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# THE GLOBAL STABILITY OF THE MINKOWSKI SPACETIME SOLUTION TO THE EINSTEIN-NONLINEAR SYSTEM IN WAVE COORDINATES

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We study the coupling of the Einstein field equations of general relativity to a family of nonlinear electromagnetic field equations. The family comprises all covariant electromagnetic models that satisfy the following criteria: (i) they are derivable from a sufficiently regular Lagrangian; (ii) they reduce to the standard Maxwell model in the weak-field limit; (iii) their corresponding energy-momentum tensors satisfy the dominant energy condition. Our main result is a proof of the global nonlinear stability of the $(1+3)$-dimensional Minkowski spacetime solution to the coupled system for any member of the family, which includes the standard Maxwell model. This stability result is a consequence of a small-data global existence result for a reduced system of equations that is equivalent to the original system in our wave-coordinate gauge. Our analysis of the spacetime metric components is based on a framework recently developed by Lindblad and Rodnianski, which allows us to derive suitable estimates for tensorial systems of quasilinear wave equations with nonlinearities that satisfy the weak null condition. Our analysis of the electromagnetic fields, which satisfy quasilinear first-order equations that have a special null structure, is based on an extension of a geometric energy-method framework developed by Christodoulou together with a collection of pointwise decay estimates for the Faraday tensor developed in the article. We work directly with the electromagnetic fields and thus avoid the use of electromagnetic potentials.

## 1. Introduction

The Einstein field equations are the fundamental equations of general relativity. They connect the *Einstein tensor* $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, which contains information about the curvature of spacetime[1] $(\mathfrak{M}, g_{\mu\nu})$, to the energy-momentum-stress-density tensor (energy-momentum tensor for short) $T_{\mu\nu}$, which contains information about the matter present in $\mathfrak{M}$. Here, $g_{\mu\nu}$ is the *spacetime metric*, $R_{\mu\nu}$ is the *Ricci curvature tensor* of $g_{\mu\nu}$, and $R = (g^{-1})^{\kappa\lambda}R_{\kappa\lambda}$ is the *scalar curvature* of $g_{\mu\nu}$. In this article, we show the stability of

[1]By spacetime, we mean a four-dimensional time-orientable Lorentzian manifold $\mathfrak{M}$ together with a Lorentzian metric $g_{\mu\nu}$ of signature $(-, +, +, +)$.

the $(1+3)$-dimensional Minkowski spacetime solution of the Einstein-nonlinear electromagnetic system, which takes the following form relative to an arbitrary coordinate system:

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \qquad (\mu, \nu = 0, 1, 2, 3), \tag{1.0.1a}$$

$$(d\mathscr{F})_{\lambda\mu\nu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{1.0.1b}$$

$$(d\mathscr{M})_{\lambda\mu\nu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3). \tag{1.0.1c}$$

Above, $T_{\mu\nu}$ (see (3.5.4a)) is one of the energy-momentum tensors corresponding to a family of nonlinear models of electromagnetism, $d$ denotes the exterior derivative operator, the two-form $\mathscr{F}_{\mu\nu}$ denotes the *Faraday tensor*, the two-form $\mathscr{M}_{\mu\nu}$ denotes the *Maxwell tensor*, and $\mathscr{M}_{\mu\nu}$ is connected to $(g_{\mu\nu}, \mathscr{F}_{\mu\nu})$ through a constitutive relation (see (3.2.4)). We make the following three assumptions concerning the electromagnetic matter model:

(1) Its Lagrangian $^{\star}\mathscr{L}$ is a scalar-valued function of the two electromagnetic invariants[2]

$$\tfrac{1}{\ell}{}_{(1)} \stackrel{\text{def}}{=} \tfrac{1}{2}(g^{-1})^{\kappa\mu}(g^{-1})^{\lambda\nu}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\mu\nu} \quad \text{and} \quad \tfrac{1}{\ell}{}_{(2)} \stackrel{\text{def}}{=} \tfrac{1}{4}(g^{-1})^{\kappa\mu}(g^{-1})^{\lambda\nu}\mathscr{F}_{\kappa\lambda}{}^{\star}\mathscr{F}_{\mu\nu},$$

where $\star$ denotes the Hodge duality operator corresponding to $g_{\mu\nu}$.

(2) The energy-momentum tensor $T_{\mu\nu}$ corresponding to $^{\star}\mathscr{L}$ satisfies the *dominant energy condition* (sufficient conditions on $^{\star}\mathscr{L}$ are given in (3.3.4a)–(3.3.4b) below).

(3) $^{\star}\mathscr{L}$ is a sufficiently differentiable function of $(\tfrac{1}{\ell}{}_{(1)}, \tfrac{1}{\ell}{}_{(2)})$, and its Taylor expansion around $(0, 0)$ agrees with that of the linear[3] Maxwell–Maxwell[4] equations to first order; i.e., $^{\star}\mathscr{L}(\tfrac{1}{\ell}{}_{(1)}, \tfrac{1}{\ell}{}_{(2)}) = -\tfrac{1}{2}\tfrac{1}{\ell}{}_{(1)} + O^{\ell+2}(|(\tfrac{1}{\ell}{}_{(1)}, \tfrac{1}{\ell}{}_{(2)})|^2)$, where $\ell \geq 10$ is an integer; see Section 2.13 regarding the notation $O^{\ell+2}(\,\cdot\,)$.

The fundamental results in [Fourès-Bruhat 1952; Choquet-Bruhat and Geroch 1969] together imply that the system (1.0.1a)–(1.0.1c) has an initial-value problem formulation in which suitably regular initial data launch a unique maximal globally hyperbolic development. Roughly speaking, the maximal globally hyperbolic development, which is uniquely determined up to isomorphism, is the largest possible solution to the equations that is uniquely determined by the data. However, the results cited are abstract in the sense that they do not provide any detailed quantitative information about the global structure of the maximal globally hyperbolic development. In particular, the results do not address the question of whether the resulting spacetime $(\mathfrak{M}, g_{\mu\nu})$ is geodesically complete. The main goal of this article is to provide a detailed qualitative and quantitative description of the global structure of maximal globally hyperbolic developments launched by data near that of the most fundamental solution to (1.0.1a)–(1.0.1c): the vacuum Minkowski spacetime. We briefly summarize our main results here. They are rigorously stated and proved in Section 16.

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[2]Throughout the article, we use Einstein's summation convention in that repeated indices are summed over.

[3]By "linear", we mean that the familiar electromagnetic equations of Maxwell are linear on any *fixed* spacetime background $(\mathcal{M}, g_{\mu\nu})$; the coupled Einstein–Maxwell system is highly nonlinear.

[4]Throughout the article, we use the terminology "Maxwell–Maxwell" equations in place of the more common terminology "Maxwell" equations. The justification is that Maxwell's theory is based on the electromagnetic equations (1.0.1b)–(1.0.1c) *and* the constitutive relation $\mathscr{M} = {}^{\star}\mathscr{F}$; in a general covariant nonlinear electromagnetic theory, such as the ones considered in this article, the equations (1.0.1b)–(1.0.1c) survive while the constitutive relation differs from that of Maxwell.

**Main results.** The vacuum Minkowski spacetime solution $\tilde{g}_{\mu\nu} \overset{\text{def}}{=} \text{diag}(-1, 1, 1, 1)$ and $\widetilde{\mathscr{F}}_{\mu\nu} \overset{\text{def}}{=} 0$ ($\mu, \nu = 0, 1, 2, 3$) to the system (1.0.1a)–(1.0.1c) is globally stable. In particular, small perturbations of the trivial initial data corresponding to $(\tilde{g}_{\mu\nu}, \widetilde{\mathscr{F}}_{\mu\nu})$ have maximal globally hyperbolic developments that are geodesically complete. Furthermore, the perturbed solution converges to the Minkowski spacetime solution as the evolution progresses. These conclusions are consequences of a small-data global existence result plus decay estimates for solutions to the *reduced* system (3.7.1a)–(3.7.1c) under the wave-coordinate gauge condition $(g^{-1})^{\kappa\lambda}\Gamma_{\kappa}{}^{\mu}{}_{\lambda} = 0$ ($\mu = 0, 1, 2, 3$), where $(g^{-1})^{\kappa\lambda}\Gamma_{\kappa}{}^{\mu}{}_{\lambda}$ is a contracted Christoffel symbol of $g_{\mu\nu}$. Furthermore, relative to the wave-coordinate system that we construct (i.e., a coordinate system $\{x^{\mu}\}_{\mu=0,1,2,3}$ such that $(g^{-1})^{\kappa\lambda}\Gamma_{\kappa}{}^{\mu}{}_{\lambda} = 0$ ($\mu = 0, 1, 2, 3$)), the system (1.0.1a)–(1.0.1c) is equivalent to the reduced system.

We recall the following standard facts (see, e.g., [Christodoulou 2008; Wald 1984]) concerning the initial data for the system (1.0.1a)–(1.0.1c), which we refer to as "abstract" initial data. The abstract initial data consist of a three-dimensional manifold $\Sigma_0$ together with the following fields on $\Sigma_0$: a Riemannian metric $\mathring{\underline{g}}_{jk}$, a symmetric type-$\binom{0}{2}$ tensor field $\mathring{K}_{jk}$, and a pair of electromagnetic one-forms $\mathring{\mathfrak{D}}_j$ and $\mathring{\mathfrak{B}}_j$ ($j, k = 1, 2, 3$). Furthermore, viable data must satisfy the *Gauss*, *Codazzi*, and *electromagnetic constraint* equations, which are respectively given by

$$\mathring{\underline{R}} - \mathring{K}_{ab}\mathring{K}^{ab} + [(\mathring{\underline{g}}^{-1})^{ab}\mathring{K}_{ab}]^2 = 2T(\widehat{N}, \widehat{N})|_{\Sigma_0}, \tag{1.0.2a}$$

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathcal{D}}}_a\mathring{K}_{bj} - (\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathcal{D}}}_j\mathring{K}_{ab} = T\left(\widehat{N}, \frac{\partial}{\partial x^j}\right)\Big|_{\Sigma_0} \quad (j = 1, 2, 3), \tag{1.0.2b}$$

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathcal{D}}}_a\mathring{\mathfrak{D}}_b = 0, \tag{1.0.3a}$$

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathcal{D}}}_a\mathring{\mathfrak{B}}_b = 0. \tag{1.0.3b}$$

In the above expressions, the indices are lowered and raised with $\mathring{\underline{g}}_{jk}$ and $(\mathring{\underline{g}}^{-1})^{jk}$, $\mathring{\underline{R}}$ denotes the scalar curvature of $\mathring{\underline{g}}_{jk}$, $\mathring{\underline{\mathcal{D}}}$ denotes the Levi-Civita connection corresponding to $\mathring{\underline{g}}_{jk}$, and $\widehat{N}^{\mu}$ is the future-directed unit $g$-normal to $\Sigma_0$ (viewed as an embedded Riemannian submanifold of $(\mathfrak{M}, g_{\mu\nu})$). The one-forms $\mathring{\mathfrak{D}}_j$ and $\mathring{\mathfrak{B}}_j$ together form a geometric decomposition of $\mathscr{F}_{\mu\nu}|_{\Sigma_0}$, and the right-hand sides of (1.0.2a)–(1.0.2b) can be computed (in principle) in terms of $\mathring{\underline{g}}_{jk}$, $\mathring{\mathfrak{D}}_j$, and $\mathring{\mathfrak{B}}_j$ alone; see Section 9.2 for more details concerning the relationship of $\mathring{\mathfrak{D}}_j$ and $\mathring{\mathfrak{B}}_j$ to $\mathscr{F}_{\mu\nu}|_{\Sigma_0}$. The dominant energy condition manifests itself along $\Sigma_0$ as the inequalities $T(\widehat{N}, \widehat{N}) \geq 0$ and $T(\widehat{N}, \widehat{N})^2 - (\mathring{\underline{g}}^{-1})^{ab}T(\widehat{N}, \partial/\partial x^a)T(\widehat{N}, \partial/\partial x^b) \geq 0$.

In this article, we consider the case $\Sigma_0 = \mathbb{R}^3$. We will construct spacetimes of the form $\mathfrak{M} = I \times \mathbb{R}^3$, where $I$ is a time interval and $\Sigma_0$ is a spacelike Cauchy hypersurface in $(\mathfrak{M}, g_{\mu\nu})$. The constraints (1.0.2a)–(1.0.2b) are necessary to ensure that (1.0.1a) can be satisfied along $\Sigma_0$ while the constraints (1.0.3a)–(1.0.3b) are necessary to ensure that the electromagnetic equations (1.0.1b)–(1.0.1c) can be satisfied along $\Sigma_0$. Our stability criteria for the abstract initial data include both decay assumptions at spatial infinity and smallness assumptions. We provide here a description of our decay assumptions at spatial infinity, which are based on the assumptions of [Lindblad and Rodnianski 2010]. Our smallness assumptions will be discussed in detail in Section 10.

***Assumptions on the abstract initial data.*** We assume that there exists a global coordinate chart $x = (x^1, x^2, x^3)$ on $\Sigma_0 = \mathbb{R}^3$, a real number $\kappa > 0$, and an integer $\ell \geq 10$ such that (with $r \stackrel{\text{def}}{=} |x| \stackrel{\text{def}}{=} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ and $j, k = 1, 2, 3$)

$$\mathring{\underline{g}}_{jk} = \delta_{jk} + \mathring{\underline{h}}_{jk}^{(0)} + \mathring{\underline{h}}_{jk}^{(1)}, \tag{1.0.4a}$$

$$\mathring{\underline{h}}_{jk}^{(0)} = \chi(r) \frac{2M}{r} \delta_{jk}, \tag{1.0.4b}$$

$$\mathring{\underline{h}}_{jk}^{(1)} = o^{\ell+1}(r^{-1-\kappa}) \qquad \text{as } r \to \infty, \tag{1.0.4c}$$

$$\mathring{K}_{jk} = o^{\ell}(r^{-2-\kappa}) \qquad \text{as } r \to \infty, \tag{1.0.4d}$$

$$\mathring{\mathfrak{D}}_j = o^{\ell}(r^{-2-\kappa}) \qquad \text{as } r \to \infty, \tag{1.0.4e}$$

$$\mathring{\mathfrak{B}}_j = o^{\ell}(r^{-2-\kappa}) \qquad \text{as } r \to \infty, \tag{1.0.4f}$$

where the meaning of $o^{\ell}(\cdot)$ is described in Section 2.13. The cut-off function $\chi(\cdot)$ in (1.0.4b) is defined in (4.2.1).

The parameter $M$ in (1.0.4a), which is known as the *ADM mass*, is constrained by the following requirements: according to the *positive mass theorem* of Schoen and Yau [1979; 1981] and Witten [1981], under the assumption that $T_{\mu\nu}$ satisfies the dominant energy condition, the only solutions $\mathring{\underline{g}}_{jk}$ and $\mathring{K}_{jk}$ to the constraint equations (1.0.2a)–(1.0.2b) that have an expansion of the form (1.0.4a) with the asymptotic behavior (1.0.4b)–(1.0.4d) either have (i) $M > 0$ or (ii) $M = 0$, in which case the Riemannian manifold $(\Sigma_0, \mathring{\underline{g}}_{jk})$ embeds isometrically into Minkowski spacetime with second fundamental form $\mathring{K}_{jk}$. The groundbreaking work of Christodoulou and Klainerman [1993] (which is discussed further in Section 1.1.1) demonstrated the stability of the Minkowski spacetime solution to the Einstein-vacuum equations in the case that the initial data are *strongly asymptotically flat*, which corresponds to the parameter range $\kappa \geq \frac{1}{2}$ in the above expansions. Our work here, which relies on the alternate framework developed by Lindblad and Rodnianski [2010] (see Section 1.1.1), allows for the parameter range $\kappa > 0$.

In this article, we do not consider the issue of solving the constraint equations. The standard method for solving the constraint equations is called the conformal method. For a detailed discussion of this method, see, e.g., [Choquet-Bruhat and York 1980]. Roughly speaking, in this approach, part of the data can be specified freely, and the constraint equations imply nonlinear elliptic PDEs for the remaining part. To the best of our knowledge, under the restrictions on $^{\star}\mathscr{L}$ described at the beginning of Section 1, there are presently no rigorous results concerning the construction of initial data on the manifold $\mathbb{R}^3$ that satisfy the constraints. However, we remark that, for the Einstein-vacuum equations $T_{\mu\nu} \equiv 0$, initial data that satisfy the constraints and that coincide with the standard Schwarzschild data (written here relative to isotropic coordinates)

$$\mathring{\underline{g}}_{jk} = \left(1 + \frac{M}{2r}\right)^4 \delta_{jk} \quad (j, k = 1, 2, 3), \tag{1.0.5a}$$

$$\mathring{K}_{jk} = 0 \qquad\qquad (j, k = 1, 2, 3) \tag{1.0.5b}$$

outside of the unit ball centered at the origin were shown to exist in [Chruściel and Delay 2002a; 2002b; Corvino 2000]. We remark that the stability of the Minkowski spacetime solution to the Einstein-vacuum equations for such data follows from the methods of the aforementioned works [Christodoulou and Klainerman 1993], [Lindblad and Rodnianski 2010] (and its precursor [2005]), and also from the *conformal method* approach of Friedrich [1986] (this is not the same conformal method that was mentioned above in connection with the constraint equations).

**Remark 1.1.** The only role of the dominant energy condition in this article is to ensure the physical condition $M \geq 0$; we assume this physical condition throughout the article. However, only the smallness of $|M|$ is needed to prove our global stability result; the sign of $M$ does not enter into our stability analysis for solutions to the evolution equations. In particular, if there existed small initial data with small negative ADM mass $M$, we would still be able to prove that the corresponding solution to the evolution equations exists globally. Similarly, if we made the replacement $T_{\mu\nu} \to -T_{\mu\nu}$ in the reduced equations (3.7.1a)–(3.7.1c), we could still prove a small-data global existence result.

## 1.1. *Comparison with previous work.*

**1.1.1.** *Mathematical comparisons.* Our result is an extension of a large and growing hierarchy of global stability results for the $(1 + 3)$-dimensional Minkowski spacetime solution to the Einstein equations. The hierarchy began with the celebrated work of Christodoulou and Klainerman [1993], who proved stability in the case of the Einstein-vacuum equations (i.e., $T_{\mu\nu} \equiv 0$). Klainerman and Nicolò [2003] gave a second proof of this result using alternate (but related) techniques. Both of these proofs used a manifestly covariant framework for the formulation of the equations and the derivation of estimates. However, mathematically speaking, the closest relatives to the present article are the seminal works [2005; 2010], in which Lindblad and Rodnianski developed a technically simpler framework for showing the stability of the Minkowski spacetime solution of the Einstein-scalar field system using a *wave-coordinate* gauge. As we previously mentioned, a wave-coordinate gauge is a coordinate system in which the contracted Christoffel symbols $(g^{-1})^{\kappa\lambda}\Gamma_{\kappa}{}^{\mu}{}_{\lambda}$ completely vanish. Relative to such a coordinate system, the Einstein-vacuum equations are equivalent to a *reduced* system comprising quasilinear wave equations for the components $g_{\mu\nu}$; in the present article, the analogous equation is (3.7.1a). In her celebrated result [1952], Choquet-Bruhat used wave coordinates to prove local well-posedness for the Einstein equations. However, because of the logarithmic divergences discussed below in Section 1.2.4 and because of the delicate nonlinearities in the Einstein equations, it was unexpected (see, e.g., [Choquet-Bruhat 1973]) that the wave-coordinate approach of [Lindblad and Rodnianski 2005; 2010] for proving the global stability of Minkowski spacetime is in fact viable. We remark that although the decay estimates of [Lindblad and Rodnianski 2005; 2010] are not as precise as those of [Christodoulou and Klainerman 1993; Klainerman and Nicolò 2003], these works are *much* shorter than their predecessors yet are robust enough to allow for modifications, including the presence of the nonlinear electromagnetic fields examined in this article. We also remark that many of the technical results we need are contained in [Lindblad and Rodnianski 2005; 2010], and we will often direct the reader to these works for their proofs.

Other stability results in this vein include [2000], in which Zipser extended the framework of [Christodoulou and Klainerman 1993] to show the stability of the Minkowski spacetime solution to the Einstein–Maxwell system, and [2007], in which Bieri weakened the assumptions of [Christodoulou and Klainerman 1993] on the decay of the initial data at spatial infinity. We also mention the work [2008] (see also [2006; 2009]), in which Loizelet used the framework of [Lindblad and Rodnianski 2005; 2010] to demonstrate the stability of the Minkowski spacetime solution of the Einstein-scalar field-Maxwell system in $1 + n$ ($n \geq 3$) dimensions. Moreover, in spacetimes of dimension $1 + n$, with $n \geq 5$ odd, it has been shown [Choquet-Bruhat et al. 2006] that a conformal method (distinct from the one used by Friedrich) can be used to show the stability of the Minkowski spacetime solution to the Einstein–Maxwell system for initial data that coincide with the standard Schwarzschild data outside of a compact set. Roughly speaking, a conformal method is a way of mapping a global existence problem into a local existence problem by working with rescaled solution variables. When a conformal method is viable, it tends to give very precise information concerning the asymptotics of the global solutions. In particular, the results of [Choquet-Bruhat et al. 2006] provide a more detailed description of the asymptotics than the results of [Loizelet 2008].

We now compare the amount of regularity and decay that we require on the data to the amount required in the alternate frameworks. The Christodoulou and Klainerman [1993], Zipser [2000], and Klainerman and Nicolò [2003] proofs required two derivatives on the curvature (i.e., four derivatives on the metric). Furthermore, the initial metric was required to be strongly asymptotically flat in the sense described above. Zipser's proof required (in addition) three derivatives on the Faraday tensor. Bieri's [2007] proof required only one derivative on the curvature (i.e., three derivatives on the metric), and it allowed for very slow decay of the data at spatial infinity: $\mathring{\underline{g}}_{jk} = \delta_{jk} + o^3(r^{-1/2})$ and $\mathring{K}_{jk} = o^2(r^{-3/2})$. The present article is less efficient: we require 11 derivatives on the metric and 10 derivatives on the Faraday tensor. We also require asymptotic flatness in the sense of (1.0.4a)–(1.0.4f), which is in between the decay required by Christodoulou and Klainerman and Bieri. Our assumptions are similar to the ones made by Lindblad and Rodnianski [2010] and Loizelet [2008]. For example, in $n \geq 3$ spatial dimensions, Loizelet's proof required $7 + 2\lfloor (n+2)/2 \rfloor$ derivatives of the metric. The main focus of the Lindblad–Rodnianski wave-coordinate approach is on providing a technically simpler approach to the proof of stability as opposed to a proof that closes at a low regularity level. There are at least two ways in which the wave-coordinate approach is suboptimal from the point of view of the number of derivatives. The first is that all product nonlinearities are estimated in $L^2$ on constant-time hypersurfaces from only $L^2 - L^\infty$ estimates with no use of intermediate $L^p$ norms, norms on other hypersurfaces,[5] or Calderón–Zygmund theory. That is, all nonlinear products are estimated in spatial $L^2$ by bounding the factor with the most derivatives on it in $L^2$ and all other factors in $L^\infty$. For quadratic terms, this means that we must be able to bound approximately half of the total number of derivatives in $L^\infty$. This approach stands in contrast to the approaches of [Christodoulou and Klainerman 1993; Zipser 2000; Klainerman and Nicolò 2003; Bieri 2007], where, e.g., intermediate $L^p$ norms and other hypersurface integrals played an important role in the analysis. The second source of suboptimality comes from the version of the weighted Klainerman–Sobolev inequality that we use (see Section 1.2.7 and (1.2.10)). This

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[5]As is explained in Section 1.2.6, our proof of global stability also makes use of the positivity of certain time integrals of the $L^2$ integrals (i.e., positive spacetime integrals) that arise in our energy identities.

inequality allows one to estimate a weighted $L^\infty$ norm of a function by weighted $L^2$ norms of *up-to-order-three* weighted derivatives. The reason that three derivatives are used (instead of the familiar two derivatives of standard Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$) is that this allows one to avoid putting more than one derivative on the weight function (the at-most-two other derivatives are rotations, which pass through the weight function); see the proof given in [Lindblad and Rodnianski 2010, Proposition 14.1].

We also emphasize the following point: the techniques used in this article to analyze the electromagnetic fields differ in a fundamental way from those used by Loizelet [2008]. Our methods are closer in spirit to (though distinct from) the methods used by Zipser [2000]. More specifically, Loizelet [2008] analyzed the standard Maxwell–Maxwell equations through the use of a four-potential[6] $A_\mu$ satisfying the *Lorenz gauge* condition $(g^{-1})^{\kappa\lambda}\mathscr{D}_\kappa A_\lambda = 0$, where $\mathscr{D}$ is the Levi-Civita connection corresponding to $g_{\mu\nu}$. In Loizelet's analysis of the Maxwell–Maxwell equations, the Lorenz gauge leads to a diagonal system of semilinear-in-$A_\mu$ wave equations for the components $A_\mu$. Furthermore, these equations can be analyzed by using the same techniques that are used in the study of the components of the metric (see (3.7.1a)) and the scalar field. In particular, in Loizelet's analysis, Lemma 12.2 can be used to deduce suitable weighted energy estimates for the components $\nabla_\mu A_\nu$. In contrast, as discussed in [Speck 2012], it is not clear that the Lorenz gauge can be used to analyze the kinds of quasilinear-in-$\mathscr{F}$ electromagnetic field equations (1.0.1c) studied in this article. More specifically, it is not clear that the Lorenz gauge in general leads to a hyperbolic formulation of the electromagnetic equations that is suitable for deriving the kinds of $L^2$ energy estimates needed for our analysis. For this reason, throughout this article, we work directly with the Faraday tensor. In particular, as described in detail in Section 8, we use Christodoulou's [2000] geometric framework to construct *energy currents* that can be used to derive the kinds of $L^2$ estimates needed in our analysis. Using these methods, we prove Lemma 12.1, which compensates for the fact that Lemma 12.2 is not generally available for controlling the electromagnetic quantities. We remark that there is another advantage to working directly with the Faraday tensor: *our smallness condition for stability depends only on the physical field variables and not on auxiliary mathematical quantities such as the components $\nabla_\mu A_\nu$.*

Now roughly speaking, the reason that we are able to prove our main stability result is because, in our wave-coordinate gauge, the nonlinear terms have a special algebraic structure, which Lindblad and Rodnianski [2003] have labeled *the weak null condition*; see Section 1.2.5 for additional details. We remark that, in order for small-data global existence to hold, it is essential that the quadratic nonlinearities have special structure: John's [1981] blow-up result shows that quadratic perturbations[7] of the homogeneous linear wave equation in $(1 + 3)$-dimensional Minkowski spacetime (of which our equations (1.2.4a) below are an example) *can sometimes lead to finite-time blow-up even for arbitrarily small data*. Now by definition, a system of PDEs satisfies the weak null condition if the corresponding *asymptotic system* has small-data global solutions. Roughly, the asymptotic system is obtained by keeping only the quadratic terms with both factors involving derivatives that are transversal to the outgoing Minkowskian null cones and the related linear term that drives their evolution along those cones (see the discussion in Section 1.2.4); the discarded terms are expected to decay faster than the remaining terms. The general philosophy is that,

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[6]Recall that a four-potential is a one-form $A_\mu$ such that $\mathscr{F}_{\mu\nu} = (dA)_{\mu\nu}$.

[7]In [John 1981], it was shown that both semilinear and quasilinear quadratic perturbations can lead to small-data blow-up.

if the asymptotic system has small-data global existence, then one should be hopeful that the original system does too. Lindblad and Rodnianski [2010] showed that the asymptotic system corresponding to the Einstein-scalar field system in wave coordinates has global solutions for small (i.e., near-Minkowskian) data. Although we do not carry out such an analysis in this article, we remark that it can be checked that the asymptotic system[8] corresponding to the Einstein-nonlinear electromagnetic system in wave coordinates also has global solutions for small data. This was our original motivation for pursuing the present work.

The aforementioned weak null condition is a generalization of the classic *null condition* of Klainerman [1986] (see also [Christodoulou 1986]), in which the quadratic nonlinearities are *standard null forms* (which are defined below in the statement of Lemma 3.8). We remark that standard null forms have a very favorable structure and are completely discarded when one forms the asymptotic system. By now, there is a very large body of global existence and almost-global existence results that are based on the analysis of nonlinearities that satisfy generalizations of Klainerman's null condition. This includes the global stability results for the Einstein equations mentioned above but also many other results; there are far too many to list exhaustively, but we mention the following as examples: [Katayama 2005; Klainerman and Sideris 1996; Lindblad 2004; 2008; Metcalfe and Sogge 2007; Metcalfe et al. 2005; Sideris 1996; Speck 2012].

**1.1.2.** *Connections to the "divergence" problem.* One of the most important unresolved issues in physics is that of the so-called "divergence problem". In the setting of classical electrodynamics on the Minkowski spacetime background, this problem manifests itself as the unhappy fact that the standard Maxwell–Maxwell equations with *point-charge* sources (i.e., delta-function source terms modeling the point charges) together with the *Lorentz force law*[9] (which is supposed to drive the motion of the point charges) do not form a well-defined system of equations. This is because the theory dictates that the Lorentz force at the location of a point charge is "infinite in all directions" so that the charge's motion is ill-defined. A further symptom of the divergence problem in this theory is that the energy of a static point charge is infinite. Moreover, our present-day flagship model of quantum electrodynamics (QED), which is based on a quantization of the classical Maxwell–Dirac field equations, has not yet fixed the crux of the problem; similar manifestations of the divergence problem arise in QED; see [Kiessling 2004a; 2004b] for a detailed discussion of these issues.

Now in [2004a; 2004b], Kiessling has taken a preliminary step in the direction of resolving the divergence problem by reconsidering classical electrodynamics in Minkowski spacetime. One of Kiessling's primary strategies is to follow the lead of Max Born [1933] by replacing the standard Maxwell–Maxwell equations with a suitable nonlinear system, the hope being that it will be possible to make rigorous mathematical sense of the motion of point charges in the nonlinear theory. Kiessling's leading candidate is the Maxwell–Born–Infeld (MBI) model of classical electromagnetism, which was proposed by Born and Infeld [1934] based on Born's [1933] earlier ideas. The electromagnetic Lagrangian for this model is

$$^{\star}\mathscr{L}_{(\text{MBI})} \overset{\text{def}}{=} \frac{1}{\beta^4} - \frac{1}{\beta^4}\big(1 + \beta^4 \mathscr{I}_{(1)} - \beta^8 \mathscr{I}_{(2)}^2\big)^{1/2} = \frac{1}{\beta^4} - \frac{1}{\beta^4}(\det_g(g + \beta^2 \mathscr{F}))^{1/2}, \tag{1.1.1}$$

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[8]To obtain this asymptotic system, one also discards the quadratic terms containing the fast-decaying null components $\alpha[\mathscr{F}]$, $\rho[\mathscr{F}]$, and $\sigma[\mathscr{F}]$ of the Faraday tensor; see Section 1.2.4.

[9]Recall that the Lorentz force is $F_{\text{Lorentz}} = q[E + v \times B]$, where $q$ is the charge associated to the point charge, $E$ is the electric field, $v$ is the instantaneous point charge velocity, and $B$ is the magnetic induction.

where $\beta > 0$ denotes *Born's "aether" constant*. We point out that, as verified in, e.g., [Speck 2012], this Lagrangian satisfies the assumptions (3.3.3a) and (3.3.4a)–(3.3.4b) so that the main results of this article apply to the MBI model. Now it turns out that it was not enough for Kiessling to simply replace the standard Maxwell–Maxwell equations with the Maxwell–Born–Infeld equations, for such a modification fails to fix the problem of the Lorentz force being ill-defined at the location of the point charge. On the other hand, in MBI theory on the Minkowski spacetime background, there exist *Lipschitz-continuous* electromagnetic potentials corresponding to solutions to the field equations with a single static point-charge source. Kiessling observed that this level of regularity is (just barely) sufficient for a relativistic version of Hamilton–Jacobi theory to be well-defined. He thus proposed a new relativistic Hamilton–Jacobi "guiding law" of motion for the point charges (see [Kiessling 2004a] for the details).

Kiessling's interest in the Maxwell–Born–Infeld system was further motivated by results contained in [Boillat 1970; Plebański 1970], which show that it is the unique[10] theory of classical electromagnetism that is derivable from a Lagrangian and that satisfies the following five postulates (see also the discussions in [Białynicki-Birula 1983; Kiessling 2004a]):

 (i)  The field equations transform covariantly under the Poincaré group.

 (ii)  The field equations are covariant under a Weyl (gauge) group.

(iii)  The electromagnetic energy surrounding a stationary point charge is finite.

(iv)  The field equations reduce to the standard Maxwell–Maxwell equations in the weak field limit.

 (v)  The solutions to the field equations are not birefringent.

We remark that the standard Maxwell–Maxwell system satisfies all of the above postulates except for (iii) and that the MBI system was shown to satisfy (iii) by Born [1933]. Physically, postulate (v) is equivalent to the statement that the "speed of light propagation" is independent of the polarization of the wave fields. Mathematically, this is the postulate that there is only a single *null cone*[11] associated to the electromagnetic equations; in a typical theory of classical electromagnetism, the causal structure of the electromagnetic equations is more complicated than the structure corresponding to a single null cone (see [Speck 2012] for a detailed discussion of this issue in the context of the Maxwell–Born–Infeld equations on the Minkowski spacetime background).

We can now clarify the connection of the present article to Kiessling's work. First, as noted in [2004a], Kiessling expects that his theory can be generalized to the case of a curved spacetime through a coupling to the Einstein equations. Next, we mention that although the Maxwell–Born–Infeld system is Kiessling's leading candidate for an electromagnetic model, he is also considering other models. In particular, by relaxing postulate (v) above, a relaxation that in principle could be supported by experimental evidence, one is led to consider a larger family of electromagnetic models. Now one basic criterion for any viable electromagnetic model is that small, nearly linear-Maxwellian electromagnetic fields in near-Minkowski spacetimes should not lead to a severe breakdown in the structure of spacetime or other degenerate

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[10]More precisely, there is a one-parameter family of such theories indexed by $\beta > 0$.

[11]In general, this "light cone" does not have to coincide with the gravitational null cone although it *does* in the case of the standard Maxwell–Maxwell equations.

behavior. *The present work confirms this criterion* for a large family of electromagnetic models coupled to the Einstein equations, including the Maxwell–Born–Infeld system and many other models that fall under the scope of Kiessling's program.

## 1.2. *Discussion of the analysis.*

**1.2.1.** *The splitting of the spacetime metric and setting up the equations.* As in [Lindblad and Rodnianski 2005; 2010], in order to analyze the spacetime metric, we split it into the following three pieces (where we view $h_{\mu\nu}^{(1)}$ as the "new unknown metric variable"):

$$g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu} \qquad (\mu, \nu = 0, 1, 2, 3), \qquad (1.2.1a)$$

$$h_{\mu\nu} = h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)} \qquad (\mu, \nu = 0, 1, 2, 3), \qquad (1.2.1b)$$

$$h_{\mu\nu}^{(0)} \overset{\text{def}}{=} \chi\left(\frac{r}{t}\right)\chi(r)\frac{2M}{r}\delta_{\mu\nu} \qquad (\mu, \nu = 0, 1, 2, 3), \qquad (1.2.1c)$$

$$\left(h_{\mu\nu}^{(0)}\big|_{t=0} = \chi(r)\frac{2M}{r}\delta_{\mu\nu}, \ \partial_t h_{\mu\nu}^{(0)}\big|_{t=0} = 0\right),$$

where $m_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric and the function $\chi$ plays several roles that will be discussed in Section 1.2.9. Above and throughout, $\chi(z)$ is a fixed cut-off function that satisfies

$$\chi \in C^\infty, \qquad \chi \equiv 1 \text{ for } z \geq \tfrac{3}{4}, \quad \text{and} \quad \chi \equiv 0 \text{ for } z \leq \tfrac{1}{2}. \qquad (1.2.2)$$

*We remark that, here and throughout the rest of the article, unless we explicitly indicate otherwise, all indices on all tensors are lowered and raised with the Minkowski metric $m_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and its inverse $(m^{-1})^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ (as is explained in Section 2.2, we use the symbol # whenever we raise indices with $g^{-1}$).* Furthermore, as in [Lindblad and Rodnianski 2005; 2010], we work in a wave-coordinate system, which is a coordinate system in which the contracted Christoffel symbols $\Gamma^\mu \overset{\text{def}}{=} (g^{-1})^{\kappa\lambda}\Gamma_{\kappa}{}^{\mu}{}_{\lambda}$ (see (3.0.2d)) of $g_{\mu\nu}$ satisfy

$$\Gamma^\mu = 0 \quad (\mu = 0, 1, 2, 3). \qquad (1.2.3)$$

We remark that several equivalent definitions of the wave-coordinate gauge (1.2.3) are discussed in Section 3.1 and that the viability of the wave-coordinate gauge for proving local well-posedness for the system (1.0.1a)–(1.0.1c) (which is a rather standard result based on the fundamental ideas of [Fourès-Bruhat 1952]) is discussed in Section 4.3.

As is discussed in detail in Section 3.7, in a wave-coordinate system $(t, x)$, the equations (1.0.1a)–(1.0.1c) are equivalent to the *reduced equations*

$$\widetilde{\Box}_g h_{\mu\nu}^{(1)} = \mathfrak{H}_{\mu\nu} - \widetilde{\Box}_g h_{\mu\nu}^{(0)} \qquad (\mu, \nu = 0, 1, 2, 3), \qquad (1.2.4a)$$

$$\nabla_\lambda \mathscr{F}_{\mu\nu} + \nabla_\mu \mathscr{F}_{\nu\lambda} + \nabla_\nu \mathscr{F}_{\lambda\mu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3), \qquad (1.2.4b)$$

$$N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda} = \mathfrak{F}^\nu \qquad (\nu = 0, 1, 2, 3), \qquad (1.2.4c)$$

where $\widetilde{\Box}_g = (g^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$ is the reduced wave operator corresponding to $g_{\mu\nu}$, $\nabla$ is the Levi-Civita connection corresponding to the *Minkowski metric $m_{\mu\nu}$,*

$$N^{\#\mu\nu\kappa\lambda} \overset{\text{def}}{=} \tfrac{1}{2}\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa} - h^{\mu\kappa}(m^{-1})^{\nu\lambda}$$
$$+ h^{\mu\lambda}(m^{-1})^{\nu\kappa} - (m^{-1})^{\mu\kappa}h^{\nu\lambda} + (m^{-1})^{\mu\lambda}h^{\nu\kappa}\big) + N_{\triangle}^{\#\mu\nu\kappa\lambda},$$

$N_{\triangle}^{\#\mu\nu\kappa\lambda} = O^{\ell}(|(h,\mathscr{F})|^2)$ is a quadratic error term that depends on the chosen model of nonlinear electromagnetism, and $\mathfrak{H}_{\mu\nu}$ and $\mathfrak{F}^{\nu}$ are inhomogeneous terms that depend in part on the chosen model of nonlinear electromagnetism.

The question of the stability of the Minkowski spacetime solution to (1.0.1a)–(1.0.1c) has thus been reduced to two subquestions: (i) show that the reduced system (1.2.4a)–(1.2.4c), where the unknowns are viewed to be $(h_{\mu\nu}^{(1)}, \mathscr{F}_{\mu\nu})$, has small-data global existence (if the ADM mass $M$ is sufficiently small) and (ii) show that the resulting spacetime $(\mathbb{R}^{1+3}, g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)})$ is geodesically complete. The second question is very much related to the first, for as in [Lindblad and Rodnianski 2005, Section 16] and [Loizelet 2008, Section 9], the question of geodesic completeness can be answered if one has sufficiently detailed information about the asymptotic behavior of $h_{\mu\nu}^{(1)}$; our stability theorem (see Section 16) provides sufficient information. Therefore, the main focus of this article is (i).

**1.2.2.** *The smallness condition.* Our smallness condition on the abstract initial data is stated in terms of the ADM mass $M$ and a weighted Sobolev norm of the field data $\underline{\nabla}_i \underline{\mathring{h}}_{jk}^{(1)}$, $\mathring{K}_{jk}$, $\mathring{\mathfrak{D}}_j$, and $\mathring{\mathfrak{B}}_k$. More specifically, in order to deduce global existence, we will require that

$$E_{\ell;\gamma}(0) + M < \varepsilon_{\ell}, \tag{1.2.5}$$

where $\varepsilon_{\ell} > 0$ is a sufficiently small positive number, $E_{\ell;\gamma}(0) \geq 0$ is defined by

$$E_{\ell;\gamma}^2(0) \overset{\text{def}}{=} \|\underline{\nabla}\underline{\mathring{h}}^{(1)}\|_{H_{1/2+\gamma}^{\ell}}^2 + \|\mathring{K}\|_{H_{1/2+\gamma}^{\ell}}^2 + \|\mathring{\mathfrak{D}}\|_{H_{1/2+\gamma}^{\ell}}^2 + \|\mathring{\mathfrak{B}}\|_{H_{1/2+\gamma}^{\ell}}^2, \tag{1.2.6}$$

the weighted Sobolev norm $\|\cdot\|_{H_{1/2+\gamma}^{\ell}}$ is defined in Definition 10.1 below, $0 < \gamma < \tfrac{1}{2}$ is a constant, and $\ell \geq 10$ is an integer. The condition $\ell \geq 10$ is needed for various weighted Sobolev embedding results, including the weighted Klainerman–Sobolev inequality (1.2.10), and the results stated in Appendix A. In the above expressions, $\underline{\nabla}$ is the Levi-Civita connection corresponding to the Euclidean metric[12] $\underline{m}_{jk} \overset{\text{def}}{=} \text{diag}(1,1,1)$. Note that the assumed fall-off conditions (1.0.4c)–(1.0.4f) guarantee the existence of a constant $0 < \gamma < \tfrac{1}{2}$ such that $E_{\ell;\gamma}(0) < \infty$.

Although the norm (1.2.6) is useful for expressing the small-data global existence condition in terms of quantities inherent to the data, from the perspective of analysis, a more useful quantity is the energy $\mathscr{E}_{\ell;\gamma;\mu}(t) \geq 0$, which is defined by

$$\mathscr{E}_{\ell;\gamma;\mu}^2(t) \overset{\text{def}}{=} \sup_{0 \leq \tau \leq t} \sum_{|I| \leq \ell} \int_{\Sigma_{\tau}} \big\{|\nabla\nabla_{\mathscr{L}}^I h^{(1)}|^2 + |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|^2\big\} w(q) \, d^3x, \tag{1.2.7}$$

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[12]Throughout the article, we use the symbol $\underline{m}$ to denote both the Euclidean metric $\underline{m}_{jk} \overset{\text{def}}{=} \text{diag}(1,1,1)$ on $\mathbb{R}^3$ and the first fundamental form $\underline{m}_{\mu\nu} \overset{\text{def}}{=} \text{diag}(0,1,1,1)$ of the constant time hypersurfaces $\Sigma_t$ viewed as embedded hypersurfaces of Minkowski spacetime; this double-use of notation should not cause any confusion.

where $\nabla$ denotes the Levi-Civita connection corresponding to the *full Minkowski spacetime metric*, $q \overset{\text{def}}{=} |x| - t$ is a null coordinate, the weight function $w(q)$ is defined by

$$w = w(q) = \begin{cases} 1 + (1 + |q|)^{1+2\gamma} & \text{if } q > 0, \\ 1 + (1 + |q|)^{-2\mu} & \text{if } q < 0, \end{cases} \tag{1.2.8}$$

$\gamma$ is from (1.2.6), and $0 < \mu < \frac{1}{2}$ is a fixed constant. In the above expression, $\mathscr{Z} \overset{\text{def}}{=} \{\partial_\mu, x_\mu \partial_\nu - x_\nu \partial_\mu, x^\kappa \partial_\kappa\}_{0 \le \mu \le \nu \le 3}$ is a subset of the conformal Killing fields of Minkowski spacetime, $I$ is a vector field multi-index, $\nabla_{\mathscr{Z}}^I$ represents iterated Minkowski covariant differentiation with respect to vector fields in $\mathscr{Z}$, and $\mathscr{L}_{\mathscr{Z}}^I$ represents iterated Lie differentiation with respect to vector fields in $\mathscr{Z}$. The significance of the set $\mathscr{Z}$ is that it is needed for the weighted Klainerman–Sobolev inequality (1.2.10), which is discussed below.

**Remark 1.2.** The presence of the parameter $\mu > 0$ in (1.2.8) might seem unnecessary as $1 + (1 + |q|)^{-2\mu} \approx 1$. However, as is explained in Section 1.2.6, the presence of $\mu > 0$ ensures that $w'(q) > 0$, an inequality that plays a key role in our energy estimates.

**1.2.3.** *Overall strategy of the proof.* The overall strategy is to deduce a hierarchy of Gronwall-amenable inequalities for the energies $\mathscr{E}_{k;\gamma;\mu}(t)$ ($0 \le k \le \ell$); this is accomplished in (16.2.5) below. The net effect is that, under the assumptions that $E_{\ell;\gamma}(0) + M \le \varepsilon$ and $\varepsilon$ is sufficiently small, we are able to deduce the following a priori estimate for the solution, which is valid during its classical lifetime:

$$\mathscr{E}_{\ell;\gamma;\mu}(t) \le c_\ell \varepsilon (1 + t)^{\tilde{c}_\ell \varepsilon}. \tag{1.2.9}$$

In the above inequality, $c_\ell$ and $\tilde{c}_\ell$ are positive constants. Now it is a standard result in the theory of hyperbolic PDEs that, if $\varepsilon$ is sufficiently small, then an a priori estimate of the form (1.2.9) implies that the solution exists for $(t, x) \in (-\infty, \infty) \times \mathbb{R}^3$; see Proposition 14.1 for more details. Furthermore, as shown in [Lindblad and Rodnianski 2005; Loizelet 2008], if $\varepsilon$ is sufficiently small, then it also follows that the spacetime $(\mathbb{R}^{1+3}, g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)})$ is geodesically complete. *The main goal of this article is therefore to derive* (1.2.9).

**1.2.4.** *Geometry and null decompositions.* Let us now describe the tools used to derive (1.2.9). First and foremost, as mentioned above in Section 1.1.1, the reason we are able to prove our stability result is that the reduced equations (1.2.4a)–(1.2.4c) have a special algebraic structure and satisfy (in the language of Lindblad and Rodnianski) the *weak null condition*. Now in order to see the special structure of the terms in the reduced equations, we use the strategy of Lindblad and Rodnianski and decompose them into their *Minkowskian null components*; we refer to this as a *Minkowskian null decomposition*. We emphasize the following point: *the Minkowskian geometry is not the "correct" geometry to use for analyzing the equations, for the actual characteristics of the system correspond to the null cones of the spacetime metric $g_{\mu\nu}$ and the characteristics of the nonlinear electromagnetic equations (which in general do not have to coincide with the gravitational null cones). However, the errors that we make in using the Minkowskian geometry (which has the advantage of being simple) are controllable.*

We stress that the strategy of using the Minkowskian geometry to prove a global stability result for the Minkowski spacetime solution (in wave coordinates) was a novel (and unexpectedly viable) feature of [Lindblad and Rodnianski 2005; 2010]. The previous works [Christodoulou and Klainerman 1993; Zipser

2000; Klainerman and Nicolò 2003; Bieri 2007] used foliations of spacetime built out of outgoing null cones of the actual spacetime metric $g_{\mu\nu}$, which logarithmically diverge from the corresponding outgoing cones of Minkowski spacetime as $t \to \infty$. The use of the actual geometry allowed these authors to derive sharp estimates for the asymptotic behavior of the perturbed solution. However, this approach required an enormous effort. In addition to (i) constructing geometric foliations, the authors also had to (ii) carefully decompose every term relative to a $g$-null frame, (iii) construct vector fields (with controllable deformation tensors) for commuting the equations, and (iv) use an elaborate collection of elliptic estimates to control the foliations. At the expense of reduced precision, the Lindblad–Rodnianski approach eliminates many of these difficulties: the Minkowskian geometry is very easy to "construct", one only has to carefully decompose "the important terms" relative to the Minkowskian null frame, the vector field differential operators are prespecified, and no elliptic estimates are needed since the foliations are prespecified.

Let us briefly recall the meaning of a Minkowskian null decomposition; a more detailed description is offered in Section 5. The notion of a Minkowskian null decomposition is intimately connected to the following spacetime subsets: the *outgoing Minkowskian null cones* $C_q^+ \stackrel{\text{def}}{=} \{(\tau, y) \mid |y| - \tau = q\}$, the *ingoing Minkowskian null cones* $C_s^- \stackrel{\text{def}}{=} \{(\tau, y) \mid |y| + \tau = s\}$, the *constant time slices* $\Sigma_t \stackrel{\text{def}}{=} \{(\tau, y) \mid \tau = t\}$, and the *Euclidean spheres* $S_{r,t} \stackrel{\text{def}}{=} \{(\tau, y) \mid t = \tau, |y| = r\}$. Observe that the *null coordinate* $q \stackrel{\text{def}}{=} |x| - t$ associated to the spacetime point with coordinates $(t, x)$ is constant on the outgoing cones and the null coordinate $s \stackrel{\text{def}}{=} |x| + t$ is constant on the ingoing cones. These coordinates will be used throughout the article to describe the rates of decay of various quantities. With $\omega^j \stackrel{\text{def}}{=} x^j/r$ ($j = 1, 2, 3$), we also define the *ingoing Minkowskian null geodesic vector field* $\underline{L}^\mu \stackrel{\text{def}}{=} (1, -\omega^1, -\omega^2, -\omega^3)$, which satisfies $m_{\kappa\lambda}\underline{L}^\kappa \underline{L}^\lambda = 0$ and is tangent to the $C_s^-$, and the *outgoing Minkowskian null geodesic vector field* $L^\mu \stackrel{\text{def}}{=} (1, \omega^1, \omega^2, \omega^3)$, which satisfies $m_{\kappa\lambda}L^\kappa L^\lambda = 0$ and $m_{\kappa\lambda}\underline{L}^\kappa L^\lambda = -2$ and is tangent to the $C_q^+$. Furthermore, in a neighborhood of each nonzero spacetime point $p$, there exists a locally defined pair of $m$-orthonormal vector fields $e_1$ and $e_2$ that are tangent to the family of Euclidean spheres and $m$-orthogonal to $\underline{L}$ and $L$. The set $\mathcal{N} \stackrel{\text{def}}{=} \{\underline{L}, L, e_1, e_2\}$, which spans the tangent space at each point, is known as a *Minkowskian null frame*. In the discussion that follows, we will also make use of the set $\mathcal{T} \stackrel{\text{def}}{=} \{L, e_1, e_2\}$, which is the subset consisting of only those frame vectors tangent to the $C_q^+$, and the set $\mathcal{L} \stackrel{\text{def}}{=} \{L\}$.

Given any two-form $\mathcal{F}$, we can decompose it into its Minkowskian null components $\underline{\alpha}[\mathcal{F}]$, $\alpha[\mathcal{F}]$, $\rho[\mathcal{F}]$, and $\sigma[\mathcal{F}]$, where $\underline{\alpha}$ and $\alpha$ are two-forms $m$-tangent[13] to the spheres $S_{r,t}$ and $\rho$ and $\sigma$ are scalars. More specifically, we define $\underline{\alpha}_A = \mathcal{F}_{A\underline{L}}$, $\alpha_A = \mathcal{F}_{AL}$, $\rho = \frac{1}{2}\mathcal{F}_{\underline{L}L}$, and $\sigma = \mathcal{F}_{12}$, where $A \in \{1, 2\}$ and we have abbreviated $\mathcal{F}_{A\underline{L}} \stackrel{\text{def}}{=} e_A^\kappa \underline{L}^\lambda \mathcal{F}_{\kappa\lambda}$, etc. Similarly, we can decompose the tensor $h_{\mu\nu}$ into its null components $h_{LL}$, $h_{\underline{L}L}$, $h_{LT}$, etc., where $T$ stands for any of the vectors in $\mathcal{T}$. We are now ready to discuss one of the major themes running throughout this article: the rates of decay of the various null components of $\mathcal{F}$ and $h$ are distinguished by the kinds of contractions taken against the null frame vectors. In particular, contractions against $L$, $e_1$, and $e_2$ are associated with favorable decay, with $L$ being the most favorable, while contractions against $\underline{L}$ are associated with unfavorable decay. Similarly, differentiation in the directions $L$, $e_1$, and $e_2$ are associated with creating *additional favorable decay* in the null coordinate $s$ while differentiation in the direction $\underline{L}$ is associated with creating less favorable additional decay in $q$

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[13]By $m$-tangent, we mean that their vector duals relative to the Minkowski metric are tangent to the $S_{r,t}$.

(see Lemma 6.16 for a precise version of this claim). Equivalently, the operator $\overline{\nabla}$ creates favorable decay in $s$ while $\nabla$ only creates decay in $q$. Here and throughout, $\overline{\nabla}$ is the null frame projection (of the derivative component only when $\overline{\nabla}$ is applied to a tensor field) of the Minkowski connection $\nabla$ onto the outgoing Minkowski null cones (i.e., $\overline{\nabla}$ projects away the $\underline{L}$ component of $\nabla$). From this point of view, the most dangerous terms in the equations are $\underline{\alpha}$ and $h_{\underline{L}\underline{L}}$ and the $\partial_q \sim \nabla_L$ derivatives (see Section 2.7) of these quantities. We recommend that at this point the reader should examine the conclusions of Propositions 15.6 and 15.7 to get a feel for the kind of decay properties possessed by the various null components.

The main idea behind the Minkowskian null decomposition is that it can be used to show the following fact: *the worst possible combinations of terms, from the point of view of decay rates, are not present in the reduced equations* (1.2.4a)–(1.2.4c). This special algebraic structure, which is of central importance in our small-data global existence proof, is examined in detail in Propositions 11.1–11.4. As revealed in [Lindblad and Rodnianski 2003; 2005; 2010], this special algebraic structure is highly tensorial in nature. A related fact is that various null components of the lower-order derivatives of the solution exhibit a partially decoupled behavior. Moreover, this partial decoupling allows us to derive a hierarchy of "upgraded pointwise decay" estimates for the lower-order derivatives. These estimates, which play an essential role in the proof of our main theorem, provide bounds that *are stronger than the bounds implied by the size of* $\mathscr{E}_{\ell;\gamma;\mu}(t)$. This critical issue is discussed in more detail in Section 1.2.11.

