ANALYSIS & PDE

Volume 7 No. 2 2014

MATTHEW KENNEDY AND DILIAN YANG

A NON-SELF-ADJOINT LEBESGUE DECOMPOSITION

msp

ANALYSIS AND PDE
Vol. 7, No. 2, 2014
dx.doi.org/10.2140/apde.2014.7.497

A NON-SELF-ADJOINT LEBESGUE DECOMPOSITION

MATTHEW KENNEDY AND DILIAN YANG

We study the structure of bounded linear functionals on a class of non-self-adjoint operator algebras that includes the multiplier algebra of every complete Nevanlinna–Pick space, and in particular the multiplier algebra of the Drury–Arveson space. Our main result is a Lebesgue decomposition expressing every linear functional as the sum of an absolutely continuous (i.e., weak-* continuous) linear functional and a singular linear functional that is far from being absolutely continuous. This is a non-self-adjoint analogue of Takesaki's decomposition theorem for linear functionals on von Neumann algebras. We apply our decomposition theorem to prove that the predual of every algebra in this class is (strongly) unique.

1. Introduction

The main result in this paper is a decomposition theorem for bounded linear functionals on a class of operator algebras that includes the multiplier algebra of every complete Nevanlinna–Pick space. Results of this kind can be seen as a noncommutative generalization of the Yosida–Hewitt decomposition of a measure into completely additive and purely finitely additive parts, or more classically, the Lebesgue decomposition of a measure into absolutely continuous and singular parts.

Takesaki [1958] proved that a bounded linear functional on a von Neumann algebra can be decomposed uniquely into the sum of a normal (i.e., weak-* continuous) linear functional and a singular linear functional that is far from being normal. Ando [1978] proved a direct analogue of Takesaki's decomposition theorem for linear functionals on the algebra H^∞, of bounded analytic functions on the complex unit disk $\mathbb{D}$. More recently, Ueda [2009; 2011] proved a generalization of Ando's result for finite maximal subdiagonal algebras, which are "analytic" subalgebras of finite von Neumann algebras introduced by Arveson [1967] as a noncommutative generalization of the algebra H^∞.

A compelling case can be made that the natural function-theoretic generalization of H^∞ is the algebra H^∞_d of multipliers on the Drury–Arveson space H^2_d. The algebra H^∞_d is contained in the algebra $H^\infty(\mathbb{B}_d)$ of bounded analytic functions on the complex unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$, but for $d \geq 2$ this inclusion is proper, and H^∞_d is seemingly much more tractable than $H^\infty(\mathbb{B}_d)$ (see, for example, [Arveson 1998]). The Drury–Arveson space H^2_d and the multiplier algebra H^∞_d are universal in the following sense: Every irreducible complete Nevanlinna–Pick space embeds into H^2_d, and the corresponding multiplier algebra arises as the compression of H^∞_d onto this embedding (see [Agler and McCarthy 2000] for details).

Both authors are partially supported by NSERC.
MSC2010: 46B04, 47B32, 47L50, 47L55.
Keywords: Lebesgue decomposition, extended F. and M. Riesz theorem, unique predual, Drury–Arveson space.

Examples of complete Nevanlinna–Pick spaces include the Hardy space and the Dirichlet space on the disk, the Drury–Arveson space itself, and more generally the class of Besov–Sobolev spaces on $\mathbb{B}_d$.

One explanation for the tractability of H_d^∞ is the fact that H_d^∞ arises as a quotient of the noncommutative analytic Toeplitz algebra F_d^∞ (see, for example, [Davidson and Pitts 1998b; Arias and Popescu 2000]). This algebra, introduced in [Popescu 1989a], can be viewed as an algebra of noncommutative analytic functions acting by left multiplication on a Hardy space F_d^2 of noncommutative analytic functions. The operator-algebraic structure of F_d^∞, which is now well understood, turns out to be strikingly similar to that of H^∞ (see, for example, [Popescu 1989a; 1989b; 1991; 1995; Arias and Popescu 2000; Davidson and Pitts 1998a; 1998b; 1999; Davidson and Yang 2008]).

For a weak-* closed two-sided ideal $\mathcal{J}$ of F_d^∞, we let $\mathcal{A}_\mathcal{J}$ denote the algebra $\mathcal{A}_\mathcal{J} = F_d^\infty/\mathcal{J}$. These algebras are the main objects of interest in this paper, for the following reason: The multiplier algebra of every irreducible complete Nevanlinna–Pick space arises as the compression of F_d^∞ to a coinvariant subspace, and this compression is completely isometrically isomorphic and weak-* to weak-* homeomorphic to a quotient of F_d^∞ by a two-sided ideal (see [Davidson and Pitts 1998b; Arias and Popescu 2000] for details).

Our main result is the following decomposition theorem for linear functionals on quotients of F_d^∞. A functional is said to be absolutely continuous if it is weak-* continuous, and singular if it is, roughly speaking, far from being weak-* continuous (we give a precise definition below).

Theorem 1.1 (Lebesgue decomposition for quotients of F_d^∞). *Let $\mathcal{J}$ be a weak-* closed two-sided ideal of F_d^∞, and let ϕ be a bounded linear functional on $\mathcal{A}_\mathcal{J}$. Then there are unique linear functionals ϕ_a and ϕ_s on $\mathcal{A}_\mathcal{J}$ such that $\phi = \phi_a + \phi_s$, where ϕ_a is absolutely continuous and ϕ_s is singular, and such that*

$$\|\phi\| \le \|\phi_a\| + \|\phi_s\| \le \sqrt{2}\|\phi\|.$$

If $d = 1$, then the constant $\sqrt{2}$ can be replaced with the constant 1. Moreover, these constants are optimal.

The following result for multiplier algebras of complete Nevanlinna–Pick spaces is an immediate consequence of Theorem 1.1.

Corollary 1.2 (Lebesgue decomposition for multiplier algebras). *Let $\mathcal{A}$ be the multiplier algebra of a complete Nevanlinna–Pick space, and let ϕ be a bounded linear functional on $\mathcal{A}$. Then there are unique linear functionals ϕ_a and ϕ_s on $\mathcal{A}$ such that $\phi = \phi_a + \phi_s$, where ϕ_a is absolutely continuous and ϕ_s is singular, and such that*

$$\|\phi\| \le \|\phi_a\| + \|\phi_s\| \le \sqrt{2}\|\phi\|.$$

We first prove that Theorem 1.1 holds for F_d^∞. The proof for quotients of F_d^∞ requires the following generalization of the classical F. and M. Riesz theorem, which is similar in spirit to the noncommutative F. and M. Riesz-type theorems proved in [Exel 1990] for operator algebras with the Dirichlet property and in [Blecher and Labuschagne 2007; Ueda 2009] for maximal subdiagonal algebras.

Theorem 1.3 (extended F. and M. Riesz theorem). *Let ϕ be a bounded linear functional on F_d^∞, and let $\phi = \phi_a + \phi_s$ be the Lebesgue decomposition of ϕ into absolutely continuous and singular parts as in*

Theorem 1.1. Let $\mathcal{J}$ be a weak- closed two-sided ideal of F_d^∞. If ϕ is zero on $\mathcal{J}$, then ϕ_a and ϕ_s are both zero on $\mathcal{J}$.*

Grothendieck [1955] proved that L^1 is the unique predual of L^∞ (up to isometric isomorphism). Sakai [1956] generalized Grothendieck's result by proving that the predual of every von Neumann algebra is unique. In fact, this latter result follows from the proof of Sakai's characterization of von Neumann algebras as C*-algebras which are dual spaces.

