rho^2)u''(\rho) - \frac{2}{\rho}u'(\rho) + 2(\lambda+2)\rho u'(\rho) + [(\lambda+1)(\lambda+2)-6]u(\rho) + \frac{\ell(\ell+1)}{\rho^2}u(\rho) = 0 \quad (4\text{-}25)$$

for all $\rho \in (0,1)$.

Proof. Let $\boldsymbol{u} \in \mathfrak{D}(\boldsymbol{L}) = \mathfrak{D}(\boldsymbol{L}_0)$ be an eigenvector associated to the eigenvalue $\lambda \in \sigma_p(\boldsymbol{L})$. The spectral equation $(\lambda - \boldsymbol{L})\boldsymbol{u} = \boldsymbol{0}$ implies $u_2(\xi) = \xi^j\partial_j u_1(\xi) + (\lambda+1)u_1(\xi)$ and

$$-(\delta^{jk} - \xi^j\xi^k)\partial_j\partial_k u_1(\xi) + 2(\lambda+2)\xi^j\partial_j u_1(\xi) + [(\lambda+1)(\lambda+2)-6]u_1(\xi) = 0 \quad (4\text{-}26)$$

(compare (4-21)), but this time the derivatives have to be interpreted in the weak sense since a priori we merely have $u_1 \in H^2(B_{1-\delta}) \cap H^1(B)$ and $u_2 \in H^1(B_{1-\delta}) \cap L^2(B)$ by Lemma 4.4. However, by invoking elliptic regularity theory [Evans 1998, p. 316, Theorem 3] we see that in fact $u_1 \in C^\infty(B) \cap H^1(B)$. As always, we write $\rho = |\xi|$ and $\omega = \xi/|\xi|$. We expand u_1 in spherical harmonics, i.e.,

$$u_1(\rho\omega) = \sum_{\ell,m}^{\infty} u_{\ell,m}(\rho)Y_{\ell,m}(\omega) \quad (4\text{-}27)$$

with $u_{\ell,m}(\rho) = (u_1(\rho\,\cdot)|Y_{\ell,m})_{L^2(S^2)}$ and for each fixed $\rho \in (0,1)$, the sum converges in $L^2(S^2)$. By dominated convergence and $u_1 \in C^\infty(B)$ it follows that $u_{\ell,m} \in C^\infty(0,1)$. Similarly, we may expand $\partial_\rho u_1(\rho\omega)$ in spherical harmonics. The corresponding expansion coefficients are given by

$$(\partial_\rho u_1(\rho\,\cdot) \mid Y_{\ell,m})_{L^2(S^2)} = \partial_\rho(u_1(\rho\,\cdot) \mid Y_{\ell,m})_{L^2(S^2)} = \partial_\rho u_{\ell,m}(\rho)$$

where we used dominated convergence and the smoothness of u_1 to pull out the derivative ∂_ρ of the inner product. In other words, we may interchange the operator ∂_ρ with the sum in (4-27). Analogously, we may expand $\Delta_{S^2}u_1(\rho\,\cdot)$ and the corresponding expansion coefficients are

$$(\Delta_{S^2}u_1(\rho\,\cdot) \mid Y_{\ell,m})_{L^2(S^2)} = (u_1(\rho\,\cdot) \mid \Delta_{S^2}Y_{\ell,m})_{L^2(S^2)} = -\ell(\ell+1)u_{\ell,m}(\rho).$$

Thus, the operator Δ_{S^2} commutes with the sum in (4-27). All differential operators that appear in (4-26) are composed of ∂_ρ and Δ_{S^2} and it is therefore a consequence of (4-26) that each $u_{\ell,m}$ satisfies (4-25) for all $\rho \in (0,1)$. Since at least one $u_{\ell,m}$ is nonzero, we obtain the desired function $u \in C^\infty(0,1)$. To complete the proof, it remains to show that $u_{\ell,m} \in H^1_{\mathrm{rad}}(0,1)$. We have

$$|u_{\ell,m}(\rho)| = \left|(u_1(\rho\,\cdot) \mid Y_{\ell,m})_{L^2(S^2)}\right| \le \|u_1(\rho\,\cdot)\|_{L^2(S^2)}$$

and thus,

$$\int_0^1 |u_{\ell,m}(\rho)|^2 \rho^2 \, d\rho \le \int_0^1 \|u_1(\rho\,\cdot\,)\|^2_{L^2(S^2)} \rho^2 \, d\rho = \|u_1\|^2_{L^2(B)}.$$

Similarly, by dominated convergence,

$$|\partial_\rho u_{\ell,m}(\rho)| = \left|(\partial_\rho u_1(\rho\,\cdot\,) \mid Y_{\ell,m})_{L^2(S^2)}\right| \lesssim \|\nabla u_1(\rho\,\cdot\,)\|_{L^2(S^2)}$$

and thus,

$$\int_0^1 |u'_{\ell,m}(\rho)|^2 \rho^2 \, d\rho \lesssim \|\nabla u_1\|^2_{L^2(B)}.$$

Consequently, $u_1 \in H^1(B)$ implies $u_{\ell,m} \in H^1_{\mathrm{rad}}(0, 1)$. □

Proposition 4.6. *For the spectrum of* $\boldsymbol{L}$ *we have*

$$\sigma(\boldsymbol{L}) \subset \{z \in \mathbb{C} : \operatorname{Re} z \le -\tfrac{1}{2}\} \cup \{0, 1\}.$$

Furthermore, $\{0, 1\} \subset \sigma_p(\boldsymbol{L})$ *and the (geometric) eigenspace of the eigenvalue* 1 *is one-dimensional and spanned by*

$$\boldsymbol{u}(\xi; 1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

whereas the (geometric) eigenspace of the eigenvalue 0 *is three-dimensional and spanned by*

$$\boldsymbol{u}_j(\xi; 0) = \begin{pmatrix} \xi^j \\ 2\xi^j \end{pmatrix}, \quad j \in \{1, 2, 3\}.$$

Proof. First of all, it is a simple exercise to check that $\boldsymbol{L}\boldsymbol{u}(\xi; 1) = \boldsymbol{u}(\xi; 1)$ and $\boldsymbol{L}\boldsymbol{u}_j(\xi; 0) = \boldsymbol{0}$ for $j = 1, 2, 3$. Since obviously $\boldsymbol{u}(\,\cdot\,; 1), \boldsymbol{u}_j(\,\cdot\,; 0) \in \mathfrak{D}(\tilde{\boldsymbol{L}}_0)$, this implies $\{0, 1\} \subset \sigma_p(\boldsymbol{L})$.

In order to prove the first assertion, let $\lambda \in \sigma(\boldsymbol{L})$ and assume $\operatorname{Re}\lambda > -\frac{1}{2}$. By Proposition 4.1 we have $\lambda \notin \sigma(\boldsymbol{L}_0)$ and thus, Lemma 4.3 implies $\lambda \in \sigma_p(\boldsymbol{L})$. From Lemma 4.5 we infer the existence of a nonzero $u \in C^\infty(0, 1) \cap H^1_{\mathrm{rad}}(0, 1)$ satisfying (4-25) for $\rho \in (0, 1)$. As before, we reduce (4-25) to the hypergeometric differential equation by setting $u(\rho) = \rho^\ell v(\rho^2)$. This yields

$$z(1-z)v''(z) + [c - (a+b+1)z]v'(z) - abv(z) = 0, \tag{4-28}$$

with $a = \frac{1}{2}(-1+\ell+\lambda)$, $b = \frac{1}{2}(4+\ell+\lambda)$, $c = \frac{3}{2}+\ell$, and $z = \rho^2$. A fundamental system of (4-28) is given by[3]

$$\begin{aligned} \phi_{1,\ell}(z; \lambda) &= {}_2F_1(a, b, a+b+1-c; 1-z), \\ \tilde{\phi}_{1,\ell}(z; \lambda) &= (1-z)^{c-a-b}{}_2F_1(c-a, c-b, c-a-b+1; 1-z) \end{aligned}$$

and thus, there exist constants $c_\ell(\lambda)$ and $\tilde{c}_\ell(\lambda)$ such that

$$v(z) = c_\ell(\lambda)\phi_{1,\ell}(z; \lambda) + \tilde{c}_\ell(\lambda)\tilde{\phi}_{1,\ell}(z; \lambda).$$

[3] Strictly speaking, this is only true for $c - a - b = -\lambda \ne 0$. In the case $\lambda = 0$ there exists a solution $\tilde{\phi}_{1,\ell}$ which behaves like $\log(1-z)$ as $z \to 1-$.

The function $\phi_{1,\ell}(z;\lambda)$ is analytic around $z=1$ whereas $\tilde{\phi}_{1,\ell}(z;\lambda)\sim(1-z)^{-\lambda}$ as $z\to 1-$ provided $\lambda\neq 0$. In the case $\lambda=0$ we have $\tilde{\phi}_{1,\ell}(z;\lambda)\sim\log(1-z)$ as $z\to 1-$. Since $u\in H^1_{\mathrm{rad}}(0,1)$ implies $v\in H^1\big(\frac{1}{2},1\big)$ and we assume $\operatorname{Re}\lambda>-\frac{1}{2}$, it follows that $\tilde{c}_\ell(\lambda)=0$. Another fundamental system of (4-28) is given by

$$\begin{aligned}\phi_{0,\ell}(z;\lambda)&={}_2F_1(a,b,c;z),\\ \tilde{\phi}_{0,\ell}(z;\lambda)&=z^{1-c}{}_2F_1(a-c+1,b-c+1,2-c;z),\end{aligned}$$

and since $\tilde{\phi}_{0,\ell}(z;\lambda)\sim z^{-\ell-\frac{1}{2}}$ as $z\to 0+$, we see that the function $\rho\mapsto\rho^\ell\tilde{\phi}_{\ell,0}(\rho^2)$ does not belong to $H^1_{\mathrm{rad}}(0,\frac{1}{2})$. As a consequence, we must have $v(z)=d_\ell(\lambda)\phi_{0,\ell}(z;\lambda)$ for some suitable $d_\ell(\lambda)\in\mathbb{C}$. In summary, we conclude that the functions $\phi_{0,\ell}(\,\cdot\,;\lambda)$ and $\phi_{1,\ell}(\,\cdot\,;\lambda)$ are linearly dependent and in view of the connection formula [Olver et al. 2010]

$$\phi_{1,\ell}(z;\lambda)=\frac{\Gamma(1-c)\Gamma(a+b+1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}\phi_{0,\ell}(z;\lambda)+\frac{\Gamma(c-1)\Gamma(a+b+1-c)}{\Gamma(a)\Gamma(b)}\tilde{\phi}_{0,\ell}(z;\lambda)$$

this is possible only if a or b is a pole of the Γ-function. This yields $-a\in\mathbb{N}_0$ or $-b\in\mathbb{N}_0$ and thus, $\frac{1}{2}(1-\ell-\lambda)\in\mathbb{N}_0$ or $-\frac{1}{2}(4+\ell+\lambda)\in\mathbb{N}_0$. The latter condition is not satisfied for any $\ell\in\mathbb{N}_0$ and the former one is satisfied only if $(\ell,\lambda)=(0,1)$ or $(\ell,\lambda)=(1,0)$, which proves $\sigma(\boldsymbol{L})\subset\{z\in\operatorname{Re}z\leq-\frac{1}{2}\}\cup\{0,1\}$. Furthermore, the above argument and the derivation in the proof of Lemma 4.5 also show that the geometric eigenspaces of the eigenvalues 0 and 1 are at most three- and one-dimensional, respectively. ☐

Remark 4.7. According to the discussion at the beginning of Section 4, the two unstable eigenvalues 1 and 0 emerge from the time translation and Lorentz invariance of the wave equation.

Spectral projections. In order to force convergence to the attractor, we need to "remove" the eigenvalues 0 and 1 from the spectrum of $\boldsymbol{L}$. This is achieved by the spectral projection

$$\boldsymbol{P}:=\frac{1}{2\pi i}\int_\gamma(z-\boldsymbol{L})^{-1}\,dz, \tag{4-29}$$

where the contour γ is given by the curve $\gamma(s)=\frac{1}{2}+\frac{3}{4}e^{2\pi is}$, $s\in[0,1]$. By Proposition 4.6 it follows that $\gamma(s)\in\rho(\boldsymbol{L})$ for all $s\in[0,1]$ and thus, the integral in (4-29) is well-defined as a Riemann integral over a continuous function (with values in a Banach space, though). Furthermore, the contour γ encloses the two unstable eigenvalues 0 and 1. The operator $\boldsymbol{L}$ decomposes into two parts:

$$\begin{aligned}\boldsymbol{L}_u&:\operatorname{rg}\boldsymbol{P}\cap\mathfrak{D}(\boldsymbol{L})\to\operatorname{rg}\boldsymbol{P}, &\quad \boldsymbol{L}_u\boldsymbol{u}&=\boldsymbol{L}\boldsymbol{u},\\ \boldsymbol{L}_s&:\ker\boldsymbol{P}\cap\mathfrak{D}(\boldsymbol{L})\to\ker\boldsymbol{P}, &\quad \boldsymbol{L}_s\boldsymbol{u}&=\boldsymbol{L}\boldsymbol{u},\end{aligned}$$

and for the spectra we have $\sigma(\boldsymbol{L}_u)=\{0,1\}$ as well as $\sigma(\boldsymbol{L}_s)=\sigma(\boldsymbol{L})\backslash\{0,1\}$. We also emphasize the crucial fact that $\boldsymbol{P}$ commutes with the semigroup $\boldsymbol{S}(\tau)$ and thus, the subspaces $\operatorname{rg}\boldsymbol{P}$ and $\ker\boldsymbol{P}$ of $\mathcal{H}$ are invariant under the linearized flow. We refer to [Kato 1995] and [Engel and Nagel 2000] for these standard facts. However, it is important to keep in mind that $\boldsymbol{P}$ is not an orthogonal projection since $\boldsymbol{L}$ is not self-adjoint. Consequently, the following statement on the dimension of $\operatorname{rg}\boldsymbol{P}$ is not trivial.

Lemma 4.8. *The algebraic multiplicities of the eigenvalues* $0,1\in\sigma_p(\boldsymbol{L})$ *equal their geometric multiplicities. In particular*, *we have* $\dim\operatorname{rg}\boldsymbol{P}=4$.

Proof. We define the two spectral projections $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$ by

$$\boldsymbol{P}_n = \frac{1}{2\pi i}\int_{\gamma_n}(z-\boldsymbol{L})^{-1}\,dz, \quad n\in\{0,1\}$$

where $\gamma_0(s)=\frac{1}{2}e^{2\pi i s}$ and $\gamma_1(s)=1+\frac{1}{2}e^{2\pi i s}$ for $s\in[0,1]$. Note that $\boldsymbol{P}=\boldsymbol{P}_0+\boldsymbol{P}_1$ and $\boldsymbol{P}_0\boldsymbol{P}_1=\boldsymbol{P}_1\boldsymbol{P}_0=\boldsymbol{0}$; see [Kato 1995]. By definition, the algebraic multiplicity of the eigenvalue $n\in\sigma_p(\boldsymbol{L})$ equals $\dim \operatorname{rg}\boldsymbol{P}_n$. First, we exclude the possibility $\dim\operatorname{rg}\boldsymbol{P}_n=\infty$. Suppose this is true. Then n belongs to the essential spectrum of $\boldsymbol{L}$, i.e., $n-\boldsymbol{L}$ fails to be semi-Fredholm [Kato 1995, p. 239, Theorem 5.28]. Since the essential spectrum is invariant under compact perturbations (see [Kato 1995, p. 244, Theorem 5.35]), we infer $n\in\sigma(\boldsymbol{L}_0)$, which contradicts the spectral statement in Proposition 4.1. Consequently, $\dim\operatorname{rg}\boldsymbol{P}_n<\infty$. We conclude that the operators $\boldsymbol{L}_{(n)}:=\boldsymbol{L}|_{\operatorname{rg}\boldsymbol{P}_n\cap\mathfrak{D}(\boldsymbol{L})}$ are in fact finite-dimensional and $\sigma(\boldsymbol{L}_{(n)})=\{n\}$. This implies that $n-\boldsymbol{L}_{(n)}$ is nilpotent and thus, there exist $m_n\in\mathbb{N}$ such that $(n-\boldsymbol{L}_{(n)})^{m_n}=\boldsymbol{0}$. We assume m_n to be minimal with this property. If $m_n=1$ we are done. Thus, assume $m_n\geq 2$. We first consider $\boldsymbol{L}_{(0)}$. Since $\ker\boldsymbol{L}$ is spanned by $\{\boldsymbol{u}_j(\,\cdot\,;0): j=1,2,3\}$ by Proposition 4.6, it follows that there exists a $\boldsymbol{u}\in\operatorname{rg}\boldsymbol{P}_0\cap\mathfrak{D}(\boldsymbol{L})$ and constants $c_1,c_2,c_3\in\mathbb{C}$, not all of them zero, such that

$$\boldsymbol{L}_{(0)}\boldsymbol{u}(\xi)=\boldsymbol{L}\boldsymbol{u}(\xi)=\sum_{j=1}^{3}c_j\boldsymbol{u}_j(\xi;0)=\begin{pmatrix}c_j\xi^j\\ 2c_j\xi^j\end{pmatrix}.$$

This implies $u_2(\xi)=\xi^j\partial_j u_1(\xi)+u_1(\xi)+c_j\xi^j$ and thus,

$$-(\delta^{jk}-\xi^j\xi^k)\partial_j\partial_k u_1(\xi)+4\xi^j\partial_j u_1(\xi)-4u_1(\xi)=-5c_j\xi^j=|\xi|\sum_{m=-1}^{1}\tilde{c}_m Y_{1,m}\Big(\frac{\xi}{|\xi|}\Big).$$

As before in the proof of Lemma 4.5, we expand u_1 as

$$u_1(\xi)=\sum_{\ell,m}^{\infty}u_{\ell,m}(|\xi|)Y_{\ell,m}\Big(\frac{\xi}{|\xi|}\Big)$$

and find

$$-(1-\rho^2)u_{1,m}''(\rho)-\frac{2}{\rho}u_{1,m}'(\rho)+4\rho u_{1,m}'(\rho)-4u_{1,m}(\rho)+\frac{2}{\rho^2}u_{1,m}(\rho)=\tilde{c}_m\rho. \tag{4-30}$$

