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GLOBAL UNIQUENESS FOR AN IBVP FOR THE TIME-HARMONIC MAXWELL EQUATIONS

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In this paper we prove uniqueness for an inverse boundary value problem (IBVP) arising in electrodynamics. We assume that the electromagnetic properties of the medium, namely the magnetic permeability, the electric permittivity, and the conductivity, are described by continuously differentiable functions.

1. Introduction

Let Ω be a bounded nonempty open subset of $\mathbb{R}^3$ with boundary denoted by $\partial\Omega$. Consider functions $\mu, \varepsilon, \sigma \in L^\infty(\Omega)$, representing magnetic permeability, electric permittivity, and conductivity, respectively, such that $\mu(x) \geq \mu_0$, $\varepsilon(x) \geq \varepsilon_0$, and $\sigma(x) \geq 0$ almost everywhere in Ω for positive constants μ_0 and ε_0. At frequency $\omega > 0$, for each medium characterized by $(\mu, \varepsilon, \sigma)$, we have access to all available data of the boundary tangential components of electric and magnetic fields. More specifically, we have access to the *Cauchy data set* $C(\mu, \varepsilon, \sigma; \omega)$ consisting of all boundary graded forms $f^1 + f^2 \in TH^\delta\big(\partial\Omega; \Lambda^1\mathbb{R}^3\big) \oplus TH^d\big(\partial\Omega; \Lambda^2\mathbb{R}^3\big)$ (see the Appendix for the definitions of these spaces and results related to l-forms) such that there exists $u^1 + u^2 \in H^d\big(\Omega; \Lambda^1\mathbb{R}^3\big) \oplus H^\delta(\Omega; \Lambda^2\mathbb{R}^3)$ satisfying

$$\delta u^2 + i\omega\varepsilon u^1 - du^1 + i\omega\mu u^2 = \sigma u^1 \tag{1-1}$$

almost everywhere in Ω and

$$\delta \operatorname{tr} u^2 + d \operatorname{tr} u^1 = f^1 + f^2 \tag{1-2}$$

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in the sense of $TH^{\delta}(\partial\Omega;\Lambda^1\mathbb{R}^3)\oplus TH^{d}(\partial\Omega;\Lambda^2\mathbb{R}^3)$. Here u^1 is the 1-form representation of the electric field and u^2 is the 2-form representation of the magnetic field. It is worth pointing out that the graded equations (1-1) and (1-2) are equivalent to the following systems of time-harmonic Maxwell equations:

$$\begin{cases}\delta u^2 + i\omega\varepsilon u^1 = \sigma u^1,\\ du^1 - i\omega\mu u^2 = 0\end{cases}$$

almost everywhere in Ω and

$$\begin{cases}\delta \operatorname{tr} u^2 = f^1,\\ d \operatorname{tr} u^1 = f^2\end{cases}$$

in the sense of the space $TH^{\delta}(\partial\Omega;\Lambda^1\mathbb{R}^3)$ for the 1-form equation and in the sense of $TH^{d}(\partial\Omega;\Lambda^2\mathbb{R}^3)$ for the 2-form equation. Throughout this paper, for convenience, we follow the graded form notation rather than the l-form system.

We are interested in the inverse boundary value problem (IBVP) of recovering $\mu,\varepsilon,\sigma\in L^\infty(\Omega)$ from the knowledge of $C(\mu,\varepsilon,\sigma;\omega)$. This problem is just a reformulation in differential forms of the usual IBVP for the time-harmonic Maxwell equations proposed in [Somersalo et al. 1992], where $\partial\Omega$ was smooth enough, the electromagnetic fields $(\mathbf{E},\mathbf{H})$ satisfied

$$\begin{cases}\nabla\times\mathbf{E} - i\omega\mu\mathbf{H} = 0,\\ \nabla\times\mathbf{H} + i\omega(\varepsilon + i\sigma/\omega)\mathbf{E} = 0\end{cases}$$

almost everywhere in Ω, and the Cauchy set $C(\mu,\varepsilon,\sigma;\omega)$ consisted of pairs

$$(\nu\times\mathbf{E}|_{\partial\Omega},\ \nu\times\mathbf{H}|_{\partial\Omega})\in TH^{1/2}_{\mathrm{Div}}(\partial\Omega)\times TH^{1/2}_{\mathrm{Div}}(\partial\Omega)$$

(see [Somersalo et al. 1992] for precise definitions) with ν denoting the unit outer normal vector to $\partial\Omega$. The uniqueness question associated to this problem is as follows. Given a frequency $\omega>0$ and two sets of parameters $\{\mu_j,\varepsilon_j,\sigma_j\}\subset L^\infty(\Omega)$ with $j\in\{1,2\}$ such that $\mu_j(x)\ge\mu_0$, $\varepsilon_j(x)\ge\varepsilon_0$, and $\sigma_j(x)\ge 0$ almost everywhere in Ω, does $C(\mu_1,\varepsilon_1,\sigma_1;\omega)=C(\mu_2,\varepsilon_2,\sigma_2;\omega)$ imply $\mu_1=\mu_2$, $\varepsilon_1=\varepsilon_2$, and $\sigma_1=\sigma_2$?

In this paper we provide the answer to this question in the case where Ω is locally described by the graph of a Lipschitz function and μ, ε, and σ are continuously differentiable in Ω. This is stated in our main theorem as follows.

Theorem 1.1. *Let Ω be a bounded nonempty open subset of $\mathbb{R}^3$. Assume that $\partial\Omega$ is locally described by the graph of a Lipschitz function. Let μ_j, ε_j, and σ_j with $j\in\{1,2\}$ belong to $C^1(\bar\Omega)$. At frequency $\omega>0$, suppose $\partial^\alpha\mu_1(x)=\partial^\alpha\mu_2(x)$, $\partial^\alpha\varepsilon_1(x)=\partial^\alpha\varepsilon_2(x)$, and $\partial^\alpha\sigma_1(x)=\partial^\alpha\sigma_2(x)$ for $\alpha\in\mathbb{N}^3$ with $|\alpha|\le 1$ and all $x\in\partial\Omega$. Then*

$$C(\mu_1,\varepsilon_1,\sigma_1,\omega)=C(\mu_2,\varepsilon_2,\sigma_2,\omega)\Longrightarrow \mu_1=\mu_2,\ \varepsilon_1=\varepsilon_2 \text{ and } \sigma_1=\sigma_2.$$

A precise definition of the space denoted by $C^1(\bar\Omega)$ is given at the beginning of Section 3. Our result assumes the coefficients to be equal up to order one on the boundary. This is required to extend them identically outside the domain. As far as we know, the only available results about uniqueness on the boundary in this context are due to Joshi and McDowall [McDowall 1997; Joshi and McDowall 2000], where $\partial\Omega$ is assumed to be locally described by a smooth function and the Cauchy data sets are given by the graph of a bounded map.

The IBVP considered in this paper was first proposed by Somersalo, Isaacson, and Cheney [Somersalo et al. 1992]. Lassas [1997] found a relation between this IBVP and the inverse conductivity problem proposed by Calderón [2006]. In general terms, the latter problem can be seen as the low-frequency limit of the former. Calderón's problem in electrical impedance tomography consists of reconstructing the conductivity of a domain by measuring electric voltages and currents on the boundary. The uniqueness question arising in this problem is whether the conductivity σ ($\sigma \in L^\infty(\Omega)$ and $\sigma(x) \geq \sigma_0 > 0$ for almost every $x \in \Omega$), in a divergence type equation $\nabla \cdot (\sigma \nabla u) = 0$ in Ω, can be determined uniquely by the boundary Dirichlet-to-Neumann map $\Lambda_\sigma : H^1(\Omega)/H^1_0(\Omega) \longrightarrow (H^1(\Omega)/H^1_0(\Omega))^*$ defined as

$$\langle \Lambda_\sigma f \mid g \rangle = \int_\Omega \sigma \nabla u \cdot \nabla v \, dx$$

for any $f, g \in H^1(\Omega)/H^1_0(\Omega)$, where $u \in H^1(\Omega)$ is the weak solution of the conductivity equation $\nabla \cdot (\sigma \nabla u) = 0$ in Ω with $u|_{\partial\Omega} = f$ and $v \in H^1(\Omega)$ with $v|_{\partial\Omega} = g$. A significant number of works have been devoted to answering not only the question of uniqueness but also the questions of reconstruction and stability. The most successful approach to treat this problem was introduced by Sylvester and Uhlmann [1987] and it is based on the construction of complex geometrical optics (CGO) solutions. In dimension 2, the problem is rather well understood and some important results can be found in [Astala and Päivärinta 2006; Clop et al. 2010; Nachman 1996]. In dimension greater than 2, there are still many open questions about the sharp smoothness to ensure uniqueness, stability, and reconstruction. Some important results can be found in [Haberman and Tataru 2013; Sylvester and Uhlmann 1987; Nachman 1988; Alessandrini 1988]. Some recent results are [Caro et al. 2013; García and Zhang 2012]. For a more complete list of papers on this problem, we refer to the survey papers [Uhlmann 2009; 2008].

The literature for the IBVP in electrodynamics under consideration is not as extensive as for Calderón's problem. [Somersalo et al. 1992] contains the first partial results for the linearization of the problem at constant electromagnetic parameters, and Sun and Uhlmann [1992] provided a local uniqueness theorem. The first global uniqueness result is due to Ola, Päivärinta, and Somersalo [Ola et al. 1993], where the authors assume that the electromagnetic coefficients are C^3-functions and $\partial\Omega$ is of class $C^{1,1}$. They also provided a reconstruction algorithm to recover the coefficients. The arguments in [Ola et al. 1993] are rather complicated, since the method developed by Sylvester and Uhlmann [1987] does not immediately apply. The lack of ellipticity of Maxwell's equations makes the problem more complicated than Calderón's. Ola and Somersalo [1996] simplified the proof in [Ola et al. 1993] by establishing a relation between Maxwell's equations and a matrix Helmholtz equation with a potential. This relation helps to deal with the lack of ellipticity, allowing them to produce exponentially growing solutions for Maxwell's equations from the CGOs for the matrix Helmholtz equation. This idea has been extensively used in proving many other results and it will be used in this paper as well. There are other results related to the IBVP under consideration in the literature. Kenig, Salo, and Uhlmann [Kenig et al. 2011] proved uniqueness for the corresponding IBVP in some noneuclidean geometries. With certain types of partial boundary data, the uniqueness was addressed by Caro, Ola, and Salo [Caro et al. 2009]; see also [Caro 2011]. The question of stability has been studied in [Caro 2010] assuming full data and in [Caro 2011] assuming partial data. Zhou [2010] used the enclosure method to reconstruct electromagnetic obstacles.

In our paper, Theorem 1.1 lowers significantly the regularity of the coefficients and the smoothness of the boundary of Ω assumed in previous results (despite the fact that domains with Lipschitz boundaries were already considered in [Caro 2010]) and it matches the regularity assumptions made in [Haberman and Tataru 2013] for Calderón's problem.

The general line of our paper follows the argument in [Ola and Somersalo 1996], relating (1-1) with an equation given by a compactly supported zeroth order perturbation of the graded Hodge–Helmholtz operator, namely

$$(\delta d + d\delta - \omega^2 \mu_0 \varepsilon_0) w_j + Q_j w_j = 0, \tag{1-3}$$

where $Q_j = Q(\varepsilon_j + i\sigma_j/\omega, \mu_j, \omega)$ with $j \in \{1, 2\}$ has to be thought of as a weak potential containing second partial derivatives of μ_j, ε_j, and σ_j. Using this relation, we are able to prove the integral formula

$$\langle (Q_2 - Q_1) w_1 \mid v_2 \rangle = 0, \tag{1-4}$$

where w_1 is a solution to (1-3) that produces a solution to (1-1) and v_2 is a solution to a first order elliptic equation (see Section 3 for more details). This integral formula, with CGOs w_1 and v_2 as inputs, will be the starting point of our proof.

To lower the regularity of the electromagnetic parameters, we adopt a recent improvement of Sylvester and Uhlmann's method that Haberman and Tataru developed [2013] to prove uniqueness of the Calderón problem with continuously differentiable conductivities. For such regularity, solving a conductivity equation can be reduced to solving a Schrödinger equation, $-\Delta v + m_q v = 0$, where m_q denotes the multiplication operator by the compactly supported weak potential $q = \Delta\sqrt{\sigma}/\sqrt{\sigma}$. Note that this reduction was first used by Sylvester and Uhlmann [1987] for smooth conductivities and later by Brown [1996] for less regular conductivities, all followed by the construction of CGOs in proper function spaces. Haberman and Tataru [2013] proved the existence of CGO solutions $v(x) = e^{x\cdot\zeta}(1 + \psi_\zeta(x))$ with $\zeta \in \mathbb{C}^n$ and $\zeta \cdot \zeta = 0$ to the Schrödinger equation. Roughly speaking, the construction is based on solving the equation $-(\Delta + 2\zeta \cdot \nabla)\psi_\zeta + m_q \psi_\zeta = 0$ in a Bourgain-type space $\dot{X}^b_\zeta$ whose norm includes the potential $|p_\zeta(\xi)|^{2b} = ||\xi|^2 - 2i\zeta \cdot \xi|^{2b}$ as a weight. In this way, the ζ-dependence is transferred into the space norms and it is shown [Haberman and Tataru 2013] that

$$\|(\Delta + 2\zeta \cdot \nabla)^{-1}\|_{\dot{X}^{-1/2}_\zeta \to \dot{X}^{1/2}_\zeta} = 1, \quad \|m_q\|_{\dot{X}^{1/2}_\zeta \to \dot{X}^{-1/2}_\zeta} < 1,$$

which guarantee the convergence of the Neumann series for ψ_ζ. Furthermore, Haberman and Tataru obtained an average decaying property for $\|\psi_\zeta\|_{\dot{X}^{1/2}_\zeta}$, from which they deduced the existence of a sequence $\{\zeta^m\}$ such that $\{\psi_{\zeta^m}\}$ vanishes as m grows.

In this paper, we adopt the idea and several of the estimates in [Haberman and Tataru 2013] to construct the CGOs w_1 and v_2 with desired properties. Nevertheless, we avoid the argument of extracting the sequence of $\{\zeta^m\}$, and directly use the decay in average. This has been previously done by Caro, García, and Reyes [Caro et al. 2013] to prove stability of the Calderón problem for $C^{1,\epsilon}$-conductivities. When plugging in the CGOs w_1 and v_2, the output of (1-4) will be certain nonlinear relations of $\varepsilon_1 + i\sigma_1/\omega$, μ_1, $\varepsilon_2 + i\sigma_2/\omega$, and μ_2 involving second weak partial derivatives of the coefficients. Thus, to conclude the proof of our theorem we will need a unique continuation property for a system of the form

$$-\Delta f + V f + a f + b g = 0,$$
$$-\Delta g + W g + c g + d f = 0,$$

where a, b, c, and d are compactly supported and belong to $L^\infty(\mathbb{R}^3)$, while V and W are again weak potentials. We will again apply the argument with Bourgain-type spaces to prove the required unique continuation property, which seems not to be available in the literature.

The paper is organized as follows. In Section 2 we show the relation between (1-1) and (1-3). The proof of the integral formula (1-4) is given in Section 3. The CGO solutions are constructed in Section 4, where we will directly refer several times to the estimates proven in [Haberman and Tataru 2013] rather than listing them in the paper. In Section 5, we complete our proof by plugging the CGOs into (1-4) and using the unique continuation principle that we will derive. An appendix is provided at the end of the paper, gathering basic facts and notations in the framework of differential forms, as well as including some technical computations for the electromagnetic IBVP.