**1.2.5.** *The special structure of the nonlinearities involving the Faraday tensor.* We now briefly summarize the special structures that allow us to extend the results of [Lindblad and Rodnianski 2010] to include small electromagnetic fields. We emphasize the following point: because of our assumptions on the electromagnetic Lagrangian, all of the important nonlinearities (from the point of view of small-data global existence) are the quadratic ones that are present in the case of the standard Maxwell–Maxwell Lagrangian $^\star\mathscr{L}_{(\mathrm{Maxwell})} = -\frac{1}{2}\mathscr{F}_{(1)}$; all of the other electromagnetic theories that are covered by our main theorem introduce cubic and higher-order nonlinearities into the PDEs that are relatively easy to control. We first discuss how the electromagnetic fields couple into the equations for the components of the metric term $h_{\mu\nu} = g_{\mu\nu} - m_{\mu\nu}$. The presence of the electromagnetic fields introduces only one important nonlinear term into these equations: the main $\mathscr{F}$-containing quadratic term $\mathfrak{Q}^{(2;h)}_{\mu\nu}(\mathscr{F}, \mathscr{F})$ on the right-hand side of (3.7.2a). A null decomposition reveals that this term has only one dangerous component involving the product $|\underline{\alpha}|^2$: $\mathfrak{Q}^{(2;h)}_{\underline{L}\underline{L}}$, which will be shown to decay like $|\mathfrak{Q}^{(2;h)}|_{\underline{L}\underline{L}} \lesssim \varepsilon^2(1+t)^{-2}$ (see inequalities (11.2.7e) and (15.3.3)). All other null components of $\mathfrak{Q}^{(2;h)}$ have a negligible effect on the dynamics because at least one of their factors is a "good" null component of $\mathscr{F}$ (see inequalities (11.2.7d) and (15.3.4c)); these quadratic terms therefore decay rapidly. Furthermore, a null decomposition of the wave equations (3.7.1a) reveals that the dangerous component only directly influences the behavior of the metric perturbation component $|\nabla h|_{\underline{L}\underline{L}}$. The main point is that Lindblad and Rodnianski [2010] were able to close their proof even though they allowed $|\nabla h|_{\underline{L}\underline{L}}$ to decay at a slower rate than the other null components of $\nabla h$. The decay rate $|\mathfrak{Q}^{(2;h)}|_{\underline{L}\underline{L}} \lesssim \varepsilon^2(1+t)^{-2}$, though relatively slow, still allows us to prove the same estimates for $|\nabla h|_{\underline{L}\underline{L}}$ and $|h|_{\underline{L}\underline{L}}$ as in [ibid.] (see Proposition 15.6 and note the presence of the growing $\ln(1+t)$ factor in (15.3.2b) compared to the other estimates).

We now discuss the nonlinearities present in the electromagnetic field equations for the components of $\mathscr{F}$. There are three important nonlinear terms: the main quadratic terms $\mathscr{P}^{\nu}_{(\mathscr{F})}(h, \nabla\mathscr{F})$ and $\mathscr{Q}^{\nu}_{(1;\mathscr{F})}(h, \nabla\mathscr{F})$ from (3.7.3a) and the main quadratic term $\mathscr{Q}^{\nu}_{(2;\mathscr{F})}(\nabla h, \mathscr{F})$ from (3.7.2b). The terms $\mathscr{Q}^{\nu}_{(1;\mathscr{F})}(h, \nabla\mathscr{F})$ and $\mathscr{Q}^{\nu}_{(2;\mathscr{F})}(\nabla h, \mathscr{F})$ have a very favorable null structure (all quadratic factors involve either a good tangential derivative $\overline{\nabla}$ or a good component of $\mathscr{F}$) and therefore have a negligible effect on the dynamics (see inequalities (11.2.7h)–(11.2.7i)). Furthermore, this special structure survives upon commuting the equations with $\mathscr{L}^{I}_{\mathscr{Z}}$. In contrast, the term $\mathscr{P}^{\nu}_{(\mathscr{F})}(h, \nabla\mathscr{F})$ has a less favorable null structure and must be handled with care. For example, if $X$ is any one-form, then in order to bound $|X_{\nu}\mathscr{P}^{\nu}_{(\mathscr{F})}(h, \nabla\mathscr{F})|$, one must in particular bound $|X||h|_{\mathscr{L}\mathscr{L}}|\nabla\mathscr{F}|$ (see inequality (11.2.7f)). The product $|h|_{\mathscr{L}\mathscr{L}}|\nabla\mathscr{F}|$ is only expected to decay like $\varepsilon^2(1+t)^{-2}$ thanks to the presence of the worst null components of $\nabla\mathscr{F}$ (the worst null component combination in the product $|h|_{\mathscr{L}\mathscr{L}}|\nabla\mathscr{F}|$ is the magnitude of the product $\frac{1}{4}h_{LL}\nabla_{\underline{L}}\underline{\alpha}_{\nu}$, which is discussed below in the third paragraph of Section 1.2.11). The main reason that we are able to handle the difficult term $\mathscr{P}^{\nu}_{(\mathscr{F})}(h, \nabla\mathscr{F})$ is that the wave-coordinate condition allows one to derive independent estimates for $|h|_{\mathscr{L}\mathscr{L}}$ that are just good enough to close the proof of stability; this is discussed in more detail in Section 1.2.10. Another difficulty is that some of this structure is destroyed after one commutes $\mathscr{P}^{\nu}_{(\mathscr{F})}(h, \nabla\mathscr{F})$ with $\mathscr{L}^{I}_{\mathscr{Z}}$. In particular, the commuted term $|\mathscr{L}^{I}_{\mathscr{Z}}h|_{\mathscr{L}\mathscr{L}}$ must be carefully analyzed, for Lie differentiation results in the presence of some potentially dangerous lower-order terms. These terms are discussed in more detail at the end of Section 1.2.12.

**1.2.6.** *Energy inequalities and the canonical stress.* The first major analytical step in deriving the all-important Gronwall-amenable estimate (16.2.5) (which is the main ingredient in the derivation of the a priori estimate (1.2.9)) is to deduce the energy inequalities of Lemmas 12.1 and 12.2, which respectively provide $L^2$ estimates for solutions to the electromagnetic *equations of variation* and $L^2$ estimates for solutions to quasilinear wave equations whose principal operator agrees with that of (1.2.4a) (i.e., whose principal operator is $\widetilde{\Box}_g$). The equations of variation are linear (in the principal term) PDEs that are satisfied by the derivatives of solutions $\mathscr{F}$ to (1.2.4b)–(1.2.4c). Specifically, the equations of variation are the PDEs (8.1.1a)–(8.1.1b). As is explained below, these equations come into play because we require $L^2$ estimates for higher-order derivatives of $h^{(1)}$ and $\mathscr{F}$ in order to close our global existence argument. We will comment mainly on the estimates for the electromagnetic equations of variation since the estimates of Lemma 12.2 are perhaps more familiar to the reader and in any case are explained in detail in [Lindblad and Rodnianski 2010, Lemma 6.1 and Proposition 6.2]. Our proof of Lemma 12.1 is based on the construction of a suitable *energy current* $\dot{J}^{\mu} \overset{\text{def}}{=} -\dot{Q}^{\mu}{}_{\nu}X^{\nu}$, where $\dot{Q}^{\mu}{}_{\nu}$ is the *canonical stress*. $\dot{Q}^{\mu}{}_{\nu} = \dot{Q}^{\mu}{}_{\nu}[\dot{\mathscr{F}}, \dot{\mathscr{F}}]$ is a tensor field that depends quadratically on the *variations* $\dot{\mathscr{F}}_{\mu\nu} \overset{\text{def}}{=} \mathscr{L}^{I}_{\mathscr{Z}}\mathscr{F}$, $X^{\nu} \overset{\text{def}}{=} w(q)\delta^{\nu}_0$ ($\nu = 0, 1, 2, 3$) is a "multiplier vector field", and $w(q)$ is the weight function defined in (1.2.8). The end result is provided by inequality (12.2.1) below. Although at first glance inequality (12.2.1) may appear to be a standard energy inequality, one of the most important features of this particular energy current is that it provides the *additional positive* spacetime integral $\int_{t_1}^{t_2}\int_{\Sigma_{\tau}}(|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2)w'(q)\,d^3x\,d\tau$ on the left-hand side of (12.2.1); here, $\dot{\alpha}$, $\dot{\rho}$, and $\dot{\sigma}$ are the "favorable" null components of the two-form $\dot{\mathscr{F}}$. This additional positive quantity, which is analogous to the quantity $\int_{t_1}^{t_2}\int_{\Sigma_{\tau}}|\overline{\nabla}\phi|^2w'(q)\,d^3x\,d\tau$ on the left-hand side of (12.2.4) that was exploited by Lindblad and Rodnianski, is one of the key advantages afforded

by our use of a weight function of the form (1.2.8). Its availability is directly related to the fact that we have better integrated control over the quadratic terms $|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2$ than we do over the term $|\underline{\dot{\alpha}}|^2$. The spacetime integral plays a key role in our derivation of the energy inequality (16.2.5).

Let us now make a few comments concerning the canonical stress and the construction of the energy current $\dot{J}^{\mu}$ introduced above. A very detailed description is located in [Christodoulou 2000; Speck 2012], so we confine ourselves here to its two most salient features. The canonical stress (see (8.2.2)) plays the role of an energy-momentum-type tensor for the electromagnetic equations of variation. Because these (linear-in-$\dot{\mathscr{F}}$) equations depend on the "background" $\mathscr{F}_{\mu\nu}$ in addition to the linearized variables $\dot{\mathscr{F}}_{\mu\nu}$, it is *not* the case that $\mathscr{D}_{\mu}(\dot{Q}^{\mu}{}_{\nu}[\dot{\mathscr{F}}, \dot{\mathscr{F}}]) = 0$; this is in contrast to the property $(g^{-1})^{\kappa\lambda}\mathscr{D}_{\kappa}T_{\lambda\nu} = 0$ (see (3.5.3)) enjoyed by the energy-momentum tensor. However, we now point out the first key property of the canonical stress: $\nabla_{\mu}(\dot{Q}^{\mu}{}_{\nu}[\dot{\mathscr{F}}, \dot{\mathscr{F}}])$ is lower-order in the sense that it does not depend on $\nabla_{\lambda}\dot{\mathscr{F}}_{\mu\nu}$; by using the equations of variation for substitution, the $\nabla_{\lambda}\dot{\mathscr{F}}_{\mu\nu}$ terms can be replaced with inhomogeneous terms (see (8.2.4)). It is already important to appreciate the availability of this nontrivial quadratic-in-$\dot{\mathscr{F}}$ quantity whose divergence can be expressed in terms of only $\mathscr{F}$, $\nabla\mathscr{F}$, $\dot{\mathscr{F}}$, and inhomogeneous terms. The availability of such a quantity is not a feature inherent to all systems of equations,[14] but is instead related to the symmetry properties of the indices of the principal terms (i.e., the terms on the left-hand side) in equations (8.1.1a)–(8.1.1b), which themselves are related to the fact that the original nonlinear electromagnetic equations are derivable from a Lagrangian.

The second key property enjoyed by the canonical stress is that of integrated positivity upon contraction against certain pairs $(\xi, X)$ consisting of a one-form $\xi$ and a vector field $X$. More precisely, for certain hypersurfaces $\Sigma$, there exist choices of $(\xi, X)$ such that $\xi$ is normal to $\Sigma$ (in the sense of covector-vector annihilation) and such that the quantity $\int_{\Sigma} \dot{Q}^{\mu}{}_{\nu}\xi_{\mu}X^{\nu}[\dot{\mathscr{F}}, \dot{\mathscr{F}}]\, d\Sigma$ is bounded from below by the square of an $L^2$-type norm of $\dot{\mathscr{F}}$ along $\Sigma$. This is a general fact that holds for all electromagnetic equations of variation that are *regularly hyperbolic* in the sense of [Christodoulou 2000]. However, in the present article, a stronger condition than integrated positivity holds: for certain pairs $(\xi, X)$, $\dot{Q}^{\mu}{}_{\nu}\xi_{\mu}X^{\nu}[\,\cdot\,,\,\cdot\,]$ is in fact a *positive-definite* quadratic form in $\dot{\mathscr{F}}$. We remark that this stronger property concerns the structure of the quadratic form $\dot{Q}^{\mu}{}_{\nu}\xi_{\mu}X^{\nu}[\,\cdot\,,\,\cdot\,]$ and therefore has nothing to do with whether $\dot{\mathscr{F}}$ satisfies the equations of variation.

The two key properties are analogous to (but distinct from) the positivity properties of an energy-momentum tensor satisfying the dominant energy condition and the positivity properties of the Bel–Robinson tensor (which played a central role in [Christodoulou and Klainerman 1993; Zipser 2000; Klainerman and Nicolò 2003; Bieri 2007]). As is explained in [Christodoulou 2000; Speck 2012], the set of pairs $(\xi, X)$ leading to integrated positivity is intimately connected to the *hyperbolicity of and the geometry of the electromagnetic equations* and to the speeds and directions of propagation in the system. In this article, the only hypersurfaces that we integrate over are the constant-time hypersurfaces $\Sigma_t$ and the only pair $(\xi, X)$ that we use is $\xi_{\mu} = -\delta^0_{\mu}$, and $X^{\nu} = w(q)\delta^{\nu}_0$. The special positivity properties stemming from this choice of $(\xi, X)$, and in particular the availability of the additional positive spacetime integral $\int_{t_1}^{t_2}\int_{\Sigma_{\tau}} (|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2)w'(q)\, d^3x\, d\tau$ mentioned above, are derived in Lemma 12.1. We emphasize that

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[14]However, such quantities do in fact exist for all scalar wave equations.

our derivation of this additional spacetime integral is not just a consequence of the second key property; rather, our derivation requires $\dot{\mathscr{F}}$ to be a solution to the equations of variation.

**1.2.7.** *Weighted Klainerman–Sobolev inequalities.* Based on the energy inequalities of Proposition 12.3, which are relatively straightforward consequences of Lemmas 12.1 and 12.2, it is clear that most of the hard work in deriving the estimate (16.2.5) goes into estimating the integrals involving the inhomogeneous terms $\mathfrak{I}$ and $\dot{\mathfrak{F}}$ on the right-hand sides of (12.2.6) and (12.2.8). In particular, we attempt to summarize here the origin of the factors $(1+\tau)^{-1}$ and $(1+\tau)^{-1+C\varepsilon}$ that appear in (16.2.5) and that are of central importance in our derivation of the fundamental a priori energy estimate (1.2.9). Roughly speaking, these factors arise from a collection of pointwise decay estimates that we will soon explain. The first tools of interest to us along these lines are the weighted Klainerman–Sobolev inequalities, which allow us to deduce pointwise decay estimates for functions $\phi \in C_0^\infty(\mathbb{R}^3)$ in terms of weighted $L^2$ estimates for $\phi$ and its Minkowskian covariant derivatives with respect to vector fields $Z \in \mathscr{Z}$. More specifically (see also Appendix B), the weighted Klainerman–Sobolev inequalities state that (with $q \stackrel{\text{def}}{=} |x| - t$)

$$(1+t+|x|)[(1+|q|)w(q)]^{1/2}|\phi(t,x)| \le C \sum_{|I| \le 3} \|w^{1/2}(q)\nabla_{\mathscr{Z}}^I \phi(t,\cdot)\|_{L^2}. \qquad (1.2.10)$$

We refer to these estimates as "*weak pointwise decay estimates*" since they have nothing to do with the special structure of the Einstein-nonlinear electromagnetic equations; a major theme permeating this article is that, in order to close our global existence bootstrap argument, the estimate (1.2.10) needs to be upgraded using the special structure of the equations. Inequality (1.2.10) can therefore be viewed as a preliminary estimate that will play a role in the proof of the upgraded estimates.

The form of the inequalities (1.2.10) raises several important issues. First, in order to apply the weighted Klainerman–Sobolev inequalities to $h^{(1)}$, we have to achieve $L^2$ control over the quantities $w^{1/2}(q)\nabla_{\mathscr{Z}}^I h^{(1)}$. To this end, we have to study the equations satisfied by the quantities $\nabla_{\mathscr{Z}}^I h^{(1)}$. In order to derive these equations, we have to commute the operator $\nabla_{\mathscr{Z}}^I$ through the reduced wave operator term $\widetilde{\Box}_g h^{(1)}$. Lindblad and Rodnianski accomplished this commutation through the use of *modified covariant derivatives* $\widehat{\nabla}_Z$, which are equal to ordinary covariant derivatives plus a scalar multiple (depending on $Z \in \mathscr{Z}$) of the identity; see Definition 6.5. The main advantage of these operators is that $\widehat{\nabla}_Z \Box_m - \Box_m \nabla_Z = 0$, where $\Box_m \stackrel{\text{def}}{=} (m^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$ denotes the wave operator of the Minkowski metric; see Lemma 6.13. Therefore, $\nabla_{\mathscr{Z}}^I h^{(1)}$ is a solution to the equation $\widetilde{\Box}_g \nabla_{\mathscr{Z}}^I h^{(1)} = \widehat{\nabla}_{\mathscr{Z}}^I \widetilde{\Box}_g h^{(1)} + H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda\nabla_{\mathscr{Z}}^I h^{(1)} - \widehat{\nabla}_{\mathscr{Z}}^I(H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda h^{(1)})$, where $\widetilde{\Box}_g h^{(1)}$ is equal to the inhomogeneous term on the right-hand side of (1.2.4a) above and $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu} = -h^{\mu\nu} + O(|h|^2)$. We remark that the analysis of the commutator term $H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda\nabla_{\mathscr{Z}}^I h^{(1)} - \widehat{\nabla}_{\mathscr{Z}}^I(H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda h^{(1)})$, which was performed in [Lindblad and Rodnianski 2010] (see also Proposition 7.1 and Lemma 16.11), is among the most challenging work encountered. Rather than repeat this analysis and the discussion behind it, which is thoroughly explained and carried out in [Lindblad and Rodnianski 2010], we will instead focus on the analogous difficulties that arise in our analysis of $\mathscr{F}$. We do, however, point out the role that the Hardy inequalities of Proposition C.1 play in the analysis of $h^{(1)}$: they are used to estimate a weighted $L^2$ norm of $\nabla_{\mathscr{Z}}^I h^{(1)}$ by a weighted $L^2$ norm of $\nabla\nabla_{\mathscr{Z}}^I h^{(1)}$. The main point is that $\nabla_{\mathscr{Z}}^I h^{(1)}$

is *not* directly controlled in $L^2$ by the energy while $\nabla \nabla_{\mathscr{L}}^I h^{(1)}$ is. The cost of applying the Hardy inequalities is powers of $1 + |q|$, which are always sufficiently available thanks to our use of the weight $w(q)$.

**1.2.8.** *The role of Lie derivatives.* The next important issue concerning the weighted Klainerman–Sobolev inequality (1.2.10) is that it is more convenient to work with Lie derivatives of $\mathscr{F}$ rather than covariant derivatives of $\mathscr{F}$; note that our definition (1.2.7) of our energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$ involves Lie derivatives of $\mathscr{F}$. According to inequality (6.5.22) below, inequality (1.2.10) remains valid if we replace the operators $\nabla_{\mathscr{L}}^I$ with $\mathscr{L}_{\mathscr{L}}^I$. However, as in the case of the $\nabla_{\mathscr{L}}^I h^{(1)}$, we have to study the equations satisfied by the $\mathscr{L}_{\mathscr{L}}^I \mathscr{F}$. Now on the one hand, Lemma 6.8 shows that the operator $\mathscr{L}_Z$ can be commuted through the Minkowski connection $\nabla$ in (1.2.4b). On the other hand, to commute Lie derivatives through (1.2.4c), it is convenient to work with *modified Lie derivatives* $\widehat{\mathscr{L}}_Z$, which are equal to ordinary Lie derivatives plus a scalar multiple[15] (depending on $Z \in \mathscr{Z}$) of the identity; see Definition 6.5. Unlike covariant derivatives, these operators have favorable commutation properties with the linear Maxwell–Maxwell term $\nabla_\mu \mathscr{F}^{\mu\nu}$, which is the leading term in (1.2.4c). More specifically, $\widehat{\mathscr{L}}_Z[((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa})\nabla_\mu \mathscr{F}_{\kappa\lambda}] = [(m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}]\nabla_\mu \mathscr{L}_Z \mathscr{F}_{\kappa\lambda}$; see Lemma 6.14. As is captured by Proposition 8.1, these operators are also useful for differentiating the nonlinear equation (1.2.4c); the error terms generated have a favorable null structure that is captured in Proposition 11.4.

**1.2.9.** *The tensorfield $h_{\mu\nu}^{(0)}$.* Let us now discuss the ideas behind the Lindblad–Rodnianski splitting of the metric defined in (1.2.1a)–(1.2.1c). We first note that because of the $2M/r$ ADM mass term present in $h_{\mu\nu}^{(0)}$, substituting the tensor field $h_{\mu\nu} \stackrel{\text{def}}{=} h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}$ in place of $h_{\mu\nu}^{(1)}$ in the definition of the energy would lead to $\mathscr{E}_{\ell;\gamma;\mu}(0) = \infty$. Thus, as a practical matter, the introduction of $h_{\mu\nu}^{(1)}$ allows us to work with a quantity of finite energy. Now according to the discussion in [Lindblad and Rodnianski 2010], the precise form $h_{\mu\nu}^{(0)} = \chi(r/t)\chi(r)(2M/r)\delta_{\mu\nu}$ was determined by making an "educated" guess concerning the contribution of the ADM mass term $\chi(r)(2M/r)\delta_{jk}$, which is present in the data, to the solution. The term $h_{\mu\nu}^{(0)}$ manifests itself in the reduced equations as the $\widetilde{\Box}_g h_{\mu\nu}^{(0)}$ inhomogeneous term on the right-hand side of the reduced equation (1.2.4a). Because of the identity $\Box_m(1/r) = 0$ for $r > 0$, where $\Box_m = (m^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$ is the Minkowski wave operator, it follows that the main contribution of the term $\widetilde{\Box}_g h_{\mu\nu}^{(0)}$ comes from the "interior" region $\left\{(t, x) \,\middle|\, \frac{1}{2} < r/t < \frac{3}{4}\right\}$; this is because the derivatives of $\chi(z)$ are supported in the interval $\left[\frac{1}{2}, \frac{3}{4}\right]$. Now in the interior region, the quantities $1 + |q|$ and $1 + s$ are uniformly comparable. Thus, the weighted Klainerman–Sobolev inequality (1.2.10) predicts strong decay for the solution in this region, and consequently, one can derive suitable weighted Sobolev bounds for the inhomogeneity $\widetilde{\Box}_g h_{\mu\nu}^{(0)}$; see Lemma 16.10 for a precise statement of this estimate.

**1.2.10.** *The wave-coordinate condition.* Before expanding our discussion of the pointwise decay estimates, we will discuss the analytic role of the *wave-coordinate* condition $\nabla_\nu[\sqrt{|\det g|}(g^{-1})^{\mu\nu}] = 0$ ($\mu = 0, 1, 2, 3$), which plays *multiple* roles in this article. First, it hyperbolizes the Einstein equations. Second, it allows us to replace certain unfavorable nonlinear terms from the equations (1.0.1a)–(1.0.1c) with more favorable ones; the culmination of this procedure is exactly the reduced system (1.2.4a)–(1.2.4c). Finally, the wave-coordinate condition also allows us to deduce several *independent and improved estimates*, both

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[15]The multiple is $2c_Z$, where $c_Z$ is the multiple corresponding to the modified covariant derivative $\widehat{\nabla}_Z$.

at both the pointwise level and the $L^2$ level, for the components $h_{LL}$ and $h_{LT}$. As we will see, these improved estimates are central to the structure of the proof of Theorem 16.1, and our stability argument would not close without them. More specifically, as shown in [Lindblad and Rodnianski 2010], a null decomposition of the wave-coordinate condition leads to the algebraic inequalities

$$|\nabla h|_{\mathscr{L}\mathscr{T}} \lesssim |\overline{\nabla} h| + |h||\nabla h|, \tag{1.2.11a}$$

$$|\nabla \nabla_Z h|_{\mathscr{L}\mathscr{L}} \lesssim |\overline{\nabla} \nabla_Z h| + \sum_{|I_1|+|I_2|\leq 1} |\nabla_{\mathscr{L}}^{I_1} h||\nabla \nabla_{\mathscr{L}}^{I_2} h| \quad (Z \in \mathscr{Z}), \tag{1.2.11b}$$

where $\overline{\nabla}$ is the null frame projection of $\nabla$ (the derivative component only) onto the outgoing Minkowski cones. Note that the right-hand side of (1.2.11a) involves only favorable derivatives of $h$ and quadratic error terms while the left-hand side involves *all* derivatives of $h$, including the dangerous $\nabla_L$ derivative. Generalizations of (1.2.11a) for $\nabla_{\mathscr{L}}^I h$ are stated in Proposition 11.1. We remark that it is important to note in these generalizations that the estimates for $|\nabla \nabla_Z h|_{\mathscr{L}\mathscr{L}}$ are stronger than what can be proved for $|\nabla \nabla_Z h|_{\mathscr{L}\mathscr{T}}$.

**1.2.11.** *Upgraded pointwise decay estimates.* We now discuss the full collection of *upgraded pointwise decay estimates* (see Propositions 15.5–15.7 below), which are of central importance in closing the global existence bootstrap argument. For as mentioned above, the weighted Klainerman–Sobolev estimates (1.2.10) are not sufficient to close the argument. We remark that the reasons that we truly need the upgraded pointwise decay estimates are discussed in more detail at beginning of Section 15. Aside from the components $h_{LL}$ and $h_{LT}$, which are controlled by the wave-coordinate condition, there is a relatively strong coupling between the evolution of the remaining components of $h$ and the evolution of the dangerous $\underline{\alpha}[\mathscr{F}]$ component of the Faraday tensor. Therefore, our proofs of the upgraded estimates (and Proposition 15.7 in particular) have a hierarchical structure; i.e., the order in which they are proved is very important. Although we don't provide a complete description of all of the subtleties of this hierarchy in this introduction, we do provide a preliminary description of some of its salient features. We first emphasize the following important feature: most null components of $h$, the $\underline{\alpha}$ null component of $\mathscr{F}$, and the components $\nabla_Z h_{LL}$ (for $Z \in \mathscr{Z}$) have better $t$-decay properties than their higher-order-derivative counterparts; this is the content of Proposition 15.6. Roughly speaking, the reason for this discrepancy is that the nondifferentiated reduced equations have a more favorable algebraic structure than the differentiated reduced equations. This feature will be particularly important during our global existence argument, for the principal terms (from the point of view of differentiability) in the Leibniz expansion of the operator $\nabla_{\mathscr{L}}^I$ acting on a quadratic term are of the form $u\nabla_{\mathscr{L}}^I v$ and similarly for the operator $\mathscr{L}_{\mathscr{L}}^I$. Consequently, the strong pointwise decay property of the nondifferentiated quantity, which is represented by $u$, is a crucially important ingredient of the derivation of the $C\varepsilon \int_0^t (1+\tau)^{-1} \mathscr{E}_{k;\gamma;\mu}^2(\tau)\, d\tau$ term on the right-hand side of (16.2.5). We emphasize that our stability proof would not go through if this term were replaced with $C\varepsilon \int_0^t (1+\tau)^{-1+C\varepsilon} \mathscr{E}_{k;\gamma;\mu}^2(\tau)\, d\tau$.

The derivation of the upgraded pointwise decay estimates for the Faraday tensor begins with Proposition 9.3, which provides a null decomposition of the electromagnetic equations of variation, and Proposition 11.4, which provides a null decomposition of the inhomogeneous terms that result after differentiating the reduced electromagnetic equations with modified Lie derivatives. The net effect is that the

null components of the *lower-order* Lie derivatives of $\mathscr{F}$ satisfy ordinary differential inequalities[16] (which we loosely refer to as ODEs) along ingoing and outgoing cones (see Proposition 11.5), and furthermore, the inhomogeneous terms appearing on the right-hand side of the ODEs can be inductively controlled (see the proofs of Propositions 15.5–15.7). We remark that this analysis of the lower-order derivatives of $\mathscr{F}$ involves a loss of several derivatives because the right-hand sides of the ODEs depend on the higher-order derivatives of $\mathscr{F}$, which are pointwise bounded via the weighted Klainerman–Sobolev estimates (1.2.10). We stress that this loss of differentiability is not a concern because we only need to analyze the lower-order derivatives of $\mathscr{F}$ in this fashion. Similar remarks apply for our analysis of the upgraded pointwise decay estimates for $h$, which are briefly described below. It is important to distinguish between two classes of ODEs that play a role in this analysis. The first class consists of ODEs for rescaled versions of the null components $(\dot{\alpha}, \dot{\rho}, \dot{\sigma}) \stackrel{\text{def}}{=} (\alpha[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}], \rho[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}], \sigma[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}])$ and involves differentiation in the direction of the null generators of the *ingoing Minkowskian cones*; i.e., the principal part of the ODEs is $\nabla_{\underline{L}}$. We remark that this point of view represents a rather crude treatment of (9.1.8b)–(9.1.8d), but because of the favorable decay properties of the inhomogeneities, this approach is sufficient to conclude the desired estimates: by integrating back towards the Cauchy hypersurface $\Sigma_0$ in the direction $-\underline{L}$, we are able to deduce $t$-decay for $\alpha[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}]$, $\rho[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}]$, and $\sigma[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}]$ from $t$-decay of the inhomogeneous terms at the expense of a loss of decay in $q$. We remark that the proof of the upgraded estimates for these components happens in two stages. We refer to the first-stage estimates, which are proved in Proposition 15.5, as the "initial upgraded" pointwise decay estimates. These first-stage estimates follow from using the weighted Klainerman–Sobolev estimates to bound the inhomogeneous terms in the ODEs. The second-stage upgraded estimates, which we refer to as "fully upgraded" pointwise decay estimates, are proved at the end of Proposition 15.7 after all of the other upgraded pointwise decay estimates for the remaining components of the lower-order derivatives of $h$ and $\mathscr{F}$ have been proved. For at this point in the upgraded hierarchy, we will have better pointwise control over the inhomogeneous terms in the ODEs than that afforded by the weighted Klainerman–Sobolev estimates.

The next class consists of ODEs for rescaled versions of the null component $\dot{\underline{\alpha}} \stackrel{\text{def}}{=} \underline{\alpha}[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}]$. Notice that (see (9.1.8a)), unlike the other null components, $\dot{\underline{\alpha}}$ does *not* satisfy an ODE that to 0-th order involves differentiation in the direction of $\underline{L}$. Instead, at first sight, it might appear that one should reason in analogy with the first class and view (9.1.8a) as an ODE in the direction of $L$ with inhomogeneous terms. However, the desired decay estimates do *not* close at this level. Instead, one must also consider the effect of the quadratic term $-\slashed{h}_\nu{}^\lambda h^{\mu\kappa} \nabla_\mu \dot{\mathscr{F}}_{\kappa\lambda}$. A null decomposition of this term reveals that it contains the dangerous term $\frac{1}{4} h_{LL} \nabla_{\underline{L}} \dot{\underline{\alpha}}_\nu$, which decays too slowly to be treated as an inhomogeneous term in the ODE satisfied by $\dot{\underline{\alpha}}$. To remedy this difficulty, we introduce the vector field $\Lambda = L + \frac{1}{4} h_{LL} \underline{L}$, which can be viewed as a first-order correction to the Minkowski outgoing null direction arising from the presence of a nonzero tensor field $h$ in the expansion $g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}$. Note that, for these upgraded pointwise decay estimates for the lower-order Lie derivatives, we do not bother to correct for the fact that the electromagnetic model is not necessarily the Maxwell–Maxwell model; the deviation from the Maxwell–Maxwell model comprises cubic terms, which we can treat as small inhomogeneities. We may thus view (9.1.8a) as an ODE in the direction of $\Lambda$ with inhomogeneous terms; this is exactly the point of

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[16]More precisely, the null components satisfy transport equations with small sources.

view emphasized in Proposition 11.5. Because we have a sufficiently strong independent decay estimates for $h_{LL}$ and for the inhomogeneities, this approach is sufficient to achieve the desired estimates.

Our analysis of the upgraded pointwise decay estimates for the metric-related quantities $h$ and $h^{(1)}$ closely mirrors the analysis in [Lindblad and Rodnianski 2010]. Hence, we will not discuss them in full detail here but instead refer the reader to the discussion in [ibid.]. The estimates can be divided into three classes, the first one being the estimates (15.3.1a) and (15.3.1b) for $|\nabla h|_{\mathscr{L}\mathscr{T}}$, $|\nabla \nabla_Z h|_{\mathscr{L}\mathscr{L}}$, $|h|_{\mathscr{L}\mathscr{T}}$, and $|\nabla_Z h|_{\mathscr{L}\mathscr{L}}$. As was suggested above, the first-class estimates are consequences of the additional special algebraic structure that follows from the wave-coordinate condition together with the weighted Klainerman–Sobolev inequality. The second class consists of the estimates (15.3.2a) and (15.3.2b) for $|\nabla h|_{\mathscr{T}\mathscr{N}}$ and $|\nabla h|$. These estimates heavily rely on the decay estimates of Lemma 13.2 and Corollary 13.3 below, which were proved in [Lindblad and Rodnianski 2010] and which are of independent interest. The lemma and its corollary can be viewed as a second-order counterpart to the ODE estimates for the Faraday tensor discussed in the previous paragraphs. It is important to note that the hypotheses of the lemma and its corollary are satisfied *as a consequence* of the independent upgraded pointwise decay estimates provided by the wave-coordinate condition. The third class consists of the estimates (15.3.4a), (15.3.4b), and (15.3.4c) for $|\nabla \nabla_{\mathscr{L}}^I h^{(1)}|$, $|\nabla_{\mathscr{L}}^I h^{(1)}|$, and $|\overline{\nabla} \nabla_{\mathscr{L}}^I h^{(1)}|$ (related estimates for the tensor field $h$ also hold). Their derivation is similar in spirit to the derivation of the second-class estimates, but the inductive proof we give is highly coupled to the simultaneous derivation of analogous upgraded pointwise decay estimates for $|\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|$, which were discussed two paragraphs ago.

**1.2.12.** *Lie differentiation, Minkowski-covariant differentiation, and null structure.* We make some final comments concerning the relationship between Lie derivatives and Minkowski-covariant derivatives. On the one hand, because we commute the equations satisfied by $h^{(1)}$ with the operators $\nabla_{\mathscr{L}}^I$, our analysis of $h^{(1)}$ naturally requires us to estimate the quantities $|\nabla_{\mathscr{L}}^I h|$, $|\nabla_{\mathscr{L}}^I h|_{\mathscr{L}\mathscr{L}}$, $|\nabla_{\mathscr{L}}^I h|_{\mathscr{L}\mathscr{T}}$, etc. Furthermore, as discussed above, the quantities $|\nabla_{\mathscr{L}}^I h|_{\mathscr{L}\mathscr{L}}$ and $|\nabla_{\mathscr{L}}^I h|_{\mathscr{L}\mathscr{T}}$ have a distinguished role in view of their connection to the wave-coordinate condition. On the other hand, because we commute the electromagnetic equations with Lie derivatives, we will have to confront the terms $|\mathscr{L}_{\mathscr{L}}^I h|$, $|\mathscr{L}_{\mathscr{L}}^I h|_{\mathscr{L}\mathscr{L}}$, $|\mathscr{L}_{\mathscr{L}}^I h|_{\mathscr{L}\mathscr{T}}$, etc. In order to bridge the gap between Lie derivative estimates and covariant derivative estimates, we provide Proposition 6.19, the proof of which relies on the special algebraic-geometric structure of the vector fields in $\mathscr{L}$. Proposition 6.19 is an especially important ingredient in the null decomposition estimate (11.1.11b). As an example of the role played by this proposition, we cite the estimate (6.5.23c), which reads

$$|\mathscr{L}_{\mathscr{L}}^I h|_{\mathscr{L}\mathscr{L}} \lesssim |\nabla_{\mathscr{L}}^I h|_{\mathscr{L}\mathscr{L}} + \underbrace{\sum_{|J| \leq |I|-1} |\nabla_{\mathscr{L}}^J h|_{\mathscr{L}\mathscr{T}}}_{\text{absent if } |I| = 0} + \underbrace{\sum_{|J'| \leq |I|-2} |\nabla_{\mathscr{L}}^{J'} h|}_{\text{absent if } |I| \leq 1}.$$

This shows that, in the translation from Lie derivatives to covariant derivatives, the error terms that arise in the analysis of the $|\cdot|_{\mathscr{L}\mathscr{L}}$ seminorm are either 1 degree lower in order *and* controllable by the wave-coordinate condition (i.e., the terms with $|J| \leq |I|-1$) or are 2 degrees lower in order (i.e., the terms with $|J'| \leq |I|-2$). This fact, and others similar to it, play an essential role in allowing our hierarchy of estimates to unfold in a viable order.

**1.3. *Outline of the article.*** The remainder of the article is organized as follows.

• In Section 2, we provide for convenience a summary of the notation that is used throughout the article.

• In Section 3, we discuss the Einstein-nonlinear electromagnetic equations in detail. We also introduce our wave-coordinate condition and our assumptions on the electromagnetic Lagrangian. Next, we derive a reduced system of equations, which is equivalent to the system of interest in our wave-coordinate gauge. In Section 3.7, we summarize the version of the reduced equations that we work with for most of the article.

• In Section 4, we construct initial data for the reduced system from the abstract initial data in a manner compatible with the wave-coordinate condition. We also sketch a proof of the fact that the wave-coordinate condition is preserved by the flow of the reduced equations.

• In Section 5, we introduce the notion of a Minkowskian null frame and discuss the corresponding null decomposition of various tensor fields.

• In Section 6, we introduce the differential operators that will be used throughout the remainder of the article, including modified Lie derivatives and modified covariant derivatives with respect to a special subset $\mathcal{Z}$ of Minkowskian conformal Killing fields. We also provide a collection of lemmas that relate the various operators.

• In Section 7, we provide a preliminary algebraic expression for the equations satisfied by $\nabla_{\mathcal{Z}}^{I} h^{(1)}$, where $h^{(1)}$ is a solution to the reduced equations.

• In Section 8, we introduce the electromagnetic equations of variation, which are a linearized version of the electromagnetic equations. We also provide a preliminary algebraic expression for the inhomogeneous terms in the equations of variation satisfied by $\mathscr{L}_{\mathcal{Z}}^{I} \mathscr{F}$, where $\mathscr{F}$ is a solution to the reduced equations. We then introduce the canonical stress tensor and use it to construct an energy current that will be used to control weighted Sobolev norms of $\mathscr{L}_{\mathcal{Z}}^{I} \mathscr{F}$.

• In Section 9, we perform two decompositions of the electromagnetic equations, including a null decomposition of the electromagnetic equations of variation and a decomposition of the electromagnetic equations into constraint equations and evolution equations for the Minkowskian one-forms $E$, $D$, $B$, and $H$. In order to connect these one-forms to the abstract initial data, we also introduce the geometric electromagnetic one-forms $\mathfrak{E}$, $\mathfrak{D}$, $\mathfrak{B}$, and $\mathfrak{H}$.

• In Section 10, we introduce our smallness condition on the abstract initial data. We then prove that this smallness condition guarantees that the energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$ of the corresponding solution to the reduced equations is small at $t = 0$; it is this smallness of $\mathscr{E}_{\ell;\gamma;\mu}(0)$ that will lead to a global solution of the reduced equations.

• In Section 11, we provide algebraic estimates for the inhomogeneities in the reduced equations under the assumption that the wave-coordinate condition holds. We also derive ordinary differential inequalities for the null components of $\mathscr{L}_{\mathcal{Z}}^{I} \mathscr{F}$ and provide algebraic estimates for the corresponding inhomogeneities.

• In Section 12, we prove weighted energy estimates for solutions to the electromagnetic equations of variation. We also recall some results of [Lindblad and Rodnianski 2010] that provide analogous weighted

energy estimates for both scalar wave equations and tensorial systems of wave equations with principal part $(g^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$.

• In Section 13, we recall some results of [Lindblad and Rodnianski 2010] that provide pointwise decay estimates for both scalar wave equations and tensorial systems of wave equations with principal part $(g^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$.

• In Section 14, we state a basic local well-posedness result and continuation principle for the reduced equations. The continuation principle will be used in Section 16 in order to deduce small-data global existence for the reduced equations from a suitable bound on the energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$.

• In Section 15, we introduce our bootstrap assumption on the energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$. We then use this assumption to deduce a collection of pointwise decay estimates for solutions to the reduced equations under the assumption that the wave-coordinate condition holds.

• In Section 16, we prove our main results. The results are separated into two theorems. In Theorem 16.1, we use the decay estimates proved in Section 15 to derive a "strong" a priori estimate for the energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$; *the proof of this theorem is the centerpiece of the article*. Theorem 16.3, which is our main theorem demonstrating the stability of Minkowski spacetime, is then an easy consequence of Theorem 16.1 and the continuation principle of Section 14. Both of these theorems rely upon the assumption that the wave-coordinate condition holds.

## 2. Notation

For convenience, in this section, we collect some of the important notation that is introduced throughout the article.

**2.1. *Constants.*** We use the symbols $c$, $\tilde{c}$, $C$, and $\widetilde{C}$ to denote generic *positive* constants that are free to vary from line to line. In general, they can depend on many quantities, but in the small-solution regime that we consider in this article, they can be chosen uniformly. Sometimes it is illuminating to explicitly indicate one of the quantities $\mathfrak{Q}$ that a constant depends on; we do by writing, e.g., $C_\mathfrak{Q}$. If $A$ and $B$ are two quantities, then we often write

$$A \lesssim B$$

to mean that "there exists a uniform constant $C > 0$ such that $A \leq CB$". Furthermore, if $A \lesssim B$ and $B \lesssim A$, then we often write

$$A \approx B.$$

**2.2. *Indices.***

• Lowercase Latin indices $a$, $b$, $j$, $k$, etc., take on the values 1, 2, or 3.

• Greek indices $\kappa$, $\lambda$, $\mu$, $\nu$, etc., take on the values 0, 1, 2, or 3.

• Primed indices $\kappa'$, $\lambda'$, etc., are used in the same way as unprimed indices.

• Uppercase Latin indices $A$, $B$, etc., take on the values 1 or 2 and are used to enumerate the two Minkowski-orthonormal null frame vectors tangent to the spheres $S_{r,t}$.

• As a convention, the tensor fields $\mathscr{F}_{\mu\nu}$, $\mathscr{M}_{\mu\nu}$, $R_{\mu\nu}$, $T_{\mu\nu}$, $\epsilon_{\mu\nu\kappa\lambda}$, and $N_{\mu\nu\kappa\lambda}$ are assumed to "naturally" have all of their indices downstairs, and unless indicated otherwise, all indices on all tensors are lowered and raised with the Minkowski metric $m_{\mu\nu}$ and its inverse $(m^{-1})^{\mu\nu}$; e.g., $T^{\mu\nu} = (m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda}T_{\kappa\lambda}$.

• The symbol # is used to indicate that all indices of a given tensor field have been raised with $g^{-1}$; e.g., $T^{\#\mu\nu} = (g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda}T_{\kappa\lambda}$.

• Repeated indices are summed over.

### 2.3. *Coordinates.*

• $\{x^\mu\}_{\mu=0,1,2,3}$ denotes the wave-coordinate system.

• $t = x^0$, $x = (x^1, x^2, x^3)$.

• $q = r - t$ and $s = r + t$ are the null coordinates of the spacetime point $(t, x)$, where $r = |x|$.

• $q_- = 0$ if $q \geq 0$, and $q_- = |q|$ if $q < 0$.

• $\omega^j = x^j/r$ ($j = 1, 2, 3$).

### 2.4. *Surfaces.* Relative to the wave-coordinate system:

• $C_s^- = \{(\tau, y) \mid |y| + \tau = s\}$ are the ingoing Minkowskian null cones.

• $C_q^+ = \{(\tau, y) \mid |y| - \tau = q\}$ are the outgoing Minkowskian null cones.

• $\Sigma_t = \{(\tau, y) \mid \tau = t\}$ are the constant Minkowskian time slices.

• $S_{r,t} = \{(\tau, y) \mid \tau = t, \ |y| = r\}$ are the Euclidean spheres.

### 2.5. *Metrics and volume forms.*

• $m_{\mu\nu}$ denotes the standard Minkowski metric on $\mathbb{R}^{1+3}$; $m_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ in our wave-coordinate system.

• $\underline{m}$ denotes the Minkowskian first fundamental form of $\Sigma_t$; $\underline{m}_{\mu\nu} = \mathrm{diag}(0, 1, 1, 1)$ in our wave-coordinate system.

• $\displaystyle\not{h}$ denotes the Minkowskian first fundamental form of $S_{r,t}$; relative to an arbitrary coordinate system, $\not{h}_{\mu\nu} = m_{\mu\nu} + \frac{1}{2}(L_\mu\underline{L}_\nu + \underline{L}_\mu L_\nu)$, where $\underline{L}$ and $L$ are defined in Section 2.9.

• $g_{\mu\nu}$ denotes the spacetime metric.

• $g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}$ is the splitting of the spacetime metric into the Minkowski metric $m_{\mu\nu}$, the Schwarzschild tail $h_{\mu\nu}^{(0)} = \chi(r/t)\chi(r)(2M/r)\delta_{\mu\nu}$, and the remainder $h_{\mu\nu}^{(1)}$.

• $h_{\mu\nu} = h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}$.

• $(g^{-1})^{\mu\nu} = (m^{-1})^{\mu\nu} + H_{(0)}^{\mu\nu} + H_{(1)}^{\mu\nu}$ is the splitting of the inverse spacetime metric into the inverse Minkowski metric $(m^{-1})^{\mu\nu}$, the Schwarzschild tail $H_{(0)}^{\mu\nu} = -\chi(r/t)\chi(r)(2M/r)\delta^{\mu\nu}$, and the remainder $H_{(1)}^{\mu\nu}$.

• $H^{\mu\nu} = H_{(0)}^{\mu\nu} + H_{(1)}^{\mu\nu}$.

• $\mathring{g}$ denotes the first fundamental form of the Cauchy hypersurface $\Sigma_0$ relative to the spacetime metric $g$.

- $\mathring{\underline{g}}_{jk} = \delta_{jk} + \chi(r)(2M/r)\delta_{jk} + \mathring{\underline{h}}^{(1)}_{jk}$ is the splitting of $\mathring{\underline{g}}_{jk}$ into the Euclidean metric $\delta_{jk}$, the Schwarzschild tail $\chi(r)(2M/r)\delta_{jk}$, and the remainder $\mathring{\underline{h}}^{(1)}_{jk}$.

- $\upsilon_{\mu\nu\kappa\lambda} = |\det m|^{1/2}[\mu\nu\kappa\lambda]$ denotes the volume form of the Minkowski metric $m$; $[\mu\nu\kappa\lambda]$ is totally antisymmetric with normalization $[0123] = 1$; $|\det m|^{1/2} = 1$ in our wave-coordinate system.

- $\epsilon_{\mu\nu\kappa\lambda} = |\det g|^{1/2}[\mu\nu\kappa\lambda]$ denotes the volume form of the spacetime metric $g$.

- $\epsilon^{\#\mu\nu\kappa\lambda} = -|\det g|^{-1/2}[\mu\nu\kappa\lambda]$ denotes the volume form of the spacetime metric $g$ with all of the indices raised with $g^{-1}$.

- $\underline{\upsilon}_{\nu\kappa\lambda} = [0\nu\kappa\lambda]$ denotes the Euclidean volume form of the surfaces $\Sigma_t$ viewed as embedded Riemannian submanifolds of Minkowski spacetime equipped with the wave-coordinate system.

- $\upsilon_{ijk} = [ijk]$ denotes the Euclidean volume form of the surfaces $\Sigma_t$ viewed as a Riemannian 3-manifold equipped with the standard Euclidean coordinate system.

- $\psi_{\mu\nu} = \upsilon_{\mu\nu\kappa\lambda}\underline{L}^\kappa L^\lambda$ denotes the Euclidean volume form of the spheres $S_{r,t}$.

## 2.6. *Hodge duals.*  For an arbitrary two-form $\mathscr{F}_{\mu\nu}$:

- $^\star\mathscr{F}_{\mu\nu} = \frac{1}{2}g_{\mu\mu'}g_{\nu\nu'}\epsilon^{\#\mu'\nu'\kappa\lambda}\mathscr{F}_{\kappa\lambda} = -\frac{1}{2}|\det g|^{-1/2}g_{\mu\mu'}g_{\nu\nu'}[\mu'\nu'\kappa\lambda]\mathscr{F}_{\kappa\lambda}$ denotes the Hodge dual of $\mathscr{F}_{\mu\nu}$ with respect to the spacetime metric $g_{\mu\nu}$.

- $^\circledast\mathscr{F}_{\mu\nu} = \frac{1}{2}\upsilon_{\mu\nu}{}^{\kappa\lambda}\mathscr{F}_{\kappa\lambda} = -\frac{1}{2}|\det m|^{-1/2}m_{\mu\mu'}m_{\nu\nu'}[\mu'\nu'\kappa\lambda]\mathscr{F}_{\kappa\lambda}$ denotes the Hodge dual of $\mathscr{F}_{\mu\nu}$ with respect to the Minkowski metric $m_{\mu\nu}$. In our wave-coordinate system, $|\det m|^{-1/2} = 1$.

## 2.7. *Derivatives.*

- $\nabla$ denotes the Levi-Civita connection corresponding to $m$.

- $\mathscr{D}$ denotes the Levi-Civita connection corresponding to $g$.

- $\mathring{\mathscr{D}}$ denotes the Levi-Civita connection corresponding to $\mathring{g}$.

- $\underline{\nabla}$ denotes the Levi-Civita connection corresponding to $\underline{m}$.

- $\slashed{\nabla}$ denotes the Levi-Civita connection corresponding to $\slashed{m}$.

- $\overline{\nabla}$ denotes the null frame projection of $\nabla$ onto the outgoing Minkowski null cones; i.e., $\overline{\nabla}_\mu = \bar{\pi}_\mu{}^\kappa\nabla_\kappa$, where $\bar{\pi}_\mu{}^\nu = \delta_\mu^\nu + \frac{1}{2}L_\mu\underline{L}^\nu$ projects vectors $X^\mu$ onto the outgoing Minkowski null cones.

- In our wave-coordinate system $\{x^\mu\}_{\mu=0,1,2,3}$, $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\nabla_\mu = \nabla_{(\partial/\partial x^\mu)}$.

- In our wave-coordinate system, $\partial_r = \omega^a\partial_a$ denotes the radial derivative, where $\omega^j = x^j/r$.

- In our wave-coordinate system, $\partial_s = \frac{1}{2}(\partial_r + \partial_t)$ and $\partial_q = \frac{1}{2}(\partial_r - \partial_t)$ denote the null derivatives; $\partial_q$ denotes partial differentiation at fixed $s$ and fixed angle $x/|x|$ while $\partial_s$ denotes partial differentiation at fixed $q$ and fixed angle $x/|x|$.

- If $X$ is a vector field and $\phi$ is a function, then $X\phi = X^\kappa\partial_\kappa\phi$.

- $\nabla_X$ denotes the differential operator $X^\kappa\nabla_\kappa$.

- $\underline{\nabla}_X$ denotes the differential operator $X^\kappa\underline{\nabla}_\kappa$.

- $\slashed{\nabla}_X$ denotes the differential operator $X^\kappa\slashed{\nabla}_\kappa$.

- $\mathscr{L}_X$ denotes the Lie derivative with respect to the vector field $X$.

- $[X, Y]^\mu = (\mathscr{L}_X Y)^\mu = X^\kappa \, \partial_\kappa Y^\mu - Y^\kappa \, \partial_\kappa X^\mu$ denotes the Lie bracket of the vector fields $X$ and $Y$.

- For $Z \in \mathscr{Z}$, $\widehat{\nabla}_Z = \nabla_Z + c_Z$ denotes the modified covariant derivative, where the constant $c_Z$ is defined in Section 2.8.

- For $Z \in \mathscr{Z}$, $\widehat{\mathscr{L}}_Z = \mathscr{L}_Z + 2c_Z$ denotes the modified Lie derivative, where the constant $c_Z$ is defined in Section 2.8.

- $\nabla^I U$, $\underline{\nabla}^I U$, $\nabla^I_{\mathscr{Z}} U$, $\widehat{\nabla}^I_{\mathscr{Z}} U$, $\mathscr{L}^I_{\mathscr{Z}} U$, and $\widehat{\mathscr{L}}^I_{\mathscr{Z}} U$ respectively denote an $|I|$-th order iterated Minkowski covariant derivative, iterated Euclidean (spatial) covariant derivative, iterated Minkowski $\mathscr{Z}$-covariant derivative, iterated modified Minkowski $\mathscr{Z}$-covariant derivative, iterated $\mathscr{Z}$-Lie derivative, and iterated modified $\mathscr{Z}$-Lie derivative of the tensor field $U$.

- $\Box_m = (m^{-1})^{\kappa\lambda} \nabla_\kappa \nabla_\lambda$ denotes the standard Minkowski wave operator.

- $\widetilde{\Box}_g = (g^{-1})^{\kappa\lambda} \nabla_\kappa \nabla_\lambda$ denotes the reduced wave operator corresponding to the spacetime metric $g$. Note that $\nabla$ is the *Minkowskian* connection.

### 2.8. *Minkowskian conformal Killing fields.* Relative to the wave-coordinate system $\{x^\mu\}_{\mu=0,1,2,3} = (t, x)$:

- $\partial_\mu = \frac{\partial}{\partial x^\mu}$ ($\mu = 0, 1, 2, 3$) denotes a translation vector field.

- $\Omega_{jk} = x_j \frac{\partial}{\partial x^k} - x_k \frac{\partial}{\partial x^j}$ ($1 \le j < k \le 3$) denotes a rotation vector field.

- $\Omega_{0j} = -t \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial t}$ ($j = 1, 2, 3$) denotes a Lorentz boost vector field.

- $S = x^\kappa \frac{\partial}{\partial x^\kappa}$ denotes the scaling vector field.

- $\mathbb{O} = \{\Omega_{jk}\}_{1 \le j < k \le 3}$ are the rotational Minkowskian Killing fields.

- $\mathscr{Z} = \left\{\frac{\partial}{\partial x^\mu}, \Omega_{\mu\nu}, S\right\}_{0 \le \mu \le \nu \le 3}$.

- For $Z \in \mathscr{Z}$, $^{(Z)}\pi_{\mu\nu} = \nabla_\mu Z_\nu + \nabla_\nu Z_\mu = c_Z m_{\mu\nu}$ is the Minkowskian deformation tensor of $Z$, where $c_Z$ is a constant.

- Commutation properties with the Maxwell–Maxwell term:

$$\widehat{\mathscr{L}}^I_{\mathscr{Z}}\left(\left((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\right)\nabla_\mu \mathscr{F}_{\kappa\lambda}\right) = \left((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\right)\nabla_\mu \mathscr{L}^I_{\mathscr{Z}} \mathscr{F}_{\kappa\lambda}.$$

- Commutation properties with the Minkowski wave operator $\Box_m = (m^{-1})^{\kappa\lambda} \nabla_\kappa \nabla_\lambda$:

$$[\Box_m, \partial_\mu] = [\Box_m, \Omega_{\mu\nu}] = 0, \qquad [\Box_m, S] = 2\Box_m, \qquad [\nabla_Z, \Box_m] = -c_Z \Box_m, \quad \text{and} \quad \Box_m \nabla_Z \phi = \widehat{\nabla} \Box_m \phi.$$

### 2.9. *Minkowskian null frames.*

- $\underline{L} = \partial_t - \partial_r$ denotes the Minkowskian null geodesic vector field transversal to the $C_q^+$; it generates the cones $C_s^-$.

- $L = \partial_t + \partial_r$ denotes the Minkowskian null geodesic vector field generating the cones $C_q^+$.

- $e_A$ ($A = 1, 2$) denotes Minkowski-orthonormal vector fields spanning the tangent space of the spheres $S_{r,t}$.

- The set $\mathscr{L} = \{L\}$ contains only $L$.

- The set $\mathcal{T} = \{L, e_1, e_2\}$ denotes the frame vector fields tangent to the $C_q^+$.
- The set $\mathcal{N} = \{\underline{L}, L, e_1, e_2\}$ denotes the entire Minkowski null frame.

