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DECOMPOSITION RANK OF $\mathcal{Z}$-STABLE C*-ALGEBRAS

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We show that C*-algebras of the form $C(X)\otimes\mathcal{Z}$, where X is compact and Hausdorff and $\mathcal{Z}$ denotes the Jiang–Su algebra, have decomposition rank at most 2. This amounts to a dimension reduction result for C*-bundles with sufficiently regular fibres. It establishes an important case of a conjecture on the fine structure of nuclear C*-algebras of Toms and Winter, even in a nonsimple setting, and gives evidence that the topological dimension of noncommutative spaces is governed by fibres rather than base spaces.

1. Introduction

The structure and classification theory of nuclear C*-algebras has seen rapid progress in recent years, largely spurred by the subtle interplay between certain topological and algebraic regularity properties, such as finite topological dimension, tensorial absorption of suitable strongly self-absorbing C*-algebras, and order completeness of homological invariants; see [Elliott and Toms 2008] for an overview. In the simple and unital case, these relations were formalized by A. Toms and W. Winter as follows.

Conjecture 1.1. For a separable, simple, unital, nonelementary, stably finite and nuclear C*-algebra A, the following are equivalent:

(i) A has finite decomposition rank: in symbols, $\operatorname{dr} A<\infty$.

(ii) A is $\mathcal{Z}$-stable: $A\cong A\otimes\mathcal{Z}$.

(iii) A has strict comparison of positive elements.

Here, decomposition rank is a notion of noncommutative topological dimension introduced in [Kirchberg and Winter 2004], $\mathcal{Z}$ denotes the Jiang–Su algebra introduced in [Jiang and Su 1999], and strict comparison essentially means that positive elements may be compared in terms of tracial values of their support projections; compare [Rørdam 2006]. If one drops the finiteness assumption on A, one should replace (i) by

(i′) A has finite nuclear dimension, $\dim_{\mathrm{nuc}} A<\infty$,

where nuclear dimension [Winter and Zacharias 2010] is a variation of the decomposition rank that can have finite values also for infinite C*-algebras.

The conjecture still makes sense in the nonsimple situation, provided one asks A to have no elementary

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subquotients (this is a minimal requirement for $\mathcal{Z}$-stability); one also has to be slightly more careful about the definition of comparison in this case.

Nuclearity in this context manifests itself most prominently via approximation properties with particularly nice completely positive maps [Christensen et al. 2012; Hirshberg et al. 2012a].

Conjecture 1.1 has a number of important consequences for the structure of nuclear C^*-algebras and it has turned out to be pivotal for many recent classification results, especially in view of the examples given in [Villadsen 1999; Rørdam 2003; Toms 2008]. Moreover, it highlights the striking analogy between the classification program for nuclear C^*-algebras (see [Elliott 1995]) and Connes' [1976] celebrated classification of injective II_1 factors.

Implications (i), (i′) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) of Conjecture 1.1 are by now known to hold in full generality [Rørdam 2004; Winter 2010; 2012]; (iii) $\Longrightarrow$ (ii) has been established under certain additional structural hypotheses [Matui and Sato 2012; Winter 2012], all of which, in particular, guarantee sufficient divisibility properties.

Arguably, it is (ii) $\Longrightarrow$ (i) which remains the least well understood of these implications. While there are promising partial results available [Winter and Zacharias 2010; Lin 2011b; Winter 2012], all of these factorize through classification theorems of some sort. This in turn makes it hard to explicitly identify the origin of finite dimensionality.[1]

In the simple purely infinite (hence $\mathcal{O}_\infty$-stable, hence $\mathcal{Z}$-stable [Kirchberg and Rørdam 2002; Kirchberg 2006]) case, one has to use Kirchberg–Phillips classification [Kirchberg 1995; Kirchberg and Phillips 2000] as well as a range result providing models to exhaust the invariant [Rørdam 2002] and then again Kirchberg–Phillips classification to show that these models have finite nuclear dimension [Winter and Zacharias 2010].[2]

In the simple stably finite case, at this point only approximately homogeneous (AH) algebras or approximately subhomogeneous (ASH) algebras for which projections separate traces are covered [Lin 2011a; Lin and Niu 2008; Winter 2004; 2007]. (This approach also includes crossed products associated to uniquely ergodic minimal dynamical systems [Toms and Winter 2009; 2013].) While both of these classes after stabilizing with $\mathcal{Z}$ can by now be shown directly to consist of TAI and TAF algebras [Lin 2011b], again finite topological dimension will only follow from classification results [Elliott et al. 2007; Winter 2010; Lin 2011a; Toms 2011] and after comparing to models which exhaust the invariant [Elliott 1996; Villadsen 1998]; see also [Rørdam 2002] for an overview. (Note that certain crossed products are shown directly to have finite nuclear dimension, or even finite decomposition rank [Hirshberg et al. 2012b; Szabo 2013]; however, $\mathcal{Z}$-stability is not assumed in these cases.)

[1] After this article appeared on the arXiv, Matui and Sato [2013] posted a very nice paper in which they prove finite decomposition rank for separable, simple, unital, nuclear, and $\mathcal{Z}$-stable C^*-algebras provided these are quasidiagonal and have a unique tracial state. While this result is restricted to the simple and monotracial case (conditions we do not need at all), it only uses quasidiagonality as additional structural hypothesis (and this is of course much more general than our local homogeneity); see also [Sato et al. 2014]. Matui and Sato's approach heavily relies on deep results of Connes and of Haagerup and, in a sense, is almost perpendicular to ours; we believe that the two methods nicely complement each other.

[2] [Matui and Sato 2013] also contains a proof of finite nuclear dimension for simple purely infinite C^*-algebras, which does not rely on classification; see also [Barlak et al. 2014].

Once again, the classification procedure does not make it entirely transparent where the finite topological dimension comes from, but at least Elliott–Gong–Li classification of simple AH algebras (of very slow dimension growth — later shown to be equivalent to slow dimension growth and to $\mathcal{Z}$-stability [Winter 2012]) heavily relies on Gong's deep dimension reduction theorem [2002]. Gong gives an essentially explicit way of replacing a given AH limit decomposition with one of low topological dimension. However, this method is technically very involved and requires both simplicity and the given inductive limit decomposition. It does not fully explain to what extent the two are necessary; in particular, it is in principle conceivable that a decomposition similar to that of Gong exists for algebras of the form $C(X) \otimes \mathcal{Q}$, where $\mathcal{Q}$ is the universal UHF algebra.[3]

In this article we show how finite topological dimension indeed arises for algebras of this type; in fact, we are able to cover algebras of the form $C(X) \otimes \mathcal{Z}$, and hence also locally homogeneous $\mathcal{Z}$-stable C*-algebras (not necessarily simple, or with a prescribed inductive limit structure). We hope our argument will shed new light on the conceptual reasons why finite topological dimension should arise in the presence of sufficient C*-algebraic regularity. Our method is based on approximately embedding the cone over the Cuntz algebra $\mathcal{O}_2$ into tracially small subalgebras of the algebra in question; these play a similar role as the small corners used in the definition of TAF algebras [Lin 2004] or the small hereditary subalgebras in property SI [Matui and Sato 2012]. We mention that we only obtain (a strong version of) finite decomposition rank, whereas Gong's reduction theorem yields an inductive limit decomposition; however, for many purposes, finite decomposition rank is sufficient; see [Winter 2010; Toms and Winter 2013].

In [Kirchberg and Rørdam 2005], algebras of the form $C(X) \otimes \mathcal{O}_2$ were shown to be approximated by algebras of the form $C(\Gamma) \otimes \mathcal{O}_2$ with Γ one-dimensional. Since $\mathcal{O}_2$ is by now known to have finite nuclear dimension [Winter and Zacharias 2010], this may be regarded as strong evidence that the topological dimension of a C*-bundle depends on the noncommutative size of the fibres more than the size of the base space. (A somewhat similar phenomenon was already observed for stable rank by Rieffel [1983].)

It is remarkable that [Kirchberg and Rørdam 2005] does not rely on a classification result in any way. It does, however, mix commutativity (of the structure algebra) and pure infiniteness (of the fibres).

It is not clear from [Kirchberg and Rørdam 2005] whether such a dimension type reduction also occurs in the setting of stably finite fibres. In the present article we show that it does, by developing a method to transport [Kirchberg and Rørdam 2005] to the situation where the fibres are UHF algebras (to pass to the case where each fibre is $\mathcal{Z}$ then requires a certain amount of additional machinery — at least if one wants to increase the dimension by no more than one). The crucial concept to link purely infinite and stably finite fibres is quasidiagonality of the cone over $\mathcal{O}_2$, discovered by Voiculescu [1993] and Kirchberg [1991]. In many ways it is most interesting just to know that the $\mathcal{Z}$-stable C*-algebras in our main result have finite decomposition rank, and the very small bound that we are able to derive is secondary. Certain technicalities can be circumvented, using [Carrión 2011, Lemma 3.1] in order to prove just finite decomposition rank, as we describe in Remarks 4.8 and 4.9. We are indebted to one of the referees for suggesting this shortcut.

[3] Added in proof: It turns out that this is not, in fact, conceivable; see [Tikuisis 2014].

One should mention that the fact that the fibres are specific strongly self-absorbing algebras in both [Kirchberg and Rørdam 2005] and in our result plays an important but in some sense secondary role: In [Kirchberg and Rørdam 2005] (combined with [Winter and Zacharias 2010]) one can replace $\mathcal{O}_2$ with $\mathcal{O}_\infty$, or in fact with any UCT Kirchberg algebra, and still arrive at finite nuclear dimension. More generally, our result yields the respective statement if the fibres have finite nuclear dimension and are $\mathcal{Z}$-stable, for example, in the simple, nuclear, classifiable case.

While at the current stage we only cover the case of highly homogeneous bundles, it will be an important task to handle bundles with non-Hausdorff spectrum, for example, $B \otimes \mathcal{Z}$ with B subhomogeneous, in order to also cover transformation group C*-algebras. This will be pursued in subsequent work by combining our technical Lemma 4.7, with the methods of [Winter 2004]; in preparation, we have stated Lemma 4.7 in a form slightly more general than necessary for the current main result Theorem 4.1. One of the referees has raised the question of whether (local) triviality of $C(X) \otimes \mathcal{Z}$ is needed to show that it has finite decomposition rank, particularly in light of the interesting examples of $C(X)$-algebras appearing in [Dadarlat 2009b; Hirshberg et al. 2007]; in response, we have added Section 5, in which we show that our result easily extends to nontrivial bundles such as these examples.

We would like to take this opportunity to thank both referees for their careful proofreading and inspiring comments.