The uniqueness of the predual of a von Neumann algebra can also be proved using Takesaki's decomposition theorem [1958] (see, for example, the proof of Corollary 3.9 of [Takesaki 2002]). A similar idea was used by Ando [1978] to prove the uniqueness of the predual of H^∞, and more recently by Ueda [2009] to prove that the predual of every maximal subdiagonal algebra is unique.

Inspired by these results, we apply Theorem 1.3 to prove that the predual of every quotient $\mathcal{A}_\mathcal{J}$ is (strongly) unique.

Theorem 1.4. *Let $\mathcal{J}$ be a weak-* closed two-sided ideal of F_d^∞. Then the algebra $\mathcal{A}_\mathcal{J}$ has a strongly unique predual.*

It follows immediately from Theorem 1.4 that the multiplier algebra of every complete Nevanlinna–Pick space has a unique predual.

Corollary 1.5. *The multiplier algebra of every complete Nevanlinna–Pick space has a strongly unique predual.*

In particular, Corollary 1.5 implies that the multiplier algebra H_d^∞ on the Drury–Arveson space has a unique predual. We believe this result is especially interesting in light of the fact that, for $d \geq 2$, the uniqueness of the predual of $H^\infty(\mathbb{B}_d)$ is an open problem.

In addition to this introduction, this paper has five other sections. In Section 2, we provide a brief review of the requisite background material. In Section 3, we prove the Lebesgue decomposition for F_d^∞, and give an example showing that the constant in the statement of the theorem is optimal. In Section 4, we prove the extended F. and M. Riesz theorem. In Section 5, we prove the Lebesgue decomposition theorem for quotients of F_d^∞, and hence for multiplier algebras of complete Nevanlinna–Pick spaces. In Section 6, we use the Lebesgue decomposition theorem to prove that the predual of every quotient of F_d^∞ is unique, and hence that the predual of the multiplier algebra of every complete Nevanlinna–Pick space is unique.

2. Preliminaries

The noncommutative analytic Toeplitz algebra. For fixed $1 \leq d \leq \infty$, let $\mathbb{C}\langle Z\rangle = \mathbb{C}\langle Z_1, \dots, Z_d\rangle$ denote the algebra of noncommutative polynomials in the variables $Z_1, \dots, Z_d$. As a vector space, $\mathbb{C}\langle Z\rangle$ is spanned by the set of monomials

$$\{Z_w = Z_{w_1} \cdots Z_{w_n} \mid w = w_1 \cdots w_n \in \mathbb{F}_d^*,\ n \geq 0\},$$

where $\mathbb{F}_d^*$ denotes the free semigroup generated by $\{1, \dots, d\}$. The noncommutative Hardy space F_d^2 is the Hilbert space obtained by completing $\mathbb{C}\langle Z\rangle$ in the natural inner product

$$\langle Z_w, Z_{w'}\rangle = \delta_{w,w'}, \quad w, w' \in \mathbb{F}_d^*.$$

Equivalently, F_d^2 is the Hilbert space consisting of noncommutative power series with square summable coefficients,

$$F_d^2 = \left\{ \sum_{w\in\mathbb{F}_d^*} a_w Z_w \;\middle|\; \sum_{w\in\mathbb{F}_d^*} |a_w|^2 < \infty \right\}.$$

We think of the elements of F_d^2 as noncommutative analytic functions.

Every element in F_d^2 gives rise to a multiplication operator on F_d^2 in the following way (note that in this noncommutative setting, it is necessary to specify whether multiplication occurs on the left or the right). For F in F_d^2, the left multiplication operator L_F is defined by

$$L_F G = FG, \quad G \in F_d^2.$$

The operator L_F is not necessarily bounded in general, simply because the product of two elements in F_d^2 is not necessarily contained in F_d^2. However, it is always densely defined on $\mathbb{C}\langle Z\rangle$.

The noncommutative analytic Toeplitz algebra F_d^∞ is the noncommutative multiplier algebra of F_d^2. It consists precisely of the functions F in F_d^2 such that the corresponding left multiplication operator is bounded,

$$F_d^\infty = \{F \in F_d^2 \mid FG \in F_d^2 \text{ for all } G \in F_d^2\}.$$

Equivalently, if we identity F in F_d^∞ with the left multiplication operator L_F on the Hilbert space F_d^2, then F_d^∞ is obtained as the closure of $\mathbb{C}\langle Z\rangle$ in the weak-* topology on $\mathcal{B}(F_d^2)$. The noncommutative disk algebra A_d is the closure of $\mathbb{C}\langle Z\rangle$ in the norm topology. Note that it is properly contained in F_d^∞.

The algebras A_d and F_d^∞ were introduced by Popescu in [1996] and [1995], respectively. For $d = 1$, F_d^2 can be identified with the classical Hardy space H^2, F_d^∞ can be identified with the classical algebra of bounded analytic functions H^∞, and A_d can be identified with the classical disk algebra of functions that are analytic on $\mathbb{D}$ with continuous extensions to the boundary.

The structure of an isometric tuple.

Definition 2.1. Let $V = (V_1, \dots, V_d)$ be an isometric tuple.

(1) V is a *unilateral shift* if it is unitarily equivalent to a multiple of $L_Z = (L_{Z_1}, \dots, L_{Z_d})$.

(2) V is *absolutely continuous* if the unital weak operator closed algebra $\mathrm{W}(V_1, \dots, V_d)$ generated by $V_1, \dots, V_d$ is algebraically isomorphic to the noncommutative analytic Toeplitz algebra F_d^∞.

(3) V is *singular* if the weakly closed algebra $\mathrm{W}(V_1, \dots, V_d)$ is a von Neumann algebra.

(4) V is of *dilation type* if it has no summand that is absolutely continuous or singular.

Theorem 2.2 (Lebesgue–von Neumann–Wold decomposition [Kennedy 2013]). *Let $V = (V_1, \dots, V_d)$ be an isometric d-tuple. Then V can be decomposed as*

$$V = V_u \oplus V_a \oplus V_s \oplus V_d,$$

where V_u is a unilateral d-shift, V_a is an absolutely continuous unitary d-tuple, V_s is a singular unitary d-tuple and V_d is a unitary d-tuple of dilation type.

Theorem 2.3 (structure theorem for free semigroup algebras [Davidson et al. 2001]). *Let $V = (V_1, \dots, V_d)$ be an isometric d-tuple, and let $\mathcal{V} = \mathrm{W}(V_1, \dots, V_d)$ denote the unital weak operator closed algebra generated by $V_1, \dots, V_d$. Then there is a maximal projection P in $\mathcal{V}$ with the range of P coinvariant for $\mathcal{V}$ such that*

(1) $\mathcal{V}P = \bigcap_{k\geq 1} (\mathcal{V}_0)^k$, *where $(\mathcal{V}_0)^k$ denotes the ideal $(\mathcal{V}_0)^k = \sum_{|w|=k} V_w \mathcal{V}$.*

(2) *If $P^\perp \neq 0$, then the restriction of $\mathcal{V}$ to the range of $P^\perp$ is an analytic free semigroup algebra.*

(3) *The compression of $\mathcal{V}$ to the range of P is a von Neumann algebra.*

(4) $\mathcal{V} = P^\perp \mathcal{V} P^\perp + \mathrm{W}^*(V)P$.