### 2.10. *Minkowskian null frame decomposition.*

- For an arbitrary vector field $X$ and frame vector field $N \in \mathcal{N}$, we define $X_N = X_\kappa N^\kappa$, where $X_\mu = m_{\mu\kappa} X^\kappa$.
- For an arbitrary vector field $X$, $X = X^\kappa \partial_\kappa = X^L L + X^{\underline{L}} \underline{L} + X^A e_A$, where $X^L = -\frac{1}{2} X_{\underline{L}}$, $X^{\underline{L}} = -\frac{1}{2} X_L$, and $X^A = X_A$.
- For an arbitrary pair of vector fields $X$ and $Y$,

$$m(X, Y) = m_{\kappa\lambda} X^\kappa X^\lambda = X^\kappa Y_\kappa = -\frac{1}{2} X_L Y_{\underline{L}} - \frac{1}{2} X_{\underline{L}} Y_L + X_A Y_A.$$

If $\mathscr{F}_{\mu\nu}$ is any two-form, its Minkowskian null components are:

- $\underline{\alpha}_\mu = \not{h}_\mu{}^\nu \mathscr{F}_{\nu\lambda} \underline{L}^\lambda$.
- $\alpha_\mu = \not{h}_\mu{}^\nu \mathscr{F}_{\nu\lambda} L^\lambda$.
- $\rho = \frac{1}{2} \mathscr{F}_{\kappa\lambda} \underline{L}^\kappa L^\lambda$.
- $\sigma = \frac{1}{2} \not{\epsilon}^{\kappa\lambda} \mathscr{F}_{\kappa\lambda}$.

### 2.11. *Electromagnetic decompositions.* If $\mathscr{F}_{\mu\nu}$ is any two-form, ${}^\star\!\mathcal{M}_{\mu\nu} = g_{\mu\kappa} g_{\nu\lambda} \left( \frac{\partial^\star \mathscr{L}}{\partial \mathscr{F}_{\kappa\lambda}} - \frac{\partial^\star \mathscr{L}}{\partial \mathscr{F}_{\lambda\kappa}} \right)$ and $\widehat{N}^\mu$ is the future-directed unit $g$-normal to $\Sigma_t$, then its electromagnetic components are:

- $\mathfrak{E}_\mu = \mathscr{F}_{\mu\kappa} \widehat{N}^\kappa$.
- $\mathfrak{B}_\mu = -{}^\star\!\mathscr{F}_{\mu\kappa} \widehat{N}^\kappa$.
- $\mathfrak{D}_\mu = -{}^\star\!\mathcal{M}_{\mu\kappa} \widehat{N}^\kappa$.
- $\mathfrak{H}_\mu = -\mathcal{M}_{\mu\kappa} \widehat{N}^\kappa$.

If $\mathscr{F}_{\mu\nu}$ is any two-form, then relative to the wave-coordinate system, its Minkowskian electromagnetic components are:

- $E_\mu = \mathscr{F}_{\mu 0}$.
- $B_\mu = -{}^\circledast \mathscr{F}_{\mu 0}$.
- $D_\mu = -{}^\circledast \mathcal{M}_{\mu 0}$.
- $H_\mu = -\mathcal{M}_{\mu 0}$.

### 2.12. *Seminorms and energies.* For an arbitrary type-$\binom{0}{2}$ tensor field $P_{\mu\nu}$ and $\mathcal{V}, \mathcal{W} \in \{\mathscr{L}, \mathcal{T}, \mathcal{N}\}$:

- $|P|_{\mathcal{V}\mathcal{W}} = \sum_{V \in \mathcal{V}, W \in \mathcal{W}} |V^\kappa W^\lambda P_{\kappa\lambda}|$.
- $|\nabla P|_{\mathcal{V}\mathcal{W}} = \sum_{N \in \mathcal{N}, V \in \mathcal{V}, W \in \mathcal{W}} |V^\kappa W^\lambda N^\gamma \nabla_\gamma P_{\kappa\lambda}|$.
- $|\overline{\nabla} P|_{\mathcal{V}\mathcal{W}} = \sum_{T \in \mathcal{T}, V \in \mathcal{V}, W \in \mathcal{W}} |V^\kappa W^\lambda T^\gamma \nabla_\gamma P_{\kappa\lambda}|$.
- $|P| = |P|_{\mathcal{N}\mathcal{N}}$.
- $|\nabla P| = |\nabla P|_{\mathcal{N}\mathcal{N}}$.
- $|\overline{\nabla} P| = |\overline{\nabla} P|_{\mathcal{N}\mathcal{N}}$.

- We use similar notation for an arbitrary tensor field $U$ of type $\binom{n}{m}$.

For an arbitrary tensor field $U$ defined on the Euclidean space $\Sigma_0$ with Euclidean coordinate system $x = (x^1, x^2, x^3)$:

- $\|U\|_{L^2}^2 = \int_{x \in \mathbb{R}^3} |U(x)|^2 \, d^3x$ is the square of the standard spatial $L^2$ norm of $U$.
- $\|U\|_{L^\infty} = \operatorname{ess\,sup}_{x \in \mathbb{R}^3} |U(x)|$ is the standard spatial $L^\infty$ norm of $U$.
- $\|U\|_{H_\eta^\ell}^2 = \sum_{|I| \leq \ell} \int_{x \in \mathbb{R}^3} (1 + |x|^2)^{(\eta + |I|)} |\underline{\nabla}^I U(x)|^2 \, d^3x$ is the square of a weighted Sobolev norm of $U$.
- $\|U\|_{C_\eta^\ell}^2 = \sum_{|I| \leq \ell} \operatorname{ess\,sup}_{x \in \mathbb{R}^3} (1 + |x|^2)^{(\eta + |I|)} |\underline{\nabla}^I U(x)|^2$ is the square of a weighted $L^\infty$ norm of $U$.

For arbitrary abstract initial data $(\mathring{\underline{h}}_{jk}^{(1)}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ on the manifold $\mathbb{R}^3$:

- $E_{\ell;\gamma}^2(0) = \|\underline{\nabla}\mathring{\underline{h}}^{(1)}\|_{H_{1/2+\gamma}^\ell}^2 + \|\mathring{K}\|_{H_{1/2+\gamma}^\ell}^2 + \|\mathring{\mathfrak{D}}\|_{H_{1/2+\gamma}^\ell}^2 + \|\mathring{\mathfrak{B}}\|_{H_{1/2+\gamma}^\ell}^2$ is the square of the norm of the abstract initial data.

For an arbitrary symmetric type-$\binom{0}{2}$ tensor field $h_{\mu\nu}^{(1)}$ and an arbitrary two-form $\mathscr{F}_{\mu\nu}$:

- $\mathscr{E}_{\ell;\gamma;\mu}^2(t) = \sup_{0 \leq \tau \leq t} \sum_{|I| \leq \ell} \int_{\Sigma_\tau} (|\nabla \nabla_{\mathscr{L}}^I h^{(1)}|^2 + |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|^2) w(q) \, d^3x$ is the square of the energy of the pair $(h_{\mu\nu}^{(1)}, \mathscr{F}_{\mu\nu})$.

## 2.13. $O^\ell(\,\cdot\,)$ and $o^\ell(\,\cdot\,)$.

- Given an $\ell$-times continuously differentiable function $f(\mathfrak{Q}_1, \ldots, \mathfrak{Q}_m)$ depending on the tensorial quantities $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_m$, we write $f(\mathfrak{Q}_1, \ldots, \mathfrak{Q}_m) = O^\ell(|\mathfrak{Q}_1|^{p_1} \cdots |\mathfrak{Q}_k|^{p_k}; \mathfrak{Q}_{k+1}, \ldots, \mathfrak{Q}_m)$ if we can decompose $f(\mathfrak{Q}_1, \ldots, \mathfrak{Q}_m) = \sum_{i=1}^n P_i(\mathfrak{Q}_1, \ldots, \mathfrak{Q}_k) \tilde{f}_i(\mathfrak{Q}_1, \ldots, \mathfrak{Q}_m)$, where $n$ is a positive integer, each $P_i(\mathfrak{Q}_1, \ldots, \mathfrak{Q}_k)$ is a polynomial in the components of $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_k$ that satisfies $|P_i(\mathfrak{Q}_1, \ldots, \mathfrak{Q}_k)| \lesssim |\mathfrak{Q}_1|^{p_1} \cdots |\mathfrak{Q}_k|^{p_k}$ on a neighborhood of the origin, and $\tilde{f}_i(\,\cdot\,)$ is $\ell$-times continuously differentiable on a neighborhood of the origin.
- Given an $\ell$-times continuously differentiable function $f(x)$, if $\lim_{r \to \infty} |\nabla^I f(x)|/r^{a+|I|} = 0$ for $|I| \leq \ell$, we write $f(x) = o^\ell(r^{-a})$.

## 2.14. Fixed constants.
The fixed constants $\ell, \delta, \gamma, \mu, \gamma'$, and $\mu'$ are subject to the following constraints:

- To prove our global stability theorem, we assume that $\ell$ is an integer satisfying $\ell \geq 10$.
- $0 < \delta < \frac{1}{4}$.
- $0 < \delta < \gamma < \frac{1}{2}$.
- $0 < \gamma' < \gamma - \delta$.
- $0 < \delta < \mu' < \frac{1}{2}$.
- $0 < \mu < \frac{1}{2} - \mu'$.

## 2.15. Weights.

- $w = w(q) = \begin{cases} 1 + (1 + |q|)^{1+2\gamma} & \text{if } q > 0, \\ 1 + (1 + |q|)^{-2\mu} & \text{if } q < 0 \end{cases}$ is the energy estimate weight function.
- $\varpi = \varpi(q) = \begin{cases} (1 + |q|)^{1+\gamma'} & \text{if } q > 0, \\ (1 + |q|)^{1/2-\mu'} & \text{if } q < 0 \end{cases}$ is the pointwise decay estimate weight function.

### 3. The Einstein-nonlinear electromagnetic system in wave coordinates

In this section, we discuss (1.0.1a)–(1.0.1c) in detail. We also discuss our assumptions on the electromagnetic Lagrangian and introduce our wave-coordinate gauge. We then derive a reduced system of equations that is equivalent to the system (1.0.1a)–(1.0.1c) in the wave-coordinate gauge. Finally, we summarize the results by providing the version (3.7.1a)–(3.7.1c) of the reduced equations, which will be used throughout the remainder of the article. In particular, in this version, we distinguish between principal terms, which require a careful treatment, and "error terms", which are, from the point of view of decay rates, relatively easy to estimate.

In this article, we consider the $(1+3)$-dimensional electrogravitational system (1.0.1a)–(1.0.1c), which we restate here for convenience:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \qquad (\mu, \nu = 0, 1, 2, 3), \tag{3.0.1a}$$

$$(d\mathscr{F})_{\lambda\mu\nu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{3.0.1b}$$

$$(d\mathscr{M})_{\lambda\mu\nu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3). \tag{3.0.1c}$$

We remark that the spacetimes we consider will always have the manifold structure $I \times \mathbb{R}^3$ for some "time" interval $I$. The energy-momentum tensor $T_{\mu\nu}$ is given below in (3.5.4a), while $\mathscr{M}_{\mu\nu}$ is related to $(g_{\mu\nu}, \mathscr{F}_{\mu\nu})$ via the constitutive relation (3.2.4). The precise forms of $T_{\mu\nu}$ and $\mathscr{M}_{\mu\nu}$ depend on the chosen model of electromagnetism, which, as is discussed in detail in Section 3.2, we assume is a Lagrangian-derived model subject to the restrictions (3.3.3a) and (3.3.4a)–(3.3.4b) below. We recall (see, e.g., [Christodoulou 2008; Wald 1984]) the following relationships between the *spacetime metric* $g_{\mu\nu}$, the *Riemann curvature tensor*[17] $R_{\mu\kappa\nu}{}^{\lambda}$, the *Ricci tensor* $R_{\mu\nu}$, the *scalar curvature* $R$, and the *Christoffel symbols* $\Gamma_{\mu}{}^{\kappa}{}_{\nu}$, which are valid in an arbitrary coordinate system:

$$R_{\mu\kappa\nu}{}^{\lambda} \overset{\text{def}}{=} \partial_\kappa \Gamma_\mu{}^\lambda{}_\nu - \partial_\mu \Gamma_\kappa{}^\lambda{}_\nu + \Gamma_\kappa{}^\lambda{}_\beta \Gamma_\mu{}^\beta{}_\nu - \Gamma_\mu{}^\lambda{}_\beta \Gamma_\kappa{}^\beta{}_\nu, \tag{3.0.2a}$$

$$R_{\mu\nu} \overset{\text{def}}{=} R_{\mu\kappa\nu}{}^{\kappa} = \partial_\kappa \Gamma_\mu{}^\kappa{}_\nu - \partial_\mu \Gamma_\kappa{}^\kappa{}_\nu + \Gamma_\kappa{}^\kappa{}_\lambda \Gamma_\mu{}^\lambda{}_\nu - \Gamma_\mu{}^\lambda{}_\kappa \Gamma_\lambda{}^\kappa{}_\nu, \tag{3.0.2b}$$

$$R \overset{\text{def}}{=} (g^{-1})^{\kappa\lambda} R_{\kappa\lambda}, \tag{3.0.2c}$$

$$\Gamma_\mu{}^\kappa{}_\nu \overset{\text{def}}{=} \tfrac{1}{2} (g^{-1})^{\kappa\lambda} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}). \tag{3.0.2d}$$

We also recall the following symmetry properties:

$$R_{\mu\nu} = R_{\nu\mu}, \tag{3.0.3}$$

$$\Gamma_\mu{}^\kappa{}_\nu = \Gamma_\nu{}^\kappa{}_\mu. \tag{3.0.4}$$

We note for future use that taking the trace with respect to $g$ of each side of (3.0.1a) implies that

$$R = -(g^{-1})^{\kappa\lambda} T_{\kappa\lambda}. \tag{3.0.5}$$

Hence, (3.0.1a) is equivalent to

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[17]Under our sign convention, $\mathscr{D}_\mu \mathscr{D}_\nu X_\kappa - \mathscr{D}_\nu \mathscr{D}_\mu X_\kappa = R_{\mu\nu\kappa}{}^{\lambda} X_\lambda$.

$$R_{\mu\nu} = T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}(g^{-1})^{\kappa\lambda}T_{\kappa\lambda}. \qquad (3.0.1a')$$

Furthermore, we note that the twice-contracted Bianchi identities (see, e.g., [Wald 1984]) are the relation (see Section 2.2 concerning our use of the notation #)

$$\mathscr{D}_{\mu}(R^{\#\mu\nu} - \tfrac{1}{2}(g^{-1})^{\mu\nu}R) = 0 \quad (\nu = 0, 1, 2, 3) \qquad (3.0.6)$$

so that by (3.0.1a) $T_{\mu\nu}$ necessarily satisfies the following divergence-free condition:

$$\mathscr{D}_{\mu}T^{\#\mu\nu} = 0 \quad (\nu = 0, 1, 2, 3). \qquad (3.0.7)$$

In the above expressions, $\mathscr{D}$ denotes the Levi-Civita connection corresponding to $g_{\mu\nu}$.

### 3.1. *Wave coordinates.*

In this article, we use the framework developed in [Lindblad and Rodnianski 2005; 2010] and work in a *wave-coordinate* system, which is defined to be a coordinate system in which

$$\Gamma^{\mu} \overset{\text{def}}{=} (g^{-1})^{\kappa\lambda}\Gamma_{\kappa}{}^{\mu}{}_{\lambda} = 0 \quad (\mu = 0, 1, 2, 3). \qquad (3.1.1a)$$

The condition (3.1.1a) is also known as *harmonic gauge* or *de Donder gauge*. It is easy to check that the condition (3.1.1a) is equivalent to the conditions

$$g_{\mu\nu}(g^{-1})^{\kappa\lambda}\Gamma_{\kappa}{}^{\nu}{}_{\lambda} = 0 \quad (\mu = 0, 1, 2, 3), \qquad (3.1.1b)$$

$$(g^{-1})^{\kappa\lambda}\,\partial_{\kappa}g_{\lambda\mu} - \tfrac{1}{2}(g^{-1})^{\kappa\lambda}\,\partial_{\mu}g_{\kappa\lambda} = 0 \quad (\mu = 0, 1, 2, 3), \qquad (3.1.1c)$$

$$\partial_{\nu}\big[\sqrt{|\det g|}(g^{-1})^{\mu\nu}\big] = 0 \quad (\mu = 0, 1, 2, 3). \qquad (3.1.1d)$$

We also note that condition (3.1.1d) follows from the identity

$$\Gamma^{\mu} \overset{\text{def}}{=} (g^{-1})^{\kappa\lambda}\Gamma_{\kappa}{}^{\mu}{}_{\lambda} = -\frac{1}{\sqrt{|\det g|}}\,\partial_{\nu}\big[\sqrt{|\det g|}(g^{-1})^{\mu\nu}\big] \quad (\mu = 0, 1, 2, 3), \qquad (3.1.2)$$

which holds in any coordinate system. Furthermore, if the wave-coordinate system is also interpreted to be a coordinate system in which the Minkowski metric takes the form $m_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, then all coordinate derivatives $\partial$ can be interpreted as covariant derivatives $\nabla$, where $\nabla$ is the Levi-Civita connection corresponding to the Minkowski metric. *Throughout the article, we will often take this point of view because it allows for a covariant interpretation of all of our equations.*

We remark that the use of wave coordinates in the study of the Einstein equations goes back at least to the work of de Donder [1921]. However, it was not until Choquet-Bruhat's [1952] fundamental work that it became clear that the Einstein equations are fundamentally hyperbolic in nature and that wave coordinates can be used to prove local well-posedness. See Section 4.3 for further discussion on the viability of using wave coordinates to analyze the system (3.0.1a)–(3.0.1c).

### 3.2. *The Lagrangian formulation of nonlinear electromagnetism.*

In this section, we recall some standard facts concerning a classical electromagnetic field theory in a Lorentzian spacetime ($\mathbb{R}^{1+3}$, $g_{\mu\nu}$). Our goal is to explain the origin of (3.0.1b)–(3.0.1c). We remark that, for our purposes in this section, we may assume that the spacetime is known. The fundamental quantity in such a classical electromagnetic field

theory is the *Faraday tensor* $\mathscr{F}_{\mu\nu}$, which is an antisymmetric type-$\binom{0}{2}$ tensor field (i.e., a two-form). We assume the *Faraday–Maxwell law*, which is the postulate that $\mathscr{F}_{\mu\nu}$ is closed:

$$(d\mathscr{F})_{\lambda\mu\nu} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{3.2.1}$$

where $d$ denotes the exterior derivative operator.

We restrict our attention to covariant theories of nonlinear electromagnetism arising from a Lagrangian $\mathscr{L}$. In such a theory, the Hodge dual[18] $^\star\mathscr{L}$ of $\mathscr{L}$ is a scalar-valued function of the two invariants of the Faraday tensor, which we denote by $\mathcal{F}_{(1)}$ and $\mathcal{F}_{(2)}$:

$$^\star\mathscr{L} = {}^\star\mathscr{L}(\mathcal{F}_{(1)}, \mathcal{F}_{(2)}), \tag{3.2.2a}$$

$$\mathcal{F}_{(1)} = \mathcal{F}_{(1)}[\mathscr{F}] \overset{\text{def}}{=} \tfrac{1}{2}(g^{-1})^{\kappa\mu}(g^{-1})^{\lambda\nu}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\mu\nu}, \tag{3.2.2b}$$

$$\mathcal{F}_{(2)} = \mathcal{F}_{(2)}[\mathscr{F}] \overset{\text{def}}{=} \tfrac{1}{4}(g^{-1})^{\kappa\mu}(g^{-1})^{\lambda\nu}\mathscr{F}_{\kappa\lambda}{}^\star\mathscr{F}_{\mu\nu} = \tfrac{1}{8}\epsilon^{\#\kappa\lambda\mu\nu}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\mu\nu}. \tag{3.2.2c}$$

Throughout the article, we use $\star$ to denote the Hodge duality operator corresponding to the spacetime metric $g_{\mu\nu}$:

$$^\star\mathscr{F}^{\#\mu\nu} \overset{\text{def}}{=} \tfrac{1}{2}\epsilon^{\#\mu\nu\kappa\lambda}\mathscr{F}_{\kappa\lambda}. \tag{3.2.3}$$

Here, $\epsilon^{\#\mu\nu\kappa\lambda}$ is totally antisymmetric with normalization $\epsilon^{\#0123} = -|\det g|^{-1/2}$ while $\epsilon_{\mu\nu\kappa\lambda}$ is totally antisymmetric with normalization $\epsilon_{0123} = |\det g|^{1/2}$. See Section 2.2 concerning our use of the notation #. We remind the reader that our main results are derived for a class of Lagrangians that satisfy certain assumptions; these assumptions are listed in (3.3.3a) and (3.3.4a)–(3.3.4b) below.

We now introduce the *Maxwell tensor* $\mathcal{M}_{\mu\nu}$, a two-form whose Hodge dual $^\star\mathcal{M}_{\mu\nu}$ is defined by

$$^\star\mathcal{M}^{\#\mu\nu} \overset{\text{def}}{=} \frac{\partial^\star\mathscr{L}}{\partial\mathscr{F}_{\mu\nu}} - \frac{\partial^\star\mathscr{L}}{\partial\mathscr{F}_{\nu\mu}}. \tag{3.2.4}$$

We also postulate that $\mathcal{M}_{\mu\nu}$ is closed:

$$(d\mathcal{M})_{\lambda\mu\nu} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3). \tag{3.2.5}$$

Taken together, (3.2.1) and (3.2.5) are the electromagnetic equations for $\mathscr{F}_{\mu\nu}$ corresponding to $^\star\mathscr{L}$.

We remark for future use that it is straightforward to verify that (3.2.1) is equivalent to any of

$$\mathscr{D}_\lambda\mathscr{F}_{\mu\nu} + \mathscr{D}_\nu\mathscr{F}_{\lambda\mu} + \mathscr{D}_\mu\mathscr{F}_{\nu\lambda} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{3.2.6a}$$

$$\nabla_\lambda\mathscr{F}_{\mu\nu} + \nabla_\nu\mathscr{F}_{\lambda\mu} + \nabla_\mu\mathscr{F}_{\nu\lambda} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{3.2.6b}$$

$$\mathscr{D}_\mu{}^\star\mathscr{F}^{\#\mu\nu} = 0 \qquad (\nu = 0, 1, 2, 3), \tag{3.2.6c}$$

$$\nabla_\mu{}^\circledast\mathscr{F}^{\mu\nu} = 0 \qquad (\nu = 0, 1, 2, 3) \tag{3.2.6d}$$

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[18]For brevity, we often refer to $^\star\mathscr{L}$ as the Lagrangian.

and that (3.2.5) is equivalent to any of

$$\mathcal{D}_\lambda \mathcal{M}_{\mu\nu} + \mathcal{D}_\nu \mathcal{M}_{\lambda\mu} + \mathcal{D}_\mu \mathcal{M}_{\nu\lambda} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{3.2.7a}$$

$$\nabla_\lambda \mathcal{M}_{\mu\nu} + \nabla_\nu \mathcal{M}_{\lambda\mu} + \nabla_\mu \mathcal{M}_{\nu\lambda} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{3.2.7b}$$

$$\mathcal{D}_\mu {}^\star\!\mathcal{M}^{\#\mu\nu} = 0 \qquad (\nu = 0, 1, 2, 3), \tag{3.2.7c}$$

$$\nabla_\mu {}^{\circledast}\!\mathcal{M}^{\mu\nu} = 0 \qquad (\nu = 0, 1, 2, 3). \tag{3.2.7d}$$

In the above formulas, $\circledast$ denotes the Hodge duality operator corresponding to the Minkowski metric $m_{\mu\nu}$; this operator is defined in Section 2.6.

We state as a lemma the following identities, which will be used for various computations. We leave the proof as a simple exercise for the reader.

**Lemma 3.1** (Basic identities). *The following identities hold*:

$$\frac{\partial |\det g|}{\partial g_{\mu\nu}} = |\det g| (g^{-1})^{\mu\nu}, \tag{3.2.8a}$$

$$\frac{\partial (g^{-1})^{\kappa\lambda}}{\partial g_{\mu\nu}} = -(g^{-1})^{\kappa\mu}(g^{-1})^{\lambda\nu}, \tag{3.2.8b}$$

$$\zeta_{(2)}^2 = |\det \mathcal{F}||\det g|^{-1}, \tag{3.2.8c}$$

$$(g^{-1})^{\kappa\lambda}\mathcal{F}_{\mu\kappa}\mathcal{F}_{\nu\lambda} - (g^{-1})^{\kappa\lambda}{}^\star\!\mathcal{F}_{\mu\kappa}{}^\star\!\mathcal{F}_{\nu\lambda} = \zeta_{(1)} g_{\mu\nu}, \tag{3.2.8d}$$

$$(g^{-1})^{\kappa\lambda}\mathcal{F}_{\mu\kappa}{}^\star\!\mathcal{F}_{\nu\lambda} = \zeta_{(2)} g_{\mu\nu}, \tag{3.2.8e}$$

$$\frac{\partial \zeta_{(1)}}{\partial g_{\mu\nu}} = -g_{\kappa\lambda}\mathcal{F}^{\#\mu\kappa}\mathcal{F}^{\#\nu\lambda}, \tag{3.2.8f}$$

$$\frac{\partial \zeta_{(2)}}{\partial g_{\mu\nu}} = -\tfrac{1}{2}\zeta_{(2)}(g^{-1})^{\mu\nu}, \tag{3.2.8g}$$

$$\frac{\partial \zeta_{(1)}}{\partial \mathcal{F}_{\mu\nu}} = \mathcal{F}^{\#\mu\nu}, \tag{3.2.8h}$$

$$\frac{\partial \zeta_{(2)}}{\partial \mathcal{F}_{\mu\nu}} = \tfrac{1}{2}{}^\star\!\mathcal{F}^{\#\mu\nu}, \tag{3.2.8i}$$

$$\frac{\partial \mathcal{F}^{\#\mu\nu}}{\partial \mathcal{F}_{\kappa\lambda}} = (g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda}, \tag{3.2.8j}$$

$$\frac{\partial {}^\star\!\mathcal{F}^{\#\mu\nu}}{\partial \mathcal{F}_{\kappa\lambda}} = \tfrac{1}{2}\epsilon^{\#\mu\nu\kappa\lambda}, \tag{3.2.8k}$$

$$\mathcal{D}_\mu \zeta_{(1)} = \mathcal{F}^{\#\kappa\lambda}\mathcal{D}_\mu \mathcal{F}_{\kappa\lambda} \qquad (\mu = 0, 1, 2, 3), \tag{3.2.8l}$$

$$\mathcal{D}_\mu \zeta_{(2)} = \tfrac{1}{2}{}^\star\!\mathcal{F}^{\#\kappa\lambda}\mathcal{D}_\mu \mathcal{F}_{\kappa\lambda} \qquad (\mu = 0, 1, 2, 3), \tag{3.2.8m}$$

$${}^\star\!\mathcal{M}^{\#\mu\nu} = 2\frac{\partial {}^\star\!\mathcal{L}}{\partial \zeta_{(1)}}\mathcal{F}^{\#\mu\nu} + \frac{\partial {}^\star\!\mathcal{L}}{\partial \zeta_{(2)}}{}^\star\!\mathcal{F}^{\#\mu\nu}. \tag{3.2.8n}$$

**3.3.** *Assumptions on the electromagnetic Lagrangian.* The standard Maxwell–Maxwell equations correspond to the Lagrangian

$$^\star\mathscr{L}_{(\text{Maxwell})} = -\tfrac{1}{2}\slashed{7}_{(1)}, \tag{3.3.1}$$

which by (3.2.8n) leads to the relationship

$$\mathscr{M}_{\mu\nu}^{(\text{Maxwell})} = {}^\star\mathscr{F}_{\mu\nu}. \tag{3.3.2}$$

Roughly speaking, we will assume that our electromagnetic Lagrangian is a covariant perturbation of $^\star\mathscr{L}_{(\text{Maxwell})}$. More precisely, we make the following assumptions concerning our Lagrangian $^\star\mathscr{L}$:

**Assumptions.** We assume that, in a neighborhood of $(0, 0)$, $^\star\mathscr{L}$ is an $(\ell + 2)$-times (where $\ell \geq 10$) continuously differentiable function of the invariants $(\slashed{7}_{(1)}, \slashed{7}_{(2)})$ that can be expanded as follows:

$$^\star\mathscr{L} = {}^\star\mathscr{L}_{(\text{Maxwell})} + O^{\ell+2}(|(\slashed{7}_{(1)}, \slashed{7}_{(2)})|^2). \tag{3.3.3a}$$

The notation $O^{\ell+2}(\,\cdot\,)$ is defined in Section 2.13.

We also assume that the corresponding energy-momentum tensor $T_{\mu\nu}$, which is defined below in (3.5.1), satisfies the *dominant energy condition*, which is the assumption that

$$T_{\kappa\lambda}X^\kappa Y^\lambda \geq 0 \tag{3.3.3b}$$

whenever the following conditions are satisfied:

- $X$ and $Y$ are both timelike (i.e., $g_{\kappa\lambda}X^\kappa X^\lambda < 0$ and $g_{\kappa\lambda}Y^\kappa Y^\lambda < 0$).

- $X$ and $Y$ are $g$-future-directed.

As discussed in, e.g., [Gibbons and Herdeiro 2001], sufficient conditions for the dominant energy condition to hold are

$$\frac{\partial\,^\star\mathscr{L}}{\partial\slashed{7}_{(1)}} < 0, \tag{3.3.4a}$$

$$^\star\mathscr{L} - \slashed{7}_{(1)}\frac{\partial\,^\star\mathscr{L}}{\partial\slashed{7}_{(1)}} - \slashed{7}_{(2)}\frac{\partial\,^\star\mathscr{L}}{\partial\slashed{7}_{(2)}} \leq 0. \tag{3.3.4b}$$

We remark that it is straightforward to verify the sufficiency of these conditions by using (3.5.4b) below and that condition (3.3.4b) is equivalent to the nonpositivity of the trace of the energy-momentum tensor corresponding to $^\star\mathscr{L}$. Furthermore, we recall that the trace vanishes in the case of the standard Maxwell–Maxwell model.

**Remark 3.2.** We make the $(\ell+2)$-times differentiability assumption because we will need to differentiate the equations (3.3.7) below $\ell$ times in order to prove our main stability theorem.

We will now derive an equivalent version of the electromagnetic equations that will be used throughout the remainder of the article. The final form, which is valid only in a wave-coordinate system, is given

below in Lemma 3.4. To begin, we use (3.2.6c), (3.2.7c), and (3.2.8n) to compute that the following equation holds:

$$-2\frac{\partial^\star\mathscr{L}}{\partial \cancel{\iota}_{(1)}}\mathscr{D}_\mu\mathscr{F}^{\#\mu\nu} - 2\mathscr{F}^{\#\mu\nu}\mathscr{D}_\mu\Big(\frac{\partial^\star\mathscr{L}}{\partial \cancel{\iota}_{(1)}}\Big) - {}^\star\mathscr{F}^{\#\mu\nu}\mathscr{D}_\mu\Big(\frac{\partial^\star\mathscr{L}}{\partial \cancel{\iota}_{(2)}}\Big) = 0. \qquad (3.3.5)$$

Furthermore, from the chain rule and the fact that $\mathscr{D}_\mu\phi = \nabla_\mu\phi$ for scalar-valued functions $\phi$, it follows from (3.3.5) and (3.2.8l)–(3.2.8m) that

$$-2\frac{\partial^\star\mathscr{L}}{\partial \cancel{\iota}_{(1)}}\mathscr{D}_\mu\mathscr{F}^{\#\mu\nu} - \Big(2\mathscr{F}^{\#\mu\nu}\frac{\partial^{2\star}\mathscr{L}}{\partial \cancel{\iota}_{(1)}^2} + {}^\star\mathscr{F}^{\#\mu\nu}\frac{\partial^{2\star}\mathscr{L}}{\partial \cancel{\iota}_{(1)}\,\partial \cancel{\iota}_{(2)}}\Big)\nabla_\mu \cancel{\iota}_{(1)}$$

$$-\Big(2\mathscr{F}^{\#\mu\nu}\frac{\partial^{2\star}\mathscr{L}}{\partial \cancel{\iota}_{(1)}\,\partial \cancel{\iota}_{(2)}} + {}^\star\mathscr{F}^{\#\mu\nu}\frac{\partial^{2\star}\mathscr{L}}{\partial \cancel{\iota}_{(2)}^2}\Big)\nabla_\mu \cancel{\iota}_{(2)} = 0. \quad (3.3.6)$$

We note for future use that (3.3.6) can be expressed as

$$N^{\#\mu\nu\kappa\lambda}\mathscr{D}_\mu\mathscr{F}_{\kappa\lambda} = 0 \quad (\nu = 0, 1, 2, 3), \qquad (3.3.7)$$

where the tensor field $N^{\#\mu\nu\kappa\lambda}$ is defined by

$$N^{\#\mu\nu\kappa\lambda} \overset{\text{def}}{=} -\frac{\partial^\star\mathscr{L}}{\partial \cancel{\iota}_{(1)}}\big((g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda} - (g^{-1})^{\mu\lambda}(g^{-1})^{\nu\kappa}\big) - 2\frac{\partial^{2\star}\mathscr{L}}{\partial \cancel{\iota}_{(1)}^2}\mathscr{F}^{\#\mu\nu}\mathscr{F}^{\#\kappa\lambda}$$

$$-\frac{\partial^{2\star}\mathscr{L}}{\partial \cancel{\iota}_{(1)}\partial \cancel{\iota}_{(2)}}\big(\mathscr{F}^{\#\mu\nu}{}^\star\mathscr{F}^{\#\kappa\lambda} + {}^\star\mathscr{F}^{\#\mu\nu}\mathscr{F}^{\#\kappa\lambda}\big) - \frac{1}{2}\frac{\partial^{2\star}\mathscr{L}}{\partial \cancel{\iota}_{(2)}^2}{}^\star\mathscr{F}^{\#\mu\nu}{}^\star\mathscr{F}^{\#\kappa\lambda}. \quad (3.3.8)$$

We also note that $N^{\#\mu\nu\kappa\lambda}$ has the following symmetry properties, which will play an important role during our construction of suitable energies for $\mathscr{F}_{\mu\nu}$ (and in particular during our proof of Lemma 8.5):

$$N^{\#\nu\mu\kappa\lambda} = -N^{\#\mu\nu\kappa\lambda} \quad (\kappa, \lambda, \mu, \nu = 0, 1, 2, 3), \qquad (3.3.9\text{a})$$

$$N^{\#\mu\nu\lambda\kappa} = -N^{\#\mu\nu\kappa\lambda} \quad (\kappa, \lambda, \mu, \nu = 0, 1, 2, 3), \qquad (3.3.9\text{b})$$

$$N^{\#\kappa\lambda\mu\nu} = N^{\#\mu\nu\kappa\lambda} \quad (\kappa, \lambda, \mu, \nu = 0, 1, 2, 3). \qquad (3.3.9\text{c})$$

The moral reason that the above properties are satisfied is that $N^{\#\mu\nu\kappa\lambda}$ is closely related to the Hessian of ${}^\star\mathscr{L}$ (with respect to $\mathscr{F}$):

$$N^{\#\mu\nu\kappa\lambda} = -\frac{1}{2}\frac{\partial^{2\star}\mathscr{L}}{\partial\mathscr{F}_{\mu\nu}\,\partial\mathscr{F}_{\kappa\lambda}} + \frac{1}{2}\frac{\partial^\star\mathscr{L}}{\partial \cancel{\iota}_{(2)}}\epsilon^{\#\mu\nu\kappa\lambda}. \qquad (3.3.10)$$

We have added the last term on the right-hand side of (3.3.10) in order to cancel a term appearing in the Hessian; this is permissible because (3.2.6a) implies that this term does not contribute to (3.3.7).

Our next goal is to formulate a "reduced" electromagnetic equation that is equivalent to (3.3.7) in a wave-coordinate system. We also decompose the reduced equation into the principal terms and error terms of an equation involving the Minkowski connection $\nabla$. This is accomplished in Lemma 3.4 below. Before proving this lemma, we first provide the following preliminary lemma, whose simple proof is left to the reader:

**Lemma 3.3** (Expansions). *Assume that the electromagnetic Lagrangian $^\star\mathscr{L}$ satisfies (3.3.3a). Then in terms of the expansion $h_{\mu\nu} \overset{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$ from (1.2.1a) and with $H^{\mu\nu} \overset{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$, we have*

$$H^{\mu\nu} = -h^{\mu\nu} + O^\infty(|h|^2)$$
$$= -h^{\mu\nu} + O^\infty(|H|^2), \tag{3.3.11a}$$

$$\nabla_\lambda (g^{-1})^{\mu\nu} = -(g^{-1})^{\mu\mu'}(g^{-1})^{\nu\nu'}\nabla_\lambda h_{\mu'\nu'}$$
$$= -(m^{-1})^{\mu\mu'}(m^{-1})^{\nu\nu'}\nabla_\lambda h_{\mu'\nu'} + O^\infty(|h||\nabla h|), \tag{3.3.11b}$$

$$|\det g| = 1 + (m^{-1})^{\kappa\lambda}h_{\kappa\lambda} + O^\infty(|h|^2)$$
$$= 1 - m_{\kappa\lambda}H^{\kappa\lambda} + O^\infty(|H|^2), \tag{3.3.11c}$$

$$|\det g|^{1/2} = 1 + \tfrac{1}{2}(m^{-1})^{\kappa\lambda}h_{\kappa\lambda} + O^\infty(|h|^2)$$
$$= 1 - \tfrac{1}{2}m_{\kappa\lambda}H^{\kappa\lambda} + O^\infty(|H|^2), \tag{3.3.11d}$$

$$|\det g|^{-1/2} = 1 - \tfrac{1}{2}(m^{-1})^{\kappa\lambda}h_{\kappa\lambda} + O^\infty(|h|^2)$$
$$= 1 + \tfrac{1}{2}m_{\kappa\lambda}H^{\kappa\lambda} + O^\infty(|H|^2), \tag{3.3.11e}$$

$$\epsilon^{\#\mu\nu\kappa\lambda} = -(1 + O^\infty(|h|))[\mu\nu\kappa\lambda], \tag{3.3.11f}$$

$$\epsilon_{\mu\nu\kappa\lambda} = (1 + O^\infty(|h|))[\mu\nu\kappa\lambda], \tag{3.3.11g}$$

$$\mathscr{F}^{\#\mu\nu} = \mathscr{F}^{\mu\nu} + O^\infty(|h||\mathscr{F}|) \overset{\text{def}}{=} (m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda}\mathscr{F}_{\kappa\lambda} + O^\infty(|h||\mathscr{F}|), \tag{3.3.11h}$$

$$^\star\mathscr{F}_{\mu\nu} = {}^\circledast\mathscr{F}_{\mu\nu} + O^\infty(|h||\mathscr{F}|) \overset{\text{def}}{=} -\tfrac{1}{2}m_{\mu\mu'}m_{\nu\nu'}[\mu'\nu'\kappa\lambda]\mathscr{F}_{\kappa\lambda} + O^\infty(|h||\mathscr{F}|), \tag{3.3.11i}$$

$$\mathscr{G}_{(1)} = \tfrac{1}{2}(m^{-1})^{\kappa\mu}(m^{-1})^{\lambda\nu}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\mu\nu} + O^\infty(|h||\mathscr{F}|^2), \tag{3.3.11j}$$

$$\mathscr{G}_{(2)} = -\tfrac{1}{8}[\mu\nu\kappa\lambda]\mathscr{F}_{\mu\nu}\mathscr{F}_{\kappa\lambda} + O^\infty(|h||\mathscr{F}|^2), \tag{3.3.11k}$$

$$^\star\mathscr{L} = -\tfrac{1}{4}(m^{-1})^{\eta\kappa}(m^{-1})^{\zeta\lambda}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\eta\zeta} + O^{\ell+2}(|h||\mathscr{F}|^2) + O^{\ell+2}(|\mathscr{F}|^4; h), \tag{3.3.11l}$$

$$\nabla\mathscr{G}_{(i)} = O^\infty(|\mathscr{F}||\nabla\mathscr{F}|) + O^\infty(|\nabla h||\mathscr{F}|^2; h) + O^\infty(|h||\mathscr{F}||\nabla\mathscr{F}|), \tag{3.3.11m}$$

$$\mathscr{M}_{\mu\nu} = {}^\circledast\mathscr{F}_{\mu\nu} + O^{\ell+1}(|h||\mathscr{F}|) + O^{\ell+1}(|\mathscr{F}|^3; h). \tag{3.3.11n}$$

*In (3.3.11f)–(3.3.11g), $[\mu\nu\kappa\lambda]$ is totally antisymmetric with normalization $[0123] = 1$, $\star$ denotes the Hodge duality operator corresponding to the spacetime metric $g_{\mu\nu}$, and $\circledast$ denotes the Hodge duality operator corresponding to the Minkowski metric $m_{\mu\nu}$. Furthermore, the notation $O(\cdot)$ is defined in Section 2.13.*

### 3.4. *The reduced electromagnetic equations.*  In this section, we provide the aforementioned decomposition of the reduced electromagnetic equations.

**Lemma 3.4** (The reduced electromagnetic equations). *Assume that the wave-coordinate condition (3.1.1a) holds. Then in terms of the expansion (1.2.1a), the system of electromagnetic equations (3.2.1) and (3.3.7) is equivalent to the following reduced system of equations:*

$$\nabla_\lambda \mathscr{F}_{\mu\nu} + \nabla_\mu \mathscr{F}_{\nu\lambda} + \nabla_\nu \mathscr{F}_{\lambda\mu} = 0, \tag{3.4.1a}$$

$$N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda} = \mathscr{Q}^\nu_{(2;\mathscr{F})}(\nabla h, \mathscr{F}) + O^\ell(|h||\nabla h||\mathscr{F}|) + O^\ell(|\nabla h||\mathscr{F}|^2; h), \tag{3.4.1b}$$

*where*

$$\begin{aligned}
N^{\#\mu\nu\kappa\lambda} = &\tfrac{1}{2}\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\\
&+ \tfrac{1}{2}\big(-h^{\mu\kappa}(m^{-1})^{\nu\lambda} + h^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\\
&+ \tfrac{1}{2}\big(-(m^{-1})^{\mu\kappa}h^{\nu\lambda} + (m^{-1})^{\mu\lambda}h^{\nu\kappa}\big) + N^{\#\mu\nu\kappa\lambda}_\triangle,
\end{aligned} \tag{3.4.2}$$

$$\mathscr{Q}^\nu_{(2;\mathscr{F})}(\nabla h, \mathscr{F}) = (m^{-1})^{\mu\kappa}(m^{-1})^{\nu\nu'}(m^{-1})^{\lambda\lambda'}(\nabla_\mu h_{\nu'\lambda'})\mathscr{F}_{\kappa\lambda}. \tag{3.4.3}$$

*Furthermore,*

$$N^{\#\mu\nu\kappa\lambda}_\triangle = O^\ell\big(|(h, \mathscr{F})|^2\big), \tag{3.4.4}$$

*and like* $N^{\#\mu\nu\kappa\lambda}$*, the tensor field* $N^{\#\mu\nu\kappa\lambda}_\triangle$ *also possesses the symmetry properties* (3.3.9a)–(3.3.9c).

**Remark 3.5.** Equations (3.4.1a)–(3.4.3) are equivalent to (3.2.1) and (3.3.7) only in a wave-coordinate system. Hence, we refer to (3.4.1a)–(3.4.3) as the "reduced" electromagnetic equations.

*Proof.* We use the assumption (3.3.3a) and the Leibniz rule to expand (3.3.6) and apply the results of Lemma 3.3, arriving at the following expansion:

$$\mathscr{D}_\mu \mathscr{F}^{\#\mu\nu} + \widetilde{N}^{\mu\nu\kappa\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda} = O^\ell(|h||\nabla h||\mathscr{F}|) + O^\ell(|\nabla h||\mathscr{F}|^2; h), \tag{3.4.5}$$

where $\widetilde{N}^{\mu\nu\kappa\lambda} = O^\ell(|(h, \mathscr{F})|^2)$. Let us now decompose the $\mathscr{D}_\mu \mathscr{F}^{\#\mu\nu}$ term. Using the antisymmetry of $\mathscr{F}^{\#\mu\nu}$, the symmetry of the Christoffel symbol $\Gamma_{\mu\ \lambda}^{\ \nu}$ under the exchanges $\mu \leftrightarrow \lambda$, the identity $\Gamma_{\kappa\ \mu}^{\ \kappa} = (1/\sqrt{|\det g|})\nabla_\mu(\sqrt{|\det g|})$, and the wave-coordinate condition $\nabla_\mu[\sqrt{|\det g|}(g^{-1})^{\mu\kappa}] = 0$ ($\kappa = 0, 1, 2, 3$), we have that

$$\begin{aligned}
\mathscr{D}_\mu \mathscr{F}^{\#\mu\nu} &= \nabla_\mu \mathscr{F}^{\#\mu\nu} + \Gamma_{\kappa\ \mu}^{\ \kappa}\mathscr{F}^{\#\mu\nu} + \Gamma_{\mu\ \lambda}^{\ \nu}\mathscr{F}^{\#\mu\lambda}\\
&= \nabla_\mu\big[(g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda}\mathscr{F}_{\kappa\lambda}\big] + \left[\frac{1}{\sqrt{|\det g|}}\nabla_\mu(\sqrt{|\det g|})\right](g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda}\mathscr{F}_{\kappa\lambda}\\
&= \frac{1}{\sqrt{|\det g|}}\nabla_\mu\big[\sqrt{|\det g|}(g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda}\mathscr{F}_{\kappa\lambda}\big]\\
&= (g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda} + \big[(g^{-1})^{\mu\kappa}\nabla_\mu(g^{-1})^{\nu\lambda}\big]\mathscr{F}_{\kappa\lambda}. \tag{3.4.6}
\end{aligned}$$

Using (3.3.11a), we conclude that the term $(g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda}$ on the right-hand side of (3.4.6) can be expressed as the terms in parentheses on the right-hand side of (3.4.2) plus $O^\ell(|h^2|)\nabla_\mu \mathscr{F}_{\kappa\lambda}$.

Similarly, using (3.3.11b), we conclude that the term $[(g^{-1})^{\mu\kappa}\nabla_\mu(g^{-1})^{\nu\lambda}]\mathscr{F}_{\kappa\lambda}$ on the right-hand side of (3.4.6) is equal to $-\mathscr{Q}^\nu_{(2;\mathscr{F})}(\nabla h, \mathscr{F}) + O^\ell(|h||\nabla h||\mathscr{F}|)$, where $\mathscr{Q}^\nu_{(2;\mathscr{F})}(\nabla h, \mathscr{F})$ is defined in (3.4.3). Combining these expansions with (3.4.5), we arrive at (3.4.1b)–(3.4.4).

The fact that $N^{\#\mu\nu\kappa\lambda}_\triangle$ possesses the symmetry properties (3.3.9a)–(3.3.9c) follows trivially from the fact that both $N^{\#\mu\nu\kappa\lambda}$ and the term in parentheses on the right-hand side of (3.4.2) have these properties. $\square$

**Remark 3.6.** With the help of the identity (3.1.2), the above proof shows that the reduced equation (3.4.1b) is obtained by adding the inhomogeneous term $-\Gamma^{\kappa}(g^{-1})^{\nu\lambda}\mathscr{F}_{\kappa\lambda}$ to the right-hand side of (3.3.7). That is, (3.4.1b) is equivalent to

$$N^{\#\mu\nu\kappa\lambda}\mathscr{D}_{\mu}\mathscr{F}_{\kappa\lambda} = -\Gamma^{\kappa}(g^{-1})^{\nu\lambda}\mathscr{F}_{\kappa\lambda}. \tag{3.4.7}$$

We will use this fact in our proof of Proposition 4.2.

**3.5.** *The energy-momentum tensor.* In this section, we discuss the energy-momentum tensor $T_{\mu\nu}$ appearing on the right-hand side of (3.0.1a). We recall that the energy-momentum tensor for an electromagnetic Lagrangian field theory is defined as follows:

$$T^{\#\mu\nu} \stackrel{\text{def}}{=} 2\frac{\partial^{\star}\mathscr{L}}{\partial g_{\mu\nu}} + (g^{-1})^{\mu\nu\star}\mathscr{L}. \tag{3.5.1}$$

It follows trivially from the definition (3.5.1) that $T_{\mu\nu}$ is symmetric:

$$T_{\mu\nu} = T_{\nu\mu} \quad (\mu, \nu = 0, 1, 2, 3). \tag{3.5.2}$$

Furthermore, we recall that, if $\mathscr{F}_{\mu\nu}$ is a solution to the (nonreduced) electromagnetic equations (3.0.1b)–(3.0.1c), then

$$\mathscr{D}_{\mu}T^{\#\mu\nu} = 0 \quad (\nu = 0, 1, 2, 3). \tag{3.5.3}$$

For the class of electromagnetic energy-momentum tensors considered in this article, we can use the chain rule and Lemma 3.1 to express $T_{\mu\nu}$ as follows:

$$T_{\mu\nu} = -2\frac{\partial^{\star}\mathscr{L}}{\partial\mathscr{I}_{(1)}}(g^{-1})^{\kappa\lambda}\mathscr{F}_{\mu\kappa}\mathscr{F}_{\nu\lambda} - \mathscr{I}_{(2)}\frac{\partial^{\star}\mathscr{L}}{\partial\mathscr{I}_{(2)}}g_{\mu\nu} + g_{\mu\nu}{}^{\star}\mathscr{L} \tag{3.5.4a}$$

$$= -2\frac{\partial^{\star}\mathscr{L}}{\partial\mathscr{I}_{(1)}}T_{\mu\nu}^{(\text{Maxwell})} + \tfrac{1}{4}Tg_{\mu\nu}, \tag{3.5.4b}$$

where

$$T_{\mu\nu}^{(\text{Maxwell})} \stackrel{\text{def}}{=} (g^{-1})^{\kappa\lambda}\mathscr{F}_{\mu\kappa}\mathscr{F}_{\nu\lambda} - \tfrac{1}{2}\mathscr{I}_{(1)}g_{\mu\nu} \tag{3.5.5}$$

is the energy-momentum tensor corresponding to the standard Maxwell–Maxwell equations and

$$T \stackrel{\text{def}}{=} (g^{-1})^{\kappa\lambda}T_{\kappa\lambda} = 4\left({}^{\star}\mathscr{L} - \mathscr{I}_{(1)}\frac{\partial^{\star}\mathscr{L}}{\partial\mathscr{I}_{(1)}} - \mathscr{I}_{(2)}\frac{\partial^{\star}\mathscr{L}}{\partial\mathscr{I}_{(2)}}\right) \tag{3.5.6}$$

is the trace of $T_{\mu\nu}$ with respect to $g_{\mu\nu}$. Furthermore, from (3.5.4a) and the expansions of Lemma 3.3, it follows that

$$T_{\mu\nu} = (m^{-1})^{\kappa\lambda}\mathscr{F}_{\mu\kappa}\mathscr{F}_{\nu\lambda} - \tfrac{1}{4}m_{\mu\nu}(m^{-1})^{\kappa\eta}(m^{-1})^{\lambda\xi}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\eta\xi} + O^{\ell+1}(|h||\mathscr{F}|^2) + O^{\ell+1}(|\mathscr{F}|^3; h). \tag{3.5.7}$$

We now compute the right-hand side of (3.0.1a′). First, taking the trace of (3.5.7) with respect to $g$, we compute that

$$(g^{-1})^{\kappa\lambda}T_{\kappa\lambda} = O^{\ell+1}(|h||\mathscr{F}|^2) + O^{\ell+1}(|\mathscr{F}|^3; h). \tag{3.5.8}$$

Combining (3.5.7) and (3.5.8) and using the expansion (1.2.1a), we have that the right-hand side of (3.0.1a') can be expressed as follows:

$$T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}(g^{-1})^{\kappa\lambda}T_{\kappa\lambda} = (m^{-1})^{\kappa\lambda}\mathscr{F}_{\mu\kappa}\mathscr{F}_{\nu\lambda} - \tfrac{1}{4}m_{\mu\nu}(m^{-1})^{\kappa\eta}(m^{-1})^{\lambda\zeta}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\eta\zeta}$$
$$+ O^{\ell+1}(|h||\mathscr{F}|^2) + O^{\ell+1}(|\mathscr{F}|^3; h). \quad (3.5.9)$$

To conclude this section, we note for future use that, if $\mathscr{F}_{\mu\nu}$ is a solution to the inhomogeneous system

$$\nabla_\lambda \mathscr{F}_{\mu\nu} + \nabla_\mu \mathscr{F}_{\nu\lambda} + \nabla_\nu \mathscr{F}_{\lambda\mu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3), \qquad (3.5.10\text{a})$$

$$N^{\#\mu\nu\kappa\lambda}\mathscr{D}_\mu \mathscr{F}_{\kappa\lambda} = \mathfrak{I}^\nu \qquad\qquad (\nu = 0, 1, 2, 3), \qquad (3.5.10\text{b})$$

then with the help of Lemma 3.1, it can be shown that the following identity holds:

$$(g^{-1})^{\kappa\lambda}\mathscr{D}_\kappa T_{\lambda\nu} = \mathfrak{I}^\kappa \mathscr{F}_{\nu\kappa} \quad (\nu = 0, 1, 2, 3). \qquad (3.5.11)$$

We will use this fact in our proof of Proposition 4.2 (which shows that the wave-coordinate gauge is preserved by the flow of the reduced equations), where $\mathfrak{I}^\nu$ will be equal to the right-hand side of (3.4.7). We also remark that (3.5.3) corresponds to the special case $\mathfrak{I}^\nu = 0$ ($\nu = 0, 1, 2, 3$).

**3.6. *The modified Ricci tensor.*** Throughout the remainder of this article, we perform the standard wave-coordinate system procedure (see, e.g., [Wald 1984]) of replacing the Ricci tensor $R_{\mu\nu}$ in the Einstein field equation (3.0.1a) with a modified Ricci tensor $\widetilde{R}_{\mu\nu}$. As we will soon see, this replacement transforms equations (3.0.1a) into a system of quasilinear wave equations.

**Definition 3.7.** We define the *modified Ricci tensor* $\widetilde{R}_{\mu\nu}$ of the metric $g_{\mu\nu}$ as follows:

$$\widetilde{R}_{\mu\nu} \overset{\text{def}}{=} R_{\mu\nu} - \tfrac{1}{2}(g_{\kappa\nu}\mathscr{D}_\mu \Gamma^\kappa + g_{\kappa\mu}\mathscr{D}_\nu \Gamma^\kappa) + u_{\mu\nu\kappa}(g, g^{-1}, \partial g)\Gamma^\kappa, \qquad (3.6.1)$$

where the Ricci tensor $R_{\mu\nu}$ is defined in (3.0.2b) and the "gauge term" $u_{\mu\nu\kappa}(g, g^{-1}, \partial g)\Gamma^\kappa$ is a smooth function of $g$, $g^{-1}$, and $\partial g$ that will be discussed in Lemma 3.8. We remark that, for purposes of covariant differentiation by $\mathscr{D}$ in (3.6.1), the $\Gamma^\mu$ are treated as the components of a vector field.

In the next lemma, we provide an algebraic decomposition of the modified Ricci tensor.