2. An auxiliary graded equation

In this section we establish a relation between

$$\delta u^2 + i\omega\varepsilon u^1 - du^1 + i\omega\mu u^2 = \sigma u^1$$

and an auxiliary graded Hodge–Helmholtz equation with zeroth order perturbation (following the idea in [Ola and Somersalo 1996]), which allows the construction of CGOs. For our purposes, it would be enough to have solutions in Ω, but for convenience we will conduct our analysis in the whole $\mathbb{R}^3$. This gives us certain freedoms in extending the coefficients outside Ω. Thus, set $B = \{x \in \mathbb{R}^3 : |x| < R\}$ with $R > 0$ such that $\bar{\Omega} \subset B$. Let ω, μ_0, and ε_0 be three positive constants. At this point, we consider μ, ε, and σ in $W^{1,\infty}(\mathbb{R}^3)$, the space of measurable functions modulo those vanishing almost everywhere such that they and their first weak partial derivatives are essentially bounded in $\mathbb{R}^3$. Furthermore, we assume that μ, ε, and σ are real-valued,

$$\operatorname{supp}(\mu - \mu_0) \subset B, \quad \operatorname{supp}(\varepsilon - \varepsilon_0) \subset B, \quad \operatorname{supp}(\sigma) \subset B,$$

and $\mu(x) \geq \mu_0$, $\varepsilon(x) \geq \varepsilon_0$, and $\sigma(x) \geq 0$ for almost every x in $\mathbb{R}^3$. For simplicity, write $\gamma = \varepsilon + i\sigma/\omega$. It is sufficient for us to produce weak solutions to

$$\delta u^2 + i\omega\gamma u^1 - du^1 + i\omega\mu u^2 = 0 \tag{2-1}$$

in $\mathbb{R}^3$, namely, forms $u^1 + u^2$ with $u^l \in L^2_{\mathrm{loc}}\big(\mathbb{R}^3; \Lambda^l \mathbb{R}^3\big)$ satisfying

$$\langle \delta u^2 + i\omega\gamma u^1 - du^1 + i\omega\mu u^2 \mid \varphi^1 + \varphi^2 \rangle = 0$$

for all $\varphi^1+\varphi^2$ with $\varphi^l \in C_0^\infty\big(\mathbb{R}^3;\, \Lambda^l\mathbb{R}^3\big)$. Here $\langle\,\cdot\mid\cdot\,\rangle$ denotes the duality bracket for distributions.

In order to derive the auxiliary equation, we augment (2-1) by adding

$$-\gamma^{-1}\delta(\gamma u^1)+\mu^{-1}d(\mu u^2)=0,$$

which is derived directly from (2-1).

Next, we consider an equation of the graded form $\sum_{l=0}^3 u^l$ where $u^l\in L^2_{\mathrm{loc}}\big(\mathbb{R}^3;\, \Lambda^l\mathbb{R}^3\big)$:

$$-\gamma^{-1}\delta(\gamma u^1)+i\omega\mu u^0+\mu^{-1}d(\mu u^0)+\delta u^2+i\omega\gamma u^1-\gamma^{-1}\delta(\gamma u^3)-du^1+i\omega\mu u^2+\mu^{-1}d(\mu u^2)+i\omega\gamma u^3=0.$$

Multiplying 0, 2-forms by $\gamma^{1/2}$ and 1, 3-forms by $\mu^{1/2}$, we obtain

$$\begin{aligned}-\gamma^{-1/2}\delta(\gamma u^1)+i\omega\gamma^{1/2}\mu u^0+\mu^{-1/2}d(\mu u^0)+\mu^{1/2}\delta u^2+i\omega\gamma\mu^{1/2}u^1&\\ -\gamma^{-1/2}\delta(\gamma u^3)-\gamma^{1/2}du^1+i\omega\gamma^{1/2}\mu u^2+\mu^{-1/2}d(\mu u^2)+i\omega\gamma\mu^{1/2}u^3&=0.\end{aligned}$$

Throughout this paper $(\,\cdot\,)^{1/2}$ will denote the principal branch of the square root, and the same convention will apply to log. If we now set

$$v=\sum_{l=0}^3 v^l=\mu^{1/2}u^0+\gamma^{1/2}u^1+\mu^{1/2}u^2+\gamma^{1/2}u^3,$$

we end up with the equation

$$P(d+\delta;\,\gamma,\mu,\omega)v=0, \tag{2-2}$$

where

$$\begin{aligned}&P(d+\delta;\,\gamma,\mu,\omega)v\\ &\quad=(d+\delta)\sum_{l=0}^3(-1)^l v^l+da\wedge v^1+da\vee(v^1+v^3)+db\wedge(v^0+v^2)-db\vee v^2+i\omega\gamma^{1/2}\mu^{1/2}v,\end{aligned}$$

$a=\frac{1}{2}\log\gamma$ and $b=\frac{1}{2}\log\mu$. The key point of this derivation to take note of is that $v=\sum_0^3 v^l$ with $v^l\in L^2_{\mathrm{loc}}\big(\mathbb{R}^3;\,\Lambda^l\mathbb{R}^3\big)$ is a weak solution of (2-2) in $\mathbb{R}^3$ (that is, for every $\varphi=\sum_0^3\varphi^l$ with $\varphi^l\in C_0^\infty\big(\mathbb{R}^3;\,\Lambda^l\mathbb{R}^3\big)$, $\langle P(d+\delta;\,\gamma,\mu,\omega)v\mid\varphi\rangle=0$ with $\langle\cdot\mid\cdot\rangle$ denoting the duality bracket for distributions) and $v^0+v^3=0$ if and only if $u^1+u^2=\gamma^{-1/2}v^1+\mu^{-1/2}v^2$ with $u^l\in L^2_{\mathrm{loc}}\big(\mathbb{R}^3;\,\Lambda^l\mathbb{R}^3\big)$ is a weak solution of (2-1) in $\mathbb{R}^3$. For convenience, let us define an operator

$$\begin{aligned}&P(d+\delta;\,\gamma,\mu,\omega)^t w\\ &:=(d+\delta)\sum_{l=0}^3(-1)^{l+1}w^l+db\wedge w^1+db\vee(w^1+w^3)+da\wedge(w^0+w^2)-da\vee w^2+i\omega\gamma^{1/2}\mu^{1/2}w\end{aligned}\tag{2-3}$$

for $w=\sum_0^3 w^l$ with $w^l\in H^\delta_{\mathrm{loc}}\big(\mathbb{R}^3;\,\Lambda^l\mathbb{R}^3\big)\cap H^d_{\mathrm{loc}}\big(\mathbb{R}^3;\,\Lambda^l\mathbb{R}^3\big)$. Note that $P(d+\delta;\,\gamma,\mu,\omega)^t$ is the formal transpose of $P(d+\delta;\,\gamma,\mu,\omega)$.

Due to the rescaling by $\gamma^{1/2}$ and $\mu^{1/2}$ that we chose, it can be verified that $P(d+\delta;\,\gamma,\mu,\omega)\circ P(d+\delta;\,\gamma,\mu,\omega)^t$ is a zeroth order perturbation of the graded Hodge–Helmholtz operator. For any graded forms

$w = \sum_0^3 w^l$ and $\varphi = \sum_0^3 \varphi^l$ with $w^l, \varphi^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$, set

$$\begin{aligned}\langle Q(\gamma, \mu, \omega) w \mid \varphi\rangle = &-\int_{\mathbb{R}^3} \omega^2(\gamma\mu - \varepsilon_0\mu_0)\langle w, \varphi\rangle\, dx \\ &+ \int_{\mathbb{R}^3} \langle i2\omega d(\gamma^{1/2}\mu^{1/2}) \vee (w^1 + w^3) + i2\omega d(\gamma^{1/2}\mu^{1/2}) \wedge (w^0 + w^2), \varphi\rangle\, dx \\ &+ \int_{\mathbb{R}^3} \langle da, da\rangle\langle w^0 + w^2, \varphi^0 + \varphi^2\rangle + \langle db, db\rangle\langle w^1 + w^3, \varphi^1 + \varphi^3\rangle\, dx \\ &+ \int_{\mathbb{R}^3} \langle da, d\langle -w^0 + w^2, \varphi^0 + \varphi^2\rangle\rangle + \langle db, d\langle w^1 - w^3, \varphi^1 + \varphi^3\rangle\rangle\, dx \\ &+ \int_{\mathbb{R}^3} \langle db, D^*(w^1 \odot \varphi^1)\rangle\, dx + \int_{\mathbb{R}^3} \langle da, D^*(*w^2 \odot *\varphi^2)\rangle\, dx. \end{aligned} \tag{2-4}$$

Proposition 2.1. *Let $w = \sum_0^3 w^l$ be a graded form with $w^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$ and assume that*

$$\int_{\mathbb{R}^3} \langle \delta w, \delta\varphi\rangle + \langle dw, d\varphi\rangle - \omega^2\varepsilon_0\mu_0\langle w, \varphi\rangle\, dx + \langle Q(\gamma, \mu, \omega) w \mid \varphi\rangle = 0 \tag{2-5}$$

for all $\varphi = \sum_0^3 \varphi^l$ with $\varphi^l \in C_0^\infty(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$. Then $v = \sum_0^3 v^l$ defined by

$$v = P(d + \delta; \gamma, \mu, \omega)^t w \tag{2-6}$$

is a weak solution to (2-2) *in $\mathbb{R}^3$ and $v^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$.*

Proof. We first prove that v is a weak solution to (2-2). Since $v^l \in L^2_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$, it is enough to show that

$$\begin{aligned}&\int_{\mathbb{R}^3} \langle P(d + \delta; \gamma, \mu, \omega)^t w, P(d + \delta; \gamma, \mu, \omega)^t \varphi\rangle\, dx \\ &\qquad = \int_{\mathbb{R}^3} \langle \delta w, \delta\varphi\rangle + \langle dw, d\varphi\rangle - \omega^2\varepsilon_0\mu_0\langle w, \varphi\rangle\, dx + \langle Q(\gamma, \mu, \omega) w \mid \varphi\rangle.\end{aligned} \tag{2-7}$$

To check this, by direct computation, the first four terms on the left-hand side are

$$\int_{\mathbb{R}^3} \left\langle (d + \delta) \sum_{l=0}^3 (-1)^{l+1} w^l, (d + \delta) \sum_{l=0}^3 (-1)^{l+1} \varphi^l \right\rangle dx = \int_{\mathbb{R}^3} \langle \delta w, \delta\varphi\rangle + \langle dw, d\varphi\rangle\, dx, \tag{2-8}$$

$$\int_{\mathbb{R}^3} \langle i\omega\gamma^{1/2}\mu^{1/2} w, i\omega\gamma^{1/2}\mu^{1/2}\varphi\rangle\, dx = -\int_{\mathbb{R}^3} \omega^2\gamma\mu\langle w, \varphi\rangle\, dx, \tag{2-9}$$

$$\begin{aligned}&\int_{\mathbb{R}^3} \left\langle (d + \delta) \sum_{l=0}^3 (-1)^{l+1} w^l, i\omega\gamma^{1/2}\mu^{1/2}\varphi \right\rangle dx + \int_{\mathbb{R}^3} \left\langle i\omega\gamma^{1/2}\mu^{1/2} w, (d + \delta) \sum_{l=0}^3 (-1)^{l+1} \varphi^l \right\rangle dx \\ &\quad = \int_{\mathbb{R}^3} \langle i\omega d(\gamma^{1/2}\mu^{1/2}) \vee (w^1 + w^3) + i\omega d(\gamma^{1/2}\mu^{1/2}) \wedge (w^0 + w^2), \varphi\rangle\, dx \\ &\qquad + \int_{\mathbb{R}^3} \langle i\omega d(\gamma^{1/2}\mu^{1/2}) \vee w^2 - i\omega d(\gamma^{1/2}\mu^{1/2}) \wedge w^1, \varphi\rangle\, dx,\end{aligned} \tag{2-10}$$

and

$$\int_{\mathbb{R}^3} \langle db \wedge w^1 + db \vee (w^1 + w^3) + da \wedge (w^0 + w^2) - da \vee w^2, i\omega\gamma^{1/2}\mu^{1/2}\varphi\rangle \\ + \langle i\omega\gamma^{1/2}\mu^{1/2} w, db \wedge \varphi^1 + db \vee (\varphi^1 + \varphi^3) + da \wedge (\varphi^0 + \varphi^2) - da \vee \varphi^2\rangle \, dx \\ = \int_{\mathbb{R}^3} \langle i\omega d(\gamma^{1/2}\mu^{1/2}) \vee (w^1 + w^3) + i\omega d(\gamma^{1/2}\mu^{1/2}) \wedge (w^0 + w^2), \varphi\rangle \, dx \\ - \int_{\mathbb{R}^3} \langle i\omega d(\gamma^{1/2}\mu^{1/2}) \vee w^2 - i\omega d(\gamma^{1/2}\mu^{1/2}) \wedge w^1, \varphi\rangle \, dx. \quad (2\text{-}11)$$

By Corollary A.2, the fifth term gives

$$\int_{\mathbb{R}^3} \langle db \wedge w^1 + db \vee (w^1 + w^3) + da \wedge (w^0 + w^2) \\ - da \vee w^2, db \wedge \varphi^1 + db \vee (\varphi^1 + \varphi^3) + da \wedge (\varphi^0 + \varphi^2) - da \vee \varphi^2\rangle \, dx \\ = \int_{\mathbb{R}^3} \langle da, da\rangle \langle w^0 + w^2, \varphi^0 + \varphi^2\rangle + \langle db, db\rangle \langle w^1 + w^3, \varphi^1 + \varphi^3\rangle \, dx. \quad (2\text{-}12)$$

By Proposition A.6, the last term yields

$$\int_{\mathbb{R}^3} \Big\langle db \wedge w^1 + db \vee (w^1 + w^3) + da \wedge (w^0 + w^2) - da \vee w^2, (d+\delta) \sum_{l=0}^{3} (-1)^{l+1} \varphi^l \Big\rangle \\ + \Big\langle (d+\delta) \sum_{l=0}^{3} (-1)^{l+1} w^l, db \wedge \varphi^1 + db \vee (\varphi^1 + \varphi^3) + da \wedge (\varphi^0 + \varphi^2) - da \vee \varphi^2 \Big\rangle \, dx \\ = \int_{\mathbb{R}^3} \langle da, d\langle -w^0 + w^2, \varphi^0 + \varphi^2\rangle\rangle + \langle db, d\langle w^1 - w^3, \varphi^1 + \varphi^3\rangle\rangle \, dx + \int_{\mathbb{R}^3} \langle db, D^*(w^1 \odot \varphi^1)\rangle \, dx \\ + \int_{\mathbb{R}^3} \langle da, D^*({*w^2} \odot {*\varphi^2})\rangle \, dx = 0. \quad (2\text{-}13)$$

Summing up identities (2-8) through (2-13) gives identity (2-7).

It remains to prove that $v^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$. Since $v^l \in L^2_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$, we have

$$(d+\delta) \sum_{0}^{3} (-1)^l v^l \in \bigoplus_{0}^{3} L^2_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$$

by (2-2). Therefore, Lemma A.7 allows us to conclude the proof. □

Remark 2.2. Identity (2-7) holds even for $\varphi = \sum_{0}^{3} \varphi^l$ with $\varphi^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$.

Similar calculations verify that the same property holds for $P(d+\delta; \gamma, \mu, \omega)^t \circ P(d+\delta; \gamma, \mu, \omega)$ as stated in Proposition 2.3. Define

$$
\begin{aligned}
\langle \widetilde{Q}(\gamma, \mu, \omega) w \mid \varphi \rangle = -\int_{\mathbb{R}^3} \omega^2 (\gamma\mu - \varepsilon_0\mu_0) \langle w, \varphi \rangle \, dx \\
+ \int_{\mathbb{R}^3} \langle -i2\omega d(\gamma^{1/2}\mu^{1/2}) \vee w^2 + i2\omega d(\gamma^{1/2}\mu^{1/2}) \wedge w^1, \varphi \rangle \, dx \\
+ \int_{\mathbb{R}^3} \langle db, db \rangle \langle w^0 + w^2, \varphi^0 + \varphi^2 \rangle + \langle da, da \rangle \langle w^1 + w^3, \varphi^1 + \varphi^3 \rangle \, dx \\
+ \int_{\mathbb{R}^3} \langle db, d\langle w^0 - w^2, \varphi^0 + \varphi^2 \rangle \rangle + \langle da, d\langle -w^1 + w^3, \varphi^1 + \varphi^3 \rangle \rangle \, dx \\
- \int_{\mathbb{R}^3} \langle da, D^*(w^1 \odot \varphi^1) \rangle \, dx - \int_{\mathbb{R}^3} \langle db, D^*(*w^2 \odot *\varphi^2) \rangle \, dx \quad \text{(2-14)}
\end{aligned}
$$

for $w = \sum_0^3 w^l$ and $\varphi = \sum_0^3 \varphi^l$ with $w^l, \varphi^l \in H^1_{\text{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$.