Let $V = V_u \oplus V_a \oplus V_s \oplus V_d$ be the Lebesgue–von Neumann–Wold decomposition of an isometric tuple V, as in Theorem 2.2, where V_u is a unilateral n-shift, V_a is an absolutely continuous unitary n-tuple, V_s is a singular unitary n-tuple and V_d is a unitary n-tuple of dilation type. Suppose that V is defined on a Hilbert space H, and let $H = H_u \oplus H_a \oplus H_s \oplus H_d$ denote the corresponding decomposition of H. By Corollary 2.7 of [Davidson et al. 2001], there is a maximal invariant subspace K for V_d such that the restriction of V_d to K is analytic. The projection P in Theorem 2.3 is determined by $P^\perp = P_{H_u} \oplus P_{H_a} \oplus P_K$.

Remark 2.4. For $d = 1$, an isometry is the direct sum of a unilateral shift, an absolutely continuous unitary and a singular unitary. Theorem 2.3 implies that, in this case, the structure projection P is the projection onto the singular unitary part. In particular, this implies that P is reducing. For $d \geq 2$, the proof of Theorem 2.3 shows that P is reducing if and only if there is no summand of dilation type.

The universal representation. We require the universal representation $\pi_u : F_d^\infty \to \mathcal{B}(H_u)$ of F_d^∞. This can be constructed as in 2.4.4 of [Blecher and Le Merdy 2004], as the restriction of the universal representation of $\mathrm{C}^*_{\max}(F_d^\infty)$. By [ibid., 3.2.12], we can identify the double dual $(F_d^\infty)^{**}$ of F_d^∞ with the algebra obtained as the weak-* closure of $\pi_u(F_d^\infty)$. We will require the operator algebra structure on $(F_d^\infty)^{**}$ provided by this identification. By replacing π_u by $\pi_u^{(\infty)}$ if necessary, we can suppose that π_u has infinite multiplicity, and hence that the weak operator topology coincides with the weak-* topology on $(F_d^\infty)^{**}$.

Let ϕ be a bounded linear functional on F_d^∞. By the Hahn–Banach theorem, we can extend ϕ to a functional on $\mathrm{C}^*_{\max}(F_d^\infty)$ with the same norm. Hence by the construction of the universal representation of $\mathrm{C}^*_{\max}(F_d^\infty)$, there are vectors x and y in H_u with $\|x\|\|y\| = \|\phi\|$ such that

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

If we identify F_d^∞ with its image $\pi_u(F_d^\infty)$ in $(F_d^\infty)^{**}$, then the functional ϕ has a unique weak-* continuous extension to a functional on $(F_d^\infty)^{**}$ with the same norm. We will use this fact repeatedly.

Since π_u is the restriction of a *-homomorphism of $\mathrm{C}^*_{\max}(F_d^\infty)$, and since the d-tuple $(L_{Z_1}, \dots, L_{Z_d})$ is isometric, it follows that the d-tuple $(\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d}))$ is also isometric. Since $(F_d^\infty)^{**}$ contains $\pi_u(A_d)$, it necessarily contains the weak operator closed algebra generated by $(\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d}))$. Let P_u denote the projection in $(F_d^\infty)^{**}$ guaranteed by Theorem 2.3. We will refer to P_u as the universal structure projection in $(F_d^\infty)^{**}$.

Remark 2.5. Let $\mathcal{S}$ denote the unital weak operator closed algebra generated by $\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d})$. From above we have $\mathcal{S} \subseteq (F_d^\infty)^{**}$, and one might guess that $\mathcal{S} = (F_d^\infty)^{**}$. However, this is not the case. Indeed, let ϕ be a bounded nonzero functional on F_d^∞ that is zero on the noncommutative disk algebra A_d. Then as above, there are vectors x and y in H_u such that

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

Let ψ denote the weak operator continuous functional on $\mathcal{S}$ defined by

$$\psi(S) = \langle Sx, y\rangle \quad \text{for all } S \in \mathcal{S}.$$

Since ϕ is zero on A_d, ψ must be zero on $\pi_u(A_d)$. Then, since $\pi_u(A_d)$ is weak operator dense in $\mathcal{S}$, it follows that $\psi(S) = \langle Sx, y\rangle = 0$ for all S in $\mathcal{S}$. But, by assumption, there is A in F_d^∞ such that $\phi(A) = \langle \pi_u(A)x, y\rangle \neq 0$. So we see that $\pi_u(A) \notin \mathcal{S}$, and hence that the inclusion $\mathcal{S} \subseteq (F_d^\infty)^{**}$ is proper.

3. The Lebesgue decomposition

In this section, we introduce the definitions of absolutely continuous and singular linear functionals on the noncommutative analytic Toeplitz algebra F_d^∞, and establish the first version of the Lebesgue decomposition. In [Davidson et al. 2005], Davidson, Li and Pitts proved a Lebesgue-type decomposition for functionals on the noncommutative disk algebra A_d. Although the algebra F_d^∞ is bigger than A_d, the next definition is closely related to (and directly inspired by) the corresponding definition for A_d.

Definition 3.1. Let ϕ be a bounded linear functional on F_d^∞. Then

(1) ϕ is *absolutely continuous* if it is weak-* continuous, and

(2) ϕ is *singular* if $\|\phi\| = \|\phi^k\|$ for every $k \geq 1$, where ϕ^k denotes the restriction of ϕ to the ideal of F_d^∞ generated by $\{L_{Z_w} \mid |w| = k\}$.

Let ϕ be a bounded linear functional on F_d^∞. Then as in Section 2, there are vectors x and y in H_u with $\|x\|\|y\| = \|\phi\|$ such that

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

We will write $P_u\phi$ and $P_u^\perp\phi$ for the linear functionals defined on F_d^∞ by

$$(P_u\phi)(A) = \langle \pi_u(A)P_ux, y\rangle \quad \text{for all } A \in F_d^\infty,$$
$$(P_u^\perp\phi)(A) = \langle \pi_u(A)P_u^\perp x, y\rangle \quad \text{for all } A \in F_d^\infty,$$

where P_u denotes the universal structure projection from Section 2. The purpose of the next result is to verify that $P_u\phi$ and $P_u^\perp\phi$ are well defined.

Lemma 3.2. *Let ϕ be a bounded linear functional on F_d^∞. Then the functionals $P_u\phi$ and $P_u^\perp\phi$, as defined above, do not depend on the choice of vectors x and y.*

Proof. Let x_1, y_1 and x_2, y_2 be pairs of vectors in H_u such that

$$\langle \pi_u(A)x_1, y_1\rangle = \langle \pi_u(A)x_2, y_2\rangle \quad \text{for all } A \in F_d^\infty.$$

Since $\pi_u(F_d^\infty)$ is weak-* dense in the algebra $(F_d^\infty)^{**}$, which contains P_u, it follows immediately that

$$\langle \pi_u(A)P_u x_1, y_1\rangle = \langle \pi_u(A)P_u x_2, y_2\rangle \quad \text{for all } A \in F_d^\infty,$$

and similarly that

$$\langle \pi_u(A)P_u^\perp x_1, y_1\rangle = \langle \pi_u(A)P_u^\perp x_2, y_2\rangle \quad \text{for all } A \in F_d^\infty. \qquad \square$$

Proposition 3.3. *A bounded functional ϕ on F_d^∞ is singular if and only if $\phi = P_u\phi$.*

Proof. Let ϕ be a singular functional on F_d^∞. We can assume that $\|\phi\| = 1$. As in Section 2, there are vectors x and y in H_u such that $\|x\|\|y\| = 1$ and