**Lemma 3.8** (Decomposition of the modified Ricci tensor [Lindblad and Rodnianski 2005, Lemmas 3.1 and 3.2]). *For a suitable choice of the gauge term $u_{\mu\nu\kappa}(g, g^{-1}, \partial g)\Gamma^\kappa$, the modified Ricci tensor $\widetilde{R}_{\mu\nu}$ of the metric $g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}$ can be decomposed as follows*:

$$\widetilde{R}_{\mu\nu} = -\tfrac{1}{2}\big(\widetilde{\Box}_g g_{\mu\nu} - \mathscr{P}(\nabla_\mu h, \nabla_\nu h) - \mathscr{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h)\big) + O^\infty(|h||\nabla h|^2), \qquad (3.6.2)$$

*where*

$$\widetilde{\Box}_g \overset{\text{def}}{=} (g^{-1})^{\kappa\lambda}\nabla_\kappa \nabla_\lambda \qquad (3.6.3)$$

*is the* reduced wave operator *corresponding to $g_{\mu\nu}$ and the quadratic terms $\mathscr{P}(\nabla_\mu \cdot, \nabla_\nu \cdot)$ and $\mathscr{Q}_{\mu\nu}^{(1;h)}(\cdot, \cdot)$ are defined by their action on tensor fields $\Pi_{\mu\nu}$, $\Theta_{\mu\nu}$, and $h_{\mu\nu}$ as follows*:

$$\mathscr{P}(\nabla_\mu \Pi, \nabla_\nu \Theta) \stackrel{\text{def}}{=} \tfrac{1}{4}(\nabla_\mu \Pi_\kappa{}^\kappa)(\nabla_\nu \Theta_\lambda{}^\lambda) - \tfrac{1}{2}(\nabla_\mu \Pi^{\kappa\lambda})(\nabla_\nu \Theta_{\kappa\lambda}), \tag{3.6.4}$$

$$\begin{aligned}
\mathscr{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h) \stackrel{\text{def}}{=}\ & (m^{-1})^{\lambda\lambda'}\mathscr{Q}_0(\nabla h_{\lambda\mu}, \nabla h_{\lambda'\nu}) \\
& - (m^{-1})^{\kappa\kappa'}(m^{-1})^{\lambda\lambda'}\mathscr{Q}_{\kappa\lambda'}(\nabla h_{\lambda\mu}, \nabla h_{\kappa'\nu}) \\
& + (m^{-1})^{\kappa\kappa'}(m^{-1})^{\lambda\lambda'}\mathscr{Q}_{\mu\kappa}(\nabla h_{\kappa'\lambda'}, \nabla h_{\lambda\nu}) \\
& + (m^{-1})^{\kappa\kappa'}(m^{-1})^{\lambda\lambda'}\mathscr{Q}_{\nu\kappa}(\nabla h_{\kappa'\lambda'}, \nabla h_{\lambda\mu}) \\
& + \tfrac{1}{2}(m^{-1})^{\kappa\kappa'}(m^{-1})^{\lambda\lambda'}\mathscr{Q}_{\lambda'\mu}(\nabla h_{\kappa\kappa'}, \nabla h_{\lambda\nu}) \\
& + \tfrac{1}{2}(m^{-1})^{\kappa\kappa'}(m^{-1})^{\lambda\lambda'}\mathscr{Q}_{\lambda'\nu}(\nabla h_{\kappa\kappa'}, \nabla h_{\lambda\mu}).
\end{aligned} \tag{3.6.5}$$

*The bilinear forms $\mathscr{Q}_0(\,\cdot\,,\cdot\,)$ and $\mathscr{Q}_{\mu\nu}(\,\cdot\,,\cdot\,)$, which appear on the right-hand side of* (3.6.5), *are known as the* standard null forms. *They are defined through their action on the derivatives of scalar-valued functions $\psi$ and $\chi$ by*

$$\mathscr{Q}_0(\nabla\psi, \nabla\chi) \stackrel{\text{def}}{=} (m^{-1})^{\kappa\lambda}(\nabla_\kappa \psi)(\nabla_\lambda \chi), \tag{3.6.6a}$$

$$\mathscr{Q}_{\mu\nu}(\nabla\psi, \nabla\chi) \stackrel{\text{def}}{=} (\nabla_\mu \psi)(\nabla_\nu \chi) - (\nabla_\nu \psi)(\nabla_\mu \chi). \tag{3.6.6b}$$

*Proof.* This decomposition is carried out in Lemmas 3.1 and 3.2 of [Lindblad and Rodnianski 2005]. □

We conclude this section by observing that (3.0.1a′), (3.5.9), and (3.6.2) together imply that under the wave-coordinate condition (3.1.1a), and under the assumption (3.3.3a) on the Lagrangian, the Einstein field equation (3.0.1a) is equivalent to the following equation:

$$\widetilde{\Box}_g g_{\mu\nu} = \mathscr{P}(\nabla_\mu h, \nabla_\nu h) + \mathscr{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h) - 2(m^{-1})^{\kappa\lambda}\mathscr{F}_{\mu\kappa}\mathscr{F}_{\nu\lambda} + \tfrac{1}{2}m_{\mu\nu}(m^{-1})^{\kappa\eta}(m^{-1})^{\lambda\zeta}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\eta\zeta}$$
$$+ O^\infty(|h||\nabla h|^2) + O^{\ell+1}(|h||\mathscr{F}|^2) + O^{\ell+1}(|\mathscr{F}|^3; h). \tag{3.6.7}$$

**3.7. *Summary of the reduced system.*** In this section, we summarize the above results by stating the form of the reduced Einstein-nonlinear electromagnetic system that we work with for most of the remainder of the article, namely (3.7.1a)–(3.7.1c); the derivation of this version of the reduced equations follows easily from the previous results of Section 3. We remind the reader that the reduced equations are obtained by adding the inhomogeneous term $-\Gamma^\kappa (g^{-1})^{\nu\lambda}\mathscr{F}_{\kappa\lambda}$ to the right-hand side of (3.3.7) and by substituting the modified Ricci tensor in place of the Ricci tensor in (3.0.1a). Furthermore, in a wave-coordinate system, the reduced system is equivalent to the system (3.0.1a)–(3.0.1c) (see Proposition 4.2).

**Reduced system.** The reduced system (where $g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}$ and the unknowns are viewed to be $(h_{\mu\nu}^{(1)}, \mathscr{F}_{\mu\nu})$) can be expressed as

$$\widetilde{\Box}_g h_{\mu\nu}^{(1)} = \mathfrak{H}_{\mu\nu} - \widetilde{\Box}_g h_{\mu\nu}^{(0)} \qquad (\mu, \nu = 0, 1, 2, 3), \tag{3.7.1a}$$

$$\nabla_\lambda \mathscr{F}_{\mu\nu} + \nabla_\mu \mathscr{F}_{\nu\lambda} + \nabla_\nu \mathscr{F}_{\lambda\mu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{3.7.1b}$$

$$N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda} = \mathfrak{F}^\nu \qquad (\nu = 0, 1, 2, 3), \tag{3.7.1c}$$

where $\widetilde{\Box}_g \stackrel{\text{def}}{=} (g^{-1})^{\kappa\lambda}\nabla_\kappa \nabla_\lambda$ is the *reduced wave operator* corresponding to $g_{\mu\nu}$.

The quantities $\mathfrak{H}_{\mu\nu}$, $N^{\#\mu\nu\kappa\lambda}$, and $\mathfrak{F}^\nu$ can be decomposed into principal terms and error terms (which are denoted with a "$\triangle$") as follows:

$$\mathfrak{H}_{\mu\nu} = \mathscr{P}(\nabla_\mu h, \nabla_\nu h) + \mathscr{Q}^{(1;h)}_{\mu\nu}(\nabla h, \nabla h) + \mathscr{Q}^{(2;h)}_{\mu\nu}(\mathscr{F}, \mathscr{F}) + \mathfrak{H}^\triangle_{\mu\nu}, \tag{3.7.2a}$$

$$\mathfrak{F}^\nu = \mathscr{Q}^\nu_{(2;\mathscr{F})}(\nabla h, \mathscr{F}) + \mathfrak{F}^\nu_\triangle, \tag{3.7.2b}$$

$$\begin{aligned}
N^{\#\mu\nu\kappa\lambda} = {} & \tfrac{1}{2}((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}) \\
& + \tfrac{1}{2}(-h^{\mu\kappa}(m^{-1})^{\nu\lambda} + h^{\mu\lambda}(m^{-1})^{\nu\kappa}) \\
& + \tfrac{1}{2}(-(m^{-1})^{\mu\kappa}h^{\nu\lambda} + (m^{-1})^{\mu\lambda}h^{\nu\kappa}) + N^{\#\mu\nu\kappa\lambda}_\triangle,
\end{aligned} \tag{3.7.2c}$$

where $\mathscr{P}(\nabla_\mu h, \nabla_\nu h)$ is defined in (3.6.4), $\mathscr{Q}^{(1;h)}_{\mu\nu}(\nabla h, \nabla h)$ is defined in (3.6.5), and

$$\mathscr{Q}^{(2;h)}_{\mu\nu}(\mathscr{F}, \mathscr{G}) = -2(m^{-1})^{\kappa\lambda}\mathscr{F}_{\mu\kappa}\mathscr{G}_{\nu\lambda} + \tfrac{1}{2}m_{\mu\nu}(m^{-1})^{\kappa\lambda}(m^{-1})^{\lambda\kappa}\mathscr{F}_{\kappa\lambda}\mathscr{G}_{\kappa\lambda}, \tag{3.7.2d}$$

$$\mathscr{Q}^\nu_{(2;\mathscr{F})}(\nabla h, \mathscr{F}) = (m^{-1})^{\mu\kappa}(m^{-1})^{\lambda\lambda'}(m^{-1})^{\nu\nu'}(\nabla_\mu h_{\nu'\lambda'})\mathscr{F}_{\kappa\lambda}, \tag{3.7.2e}$$

$$\mathfrak{H}^\triangle_{\mu\nu} = O^\infty(|h||\nabla h|^2) + O^{\ell+1}(|h||\mathscr{F}|^2) + O^{\ell+1}(|\mathscr{F}|^3; h), \tag{3.7.2f}$$

$$\mathfrak{F}^\nu_\triangle = O^\ell(|h||\nabla h||\mathscr{F}|) + O^\ell(|\nabla h||\mathscr{F}|^2; h), \tag{3.7.2g}$$

$$N^{\#\mu\nu\kappa\lambda}_\triangle = O^\ell(|(h, \mathscr{F})|^2). \tag{3.7.2h}$$

Furthermore, the left-hand side of (3.7.1c) can be expressed as

$$\begin{aligned}
N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda} = {} & \tfrac{1}{2}\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu \mathscr{F}_{\kappa\lambda} \\
& - \mathscr{P}^\nu_{(\mathscr{F})}(h, \nabla\mathscr{F}) - \mathscr{Q}^\nu_{(1;\mathscr{F})}(h, \nabla\mathscr{F}) + N^{\#\mu\nu\kappa\lambda}_\triangle \nabla_\mu \mathscr{F}_{\kappa\lambda},
\end{aligned} \tag{3.7.3a}$$

where

$$\mathscr{P}^\nu_{(\mathscr{F})}(h, \nabla\mathscr{F}) = (m^{-1})^{\mu\mu'}(m^{-1})^{\kappa\kappa'}(m^{-1})^{\nu\lambda}h_{\mu'\kappa'}\nabla_\mu \mathscr{F}_{\kappa\lambda}, \tag{3.7.3b}$$

$$\mathscr{Q}^\nu_{(1;\mathscr{F})}(h, \nabla\mathscr{F}) = (m^{-1})^{\mu\kappa}(m^{-1})^{\nu\nu'}(m^{-1})^{\lambda\lambda'}h_{\nu'\lambda'}\nabla_\mu \mathscr{F}_{\kappa\lambda}. \tag{3.7.3c}$$

More precisely, (3.7.1a) follows from (3.6.7) and the expansions (1.2.1a)–(1.2.1b) while (3.7.1b)–(3.7.1c) were derived in Lemma 3.4.

# 4. The initial-value problem

In this section, we discuss the abstract initial data and the constraint equations for the Einstein-nonlinear electromagnetic system. We then use the abstract initial data to construct initial data for the reduced equations that satisfy the wave-coordinate condition at $t = 0$. Finally, we sketch a proof of the well-known fact that the wave-coordinate condition is satisfied by the solution to the reduced equations launched by this data; this result shows that the wave-coordinate gauge is a viable gauge for studying the Einstein-nonlinear electromagnetic system.

**4.1. *The abstract initial data.*** The initial-value problem formulation of the Einstein equations goes back to the seminal work by Fourès-Bruhat [1952]. In this article, initial data for the Einstein-nonlinear

electromagnetic system consist of the 3-dimensional manifold $\Sigma_0 = \mathbb{R}^3$ together with the following fields on $\Sigma_0$: a Riemannian metric $\mathring{\underline{g}}_{jk}$, a symmetric two-tensor $\mathring{K}_{jk}$, and a pair of one-forms $\mathring{\mathfrak{D}}_j$ and $\mathring{\mathfrak{B}}_j$. After we construct the ambient Lorentzian spacetime $(\mathfrak{M}, g_{\mu\nu})$, $\mathring{\underline{g}}_{jk}$ and $\mathring{K}_{jk}$ will respectively be the first and second fundamental forms of $\Sigma_0$ while $\mathring{\mathfrak{D}}_j$ and $\mathring{\mathfrak{B}}_j$, which are defined below in Section 9.2, will be an electromagnetic decomposition of $\mathscr{F}_{\mu\nu}|_{\Sigma_0}$ into a pair of one-forms that are both $m$-tangent and $g$-tangent to $\Sigma_0$.

It is well-known that one cannot consider arbitrary data for the Einstein-nonlinear electromagnetic system. The data are subject to the following constraints:

$$\mathring{\underline{R}} - \mathring{K}_{ab}\mathring{K}^{ab} + \left[(\mathring{\underline{g}}^{-1})^{ab}\mathring{K}_{ab}\right]^2 = 2T(\widehat{N}, \widehat{N})|_{\Sigma_0}, \tag{4.1.1a}$$

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathfrak{D}}}_a \mathring{K}_{bj} - (\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathfrak{D}}}_j \mathring{K}_{ab} = T\left(\widehat{N}, \frac{\partial}{\partial x^j}\right)\Big|_{\Sigma_0} \quad (j = 1, 2, 3), \tag{4.1.1b}$$

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathfrak{D}}}_a \mathring{\mathfrak{D}}_b = 0, \tag{4.1.2a}$$

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathfrak{D}}}_a \mathring{\mathfrak{B}}_b = 0, \tag{4.1.2b}$$

where $\mathring{\underline{\mathfrak{D}}}$ is the Levi-Civita connection corresponding to $\mathring{\underline{g}}_{jk}$, $\mathring{\underline{R}}$ is the scalar curvature of $\mathring{\underline{g}}_{jk}$, $T_{\mu\nu}$ is defined in (3.5.4a), and $\widehat{N}^\mu$ is the future-directed unit $g$-normal to $\Sigma_0$. The right-hand sides of (4.1.1a)–(4.1.1b) can (in principle) be computed in terms of and $\mathring{\underline{g}}_{jk}$, $\mathring{\mathfrak{D}}_j$, and $\mathring{\mathfrak{B}}_j$ with the help of the relations (9.2.3), which connect these quantities to $\mathscr{F}_{\mu\nu}|_{\Sigma_0}$. In (4.1.1a)–(4.1.1b), indices are lowered and raised with the Riemannian metric $\mathring{\underline{g}}_{jk}$ and its inverse $(\mathring{\underline{g}}^{-1})^{jk}$. The constraints (4.1.1a)–(4.1.1b) are respectively known as the *Gauss* and *Codazzi* equations while (4.1.2a)–(4.1.2b) are known as the *electromagnetic constraints*. They relate the fields present in the ambient spacetime $(\mathfrak{M}, g_{\mu\nu}, \mathscr{F}_{\mu\nu})$ (which has to be constructed) to the fields induced on an embedded Riemannian hypersurface (which will be $(\Sigma_0, \mathring{\underline{g}}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ after construction). Without providing the rather standard details (see, e.g., [Christodoulou 2008]), we remark that they are consequences of the following assumptions:

- $\Sigma_0$ is a spacelike submanifold of the spacetime manifold $\mathfrak{M}$.

- $\mathring{\underline{g}}_{jk}$ is the first fundamental form of $\Sigma_0$, and $\mathring{K}_{jk}$ is the second fundamental form of $\Sigma_0$.

- The Einstein-nonlinear electromagnetic system is satisfied along $\Sigma_0$.

- Along $\Sigma_0$ (viewed as a subset of $\mathfrak{M}$), $\mathfrak{B}_\mu = -^\star\mathscr{F}_{\mu\kappa}\widehat{N}^\kappa$ and $\mathfrak{D}_\mu = -^\star\mathscr{M}_{\mu\kappa}\widehat{N}^\kappa$.

We recall that, under the above assumptions, $\mathring{\underline{g}}$ and $\mathring{K}$ are defined by

$$\mathring{\underline{g}}|_p(X, Y) = g|_p(X, Y) \qquad \forall X, Y \in T_p\Sigma_0, \tag{4.1.3}$$

$$\mathring{K}|_p(X, Y) = g|_p(\mathscr{D}_X\widehat{N}, Y) \quad \forall X, Y \in T_p\Sigma_0, \tag{4.1.4}$$

where $\widehat{N}$ is the future-directed unit $g$-normal[19] to $\Sigma_0$ at $p$ and $\mathscr{D}$ is the Levi-Civita connection corresponding to $g$. Furthermore, if $X$ and $Y$ are vector fields tangent to $\Sigma_0$, then

$$\mathscr{D}_X Y = \mathring{\underline{\mathscr{D}}}_X Y + \mathring{K}(X, Y)\widehat{N}. \tag{4.1.5}$$

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[19]Under the assumptions of Section 4.2, it follows that, at every point $p \in \Sigma_0$, $\widehat{N}^\mu = (A^{-1}, 0, 0, 0)$, where $A$ is defined by (4.2.2).

We also remind the reader that our stability theorem requires the hypothesis that the abstract initial data decay at spatial infinity according to the rates (1.0.4a)–(1.0.4f).

**4.2. *The initial data for the reduced equations.*** We assume that we are given "abstract" initial data $(\mathring{\underline{g}}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ $(j, k = 1, 2, 3)$ on the manifold $\mathbb{R}^3$ for the Einstein equations as discussed in the previous section. In this section, we will use this data to construct data $(g_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0}, \mathscr{F}_{\mu\nu}|_{t=0})$ $(\mu, \nu = 0, 1, 2, 3)$ for the reduced equations (3.7.1a)–(3.7.1c) that satisfy the wave-coordinate condition $\Gamma^\mu|_{t=0} = 0$. We begin by recalling that $\chi(z)$ is a fixed cut-off function with the following properties:

$$\chi \in C^\infty, \qquad \chi \equiv 1 \text{ for } z \geq \tfrac{3}{4}, \quad \text{and} \quad \chi \equiv 0 \text{ for } z \leq \tfrac{1}{2}. \tag{4.2.1}$$

We then define the function $A(x^1, x^2, x^3) \geq 0$ by

$$A^2 \overset{\text{def}}{=} 1 - \frac{2M}{r}\chi(r) \quad \text{and} \quad r \overset{\text{def}}{=} |x|. \tag{4.2.2}$$

We define the data for the spacetime metric $g_{\mu\nu}$ by

$$g_{00}|_{t=0} = -A^2, \qquad g_{0j}|_{t=0} = 0, \qquad g_{jk}|_{t=0} = \mathring{\underline{g}}_{jk}, \tag{4.2.3a}$$

$$\partial_t g_{00}|_{t=0} = 2A^3(\mathring{\underline{g}}^{-1})^{ab}\mathring{K}_{ab},$$

$$\partial_t g_{0j}|_{t=0} = A^2(\mathring{\underline{g}}^{-1})^{ab}\,\partial_a \mathring{\underline{g}}_{bj} - \tfrac{1}{2}A^2(\mathring{\underline{g}}^{-1})^{ab}\,\partial_j \mathring{\underline{g}}_{ab} - A\,\partial_j A, \tag{4.2.3b}$$

$$\partial_t g_{jk}|_{t=0} = 2A\mathring{K}_{jk}$$

and the data for the Faraday tensor $\mathscr{F}_{\mu\nu}$ by

$$\mathscr{F}_{j0}|_{t=0} = \mathring{E}_j \quad \text{and} \quad \mathscr{F}_{jk}|_{t=0} = [ijk]\mathring{B}_i. \tag{4.2.4}$$

The one-forms $\mathring{E}_j$ and $\mathring{B}_j$ can be expressed in terms of $\mathring{\underline{h}}_{jk}$ and the one-forms $\mathring{\mathfrak{D}}_j$ and $\mathring{\mathfrak{B}}_j$ appearing in the constraint equations (4.1.2a)–(4.1.2b) by using the relations (9.2.3) and (9.2.4) below. The precise form of these relations depends on the choice of Lagrangian $^\star\mathscr{L}$, but in the small-data regime, the estimates (9.2.7) (9.2.8a), and (9.2.8b) hold.

We now state the main result of this section.

**Lemma 4.1** (Wave-coordinate condition holds at $t = 0$). *Suppose that the initial data $(g_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0})$ $(\mu, \nu = 0, 1, 2, 3)$ for the reduced equations are constructed from abstract initial data $(\mathring{\underline{g}}_{jk}, \mathring{K}_{jk})$ $(j, k = 1, 2, 3)$ as described above. Then the wave-coordinate condition holds initially:*

$$\Gamma^\mu|_{t=0} \quad (\mu = 0, 1, 2, 3). \tag{4.2.5}$$

*Proof.* Lemma 4.1 follows from the expression (3.1.1c), the definitions (4.2.3a)–(4.2.3b), and straightforward calculations. $\qquad\square$

Note that the above definitions induce the following data for the spacetime metric "remainder" piece $h_{\mu\nu}^{(1)}$, which is defined by (1.2.1a)–(1.2.1c):

$$h^{(1)}_{00}|_{t=0} = 0, \qquad h^{(1)}_{0j}|_{t=0} = 0, \qquad h^{(1)}_{jk}|_{t=0} = \mathring{\underline{h}}^{(1)}_{jk}, \tag{4.2.6a}$$

$$\partial_t h^{(1)}_{00}|_{t=0} = 2A^3(\mathring{\underline{g}}^{-1})^{ab}\mathring{K}_{ab},$$

$$\partial_t h^{(1)}_{0j}|_{t=0} = A^2(\mathring{\underline{g}}^{-1})^{ab}\,\partial_a\mathring{\underline{g}}_{bj} - \tfrac{1}{2}A^2(\mathring{\underline{g}}^{-1})^{ab}\,\partial_j\mathring{\underline{g}}_{ab} - A\,\partial_j A, \tag{4.2.6b}$$

$$\partial_t h^{(1)}_{jk}|_{t=0} = 2A\mathring{K}_{jk}.$$

Similarly, the following data are induced in $h_{\mu\nu} = h^{(0)}_{\mu\nu} + h^{(1)}_{\mu\nu}$, which is defined in (1.2.1b):

$$h_{00}|_{t=0} = \chi(r)\frac{2M}{r}, \qquad h_{0j}|_{t=0} = 0, \qquad h_{jk}|_{t=0} = \chi(r)\frac{2M}{r}\delta_{jk} + \mathring{\underline{h}}^{(1)}_{jk}, \tag{4.2.7a}$$

$$\partial_t h_{00}|_{t=0} = 2A^3(\mathring{\underline{g}}^{-1})^{ab}\mathring{K}_{ab},$$

$$\partial_t h_{0j}|_{t=0} = A^2(\mathring{\underline{g}}^{-1})^{ab}\,\partial_a\mathring{\underline{g}}_{bj} - \tfrac{1}{2}A^2(\mathring{\underline{g}}^{-1})^{ab}\,\partial_j\mathring{\underline{g}}_{ab} - A\,\partial_j A, \tag{4.2.7b}$$

$$\partial_t h_{jk}|_{t=0} = 2A\mathring{K}_{jk}.$$

We will make use of these facts in our proof of Proposition 10.4 below.

**4.3.** *Preservation of the wave-coordinate gauge.* In this section, we sketch a proof of the fact that, if the reduced data are constructed from abstract data as described in Section 4.2, then the wave-coordinate condition $\Gamma^\mu = 0$ is preserved by the flow of the reduced equations. This result requires the assumption that the abstract data satisfy the constraints (4.1.1a)–(4.1.2b). To simplify the discussion, we assume in this section that the data are smooth. However, the result also holds in the regularity class we use during our global existence proof. We remark that this result is quite standard and that we have included it only for convenience.

**Proposition 4.2** (Preservation of the wave-coordinate gauge). *Suppose that* $(g_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0}, \mathscr{F}_{\mu\nu}|_{t=0})$ $(\mu, \nu = 0, 1, 2, 3)$ *are smooth initial data for the reduced equations* (3.7.1a)–(3.7.1c) *that are constructed from abstract initial data satisfying the constraints* (4.1.1a)–(4.1.2b) *as described in Section 4.2. In particular, by Lemma 4.1, the wave-coordinate condition* $\Gamma^\mu|_{t=0}$ *holds. Assume further that the reduced data are small enough so that they lie within the regime of hyperbolicity*[20] *of the reduced equations. Let* $(g_{\mu\nu}, \mathscr{F}_{\mu\nu})$ *be the corresponding smooth solution to the reduced equations that is launched by the data. Let* $T > 0$, *and assume that the reduced solution exists on the slab* $[0, T) \times \mathbb{R}^3$ *and lies within the regime of hyperbolicity of the reduced equations. Then* $\Gamma^\mu \equiv 0$ *for* $[t, x] \in [0, T) \times \mathbb{R}^3$.

*Sketch of proof.* Our goal is to show that under the assumptions of the proposition, whenever we have a smooth solution to the reduced equations (3.7.1a)–(3.7.1c) on $[0, T) \times \mathbb{R}^3$, the corresponding $\Gamma^\mu$ satisfy a *homogeneous-in-*$\Gamma^\mu$ system of wave equations with principal part equal to $(g^{-1})^{\kappa\lambda}\,\partial_\kappa\,\partial_\lambda$ and with trivial initial data $\Gamma^\mu|_{t=0} = \partial_t \Gamma^\mu|_{t=0} = 0$. The conclusion that $\Gamma^\mu \equiv 0$ for $(t, x) \in [0, T) \times \mathbb{R}^3$ then follows from a standard uniqueness theorem for such wave equations that is based on energy estimates (see, e.g., [Hörmander 1997; Sogge 2008] for ideas on how to prove such a theorem). To derive the equations satisfied by the $\Gamma^\mu$, we will view $\Gamma^\mu$ as a vector field for purposes of covariant

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[20]Since our electromagnetic equations are perturbations of the standard Maxwell–Maxwell equations, there will always be such a regime.

differentiation. We first recall (see Remark 3.6) that (3.6.2) is obtained by adding the gauge term $-\frac{1}{2}(g_{\kappa\nu}\mathscr{D}_\mu\Gamma^\kappa + g_{\kappa\mu}\mathscr{D}_\nu\Gamma^\kappa) + u_{\mu\nu\kappa}(g, g^{-1}, \partial g)\Gamma^\kappa$ to the expression (3.0.2b) for $R_{\mu\nu}$. Consequently, it follows that, for a solution to the reduced equations (3.7.1a)–(3.7.1c), we have that

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} - T_{\mu\nu} = \tfrac{1}{2}(g_{\kappa\nu}\mathscr{D}_\mu\Gamma^\kappa + g_{\kappa\mu}\mathscr{D}_\nu\Gamma^\kappa) - u_{\mu\nu\kappa}(g, g^{-1}, \partial g)\Gamma^\kappa$$
$$- \tfrac{1}{2}g_{\mu\nu}\mathscr{D}_\kappa\Gamma^\kappa + \tfrac{1}{2}g_{\mu\nu}(g^{-1})^{\kappa\lambda}u_{\kappa\lambda\delta}(g, g^{-1}, \partial g)\Gamma^\delta. \quad (4.3.1)$$

We note that the left-hand side of (4.3.1) is simply the difference of the left-hand and right-hand sides of the Einstein equation (1.0.1a).

We now apply $(g^{-1})^{\nu\lambda}\mathscr{D}_\lambda$ to each side of (4.3.1), use the Bianchi identity $(g^{-1})^{\nu\lambda}\mathscr{D}_\lambda(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = 0$, the fact that $(g^{-1})^{\nu\lambda}\mathscr{D}_\lambda T_{\mu\nu} = -\Gamma^\kappa(g^{-1})^{\beta\lambda}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\mu\beta}$ (see Remark 3.6 and (3.5.11)), and the curvature relation $\mathscr{D}_\mu\mathscr{D}_\kappa\Gamma^\kappa = \mathscr{D}_\kappa\mathscr{D}_\mu\Gamma^\kappa - R_{\mu\kappa}\Gamma^\kappa$, and expand the covariant derivatives in terms of coordinate derivatives and Christoffel symbols to deduce that the $\Gamma^\mu$ are solutions to the following *hyperbolic* system of wave equations that is *homogeneous in* $\Gamma^\mu$:

$$(g^{-1})^{\kappa\lambda}\partial_\kappa\partial_\lambda\Gamma^\mu = A^{\mu\kappa}{}_\lambda(g, g^{-1}, \partial g, \partial\partial g)\partial_\kappa\Gamma^\lambda + B^\mu{}_\kappa(g, g^{-1}, \partial g, \partial\partial g, \mathscr{F})\Gamma^\kappa \quad (\mu = 0, 1, 2, 3), \quad (4.3.2)$$

where the $A^{\mu\kappa}{}_\lambda(g(t, x), g^{-1}(t, x), \partial g(t, x), \partial\partial g(t, x))$ and $B^\mu{}_\kappa(g(t, x), g^{-1}(t, x), \partial g(t, x), \partial\partial g(t, x), \mathscr{F}(t, x))$ are smooth functions of $(t, x)$.

To complete our sketch of the proof, it remains to show that $\partial_t\Gamma^\mu|_{t=0} = 0$. Since the abstract initial data $(\mathring{g}_{jk}, \mathring{K}_{jk}, \mathring{\mathscr{D}}_j, \mathring{\mathscr{B}}_j)$ $(j, k = 1, 2, 3)$ are assumed to satisfy the constraint equations (4.1.1a)–(4.1.1b), it follows that the left-hand side of (4.3.1) is equal to 0 at $t = 0$ after contracting[21] against $\widehat{N}^\mu\widehat{N}^\nu$ or $\widehat{N}^\mu X^\nu$, where $\widehat{N}^\mu$ is the future-directed unit $g$-normal to $\Sigma_0$ and $X^\mu$ is any vector tangent to $\Sigma_0$.

Recalling that $\widehat{N}^\mu|_{t=0} = A^{-1}\delta_0^\mu$ and choosing $X^\nu = \delta_j^\nu$, it therefore follows that the right-hand side must also be equal to 0 at $t = 0$ upon contraction (where $j = 1, 2, 3$ in (4.3.3b)):

$$\big(g_{\kappa 0}\mathscr{D}_t\Gamma^\kappa - u_{00\kappa}(g, g^{-1}, \partial g)\Gamma^\kappa - \tfrac{1}{2}g_{00}\mathscr{D}_\lambda\Gamma^\lambda$$
$$+ \tfrac{1}{2}g_{00}(g^{-1})^{\kappa\lambda}u_{\kappa\lambda\delta}(g, g^{-1}, \partial g)\Gamma^\delta\big)\big|_{t=0} = 0, \quad (4.3.3a)$$
$$\big(\tfrac{1}{2}(g_{\kappa j}\mathscr{D}_t\Gamma^\kappa + g_{\kappa 0}\mathscr{D}_j\Gamma^\kappa) - u_{0j\kappa}(g, g^{-1}, \partial g)\Gamma^\kappa - \tfrac{1}{2}g_{0j}\mathscr{D}_\lambda\Gamma^\lambda$$
$$+ \tfrac{1}{2}g_{0j}(g^{-1})^{\kappa\lambda}u_{\kappa\lambda\delta}(g, g^{-1}, \partial g)\Gamma^\delta\big)\big|_{t=0} = 0. \quad (4.3.3b)$$

Expanding the covariant differentiation in (4.3.3a)–(4.3.3b) in terms of coordinate derivatives and Christoffel symbols and using (4.2.3a) plus the fact that the initial data were constructed so as to satisfy $\Gamma^\mu|_{t=0} = 0$, it is straightforward to verify that $\partial_t\Gamma^\mu$ must *also necessarily* be trivial at $t = 0$:

$$\partial_t\Gamma^\mu|_{t=0} = 0 \quad (\mu = 0, 1, 2, 3). \quad (4.3.4)$$

This completes our sketch of a proof of the proposition. $\qquad\qquad\qquad\qquad\qquad\square$

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[21] In fact, one derives the constraint equations by assuming that these contractions are 0 at $t = 0$.

## 5. Geometry and the Minkowskian null frame

In this section, we introduce the families of ingoing Minkowskian null cones $C_s^-$, outgoing Minkowskian light cones $C_q^+$, constant Minkowskian time slices $\Sigma_t$, and Euclidean spheres $S_{r,t}$. We then discuss the well-known notion of a Minkowskian null frame, which allows us to geometrically decompose the tangent space at $p$ as a direct sum $T_p \mathbb{R}^{1+3} = \text{span}\{\underline{L}|_p\} \oplus \text{span}\{L|_p\} \oplus T_p S_{r,t}$. These decompositions allow us to geometrically decompose tensor fields. In Section 5.3, we provide a full description of the null decomposition of a two-form $\mathscr{F}$ into its *Minkowskian null components*. This decomposition will be essential to our subsequent analysis of the decay properties of the Faraday tensor. In Section 9.1, we will derive equations for these null components under the assumption that $\mathscr{F}$ is a solution to the reduced electromagnetic equations (3.7.1b)–(3.7.1c). In Section 15, we will use the equations for the null components to deduce "upgraded" pointwise decay estimates for the lower-order Lie derivatives of $\mathscr{F}$; these estimates are essential for closing our global existence bootstrap argument in Section 16. We refer the reader to Section 1.2.4 for discussion on how our use of Minkowskian decompositions compares and contrasts against other decompositions that have been used by other authors in the context of the stability of Minkowski spacetime.

**5.1.** *The Minkowskian null frame.*  Before proceeding, we introduce the subsets $C_q^+$, $C_s^-$, $\Sigma_t$, and $S_{r,t}$.

**Definition 5.1.** In our wave-coordinate system $(t, x)$, we define the *outgoing Minkowski null cones $C_q^+$*, *ingoing Minkowski null cones $C_s^-$*, *constant Minkowskian time slices $\Sigma_t$*, and Euclidean spheres $S_{r,t}$ as

$$C_q^+ \overset{\text{def}}{=} \{(\tau, y) \mid |y| - \tau = q\}, \tag{5.1.1a}$$

$$C_s^- \overset{\text{def}}{=} \{(\tau, y) \mid |y| + \tau = s\}, \tag{5.1.1b}$$

$$\Sigma_t \overset{\text{def}}{=} \{(\tau, y) \mid \tau = t\}, \tag{5.1.1c}$$

$$S_{r,t} \overset{\text{def}}{=} \{(\tau, y) \mid \tau = t, \ |y| = r\}. \tag{5.1.1d}$$

In the above formulas, $y \overset{\text{def}}{=} (y^1, y^2, y^3)$ and $|y| \overset{\text{def}}{=} \sqrt{(y^1)^2 + (y^2)^2 + (y^2)^2}$.

We also introduce the following vector fields, which play a fundamental role throughout this article:

**Definition 5.2.** We define the *ingoing Minkowski-null geodesic vector field $\underline{L}$* and the *outgoing Minkowski-null geodesic vector field $L$* by

$$\underline{L}^\mu = (1, -\omega^1, -\omega^2, -\omega^3), \tag{5.1.2a}$$

$$L^\mu = (1, \omega^1, \omega^2, \omega^3), \tag{5.1.2b}$$

where $\omega^j \overset{\text{def}}{=} x^j / r$. By "Minkowski-null", we mean that $m(\underline{L}, \underline{L}) = m(L, L) = 0$. Note that $\underline{L}$ is tangent to the ingoing cones $C_s^-$, that $L$ is tangent to the outgoing cones $C_q^+$, and that $\underline{L}$ and $L$ are both $m$-orthogonal to the $S_{r,t}$. By "Minkowski-geodesic", we mean that $\nabla_{\underline{L}} \underline{L} = \nabla_L L = 0$.

Note that

$$\underline{L} = \partial_t - \partial_r, \tag{5.1.3a}$$

$$L = \partial_t + \partial_r. \tag{5.1.3b}$$

We now recall the definitions of the Minkowskian first fundamental forms of the surfaces $\Sigma_t$ and $S_{r,t}$.

**Definition 5.3.** The *Minkowskian first fundamental* forms of the surfaces $\Sigma_t$ and $S_{r,t}$ are respectively defined to be the following intrinsic metrics:

$$\underline{m}_{\mu\nu} \stackrel{\text{def}}{=} \text{diag}(0, 1, 1, 1), \tag{5.1.4a}$$

$$\not{h}_{\mu\nu} \stackrel{\text{def}}{=} m_{\mu\nu} + \tfrac{1}{2}(\underline{L}_\mu L_\nu + L_\mu \underline{L}_\nu). \tag{5.1.4b}$$

Recall that $\underline{m}|_p(X, Y) = m|_p(X, Y)$ for $X, Y \in T_p\Sigma_t$ and $\not{h}(X, Y) = m(X, Y)$ for $X, Y \in T_pS_{r,t}$. Note also that the tensor fields $\underline{m}_\mu{}^\nu$ and $\not{h}_\mu{}^\nu$ respectively $m$-orthogonally project onto the $\Sigma_t$ and the $S_{r,t}$.

We now define a related tensor field corresponding to the outgoing Minkowski null cones $C_q^+$.

**Definition 5.4.** The tensor field $\bar{\pi}_\mu{}^\nu$, which projects vectors $X^\mu$ onto the outgoing cones $C_q^+$, is defined as

$$\bar{\pi}_\mu{}^\nu \stackrel{\text{def}}{=} \delta_\mu^\nu + \tfrac{1}{2}L_\mu \underline{L}^\nu. \tag{5.1.5}$$

Note in particular that $\bar{\pi}_\mu{}^\nu \underline{L}^\mu = 0$ while $\bar{\pi}_\mu{}^\nu X^\mu = X^\nu$ whenever $X$ is tangent to $C_q^+$.

Furthermore, we recall the definitions of the Minkowskian volume forms of Minkowski spacetime and of the surfaces $\Sigma_t$ and $S_{r,t}$.

**Definition 5.5.** The *Minkowskian volume forms* of Minkowski spacetime, the surfaces $\Sigma_t$, and the Euclidean spheres $S_{r,t}$ are respectively defined relative to our wave-coordinate system as follows:

$$\upsilon_{\mu\nu\kappa\lambda} \stackrel{\text{def}}{=} [\mu\nu\kappa\lambda], \tag{5.1.6a}$$

$$\underline{\upsilon}_{\nu\kappa\lambda} \stackrel{\text{def}}{=} \upsilon_{0\nu\kappa\lambda}, \tag{5.1.6b}$$

$$\not{\upsilon}_{\mu\nu} \stackrel{\text{def}}{=} \upsilon_{\mu\nu\kappa\lambda} \underline{L}^\kappa L^\lambda, \tag{5.1.6c}$$

where $[\mu\nu\kappa\lambda]$ is totally antisymmetric with normalization $[0123] = 1$.

We also recall what it means for a spacetime tensor field to be $m$-tangent to the surfaces $\Sigma_t$ or $S_{r,t}$.

**Definition 5.6.** Let $U$ be a type-$\binom{n}{m}$ spacetime tensor field. We say that $U$ is $m$-tangent to the time slices $\Sigma_t$ if

$$U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n} = \underline{m}_{\mu_1}{}^{\mu_1'} \cdots \underline{m}_{\mu_m}{}^{\mu_m'} \underline{m}_{\nu_1'}{}^{\nu_1} \cdots \underline{m}_{\nu_n'}{}^{\nu_n} U_{\mu_1'\cdots\mu_m'}{}^{\nu_1'\cdots\nu_n'}. \tag{5.1.7}$$

Equivalently, $U$ is $m$-tangent to the $\Sigma_t$ if and only if every wave-coordinate component of $U$ containing a 0 index vanishes.

Similarly, we say that $U$ is $m$-tangent to the spheres $S_{r,t}$ if

$$U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n} = \not{h}_{\mu_1}{}^{\mu_1'} \cdots \not{h}_{\mu_m}{}^{\mu_m'} \not{h}_{\nu_1'}{}^{\nu_1} \cdots \not{h}_{\nu_n'}{}^{\nu_n} U_{\mu_1'\cdots\mu_m'}{}^{\nu_1'\cdots\nu_n'}. \tag{5.1.8}$$

Equivalently, $U$ is $m$-tangent to the spheres $S_{r,t}$ if and only if any contraction of any index of $U$ with either $\underline{L}$ or $L$ vanishes.

We are now ready to introduce the notion of a *Minkowskian null frame*. We complement the vector fields $\underline{L}$ and $L$ with a locally defined pair of $m$-orthogonal vector fields $e_1$ and $e_2$ that are tangent to the spheres $S_{r,t}$ and therefore $m$-orthogonal to $\underline{L}$ and $L$. The resulting collection of vector fields $\mathcal{N} \stackrel{\text{def}}{=} \{\underline{L}, L, e_1, e_2\}$ is known as *Minkowskian null frame*. It spans the tangent space $T_p\mathbb{R}^{1+3}$ at each point $p$ where it is defined.

We leave the proof of the following lemma, which summarizes some of the important properties of the geometric quantities introduced in this section, as an exercise for the reader:

**Lemma 5.7** (Null frame field properties). *The following identities hold*:

$$\nabla_L L = \nabla_{\underline{L}}\underline{L} = 0, \tag{5.1.9a}$$

$$\nabla_L \underline{L} = \nabla_{\underline{L}} L = 0, \tag{5.1.9b}$$

$$L^\kappa \underline{L}_\kappa = -2, \tag{5.1.9c}$$

$$e_A^\kappa L_\kappa = e_A^\kappa \underline{L}_\kappa = 0 \qquad (A = 1, 2), \tag{5.1.9d}$$

$$m_{\kappa\lambda} e_A^\kappa e_B^\lambda = \delta_{AB} \qquad (A, B = 1, 2), \tag{5.1.9e}$$

$$\nabla_{\underline{L}} \not{h}_{\mu\nu} = \nabla_L \not{h}_{\mu\nu} = 0 \qquad (\mu, \nu = 0, 1, 2, 3), \tag{5.1.10}$$

$$\nabla_{\underline{L}} \not{\psi}_{\mu\nu} = \nabla_L \not{\psi}_{\mu\nu} = 0 \qquad (\mu, \nu = 0, 1, 2, 3). \tag{5.1.11}$$

*See Definition 6.4 concerning our use of notation in these formulas.*

Later in the article, we will see that the decay rates of the null components (see Section 5.3) of $h$ and $\mathscr{F}$ are distinguished according to the kinds of contractions of $\mathscr{F}$ taken against $\underline{L}$, $L$, $e_1$, and $e_2$. With these ideas in mind, we introduce the following sets of vector fields:

$$\mathscr{L} \stackrel{\text{def}}{=} \{L\}, \qquad \mathscr{T} \stackrel{\text{def}}{=} \{L, e_1, e_2\}, \quad \text{and} \quad \mathcal{N} \stackrel{\text{def}}{=} \{\underline{L}, L, e_1, e_2\}. \tag{5.1.12}$$

In order to measure the size of the contractions of various tensors and their covariant derivatives against vectors belonging to the sets $\mathscr{L}$, $\mathscr{T}$, and $\mathcal{N}$, we introduce the following definitions:

**Definition 5.8.** If $\mathscr{V}$ and $\mathscr{W}$ denote any two of the above sets and $P$ is a type-$\binom{0}{2}$ tensor, then we define the following pointwise seminorms:

$$|P|_{\mathscr{V}\mathscr{W}} \stackrel{\text{def}}{=} \sum_{V \in \mathscr{V}, \ W \in \mathscr{W}} |V^\kappa W^\lambda P_{\kappa\lambda}|, \tag{5.1.13a}$$

$$|\nabla P|_{\mathscr{V}\mathscr{W}} \stackrel{\text{def}}{=} \sum_{N \in \mathcal{N}, \ V \in \mathscr{V}, \ W \in \mathscr{W}} |V^\kappa W^\lambda N^\gamma \nabla_\gamma P_{\kappa\lambda}|, \tag{5.1.13b}$$

$$|\overline{\nabla} P|_{\mathscr{V}\mathscr{W}} \stackrel{\text{def}}{=} \sum_{T \in \mathscr{T}, \ V \in \mathscr{V}, \ W \in \mathscr{W}} |V^\kappa W^\lambda T^\gamma \nabla_\gamma P_{\kappa\lambda}|. \tag{5.1.13c}$$

We often use the abbreviations $|P| \stackrel{\text{def}}{=} |P|_{\mathcal{N}\mathcal{N}}$, $|\nabla P| \stackrel{\text{def}}{=} |\nabla P|_{\mathcal{N}\mathcal{N}}$, and $|\overline{\nabla} P| \stackrel{\text{def}}{=} |\overline{\nabla} P|_{\mathcal{N}\mathcal{N}}$.

The above definition generalizes in an obvious way to arbitrary type-$\binom{n}{m}$ tensor fields $U_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n}$. Observe that, for any such tensor field, the following inequalities hold *in our wave-coordinate system*:

$$|U| \approx \sum_{\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n = 0}^{3} |U_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n}|. \tag{5.1.14}$$

**5.2.** *Minkowskian null frame decomposition of a tensor field.* For an arbitrary vector field $X$ and frame vector field $N \in \mathcal{N}$, we define

$$X_N \overset{\text{def}}{=} X_\kappa N^\kappa, \quad \text{where } X_\mu \overset{\text{def}}{=} m_{\mu\kappa} X^\kappa. \tag{5.2.1}$$

The components $X_N$ are known as the *Minkowskian null components* of $X$. In the sequel, we often abbreviate

$$X_A \overset{\text{def}}{=} X_{e_A} \quad \text{and} \quad \nabla_A \overset{\text{def}}{=} \nabla_{e_A}, \quad \text{etc.} \tag{5.2.2}$$

It follows from (5.2.1) that

$$X = X^\kappa \, \partial_\kappa = X^L L + X^{\underline{L}} \underline{L} + X^A e_A, \tag{5.2.3}$$

$$X^L = -\tfrac{1}{2} X_{\underline{L}}, \qquad X^{\underline{L}} = -\tfrac{1}{2} X_L, \qquad X^A = X_A. \tag{5.2.4}$$

Furthermore, it is easy to check that

$$m(X, Y) \overset{\text{def}}{=} m_{\kappa\lambda} X^\kappa X^\lambda = X^\kappa Y_\kappa = -\tfrac{1}{2} X_L Y_{\underline{L}} - \tfrac{1}{2} X_{\underline{L}} Y_L + \delta^{AB} X_A Y_B. \tag{5.2.5}$$

The above null decomposition of a vector field generalizes in the obvious way to higher-order tensor fields. In the next section, we provide a detailed version of the null decomposition of two-forms $\mathscr{F}$ since this decomposition is needed for our derivation of decay estimates later in the article; see, e.g., Propositions 9.3 and 11.5.

**5.3.** *The detailed Minkowskian null decomposition of a two-form.*

**Definition 5.9.** Given any two-form $\mathscr{F}$, we define its *Minkowskian null components* to be the following pair of one-forms $\underline{\alpha}_\mu$ and $\alpha_\mu$ and the following pair of scalars $\rho$ and $\sigma$:

$$\underline{\alpha}_\mu \overset{\text{def}}{=} {\not{m}}_\mu{}^\nu \mathscr{F}_{\nu\lambda} \underline{L}^\lambda \quad (\mu = 0, 1, 2, 3), \tag{5.3.1a}$$

$$\alpha_\mu \overset{\text{def}}{=} {\not{m}}_\mu{}^\nu \mathscr{F}_{\nu\lambda} L^\lambda \quad (\mu = 0, 1, 2, 3), \tag{5.3.1b}$$

$$\rho \overset{\text{def}}{=} \tfrac{1}{2} \mathscr{F}_{\kappa\lambda} \underline{L}^\kappa L^\lambda, \tag{5.3.1c}$$

$$\sigma \overset{\text{def}}{=} \tfrac{1}{2} {\not{\psi}}^{\kappa\lambda} \mathscr{F}_{\kappa\lambda}. \tag{5.3.1d}$$

It is a simple exercise to check that $\underline{\alpha}_\mu$ and $\alpha_\mu$ are $m$-tangent to the spheres $S_{r,t}$:

$$\underline{\alpha}_\kappa \underline{L}^\kappa = 0, \qquad \underline{\alpha}_\kappa L^\kappa = 0, \tag{5.3.2a}$$

$$\alpha_\kappa \underline{L}^\kappa = 0, \qquad \alpha_\kappa L^\kappa = 0. \tag{5.3.2b}$$

Furthermore, relative to the null frame $\mathcal{N} \overset{\text{def}}{=} \{\underline{L}, L, e_1, e_2\}$, we have that

$$\underline{\alpha}_A = \mathscr{F}_{A\underline{L}} \qquad (A = 1, 2), \tag{5.3.3a}$$

$$\alpha_A = \mathscr{F}_{AL} \qquad (A = 1, 2), \tag{5.3.3b}$$

$$\rho = \tfrac{1}{2}\mathscr{F}_{\underline{L}L}, \tag{5.3.3c}$$

$$\sigma = \mathscr{F}_{12}. \tag{5.3.3d}$$

In terms of the seminorms introduced in Definition 5.8, it follows that

$$|\mathscr{F}| \approx |\mathscr{F}|_{\mathcal{N}\mathcal{N}} \approx |\underline{\alpha}| + |\alpha| + |\rho| + |\sigma|, \tag{5.3.4a}$$

$$|\mathscr{F}|_{\mathscr{L}\mathcal{N}} \approx |\alpha| + |\rho|, \tag{5.3.4b}$$

$$|\mathscr{F}|_{\mathscr{T}\mathscr{T}} \approx |\alpha| + |\sigma|. \tag{5.3.4c}$$

The null components of $^{\circledast}\mathscr{F}$ (the Minkowskian Hodge duality operator $\circledast$ is defined in Section 2.6) can be expressed in terms of the above null components of $\mathscr{F}$. Denoting the null components[22] of $^{\circledast}\mathscr{F}$ by $^{\odot}\underline{\alpha}$, $^{\odot}\alpha$, $^{\odot}\rho$, and $^{\odot}\sigma$, we leave it as a simple exercise for the reader to check that

$$^{\odot}\underline{\alpha}_A = -\underline{\alpha}^B \psi_{BA} \quad (A = 1, 2), \tag{5.3.5a}$$

$$^{\odot}\alpha_A = \alpha^B \psi_{BA} \qquad (A = 1, 2), \tag{5.3.5b}$$

$$^{\odot}\rho = \sigma, \tag{5.3.5c}$$

$$^{\odot}\sigma = -\rho. \tag{5.3.5d}$$

## 6. Differential operators

In this section, we introduce a collection of differential operators that will be used throughout the remainder of the article. In order to define these operators, we also introduce subsets $\mathbb{O}$ and $\mathscr{L}$ of Minkowskian conformal Killing fields. Finally, we prove a collection of lemmas that expose useful properties of these operators and that illustrate various relationships between them.

**6.1. *Covariant derivatives.*** As previously mentioned, throughout the article, $\nabla$ denotes the Levi-Civita connection of the Minkowski metric $m$. Let $\underline{m}$ and $\not{m}$ be the first fundamental forms of the $\Sigma_t$ and $S_{r,t}$ as defined in Definition 5.3, and let $\underline{\nabla}$ and $\not{\nabla}$ be their corresponding Levi-Civita connections. We state as a lemma the following well-known identities, which relate the connections $\underline{\nabla}$ and $\not{\nabla}$ to $\nabla$:

**Lemma 6.1** (Relationships between connections). *If $U$ is any type-$\binom{n}{m}$ tensor field $m$-tangent to the $\Sigma_t$, then*

$$\underline{\nabla}_\lambda U_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n} = \underline{m}_\lambda{}^{\lambda'} \underline{m}_{\mu_1}{}^{\mu_1'} \cdots \underline{m}_{\mu_m}{}^{\mu_m'} \underline{m}_{\nu_1'}{}^{\nu_1} \cdots \underline{m}_{\nu_n'}{}^{\nu_n} \nabla_{\lambda'} U_{\mu_1' \cdots \mu_m'}{}^{\nu_1' \cdots \nu_n'}. \tag{6.1.1}$$

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[22]We use the symbol $\odot$ in order to avoid confusion with the Minkowskian Hodge duality operator $\circledast$; i.e., it is not true that $^{\circledast}(\underline{\alpha}[\mathscr{F}]) = \underline{\alpha}[^{\circledast}\mathscr{F}]$.

*Similarly, if $U$ is any type-$\binom{n}{m}$ tensor field m-tangent to $S_{r,t}$, then*

$$\nabla\!\!\!/_{\lambda} U_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n} = h\!\!\!/_{\lambda}{}^{\lambda'} h\!\!\!/_{\mu_1}{}^{\mu_1'} \cdots h\!\!\!/_{\mu_m}{}^{\mu_m'} h\!\!\!/_{\nu_1'}{}^{\nu_1} \cdots h\!\!\!/_{\nu_n'}{}^{\nu_n} \nabla\!\!\!/_{\lambda'} U_{\mu_1' \cdots \mu_m'}{}^{\nu_1' \cdots \nu_n'}. \tag{6.1.2}$$

We recall the following fundamental properties of the connections $\nabla$, $\underline{\nabla}$, and $\nabla\!\!\!/$:

$$\nabla_{\lambda} m_{\mu\nu} = 0 = \nabla_{\lambda} (m^{-1})^{\mu\nu} \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{6.1.3a}$$

$$\underline{\nabla}_{\lambda} \underline{m}_{\mu\nu} = 0 \quad\quad\quad\quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{6.1.3b}$$

$$\nabla\!\!\!/_{\lambda} h\!\!\!/_{\mu\nu} = 0 \quad\quad\quad\quad (\lambda, \mu, \nu = 0, 1, 2, 3). \tag{6.1.3c}$$

We will also make use of the projection of the operator $\nabla$ onto the favorable directions, i.e., the directions tangent to the outgoing Minkowski cones $C_q^+$.

**Definition 6.2.** If $U$ is any type-$\binom{n}{m}$ spacetime tensor field, then we define the projected Minkowskian covariant derivative $\overline{\nabla} U$ by

$$\overline{\nabla}_{\lambda} U_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n} = \bar{\pi}_{\lambda}{}^{\lambda'} \nabla_{\lambda'} U_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n}, \tag{6.1.4}$$

where the null frame projection $\bar{\pi}_{\mu}{}^{\nu}$ is defined in (5.1.5).

**Remark 6.3.** Note that only the $\lambda$ component is projected onto the outgoing cones so that the tensor field $\overline{\nabla}_{\lambda} U_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n}$ need not be m-tangent to the outgoing Minkowski cones.

**Definition 6.4.** If $X$ is any vector field, then we define the covariant derivative operators $\nabla_X$ and $\nabla\!\!\!/_X$ by

$$\nabla_X \overset{\text{def}}{=} X^{\kappa} \nabla_{\kappa}, \tag{6.1.5a}$$

$$\nabla\!\!\!/_X \overset{\text{def}}{=} X^{\kappa} \nabla\!\!\!/_{\kappa}. \tag{6.1.5b}$$

**6.2. *Minkowskian conformal Killing fields.*** In this section, we introduce the special set of vector fields $\mathscr{Z}$ that appears in the definition (1.2.7) of our energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$ and in the weighted Klainerman–Sobolev inequality (1.2.10). We begin by recalling that a *Minkowskian conformal Killing field* is a vector field $Z$ such that

$$\nabla_{\mu} Z_{\nu} + \nabla_{\nu} Z_{\mu} = {}^{(Z)}\phi \, m_{\mu\nu} \tag{6.2.1}$$

for some function ${}^{(Z)}\phi(t, x)$. The tensor field

$$ {}^{(Z)}\pi_{\mu\nu} \overset{\text{def}}{=} \nabla_{\mu} Z_{\nu} + \nabla_{\nu} Z_{\mu} \tag{6.2.2}$$

is known as the *Minkowskian deformation tensor* of $Z$. If ${}^{(Z)}\pi_{\mu\nu} = 0$, then $Z$ is known as a *Minkowskian Killing field*. We also recall that the conformal Killing fields of the Minkowski metric $m_{\mu\nu}$ form a Lie algebra under the Lie bracket $[\,\cdot\,,\,\cdot\,]$ (see (6.3.1)). The Lie algebra is generated by the following 15 vector fields (see, e.g., [Christodoulou 2008]):

(i) the four *translations* $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$ ($\mu = 0, 1, 2, 3$),

(ii) the three *rotations* $\Omega_{jk} \overset{\text{def}}{=} x_j \frac{\partial}{\partial x^k} - x_k \frac{\partial}{\partial x^j}$ ($1 \le j < k \le 3$),

(iii) the three *Lorentz boosts* $\Omega_{0j} \overset{\text{def}}{=} -t \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial t}$ ($j = 1, 2, 3$),

(iv) the *scaling* vector field $S \stackrel{\text{def}}{=} x^\kappa \frac{\partial}{\partial x^\kappa}$, and

(v) the four *acceleration* vector fields $K_\mu \stackrel{\text{def}}{=} -2x_\mu S + g_{\kappa\lambda} x^\kappa x^\lambda \frac{\partial}{\partial x^\mu}$ ($\mu = 0, 1, 2, 3$).