Proposition 2.3. *Let $w = \sum_0^3 w^l$ be a graded form with $w^l \in H^1_{\text{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$ and assume that*

$$
\int_{\mathbb{R}^3} \langle \delta w, \delta\varphi \rangle + \langle dw, d\varphi \rangle - \omega^2 \varepsilon_0\mu_0 \langle w, \varphi \rangle \, dx + \langle \widetilde{Q}(\gamma, \mu, \omega) w \mid \varphi \rangle = 0 \quad \text{(2-15)}
$$

for all $\varphi = \sum_0^3 \varphi^l$ with $\varphi^l \in C_0^\infty(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$. Then $v = \sum_0^3 v^l$ defined by

$$
v = P(d + \delta; \gamma, \mu, \omega) w
$$

is a weak solution of

$$
P(d + \delta; \gamma, \mu, \omega)^t v = 0
$$

in $\mathbb{R}^3$ and $v^l \in H^1_{\text{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$.

Recall that $v = \sum_0^3 v^l$ with $v^l \in L^2_{\text{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$ is a weak solution to (2-2) and satisfies $v^0 + v^3 = 0$ in $\mathbb{R}^3$ if and only if $u^1 + u^2 = \gamma^{-1/2} v^1 + \mu^{-1/2} v^2$ with $u^l \in L^2_{\text{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$ is a weak solution of (2-1) in $\mathbb{R}^3$. We finish this section by singling out the equation of $v^0 + v^3$ from (2-15), which is used later to show that the CGOs we will construct in Section 4 satisfy $v^0 + v^3 = 0$.

Proposition 2.4. *Let $v = \sum_0^3 v^l$ with $v^l \in H^1_{\text{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$ satisfy*

$$
P(d + \delta; \gamma, \mu, \omega) v = 0
$$

in any bounded open subset of $\mathbb{R}^3$. For any $\varphi = \varphi^0 + \varphi^3$ with φ^l belonging to $C_0^\infty(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$, we have

$$
\int_{\mathbb{R}^3} \langle \delta(v^0 + v^3), \delta\varphi \rangle + \langle d(v^0 + v^3), d\varphi \rangle - \omega^2 \varepsilon_0\mu_0 \langle v^0 + v^3, \varphi \rangle \, dx + \langle \tilde{q}(\gamma, \mu, \omega)(v^0 + v^3) \mid \varphi \rangle = 0, \quad \text{(2-16)}
$$

where

$$
\begin{aligned}
\langle \tilde{q}(\gamma, \mu, \omega)(v^0 + v^3) \mid \varphi \rangle = -\int_{\mathbb{R}^3} \omega^2 (\gamma\mu - \varepsilon_0\mu_0) \langle v^0 + v^3, \varphi \rangle \, dx + \int_{\mathbb{R}^3} \langle db, db \rangle \langle v^0, \varphi^0 \rangle + \langle da, da \rangle \langle v^3, \varphi^3 \rangle \\
+ \langle db, d\langle v^0, \varphi^0 \rangle \rangle + \langle da, d\langle v^3, \varphi^3 \rangle \rangle \, dx.
\end{aligned}
$$

Proof. This is immediate from the proof of Proposition 2.3 and the fact that $\widetilde{Q}(\gamma, \mu, \omega)$ decouples for $v^0 + v^3$. □

3. An integral formula

In this section we provide an integral formula that serves as the starting point to prove uniqueness of the IBVP. To do this, we exploit the computations which allow us to produce solutions for (2-1) from solutions of (2-5) (see Proposition 2.1).

Let Ω be a bounded nonempty open subset in $\mathbb{R}^3$ whose boundary $\partial\Omega$ can be locally described by the graph of a Lipschitz function. Throughout the rest of the paper, we assume that μ_j, ε_j, and σ_j belong to $C^1(\bar{\Omega})$ with $j \in \{1, 2\}$ such that $\mu_j(x) \geq \mu_0$, $\varepsilon_j(x) \geq \varepsilon_0$, and $\sigma_j(x) \geq 0$ everywhere in Ω. Here we say that f is in $C^1(\bar{\Omega})$ if $f : \Omega \longrightarrow \mathbb{C}$ is continuously differentiable in Ω, its partial derivatives $\partial^\alpha f$ are uniformly continuous in Ω for $\alpha \in \mathbb{N}^3$ and $|\alpha| = 1$, and

$$|\partial^\alpha f(x)| \leq C \quad \text{for all } x \in \Omega,\ |\alpha| \leq 1, \tag{3-1}$$

for a certain positive constant C. The norm on $C^1(\bar{\Omega})$, defined as the smallest constant C for which (3-1) holds, makes $C^1(\bar{\Omega})$ a Banach space. Since $\partial\Omega$ is of Lipschitz class, f defined as above is uniformly continuous and, consequently, $\partial^\alpha f$ possesses a unique bounded continuous extension to $\bar{\Omega}$ for any $|\alpha| \leq 1$. This extension will still be denoted by f.

Consider $C_j = C(\mu_j, \varepsilon_j, \sigma_j; \omega)$, the Cauchy data set associated to μ_j, ε_j, and σ_j at frequency $\omega > 0$. Write $\gamma_j = \varepsilon_j + i\sigma_j/\omega$ and assume $\partial^\alpha\gamma_1(x) = \partial^\alpha\gamma_2(x)$ and $\partial^\alpha\mu_1(x) = \partial^\alpha\mu_2(x)$ for all $x \in \partial\Omega$ and $|\alpha| \leq 1$. We can extend[1] γ_j and μ_j to continuously differentiable functions in $\mathbb{R}^3$, still denoted by γ_j and μ_j, such that $|\partial^\alpha\gamma_j(x)| + |\partial^\alpha\mu_j(x)| \leq C$, $\mu_j(x) \geq \mu_0$, $\varepsilon_j(x) \geq \varepsilon_0$, and $\sigma_j(x) \geq 0$ for all $x \in \mathbb{R}^3$, $|\alpha| \leq 1$ and a certain constant $C > 0$,

$$\operatorname{supp}(\mu_j - \mu_0) \subset B, \quad \operatorname{supp}(\gamma_j - \varepsilon_0) \subset B,$$

where $B = \{x \in \mathbb{R}^3 : |x| < R\} \supset \bar{\Omega}$, and $\gamma_1(x) = \gamma_2(x)$ and $\mu_1(x) = \mu_2(x)$ for all $x \in \mathbb{R}^3 \setminus \Omega$. For convenience, we write $a_j = \frac{1}{2}\log\gamma_j$ and $b_j = \frac{1}{2}\log\mu_j$.

Proposition 3.1. *Let $w_1 = \sum_0^3 w_1^l$ be a graded form with $w_1^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)$ satisfying*

$$\int_{\mathbb{R}^3} \langle\delta w_1, \delta\varphi\rangle + \langle dw_1, d\varphi\rangle - \omega^2\varepsilon_0\mu_0\langle w_1, \varphi\rangle\, dx + \langle Q(\gamma_1, \mu_1, \omega)w_1 \mid \varphi\rangle = 0 \tag{3-2}$$

for all $\varphi = \sum_0^3 \varphi^l$ with $\varphi^l \in C_0^\infty(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)$. Assume that $v_1 = \sum_0^3 v_1^l$, defined by

$$v_1 = P(d + \delta; \gamma_1, \mu_1, \omega)^t w_1, \tag{3-3}$$

satisfies $v_1^0 + v_1^3 = 0$. Let $v_2 = \sum_0^3 v_2^l$ with $v_2^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)$ satisfy

$$P(d + \delta; \gamma_2, \mu_2, \omega)^t v_2 = 0 \tag{3-4}$$

[1]The extensions we want to perform here are of Whitney type. These kinds of extensions hold for functions defined on any closed subset of $\mathbb{R}^n$ whenever the functions can be approximated by certain polynomials. In order to ensure the existence of such polynomials, we use that $\partial\Omega$ is of Lipschitz class. The argument to prove the existence of such polynomials is similar to the one carried out in Section 2 of [Caro et al. 2013] for $C^{1,\varepsilon}(\bar{\Omega})$ functions with the only difference being that, where the authors referred to Chapter VI, Section 2 of [Stein 1970], we refer to Chapter VI, Section 4.7 of [Stein 1970].

in any bounded open subset of $\mathbb{R}^3$. *Then* $C_1 = C_2$ *implies*

$$\langle (Q(\gamma_2, \mu_2, \omega) - Q(\gamma_1, \mu_1, \omega))w_1 \mid v_2\rangle = 0.$$

Proof. By Remark 2.2 and because $\gamma_1(x) = \gamma_2(x)$ and $\mu_1(x) = \mu_2(x)$ for all $x \in \mathbb{R}^3 \setminus \Omega$, we know that

$$\begin{aligned}\langle (Q(\gamma_2, \mu_2, \omega) - Q(\gamma_1, \mu_1, \omega))w_1 \mid v_2\rangle &= \int_\Omega \langle P(d+\delta; \gamma_2, \mu_2, \omega)^t w_1, P(d+\delta; \gamma_2, \mu_2, \omega)^t v_2\rangle \, dx \\ &\quad - \int_\Omega \langle P(d+\delta; \gamma_1, \mu_1, \omega)^t w_1, P(d+\delta; \gamma_1, \mu_1, \omega)^t v_2\rangle \, dx \\ &= -\int_\Omega \langle v_1, P(d+\delta; \gamma_1, \mu_1, \omega)^t v_2\rangle \, dx.\end{aligned}$$

The last equality follows from (3-4) and (3-3).

Since $v_1^0 + v_1^3 = 0$, we have that $u_1^1 + u_1^2 = \gamma_1^{-1/2} v_1^1 + \mu_1^{-1/2} v_1^2$ satisfies

$$\delta u_1^2 + i\omega\gamma_1 u_1^1 - du_1^1 + i\omega\mu_1 u_1^2 = 0 \tag{3-5}$$

almost everywhere in Ω (see Section 2). The definitions of boundary traces $\delta\,\mathrm{tr}$ and $d\,\mathrm{tr}$ (see Section A3) give

$$\begin{aligned}&-\int_\Omega \langle v_1, P(d+\delta; \gamma_1, \mu_1, \omega)^t v_2\rangle \, dx \\ &\qquad = \langle \delta\,\mathrm{tr}(\gamma_1 u_1^1) \mid \gamma_1^{-1/2} v_2^0\rangle + \langle \delta\,\mathrm{tr}\, u_1^2 \mid \mu_1^{1/2} v_2^1\rangle - \langle d\,\mathrm{tr}\, u_1^1 \mid \gamma_1^{1/2} v_2^2\rangle + \langle d\,\mathrm{tr}(\mu_1 u_1^2) \mid \mu_1^{-1/2} v_2^3\rangle.\end{aligned} \tag{3-6}$$

Suppose $f = f^1 + f^2$ with $f^l \in L^2_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)$ is a weak solution to

$$\delta f^2 + i\omega\gamma_2 f^1 - df^1 + i\omega\mu_2 f^2 = 0 \tag{3-7}$$

in $\mathbb{R}^3$. Note that then $f^1 \in H^d(\Omega; \Lambda^1\mathbb{R}^3)$, $f^2 \in H^\delta(\Omega; \Lambda^2\mathbb{R}^3)$. Set $g = g^1 + g^2 = \gamma_2^{1/2} f^1 + \mu_2^{1/2} f^2$. By (3-4), we obviously have

$$\int_\Omega \langle g, P(d+\delta; \gamma_2, \mu_2, \omega)^t v_2\rangle \, dx = 0.$$

Once more by the definitions of $\delta\,\mathrm{tr}$ and $d\,\mathrm{tr}$, we have

$$\begin{aligned}0 &= \int_\Omega \langle g, P(d+\delta; \gamma_2, \mu_2, \omega)^t v_2\rangle \, dx \\ &= -\langle \delta\,\mathrm{tr}(\gamma_2 f^1) \mid \gamma_2^{-1/2} v_2^0\rangle - \langle \delta\,\mathrm{tr}\, f^2 \mid \mu_2^{1/2} v_2^1\rangle + \langle d\,\mathrm{tr}\, f^1 \mid \gamma_2^{1/2} v_2^2\rangle - \langle d\,\mathrm{tr}(\mu_2 f^2) \mid \mu_2^{-1/2} v_2^3\rangle.\end{aligned} \tag{3-8}$$

Since $\delta\,\mathrm{tr}\, u_1^2 + d\,\mathrm{tr}\, u_1^1 \in C_1 = C_2$ by assumption, there exists $u_2 = u_2^1 + u_2^2$ with $u_2^1 \in H^d(\Omega; \Lambda^1\mathbb{R}^3)$ and $u_2^2 \in H^\delta(\Omega; \Lambda^2\mathbb{R}^3)$ a solution to (3-7) in Ω such that

$$\delta\,\mathrm{tr}\, u_1^2 + d\,\mathrm{tr}\, u_1^1 = \delta\,\mathrm{tr}\, u_2^2 + d\,\mathrm{tr}\, u_2^1.$$

Define[2] $f(x)=u_2(x)$ for almost every $x\in\Omega$ and $f(x)=u_1(x)$ for almost every $x\in\mathbb{R}^3\setminus\Omega$. Using (3-6) and (3-8) and noting that $\gamma_1(x)=\gamma_2(x)$ and $\mu_1(x)=\mu_2(x)$ for all $x\in\partial\Omega$, we can conclude

$$\begin{aligned}&\langle (Q(\gamma_2,\mu_2,\omega)-Q(\gamma_1,\mu_1,\omega))w_1 \mid v_2\rangle\\ &=-\frac{1}{i\omega}\langle\delta\,\mathrm{tr}(\delta u_1^2)\,|\,\gamma_2^{-1/2}v_2^0\rangle+\frac{1}{i\omega}\langle d\,\mathrm{tr}(du_1^1)\,|\,\mu_2^{-1/2}v_2^3\rangle+\frac{1}{i\omega}\langle\delta\,\mathrm{tr}(\delta u_2^2)\,|\,\gamma_2^{-1/2}v_2^0\rangle-\frac{1}{i\omega}\langle d\,\mathrm{tr}(du_2^1)\,|\,\mu_2^{-1/2}v_2^3\rangle.\end{aligned}$$

The result follows by Lemma A.5. □

4. The construction of CGO solutions

In this section we construct the CGO solutions that will be plugged into the integral formula in Proposition 3.1. To deal with less regular electromagnetic coefficients than those in [Ola and Somersalo 1996], we adopt Bourgain-type spaces introduced by Haberman and Tataru [2013].

Let $\zeta=\sum_1^3\zeta_j\,dx^j$ be a constant 1-differential form in $\mathbb{R}^3$ and let p_ζ denote the polynomial

$$p_\zeta(\xi)=|\xi|^2-2i\langle\zeta,\xi\rangle.$$

For any $b\in\mathbb{R}$, let $\dot X_\zeta^b$ denote the space of graded forms $w=\sum_0^3 w^l$ such that $w^l\in\mathscr{S}'(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$ and its Fourier transform

$$\widehat{w^l}\in L^2(\mathbb{R}^3,|p_\zeta|^{2b}d\xi;\Lambda^l\mathbb{R}^3).$$

The functional

$$w\in\dot X_\zeta^b\longmapsto\|w\|_{\dot X_\zeta^b}=\biggl(\sum_{l=0}^3\||p_\zeta|^b\widehat{w^l}\|^2_{L^2(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)}\biggr)^{1/2}$$

makes $\dot X_\zeta^b$ a normed space. Moreover, if $b<1$, then $\dot X_\zeta^b$ is a Hilbert space. As in [Haberman and Tataru 2013], we will only use the cases where $b\in\{1/2,-1/2\}$. Note that $\dot X_\zeta^{-1/2}$ can be identified as the dual space of $\dot X_\zeta^{1/2}$. The simplest feature of these spaces is that the operator $(\Delta_\zeta+\langle\zeta,\zeta\rangle)^{-1}$ (defined by the symbol $(p_\zeta)^{-1}$) is a bounded linear operator from $\dot X_\zeta^{-1/2}$ to $\dot X_\zeta^{1/2}$ with norm

$$\|(\Delta_\zeta+\langle\zeta,\zeta\rangle)^{-1}\|_{\dot X_\zeta^{-1/2}\to\dot X_\zeta^{1/2}}=1. \tag{4-1}$$

Let Δ_ζ denote the conjugate operator $\Delta_\zeta=e_{-\zeta}(d\delta+\delta d)\circ e_\zeta$ where $e_\zeta(x)=e^{\zeta\cdot x}$ and $\zeta\cdot x=\sum_1^3\zeta_jx^j$.