For at least one $m\in\{-1,0,1\}$ we have $\tilde{c}_m\neq 0$ and by normalizing $u_{1,m}$ accordingly, we may assume $\tilde{c}_m=1$. Of course, (4-30) with $\tilde{c}_m=0$ is nothing but the spectral equation (4-25) with $\ell=1$ and $\lambda=0$. An explicit solution is therefore given by $\psi(\rho)=\rho$ which may of course also be easily checked directly. Another solution is $\tilde{\psi}(\rho):=\tilde{\psi}_{0,1}(\rho;0)=\rho\tilde{\phi}_{0,1}(\rho^2;0)$, where $\tilde{\phi}_{1,0}(\,\cdot\,;0)$ is the hypergeometric function from the proof of Proposition 4.6. We have the asymptotic behavior $\tilde{\psi}(\rho)\sim\rho^{-2}$ as $\rho\to 0+$ and $|\tilde{\psi}(\rho)|\simeq|\log(1-\rho)|$ as $\rho\to 1-$. By the variation of constants formula we infer that $u_{1,m}$ must be of the form

$$u_{1,m}(\rho)=c\psi(\rho)+\tilde{c}\tilde{\psi}(\rho)+\psi(\rho)\int_{\rho_0}^{\rho}\frac{\tilde{\psi}(s)}{W(s)}\frac{s}{1-s^2}\,ds-\tilde{\psi}(\rho)\int_{\rho_1}^{\rho}\frac{\psi(s)}{W(s)}\frac{s}{1-s^2}\,ds \tag{4-31}$$

for suitable constants $c, \tilde{c} \in \mathbb{C}$, $\rho_0, \rho_1 \in [0, 1]$ and

$$W(\rho) = W(\psi, \tilde{\psi})(\rho) = \frac{d}{\rho^2(1-\rho^2)}$$

where $d \in \mathbb{R}\backslash\{0\}$. Recall that $u_1 \in H^1(B)$ implies $u_{1,m} \in H^1_{\text{rad}}(0, 1)$ and by considering the behavior of (4-31) as $\rho \to 0+$, we see that necessarily

$$\tilde{c} = \int_{\rho_1}^{0} \frac{\psi(s)}{W(s)} \frac{s}{1-s^2}\, ds,$$

which leaves us with

$$u_{1,m}(\rho) = c\psi(\rho) + \psi(\rho) \int_{\rho_0}^{\rho} \frac{\tilde{\psi}(s)}{W(s)} \frac{s}{1-s^2}\, ds - \tilde{\psi}(\rho) \int_{0}^{\rho} \frac{\psi(s)}{W(s)} \frac{s}{1-s^2}\, ds.$$

Next, we consider the behavior as $\rho \to 1-$. Since

$$\left| \int_{\rho_0}^{\rho} \frac{\tilde{\psi}(s)}{W(s)} \frac{s}{1-s^2}\, ds \right| \lesssim 1$$

for all $\rho \in (0, 1)$ and $\tilde{\psi} \notin H^1_{\text{rad}}\big(\frac{1}{2}, 1\big)$, we must have

$$\lim_{\rho \to 1-} \int_{0}^{\rho} \frac{\psi(s)}{W(s)} \frac{s}{1-s^2}\, ds = 0.$$

This, however, is impossible since

$$\frac{\psi(s)}{W(s)} \frac{s}{1-s^2} = \frac{1}{d} s^4.$$

Thus, we arrive at a contradiction and our initial assumption $m_0 \geq 2$ must be wrong. Consequently, from Proposition 4.6 we infer $\dim \operatorname{rg} \boldsymbol{P}_0 = \dim \ker \boldsymbol{L} = 3$ as claimed. By exactly the same type of argument one proves that $\dim \operatorname{rg} \boldsymbol{P}_1 = 1$. □

Resolvent estimates. Our next goal is to obtain existence of the resolvent $\boldsymbol{R_L}(\lambda) \in \mathcal{B}(\mathcal{H})$ for $\lambda \in H_{-\frac{1}{2}+\epsilon} := \{z \in \mathbb{C} : \operatorname{Re} z \geq -\frac{1}{2} + \epsilon\}$ and $|\lambda|$ large.

Lemma 4.9. *Fix $\epsilon > 0$. Then there exists a constant $C > 0$ such that $\boldsymbol{R_L}(\lambda)$ exists as a bounded operator on $\mathcal{H}$ for all $\lambda \in H_{-\frac{1}{2}+\epsilon}$ with $|\lambda| > C$.*

Proof. From Proposition 4.1 we know that $\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda) \in \mathcal{B}(\mathcal{H})$ for all $\lambda \in H_{-\frac{1}{2}+\epsilon}$ with the bound (see [Engel and Nagel 2000, p. 55, Theorem 1.10])

$$\|\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\| \leq \frac{1}{\operatorname{Re}\lambda + \frac{1}{2}}.$$

Furthermore, recall the identity $\boldsymbol{R_L}(\lambda) = \boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)[1 - \boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)]^{-1}$. By definition of $\boldsymbol{L}'$ we have

$$\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f} = \begin{pmatrix} 0 \\ 6[\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_1 \end{pmatrix},$$

where we use the notation $[\boldsymbol{g}]_k$ for the k-th component of the vector $\boldsymbol{g}$. Set $\boldsymbol{u} = \boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}$ for a given $\boldsymbol{f} \in \mathcal{H}$. Then we have $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}_0)$ and $(\lambda - \boldsymbol{L}_0)\boldsymbol{u} = \boldsymbol{f}$, which implies $u_2(\xi) = \xi^j \partial_j u_1(\xi) + (\lambda+1)u_1(\xi) - f_1(\xi)$, or, equivalently,

$$[\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_1(\xi) = \frac{1}{\lambda+1}\Big[-\xi^j \partial_j [\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_1(\xi) + [\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_2(\xi) + f_1(\xi)\Big].$$

Consequently, we infer

$$\begin{aligned}\|[\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_1\|_{L^2(B)} &\lesssim \frac{1}{|\lambda+1|}\Big[\|[\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_1\|_{H^1(B)} + \|[\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_2\|_{L^2(B)} + \|f_1\|_{L^2(B)}\Big] \\ &\lesssim \frac{\|\boldsymbol{f}\|}{|\lambda+1|},\end{aligned}$$

which yields $\|\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\| \lesssim 1/|\lambda+1|$ for all $\lambda \in H_{-\frac{1}{2}+\epsilon}$. We conclude that the Neumann series

$$[1 - \boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)]^{-1} = \sum_{k=0}^{\infty} [\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)]^k$$

converges in norm provided $|\lambda|$ is sufficiently large. This yields the desired result. □

Estimates for the linearized evolution. Finally, we obtain improved growth estimates for the semigroup $\boldsymbol{S}$ from Lemma 4.3 which governs the linearized evolution.

Proposition 4.10. *Fix $\epsilon > 0$. Then the semigroup $\boldsymbol{S}$ from Lemma 4.3 satisfies the estimates*

$$\begin{aligned}\|\boldsymbol{S}(\tau)(1-\boldsymbol{P})\boldsymbol{f}\| &\le Ce^{(-\frac{1}{2}+\epsilon)\tau}\|(1-\boldsymbol{P})\boldsymbol{f}\|, \\ \|\boldsymbol{S}(\tau)\boldsymbol{P}\boldsymbol{f}\| &\le Ce^{\tau}\|\boldsymbol{P}\boldsymbol{f}\|,\end{aligned}$$

for all $\tau \ge 0$ and $\boldsymbol{f} \in \mathcal{H}$.

Proof. The operator $\boldsymbol{L}_s$ is the generator of the subspace semigroup $\boldsymbol{S}_s$ defined by $\boldsymbol{S}_s(\tau) := \boldsymbol{S}(\tau)|_{\ker \boldsymbol{P}}$. We have $\sigma(\boldsymbol{L}_s) \subset \{z \in \mathbb{C} : \operatorname{Re} z \le -\frac{1}{2}\}$ and the resolvent $\boldsymbol{R}_{\boldsymbol{L}_s}(\lambda)$ is the restriction of $\boldsymbol{R}_{\boldsymbol{L}}(\lambda)$ to $\ker \boldsymbol{P}$. Consequently, by Lemma 4.9 we infer $\|\boldsymbol{R}_{\boldsymbol{L}_s}(\lambda)\| \lesssim 1$ for all $\lambda \in H_{-\frac{1}{2}+\epsilon}$ and thus, the Gearhart–Prüss–Greiner theorem (see [Engel and Nagel 2000, p. 302, Theorem 1.11]) yields the semigroup decay $\|\boldsymbol{S}_s(\tau)\| \lesssim e^{(-\frac{1}{2}+\epsilon)\tau}$. The estimate for $\boldsymbol{S}(\tau)\boldsymbol{P}$ follows from the fact that $\operatorname{rg} \boldsymbol{P}$ is spanned by eigenfunctions of $\boldsymbol{L}$ with eigenvalues 0 and 1 (Proposition 4.6 and Lemma 4.8). □

5. Nonlinear perturbation theory

In this section we consider the full problem (3-5),

$$\begin{aligned}\partial_0\phi_1 &= -\xi^j\partial_j\phi_1 - \phi_1 + \phi_2, \\ \partial_0\phi_2 &= \partial_j\partial^j\phi_1 - \xi^j\partial_j\phi_2 - 2\phi_2 + 6\phi_1 + 3\sqrt{2}\phi_1^2 + \phi_1^3,\end{aligned} \tag{5-1}$$

with prescribed initial data at $\tau = 0$. An operator formulation of (5-1) is obtained by defining the nonlinearity

$$N(u) := \begin{pmatrix} 0 \\ 3\sqrt{2}u_1^2 + u_1^3 \end{pmatrix}.$$

It is an immediate consequence of the Sobolev embedding $H^1(B) \hookrightarrow L^p(B)$, $p \in [1, 6]$, that $N : \mathcal{H} \to \mathcal{H}$ and we have the estimate

$$\|N(u) - N(v)\| \lesssim \|u - v\|(\|u\| + \|v\|) \tag{5-2}$$

for all $u, v \in \mathcal{H}$ with $\|u\|, \|v\| \le 1$. The Cauchy problem for (5-1) is formally equivalent to

$$\begin{aligned} \frac{d}{d\tau}\Phi(\tau) &= L\Phi(\tau) + N(\Phi(\tau)), \\ \Phi(0) &= u, \end{aligned} \tag{5-3}$$

for a strongly differentiable function $\Phi : [0, \infty) \to \mathcal{H}$ where u are the prescribed data. In fact, we shall consider the weak version of (5-3) which reads

$$\Phi(\tau) = S(\tau)u + \int_0^\tau S(\tau - \sigma)N(\Phi(\sigma))\, d\sigma. \tag{5-4}$$

Since the semigroup S is unstable, one cannot expect to obtain a global solution of (5-4) for general data $u \in \mathcal{H}$. However, on the subspace $\ker P$, the semigroup S is stable (Proposition 4.10). In order to isolate the instability in the nonlinear context, we formally project (5-4) to the unstable subspace $\operatorname{rg} P$ which yields

$$P\Phi(\tau) = S(\tau)Pu + \int_0^\tau S(\tau - \sigma)PN(\Phi(\sigma))\, d\sigma.$$

This suggests to subtract the "bad" term

$$S(\tau)Pu + \int_0^\infty S(\tau - \sigma)PN(\Phi(\sigma))\, d\sigma$$

from (5-4) in order to force decay. We obtain the equation

$$\Phi(\tau) = S(\tau)(1 - P)u + \int_0^\tau S(\tau - \sigma)N(\Phi(\sigma))\, d\sigma - \int_0^\infty S(\tau - \sigma)PN(\Phi(\sigma))\, d\sigma. \tag{5-5}$$

First, we solve (5-5) and then we relate solutions of (5-5) to solutions of (5-4).

Solution of the modified equation. We solve (5-5) by a fixed point argument. To this end we define

$$K_u(\Phi)(\tau) := S(\tau)(1 - P)u + \int_0^\tau S(\tau - \sigma)N(\Phi(\sigma))\, d\sigma - \int_0^\infty S(\tau - \sigma)PN(\Phi(\sigma))\, d\sigma$$

and show that K_u defines a contraction mapping on (a closed subset of) the Banach space $\mathcal{X}$, given by

$$\mathcal{X} := \left\{ \Phi \in C([0, \infty), \mathcal{H}) : \sup_{\tau > 0} e^{(\frac{1}{2} - \epsilon)\tau} \|\Phi(\tau)\| < \infty \right\}$$

with norm

$$\|\Phi\|_{\mathcal{X}} := \sup_{\tau>0} e^{(\frac{1}{2}-\epsilon)\tau} \|\Phi(\tau)\|,$$

where $\epsilon \in (0, \frac{1}{2})$ is arbitrary but fixed. We further write

$$\mathcal{X}_\delta := \{\Phi \in \mathcal{X} : \|\Phi\|_{\mathcal{X}} \le \delta\}$$

for the closed ball of radius $\delta > 0$ in $\mathcal{X}$.

Proposition 5.1. *Let $\delta > 0$ be sufficiently small and suppose $\boldsymbol{u} \in \mathcal{H}$ with $\|\boldsymbol{u}\| < \delta^2$. Then $\boldsymbol{K_u}$ maps $\mathcal{X}_\delta$ to $\mathcal{X}_\delta$ and we have the estimate*

$$\|\boldsymbol{K_u}(\Phi) - \boldsymbol{K_u}(\Psi)\|_{\mathcal{X}} \le C\delta \|\Phi - \Psi\|_{\mathcal{X}}$$

for all $\Phi, \Psi \in \mathcal{X}_\delta$.

Proof. First observe that $\boldsymbol{K_u} : \mathcal{X}_\delta \to C([0,\infty), \mathcal{H})$ since $\|\boldsymbol{N}(\Phi(\tau))\| \lesssim e^{(-1+2\epsilon)\tau}$ for any $\Phi \in \mathcal{X}_\delta$. We have

$$\boldsymbol{P}\boldsymbol{K_u}(\Phi)(\tau) = -\int_\tau^\infty \boldsymbol{S}(\tau-\sigma)\boldsymbol{P}\boldsymbol{N}(\Phi(\sigma))\, d\sigma, \tag{5-6}$$

which yields

$$\begin{aligned} \big\|\boldsymbol{P}[\boldsymbol{K_u}(\Phi)(\tau) - \boldsymbol{K_u}(\Psi)(\tau)]\big\| &\lesssim \int_\tau^\infty e^{\tau-\sigma} \|\Phi(\sigma) - \Psi(\sigma)\| \big(\|\Phi(\sigma)\| + \|\Psi(\sigma)\|\big)\, d\sigma \\ &\lesssim \|\Phi - \Psi\|_{\mathcal{X}} (\|\Phi\|_{\mathcal{X}} + \|\Psi\|_{\mathcal{X}}) e^\tau \int_\tau^\infty e^{(-2+2\epsilon)\sigma}\, d\sigma \\ &\lesssim \delta e^{(-1+2\epsilon)\tau} \|\Phi - \Psi\|_{\mathcal{X}} \end{aligned}$$

for all $\Phi, \Psi \in \mathcal{X}_\delta$ by Proposition 4.10. On the stable subspace we have

$$(1-\boldsymbol{P})\boldsymbol{K_u}(\Phi)(\tau) = \boldsymbol{S}(\tau)(1-\boldsymbol{P})\boldsymbol{u} + \int_0^\tau \boldsymbol{S}(\tau-\sigma)(1-\boldsymbol{P})\boldsymbol{N}(\Phi(\sigma))\, d\sigma$$

and thus,

$$\begin{aligned} \big\|(1-\boldsymbol{P})[\boldsymbol{K_u}(\Phi)(\tau) - \boldsymbol{K_u}(\Psi)(\tau)]\big\| &\lesssim \int_0^\tau e^{(-\frac{1}{2}+\epsilon)(\tau-\sigma)} \|\Phi(\sigma) - \Psi(\sigma)\| \big(\|\Phi(\sigma)\| + \|\Psi(\sigma)\|\big)\, d\sigma \\ &\lesssim \|\Phi - \Psi\|_{\mathcal{X}} \delta e^{(-\frac{1}{2}+\epsilon)\tau} \int_0^\tau e^{(-\frac{1}{2}+\epsilon)\sigma}\, d\sigma \\ &\lesssim \delta e^{(-\frac{1}{2}+\epsilon)\tau} \|\Phi - \Psi\|_{\mathcal{X}} \end{aligned}$$

again by Proposition 4.10. We conclude that

$$\|\boldsymbol{K_u}(\Phi) - \boldsymbol{K_u}(\Psi)\|_{\mathcal{X}} \lesssim \delta \|\Phi - \Psi\|_{\mathcal{X}}$$

for all $\Phi, \Psi \in \mathcal{X}$. By a slight modification of the above argument one similarly proves $\|\boldsymbol{K_u}(\Phi)\|_{\mathcal{X}} \le \delta$ for all $\Phi \in \mathcal{X}_\delta$ (here $\|\boldsymbol{u}\| \le \delta^2$ is used). ☐

Now we can conclude the existence of a solution to (5-5) by invoking the contraction mapping principle.

Lemma 5.2. *Let $\delta > 0$ be sufficiently small. Then, for any $\boldsymbol{u} \in \mathcal{H}$ with $\|\boldsymbol{u}\| \leq \delta^2$, there exists a unique solution $\Phi_{\boldsymbol{u}} \in \mathcal{X}_\delta$ to* (5-5).