It can be checked that the translations, rotations, and Lorentz boosts are in fact Killing fields of $m_{\mu\nu}$.

Two subsets of the above conformal Killing fields will play a prominent role in the remainder of the article, namely the rotations $\mathbb{O}$ and a larger set $\mathscr{Z}$, which are defined by

$$\mathbb{O} \stackrel{\text{def}}{=} \{\Omega_{jk}\}_{1 \leq j < k \leq 3}, \tag{6.2.3a}$$

$$\mathscr{Z} \stackrel{\text{def}}{=} \left\{ \frac{\partial}{\partial x^\mu}, \Omega_{\mu\nu}, S \right\}_{0 \leq \mu \leq \nu \leq 3}. \tag{6.2.3b}$$

The vector fields in $\mathscr{Z}$ satisfy a strong version of the relation (6.2.1). That is, if $Z \in \mathscr{Z}$, then

$$\nabla_\mu Z_\nu = {}^{(Z)}c_{\mu\nu}, \tag{6.2.4}$$

where the components ${}^{(Z)}c_{\mu\nu}$ are *constants* in our wave-coordinate system. In particular, we compute for future use that

$$\nabla_\mu S_\nu = m_{\mu\nu}, \tag{6.2.5a}$$

$$\nabla_\mu (\Omega_{\kappa\lambda})_\nu = m_{\mu\kappa} m_{\nu\lambda} - m_{\mu\lambda} m_{\nu\kappa}. \tag{6.2.5b}$$

We note in addition that if $Z \in \mathscr{Z}$ then there exists a *constant* $c_Z$ such that

$$\nabla_\mu Z_\nu + \nabla_\nu Z_\mu = c_Z m_{\mu\nu}. \tag{6.2.6}$$

Furthermore, by contracting each side of (6.2.6) against $(m^{-1})^{\mu\nu}$, we deduce that

$$c_Z = \tfrac{1}{4} {}^{(Z)}\pi_\kappa{}^\kappa = \tfrac{1}{2} {}^{(Z)}c_\kappa{}^\kappa. \tag{6.2.7}$$

**6.3. *Lie derivatives.*** As mentioned in Section 1.2.3, it is convenient to use Lie derivatives to differentiate the electromagnetic equations (3.7.1b)–(3.7.1c). In this section, we recall some basic facts concerning Lie derivatives.

We recall that, if $X$ and $Y$ are any pair of vector fields, then relative to an arbitrary coordinate system their *Lie bracket* $[X, Y]$ can be expressed as

$$[X, Y]^\mu = X^\kappa \, \partial_\kappa Y^\mu - Y^\kappa \, \partial_\kappa X^\mu. \tag{6.3.1}$$

Furthermore, we have that

$$\mathscr{L}_X Y = [X, Y], \tag{6.3.2}$$

where $\mathscr{L}$ denotes the *Lie derivative operator*. Given a tensor field $U$ of type $\binom{0}{m}$ and vector fields $Y_{(1)}, \ldots, Y_{(m)}$, the Leibniz rule for $\mathscr{L}$ implies that (6.3.2) generalizes as follows:

$$(\mathscr{L}_X U)(Y_{(1)}, \ldots, Y_{(m)})$$
$$= X\{U(Y_{(1)}, \ldots, Y_{(m)})\} - \sum_{i=1}^n U(Y_{(1)}, \ldots, Y_{(i-1)}, [X, Y_{(i)}], Y_{(i+1)}, \ldots, Y_{(m)}). \tag{6.3.3}$$

Using Lemma 6.7 below, we see that the left-hand side of (6.2.6) is equal to the Lie derivative of the Minkowski metric. It therefore follows that if $Z \in \mathscr{Z}$ then

$$\mathscr{L}_Z m_{\mu\nu} = c_Z m_{\mu\nu}, \tag{6.3.4a}$$

$$(\mathscr{L}_Z m^{-1})^{\mu\nu} = -c_Z (m^{-1})^{\mu\nu}, \tag{6.3.4b}$$

where the constant $c_Z$ is defined in (6.2.6).

## 6.4. *Modified covariant and modified Lie derivatives.*
It will be convenient for us to work with *modified Minkowski covariant derivatives* $\widehat{\nabla}_Z$ and *modified Lie derivatives*[23] $\widehat{\mathscr{L}}_Z$.

**Definition 6.5.** For $Z \in \mathscr{Z}$, we define the modified Minkowski covariant derivative $\widehat{\nabla}_Z$ by

$$\widehat{\nabla}_Z \overset{\text{def}}{=} \nabla_Z + c_Z, \tag{6.4.1}$$

where $c_Z$ denotes the constant from (6.2.6).

For each vector field $Z \in \mathscr{Z}$, we define the modified Lie derivative $\widehat{\mathscr{L}}_Z$ by

$$\widehat{\mathscr{L}}_Z \overset{\text{def}}{=} \mathscr{L}_Z + 2c_Z, \tag{6.4.2}$$

where $c_Z$ denotes the constant from (6.2.6).

The crucial features of the above definitions are captured by Lemmas 6.13 and 6.14 below. The first shows that, for each $Z \in \mathscr{Z}$, $\widehat{\nabla}_Z \Box_m \phi = \Box_m \nabla_Z \phi$, where $\Box_m = (m^{-1})^{\kappa\lambda} \nabla_\kappa \nabla_\lambda$ is the *Minkowski* wave operator. The second shows that

$$\widehat{\mathscr{L}}_Z \big( \big( (m^{-1})^{\mu\kappa} (m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda} (m^{-1})^{\nu\kappa} \big) \nabla_\mu \mathscr{F}_{\kappa\lambda} \big) = \big( (m^{-1})^{\mu\kappa} (m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda} (m^{-1})^{\nu\kappa} \big) \nabla_\mu \mathscr{L}_Z \mathscr{F}_{\kappa\lambda}.$$

Furthermore, Lemma 6.8 shows that $\mathscr{L}_Z \nabla_{[\lambda} \mathscr{F}_{\mu\nu]} = \nabla_{[\lambda} \mathscr{L}_Z \mathscr{F}_{\mu\nu]}$, where $[ \, \cdot \, ]$ denotes antisymmetrization. These commutation identities suggest that the operators $\widehat{\nabla}_Z$ and $\widehat{\mathscr{L}}_Z$ are potentially useful operators for differentiating the nonlinear equations (3.7.1a) and (3.7.1b)–(3.7.1c), respectively. This suggestion is borne out in Propositions 11.4 and 11.6, which show that the inhomogeneous terms generated by differentiating the nonlinear equations have a special algebraic structure, a structure that will be exploited during our global existence bootstrap argument.

## 6.5. *Vector-field algebra.*
We introduce here some notation that will allow us to compactly express iterated derivatives. If $\mathscr{A}$ is one of the sets from (6.2.3a)–(6.2.3b), then we label the vector fields in $\mathscr{A}$ as $Z^{\iota_1}, \ldots, Z^{\iota_d}$, where $d$ is the cardinality of $\mathscr{A}$. Then for any multi-index $I = (\iota_1, \ldots, \iota_k)$ of length $k$, where each $\iota_i \in \{1, 2, \ldots, d\}$, we make the following definition:

**Definition 6.6.** The iterated derivative operators are defined by

$$\nabla_{\mathscr{A}}^I \overset{\text{def}}{=} \nabla_{Z^{\iota_1}} \circ \cdots \circ \nabla_{Z^{\iota_k}}, \tag{6.5.1a}$$

$$\widehat{\nabla}_{\mathscr{A}}^I \overset{\text{def}}{=} \widehat{\nabla}_{Z^{\iota_1}} \circ \cdots \circ \widehat{\nabla}_{Z^{\iota_k}}, \tag{6.5.1b}$$

$$\mathscr{L}_{\mathscr{A}}^I \overset{\text{def}}{=} \mathscr{L}_{Z^{\iota_1}} \circ \cdots \circ \mathscr{L}_{Z^{\iota_k}}, \tag{6.5.1c}$$

$$\widehat{\mathscr{L}}_{\mathscr{A}}^I \overset{\text{def}}{=} \widehat{\mathscr{L}}_{Z^{\iota_1}} \circ \cdots \circ \widehat{\mathscr{L}}_{Z^{\iota_k}}, \quad \text{etc.} \tag{6.5.1d}$$

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[23]Note that these are not the same modified Lie derivatives that appear in [Christodoulou and Klainerman 1993; Zipser 2000; Klainerman and Nicolò 2003; Bieri 2007].

Similarly, if $I = (\mu_1, \ldots, \mu_k)$ is a coordinate multi-index of length $k$, where $\mu_1, \ldots, \mu_k \in \{0, 1, 2, 3\}$ and $U$ is a tensor field, then we use shorthand notation such as

$$\nabla^I U \overset{\text{def}}{=} \nabla_{\mu_1} \cdots \nabla_{\mu_k} U, \quad \text{etc.} \tag{6.5.2}$$

Under the above conventions, the Leibniz rule can be written as, e.g.,

$$\mathscr{L}_{\mathscr{Z}}^I (UV) = \sum_{I_1 + I_2 = I} (\mathscr{L}_{\mathscr{Z}}^{I_1} U)(\mathscr{L}_{\mathscr{Z}}^{I_2} V), \quad \text{etc.}, \tag{6.5.3}$$

where by a sum over $I_1 + I_2 = I$ we mean a sum over all order-preserving partitions of the index $I$ into two multi-indices. That is, if $I = (\iota_1, \ldots, \iota_k)$, then $I_1 = (\iota_{i_1}, \ldots, \iota_{i_a})$ and $I_2 = (\iota_{i_{a+1}}, \ldots, \iota_{i_k})$, where $i_1, \ldots, i_k$ is any reordering of the integers $1, \ldots, k$ such that $i_1 < \cdots < i_a$ and $i_{a+1} < \cdots < i_k$.

The next standard lemma provides a useful expression relating Lie derivatives to covariant derivatives.

**Lemma 6.7** (Lie derivatives in terms of covariant derivatives [Wald 1984, p. 441]). *Let $X$ be a vector field, and let $U$ be a tensor field of type $\binom{n}{m}$. Then $\mathscr{L}_X U$ can be expressed in terms of covariant derivatives of $U$ and $X$ as follows*:

$$\mathscr{L}_X U_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n} = \nabla_X U_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n} + U_{\kappa \mu_2 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n} \nabla_{\mu_1} X^\kappa + \cdots + U_{\mu_1 \cdots \mu_{m-1} \kappa}{}^{\nu_1 \cdots \nu_n} \nabla_{\mu_m} X^\kappa$$
$$- U_{\mu_1 \cdots \mu_m}{}^{\kappa \nu_2 \cdots \nu_n} \nabla_\kappa X^{\nu_1} - \cdots - U_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_{n-1} \kappa} \nabla_\kappa X^{\nu_n}. \tag{6.5.4}$$

The next lemma shows that the operators $\mathscr{L}_Z$ and $\widehat{\mathscr{L}}_Z$ commute with $\nabla$ if $Z \in \mathscr{Z}$.

**Lemma 6.8** ($\mathscr{L}_Z$ and $\nabla$ commute). *Let $\nabla$ denote the Levi-Civita connection corresponding to the Minkowski metric $m$, and let $I$ be a $\mathscr{Z}$-multi-index. Let $\widehat{\mathscr{L}}_{\mathscr{Z}}^I$ be the iterated modified Lie derivative from Definitions 6.5 and 6.6. Then*

$$[\nabla, \mathscr{L}_{\mathscr{Z}}^I] = 0 \quad and \quad [\nabla, \widehat{\mathscr{L}}_{\mathscr{Z}}^I] = 0. \tag{6.5.5}$$

*In an arbitrary coordinate system, equations (6.5.5) are equivalent to the following relations, which hold for all type-$\binom{n}{m}$ tensor fields $U$*:

$$\nabla_\mu \{\mathscr{L}_{\mathscr{Z}}^I U_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n}\} = \mathscr{L}_{\mathscr{Z}}^I \{\nabla_\mu U_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n}\},$$
$$\nabla_\mu \{\widehat{\mathscr{L}}_{\mathscr{Z}}^I U_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n}\} = \widehat{\mathscr{L}}_{\mathscr{Z}}^I \{\nabla_\mu U_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n}\}. \tag{6.5.6}$$

*Proof.* The relation (6.5.5) can be shown via induction in $|I|$ by using (6.5.4) and the fact that $\nabla \nabla Z = 0$. $\square$

The next lemma captures the commutation properties of vector fields $Z \in \mathscr{Z}$.

**Lemma 6.9** (Lie bracket relations [Christodoulou and Klainerman 1990, p. 139]). *Relative to the wave-coordinate system $\{x^\mu\}_{\mu=0,1,2,3}$, the vector fields belonging to the subset $\mathscr{Z} \overset{\text{def}}{=} \{\frac{\partial}{\partial x^\mu}, \Omega_{\mu\nu}, S\}_{0 \leq \mu \leq \nu \leq 3}$ of the Minkowskian conformal Killing fields satisfy the following commutation relations, where ${}^{(Z)}c_\mu{}^\kappa$ is defined in (6.2.4)*:

$$\left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right] = 0 = {}^{(\partial/\partial x^\nu)}c_\mu{}^\kappa \frac{\partial}{\partial x^\kappa} \qquad (\mu, \nu = 0, 1, 2, 3), \qquad \text{(6.5.7a)}$$

$$\left[\frac{\partial}{\partial x^\lambda}, \Omega_{\mu\nu}\right] = m_{\lambda\mu}\frac{\partial}{\partial x^\nu} - m_{\lambda\nu}\frac{\partial}{\partial x^\mu} = {}^{(\Omega_{\mu\nu})}c_\lambda{}^\kappa \frac{\partial}{\partial x^\kappa} \qquad (\lambda, \mu, \nu = 0, 1, 2, 3), \qquad \text{(6.5.7b)}$$

$$\left[\frac{\partial}{\partial x^\mu}, S\right] = \frac{\partial}{\partial x^\mu} = {}^{(S)}c_\mu{}^\kappa \frac{\partial}{\partial x^\kappa} \qquad (\mu = 0, 1, 2, 3), \qquad \text{(6.5.7c)}$$

$$[\Omega_{\kappa\lambda}, \Omega_{\mu\nu}] = m_{\kappa\mu}\Omega_{\nu\lambda} - m_{\kappa\nu}\Omega_{\mu\lambda} + m_{\lambda\mu}\Omega_{\kappa\nu} - m_{\lambda\nu}\Omega_{\kappa\mu} \quad (\kappa, \lambda, \mu, \nu = 0, 1, 2, 3), \qquad \text{(6.5.7d)}$$

$$[\Omega_{\mu\nu}, S] = 0 \qquad (\mu, \nu = 0, 1, 2, 3). \qquad \text{(6.5.7e)}$$

We now provide the following simple commutation lemma:

**Lemma 6.10** ($\nabla_Z$ and $\nabla_{\partial/\partial x^\mu}$ commutation relations). *Let $Z \in \mathscr{Z}$. Then relative to the wave-coordinate system $\{x^\mu\}_{\mu=0,1,2,3}$, the differential operators $\nabla_{\partial/\partial x^\mu}$ and $\nabla_Z$ satisfy the following commutation relations*:

$$[\nabla_{\partial/\partial x^\mu}, \nabla_Z] = {}^{(Z)}c_\mu{}^\kappa \frac{\partial}{\partial x^\kappa}, \qquad \text{(6.5.8)}$$

*where ${}^{(Z)}c_\mu{}^\kappa$ is defined in* (6.2.4).

*Proof.* The relation (6.5.8) follows from Lemma 6.9 and the identity $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]}$, which holds for all pairs of vector fields $X$ and $Y$; this identity holds because of the torsion-free property of the connection $\nabla$ and because the Riemann curvature tensor of the Minkowski metric $m_{\mu\nu}$ completely vanishes. □

The next lemma shows that the operators $\nabla$ and $\nabla_{\mathscr{Z}}^I$ commute up to lower-order terms.

**Lemma 6.11** ($\nabla$ and $\nabla_{\mathscr{Z}}^I$ commutation inequalities). *Let $U$ be a type-$\binom{n}{m}$ tensor field, and let $I$ be a $\mathscr{Z}$-multi-index. Then the following inequality holds*:

$$|\nabla_{\mathscr{Z}}^I \nabla U| \lesssim |\nabla \nabla_{\mathscr{Z}}^I U| + \sum_{|J| \leq |I|-1} |\nabla \nabla_{\mathscr{Z}}^J U|. \qquad \text{(6.5.9)}$$

*Proof.* Using (5.1.14), we have that

$$|\nabla_{\mathscr{Z}}^I \nabla U| \approx \sum_{\mu=0}^{3} |\nabla_{\mathscr{Z}}^I \nabla_{\partial/\partial x^\mu} U|. \qquad \text{(6.5.10)}$$

We therefore repeatedly apply Lemma 6.10 to deduce that there exist constants $C_{I;J}^\nu$ such that

$$\nabla_{\mathscr{Z}}^I \nabla_{\partial/\partial x^\mu} U = \nabla_{\partial/\partial x^\mu} \nabla_{\mathscr{Z}}^I U + \sum_{|J| \leq |I|-1} \sum_{\nu=0}^{3} C_{I;J}^\nu \nabla_{\partial/\partial x^\nu} \nabla_{\mathscr{Z}}^J U. \qquad \text{(6.5.11)}$$

Inequality (6.5.9) now follows from applying (5.1.14) to each side of (6.5.11). □

The next lemma provides some important differential identities.

**Lemma 6.12** (Geometric differential identities). *Let $\underline{L}$ and $L$ be the Minkowski-null geodesic vector fields defined in* (5.1.2a)–(5.1.2b), *and let $O \in \mathbb{O}$. Then the vector fields $\underline{L}$, $L$, and $O$ mutually commute*:

$$[\underline{L}, L] = 0, \qquad [\underline{L}, O] = 0, \quad and \quad [L, O] = 0. \qquad \text{(6.5.12)}$$

*Furthermore, let* $\upsilon_{\kappa\lambda\mu\nu}$, $\not{m}_{\mu\nu}$, *and* $\not{\psi}_{\mu\nu}$ *denote the tensor fields defined in* (5.1.4b), (5.1.6a), *and* (5.1.6c). *Then*

$$\mathscr{L}_O \upsilon_{\kappa\lambda\mu\nu} = 0, \tag{6.5.13a}$$

$$\mathscr{L}_O \not{m}_{\mu\nu} = 0, \tag{6.5.13b}$$

$$\mathscr{L}_O \not{\psi}_{\mu\nu} = 0. \tag{6.5.13c}$$

*Proof.* Equation (6.5.12) can be checked by performing straightforward calculations and using the definitions (5.1.2a)–(5.1.2b) of $\underline{L}$ and $L$, the definitions of the rotations $O \in \mathbb{O}$ given at the beginning of Section 6.2, and the Lie bracket formula (6.3.1). Equation (6.5.13a) follows from the well-known identity $\mathscr{L}_X \upsilon_{\kappa\lambda\mu\nu} = \frac{1}{2}{}^{(X)}\pi^\beta{}_\beta \upsilon_{\kappa\lambda\mu\nu}$, where ${}^{(X)}\pi_{\mu\nu}$ is defined in (6.2.2), together with the fact that $\mathscr{L}_O m_{\mu\nu} = {}^{(O)}\pi_{\mu\nu} = 0$ (i.e., that $O$ is a Killing field of $m_{\mu\nu}$). Equations (6.5.13b) and (6.5.13c) then follow from definitions (5.1.4b) and (5.1.6c) and the identities (6.5.12)–(6.5.13a). $\qquad\square$

The next lemma shows that the modified covariant derivatives $\widehat{\nabla}_{\mathscr{L}}^I$ have favorable commutation properties with the Minkowski wave operator.

**Lemma 6.13** ($\widehat{\nabla}_{\mathscr{L}}^I$ and $\square_m$ commutation properties). *Let $I$ be a $\mathscr{L}$-multi-index, and let $\phi$ be any function. Let $\widehat{\nabla}_{\mathscr{L}}^I$ be the iterated modified Minkowski covariant derivative operator from Definitions 6.5 and 6.6, and let $\square_m \overset{\text{def}}{=} (m^{-1})^{\kappa\lambda} \nabla_\kappa \nabla_\lambda$ denote the Minkowski wave operator. Then*

$$\widehat{\nabla}_{\mathscr{L}}^I \square_m \phi = \square_m \nabla_{\mathscr{L}}^I \phi. \tag{6.5.14}$$

*Proof.* Using the symmetry of the tensor field $\nabla_\kappa \nabla_\lambda \phi$ together with (6.1.3a), (6.2.6), and definition (6.4.1), we compute that

$$\begin{aligned}
\square_m \nabla_Z \phi = (m^{-1})^{\kappa\lambda} \nabla_\kappa \nabla_\lambda (Z^\zeta \nabla_\zeta \phi) &= \nabla_Z \square_m \phi + 2(\nabla^\kappa Z^\lambda) \nabla_\lambda \nabla_\kappa \phi \\
&= \nabla_Z \square_m \phi + (\nabla^\kappa Z^\lambda + \nabla^\lambda Z^\kappa) \nabla_\kappa \nabla_\lambda \phi \\
&= \nabla_Z \square_m \phi + c_Z \square_m \phi \\
&\overset{\text{def}}{=} \widehat{\nabla}_Z \square_m \phi.
\end{aligned} \tag{6.5.15}$$

This proves (6.5.14) in the case $|I| = 1$. The general case now follows inductively. $\qquad\square$

The next lemma shows that the modified Lie derivative $\mathscr{L}_{\mathscr{L}}^I$ operator has favorable commutation properties with the linear Maxwell–Maxwell term $\nabla_\mu \mathscr{F}^{\mu\nu} = \frac{1}{2}[(m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}]\nabla_\mu \mathscr{F}_{\kappa\lambda}$.

**Lemma 6.14** (Commutation properties of $\widehat{\mathscr{L}}_{\mathscr{L}}^I$ with a linear Maxwell–Maxwell term). *Let $I$ be a $\mathscr{L}$-multi-index, and let $\mathscr{F}$ be a two-form. Let $\widehat{\mathscr{L}}_{\mathscr{L}}^I$ be the iterated modified Lie derivative from Definitions 6.5 and 6.6. Then*

$$\widehat{\mathscr{L}}_{\mathscr{L}}^I \Big( \big( (m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa} \big) \nabla_\mu \mathscr{F}_{\kappa\lambda} \Big)$$
$$= \big( (m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa} \big) \nabla_\mu \mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{\kappa\lambda}. \tag{6.5.16}$$

*Proof.* Let $Z \in \mathscr{L}$. By the Leibniz rule, (6.3.4b), and Lemma 6.8, we have that

$$\mathscr{L}_Z\big(\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu\mathscr{F}_{\kappa\lambda}\big)$$
$$= -2c_Z\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu\mathscr{F}_{\kappa\lambda}$$
$$+ \big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu\mathscr{L}_Z\mathscr{F}_{\kappa\lambda}. \quad (6.5.17)$$

It thus follows from Definition 6.5 that

$$\widehat{\mathscr{L}}_Z\big(\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu\mathscr{F}_{\kappa\lambda}\big)$$
$$= \big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu\mathscr{L}_Z\mathscr{F}_{\kappa\lambda}. \quad (6.5.18)$$

This implies (6.5.16) in the case $|I| = 1$. The general case now follows inductively. $\qquad\square$

The next lemma shows that some of the differential operators we have introduced commute with the null decomposition of a two-form.

**Lemma 6.15** (Differential operators that commute with the null decomposition). *Let $\mathscr{F}$ be a two-form, and let $\underline{\alpha}$, $\alpha$, $\rho$, and $\sigma$ be its Minkowskian null components. Let $O \in \mathbb{O}$ be any of the rotational Minkowskian Killing fields $\Omega_{jk}$ ($1 \leq j < k \leq 3$). Then $\mathscr{L}_O\underline{\alpha}[\mathscr{F}] = \underline{\alpha}[\mathscr{L}_O\mathscr{F}]$, $\mathscr{L}_O\alpha[\mathscr{F}] = \alpha[\mathscr{L}_O\mathscr{F}]$, $\mathscr{L}_O\rho[\mathscr{F}] = \rho[\mathscr{L}_O\mathscr{F}]$, and $\mathscr{L}_O\sigma[\mathscr{F}] = \sigma[\mathscr{L}_O\mathscr{F}]$. An analogous result holds for the operators $\nabla_{\underline{L}}$ and $\nabla_L$; i.e., $\mathscr{L}_O$, $\nabla_{\underline{L}}$, and $\nabla_L$ commute with the null decomposition of $\mathscr{F}$.*

*Proof.* Lemma 6.15 follows from Definition 5.9, Lemmas 5.7 and 6.12, and the fact that $\mathscr{L}_O m_{\mu\nu} = (\mathscr{L}_O m^{-1})^{\mu\nu} = 0$. $\qquad\square$

The next lemma shows that *weighted* covariant derivatives can be controlled by covariant derivatives with respect to vector fields $Z \in \mathscr{Z}$.

**Lemma 6.16** (Weighted pointwise differential operator inequalities [Lindblad and Rodnianski 2010, Lemma 5.1]). *For any tensor field $U$ and any two-tensor $\Pi$, we have the following pointwise estimates (where $|\overline{\nabla}^2 U| \overset{\text{def}}{=} |\overline{\nabla}\,\overline{\nabla} U|$):*

$$(1 + t + |q|)|\overline{\nabla} U| + (1 + |q|)|\nabla U| \lesssim \sum_{|I| \leq 1} |\nabla_{\mathscr{L}}^I U|, \quad (6.5.19a)$$

$$|\overline{\nabla}^2 U| + r^{-1}|\overline{\nabla} U| \lesssim r^{-1}(1 + t + |q|)^{-1} \sum_{|I| \leq 2} |\nabla_{\mathscr{L}}^I U|, \quad (6.5.19b)$$

$$|\Pi^{\kappa\lambda}\nabla_\kappa\nabla_\lambda U| \lesssim \big((1 + t + |q|)^{-1}|\Pi| + (1 + |q|)^{-1}|\Pi|_{\mathscr{L}\mathscr{L}}\big) \sum_{|I| \leq 1} |\nabla\nabla_{\mathscr{L}}^I U|. \quad (6.5.19c)$$

The next lemma shows that rotational Lie derivatives can be used to approximate weighted $S_{r,t}$-intrinsic covariant derivatives.

**Lemma 6.17** (Weighted covariant derivatives approximated by rotational Lie derivatives [Speck 2012, Lemma 8.0.5]). *Let $U$ be any tensor field $m$-tangent to the spheres $S_{r,t}$ and $k \geq 0$ be any integer. Then with $r \overset{\text{def}}{=} |x|$, we have that*

$$\sum_{|I| \leq k} r^{|I|}|\slashed{\nabla}^I U| \approx \sum_{|I| \leq k} |\mathscr{L}_{\mathbb{O}}^I U|. \quad (6.5.20)$$

**Corollary 6.18.** *Let $\mathcal{F}$ be a two-form, and let $\underline{\alpha}[\mathcal{F}]$, $\alpha[\mathcal{F}]$, $\rho[\mathcal{F}]$, and $\sigma[\mathcal{F}]$ denote its Minkowskian null components. Then with $r = |x|$, we have that*

$$r|\not{\nabla}\underline{\alpha}[\mathcal{F}]| \lesssim \sum_{|I| \leq 1} |\underline{\alpha}[\mathscr{L}_{\mathscr{Z}}^I \mathcal{F}]|. \tag{6.5.21}$$

*Furthermore, analogous inequalities hold for $\alpha[\mathcal{F}]$, $\rho[\mathcal{F}]$, and $\sigma[\mathcal{F}]$.*

*Proof.* Inequality (6.5.21) follows from Lemmas 6.15 and 6.17. □

Finally, the following proposition provides pointwise inequalities relating various Lie and covariant derivative operators under various contraction seminorms:

**Proposition 6.19** (Lie derivative and Minkowski covariant derivative comparison inequalities). *Let $U$ be a tensor field. Then*

$$\sum_{|I| \leq k} |\mathscr{L}_{\mathscr{Z}}^I U| \approx \sum_{|I| \leq k} |\nabla_{\mathscr{Z}}^I U|. \tag{6.5.22}$$

*Furthermore, let $P$ be a symmetric or an antisymmetric type-$\binom{0}{2}$ tensor field. Then the following inequalities hold*:

$$\sum_{|I| \leq k} |\nabla \mathscr{L}_{\mathscr{Z}}^I P| \lesssim \sum_{|I| \leq k} |\nabla \nabla_{\mathscr{Z}}^I P|, \tag{6.5.23a}$$

$$\sum_{|I| \leq k} |\bar{\nabla} \mathscr{L}_{\mathscr{Z}}^I P| \lesssim \sum_{|I| \leq k} |\bar{\nabla} \nabla_{\mathscr{Z}}^I P|, \tag{6.5.23b}$$

$$|\mathscr{L}_{\mathscr{Z}}^I P|_{\mathscr{L}\mathscr{L}} \lesssim |\nabla_{\mathscr{Z}}^I P|_{\mathscr{L}\mathscr{L}} + \underbrace{\sum_{|J| \leq |I|-1} |\nabla_{\mathscr{Z}}^J P|_{\mathscr{L}\mathscr{T}}}_{absent \ if \ |I| = 0} + \underbrace{\sum_{|J'| \leq |I|-2} |\nabla_{\mathscr{Z}}^{J'} P|}_{absent \ if \ |I| \leq 1}, \tag{6.5.23c}$$

$$|\nabla \mathscr{L}_{\mathscr{Z}}^I P|_{\mathscr{L}\mathscr{L}} \lesssim |\nabla \nabla_{\mathscr{Z}}^I P|_{\mathscr{L}\mathscr{L}} + \underbrace{\sum_{|J| \leq |I|-1} |\nabla_{\mathscr{Z}}^J P|_{\mathscr{L}\mathscr{T}}}_{absent \ if \ |I| = 0} + \underbrace{\sum_{|J'| \leq |I|-2} |\nabla \nabla_{\mathscr{Z}}^{J'} P|}_{absent \ if \ |I| \leq 1}, \tag{6.5.23d}$$

$$|\nabla P|_{\mathscr{L}\mathscr{N}} + |\nabla P|_{\mathscr{T}\mathscr{T}} \lesssim (1+|q|)^{-1} \sum_{|I| \leq 1} \left( |\mathscr{L}_{\mathscr{Z}}^I P|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}_{\mathscr{Z}}^I P|_{\mathscr{T}\mathscr{T}} \right) + (1+t+|q|)^{-1} \sum_{|I| \leq 1} |\mathscr{L}_{\mathscr{Z}}^I P|. \tag{6.5.23e}$$

*Proof.* Inequality (6.5.22) follows inductively from (6.2.4) and (6.5.4).

To prove the remaining inequalities, for each $Z \in \mathscr{Z}$, we define the contraction operator $\mathscr{C}_Z$ by

$$(\mathscr{C}_Z P)_{\mu\nu} \overset{\text{def}}{=} P_{\kappa\nu} {}^{(Z)}c_\mu{}^\kappa + P_{\mu\kappa} {}^{(Z)}c_\nu{}^\kappa, \tag{6.5.24}$$

where the covariantly constant tensor field ${}^{(Z)}c_\mu{}^\kappa$ is defined in (6.2.4). It follows from definition (6.5.24) and Lemma 6.7 that

$$\mathscr{L}_{\mathscr{Z}} P = \nabla_Z P + \mathscr{C}_Z P. \tag{6.5.25}$$

Since each $Z \in \mathscr{Z}$ is a conformal Killing field and since $L^\mu L^\nu m_{\mu\nu} = 0$, it follows that $L^\mu L_\nu {}^{(Z)}c_\mu{}^\nu = 0$. Also using the fact that each ${}^{(Z)}c_\mu{}^\nu$ is a constant, we have that

$$|\mathscr{C}_Z P|_{\mathscr{L}\mathscr{L}} \lesssim |P|_{\mathscr{L}\mathscr{T}},  \tag{6.5.26}$$

$$|\mathscr{C}_Z P| \lesssim |P|.  \tag{6.5.27}$$

If $I = (\iota_1, \ldots, \iota_k)$ is a $\mathscr{Z}$-multi-index with $1 \le |I| = k$, then using the fact that the components $^{(Z)}c_\mu{}^\kappa$ are constants, we have that

$$
\begin{aligned}
\mathscr{L}_{\mathscr{Z}}^I P &\overset{\text{def}}{=} \mathscr{L}_{Z^{\iota_1}} \circ \cdots \circ \mathscr{L}_{Z^{\iota_k}} P \\
&= (\nabla_{Z^{\iota_1}} + \mathscr{C}_{Z^{\iota_1}}) \circ \cdots \circ (\nabla_{Z^{\iota_k}} + \mathscr{C}_{Z^{\iota_k}}) P \\
&= \nabla_{\mathscr{Z}}^I P + \sum_{i=1}^k \mathscr{C}_{Z^{\iota_i}} \circ \nabla_{Z^{\iota_1}} \circ \cdots \circ \nabla_{Z^{\iota_{i-1}}} \circ \nabla_{Z^{\iota_{i+1}}} \circ \cdots \circ \nabla_{Z^{\iota_k}} P + \overbrace{\sum_{\substack{I_1+I_2=I \\ |I_2| \le k-2}} \mathscr{C}_{\mathscr{Z}}^{I_1} \nabla_{\mathscr{Z}}^{I_2} P}^{\text{absent if } k=1}.
\end{aligned}  \tag{6.5.28}
$$

Inequality (6.5.23a) now follows from applying $\nabla$ to each side of (6.5.28), from using the fact that the operator $\nabla$ commutes through the operators $\mathscr{C}_Z$, and from (6.5.27). Inequality (6.5.23b) follows from similar reasoning. Inequalities (6.5.23c) and (6.5.23d) also follow from similar reasoning together with (6.5.26).

To prove (6.5.23e), we first observe that, by (6.5.19a) and (6.5.22), we have that

$$
\begin{aligned}
|\nabla P|_{\mathscr{L}\mathscr{N}} + |\nabla P|_{\mathscr{T}\mathscr{T}} &\lesssim |\nabla_{\underline{L}} P|_{\mathscr{L}\mathscr{N}} + |\nabla_{\underline{L}} P|_{\mathscr{T}\mathscr{T}} + |\overline{\nabla} P| \\
&\lesssim |\nabla_{\underline{L}} P|_{\mathscr{L}\mathscr{N}} + |\nabla_{\underline{L}} P|_{\mathscr{T}\mathscr{T}} + (1+t+|q|)^{-1} \sum_{|I| \le 1} |\mathscr{L}_{\mathscr{Z}}^I P|.
\end{aligned}  \tag{6.5.29}
$$

Therefore, from (6.5.29), we see that to prove (6.5.23e) it suffices to prove that the following inequality holds for any symmetric or antisymmetric type-$\binom{0}{2}$ tensor field $P$:

$$|\nabla_{\underline{L}} P|_{\mathscr{L}\mathscr{N}} + |\nabla_{\underline{L}} P|_{\mathscr{T}\mathscr{T}} \lesssim (1+|q|)^{-1} \sum_{|I| \le 1} \left( |\mathscr{L}_{\mathscr{Z}}^I P|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}_{\mathscr{Z}}^I P|_{\mathscr{T}\mathscr{T}} \right).  \tag{6.5.30}$$

To this end, we use the vector fields $S = x^\kappa \partial_\kappa$ and $\Omega_{0j} = -t \partial_j - x_j \partial_t$ to decompose

$$\underline{L} = -q^{-1}(S + \omega^a \Omega_{0a}) \quad \text{and} \quad \omega^a \overset{\text{def}}{=} x^a/r,  \tag{6.5.31}$$

which implies that

$$-q\nabla_{\underline{L}} P_{\mu\nu} = \nabla_S P_{\mu\nu} + \omega^a \nabla_{\Omega_{0a}} P_{\mu\nu}.  \tag{6.5.32}$$

Using (6.2.5a), (6.2.5b), and (6.5.4), we compute that

$$\nabla_S P_{\mu\nu} = \mathscr{L}_S P_{\mu\nu} - 2 P_{\mu\nu},  \tag{6.5.33}$$

$$\omega^a \nabla_{\Omega_{0a}} P_{\mu\nu} = \omega^a \mathscr{L}_{\Omega_{0a}} P_{\mu\nu} - \tfrac{1}{2} \left( \underline{L}_\mu L^\kappa P_{\kappa\nu} - L_\mu \underline{L}^\kappa P_{\kappa\nu} + \underline{L}_\nu L^\kappa P_{\mu\kappa} - L_\nu \underline{L}^\kappa P_{\mu\kappa} \right).  \tag{6.5.34}$$

Inserting these two identities into (6.5.32), we conclude that

$$
\begin{aligned}
-q\nabla_{\underline{L}} P_{\mu\nu} = \mathscr{L}_S P_{\mu\nu} &+ \omega^a \mathscr{L}_{\Omega_{0a}} P_{\mu\nu} - 2 P_{\mu\nu} \\
&- \tfrac{1}{2} \left( \underline{L}_\mu L^\kappa P_{\kappa\nu} - L_\mu \underline{L}^\kappa P_{\kappa\nu} + \underline{L}_\nu L^\kappa P_{\mu\kappa} - L_\nu \underline{L}^\kappa P_{\mu\kappa} \right).
\end{aligned}  \tag{6.5.35}
$$

Contracting (6.5.35) against the sets $\mathscr{LN}$ and $\mathscr{TT}$, we see that

$$|q|\,|\nabla_{\underline{L}}P|_{\mathscr{LN}} + |q|\,|\nabla_{\underline{L}}P|_{\mathscr{TT}} \lesssim \sum_{|I|\leq 1} \left(|\mathscr{L}_{\mathscr{Z}}^I P|_{\mathscr{LN}} + |\mathscr{L}_{\mathscr{Z}}^I P|_{\mathscr{TT}}\right). \tag{6.5.36}$$

Furthermore, by decomposing

$$\underline{L} = \partial_t - \partial_r = \partial_t - \omega^a\,\partial_a \tag{6.5.37}$$

and using the fact that ${}^{(\partial/\partial t)}c_\mu{}^\nu = {}^{(\partial/\partial x^j)}c_\mu{}^\nu = 0$ (where ${}^{(Z)}c_{\mu\nu}$ is defined in (6.2.4)), we deduce that

$$\nabla_{\underline{L}}P_{\mu\nu} = \mathscr{L}_{\partial/\partial t}P_{\mu\nu} - \omega^a \mathscr{L}_{\partial/\partial x^a}P_{\mu\nu}. \tag{6.5.38}$$

Contracting (6.5.38) against the sets $\mathscr{LN}$ and $\mathscr{TT}$, we have that

$$|\nabla_{\underline{L}}P|_{\mathscr{LN}} + |\nabla_{\underline{L}}P|_{\mathscr{TT}} \lesssim \sum_{|I|=1} \left(|\mathscr{L}_{\mathscr{Z}}^I P|_{\mathscr{LN}} + |\mathscr{L}_{\mathscr{Z}}^I P|_{\mathscr{TT}}\right). \tag{6.5.39}$$

Adding (6.5.36) and (6.5.39), we arrive at inequality (6.5.30). This completes our proof of (6.5.23e). $\square$

## 7. The reduced equation satisfied by $\nabla_{\mathscr{Z}}^I h^{(1)}$

In this short section, we assume that $h_{\mu\nu}^{(1)}$ is a solution to the reduced equation (3.7.1a). We provide a proposition that gives a preliminary description of the inhomogeneities in the equation satisfied by $\nabla_{\mathscr{Z}}^I h_{\mu\nu}^{(1)}$.

**Proposition 7.1** (Inhomogeneities for $\nabla_{\mathscr{Z}}^I h_{\mu\nu}^{(1)}$). *Suppose that $h_{\mu\nu}^{(1)}$ is a solution to the reduced equation* (3.7.1a), *and let $I$ be any $\mathscr{Z}$-multi-index. Then $\nabla_{\mathscr{Z}}^I h_{\mu\nu}^{(1)}$ is a solution to the inhomogeneous system*

$$\widetilde{\square}_g \nabla_{\mathscr{Z}}^I h_{\mu\nu}^{(1)} = \mathfrak{H}_{\mu\nu}^{(1;I)}, \tag{7.0.1}$$

$$\begin{aligned}
\mathfrak{H}_{\mu\nu}^{(1;I)} &= \widehat{\nabla}_{\mathscr{Z}}^I \mathfrak{H}_{\mu\nu} - \widehat{\nabla}_{\mathscr{Z}}^I \widetilde{\square} h_{\mu\nu}^{(0)} - \left(\widehat{\nabla}_{\mathscr{Z}}^I \widetilde{\square}_g h_{\mu\nu}^{(1)} - \widetilde{\square}_g \nabla_{\mathscr{Z}}^I h_{\mu\nu}^{(1)}\right) \\
&= \widehat{\nabla}_{\mathscr{Z}}^I \mathfrak{H}_{\mu\nu} - \widehat{\nabla}_{\mathscr{Z}}^I \widetilde{\square} h_{\mu\nu}^{(0)} - \left(\widehat{\nabla}_{\mathscr{Z}}^I \left(H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda h_{\mu\nu}^{(1)}\right) - H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda \nabla_{\mathscr{Z}}^I h_{\mu\nu}^{(1)}\right).
\end{aligned} \tag{7.0.2}$$

*Proof.* Proposition 7.1 follows from differentiating each side of (3.7.1a) with modified covariant derivatives $\widehat{\nabla}_{\mathscr{Z}}^I$ and applying Lemma 6.13. $\square$

## 8. The equations of variation, the canonical stress, and electromagnetic energy currents

In this section, we introduce the electromagnetic equations of variation, which are linearized versions of the reduced electromagnetic equations. The significance of the equations of variation is the following: if $\mathscr{F}$ is a solution to the reduced electromagnetic equations (3.7.1b)–(3.7.1c), then $\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}$ is a solution to the equations of variation. We then provide a preliminary description of the structure of the inhomogeneous terms in the equations of variation satisfied by $\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}$. Additionally, we introduce the canonical stress tensor field and use it to construct energy currents. The energy currents are vector fields that will be used in the divergence theorem to derive weighted energy estimates for solutions to the equations of variation; this analysis is carried out in Section 12.

**8.1. *Equations of variation.*** The equations of variation in the unknowns $\dot{\mathscr{F}}_{\mu\nu}$ are the linearization[24] of (3.7.1b)–(3.7.1c) around a background $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$. More specifically, the equations of variation are the system

$$\nabla_\lambda \dot{\mathscr{F}}_{\mu\nu} + \nabla_\mu \dot{\mathscr{F}}_{\nu\lambda} + \nabla_\nu \dot{\mathscr{F}}_{\lambda\mu} = \dot{\mathfrak{F}}_{\lambda\mu\nu} \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{8.1.1a}$$

$$N^{\#\mu\nu\kappa\lambda} \nabla_\mu \dot{\mathscr{F}}_{\kappa\lambda} = \dot{\mathfrak{F}}^\nu \qquad (\nu = 0, 1, 2, 3), \tag{8.1.1b}$$

where $N^{\#\mu\nu\kappa\lambda}$ is the $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$-dependent tensor field defined in (3.7.2c) and $\dot{\mathfrak{F}}_{\lambda\mu\nu}$ and $\dot{\mathfrak{F}}^\nu$ are inhomogeneous terms that are specified in Proposition 8.1. In this article, the equations of variation will arise when we differentiate the reduced equations (3.7.1b)–(3.7.1c) with modified Lie derivatives. In particular, $\dot{\mathscr{F}}$ will be equal to $\mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{\mu\nu}$. The next proposition, which is a companion of Proposition 7.1, provides a preliminary expression of the inhomogeneous terms that arise in the study of the equations of variation satisfied by $\mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{\mu\nu}$. We remark that the proof of the proposition uses lemmas that are proved in Section 11.

**Proposition 8.1** (Inhomogeneities for $\mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{\mu\nu}$). *If $\mathscr{F}_{\mu\nu}$ is a solution to the reduced electromagnetic equations (3.7.1b)–(3.7.1c) and $I$ is a $\mathscr{L}$-multi-index, then $\dot{\mathscr{F}}_{\mu\nu} \overset{\text{def}}{=} \mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{\mu\nu}$ is a solution to the equations of variation (8.1.1a)–(8.1.1b) (corresponding to the background $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$) with inhomogeneous terms $\dot{\mathfrak{F}}_{\lambda\mu\nu} \overset{\text{def}}{=} \mathfrak{F}_{\lambda\mu\nu}^{(I)}$ and $\dot{\mathfrak{F}}^\nu \overset{\text{def}}{=} \mathfrak{F}_{(I)}^\nu$, where*

$$\mathfrak{F}_{\lambda\mu\nu}^{(I)} = 0, \tag{8.1.2a}$$

$$\mathfrak{F}_{(I)}^\nu = \widehat{\mathscr{L}}_{\mathscr{L}}^I \mathfrak{F}^\nu + \left( N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathscr{L}}^I \left( N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{F}_{\kappa\lambda} \right) \right). \tag{8.1.2b}$$

*Furthermore, there exist constants $\widetilde{C}_{1;I_1,I_2}$, $\widetilde{C}_{2;I_1,I_2}$, $\widetilde{C}_{\mathscr{P};I_1,I_2}$, $\widetilde{C}_{\mathfrak{F}_\triangle;J}$, and $\widetilde{C}_{N_\triangle^\#;I_1,I_2}$ such that*

$$\widehat{\mathscr{L}}_{\mathscr{L}}^I \mathfrak{F}^\nu = \sum_{|I_1|+|I_2| \leq |I|} \widetilde{C}_{2;I_1,I_2} \mathscr{Q}_{(2;\mathscr{F})}^\nu (\nabla \mathscr{L}_{\mathscr{L}}^{I_1} h, \mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}) + \sum_{|J| \leq |I|} \widetilde{C}_{\mathfrak{F}_\triangle;J} \mathscr{L}_{\mathscr{L}}^J \mathfrak{F}_\triangle^\nu, \tag{8.1.3a}$$

$$N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathscr{L}}^I (N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{F}_{\kappa\lambda})$$

$$= \sum_{\substack{|I_1|+|I_2| \leq |I| \\ |I_2| \leq |I|-1}} \widetilde{C}_{\mathscr{P};I_1,I_2} \mathscr{P}_{(\mathscr{F})}^\nu (\mathscr{L}_{\mathscr{L}}^{I_1} h, \nabla \mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}) + \sum_{\substack{|I_1|+|I_2| \leq |I| \\ |I_2| \leq |I|-1}} \widetilde{C}_{1;I_1,I_2} \mathscr{Q}_{(1;\mathscr{F})}^\nu (\mathscr{L}_{\mathscr{L}}^{I_1} h, \nabla \mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F})$$

$$+ \sum_{\substack{|I_1|+|I_2| \leq |I| \\ |I_2| \leq |I|-1}} \widetilde{C}_{N_\triangle^\#;I_1,I_2} (\mathscr{L}_{\mathscr{L}}^{I_1} N_\triangle^{\#\mu\nu\kappa\lambda}) \nabla_\mu \mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}_{\kappa\lambda}. \tag{8.1.3b}$$

*In the above formulas, $\mathfrak{F}_\triangle^\nu$ and $N_\triangle^{\#\mu\nu\kappa\lambda}$ are the error terms appearing in (3.7.2g) and (3.7.2h), respectively, while $\mathscr{P}_{(\mathscr{F})}^\nu(\cdot,\cdot)$ and $\mathscr{Q}_{(i;\mathscr{F})}^\nu(\cdot,\cdot)$ ($i = 1, 2$ and $\nu = 0, 1, 2, 3$) are the quadratic forms defined in (3.7.3b), (3.7.3c), and (3.7.2e), respectively.*

*Proof.* To prove (8.1.2a), we first recall (3.7.1b), which states that $\mathscr{F}_{\mu\nu}$ is a solution to $\nabla_{[\kappa} \mathscr{F}_{\mu\nu]} = 0$, where $[\cdot]$ denotes antisymmetrization. From (6.5.5), it therefore follows that

$$0 = \mathscr{L}_{\mathscr{L}}^I \nabla_{[\lambda} \mathscr{F}_{\mu\nu]} = \nabla_{[\lambda} \mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{\mu\nu]}, \tag{8.1.4}$$

which is the desired result.

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[24]More precisely, the equations of variation are linear in $\dot{\mathscr{F}}$.

To derive (8.1.2b), we conclude that $\widehat{\mathcal{L}}_{\mathcal{Z}}^{I}(N^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathscr{F}_{\kappa\lambda}) = \widehat{\mathcal{L}}_{\mathcal{Z}}^{I}\mathfrak{F}^{\nu}$ by simply differentiating each side of (8.1.1b) with $\widehat{\mathcal{L}}_{\mathcal{Z}}^{I}$. Trivial algebraic manipulation then leads to the fact that $N^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathcal{L}_{\mathcal{Z}}^{I}\mathscr{F}_{\kappa\lambda} = \mathfrak{F}_{(I)}^{\nu}$, where $\mathfrak{F}_{(I)}^{\nu}$ is defined by (8.1.2b).

Equation (8.1.3a) follows from (3.7.2b), Definition 6.5 of $\widehat{\mathcal{L}}_{Z}$, and Lemma 11.8, which is proved in Section 11.2.

To prove (8.1.3b), we first recall the decomposition (3.7.3a):

$$N^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathscr{F}_{\kappa\lambda} = \tfrac{1}{2}\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_{\mu}\mathscr{F}_{\kappa\lambda}$$
$$- \mathscr{P}_{(\mathscr{F})}^{\nu}(h, \nabla\mathscr{F}) - \mathcal{Q}_{(1;\mathscr{F})}^{\nu}(h, \nabla\mathscr{F}) + N_{\triangle}^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathscr{F}_{\kappa\lambda}. \quad (8.1.5)$$

The commutator term arising from the first term on the right-hand side of (8.1.5) vanishes. More specifically, we use (6.5.16) to conclude that

$$\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_{\mu}\mathcal{L}_{\mathcal{Z}}^{I}\mathscr{F}_{\kappa\lambda}$$
$$- \widehat{\mathcal{L}}_{\mathcal{Z}}^{I}\big(\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_{\mu}\mathscr{F}_{\kappa\lambda}\big) = 0. \quad (8.1.6)$$

Therefore, it follows from (8.1.5) and (8.1.6) that

$$N^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathcal{L}_{\mathcal{Z}}^{I}\mathscr{F}_{\kappa\lambda} - \widehat{\mathcal{L}}_{\mathcal{Z}}^{I}(N^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathscr{F}_{\kappa\lambda}) = \widehat{\mathcal{L}}_{\mathcal{Z}}^{I}\mathscr{P}_{(\mathscr{F})}^{\nu}(h, \nabla\mathscr{F}) - \mathscr{P}_{(\mathscr{F})}^{\nu}(h, \nabla\mathcal{L}_{\mathcal{Z}}^{I}\mathscr{F})$$
$$+ \widehat{\mathcal{L}}_{\mathcal{Z}}^{I}\mathcal{Q}_{(1;\mathscr{F})}^{\nu}(h, \nabla\mathscr{F}) - \mathcal{Q}_{(1;\mathscr{F})}^{\nu}(h, \nabla\mathcal{L}_{\mathcal{Z}}^{I}\mathscr{F})$$
$$+ N_{\triangle}^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathcal{L}_{\mathcal{Z}}^{I}\mathscr{F}_{\kappa\lambda} - \widehat{\mathcal{L}}_{\mathcal{Z}}^{I}(N_{\triangle}^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathscr{F}_{\kappa\lambda}). \quad (8.1.7)$$

The expression (8.1.3b) now follows from (8.1.7), the Leibniz rule, Definition 6.5 of $\widehat{\mathcal{L}}_{Z}$, Lemma 6.8, and Lemma 11.8 below.     $\square$

## 8.2. *The canonical stress.*

The notion of the *canonical stress tensor field* $\dot{Q}^{\mu}{}_{\nu}$ in the context of PDE energy estimates was introduced by Christodoulou [2000]. As explained in Section 1.2.6, from the point of view of energy estimates, it plays the role of an energy-momentum-type tensor for the equations of variation. Its two key properties are (i) its divergence is lower-order (in the sense of the number of derivatives falling on the variations $\dot{\mathscr{F}}_{\mu\nu}$) and (ii) contraction against certain pairs $(\xi, X)$ consisting of a one-form $\xi_{\mu}$ and a vector field $X^{\nu}$ leads to an energy density that can be used derive $L^{2}$ control of solutions $\dot{\mathscr{F}}_{\mu\nu}$ to the equations of variation. As we will see, property (i) is captured by Lemma 8.5 and (8.3.3) while property (ii) is captured by (8.3.2), (12.2.1), and (12.2.8). In order to explain the origin of the canonical stress, we first define the *linearized Lagrangian*; our definition is modeled after the definition given by Christodoulou [2000].

**Definition 8.2.** Given an electromagnetic Lagrangian $\mathcal{L}[\,\cdot\,]$ (as described in Section 3.2) and a "background" $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$, we define the linearized Lagrangian by

$$\dot{\mathcal{L}} \overset{\text{def}}{=} -\tfrac{1}{4}N^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}, \quad (8.2.1)$$

where $N^{\#\zeta\eta\kappa\lambda}$ is the $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$-dependent tensor field defined in (3.3.8).

**Remark 8.3.** $\dot{\mathcal{L}}$ is equal to $\frac{1}{2}(\partial^2{}^{\star}\mathcal{L}[h,\mathscr{F}]/(\partial\mathscr{F}_{\zeta\eta}\,\partial\mathscr{F}_{\kappa\lambda}))\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}$ up to a correction term $\frac{1}{4}(\partial^{\star}\mathcal{L}/\partial\mathcal{F}_{(2)})\cdot\epsilon^{\#\mu\nu\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}$ corresponding to the term $\frac{1}{2}(\partial^{\star}\mathcal{L}/\partial\mathcal{F}_{(2)})\epsilon^{\#\mu\nu\kappa\lambda}$ from (3.3.10).

The merit of Definition 8.2 is the following: the principal part (from the point of view of number of derivatives) of the Euler–Lagrange equations (assuming that we view $(h,\mathscr{F})$ as a known background and $\dot{\mathscr{F}}$ to be the unknowns and that an appropriately defined action[25] is stationary with respect to closed variations of $\dot{\mathscr{F}}$) corresponding to $\dot{\mathcal{L}}[\dot{\mathscr{F}};h,\mathscr{F}]$ is identical to the principal part of the electromagnetic equations of variation (8.1.1b); i.e., $\dot{\mathcal{L}}[\dot{\mathscr{F}};h,\mathscr{F}]$ generates the principal part of the linearized equations.