We remind the reader that the Jiang–Su algebra $\mathcal{Z}$ is an inductive limit of so-called dimension-drop C*-algebras

$$\mathcal{Z}_{p_0,p_1} := \{f \in C([0,1], M_{p_0} \otimes M_{p_1}) : f(0) \in M_{p_0} \otimes \mathbb{C} \cdot 1_{p_1} \text{ and } f(1) \in \mathbb{C} \cdot 1_{p_0} \otimes M_{p_1}\}, \tag{1-1}$$

where $p_0, p_1 \in \mathbb{N}$ are coprime, and it can be defined as the unique simple, monotracial limit of such algebras. It has also been realized as an inductive limit of generalized dimension-drop algebras, which are defined as in (1-1), but with p_0, p_1 taken to be coprime supernatural numbers (so that M_{p_i} denotes a UHF algebra) [Rørdam and Winter 2010, Theorem 3.4]. The connecting maps in this inductive limit have the crucial feature of being trace-collapsing.

2. Decomposition rank of homomorphisms

In this section we introduce the notions of decomposition rank and nuclear dimension of *-homomorphisms, building naturally on the respective notions for C*-algebras, just as nuclearity for *-homomorphisms arises from the completely positive approximation property for C*-algebras. We first recall from [Winter 2003] the notion of completely positive contractive (c.p.c.) order zero maps.

Definition 2.1. Let A, B be C*-algebras and let $\phi : A \to B$ be a c.p.c. map. We say that ϕ has order zero if it preserves orthogonality in the sense that if $a, b \in A_+$ satisfy $ab = 0$, then $\phi(a)\phi(b) = 0$.

Definition 2.2. Let $\alpha : A \to B$ be a *-homomorphism of C*-algebras. We say that α has decomposition rank at most n, and write $\mathrm{dr}(\alpha) \leq n$, if, for any finite subset $\mathcal{F} \subset A$ and any $\epsilon > 0$, there exists a finite dimensional C*-algebra F and c.p.c. maps

$$\psi : A \to F \quad \text{and} \quad \phi : F \to B$$

such that ϕ is $(n+1)$-colourable, in the sense that we can write

$$F = F^{(0)} \oplus \cdots \oplus F^{(n)}$$

and $\phi|_{F^{(i)}}$ has order zero for all i, and such that $\phi\psi$ is point-norm close to α, in the sense that, for $a \in \mathcal{F}$,

$$\|\alpha(a) - \phi\psi(a)\| < \epsilon.$$

We may define nuclear dimension of α similarly (and write $\dim_{\mathrm{nuc}}(\alpha) \le n$), where instead of requiring that ϕ is contractive, we only ask that $\phi|_{F^{(i)}}$ is contractive for each i.

Remark 2.3. The decomposition rank (respectively nuclear dimension) of a C*-algebra, as defined in [Kirchberg and Winter 2004, Definition 3.1] (respectively [Winter and Zacharias 2010, Definition 2.1]) is just the decomposition rank (respectively nuclear dimension) of the identity map.

The following generalizes some permanence properties for decomposition rank and nuclear dimension of C*-algebras. Proofs are omitted, as they are essentially the same as those found in [Kirchberg and Winter 2004; Winter 2003; Winter and Zacharias 2010].

Proposition 2.4. *Let A, B be C*-algebras and let $\alpha : A \to B$ be a *-homomorphism.*

(i) *Suppose that A is locally approximated by a family of C*-subalgebras $(A_\lambda)_\Lambda$, in the sense that, for every finite subset $\mathcal{F} \subset A$ and every tolerance $\epsilon > 0$, there exists λ such that $\mathcal{F} \subset_\epsilon A_\lambda$. Then*

$$\mathrm{dr}(\alpha) \le \sup_\Lambda \mathrm{dr}(\alpha|_{A_\lambda}) \quad \textit{and} \quad \dim_{\mathrm{nuc}}(\alpha) \le \sup_\Lambda \dim_{\mathrm{nuc}}(\alpha|_{A_\lambda}).$$

(ii) *If $C \subset A$ is a hereditary C*-subalgebra, then*

$$\mathrm{dr}(\alpha_C) \le \mathrm{dr}(\alpha) \quad \textit{and} \quad \dim_{\mathrm{nuc}}(\alpha_C) \le \dim_{\mathrm{nuc}}(\alpha),$$

where $\alpha_C := \alpha|_C : C \to \mathrm{her}(\alpha(C))$.

When computing the decomposition rank (or nuclear dimension), it is often convenient to replace the codomain by its sequence algebra, defined to be

$$A_\infty := \Big(\prod_{\mathbb{N}} A\Big) \Big/ \Big(\bigoplus_{\mathbb{N}} A\Big).$$

We shall denote by

$$\pi_\infty : \prod_{\mathbb{N}} A \to A_\infty$$

the quotient map, and by $\iota_\infty : A \to A_\infty$ the canonical embedding as constant sequences.

Proposition 2.5. *Let $\alpha : A \to B$ be a *-homomorphism.*

Then

$$\mathrm{dr}(\alpha) = \mathrm{dr}(\iota_\infty \circ \alpha) \quad \textit{and} \quad \dim_{\mathrm{nuc}}(\alpha) = \dim_{\mathrm{nuc}}(\iota_\infty \circ \alpha).$$

Proof. Straightforward, using stability of the relations defining c.p.c. order zero maps on finite dimensional domains [Kirchberg and Winter 2004]. □

Proposition 2.6. *Let $\mathscr{D}$ be a strongly self-absorbing* C*-*algebra* (as defined in [Toms and Winter 2007]), *and let A be a $\mathscr{D}$-stable* C*-*algebra.*

Then

$$\mathrm{dr}(A) = \mathrm{dr}(\mathrm{id}_A \otimes 1_{\mathscr{D}}) \quad \textit{and} \quad \dim_{\mathrm{nuc}}(A) = \dim_{\mathrm{nuc}}(\mathrm{id}_A \otimes 1_{\mathscr{D}}).$$

Proof. This follows easily from the fact that id_A has approximate factorizations of the form

$$\mathscr{D} \xrightarrow{\mathrm{id}_A \otimes 1_{\mathscr{D}}} \mathscr{D} \otimes \mathscr{D} \xrightarrow{\phi} A \otimes \mathscr{D},$$

where ϕ is a *-isomorphism. □

3. $C(X)$-algebras and decomposition rank

For a locally compact Hausdorff space X, a $C_0(X)$-algebra is a C*-algebra A equipped with a nondegenerate *-homomorphism $C_0(X) \to Z\mathcal{M}(A)$, called the structure map [Kasparov 1988, Definition 1.5]. Here $\mathcal{M}(A)$ refers to the multiplier algebra of A and $Z\mathcal{M}(A)$ to its centre; note that if A is unital, so is the structure map. In this section, we study the decomposition rank of such structure maps. Proposition 3.2 below is reminiscent of [Winter 2003, Proposition 2.19], which shows that the completely positive rank of $C(X)$ equals the covering dimension of X.

Definition 3.1. Let A be a $C_0(X)$-algebra and let $a \in A$. Define the support of a to be the smallest closed set $F \subset X$ such that $ag = 0$ whenever $g \in C_0(X \setminus F) \subset C_0(X)$. (This is easily seen to be well defined.)

We note the following property of order zero maps, which was obtained in the proof of [Kirchberg and Winter 2004, Proposition 5.1] (sixth line from the bottom of p. 79): if $\phi : A \to B$ is an order zero map and A is a unital C*-algebra, then

$$\|\phi(x)\| = \|\phi(1_A)\| \, \|x\| \quad \text{for any } x \in A. \tag{3-1}$$

Proposition 3.2. *Let X be a compact Hausdorff space, and let A be a unital $C(X)$-algebra with structure map $\iota : C(X) \to Z(A)$.*

The following are equivalent:

(i) $\mathrm{dr}(\iota) \leq n$.

(ii) $\dim_{\mathrm{nuc}}(\iota) \leq n$.

(iii) *The definition of* $\mathrm{dr}(\iota) \leq n$ *holds with the additional requirements that F is abelian and ψ is a unital *-homomorphism.*

(iv) *For any finite open cover $\mathcal{U}$ of X, any $\epsilon > 0$, and any $b \in C(X)_+$, there exists an $(n+1)$-colourable ϵ-approximate finite partition of b; that is, positive elements $b_j^{(i)} \in A$ for $i = 0, \dots, n$, $j = 1, \dots, r$, such that*

(a) *for each i, the elements $b_1^{(i)}, \dots, b_r^{(i)}$ are pairwise orthogonal,*

(b) *for each i, j, the support of $b_j^{(i)}$ is contained in some open set in the given cover $\mathcal{U}$, and*

(c) $\left\| \sum_{i,j} b_j^{(i)} - \iota(b) \right\| \leq \epsilon$.

Proof. (iii) ⇒ (i) ⇒ (ii) is obvious.

(ii) ⇒ (iv). Let us first assume $b = 1$. Let $\mathcal{F}$ be a finite partition of unity such that, for each $f \in \mathcal{F}$, there exists $U_f \in \mathcal{U}$ such that supp $f \subset U_f$. Set

$$\eta := \frac{\epsilon}{2|\mathcal{F}|(n+1)}. \tag{3-2}$$

Use $\dim_{\mathrm{nuc}}(\iota) \leq n$ to obtain

$$C(X) \xrightarrow{\psi} F^{(0)} \oplus \cdots \oplus F^{(n)} \xrightarrow{\phi} A$$

such that ψ is c.p.c., $\phi|_{F^{(i)}}$ is c.p.c. and order zero for all $i = 0, \ldots, n$, $\phi(\psi(f)) =_\eta f$ for $f \in \mathcal{F}$, and $\phi(\psi(1)) =_{\epsilon/2} 1$. Set

$$F^{(i)} := \bigoplus_{j=1}^{r_i} M_{m(i,j)}.$$

(By throwing in some zero summands if necessary, we may as well assume all the r_i to be equal.)

For each $i = 0, \ldots, n$ and $j = 1, \ldots, r_i$, we set

$$a_j^{(i)} := \left(\phi(\psi(1_{C(X)})1_{M_{m(i,j)}}) - \frac{\epsilon}{2(n+1)}\right)_+.$$

For each i, since $\phi|_{F^{(i)}}$ is order zero, $a_1^{(i)}, \ldots, a_{r_i}^{(i)}$ are orthogonal. We estimate

$$1 =_{\epsilon/2} \phi(\psi(1)) = \sum_{i=0}^{n} \sum_{j=1}^{r_i} \phi(\psi(1)1_{M_{m(i,j)}}) =_{\frac{(n+1)\epsilon}{2(n+1)}} \sum_{i,j} a_j^{(i)},$$

where the last approximation is obtained using the fact that the inner summands are orthogonal.