Remark 4.1. Given $f\in\dot X_\zeta^{-1/2}$, it is an obvious consequence of the definition of $\dot X_\zeta^{1/2}$ that there exists a unique $u\in\dot X_\zeta^{1/2}$ satisfying

$$\Delta_\zeta u+\langle\zeta,\zeta\rangle u=f.$$

Remark 4.2. If $u\in\dot X_\zeta^{1/2}$ with $u=\sum_0^3u^l$, then $u^l\in H^1_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$. This is a simple consequence of (5) and (6) in Lemma 2.2 of [Haberman and Tataru 2013] and the finite band property (sometimes called Bernstein's inequality).

[2] This definition satisfies the appropriate conditions, since $\gamma_1(x)=\gamma_2(x)$ and $\mu_1(x)=\mu_2(x)$ for all $x\in\mathbb{R}^3\setminus\Omega$.

4A. ***The construction of w_1.*** Let ζ_1 be a complex-valued constant 1-form in $\mathbb{R}^3$ satisfying $\langle \zeta_1, \zeta_1\rangle = -k^2$ where $k = \omega^{1/2}\mu_0\epsilon_0$. We are looking for $w_1 = \sum_0^3 w_1^l$ with $w_1^l \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l\mathbb{R}^3)$, the solution to (3-2) of the form

$$w_1 = e_{\zeta_1}(A_{\zeta_1} + R_{\zeta_1}) \tag{4-2}$$

with A_{ζ_1} a constant graded differential form in $\mathbb{R}^3$ and $R_{\zeta_1} \in \dot{X}^{1/2}_{\zeta_1}$. Moreover, we want R_{ζ_1} to bear a certain sense of smallness. Note that this is equivalent to finding R_{ζ_1}, which solves

$$(\Delta_{\zeta_1} - k^2)R_{\zeta_1} + Q(\gamma_1, \mu_1, \omega)R_{\zeta_1} = -Q(\gamma_1, \mu_1, \omega)A_{\zeta_1} \tag{4-3}$$

in $\dot{X}^{1/2}_{\zeta_1}$. Note that $Q(\gamma_1, \mu_1, \omega)A_{\zeta_1} \in \dot{X}^{-1/2}_{\zeta_1}$. In the scalar case, this was done in [Haberman and Tataru 2013] for such Bourgain-type spaces. In the original case of smooth coefficients, such equations were solved in weighted L^2 spaces in [Sylvester and Uhlmann 1987] for the scalar case and in [Ola and Somersalo 1996] for systems.

Lemma 4.3. *Let ζ_1 and A_{ζ_1} be as above. For $|\zeta_1|$ large enough, there exists a solution $R_{\zeta_1} \in \dot{X}^{1/2}_{\zeta_1}$ to* (4-3) *such that*

$$\|R_{\zeta_1}\|_{\dot{X}^{1/2}_{\zeta_1}} \lesssim \|Q(\gamma_1, \mu_1, \omega)A_{\zeta_1}\|_{\dot{X}^{-1/2}_{\zeta_1}}, \tag{4-4}$$

where the implicit constant (*incorporated in the symbol* $\lesssim$) *is independent of* ζ_1.

Proof. By using a Neumann series argument (see [Sylvester and Uhlmann 1987]), we can show the existence of $R_{\zeta_1} \in \dot{X}^{1/2}_{\zeta_1}$ satisfying

$$\|R_{\zeta_1}\|_{\dot{X}^{1/2}_{\zeta_1}} \le \|(I + (\Delta_{\zeta_1} - k^2)^{-1}Q(\gamma_1, \mu_1, \omega))^{-1}\|_{\dot{X}^{1/2}_{\zeta_1} \to \dot{X}^{1/2}_{\zeta_1}} \|Q(\gamma_1, \mu_1, \omega)A_{\zeta_1}\|_{\dot{X}^{-1/2}_{\zeta_1}}$$

for $|\zeta_1|$ large enough, as a simple consequence of (4-1) and

$$\|Q(\gamma_1, \mu_1, \omega)\|_{\dot{X}^{1/2}_{\zeta_1} \to \dot{X}^{-1/2}_{\zeta_1}} = o(\mathbf{1}(|\zeta_1|)). \tag{4-5}$$

Here $\mathbf{1}(t) = 1$ for any $t \in \mathbb{R}$.

To prove (4-5), let u and v belong to $\dot{X}^{1/2}_{\zeta_1}$. By a slight modification of Corollary 2.1 in [Haberman and Tataru 2013], we have that

$$\begin{aligned}
|\langle Q(\gamma_1, \mu_1, \omega)u \mid v\rangle| \lesssim |\zeta_1|^{-1}\|u\|_{\dot{X}^{1/2}_{\zeta_1}}\|v\|_{\dot{X}^{1/2}_{\zeta_1}} \\
&+ \left|\int_{\mathbb{R}^3} \langle \alpha_h, d\langle -u^0 + u^2, v^0 + v^2\rangle\rangle + \langle \beta_h, d\langle u^1 - u^3, \varphi^1 + v^3\rangle\rangle\, dx\right| \\
&+ \left|\int_{\mathbb{R}^3} \langle \beta_h, D^*(u^1 \odot v^1)\rangle\, dx\right| + \left|\int_{\mathbb{R}^3} \langle \alpha_h, D^*(*u^2 \odot *v^2)\rangle\, dx\right| \\
&+ \left|\int_{\mathbb{R}^3} \langle da_1 - \alpha_h, d\langle -u^0 + u^2, v^0 + v^2\rangle\rangle + \langle db_1 - \beta_h, d\langle u^1 - u^3, \varphi^1 + v^3\rangle\rangle\, dx\right| \\
&+ \left|\int_{\mathbb{R}^3} \langle db_1 - \beta_h, D^*(u^1 \odot v^1)\rangle\, dx\right| + \left|\int_{\mathbb{R}^3} \langle da_1 - \alpha_h, D^*(*u^2 \odot *v^2)\rangle\, dx\right|
\end{aligned}$$

where α_h and β_h are 1-forms in $\mathbb{R}^3$ defined by

$$\alpha_h = \varphi_h * da_1, \quad \beta_h = \varphi_h * db_1$$

(here $*$ denotes convolution) with $0 < h \le 1$, $\varphi_h(x) = h^{-3}\varphi(x/h)$, $\varphi \in C_0^\infty(\mathbb{R}^3)$, $0 \le \varphi(x) \le 1$ for all $x \in \mathbb{R}^3$ and $\int_{\mathbb{R}^3} \varphi\, dx = 1$. Note that the implicit constant depends on ε_0, μ_0, Ω, and the C^1-norms of γ_1 and μ_1. A further modification of Lemma 2.3 in [Haberman and Tataru 2013] gives

$$\begin{aligned}|\langle Q(\gamma_1,\mu_1,\omega)u \mid v\rangle| &\lesssim |\zeta_1|^{-1}\|u\|_{\dot X^{1/2}_{\zeta_1}}\|v\|_{\dot X^{1/2}_{\zeta_1}} + |\zeta_1|^{-1}(\|\delta\alpha_h\|_{L^\infty(\mathbb{R}^3)} + \|\delta\beta_h\|_{L^\infty(\mathbb{R}^3)})\|u\|_{\dot X^{1/2}_{\zeta_1}}\|v\|_{\dot X^{1/2}_{\zeta_1}}\\ &\qquad + (\|da_1-\alpha_h\|_{L^\infty(\mathbb{R}^3)} + \|db_1-\beta_h\|_{L^\infty(\mathbb{R}^3)})\|u\|_{\dot X^{1/2}_{\zeta_1}}\|v\|_{\dot X^{1/2}_{\zeta_1}}\\ &\lesssim (|\zeta_1|^{-1}h^{-1} + o(\mathbf{1}(h)))\|u\|_{\dot X^{1/2}_{\zeta_1}}\|v\|_{\dot X^{1/2}_{\zeta_1}}\end{aligned}$$

as h vanishes. Choosing $h = |\zeta_1|^{-1/2}$, this implies (4-1) and the lemma is proven. □

Up to this point, nothing has been said about the smallness of R_{ζ_1}. We will see in the next lemma that estimate (4-4) yields such smallness in an average sense. This idea is one of the key points in [Haberman and Tataru 2013].

Lemma 4.4. *Let $s \in \mathbb{R}$ satisfy $s \ge 1$. Given a real-valued constant 1-form ρ in $\mathbb{R}^3$, choose η_1 and η_2 also real-valued constant 1-forms such that $\langle \eta_1, \eta_2\rangle = 0$, $\langle \eta_j, \rho\rangle = 0$, and $|\eta_j| = 1$ for $j \in \{1,2\}$. Set*

$$\zeta_1 = -\sqrt{s^2 + \frac{|\rho|^2}{4}}\,\eta_1 + i\left(\frac{\rho}{2} - \sqrt{s^2+k^2}\,\eta_2\right),$$

and assume $|A_{\zeta_1}|$ is bounded as a function of s, η_1. Then the R_{ζ_1} obtained in Lemma 4.3 satisfies

$$\frac{1}{\lambda}\int_{S^1}\int_{\lambda}^{2\lambda} \|R_{\zeta_1}\|^2_{\dot X^{1/2}_{\zeta_1}}\, ds\, d\eta_1 = o(1(\lambda)) \tag{4-6}$$

as λ becomes large. Here S^1 denotes the intersection between the unit sphere in $\mathbb{R}^3$ and the plane defined by η_1 and η_2.

Proof. By the definition of $Q(\gamma_1,\mu_1,\omega)$, the identity (2-13), the fact that A_{ζ_1} is constant, and the fact that $Q(\gamma_1,\mu_1,\omega)$ is compactly supported, we have

$$|\langle Q(\gamma_1,\mu_1,\omega)A_{\zeta_1} \mid v\rangle| \lesssim \sum_{l=0}^{3}\|\chi v^l\|_{L^2(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)} + \|\chi(d+\delta)f_{\zeta_1}\|_{\dot X^{-1/2}_{\zeta_1}}\|v\|_{\dot X^{1/2}_{\zeta_1}},$$

where $v = \sum_0^3 v^l$, $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\chi(x) = 1$ for all $x \in \operatorname{supp} d\gamma_1 \cup \operatorname{supp} d\mu_1$ and

$$f_{\zeta_1} = db_1 \wedge A^1_{\zeta_1} + db_1 \vee (A^1_{\zeta_1} + A^3_{\zeta_1}) + da_1 \wedge (A^0_{\zeta_1} + A^2_{\zeta_1}) - da_1 \vee A^2_{\zeta_1}$$

with $A_{\zeta_1} = \sum_0^3 A^l_{\zeta_1}$. By (5) in Lemma 2.2 of [Haberman and Tataru 2013], this gives

$$\|Q(\gamma_1,\mu_1,\omega)A_{\zeta_1}\|_{\dot X^{-1/2}_{\zeta_1}} \lesssim s^{-1/2} + \|\chi(d+\delta)f_{\zeta_1}\|_{\dot X^{-1/2}_{\zeta_1}}.$$

Now an immediate modification of Lemma 3.1 in [Haberman and Tataru 2013] allows us to check that

$$\frac{1}{\lambda}\int_{S^1}\int_{\lambda}^{2\lambda}\|\chi(d+\delta)f_{\zeta_1}\|^2_{\dot{X}^{-1/2}_{\zeta_1}}\,ds\,d\eta_1 = o(\mathbf{1}(\lambda)),$$

which implies

$$\frac{1}{\lambda}\int_{S^1}\int_{\lambda}^{2\lambda}\|Q(\gamma_1,\mu_1,\omega)A_{\zeta_1}\|^2_{\dot{X}^{-1/2}_{\zeta_1}}\,ds\,d\eta_1 = o(\mathbf{1}(\lambda)) \tag{4-7}$$

as λ becomes large. By (4-4), we obtain (4-6). □

From the construction of $R_{\zeta_1} \in \dot{X}^{1/2}_{\zeta_1}$ solving (4-3), the existence of w_1 of the form (4-2) that solves (3-2) is immediate. However, it turns out that for such a w_1 to satisfy the condition in Proposition 3.1, the constant 1-form A_{ζ_1} has to be chosen carefully.

Lemma 4.5. *Let $w_1 = \sum_{l=0}^{3} w_1^l$ as in* (4-2) *with ζ_1, A_{ζ_1}, and R_{ζ_1} as in Lemma 4.3. Then*

$$w_1^l \in H^1_{\mathrm{loc}}\big(\mathbb{R}^3;\Lambda^l\mathbb{R}^3\big)$$

and w_1 is a solution of (3-2). *Moreover, if A_{ζ_1} satisfies the relation*

$$-\zeta_1 \vee A^1_{\zeta_1} + ikA^0_{\zeta_1} - \zeta_1 \wedge A^2_{\zeta_1} + ikA^3_{\zeta_1} = 0, \tag{4-8}$$

then $v_1 = \sum_0^3 v_1^l$ defined as in (3-3) *satisfies $v_1^0 + v_1^3 = 0$ for $|\zeta_1|$ large enough.*

Proof. We can ensure w_1^l is in $H^1_{\mathrm{loc}}\big(\mathbb{R}^3;\Lambda^l\mathbb{R}^3\big)$ since $R_{\zeta_1} \in \dot{X}^{1/2}_{\zeta_1}$ (See Remark 4.2). Additionally, w_1 is a solution of (3-2) since $R_{\zeta_1} \in \dot{X}^{1/2}_{\zeta_1}$ solves[3] (4-3).

In order to prove the second part of this lemma, note that $v_1^l \in H^1_{\mathrm{loc}}\big(\mathbb{R}^3;\Lambda^l\mathbb{R}^3\big)$ and

$$P(d+\delta;\gamma_1,\mu_1,\omega)v_1 = 0$$

in any bounded open subset of $\mathbb{R}^3$ by Proposition 2.1. Then by Proposition 2.4 we know that $v_1^0 + v_1^3$ is a weak solution to

$$(\delta d + d\delta - k^2)(v_1^0 + v_1^3) + \tilde{q}(\gamma_1,\mu_1,\omega)(v_1^0 + v_1^3) = 0$$

in $\mathbb{R}^3$. By (3-3), we can write $v_1^l = e_{\zeta_1}(B^l_{\zeta_1} + S^l_{\zeta_1})$ with $l \in \{0, 3\}$, where

$$\begin{aligned}
B^0_{\zeta_1} &= -\zeta_1 \vee A^1_{\zeta_1} + ikA^0_{\zeta_1},\\
S^0_{\zeta_1} &= -\zeta_1 \vee R^1_{\zeta_1} + \delta R^1_{\zeta_1} + db \vee (A^1_{\zeta_1} + R^1_{\zeta_1}) + i\omega\gamma_1^{1/2}\mu_1^{1/2}R^0_{\zeta_1} + i(\omega\gamma_1^{1/2}\mu_1^{1/2} - k)A^0_{\zeta_1}, && (4\text{-}9)\\
B^3_{\zeta_1} &= -\zeta_1 \wedge A^2_{\zeta_1} + ikA^3_{\zeta_1},\\
S^3_{\zeta_1} &= -\zeta_1 \wedge R^2_{\zeta_1} - dR^2_{\zeta_1} + da \wedge (A^2_{\zeta_1} + R^2_{\zeta_1}) + i\omega\gamma_1^{1/2}\mu_1^{1/2}R^3_{\zeta_1} + i(\omega\gamma_1^{1/2}\mu_1^{1/2} - k)A^3_{\zeta_1}. && (4\text{-}10)
\end{aligned}$$

Then relation (4-8) implies $B^0_{\zeta_1} + B^3_{\zeta_1} = 0$, and hence that $v_1^0 + v_1^3 = e_{\zeta_1}(S^0_{\zeta_1} + S^3_{\zeta_1})$ is a weak solution of

$$(\Delta_{\zeta_1} - k^2)(S^0_{\zeta_1} + S^3_{\zeta_1}) + \tilde{q}(\gamma_1,\mu_1,\omega)(S^0_{\zeta_1} + S^3_{\zeta_1}) = 0 \tag{4-11}$$

in $\mathbb{R}^3$.