Proof. By Proposition 5.1 we may choose $\delta > 0$ so small that

$$\|\boldsymbol{K_u}(\Phi) - \boldsymbol{K_u}(\Psi)\|_{\mathcal{X}} \leq \tfrac{1}{2}\|\Phi - \Psi\|_{\mathcal{X}}$$

for all $\Phi, \Psi \in \mathcal{X}_\delta$ and thus, the contraction mapping principle implies the existence of a unique $\Phi_{\boldsymbol{u}} \in \mathcal{X}_\delta$ with $\Phi_{\boldsymbol{u}} = \boldsymbol{K_u}(\Phi_{\boldsymbol{u}})$. By the definition of $\boldsymbol{K_u}$, $\Phi_{\boldsymbol{u}}$ is a solution to (5-5). ☐

***Solution of* (5-4).** Recall that $\operatorname{rg} \boldsymbol{P}$ is spanned by eigenfunctions of $\boldsymbol{L}$ with eigenvalues 0 and 1; see Lemma 4.8. As in the proof of Lemma 4.8 we write $\boldsymbol{P} = \boldsymbol{P}_0 + \boldsymbol{P}_1$, where $\boldsymbol{P}_n$, $n \in \{0, 1\}$, projects to the geometric eigenspace of $\boldsymbol{L}$ associated to the eigenvalue $n \in \sigma_p(\boldsymbol{L})$. Consequently, we infer $\boldsymbol{S}(\tau)\boldsymbol{P}_n = e^{n\tau}\boldsymbol{P}_n$. This shows that the “bad” term we subtracted from (5-4) may be written as

$$\boldsymbol{S}(\tau)\boldsymbol{P}\boldsymbol{u} + \int_0^\infty \boldsymbol{S}(\tau - \sigma)\boldsymbol{P}\boldsymbol{N}(\Phi_{\boldsymbol{u}}(\sigma))\, d\sigma = \boldsymbol{S}(\tau)[\boldsymbol{P}\boldsymbol{u} - \boldsymbol{F}(\boldsymbol{u})],$$

where $\boldsymbol{F}$ is given by

$$\boldsymbol{F}(\boldsymbol{u}) := -\boldsymbol{P}_0 \int_0^\infty \boldsymbol{N}(\Phi_{\boldsymbol{u}}(\sigma))\, d\sigma - \boldsymbol{P}_1 \int_0^\infty e^{-\sigma}\boldsymbol{N}(\Phi_{\boldsymbol{u}}(\sigma))\, d\sigma.$$

According to Lemma 5.2, the function $\boldsymbol{F}$ is well-defined on $\mathcal{B}_{\delta^2} := \{\boldsymbol{u} \in \mathcal{H} : \|\boldsymbol{u}\| < \delta^2\}$ with values in $\operatorname{rg} \boldsymbol{P}$ and this shows that we have effectively modified the *initial data* by adding an element of the four-dimensional subspace $\operatorname{rg} \boldsymbol{P}$ of $\mathcal{H}$. Note, however, that the modification depends on the solution itself. Consequently, if the initial data for (5-4) are of the form $\boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u})$ for $\boldsymbol{u} \in \ker \boldsymbol{P}$, (5-4) and (5-5) are equivalent and Lemma 5.2 yields the desired solution of (5-4). We also remark that $\boldsymbol{F}(\boldsymbol{0}) = \boldsymbol{0}$. The following result implies that the graph

$$\{\boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u}) : \boldsymbol{u} \in \ker \boldsymbol{P},\ \|\boldsymbol{u}\| < \delta^2\} \subset \ker \boldsymbol{P} \oplus \operatorname{rg} \boldsymbol{P} = \mathcal{H}$$

defines a Lipschitz manifold of codimension 4.

Lemma 5.3. *Let $\delta > 0$ be sufficiently small. Then the function $\boldsymbol{F} : \mathcal{B}_{\delta^2} \to \operatorname{rg} \boldsymbol{P} \subset \mathcal{H}$ satisfies*

$$\|\boldsymbol{F}(\boldsymbol{u}) - \boldsymbol{F}(\boldsymbol{v})\| \leq C\delta\|\boldsymbol{u} - \boldsymbol{v}\|.$$

Proof. First, we claim that $\boldsymbol{u} \mapsto \Phi_{\boldsymbol{u}} : \mathcal{B}_{\delta^2} \to \mathcal{X}_\delta \subset \mathcal{X}$ is Lipschitz-continuous. Indeed, since $\Phi_{\boldsymbol{u}} = \boldsymbol{K_u}(\Phi_{\boldsymbol{u}})$ we infer

$$\begin{aligned}\|\Phi_{\boldsymbol{u}} - \Phi_{\boldsymbol{v}}\|_{\mathcal{X}} &\leq \|\boldsymbol{K_u}(\Phi_{\boldsymbol{u}}) - \boldsymbol{K_u}(\Phi_{\boldsymbol{v}})\|_{\mathcal{X}} + \|\boldsymbol{K_u}(\Phi_{\boldsymbol{v}}) - \boldsymbol{K_v}(\Phi_{\boldsymbol{v}})\|_{\mathcal{X}} \\ &\lesssim \delta\|\Phi_{\boldsymbol{u}} - \Phi_{\boldsymbol{v}}\|_{\mathcal{X}} + \|\boldsymbol{u} - \boldsymbol{v}\|\end{aligned}$$

by Proposition 5.1 and the fact that

$$\|\boldsymbol{K_u}(\Phi_{\boldsymbol{v}})(\tau) - \boldsymbol{K_v}(\Phi_{\boldsymbol{v}})(\tau)\| = \|\boldsymbol{S}(\tau)(1 - \boldsymbol{P})(\boldsymbol{u} - \boldsymbol{v})\| \lesssim e^{(-\frac{1}{2}+\epsilon)\tau}\|\boldsymbol{u} - \boldsymbol{v}\|.$$

The claim now follows from $\|\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v})\| \lesssim \|\boldsymbol{u} - \boldsymbol{v}\|(\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)$. ☐

We summarize our results in a theorem.

Theorem 5.4. *Let $\delta > 0$ be sufficiently small. There exists a codimension-4 Lipschitz manifold $\mathcal{M} \subset \mathcal{H}$ with $\mathbf{0} \in \mathcal{M}$ such that for any $\boldsymbol{u} \in \mathcal{M}$, (5-4) has a solution $\Phi \in \mathcal{X}_\delta$. Moreover, Φ is unique in $C([0,\infty),\mathcal{H})$. If, in addition, $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L})$ then $\Phi \in C^1([0,\infty),\mathcal{H})$ and Φ solves (5-3) with $\Phi(0) = \boldsymbol{u}$.*

Proof. The last statement follows from standard results of semigroup theory. Uniqueness in $C([0,\infty),\mathcal{H})$ is a simple exercise. $\square$

Proof of Theorem 1.1. Theorem 1.1 is now a consequence of Theorem 5.4: (3-4) implies

$$\frac{v\circ\Phi_T(X)-v_0\circ\Phi_T(X)}{T^2-|X|^2} = \frac{1}{(-T)}\phi\Big(-\log(-T), \frac{X}{(-T)}\Big) \tag{5-7}$$

and thus,

$$\begin{aligned}|T|^{-1}\|v-v_0\|_{L^2(\Sigma_T)} &= |T|^{-2}\Big\|\phi_1\Big(-\log(-T), \frac{\cdot}{|T|}\Big)\Big\|_{L^2(B_{|T|})}\\ &= |T|^{-\frac{1}{2}}\big\|\phi_1(-\log(-T),\cdot)\big\|_{L^2(B)}\\ &\lesssim |T|^{-\epsilon}.\end{aligned}$$

Similarly, we obtain

$$\partial_{X^j}\frac{v\circ\Phi_T(X)-v_0\circ\Phi_T(X)}{T^2-|X|^2} = \frac{1}{T^2}\partial_j\phi_1\Big(-\log(-T), \frac{X}{(-T)}\Big),$$

which yields

$$\|v-v_0\|_{\dot H^1(\Sigma_T)} = T^{-2}\Big\|\nabla\phi_1\Big(-\log(-T), \frac{\cdot}{|T|}\Big)\Big\|_{L^2(B_{|T|})} \lesssim |T|^{-\epsilon}.$$

For the time derivative we infer

$$\begin{aligned}\partial_T\frac{v\circ\Phi_T(X)-v_0\circ\Phi_T(X)}{T^2-|X|^2} &= \frac{1}{T^2}\Big(\partial_0\phi+\frac{X^j}{(-T)}\partial_j\phi+\phi\Big)\Big(-\log(-T), \frac{X}{(-T)}\Big)\\ &= \frac{1}{T^2}\phi_2\Big(-\log(-T), \frac{X}{(-T)}\Big)\end{aligned}$$

and hence,

$$\|\nabla_n v-\nabla_n v_0\|_{L^2(\Sigma_T)} = T^{-2}\Big\|\phi_2\Big(-\log(-T), \frac{\cdot}{|T|}\Big)\Big\|_{L^2(B_{|T|})} \lesssim |T|^{-\epsilon}.$$

Finally, we turn to the Strichartz estimate. First, note that the modulus of the determinant of the Jacobian of $(T,X)\mapsto(t,x)$ is $(T^2-|X|^2)^{-4}$. This is easily seen by considering the transformation

$$X^\mu \mapsto y^\mu = -\frac{X^\mu}{X_\sigma X^\sigma},$$

which has the same Jacobian determinant (up to a sign) since $t=-y^0$ and $x^j=y^j$. We obtain

$$\partial_\nu y^\mu = -\frac{X_\sigma X^\sigma\delta_\nu{}^\mu - 2X_\nu X^\mu}{(X_\sigma X^\sigma)^2}$$

and hence,

$$\partial_\nu y^\mu \partial_\mu y^\lambda = \frac{\delta_\nu{}^\lambda}{(X_\sigma X^\sigma)^2}$$

which yields $|\det(\partial_\nu y^\mu)| = (X_\sigma X^\sigma)^{-4} = (T^2 - |X|^2)^{-4}$. Furthermore, note that $s \in [t, 2t]$ and $x \in B_{(1-\delta)t}$ imply

$$\begin{aligned} S &:= -\frac{s}{s^2 - |x|^2} \geq -\frac{t}{t^2 - |x|^2} \geq -\frac{c_\delta}{t}, \\ S &\leq -\frac{2t}{4t^2 - |x|^2} \leq -\frac{1}{2t}. \end{aligned}$$

Consequently, by (5-7) and Sobolev embedding we infer

$$\begin{aligned} \|v - v_0\|^4_{L^4(t,2t)L^4(B_{(1-\delta)t})} &\leq \int_{-\frac{c_\delta}{t}}^{-\frac{1}{2t}} \int_{B_{(1-\delta)|S|}} \left| \frac{v \circ \Phi_S(X) - v_0 \circ \Phi_S(X)}{S^2 - |X|^2} \right|^4 dX\, dS \\ &\lesssim \int_{-\frac{c_\delta}{t}}^{-\frac{1}{2t}} |S|^{-4} \left\| \phi\Big(-\log(-S), \frac{\cdot}{|S|}\Big) \right\|^4_{L^4(B_{|S|})} dS \\ &= \int_{-\frac{c_\delta}{t}}^{-\frac{1}{2t}} |S|^{-1} \left\| \phi(-\log(-S), \cdot) \right\|^4_{L^4(B)} dS \\ &\lesssim \int_{-\frac{c_\delta}{t}}^{-\frac{1}{2t}} |S|^{-1} \left\| \phi(-\log(-S), \cdot) \right\|^4_{H^1(B)} dS \\ &\lesssim \int_{-\frac{c_\delta}{t}}^{-\frac{1}{2t}} |S|^{1-4\epsilon}\, dS \simeq t^{-2+4\epsilon} \end{aligned}$$

as claimed.

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A NON-SELF-ADJOINT LEBESGUE DECOMPOSITION

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We study the structure of bounded linear functionals on a class of non-self-adjoint operator algebras that includes the multiplier algebra of every complete Nevanlinna–Pick space, and in particular the multiplier algebra of the Drury–Arveson space. Our main result is a Lebesgue decomposition expressing every linear functional as the sum of an absolutely continuous (i.e., weak-* continuous) linear functional and a singular linear functional that is far from being absolutely continuous. This is a non-self-adjoint analogue of Takesaki's decomposition theorem for linear functionals on von Neumann algebras. We apply our decomposition theorem to prove that the predual of every algebra in this class is (strongly) unique.

1. Introduction

The main result in this paper is a decomposition theorem for bounded linear functionals on a class of operator algebras that includes the multiplier algebra of every complete Nevanlinna–Pick space. Results of this kind can be seen as a noncommutative generalization of the Yosida–Hewitt decomposition of a measure into completely additive and purely finitely additive parts, or more classically, the Lebesgue decomposition of a measure into absolutely continuous and singular parts.

Takesaki [1958] proved that a bounded linear functional on a von Neumann algebra can be decomposed uniquely into the sum of a normal (i.e., weak-* continuous) linear functional and a singular linear functional that is far from being normal. Ando [1978] proved a direct analogue of Takesaki's decomposition theorem for linear functionals on the algebra H^∞, of bounded analytic functions on the complex unit disk $\mathbb{D}$. More recently, Ueda [2009; 2011] proved a generalization of Ando's result for finite maximal subdiagonal algebras, which are "analytic" subalgebras of finite von Neumann algebras introduced by Arveson [1967] as a noncommutative generalization of the algebra H^∞.

A compelling case can be made that the natural function-theoretic generalization of H^∞ is the algebra H^∞_d of multipliers on the Drury–Arveson space H^2_d. The algebra H^∞_d is contained in the algebra $H^\infty(\mathbb{B}_d)$ of bounded analytic functions on the complex unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$, but for $d \geq 2$ this inclusion is proper, and H^∞_d is seemingly much more tractable than $H^\infty(\mathbb{B}_d)$ (see, for example, [Arveson 1998]). The Drury–Arveson space H^2_d and the multiplier algebra H^∞_d are universal in the following sense: Every irreducible complete Nevanlinna–Pick space embeds into H^2_d, and the corresponding multiplier algebra arises as the compression of H^∞_d onto this embedding (see [Agler and McCarthy 2000] for details).

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Examples of complete Nevanlinna–Pick spaces include the Hardy space and the Dirichlet space on the disk, the Drury–Arveson space itself, and more generally the class of Besov–Sobolev spaces on $\mathbb{B}_d$.

One explanation for the tractability of H_d^∞ is the fact that H_d^∞ arises as a quotient of the noncommutative analytic Toeplitz algebra F_d^∞ (see, for example, [Davidson and Pitts 1998b; Arias and Popescu 2000]). This algebra, introduced in [Popescu 1989a], can be viewed as an algebra of noncommutative analytic functions acting by left multiplication on a Hardy space F_d^2 of noncommutative analytic functions. The operator-algebraic structure of F_d^∞, which is now well understood, turns out to be strikingly similar to that of H^∞ (see, for example, [Popescu 1989a; 1989b; 1991; 1995; Arias and Popescu 2000; Davidson and Pitts 1998a; 1998b; 1999; Davidson and Yang 2008]).

For a weak-* closed two-sided ideal $\mathcal{J}$ of F_d^∞, we let $\mathcal{A}_\mathcal{J}$ denote the algebra $\mathcal{A}_\mathcal{J} = F_d^\infty/\mathcal{J}$. These algebras are the main objects of interest in this paper, for the following reason: The multiplier algebra of every irreducible complete Nevanlinna–Pick space arises as the compression of F_d^∞ to a coinvariant subspace, and this compression is completely isometrically isomorphic and weak-* to weak-* homeomorphic to a quotient of F_d^∞ by a two-sided ideal (see [Davidson and Pitts 1998b; Arias and Popescu 2000] for details).

Our main result is the following decomposition theorem for linear functionals on quotients of F_d^∞. A functional is said to be absolutely continuous if it is weak-* continuous, and singular if it is, roughly speaking, far from being weak-* continuous (we give a precise definition below).

Theorem 1.1 (Lebesgue decomposition for quotients of F_d^∞). *Let $\mathcal{J}$ be a weak-* closed two-sided ideal of F_d^∞, and let ϕ be a bounded linear functional on $\mathcal{A}_\mathcal{J}$. Then there are unique linear functionals ϕ_a and ϕ_s on $\mathcal{A}_\mathcal{J}$ such that $\phi = \phi_a + \phi_s$, where ϕ_a is absolutely continuous and ϕ_s is singular, and such that*

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|.$$

If $d = 1$, then the constant $\sqrt{2}$ can be replaced with the constant 1. Moreover, these constants are optimal.

The following result for multiplier algebras of complete Nevanlinna–Pick spaces is an immediate consequence of Theorem 1.1.

Corollary 1.2 (Lebesgue decomposition for multiplier algebras). *Let $\mathcal{A}$ be the multiplier algebra of a complete Nevanlinna–Pick space, and let ϕ be a bounded linear functional on $\mathcal{A}$. Then there are unique linear functionals ϕ_a and ϕ_s on $\mathcal{A}$ such that $\phi = \phi_a + \phi_s$, where ϕ_a is absolutely continuous and ϕ_s is singular, and such that*

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|.$$

We first prove that Theorem 1.1 holds for F_d^∞. The proof for quotients of F_d^∞ requires the following generalization of the classical F. and M. Riesz theorem, which is similar in spirit to the noncommutative F. and M. Riesz-type theorems proved in [Exel 1990] for operator algebras with the Dirichlet property and in [Blecher and Labuschagne 2007; Ueda 2009] for maximal subdiagonal algebras.

Theorem 1.3 (extended F. and M. Riesz theorem). *Let ϕ be a bounded linear functional on F_d^∞, and let $\phi = \phi_a + \phi_s$ be the Lebesgue decomposition of ϕ into absolutely continuous and singular parts as in*

Theorem 1.1. Let $\mathcal{J}$ be a weak- closed two-sided ideal of F_d^∞. If ϕ is zero on $\mathcal{J}$, then ϕ_a and ϕ_s are both zero on $\mathcal{J}$.*

Grothendieck [1955] proved that L^1 is the unique predual of L^∞ (up to isometric isomorphism). Sakai [1956] generalized Grothendieck's result by proving that the predual of every von Neumann algebra is unique. In fact, this latter result follows from the proof of Sakai's characterization of von Neumann algebras as C*-algebras which are dual spaces.

The uniqueness of the predual of a von Neumann algebra can also be proved using Takesaki's decomposition theorem [1958] (see, for example, the proof of Corollary 3.9 of [Takesaki 2002]). A similar idea was used by Ando [1978] to prove the uniqueness of the predual of H^∞, and more recently by Ueda [2009] to prove that the predual of every maximal subdiagonal algebra is unique.

Inspired by these results, we apply Theorem 1.3 to prove that the predual of every quotient $\mathcal{A}_\mathcal{J}$ is (strongly) unique.