Lastly, we must verify that each $a_j^{(i)}$ has support contained in an open set from the cover $\mathcal{U}$. Fix i and j. Let $f_{i,j} \in \mathcal{F}$ maximize $f \mapsto \|\psi(f)1_{M_{m(i,j)}}\|$. We shall show that the support of $a_j^{(i)}$ is contained in the support of $f_{i,j}$ by showing that $a_j^{(i)}|_K = 0$, where

$$K := \{x \in X : f_{i,j} = 0\}.$$

Since $1 = \sum_{f \in \mathcal{F}} f$, we must have

$$\|\psi(f_{i,j})1_{M_{m(i,j)}}\| \geq \frac{1}{|\mathcal{F}|} \|\psi(1)1_{M_{m(i,j)}}\|. \tag{3-3}$$

Noting that

$$f_{i,j} =_\eta \phi(\psi(f_{i,j})) \geq \phi(\psi(f_{i,j})1_{M_{m(i,j)}}),$$

we must have

$$\|\phi(\psi(f_{i,j})1_{M_{m(i,j)}})|_K\| \leq \eta. \tag{3-4}$$

We get

$$\begin{aligned}\|\phi(\psi(1)1_{M_{m(i,j)}})|_K\| &\overset{\text{(3-1)}}{=} \|\phi(1_{M_{m(i,j)}})|_K\|\,\|\psi(1)1_{M_{m(i,j)}}\| \\ &\overset{\text{(3-3)}}{\leq} \|\phi(1_{M_{m(i,j)}})|_K\|\,|\mathcal{F}|\,\|\psi(f_{i,j})1_{M_{m(i,j)}}\| \\ &\overset{\text{(3-1)}}{=} \|\phi(\psi(f_{i,j})1_{M_{m(i,j)}})|_K\| \\ &\underset{\text{(3-4)}}{\overset{\text{(3-2)}}{\leq}} \frac{\epsilon}{2(n+1)};\end{aligned}$$

therefore, $a_j^{(i)}|_K = 0$, as required.

If b is not the unit, we may still assume that $\|b\| \leq 1$ and use the argument above to obtain an $(n+1)$-colourable approximate partition of unity $(a_j^{(i)})$ subordinate to $\mathcal{U}$. Then simply set $b_j^{(i)} = ba_j^{(i)}$.

(iv) $\Rightarrow$ (iii). It will suffice to prove the condition in (iii) assuming that $\mathcal{F}$ consists of self-adjoint contractions.

Take an open cover $\mathcal{U}$ of X along with points $x_U \in U$ for every $U \in \mathcal{U}$ such that, for any $f \in \mathcal{F}$, $U \in \mathcal{U}$, and $x \in U$,

$$|f(x) - f(x_U)| < \frac{\epsilon}{2}. \tag{3-5}$$

Use (iv) with $b = 1$ to find an $(n+1)$-colourable $\epsilon/2$-approximate partition of unity

$$(a_j^{(i)})_{i=0,\dots,n;\, j=1,\dots,r}$$

subordinate to $\mathcal{U}$. By a standard rescaling argument, we may assume that $\sum a_j^{(i)} \leq 1$. For each i, j, let $U(i, j) \in \mathcal{U}$ be such that $\operatorname{supp} a_j^{(i)} \subset U(i, j)$.

Define $\psi : C(X) \to (\mathbb{C}^r)^n$ by

$$\psi(f) = (f(x_{U(i,j)}))_{i=0,\dots,n;\, j=1,\dots,r}$$

and define $\phi : (\mathbb{C}^r)^n \to C(X, A)$ by

$$\phi(\lambda_{i,j})_{i=0,\dots,n;\, j=1,\dots,r} = \sum_{i,j} \lambda_{i,j} \cdot a_j^{(i)}.$$

Clearly, ψ is a *-homomorphism, while ϕ is c.p.c. and its restriction to each copy of $\mathbb{C}^r$ is order zero.

To verify that $\phi \circ \psi$ approximates θ in the appropriate sense, fix $f \in \mathcal{F}$ and $x \in X$. We shall show that $\|\phi\psi(f)(x) - f(x)\| < \epsilon$ (in the fibre $A(x)$). Let

$$S = \{(i, j) \in \{0, \dots, n\} \times \{1, \dots, r\} : x \in U(i, j)\},$$

so that

$$\phi(\psi(f))(x) = \sum_{(i,j)\in S} f(x_{U(i,j)}) \cdot a_j^{(i)}(x) \quad \text{and} \quad 1 =_{\epsilon/2} \sum_{(i,j)\in S} a_j^{(i)}(x).$$

By (3-5),

$$(f(x)-\epsilon/2)\cdot\sum_{(i,j)\in S} a_j^{(i)}(x) \le \sum_{(i,j)\in S} f(x_{U(i,j)})\cdot a_j^{(i)}(x) \le (f(x)+\epsilon/2)\cdot\sum_{(i,j)\in S} a_j^{(i)}(x).$$

It follows that

$$\phi(\psi(f)) = \sum_{(i,j)\in S} f(x_{U(i,j)})\cdot a_j^{(i)} =_{\epsilon/2} f(x)\cdot\sum_{(i,j)\in S} a_j^{(i)} =_{\epsilon/2} f(x),$$

as required. □

Proposition 3.3. *Let X be a locally compact metrizable space with finite covering dimension, and let A be a $C_0(X)$-algebra all of whose fibres are isomorphic to $\mathcal{O}_2$. Let $U\subset X$ be an open subset such that $\bar{U}$ is compact.*

Then $C_0(U)A\cong C_0(U,\mathcal{O}_2)$ as $C_0(U)$-algebras.

Proof. [Dadarlat 2009a, Theorem 1.1] says that $A|_{\bar{U}}\cong C(\bar{U},\mathcal{O}_2)$, as $C(\bar{U})$-algebras. Viewing $C_0(U)A$ as an ideal of $A|_{\bar{U}}$, the result follows. □

4. Decomposition rank of $C_0(X,\mathcal{Z})$

In this section, we prove our main result.

Theorem 4.1. *Let A be a* C*-*algebra which is locally approximated by hereditary subalgebras of* C*-*algebras of the form $C(X,\mathcal{K})$, with X compact Hausdorff.*

Then

$$\mathrm{dr}(A\otimes\mathcal{Z})\le 2.$$

In particular, any $\mathcal{Z}$-stable AH C*-*algebra has decomposition rank at most* 2.

In our proof, we will make use of the huge amount of space provided by the noncommutative fibres in two ways. First, we exhaust the identity on X by pairwise orthogonal functions up to a tracially small hereditary subalgebra. This will be designed to host an algebra of the form $C_0(Z)\otimes\mathcal{O}_2$, which is possible by quasidiagonality of the cone over $\mathcal{O}_2$. The first factor embedding of $C_0(Z)$ into the latter can be approximated by 2-colourable maps as shown by Kirchberg and Rørdam (see below). Together with the initial set of functions, we obtain a 3-colourable, hence 2-dimensional, approximation of the first factor embedding of $C(X)$ into $C(X)\otimes\mathcal{Z}$.

We will first carry out this construction with a UHF algebra in place of $\mathcal{Z}$; a slight modification will then allow us to pass to certain $C([0,1])$-algebras with UHF fibres, which immediately yields the general case.

In fact, if one is only concerned with showing that $A\otimes\mathcal{Z}$ has finite decomposition rank, our argument can be significantly shortened; using [Carrión 2011, Lemma 3.1], it suffices to show that $A\otimes U$ has finite decomposition rank, when U is an infinite dimensional, self-absorbing UHF algebra. Remarks 4.8

and 4.9 describe how one can easily modify (and skip some long technicalities in) the arguments below in order to efficiently prove that $A \otimes \mathcal{Z}$ has finite decomposition rank.

As noted above, a result of Kirchberg and Rørdam [2005, Proposition 3.7] on 1-dimensional approximations in the case of $\mathcal{O}_2$-fibred bundles is a crucial ingredient; this in turn relies on the fact that the unitary group of $C(S^1, \mathcal{O}_2)$ is connected [Cuntz 1981]. We note the following direct consequence which is more adapted to our needs.

Theorem 4.2. *For any locally compact Hausdorff space X, the decomposition rank of the first factor embedding $C_0(X) \to C_0(X, \mathcal{O}_2)$ is at most one.*

Proof. We begin with the case that X is compact and metrizable. By [Kirchberg and Rørdam 2005, Proposition 3.7], there exists a *-subalgebra $A \subset C(X, \mathcal{O}_2)$ which contains $C(X) \otimes 1_{\mathcal{O}_2}$ and is isomorphic to $C(Y)$ where Y is compact metrizable with covering dimension at most one. Therefore, the decomposition rank of the first factor embedding $C(X) \to C(X) \otimes \mathcal{O}_2$ is at most the decomposition rank of the inclusion $C(X) \otimes 1_{\mathcal{O}_2} \subset A$, which in turn is at most $\operatorname{dr} A \leq 1$.

For X compact but not metrizable, $C(X)$ is locally approximated by finitely generated unital subalgebras, which are of the form $C(Y)$ where Y is compact and metrizable. Therefore, by Proposition 2.4(i), the claim holds in this case too.

For the case that X is not compact, we let $\widetilde{X}$ denote the one-point compactification of X. Then $C_0(X, \mathcal{O}_2)$ is the hereditary subalgebra of $C(\widetilde{X}, \mathcal{O}_2)$ generated by $C_0(X)$, and therefore the result follows from Proposition 2.4(ii). ☐

Remark 4.3. The preceding result also implies that $\dim_{\mathrm{nuc}}(A \otimes \mathcal{O}_2) \leq 3$ for A as in Theorem 4.1 — this can be seen using Proposition 2.6, [Winter and Zacharias 2010, Theorem 7.4], and the analogue of [Winter and Zacharias 2010, Proposition 2.3(ii)].

In what follows, D_n denotes the diagonal subalgebra of M_n.