[3] See also (A-20).

To complete the proof, it is sufficient to show that (4-11) is uniquely solvable in $\dot{X}^{1/2}_{\zeta_1}$ for $|\zeta_1|$ large enough and $S^0_{\zeta_1} + S^3_{\zeta_1}$ belongs to $\dot{X}^{1/2}_{\zeta_1}$.

Using the same argument as in proving (4-5), we see that $\tilde{q}(\gamma_1, \mu_1, \omega)$ is a bounded linear operator from $\dot{X}^{1/2}_{\zeta_1}$ to $\dot{X}^{-1/2}_{\zeta_1}$ and its operator norm is $o(\mathbf{1}(|\zeta_1|))$. Then, by Remark 4.1, identity (4-1), and the Banach fixed-point theorem, (4-11) is uniquely solvable in $\dot{X}^{1/2}_{\zeta_1}$ for $|\zeta_1|$ large enough.

Since $e_{\zeta_1} S^l_{\zeta_1} = v^l_1 \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$ for $l \in \{0, 3\}$, we know that $\chi(S^0_{\zeta_1} + S^3_{\zeta_1}) \in \dot{X}^{1/2}_{\zeta_1}$ for $\chi \in C^\infty_0(\mathbb{R}^3)$ such that $\chi(x) = 1$ for all $x \in (\operatorname{supp} d\gamma_1 \cup \operatorname{supp} d\mu_1)$. Therefore, the right-hand side of

$$(\Delta_{\zeta_1} - k^2)(S^0_{\zeta_1} + S^3_{\zeta_1}) = -\tilde{q}(\gamma_1, \mu_1, \omega)\chi(S^0_{\zeta_1} + S^3_{\zeta_1})$$

is in $\dot{X}^{-1/2}_{\zeta_1}$. Further, it is not hard to see from (4-9) and (4-10) that $\widehat{S^l_{\zeta_1}}$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$ with $l \in \{0, 3\}$. The last two facts imply that $S^0_{\zeta_1} + S^3_{\zeta_1} \in \dot{X}^{1/2}_{\zeta_1}$. □

Remark 4.6. The condition given by (4-8) is necessary in our proof since $B^0_{\zeta_1} + B^3_{\zeta_1}$ does not belong to $\dot{X}^{1/2}_{\zeta_1}$.

As a conclusion of these lemmas, we can state the constructions of w_1 in the following theorem.

Theorem 4.7. *Let $s \in \mathbb{R}$ satisfy $s \geq 1$. Given a real-valued constant 1-form ρ in $\mathbb{R}^3$, choose η_1 and η_2 also real-valued constant 1-forms in $\mathbb{R}^3$ such that $\langle \eta_1, \eta_2 \rangle = 0$, $\langle \eta_j, \rho \rangle = 0$, and $|\eta_j| = 1$ for $j \in \{1, 2\}$. Set*

$$\zeta_1 = -\sqrt{s^2 + \frac{|\rho|^2}{4}}\,\eta_1 + i\left(\frac{\rho}{2} - \sqrt{s^2 + k^2}\,\eta_2\right)$$

and

$$A_{\zeta_1} = \frac{\sqrt{2}}{|\zeta_1|}(\zeta_1 \vee \alpha + ik\alpha + ik\beta + \zeta_1 \wedge \beta),$$

where either $\alpha = \eta_1$ and $\beta = 0$ or $\alpha = 0$ and $\beta = |\rho|^{-1}\eta_2 \wedge \rho$. Then, for $|\zeta_1|$ large enough, there exists $w_1 = \sum_0^3 w^l_1$ with $w^l_1 \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^l \mathbb{R}^3)$ of the form

$$w_1 = e_{\zeta_1}(A_{\zeta_1} + R_{\zeta_1}),$$

which is a weak solution to

$$(d\delta + \delta d - k^2)w_1 + Q(\gamma_1, \mu_1, \omega)w_1 = 0$$

in $\mathbb{R}^3$. Moreover, we have $R_{\zeta_1} \in \dot{X}^{1/2}_{\zeta_1}$ satisfies

$$\frac{1}{\lambda}\int_{S^1}\int_{\lambda}^{2\lambda} \|R_{\zeta_1}\|^2_{\dot{X}^{1/2}_{\zeta_1}}\, ds\, d\eta_1 = o(1(\lambda))$$

as λ becomes large. Here S^1 denotes the intersection between the unit sphere in $\mathbb{R}^3$ and the plane defined by η_1 and η_2. Furthermore, $v_1 = \sum_0^3 v^l_1$ defined by

$$v_1 = P(d + \delta; \gamma_1, \mu_1, \omega)^t w_1$$

satisfies $v^0_1 + v^3_1 = 0$ for $|\zeta_1|$ large enough.

4B. ***The construction of v_2.*** Let ζ_2 be a complex-valued constant 1-form in $\mathbb{R}^3$ satisfying $\langle\zeta_2,\zeta_2\rangle=-k^2$. We are looking for the solution $v_2=\sum_0^3 v_2^l$ with $v_2^l\in H^1_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$ to (3-4) in any bounded subset of $\mathbb{R}^3$ of the form

$$v_2=e_{\zeta_2}(B_{\zeta_2}+S_{\zeta_2}),\tag{4-12}$$

where B_{ζ_2} is a constant graded differential form in $\mathbb{R}^3$ and $S_{\zeta_2}\in\dot{X}^{1/2}_{\zeta_2}$. In addition, we want S_{ζ_2} to be small in the sense of (4-6). To construct such a v_2, by Proposition 2.3, we start with the construction of a solution w_2 to

$$\int_{\mathbb{R}^3}\langle\delta w_2,\delta\varphi\rangle+\langle dw_2,d\varphi\rangle-\omega^2\varepsilon_0\mu_0\langle w_2,\varphi\rangle\,dx+\langle\widetilde{Q}(\gamma_2,\mu_2,\omega)w_2\mid\varphi\rangle=0\tag{4-13}$$

for all $\varphi=\sum_0^3\varphi^l$, with $\varphi^l\in C_0^\infty(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$.

Lemma 4.8. *Let $A_{\zeta_2}=A^1_{\zeta_2}+A^2_{\zeta_2}$ be a constant graded differential form in $\mathbb{R}^3$. For $|\zeta_2|$ large enough, there exists $R_{\zeta_2}=R^1_{\zeta_2}+R^2_{\zeta_2}\in\dot{X}^{1/2}_{\zeta_2}$ such that $w_2=w_2^1+w_2^2$ with*

$$w_2^l=e_{\zeta_2}(A^l_{\zeta_2}+R^l_{\zeta_2})$$

and $w_2^l\in H^1_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$, is a solution of (4-13) *in* $\mathbb{R}^3$.

Proof. Analogous to the proof of Lemma 4.3, the existence of a general $R_{\zeta_2}=\sum_0^3 R^l_{\zeta_2}$ for a given constant $A_{\zeta_2}=\sum_0^3 A^l_{\zeta_2}$ is immediate by

$$\|\widetilde{Q}(\gamma_2,\mu_2,\omega)\|_{\dot{X}^{1/2}_{\zeta_2}\to\dot{X}^{-1/2}_{\zeta_2}}=o(\mathbf{1}(|\zeta_2|))$$

as $|\zeta_2|$ becomes large. Since $\widetilde{Q}(\gamma_2,\mu_2,\omega)$ decouples for 1 and 2 forms, we can ensure that $R_{\zeta_2}=R^1_{\zeta_2}+R^2_{\zeta_2}$ for $A_{\zeta_2}=A^1_{\zeta_2}+A^2_{\zeta_2}$. ☐

Now Proposition 2.3 states that $v_2=P(d+\delta;\gamma_2,\mu_2,\omega)w_2$ is a solution to (3-4). Moreover, we can write v_2 as in (4-12). However, we still need to show the smallness of S_{ζ_2}.

Theorem 4.9. *Let $s\in\mathbb{R}$ satisfy $s\geq 1$. Given a real-valued constant 1-form ρ in $\mathbb{R}^3$, we choose η_1 and η_2 two other real-valued constant 1-forms in $\mathbb{R}^3$ such that $\langle\eta_1,\eta_2\rangle=0$, $\langle\eta_j,\rho\rangle=0$, and $|\eta_j|=1$ for $j\in\{1,2\}$. Set*

$$\zeta_2=\sqrt{s^2+\frac{|\rho|^2}{4}}\,\eta_1+i\left(\frac{\rho}{2}+\sqrt{s^2+k^2}\,\eta_2\right)$$

and let α and β be as in Theorem 4.7. If $|\zeta_2|$ is large enough, there exists $v_2=\sum_0^3 v_2^l$ with

$$v_2^l\in H^1_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$$

of the form

$$v_2=e_{\zeta_2}(B_{\zeta_2}+S_{\zeta_2}),$$

where

$$B_{\zeta_2}=-\frac{\sqrt{2}}{|\zeta_2|}(\zeta_2\vee(\alpha+\beta)+\zeta_2\wedge(-\alpha+\beta)+ik(\alpha+\beta))\tag{4-14}$$

and $S_{\zeta_2} \in \dot{X}^{1/2}_{\zeta_2}$, *which solves*

$$P(d+\delta;\gamma_2,\mu_2,\omega)^t v_2 = 0$$

in any bounded open subset of $\mathbb{R}^3$ *and satisfies*

$$\frac{1}{\lambda}\int_{S^1}\int_{\lambda}^{2\lambda} \|S_{\zeta_2}\|^2_{\dot{X}^{1/2}_{\zeta_2}}\, ds\, d\eta_1 = o(1(\lambda)) \tag{4-15}$$

as λ *becomes large. Here* S^1 *denotes the intersection between the unit sphere in* $\mathbb{R}^3$ *and the plane defined by* η_1 *and* η_2.

Proof. Let w_2 be as in Lemma 4.8 with $A_{\zeta_2} = A^1_{\zeta_2} + A^2_{\zeta_2} = -\sqrt{2}(\alpha+\beta)$. By Proposition 2.3, we know that $v_2 = \sum_0^3 v_2^l$ defined by

$$v_2 = P(d+\delta;\gamma_2,\mu_2,\omega)w_2$$

satisfies that $v_2^l \in H^1_{\mathrm{loc}}\big(\mathbb{R}^3;\Lambda^l\mathbb{R}^3\big)$ and solves

$$P(d+\delta;\gamma_2,\mu_2,\omega)^t v_2 = 0 \tag{4-16}$$

in any bounded open subset of $\mathbb{R}^3$. One can easily write

$$v_2 = e_{\zeta_2}(B_{\zeta_2} + S_{\zeta_2})$$

and check that B_{ζ_2} is given by (4-14) and

$$\begin{aligned} S_{\zeta_2} = \frac{1}{|\zeta_2|}\big(&\zeta_2 \vee (R^1_{\zeta_2} + R^2_{\zeta_2}) + \zeta_2 \wedge (-R^1_{\zeta_2} + R^2_{\zeta_2}) + (d+\delta)(-R^1_{\zeta_2} + R^2_{\zeta_2}) \\ &+ da_2 \wedge (A^1_{\zeta_2} + R^1_{\zeta_2}) + da_2 \vee (A^1_{\zeta_2} + R^1_{\zeta_2}) + db_2 \wedge (A^2_{\zeta_2} + R^2_{\zeta_2}) - db_2 \vee (A^2_{\zeta_2} + R^2_{\zeta_2}) \\ &+ i\omega\gamma_2^{1/2}\mu_2^{1/2}(R^1_{\zeta_2} + R^2_{\zeta_2}) + i(\omega\gamma_2^{1/2}\mu_2^{1/2} - k)(A^1_{\zeta_2} + A^2_{\zeta_2})\big). \end{aligned}$$

Moreover, by (4-16) and (2-7), we know that S_{ζ_2} satisfies the familiar equation

$$(\Delta_{\zeta_2} - k^2)S_{\zeta_2} + Q(\gamma_2,\mu_2,\omega)S_{\zeta_2} = -Q(\gamma_2,\mu_2,\omega)B_{\zeta_2}. \tag{4-17}$$

Since $Q(\gamma_2,\mu_2,\omega)B_{\zeta_2} \in \dot{X}^{-1/2}_{\zeta_2}$, (4-17) is uniquely solvable in $\dot{X}^{1/2}_{\zeta_2}$. Therefore, since $S_{\zeta_2} \in \dot{X}^{1/2}_{\zeta_2}$ and $|B_{\zeta_2}| = \mathbb{O}(\mathbf{1}(|\zeta_2|))$, S_{ζ_2} satisfies (4-15). □

5. Proof of uniqueness

To complete the proof of Theorem 1.1, the final step is to plug into the integral formula given in Proposition 3.1 the w_1 and v_2 obtained in Theorem 4.7 and Theorem 4.9 and to let λ go to ∞. The output turns out to be certain nonlinear relations of γ_1, μ_1, γ_2, μ_2, and their weak partial derivatives up to the second order. Then a unique continuation principle argument can be used to conclude the uniqueness.

Throughout this section we let Q_j denote $Q(\gamma_j,\mu_j,\omega)$ with $j \in \{1,2\}$. If

$$\begin{aligned} A_1 &= -(\eta_1 + i\eta_2) \vee \alpha - (\eta_1 + i\eta_2) \wedge \beta, \\ B_2 &= -(\eta_1 + i\eta_2) \vee (\alpha+\beta) - (\eta_1 + i\eta_2) \wedge (-\alpha+\beta) \end{aligned}$$

with α and β as in Theorem 4.7, we see that, for any ρ, $|A_{\zeta_1} - A_1| + |B_{\zeta_2} - B_2| = \mathbb{O}(s^{-1})$ for s large enough and all $\eta_1, \eta_2 \in S^1$. The implicit constant (incorporated in the symbol $\mathbb{O}$) here depends on ρ. On the other hand, plugging w_1 and v_2 into Proposition 3.1, as in Theorem 4.7 and Theorem 4.9, we get

$$\langle (Q_2-Q_1)e_{i\rho}A_1 \mid B_2\rangle = \langle (Q_1-Q_2)(A_{\zeta_1}+R_{\zeta_1}) \mid e_{i\rho}(B_{\zeta_2}-B_2+S_{\zeta_2})\rangle + \langle (Q_1-Q_2)B_2 \mid e_{i\rho}(A_{\zeta_1}-A_1+R_{\zeta_1})\rangle.$$

We know that, for each ρ, Q_j is bounded from $\dot X^{1/2}_{\zeta_j}$ to $\dot X^{-1/2}_{\zeta_j}$ and its norm is $o(\mathbf{1}(s))$ for s large enough and all η_1 (see (4-5) and the same applies to Q_2). The same is true for $Q_1 - Q_2$ from $\dot X^{1/2}_{\zeta_1}$ to $\dot X^{-1/2}_{\zeta_2}$ as an immediate consequence of the proof of Lemma 2.3 in [Haberman and Tataru 2013]. Thus, for each ρ, we have

$$\begin{aligned} |\langle (Q_2 - Q_1)e_{i\rho}A_1 \mid B_2\rangle| &\lesssim \|(Q_1 - Q_2)B_2\|_{\dot X^{-1/2}_{\zeta_1}} [\|\chi(A_{\zeta_1} - A_1)\|_{\dot X^{1/2}_{\zeta_1}} + \|R_{\zeta_1}\|_{\dot X^{1/2}_{\zeta_1}}] \\ &\quad + [\|(Q_1 - Q_2)A_{\zeta_1}\|_{\dot X^{-1/2}_{\zeta_2}} + \|R_{\zeta_1}\|_{\dot X^{1/2}_{\zeta_1}}][\|\chi(B_{\zeta_2} - B_2)\|_{\dot X^{1/2}_{\zeta_2}} + \|S_{\zeta_2}\|_{\dot X^{1/2}_{\zeta_2}}], \end{aligned} \tag{5-1}$$

where $\chi \in \mathbb{C}^\infty_0(\mathbb{R}^3)$ such that $\chi(x) = 1$ for all $x \in \operatorname{supp} d\gamma_2 \cup \operatorname{supp} d\mu_2$. Here the implicit constant might depend on ρ.