Theorem 1.4. *Let $\mathcal{J}$ be a weak-* closed two-sided ideal of F_d^∞. Then the algebra $\mathcal{A}_\mathcal{J}$ has a strongly unique predual.*

It follows immediately from Theorem 1.4 that the multiplier algebra of every complete Nevanlinna–Pick space has a unique predual.

Corollary 1.5. *The multiplier algebra of every complete Nevanlinna–Pick space has a strongly unique predual.*

In particular, Corollary 1.5 implies that the multiplier algebra H_d^∞ on the Drury–Arveson space has a unique predual. We believe this result is especially interesting in light of the fact that, for $d \geq 2$, the uniqueness of the predual of $H^\infty(\mathbb{B}_d)$ is an open problem.

In addition to this introduction, this paper has five other sections. In Section 2, we provide a brief review of the requisite background material. In Section 3, we prove the Lebesgue decomposition for F_d^∞, and give an example showing that the constant in the statement of the theorem is optimal. In Section 4, we prove the extended F. and M. Riesz theorem. In Section 5, we prove the Lebesgue decomposition theorem for quotients of F_d^∞, and hence for multiplier algebras of complete Nevanlinna–Pick spaces. In Section 6, we use the Lebesgue decomposition theorem to prove that the predual of every quotient of F_d^∞ is unique, and hence that the predual of the multiplier algebra of every complete Nevanlinna–Pick space is unique.

2. Preliminaries

The noncommutative analytic Toeplitz algebra. For fixed $1 \leq d \leq \infty$, let $\mathbb{C}\langle Z\rangle = \mathbb{C}\langle Z_1, \dots, Z_d\rangle$ denote the algebra of noncommutative polynomials in the variables $Z_1, \dots, Z_d$. As a vector space, $\mathbb{C}\langle Z\rangle$ is spanned by the set of monomials

$$\{Z_w = Z_{w_1} \cdots Z_{w_n} \mid w = w_1 \cdots w_n \in \mathbb{F}_d^*,\ n \geq 0\},$$

where $\mathbb{F}_d^*$ denotes the free semigroup generated by $\{1, \dots, d\}$. The noncommutative Hardy space F_d^2 is the Hilbert space obtained by completing $\mathbb{C}\langle Z\rangle$ in the natural inner product

$$\langle Z_w, Z_{w'}\rangle = \delta_{w,w'}, \quad w, w' \in \mathbb{F}_d^*.$$

Equivalently, F_d^2 is the Hilbert space consisting of noncommutative power series with square summable coefficients,

$$F_d^2 = \left\{ \sum_{w\in\mathbb{F}_d^*} a_w Z_w \;\middle|\; \sum_{w\in\mathbb{F}_d^*} |a_w|^2 < \infty \right\}.$$

We think of the elements of F_d^2 as noncommutative analytic functions.

Every element in F_d^2 gives rise to a multiplication operator on F_d^2 in the following way (note that in this noncommutative setting, it is necessary to specify whether multiplication occurs on the left or the right). For F in F_d^2, the left multiplication operator L_F is defined by

$$L_F G = FG, \quad G \in F_d^2.$$

The operator L_F is not necessarily bounded in general, simply because the product of two elements in F_d^2 is not necessarily contained in F_d^2. However, it is always densely defined on $\mathbb{C}\langle Z\rangle$.

The noncommutative analytic Toeplitz algebra F_d^∞ is the noncommutative multiplier algebra of F_d^2. It consists precisely of the functions F in F_d^2 such that the corresponding left multiplication operator is bounded,

$$F_d^\infty = \{F \in F_d^2 \mid FG \in F_d^2 \text{ for all } G \in F_d^2\}.$$

Equivalently, if we identity F in F_d^∞ with the left multiplication operator L_F on the Hilbert space F_d^2, then F_d^∞ is obtained as the closure of $\mathbb{C}\langle Z\rangle$ in the weak-* topology on $\mathcal{B}(F_d^2)$. The noncommutative disk algebra A_d is the closure of $\mathbb{C}\langle Z\rangle$ in the norm topology. Note that it is properly contained in F_d^∞.

The algebras A_d and F_d^∞ were introduced by Popescu in [1996] and [1995], respectively. For $d = 1$, F_d^2 can be identified with the classical Hardy space H^2, F_d^∞ can be identified with the classical algebra of bounded analytic functions H^∞, and A_d can be identified with the classical disk algebra of functions that are analytic on $\mathbb{D}$ with continuous extensions to the boundary.

The structure of an isometric tuple.

Definition 2.1. Let $V = (V_1, \dots, V_d)$ be an isometric tuple.

(1) V is a *unilateral shift* if it is unitarily equivalent to a multiple of $L_Z = (L_{Z_1}, \dots, L_{Z_d})$.

(2) V is *absolutely continuous* if the unital weak operator closed algebra $\mathrm{W}(V_1, \dots, V_d)$ generated by $V_1, \dots, V_d$ is algebraically isomorphic to the noncommutative analytic Toeplitz algebra F_d^∞.

(3) V is *singular* if the weakly closed algebra $\mathrm{W}(V_1, \dots, V_d)$ is a von Neumann algebra.

(4) V is of *dilation type* if it has no summand that is absolutely continuous or singular.

Theorem 2.2 (Lebesgue–von Neumann–Wold decomposition [Kennedy 2013]). *Let $V = (V_1, \dots, V_d)$ be an isometric d-tuple. Then V can be decomposed as*

$$V = V_u \oplus V_a \oplus V_s \oplus V_d,$$

where V_u is a unilateral d-shift, V_a is an absolutely continuous unitary d-tuple, V_s is a singular unitary d-tuple and V_d is a unitary d-tuple of dilation type.

Theorem 2.3 (structure theorem for free semigroup algebras [Davidson et al. 2001]). *Let $V = (V_1, \dots, V_d)$ be an isometric d-tuple, and let $\mathcal{V} = \mathrm{W}(V_1, \dots, V_d)$ denote the unital weak operator closed algebra generated by $V_1, \dots, V_d$. Then there is a maximal projection P in $\mathcal{V}$ with the range of P coinvariant for $\mathcal{V}$ such that*

(1) $\mathcal{V}P = \bigcap_{k \geq 1} (\mathcal{V}_0)^k$, *where $(\mathcal{V}_0)^k$ denotes the ideal $(\mathcal{V}_0)^k = \sum_{|w|=k} V_w \mathcal{V}$.*

(2) *If $P^{\perp} \neq 0$, then the restriction of $\mathcal{V}$ to the range of $P^{\perp}$ is an analytic free semigroup algebra.*

(3) *The compression of $\mathcal{V}$ to the range of P is a von Neumann algebra.*

(4) $\mathcal{V} = P^{\perp}\mathcal{V}P^{\perp} + \mathrm{W}^*(V)P$.

Let $V = V_u \oplus V_a \oplus V_s \oplus V_d$ be the Lebesgue–von Neumann–Wold decomposition of an isometric tuple V, as in Theorem 2.2, where V_u is a unilateral n-shift, V_a is an absolutely continuous unitary n-tuple, V_s is a singular unitary n-tuple and V_d is a unitary n-tuple of dilation type. Suppose that V is defined on a Hilbert space H, and let $H = H_u \oplus H_a \oplus H_s \oplus H_d$ denote the corresponding decomposition of H. By Corollary 2.7 of [Davidson et al. 2001], there is a maximal invariant subspace K for V_d such that the restriction of V_d to K is analytic. The projection P in Theorem 2.3 is determined by $P^{\perp} = P_{H_u} \oplus P_{H_a} \oplus P_K$.

Remark 2.4. For $d = 1$, an isometry is the direct sum of a unilateral shift, an absolutely continuous unitary and a singular unitary. Theorem 2.3 implies that, in this case, the structure projection P is the projection onto the singular unitary part. In particular, this implies that P is reducing. For $d \geq 2$, the proof of Theorem 2.3 shows that P is reducing if and only if there is no summand of dilation type.

The universal representation. We require the universal representation $\pi_u : F_d^{\infty} \to \mathcal{B}(H_u)$ of F_d^{∞}. This can be constructed as in 2.4.4 of [Blecher and Le Merdy 2004], as the restriction of the universal representation of $\mathrm{C}^*_{\max}(F_d^{\infty})$. By [ibid., 3.2.12], we can identify the double dual $(F_d^{\infty})^{**}$ of F_d^{∞} with the algebra obtained as the weak-* closure of $\pi_u(F_d^{\infty})$. We will require the operator algebra structure on $(F_d^{\infty})^{**}$ provided by this identification. By replacing π_u by $\pi_u^{(\infty)}$ if necessary, we can suppose that π_u has infinite multiplicity, and hence that the weak operator topology coincides with the weak-* topology on $(F_d^{\infty})^{**}$.

Let ϕ be a bounded linear functional on F_d^{∞}. By the Hahn–Banach theorem, we can extend ϕ to a functional on $\mathrm{C}^*_{\max}(F_d^{\infty})$ with the same norm. Hence by the construction of the universal representation of $\mathrm{C}^*_{\max}(F_d^{\infty})$, there are vectors x and y in H_u with $\|x\| \|y\| = \|\phi\|$ such that

$$\phi(A) = \langle \pi_u(A)x, y \rangle \quad \text{for all } A \in F_d^{\infty}.$$

If we identify F_d^∞ with its image $\pi_u(F_d^\infty)$ in $(F_d^\infty)^{**}$, then the functional ϕ has a unique weak-* continuous extension to a functional on $(F_d^\infty)^{**}$ with the same norm. We will use this fact repeatedly.

Since π_u is the restriction of a *-homomorphism of $\mathrm{C}^*_{\max}(F_d^\infty)$, and since the d-tuple $(L_{Z_1}, \dots, L_{Z_d})$ is isometric, it follows that the d-tuple $(\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d}))$ is also isometric. Since $(F_d^\infty)^{**}$ contains $\pi_u(A_d)$, it necessarily contains the weak operator closed algebra generated by $(\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d}))$. Let P_u denote the projection in $(F_d^\infty)^{**}$ guaranteed by Theorem 2.3. We will refer to P_u as the universal structure projection in $(F_d^\infty)^{**}$.

Remark 2.5. Let $\mathcal{S}$ denote the unital weak operator closed algebra generated by $\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d})$. From above we have $\mathcal{S} \subseteq (F_d^\infty)^{**}$, and one might guess that $\mathcal{S} = (F_d^\infty)^{**}$. However, this is not the case. Indeed, let ϕ be a bounded nonzero functional on F_d^∞ that is zero on the noncommutative disk algebra A_d. Then as above, there are vectors x and y in H_u such that

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

Let ψ denote the weak operator continuous functional on $\mathcal{S}$ defined by

$$\psi(S) = \langle Sx, y\rangle \quad \text{for all } S \in \mathcal{S}.$$

Since ϕ is zero on A_d, ψ must be zero on $\pi_u(A_d)$. Then, since $\pi_u(A_d)$ is weak operator dense in $\mathcal{S}$, it follows that $\psi(S) = \langle Sx, y\rangle = 0$ for all S in $\mathcal{S}$. But, by assumption, there is A in F_d^∞ such that $\phi(A) = \langle \pi_u(A)x, y\rangle \neq 0$. So we see that $\pi_u(A) \notin \mathcal{S}$, and hence that the inclusion $\mathcal{S} \subseteq (F_d^\infty)^{**}$ is proper.

3. The Lebesgue decomposition

In this section, we introduce the definitions of absolutely continuous and singular linear functionals on the noncommutative analytic Toeplitz algebra F_d^∞, and establish the first version of the Lebesgue decomposition. In [Davidson et al. 2005], Davidson, Li and Pitts proved a Lebesgue-type decomposition for functionals on the noncommutative disk algebra A_d. Although the algebra F_d^∞ is bigger than A_d, the next definition is closely related to (and directly inspired by) the corresponding definition for A_d.

Definition 3.1. Let ϕ be a bounded linear functional on F_d^∞. Then

(1) ϕ is *absolutely continuous* if it is weak-* continuous, and

(2) ϕ is *singular* if $\|\phi\| = \|\phi^k\|$ for every $k \geq 1$, where ϕ^k denotes the restriction of ϕ to the ideal of F_d^∞ generated by $\{L_{Z_w} \mid |w| = k\}$.

Let ϕ be a bounded linear functional on F_d^∞. Then as in Section 2, there are vectors x and y in H_u with $\|x\|\|y\| = \|\phi\|$ such that

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

We will write $P_u\phi$ and $P_u^\perp\phi$ for the linear functionals defined on F_d^∞ by

$$(P_u\phi)(A) = \langle \pi_u(A)P_u x, y\rangle \quad \text{for all } A \in F_d^\infty,$$
$$(P_u^\perp\phi)(A) = \langle \pi_u(A)P_u^\perp x, y\rangle \quad \text{for all } A \in F_d^\infty,$$

where P_u denotes the universal structure projection from Section 2. The purpose of the next result is to verify that $P_u\phi$ and $P_u^{\perp}\phi$ are well defined.

Lemma 3.2. *Let ϕ be a bounded linear functional on F_d^∞. Then the functionals $P_u\phi$ and $P_u^{\perp}\phi$, as defined above, do not depend on the choice of vectors x and y.*

Proof. Let x_1, y_1 and x_2, y_2 be pairs of vectors in H_u such that

$$\langle \pi_u(A)x_1, y_1\rangle = \langle \pi_u(A)x_2, y_2\rangle \quad \text{for all } A \in F_d^\infty.$$

Since $\pi_u(F_d^\infty)$ is weak-* dense in the algebra $(F_d^\infty)^{**}$, which contains P_u, it follows immediately that

$$\langle \pi_u(A)P_u x_1, y_1\rangle = \langle \pi_u(A)P_u x_2, y_2\rangle \quad \text{for all } A \in F_d^\infty,$$

and similarly that

$$\langle \pi_u(A)P_u^{\perp} x_1, y_1\rangle = \langle \pi_u(A)P_u^{\perp} x_2, y_2\rangle \quad \text{for all } A \in F_d^\infty. \qquad \square$$

Proposition 3.3. *A bounded functional ϕ on F_d^∞ is singular if and only if $\phi = P_u\phi$.*

Proof. Let ϕ be a singular functional on F_d^∞. We can assume that $\|\phi\| = 1$. As in Section 2, there are vectors x and y in H_u such that $\|x\|\|y\| = 1$ and

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

By the singularity of ϕ, we can find a sequence (A_k) of elements in F_d^∞ such that $\lim \phi(A_k) \to 1$, and such that each A_k belongs to the unit ball of $(F_{d,0}^\infty)^k = \sum_{|w|=k} F_d^\infty L_{Z_w}$. Let T be an accumulation point of the sequence $(\pi_u(A_k))$ in $(F_d^\infty)^{**}$, and let $\mathcal{S}$ denote the unital weak operator closed algebra generated by $(\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d}))$. It is clear that the weak-* closure of the image $\pi_u((F_{d,0}^\infty)^k)$ of the ideal $(F_{d,0}^\infty)^k$ can be written as $(F_d^\infty)^{**}\mathcal{S}_0^k$, where $\mathcal{S}_0$ denotes the ideal in $\mathcal{S}$ generated by $\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d})$. Thus $\pi_u(A_k)$ belongs to $(F_d^\infty)^{**}\mathcal{S}_0^k$. By Theorem 2.3, $\mathcal{S}P_u = \bigcap_{k\ge 1} \mathcal{S}_0^k$. Hence T belongs to the unit ball of

$$\bigcap_{k\ge 1} (F_d^\infty)^{**}\mathcal{S}_0^k = (F_d^\infty)^{**} \bigcap_{k\ge 1} \mathcal{S}_0^k = (F_d^\infty)^{**} P_u.$$

In particular, this means that $T = TP_u$. Since $\phi(T) = 1$, this gives

$$\|x\|\|y\| = 1 = \langle Tx, y\rangle = \langle TP_u x, y\rangle \le \|P_u x\|\|y\| \le \|x\|\|y\|.$$

Hence $P_u x = x$, and it follows that $\phi = P_u\phi$.

Conversely, let ϕ be a functional on F_d^∞ such that $\phi = P_u\phi$. As before, we can assume that $\|\phi\| = 1$, and there are vectors x and y in H_u such that $\|x\|\|y\| = 1$ and

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

The fact that $P_u\phi = \phi$ implies that we can choose x satisfying $x = P_u x$, and hence that

$$\phi(A) = \langle \pi_u(A)P_u x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

Let ψ denote the functional on $(F_d^\infty)^{**}$ defined by

$$\psi(T) = \langle T P_u x, y\rangle \quad \text{for all } T \in (F_d^\infty)^{**},$$

and for $k \geq 1$, let ψ^k denote the restriction of ψ to $(F_d^\infty)^{**}\mathcal{S}_0^k$. Then as above,

$$(F_d^\infty)^{**} P_u = \bigcap_{k\geq 1} (F_d^\infty)^{**}\mathcal{S}_0^k.$$

Hence $\|\psi\| = \|\psi^k\|$ for every $k \geq 1$. It follows that $\|\phi\| = \|\phi^k\|$, where ϕ^k is defined as in Definition 3.1, and hence that ϕ is singular. □

Lemma 3.4. *The range of the projection $P_u^\perp$ is invariant for $(F_d^\infty)^{**}$.*

Proof. It suffices to show that whenever x and y are vectors in F_d^2 such that $x = P_u^\perp x$ and $y = P_u y$, and the functional ϕ on F_d^∞ is defined by

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty,$$

then $\phi = 0$. By Theorem 2.3, the range of $P_u^\perp$ is invariant for $\pi_u(A_d)$. Hence ϕ is zero on A_d. Let A be an element of F_d^∞. By Corollary 2.6 of [Davidson and Pitts 1998a], for $k \geq 1$, we can write A uniquely as

$$A = \sum_{|w|<k} a_w L_{Z_w} + A',$$

where the a_w are scalars, and A' belongs to $(F_{d,0}^\infty)^k$. The fact that ϕ is zero on A_d implies that $\phi(A) = \phi(A')$. It follows from Definition 3.1 that ϕ is singular. Hence by Proposition 3.3, $\phi = P_u\phi$, i.e.,

$$\phi(A) = \langle \pi_u(A) P_u x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

Since $x = P_u^\perp x$, it follows that $\phi = 0$, as required. □

Proposition 3.5. *Let ϕ be a bounded linear functional on F_d^∞. Then ϕ is absolutely continuous if and only if $\phi = P_u^\perp\phi$.*

Proof. Suppose first that ϕ is absolutely continuous. Then it is weak-* continuous, so there are sequences of vectors (x_k) and (y_k) in F_d^2 such that

$$\phi(A) = \sum \langle A x_k, y_k\rangle \quad \text{for all } A \in F_d^\infty.$$

Since the d-tuple $(L_{Z_1}, \ldots, L_{Z_d})$ is equivalent to a restriction of the unilateral shift part of the d-tuple $(\pi_u(L_{Z_1}), \ldots, \pi_u(L_{Z_d}))$, F_d^2 can be identified with a subspace of H_u, and it follows that $\phi = P_u^\perp\phi$.