Lemma 4.4. *Let $I_1, \dots, I_n \subset (0,1)$ be nonempty closed intervals and let $a_{1/2} \in C_0((0,1), D_n)_+$ be a function of norm 1 such that, for $t \in I_s$, the s-th diagonal entry of $a_{1/2}(t)$ is 1.*

Then there exist $a_0, a_1, e_0, e_{1/2}, e_1 \in C([0,1], D_n)_+$ such that

(i) *e_0 and e_1 are orthogonal,*

(ii) *$a_0 + a_{1/2} + a_1 = e_0 + e_{1/2} + e_1 = 1$,*

(iii) *for $i = 0, 1$, we have $a_i(i) = 1_n$,*

(iv) *e_0, e_1 act like a unit on a_0, a_1, respectively, and*

(v) *$a_{1/2}$ acts like a unit on $e_{1/2}$.*

Proof. Since $D_n \cong \mathbb{C}^n$, it suffices to work in one coordinate at a time — that is, to assume that $n = 1$. Then define

$$a_0(x) := \begin{cases} 1 - a_{1/2}(x) & \text{if } x \text{ is to the left of } I_1, \\ 0 & \text{otherwise,} \end{cases}$$

$$a_1(x) := \begin{cases} 1 - a_{1/2}(x) & \text{if } x \text{ is to the right of } I_1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that since $a_{1/2} \equiv 1$ on I_1, these are continuous. Now, we may find continuous orthogonal functions e_0, e_1 such that e_0 is 1 to the left of I_1, and e_1 is 1 to the right of I_1. Finally, set $e_{1/2} := 1-(e_0+e_1)$. Then (i), (ii), and (iii) clearly hold by construction. (iv) holds since each a_i is nonzero only on one side of I_1, and the corresponding e_i is identically 1 on that side. Likewise, (v) holds since $e_{1/2}$ is nonzero only on I_1, where $a_{1/2}$ is identically 1. □

We mention the following well-known fact explicitly for convenience. Here $\otimes$ denotes the minimal tensor product.

Proposition 4.5. *Let A_1, A_2, B_1, and B_2 be* C*-*algebras, and suppose that $\phi^{(i)} : A_i \to (B_i)_\infty$ is a* *-*homomorphism for $i = 1, 2$ with a c.p. lift $(\phi_k^{(i)})_{\mathbb{N}} : A_i \to \prod_{\mathbb{N}} B_i$.*

Then

$$\phi_1 \otimes \phi_2 = \pi_\infty \circ (\phi_k^{(1)} \otimes \phi_k^{(2)})_{\mathbb{N}} : A_1 \otimes A_2 \to (B_1 \otimes B_2)_\infty$$

is a *-*homomorphism.*

Lemma 4.6. *Let A be an infinite dimensional UHF algebra.*

Then there exist positive orthogonal contractions

$$a_0, a_1 \in C([0,1], A)_\infty,$$

a *-*homomorphism*

$$\psi : C_0(Z, \mathcal{O}_2) \to C_0((0,1), A)_\infty,$$

where $Z = (0,1]^2$, and a positive element $c \in C_c(Z, \mathbb{C} \cdot 1_{\mathcal{O}_2})$ such that $\psi(c)$ commutes with a_0, a_1,

$$a_0 + a_1 + \psi(c) = 1, \tag{4-1}$$

and $a_0(0) = a_1(1) = 1$. In addition, there exist positive contractions $e_0, e_{1/2}, e_1 \in C([0,1], A)_\infty$ such that

(i) *e_0, e_1 are orthogonal,*

(ii) *$e_0 + e_{1/2} + e_1 = 1$,*

(iii) *$\psi(c)$ acts like a unit on $e_{1/2}$,*

(iv) *e_i acts like a unit on a_i for $i = 0, 1$, and*

(v) *$e_0, e_{1/2}, e_1, a_0, a_1, \psi(c)$ all commute.*

Proof. Let $A = M_{n_1 n_2 \cdots}$ where $n_1, n_2, \ldots$ are a sequence of natural numbers ≥ 2. Since the cone over $\mathcal{O}_2$ is quasidiagonal (see [Voiculescu 1991] and [Kirchberg 1993, Theorem 5.1]) there exists a sequence of c.p.c. maps

$$\phi_k : C_0((0,1], \mathcal{O}_2) \to M_{n_1 \cdots n_k}$$

which are approximately multiplicative and approximately isometric, meaning that

$$\|\phi_k(a)\phi_k(b) - \phi_k(ab)\| \to 0 \quad \text{and} \quad \|\phi_k(a)\| \to \|a\| \quad \text{as } k \to \infty$$

for all $a, b \in C_0((0,1], \mathcal{O}_2)$. Fix a positive element

$$d \in C_c((0,1], \mathbb{C}\cdot 1_{\mathcal{O}_2})$$

of norm 1.

For each k, let λ_k denote the greatest eigenvalue of $\phi_k(d)$. Note that

$$\lambda_k = \|\phi_k(d)\| \to 1 \quad \text{as } k \to \infty.$$

Fix k for a moment and let $l = n_1 \cdots n_k$. Let

$$I_1, \dots, I_l$$

be nonempty disjoint closed intervals in $(0,1)$. Let

$$u_1, \dots, u_l \in M_l$$

be unitaries such that, for each s, $u_s \phi_k(d) u_s^*$ is a diagonal matrix whose s-th diagonal entry is λ_k. Let

$$h_1, \dots, h_l \in C_0((0,1))$$

be positive normalized functions with disjoint support, such that $h_s|_{I_s} \equiv 1$ for each s. Set $Z := (0,1]^2$ and define

$$\psi_k : C_0(Z, \mathcal{O}_2) \cong C_0((0,1]) \otimes C_0((0,1], \mathcal{O}_2) \to C([0,1]) \otimes M_l \cong C([0,1], M_l) \subset C([0,1], A)$$

by

$$\psi_k(f \otimes b) = \sum_{s=1}^{l} f(h_s) \otimes u_s \phi_k(b) u_s^*.$$

Let $f \in C_c((0,1])$ be a function satisfying $f(1) = 1$ and set

$$c = f \otimes d \in C_c(Z, \mathbb{C}\cdot 1_{\mathcal{O}_2}).$$

By construction, $\psi_k(c) \in C((0,1), D_l)_+$, and for $t \in I_s$, the s-th diagonal entry is λ_k. Let

$$c'_k \in C([0,1], D_l)_+$$

be of norm 1, such that

$$\|c'_k - \psi_k(c)\| = |1 - \lambda_k|$$

and for $t \in I_s$, the s-th diagonal entry is 1. Feeding

$$a_{1/2} := c'_k$$

to Lemma 4.4, let

$$a_{0,k}, a_{1,k}, e_{0,k}, e_{1/2,k}, e_{1,k} \in C([0,1], D_l)_+$$

be the output, satisfying (i)–(v) of Lemma 4.4.

Having found these for each k, set

$$\psi := \pi_\infty \circ (\psi_1, \psi_2, \dots) : C_0(Z, \mathcal{O}_2) \to C([0,1], A)_\infty.$$

Set

$$a_i := \pi_\infty(a_{i,1}, a_{i,2}, \dots)$$

for $i = 0, 1$ and

$$e_i := \pi_\infty(e_{i,1}, e_{i,2}, \dots)$$

for $i = 0, \frac{1}{2}, 1$.

Since all unitaries in M_l (and in particular, all the u_s) are in the same path component, ψ_k is unitarily equivalent to $\alpha \otimes \phi_k$, where

$$\alpha : C_0((0,1]) \to C([0,1])$$

is the *-homomorphism given by

$$f \mapsto f(h_1 + \cdots + h_l).$$

From this observation and Proposition 4.5, it follows that ψ is a *-homomorphism.

Notice further that

$$\psi(c) = \pi_\infty(c_1', c_2', \dots),$$

and therefore, drawing on the finite stage results, we see that

$$a_0 + a_1 + \psi(c) = 1$$

and that (i)–(v) hold. □

Lemma 4.7. *Let $p, q > 1$ be natural numbers. Let $X = [0,1]^m$ for some m and let $\epsilon > 0$.*

Then there exist positive orthogonal elements

$$h_0, \dots, h_k \in C(X, \mathcal{Z})_\infty,$$

a *-homomorphism

$$\phi : C_0(Z, \mathcal{O}_2) \to C(X, \mathcal{Z})_\infty$$

for some locally compact, metrizable, finite dimensional space Z, and a positive element $c \in C_c(Z, \mathbb{C} \cdot 1_{\mathcal{O}_2})$ such that $\phi(c)$ commutes with $h_0, \dots, h_k$,

$$h_0 + \cdots + h_k + \phi(c) = 1,$$

and the support of h_i has diameter at most ϵ for $i = 0, \dots, k$ with respect to the ℓ^∞ metric on $[0,1]^m$.

In addition, there exist positive contractions $e_0, e_{1/2}, e_1 \in C(X, \mathcal{Z})_\infty$ such that

(i) *e_0, e_1 are orthogonal,*

(ii) *$e_0 + e_{1/2} + e_1 = 1$,*

(iii) *e_j is identically 1 on $\{j\} \times [0,1]^{m-1}$, for $j = 0, 1$,*

(iv) *$\phi(c)$ acts like a unit on $e_{1/2}$,*

(v) *$e_0 + e_1$ acts like a unit on h_i for all $i = 0, \dots, k$, and*

(vi) *$e_0, e_{1/2}, e_1, h_0, \dots, h_k, \phi(c)$ all commute.*

Remark 4.8. This lemma of course holds with a self-absorbing UHF algebra in place of $\mathcal{Z}$ (since such a UHF algebra contains $\mathcal{Z}$). But, in fact, this variation is shown in Steps 1 and 2 of the proof below, and, as we will see in Remark 4.9, this variation is sufficient to prove that $A \otimes \mathcal{Z}$ as in Theorem 4.1 has finite decomposition rank. A reader only interested in showing finite decomposition rank may therefore skip the third step of the proof below.

Proof. This will be proven in three steps. In Step 1, we will prove the statement of the proposition with $\mathcal{Z}$ replaced by a UHF algebra of infinite type and with $m = 1$. In Step 2, we will still replace A by a UHF algebra of infinite type, but we will allow any $m \in \mathbb{N}$. Step 3 will be the proof of the proposition.

Step 1. Let A be a UHF algebra of infinite type. Let

$$a_0, a_1, \psi, c, e'_0, e'_{1/2}, e'_1, Z$$

be as in Lemma 4.6, with e'_i in place of e_i. Note that each a_i has a positive normalized lift

$$(a_{i,j})_{j=1}^{\infty} \in \prod_{\mathbb{N}} C([0,1], A)$$

such that $a_{i,j}(t) = \delta_{i,t}1$ for all $i, t = 0, 1$ and all j; likewise, each e'_i, $i = 0, \frac{1}{2}, 1$, has a positive normalized lift

$$(e'_{i,j})_{j=1}^{\infty} \in \prod_{\mathbb{N}} C([0,1], A)$$

such that, for $i = 0, 1$, $e'_{i,j}(i) = 1$.