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

By the singularity of ϕ, we can find a sequence (A_k) of elements in F_d^∞ such that $\lim \phi(A_k) \to 1$, and such that each A_k belongs to the unit ball of $(F_{d,0}^\infty)^k = \sum_{|w|=k} F_d^\infty L_{Z_w}$. Let T be an accumulation point of the sequence $(\pi_u(A_k))$ in $(F_d^\infty)^{**}$, and let $\mathcal{S}$ denote the unital weak operator closed algebra generated by $(\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d}))$. It is clear that the weak-* closure of the image $\pi_u((F_{d,0}^\infty)^k)$ of the ideal $(F_{d,0}^\infty)^k$ can be written as $(F_d^\infty)^{**}\mathcal{S}_0^k$, where $\mathcal{S}_0$ denotes the ideal in $\mathcal{S}$ generated by $\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d})$. Thus $\pi_u(A_k)$ belongs to $(F_d^\infty)^{**}\mathcal{S}_0^k$. By Theorem 2.3, $\mathcal{S}P_u = \bigcap_{k\geq 1} \mathcal{S}_0^k$. Hence T belongs to the unit ball of

$$\bigcap_{k\geq 1}(F_d^\infty)^{**}\mathcal{S}_0^k = (F_d^\infty)^{**}\bigcap_{k\geq 1}\mathcal{S}_0^k = (F_d^\infty)^{**}P_u.$$

In particular, this means that $T = TP_u$. Since $\phi(T) = 1$, this gives

$$\|x\|\|y\| = 1 = \langle Tx, y\rangle = \langle TP_u x, y\rangle \leq \|P_u x\|\|y\| \leq \|x\|\|y\|.$$

Hence $P_u x = x$, and it follows that $\phi = P_u\phi$.

Conversely, let ϕ be a functional on F_d^∞ such that $\phi = P_u\phi$. As before, we can assume that $\|\phi\| = 1$, and there are vectors x and y in H_u such that $\|x\|\|y\| = 1$ and

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

The fact that $P_u\phi = \phi$ implies that we can choose x satisfying $x = P_u x$, and hence that

$$\phi(A) = \langle \pi_u(A)P_u x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

Let ψ denote the functional on $(F_d^\infty)^{**}$ defined by

$$\psi(T) = \langle T P_u x, y\rangle \quad \text{for all } T \in (F_d^\infty)^{**},$$

and for $k \geq 1$, let ψ^k denote the restriction of ψ to $(F_d^\infty)^{**}\mathcal{S}_0^k$. Then as above,

$$(F_d^\infty)^{**} P_u = \bigcap_{k\geq 1} (F_d^\infty)^{**}\mathcal{S}_0^k.$$

Hence $\|\psi\| = \|\psi^k\|$ for every $k \geq 1$. It follows that $\|\phi\| = \|\phi^k\|$, where ϕ^k is defined as in Definition 3.1, and hence that ϕ is singular. □

Lemma 3.4. *The range of the projection $P_u^\perp$ is invariant for $(F_d^\infty)^{**}$.*

Proof. It suffices to show that whenever x and y are vectors in F_d^2 such that $x = P_u^\perp x$ and $y = P_u y$, and the functional ϕ on F_d^∞ is defined by

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty,$$

then $\phi = 0$. By Theorem 2.3, the range of $P_u^\perp$ is invariant for $\pi_u(A_d)$. Hence ϕ is zero on A_d. Let A be an element of F_d^∞. By Corollary 2.6 of [Davidson and Pitts 1998a], for $k \geq 1$, we can write A uniquely as

$$A = \sum_{|w|<k} a_w L_{Z_w} + A',$$

where the a_w are scalars, and A' belongs to $(F_{d,0}^\infty)^k$. The fact that ϕ is zero on A_d implies that $\phi(A) = \phi(A')$. It follows from Definition 3.1 that ϕ is singular. Hence by Proposition 3.3, $\phi = P_u\phi$, i.e.,

$$\phi(A) = \langle \pi_u(A)P_u x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

Since $x = P_u^\perp x$, it follows that $\phi = 0$, as required. □

Proposition 3.5. *Let ϕ be a bounded linear functional on F_d^∞. Then ϕ is absolutely continuous if and only if $\phi = P_u^\perp\phi$.*

Proof. Suppose first that ϕ is absolutely continuous. Then it is weak-* continuous, so there are sequences of vectors (x_k) and (y_k) in F_d^2 such that

$$\phi(A) = \sum \langle Ax_k, y_k\rangle \quad \text{for all } A \in F_d^\infty.$$

Since the d-tuple $(L_{Z_1}, \ldots, L_{Z_d})$ is equivalent to a restriction of the unilateral shift part of the d-tuple $(\pi_u(L_{Z_1}), \ldots, \pi_u(L_{Z_d}))$, F_d^2 can be identified with a subspace of H_u, and it follows that $\phi = P_u^\perp\phi$.

Conversely, suppose that $\phi = P_u^\perp\phi$. As in Section 2, there are vectors x and y in H_u with $\|x\|\|y\| = \|\phi\|$ such that

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

The fact that $\phi = P_u^\perp\phi$ implies that we can choose x satisfying $P_u^\perp x = x$. Since, by Lemma 3.4, the range of $P_u^\perp$ is invariant for $\pi_u(F_d^\infty)$, it follows that for every A in F_d^∞, we have

$$\phi(A) = \langle \pi_u(A)x, y\rangle = \langle P_u^\perp \pi_u(A) P_u^\perp x, y\rangle = \langle \pi_u(A)P_u^\perp x, P_u^\perp y\rangle.$$

Hence we can also choose y satisfying $P_u^{\perp} y = y$.

By the construction of P_u, the restriction of the operators $\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d})$ to the cyclic subspace generated by x and y is analytic. Thus, by the main result of [Kennedy 2013], the weak-* closed algebra generated by this restriction is completely isometrically isomorphic and weak-* to weak-* homeomorphic to F_d^∞. It follows that ϕ is weak-* continuous on F_d^∞. □

Theorem 3.6 (Lebesgue decomposition for F_d^∞). *Let ϕ be a bounded linear functional on F_d^∞. Then there are unique bounded linear functionals ϕ_a and ϕ_s on F_d^∞ such that $\phi = \phi_a + \phi_s$, where ϕ_a is absolutely continuous and ϕ_s is singular, and such that*

$$\|\phi\| \le \|\phi_a\| + \|\phi_s\| \le \sqrt{2}\|\phi\|.$$

If $d = 1$, then the constant $\sqrt{2}$ can be replaced with the constant 1.

Proof. As in Section 2, there are vectors x and y in H_u such that $\|x\|\|y\| = \|\phi\|$ and

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

Define ϕ_a and ϕ_s by $\phi_a = P_u^{\perp}\phi$ and $\phi_s = P_u\phi$, respectively. Then ϕ_a is absolutely continuous by Proposition 3.5, and ϕ_s is singular by Proposition 3.3. We clearly have $\phi = \phi_a + \phi_s$. To see that ϕ_a and ϕ_s are unique, suppose that

$$\phi_a + \phi_s = \psi_a + \psi_s,$$

where ψ_a is absolutely continuous and ψ_s is absolutely continuous. Then

$$\phi_a - \psi_a = \psi_s - \phi_s.$$

It is clear that the functional $\phi_a - \psi_a$ is absolutely continuous, and Proposition 3.3 implies that the functional $\psi_s - \phi_s$ is singular. Applying Proposition 3.5 and Proposition 3.3 again, we can therefore write

$$\phi_a - \psi_a = P_u^{\perp}(\phi_a - \psi_a) = P_u^{\perp}(\psi_s - \phi_s) = P_u P_u^{\perp}(\psi_s - \phi_s) = 0.$$

Hence $\phi_a = \psi_a$, and it follows similarly that $\phi_s = \psi_s$. Finally, we compute

$$\|\phi\| \le \|\phi_a\| + \|\phi_s\| \le \|Px\|\|y\| + \|P^{\perp}x\|\|y\| \le \sqrt{2}\|x\|\|y\| = \sqrt{2}\|\phi\|.$$

If $d = 1$, then Remark 2.4 implies that $(F_d^\infty)^{**}$ is the direct sum of two algebras reduced by P_u. If we identify F_d^∞ with its image in $(F_d^\infty)^{**}$, then the functionals ϕ, ϕ_a and ϕ_s extend uniquely to weak-* continuous functionals on $(F_d^\infty)^{**}$ with the same norm. Since $\phi_a = P_u^{\perp}\phi_a$ and $\phi_s = P_u\phi_s$, it follows that in this case, $\|\phi\| = \|\phi_a\| + \|\phi_s\|$. □

The next example is based on Example 5.10 from [Davidson et al. 2005]. It establishes that for $d \ge 2$, the constant $\sqrt{2}$ in the statement of Theorem 3.6 is the best possible.