Conversely, suppose that $\phi = P_u^\perp\phi$. As in Section 2, there are vectors x and y in H_u with $\|x\|\|y\| = \|\phi\|$ such that

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

The fact that $\phi = P_u^\perp\phi$ implies that we can choose x satisfying $P_u^\perp x = x$. Since, by Lemma 3.4, the range of $P_u^\perp$ is invariant for $\pi_u(F_d^\infty)$, it follows that for every A in F_d^∞, we have

$$\phi(A) = \langle \pi_u(A)x, y\rangle = \langle P_u^\perp \pi_u(A) P_u^\perp x, y\rangle = \langle \pi_u(A) P_u^\perp x, P_u^\perp y\rangle.$$

Hence we can also choose y satisfying $P_u^{\perp} y = y$.

By the construction of P_u, the restriction of the operators $\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d})$ to the cyclic subspace generated by x and y is analytic. Thus, by the main result of [Kennedy 2013], the weak-* closed algebra generated by this restriction is completely isometrically isomorphic and weak-* to weak-* homeomorphic to F_d^∞. It follows that ϕ is weak-* continuous on F_d^∞. ☐

Theorem 3.6 (Lebesgue decomposition for F_d^∞). *Let ϕ be a bounded linear functional on F_d^∞. Then there are unique bounded linear functionals ϕ_a and ϕ_s on F_d^∞ such that $\phi = \phi_a + \phi_s$, where ϕ_a is absolutely continuous and ϕ_s is singular, and such that*

$$\|\phi\| \le \|\phi_a\| + \|\phi_s\| \le \sqrt{2}\|\phi\|.$$

If $d = 1$, then the constant $\sqrt{2}$ can be replaced with the constant 1.

Proof. As in Section 2, there are vectors x and y in H_u such that $\|x\|\|y\| = \|\phi\|$ and

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

Define ϕ_a and ϕ_s by $\phi_a = P_u^{\perp}\phi$ and $\phi_s = P_u\phi$, respectively. Then ϕ_a is absolutely continuous by Proposition 3.5, and ϕ_s is singular by Proposition 3.3. We clearly have $\phi = \phi_a + \phi_s$. To see that ϕ_a and ϕ_s are unique, suppose that

$$\phi_a + \phi_s = \psi_a + \psi_s,$$

where ψ_a is absolutely continuous and ψ_s is absolutely continuous. Then

$$\phi_a - \psi_a = \psi_s - \phi_s.$$

It is clear that the functional $\phi_a - \psi_a$ is absolutely continuous, and Proposition 3.3 implies that the functional $\psi_s - \phi_s$ is singular. Applying Proposition 3.5 and Proposition 3.3 again, we can therefore write

$$\phi_a - \psi_a = P_u^{\perp}(\phi_a - \psi_a) = P_u^{\perp}(\psi_s - \phi_s) = P_u P_u^{\perp}(\psi_s - \phi_s) = 0.$$

Hence $\phi_a = \psi_a$, and it follows similarly that $\phi_s = \psi_s$. Finally, we compute

$$\|\phi\| \le \|\phi_a\| + \|\phi_s\| \le \|Px\|\|y\| + \|P^{\perp}x\|\|y\| \le \sqrt{2}\|x\|\|y\| = \sqrt{2}\|\phi\|.$$

If $d = 1$, then Remark 2.4 implies that $(F_d^\infty)^{**}$ is the direct sum of two algebras reduced by P_u. If we identify F_d^∞ with its image in $(F_d^\infty)^{**}$, then the functionals ϕ, ϕ_a and ϕ_s extend uniquely to weak-* continuous functionals on $(F_d^\infty)^{**}$ with the same norm. Since $\phi_a = P_u^{\perp}\phi_a$ and $\phi_s = P_u\phi_s$, it follows that in this case, $\|\phi\| = \|\phi_a\| + \|\phi_s\|$. ☐

The next example is based on Example 5.10 from [Davidson et al. 2005]. It establishes that for $d \ge 2$, the constant $\sqrt{2}$ in the statement of Theorem 3.6 is the best possible.

Example 3.7. Define ϕ on $\mathbb{C}\langle Z\rangle$ by setting

$$\phi(L_{Z_w}) = \begin{cases} 1/\sqrt{2} & \text{if } w = \varnothing \text{ or } w = 21^n \text{ for } n \ge 0, \\ 0 & \text{otherwise,} \end{cases}$$

and extending by linearity. We will first show that ϕ extends to a bounded linear functional on the noncommutative disk algebra A_2. Let $\mathcal{H}_\phi$ denote the Hilbert space $\mathbb{C}e \oplus F^2_2,/$ and define a 2-tuple $S = (S_1, S_2)$ on $\mathcal{H}_\phi$ by setting

$$S_1 = \begin{pmatrix} I & 0 \\ 0 & L_1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 \\ \xi_\varnothing e^* & L_2 \end{pmatrix}.$$

It is easy to check that S is isometric. By the universal property of the noncommutative disk algebra, we obtain a completely isometric representation π_ϕ of A_2 satisfying

$$\pi_\phi(L_{Z_w}) = S_{w_1} \cdots S_{w_n}, \quad w = w_1 \cdots w_n \in \mathbb{F}^*_d,$$

and we can extend ϕ to A_2 by

$$\phi(A) = \langle \pi_\phi(A)(e + \xi_\varnothing)/\sqrt{2}, \xi_\varnothing \rangle, \quad A \in A_2.$$

From this, it is easy to check that $\|\phi\| \leq 1$.

Let $\mathcal{S}$ denote the unital weakly closed algebra generated by S_1 and S_2. The structure projection from Theorem 2.3 is the projection P onto $\mathbb{C}e$, which is contained in $\mathcal{S}$. Hence $\mathcal{S}$ contains the element $B = (S_2 P + P^\perp)/\sqrt{2}$. The results of [Kennedy 2011] imply that Theorem 5.4 of [Davidson et al. 2005] applies to the unital weak operator closed algebra generated by any isometric tuple. Thus there is a net (B_λ) of elements in the unit ball of A_d such that w*-lim $\pi_\phi(B_\lambda) = B$ in $\mathcal{S}$. It is easy to check that $\|B\| = 1$ and $\langle B(e + \xi_\varnothing)/\sqrt{2}, \xi_\varnothing \rangle = 1$, so it follows that $\|\phi\| = 1$.

By the Hahn–Banach theorem, we can extend ϕ to a functional on F^∞_d with the same norm, which we continue to denote by ϕ. Let $\phi = \phi_a + \phi_s$ be the Lebesgue decomposition of ϕ into absolutely continuous and singular parts as in Theorem 3.6. Then restricted to A_d, we can write

$$\begin{aligned} \phi_a(A) &= (P^\perp \phi)(A) = \langle \pi(A)\xi_\varnothing/\sqrt{2}, \xi_\varnothing \rangle, && A \in A_2, \\ \phi_s(A) &= (P\phi)(A) = \langle \pi(A)e/\sqrt{2}, \xi_\varnothing \rangle, && A \in A_2. \end{aligned}$$

Letting B be as above, an easy computation gives

$$\langle B\xi_\varnothing/\sqrt{2}, \xi_\varnothing \rangle = \langle Be/\sqrt{2}, \xi_\varnothing \rangle = 1/\sqrt{2}.$$

Arguing as before, this implies $\|\phi_a\| \geq 1/\sqrt{2}$ and $\|\phi_s\| \geq 1/\sqrt{2}$. By Theorem 3.6, it follows that $\|\phi_a\| + \|\phi_s\| = \sqrt{2}\|\phi\|$.

Remark 3.8. It is well known that the algebra H^∞ is completely isometrically isomorphic to a subalgebra of $L^\infty(\mathbb{T})$. Ando [1978] used this fact to define a notion of absolute continuity and singularity for functionals on H^∞. Namely, a functional on H^∞ is absolutely continuous in the sense of [ibid.] if it extends to a normal functional on $L^\infty(\mathbb{T})$, and singular in the sense of [ibid.] if it extends to a singular functional on $L^\infty(\mathbb{T})$ (see Chapter 2 of [Takesaki 2002] for the definition of a singular functional on a von Neumann algebra). We now show that these definitions agree with Definition 3.1.

It is clear that a functional on H^∞ that is absolutely continuous in the sense of Definition 3.1 is also absolutely continuous in the sense of [Ando 1978]. Let ϕ be a functional on H^∞ that is singular in the sense of Definition 3.1. A Lebesgue decomposition theorem also holds for H^∞ using the definition of absolute continuity and singularity from [Ando 1978] (see, for example, [Ueda 2009]). Hence there are functionals $\tilde{\phi}_a$ and $\tilde{\phi}_s$ on F_d^∞ such that $\phi = \tilde{\phi}_a + \tilde{\phi}_s$, where $\tilde{\phi}_a$ is absolutely continuous in the sense of [Ando 1978], and $\tilde{\phi}_s$ is singular in the sense of [ibid.]. Moreover, $\|\phi\| = \|\tilde{\phi}_a\| + \|\tilde{\phi}_s\|$. Note that $\tilde{\phi}_a$ is absolutely continuous (in our sense). This implies that

$$\|\tilde{\phi}_s\| \le \|\phi\| = \limsup \|\phi^k\| \le \limsup(\|\tilde{\phi}_a^k\| + \|\tilde{\phi}_s^k\|) = \limsup \|\tilde{\phi}_s^k\| \le \|\tilde{\phi}_s\|.$$

Hence $\phi = \tilde{\phi}_s$ and ϕ is singular in the sense of [ibid.].

Now let ϕ be an arbitrary functional on H^∞, let $\phi = \phi_a + \phi_s$ be the Lebesgue decompositions of ϕ as in Theorem 3.6, and let $\phi = \tilde{\phi}_a + \tilde{\phi}_s$ be the Lebesgue decomposition of ϕ as in [ibid.]. Then $\phi_a - \tilde{\phi}_a = \tilde{\phi}_s - \phi_s$. From above, $\phi_a - \tilde{\phi}_a$ is absolutely continuous in the sense of [ibid.], and $\tilde{\phi}_s - \phi_s$ is singular in the sense of [ibid.]. Hence by the uniqueness of the Lebesgue decomposition, $\phi_a = \tilde{\phi}_a$ and $\phi_s = \tilde{\phi}_s$.

We note that Definition 3.1 gives an intrinsic characterization of singular functionals on H^∞, which answers (at least in this classical setting) a question from [Ueda 2009]. For $d \ge 2$, it would be interesting to know if there is an appropriate noncommutative analogue of $L^\infty(\mathbb{T})$ with a subalgebra that is completely isometrically isomorphic to F_d^∞.

4. The extended F. and M. Riesz theorem

The results in this section can be viewed as noncommutative generalizations of the classical results referred to as the F. and M. Riesz theorem. As mentioned in the introduction, results of this kind have been established in different settings by Exel [1990], by Blecher and Labuschagne [2007], and by Ueda [2009]. In fact, Blecher and Labuschagne seem to have anticipated that an F. and M. Riesz-type theorem should hold for F_d^∞ (see the introduction of [Blecher and Labuschagne 2007]).

Theorem 4.1 (extended F. and M. Riesz theorem). *Let ϕ be a bounded linear functional on F_d^∞, and let $\phi = \phi_a + \phi_s$ be the Lebesgue decomposition of ϕ into absolutely continuous and singular parts as in Theorem 3.6. Let $\mathcal{I}$ be a two-sided ideal of F_d^∞. If ϕ is zero on $\mathcal{I}$, then ϕ_a and ϕ_s are both zero on $\mathcal{I}$.*

Proof. As in Section 2, there are vectors x and y in H_u such that

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

By Proposition 3.5 we can write $\phi_a = P_u^\perp \phi$, and by Proposition 3.3 we can write $\phi_s = P_u \phi$. If we identify F_d^∞ with its image $\pi_u(F_d^\infty)$ in $(F_d^\infty)^{**}$, then the functionals ϕ, ϕ_a and ϕ_s each have unique weak-* continuous extensions to functionals on $(F_d^\infty)^{**}$ with the same norm.

Let $\mathcal{J}$ denote the ideal in $(F_d^\infty)^{**}$ obtained by taking the weak-* closure of $\pi_u(\mathcal{I})$. Since ϕ is zero on $\mathcal{I}$, it is zero on $\mathcal{J}$. For A in $\mathcal{I}$, $\pi_u(A)P_u^\perp$ belongs to $\mathcal{J}$, which implies

$$0 = (P_u^\perp \phi)(A) = \phi_a(A).$$

Hence ϕ_a is zero on $\mathcal{I}$, and it follows immediately that ϕ_s is also zero on $\mathcal{I}$. $\square$

Corollary 4.2 (F. and M. Riesz theorem). *Let ϕ be a bounded linear functional on F_d^∞. If ϕ is zero on $F_{d,0}^\infty$, where $F_{d,0}^\infty$ denotes the ideal of F_d^∞ generated by $L_{Z_1}, \dots, L_{Z_d}$, then ϕ is absolutely continuous.*

Proof. Let $\phi = \phi_a + \phi_s$ be the Lebesgue decomposition of ϕ into absolutely continuous and singular parts as in Theorem 3.6. By Theorem 4.1, ϕ_a and ϕ_s are both zero on $F_{d,0}^\infty$. By Definition 3.1, if ϕ_s is zero on $F_{d,0}^\infty$, it is necessarily zero on all of F_d^∞. Hence $\phi = \phi_a$ and ϕ is absolutely continuous. □

5. Quotient algebras

For a weak-* closed two-sided ideal $\mathcal{J}$ of F_d^∞, let $\mathcal{A}_I$ denote the quotient algebra $F_d^\infty/\mathcal{J}$.

Definition 5.1. Let $\mathcal{J}$ be a weak-* closed two-sided ideal of F_d^∞, and let ϕ be a bounded functional on $\mathcal{A}_\mathcal{J}$. Then

(1) ϕ is *absolutely continuous* if it is weak-* continuous, and

(2) ϕ is *singular* if $\|\phi\| = \|\phi^k\|$ for every $k \geq 1$, where ϕ^k denotes the restriction of ϕ to the ideal of $\mathcal{A}_\mathcal{J}$ generated by $\{\overline{L_{Z_w}} \mid |w| = k\}$, where for a word w in $\mathbb{F}_d^*$, $\overline{L_{Z_w}}$ denotes the image in $\mathcal{A}_\mathcal{J}$ of L_{Z_w}.

Theorem 5.2 (Lebesgue decomposition for quotients of F_d^∞). *Let $\mathcal{J}$ be a weak-* closed two-sided ideal of F_d^∞, and let ϕ be a bounded linear functional on $\mathcal{A}_\mathcal{J}$. Then there are unique linear functionals ϕ_a and ϕ_s on $\mathcal{A}_\mathcal{J}$ such that $\phi = \phi_a + \phi_s$, where ϕ_a is absolutely continuous and ϕ_s is singular, and such that*

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|.$$

If $d = 1$, then the constant $\sqrt{2}$ can be replaced with the constant 1.

Proof. By basic functional analysis, we can lift the functional ϕ to a functional ψ on F_d^∞ with the same norm. Let $\psi = \psi_a + \psi_s$ be the Lebesgue decomposition of ψ into absolutely continuous and singular parts as in Theorem 3.6. The functional ψ annihilates $\mathcal{J}$, so by Theorem 4.1, both ψ_a and ψ_s annihilate $\mathcal{J}$. Hence ψ_a and ψ_s induce functionals ϕ_a and ϕ_s on $\mathcal{A}_\mathcal{J}$, respectively, with the same norm. Clearly $\phi = \phi_a + \phi_s$, and the inequality

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|$$

follows from the corresponding inequality in Theorem 3.6. The functional ϕ_a is absolutely continuous since ψ_a is absolutely continuous on F_d^∞. To see that ϕ_s is singular, simply note that for every $k \geq 1$, the ideal $(\mathcal{A}_{\mathcal{J},0})^k$ is the image in $\mathcal{A}_\mathcal{J}$ of the ideal $(F_{d,0}^\infty)^k$. □

Corollary 5.3 (Lebesgue decomposition for multiplier algebras). *Let $\mathcal{A}$ be the multiplier algebra of a complete Nevanlinna–Pick space, and let ϕ be a bounded linear functional on $\mathcal{A}$. Then there are unique linear functionals ϕ_a and ϕ_s on $\mathcal{A}$ such that $\phi = \phi_a + \phi_s$, where ϕ_a is absolutely continuous and ϕ_s is singular, and such that*

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|.$$

6. Uniqueness of the predual

Let X and Y be Banach spaces such that $X^* = Y$. Then X is said to be a predual for Y. Every predual X of Y naturally embeds into the dual space Y^*, and a subspace X of Y^* is a predual of Y if and only if it satisfies the following properties:

(1) The subspace X norms Y, i.e., $\sup\{|x(y)| \mid x \in X,\ \|x\| \leq 1\} = \|y\|$ for all y in Y.

(2) The closed unit ball of Y is compact in the $\sigma(Y, X)$ topology.

The space Y is said to have a strongly unique predual if there is a unique subspace X of Y^* such that $Y = X^*$. For a survey on uniqueness results for preduals, we refer the reader to [Godefroy 1989].