Let $k \geq 2/\epsilon$ be a natural number. For $i = 0, \dots, k$, $j \in \mathbb{N}$, and $t \in [0,1]$, set

$$h_{i,j}(t) := \begin{cases} 0 & \text{if } t \leq \frac{i-1}{k} \text{ or } t \geq \frac{i+1}{k}, \\ a_{1,j}(kt - (i-1)) & \text{if } t \in \left[\frac{i-1}{k}, \frac{i}{k}\right], \\ a_{0,j}(kt - i) & \text{if } t \in \left[\frac{i}{k}, \frac{i+1}{k}\right]. \end{cases} \tag{4-2}$$

Note that the endpoint conditions on $a_{i,j}$ make $h_{i,j}$ well defined and continuous on $[0,1]$. Likewise, set

$$e_{i,j}(t) := \begin{cases} e'_{i,j}(0) & \text{if } t = 0, \\ e'_{i,j}(1) & \text{if } t \geq \frac{1}{k}, \\ e'_{i,j}(kt) & \text{if } t \in \left[0, \frac{1}{k}\right]. \end{cases} \tag{4-3}$$

Set

$$h_i := \pi_\infty(h_{i,1}, h_{i,2}, \dots), e_i := \pi_\infty(e_{i,1}, e_{i,2}, \dots) \in C([0,1], A)_\infty$$

for $i = 0, 1$. Choose a c.p.c. lift for ψ, that is, c.p.c. maps

$$\psi_j : C_0(Z, \mathcal{O}_2) \to C_0((0,1), A) \subset C([0,1], A)$$

such that

$$\psi := \pi_\infty \circ (\psi_1, \psi_2, \dots).$$

Define

$$\phi_j : C_0(Z, \mathcal{O}_2) \to C([0,1], A)$$

by

$$\phi_j(a)(t) = \psi_j(a)(kt - i), \tag{4-4}$$

if $i \in \mathbb{N}$ is such that $t \in [i/k, (i+1)/k]$. Note that this is well defined since the image of ψ_j is contained in $C_0((0,1), A)$. Use $(\phi_j)_{j=1}^{\infty}$ to define

$$\phi = \pi_\infty \circ (\phi_1, \phi_2, \dots) : C_0(Z, \mathcal{O}_2) \to C([0,1], A)_\infty.$$

Then ϕ is a *-homomorphism.

Let us first show that $h_0 + \cdots + h_k + \phi(c) = 1$, and then that (i)–(vi) hold. For $t \in [0,1]$, let i be such that $t \in [i/k, (i+1)/k]$. Then, by (4-2), we have, for all i,

$$h_i(t) = a_0(kt - i), \quad h_{i+1}(t) = a_1(kt - i), \quad h_j(t) = 0$$

for $j \neq i, i+1$. Thus

$$(h_0 + \cdots + h_k + \phi(c))(t) \overset{(4\text{-}4)}{=} a_0(kt - i) + a_1(kt - i) + \psi(c)(kt - i) \overset{(4\text{-}1)}{=} 1.$$

Properties (i) and (ii) hold by Lemma 4.6(i) and (ii), and since, for each $t \in [0,1]$, there exists s such that $e_j(t) = e'_j(s)$ for $j = 0, \frac{1}{2}, 1$ (by (4-3)). Property (iii) holds since $e_i(i) = e'_i(i)$ (by (4-3)) and since $a_i(i) = 1$.

(iv): $e_{1/2}$ is supported on $[0, 1/k]$, so it suffices to show that

$$(\phi(c)e_{1/2})(t) = e_{1/2}(t)$$

for $t \in [0, 1/k]$. But, for such t,

$$(\phi(c)e_{1/2})(t) \underset{(4\text{-}4)}{\overset{(4\text{-}3)}{=}} \psi(c)(kt)e'_{1/2}(kt) \overset{\text{Lemma 4.6(iii)}}{=} e'_{1/2}(kt) \overset{(4\text{-}3)}{=} e_{1/2}(t).$$

(v): By a similar computation, this time using Lemma 4.6(iv), we see that $e_0 a_0 = a_0$, while $e_1 a_i = a_i$ for $i = 1, \dots, k$.

(vi) is clear from (4-2), (4-3), (4-4), and Lemma 4.6(v).

Finally, also, for each i, the support of h_i is contained in $\left[\frac{i-1}{k}, \frac{i+1}{k}\right]$, which has diameter at most ϵ.

Step 2. From Step 1, let

$$g_0, \dots, g_{k'} \in C([0,1], A)_\infty$$

be orthogonal positive contractions,

$$\psi : C_0(Y, \mathcal{O}_2) \to C([0,1], A)_\infty$$

a *-homomorphism for some locally compact, metrizable, finite dimensional space Y, and $d \in C_c(Y, \mathbb{C}\cdot 1_{\mathcal{O}_2})$ a positive contraction such that $\psi(d)$ commutes with $g_0, \dots, g_{k'}$,

$$g_0 + \cdots + g_{k'} + \psi(d) = 1$$

and the support of g_i has diameter at most ϵ for $i = 0, \dots, k'$; furthermore, let

$$e'_0, e'_{1/2}, e'_1 \in C([0,1], A)_\infty$$

be such that

(i′) e'_0, e'_1 are orthogonal,

(ii′) $e'_0 + e'_{1/2} + e'_1 = 1$,

(iii′) e'_j is identically 1 on $\{j\} \times [0,1]^{m-1}$, for $j = 0, 1$,

(iv′) $\psi(d)$ acts like a unit on $e'_{1/2}$,

(v′) $e'_0 + e'_1$ acts like a unit on g_i for all $i = 0, \dots, k'$, and

(vi′) $e'_0, e'_{1/2}, e'_1, g_0, \dots, g_{k'}, \psi(d)$ all commute.

For $i = (i_1, \dots, i_m) \in \{0, \dots, k'\}^m$, set

$$h_i := g_{i_1} \otimes \cdots \otimes g_{i_m} \in (C([0,1], A)^{\otimes m})_\infty,$$

where we have used the canonical inclusion

$$(C([0,1], A)_\infty)^{\otimes m} \to (C([0,1], A)^{\otimes m})_\infty;$$

compare Proposition 4.5.

Then $\{h_i\}$ is a set of pairwise orthogonal positive contractions, and each one has support with diameter at most ϵ (recall that we are using the ℓ^∞ metric on $[0,1]^m$). Proposition 4.5 gives us a *-homomorphism

$$\phi' := (\psi^\sim)^{\otimes m} : C := (C_0(Y, \mathcal{O}_2)^\sim)^{\otimes m} \to (C([0,1], M_{n^\infty})^{\otimes m})_\infty.$$

Set

$$c := 1 - (1-d)^{\otimes m} \in C.$$

We can easily see that $\phi'(c)$ commutes with each h_i; a simple computation shows that

$$\sum_i h_i + \phi'(c) = 1.$$

Setting

$$e_i := e'_i \otimes 1^{\otimes(m-1)} \quad \text{for } i = 0, \tfrac{1}{2}, 1,$$

it is easy to see that (i), (ii), (iii), (v), and (vi) hold (with ϕ' in place of ϕ). To see that (iv) holds, we compute

$$\begin{aligned} \phi'(c)e_{1/2} &= (1-(1-\psi(d))^{\otimes m})(e'_{1/2} \otimes 1^{\otimes(m-1)}) \\ &= \phi'(c) - (e'_{1/2} - \psi(d)e'_{1/2}) \otimes (1-\psi(d))^{\otimes(m-1)} \\ &\overset{(\text{iv}')}{=} \phi'(c). \end{aligned}$$

We may set

$$k := (k'+1)^m - 1$$

and relabel the h_i as $h_0, \dots, h_k$.

All that remains is to modify ϕ' to make it a map whose domain is $C_0(Z, \mathcal{O}_2)$ for some Z. Set

$$Z' := (Y \amalg \{\infty\})^{\times m}.$$

Then C may be identified with a certain $C(Z')$-subalgebra of $C(Z', \mathcal{O}_2^{\otimes m})$. All of the fibres of C are isomorphic to $\mathcal{O}_2$ except for the fibre at $(\infty, \dots, \infty)$, which is $\mathbb{C}$. One can easily verify that the element c is in $C_0(U, \mathbb{C} \cdot 1_{\mathcal{O}_2^{\otimes m}})$ where U is some open subset of Z' whose closure does not contain $(\infty, \dots, \infty)$.

Let Z be an open subset of Z' such that $\overline{U} \subset Z$ and whose closure does not contain $(\infty, \dots, \infty)$; in particular, $\overline{Z}$ is a compact subset of $Z' \backslash \{(\infty, \dots, \infty)\}$. By Proposition 3.3, $C_0(Z)C \cong C_0(Z, \mathcal{O}_2)$ as $C_0(Z)$-algebras. With this identification, we have $c \in C_c(Z, \mathbb{C} \cdot 1_{\mathcal{O}_2})$ (since c is in the image of the structure map, which is fixed by the isomorphism $C_0(Z)C \cong C_0(Z, \mathcal{O}_2)$), and we may define

$$\phi := \phi'|_{C_0(Z)C} : C_0(Z, \mathcal{O}_2) \to C(X, A)_\infty.$$

<u>Step 3</u>. Let p_0, p_1 be coprime natural numbers. Since $\mathcal{Z}_{p_0^\infty, p_1^\infty}$ (as defined in [Rørdam and Winter 2010, Section 2]) embeds unitally into $\mathcal{Z}$ [Rørdam 2004, Proposition 2.2], it suffices to do this part with $\mathcal{Z}_{p_0^\infty, p_1^\infty}$ in place of $\mathcal{Z}$.

From Step 2, for $i = 0, 1$, we may find

$$h_0^{(i)}, \dots, h_k^{(i)} \in C(X, M_{p_i^\infty})_\infty,$$

a *-homomorphism

$$\phi_i : C_0(Z_i, \mathcal{O}_2) \to C(X, M_{p_i^\infty})_\infty$$

for some locally compact, metrizable, finite dimensional space Z_i, and a positive element

$$c_i \in C_c(Z_i, \mathbb{C} \cdot 1_{\mathcal{O}_2})$$

such that ϕ_i commutes with $h_0^{(i)}, \dots, h_k^{(i)}$,

$$h_0^{(i)} + \cdots + h_k^{(i)} + \phi_i(c_i) = 1,$$

and the support of $h_j^{(i)}$ has diameter at most ϵ for $j = 1, \dots, k$. We may also find $e_l^{(i)}$ for $l = 0, \frac{1}{2}, 1$ satisfying (i)–(vi).