**Definition 8.4.** Given a linearized Lagrangian $\dot{\mathcal{L}}[\dot{\mathscr{F}};h,\mathscr{F}]$, the canonical stress tensor field $\dot{Q}^{\mu}{}_{\nu}$ is defined as follows:

$$\dot{Q}^{\mu}{}_{\nu}=\dot{Q}^{\mu}{}_{\nu}[\dot{\mathscr{F}},\dot{\mathscr{F}}]\overset{\text{def}}{=}-2\frac{\partial\dot{\mathcal{L}}}{\partial\dot{\mathscr{F}}_{\mu\zeta}}\dot{\mathscr{F}}_{\nu\zeta}+\delta^{\mu}_{\nu}\dot{\mathcal{L}}=N^{\#\mu\zeta\kappa\lambda}\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{\nu\zeta}-\frac{1}{4}\delta^{\mu}_{\nu}N^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}, \tag{8.2.2}$$

where $N^{\#\mu\nu\kappa\lambda}$ is defined in (3.3.8).

Note that, in contrast to the energy-momentum tensor $T_{\mu\nu}$, $\dot{Q}_{\mu\nu}\overset{\text{def}}{=}m_{\mu\kappa}\dot{Q}^{\kappa}{}_{\nu}$ is in general not symmetric. We use the notation $\dot{Q}^{\mu}{}_{\nu}[\dot{\mathscr{F}},\dot{\mathscr{F}}]$ whenever we want to emphasize the quadratic dependence of $\dot{Q}^{\mu}{}_{\nu}$ on $\dot{\mathscr{F}}$.

Because of our assumption (3.3.3a) concerning the Lagrangian, $\dot{Q}^{\mu}{}_{\nu}$ is equal to the energy-momentum tensor (in $\dot{\mathscr{F}}$) for the standard Maxwell–Maxwell equations in Minkowski spacetime plus small corrections. More precisely, we insert the decomposition (3.7.2c) of $N^{\#\mu\zeta\kappa\lambda}$ into the right-hand side of (8.2.2) and perform simple computations, thereby arriving at the following decomposition of $\dot{Q}^{\mu}{}_{\nu}$:

$$\dot{Q}^{\mu}{}_{\nu}[\dot{\mathscr{F}},\dot{\mathscr{F}}]=\overbrace{\dot{\mathscr{F}}^{\mu\zeta}\dot{\mathscr{F}}_{\nu\zeta}-\frac{1}{4}\delta^{\mu}_{\nu}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}^{\zeta\eta}}^{\substack{\text{terms from linear Maxwell–Maxwell}\\\text{equations in Minkowski spacetime}}}\overbrace{-h^{\mu\kappa}\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_{\nu}{}^{\zeta}-h^{\kappa\lambda}\dot{\mathscr{F}}^{\mu}{}_{\kappa}\dot{\mathscr{F}}_{\nu\lambda}+\frac{1}{2}\delta^{\mu}_{\nu}h^{\kappa\lambda}\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_{\lambda}{}^{\eta}}^{\substack{\text{corrections to Minkowskian linear}\\\text{Maxwell–Maxwell equations arising from }h}}$$
$$+\underbrace{N^{\#\mu\zeta\kappa\lambda}_{\triangle}\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{\nu\zeta}-\frac{1}{4}\delta^{\mu}_{\nu}N^{\#\zeta\eta\kappa\lambda}_{\triangle}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}}_{\text{error terms}}. \tag{8.2.3}$$

The next lemma captures the lower-order divergence property enjoyed by $\dot{Q}^{\mu}{}_{\nu}$.

**Lemma 8.5** (Divergence of the canonical stress). *Let $\dot{\mathscr{F}}_{\mu\nu}$ be a solution to the equations of variation* (8.1.1a)–(8.1.1b) *corresponding to the background* $(h_{\mu\nu},\mathscr{F}_{\mu\nu})$, *and let $\dot{\mathfrak{F}}_{\lambda\mu\nu}$ and $\dot{\mathfrak{F}}^{\nu}$ be the inhomogeneous terms from the right-hand sides of* (8.1.1a)–(8.1.1b). *Let $\dot{Q}^{\mu}{}_{\nu}[\dot{\mathscr{F}},\dot{\mathscr{F}}]$ be the canonical stress tensor field defined in* (8.2.2). *Then*

$$\nabla_{\mu}\big(\dot{Q}^{\mu}{}_{\nu}[\dot{\mathscr{F}},\dot{\mathscr{F}}]\big)=-\frac{1}{2}N^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathfrak{F}}_{\nu\kappa\lambda}+\dot{\mathscr{F}}_{\nu\eta}\dot{\mathfrak{F}}^{\eta}+(\nabla_{\mu}N^{\#\mu\zeta\kappa\lambda})\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{\nu\zeta}-\frac{1}{4}(\nabla_{\nu}N^{\#\zeta\eta\kappa\lambda})\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}$$
$$=-\frac{1}{2}N^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathfrak{F}}_{\nu\kappa\lambda}+\dot{\mathscr{F}}_{\nu\eta}\dot{\mathfrak{F}}^{\eta}-(\nabla_{\mu}h^{\mu\kappa})\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_{\nu}{}^{\zeta}-(\nabla_{\mu}h^{\kappa\lambda})\dot{\mathscr{F}}^{\mu}{}_{\kappa}\dot{\mathscr{F}}_{\nu\lambda}$$
$$+\frac{1}{2}(\nabla_{\nu}h^{\kappa\lambda})\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_{\lambda}{}^{\eta}+(\nabla_{\mu}N^{\#\mu\zeta\kappa\lambda}_{\triangle})\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{\nu\zeta}-\frac{1}{4}(\nabla_{\nu}N^{\#\zeta\eta\kappa\lambda}_{\triangle})\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}. \tag{8.2.4}$$

*Proof.* To obtain (8.2.4), we use (8.1.1a)–(8.1.1b), the expansion (3.7.2c), and the properties (3.3.9a)–(3.3.9c) (which are also satisfied by the tensor field $N^{\#\mu\zeta\kappa\lambda}_{\triangle}$). $\qquad\square$

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[25]A suitable action $\mathscr{A}_{\mathfrak{C}}[\dot{\mathscr{F}}]$ is, e.g., of the form $\mathscr{A}_{\mathfrak{C}}[\dot{\mathscr{F}}]\overset{\text{def}}{=}\int_{\mathfrak{C}\in\mathfrak{M}}\dot{\mathcal{L}}[\dot{\mathscr{F}};h,\mathscr{F}]\,d^4x$, where $\mathfrak{C}$ is a compact subset of spacetime.

**8.3.** *Electromagnetic energy currents.* In this section, we introduce the energy currents that will be used to derive the weighted energy estimate (12.2.1) for a solution $\dot{\mathscr{F}}$ to the equations of variation (8.1.1a)–(8.1.1b).

**Definition 8.6.** Let $h_{\mu\nu}$ be a symmetric type-$\binom{0}{2}$ tensor field, and let $\mathscr{F}_{\mu\nu}$ and $\dot{\mathscr{F}}_{\mu\nu}$ be a pair of two-forms. Let $w(q)$ be the weight defined in (12.1.1), and let $X^\nu \overset{\text{def}}{=} w(q)\delta_0^\nu$ be the "multiplier" vector field. We define the *energy current* $\dot{J}^\mu_{(h,\mathscr{F})}[\dot{\mathscr{F}}]$ corresponding to the variation $\dot{\mathscr{F}}_{\mu\nu}$ and the background $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$ to be the vector field

$$\dot{J}^\mu_{(h,\mathscr{F})}[\dot{\mathscr{F}}] \overset{\text{def}}{=} -\dot{Q}^\mu{}_\nu[\dot{\mathscr{F}}, \dot{\mathscr{F}}]X^\nu = -w(q)\dot{Q}^\mu{}_0[\dot{\mathscr{F}}, \dot{\mathscr{F}}], \tag{8.3.1}$$

where $\dot{Q}^\mu{}_\nu[\dot{\mathscr{F}}, \dot{\mathscr{F}}]$ is the canonical stress tensor field from (8.2.2).

**Lemma 8.7** (Positivity of $\dot{J}^0_{(h,\mathscr{F})}$). *Let $\dot{J}^\mu_{(h,\mathscr{F})}[\dot{\mathscr{F}}]$ be the energy current defined in (8.3.1). Then*

$$\dot{J}^0_{(h,\mathscr{F})} = \tfrac{1}{2}|\dot{\mathscr{F}}|^2 w(q) + \left(O^\infty(|h|; \mathscr{F}) + O^\ell(|(h, \mathscr{F})|^2)\right)|\dot{\mathscr{F}}|^2 w(q). \tag{8.3.2}$$

*Furthermore, if $\dot{\mathscr{F}}_{\mu\nu}$ is a solution to the equations of variation (8.1.1a)–(8.1.1b) with inhomogeneous terms $\dot{\mathfrak{J}}_{\lambda\mu\nu} \equiv 0$, then the Minkowskian divergence of $\dot{J}_{(h,\mathscr{F})}$ can be expressed as follows:*

$$\nabla_\mu \dot{J}^\mu_{(h,\mathscr{F})} = -\tfrac{1}{2}w'(q)(|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2) - w(q)\dot{\mathscr{F}}_{0\eta}\dot{\mathfrak{s}}^\eta$$
$$- w(q)\left(-(\nabla_\mu h^{\mu\kappa})\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_0{}^\zeta - (\nabla_\mu h^{\kappa\lambda})\dot{\mathscr{F}}^\mu{}_\kappa\dot{\mathscr{F}}_{0\lambda} + \tfrac{1}{2}(\nabla_t h^{\kappa\lambda})\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_\lambda{}^\eta\right)$$
$$- w'(q)\left(-L_\mu h^{\mu\kappa}\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_0{}^\zeta - L_\mu h^{\kappa\lambda}\dot{\mathscr{F}}^\mu{}_\kappa\dot{\mathscr{F}}_{0\lambda} - \tfrac{1}{2}h^{\kappa\lambda}\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_\lambda{}^\eta\right)$$
$$- w(q)\left((\nabla_\mu N_\triangle^{\#\mu\zeta\kappa\lambda})\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} - \tfrac{1}{4}(\nabla_t N_\triangle^{\#\zeta\eta\kappa\lambda})\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\right)$$
$$- w'(q)\left(L_\mu N_\triangle^{\#\mu\zeta\kappa\lambda}\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} + \tfrac{1}{4}N_\triangle^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\right), \tag{8.3.3}$$

*where $\dot{\alpha} \overset{\text{def}}{=} \alpha[\dot{\mathscr{F}}]$, $\dot{\rho} \overset{\text{def}}{=} \rho[\dot{\mathscr{F}}]$, and $\dot{\sigma} \overset{\text{def}}{=} \sigma[\dot{\mathscr{F}}]$ are the "favorable" Minkowskian null components of $\dot{\mathscr{F}}$ defined in Section 5.3.*

**Remark 8.8.** The term $\tfrac{1}{2}w'(q)(|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2)$ appearing on the right-hand side of (8.3.3) is of central importance for closing the bootstrap argument during our global existence proof. It manifests itself as the additional positive spacetime integral $\int_0^t \int_{\Sigma_\tau} (|\dot{\mathscr{F}}|^2_{\mathscr{L}\mathscr{N}} + |\dot{\mathscr{F}}|^2_{\mathscr{J}\mathscr{J}})w'(q)\, d^3x\, d\tau$ on the left-hand side of (12.2.1) below and provides a means for controlling some of the spacetime integrals that emerge in Section 16.4.

*Proof.* Equation (8.3.2) follows from (8.2.3), simple calculations, and (3.7.2h).

To prove (8.3.3), we first recall that since $q = r - t$ it follows that $\nabla_\mu q = L_\mu$, where $L$ is defined in (5.1.2b). Hence, we have that $\nabla_\mu w(q) = w'(q)L_\mu$. Using this fact, (8.2.3), and (8.2.4), we calculate that

$$\nabla_\mu \dot{J}^\mu_{(h,\mathscr{F})} = -w(q)\dot{\mathscr{F}}_{0\eta}\dot{\mathfrak{F}}^\eta - w(q)\Big(-(\nabla_\mu h^{\mu\kappa})\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_0{}^\zeta - (\nabla_\mu h^{\kappa\lambda})\dot{\mathscr{F}}^\mu{}_\kappa\dot{\mathscr{F}}_{0\lambda} + \tfrac{1}{2}(\nabla_t h^{\kappa\lambda})\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_\lambda{}^\eta\Big)$$

$$- w(q)\Big((\nabla_\mu N_\triangle^{\#\mu\zeta\kappa\lambda})\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} - \tfrac{1}{4}(\nabla_t N_\triangle^{\#\zeta\eta\kappa\lambda})\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\Big)$$

$$- w'(q)\underbrace{\Big(L_\mu\dot{\mathscr{F}}^{\mu\zeta}\dot{\mathscr{F}}_{0\zeta} + \tfrac{1}{4}\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}^{\kappa\lambda}\Big)}_{(|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2)/2}$$

$$- w'(q)\Big(-L_\mu h^{\mu\kappa}\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_0{}^\zeta - L_\mu h^{\kappa\lambda}\dot{\mathscr{F}}^\mu{}_\kappa\dot{\mathscr{F}}_{0\lambda} - \tfrac{1}{2}h^{\kappa\lambda}\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_\lambda{}^\eta\Big)$$

$$- w'(q)\Big(L_\mu N_\triangle^{\#\mu\zeta\kappa\lambda}\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} + \tfrac{1}{4}N_\triangle^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\Big). \qquad (8.3.4)$$

The expression (8.3.3) thus follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

## 9. Decompositions of the electromagnetic equations

In this section we perform two decompositions of the electromagnetic equations. The first is a null decomposition of the equations of variation, which will be used in Section 15 to derive pointwise decay estimates for the lower-order Lie derivatives of $\mathscr{F}_{\mu\nu}$. The second is a decomposition of the electromagnetic equations into constraint and evolution equations for the Minkowskian one-forms $E_\mu$ and $B_\mu$, which are respectively known as the electric field and magnetic induction. This decomposition will be used in Section 10 to prove that our smallness condition on the abstract data necessarily implies a smallness condition on the initial energy $\mathscr{E}_{\ell;\gamma;\mu}(0)$ of the corresponding solution to the reduced equations. We remark that the Minkowskian one-forms $D_\mu$ and $H_\mu$, which are respectively known as the electric displacement and the magnetic field, and also the geometric electromagnetic one-forms $\mathfrak{E}_\mu$, $\mathfrak{B}_\mu$, $\mathfrak{D}_\mu$, and $\mathfrak{H}_\mu$ will play a role in the discussion.

**9.1.** *The Minkowskian null decomposition of the electromagnetic equations of variation.* In this section, we decompose the equations of variation into equations for the null components of $\dot{\mathscr{F}}$. The main advantage of our decomposition, which is given in Proposition 9.3, is that the terms in each equation can be separated into two classes: (i) a derivative of a null component in a "nearly Minkowski-null" direction[26] and (ii) the error terms. Although from the point of view of differentiability some of the error terms are higher-order, it will turn out that all error terms are lower-order in terms of decay rates. In this way, the equations can be viewed as *ordinary differential inequalities* with inhomogeneous terms (which we loosely refer to as ODEs) for the null components of $\dot{\mathscr{F}}$. This point of view is realized in Proposition 11.5. The key point is that the ODEs we derive are amenable to Gronwall estimates. In Section 15, we will use this line of argument to derive pointwise decay estimates for the null components of the lower-order Lie derivatives of a solution $\mathscr{F}$ to the electromagnetic equations (3.7.1b)–(3.7.1c). These estimates will be an improvement over what can be deduced from the weighted Klainerman–Sobolev inequality (B.4) alone; see the beginning of Section 15 for additional details regarding this improvement.

We begin the analysis by using (3.7.2c) to write the equations of variation (8.1.1a)–(8.1.1b) in the following form:

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[26]By "nearly Minkowski-null", we mean vectors that are nearly parallel to $\underline{L}$ or $L$ with some corrections coming from the presence of a nonzero $h$ in the case of the vector field $L$.

$$\nabla_\lambda \dot{\mathscr{F}}_{\mu\nu} + \nabla_\mu \dot{\mathscr{F}}_{\nu\lambda} + \nabla_\nu \dot{\mathscr{F}}_{\lambda\mu} = 0, \tag{9.1.1a}$$

$$\frac{1}{2}\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu \dot{\mathscr{F}}_{\kappa\lambda}$$

$$+ \frac{1}{2}\big(-h^{\mu\kappa}(m^{-1})^{\nu\lambda} + h^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu \dot{\mathscr{F}}_{\kappa\lambda}$$

$$+ \frac{1}{2}\big(-(m^{-1})^{\mu\kappa}h^{\nu\lambda} + (m^{-1})^{\mu\lambda}h^{\nu\kappa}\big)\nabla_\mu \dot{\mathscr{F}}_{\kappa\lambda} + N_\triangle^{\#\mu\nu\kappa\lambda}\nabla_\mu \dot{\mathscr{F}}_{\kappa\lambda} = \dot{\mathfrak{F}}^\nu. \tag{9.1.1b}$$

In our calculations below, we will make use of the identities

$$\nabla_A \underline{L} = -r^{-1}e_A \quad \text{and} \quad \nabla_A L = r^{-1}e_A, \tag{9.1.2}$$

which can be directly calculated in our wave-coordinate system by using (5.1.2a)–(5.1.2b). We will also make use of the identity

$$\slashed{\nabla}_A e_B = \nabla_A e_B + \tfrac{1}{2}m(\nabla_A e_B, \underline{L})L + \tfrac{1}{2}m(\nabla_A e_B, L)\underline{L}$$

$$= \nabla_A e_B - \tfrac{1}{2}m(e_B, \nabla_A \underline{L})L - \tfrac{1}{2}m(e_B, \nabla_A L)\underline{L}$$

$$= \nabla_A e_B + \tfrac{1}{2}r^{-1}\delta_{AB}(L - \underline{L}), \tag{9.1.3}$$

which follows from (6.1.2) and (9.1.2).

Furthermore, if $U$ is a type-$\binom{0}{m}$ tensor field and $X_{(i)}$ $(1 \le i \le m)$ and $Y$ are vector fields, then by the Leibniz rule we have that

$$\nabla_Y\{U(X_{(1)}, \ldots, X_{(m)})\} = (\nabla_Y U)(X_{(1)}, \ldots, X_{(m)}) + U(\nabla_Y X_{(1)}, X_{(2)}, \ldots, X_{(m)})$$

$$+ \cdots + U(X_{(1)}, X_{(2)}, \ldots, \nabla_Y X_{(m)}). \tag{9.1.4}$$

Similarly, if $U$ is $m$-tangent to the spheres $S_{r,t}$, then

$$\slashed{\nabla}_{e_A}\{U(e_{B_{(1)}}, \ldots, e_{B_{(m)}})\} = (\slashed{\nabla}_A U)(e_{B_{(1)}}, \ldots, e_{B_{(m)}}) + U(\slashed{\nabla}_A e_{B_{(1)}}, e_{B_{(2)}}, \ldots, e_{B_{(m)}})$$

$$+ \cdots + U(e_{B_{(1)}}, e_{B_{(2)}}, \ldots, \slashed{\nabla}_A e_{B_{(m)}}). \tag{9.1.5}$$

Applying (9.1.4) and (9.1.5) to $\mathscr{F}$ and using (9.1.2), (9.1.3), and (5.3.5a)–(5.3.5d), we compute (as in [Christodoulou and Klainerman 1990, p. 161]) the following identities, which we state as a lemma:

**Lemma 9.1** (Contracted derivatives expressed in terms of the null components [Christodoulou and Klainerman 1990, p. 161]). *Let $\mathscr{F}$ be a two-form, and let $\underline{\alpha}$, $\alpha$, $\rho$, and $\sigma$ be its Minkowskian null components. Then the following identities hold*:

$$\nabla_A \mathscr{F}_{B\underline{L}} = \slashed{\nabla}_A \underline{\alpha}_B - r^{-1}(\rho \delta_{AB} + \sigma \slashed{\psi}_{AB}), \tag{9.1.6a}$$

$$\nabla_A \mathscr{F}_{BL} = \slashed{\nabla}_A \alpha_B - r^{-1}(\rho \delta_{AB} - \sigma \slashed{\psi}_{AB}), \tag{9.1.6b}$$

$$\nabla_A {}^{\circledast}\mathscr{F}_{B\underline{L}} = -\slashed{\psi}_{CB}\slashed{\nabla}_A \underline{\alpha}_C - r^{-1}(\sigma \delta_{AB} - \rho \slashed{\psi}_{AB}), \tag{9.1.6c}$$

$$\nabla_A {}^{\circledast}\mathscr{F}_{BL} = \slashed{\psi}_{CB}\slashed{\nabla}_A \alpha_C - r^{-1}(\sigma \delta_{AB} + \rho \slashed{\psi}_{AB}), \tag{9.1.6d}$$

$$\tfrac{1}{2}\nabla_A \mathscr{F}_{\underline{L}L} = \slashed{\nabla}_A \rho + \tfrac{1}{2}r^{-1}(\underline{\alpha}_A + \alpha_A), \tag{9.1.6e}$$

$$\tfrac{1}{2}\nabla_A {}^{\circledast}\mathscr{F}_{\underline{L}L} = \slashed{\nabla}_A \sigma + \tfrac{1}{2}r^{-1}(-\slashed{\psi}_{BA}\underline{\alpha}_B + \slashed{\psi}_{BA}\alpha_B), \tag{9.1.6f}$$

$$\nabla_A \mathscr{F}_{BC} = \slashed{\psi}_{BC}\big(\slashed{\nabla}_A \sigma + \tfrac{1}{2}r^{-1}(-\slashed{\psi}_{DA}\underline{\alpha}_D + \slashed{\psi}_{DA}\alpha_D)\big). \tag{9.1.6g}$$

*In all of our expressions, contractions are taken after differentiating; e.g.,* $\nabla_A \mathscr{F}_{BL} \stackrel{\text{def}}{=} e_A^\mu e_B^\kappa \underline{L}^\lambda \nabla_\mu \mathscr{F}_{\kappa\lambda}$.

**Remark 9.2.** The identities in Lemma 9.1 can be reinterpreted as identities for spacetime tensors that are $m$-tangent to the spheres $S_{r,t}$. That is, they can be rephrased in terms of our wave-coordinate frame with the help of the projection $\slashed{h}_\mu{}^\nu$ and the spherical volume form $\slashed{\psi}_\mu{}^\nu$ defined in (5.1.4b) and (5.1.6c), respectively. For example, (9.1.6a) is equivalent to the following equation:

$$\slashed{h}_\mu{}^{\mu'}\slashed{h}_\nu{}^{\nu'}\underline{L}^\kappa\nabla_\kappa\mathscr{F}_{\nu'\kappa} = \slashed{h}_\mu{}^{\mu'}\slashed{h}_\nu{}^{\nu'}\nabla_{\mu'}\underline{\alpha}_{\nu'} - r^{-1}(\rho\,\slashed{h}_{\mu\nu} + \sigma\,\slashed{\psi}_{\mu\nu}). \tag{9.1.7}$$

We will use the spacetime-coordinate-frame version of the identities in our proof of Proposition 9.3.

We now derive equations for the null components of a solution $\dot{\mathscr{F}}$ to (9.1.1a)–(9.1.1b).

**Proposition 9.3** (Minkowskian null decomposition of the equations of variation). *Let $\dot{\mathscr{F}}$ be a solution to the equations of variation (9.1.1a)–(9.1.1b), and let $\underline{\dot{\alpha}} \overset{\text{def}}{=} \underline{\alpha}[\dot{\mathscr{F}}]$, $\dot{\alpha} \overset{\text{def}}{=} \alpha[\dot{\mathscr{F}}]$, $\dot{\rho} \overset{\text{def}}{=} \rho[\dot{\mathscr{F}}]$, and $\dot{\sigma} \overset{\text{def}}{=} \sigma[\dot{\mathscr{F}}]$ denote its Minkowskian null components. Assume that the source term $\tilde{\mathfrak{F}}_{\lambda\mu\nu}$ on the right-hand side of (9.1.1a) vanishes.[27] Then the following equations are satisfied by the null components:*

$$\nabla_L\underline{\dot{\alpha}}_\nu + r^{-1}\underline{\dot{\alpha}}_\nu + \slashed{h}_\nu{}^\kappa\nabla_\kappa\dot{\rho} - \slashed{\psi}_\nu{}^\kappa\nabla_\kappa\dot{\sigma} - \overbrace{\slashed{h}_\nu{}^\lambda h^{\mu\kappa}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{\slashed{h}_{\nu\lambda}\mathscr{P}^\lambda_{(\mathscr{F})}(h,\nabla\dot{\mathscr{F}})} - \overbrace{\slashed{h}_{\nu\nu'}(m^{-1})^{\mu\kappa}h^{\nu'\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{\slashed{h}_{\nu\lambda}\mathcal{Q}^\lambda_{(1;\mathscr{F})}(h,\nabla\dot{\mathscr{F}})}$$
$$+\slashed{h}_{\nu\nu'}N_\triangle^{\#\mu\nu'\kappa\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda} = \slashed{h}_{\nu\nu'}\dot{\mathfrak{F}}^{\nu'}, \quad (9.1.8\text{a})$$

$$\nabla_{\underline{L}}\dot{\alpha}_\nu - r^{-1}\dot{\alpha}_\nu - \slashed{h}_\nu{}^\kappa\nabla_\kappa\dot{\rho} - \slashed{\psi}_\nu{}^\kappa\nabla_\kappa\dot{\sigma} - \overbrace{\slashed{h}_\nu{}^\lambda h^{\mu\kappa}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{\slashed{h}_{\nu\lambda}\mathscr{P}^\lambda_{(\mathscr{F})}(h,\nabla\dot{\mathscr{F}})} - \overbrace{\slashed{h}_{\nu\nu'}(m^{-1})^{\mu\kappa}h^{\nu'\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{\slashed{h}_{\nu\lambda}\mathcal{Q}^\lambda_{(1;\mathscr{F})}(h,\nabla\dot{\mathscr{F}})}$$
$$+\slashed{h}_{\nu\nu'}N_\triangle^{\#\mu\nu'\kappa\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda} = \slashed{h}_{\nu\nu'}\dot{\mathfrak{F}}^{\nu'}, \quad (9.1.8\text{b})$$

$$\nabla_{\underline{L}}\dot{\rho} - 2r^{-1}\dot{\rho} + \slashed{h}^{\mu\nu}\nabla_\mu\underline{\dot{\alpha}}_\nu - \overbrace{\underline{L}^\lambda h^{\mu\kappa}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{\underline{L}_\lambda\mathscr{P}^\lambda_{(\mathscr{F})}(h,\nabla\dot{\mathscr{F}})} - \overbrace{\underline{L}_\nu(m^{-1})^{\mu\kappa}h^{\nu\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{\underline{L}_\nu\mathcal{Q}^\nu_{(1;\mathscr{F})}(h,\nabla\dot{\mathscr{F}})}$$
$$+\underline{L}_\nu N_\triangle^{\#\mu\nu\kappa\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda} = \underline{L}_\lambda\dot{\mathfrak{F}}^\lambda, \quad (9.1.8\text{c})$$

$$\nabla_{\underline{L}}\dot{\sigma} - 2r^{-1}\dot{\sigma} + \slashed{\psi}^{\mu\nu}\nabla_\mu\underline{\dot{\alpha}}_\nu = 0, \quad (9.1.8\text{d})$$

$$\nabla_L\dot{\rho} + 2r^{-1}\dot{\rho} - \slashed{h}^{\mu\nu}\nabla_\mu\dot{\alpha}_\nu + \overbrace{L^\lambda h^{\mu\kappa}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{L_\lambda\mathscr{P}^\lambda_{(\mathscr{F})}(h,\nabla\dot{\mathscr{F}})} + \overbrace{L_\nu(m^{-1})^{\mu\kappa}h^{\nu\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{L_\nu\mathcal{Q}^\nu_{(1;\mathscr{F})}(h,\nabla\dot{\mathscr{F}})}$$
$$-L_\nu N_\triangle^{\#\mu\nu\kappa\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda} = -L_\lambda\dot{\mathfrak{F}}^\lambda, \quad (9.1.8\text{e})$$

$$\nabla_L\dot{\sigma} + 2r^{-1}\dot{\sigma} + \slashed{\psi}^{\mu\nu}\nabla_\mu\underline{\dot{\alpha}}_\nu = 0. \quad (9.1.8\text{f})$$

*In the above expressions, the quadratic terms $\mathscr{P}^\lambda_{(\mathscr{F})}(h, \nabla\dot{\mathscr{F}})$ and $\mathcal{Q}^\lambda_{(1;\mathscr{F})}(h, \nabla\dot{\mathscr{F}})$ are as defined in Section 3.7.*

**Remark 9.4.** Note that in the above equations, we have that, e.g., $\slashed{h}_\nu{}^\kappa\nabla_\kappa = \slashed{h}_\nu{}^\kappa\slashed{\nabla}_\kappa$ and $\slashed{\psi}_\nu{}^\kappa\nabla_\kappa = \slashed{\psi}_\nu{}^\kappa\slashed{\nabla}_\kappa$ so that these operators only involve favorable angular derivatives.

*Proof.* To obtain (9.1.8a) and (9.1.8b), we contract (9.1.1a) against $\underline{L}^\lambda L^\mu e_A^\nu$ and (9.1.1b) against $(e_A)_\nu$ and use Lemma 9.1 plus Remark 9.2 to deduce that

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[27] By Proposition 8.1, this assumption holds for the variations $\dot{\mathscr{F}}$ of interest in this article.

$$\nabla_L \underline{\alpha}_\nu - \nabla_{\underline{L}} \alpha_\nu + 2 \slashed{h}_\nu^{\,\nu'} \nabla_{\nu'} \rho + r^{-1} (\underline{\alpha}_\nu + \alpha_\nu) = 0, \qquad (9.1.9)$$

$$\nabla_L \underline{\alpha}_\nu + \nabla_{\underline{L}} \alpha_\nu - 2 \slashed{\psi}_\nu^{\,\kappa} \nabla_\kappa \sigma + r^{-1} (\underline{\alpha}_\nu - \alpha_\nu)$$
$$- 2 \slashed{h}_\nu^{\,\lambda} h^{\mu\kappa} \nabla_\mu \dot{\mathscr{F}}_{\kappa\lambda} - 2 \slashed{h}_{\nu\nu'} (m^{-1})^{\mu\kappa} h^{\nu'\lambda} \nabla_\mu \dot{\mathscr{F}}_{\kappa\lambda} + \slashed{h}_{\nu\nu'} N_\triangle^{\#\mu\nu'\kappa\lambda} \nabla_\mu \dot{\mathscr{F}}_{\kappa\lambda} = 2 \slashed{h}_{\nu\nu'} \dot{\tilde{\mathfrak{F}}}^{\nu'}. \quad (9.1.10)$$

Adding the two above equations gives (9.1.8a) while subtracting the first from the second gives (9.1.8b).

Similarly, to deduce (9.1.8d), we contract (9.1.1a) against $\underline{L}^\lambda e_A^{\,\mu} e_B^{\,\nu}$ and then contract against $\slashed{\psi}_{AB}$; to deduce (9.1.8f), we contract (9.1.1a) against $L^\lambda e_A^{\,\mu} e_B^{\,\nu}$ and then against $\slashed{\psi}_{AB}$; to deduce (9.1.8c), we contract (9.1.1b) against $\underline{L}_\nu$; and to deduce (9.1.8e), we contract (9.1.1b) against $-L_\nu$. $\qquad\square$

**9.2. *Electromagnetic one-forms.*** In this section, we introduce the one-forms $\mathfrak{E}$, $\mathfrak{B}$, $\mathfrak{D}$, and $\mathfrak{H}$, which are derived from a geometric decomposition of $\mathscr{F}$ that depends on the spacetime metric $g_{\mu\nu}$. We also introduce the one-forms $E$, $B$, $D$, and $H$, which are derived from a Minkowskian decomposition of $\mathscr{F}$. We then derive an equivalent version of the electromagnetic equations, namely constraint and electromagnetic evolution equations for the Minkowskian one-forms. These quantities play a role only in Section 10, where they are used to connect the smallness of the abstract initial data to the smallness of the energy of the corresponding reduced solution at $t = 0$. Furthermore, we show that the abstract one-forms $\mathring{\mathfrak{D}}$ and $\mathring{\mathfrak{B}}$ satisfy the constraints (1.0.3a)–(1.0.3b) if and only if the corresponding Minkowskian one-forms $\mathring{D}$ and $\mathring{B}$ satisfy a Minkowskian version of the constraints.

We will perform our electromagnetic decompositions of the equations with the help of two versions of the (nonreduced) electromagnetic equations, namely (3.2.6a) and (3.2.7a) and (3.2.6b) and (3.2.7b). We restate them here for convenience:

$$\mathscr{D}_\lambda \mathscr{F}_{\mu\nu} + \mathscr{D}_\mu \mathscr{F}_{\nu\lambda} + \mathscr{D}_\nu \mathscr{F}_{\lambda\mu} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \qquad (9.2.1a)$$

$$\mathscr{D}_\lambda \mathscr{M}_{\mu\nu} + \mathscr{D}_\mu \mathscr{M}_{\nu\lambda} + \mathscr{D}_\nu \mathscr{M}_{\lambda\mu} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \qquad (9.2.1b)$$

$$\nabla_\lambda \mathscr{F}_{\mu\nu} + \nabla_\mu \mathscr{F}_{\nu\lambda} + \nabla_\nu \mathscr{F}_{\lambda\mu} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \qquad (9.2.2a)$$

$$\nabla_\lambda \mathscr{M}_{\mu\nu} + \nabla_\mu \mathscr{M}_{\nu\lambda} + \nabla_\nu \mathscr{M}_{\lambda\mu} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3). \qquad (9.2.2b)$$

Before decomposing the equations, we first define the aforementioned geometric electromagnetic one-forms.

**Definition 9.5.** Let $\widehat{N}^\mu = \widehat{N}^\mu(t, x)$ denote the future-directed unit $g$-normal to the hypersurface $\Sigma_t$. Then in components relative to an arbitrary coordinate system, we define the following one-forms:

$$\mathfrak{E}_\mu = \mathscr{F}_{\mu\kappa} \widehat{N}^\kappa, \qquad \mathfrak{B}_\mu = -^\star \mathscr{F}_{\mu\kappa} \widehat{N}^\kappa, \qquad \mathfrak{D}_\mu = -^\star \mathscr{M}_{\mu\kappa} \widehat{N}^\kappa, \quad \text{and} \quad \mathfrak{H}_\mu = -\mathscr{M}_{\mu\kappa} \widehat{N}^\kappa. \qquad (9.2.3)$$

*Note that, in the above expressions, $\star$ denotes the Hodge duality operator corresponding to the spacetime metric $g$.*

We now define the Minkowskian electromagnetic one-forms.

**Definition 9.6.** In components relative to the wave-coordinate system $\{x^\mu\}_{\mu=0,1,2,3}$, we define the *electric field $E$*, the *magnetic induction $B$*, the *electric displacement $D$*, and the *magnetic field $H$* by

$$E_\mu = \mathscr{F}_{\mu 0}, \qquad B_\mu = -^\circledast \mathscr{F}_{\mu 0}, \qquad D_\mu = -^\circledast \mathscr{M}_{\mu 0}, \quad \text{and} \quad H_\mu = -\mathscr{M}_{\mu 0}. \qquad (9.2.4)$$

*Note that in the above expressions, $\circledast$ denotes the Hodge duality operator corresponding to the Minkowski metric $m$.*

Observe that (9.2.4) implies that

$$\mathscr{F}_{jk} = [ijk]B_i, \qquad B_j = \tfrac{1}{2}[jab]\mathscr{F}_{ab}, \quad \text{and} \quad D_j = \tfrac{1}{2}[jab]\mathcal{M}_{ab} \qquad (j, k = 1, 2, 3). \tag{9.2.5}$$

**Remark 9.7.** Our definition of $B$ coincides with the one commonly found in the physics literature, but it has the opposite sign convention of the definition given in [Christodoulou and Klainerman 1990].

It follows from the antisymmetry of $\mathscr{F}_{\mu\nu}$ and $\mathcal{M}_{\mu\nu}$ that $E_\mu$, $B_\mu$, $D_\mu$, and $H_\mu$ are $m$-tangent to the hyperplanes $\Sigma_t$; i.e., we have that $E_0 = B_0 = D_0 = H_0 = 0$. We may therefore view these four quantities as one-forms that are intrinsic to $\Sigma_t$. Similarly, we have that $\mathfrak{E}_\mu \widehat{N}^\mu = \mathfrak{B}_\mu \widehat{N}^\mu = \mathfrak{D}_\mu \widehat{N}^\mu = \mathfrak{H}_\mu \widehat{N}^\mu = 0$.

From the assumption (3.3.3a) on the electromagnetic Lagrangian, (3.3.11n), Definition 9.6, (9.2.5), and the implicit-function theorem, we deduce that, when all of the fields are sufficiently small, we have (see Section 2.13 for the definition of $O^{\ell+1}(\,\cdot\,)$):

$$D = E + O^{\ell+1}(|h||(E, B)|) + O^{\ell+1}(|(E, B)|^3; h), \tag{9.2.6a}$$

$$H = B + O^{\ell+1}(|h||(E, B)|) + O^{\ell+1}(|(E, B)|^3; h), \tag{9.2.6b}$$

$$E = D + O^{\ell+1}(|h||(D, B)|) + O^{\ell+1}(|(D, B)|^3; h), \tag{9.2.6c}$$

$$H = B + O^{\ell+1}(|h||(D, B)|) + O^{\ell+1}(|(D, B)|^3; h). \tag{9.2.6d}$$

We now assume that the reduced initial data $(g_{\mu\nu}|_{\Sigma_0}, \partial_t g_{\mu\nu}|_{\Sigma_0}, \mathscr{F}_{0j}|_{\Sigma_0} = \mathring{E}_j, \mathscr{F}_{jk}|_{\Sigma_0} = [ijk]\mathring{B}_i)$ have been constructed from the abstract initial data $(\mathring{\underline{g}}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ in the manner described in Section 4.2. In particular, we recall that $\widehat{N}^\nu|_{\Sigma_0} = A^{-1}\delta_0^\nu$, where $A \stackrel{\text{def}}{=} \sqrt{1 - (2M/r)\chi(r)}$. Consequently, we can use (3.3.11i) and (4.2.7a) to deduce that

$$\mathring{E} = \mathring{D} + O^{\ell+1}\big(|\mathring{h}^{(1)}||(\mathring{D}, \mathring{B})|; \chi(r)M/r\big)$$
$$+ O^{\ell+1}\big(|\chi(r)M/r||(\mathring{D}, \mathring{B})|; \mathring{h}^{(1)}\big) + O^{\ell+1}\big(|(\mathring{D}, \mathring{B})|^3; \chi(r)M/r; \mathring{h}^{(1)}\big). \tag{9.2.7}$$

Using also Definitions 9.5 and 9.6, we infer that the following relations hold:

$$\mathring{B} = \mathring{\mathfrak{B}} + O^{\ell+1}\big(|\chi(r)M/r||(\mathring{\mathfrak{D}}, \mathring{\mathfrak{B}})|; \mathring{h}^{(1)}\big) + O^{\ell+1}\big(|\mathring{h}^{(1)}||(\mathring{\mathfrak{D}}, \mathring{\mathfrak{B}})|; \chi(r)M/r\big), \tag{9.2.8a}$$

$$\mathring{D} = \mathring{\mathfrak{D}} + O^{\ell+1}\big(|\chi(r)M/r||(\mathring{\mathfrak{D}}, \mathring{\mathfrak{B}})|; \mathring{h}^{(1)}\big) + O^{\ell+1}\big(|\mathring{h}^{(1)}||(\mathring{\mathfrak{D}}, \mathring{\mathfrak{B}})|; \chi(r)M/r\big), \tag{9.2.8b}$$

$$\mathring{\mathfrak{B}} = \mathring{B} + O^{\ell+1}\big(|\chi(r)M/r||(\mathring{D}, \mathring{B})|; \mathring{h}^{(1)}\big) + O^{\ell+1}\big(|\mathring{h}^{(1)}||(\mathring{D}, \mathring{B})|; \chi(r)M/r\big), \tag{9.2.8c}$$

$$\mathring{\mathfrak{D}} = \mathring{D} + O^{\ell+1}\big(|\chi(r)M/r||(\mathring{D}, \mathring{B})|; \mathring{h}^{(1)}\big) + O^{\ell+1}\big(|\mathring{h}^{(1)}||(\mathring{D}, \mathring{B})|; \chi(r)M/r\big). \tag{9.2.8d}$$

**Remark 9.8.** Logically speaking, the ADM mass $M$ (and hence also the components of the unit normal vector $\widehat{N}|_{\Sigma_0}$) is only well-defined *after* one has solved the abstract Einstein constraint equations (1.0.2a)–(1.0.3b).

The main goal of this section is to deduce the following proposition, which is a decomposition of the electromagnetic equations into *constraint* equations and *evolution* equations:

**Proposition 9.9** (Electromagnetic constraint and evolution equations). *Under the assumption* (3.3.3a) *on* $^\star\mathscr{L}$, *the* (*nonreduced*) *electromagnetic equations* (9.2.2a)–(9.2.2b) *are equivalent to pairs of constraint equations and evolution equations that have the following structure* (*the precise details depend on the choice of electromagnetic Lagrangian* $^\star\mathscr{L}$):

Constraint equations

$$(\underline{m}^{-1})^{ab}\underline{\nabla}_a D_b = 0, \tag{9.2.9a}$$

$$(\underline{m}^{-1})^{ab}\underline{\nabla}_a B_b = 0, \tag{9.2.9b}$$

Evolution equations

$$\partial_t B_j = -[jab]\underline{\nabla}_a E_b, \tag{9.2.10a}$$

$$\partial_t E_j = [jab]\underline{\nabla}_a B_b + O^\ell\big(|h||\underline{\nabla}(E, B)|; (E, B)\big)$$
$$+ O^\ell\big(|(E, B)|^2|\underline{\nabla}(E, B)|; h\big) + O^\ell\big(|\nabla h||(E, B)|; h\big). \tag{9.2.10b}$$

*Furthermore, assume that the reduced initial data* $(g_{\mu\nu}|_{\Sigma_0}, \partial_t g_{\mu\nu}|_{\Sigma_0}, \mathscr{F}_{0j}|_{\Sigma_0} = \mathring{E}_j, \mathscr{F}_{jk}|_{\Sigma_0} = [ijk]\mathring{B}_i)$ *have been constructed from the abstract initial data* $(\mathring{\underline{g}}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ *in the manner described in Section 4.2. Then* (9.2.9a)–(9.2.9b) *hold for* $\mathring{D}$ *and* $\mathring{B}$ *along* $\Sigma_0$ *if and only if the following equations hold along* $\Sigma_0$:

Abstract constraint equations

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathscr{D}}}_a\mathring{\mathfrak{D}}_b = 0, \tag{9.2.11a}$$

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathscr{D}}}_a\mathring{\mathfrak{B}}_b = 0. \tag{9.2.11b}$$

*In the above expressions,* $\mathring{\underline{g}}_{jk}$ *is the first fundamental form of* $\Sigma_0$ *and* $\mathring{\underline{\mathscr{D}}}$ *is the Levi-Civita connection corresponding to* $\mathring{\underline{g}}_{jk}$.

**Remark 9.10.** In (9.2.9a)–(9.2.9b), $(\underline{m}^{-1})^{ab}\underline{\nabla}_a$ is the standard Euclidean divergence operator while in equations (9.2.10a)–(9.2.10b) $[jab]\underline{\nabla}_a$ is the standard Euclidean curl operator.

**Remark 9.11.** With the help of (9.2.16)–(9.2.17) below, it is straightforward to check that, if a classical solution to the evolution equations satisfies the constraints at $t = 0$, then it necessarily satisfies the constraints (9.2.9a)–(9.2.9b) at all later times (as long as it persists).

*Proof.* We first show that (9.2.9b) and (9.2.11b) follow from either (9.2.1a) or (9.2.2a) (which are equivalent) and that (9.2.9b) holds if and only if (9.2.11b) holds. To this end, we first note that, since $\widehat{N}^\mu$ is the future-directed unit $g$-normal to $\Sigma_t$ and $g_{\mu\nu} = \mathring{\underline{g}}_{\mu\nu} - \widehat{N}_\mu\widehat{N}_\nu$ along $\Sigma_0$, the following identities hold for any one-form $X_\mu$ $g$-tangent to $\Sigma_0$ and any two-form $P_{\mu\nu}$:

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathscr{D}}}_a X_b = (g^{-1})^{\kappa\lambda}\mathscr{D}_\kappa X_\lambda - X_\lambda\widehat{N}^\kappa\mathscr{D}_\kappa\widehat{N}^\lambda, \tag{9.2.12}$$

$$(g^{-1})^{\kappa\lambda}P_{\lambda\nu}\mathscr{D}_\kappa\widehat{N}^\nu = P_{\lambda\nu}\widehat{N}^\nu\widehat{N}^\kappa\mathscr{D}_\kappa\widehat{N}^\lambda. \tag{9.2.13}$$

Using (9.2.12) and (9.2.13) with $X_\mu \stackrel{\text{def}}{=} \mathfrak{B}_\mu$ and $P_{\mu\nu} \stackrel{\text{def}}{=} {}^\star \mathscr{F}_{\mu\nu}$, we compute that the following identities hold along $\Sigma_0$:

$$
\begin{aligned}
(\mathring{\underline{g}}^{-1})^{ab} \mathring{\underline{\mathscr{D}}}_a \mathfrak{B}_b &= (g^{-1})^{\kappa\lambda} \mathscr{D}_\kappa \mathfrak{B}_\lambda - \mathfrak{B}_\lambda \widehat{N}^\kappa \mathscr{D}_\kappa \widehat{N}^\lambda \\
&= -(g^{-1})^{\kappa\lambda} \mathscr{D}_\kappa ({}^\star \mathscr{F}_{\lambda\nu} \widehat{N}^\nu) + {}^\star \mathscr{F}_{\lambda\nu} \widehat{N}^\nu \widehat{N}^\kappa \mathscr{D}_\kappa \widehat{N}^\lambda \\
&= -\tfrac{1}{2} g_{\nu\nu'} \widehat{N}^{\nu'} \epsilon^{\#\mu\nu\kappa\lambda} \mathscr{D}_\mu \mathscr{F}_{\kappa\lambda}.
\end{aligned}
\tag{9.2.14}
$$

Identities analogous to (9.2.14) hold if we make the replacements $(\mathring{\underline{g}}^{-1}, g, \mathring{\underline{\mathscr{D}}}, \mathscr{D}, \star, \widehat{N}^\mu, \epsilon^{\#\mu\nu\kappa\lambda}, \mathfrak{B}) \to (\underline{m}^{-1}, m, \underline{\nabla}, \nabla, \circledast, \hat{n}^\mu, \upsilon^{\mu\nu\kappa\lambda}, B)$, where $\hat{n}^\mu(t, x)$ is the future-directed Minkowskian unit normal to $\Sigma_t$. Now by (9.2.14) and the Minkowskian analogy of (9.2.14), (9.2.9b) and (9.2.11b) follow from either (9.2.1a) or (9.2.2a) since either (9.2.1a) or (9.2.2a) is sufficient to guarantee that the right-hand side of (9.2.14) is 0. Furthermore, since $g_{\nu\nu'} \widehat{N}^{\nu'}$ and $m_{\nu\nu'} \hat{n}^{\nu'}$ are proportional along $\Sigma_0$, since $\epsilon^{\#\mu\nu\kappa\lambda}$ and $\upsilon^{\mu\nu\kappa\lambda}$ are proportional, and since the Christoffel symbols of $\mathscr{D}$ and $\nabla$ are symmetric in their two lower indices, it follows that

$$
g_{\nu\nu'} \widehat{N}^{\nu'} \epsilon^{\#\mu\nu\kappa\lambda} \mathscr{D}_\mu \mathscr{F}_{\kappa\lambda}|_{\Sigma_0} = 0 \iff m_{\nu\nu'} \hat{n}^{\nu'} \upsilon^{\mu\nu\kappa\lambda} \nabla_\mu \mathscr{F}_{\kappa\lambda}|_{\Sigma_0} = 0.
\tag{9.2.15}
$$

Hence, (9.2.9b) holds along $\Sigma_0$ if and only if (9.2.11b) holds along $\Sigma_0$. The derivation of (9.2.9a) and (9.2.11a) along $\Sigma_0$ from (9.2.1b) or (9.2.2b) and the proof of the equivalence of (9.2.9a) and (9.2.11a) along $\Sigma_0$ are similar.

We now set $\lambda = 0$, $\mu = a$, and $\nu = b$ in (9.2.2a), contract against the Euclidean volume form $[jab]$, and use (9.2.4)–(9.2.5) to deduce that

$$
\partial_t B_j = -[jab] \underline{\nabla}_a E_b.
\tag{9.2.16}
$$

Similarly, we set $\lambda = 0$, $\mu = a$, and $\nu = b$ in (9.2.2b), contract against $[jab]$, and use (9.2.4)–(9.2.5) to deduce that

$$
\partial_t D_j = [jab] \underline{\nabla}_a H_b.
\tag{9.2.17}
$$

Finally, we use (9.2.16), (9.2.17), and (9.2.6a)–(9.2.6b) to deduce (9.2.10a)–(9.2.10b).  $\square$

## 10. The smallness condition on the abstract data

In this section, we assume that we are given abstract initial data $(\mathring{\underline{g}}_{jk} = \delta_{jk} + \mathring{\underline{h}}^{(0)}_{jk} + \mathring{\underline{h}}^{(1)}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ $(j, k = 1, 2, 3)$ on the manifold $\mathbb{R}^3$ satisfying the constraint equations (4.1.1a)–(4.1.2b). Our goal is to describe in detail the smallness condition on $(\mathring{\underline{h}}^{(0)}_{jk}, \mathring{\underline{h}}^{(1)}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ that will lead to global existence for the reduced system (3.7.1a)–(3.7.1c) under the assumption that its initial data $(g_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0}, \mathscr{F}_{\mu\nu}|_{t=0})$ $(\mu, \nu = 0, 1, 2, 3)$ are constructed from the abstract initial data as described in Section 4.2. Recall that our global existence argument is heavily based on the analysis of $\mathscr{E}_{\ell;\gamma;\mu}(t)$, which is the energy defined in (1.2.7). In particular, $\mathscr{E}_{\ell;\gamma;\mu}(0)$ must be sufficiently small in order for us to close the argument. The energy depends on *both normal and tangential* Minkowskian covariant derivatives of the quantities $(\nabla_\lambda h^{(1)}_{\mu\nu}, \mathscr{F}_{\mu\nu})$ at $t = 0$. On the other hand, our smallness condition will be expressed in terms of the ADM mass $M$ and $E_{\ell;\gamma}(0)$, which is a weighted Sobolev norm of $(\underline{\nabla}_i \mathring{\underline{h}}^{(1)}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ depending only on *tangential* derivatives of the abstract data. More specifically, our smallness condition is expressed

in terms of the weighted Sobolev norms $\| \cdot \|_{H^\ell_{1/2+\gamma}}$ introduced in Definition 10.1. The main result of this section is contained in Proposition 10.4, which shows that, if $E_{\ell;\gamma}(0) + M$ is sufficiently small and $(h^{(1)}_{\mu\nu}, \mathscr{F}_{\mu\nu})$ is the corresponding solution to the reduced equations, then $\mathscr{E}_{\ell;\gamma;\mu}(0) \lesssim E_{\ell;\gamma}(0) + M$. Thus, Proposition 10.4 allows us to deduce the smallness of $\mathscr{E}_{\ell;\gamma;\mu}(0)$ from the smallness of quantities that depend exclusively on the abstract initial data.

We begin by introducing the weighted Sobolev norm discussed in the above paragraph.

**Definition 10.1.** Let $U(x)$ be a tensor field defined along the Euclidean space $\mathbb{R}^3$. Then for any integer $\ell \geq 0$ and any real number $\eta$, we define the $H^\ell_\eta$ norm of $U$ by

$$\|U\|^2_{H^\ell_\eta} \overset{\text{def}}{=} \sum_{|I| \leq \ell} \int_{x \in \mathbb{R}^3} (1 + |x|^2)^{(\eta + |I|)} |\underline{\nabla}^I U(x)|^2 \, d^3 x. \tag{10.0.1}$$

We also introduce the following norm, which can be controlled in terms of a suitable $H^\ell_\eta$ norm via a Sobolev embedding result; see Proposition A.1.

**Definition 10.2.** Let $U(x)$ be a tensor field defined along the Euclidean space $\mathbb{R}^3$. Then for any integer $\ell \geq 0$ and any real number $\eta$, we define the $C^\ell_\eta$ norm of $U$ by

$$\|U\|^2_{C^\ell_\eta} \overset{\text{def}}{=} \sum_{|I| \leq \ell} \operatorname*{ess\,sup}_{x \in \mathbb{R}^3} (1 + |x|^2)^{(\eta + |I|)} |\underline{\nabla}^I U(x)|^2. \tag{10.0.2}$$

We are now ready to introduce our norm $E_{\ell;\gamma}(0) \geq 0$ on the abstract initial data. Recall that, as discussed in Section 4.1, the data are the following four fields on $\mathbb{R}^3$: $(\mathring{\underline{g}}_{jk} = \delta_{jk} + \underbrace{\mathring{\underline{h}}^{(0)}_{jk} + \mathring{\underline{h}}^{(1)}_{jk}}_{\mathring{\underline{h}}_{jk}}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ $(j, k = 1, 2, 3)$.

**Definition 10.3.** The norm $E_{\ell;\gamma}(0) \geq 0$ of the abstract initial data is defined by

$$E^2_{\ell;\gamma}(0) \overset{\text{def}}{=} \|\underline{\nabla} \mathring{\underline{h}}^{(1)}\|^2_{H^\ell_{1/2+\gamma}} + \|\mathring{K}\|^2_{H^\ell_{1/2+\gamma}} + \|\mathring{\mathfrak{D}}\|^2_{H^\ell_{1/2+\gamma}} + \|\mathring{\mathfrak{B}}\|^2_{H^\ell_{1/2+\gamma}}. \tag{10.0.3}$$

***The smallness condition.*** Our smallness condition for global existence is

$$E_{\ell;\gamma}(0) + M \leq \varepsilon_\ell, \tag{10.0.4}$$

where $\varepsilon_\ell$ is a sufficiently small positive number.