In the operator-theoretic setting, the results of Sakai [1956], Ando [1978] and Ueda [2009] mentioned in the introduction established that von Neumann algebras and maximal subdiagonal algebras have unique preduals. Ruan [1992] proved that an operator algebra with a weak-* dense subalgebra of compact operators has a unique predual, which applies to, for example, nest algebras and atomic CSL algebras. Effros, Ozawa and Ruan proved in [Effros et al. 2001] that a W*TRO (i.e., a corner of a von Neumann algebras) has a unique predual. More recently, Davidson and Wright [2011] proved that a free semigroup algebra has a unique predual. Note that Davidson and Wright's result applies to F_d^∞, but not to quotients of F_d^∞.

The following definition is closely related to the notion of an M-ideal in a Banach space (see [Harmand et al. 1993] for more information).

Definition 6.1. A Banach space X is L-embedded if there is a projection P on the bidual X^{**} with range X such that

$$\|x\| = \|Px\| + \|x - Px\| \quad \text{for all } x \in X^{**}.$$

The following result of Pfitzner implies that every separable L-embedded space has Godefroy and Talagrand's property (X), and hence by a result of Godefroy and Talagrand [1981], that it is the unique predual of its dual.

Theorem 6.2 [Pfitzner 2007]. *Separable L-embedded spaces have property* (X).

The results of Sakai, Ando and Ueda on decompositions of linear functionals imply that the preduals of von Neumann algebras and maximal subdiagonal algebras are L-embedded, and hence by Pfitzner's theorem, that they are unique. However, Example 3.7 shows that preduals of quotients of F_d^∞ are not, in general, L-embedded, so we are unable to use Pfitzner's result. Instead, we give a direct proof that quotients of F_d^∞ have (strongly) unique preduals.

Theorem 6.3. *Let $\mathcal{J}$ be a weak-* closed two-sided ideal of F_d^∞. Then the algebra $\mathcal{A}_\mathcal{J}$ has a strongly unique predual.*

Proof. Suppose E is a predual for $\mathcal{A}_\mathcal{J}$, identified with a subspace of $(\mathcal{A}_\mathcal{J})^*$. By Theorem 5.2,

$$(\mathcal{A}_\mathcal{J})^* = (\mathcal{A}_\mathcal{J})^*_a \oplus (\mathcal{A}_\mathcal{J})^*_s,$$

where $(\mathcal{A}_{\mathcal{J}})^*_a$ and $(\mathcal{A}_{\mathcal{J}})^*_s$ denote the set of absolutely continuous and singular functionals on $\mathcal{A}_{\mathcal{J}}$, respectively. We want to prove that $E = (\mathcal{A}_{\mathcal{J}})^*_a$.

Let ϕ be a functional in E, and let $\phi = \phi_a + \phi_s$ be the Lebesgue decomposition of ϕ as in Theorem 5.2. We will prove that $\phi_s = 0$. Suppose to the contrary that $\phi_s \neq 0$. By basic functional analysis, we can lift the functional ϕ to a functional ψ on F_d^∞ that is zero on $\mathcal{J}$. Let $\psi = \psi_a + \psi_s$ be the Lebesgue decomposition of ψ as in Theorem 3.6. By Theorem 4.1, ψ_a and ψ_s are both zero on $\mathcal{J}$, and by construction they induce the functionals ϕ_a and ϕ_s, respectively, on the quotient $\mathcal{A}_{\mathcal{J}}$.

It follows from the results of [Kennedy 2011] that Theorem 5.4 of [Davidson et al. 2005] applies to the unital weak operator closed algebra generated by any isometric tuple. Thus there is a net (B_λ) of elements in the unit ball of F_d^∞ such that w*-lim $\pi_u(B_\lambda) = P_u$ in $(F_d^\infty)^{**}$. Since the net (B_λ) is weak-* convergent in $(F_d^\infty)^{**}$, it is weakly Cauchy in F_d^∞. Since the closed unit ball of F_d^∞ is compact in the weak-* topology, and in particular is complete, this implies that there is B in the closed unit ball of F_d^∞ such that w*-lim $B_\lambda = B$ in F_d^∞. For every weak-* continuous functional τ on F_d^∞, Proposition 3.5 implies that

$$\tau(B) = \lim_\lambda \tau(B_\lambda) = (P_u \tau)(1) = 0.$$

Hence $B = 0$.

Let A be an element in the unit ball of F_d^∞ such that $\psi_s(A) \neq 0$. Since the net (B_λ) is weakly Cauchy in F_d^∞, the image $(\overline{B_\lambda})$ is weakly Cauchy in $\mathcal{A}_{\mathcal{J}}$. It follows that the net $(\overline{AB_\lambda})$ is also weakly Cauchy in $\mathcal{A}_{\mathcal{J}}$. Since E is a predual of $\mathcal{A}_{\mathcal{J}}$, the closed unit ball of $\mathcal{A}_{\mathcal{J}}$ is compact in the $\sigma(\mathcal{A}_{\mathcal{J}}, E)$ topology, and in particular is complete. Thus, the net $(\overline{AB_\lambda})$ converges in the $\sigma(\mathcal{A}_{\mathcal{J}}, E)$ topology to an element C in the unit ball of $\mathcal{A}_{\mathcal{J}}$. By Proposition 3.3, we have

$$\phi(C) = \lim_\lambda \phi(\overline{AB_\lambda}) = \lim_\lambda \psi(AB_\lambda) = (P_u \psi)(A) = \psi_s(A) \neq 0,$$

so that $C \neq 0$. But since w*-lim $B_\lambda = 0$ in F_d^∞, it follows that w*-lim $\overline{AB_\lambda} = 0$ in $\mathcal{A}_{\mathcal{J}}$. So for every τ in $(\mathcal{A}_{\mathcal{J}})^*_a$, we necessarily have

$$\tau(C) = \lim_\lambda \tau(\overline{AB_\lambda}) = 0.$$

Since $(\mathcal{A}_{\mathcal{J}})^*_a$ separates points, this implies that $C = 0$, which gives a contradiction. Thus $\phi = \phi_a$, meaning ϕ is absolutely continuous.

Since ϕ was arbitrary, it follows from above that every functional in E is absolutely continuous, i.e., that E is contained in $(\mathcal{A}_{\mathcal{J}})^*_a$. If it were the case that $E \neq (\mathcal{A}_{\mathcal{J}})^*_a$, then we could apply the Hahn–Banach theorem to separate E from $(\mathcal{A}_{\mathcal{J}})^*_a$ with an element of $\mathcal{A}_{\mathcal{J}}$. But the fact that E is a predual of $\mathcal{A}_{\mathcal{J}}$ means in particular it must norm $\mathcal{A}_{\mathcal{J}}$, so this is impossible. Therefore, we conclude that $E = (\mathcal{A}_{\mathcal{J}})^*_a$, and hence that $(\mathcal{A}_{\mathcal{J}})^*_a$ is the unique predual of $\mathcal{A}_{\mathcal{J}}$. □

Corollary 6.4. *The multiplier algebra of every complete Nevanlinna–Pick space has a strongly unique predual.*

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BOHR'S ABSOLUTE CONVERGENCE PROBLEM FOR $\mathcal{H}_p$-DIRICHLET SERIES IN BANACH SPACES

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The Bohr–Bohnenblust–Hille theorem states that the width of the strip in the complex plane on which an ordinary Dirichlet series $\sum_n a_n n^{-s}$ converges uniformly but not absolutely is less than or equal to $\frac{1}{2}$, and this estimate is optimal. Equivalently, the supremum of the absolute convergence abscissas of all Dirichlet series in the Hardy space $\mathcal{H}_\infty$ equals $\frac{1}{2}$. By a surprising fact of Bayart the same result holds true if $\mathcal{H}_\infty$ is replaced by any Hardy space $\mathcal{H}_p$, $1 \leq p < \infty$, of Dirichlet series. For Dirichlet series with coefficients in a Banach space X the maximal width of Bohr's strips depend on the geometry of X; Defant, García, Maestre and Pérez-García proved that such maximal width equals $1 - 1/\operatorname{Cot} X$, where $\operatorname{Cot} X$ denotes the maximal cotype of X. Equivalently, the supremum over the absolute convergence abscissas of all Dirichlet series in the vector-valued Hardy space $\mathcal{H}_\infty(X)$ equals $1 - 1/\operatorname{Cot} X$. In this article we show that this result remains true if $\mathcal{H}_\infty(X)$ is replaced by the larger class $\mathcal{H}_p(X)$, $1 \leq p < \infty$.

1. Main result and its motivation

Given a Banach space X, an ordinary Dirichlet series in X is a series of the form $D = \sum_n a_n n^{-s}$, where the coefficients a_n are vectors in X and s is a complex variable. Maximal domains where such Dirichlet series converge conditionally, uniformly or absolutely are half planes $[\mathrm{Re} > \sigma]$, where $\sigma = \sigma_c$, σ_u or σ_a are called the abscissa of conditional, uniform or absolute convergence, respectively. More precisely, $\sigma_\alpha(D)$ is the infimum of all $r \in \mathbb{R}$ such that on $[\mathrm{Re} > r]$ we have convergence of D of the requested type $\alpha = c, u$ or a. Clearly, we have $\sigma_c(D) \leq \sigma_u(D) \leq \sigma_a(D)$, and it can be easily shown that $\sup \sigma_a(D) - \sigma_c(D) = 1$, where the supremum is taken over all Dirichlet series D with coefficients in X. To determine the maximal width of the strip on which a Dirichlet series in X converges uniformly but not absolutely is more complicated. The main result of [Defant et al. 2008] states, with the notation given below, that

$$S(X) := \sup \sigma_a(D) - \sigma_u(D) = 1 - \frac{1}{\operatorname{Cot} X}\,. \tag{1}$$

Recall that a Banach space X is of cotype q, $2 \leq q < \infty$, whenever there is a constant $C \geq 0$ such that for each choice of finitely many vectors $x_1, \ldots, x_N \in X$ we have

$$\left(\sum_{k=1}^{N} \|x_k\|_X^q\right)^{1/q} \leq C\left(\int_{\mathbb{T}^N} \left\|\sum_{k=1}^{N} x_k z_k\right\|_X^2 dz\right)^{1/2}, \tag{2}$$

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where $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ and $\mathbb{T}^N$ is endowed with the N-th product of the normalized Lebesgue measure on $\mathbb{T}$; we denote the best of such constants C by $C_q(X)$. As usual we write

$$\operatorname{Cot} X := \inf\{2 \le q < \infty \mid X \text{ is of cotype } q\},$$

and, although this infimum in general is not attained, we call it the optimal cotype of X. If there is no $2 \le q < \infty$ for which X has cotype q, then X is said to have no finite cotype, and we put $\operatorname{Cot} X = \infty$. To see an example,

$$\operatorname{Cot} \ell_q = \begin{cases} q & \text{for } 2 \le q \le \infty, \\ 2 & \text{for } 1 \le q \le 2\,. \end{cases}$$

The scalar case $X = \mathbb{C}$ in (1) was first studied over a hundred years ago: Bohr [1913a] proved that $S(\mathbb{C}) \le \frac{1}{2}$, and Bohnenblust and Hille [1931] that $S(\mathbb{C}) \ge \frac{1}{2}$. Clearly, the equality

$$S(\mathbb{C}) = \tfrac{1}{2}, \tag{3}$$

nowadays called the *Bohr–Bohnenblust–Hille theorem*, fits with (1). Let us give a second formulation of (1). Define the vector space $\mathcal{H}_\infty(X)$ of all Dirichlet series $D = \sum_n a_n n^{-s}$ in X such that

- $\sigma_c(D) \le 0$,
- the function $D(s) = \sum_n a_n(1/n^s)$ on $\operatorname{Re} s > 0$ is bounded.

Then $\mathcal{H}_\infty(X)$ together with the norm

$$\|D\|_{\mathcal{H}_\infty(X)} = \sup_{\operatorname{Re} s > 0} \Bigl\| \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \Bigr\|_X$$

forms a Banach space. For any Dirichlet series D in X we have

$$\sigma_u(D) = \inf\Bigl\{\sigma \in \mathbb{R} \Bigm| \sum_n \frac{a_n}{n^\sigma} \frac{1}{n^s} \in \mathcal{H}_\infty(X)\Bigr\}. \tag{4}$$

In the scalar case $X = \mathbb{C}$, this is (what we call) *Bohr's fundamental theorem* [1913b], and for Dirichlet series in arbitrary Banach spaces the proof follows similarly. Together with (4) a simply translation argument gives the following reformulation of (1):

$$S(X) = \sup_{D \in \mathcal{H}_\infty(X)} \sigma_a(D) = 1 - \frac{1}{\operatorname{Cot} X}. \tag{5}$$

Following an ingenious idea of Bohr each Dirichlet series may be identified with a power series in infinitely many variables. More precisely, fix a Banach space X and denote by $\mathfrak{P}(X)$ the vector space of all formal power series $\sum_\alpha c_\alpha z^\alpha$ in X and by $\mathfrak{D}(X)$ the vector space of all Dirichlet series $\sum_n a_n n^{-s}$ in X. Let as usual $(p_n)_n$ be the sequence of prime numbers. Since each integer n has a unique prime number decomposition $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} = p^\alpha$ with $\alpha_j \in \mathbb{N}_0,\ 1 \le j \le k$, the linear mapping

$$\begin{aligned} \mathfrak{B}_X : \mathfrak{P}(X) &\to \mathfrak{D}(X), \\ \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha &\rightsquigarrow \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{if } a_{p^\alpha} = c_\alpha, \end{aligned} \tag{6}$$

is bijective; we call $\mathfrak{B}_X$ the *Bohr transform in* X. As discovered by Bayart [2002] this (a priori *very*) formal identification allows us to develop a theory of Hardy spaces of scalar–valued Dirichlet series.

Similarly, we now define Hardy spaces of X-valued Dirichlet series. Denote by dw the normalized Lebesgue measure on the infinite-dimensional polytorus $\mathbb{T}^\infty = \prod_{k=1}^\infty \mathbb{T}$, that is, the countable product measure of the normalized Lebesgue measure on $\mathbb{T}$. For any multiindex $\alpha = (\alpha_1, \dots, \alpha_n, 0, \dots) \in \mathbb{Z}^{(\mathbb{N})}$ (all finite sequences in $\mathbb{Z}$) the α-th Fourier coefficient $\hat{f}(\alpha)$ of $f \in L_1(\mathbb{T}^\infty, X)$ is given by

$$\hat{f}(\alpha) = \int_{\mathbb{T}^\infty} f(w) w^{-\alpha}\, dw,$$

where we as usual write w^α for the monomial $w_1^{\alpha_1} \cdots w_n^{\alpha_n}$. Then, given $1 \le p < \infty$, the X-valued Hardy space on $\mathbb{T}^\infty$ is the subspace of $L_p(\mathbb{T}^\infty, X)$ defined as

$$H_p(\mathbb{T}^\infty, X) = \{ f \in L_p(\mathbb{T}^\infty, X) \mid \hat{f}(\alpha) = 0 \text{ for all } \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})} \}. \tag{7}$$

Assigning to each $f \in H_p(\mathbb{T}^\infty, X)$ its unique formal power series $\sum_\alpha \hat{f}(\alpha) z^\alpha$ we may consider $H_p(\mathbb{T}^\infty, X)$ as a subspace of $\mathfrak{P}(X)$. We denote the image of this subspace under the Bohr transform $\mathfrak{B}_X$ by

$$\mathcal{H}_p(X).$$

This vector space of all (so-called) $\mathcal{H}_p(X)$-Dirichlet series D together with the norm

$$\|D\|_{\mathcal{H}_p(X)} = \|\mathfrak{B}_X^{-1}(D)\|_{H_p(\mathbb{T}^\infty, X)}$$

forms a Banach space; in other words, through Bohr's transform $\mathfrak{B}_X$ from (6) we by definition identify

$$\mathcal{H}_p(X) = H_p(\mathbb{T}^\infty, X), \quad 1 \le p < \infty.$$

For $p = \infty$ we this way of course could also define a Banach space $\mathcal{H}_\infty(X)$, and it turns out that at least in the scalar case $X = \mathbb{C}$ this definition then coincides with the one given above; but we remark that these two $\mathcal{H}_\infty(X)$'s are different for arbitrary X. It is important to note that by the Birkhoff–Khinchin ergodic theorem the following internal description of the $\mathcal{H}_p(X)$-norm for finite Dirichlet polynomials $D = \sum_{k=1}^n a_k k^{-s}$ holds:

$$\|D\|_{\mathcal{H}_p(X)} = \lim_{T \to \infty} \left(\frac{1}{2T} \int_{-T}^{T} \left\| \sum_{k=1}^{n} a_k \frac{1}{k^t} \right\|_X^p dt \right)^{1/p}$$

(see, for example, Bayart [2002] for the scalar case, and the vector-valued case follows exactly the same way).

Motivated by (4) we define for $D \in \mathfrak{D}(X)$ and $1 \le p < \infty$

$$\sigma_{\mathcal{H}_p(X)}(D) := \inf\left\{\sigma \in \mathbb{R} \,\middle|\, \sum_n \frac{a_n}{n^\sigma}\frac{1}{n^s} \in \mathcal{H}_p(X)\right\}, \tag{8}$$

the so-called $\mathcal{H}_p(X)$-abscissa of D. In [Aleman et al. $\ge$ 2014], Aleman, Olsen, and Saksman prove that the sequence (of Dirichlet series) $1/n^s$, $n \in \mathbb{N}$ is a Schauder basis in $\mathcal{H}_p(\mathbb{C})$ for $1 < p < \infty$. Hence, for $1 < p < \infty$ and any Dirichlet series $D \in \mathfrak{D}(\mathbb{C})$ we have

$$\sigma_{\mathcal{H}_p(\mathbb{C})}(D) = \inf\left\{\sigma \in \mathbb{R} \,\middle|\, \left(\sum_{n=1}^N \frac{a_n}{n^\sigma}\frac{1}{n^s}\right)_N \text{ is Cauchy in } \mathcal{H}_p(\mathbb{C})\right\}, \tag{9}$$

which (in the scalar case) is the perfect analog of Bohr's fundamental theorem (i.e., the case $p = \infty$ from (4), where uniform convergence is precisely being Cauchy in $\mathcal{H}_p(\mathbb{C})$). In [Defant 2013] it is shown that (9) also holds true for $p = 1$ (although in this case the $1/n^s$ are definitely no Schauder basis in $\mathcal{H}_1(\mathbb{C})$), and even more: The arguments given in [Defant 2013] (inspired by Bohr's original ideas [1913b]) prove that (9) even holds for any $1 \le p \le \infty$ and any X-valued Dirichlet series $D \in \mathcal{H}_p(X)$. In view of (1) and (5), it therefore seems natural to study

$$S_p(X) := \sup_{D \in \mathfrak{D}(X)} \sigma_a(D) - \sigma_{\mathcal{H}_p(X)}(D) = \sup_{D \in \mathcal{H}_p(X)} \sigma_a(D)$$

(for the second equality use again a simple translation argument). The scalar case is completely understood since, by a result of Bayart [2002],

$$S_p(\mathbb{C}) = \tfrac{1}{2} \quad \text{for every } 1 \le p < \infty, \tag{10}$$

which according to Helson [2005] is surprising since $\mathcal{H}_\infty(\mathbb{C})$ is much smaller than $\mathcal{H}_p(\mathbb{C})$.