If $\alpha = \eta_1$ and $\beta = 0$, then $A_1 = -1$, $B_2 = -1 + i\eta_2 \wedge \eta_1$, and the left-hand side of (5-1) gives

$$\begin{aligned} &\langle (Q_2 - Q_1)e_{i\rho}A_1 \mid B_2\rangle \\ &\quad = \int_{\mathbb{R}^3} \langle d(a_1 - a_2), de_{i\rho}\rangle\, dx + \int_{\mathbb{R}^3} \langle d(a_1 + a_2), d(a_2 - a_1)\rangle e_{i\rho}\, dx + \int_{\mathbb{R}^3} \omega^2(\gamma_1\mu_1 - \gamma_2\mu_2)e_{i\rho}\, dx. \end{aligned} \tag{5-2}$$

If $\alpha = 0$ and $\beta = |\rho|^{-1}\eta_2 \wedge \rho$, then

$$A_1 = -|\rho|^{-1}\eta_1 \wedge \eta_2 \wedge \rho, \quad B_2 = -|\rho|^{-1}(\eta_1 + i\eta_2) \vee (\eta_2 \wedge \rho) - |\rho|^{-1}\eta_1 \wedge \eta_2 \wedge \rho,$$

and we have

$$\begin{aligned} &\langle (Q_2 - Q_1)e_{i\rho}A_1 \mid B_2\rangle \\ &\quad = \int_{\mathbb{R}^3} \langle d(b_1 - b_2), de_{i\rho}\rangle\, dx + \int_{\mathbb{R}^3} \langle d(b_1 + b_2), d(b_2 - b_1)\rangle e_{i\rho}\, dx + \int_{\mathbb{R}^3} \omega^2(\gamma_1\mu_1 - \gamma_2\mu_2)e_{i\rho}\, dx. \end{aligned} \tag{5-3}$$

Meanwhile, by the choice of A_1 and B_2 above, we have

$$\begin{aligned} \Big(\frac{1}{\lambda}\int_{S^1}\int_\lambda^{2\lambda} \|\chi(A_{\zeta_1} - A_1)\|^2_{\dot X^{1/2}_{\zeta_1}}\, ds\, d\eta_1\Big)^{1/2} &= \mathbb{O}(\mathbf{1}(\lambda)), \\ \Big(\frac{1}{\lambda}\int_{S^1}\int_\lambda^{2\lambda} \|\chi(B_{\zeta_2} - B_2)\|^2_{\dot X^{1/2}_{\zeta_2}}\, ds\, d\eta_1\Big)^{1/2} &= \mathbb{O}(\mathbf{1}(\lambda)). \end{aligned}$$

Then, after averaging (5-1) on $(s, \eta_1) \in [\lambda, 2\lambda] \times S^1$ and using the Cauchy–Schwartz inequality, we get

$$\begin{aligned} |\langle (Q_2 - Q_1)e_{i\rho}A_1 \mid B_2\rangle| &\lesssim [\mathbb{O}(\mathbf{1}(\lambda)) + o(\mathbf{1}(\lambda))]\Big(\frac{1}{\lambda}\int_{S^1}\int_\lambda^{2\lambda} \|(Q_1 - Q_2)B_2\|^2_{\dot X^{-1/2}_{\zeta_1}}\, ds\, d\eta_1\Big)^{1/2} \\ &\quad + [\mathbb{O}(\mathbf{1}(\lambda)) + o(\mathbf{1}(\lambda))]\Big[\Big(\frac{1}{\lambda}\int_{S^1}\int_\lambda^{2\lambda} \|(Q_1 - Q_2)A_{\zeta_1}\|^2_{\dot X^{-1/2}_{\zeta_2}}\, ds\, d\eta_1\Big)^{1/2} + o(\mathbf{1}(\lambda))\Big], \end{aligned}$$

where Theorems 4.7 and 4.9 are used. It is not hard to see this converges to zero as λ goes to ∞ by the same argument we used in proving (4-7) and by noticing that the left-hand side is independent of λ. Thus, by (5-2) and (5-3), we arrive at

$$\int_{\mathbb{R}^3} \langle d(a_2-a_1), de_{i\rho}\rangle \, dx - \int_{\mathbb{R}^3} \langle d(a_1+a_2), d(a_2-a_1)\rangle e_{i\rho}\, dx + \int_{\mathbb{R}^3} \omega^2(\gamma_2\mu_2-\gamma_1\mu_1)e_{i\rho}\, dx = 0 \tag{5-4}$$

and

$$\int_{\mathbb{R}^3} \langle d(b_2-b_1), de_{i\rho}\rangle \, dx - \int_{\mathbb{R}^3} \langle d(b_1+b_2), d(b_2-b_1)\rangle e_{i\rho}\, dx + \int_{\mathbb{R}^3} \omega^2(\gamma_2\mu_2-\gamma_1\mu_1)e_{i\rho}\, dx = 0 \tag{5-5}$$

for any ρ. So far, this shows that

$$\begin{cases} \delta d(a_2-a_1) - \langle d(a_1+a_2), d(a_2-a_1)\rangle + \omega^2(\gamma_2\mu_2-\gamma_1\mu_1) = 0, \\ \delta d(b_2-b_1) - \langle d(b_1+b_2), d(b_2-b_1)\rangle + \omega^2(\gamma_2\mu_2-\gamma_1\mu_1) = 0, \end{cases}$$

a system that has to be understood in the weak sense. Finally, some simple computations yield a system of second order equations of the form

$$\begin{cases} -\Delta(\gamma_2^{1/2}-\gamma_1^{1/2}) + V(\gamma_2^{1/2}-\gamma_1^{1/2}) + a(\gamma_2^{1/2}-\gamma_1^{1/2}) + b(\mu_2^{1/2}-\mu_1^{1/2}) = 0, \\ -\Delta(\mu_2^{1/2}-\mu_1^{1/2}) + W(\mu_2^{1/2}-\mu_1^{1/2}) + c(\mu_2^{1/2}-\mu_1^{1/2}) + d(\gamma_2^{1/2}-\gamma_1^{1/2}) = 0, \end{cases}$$

again in the weak sense with

$$V = -\frac{\delta d(\gamma_1^{1/2}+\gamma_2^{1/2})}{\gamma_1^{1/2}+\gamma_2^{1/2}}, \quad W = -\frac{\delta d(\mu_1^{1/2}+\mu_2^{1/2})}{\mu_1^{1/2}+\mu_2^{1/2}}$$

and

$$a = \mathbf{1}_\Omega \omega^2 \gamma_1^{1/2}\gamma_2^{1/2}(\mu_1+\mu_2), \quad b = -\mathbf{1}_\Omega \omega^2 \gamma_1^{1/2}\gamma_2^{1/2}(\gamma_1+\gamma_2)\frac{\mu_1^{1/2}+\mu_2^{1/2}}{\gamma_1^{1/2}+\gamma_2^{1/2}},$$
$$c = \mathbf{1}_\Omega \omega^2 \mu_1^{1/2}\mu_2^{1/2}(\gamma_1+\gamma_2), \quad d = -\mathbf{1}_\Omega \omega^2 \mu_1^{1/2}\mu_2^{1/2}(\mu_1+\mu_2)\frac{\gamma_1^{1/2}+\gamma_2^{1/2}}{\mu_1^{1/2}+\mu_2^{1/2}},$$

where $\mathbf{1}_\Omega$ is the characteristic function of Ω. Note that $\gamma_2^{1/2}-\gamma_1^{1/2}$ and $\mu_2^{1/2}-\mu_1^{1/2}$ belong to $H^1(\mathbb{R}^3)$ and they are compactly supported. Thus the next unique continuation result implies that $\gamma_2=\gamma_1$ and $\mu_2=\mu_1$.

Lemma 5.1. *Let f and g belong to $H^1(\mathbb{R}^3)$ and assume that they are compactly supported. Then f and g vanish if and only if they satisfy*

$$\begin{cases} -\Delta f + Vf + af + bg = 0, \\ -\Delta g + Wg + cg + df = 0. \end{cases} \tag{5-6}$$

Proof. Let $\zeta \in \mathbb{C}^n$ satisfies $\zeta\cdot\zeta = 0$. Set $u(x) = e^{\zeta\cdot x}f(x)$ and $v(x) = e^{\zeta\cdot x}g(x)$. Since f and g belong to $H^1(\mathbb{R}^3)$ and they are compactly supported, u and v also belong to $H^1(\mathbb{R}^3)$ and, consequently, to $\dot{X}_\zeta^{1/2}$. Moreover, u and v solve

$$\begin{cases} -(\Delta + 2\zeta\cdot\nabla)u + Vu + au + bv = 0, \\ -(\Delta + 2\zeta\cdot\nabla)v + Wv + cv + du = 0. \end{cases} \tag{5-7}$$

Let $w = w^0 + w^3$ be the graded form given by $w^0 = u$ and $w^3 = *v$ and define

$$\langle Qw \mid \varphi\rangle = -\int_{\mathbb{R}^3}\Big\langle d(\gamma_1^{1/2}+\gamma_2^{1/2}), d\frac{\langle w^0,\varphi^0\rangle}{\gamma_1^{1/2}+\gamma_2^{1/2}}\Big\rangle\,dx + \int_{\mathbb{R}^3}\langle aw^0+bw^3,\varphi^0\rangle\,dx \\ -\int_{\mathbb{R}^3}\Big\langle d(\mu_1^{1/2}+\mu_2^{1/2}), d\frac{\langle w^3,\varphi^3\rangle}{\mu_1^{1/2}+\mu_2^{1/2}}\Big\rangle\,dx + \int_{\mathbb{R}^3}\langle dw^0+cw^3,\varphi^3\rangle\,dx$$

for any $\varphi = \varphi^0+\varphi^3$ with $\varphi^l \in H^1(\mathbb{R}^3;\Lambda^l\mathbb{R}^3)$. Then $w \in \dot{X}^{1/2}_\zeta$ and (5-7) reads

$$\Delta_\zeta w + Qw = 0. \tag{5-8}$$

Here we have identified ζ with a 1-form also denoted by ζ. Following the same argument as in Lemma 4.3, we can prove

$$\|Q\|_{\dot{X}^{1/2}_\zeta\to\dot{X}^{-1/2}_\zeta} = o(\mathbf{1}(|\zeta|)) \tag{5-9}$$

as $|\zeta|$ becomes large. Then Remark 4.1, identity (4-1), (5-9), and the Banach fixed-point theorem imply that (5-8) has a unique solution belonging to $\dot{X}^{1/2}_\zeta$. Therefore, $w = 0$, which in turn implies $f = g = 0$. □

Appendix: The framework of differential forms

Since the tools used in this paper are scattered throughout the literature, to make the paper more self-contained, we summarized them in this appendix. We start with collecting several basics required in the framework of differential forms (see [Taylor 1996] and [Federer 1969] for some details of differential forms and Grassman graded algebra), and the basic functional spaces and properties for the current discussion of PDEs. Then we show a useful identity used in the paper, and end our discussion with recalling basic facts about the Fourier transform of graded forms.

A1. ***Tools of multivariable calculus.*** For $x \in \mathbb{R}^n$ and $n \in \mathbb{N}\backslash\{0\}$, let $T_x\mathbb{R}^n$ denote the complex vector space of distributions X of order one in $\mathbb{R}^n$ satisfying $\operatorname{supp} X = \{x\}$ and $\langle X \mid c\rangle = 0$ for any constant function c (See Theorem 2.3.4 in [Hörmander 1983] for the justification of this definition). Such X can be uniquely extended to a linear form on $C^1(\mathbb{R}^n)$, the space of continuously differentiable functions in $\mathbb{R}^n$. Let $\partial_{x^j}|_x$ denote the distribution given by

$$\langle \partial_{x^j}|_x \mid \phi\rangle = \partial_{x^j}\phi(x)$$

for any $\phi \in C^1(\mathbb{R}^n)$. The set $\{\partial_{x^1}|_x,\dots,\partial_{x^n}|_x\}$ is a base of $T_x\mathbb{R}^n$. Let $T^*_x\mathbb{R}^n$ denote the dual vector space of $T_x\mathbb{R}^n$ with $\{dx^1|_x,\dots,dx^n|_x\}$ being the dual base. We define on $T^*_x\mathbb{R}^n$ the inner product $\langle\cdot,\cdot\rangle$ given by the bilinear extension of $\langle dx^j|_x, dx^k|_x\rangle = \delta_{jk}$ (Kronecker delta). Note that it is not a Hermitian product.

A1.1. *Differential forms.* Let $\Lambda^l\mathbb{R}^n$ with $l \in \{0,1,\dots,n\}$ and $n \ge 2$ denote the smooth complex vector bundle over $\mathbb{R}^n$ whose fiber at $x \in \mathbb{R}^n$ consists of $\Lambda^l T^*_x\mathbb{R}^n$, the l-fold exterior product of $T^*_x\mathbb{R}^n$. By convention, a 0-fold is just a complex number and a 1-fold is an element of $T^*_x\mathbb{R}^n$. Let E be a nonempty subset of $\mathbb{R}^n$; an l-form on E is a section u of $\Lambda^l\mathbb{R}^n$ over E, so $u(x) = u|_x \in \Lambda^l T^*_x\mathbb{R}^n$ for any $x \in E$. Any

l-form on E with $l \in \{1, \dots, n\}$ can be written as

$$u = \sum_{\alpha \in S^l} u_\alpha \, dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_l}$$

with $S^l = \{(\alpha_1, \dots, \alpha_l) \in \{1, \dots, n\}^l : \alpha_1 < \cdots < \alpha_l\}$ and $u_\alpha : E \longrightarrow \mathbb{C}$. It is convenient to call u_α with $\alpha \in S^l$ the component functions of u.

The exterior product of an l-form u and an m-form v, both on E, is denoted by $(u \wedge v)(x) = u|_x \wedge v|_x$ for any $x \in E$. Recall that the exterior product is bilinear, associative and anticommutative:

$$u \wedge v = (-1)^{lm} v \wedge u. \tag{A-1}$$

Since a 0-form v on E is nothing but a map from E to $\mathbb{C}$, it holds that $u \wedge v = v \wedge u = vu$ for any l-form u on E.

The inner product of two l-forms on E with $l \in \{2, \dots, n\}$ can be defined at each point $x \in E$ as the bilinear extension of

$$\langle (dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_l})|_x, (dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_l})|_x \rangle = \det \langle dx^{\alpha_j}|_x, dx^{\beta_k}|_x \rangle,$$

where the right-hand side stands for the determinant of the matrix

$$(\langle dx^{\alpha_j}|_x, dx^{\beta_k}|_x \rangle)_{jk}.$$

The inner product of two 0-forms is just the usual product of functions. The inner product on l-forms can be immediately extended to graded forms $u(x) = \sum_0^n u^l(x)$ and $v(x) = \sum_0^n v^l(x)$ on E, with u^l and v^l l-forms on E, as follows:

$$\langle u, v \rangle(x) = \sum_{l=0}^{n} \langle u^l|_x, v^l|_x \rangle.$$

Associated to this inner product, we consider the norm satisfying $|u|^2 = \langle u, \bar{u} \rangle$.