Recall that the energy $\mathscr{E}_{\ell;\gamma;\mu}(t) \geq 0$ is defined by

$$\mathscr{E}^2_{\ell;\gamma;\mu}(t) \overset{\text{def}}{=} \sup_{0 \leq \tau \leq t} \sum_{|I| \leq \ell} \int_{\Sigma_\tau} \left( |\nabla \nabla^I_{\mathscr{L}} h^{(1)}|^2 + |\mathscr{L}^I_{\mathscr{L}} \mathscr{F}|^2 \right) w(q) \, d^3 x, \tag{10.0.5}$$

where $\nabla$ denotes the *full Minkowski spacetime* covariant derivative operator and the weight $w(q)$ is defined in (12.1.1). The dependence on $\gamma$ and $\mu$ in $\mathscr{E}_{\ell;\gamma;\mu}$ is through $w(q)$. The next proposition, which is the main result of this section, shows that the smallness of $\mathscr{E}_{\ell;\gamma;\mu}(0)$ follows from the smallness of $E_{\ell;\gamma}(0) + M$:

**Proposition 10.4** (The smallness of the initial energy). *Let* $(\mathring{\underline{g}}_{jk} = \delta_{jk} + \mathring{\underline{h}}^{(0)}_{jk} + \mathring{\underline{h}}^{(1)}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ $(j, k = 1, 2, 3)$ *be abstract initial data on the manifold* $\mathbb{R}^3$ *for the Einstein-nonlinear electromagnetic system* (1.0.1a)–(1.0.1c). *Assume that the abstract initial data satisfy the constraints* (1.0.2a)–(1.0.3b) *and*

*that they are asymptotically flat in the sense that* (1.0.4a)–(1.0.4f) *hold. Let* $(g_{\mu\nu}|_{t=0} = m_{\mu\nu} + h^{(0)}_{\mu\nu}|_{t=0} + h^{(1)}_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0} = \partial_t h^{(0)}_{\mu\nu}|_{t=0} + \partial_t h^{(1)}_{\mu\nu}|_{t=0}, \mathscr{F}_{\mu\nu}|_{t=0})$ $(\mu, \nu = 0, 1, 2, 3)$ *be the corresponding initial data for the reduced system* (3.7.1a)–(3.7.1c) *as defined in Section 4.2, and let* $(g_{\mu\nu} = m_{\mu\nu} + h^{(0)}_{\mu\nu} + h^{(1)}_{\mu\nu}, \mathscr{F}_{\mu\nu})$ *be the solution to the reduced system launched by this data. Let* $\ell \geq 10$ *be an integer. In particular, by Proposition 4.2, the wave-coordinate condition* (3.1.1a) *is satisfied by the reduced solution. Then there exist a constant* $\varepsilon_0 > 0$ *and a constant* $C_\ell > 0$ *such that, if* $E_{\ell;\gamma}(0) + M \leq \varepsilon \leq \varepsilon_0$, *then*

$$\mathscr{E}_{\ell;\gamma;\mu}(0) \leq C_\ell\{E_{\ell;\gamma}(0) + M\} \leq C_\ell \varepsilon. \tag{10.0.6}$$

**Remark 10.5.** Note that $q \geq 0$ holds at $t = 0$. Therefore, $\mathscr{E}_{\ell;\gamma;\mu}(0)$ does not depend on the constant $\mu$.

The proof of Proposition 10.4 starts on page 845. We first establish some technical lemmas.

**Lemma 10.6** (Energy in terms of $h^{(1)}$, $E$, and $B$). *Let* $\mathscr{F}_{\mu\nu}$ *be a two-form, let the pair of one-forms* $(E_\mu, B_\mu)$ *be its Minkowskian electromagnetic decomposition as defined in Section 9.2, and let* $h^{(1)}_{\mu\nu}$ *be an arbitrary type-$\binom{0}{2}$ tensor field. Let* $\mathscr{E}_{\ell;\gamma;\mu}(t)$ *be the energy defined in* (10.0.5). *Then*

$$\mathscr{E}^2_{\ell;\gamma;\mu}(t) \approx \sup_{0 \leq \tau \leq t} \sum_{|I| \leq \ell} \int_{\Sigma_\tau} \left(|\nabla \nabla^I_{\mathscr{L}} h^{(1)}|^2 + |\nabla^I_{\mathscr{L}} E|^2 + |\nabla^I_{\mathscr{L}} B|^2\right) w(q)\, d^3x. \tag{10.0.7}$$

*Proof.* Equation (10.0.7) easily follows from (6.5.22) and the identity $|\nabla^I_{\mathscr{L}} \mathscr{F}|^2 = 2|\nabla^I_{\mathscr{L}} E|^2 + 2|\nabla^I_{\mathscr{L}} B|^2$, the verification of which we leave to the reader. $\qquad\square$

**Lemma 10.7.** *The following estimates hold for any $\ell$-times differentiable spacetime tensor field* $U(t, x)$ *defined in a neighborhood of* $\Sigma_0 \overset{\text{def}}{=} \{(t, x) \mid t = 0\}$, *where* $w(q)$ *is the weight defined in* (12.1.1):

$$\left(\sum_{|I| \leq \ell} w^{1/2}(q)|\nabla^I_{\mathscr{L}} U|\right)\bigg|_{\Sigma_0} \approx \left(\sum_{|I| \leq \ell} (1+r)^{1/2+\gamma+|I|}|\nabla^I U|\right)\bigg|_{\Sigma_0}$$

$$\approx \left(\sum_{|J|+k \leq \ell} (1+r)^{1/2+\gamma+|J|+k}|\partial^k_t \underline{\nabla}^J U|\right)\bigg|_{\Sigma_0}. \tag{10.0.8}$$

*The same estimates hold if* $\nabla^I_{\mathscr{L}}$ *is replaced with* $\mathscr{L}^I_{\mathscr{L}}$. *The notation* $|_{\Sigma_0}$ *is meant to indicate that the estimates only hold along* $\Sigma_0$.

*Proof.* By iterating the identity $\frac{\partial}{\partial x^\mu} = (x^\kappa \Omega_{\kappa\mu} + x_\mu S)/qs$ and noting that $q = r = s$ along $\Sigma_0$, we deduce that

$$(1+r)^{|I|}|\nabla^I U| \lesssim \sum_{|J| \leq |I|} |\nabla^J_{\mathscr{L}} U|. \tag{10.0.9}$$

It thus follows from the definition (12.1.1) of $w(q)$ that

$$\left(\sum_{|I| \leq \ell} (1+r)^{1/2+\gamma+|I|}|\nabla^I U|\right)\bigg|_{\Sigma_0} \lesssim \left(\sum_{|I| \leq \ell} w^{1/2}(q)|\nabla^I_{\mathscr{L}} U|\right)\bigg|_{\Sigma_0}. \tag{10.0.10}$$

On the other hand, the opposite inequality follows easily from expanding the operator $\nabla^I_{\mathscr{L}}$ and using the Leibniz rule plus (6.2.4). This proves the first $\approx$ in (10.0.8). The second $\approx$ is trivial. We have thus

established (10.0.8). To establish the same estimates with the operator $\mathscr{L}_{\mathscr{L}}^I$ in place of $\nabla_{\mathscr{L}}^I$, we simply use (6.5.22). □

**Corollary 10.8.** *Under the assumptions of Lemma 10.6, we have that*

$$\mathscr{E}_{\ell;\gamma;\mu}^2(0) \approx \sum_{k+|I|\leq\ell} \int_{\mathbb{R}^3} (1+|x|)^{1+2\gamma+2(k+|I|)} \left( |\partial_t^k \nabla^I \partial_t h^{(1)}|^2(0,x) + |\nabla^I \nabla h^{(1)}|^2(0,x) \right) d^3x$$

$$+ \int_{\mathbb{R}^3} (1+|x|)^{1+2\gamma+2(k+|I|)} \left( |\partial_t^k \nabla^I E|^2(0,x) + |\partial_t^k \nabla^I B|^2(0,x) \right) d^3x. \quad (10.0.11)$$

*Proof.* Corollary 10.8 follows easily from Lemmas 10.6 and 10.7. □

**Lemma 10.9.** *Assume the hypotheses of Proposition 10.4. Let $k \geq 1$ and $\ell \geq 10$ be integers, and let $J$ be a $\nabla$-multi-index. Assume that $|J| + k \leq \ell$. Define the arrays $V$, $V^{(0)}$, $V^{(1)}$, $W$, $W^{(0)}$, and $W^{(1)}$ by*

$$V \overset{\text{def}}{=} (h, \nabla h, \partial_t h, E, B) = V^{(0)} + V^{(1)}, \quad (10.0.12a)$$

$$V^{(0)} \overset{\text{def}}{=} (h^{(0)}, \nabla h^{(0)}, \partial_t h^{(0)}, 0, 0), \quad (10.0.12b)$$

$$V^{(1)} \overset{\text{def}}{=} (h^{(1)}, \nabla h^{(1)}, \partial_t h^{(1)}, E, B), \quad (10.0.12c)$$

$$W \overset{\text{def}}{=} (0, \nabla h, \partial_t h, E, B) = W^{(0)} + W^{(1)}, \quad (10.0.12d)$$

$$W^{(0)} \overset{\text{def}}{=} (0, \nabla h^{(0)}, \partial_t h^{(0)}, 0, 0), \quad (10.0.12e)$$

$$W^{(1)} \overset{\text{def}}{=} (0, \nabla h^{(1)}, \partial_t h^{(1)}, E, B). \quad (10.0.12f)$$

*In the above expressions, the tensor fields $h_{\mu\nu}^{(0)}$ and $h_{\mu\nu}^{(1)}$ are defined by (1.2.1a)–(1.2.1c) while the electromagnetic one-forms $E_\mu$ and $B_\mu$ are defined in (9.2.4). Assume further that $|V^{(1)}| + M \leq \varepsilon$. Then if $\varepsilon$ is sufficiently small, $\partial_t^k \nabla^J W^{(1)}$ can be written as the following finite linear combination:*

$$\nabla^J \partial_t^k W^{(1)} = \sum \text{terms}, \quad (10.0.13)$$

*where each* term *can be written as*

$$\text{term} = \sum_{\substack{|I_1|+\cdots+|I_s|\leq|J|+k \\ 0\leq|I_1|,...,|I_s|}} F_{(I_1,...,I_s;J;k;s)}(t,x) \times \mathcal{M}_{(I_1,...,I_s;J;k;s)}(V)[\nabla^{I_1} W^{(1)}, \ldots, \nabla^{I_s} W^{(1)}], \quad (10.0.14)$$

*and*:

(i) *The array-valued functions $\mathcal{M}_{(I_1,...,I_s;J;k;s)}(V)[\nabla^{I_1} W^{(1)}, \ldots, \nabla^{I_s} W^{(1)}]$ are continuous in a neighborhood of $V = 0$ and are multilinear in the arguments $[\nabla^{I_1} W^{(1)}, \ldots, \nabla^{I_s} W^{(1)}]$.*

(ii) *If $s = 0$ (i.e., if there are no multilinear arguments $[\,\cdot\,]$), the array-valued functions $F_{(I_1,...,I_s;J;k;s)}(t,x)$ are smooth and satisfy $|F_{(I_1,...,I_s;J;k;s)}(t,x)| \lesssim M(1+t+|x|)^{-(3+|J|+k)}$, where $M$ is the ADM mass.*

(iii) *When $s \geq 1$, $|F_{(I_1,...,I_s;J;k;s)}(t,x)| \lesssim (1+t+|x|)^{-d}$, where $d \geq |J|+k - (|I_1|+\cdots+|I_s|) - (s-1)$.*

*Proof.* We first claim that we can write the reduced system (3.7.1a)–(3.7.1c) as a finite linear combination

$$\partial_t W^{(1)} = \sum \text{terms}, \quad (10.0.15a)$$

where each term can be written in the form

$$\text{term} = \sum_{|I|=1} \mathcal{M}_{(I;0;1;1)}(V)[\underline{\nabla}^I W^{(1)}] + \mathcal{M}_{(0;0;1;2)}(V)[W^{(1)}, W^{(1)}]$$
$$+ f_{(0;0;1;1)}(t,x)\, \mathcal{M}_{(0;0;1;1)}(V)[W^{(1)}] + f_{(0;0;1;0)}(t,x)\, \mathcal{M}_{(0;0;1;0)}(V). \quad (10.0.15b)$$

Above, the functions $\mathcal{M}_{(\cdot)}(V)[\,\cdot\,]$, which depend on the $(\ell+2)$-times continuously differentiable Lagrangian $^\star\mathscr{L}$ for the electromagnetic equations, have the properties stated in the conclusions of the theorem. In addition, $f_{(0;0;1;1)}(t,x)$ and $f_{(0;0;1;0)}(t,x)$ are smooth functions satisfying $|\nabla^I f_{(0;0;1;1)}(t,x)| \lesssim (1+t+|x|)^{-(2+|I|)}$ and $|\nabla^I f_{(0;0;1;0)}(t,x)| \lesssim M(1+t+|x|)^{-(3+|I|)}$ for any $\nabla$-multi-index $I$. Let us accept the claim (10.0.15b) for now; we will briefly discuss the derivation of (10.0.15b) at the end of the proof. We also note that

$$\partial_t V = \partial_t W^{(1)} + \Pi_1 W^{(1)} + \partial_t V^{(0)}, \quad (10.0.16)$$

$$\underline{\nabla} V = \underline{\nabla} W^{(1)} + \Pi_2 W^{(1)} + \underline{\nabla} V^{(0)}, \quad (10.0.17)$$

where $V^{(0)}(t,x)$ satisfies $|\nabla^I \partial_t V^{(0)}(t,x)| + |\nabla^I \underline{\nabla} V^{(0)}(t,x)| \lesssim (1+t+|x|)^{-(2+|I|)}$ for any $\nabla$-multi-index $I$ (see Lemma 15.1), $\Pi_1 W^{(1)} \overset{\text{def}}{=} (\partial_t h^{(1)}, 0, 0, 0, 0)$, and $\Pi_2 W^{(1)} \overset{\text{def}}{=} (\underline{\nabla} h^{(1)}, 0, 0, 0, 0)$. Now with the help of (10.0.16)–(10.0.17), the chain rule, and the Leibniz rule, we repeatedly partially differentiate (10.0.15b) with respect to time and spatial derivatives, using the resulting equations to replace time derivatives with spatial derivatives, thereby inductively arriving at an expression of the form (10.0.14) verifying the properties (i)–(iii). The properties (ii)–(iii) capture the fact that each additional differentiation of $\partial_t W^{(1)}$ either (a) creates an additional decay factor of $(1+t+|x|)^{-1}$ (when the derivative falls on one of the $f_{...}(t,x)$), (b) increases one of the powers $|I_j|$ (when the derivative is spatial and falls on one of the multilinear factors $[\ldots, \underline{\nabla}^{I_j} W^{(1)}, \ldots]$), or (c) increases $s$ by one (when the derivative falls on $\mathcal{M}(V)$, thereby creating an additional multilinear factor of $\nabla W^{(1)}$ via the chain rule).

We now return to the issue of expressing $\partial_t W^{(1)}$ in the form (10.0.15a)–(10.0.15b). We will make repeated use of the splitting $h = h^{(0)} + h^{(1)}$, where $h^{(0)}$ is the smooth function of $(t,x)$ with the decay properties (15.1.1a), which are proved in Section 15.1. We first note that $\partial_t E$ and $\partial_t B$ can be expressed in the desired form by using (9.2.10a)–(9.2.10b) together with the splitting of $h$ and the properties (15.1.1a). We remark that, although (9.2.10a)–(9.2.10b) are *nonreduced* electromagnetic equations, they are nonetheless satisfied by virtue of the fact that the wave-coordinate condition holds and the fact that the reduced and nonreduced equations are equivalent under that condition. Next, we note that the quantities $\partial_t \underline{\nabla} h^{(1)}_{\mu\nu}$ can be expressed in the desired form through the trivial identity $\partial_t \underline{\nabla} h^{(1)}_{\mu\nu} = \underline{\nabla} \partial_t h^{(1)}_{\mu\nu}$. The quantities $\partial_t^2 h^{(1)}_{\mu\nu}$ can be expressed in the desired form by using (3.7.1a) to isolate them. We remark that the $\mathcal{M}_{I;0;1;1}(V)[\underline{\nabla}^I W^{(1)}]$ term on the right-hand side of (10.0.15b) arises from the spatial derivatives and mixed spacetime derivatives of $h^{(1)}$ contained in the term $\widetilde{\Box}_g h^{(1)}_{\mu\nu}$ on the left-hand side of (3.7.1a). Furthermore, the $\mathcal{M}_{0;0;1;2}(V)[W^{(1)}, W^{(1)}]$ term on the right-hand side of (10.0.15b) arises from the quadratic and higher-order-in-$W^{(1)}$ terms on the right-hand sides of (3.7.1a) and (9.2.10b) while the $f_{0;0;1;1}(t,x)\mathcal{M}_{0;0;1;1}(V)[W^{(1)}]$ term on the right-hand side of (10.0.15b) arises from the $h^{(0)}$- and $\nabla h^{(0)}$-containing factors that arise from the terms on the right-hand sides of (3.7.1a) and (9.2.10b)

that contain a linear factor of $h$ or $\nabla h$. Finally, the $f_{0;0;1;0}(t,x)\mathcal{M}_{0;0;1;0}(V)$ term on the right-hand side of (10.0.15b) arises from the $\tilde{\Box}_g h^{(0)}_{\mu\nu}$ term on the right-hand side of (3.7.1a) and from the $O(|\nabla h^{(0)}|^2)$ terms arising from splitting the $O(|\nabla h|^2)$ terms on the right-hand side of (3.7.1a). $\qquad\square$

**Corollary 10.10.** *Assume the hypotheses of Proposition 10.4, which include the smallness condition $E_{\ell;\gamma}(0) + M \leq \varepsilon$. Let $k \geq 0$ be an integer, let $J$ be a $\nabla$ multi-index, and assume that $|J| + k \leq \ell$. Let $V(t,x), \ldots, W^{(1)}(t,x)$ be the array-valued functions defined in (10.0.12a)–(10.0.12f), let $\mathring{V}(x) = V(0,x), \ldots, \mathring{W}^{(1)}(x) = W^{(1)}(0,x)$, and assume that $\|\mathring{V}^{(1)}\|_{L^\infty} + \|\mathring{W}^{(1)}\|_{H^\ell_{1/2+\gamma}} \leq \varepsilon$. Then if $\varepsilon$ is sufficiently small, the following inequality holds:*

$$\left\| (1+|x|)^{1/2+\gamma+|J|+k} \nabla^J \partial_t^k W^{(1)}(0,x) \right\|_{L^2} \lesssim \|\mathring{W}^{(1)}\|_{H^\ell_{1/2+\gamma}} + M. \tag{10.0.18}$$

*Proof.* We first consider the case $s = 0$ in (10.0.14). Then using that $|F_{(0;J;k;0)}(t,x)| \lesssim M(1+|x|)^{-(3+|J|+k)}$ (i.e., property (ii) from the conclusions of Lemma 10.9) and recalling that $0 < \gamma < \frac{1}{2}$, we deduce that

$$\left\| (1+|x|)^{1/2+\gamma+|J|+k} F_{(0;J;k;0)}(0,x)\mathcal{M}_{(0;J;k;0)}(\mathring{V}(x)) \right\|_{L^2}^2$$
$$= \int_{x\in\mathbb{R}^3} (1+|x|)^{1+2\gamma+2|J|+2k} |F_{(0;J;k;0)}(0,x)\mathcal{M}_{(0;J;k;0)}(\mathring{V}(x))|^2 \, d^3 x$$
$$\lesssim M^2 \int_{x\in\mathbb{R}^3} (1+|x|)^{2\gamma-5} \, d^3 x \lesssim M^2. \tag{10.0.19}$$

For the case $s \geq 1$, we first use Proposition A.1 to deduce that, for all $\nabla$-indices $K$ with $|K| \leq \ell - 2$, we have

$$|\nabla^K \mathring{W}^{(1)}(x)| \lesssim (1+|x|)^{-(|K|+1)} \|\mathring{W}^{(1)}\|_{H^{|K|+2}_{1/2+\gamma}}. \tag{10.0.20}$$

Then (without loss of generality assuming $|I_1| \leq |I_2| \leq \cdots \leq |I_s|$) we use $|F_{(I_1,\ldots,I_s;J;k;s)}(t,x)| \lesssim (1+t+|x|)^{-(|J|+k-(|I_1|+\cdots+|I_s|)-(s-1))}$ (i.e., property (iii)), together with (10.0.20), to deduce

$$\left\| (1+|x|)^{1/2+\gamma+|J|+k} F_{(I_1,\ldots,I_s;J;k;s)}(0,x) \times \mathcal{M}_{(I_1,\ldots,I_s;J;k;s)}(\mathring{V}(x))[\nabla^{I_1}\mathring{W}^{(1)}(x), \ldots, \nabla^{I_s}\mathring{W}^{(1)}(x)] \right\|_{L^2}$$
$$\lesssim \left\| (1+|x|)^{|I_1|+\cdots+|I_{s-1}|+(s-1)} \prod_{i=1}^{s-1} \nabla^{I_i}\mathring{W}^{(1)}(x) \right\|_{L^\infty} \times \left\| (1+|x|)^{1/2+\gamma+|I_s|}\nabla^{I_s}\mathring{W}^{(1)}(x) \right\|_{L^2}$$
$$\lesssim \left\| (1+|x|)^{1/2+\gamma+|I_s|}\nabla^{I_s}\mathring{W}^{(1)}(x) \right\|_{L^2} \lesssim \|\mathring{W}^{(1)}\|_{H^\ell_{1/2+\gamma}}. \tag{10.0.21}$$

Combining (10.0.19) and (10.0.21), we arrive at (10.0.18). $\qquad\square$

We are now ready for the proof of the proposition.

*Proof of Proposition 10.4.* We first stress that *the estimates derived in this proof are valid under the assumption that $\varepsilon$ is sufficiently small*. Recall that $g_{\mu\nu}(t,x) = m_{\mu\nu} + \chi(r/t)\chi(r)(2M/r)\delta_{\mu\nu} + h^{(1)}_{\mu\nu}(t,x)$. Also recall that, according to the assumptions of the proposition, we have (see (4.2.6a)–(4.2.6b)) the

following relations (where we slightly abuse matrix notation):

$$h^{(1)}(0,x) = \begin{pmatrix} 0 & 0 \\ 0 & \mathring{\underline{h}}_{jk}^{(1)} \end{pmatrix}, \tag{10.0.22a}$$

$$\partial_t h^{(1)}(0,x) = \begin{pmatrix} 2A^3(\mathring{\underline{g}}^{-1})^{ab}\mathring{K}_{ab} & A^2(\mathring{\underline{g}}^{-1})^{ab}\,\partial_a\mathring{\underline{g}}_{bj} - \frac{1}{2}A^2(\mathring{\underline{g}}^{-1})^{ab}\,\partial_j\mathring{\underline{g}}_{ab} - A\,\partial_j A \\ A^2(\mathring{\underline{g}}^{-1})^{ab}\,\partial_a\mathring{\underline{g}}_{bj} - \frac{1}{2}A^2(\mathring{\underline{g}}^{-1})^{ab}\,\partial_j\mathring{\underline{g}}_{ab} - A\,\partial_j A & 2A\mathring{K}_{jk} \end{pmatrix}, \tag{10.0.22b}$$

where $A(x) = \sqrt{1-(2M/r)\chi(r)}$ and $\mathring{\underline{g}}_{jk}(x) = \delta_{jk} + (2M/r)\chi(r)\delta_{jk} + \mathring{\underline{h}}_{jk}^{(1)}(x)$. Note that $(\mathring{\underline{g}}^{-1})^{jk} = \delta^{jk} + O^{\infty}(|(M/r)\chi(r)|;\mathring{\underline{h}}^{(1)}) + O^{\infty}(|\mathring{\underline{h}}^{(1)}|;(M/r)\chi(r))$. Our immediate objectives are to relate $\|\mathring{\underline{E}}\|_{H_{1/2+\gamma}^{\ell}}$ and $\|\partial_t h^{(1)}(0,\cdot)\|_{H_{1/2+\gamma}^{\ell}}$ to the inherent quantities $\|\underline{\nabla}\mathring{\underline{h}}\|_{H_{1/2+\gamma}^{\ell}}$, $\|\mathring{K}\|_{H_{1/2+\gamma}^{\ell}}$, $\|\mathring{\mathfrak{D}}\|_{H_{1/2+\gamma}^{\ell}}$, $\|\mathring{\mathfrak{B}}\|_{H_{1/2+\gamma}^{\ell}}$, and $M$. To this end, we first observe that the following estimates hold for sufficiently small $M$:

$$\left|\underline{\nabla}^I\left(\frac{M}{r}\chi(r)\right)\right| \lesssim M(1+r)^{-(1+|I|)}, \tag{10.0.23}$$

$$|A(x)| \lesssim 1, \tag{10.0.24}$$

$$|\underline{\nabla}^I A(x)| \lesssim M(1+r)^{-(1+|I|)} \quad (|I| \geq 1). \tag{10.0.25}$$

With the help of (10.0.22a)–(10.0.22b), the decay estimates (10.0.23)–(10.0.25), the Leibniz rule, Corollary A.4, the definition of $\|\cdot\|_{H_{1/2+\gamma}^{\ell}}$, and the assumption $0 < \gamma < \frac{1}{2}$, it is straightforward to check that

$$\|\partial_t h^{(1)}(0,\cdot)\|_{H_{1/2+\gamma}^{\ell}} \lesssim \|\underline{\nabla}\mathring{\underline{h}}^{(1)}\|_{H_{1/2+\gamma}^{\ell}} + \|\mathring{K}\|_{H_{1/2+\gamma}^{\ell}} + M. \tag{10.0.26}$$

Furthermore, from (9.2.8a)–(9.2.8d) and Corollary A.4, it follows that

$$\|\mathring{D}\|_{H_{1/2+\gamma}^{\ell}} + \|\mathring{B}\|_{H_{1/2+\gamma}^{\ell}} \approx \|\mathring{\mathfrak{D}}\|_{H_{1/2+\gamma}^{\ell}} + \|\mathring{\mathfrak{B}}\|_{H_{1/2+\gamma}^{\ell}}. \tag{10.0.27}$$

Similarly, from (9.2.6a) and (9.2.6c), we have that

$$\|\mathring{E}\|_{H_{1/2+\gamma}^{\ell}} + \|\mathring{B}\|_{H_{1/2+\gamma}^{\ell}} \approx \|\mathring{D}\|_{H_{1/2+\gamma}^{\ell}} + \|\mathring{B}\|_{H_{1/2+\gamma}^{\ell}}. \tag{10.0.28}$$

By (10.0.26), (10.0.27), (10.0.28), and Proposition A.1, it follows that, if $E_{\ell;\gamma}(0) + M$ is sufficiently small, then the smallness conditions[28] for $\|\mathring{V}^{(1)}\|_{L^{\infty}}$ and $\|\mathring{W}^{(1)}\|_{H_{1/2+\gamma}^{\ell}}$ in the hypotheses of Lemma 10.9 and Corollary 10.10 hold. Therefore, combining Corollaries 10.8 and 10.10, (10.0.26), (10.0.27), and (10.0.28), we deduce that, if $\varepsilon$ is sufficiently small, then

$$\begin{aligned} \mathscr{E}_{\ell;\gamma;\mu}^2(0) &\lesssim \|\underline{\nabla}\mathring{\underline{h}}^{(1)}\|_{H_{1/2+\gamma}^{\ell}}^2 + \|\partial_t h^{(1)}(0,\cdot)\|_{H_{1/2+\gamma}^{\ell}}^2 + \|\mathring{E}\|_{H_{1/2+\gamma}^{\ell}}^2 + \|\mathring{B}\|_{H_{1/2+\gamma}^{\ell}}^2 + M^2 \\ &\lesssim \|\underline{\nabla}\mathring{\underline{h}}^{(1)}\|_{H_{1/2+\gamma}^{\ell}}^2 + \|\mathring{K}\|_{H_{1/2+\gamma}^{\ell}}^2 + \|\mathring{\mathfrak{D}}\|_{H_{1/2+\gamma}^{\ell}}^2 + \|\mathring{\mathfrak{B}}\|_{H_{1/2+\gamma}^{\ell}}^2 + M^2 \\ &\overset{\text{def}}{=} E_{\ell;\gamma}^2(0) + M^2. \end{aligned} \tag{10.0.29}$$

This concludes our proof of Proposition 10.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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[28]As in the Lindblad–Rodnianski proof of Corollary 15.3 below, the *smallness* condition $|h^{(1)}(0,x)| \lesssim \varepsilon(1+r)^{-1-\gamma}$ follows from integrating the smallness condition $|\partial_r h^{(1)}(0,x)| \lesssim \varepsilon(1+r)^{-2-\gamma}$, which is a consequence of Proposition A.1, from spatial infinity, and from using the decay assumption (1.0.4c) for $|\mathring{h}^{(1)}(x)|$ at spatial infinity.

## 11. Algebraic estimates of the nonlinearities

In this section, we provide algebraic estimates for the inhomogeneous terms that arise from commuting the reduced equations (3.7.1a)–(3.7.1c) with various differential operators. We also use the equations of Proposition 9.3 to derive ordinary differential inequalities for the null components of $\dot{\mathscr{F}} = \mathscr{L}_{\mathscr{L}}^I \mathscr{F}$. Furthermore, we provide algebraic estimates for the inhomogeneous terms appearing on the right-hand sides of these inequalities. Many of the estimates derived in this section rely on the wave coordinate condition.

**11.1. *Statement and proofs of the propositions.*** The proofs of the propositions given in this section use the results of a collection of technical null-structure lemmas, which we relegate to the end of the section. We begin by quoting the following proposition, which is central to many of the estimates. The basic idea is the following: many of our estimates would break down if we could not achieve good control of the components $h_{LL}$ and $h_{LT}$. Amazingly, as shown in [Lindblad and Rodnianski 2005; 2010], the wave-coordinate condition allows for *independent, improved* estimates of exactly these components.

**Proposition 11.1** (Algebraic consequences of the wave coordinate condition [Lindblad and Rodnianski 2010, Proposition 8.2]). *Let $g$ be a Lorentzian metric satisfying the wave-coordinate condition* (3.1.1a) *relative to the coordinate system $\{x^\mu\}_{\mu=0,1,2,3}$. Let $I$ be a $\mathscr{L}$-multi-index, and assume that $|\nabla_{\mathscr{L}}^J h| \leq \varepsilon$ holds for all $\mathscr{L}$-multi-indices $J$ satisfying $|J| \leq \lfloor |I|/2 \rfloor$, where $h_{\mu\nu} \stackrel{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$. Then if $\varepsilon$ is sufficiently small, the following pointwise estimates hold for the tensor $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$:*

$$|\nabla \nabla_{\mathscr{L}}^I H|_{\mathscr{L}\mathscr{T}} \lesssim \sum_{|J| \leq |I|} |\overline{\nabla} \nabla_{\mathscr{L}}^J H| + \underbrace{\sum_{|J| \leq |I|-1} |\nabla \nabla_{\mathscr{L}}^J H|}_{\text{absent if } |I|=0} + \sum_{|I_1|+|I_2| \leq |I|} |\nabla_{\mathscr{L}}^{I_1} H||\nabla \nabla_{\mathscr{L}}^{I_2} H|, \qquad (11.1.1a)$$

$$|\nabla \nabla_{\mathscr{L}}^I H|_{\mathscr{L}\mathscr{L}} \lesssim \sum_{|J| \leq |I|} |\overline{\nabla} \nabla_{\mathscr{L}}^J H| + \underbrace{\sum_{|J| \leq |I|-2} |\nabla \nabla_{\mathscr{L}}^J H|}_{\text{absent if } |I| \leq 1} + \sum_{|I_1|+|I_2| \leq |I|} |\nabla_{\mathscr{L}}^{I_1} H||\nabla \nabla_{\mathscr{L}}^{I_2} H|. \qquad (11.1.1b)$$

*Furthermore, analogous estimates hold for the tensor $h_{\mu\nu}$.*

The next lemma provides an analogous version of the proposition for the "remainder" pieces of $(g^{-1})^{\mu\nu}$ and $g_{\mu\nu}$.

**Lemma 11.2** (Algebraic/analytic consequences of the wave-coordinate condition; slight extension of [Lindblad and Rodnianski 2010, Lemma 15.4]). *Let $g$ be a Lorentzian metric satisfying the wave-coordinate condition* (3.1.1a) *relative to the coordinate system $\{x^\mu\}_{\mu=0,1,2,3}$, and let $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$. Let $k \geq 0$ be an integer, and assume that there is a constant $\varepsilon$ such that $|\nabla_{\mathscr{L}}^J h| \leq \varepsilon$ holds for all $\mathscr{L}$-multi-indices $J$ satisfying $|J| \leq \lfloor k/2 \rfloor$, where $h_{\mu\nu} \stackrel{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$. Let*

$$H_{(1)}^{\mu\nu} \stackrel{\text{def}}{=} H^{\mu\nu} - H_{(0)}^{\mu\nu} \quad \text{and} \quad H_{(0)}^{\mu\nu} \stackrel{\text{def}}{=} -\chi\left(\frac{r}{t}\right)\chi(r)\frac{2M}{r}\delta^{\mu\nu}, \qquad (11.1.2)$$

*where $H_{(1)}^{\mu\nu}$ is the tensor obtained by subtracting the Schwarzschild part $H_{(0)}^{\mu\nu}$ from $H^{\mu\nu}$, and $\chi_0\left(\frac{1}{2} < z < \frac{3}{4}\right)$ denotes the characteristic function of the interval $\left[\frac{1}{2}, \frac{3}{4}\right]$. Assume further that $M \leq \varepsilon$. Then if $\varepsilon$ is sufficiently*

*small*, *the following pointwise estimates hold*:

$$\sum_{|I|\leq k}|\nabla\nabla_{\mathscr{L}}^I H_{(1)}|_{\mathscr{L}\mathscr{L}} + \sum_{|J|\leq k-1}|\nabla\nabla_{\mathscr{L}}^J H_{(1)}|_{\mathscr{L}\mathscr{T}}$$

$$\lesssim \sum_{|I|\leq k}|\overline{\nabla}\nabla_{\mathscr{L}}^I H_{(1)}| + \varepsilon\sum_{|I|\leq k}(1+t+|q|)^{-1} + \varepsilon\sum_{|I|\leq k}(1+t+|q|)^{-2}|\nabla_{\mathscr{L}}^I H_{(1)}||\nabla\nabla_{\mathscr{L}}^I H_{(1)}|$$

$$+ \sum_{|I_1|+|I_2|\leq k}|\nabla_{\mathscr{L}}^{I_1} H_{(1)}||\nabla\nabla_{\mathscr{L}}^{I_2} H_{(1)}| + \underbrace{\sum_{|J'|\leq k-2}|\nabla\nabla_{\mathscr{L}}^{J'} H_{(1)}|}_{absent\ if\ k\,\leq\,1}$$

$$+ \varepsilon(1+t+|q|)^{-2}\chi_0\left(\frac{1}{2} < \frac{r}{t} < \frac{3}{4}\right) + \varepsilon^2(1+t+|q|)^{-3}. \quad (11.1.3)$$

*Additionally, let*

$$h_{\mu\nu}^{(1)} \stackrel{\text{def}}{=} h_{\mu\nu} - h_{\mu\nu}^{(0)} \quad and \quad h_{\mu\nu}^{(0)} \stackrel{\text{def}}{=} \chi\left(\frac{r}{t}\right)\chi(r)\frac{2M}{r}\delta_{\mu\nu}, \quad (11.1.4)$$

*where $h_{\mu\nu}^{(1)}$ is the tensor field obtained by subtracting the Schwarzschild part $h_{\mu\nu}^{(0)}$ from $h_{\mu\nu}$. Then an estimate analogous to* (11.1.2) *holds if we replace the tensor field $H_{(1)}$ with the tensor field $h^{(1)}$.*

*Proof.* The estimates for the tensor field $H_{(1)}^{\mu\nu}$ were proved as [Lindblad and Rodnianski 2010, Lemma 15.4]. The analogous estimates for the tensor field $h_{\mu\nu}^{(1)}$ follow from those for $H_{(1)}^{\mu\nu}$ together with the fact that $H_{(1);\mu\nu} = -h_{\mu\nu}^{(1)} + O^\infty(|h^{(0)} + h^{(1)}|^2)$ and the decay estimates for $h^{(0)}$ provided by Lemma 15.1 below. $\square$

We now provide the following proposition, which captures the algebraic structure of the inhomogeneous term $\mathfrak{H}_{\mu\nu}$ appearing on the right-hand side of the reduced equation (3.7.1a).

**Proposition 11.3** (Algebraic estimates of $\mathfrak{H}_{\mu\nu}$ and $\nabla_{\mathscr{L}}^I\mathfrak{H}_{\mu\nu}$; extension of [Lindblad and Rodnianski 2010, Proposition 9.8]). *Let $\mathfrak{H}_{\mu\nu}$ be the inhomogeneous term on the right-hand side of the reduced equation* (3.7.1a), *and assume that the wave-coordinate condition* (3.1.1a) *holds. Then*

$$|\mathfrak{H}|_{\mathscr{T}\mathscr{N}} \lesssim |\overline{\nabla}h||\nabla h| + (|\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{F}|_{\mathscr{T}\mathscr{T}})|\mathscr{F}| + O^\infty(|h||\nabla h|^2) + O^{\ell+1}(|h||\mathscr{F}|^2) + O^{\ell+1}(|\mathscr{F}|^3; h), \quad (11.1.5a)$$

$$|\mathfrak{H}| \lesssim |\nabla h|_{\mathscr{T}\mathscr{N}}^2 + |\overline{\nabla}h||\nabla h| + |\mathscr{F}|^2 + O^\infty(|h||\nabla h|^2) + O^{\ell+1}(|h||\mathscr{F}|^2) + O^{\ell+1}(|\mathscr{F}|^3; h). \quad (11.1.5b)$$

*In addition, assume that there exists an $\varepsilon > 0$ such that $|\nabla_{\mathscr{L}}^J h| + |\mathscr{L}_{\mathscr{L}}^J\mathscr{F}| \leq \varepsilon$ holds for all $\mathscr{L}$-multi-indices $|J| \leq \lfloor|I|/2\rfloor$. Then if $\varepsilon$ is sufficiently small, the following pointwise estimates hold:*

$$|\nabla_{\mathscr{L}}^I\mathfrak{H}| \lesssim \sum_{|I_1|+|I_2|\leq|I|}\left(|\nabla\nabla_{\mathscr{L}}^{I_1}h|_{\mathscr{T}\mathscr{N}}|\nabla\nabla_{\mathscr{L}}^{I_2}h|_{\mathscr{T}\mathscr{N}} + |\overline{\nabla}\nabla_{\mathscr{L}}^{I_1}h||\nabla\nabla_{\mathscr{L}}^{I_2}h|\right) + \sum_{|I_1|+|I_2|\leq|I|}|\mathscr{L}_{\mathscr{L}}^{I_1}\mathscr{F}||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|$$

$$+ \underbrace{\sum_{|I_1|+|I_2|\leq|I|-2}|\nabla\nabla_{\mathscr{L}}^{I_1}h||\nabla\nabla_{\mathscr{L}}^{I_2}h|}_{absent\ if\ |I|\,\leq\,1} + \sum_{|I_1|+|I_2|+|I_3|\leq|I|}|\nabla\nabla_{\mathscr{L}}^{I_1}h||\nabla\nabla_{\mathscr{L}}^{I_2}h||\nabla\nabla_{\mathscr{L}}^{I_3}h|$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\leq|I|}|\nabla_{\mathscr{L}}^{I_1}h||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}||\mathscr{L}_{\mathscr{L}}^{I_3}\mathscr{F}| + \sum_{|I_1|+|I_2|+|I_3|\leq|I|}|\mathscr{L}_{\mathscr{L}}^{I_1}\mathscr{F}||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}||\mathscr{L}_{\mathscr{L}}^{I_3}\mathscr{F}|. \quad (11.1.5c)$$

*Proof.* Using (3.7.2a), we can decompose $\mathfrak{H}_{\mu\nu}$ into

$$\mathfrak{H}_{\mu\nu} = \text{(i)}_{\mu\nu} + \text{(ii)}_{\mu\nu} + \text{(iii)}_{\mu\nu} + \text{(iv)}_{\mu\nu}, \tag{11.1.6}$$

where

$$\text{(i)}_{\mu\nu} \overset{\text{def}}{=} \mathscr{P}(\nabla_\mu h, \nabla_\nu h), \tag{11.1.7}$$

$$\text{(ii)}_{\mu\nu} \overset{\text{def}}{=} \mathcal{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h), \tag{11.1.8}$$

$$\text{(iii)}_{\mu\nu} \overset{\text{def}}{=} \mathcal{Q}_{\mu\nu}^{(2;h)}(\mathscr{F}, \mathscr{F}), \tag{11.1.9}$$

$$\text{(iv)}_{\mu\nu} \overset{\text{def}}{=} O^\infty(|h||\nabla h|^2) + O^{\ell+1}(|h||\mathscr{F}|^2) + O^{\ell+1}(|\mathscr{F}|^3; h). \tag{11.1.10}$$

We will analyze each of the four pieces separately.

The facts that $|\text{(i)}|_{\mathcal{T}\mathcal{N}} \lesssim$ the right-hand side of (11.1.5a) and that $|\text{(i)}| \lesssim$ the right-hand side of (11.1.5b) follow from Proposition 11.1, (11.2.7a), and (11.2.7b). The fact that $|\nabla_{\mathscr{L}}^I \text{(i)}| \lesssim$ the right-hand side of (11.1.5c) follows from Proposition 11.1, (11.2.2c), and (11.2.7a).

The facts that $|\text{(ii)}|_{\mathcal{T}\mathcal{N}} \lesssim$ the right-hand side of (11.1.5a) and that $|\text{(ii)}| \lesssim$ the right-hand side of (11.1.5b) both follow from (11.2.7c). The fact that $|\nabla_{\mathscr{L}}^I \text{(ii)}| \lesssim$ the right-hand side of (11.1.5c) follows from (11.2.2a) and (11.2.7c).

The fact that $|\text{(iii)}|_{\mathcal{T}\mathcal{N}} \lesssim$ the right-hand side of (11.1.5a) follows from (11.2.7d) while the fact that $|\text{(iii)}| \lesssim$ the right-hand side of (11.1.5b) follows from (11.2.7e). The fact that $|\nabla_{\mathscr{L}}^I \text{(iii)}| \lesssim$ the right-hand side of (11.1.5c) follows from (6.5.22), (11.2.2b), and (11.2.7e).

The desired estimates for term (iv) follow easily with the help of the Leibniz rule and (6.5.22). $\qquad\square$

The next proposition captures the special algebraic structure of the reduced inhomogeneous term $\mathfrak{F}_{(I)}^\nu$ defined in (8.1.2b).

**Proposition 11.4** (Algebraic estimates of $\mathfrak{F}_{(I)}^\nu$). *Let $\mathfrak{F}^\nu$ be the inhomogeneous term* (3.7.2b) *in the reduced electromagnetic equations, let $I$ be a $\mathscr{L}$-multi-index with $|I| = k$, and let $X_\nu$ be any one-form. In addition, assume that there exists an $\varepsilon > 0$ such that $|\nabla_{\mathscr{L}}^J h| + |\mathscr{L}_{\mathscr{L}}^J \mathscr{F}| \le \varepsilon$ holds for all $\mathscr{L}$-multi-indices $|J| \le \lfloor k/2 \rfloor$. Then if $\varepsilon$ is sufficiently small, the following pointwise estimates hold:*

$$|X_\nu \widehat{\mathscr{L}}_{\mathscr{L}}^I \mathfrak{F}^\nu|$$

$$\lesssim \sum_{|I_1|+|I_2|\le k} |X||\overline{\nabla}\nabla_{\mathscr{L}}^{I_1} h||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}| + \sum_{|I_1|+|I_2|\le k} |X||\nabla\nabla_{\mathscr{L}}^{I_1} h|\big(|\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|_{\mathscr{L}\mathcal{N}} + |\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|_{\mathcal{T}\mathcal{T}}\big)$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\le k} |X||\nabla_{\mathscr{L}}^{I_1} h||\nabla\nabla_{\mathscr{L}}^{I_2} h||\mathscr{L}_{\mathscr{L}}^{I_3}\mathscr{F}| + \sum_{|I_1|+|I_2|+|I_3|\le k} |X||\nabla\nabla_{\mathscr{L}}^{I_1} h||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}||\mathscr{L}_{\mathscr{L}}^{I_3}\mathscr{F}|$$

$$\lesssim (1+t+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\le k+1 \\ |I_2|\le k}} |X||\nabla_{\mathscr{L}}^{I_1} h||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}| + (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\le k+1 \\ |I_2|\le k}} |X||\nabla_{\mathscr{L}}^{I_1} h|\big(|\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|_{\mathscr{L}\mathcal{N}} + |\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|_{\mathcal{T}\mathcal{T}}\big)$$

$$+ (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|+|I_3|\le k+1 \\ |I_2|,|I_3|\le k}} |X||\nabla_{\mathscr{L}}^{I_1} h||\nabla_{\mathscr{L}}^{I_2} h||\mathscr{L}_{\mathscr{L}}^{I_3}\mathscr{F}|$$

$$+ (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|+|I_3|\le k+1 \\ |I_2|,|I_3|\le k}} |X||\nabla_{\mathscr{L}}^{I_1} h||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}||\mathscr{L}_{\mathscr{L}}^{I_3}\mathscr{F}|. \tag{11.1.11a}$$

*In addition, the same estimates hold if we replace modified Lie derivatives $\widehat{\mathscr{L}}_{\mathscr{L}}^{I}$ with standard Lie derivatives $\mathscr{L}_{\mathscr{L}}^{I}$.*

*Furthermore, let $N^{\#\mu\nu\kappa\lambda}$ be the tensor field from the reduced electromagnetic equation* (3.7.1c). *Then if $\varepsilon$ is sufficiently small and $k \geq 1$, the following pointwise commutator estimate holds:*

$$
\left| X_\nu \left( N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{L}_{\mathscr{L}}^{I} \mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathscr{L}}^{I} (N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{F}_{\kappa\lambda}) \right) \right|
$$

$$
\lesssim (1+|q|)^{-1} \sum_{|I'|=k,\ |J|\leq 1} |X| |\nabla_{\mathscr{L}}^{I'} h|_{\mathscr{L}\mathscr{L}} |\mathscr{L}_{\mathscr{L}}^{J} \mathscr{F}|
$$

$$
+ (1+|q|)^{-1} \sum_{|J|\leq 1,\ |I'|=k} |X| |\nabla_{\mathscr{L}}^{J} h|_{\mathscr{L}\mathscr{L}} |\mathscr{L}_{\mathscr{L}}^{I'} \mathscr{F}|
$$

$$
+ (1+|q|)^{-1} \sum_{|I'|=k} |X| |h|_{\mathscr{L}\mathscr{T}} |\mathscr{L}_{\mathscr{L}}^{I'} \mathscr{F}|
$$

$$
+ (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\leq k+1 \\ |I_1|,|I_2|\leq k}} |X| |\nabla_{\mathscr{L}}^{I_1} h| \left( |\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}|_{\mathscr{T}\mathscr{T}} \right)
$$

$$
+ (1+t+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\leq k+1 \\ |I_1|,|I_2|\leq k}} |X| |\nabla_{\mathscr{L}}^{I_1} h| |\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}|
$$

$$
+ (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\leq k+1 \\ |I_1|,|I_2|\leq k}} |X|_{\mathscr{L}} |\nabla_{\mathscr{L}}^{I_1} h| |\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}|
$$

$$
+ (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\leq k+1 \\ |I_1|\leq k-1,\ |I_2|\leq k-1}} |X| |\nabla_{\mathscr{L}}^{I_1} h|_{\mathscr{L}\mathscr{L}} |\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}|
$$

$$
+ (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\leq k \\ |I_1|\leq k-1,\ |I_2|\leq k-1}} |X| |\nabla_{\mathscr{L}}^{I_1} h|_{\mathscr{L}\mathscr{T}} |\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}|
$$

$$
+ \underbrace{(1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\leq k-1 \\ |I_1|\leq k-2,\ |I_2|\leq k-1}} |X| |\nabla_{\mathscr{L}}^{I_1} h| |\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}|}_{\text{absent if } k=1}
$$

$$
+ (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|+|I_3|\leq k+1 \\ |I_1|,|I_2|,|I_3|\leq k}} |X| |\nabla_{\mathscr{L}}^{I_1} h| |\nabla_{\mathscr{L}}^{I_2} h| |\mathscr{L}_{\mathscr{L}}^{I_3} \mathscr{F}|
$$

$$
+ (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|+|I_3|\leq k+1 \\ |I_1|,|I_2|,|I_3|\leq k}} |X| |\nabla_{\mathscr{L}}^{I_1} h| |\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}| |\mathscr{L}_{\mathscr{L}}^{I_3} \mathscr{F}|
$$

$$
+ (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|+|I_3|\leq k+1 \\ |I_1|,|I_2|,|I_3|\leq k}} |X| |\mathscr{L}_{\mathscr{L}}^{I_1} \mathscr{F}| |\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}| |\mathscr{L}_{\mathscr{L}}^{I_3} \mathscr{F}|. \quad (11.1.11b)
$$

*Proof.* To derive (11.1.11a), we first appeal to the relation (8.1.3a), which shows that we have to estimate principal terms of the form $X_\nu \mathscr{Q}^\nu_{(2;\mathscr{F})}(\nabla \mathscr{L}^{I_1}_{\mathscr{Z}} h, \mathscr{L}^{I_2}_{\mathscr{Z}} \mathscr{F})$ and error terms of the form $X_\nu \mathscr{L}^J_{\mathscr{Z}} \mathfrak{F}^\nu_\triangle$. The desired estimates for the principal terms follow from the null-structure estimate (11.2.7i) together with inequalities (6.5.22), (6.5.23a), and (6.5.23b), which allow us to estimate Lie derivatives of $h$ in terms of covariant derivatives of $h$. The error terms can easily be bounded by the right-hand side of (11.1.11a), where we use Lemma 6.16 to derive the second inequality in (11.1.11a).

Inequality (11.1.11b) can be proved in a similar fashion with the help of the relation (8.1.3b). In this case, there are two kinds of principal terms that have to be estimated: $X_\nu \mathscr{P}^\nu_{(\mathscr{F})}(\mathscr{L}^{I_1}_{\mathscr{Z}} h, \nabla \mathscr{L}^{I_2}_{\mathscr{Z}} \mathscr{F})$ and $X_\nu \mathscr{Q}^\nu_{(1;\mathscr{F})}(\mathscr{L}^{I_1}_{\mathscr{Z}} h, \nabla \mathscr{L}^{I_2}_{\mathscr{Z}} \mathscr{F})$ while the error terms are of the form $X_\nu(\mathscr{L}^{I_1}_{\mathscr{Z}} N^{\#\mu\nu\kappa\lambda}_\triangle)\nabla_\mu \mathscr{L}^{I_2}_{\mathscr{Z}} \mathscr{F}_{\kappa\lambda}$. The error terms can be estimated as in the previous paragraph. The principal terms can be bounded by using the null-structure estimates (11.2.7f) and (11.2.7h). As in the previous paragraph, we use (6.5.22) and (6.5.23c) to estimate Lie derivatives of $h$ in terms of covariant derivatives of $h$. □

As discussed at the beginning of Section 9.1, the null components of the lower-order Lie derivatives of $\mathscr{F}$ satisfy ordinary differential inequalities with controllable inhomogeneous terms. The next proposition provides convenient algebraic expressions for the inhomogeneities. In Section 15, these algebraic expressions will be combined with preliminary pointwise decay estimates to deduce upgraded pointwise decay estimates for the null components of $\mathscr{F}$ and its lower-order Lie derivatives.