The following theorem unifies and generalizes (1), (3) as well as (10), and it is our main result.

Theorem 1.1. *For every $1 \le p \le \infty$ and every Banach space X we have*

$$S_p(X) = 1 - \frac{1}{\operatorname{Cot} X}.$$

The proof will be given in Section 3. But before we start let us give an interesting reformulation in terms of the monomial convergence of X-valued H_p-functions on $\mathbb{T}^\infty$. Fix a Banach space X and $1 \le p \le \infty$, and define the set of monomial convergence of $H_p(\mathbb{T}^\infty, X)$:

$$\operatorname{mon} H_p(\mathbb{T}^\infty, X) = \left\{z \in B_{c_0} \,\middle|\, \sum_\alpha \|\hat{f}(\alpha) z^\alpha\|_X < \infty \text{ for all } f \in H_p(\mathbb{T}^\infty, X)\right\}.$$

Philosophically, this is the largest set M on which for each $f \in H_p(\mathbb{T}^\infty, X)$ the definition $g(z) = \sum_\alpha \hat{f}(\alpha) z^\alpha$, $z \in M$ leads to an extension of f from the distinguished boundary $\mathbb{T}^\infty$ to its "interior" B_{c_0} (the open unit ball of the Banach space c_0 of all null sequences). For a detailed study of sets of monomial convergence in the scalar case $X = \mathbb{C}$ see [Defant et al. 2009], and in the vector-valued case [Defant and Sevilla-Peris 2011].

We later need the following two basic properties of monomial domains (in the scalar case see [Defant et al. 2008, p. 550; 2014, Lemma 4.3], and in the vector-valued case the proofs follow similar lines).

Remark 1.2. (1) Let $z \in \operatorname{mon} H_p(\mathbb{T}^\infty, X)$. Then $u = (z_{\sigma(n)})_n \in \operatorname{mon} H_p(\mathbb{T}^\infty, X)$ for every permutation σ of $\mathbb{N}$.

(2) Let $z \in \operatorname{mon} H_p(\mathbb{T}^\infty, X)$ and $x = (x_n)_n \in \mathbb{D}^\infty$ be such that $|x_n| \leq |z_n|$ for all but finitely many n's. Then $x \in \operatorname{mon} H_p(\mathbb{T}^\infty, X)$.

Given $1 \leq p \leq \infty$ and a Banach space X, the following number measures the size of $\operatorname{mon} H_p(\mathbb{T}^\infty, X)$ within the scale of ℓ_r-spaces:

$$M_p(X) = \sup\{1 \leq r \leq \infty \mid \ell_r \cap B_{c_0} \subset \operatorname{mon} H_p(\mathbb{T}^\infty, X)\}.$$

The following result is a reformulation of Theorem 1.1 in terms of vector-valued H_p-functions on $\mathbb{T}^\infty$ through Bohr's transform $\mathfrak{B}_X$. The proof is modeled along ideas from Bohr's seminal article [1913a, Satz IX].

Corollary 1.3. *For each Banach space X and $1 \leq p \leq \infty$ we have*

$$M_p(X) = \frac{\operatorname{Cot} X}{\operatorname{Cot} X - 1}\,.$$

Proof. We are going to prove that $S_p(X) = 1/M_p(X)$, and as a consequence the conclusion follows from Theorem 1.1. We begin by showing that $S_p(X) \leq 1/M_p(X)$. We fix $q < M_p(X)$ and $r > 1/q$; then we have that $(1/p_n^r)_n \in \ell_q \cap B_{c_0}$ and, by the very definition of $M_p(X)$, $\sum_\alpha \|\hat{f}(\alpha)(1/p^r)^\alpha\|_X < \infty$ converges absolutely for every $f \in H_p(\mathbb{T}^\infty, X)$. We choose now an arbitrary Dirichlet series

$$D = \mathfrak{B}_X f = \sum_n a_n n^{-s} \in \mathcal{H}_p(X) \quad \text{with } f \in H_p(\mathbb{T}^\infty, X).$$

Then

$$\sum_n \|a_n\|_X \frac{1}{n^r} = \sum_\alpha \|a_{p^\alpha}\|_X \Big(\frac{1}{p^\alpha}\Big)^r = \sum_\alpha \|\hat{f}(\alpha)\|_X \Big(\frac{1}{p^r}\Big)^\alpha < \infty.$$

Clearly, this implies that $S_p(X) \leq r$. Since this holds for each $r > 1/q$, we get that $S_p(X) \leq 1/q$, and since this now holds for each $q < M_p(X)$, we have $S_p(X) \leq 1/M_p(X)$. Conversely, let us take some $q > M_p(X)$; then there is $z \in \ell_q \cap B_{c_0}$ and $f \in H_p(\mathbb{T}^\infty, X)$ such that $\sum_\alpha \hat{f}(\alpha) z^\alpha$ does not converge absolutely. By Remark 1.2 we may assume that z is decreasing, and hence $(z_n n^{1/q})_n$ is bounded. We choose now $r > q$ and define $w_n = 1/p_n^{1/r}$. By the prime number theorem we know that there is a universal constant $C > 0$ such that

$$0 < \frac{z_n}{w_n} = z_n p_n^{1/r} = z_n n^{1/q} \frac{p_n^{1/r}}{n^{1/q}} = z_n n^{1/q} \Big(\frac{p_n}{n}\Big)^{1/r} \frac{1}{n^{1/q-1/r}} \leq C z_n n^{1/q} \frac{(\log n)^{1/r}}{n^{1/q-1/r}}.$$

The last term tends to 0 as $n \to \infty$; hence $z_n \leq w_n$ but for a finite number of n's. By Remark 1.2 this implies that $\sum_\alpha \hat{f}(\alpha) w^\alpha$ does not converge absolutely. But then $D = \mathfrak{B}_X f = \sum_n a_n n^{-s} \in \mathcal{H}_p(X)$

satisfies

$$\sum_n \|a_n\|_X \frac{1}{n^{1/r}} = \sum_\alpha \|a_{p^\alpha}\|_X \Big(\frac{1}{p^{1/r}}\Big)^\alpha = \sum_\alpha \|\hat{f}(\alpha)\|_X w^\alpha = \infty.$$

This gives that $\sigma_a(D) \geq 1/r$ for every $r > q$, hence $\sigma_a(D) \geq 1/q$. Since this holds for every $q > M_p(X)$, we finally have $S_p(X) \geq 1/M_p(X)$. □

We shall use standard notation and notions from Banach space theory, as presented, for example, in [Lindenstrauss and Tzafriri 1977; 1979]. For everything needed on polynomials in Banach spaces see, for example, [Dineen 1999; Floret 1997].

2. Relevant inequalities

The main aim here is to prove a sort of polynomial extension of the notion of cotype. Recall the definition of $C_q(X)$ from (2). Moreover, from Kahane's inequality we know that there is a (best) constant $K \geq 1$ such that, for each Banach space X and each choice of finitely many vectors $x_1, \dots, x_N \in X$,

$$\Big(\int_{\mathbb{T}^N} \Big\|\sum_{k=1}^N x_k z_k\Big\|_X^2 dz\Big)^{1/2} \leq K \int_{\mathbb{T}^N} \Big\|\sum_{k=1}^N x_k z_k\Big\|_X dz.$$

As usual we write $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $\alpha! = \alpha_1! \cdots \alpha_N!$ for every multiindex $\alpha \in \mathbb{N}_0^N$.

Proposition 2.1. *Let X be a Banach space of cotype q, $2 \leq q < \infty$, and*

$$P : \mathbb{C}^N \to X, \quad P(z) = \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha|=m}} c_\alpha z^\alpha$$

be an m-homogeneous polynomial. Let

$$T : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \to X, \quad T(z^{(1)}, \dots, z^{(m)}) = \sum_{i_1,\dots,i_m=1}^N a_{i_1,\dots,i_m} z_{i_1}^{(1)} \cdots z_{i_m}^{(m)}$$

be the unique m-linear symmetrization of P. Then

$$\Big(\sum_{i_1,\dots,i_m} \|a_{i_1,\dots,i_m}\|_X^q\Big)^{1/q} \leq (C_q(X)K)^m \frac{m^m}{m!} \int_{\mathbb{T}^N} \|P(z)\|_X \, dz.$$

Before we give the proof let us note that [Bombal et al. 2004, Theorem 3.2] is an m-linear result that, combined with polarization, gives (with the previous notation)

$$\Big(\sum_{i_1,\dots,i_m} \|a_{i_1,\dots,i_m}\|_X^q\Big)^{1/q} \leq C_q(X)^m \frac{m^m}{m!} \sup_{z \in \mathbb{D}^N} \|P(z)\|.$$

Our result allows us to replace (up to the constant K) the $\|\ \|_\infty$ norm with the smaller norm $\|\ \|_1$. We prepare the proof of Proposition 2.1 with three lemmas. The first one is a complex version of [Defant et al. 2010, Lemma 2.2] with essentially the same proof; we include it for the sake of completeness.

Lemma 2.2. *Let X be a Banach space of cotype q, $2 \le q < \infty$. Then, for every m-linear form*

$$T : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \to X, \quad T(z^{(1)}, \dots, z^{(m)}) = \sum_{i_1,\dots,i_m=1}^{N} a_{i_1,\dots,i_m} z^{(1)}_{i_1} \cdots z^{(m)}_{i_m},$$

we have

$$\Big(\sum_{i_1,\dots,i_m=1}^{N} \|a_{i_1,\dots,i_m}\|_X^q \Big)^{1/q} \le (C_q(X)\, K)^m \int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N} \|T(z^{(1)}, \dots, z^{(m)})\|_X \, dz^{(1)} \cdots dz^{(m)}.$$

Proof. We prove this result by induction on the degree m. For $m = 1$ the result is an immediate consequence of the definition of cotype q and Kahane's inequality. Assume that the result holds for $m-1$. By the continuous Minkowski inequality we then conclude that for every choice of finitely many vectors $a_{i_1,\dots,i_m} \in X$ with $1 \le i_j \le N$, $1 \le j \le m$ we have

$$\begin{aligned}
\sum_{i_1,\dots,i_m} \|a_{i_1,\dots,i_m}\|_X^q &= \sum_{i_1,\dots,i_{m-1}} \sum_{i_m} \|a_{i_1,\dots,i_m}\|_X^q \\
&\le C_q(X)^q K^q \Big(\sum_{i_1,\dots,i_{m-1}} \Big(\int_{\mathbb{T}^N} \Big\| \sum_{i_m} a_{i_1,\dots,i_m} z^{(m)}_{i_m} \Big\|_X \, dz^{(m)} \Big)^q \Big)^{q/q} \\
&\le C_q(X)^q K^q \Big(\int_{\mathbb{T}^N} \Big(\sum_{i_1,\dots,i_{m-1}} \Big\| \sum_{i_m} a_{i_1,\dots,i_m} z^{(m)}_{i_m} \Big\|_X^q \Big)^{1/q} dz^{(m)} \Big)^q \\
&\le C_q(X)^{qm} K^{qm} \Big(\int_{\mathbb{T}^N} \underbrace{\int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N}}_{m-1} \Big\| \sum_{i_1,\dots,i_{m-1}} a_{i_1,\dots,i_{m-1}} z^{(1)}_{i_1}, \dots, z^{(m-1)}_{i_{m-1}} \Big\|_X dz^{(1)} \cdots dz^{(m-1)} \, dz^{(m)} \Big)^q,
\end{aligned}$$

which is the conclusion. □

The following two lemmas are needed to produce a polynomial analog of the preceding result.

Lemma 2.3. *Let X be a Banach space, and $f : \mathbb{C} \to X$ a holomorphic function. Then for R_1, R_2, $R \ge 0$ with $R_1 + R_2 \le R$ we have*

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 + R_2 z_2)\|_X \, dz_1 \, dz_2 \le \int_{\mathbb{T}} \|f(Rz)\|_X \, dz.$$

Proof. By the rotation invariance of the normalized Lebesgue measure on $\mathbb{T}$ we get

$$\begin{aligned}
\int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 + R_2 z_2)\|_X \, dz_1 \, dz_2 &= \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 z_2 + R_2 z_2)\|_X \, dz_1 \, dz_2 \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2(R_1 z_1 + R_2))\|_X \, dz_1 \, dz_2 = \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2 |R_1 z_1 + R_2|)\|_X \, dz_2 \, dz_1 \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2 r(z_1) R)\|_X \, dz_2 \, dz_1 = \int_0^{2\pi} \int_0^{2\pi} \|f(r(e^{is}) R e^{it})\|_X \, \frac{dt}{2\pi} \, \frac{ds}{2\pi},
\end{aligned}$$

where $r(z) = (1/R)|R_1 z + R_2|$, $z \in \mathbb{T}$. We know that for each holomorphic function $h : \mathbb{C} \to X$ we have

$$\int_{\mathbb{T}} \|h(z)\|_X \, dz = \sup_{0 \le r \le 1} \int_0^{2\pi} \|h(re^{it})\|_X \frac{dt}{2\pi}$$

(see, for example, Blasco and Xu [1991, p. 338]). Define now $h(z) = f\big(Rz\big)$, and note that $0 \le r(z) \le 1$ for all $z \in \mathbb{T}$. Then

$$\begin{aligned}\int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 + R_2 z_2)\|_X \, dz_1 \, dz_2 &= \int_0^{2\pi} \int_0^{2\pi} \|h(r(e^{is})e^{it})\|_X \frac{dt}{2\pi} \frac{ds}{2\pi} \\ &\le \int_0^{2\pi} \int_{\mathbb{T}} \|h(z)\|_X \, dz \frac{ds}{2\pi} = \int_{\mathbb{T}} \|f(Rz)\|_X \, dz.\end{aligned}$$

This completes the proof. □

A sort of iteration of the preceding result leads to the next:

Lemma 2.4. *Let X be a Banach space, and $f : \mathbb{C}^N \to X$ a holomorphic function. Then, for every m,*

$$\int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N} \|f(z^{(1)} + \cdots + z^{(m)})\|_X \, dz^{(1)} \cdots dz^{(m)} \le \int_{\mathbb{T}^N} \|f(mz)\|_X \, dz.$$

Proof. We fix some m, and do induction with respect to N. For $N = 1$ we obtain from Lemma 2.3 that

$$\begin{aligned}&\underbrace{\int_{\mathbb{T}} \cdots \int_{\mathbb{T}}}_{m-2} \int_{\mathbb{T}} \int_{\mathbb{T}} \| \underbrace{f(z^{(1)} + \cdots + z^{(m-2)} + z^{(m-1)} + z^{(m)})}_{=: g_{z^{(1)},\ldots,z^{(m-2)}}(z^{(m-1)} + z^{(m)})} \|_X \, dz^{(m-1)} \, dz^{(m)} \, dz^{(1)} \cdots dz^{(m-2)} \\ &\qquad \le \underbrace{\int_{\mathbb{T}} \cdots \int_{\mathbb{T}}}_{m-2} \int_{\mathbb{T}} \|g_{z^{(1)},\ldots,z^{(m-2)}}(2w)\|_X \, dw \, dz^{(1)} \cdots dz^{(m-2)} \\ &\qquad = \underbrace{\int_{\mathbb{T}} \cdots \int_{\mathbb{T}}}_{m-3} \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z^{(1)} + \cdots + z^{(m-2)} + 2w)\|_X \, dw \, dz^{(m-2)} \, dz^{(1)} \cdots dz^{(m-3)} \\ &\qquad \le \underbrace{\int_{\mathbb{T}} \cdots \int_{\mathbb{T}}}_{m-3} \int_{\mathbb{T}} \|f(z^{(1)} + \cdots + z^{(m-3)} + 3w)\|_X \, dz^{(1)} \cdots dz^{(m-3)} \, dw \\ &\qquad \le \cdots \le \int_{\mathbb{T}} \|f(mz)\|_X \, dz.\end{aligned}$$

We now assume that the conclusion holds for $N - 1$ and write each $z \in \mathbb{T}^N$ as $z = (u, w)$, with $u \in \mathbb{T}^{N-1}$ and $w \in \mathbb{T}$. Then, using the case $N = 1$ in the first inequality and the inductive hypothesis in the second, we have

$$\begin{aligned}
&\int_{\mathbb{T}^N}\cdots\int_{\mathbb{T}^N}\|f(z^{(1)}+\cdots+z^{(m)})\|_X\,dz^{(1)}\cdots dz^{(m)}\\
&\qquad=\int_{\mathbb{T}^{N-1}}\cdots\int_{\mathbb{T}^{N-1}}\biggl(\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\|f((u^{(1)},w_1)+\cdots+(u^{(m)},w_m))\|_X\,dw_1\cdots dw_N\biggr)du^{(1)}\cdots du^{(m)}\\
&\qquad\le\int_{\mathbb{T}^{N-1}}\cdots\int_{\mathbb{T}^{N-1}}\biggl(\int_{\mathbb{T}}\|f((u^{(1)},mw)+\cdots+(u^{(m)},mw))\|_X\,dw\biggr)du^{(1)}\cdots du^{(m)}\\
&\qquad=\int_{\mathbb{T}}\biggl(\int_{\mathbb{T}^{N-1}}\cdots\int_{\mathbb{T}^{N-1}}\|f((u^{(1)},mw)+\cdots+(u^{(m)},mw))\|_X\,du^{(1)}\cdots du^{(m)}\biggr)dw\\
&\qquad\le\int_{\mathbb{T}}\biggl(\int_{\mathbb{T}^{N-1}}\|f((mu,mw)+\cdots+(mu,mw))\|_X\,du\biggr)dw\\
&\qquad=\int_{\mathbb{T}^N}\|f(mz)\|_X\,dz,
\end{aligned}$$

as desired. □

Proof of the inequality from Proposition 2.1. By the polarization formula we know that for every choice of $z^{(1)},\dots,z^{(m)}\in\mathbb{T}^N$ we have

$$T(z^{(1)},\dots,z^{(m)})=\frac{1}{2^m m!}\sum_{\varepsilon_i=\pm1}\varepsilon_i\cdots\varepsilon_m P\biggl(\sum_{i=1}^N\varepsilon_i z^{(i)}\biggr)$$