Example 3.7. Define ϕ on $\mathbb{C}\langle Z\rangle$ by setting

$$\phi(L_{Z_w}) = \begin{cases} 1/\sqrt{2} & \text{if } w = \varnothing \text{ or } w = 21^n \text{ for } n \ge 0, \\ 0 & \text{otherwise,} \end{cases}$$

and extending by linearity. We will first show that ϕ extends to a bounded linear functional on the noncommutative disk algebra A_2. Let $\mathcal{H}_\phi$ denote the Hilbert space $\mathbb{C}e \oplus F^2_2/\mathcal{J}$ and define a 2-tuple $S = (S_1, S_2)$ on $\mathcal{H}_\phi$ by setting

$$S_1 = \begin{pmatrix} I & 0 \\ 0 & L_1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 \\ \xi_\varnothing e^* & L_2 \end{pmatrix}.$$

It is easy to check that S is isometric. By the universal property of the noncommutative disk algebra, we obtain a completely isometric representation π_ϕ of A_2 satisfying

$$\pi_\phi(L_{Z_w}) = S_{w_1} \cdots S_{w_n}, \quad w = w_1 \cdots w_n \in \mathbb{F}^*_d,$$

and we can extend ϕ to A_2 by

$$\phi(A) = \langle \pi_\phi(A)(e + \xi_\varnothing)/\sqrt{2}, \xi_\varnothing \rangle, \quad A \in A_2.$$

From this, it is easy to check that $\|\phi\| \le 1$.

Let $\mathcal{S}$ denote the unital weakly closed algebra generated by S_1 and S_2. The structure projection from Theorem 2.3 is the projection P onto $\mathbb{C}e$, which is contained in $\mathcal{S}$. Hence $\mathcal{S}$ contains the element $B = (S_2 P + P^\perp)/\sqrt{2}$. The results of [Kennedy 2011] imply that Theorem 5.4 of [Davidson et al. 2005] applies to the unital weak operator closed algebra generated by any isometric tuple. Thus there is a net (B_λ) of elements in the unit ball of A_d such that w*-lim $\pi_\phi(B_\lambda) = B$ in $\mathcal{S}$. It is easy to check that $\|B\| = 1$ and $\langle B(e + \xi_\varnothing)/\sqrt{2}, \xi_\varnothing \rangle = 1$, so it follows that $\|\phi\| = 1$.

By the Hahn–Banach theorem, we can extend ϕ to a functional on F^∞_d with the same norm, which we continue to denote by ϕ. Let $\phi = \phi_a + \phi_s$ be the Lebesgue decomposition of ϕ into absolutely continuous and singular parts as in Theorem 3.6. Then restricted to A_d, we can write

$$\phi_a(A) = (P^\perp \phi)(A) = \langle \pi(A)\xi_\varnothing/\sqrt{2}, \xi_\varnothing \rangle, \quad A \in A_2,$$
$$\phi_s(A) = (P\phi)(A) = \langle \pi(A)e/\sqrt{2}, \xi_\varnothing \rangle, \quad A \in A_2.$$

Letting B be as above, an easy computation gives

$$\langle B\xi_\varnothing/\sqrt{2}, \xi_\varnothing \rangle = \langle Be/\sqrt{2}, \xi_\varnothing \rangle = 1/\sqrt{2}.$$

Arguing as before, this implies $\|\phi_a\| \ge 1/\sqrt{2}$ and $\|\phi_s\| \ge 1/\sqrt{2}$. By Theorem 3.6, it follows that $\|\phi_a\| + \|\phi_s\| = \sqrt{2}\|\phi\|$.

Remark 3.8. It is well known that the algebra H^∞ is completely isometrically isomorphic to a subalgebra of $L^\infty(\mathbb{T})$. Ando [1978] used this fact to define a notion of absolute continuity and singularity for functionals on H^∞. Namely, a functional on H^∞ is absolutely continuous in the sense of [ibid.] if it extends to a normal functional on $L^\infty(\mathbb{T})$, and singular in the sense of [ibid.] if it extends to a singular functional on $L^\infty(\mathbb{T})$ (see Chapter 2 of [Takesaki 2002] for the definition of a singular functional on a von Neumann algebra). We now show that these definitions agree with Definition 3.1.

It is clear that a functional on H^∞ that is absolutely continuous in the sense of Definition 3.1 is also absolutely continuous in the sense of [Ando 1978]. Let ϕ be a functional on H^∞ that is singular in the sense of Definition 3.1. A Lebesgue decomposition theorem also holds for H^∞ using the definition of absolute continuity and singularity from [Ando 1978] (see, for example, [Ueda 2009]). Hence there are functionals $\tilde{\phi}_a$ and $\tilde{\phi}_s$ on F_d^∞ such that $\phi = \tilde{\phi}_a + \tilde{\phi}_s$, where $\tilde{\phi}_a$ is absolutely continuous in the sense of [Ando 1978], and $\tilde{\phi}_s$ is singular in the sense of [ibid.]. Moreover, $\|\phi\| = \|\tilde{\phi}_a\| + \|\tilde{\phi}_s\|$. Note that $\tilde{\phi}_a$ is absolutely continuous (in our sense). This implies that

$$\|\tilde{\phi}_s\| \le \|\phi\| = \limsup \|\phi^k\| \le \limsup(\|\tilde{\phi}_a^k\| + \|\tilde{\phi}_s^k\|) = \limsup \|\tilde{\phi}_s^k\| \le \|\tilde{\phi}_s\|.$$

Hence $\phi = \tilde{\phi}_s$ and ϕ is singular in the sense of [ibid.].

Now let ϕ be an arbitrary functional on H^∞, let $\phi = \phi_a + \phi_s$ be the Lebesgue decompositions of ϕ as in Theorem 3.6, and let $\phi = \tilde{\phi}_a + \tilde{\phi}_s$ be the Lebesgue decomposition of ϕ as in [ibid.]. Then $\phi_a - \tilde{\phi}_a = \tilde{\phi}_s - \phi_s$. From above, $\phi_a - \tilde{\phi}_a$ is absolutely continuous in the sense of [ibid.], and $\tilde{\phi}_s - \phi_s$ is singular in the sense of [ibid.]. Hence by the uniqueness of the Lebesgue decomposition, $\phi_a = \tilde{\phi}_a$ and $\phi_s = \tilde{\phi}_s$.

We note that Definition 3.1 gives an intrinsic characterization of singular functionals on H^∞, which answers (at least in this classical setting) a question from [Ueda 2009]. For $d \ge 2$, it would be interesting to know if there is an appropriate noncommutative analogue of $L^\infty(\mathbb{T})$ with a subalgebra that is completely isometrically isomorphic to F_d^∞.