Now let $T_x^* \mathbb{R}^n$ be endowed with an orientation. The Hodge star operator of an l-form on E with $l \in \{1, \dots, n-1\}$ is defined at each point $x \in E$ as the linear extension of

$$*(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_l})|_x = (dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_{n-l}})|_x,$$

where $(\beta_1, \dots, \beta_{n-l}) \in \{1, \dots, n\}^{n-l}$ is chosen such that

$$\{dx^{\alpha_1}, \dots dx^{\alpha_l}, dx^{\beta_1}, \dots, dx^{\beta_{n-l}}\}$$

is a positive base of $T_x^* \mathbb{R}^n$. The case of 0-forms and n-forms follows from

$$*1|_x = (dx^1 \wedge \cdots \wedge dx^n)|_x, \quad *(dx^1 \wedge \cdots \wedge dx^n)|_x = 1|_x,$$

where 1 denotes the constant function taking the value 1 at any point. Now, if u and v are l-forms on E,

$$**u(x) = (-1)^{l(n-l)}u(x), \tag{A-2}$$

$$\langle u, v\rangle(x) = *(u|_x \wedge *v|_x) = *(v|_x \wedge *u|_x), \tag{A-3}$$

$$\langle u, v\rangle = \langle *u, *v\rangle. \tag{A-4}$$

Let u be an l-form on E and let v be an m-form on E. The vee product of v and u at each point $x \in E$ is defined as

$$(v \vee u)(x) = (-1)^{(n+m-l)(l-m)}*(v|_x \wedge *u|_x). \tag{A-5}$$

Note that whenever $m > l$, $(v \vee u)(x) = 0$ for all $x \in E$. The vee product is bilinear, but it is neither associative nor commutative. The product satisfies

$$\langle w \wedge v, u\rangle = \langle w, v \vee u\rangle \tag{A-6}$$

for any k-form w on E.

Proposition A.1. *If u and v are 1-forms and w is an l-form with $l \in \{0, \dots, n\}$, then*

$$u \vee (v \wedge w) - v \wedge (u \vee w) = (-1)^l \langle u, v\rangle w. \tag{A-7}$$

Corollary A.2. *If u^1 and v^1 are 1-forms and u^l and v^l are l-forms with $l \in \{0, \dots, n\}$, then*

$$\langle u^1 \vee u^l, v^1 \vee v^l\rangle + \langle v^1 \wedge u^l, u^1 \wedge v^l\rangle = \langle u^1, v^1\rangle\langle u^l, v^l\rangle.$$

Proof. Since

$$\langle u^1 \vee u^l, v^1 \vee v^l\rangle + \langle v^1 \wedge u^l, u^1 \wedge v^l\rangle = (-1)^l \langle u^1 \vee (v^1 \wedge u^l) - v^1 \wedge (u^1 \vee u^l), v^l\rangle,$$

the identity follows from (A-7). □

Let G be a nonempty open subset of $\mathbb{R}^n$ and k a positive integer. An l-form u on G with $l \in \{1, \dots, n\}$ is said to be k-times continuously differentiable if its component functions are k-times continuously differentiable in G. We write $u \in C^k(G; \Lambda^l\mathbb{R}^n)$. If $u \in C^k(G; \Lambda^l\mathbb{R}^n)$ for any positive integer k, we say that u is smooth and we write $u \in C^\infty(G; \Lambda^l\mathbb{R}^n)$. Furthermore, $u \in C^k(G; \Lambda^l\mathbb{R}^n)$ (respectively $u \in C^\infty(G; \Lambda^l\mathbb{R}^n)$) is said to be compactly supported if its component functions are compactly supported in G, in which case we write $u \in C_0^\infty(G; \Lambda^l\mathbb{R}^n)$ (respectively $u \in C_0^\infty(G; \Lambda^l\mathbb{R}^n)$). These definitions are naturally generalized to 0-forms, where the conventional function space notations are also used.

The exterior derivative of $u \in C^1(G; \Lambda^0\mathbb{R}^n)$ is a 1-form defined by

$$du|_x(X) = \langle X \mid \chi_x u\rangle$$

for each $x \in G$ and $X \in T_x\mathbb{R}^n$. Here $\chi_x \in C_0^\infty(G)$ with $\chi_x(x) = 1$ on G, and $\chi_x u$ is understood as the extension of u by zero outside G. The exterior derivative of $u \in C^1(G; \Lambda^l\mathbb{R}^n)$ with $l \in \{1, \dots, n\}$ is defined by

$$du = \sum_{\alpha \in S^l} du_\alpha \wedge dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_l}.$$

Recall that $d(du) = 0$ for any $u \in C^2(G; \Lambda^l\mathbb{R}^3)$ and

$$d(u \wedge v) = du \wedge v + (-1)^l u \wedge dv, \tag{A-8}$$

for any $u \in C^1(G; \Lambda^l\mathbb{R}^3)$ and $v \in C^1(G; \Lambda^m\mathbb{R}^3)$.

A1.2. *Symmetric tensors.* Let $\Sigma^l\mathbb{R}^n$ with $l \in \mathbb{N}$ and $n \geq 2$ denote the smooth complex vector bundle over $\mathbb{R}^n$ whose fiber at $x \in \mathbb{R}^n$ consists of $\Sigma^l T_x^*\mathbb{R}^n$, the l-fold symmetric tensor product of $T_x^*\mathbb{R}^n$. By convention, a 0-fold is just a complex number and a 1-fold is an element of $T_x^*\mathbb{R}^n$. Let E be a nonempty subset of $\mathbb{R}^n$; an l-symmetric tensor on E is a section u of $\Sigma^l\mathbb{R}^n$ over E, so $u(x) = u|_x \in \Sigma^l T_x^*\mathbb{R}^n$ for any $x \in E$. Any l-symmetric tensor on E with $l \in \{1, \dots, n\}$ can be written as

$$u = \sum_{\alpha \in T^l} u_\alpha \, dx^{\alpha_1} \odot \cdots \odot dx^{\alpha_l}$$

with $T^l = \{(\alpha_1, \dots, \alpha_l) \in \{1, \dots, n\}^l : \alpha_1 \leq \cdots \leq \alpha_l\}$ and $u_\alpha : E \longrightarrow \mathbb{C}$. It is convenient to call u_α with $\alpha \in T^l$ the component functions of u and to point out that $\Sigma^l T_x^*\mathbb{R}^n = \Lambda^l T_x^*\mathbb{R}^n$ for $l \in \{0, 1\}$, which in turn implies $\Sigma^l\mathbb{R}^n = \Lambda^l\mathbb{R}^n$ for $l \in \{0, 1\}$.

The symmetric tensor product of an l-symmetric tensor u and an m-symmetric tensor v, both on E, is denoted by $(u \odot v)(x) = u|_x \odot v|_x$ for any $x \in E$. Recall that the symmetric tensor product is bilinear, associative, and commutative. Moreover, if u and v are 1-symmetric tensors,

$$u \odot v = \tfrac{1}{2}(u \otimes v + v \otimes u).$$

The inner product of two l-symmetric tensors on E with $l \in \mathbb{N} \setminus \{0, 1\}$ can be defined at each point $x \in E$ as the bilinear extension of

$$\langle (dx^{\alpha_1} \odot \cdots \odot dx^{\alpha_l})|_x, (dx^{\beta_1} \odot \cdots \odot dx^{\beta_l})|_x \rangle = |\det \langle dx^{\alpha_j}|_x, dx^{\beta_k}|_x \rangle|.$$

Let G be a nonempty open subset of $\mathbb{R}^n$. An l-symmetric tensor u on G with $l \in \mathbb{N}$ is said to be k-times continuously differentiable if its component functions are k-times continuously differentiable in G, and we write $u \in C^k(G; \Sigma^l\mathbb{R}^n)$. Furthermore, $u \in C^k(G; \Sigma^l\mathbb{R}^n)$ with $l \in \mathbb{N}$ is said to be compactly supported if its component functions are compactly supported in G, and we write $u \in C_0^k(G; \Sigma^l\mathbb{R}^n)$. These definitions extend naturally to 0-symmetric tensors on G.

The symmetric derivative of a smooth l-symmetric tensor u on G with $l \in \mathbb{N} \setminus \{0\}$ is defined by

$$i\,Du = \sum_{\alpha \in T^l} du_\alpha \odot dx^{\alpha_1} \odot \cdots \odot dx^{\alpha_l}.$$

A2. ***Functional spaces.*** Let $L^1_{\mathrm{loc}}(E; \Lambda^l\mathbb{R}^n)$ denote the space of locally integrable l-forms (whose component functions are in $L^1_{\mathrm{loc}}(E)$) modulo those which vanish almost everywhere (a.e.) in E. The space $L^p(E; \Lambda^l\mathbb{R}^n)$, with $p \in [1, +\infty)$, consists of all $u \in L^1_{\mathrm{loc}}(E; \Lambda^l\mathbb{R}^n)$ such that

$$\int_E \langle u, \bar{u} \rangle^{p/2} \, dx < +\infty.$$

Endowed with the norm

$$\|u\|_{L^p(E;\Lambda^l\mathbb{R}^n)} = \left(\int_E \langle u, \bar{u}\rangle^{p/2}\,dx\right)^{1/p},$$

$L^p(E; \Lambda^l\mathbb{R}^n)$ is a Banach space. Moreover, $L^2(E; \Lambda^l\mathbb{R}^n)$ is a Hilbert space.

Let $u \in L^1_{\mathrm{loc}}(G; \Lambda^l\mathbb{R}^n)$ with $l \in \{1, \dots, n\}$. We say that $v \in L^1_{\mathrm{loc}}(G; \Lambda^{l-1}\mathbb{R}^n)$ is the formal adjoint derivative of u, denoted by $v = \delta u$, if

$$\int_G \langle v, w\rangle\,dx = \int_G \langle u, dw\rangle\,dx$$

for any $w \in C^1_0(G; \Lambda^{l-1}\mathbb{R}^n)$. If $u \in L^1_{\mathrm{loc}}(G; \Lambda^0\mathbb{R}^n)$, we define $\delta u = 0$. For all $u \in L^1_{\mathrm{loc}}(G; \Lambda^l\mathbb{R}^n)$ with $l \in \{0, \dots, n\}$ such that $\delta u \in L^1_{\mathrm{loc}}(G; \Lambda^{l-1}\mathbb{R}^n)$, one has $\delta(\delta u) = 0$. Moreover, if $u \in C^1(G; \Lambda^l\mathbb{R}^n)$, then

$$\delta u = (-1)^{n(l+1)+1} {*d*u}. \tag{A-9}$$

Proposition A.3. *Consider $u \in L^1_{\mathrm{loc}}(G; \Lambda^l\mathbb{R}^n)$ and $v \in C^1(G; \Lambda^m\mathbb{R}^n)$. If $\delta u \in L^1_{\mathrm{loc}}(G; \Lambda^{l-1}\mathbb{R}^n)$, then $\delta(v \vee u) \in L^1_{\mathrm{loc}}(G; \Lambda^{l-m-1}\mathbb{R}^n)$ and*

$$\delta(v \vee u) = (-1)^{l-m} dv \vee u + v \vee \delta u. \tag{A-10}$$

Let $u \in L^1_{\mathrm{loc}}(G; \Lambda^l\mathbb{R}^n)$ with $l \in \{0, \dots, (n-1)\}$. We say that $v \in L^1_{\mathrm{loc}}(G; \Lambda^{l+1}\mathbb{R}^n)$ is the (weak) exterior derivative of u, denoted by $v = du$, if

$$\int_G \langle v, w\rangle\,dx = \int_G \langle u, \delta w\rangle\,dx$$

for any $w \in C^1_0(G; \Lambda^{l+1}\mathbb{R}^n)$. If $u \in L^1_{\mathrm{loc}}(G; \Lambda^n\mathbb{R}^n)$, we define $du = 0$. For all $u \in L^1_{\mathrm{loc}}(G; \Lambda^l\mathbb{R}^n)$ with $l \in \{0, \dots, n\}$ such that $du \in L^1_{\mathrm{loc}}(G; \Lambda^{l+1}\mathbb{R}^n)$, one has $d(du) = 0$.

Proposition A.4. *Let $u \in L^1_{\mathrm{loc}}(G; \Lambda^l\mathbb{R}^n)$ such that $\delta u \in L^1_{\mathrm{loc}}(G; \Lambda^{l-1}\mathbb{R}^n)$. Then $*d*u \in L^1_{\mathrm{loc}}(G; \Lambda^{l-1}\mathbb{R}^n)$ and*

$$\delta u = (-1)^{n(l+1)+1} {*d*u}. \tag{A-11}$$

We now present certain Sobolev spaces of forms, in which our PDEs are discussed. Let $H^d(G; \Lambda^l\mathbb{R}^n)$ (respectively $H^\delta(G; \Lambda^l\mathbb{R}^n)$) denote the space of $u \in L^2(G; \Lambda^l\mathbb{R}^n)$ such that $du \in L^2(G; \Lambda^{l+1}\mathbb{R}^n)$ (respectively $\delta u \in L^2(G; \Lambda^{l-1}\mathbb{R}^n)$), endowed with the norm

$$\|u\|_{H^d(G;\Lambda^l\mathbb{R}^n)} = (\|u\|^2_{L^2(G;\Lambda^l\mathbb{R}^n)} + \|du\|^2_{L^2(G;\Lambda^{l+1}\mathbb{R}^n)})^{1/2}$$
$$\left(\text{respectively } \|u\|_{H^\delta(G;\Lambda^l\mathbb{R}^n)} = (\|u\|^2_{L^2(G;\Lambda^l\mathbb{R}^n)} + \|\delta u\|^2_{L^2(G;\Lambda^{l-1}\mathbb{R}^n)})^{1/2}\right).$$

It is observed that $H^d(G; \Lambda^l\mathbb{R}^n)$ and $H^\delta(G; \Lambda^l\mathbb{R}^n)$ are Hilbert spaces and $C^1_0(\mathbb{R}^n; \Lambda^l\mathbb{R}^n)$ is dense in them. Let $H^d_{\mathrm{loc}}(\mathbb{R}^n; \Lambda^l\mathbb{R}^n)$ and $H^\delta_{\mathrm{loc}}(\mathbb{R}^n; \Lambda^l\mathbb{R}^n)$ denote the spaces of $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \Lambda^l\mathbb{R}^n)$ such that $u|_U \in H^d(U; \Lambda^l\mathbb{R}^n)$ and $u|_U \in H^\delta(U; \Lambda^l\mathbb{R}^n)$, respectively, for any bounded nonempty open subset U in $\mathbb{R}^n$.

Finally, by a density argument, we have

$$\int_{\mathbb{R}^n} \langle du, v\rangle \, dx = \int_{\mathbb{R}^n} \langle u, \delta v\rangle \, dx \tag{A-12}$$

for all $u \in H^d(\mathbb{R}^n; \Lambda^{l-1}\mathbb{R}^n)$ and $v \in H^\delta(\mathbb{R}^n; \Lambda^l\mathbb{R}^n)$ with $l \in \{1, \dots, n\}$.

A3. ***Traces.***[4]

Let U be a nonempty bounded open subset of $\mathbb{R}^n$, and let $H^1(U; \Lambda^l\mathbb{R}^n)$ denote the space of all $u \in L^2(U; \Lambda^l\mathbb{R}^n)$ whose component functions u_α satisfy $du_\alpha \in L^2(U; \Lambda^1\mathbb{R}^n)$ for all $\alpha \in S^l$, endowed with the norm

$$\|u\|_{H^1(U;\Lambda^l\mathbb{R}^n)} = \Big(\|u\|^2_{L^2(U;\Lambda^l\mathbb{R}^n)} + \sum_{\alpha\in S^l} \|du_\alpha\|^2_{L^2(U;\Lambda^1\mathbb{R}^n)} \Big)^{1/2}. \tag{A-13}$$

Given G, a nonempty open subset of $\mathbb{R}^n$, by $H^1_{\mathrm{loc}}(G; \Lambda^l\mathbb{R}^n)$ we denote the space of $u \in L^1_{\mathrm{loc}}(G; \Lambda^l\mathbb{R}^n)$ such that $u|_U \in H^1(U; \Lambda^l\mathbb{R}^n)$ for any bounded nonempty open subset U of G.