From Lemma 4.6, let

$$a_0, a_1, e_0', e_{1/2}', e_1' \in C\left(\left[\tfrac{1}{3}, \tfrac{2}{3}\right], A\right)_\infty$$

be positive orthogonal contractions, let

$$\psi : C_0(Y, \mathcal{O}_2) \to C_0\left(\left(\tfrac{1}{3}, \tfrac{2}{3}\right), M_{(p_0 p_1)^\infty}\right)_\infty$$

be a *-homomorphism for some locally compact, metrizable, finite dimensional space Y, and let

$$d \in C_c(Z, \mathbb{C} \cdot 1_{\mathcal{O}_2})$$

be positive such that $\psi(d)$ commutes with a_0, a_1,

$$a_0 + a_1 + \psi(d) = 1,$$

$a_0\big(\frac{1}{3}\big) = a_1\big(\frac{2}{3}\big) = 1$, and such that (i)–(v) of Lemma 4.6 hold. We continuously extend a_0, a_1, e'_0, $e'_{1/2}$, e'_1 to $[0,1]$ by allowing them to be constant on $\big[0, \frac{1}{3}\big]$ and on $\big[\frac{2}{3}, 1\big]$.

Upon choosing an isomorphism

$$M_{(p_0p_1)^\infty} \otimes M_{p_0^\infty} \otimes M_{p_1^\infty} \cong M_{p_0^\infty} \otimes M_{p_1^\infty}$$

and using the diagonal restriction $C(X, M_{p_0^\infty})_\infty \otimes C(X, M_{p_1^\infty})_\infty \to C(X, M_{p_0^\infty} \otimes M_{p_1^\infty})_\infty$, we obtain a *-homomorphism

$$\begin{aligned}\rho : C([0,1], M_{(p_0p_1)^\infty})_\infty \otimes C(X, M_{p_0^\infty})_\infty \otimes C(X, M_{p_1^\infty})_\infty &\to C([0,1]\times X, M_{(p_0p_1)^\infty} \otimes M_{p_0^\infty} \otimes M_{p_1^\infty})_\infty \\ &\cong C([0,1]\times X, M_{p_0^\infty} \otimes M_{p_1^\infty})_\infty,\end{aligned}$$

and define

$$\hat{h}_{0,j} := \rho\big(a_0 \otimes h_j^{(0)} \otimes 1_{C(X,M_{p_1^\infty})_\infty}\big) \quad \text{and} \quad \hat{h}_{1,j} := \rho\big(a_1 \otimes 1_{C(X,M_{p_0^\infty})_\infty} \otimes h_j^{(1)}\big)$$

for $j = 1, \dots, k$. Note that a_i has a lift

$$(a_{i,k})_{k=1}^\infty \in \textstyle\prod_{\mathbb{N}} C([0,1], M_{(p_0p_1)^\infty})$$

such that $a_{i,k}(t) \in \mathbb{C}\cdot 1$ for $t = 0, 1$, and, consequently,

$$\hat{h}_{i,j} \in C(X, \mathscr{Z}_{p_0^\infty, p_1^\infty})_\infty.$$

Define a *-homomorphism

$$\phi : C([0,1]) \otimes C_0(Y, \mathcal{O}_2)^\sim \otimes C_0(Z_0, \mathcal{O}_2)^\sim \otimes C_0(Z_1, \mathcal{O}_2)^\sim \to C([0,1] \times X, M_{p_0^\infty} \otimes M_{p_1^\infty})_\infty$$

by

$$\phi := \rho \circ \big(\mathrm{id}_{C([0,1])} \otimes (\psi^\sim) \otimes (\phi_0^\sim) \otimes (\phi_1^\sim)\big).$$

Let

$$Y' := \{y \in Y : d(y) > 0\} \quad \text{and} \quad Z'_i := \{z \in Z_i : c_i(z) \neq 0\},$$

and, using these, set

$$\begin{aligned}C := \mathrm{C}^*\big(&C_0[0,1) \otimes 1_{C_0(Y,\mathcal{O}_2)^\sim} \otimes C_0(Z'_0, \mathcal{O}_2) \otimes 1_{C_0(Z_1,\mathcal{O}_2)^\sim}, \\ &C_0(0,1] \otimes 1_{C_0(Y,\mathcal{O}_2)^\sim} \otimes 1_{C_0(Z_0,\mathcal{O}_2)^\sim} \otimes C_0(Z'_1, \mathcal{O}_2), \\ &1_{C([0,1])} \otimes C_0(Y', \mathcal{O}_2) \otimes 1_{C_0(Z_0,\mathcal{O}_2)^\sim} \otimes 1_{C_0(Z_1,\mathcal{O}_2)^\sim}\big).\end{aligned}$$

Using Proposition 3.3 as in Step 2, C is a subalgebra of some $C_0(Z)$-algebra

$$D \subset C[0,1] \otimes C_0(Y, \mathcal{O}_2)^\sim \otimes C_0(Z_0, \mathcal{O}_2) \otimes C_0(Z_1, \mathcal{O}_2)$$

for some open subset Z of

$$[0,1] \times (Y' \cup \{\infty\}) \times (Z'_0 \cup \infty) \times (Z'_1 \cup \infty),$$

and D is isomorphic, as a $C_0(Z)$-algebra, to $C_0(Z, \mathcal{O}_2)$, via an isomorphism taking C into $C_c(Z, \mathcal{O}_2)$. One easily sees that $\phi(C) \subset C(X, \mathcal{Z}_{p_0,p_1})$.

Let $f_0 \in C_0[0,1)_+$ be identically 1 on $\left[0, \frac{2}{3}\right]$, and let $f_1 \in C_0(0,1]_+$ be identically 1 on $\left[\frac{1}{3}, 1\right]$. Set

$$\hat{c} := f_0 \otimes 1 \otimes c_0 \otimes 1 + f_1 \otimes 1 \otimes 1 \otimes c_1 + 1 \otimes d \otimes 1 \otimes 1 \in C.$$

Identifying D with $C_0(Z, \mathcal{O}_2)$, we see that $\hat{c} \in C_c(Z, \mathbb{C} \cdot 1_{\mathcal{O}_2})$. It is straightforward to check that $\phi(\hat{c})$ commutes with $\hat{h}_{i,j}$ for all i, j, and we may easily compute

$$\phi(\hat{c}) + \sum_{i,j} \hat{h}_{i,j} \geq 1.$$

Let $g \in C_0(0, \infty]$ be the function $g(t) = \max\{t, 1\}$ and set

$$c := g(\hat{c}).$$

Then, by commutativity, it follows that

$$\phi(c) + \sum_{i,j} \hat{h}_{i,j} \geq 1. \tag{4-5}$$

Let $g_0, g_{1/2}, g_1 \in C(X)_+$ be a partition of unity such that g_j is identically 1 on $\{j\} \times [0,1]^{m-1}$ for $j = 0, 1$, g_0 is supported on $\left[0, \frac{1}{3}\right] \times [0,1]^{m-1}$, and g_1 is supported on $\left[\frac{2}{3}, 1\right] \times [0,1]^{m-1}$. Let us define

$$e_j := \rho(e'_0 \otimes e_j^{(0)} \otimes 1 + e'_1 \otimes 1 \otimes e_j^{(1)}) + g_j \rho(e'_{1/2} \otimes 1 \otimes 1) \tag{4-6}$$

for $j = 0, \frac{1}{2}, 1$. It is clear by their definitions that $e_0, e_{1/2}, e_1, \hat{h}_0, \dots, \hat{h}_k, \phi(c)$ all commute.

Let us now check that $(e_0 + e_1)\hat{h}_{i,j} = \hat{h}_{i,j}$. Certainly

$$\begin{aligned}
&(e_0+e_1)\hat{h}_{0,j} \\
&\overset{(4\text{-}6)}{=} \big(\rho(e'_0 \otimes (e_0^{(0)} + e_1^{(0)}) \otimes 1 + e'_1 \otimes 1 \otimes (e_0^{(1)} \otimes e_1^{(1)})) + (g_0+g_1)\rho(e'_{1/2} \otimes 1 \otimes 1)\big)\rho(a_0 \otimes h_j^{(0)} \otimes 1) \\
&\underset{4.6(\text{ii,iv})}{\overset{\text{Lemma}}{=}} \rho(a_0 \otimes ((e_0^{(0)} + e_1^{(0)})h_j^{(0)}) \otimes 1) \\
&\overset{\text{Step 2(v)}}{=} \rho(a_0 \otimes h_j^{(0)} \otimes 1) = \hat{h}_{0,j},
\end{aligned}$$

and likewise, $(e_0 + e_1)\hat{h}_{1,j} = \hat{h}_{1,j}$ as required.

Since all terms in (4-5) commute, it is easy to see that for any $\epsilon > 0$, there exist orthogonal elements $\tilde{h}_{i,j} \leq \hat{h}_{i,j}$ which commute with $e_0, e_{1/2}, e_1$ and $\phi(c)$, such that

$$\phi(c) + \sum_{i,j} \tilde{h}_{i,j} =_\epsilon 1.$$

Then, by a diagonal sequence argument, it follows that there exist orthogonal elements $h_{i,j}$ with supports contained in those of $\hat{h}_{i,j}$, commuting with $e_0, e_{1/2}, e_1$ and $\phi(c)$, and such that

$$\phi(c) + \sum_{i,j} h_{i,j} = 1 \quad \text{and} \quad (e_0 + e_1)h_{i,j} = h_{i,j}.$$

Hence (v) holds.

Now let us verify (i)–(iv).

(i) holds using the following orthogonalities:

$$\begin{aligned} e_0^{(i)} &\perp e_1^{(i)}, \qquad i=0,1,\\ g_0 &\perp g_1,\\ e_0' &\perp e_1',\\ \rho(1\otimes e_j^{(0)}\otimes 1) &\perp g_{1-j}, \quad j=0,1,\\ \rho(1\otimes 1\otimes e_j^{(1)}) &\perp g_{1-j}, \quad j=0,1. \end{aligned}$$

(ii): We compute

$$\begin{aligned} \epsilon_0+e_{1/2}+e_1 &\overset{(4\text{-}6)}{=} \rho\big(e_0'\otimes(e_0^{(0)}+e_{1/2}^{(0)}+e_1^{(0)})\otimes 1+e_1'\otimes 1\otimes(e_0^{(1)}+e_{1/2}^{(1)}+e_1^{(1)})\big)\\ &\qquad\qquad + (g_0+g_{1/2}+g_1)\rho(e_{1/2}'\otimes 1\otimes 1)\\ &\overset{\text{Step 2(ii)}}{=} \rho((e_0'+e_1'+)\otimes 1\otimes 1) \underset{4.6\text{(ii)}}{\overset{\text{Lemma}}{=}} 1. \end{aligned}$$

(iii): For $x\in\{j\}\times[0,1]^{m-1}$,

$$e_j(x) \overset{\text{Step 2(iii)}}{=} e_0'+e_1'+g_j(x)e_{1/2}' \underset{4.6\text{(ii)}}{\overset{\text{Lemma}}{=}} 1.$$

(iv) follows from the fact that $\phi(\hat{c})e_{1/2}=e_{1/2}\phi(\hat{c})\ge e_{1/2}$, by considering irreducible representations of $C^*(\phi(\hat{c}),e_{1/2})$. □

Proof of Theorem 4.1. By Proposition 2.4(i) and [Kirchberg and Winter 2004, Proposition 3.8], it suffices to verify the theorem for C*-algebras A of the form $C(X,\mathcal{K})$, where X is compact and Hausdorff. By [ibid., (3.5)], it suffices to prove it for $A=C(X)$. Again by Proposition 2.4(i), it suffices to assume that $C(X)$ is finitely generated. Finally, when $C(X)$ is finitely generated, it is a quotient of $C([0,1]^m)$ for some m, and so, by [ibid., (3.3)], the result reduces to showing that $\operatorname{dr} C(X,\mathcal{Z})\le 2$ for $X=[0,1]^m$. By Proposition 2.6, we must show that the first factor embedding $C(X,\mathcal{Z})\to C(X,\mathcal{Z})\otimes\mathcal{Z}$ has decomposition rank at most 2.