4. The extended F. and M. Riesz theorem

The results in this section can be viewed as noncommutative generalizations of the classical results referred to as the F. and M. Riesz theorem. As mentioned in the introduction, results of this kind have been established in different settings by Exel [1990], by Blecher and Labuschagne [2007], and by Ueda [2009]. In fact, Blecher and Labuschagne seem to have anticipated that an F. and M. Riesz-type theorem should hold for F_d^∞ (see the introduction of [Blecher and Labuschagne 2007]).

Theorem 4.1 (extended F. and M. Riesz theorem). *Let ϕ be a bounded linear functional on F_d^∞, and let $\phi = \phi_a + \phi_s$ be the Lebesgue decomposition of ϕ into absolutely continuous and singular parts as in Theorem 3.6. Let $\mathcal{J}$ be a two-sided ideal of F_d^∞. If ϕ is zero on $\mathcal{J}$, then ϕ_a and ϕ_s are both zero on $\mathcal{J}$.*

Proof. As in Section 2, there are vectors x and y in H_u such that

$$\phi(A) = \langle \pi_u(A)x, y\rangle \quad \text{for all } A \in F_d^\infty.$$

By Proposition 3.5 we can write $\phi_a = P_u^\perp \phi$, and by Proposition 3.3 we can write $\phi_s = P_u \phi$. If we identify F_d^∞ with its image $\pi_u(F_d^\infty)$ in $(F_d^\infty)^{**}$, then the functionals ϕ, ϕ_a and ϕ_s each have unique weak-* continuous extensions to functionals on $(F_d^\infty)^{**}$ with the same norm.

Let $\mathfrak{J}$ denote the ideal in $(F_d^\infty)^{**}$ obtained by taking the weak-* closure of $\pi_u(\mathcal{J})$. Since ϕ is zero on $\mathcal{J}$, it is zero on $\mathfrak{J}$. For A in $\mathcal{J}$, $\pi_u(A)P_u^\perp$ belongs to $\mathfrak{J}$, which implies

$$0 = (P_u^\perp \phi)(A) = \phi_a(A).$$

Hence ϕ_a is zero on $\mathcal{J}$, and it follows immediately that ϕ_s is also zero on $\mathcal{J}$. □

Corollary 4.2 (F. and M. Riesz theorem). *Let ϕ be a bounded linear functional on F_d^∞. If ϕ is zero on $F_{d,0}^\infty$, where $F_{d,0}^\infty$ denotes the ideal of F_d^∞ generated by $L_{Z_1}, \dots, L_{Z_d}$, then ϕ is absolutely continuous.*

Proof. Let $\phi = \phi_a + \phi_s$ be the Lebesgue decomposition of ϕ into absolutely continuous and singular parts as in Theorem 3.6. By Theorem 4.1, ϕ_a and ϕ_s are both zero on $F_{d,0}^\infty$. By Definition 3.1, if ϕ_s is zero on $F_{d,0}^\infty$, it is necessarily zero on all of F_d^∞. Hence $\phi = \phi_a$ and ϕ is absolutely continuous. □

5. Quotient algebras

For a weak-* closed two-sided ideal $\mathcal{J}$ of F_d^∞, let $\mathcal{A}_I$ denote the quotient algebra $F_d^\infty / \mathcal{J}$.

Definition 5.1. Let $\mathcal{J}$ be a weak-* closed two-sided ideal of F_d^∞, and let ϕ be a bounded functional on $\mathcal{A}_\mathcal{J}$. Then

(1) ϕ is *absolutely continuous* if it is weak-* continuous, and

(2) ϕ is *singular* if $\|\phi\| = \|\phi^k\|$ for every $k \geq 1$, where ϕ^k denotes the restriction of ϕ to the ideal of $\mathcal{A}_\mathcal{J}$ generated by $\{\overline{L_{Z_w}} \mid |w| = k\}$, where for a word w in $\mathbb{F}_d^*$, $\overline{L_{Z_w}}$ denotes the image in $\mathcal{A}_\mathcal{J}$ of L_{Z_w}.

Theorem 5.2 (Lebesgue decomposition for quotients of F_d^∞). *Let $\mathcal{J}$ be a weak-* closed two-sided ideal of F_d^∞, and let ϕ be a bounded linear functional on $\mathcal{A}_\mathcal{J}$. Then there are unique linear functionals ϕ_a and ϕ_s on $\mathcal{A}_\mathcal{J}$ such that $\phi = \phi_a + \phi_s$, where ϕ_a is absolutely continuous and ϕ_s is singular, and such that*

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|.$$

If $d = 1$, then the constant $\sqrt{2}$ can be replaced with the constant 1.

Proof. By basic functional analysis, we can lift the functional ϕ to a functional ψ on F_d^∞ with the same norm. Let $\psi = \psi_a + \psi_s$ be the Lebesgue decomposition of ψ into absolutely continuous and singular parts as in Theorem 3.6. The functional ψ annihilates $\mathcal{J}$, so by Theorem 4.1, both ψ_a and ψ_s annihilate $\mathcal{J}$. Hence ψ_a and ψ_s induce functionals ϕ_a and ϕ_s on $\mathcal{A}_\mathcal{J}$, respectively, with the same norm. Clearly $\phi = \phi_a + \phi_s$, and the inequality

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|$$

follows from the corresponding inequality in Theorem 3.6. The functional ϕ_a is absolutely continuous since ψ_a is absolutely continuous on F_d^∞. To see that ϕ_s is singular, simply note that for every $k \geq 1$, the ideal $(\mathcal{A}_{\mathcal{J},0})^k$ is the image in $\mathcal{A}_\mathcal{J}$ of the ideal $(F_{d,0}^\infty)^k$. □

Corollary 5.3 (Lebesgue decomposition for multiplier algebras). *Let $\mathcal{A}$ be the multiplier algebra of a complete Nevanlinna–Pick space, and let ϕ be a bounded linear functional on $\mathcal{A}$. Then there are unique linear functionals ϕ_a and ϕ_s on $\mathcal{A}$ such that $\phi = \phi_a + \phi_s$, where ϕ_a is absolutely continuous and ϕ_s is singular, and such that*

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|.$$

6. Uniqueness of the predual

Let X and Y be Banach spaces such that $X^* = Y$. Then X is said to be a predual for Y. Every predual X of Y naturally embeds into the dual space Y^*, and a subspace X of Y^* is a predual of Y if and only if it satisfies the following properties:

(1) The subspace X norms Y, i.e., $\sup\{|x(y)| \mid x \in X,\ \|x\| \le 1\} = \|y\|$ for all y in Y.

(2) The closed unit ball of Y is compact in the $\sigma(Y, X)$ topology.

The space Y is said to have a strongly unique predual if there is a unique subspace X of Y^* such that $Y = X^*$. For a survey on uniqueness results for preduals, we refer the reader to [Godefroy 1989].

In the operator-theoretic setting, the results of Sakai [1956], Ando [1978] and Ueda [2009] mentioned in the introduction established that von Neumann algebras and maximal subdiagonal algebras have unique preduals. Ruan [1992] proved that an operator algebra with a weak-* dense subalgebra of compact operators has a unique predual, which applies to, for example, nest algebras and atomic CSL algebras. Effros, Ozawa and Ruan proved in [Effros et al. 2001] that a W*TRO (i.e., a corner of a von Neumann algebras) has a unique predual. More recently, Davidson and Wright [2011] proved that a free semigroup algebra has a unique predual. Note that Davidson and Wright's result applies to F_d^∞, but not to quotients of F_d^∞.