(see, for example, [Dineen 1999] or [Floret 1997]). Hence we deduce from Lemma 2.4

$$\begin{aligned}
\int_{\mathbb{T}^N}\cdots\int_{\mathbb{T}^N}\|T(z^{(1)},\dots,z^{(m)})\|_X\,dz^{(1)}\cdots dz^{(m)}&\le\frac{1}{2^m m!}\sum_{\varepsilon_i=\pm1}\int_{\mathbb{T}^N}\cdots\int_{\mathbb{T}^N}\biggl\|P\biggl(\sum_{i=1}^N\varepsilon_i z^{(i)}\biggr)\biggr\|_X dz^{(1)}\cdots dz^{(m)}\\
&=\frac{1}{2^m m!}\sum_{\varepsilon_i=\pm1}\int_{\mathbb{T}^N}\cdots\int_{\mathbb{T}^N}\biggl\|P\biggl(\sum_{i=1}^N z^{(i)}\biggr)\biggr\|_X dz^{(1)}\cdots dz^{(m)}\\
&=\frac{1}{m!}\int_{\mathbb{T}^N}\cdots\int_{\mathbb{T}^N}\biggl\|P\biggl(\sum_{i=1}^N z^{(i)}\biggr)\biggr\|_X dz^{(1)}\cdots dz^{(m)}\\
&\le\frac{1}{m!}\int_{\mathbb{T}^N}\|P(mz)\|_X\,dz=\frac{m^m}{m!}\int_{\mathbb{T}^N}\|P(z)\|_X\,dz.
\end{aligned}$$

Then by Lemma 2.2 we obtain

$$\begin{aligned}
\biggl(\sum_{i_1,\dots,i_m}^N\|a_{i_1,\dots,i_m}\|_X^q\biggr)^{1/q}&\le(C_q(X)K)^m\int_{\mathbb{T}^\infty}\cdots\int_{\mathbb{T}^\infty}\|T(z^{(1)},\dots,z^{(m)})\|_X\,dz^{(1)}\cdots dz^{(m)}\\
&=(C_q(X)K)^m\frac{m^m}{m!}\int_{\mathbb{T}^N}\|P(z)\|_X\,dz,
\end{aligned}$$

which completes the proof of Proposition 2.1. □

A second proposition is needed which allows us to reduce the proof of our main result (Theorem 1.1) to the homogeneous case. It is a vector-valued version of a result of [Cole and Gamelin 1986, Theorem 9.2] with a similar proof (here only given for the sake of completeness).

Proposition 2.5. *There is a contractive projection*

$$\Phi_m : H_p(\mathbb{T}^N, X) \to H_p(\mathbb{T}^N, X), \quad f \mapsto f_m,$$

such that, for all $f \in H_p(\mathbb{T}^N, X)$,

$$\hat{f}(\alpha) = \hat{f}_m(\alpha) \quad \text{for all } \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| = m. \tag{11}$$

Proof. Let $\mathcal{P}(\mathbb{C}^N, X) \subset H_p(\mathbb{T}^N, X)$ be the subspace of all finite polynomials $f = \sum_{\alpha\in\Lambda} c_\alpha z^\alpha$; here Λ is a finite set of multiindices in $\mathbb{N}_0^N$ and the coefficients $c_\alpha \in X$. Define the linear projection Φ_m^0 on $\mathcal{P}(\mathbb{C}^N, X)$ by

$$\Phi_m^0(f)(z) = f_m(z) = \sum_{\alpha\in\Lambda, |\alpha|=m} \hat{f}(\alpha) z^\alpha\,;$$

clearly, we have (11). In order to show that Φ_m^0 is a contraction on $\big(\mathcal{P}(\mathbb{C}^N, X), \|\cdot\|_p\big)$ fix some function $f \in \mathcal{P}(\mathbb{C}^N, X)$ and $z \in \mathbb{T}^N$, and define

$$f(z\,\cdot\,) : \mathbb{T} \to X, \quad w \mapsto f(zw).$$

Clearly, we have

$$f(zw) = \sum_k f_k(z) w^k,$$

and hence

$$f_m(z) = \int_{\mathbb{T}} f(zw) w^{-m}\, dw.$$

Integration, Hölder's inequality and the rotation invariance of the normalized Lebesgue measure on $\mathbb{T}^N$ give

$$\begin{aligned}
\int_{\mathbb{T}^N} \|f_m(z)\|_X^p\, dz &= \int_{\mathbb{T}^N} \left\| \int_{\mathbb{T}} f(zw) w^{-m}\, dw \right\|_X^p dz \\
&\le \int_{\mathbb{T}^N} \left(\int_{\mathbb{T}} \|f(zw)\|_X\, dw \right)^p dz \\
&\le \int_{\mathbb{T}} \int_{\mathbb{T}^N} \|f(zw)\|_X^p\, dz\, dw = \int_{\mathbb{T}^N} \|f(z)\|_X^p\, dz,
\end{aligned}$$

which proves that Φ_m^0 is a contraction on $(\mathcal{P}(\mathbb{C}^N, X), \|\cdot\|_p)$. By Fejér's theorem (vector-valued) we know that $\mathcal{P}(\mathbb{C}^N, X)$ is a dense subspace of $H_p(\mathbb{T}^N, X)$. Hence Φ_m^0 extends to a contractive projection Φ_m on $H_p(\mathbb{T}^N, X)$. This extension Φ_m still satisfies (11) since the mapping $H_p(\mathbb{T}^N, X) \to X$, $f \mapsto \hat{f}(\alpha)$ is continuous for each multiindex α. ☐

3. Proof of the main result

We are now ready to prove Theorem 1.1. Let $1 \le p < \infty$, and recall from (1) that

$$1 - \frac{1}{\operatorname{Cot} X} = S_\infty(X) \le S_p(X);$$

see Remark 3.1 for a direct argument. Hence it suffices to concentrate on the upper estimate in Theorem 1.1: Since we obviously have $S_p(X) \le S_1(X)$, we are going to prove that

$$S_1(X) \le 1 - \frac{1}{\operatorname{Cot} X}. \tag{12}$$

Suppose first that X has no finite cotype, i.e., $\operatorname{Cot} X = \infty$. For $D = \sum_n a_n n^{-s} \in \mathcal{H}_1(X)$ we take $f \in H_1(\mathbb{T}^\infty, X)$ with $D = \mathfrak{B}_X f$. Note that

$$\|\hat{f}(\alpha)\|_X \le \int_{\mathbb{T}^\infty} \|f(w) w^{-\alpha}\|_X \, dw = \|f\|_{L_1(\mathbb{T}^\infty, X)} < \infty;$$

hence, by the definition of $\mathfrak{B}_X$, the coefficients of D are also bounded by $\|f\|_{L_1(\mathbb{T}^\infty, X)}$. As a consequence, for every $\sigma > 1$ we have

$$\sum_{n=1}^\infty \|a_n\|_X \frac{1}{n^\sigma} \le \sum_{n=1}^\infty \|f\|_{L_1(\mathbb{T}^\infty, X)} \frac{1}{n^\sigma} < \infty.$$

This means that $S_1(X) \le 1$ and as a consequence (12) holds.

Now if X has finite cotype, take $q > \operatorname{Cot} X$ and $\varepsilon > 0$, and put $s = (1 - 1/q)(1 + 2\varepsilon)$. Choose an integer k_0 such that $p_{k_0}^{\varepsilon/q'} > e C_q(X) K \big(\sum_{j=1}^\infty 1/p_j^{1+\varepsilon}\big)^{1/q'}$ and define

$$\tilde{p} = (\underbrace{p_{k_0}, \dots, p_{k_0}}_{k_0 \text{ times}}, p_{k_0+1}, p_{k_0+2}, \dots).$$

We are going to show that there is a constant $C(q, X, \varepsilon) > 0$ such that for every $f \in H_1(\mathbb{T}^\infty, X)$ we have

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} \le C(q, X, \varepsilon) \|f\|_{H_1(\mathbb{T}^\infty, X)}. \tag{13}$$

This finishes the argument: By Remark 1.2 the sequence $1/p^s$ is in mon $H_1(\mathbb{T}^\infty, X)$. But in view of Bohr's transform from (6), this means that for every Dirichlet series $D = \sum_n a_n n^{-s} = \mathfrak{B}_X f \in \mathcal{H}_1(X)$ with $f \in H_1(\mathbb{T}^\infty, X)$ we have

$$\sum_{n=1}^\infty \|a_n\|_X \frac{1}{n^s} = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \|\hat{f}(\alpha)\|_X \frac{1}{p^{s\alpha}} < \infty.$$

Therefore $\sigma_a(D) \le (1 - 1/q)(1 + 2\varepsilon)$ for each such D which, since $\varepsilon > 0$ was arbitrary, is what we wanted to prove.

It remains to check (13); the idea is to show first that (13) holds for all X-valued H_1-functions which only depend on N variables: There is a constant $C(q, X, \varepsilon) > 0$ such that for all N and every

$f \in H_1(\mathbb{T}^N, X)$ we have

$$\sum_{\alpha \in \mathbb{N}_0^N} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} \le C(q, X, \varepsilon)\|f\|_{H_1(\mathbb{T}^N, X)}. \tag{14}$$

In order to understand that (14) implies (13) (and hence the conclusion), assume that (14) holds and take some $f \in H_1(\mathbb{T}^\infty, X)$. Given an arbitrary N, define

$$f_N : \mathbb{T}^N \to X, \quad f_N(w) = \int_{\mathbb{T}^\infty} f(w, \tilde{w})\, d\tilde{w}.$$

Then it can be easily shown that $f_N \in L_1(\mathbb{T}^N, X)$, $\|f_N\|_1 \le \|f\|_1$, and $\hat{f_N}(\alpha) = \hat{f}(\alpha)$ for all $\alpha \in \mathbb{Z}^N$. If we now apply (14) to this f_N, we get

$$\sum_{\alpha \in \mathbb{N}_0^N} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} \le C(q, X, \varepsilon)\|f\|_{H_1(\mathbb{T}^\infty, X)},$$

which, after taking the supremum over all possible N on the left side, leads to (13).

We turn to the proof of (14), and here in a first step will show the following: For every N, every m-homogeneous polynomial $P : \mathbb{C}^N \to X$ and every $u \in \ell_{q'}$ we have

$$\sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| = m}} \|\hat{P}(\alpha)u^\alpha\|_X \le (eC_q(X)K)^m \int_{\mathbb{T}^N} \|P(z)\|_X\, dz \left(\sum_{j=1}^\infty |u_j|^{q'}\right)^{m/q'}. \tag{15}$$

Indeed, take such a polynomial $P(z) = \sum_{\alpha \in \mathbb{N}_0^N, |\alpha|=m} \hat{P}(\alpha) z^\alpha,\ z \in \mathbb{T}^N$, and look at its unique m-linear symmetrization

$$T : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \to X, \quad T(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1,\ldots,i_m=1}^N a_{i_1,\ldots,i_m} z_{i_1}^{(1)}, \ldots, z_{i_m}^{(m)}.$$

Then we know from Proposition 2.1 that

$$\left(\sum_{i_1,\ldots,i_m=1}^N \|a_{i_1,\ldots,i_m}\|_X^q\right)^{1/q} \le (eC_q(X)K)^m \int_{\mathbb{T}^N} \|P(z)\|_X\, dz.$$

Hence (15) follows by Hölder's inequality:

$$\sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| = m}} \|\hat{P}(\alpha)u^\alpha\|_X = \sum_{i_1,\ldots,i_m=1}^N \|a_{i_1,\ldots,i_m}\|_X |u_{i_1} \cdots u_{i_N}| \le (eC_q(X)K)^m \int_{\mathbb{T}^N} \|P(z)\|_X\, dz \left(\sum_{j=1}^\infty |u_j|^{q'}\right)^{m/q'}.$$

We finally give the proof of (14): Take $f \in H_1(\mathbb{T}^N, X)$, and recall from Proposition 2.5 that for each integer m there is an m-homogeneous polynomial $P_m : \mathbb{C}^N \to X$ such that $\|P_m\|_{H_1(\mathbb{T}^N, X)} \le \|f\|_{H_1(\mathbb{T}^N, X)}$

and $\hat{P}_m(\alpha) = \hat{f}(\alpha)$ for all $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = m$. From (15), the definition of s, and the fact that $\max\{p_{k_0}, p_j\} \le \tilde{p}_j$ for all j we have

$$\begin{aligned}
\sum_{\alpha\in\mathbb{N}_0^N} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} &= \sum_{m=1}^{\infty} \sum_{\substack{\alpha\in\mathbb{N}_0^N \\ |\alpha|=m}} \|\hat{P}_m(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} \\
&\le \sum_{m=1}^{\infty} (eC_q(X)K)^m \|P_m\|_{H_1(\mathbb{T}^N,X)} \left(\sum_{j=1}^{\infty} \frac{1}{\tilde{p}_j^{sq'}}\right)^{m/q'} \\
&= \sum_{m=1}^{\infty} (eC_q(X)K)^m \|f\|_{H_1(\mathbb{T}^N,X)} \left(\sum_{j=1}^{\infty} \frac{1}{\tilde{p}_j^{1+2\varepsilon}}\right)^{m/q'} \\
&= \sum_{m=1}^{\infty} (eC_q(X)K)^m \|f\|_{H_1(\mathbb{T}^N,X)} \left(\sum_{j=1}^{\infty} \frac{1}{\tilde{p}_j^{1+\varepsilon}} \frac{1}{\tilde{p}_j^{\varepsilon}}\right)^{m/q'} \\
&\le \|f\|_{H_1(\mathbb{T}^N,X)} \sum_{m=1}^{\infty} \underbrace{\left(\frac{eC_q(X)K\left(\sum_{j=1}^{\infty} p_j^{-(1+\varepsilon)}\right)^{1/q'}}{p_{k_0}^{\varepsilon/q'}}\right)^m}_{<1}.
\end{aligned}$$

This completes the proof of Theorem 1.1. □

Remark 3.1. We end this note with a direct proof of the fact

$$1 - \frac{1}{\operatorname{Cot} X} \le S_p(X), \quad 1 \le p < \infty, \tag{16}$$

in which we do not use the inequality

$$1 - \frac{1}{\operatorname{Cot} X} \le S_\infty(X) \tag{17}$$

from [Defant et al. 2008] (here repeated in (1)). The proof of (17) given in that reference shows in a first step that $1 - 1/\Pi(X) \le S_\infty(X)$ where

$$\Pi(X) = \inf\{r \ge 2 \mid \mathrm{id}_X \text{ is } (r, 1)\text{-summing}\},$$

and then, in a second step, applies a fundamental theorem of Maurey and Pisier stating that $\Pi(X) = \operatorname{Cot} X$.

The following argument for (16) is very similar to the original one from [Defant et al. 2008] but does not use the Maurey–Pisier theorem (since we here consider $\mathcal{H}_p(X)$, $1 \le p < \infty$ instead of $\mathcal{H}_\infty(X)$): By the proof of Corollary 1.3, inequality (16) is equivalent to

$$M_p(X) \le \frac{\operatorname{Cot} X}{\operatorname{Cot} X - 1}.$$

Take $r < M_p(X)$, so that $\ell_r \cap B_{c_0} \subset \operatorname{mon} H_p(\mathbb{T}^\infty, X)$. Let $H_p^1(\mathbb{T}^\infty, X)$ be the subspace of $H_p(\mathbb{T}^\infty, X)$ formed by all 1-homogeneous polynomials (i.e., linear operators). We can define a bilinear operator

$\ell_r \times H_p^1(\mathbb{T}^\infty, X) \to \ell_1(X)$ by $(z, f) \mapsto (z_j f(e_j))_j$ which, by a closed graph argument, is continuous. Therefore, there is a constant M such that for all $z \in \ell_r$ and all $f \in H_p^1(\mathbb{T}^\infty, X)$ we have

$$\sum_j |z_j| \|f(e_j)\|_X \le M \|z\|_{\ell_r} \|f\|_{H_p(\mathbb{T}^\infty, X)}.$$

Taking the supremum over all $z \in B_{\ell_r}$ we obtain for all $f \in H_p^1(\mathbb{T}^\infty, X)$

$$\Big(\sum_j \|f(e_j)\|_X^{r'}\Big)^{1/r'} \le M \|f\|_{H_p(\mathbb{T}^\infty, X)}.$$

Now, take $x_1, \dots, x_N \in X$, define $f \in H_p^1(\mathbb{T}^\infty, X)$ by

$$f(e_j) = \begin{cases} x_j & \text{if } 1 \le j \le N, \\ 0 & \text{if } j > N \end{cases}$$

and extend it by linearity. By the previous inequality and Proposition 2.5 we have

$$\Big(\sum_{j=1}^N \|x_j\|_X^{r'}\Big)^{1/r'} \le M \Big(\int_{\mathbb{T}^N} \Big\| \sum_{j=1}^N x_j z_j \Big\|_X^{r'} dz\Big)^{1/r'}.$$

By Kahane's inequality, X has cotype r', which means that $r' > \operatorname{Cot} X$ or, equivalently, $r < \dfrac{\operatorname{Cot} X}{\operatorname{Cot} X - 1}$. Since $r < M_p(X)$ was arbitrary, we obtain (16).

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