It is a consequence of (A-11) that, for any $u \in H^1(U; \Lambda^l\mathbb{R}^n)$, one has

$$\|u\|_{H^\delta(U;\Lambda^l\mathbb{R}^n)} \le \|u\|_{H^1(U;\Lambda^l\mathbb{R}^n)}. \tag{A-14}$$

Let $H^1_0(U; \Lambda^l\mathbb{R}^n)$ denote the closure in $H^1(U; \Lambda^l\mathbb{R}^n)$ of $C_0^\infty(U; \Lambda^l\mathbb{R}^n)$ modulo those vanishing a.e. in U. We then define the space

$$TH^1(\partial U; \Lambda^l\mathbb{R}^n) = H^1(U; \Lambda^l\mathbb{R}^n)/H^1_0(U; \Lambda^l\mathbb{R}^n).$$

If $f \in TH^1(\partial U; \Lambda^l\mathbb{R}^n)$, let $u_f \in H^1(U; \Lambda^l\mathbb{R}^n)$ denote a representative of f. This space can be endowed with the norm

$$\|f\|_{TH^1(\partial U;\Lambda^l\mathbb{R}^n)} = \inf\{\|u\|_{H^1(U;\Lambda^l\mathbb{R}^n)} : u - u_f \in H^1_0(U; \Lambda^l\mathbb{R}^n)\}.$$

Let $TH^1(\partial U; \Lambda^l\mathbb{R}^n)^*$ denote the dual space of $TH^1(\partial U; \Lambda^l\mathbb{R}^n)$ with the functional $\|\cdot\|_{TH^1(\partial U;\Lambda^l\mathbb{R}^n)^*}$ standing for the dual norm.

The latter spaces will be used as auxiliary spaces to define certain traces on $H^d(U; \Lambda^l\mathbb{R}^n)$ and $H^\delta(U; \Lambda^l\mathbb{R}^n)$. Firstly, define the d-trace of $v \in H^d(U; \Lambda^l\mathbb{R}^n)$ with $l \in \{0, \dots, n-1\}$ as

$$\langle d \operatorname{tr} v \mid f\rangle = \int_U \langle dv, u\rangle \, dx - \int_U \langle v, \delta u\rangle \, dx$$

for any $f \in TH^1(\partial U; \Lambda^{l+1}\mathbb{R}^n)$ where $u \in H^1(U; \Lambda^{l+1}\mathbb{R}^n)$ such that $u - u_f \in H^1_0(U; \Lambda^{l+1}\mathbb{R}^n)$. Since (A-14) holds, we have

$$\langle d \operatorname{tr} v \mid f\rangle \le \|v\|_{H^d(U;\Lambda^l\mathbb{R}^n)} \|u\|_{H^1(U;\Lambda^{l+1}\mathbb{R}^n)}$$

for all $u \in H^1(U; \Lambda^{l+1}\mathbb{R}^n)$ such that $u - u_f \in H^1_0(U; \Lambda^{l+1}\mathbb{R}^n)$. Hence $d \operatorname{tr} v \in TH^1(\partial U; \Lambda^{l+1}\mathbb{R}^n)^*$ and

$$\|d \operatorname{tr} v\|_{TH^1(\partial U;\Lambda^{l+1}\mathbb{R}^n)^*} \le \|v\|_{H^d(U;\Lambda^l\mathbb{R}^n)}.$$

[4]For more details on traces see [Mitrea 2004; Schwarz 1995].

This motivates the definition of $TH^d(\partial U;\,\Lambda^{l+1}\mathbb{R}^n)$ to be the space of all $g \in TH^1(\partial U;\,\Lambda^{l+1}\mathbb{R}^n)^*$ such that $d\,\mathrm{tr}\,v = g$ for some $v \in H^d(U;\,\Lambda^l\mathbb{R}^n)$. The endowed norm is then given by

$$\|g\|_{TH^d(\partial U;\Lambda^{l+1}\mathbb{R}^n)} = \inf\{\|v\|_{H^d(U;\Lambda^l\mathbb{R}^n)} : d\,\mathrm{tr}\,v = g\}.$$

Finally, we define the δ-trace of $v \in H^\delta(U;\,\Lambda^l\mathbb{R}^n)$ with $l \in \{1,\dots,n\}$ as

$$\langle \delta\,\mathrm{tr}\,v \mid f\rangle = (-1)^l \int_U \langle \delta v, u\rangle\,dx - (-1)^l \int_U \langle v, du\rangle\,dx$$

for any $f \in TH^1(\partial U;\,\Lambda^{l-1}\mathbb{R}^n)$ where $u \in H^1(U;\,\Lambda^{l-1}\mathbb{R}^n)$ such that $u - u_f \in H^1_0(U;\,\Lambda^{l-1}\mathbb{R}^n)$. Similarly we would have $\delta\,\mathrm{tr}\,v \in TH^1(\partial U;\,\Lambda^{l-1}\mathbb{R}^n)^*$ and

$$\|\delta\,\mathrm{tr}\,v\|_{TH^1(\partial U;\Lambda^{l-1}\mathbb{R}^n)^*} \le \|v\|_{H^\delta(U;\Lambda^l\mathbb{R}^n)}.$$

Moreover, we define $TH^\delta(\partial U;\,\Lambda^{l-1}\mathbb{R}^n)$, the space consisting of all g belonging to $TH^1(\partial U;\,\Lambda^{l-1}\mathbb{R}^n)^*$, such that there exists $v \in H^\delta(U;\,\Lambda^l\mathbb{R}^n)$ with $\delta\,\mathrm{tr}\,v = g$ with norm

$$\|g\|_{TH^\delta(\partial U;\Lambda^{l-1}\mathbb{R}^n)} = \inf\{\|v\|_{H^\delta(U;\Lambda^l\mathbb{R}^n)} : \delta\,\mathrm{tr}\,v = g\}.$$

Then we will need the following lemma about these spaces.

Lemma A.5. *Given the definitions above*,

(a) *if* $u \in H^d(U;\,\Lambda^l\mathbb{R}^n)$ *with* $l \in \{0,\dots,n-2\}$ *and* $d\,\mathrm{tr}\,u = 0$, *then* $d\,\mathrm{tr}(du) = 0$;

(b) *if* $u \in H^\delta(U;\,\Lambda^l\mathbb{R}^n)$ *with* $l \in \{2,\dots,n\}$ *and* $\delta\,\mathrm{tr}\,u = 0$, *then* $\delta\,\mathrm{tr}(\delta u) = 0$.

Proof. In order to prove (a), let us consider the bounded linear operator

$$(\nu \vee \cdot\,) : TH^d(\partial U;\,\Lambda^l\mathbb{R}^n) \longrightarrow TH^\delta(\partial U;\,\Lambda^{l-1}\mathbb{R}^n)^*$$

given by

$$\langle \nu \vee f \mid g\rangle = \int_U \langle du, v\rangle\,dx - \int_U \langle u, \delta v\rangle\,dx,$$

where $u \in H^d(U;\,\Lambda^{l-1}\mathbb{R}^n)$, $v \in H^\delta(U;\,\Lambda^l\mathbb{R}^n)$, $d\,\mathrm{tr}\,u = f$, and $\delta\,\mathrm{tr}\,v = g$. Here $TH^\delta(\partial U;\,\Lambda^{l-1}\mathbb{R}^n)^*$ denotes the dual of $TH^\delta(\partial U;\,\Lambda^{l-1}\mathbb{R}^n)$. Let u be as in (a) and $g \in TH^1(U;\,\Lambda^{l+2}\mathbb{R}^n)$. Then

$$\langle d\,\mathrm{tr}(du) \mid g\rangle = -\langle \nu \vee d\,\mathrm{tr}\,u \mid \delta\,\mathrm{tr}(\delta v_g)\rangle,$$

where $v_g \in H^1(U;\,\Lambda^{l+2}\mathbb{R}^n)$ denotes a representative of g. Therefore (a) holds.

A similar proof applies to (b) by considering the operator

$$(\nu \vee \cdot\,) : TH^\delta(\partial U;\,\Lambda^l\mathbb{R}^n) \longrightarrow TH^d(\partial U;\,\Lambda^{l+1}\mathbb{R}^n)^*$$

defined by

$$\langle \nu \wedge f \mid g\rangle = (-1)^{l+1} \int_U \langle \delta u, v\rangle\,dx - (-1)^{l+1} \int_U \langle u, dv\rangle\,dx,$$

where $u \in H^\delta(U;\,\Lambda^{l+1}\mathbb{R}^n)$, $v \in H^d(U;\,\Lambda^l\mathbb{R}^n)$, $\delta\,\mathrm{tr}\,u = f$, and $d\,\mathrm{tr}\,v = g$. We leave the proof to the readers. $\square$

A4. ***A useful identity.*** Given G, a nonempty open subset of $\mathbb{R}^n$, let $L^1_{\mathrm{loc}}(G;\Sigma^l\mathbb{R}^n)$ denote the space of locally integrable l-symmetric tensors (whose component functions are in $L^1_{\mathrm{loc}}(E)$) modulo those which vanish a.e. in E.

For $u\in L^1_{\mathrm{loc}}(G;\Sigma^l\mathbb{R}^n)$ with $l\in\mathbb{N}\setminus\{1,2\}$, we say that $v\in L^1_{\mathrm{loc}}(G;\Sigma^{l-1}\mathbb{R}^n)$ is the formal adjoint (symmetric) derivative of u, denoted by $v=D^*u$, if

$$\int_G \langle v,w\rangle\,dx=\int_G\langle u,Dw\rangle\,dx$$

for any $w\in C^1_0(G;\Sigma^{l-1}\mathbb{R}^n)$.

Note that if

$$u=\sum_{j=1}^n u_j dx^j \quad\text{and}\quad v=\sum_{j=1}^n v_j dx^j$$

are such that $u\odot v\in L^1_{\mathrm{loc}}(G;\Sigma^2\mathbb{R}^n)$ and $D^*(u\odot v)\in L^1_{\mathrm{loc}}(G;\Sigma^1\mathbb{R}^n)$, then

$$D^*(u\odot v)=-\sum_{k=1}^n\biggl(\sum_{j=1}^n \partial_{x^j}(u_jv_k+u_kv_j)\biggr)\,dx^k. \tag{A-15}$$

Proposition A.6. *Given u and v in $H^1_{\mathrm{loc}}(G;\Lambda^1\mathbb{R}^n)$, we have that $d\langle u,v\rangle$ and $D^*(u\odot v)$ belong to $L^1_{\mathrm{loc}}(G;\Lambda^1\mathbb{R}^n)$ and the following identity holds*:

$$u\vee dv+v\vee du+\delta u\vee v+\delta v\vee u=d\langle u,v\rangle+D^*(u\odot v).$$

A5. ***Local regularity.*** Here we prove a local regularity lemma for the operator $(d+\delta)\sum_0^n(-1)^l$.

Lemma A.7. *Let $v=\sum_0^n v^l$ be such that $v^l\in L^2_{\mathrm{loc}}(\mathbb{R}^n;\Lambda^l\mathbb{R}^n)$ and*

$$(d+\delta)\sum_{l=0}^n(-1)^l v^l\in\bigoplus_{l=0}^n L^2_{\mathrm{loc}}(\mathbb{R}^n;\Lambda^l\mathbb{R}^n).$$

Then $v^l\in H^1_{\mathrm{loc}}(\mathbb{R}^n;\Lambda^l\mathbb{R}^n)$ for $l\in\{0,\dots,n\}$.

Proof. By using Corollary A.2 and the identity

$$\langle\xi\wedge\widehat{\phi^{l-1}}(\xi),\overline{\xi\vee\widehat{\phi^{l+1}}(\xi)}\rangle=0,$$

we can check that

$$\|\phi\|^2_{L^2}=\|\phi\|^2_{H^{-1}}+\biggl\|(d+\delta)\sum_{l=0}^n(-1)^l\phi^l\biggr\|^2_{H^{-1}} \tag{A-16}$$

for all $\phi=\sum_0^n\phi^l$ such that $\phi^l\in L^2(\mathbb{R}^n;\Lambda^l\mathbb{R}^n)$. Here we are using the notation $\|\varphi\|^2_Y=\sum_0^n\|\varphi^l\|^2_{Y(\mathbb{R}^n;\Lambda^l\mathbb{R}^n)}$ for $\varphi=\sum_0^n\varphi^l$ with $\varphi^l\in Y(\mathbb{R}^n;\Lambda^l\mathbb{R}^n)$, where Y denotes either L^2 or H^{-1}. Recall that $\|\varphi^l\|^2_{H^{-1}(\mathbb{R}^n;\Lambda^l\mathbb{R}^n)}=\int_{\mathbb{R}^n}(1+|\xi|^2)^{-1}|\widehat{\varphi^l}(\xi)|^2\,d\xi$.

Let ψ be a compactly supported smooth function in $\mathbb{R}^n$ and let $\Delta_h^j\phi$ be defined as

$$\Delta_h^j\phi(x) = \frac{1}{h}(\phi(x+he_j) - \phi(x))$$

with ϕ as in (A-16), h a positive parameter, and e_j the j-th element of the orthonormal basis of $\mathbb{R}^n$. By (A-16) and the commutativity between Δ_h^j and $(d+\delta)\sum_0^n(-1)^l$, we have

$$\|\Delta_h^j(\psi v)\|_{L^2}^2 = \|\Delta_h^j(\psi v)\|_{H^{-1}}^2 + \|\Delta_h^j(d+\delta)\sum_{l=0}^n(-1)^l(\psi v^l)\|_{H^{-1}}^2. \tag{A-17}$$

Since

$$(d+\delta)\sum_{l=0}^n(-1)^l(\psi v^l) = \psi(d+\delta)\sum_{l=0}^n(-1)^l v^l + \sum_{l=0}^n(-1)^l d\psi\wedge v^l + d\psi\vee v^l$$

and v and $(d+\delta)\sum_0^n(-1)^l v^l$ belong to $\bigoplus_0^n L^2_{\mathrm{loc}}\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big)$, the statement of the result follows by making the parameter h go to zero in the identity (A-17)[5]. □

A6. ***Fourier transform of forms and operator Δ_ζ.*** An l-form u with $l\in\{0,\dots,n\}$ is said to belong to the Schwartz space $\mathcal{S}\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big)$ if its component functions u_α ($\alpha\in S^l$) are in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. We can define the space $\mathcal{S}'\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big)$ of l-form-valued tempered distributions similarly. The Fourier Transform of $u\in\mathcal{S}\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big)$ is then defined by

$$\hat{u} = \sum_{\alpha\in S^l}\widehat{u_\alpha}\,d\xi^{\alpha_1}\wedge\cdots\wedge d\xi^{\alpha_l}\in\mathcal{S}\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big).$$

The Fourier Transform $\hat{u}$ for $u\in\mathcal{S}'\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big)$ can be defined by duality. One can easily verify the following identities for $u\in\mathcal{S}\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big)$;

$$\widehat{du}(\xi) = i\xi\wedge\hat{u}(\xi),\quad \widehat{\delta u}(\xi) = i(-1)^l\xi\vee\hat{u}(\xi), \tag{A-18}$$

where $\xi\in\mathbb{R}^3\setminus\{0\}$ can be viewed as a 1-form. For $u, v\in L^2\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big)$, we have

$$\int_{\mathbb{R}^n}\langle u,\bar{v}\rangle\,dx = \int_{\mathbb{R}^n}\langle\hat{u},\bar{\hat{v}}\rangle\,dx, \tag{A-19}$$

making the Fourier transform a unitary map on $L^2\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big)$.

Given $\zeta=\sum_1^n\zeta_j\,dx^j$, a constant 1-differential form in $\mathbb{R}^n$, consider the conjugated Hodge–Laplacian operator $\Delta_\zeta = e_{-\zeta}(d\delta+\delta d)\circ e_\zeta$, where $e_\zeta(x)=e^{\zeta\cdot x}$ and $\zeta\cdot x=\sum_1^n\zeta_j x^j$. When acting on an l-form $u\in H^d\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big)\cap H^\delta\big(\mathbb{R}^n;\Lambda^l\mathbb{R}^n\big)$, it reads

$$\Delta_\zeta u = (d\delta+\delta d)u + (-1)^l d(\zeta\vee u) + \zeta\wedge\delta u + \delta(\zeta\wedge u) + (-1)^{l+1}\zeta\vee du - \langle\zeta,\zeta\rangle u, \tag{A-20}$$

(understood in the weak sense). Moreover, it is easy to verify that the symbol of Δ_ζ is $|\xi|^2 - 2i\langle\zeta,\xi\rangle - \langle\zeta,\zeta\rangle$ by (A-18).

[5] See Theorem (6.19) of [Folland 1995] for more details.

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