We will do this in two steps. In Step 1, we will use Lemma 4.7 to show that the first factor embedding $\iota_0 : C(X)\to C(X)\otimes\mathcal{Z}$ has decomposition rank at most 2. In Step 2, we will use Step 1, with X replaced by $X\times[0,1]$, to prove the theorem.

Step 1. Due to Proposition 2.5, it suffices to replace ι_0 by its composition with the inclusion $C(X)\otimes\mathcal{Z}\subset(C(X)\otimes\mathcal{Z})_\infty$, that is, ι_0 is now

$$C(X)\cong C(X)\otimes 1_{\mathcal{Z}}\subset C(X)\otimes\mathcal{Z}\subset(C(X)\otimes\mathcal{Z})_\infty.$$

To show that $\operatorname{dr}\iota_0\le 2$, we verify condition (iv) of Proposition 3.2. Let $\mathcal{U}$ be an open cover of X and let $\epsilon>0$. By the Lebesgue covering lemma, we may possibly reduce ϵ so that $\mathcal{U}$ is refined by the set of all open sets of diameter at most ϵ. Then, it suffices to assume that $\mathcal{U}$ is in fact the set of all open sets of diameter at most ϵ.

Let $h_0, \dots, h_k, \phi, c$ be as in Lemma 4.7. By Theorem 4.2 and condition (iv) of Proposition 3.2, we may find

$$b_j^{(i)} \in C_0(X \times Z, \mathcal{O}_2) \cong C(X) \otimes C_0(Z) \otimes \mathcal{O}_2$$

for $i = 0, 1$, $j = 0, \dots, r$ such that

(i) for each $i = 0, 1$, the elements $b_0^{(i)}, \dots, b_r^{(i)}$ are pairwise orthogonal,

(ii) for each i, j, the support of $b_j^{(i)}$ is contained in $U \times Z$ for some $U \in \mathscr{U}$, and

(iii) $\big\| \sum_{i,j} b_j^{(i)} - 1_{C(X)} \otimes c \big\| < \epsilon$ (note that $c \in C_0(Z) \otimes 1_{\mathcal{O}_2}$).

Define

$$\hat{\phi} : C_0(X \times Z, \mathcal{O}_2) \cong C(X) \otimes C_0(Z, \mathcal{O}_2) \to C(X, \mathscr{Z})_\infty$$

by $\hat{\phi}(f \otimes a) = f\phi(a)$. This is a *-homomorphism. For $i = 0, 1$ and $j = 0, \dots, r$, set

$$a_j^{(i)} := \hat{\phi}(b_j^{(i)}),$$

and, for $j = 0, 1 \dots, k$, set

$$a_j^{(2)} := h_j.$$

Since $\hat{\phi}$ is a homomorphism, $a_0^{(i)}, \dots, a_r^{(i)}$ are pairwise orthogonal for $i = 0, 1$. Also, by the definition of $\hat{\phi}$ and the choice of $b_j^{(i)}$, the support of each $a_j^{(i)}$ is contained in some set in $\mathscr{U}$, for $i = 0, 1$. Since the supports of the h_j have diameter at most ϵ, the respective statement holds for the $a_j^{(2)}$ as well. Finally,

$$\sum_{i,j} a_j^{(i)} = \hat{\phi}\Big(\sum_{i=0,1} \sum_{j=0}^{k} b_j^{(i)} \Big) + \sum_{j=0}^{k} h_j =_\epsilon \hat{\phi}(1 \otimes c) + \sum_{j=0}^{k} h_j = \phi(c) + \sum_{j=0}^{k} h_j = 1,$$

as required.

Step 2. Since $\mathscr{Z}$ is an inductive limit of algebras of the form $\mathscr{Z}_{p,q}$ (for $p, q \in \mathbb{N}$), by Proposition 2.4(i), it suffices to show that the decomposition rank of the first factor embedding

$$\iota := \mathrm{id}_{C(X, \mathscr{Z}_{p,q})} \otimes 1_{\mathscr{Z}} : C(X, \mathscr{Z}_{p,q}) \to C(X, \mathscr{Z}_{p,q}) \otimes \mathscr{Z} \tag{4-7}$$

is at most 2. The proof will combine Step 1 with the idea of Proposition 3.2(iv) $\Rightarrow$ (iii).

For $t \in [0, 1]$, we let $\mathrm{ev}_t : \mathscr{Z}_{p,q} \to M_p \otimes M_q$ denote the point-evaluation at t, while we also let

$$\overline{\mathrm{ev}}_0 : \mathscr{Z}_{p,q} \to M_p \quad \text{and} \quad \overline{\mathrm{ev}}_1 : \mathscr{Z}_{p,q} \to M_q$$

denote the irreducible representations which satisfy

$$\mathrm{ev}_0(\,\cdot\,) = \overline{\mathrm{ev}}_0(\,\cdot\,) \otimes 1_{M_q} \quad \text{and} \quad \mathrm{ev}_1(\,\cdot\,) = 1_{M_p} \otimes \overline{\mathrm{ev}}_1(\,\cdot\,).$$

Let $\mathscr{F} \subset C(X, \mathscr{Z}_{p,q})$ be the finite set to approximate, and let $\epsilon > 0$ be the tolerance. Let us assume that $\mathscr{F}$ consists of contractions. Let $\mathscr{U}$ be an open cover of $X \times [0, 1]$, such that, for all $f \in \mathscr{F}$ and $U \in \mathscr{U}$, if

$(x,t),(x',t') \in U$, then

$$\|\mathrm{ev}_t(f(x)) - \mathrm{ev}_{t'}(f(x'))\| < \epsilon/2.$$

Let us also assume that no $U \in \mathcal{U}$ intersects both $X \times \{0\}$ and $X \times \{1\}$.

Using Step 1 (with $X \times [0,1]$ in place of X) and Proposition 3.2(iv), we may find a 3-colourable $\epsilon/2$-approximate partition of unity

$$(a_j^{(i)})_{i=0,1,2;j=0,\dots,r} \subset C(X \times [0,1]) \otimes \mathcal{Z}$$

subordinate to $\mathcal{U}$, and such that

$$\sum a_j^{(i)} \leq 1.$$

Upon replacing $\mathcal{U}$ by a subcover, we may clearly assume that $\mathcal{U}$ is of the form $(U_j^{(i)})_{i=0,1,2;j=0,\dots,r}$, with the support of each $a_j^{(i)}$ being contained in $U_j^{(i)}$.

For each i, j, we shall choose a matrix algebra $F_j^{(i)}$ and produce maps

$$C(X, \mathcal{Z}_{p,q}) \xrightarrow{\psi_j^{(i)}} F_j^{(i)} \xrightarrow{\phi_j^{(i)}} C(X, \mathcal{Z}_{p,q}) \otimes \mathcal{Z}.$$

We distinguish three cases, depending on properties of the set $U_j^{(i)} \in \mathcal{U}$. In every case, we arrange that

$$\phi_j^{(i)}\psi_j^{(i)}(f) = a_j^{(i)} \otimes \mathrm{ev}_{t_j^{(i)}}(f(x_j^{(i)})),$$

where $(x_j^{(i)}, t_j^{(i)})$ is a point from $U_j^{(i)}$, and we make sense of the right-hand side by using the canonical identification of $C(X, \mathcal{Z}_{p,q}) \otimes \mathcal{Z}$ with a subalgebra of

$$C(X \times [0,1]) \otimes \mathcal{Z} \otimes M_p \otimes M_q$$

(determined by boundary conditions at $X \times \{0\}$ and at $X \times \{1\}$).

Case 1. If $U_j^{(i)} \cap (X \times \{0\}) \neq \varnothing$, let $(x_j^{(i)}, t_j^{(i)} = 0)$ be a point in this intersection. We set $F_j^{(i)} := M_p$ and define

$$\psi_j^{(i)}(f) := \overline{\mathrm{ev}}_0(f(x_j^{(i)})),$$
$$\phi_j^{(i)}(T) := a_j^{(i)} \otimes T \otimes 1_{M_q}.$$

By assumption, $U_j^{(i)} \cap (X \times \{1\}) = \varnothing$, so, for all $x \in X$,

$$\mathrm{ev}_1(\phi_j^{(i)}(T)(x)) = 0,$$

and therefore, the range of $\phi_j^{(i)}$ lies in $C(X, \mathcal{Z}_{p,q}) \otimes \mathcal{Z}$.

Case 2. If $U_j^{(i)} \cap (X \times \{1\}) \neq \varnothing$, as in Case 1, let $(x_j^{(i)}, t_j^{(i)} = 1)$ be a point in this intersection. We set $F_j^{(i)} := M_q$ and define

$$\psi_j^{(i)}(f) := \overline{\mathrm{ev}}_1(f(x_j^{(i)})) \quad \text{and} \quad \phi_j^{(i)}(T) := a_j^{(i)} \otimes 1_{M_p} \otimes T.$$

Case 3. If $U_j^{(i)} \cap (X \times \{0\}) = \varnothing$ and $U_j^{(i)} \cap (X \times \{1\}) = \varnothing$, then let $(x_j^{(i)}, t_j^{(i)})$ be any point in $U_j^{(i)}$. We set $F_j^{(i)} := M_p \otimes M_q$ and define

$$\psi_j^{(i)}(f) := \mathrm{ev}_{t_j^{(i)}}(f(x_j^{(i)})) \quad \text{and} \quad \phi_j^{(i)}(T) := a_j^{(i)} \otimes T.$$

We now set $F := \bigoplus_{i,j} F_j^{(i)}$ and use $(\psi_j^{(i)})$ and $(\phi_j^{(i)})$ to define

$$C(X, \mathcal{Z}_{p,q}) \xrightarrow{\psi} F \xrightarrow{\phi} C(X, \mathcal{Z}_{p,q}) \otimes \mathcal{Z}.$$

We have that ψ is c.p.c. since all of its components are. Each $\phi_j^{(i)}$ is c.p. and order zero. For each i, j_1, j_2, if $j_1 \neq j_2$, the images of $\phi_{j_1}^{(i)}$ and $\phi_{j_2}^{(i)}$ are orthogonal. Thus, for each i,

$$\phi|_{\bigoplus_j F_j^{(i)}}$$

is order zero. Also, $\phi(1) = \sum a_j^{(i)} \leq 1$, so that ϕ is contractive.