The following definition is closely related to the notion of an M-ideal in a Banach space (see [Harmand et al. 1993] for more information).

Definition 6.1. A Banach space X is L-embedded if there is a projection P on the bidual X^{**} with range X such that

$$\|x\| = \|Px\| + \|x - Px\| \quad \text{for all } x \in X^{**}.$$

The following result of Pfitzner implies that every separable L-embedded space has Godefroy and Talagrand's property (X), and hence by a result of Godefroy and Talagrand [1981], that it is the unique predual of its dual.

Theorem 6.2 [Pfitzner 2007]. *Separable L-embedded spaces have property* (X).

The results of Sakai, Ando and Ueda on decompositions of linear functionals imply that the preduals of von Neumann algebras and maximal subdiagonal algebras are L-embedded, and hence by Pfitzner's theorem, that they are unique. However, Example 3.7 shows that preduals of quotients of F_d^∞ are not, in general, L-embedded, so we are unable to use Pfitzner's result. Instead, we give a direct proof that quotients of F_d^∞ have (strongly) unique preduals.

Theorem 6.3. *Let $\mathcal{J}$ be a weak-* closed two-sided ideal of F_d^∞. Then the algebra $\mathcal{A}_{\mathcal{J}}$ has a strongly unique predual.*

Proof. Suppose E is a predual for $\mathcal{A}_{\mathcal{J}}$, identified with a subspace of $(\mathcal{A}_{\mathcal{J}})^*$. By Theorem 5.2,

$$(\mathcal{A}_{\mathcal{J}})^* = (\mathcal{A}_{\mathcal{J}})^*_a \oplus (\mathcal{A}_{\mathcal{J}})^*_s,$$

where $(\mathcal{A}_{\mathcal{J}})^*_a$ and $(\mathcal{A}_{\mathcal{J}})^*_s$ denote the set of absolutely continuous and singular functionals on $\mathcal{A}_{\mathcal{J}}$, respectively. We want to prove that $E = (\mathcal{A}_{\mathcal{J}})^*_a$.

Let ϕ be a functional in E, and let $\phi = \phi_a + \phi_s$ be the Lebesgue decomposition of ϕ as in Theorem 5.2. We will prove that $\phi_s = 0$. Suppose to the contrary that $\phi_s \neq 0$. By basic functional analysis, we can lift the functional ϕ to a functional ψ on F_d^∞ that is zero on $\mathcal{J}$. Let $\psi = \psi_a + \psi_s$ be the Lebesgue decomposition of ψ as in Theorem 3.6. By Theorem 4.1, ψ_a and ψ_s are both zero on $\mathcal{J}$, and by construction they induce the functionals ϕ_a and ϕ_s, respectively, on the quotient $\mathcal{A}_{\mathcal{J}}$.

It follows from the results of [Kennedy 2011] that Theorem 5.4 of [Davidson et al. 2005] applies to the unital weak operator closed algebra generated by any isometric tuple. Thus there is a net (B_λ) of elements in the unit ball of F_d^∞ such that w*-lim $\pi_u(B_\lambda) = P_u$ in $(F_d^\infty)^{**}$. Since the net (B_λ) is weak-* convergent in $(F_d^\infty)^{**}$, it is weakly Cauchy in F_d^∞. Since the closed unit ball of F_d^∞ is compact in the weak-* topology, and in particular is complete, this implies that there is B in the closed unit ball of F_d^∞ such that w*-lim $B_\lambda = B$ in F_d^∞. For every weak-* continuous functional τ on F_d^∞, Proposition 3.5 implies that

$$\tau(B) = \lim_\lambda \tau(B_\lambda) = (P_u\tau)(1) = 0.$$

Hence $B = 0$.

Let A be an element in the unit ball of F_d^∞ such that $\psi_s(A) \neq 0$. Since the net (B_λ) is weakly Cauchy in F_d^∞, the image $(\overline{B_\lambda})$ is weakly Cauchy in $\mathcal{A}_{\mathcal{J}}$. It follows that the net $(\overline{AB_\lambda})$ is also weakly Cauchy in $\mathcal{A}_{\mathcal{J}}$. Since E is a predual of $\mathcal{A}_{\mathcal{J}}$, the closed unit ball of $\mathcal{A}_{\mathcal{J}}$ is compact in the $\sigma(\mathcal{A}_{\mathcal{J}}, E)$ topology, and in particular is complete. Thus, the net $(\overline{AB_\lambda})$ converges in the $\sigma(\mathcal{A}_{\mathcal{J}}, E)$ topology to an element C in the unit ball of $\mathcal{A}_{\mathcal{J}}$. By Proposition 3.3, we have

$$\phi(C) = \lim_\lambda \phi(\overline{AB_\lambda}) = \lim_\lambda \psi(AB_\lambda) = (P_u\psi)(A) = \psi_s(A) \neq 0,$$

so that $C \neq 0$. But since w*-lim $B_\lambda = 0$ in F_d^∞, it follows that w*-lim $\overline{AB_\lambda} = 0$ in $\mathcal{A}_{\mathcal{J}}$. So for every τ in $(\mathcal{A}_{\mathcal{J}})^*_a$, we necessarily have

$$\tau(C) = \lim_\lambda \tau(\overline{AB_\lambda}) = 0.$$

Since $(\mathcal{A}_{\mathcal{J}})^*_a$ separates points, this implies that $C = 0$, which gives a contradiction. Thus $\phi = \phi_a$, meaning ϕ is absolutely continuous.

Since ϕ was arbitrary, it follows from above that every functional in E is absolutely continuous, i.e., that E is contained in $(\mathcal{A}_{\mathcal{J}})^*_a$. If it were the case that $E \neq (\mathcal{A}_{\mathcal{J}})^*_a$, then we could apply the Hahn–Banach theorem to separate E from $(\mathcal{A}_{\mathcal{J}})^*_a$ with an element of $\mathcal{A}_{\mathcal{J}}$. But the fact that E is a predual of $\mathcal{A}_{\mathcal{J}}$ means in particular it must norm $\mathcal{A}_{\mathcal{J}}$, so this is impossible. Therefore, we conclude that $E = (\mathcal{A}_{\mathcal{J}})^*_a$, and hence that $(\mathcal{A}_{\mathcal{J}})^*_a$ is the unique predual of $\mathcal{A}_{\mathcal{J}}$. □

Corollary 6.4. *The multiplier algebra of every complete Nevanlinna–Pick space has a strongly unique predual.*

Acknowledgements

We are grateful to Ken Davidson and Adam Fuller for their helpful comments and suggestions. We would also like to thank the anonymous referees for their suggestions.