Finally, let $f \in \mathcal{F}$ and let us check that $\phi\psi(f) =_\epsilon f$. As in the proof of Proposition 3.2(iv) $\Rightarrow$ (iii), we have for each i, j that if $x \in U_j^{(i)}$, then

$$\mathrm{ev}_{t_j^{(i)}}(f(x_j^{(i)})) =_{\epsilon/2} \mathrm{ev}_t(f(x)),$$

and therefore,

$$\mathrm{ev}_t(f(x)) - \frac{\epsilon}{2} \cdot 1_{M_p \otimes M_q} \leq \mathrm{ev}_{t_j^{(i)}}(f(x_j^{(i)})) \leq \mathrm{ev}_t(f(x)) + \frac{\epsilon}{2} \cdot 1_{M_p \otimes M_q}.$$

Since $a_j^{(i)}$ commutes with f, this gives

$$\begin{aligned} a_j^{(i)}(x,t)\Big(\mathrm{ev}_t(f(x)) - \tfrac{\epsilon}{2} \cdot 1_{M_p \otimes M_q}\Big) &\leq a_j^{(i)}(x,t)\mathrm{ev}_{t_j^{(i)}}(f(x_j^{(i)})) \\ &\leq a_j^{(i)}(x,t)\Big(\mathrm{ev}_t(f(x)) + \tfrac{\epsilon}{2} \cdot 1_{M_p \otimes M_q}\Big). \end{aligned}$$

Moreover, since $a_j^{(i)}$ vanishes outside of $U_j^{(i)}$, these inequalities continue to hold for all $x \in X$ and all $t \in [0, 1]$.

Summing over i, j, we find that

$$\begin{aligned} \sum_{i,j} a_j^{(i)}(x,t)\Big(\mathrm{ev}_t(f(x)) - \tfrac{\epsilon}{2} \cdot 1_{M_p \otimes M_q}\Big) &\leq \sum_{i,j} a_j^{(i)}(x,t)\mathrm{ev}_{t_j^{(i)}}(f(x_j^{(i)})) \\ &\leq \sum_{i,j} a_j^{(i)}(x,t)\Big(\mathrm{ev}_t(f(x)) + \tfrac{\epsilon}{2} \cdot 1_{M_p \otimes M_q}\Big), \end{aligned}$$

and therefore

$$\mathrm{ev}_t(f(x)) =_{\epsilon/2} \sum_{i,j} a_j^{(i)}(x,t)\mathrm{ev}_t(f(x)) =_{\epsilon/2} \sum_{i,j} a_j^{(i)}(x,t)\mathrm{ev}_{t_j^{(i)}}(f(x_j^{(i)})) = \mathrm{ev}_t(\phi\psi(f)(x)).$$

Since this holds for all $x \in X, t \in [0, 1]$, this means that $\|f - \phi\psi(f)\| < \epsilon$, as required. □

Remark 4.9. Here we describe how one can give a shorter proof that $A \otimes \mathcal{Z}$ has decomposition rank at most 5, for A as in Theorem 4.1. Since $A \otimes \mathcal{Z}$ is an inductive limit of $A \otimes \mathcal{Z}_{p^\infty,q^\infty}$, it suffices to show that the latter has decomposition rank at most 5. This algebra is a $C([0,1])$-algebra whose fibres are all of the form $A \otimes U$, where U is a self-absorbing UHF algebra. Hence, by [Carrión 2011, Lemma 3.1], $A \otimes \mathcal{Z}_{p^\infty,q^\infty}$ has decomposition rank at most $5 = (\dim[0,1]+1)(2+1)-1$ if we show that $A \otimes U$ has decomposition rank at most 2 for every infinite dimensional, self-absorbing UHF algebra.

As in the first paragraph of the proof above, it suffices to show that the first-factor embedding $C(X,U) \to C(X,U) \otimes U$ has decomposition rank at most 2, when $X = [0,1]^m$. Since U is a limit of finite dimensional C*-algebras, by Proposition 2.4(i), the decomposition rank of this first-factor embedding agrees with the decomposition rank of the first-factor embedding $\iota_0 : C(X) \to C(X) \otimes U$. Then following Step 1 of the above proof verbatim, except with U in place of $\mathcal{Z}$, shows that this ι_0 has decomposition rank at most 2; moreover, we only need to use the variation of Lemma 4.7 where $\mathcal{Z}$ is replaced by U, and, as explained in Remark 4.8, the proof of that lemma can be simplified in that case.

5. $\mathcal{Z}$-stable $C(X)$-algebras

The proof of [Carrión 2011, Lemma 3.1] actually shows the following.

Lemma 5.1. *Let X be a compact metric space, let A be a $C(X)$-algebra, and let B be a unital* C**-algebra. Denote by $\iota_{C(X)} : C(X) \to C(X) \otimes B$ and $\iota_A : A \to A \otimes B$ the first-factor embeddings. Then*

$$\operatorname{dr} \iota_A \le (\operatorname{dr} \iota_{C(X)} + 1)\big(\max_{x \in X} \operatorname{dr} A(x) + 1\big) - 1 \tag{5-1}$$

and

$$\dim_{\mathrm{nuc}} \iota_A \le (\dim_{\mathrm{nuc}} \iota_{C(X)} + 1)\big(\max_{x \in X} \dim_{\mathrm{nuc}} A(x) + 1\big) - 1. \tag{5-2}$$

Proof. Although this is essentially the same as the proof of [Carrión 2011, Lemma 3.1], we provide a detailed proof of (5-1) for the reader's convenience.

Set $k := \max_{x \in X} \operatorname{dr} A(x)$ and $l := \operatorname{dr} \iota_{C(X)}$. Let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$. Without loss of generality, $\mathcal{F}$ consists of self-adjoint contractions. As shown in the proof of [Carrión 2011, Lemma 3.1], there exists an open cover $\mathcal{U}$ of X such that, for each $U \in \mathcal{U}$, there exists a finite dimensional C*-algebra F_U and c.p.c. maps $\psi_U : A \to F_U$, $\phi_U : F_U \to A$ such that ϕ_U is $(k+1)$-colourable and $\phi_U \psi_U(a)(x) =_{\epsilon/2} a(x)$ for all $a \in \mathcal{F}$ and all $x \in U$. By Proposition 3.2(iv), let $(b_j^{(i)})_{j=1,\dots,r;\, i=0,\dots,l} \subset C(X) \otimes B$ be an $(l+1)$-colourable, $\epsilon/2$-approximate partition of 1, subordinate to $\mathcal{U}$, and, by a rescaling argument, we may assume $b_j^{(i)} \le 1$ for each i, j. Hence, for each i, j, we may pick some $U_j^{(i)} \in \mathcal{U}$ containing the support of $b_j^{(i)}$. Define

$$\psi := \bigoplus_{i,j} \psi_{U_j^{(i)}} : A \to \bigoplus_{i,j} F_{U_j^{(i)}}$$

and $\phi : \bigoplus_{i,j} F_{U_j^{(i)}} \to A \otimes B$ by

$$\phi((a_j^{(i)})) := \sum_{i,j} \phi_{U_j^{(i)}}(a_j^{(i)}) \otimes b_j^{(i)}.$$

One readily verifies that ϕ is $(k+1)(l+1)$-colourable, and, as in the proof of Proposition 3.2(iv) $\Rightarrow$ (iii), that $\phi\psi(a) =_\epsilon a \otimes 1_B$ for all $a \in \mathcal{F}$. ☐

Corollary 5.2. *If A is a $\mathcal{Z}$-stable $C(X)$-algebra whose fibres have decomposition rank* (*respectively nuclear dimension*) *bounded by M, the decomposition rank* (*respectively nuclear dimension*) *of A is at most* $3(M+1)-1$.

Proof. We shall apply Lemma 5.1 with $\mathcal{Z}$ in place of B. Using the notation of Lemma 5.1, Theorem 4.1 tells us that $\dim_{\text{nuc}} \iota_{C(X)} \leq \text{dr}\, \iota_{C(X)} \leq 2$. Thus, if the fibres of A have decomposition rank at most M, then, by Lemma 5.1, $\text{dr}\, \iota_A \leq (2+1)(M+1)-1$. ☐

This shows in particular that the $C(X)$-algebra in [Hirshberg et al. 2007, Example 4.7] (which is $\mathcal{Z}$-stable by [Dadarlat and Toms 2009]) has decomposition rank at most 2, and that the $C(X)$-algebra E in [Dadarlat 2009b, Section 3] (which is $\mathcal{Z}$-stable since it is an extension of patently $\mathcal{Z}$-stable C*-algebras) has nuclear dimension at most 5. On the other hand, the $C(X)$-algebra in [Hirshberg et al. 2007, Example 4.8] is not $\mathcal{Z}$-stable, and it is shown in [Robert and Tikuisis 2013, Section 7.4] that it does not have finite nuclear dimension.

Another immediate application is the following strengthening of Theorem 4.1. See [Dadarlat and Pennig 2013] for a discussion of $C(X)$-algebras with fibres $\mathcal{D} \otimes \mathcal{K}$, where $\mathcal{D}$ is either $\mathcal{Z}$ or an infinite dimensional UHF algebra.

Corollary 5.3. *If A is a $\mathcal{Z}$-stable $C(X)$-algebra whose fibres are all AF algebras tensored by $\mathcal{Z}$, then* $\text{dr}\, A \leq 2$.

Proof. It suffices to show that $B := A \otimes \mathcal{Z}_{p^\infty, q^\infty}$ has decomposition rank at most 2. Note that B is a $\mathcal{Z}$-stable $C(X \times [0, 1])$-algebra with AF fibres. Therefore, by Corollary 5.2, $\text{dr}\, B$ is at most $3(0+1)-1 = 2$, as required. ☐

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