References

[Agler and McCarthy 2000] J. Agler and J. E. McCarthy, "Complete Nevanlinna–Pick kernels", *J. Funct. Anal.* **175**:1 (2000), 111–124. MR 2001h:47019 Zbl 0957.47013

[Ando 1978] T. Ando, "On the predual of H^∞", *Comment. Math. Special Issue* **1** (1978), 33–40. MR 80c:46063 Zbl 0384.46035

[Arias and Popescu 2000] A. Arias and G. Popescu, "Noncommutative interpolation and Poisson transforms", *Israel J. Math.* **115** (2000), 205–234. MR 2001i:47021 Zbl 0967.47045

[Arveson 1967] W. B. Arveson, "Analyticity in operator algebras", *Amer. J. Math.* **89** (1967), 578–642. MR 36 #6946 Zbl 0183.42501

[Arveson 1998] W. Arveson, "Subalgebras of C^*-algebras, III: Multivariable operator theory", *Acta Math.* **181**:2 (1998), 159–228. MR 2000e:47013 Zbl 0952.46035

[Blecher and Labuschagne 2007] D. P. Blecher and L. E. Labuschagne, "Noncommutative function theory and unique extensions", *Studia Math.* **178**:2 (2007), 177–195. MR 2007m:46102 Zbl 1121.46048

[Blecher and Le Merdy 2004] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules: An operator space approach*, London Mathematical Society Monographs. New Series **30**, Clarendon Press, Oxford, 2004. MR 2006a:46070 Zbl 1061.47002

[Davidson and Pitts 1998a] K. R. Davidson and D. R. Pitts, "The algebraic structure of non-commutative analytic Toeplitz algebras", *Math. Ann.* **311**:2 (1998), 275–303. MR 2001c:47082 Zbl 0939.47060

[Davidson and Pitts 1998b] K. R. Davidson and D. R. Pitts, "Nevanlinna–Pick interpolation for non-commutative analytic Toeplitz algebras", *Integral Equations Operator Theory* **31**:3 (1998), 321–337. MR 2000g:47016 Zbl 0917.47017

[Davidson and Pitts 1999] K. R. Davidson and D. R. Pitts, "Invariant subspaces and hyper-reflexivity for free semigroup algebras", *Proc. London Math. Soc.* (3) **78**:2 (1999), 401–430. MR 2000k:47005 Zbl 0997.46042

[Davidson and Wright 2011] K. R. Davidson and A. Wright, "Operator algebras with unique preduals", *Canad. Math. Bull.* **54**:3 (2011), 411–421. MR 2012h:47153 Zbl 1244.47062

[Davidson and Yang 2008] K. R. Davidson and D. Yang, "A note on absolute continuity in free semigroup algebras", *Houston J. Math.* **34**:1 (2008), 283–288. MR 2009c:47122 Zbl 1147.47057

[Davidson et al. 2001] K. R. Davidson, E. Katsoulis, and D. R. Pitts, "The structure of free semigroup algebras", *J. Reine Angew. Math.* **533** (2001), 99–125. MR 2002a:47107 Zbl 0967.47047

[Davidson et al. 2005] K. R. Davidson, J. Li, and D. R. Pitts, "Absolutely continuous representations and a Kaplansky density theorem for free semigroup algebras", *J. Funct. Anal.* **224**:1 (2005), 160–191. MR 2006f:47088 Zbl 1084.46042

[Effros et al. 2001] E. G. Effros, N. Ozawa, and Z.-J. Ruan, "On injectivity and nuclearity for operator spaces", *Duke Math. J.* **110**:3 (2001), 489–521. MR 2002k:46151 Zbl 1010.46060

[Exel 1990] R. Exel, "The F. and M. Riesz theorem for C^*-algebras", *J. Operator Theory* **23**:2 (1990), 351–368. MR 91m:46093 Zbl 0772.46027

[Godefroy 1989] G. Godefroy, "Existence and uniqueness of isometric preduals: A survey", pp. 131–193 in *Banach space theory* (Iowa City, IA, 1987), edited by B.-L. Lin, Contemp. Math. **85**, Amer. Math. Soc., Providence, RI, 1989. MR 90b:46035 Zbl 0674.46010

[Godefroy and Talagrand 1981] G. Godefroy and M. Talagrand, "Nouvelles classes d'espaces de Banach á predual unique", pp. 1–28 in *Séminaire d'analyse fonctionnelle* (Palaiseau, France, 1980–1981), École Polytechnique Centre de Mathématiques, Palaiseau, 1981. MR 83d:46003 Zbl 0475.46013

[Grothendieck 1955] A. Grothendieck, "Une caractérisation vectorielle-métrique des espaces L^1", *Canad. J. Math.* **7** (1955), 552–561. MR 17,877d Zbl 0065.34503

[Harmand et al. 1993] P. Harmand, D. Werner, and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Mathematics **1547**, Springer, Berlin, 1993. MR 94k:46022 Zbl 0789.46011

[Kennedy 2011] M. Kennedy, "Wandering vectors and the reflexivity of free semigroup algebras", *J. Reine Angew. Math.* **653** (2011), 47–73. MR 2012g:47220 Zbl 1218.47136

[Kennedy 2013] M. Kennedy, "The structure of an isometric tuple", *Proc. Lond. Math. Soc.* (3) **106**:5 (2013), 1157–1177. MR 3066752 Zbl 06176898

[Pfitzner 2007] H. Pfitzner, "Separable L-embedded Banach spaces are unique preduals", *Bull. Lond. Math. Soc.* **39**:6 (2007), 1039–1044. MR 2009a:46030 Zbl 1147.46008

[Popescu 1989a] G. Popescu, "Multi-analytic operators and some factorization theorems", *Indiana Univ. Math. J.* **38**:3 (1989), 693–710. MR 90k:47019 Zbl 0661.47020

[Popescu 1989b] G. Popescu, "Characteristic functions for infinite sequences of noncommuting operators", *J. Operator Theory* **22**:1 (1989), 51–71. MR 91m:47012 Zbl 0703.47009

[Popescu 1991] G. Popescu, "von Neumann inequality for $(B(\mathcal{H})^n)_1$", *Math. Scand.* **68**:2 (1991), 292–304. MR 92k:47073 Zbl 0774.46033

[Popescu 1995] G. Popescu, "Multi-analytic operators on Fock spaces", *Math. Ann.* **303**:1 (1995), 31–46. MR 96k:47049 Zbl 0835.47015

[Popescu 1996] G. Popescu, "Non-commutative disc algebras and their representations", *Proc. Amer. Math. Soc.* **124**:7 (1996), 2137–2148. MR 96k:47077 Zbl 0864.46043

[Ruan 1992] Z.-J. Ruan, "On the predual of dual algebras", *J. Operator Theory* **27**:1 (1992), 179–192. MR 94k:47070 Zbl 0846.47032

[Sakai 1956] S. Sakai, "A characterization of W^*-algebras", *Pacific J. Math.* **6** (1956), 763–773. MR 18,811f Zbl 0072.12404

[Takesaki 1958] M. Takesaki, "On the conjugate space of operator algebra", *Tôhoku Math. J.* (2) **10** (1958), 194–203. MR 20 #7227 Zbl 0089.10703

[Takesaki 2002] M. Takesaki, *Theory of operator algebras, I*, Encyclopaedia of Mathematical Sciences **124**, Springer, Berlin, 2002. MR 2002m:46083 Zbl 0990.46034

[Ueda 2009] Y. Ueda, "On peak phenomena for non-commutative H^∞", *Math. Ann.* **343**:2 (2009), 421–429. MR 2011a:46100 Zbl 1171.46042

[Ueda 2011] Y. Ueda,"On the predual of non-commutative H^∞", *Bull. Lond. Math. Soc.* **43**:5 (2011), 886–896. MR 2012m:46071 Zbl 1234.46051

Received 4 Jul 2013. Revised 27 Oct 2013. Accepted 27 Nov 2013.

MATTHEW KENNEDY: mkennedy@math.carleton.ca
School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, Ottawa, ON K1S 5B6, Canada

DILIAN YANG: dyang@uwindsor.ca
Department of Mathematics and Statistics, University of Windsor, 401 Sunset Avenue, Windsor, ON N9B 3P4, Canada

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2014 is US $180/year for the electronic version, and $355/year (+$50, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 7 No. 2 2014

2157-5045(2014)7:2;1-E