**Proposition 11.5** (Ordinary differential inequalities for $\underline{\alpha}[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]$, $\alpha[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]$, $\rho[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]$, and $\sigma[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]$). *Let $\mathscr{F}$ be a solution to the reduced electromagnetic equations (3.7.1b)–(3.7.1c), and let $\underline{\alpha}$, $\alpha$, $\rho$, and $\sigma$ denote its Minkowskian null components. Let $\Lambda \overset{\text{def}}{=} L + \frac{1}{4}h_{LL}\underline{L}$, and assume that $|h| + |\mathscr{F}| \leq \varepsilon$ holds. Then if $\varepsilon$ is sufficiently small, the following pointwise estimate holds:*

$$r^{-1}|\nabla_\Lambda(r\underline{\alpha})| \lesssim r^{-1}|h|_{\mathscr{L}\mathscr{L}}|\underline{\alpha}| + \sum_{|I|\leq 1} r^{-1}\big(|\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}|_{\mathscr{T}\mathscr{T}}\big) + \sum_{|I_1|+|I_2|\leq 1} r^{-1}|\nabla^{I_1}_{\mathscr{Z}}h||\mathscr{L}^{I_2}_{\mathscr{Z}}\mathscr{F}|$$

$$+ \sum_{|I|\leq 1}(1+|q|)^{-1}|h|\big(|\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}|_{\mathscr{T}\mathscr{T}}\big)$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\leq 1}(1+|q|)^{-1}|\nabla^{I_1}_{\mathscr{Z}}h||\nabla^{I_2}_{\mathscr{Z}}h||\mathscr{L}^{I_3}_{\mathscr{Z}}\mathscr{F}|$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\leq 1}(1+|q|)^{-1}|\nabla^{I_1}_{\mathscr{Z}}h||\mathscr{L}^{I_2}_{\mathscr{Z}}\mathscr{F}||\mathscr{L}^{I_3}_{\mathscr{Z}}\mathscr{F}|$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\leq 1}(1+|q|)^{-1}|\mathscr{L}^{I_1}_{\mathscr{Z}}\mathscr{F}||\mathscr{L}^{I_2}_{\mathscr{Z}}\mathscr{F}||\mathscr{L}^{I_3}_{\mathscr{Z}}\mathscr{F}|. \quad (11.1.12)$$

*Similarly, for each $\mathscr{Z}$-multi-index $I$, let $\underline{\alpha}[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]$, $\alpha[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]$, $\rho[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]$, and $\sigma[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]$ denote the Minkowskian null components of $\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}$. Furthermore, let $\varpi(q)$ be any differentiable function of $q$. Assume that $|\nabla^I_{\mathscr{Z}}h| + |\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}| \leq \varepsilon$ holds for $|I| \leq \lfloor k/2 \rfloor$. Then if $\varepsilon$ is sufficiently small, the following pointwise estimates also hold:*

$$\sum_{|I|\le k} r^{-1}\big|\nabla_\Lambda\big(r\varpi(q)\underline{\alpha}[\mathscr{L}_{\mathscr{L}}^I\mathscr{F}]\big)\big| \lesssim \sum_{|I|\le k} r^{-1}\varpi(q)|h|_{\mathscr{L}\mathscr{L}}\big|\underline{\alpha}[\mathscr{L}_{\mathscr{L}}^I\mathscr{F}]\big| + \sum_{|I|\le k}\varpi'(q)|h|_{\mathscr{L}\mathscr{L}}\big|\underline{\alpha}[\mathscr{L}_{\mathscr{L}}^I\mathscr{F}]\big|$$

$$+ \underbrace{\sum_{|I|\le k,\,|J|\le 1}\varpi(q)(1+|q|)^{-1}|\nabla_{\mathscr{L}}^I h|_{\mathscr{L}\mathscr{L}}\big|\underline{\alpha}[\mathscr{L}_{\mathscr{L}}^J\mathscr{F}]\big|}_{\textit{absent if }k\le 1}$$

$$+ \underbrace{\sum_{|J|\le 1,\,|I|\le k}\varpi(q)(1+|q|)^{-1}|\nabla_{\mathscr{L}}^J h|_{\mathscr{L}\mathscr{L}}\big|\underline{\alpha}[\mathscr{L}_{\mathscr{L}}^I\mathscr{F}]\big|}_{\textit{absent if }k=0}$$

$$+ \underbrace{\sum_{|I|\le k}\varpi(q)(1+|q|)^{-1}|h|_{\mathscr{L}\mathscr{T}}\big|\underline{\alpha}[\mathscr{L}_{\mathscr{L}}^I\mathscr{F}]\big|}_{\textit{absent if }k=0}$$

$$+ \underbrace{\sum_{\substack{|I_1|+|I_2|\le k+1\\ |I_1|\le k-1,\,|I_2|\le k-1}}\varpi(q)(1+|q|)^{-1}|\nabla_{\mathscr{L}}^{I_1} h|\big|\underline{\alpha}[\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}]\big|}_{\textit{absent if }k=0}$$

$$+ \sum_{|I|\le |k|+1}\varpi(q)r^{-1}\big(|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|_{\mathscr{L}\mathscr{N}}+|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|_{\mathscr{T}\mathscr{T}}\big)$$

$$+ \sum_{|I_1|+|I_2|\le k+1}\varpi(q)(1+|q|)^{-1}|\nabla_{\mathscr{L}}^{I_1} h|\big(|\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|_{\mathscr{L}\mathscr{N}}+|\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|_{\mathscr{T}\mathscr{T}}\big)$$

$$+ \sum_{|I_1|+|I_2|\le k+1}\varpi(q)(1+t+|q|)^{-1}|\nabla_{\mathscr{L}}^{I_1} h||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\le k+1}\varpi(q)(1+|q|)^{-1}|\nabla_{\mathscr{L}}^{I_1} h||\nabla_{\mathscr{L}}^{I_2} h||\mathscr{L}_{\mathscr{L}}^{I_3}\mathscr{F}|$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\le k+1}\varpi(q)(1+|q|)^{-1}|\nabla_{\mathscr{L}}^{I_1} h||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}||\mathscr{L}_{\mathscr{L}}^{I_3}\mathscr{F}|$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\le k+1}\varpi(q)(1+|q|)^{-1}|\mathscr{L}_{\mathscr{L}}^{I_1}\mathscr{F}||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}||\mathscr{L}_{\mathscr{L}}^{I_3}\mathscr{F}|,\qquad (11.1.13\text{a})$$

$$\sum_{|I|\le k} r^{-1}\big|\nabla_{\underline{L}}(r\alpha[\mathscr{L}_{\mathscr{L}}^I\mathscr{F}])\big| \lesssim \sum_{|I|\le k+1} r^{-1}|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}| + \sum_{\substack{|I_1|+|I_2|\le k+1\\ |I_1|\le k}}(1+|q|)^{-1}|\nabla_{\mathscr{L}}^{I_1} h||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\le k+1}(1+|q|)^{-1}|\nabla_{\mathscr{L}}^{I_1} h||\nabla_{\mathscr{L}}^{I_2} h||\mathscr{L}_{\mathscr{L}}^{I_3}\mathscr{F}|$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\le k+1}(1+|q|)^{-1}|\nabla_{\mathscr{L}}^{I_1} h||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}||\mathscr{L}_{\mathscr{L}}^{I_3}\mathscr{F}|$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\le k+1}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{L}}^{I_1}\mathscr{F}||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}||\mathscr{L}_{\mathscr{L}}^{I_3}\mathscr{F}|,\qquad (11.1.13\text{b})$$

$$\sum_{|I|\leq k} r^{-2}\big|\nabla_{\underline{L}}(r^2\rho[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}])\big| \lesssim \sum_{|I|\leq k+1} r^{-1}|\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}| + \sum_{\substack{|I_1|+|I_2|\leq k+1 \\ |I_1|\leq k}} (1+|q|)^{-1}|\nabla^{I_1}_{\mathscr{Z}}h||\mathscr{L}^{I_2}_{\mathscr{Z}}\mathscr{F}|$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\leq k+1} (1+|q|)^{-1}|\nabla^{I_1}_{\mathscr{Z}}h||\nabla^{I_2}_{\mathscr{Z}}h||\mathscr{L}^{I_3}_{\mathscr{Z}}\mathscr{F}|$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\leq k+1} (1+|q|)^{-1}|\nabla^{I_1}_{\mathscr{Z}}h||\mathscr{L}^{I_2}_{\mathscr{Z}}\mathscr{F}||\mathscr{L}^{I_3}_{\mathscr{Z}}\mathscr{F}|$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\leq k+1} (1+|q|)^{-1}|\mathscr{L}^{I_1}_{\mathscr{Z}}\mathscr{F}||\mathscr{L}^{I_2}_{\mathscr{Z}}\mathscr{F}||\mathscr{L}^{I_3}_{\mathscr{Z}}\mathscr{F}|, \quad (11.1.13c)$$

$$\sum_{|I|\leq k} r^{-2}\big|\nabla_{\underline{L}}(r^2\sigma[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}])\big| \lesssim \sum_{|I|\leq k+1} r^{-1}|\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}|. \tag{11.1.13d}$$

*Proof.* Our proof of (11.1.12) is based on decomposing the terms in (9.1.8a), where $\underline{\dot{\alpha}}_\nu \overset{\text{def}}{=} \underline{\alpha}_\nu[\mathscr{F}]$, $\dot{\mathfrak{F}}^{\nu'} = \mathfrak{F}^{\nu'}$, etc., in the equation. We remind the reader that this equation is a consequence of performing a Minkowskian null decomposition on the electromagnetic equations (3.7.1b)–(3.7.1c). Here, $\mathfrak{F}^{\nu'}$ is defined in (3.7.2b). We begin by noting that the first two terms in (9.1.8a) can be written as $r^{-1}\nabla_L(r\underline{\alpha})$. We then remove the dangerous $-\frac{1}{4}h_{LL}\nabla_{\underline{L}}\underline{\alpha}_\nu$ component from the quadratic term $\slashed{h}_{\nu\lambda}\mathscr{P}^\lambda_{(\mathscr{F})}(h,\nabla\mathscr{F}) \overset{\text{def}}{=} \slashed{h}_\nu{}^\lambda h^{\mu\kappa}\nabla_\mu\mathscr{F}_{\kappa\lambda}$ on the left-hand side of (9.1.8a) and add it to the $r^{-1}\nabla_L(r\underline{\alpha}_\nu)$ term. From the fact that $\nabla_\Lambda r = 1 - \frac{1}{4}h_{LL}$, it follows that the resulting sum can be written as $r^{-1}\nabla_\Lambda(r\underline{\alpha}_\nu) + \frac{1}{4}r^{-1}h_{LL}\underline{\alpha}_\nu$. We then put the $\frac{1}{4}r^{-1}h_{LL}\underline{\alpha}_\nu$ term on the right-hand side of (11.1.12) as the first inhomogeneous term; all the remaining terms in (9.1.8a) will also be placed on the right-hand side of (11.1.12). The left-over terms in $\mathscr{P}^\nu_{(\mathscr{F})}(h,\nabla\mathscr{F})$ (after the dangerous component $\frac{1}{4}h_{LL}\nabla_{\underline{L}}\underline{\alpha}^\nu$ has been removed) are denoted by $\widetilde{\mathscr{P}}^\nu_{(\mathscr{F})}(h,\nabla\mathscr{F})$ in Lemma 11.10 below. Now by (11.2.7g), with $X_{\nu'} \overset{\text{def}}{=} \slashed{h}_{\nu\nu'}$ (so that $|X|_{\mathscr{L}} = 0$), it follows that the left-over terms $X_{\nu'}\widetilde{\mathscr{P}}^{\nu'}_{(\mathscr{F})}(h,\nabla\mathscr{F})$ are bounded by the right-hand side of (11.1.12). The terms $\slashed{\nabla}\rho$ and $\slashed{\nabla}\sigma$ appearing on the left-hand side of (9.1.8a) (see Remark 9.4) can be bounded by the second term on the right-hand side of (11.1.12) via Corollary 6.18. The remaining terms in (11.1.12) that need to be bounded can be expressed as $X_{\nu'}\mathscr{Q}^{\nu'}_{(1;\mathscr{F})}(h,\nabla\mathscr{F})$, $X_{\nu'}N_\triangle^{\#\beta\nu'\kappa\lambda}\nabla_\beta\mathscr{F}_{\kappa\lambda}$, and $X_{\nu'}\mathfrak{F}^{\nu'}$. The first of these can be bounded by using (11.2.7h) and the third with (11.1.11a) (in the case $|I| = 0$) while the second (with the help of Lemma 6.16) contributes to the cubic terms on the right-hand side of (11.1.12).

Our proof of (11.1.13a) is similar but more elaborate. To begin, we differentiate the electromagnetic equations with the iterated modified Lie derivative $\widehat{\mathscr{L}}^I_{\mathscr{Z}}$ to obtain the equations of variation (8.1.1a)–(8.1.1b) for $\dot{\mathscr{F}}_{\mu\nu} \overset{\text{def}}{=} \mathscr{L}^I_{\mathscr{Z}}\mathscr{F}_{\mu\nu}$ with inhomogeneous terms $\dot{\mathfrak{F}}^\nu = \mathfrak{F}^\nu_{(I)}$, where $\mathfrak{F}^\nu_{(I)}$ is defined in (8.1.2b). We then perform a null decomposition of the equations of variation, obtaining (9.1.8a) with $\underline{\dot{\alpha}}_\nu \overset{\text{def}}{=} \underline{\alpha}_\nu[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]$, $\dot{\mathfrak{F}}^{\nu'} \overset{\text{def}}{=} \mathfrak{F}^{\nu'}_{(I)}$, etc. Next, we multiply (9.1.8a) by $\varpi(q)$, use the identities $\nabla_\Lambda r = 1 - \frac{1}{4}h_{LL}$ and $\nabla_\Lambda q = -\frac{1}{2}h_{LL}$, and argue as above, removing the dangerous $-\frac{1}{4}h_{LL}\nabla_{\underline{L}}\underline{\alpha}_\nu[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]$ component from the quadratic term $\slashed{h}_{\nu\lambda}\mathscr{P}^\lambda_{(\mathscr{F})}(h,\nabla\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}) \overset{\text{def}}{=} \slashed{h}_\nu{}^\lambda h^{\mu\kappa}\nabla_\mu\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}_{\kappa\lambda}$ and denoting the remaining terms by $\slashed{h}_{\nu\lambda}\widetilde{\mathscr{P}}^\lambda_{(\mathscr{F})}(h,\nabla\mathscr{L}^I_{\mathscr{Z}}\mathscr{F})$, to deduce that $\varpi(q)(\nabla_{\underline{L}}\underline{\alpha}_\nu[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}] + \frac{1}{4}h_{LL}\nabla_{\underline{L}}\underline{\alpha}_\nu[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}] + r^{-1}\underline{\alpha}_\nu[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]) = r^{-1}\nabla_\Lambda(r\varpi(q)\underline{\alpha}_\nu[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]) + \frac{1}{4}r^{-1}\varpi(q)h_{LL}\underline{\alpha}_\nu[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}] + \frac{1}{2}\varpi'(q)h_{LL}\underline{\alpha}_\nu[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]$. The first of these three terms is the only term on the left-hand side of (11.1.13a) while the last two are brought over to the right-hand side of (11.1.13a). To bound $\slashed{h}_{\nu\nu'}\mathfrak{F}^{\nu'}_{(I)}$ by the right-hand side of (11.1.13a), we again set $X_{\nu'} \overset{\text{def}}{=} \slashed{h}_{\nu\nu'}$ (so that $|X|_{\mathscr{L}} = 0$); the

desired bound then follows from (11.1.11a) and (11.1.11b) together with repeated use of the inequality $|\mathcal{L}_{\mathscr{L}}^I \mathcal{F}| \lesssim |\underline{\alpha}[\mathcal{L}_{\mathscr{L}}^I \mathcal{F}]| + |\mathcal{L}_{\mathscr{L}}^I \mathcal{F}|_{\mathscr{L}\mathscr{N}} + |\mathcal{L}_{\mathscr{L}}^I \mathcal{F}|_{\mathscr{T}\mathscr{T}}$. The terms $\varpi(q)\, \slashed{\nabla}\rho[\mathcal{L}_{\mathscr{L}}^I \mathcal{F}]$ and $\varpi(q)\, \slashed{\nabla}\sigma[\mathcal{L}_{\mathscr{L}}^I \mathcal{F}]$ appearing on the left-hand side of (9.1.8a) (see Remark 9.4) can be bounded by the seventh sum on the right-hand side of (11.1.13a) with the help of Corollary 6.18. The remaining three terms on the left-hand side of (9.1.8a) to be estimated are $X_{\nu'}\widetilde{\mathscr{P}}_{(\mathscr{F})}^{\nu'}(h, \nabla\mathcal{L}_{\mathscr{L}}^I \mathcal{F})$, $X_{\nu'}\mathscr{Q}_{(1;\mathscr{F})}^{\nu'}(h, \nabla\mathcal{L}_{\mathscr{L}}^I \mathcal{F})$, and $X_{\nu'}N_{\triangle}^{\#\beta\nu'\kappa\lambda}\nabla_\beta\mathcal{L}_{\mathscr{L}}^I \mathcal{F}_{\kappa\lambda}$. The first of these can be bounded by using (11.2.7g) and the second with (11.2.7h) while the third (with the help of Lemma 6.16) contributes to the cubic terms on the right-hand side of (11.1.12).

The proofs of (11.1.13b)–(11.1.13d), which are based on an analysis of (9.1.8b)–(9.1.8d), are similar but much simpler. We will provide a brief overview of how to derive (11.1.13b); we then leave the remaining details to the reader. To begin, as in the previous paragraph, we differentiate the electromagnetic equations with the iterated modified Lie derivative $\widehat{\mathcal{L}}_{\mathscr{L}}^I$ and null-decompose the equations of variation. We use the same notation as in the previous paragraph and also the notation $\dot{\alpha}_\nu \overset{\text{def}}{=} \alpha_\nu[\mathcal{L}_{\mathscr{L}}^I \mathcal{F}]$. To derive inequality (11.1.13b), we will manipulate the equation (9.1.8b) satisfied by $\dot{\alpha}_\nu$. First, we rewrite the first two terms on the left-hand side of (9.1.8b) as $r^{-1}\nabla_{\underline{L}}(r\dot{\alpha})$. This term is the only one that appears on the left-hand side of (11.1.13b); all other terms are placed on the right-hand side. The only thing that remains to be discussed is how to bound these other terms from (9.1.8b) by the right-hand side of (11.1.13b). These terms separate into two classes: the linear terms involving angular derivatives $\slashed{\nabla}$ and the remaining nonlinear terms. As in the previous paragraph, the linear terms can be suitably bounded by the first sum on the right-hand side of (11.1.13b) thanks to Corollary 6.18. With the help of Lemma 6.16, the nonlinear terms can all bounded in the crudest possible fashion by estimates of, e.g., the form

$$\sum_{|I| \le k} |\nabla_{\mathscr{L}}^I(U\nabla V)| \lesssim (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2| \le k+1 \\ |I_1| \le k}} |\nabla_{\mathscr{L}}^{I_1} U||\nabla_{\mathscr{L}}^{I_2} V|. \qquad \square$$

The next proposition provides pointwise estimates for the challenging commutator term $\widetilde{\square}_g \nabla_{\mathscr{L}}^I h^{(1)} - \widehat{\nabla}_{\mathscr{L}}^I \widetilde{\square}_g h^{(1)}$ from the right-hand side of (7.0.1).

**Proposition 11.6** (Algebraic estimates of $[\widetilde{\square}_g, \nabla_{\mathscr{L}}^I]$ [Lindblad and Rodnianski 2010, Proposition 5.3]). *Let $g_{\mu\nu}$ be a Lorentzian metric, $h_{\mu\nu} \overset{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$, and $H^{\mu\nu} \overset{\text{def}}{=} (g^{-1})^{\mu\nu} - m^{\mu\nu}$. Let $\widetilde{\square}_g \overset{\text{def}}{=} \square_m + H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$, and let $I$ be a $\mathscr{L}$-multi-index with $1 \le |I|$. Let $\widehat{\nabla}_{\mathscr{L}}^I$ denote the modified Minkowskian covariant derivative operator defined in (6.4.1). Assume that there is a constant $\varepsilon$ such that $|\nabla_{\mathscr{L}}^J h| \le \varepsilon$ holds for all $\mathscr{L}$-multi-indices $J$ satisfying $|J| \le \lfloor |I|/2 \rfloor$. Then if $\varepsilon$ is sufficiently small, the following pointwise estimate holds:*

$$|\widetilde{\square}_g \nabla_{\mathscr{L}}^I h^{(1)} - \widehat{\nabla}_{\mathscr{L}}^I \widetilde{\square}_g h^{(1)}| \lesssim (1+t+|q|)^{-1} \sum_{|K| \le |I|} \sum_{|J|+(|K|-1)_+ \le |I|} |\nabla\nabla_{\mathscr{L}}^K h^{(1)}||\nabla_{\mathscr{L}}^J H|$$

$$+ (1+|q|)^{-1} \sum_{|K| \le |I|} \sum_{|J|+(|K|-1)_+ \le |I|} |\nabla\nabla_{\mathscr{L}}^K h^{(1)}||\nabla_{\mathscr{L}}^J H|_{\mathscr{L}\mathscr{L}}$$

$$+ (1+|q|)^{-1} \sum_{|K| \le |I|} \sum_{|J'|+(|K|-1)_+ \le |I|-1} |\nabla\nabla_{\mathscr{L}}^K h^{(1)}||\nabla_{\mathscr{L}}^{J'} H|_{\mathscr{L}\mathscr{T}}$$

$$+ (1+|q|)^{-1} \underbrace{\sum_{|K| \le |I|} \sum_{|J''|+(|K|-1)_+ \le |I|-2} |\nabla\nabla_{\mathscr{L}}^K h^{(1)}||\nabla_{\mathscr{L}}^{J''} H|}_{\text{absent if } |I| \le 1 \text{ or } |K| = |I|}, \qquad (11.1.15)$$

*where* $(|K| - 1)_+ \overset{\text{def}}{=} 0$ *if* $|K| = 0$ *and* $(|K| - 1)_+ \overset{\text{def}}{=} |K| - 1$ *if* $|K| \geq 1$.

**Corollary 11.7** (Algebraic estimates of $|\widetilde{\Box}_g \nabla_{\mathscr{L}}^I h^{(1)}|$). *Assume that* $h_{\mu\nu}^{(1)}$ ($\mu$, $\nu = 0, 1, 2, 3$) *is a solution to the reduced equation* (3.7.1a). *Then under the assumptions of Proposition 11.6, we have that*

$$
\begin{aligned}
|\widetilde{\Box}_g \nabla_{\mathscr{L}}^I h^{(1)}| \lesssim |\widehat{\nabla}_{\mathscr{L}}^I \mathfrak{H}| &+ |\widehat{\nabla}_{\mathscr{L}}^I \widetilde{\Box}_g h^{(0)}| + (1 + t + |q|)^{-1} \sum_{|K| \leq |I|} \sum_{|J| + (|K| - 1)_+ \leq |I|} |\nabla \nabla_{\mathscr{L}}^K h^{(1)}||\nabla_{\mathscr{L}}^J H| \\
&+ (1 + |q|)^{-1} \sum_{|K| \leq |I|} \sum_{|J| + (|K| - 1)_+ \leq |I|} |\nabla \nabla_{\mathscr{L}}^K h^{(1)}||\nabla_{\mathscr{L}}^J H|_{\mathscr{L}\mathscr{L}} \\
&+ (1 + |q|)^{-1} \sum_{|K| \leq |I|} \sum_{|J'| + (|K| - 1)_+ \leq |I| - 1} |\nabla \nabla_{\mathscr{L}}^K h^{(1)}||\nabla_{\mathscr{L}}^{J'} H|_{\mathscr{L}\mathscr{T}} \\
&+ (1 + |q|)^{-1} \underbrace{\sum_{|K| \leq |I|} \sum_{|J''| + (|K| - 1)_+ \leq |I| - 2} |\nabla \nabla_{\mathscr{L}}^K h^{(1)}||\nabla_{\mathscr{L}}^{J''} H|}_{\text{absent if } |I| \leq 1 \text{ or } |K| = |I|}.
\end{aligned}
\tag{11.1.16}
$$

*Proof.* Simply use Proposition 7.1 to decompose $\widetilde{\Box}_g \nabla_{\mathscr{L}}^I h^{(1)} = \widehat{\nabla}_{\mathscr{L}}^I \mathfrak{H} - \widehat{\nabla}_{\mathscr{L}}^I \widetilde{\Box}_g h^{(0)} + (\widetilde{\Box}_g \nabla_{\mathscr{L}}^I h^{(1)} - \widehat{\nabla}_{\mathscr{L}}^I \widetilde{\Box}_g h^{(1)})$ and apply Proposition 11.6. $\qquad\square$

**11.2. *Null-structure lemmas.*** In this section, we provide the lemmas that are used in the proofs of some of the previous propositions. We will make repeated use of the following decompositions of the Minkowski metric and its inverse:

$$
m_{\mu\nu} = -\tfrac{1}{2} L_\mu \underline{L}_\nu - \tfrac{1}{2} \underline{L}_\mu L_\nu + \slashed{m}_{\mu\nu},
\tag{11.2.1a}
$$

$$
(m^{-1})^{\mu\nu} = -\tfrac{1}{2} L^\mu \underline{L}^\nu - \tfrac{1}{2} \underline{L}^\mu L^\nu + \slashed{m}^{\mu\nu},
\tag{11.2.1b}
$$

where $\slashed{m}_{\mu\nu}$ is the Euclidean first fundamental form of the spheres $S_{r,t}$ defined in (5.1.4b).

We begin with a lemma that shows that the essential algebraic structure of the quadratic terms appearing on the right-hand sides of the reduced equations (3.7.1a)–(3.7.1c) is preserved under differentiation.

**Lemma 11.8** (Leibniz rules for the quadratic terms). *Let* $\mathfrak{Q}_0(\nabla\psi, \nabla\chi)$ *and* $\mathfrak{Q}_{\mu\nu}(\nabla\psi, \nabla\chi)$ *denote the standard null forms defined in* (3.6.6a)–(3.6.6b), *and let* $\mathfrak{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h)$, $\mathfrak{Q}_{\mu\nu}^{(2;h)}(\mathscr{F}, \mathscr{F})$, $\mathscr{P}(\nabla_\mu h, \nabla_\nu h)$, $\mathscr{P}_{(\mathscr{F})}^\nu(\nabla h, \mathscr{F})$, $\mathfrak{Q}_{(1;\mathscr{F})}^\nu(h, \nabla\mathscr{F})$, *and* $\mathfrak{Q}_{(2;\mathscr{F})}^\nu(h, \nabla\mathscr{F})$ *denote the quadratic terms defined in* (3.6.5), (3.7.2d), (3.6.4), (3.7.3b), (3.7.3c), *and* (3.7.2e), *respectively. Let* $I$ *be a* $\mathscr{L}$-*multi-index. Then there exist constants* $C_{I_1, I_2; \mu\nu}^{\kappa\lambda\gamma\gamma'\delta\delta'}$, $C_{I_1, I_2; \mu\nu}^{0;\gamma\gamma'\delta\delta'}$, $C_{I_1, I_2}$, $C_{\mathscr{P}; I_1, I_2; \mu\nu}^{\kappa\lambda}$, $C_{\mathscr{P}; I_1, I_2}$, *and* $C_{i; I_1, I_2}$ *such that*

$$
\begin{aligned}
\nabla_{\mathscr{L}}^I \mathfrak{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h) = \sum_{|I_1| + |I_2| \leq |I|} &C_{I_1, I_2; \mu\nu}^{\kappa\lambda\gamma\gamma'\delta\delta'} \mathfrak{Q}_{\kappa\lambda}(\nabla\nabla_{\mathscr{L}}^{I_1} h_{\gamma\gamma'}, \nabla\nabla_{\mathscr{L}}^{I_2} h_{\delta\delta'}) \\
&+ \sum_{|I_1| + |I_2| < |I|} C_{I_1, I_2; \mu\nu}^{0;\gamma\gamma'\delta\delta'} \mathfrak{Q}_0(\nabla\nabla_{\mathscr{L}}^{I_1} h_{\gamma\gamma'}, \nabla\nabla_{\mathscr{L}}^{I_2} h_{\delta\delta'}),
\end{aligned}
\tag{11.2.2a}
$$

$$
\nabla_{\mathscr{L}}^I \mathfrak{Q}_{\mu\nu}^{(2;h)}(\mathscr{F}, \mathscr{F}) = \sum_{|I_1| + |I_2| \leq |I|} C_{I_1, I_2} \mathfrak{Q}_{\mu\nu}^{(2;h)}(\nabla_{\mathscr{L}}^{I_1} \mathscr{F}, \nabla_{\mathscr{L}}^{I_2} \mathscr{F}),
\tag{11.2.2b}
$$

$$
\nabla_{\mathscr{L}}^I \mathscr{P}(\nabla_\mu h, \nabla_\nu h) = \sum_{|I_1| + |I_2| \leq |I|} C_{\mathscr{P}; I_1, I_2; \mu\nu}^{\kappa\lambda} \mathscr{P}(\nabla_\kappa \nabla_{\mathscr{L}}^{I_1} h, \nabla_\lambda \nabla_{\mathscr{L}}^{I_2} h),
\tag{11.2.2c}
$$

$$\mathscr{L}_{\mathscr{Z}}^{I}\mathscr{P}_{(\mathscr{F})}^{\nu}(\nabla h, \mathscr{F}) = \sum_{|I_1|+|I_2|\le|I|} C_{\mathscr{P};I_1,I_2}\mathscr{P}_{(\mathscr{F})}^{\nu}(\nabla\mathscr{L}_{\mathscr{Z}}^{I_1}h, \mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}), \tag{11.2.2d}$$

$$\mathscr{L}_{\mathscr{Z}}^{I}\mathscr{Q}_{(i;\mathscr{F})}^{\nu}(h, \nabla\mathscr{F}) = \sum_{|I_1|+|I_2|\le|I|} C_{i;I_1,I_2}\mathscr{Q}_{(i;\mathscr{F})}^{\nu}(\mathscr{L}_{\mathscr{Z}}^{I_1}h, \nabla\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}) \quad (i = 1, 2). \tag{11.2.2e}$$

*Proof.* By pure calculation, if $Z \in \mathscr{Z}$, then the following identity holds for the standard null form $\mathscr{Q}_{\mu\nu}(\nabla\psi, \nabla\chi)$:

$$\nabla_Z\mathscr{Q}_{\mu\nu}(\nabla\psi, \nabla\chi) = \mathscr{Q}_{\mu\nu}(\nabla\nabla_Z\psi, \nabla\chi) + \mathscr{Q}_{\mu\nu}(\nabla\psi, \nabla\nabla_Z\chi)$$
$$- {}^{(Z)}c_\mu{}^\kappa\mathscr{Q}_{\kappa\nu}(\nabla\psi, \nabla\chi) - {}^{(Z)}c_\nu{}^\kappa\mathscr{Q}_{\mu\kappa}(\nabla\psi, \nabla\chi), \tag{11.2.3}$$

where ${}^{(Z)}c_{\mu\nu}$ is the covariantly constant tensor field defined in (6.2.4). A similar identity holds for the standard null form $\mathscr{Q}_0(\nabla\psi, \nabla\chi)$. Equation (11.2.2a) now follows inductively from these facts and the Leibniz rule since $\mathscr{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h)$ is a linear combination of standard null forms. Equation (11.2.2c) follows similarly. Equation (11.2.2b) follows trivially from definition (3.7.2d) and the Leibniz rule. Equations (11.2.2d) and (11.2.2e) follow from (6.3.4b), Lemma 6.8, and the Leibniz rule. $\square$

The next lemma concerns the null structure of the standard null forms.

**Lemma 11.9** (Null structure estimates for the standard null forms). *Let* $\mathscr{Q}_0(\nabla\psi, \nabla\chi) \overset{\text{def}}{=} (m^{-1})^{\kappa\lambda}(\nabla_\kappa\psi) \cdot (\nabla_\lambda\chi)$ *and* $\mathscr{Q}_{\mu\nu}(\nabla\psi, \nabla\chi) \overset{\text{def}}{=} (\nabla_\mu\psi)(\nabla_\nu\chi) - (\nabla_\nu\psi)(\nabla_\mu\chi)$ *denote the standard null forms defined in* (3.6.6a)–(3.6.6b). *Then*

$$|\mathscr{Q}_0(\nabla\psi, \nabla\chi)| + |\mathscr{Q}_{\mu\nu}(\nabla\psi, \nabla\chi)| \lesssim |\overline{\nabla}\psi||\nabla\chi| + |\overline{\nabla}\chi||\nabla\psi|. \tag{11.2.4}$$

*Proof.* The estimate (11.2.4) for $\mathscr{Q}_0$ easily follows from using (11.2.1b) to decompose $(m^{-1})^{\kappa\lambda}$. To obtain the estimates for $\mathscr{Q}_{\mu\nu}(\nabla\psi, \nabla\chi)$, we first consider the $\mathscr{Q}_{\mu\nu}(\nabla\psi, \nabla\chi)$ to be components of a 2-covariant tensor $\mathscr{Q}(\nabla\psi, \nabla\chi)$. Inequality (11.2.4) is equivalent to the following inequality:

$$|\mathscr{Q}(\nabla\psi, \nabla\chi)|_{\mathscr{N}\mathscr{N}} \lesssim |\overline{\nabla}\psi||\nabla\chi| + |\overline{\nabla}\chi||\nabla\psi|. \tag{11.2.5}$$

Contracting $\mathscr{Q}_{\mu\nu}(\nabla\psi, \nabla\chi)$ against frame vectors $N^\mu, N^\nu \in \mathscr{N}$, we see that the only component on the left-hand side of (11.2.5) that could pose any difficulty is $\underline{L}^\mu\underline{L}^\nu\mathscr{Q}_{\mu\nu}(\nabla\psi, \nabla\chi)$. But the antisymmetry of the $\mathscr{Q}_{\mu\nu}(\cdot, \cdot)$ implies that this component is 0. $\square$

The next lemma addresses the null structure of some of the terms appearing in the reduced equations (3.7.1a)–(3.7.1c).

**Lemma 11.10** (Null structure estimates for the reduced equations). *Let* $\mathscr{P}(\nabla_\mu\Pi, \nabla_\nu\Theta), \mathscr{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h),$ $\mathscr{Q}_{\mu\nu}^{(2;h)}(\mathscr{F}, \mathscr{G}), \mathscr{P}_{(\mathscr{F})}^{\nu}(h, \nabla\mathscr{F}), \mathscr{Q}_{(1;\mathscr{F})}^{\nu}(h, \nabla\mathscr{F}),$ *and* $\mathscr{Q}_{(2;\mathscr{F})}^{\nu}(\nabla h, \mathscr{F})$ *be the quadratic forms defined in Section 3.7, and define the quadratic form* $\widetilde{\mathscr{P}}_{(\mathscr{F})}^{\nu}(h, \nabla\mathscr{F})$ *by removing the* $\nabla_{\underline{L}}\underline{\alpha}^\nu[\mathscr{F}]$*-containing component of* $\mathscr{P}_{(\mathscr{F})}^{\nu}(h, \nabla\mathscr{F})$:

$$\widetilde{\mathscr{P}}_{(\mathscr{F})}^{\nu}(h, \nabla\mathscr{F}) \overset{\text{def}}{=} \mathscr{P}_{(\mathscr{F})}^{\nu}(h, \nabla\mathscr{F}) - \tfrac{1}{4}h_{LL}\slashed{h}^{\nu\nu'}\nabla_{\underline{L}}\mathscr{F}_{\underline{L}\nu'}$$
$$= \mathscr{P}_{(\mathscr{F})}^{\nu}(h, \nabla\mathscr{F}) + \tfrac{1}{4}h_{LL}\nabla_{\underline{L}}\underline{\alpha}^\nu[\mathscr{F}]. \tag{11.2.6}$$

*Let $X_\nu$ be any one-form, let $\Pi_{\mu\nu}$ and $\Theta_{\mu\nu}$ be symmetric type-$\binom{0}{2}$ tensor fields, and let $\mathscr{F}_{\mu\nu}$ and $\mathscr{G}_{\mu\nu}$ be two-forms. Then the following pointwise inequalities hold:*

$$|\mathscr{P}(\nabla_\mu\Pi, \nabla_\nu\Theta)| \lesssim |\nabla\Pi|_{\mathscr{T}\mathscr{N}}|\nabla\Theta|_{\mathscr{T}\mathscr{N}}$$
$$+ |\nabla\Pi|_{\mathscr{L}\mathscr{L}}|\nabla\Theta| + |\Pi||\nabla\Theta|_{\mathscr{L}\mathscr{L}} \quad (\mu, \nu = 0, 1, 2, 3), \quad (11.2.7\text{a})$$

$$\sum_{T\in\mathscr{T},\ N\in\mathscr{N}} |T^\mu N^\nu \mathscr{P}(\nabla_\mu\Pi, \nabla_\nu\Theta)| \lesssim |\overline{\nabla}\Pi||\nabla\Theta|, \quad (11.2.7\text{b})$$

$$|\mathscr{Q}^{(1;h)}_{\mu\nu}(\nabla\Pi, \nabla\Theta)| \lesssim |\overline{\nabla}\Pi||\nabla\Theta| + |\nabla\Pi||\overline{\nabla}\Theta| \quad (\mu, \nu = 0, 1, 2, 3), \quad (11.2.7\text{c})$$

$$\sum_{T\in\mathscr{T},\ N\in\mathscr{N}} |T^\mu N^\nu \mathscr{Q}^{(2;h)}_{\mu\nu}(\mathscr{F}, \mathscr{G})| \lesssim (|\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{F}|_{\mathscr{T}\mathscr{T}})|\mathscr{G}| + |\mathscr{F}|(|\mathscr{G}|_{\mathscr{L}\mathscr{N}} + |\mathscr{G}|_{\mathscr{T}\mathscr{T}}), \quad (11.2.7\text{d})$$

$$|\mathscr{Q}^{(2;h)}_{\mu\nu}(\mathscr{F}, \mathscr{G})| \lesssim |\mathscr{F}||\mathscr{G}| \quad (\mu, \nu = 0, 1, 2, 3), \quad (11.2.7\text{e})$$

$$|X_\nu \mathscr{P}^\nu_{(\mathscr{F})}(h, \nabla\mathscr{F})| \lesssim |X||h||\overline{\nabla}\mathscr{F}| + |X||h|(|\nabla\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\nabla\mathscr{F}|_{\mathscr{T}\mathscr{T}}) + |X||h|_{\mathscr{L}\mathscr{L}}|\nabla\mathscr{F}| + |X|_{\mathscr{L}}|h||\nabla\mathscr{F}|$$
$$\lesssim (1 + t + |q|)^{-1} \sum_{|I|\leq 1} |X||h||\mathscr{L}^I_{\mathscr{L}}\mathscr{F}|$$
$$+ (1 + |q|)^{-1} \sum_{|I|\leq 1} |X||h|(|\mathscr{L}^I_{\mathscr{L}}\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}^I_{\mathscr{L}}\mathscr{F}|_{\mathscr{T}\mathscr{T}})$$
$$+ (1 + |q|)^{-1} \sum_{|I|\leq 1} |X||h|_{\mathscr{L}\mathscr{L}}|\mathscr{L}^I_{\mathscr{L}}\mathscr{F}|$$
$$+ (1 + |q|)^{-1} \sum_{|I|\leq 1} |X|_{\mathscr{L}}|h||\mathscr{L}^I_{\mathscr{L}}\mathscr{F}|, \quad (11.2.7\text{f})$$

$$|X_\nu \widetilde{\mathscr{P}}^\nu_{(\mathscr{F})}(h, \nabla\mathscr{F})| \lesssim |X||h||\overline{\nabla}\mathscr{F}| + |X||h|(|\nabla\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\nabla\mathscr{F}|_{\mathscr{T}\mathscr{T}}) + |X|_{\mathscr{L}}|h||\nabla\mathscr{F}|$$
$$\lesssim (1 + t + |q|)^{-1} \sum_{|I|\leq 1} |X||h||\mathscr{L}^I_{\mathscr{L}}\mathscr{F}|$$
$$+ (1 + |q|)^{-1} \sum_{|I|\leq 1} |X||h|(|\mathscr{L}^I_{\mathscr{L}}\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}^I_{\mathscr{L}}\mathscr{F}|_{\mathscr{T}\mathscr{T}})$$
$$+ (1 + |q|)^{-1} \sum_{|I|\leq 1} |X|_{\mathscr{L}}|h||\mathscr{L}^I_{\mathscr{L}}\mathscr{F}|, \quad (11.2.7\text{g})$$

$$|X_\nu \mathscr{Q}^\nu_{(1;\mathscr{F})}(h, \nabla\mathscr{F})| \lesssim |X||h||\overline{\nabla}\mathscr{F}| + |X||h|(|\nabla\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\nabla\mathscr{F}|_{\mathscr{T}\mathscr{T}})$$
$$\lesssim (1 + t + |q|)^{-1} \sum_{|I|\leq 1} |X||h||\mathscr{L}^I_{\mathscr{L}}\mathscr{F}|$$
$$+ (1 + |q|)^{-1} \sum_{|I|\leq 1} |X||h|(|\mathscr{L}^I_{\mathscr{L}}\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}^I_{\mathscr{L}}\mathscr{F}|_{\mathscr{T}\mathscr{T}}), \quad (11.2.7\text{h})$$

$$|X_\nu \mathscr{Q}^\nu_{(2;\mathscr{F})}(\nabla h, \mathscr{F})| \lesssim |X||\overline{\nabla}h||\mathscr{F}| + |X||\nabla h||\mathscr{F}|_{\mathscr{L}\mathscr{N}}$$
$$\lesssim (1 + t + |q|)^{-1} \sum_{|I|\leq 1} |X||\nabla^I_{\mathscr{L}}h||\mathscr{F}|$$
$$+ (1 + |q|)^{-1} \sum_{|I|\leq 1} |X||\nabla^I_{\mathscr{L}}h|(|\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{F}|_{\mathscr{T}\mathscr{T}}). \quad (11.2.7\text{i})$$

*Proof.* Inequality (11.2.7c) follows directly from Lemma 11.9 since $\mathfrak{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h)$ is a linear combination of standard null forms. Inequality (11.2.7e) is trivial while (11.2.7a), (11.2.7b), and the first inequalities in (11.2.7d)–(11.2.7i) are straightforward to verify by using (11.2.1a)–(11.2.1b). The second inequalities in (11.2.7d)–(11.2.7i) then follow from the first ones, Lemma 6.16, and Proposition 6.19. $\qquad\square$

The next lemma addresses the null structure of some of the cubic terms on the right-hand side of (12.2.4).

**Lemma 11.11** (Null structure estimates for quasilinear wave equations [Lindblad and Rodnianski 2010, Lemma 4.2]). *Let $\Pi$ be a type-$\binom{0}{2}$ tensor field, and let $\phi$ be a scalar function. Then the following inequalities hold*:

$$|\Pi^{\kappa\lambda}(\nabla_\kappa\phi)(\nabla_\lambda\phi)| \lesssim |\Pi|_{\mathscr{L}\mathscr{L}}|\nabla\phi|^2 + |\Pi||\overline{\nabla}\phi||\nabla\phi|, \tag{11.2.8a}$$

$$|L_\kappa\Pi^{\kappa\lambda}\nabla_\lambda\phi| \lesssim |\Pi|_{\mathscr{L}\mathscr{L}}|\nabla\phi| + |\Pi||\overline{\nabla}\phi|, \tag{11.2.8b}$$

$$|(\nabla_\kappa\Pi^{\kappa\lambda})\nabla_\lambda\phi| \lesssim |\nabla\Pi|_{\mathscr{L}\mathscr{L}}|\nabla\phi| + |\overline{\nabla}\Pi||\nabla\phi| + |\nabla\Pi||\overline{\nabla}\phi|, \tag{11.2.8c}$$

$$|\Pi^{\kappa\lambda}\nabla_\kappa\nabla_\lambda\phi| \lesssim |\Pi|_{\mathscr{L}\mathscr{L}}|\nabla\nabla\phi| + |\overline{\nabla}\nabla\phi|. \tag{11.2.8d}$$

The following lemma addresses the null structure of some of the cubic terms on the right-hand side of (12.2.8):

**Lemma 11.12** (Null structure estimates for the terms appearing in the divergence of the electromagnetic energy currents). *Let $h_{\mu\nu}$ be a type-$\binom{0}{2}$ tensor field, and let $\mathscr{F}_{\mu\nu}$ be a two-form. Then the following inequalities hold*:

$$|(\nabla_\mu h^{\mu\kappa})\mathscr{F}_{\kappa\zeta}\mathscr{F}_0^{\ \zeta}| \lesssim |\nabla h|_{\mathscr{L}\mathscr{L}}|\mathscr{F}|^2 + |\overline{\nabla}h||\mathscr{F}|^2 + |\nabla h||\mathscr{F}|(|\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{F}|_{\mathscr{T}\mathscr{T}}), \tag{11.2.9a}$$

$$|(\nabla_\mu h^{\kappa\lambda})\mathscr{F}^\mu_{\ \kappa}\mathscr{F}_{0\lambda}| \lesssim |\overline{\nabla}h||\mathscr{F}|^2 + |\nabla h||\mathscr{F}|(|\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{F}|_{\mathscr{T}\mathscr{T}}), \tag{11.2.9b}$$

$$|(\nabla_t h^{\kappa\lambda})\mathscr{F}_{\kappa\eta}\mathscr{F}_\lambda^{\ \eta}| \lesssim |\nabla h|_{\mathscr{L}\mathscr{L}}|\mathscr{F}|^2 + |\nabla h||\mathscr{F}|(|\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{F}|_{\mathscr{T}\mathscr{T}}), \tag{11.2.9c}$$

$$|L_\mu h^{\mu\kappa}\mathscr{F}_{\kappa\zeta}\mathscr{F}_0^{\ \zeta}| \lesssim |h|_{\mathscr{L}\mathscr{L}}|\mathscr{F}|^2 + |h||\mathscr{F}|(|\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{F}|_{\mathscr{T}\mathscr{T}}), \tag{11.2.9d}$$

$$|L_\mu h^{\kappa\lambda}\mathscr{F}^\mu_{\ \kappa}\mathscr{F}_{0\lambda}| \lesssim |h||\mathscr{F}||\mathscr{F}|_{\mathscr{L}\mathscr{N}}, \tag{11.2.9e}$$

$$|h^{\kappa\lambda}\mathscr{F}_{\kappa\eta}\mathscr{F}_\lambda^{\ \eta}| \lesssim |h|_{\mathscr{L}\mathscr{L}}|\mathscr{F}|^2 + |h||\mathscr{F}|(|\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{F}|_{\mathscr{T}\mathscr{T}}). \tag{11.2.9f}$$

*Proof.* It is straightforward to derive inequalities (11.2.9a)–(11.2.9f) by using (11.2.1a). $\qquad\square$

## 12. Weighted energy estimates for the electromagnetic equations of variation and for systems of nonlinear wave equations in a curved spacetime

In this section, we prove weighted energy estimates for the electromagnetic equations of variation

$$\nabla_\lambda\dot{\mathscr{F}}_{\mu\nu} + \nabla_\mu\dot{\mathscr{F}}_{\nu\lambda} + \nabla_\nu\dot{\mathscr{F}}_{\lambda\mu} = \dot{\mathfrak{F}}_{\lambda\mu\nu} \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{12.0.1a}$$

$$N^{\#\mu\nu\kappa\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda} = \dot{\mathfrak{F}}^\nu \qquad (\nu = 0, 1, 2, 3). \tag{12.0.1b}$$

Our estimates complement the weighted energy estimates proved in [Lindblad and Rodnianski 2010] for the inhomogeneous wave equation

$$\widetilde{\Box}_g\phi = \mathfrak{I} \tag{12.0.2}$$

and for tensorial systems of inhomogeneous wave equations with principal part $\widetilde{\Box}_g$:

$$\widetilde{\Box}_g \phi_{\mu\nu} = \mathfrak{I}_{\mu\nu} \quad (\mu, \nu = 0, 1, 2, 3). \tag{12.0.3}$$

**12.1. *The energy estimate weight function $w(q)$*.** As in [Lindblad and Rodnianski 2010], our energy estimates will involve the weight function $w(q)$ defined by

$$w = w(q) = \begin{cases} 1 + (1 + |q|)^{1+2\gamma} & \text{if } q > 0, \\ 1 + (1 + |q|)^{-2\mu} & \text{if } q < 0, \end{cases} \tag{12.1.1}$$

where the constants $\gamma$ and $\mu$ are subject to the restrictions stated in Section 2.14.

Observe that the following inequalities follow from the definition (12.1.1):

$$w' \leq 4(1 + |q|)^{-1} w \leq 16\mu^{-1}(1 + q_-)^{2\mu} w', \tag{12.1.2}$$

where $q_- = 0$ if $q \geq 0$ and $q_- = |q|$ if $q < 0$.

**12.2. *Weighted energy estimates*.** We begin by deriving weighted energy estimates for the electromagnetic equations of variation.

**Lemma 12.1** (Weighted energy estimates for $\dot{\mathscr{F}}$). *Assume that $\dot{\mathscr{F}}_{\mu\nu}$ is a solution to the equations of variation* (8.1.1a)–(8.1.1b) *corresponding to the background* $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$, *where* $h_{\mu\nu} \overset{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$. *Let* $\dot{\alpha} \overset{\text{def}}{=} \alpha[\dot{\mathscr{F}}]$, $\dot{\rho} \overset{\text{def}}{=} \rho[\dot{\mathscr{F}}]$, *and* $\dot{\sigma} \overset{\text{def}}{=} \sigma[\dot{\mathscr{F}}]$ *denote the "favorable" Minkowskian null components of $\dot{\mathscr{F}}$ as defined in Definition 5.9. Assume that $|h| + |\mathscr{F}| \leq \varepsilon$. Then if $\varepsilon$ is sufficiently small and $t_1 \leq t_2$, the following integral inequality holds*:

$$\int_{\Sigma_{t_2}} |\dot{\mathscr{F}}|^2 w(q) \, d^3 x + \int_{t_1}^{t_2} \int_{\Sigma_\tau} \left( |\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2 \right) w'(q) \, d^3 x \, d\tau$$

$$\lesssim \int_{\Sigma_{t_1}} |\dot{\mathscr{F}}|^2 w(q) \, d^3 x + \int_{t_1}^{t_2} \int_{\Sigma_\tau} \left| \dot{\mathscr{F}}_{0\eta} \dot{\mathfrak{F}}^\eta \right| w(q) \, d^3 x \, d\tau$$

$$+ \int_{t_1}^{t_2} \int_{\Sigma_\tau} \left| -(\nabla_\mu h^{\mu\kappa}) \dot{\mathscr{F}}_{\kappa\zeta} \dot{\mathscr{F}}_0{}^\zeta - (\nabla_\mu h^{\kappa\lambda}) \dot{\mathscr{F}}^\mu{}_\kappa \dot{\mathscr{F}}_{0\lambda} + \tfrac{1}{2}(\nabla_t h^{\kappa\lambda}) \dot{\mathscr{F}}_{\kappa\eta} \dot{\mathscr{F}}_\lambda{}^\eta \right| w(q) \, d^3 x \, d\tau$$

$$+ \int_{t_1}^{t_2} \int_{\Sigma_\tau} \left| L_\mu h^{\mu\kappa} \dot{\mathscr{F}}_{\kappa\zeta} \dot{\mathscr{F}}_0{}^\zeta + L_\mu h^{\kappa\lambda} \dot{\mathscr{F}}^\mu{}_\kappa \dot{\mathscr{F}}_{0\lambda} + \tfrac{1}{2} h^{\kappa\lambda} \dot{\mathscr{F}}_{\kappa\eta} \dot{\mathscr{F}}_\lambda{}^\eta \right| w'(q) \, d^3 x \, d\tau$$

$$+ \int_{t_1}^{t_2} \int_{\Sigma_\tau} \left| (\nabla_\mu N_\triangle^{\#\mu\zeta\kappa\lambda}) \dot{\mathscr{F}}_{\kappa\lambda} \dot{\mathscr{F}}_{0\zeta} - \tfrac{1}{4}(\nabla_t N_\triangle^{\#\zeta\eta\kappa\lambda}) \dot{\mathscr{F}}_{\zeta\eta} \dot{\mathscr{F}}_{\kappa\lambda} \right| w(q) \, d^3 x \, d\tau$$

$$+ \int_{t_1}^{t_2} \int_{\Sigma_\tau} \left| L_\mu N_\triangle^{\#\mu\zeta\kappa\lambda} \dot{\mathscr{F}}_{\kappa\lambda} \dot{\mathscr{F}}_{0\zeta} + \tfrac{1}{4} N_\triangle^{\#\zeta\eta\kappa\lambda} \dot{\mathscr{F}}_{\zeta\eta} \dot{\mathscr{F}}_{\kappa\lambda} \right| w'(q) \, d^3 x \, d\tau. \tag{12.2.1}$$

*Proof.* It follows from (8.3.2) that, if $\varepsilon$ is sufficiently small, we have that

$$\tfrac{1}{4} |\dot{\mathscr{F}}|^2 w(q) \leq j^0_{(h,\mathscr{F})} \leq |\dot{\mathscr{F}}|^2 w(q). \tag{12.2.2}$$

From (8.3.3) and the divergence theorem, it follows that

$$\int_{\Sigma_{t_2}} j^0_{(h,\mathscr{F})}\, d^3x + \frac{1}{2}\int_{t_1}^{t_2}\int_{\Sigma_\tau} w'(q)(|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2)\, d^3x\, d\tau$$

$$= \int_{\Sigma_{t_1}} j^0_{(h,\mathscr{F})}\, d^3x - \int_{t_1}^{t_2}\int_{\Sigma_\tau} w(q)\dot{\bar{\mathscr{F}}}_{0\eta}\dot{\mathfrak{s}}^\eta\, d^3x\, d\tau$$

$$- \int_{t_1}^{t_2}\int_{\Sigma_\tau} w(q)\big\{-(\nabla_\mu h^{\mu\kappa})\dot{\bar{\mathscr{F}}}_{\kappa\zeta}\dot{\mathscr{F}}_0{}^\zeta - (\nabla_\mu h^{\kappa\lambda})\dot{\bar{\mathscr{F}}}^\mu{}_\kappa\dot{\mathscr{F}}_{0\lambda} + \tfrac{1}{2}(\nabla_t h^{\kappa\lambda})\dot{\bar{\mathscr{F}}}_{\kappa\eta}\dot{\mathscr{F}}_\lambda{}^\eta\big\}\, d^3x\, d\tau$$

$$- \int_{t_1}^{t_2}\int_{\Sigma_\tau} w'(q)\big\{-L_\mu h^{\mu\kappa}\dot{\bar{\mathscr{F}}}_{\kappa\zeta}\dot{\mathscr{F}}_0{}^\zeta - L_\mu h^{\kappa\lambda}\dot{\bar{\mathscr{F}}}^\mu{}_\kappa\dot{\mathscr{F}}_{0\lambda} - \tfrac{1}{2}h^{\kappa\lambda}\dot{\bar{\mathscr{F}}}_{\kappa\eta}\dot{\mathscr{F}}_\lambda{}^\eta\big\}\, d^3x\, d\tau$$

$$- \int_{t_1}^{t_2}\int_{\Sigma_\tau} w(q)\big\{(\nabla_\mu N_\triangle^{\#\mu\zeta\kappa\lambda})\dot{\bar{\mathscr{F}}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} - \tfrac{1}{4}(\nabla_t N_\triangle^{\#\zeta\eta\kappa\lambda})\dot{\bar{\mathscr{F}}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\big\}\, d^3x\, d\tau$$

$$- \int_{t_1}^{t_2}\int_{\Sigma_\tau} w'(q)\big\{L_\mu N_\triangle^{\#\mu\zeta\kappa\lambda}\dot{\bar{\mathscr{F}}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} + \tfrac{1}{4}N_\triangle^{\#\zeta\eta\kappa\lambda}\dot{\bar{\mathscr{F}}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\big\}\, d^3x\, d\tau, \qquad (12.2.3)$$

which, with the help of (12.2.2), implies (12.2.1). □

We now recall the analogous lemma proved in [Lindblad and Rodnianski 2010] for solutions to the inhomogeneous wave equation in curved spacetime.

**Lemma 12.2** (Weighted energy estimates for a scalar wave equation [Lindblad and Rodnianski 2010, Lemma 6.1]). *Assume that the scalar-valued function $\phi$ is a solution to the equation $\widetilde{\Box}_g\phi = \mathfrak{I}$, and let $H^{\mu\nu}\overset{\text{def}}{=}(g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$. Assume that the metric $g_{\mu\nu}$ is such that $|H|\leq\frac{1}{2}$. Then*

$$\int_{\Sigma_{t_2}}|\nabla\phi|^2 w(q)\, d^3x + 2\int_{t_1}^{t_2}\int_{\Sigma_\tau}|\overline{\nabla}\phi|^2 w'(q)\, d^3x\, d\tau$$

$$\leq 4\int_{\Sigma_{t_1}}|\nabla\phi|^2 w(q)\, d^3x + 4\int_{t_1}^{t_2}\int_{\Sigma_\tau}|\mathfrak{I}_\kappa\nabla_t\phi^\kappa|w(q)\, d^3x\, d\tau$$

$$+ 4\int_{t_1}^{t_2}\int_{\Sigma_\tau}\big|(\nabla_\nu H^{\nu\lambda})(\nabla_\lambda\phi)(\nabla_t\phi) - \tfrac{1}{2}(\nabla_t H^{\lambda\kappa})(\nabla_\lambda\phi)(\nabla_\kappa\phi)\big|w(q)\, d^3x\, d\tau$$

$$+ 4\int_{t_1}^{t_2}\int_{\Sigma_\tau}\big|\underbrace{(\omega_j H^{j\lambda} - H^{0\lambda})}_{L_\kappa H^{\kappa\lambda}}(\nabla_t\phi)(\nabla_\lambda\phi) + \tfrac{1}{2}H^{\lambda\kappa}(\nabla_\lambda\phi)(\nabla_\kappa\phi)\big|w'(q)\, d^3x\, d\tau. \quad (12.2.4)$$

We now extend the results of the previous lemmas by estimating (under assumptions that are compatible with our global stability theorem) some of the cubic terms on the right-hand sides of (12.2.1) and (12.2.4).

**Proposition 12.3** (Weighted energy estimates for the reduced equations; extension of [Lindblad and Rodnianski 2010, Proposition 6.2]). *Let $\phi$ be a solution to $\widetilde{\Box}_g\phi = \mathfrak{I}$ for the metric $g_{\mu\nu}$, and let $H^{\mu\nu}\overset{\text{def}}{=}(g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$. Let $\gamma$ and $\mu$ be positive constants satisfying the restrictions described*

*in Section 2.14. Assume that the following pointwise estimates hold for $(t, x) \in [0, T) \times \mathbb{R}^3$:*

$$(1+|q|)^{-1}|H|_{LL} + |\nabla H|_{LL} + |\overline{\nabla} H| \le C\varepsilon(1+t+|q|)^{-1}, \tag{12.2.5a}$$

$$(1+|q|)^{-1}|H| + |\nabla H| \le C\varepsilon(1+t+|q|)^{-1/2}(1+|q|)^{-1/2}(1+q_-)^{-\mu}, \tag{12.2.5b}$$

*where $q_- = 0$ if $q \ge 0$ and $q_- = |q|$ if $q < 0$. Then there exists a constant $C_1 > 0$ such that, if $0 < \varepsilon \le \mu/C_1$, then the following integral inequality holds for $t \in [0, T)$:*

$$\int_{\Sigma_t} |\nabla\phi|^2 w(q)\, d^3x + \int_0^t \int_{\Sigma_\tau} |\overline{\nabla}\phi|^2 w'(q)\, d^3x\, d\tau$$

$$\lesssim \int_{\Sigma_0} |\nabla\phi|^2 w(q)\, d^3x + \int_0^t \int_{\Sigma_\tau} \left( \frac{C\varepsilon|\nabla\phi|^2}{1+\tau} + |\mathfrak{I}||\nabla\phi| \right) w(q)\, d^3x\, d\tau. \tag{12.2.6}$$

*Furthermore, let $\dot{\mathscr{F}}_{\mu\nu}$ be a solution to the electromagnetic equations of variation (8.1.1a)–(8.1.1b) corresponding to the background $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$, where $h_{\mu\nu} \overset{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$. Assume that the following pointwise estimates hold for $(t, x) \in [0, T) \times \mathbb{R}^3$:*

$$(1+|q|)^{-1}|h|_{\mathscr{L}\mathscr{L}} + |\nabla h|_{\mathscr{L}\mathscr{L}} + |\overline{\nabla} h| + |\mathscr{F}| \le C\varepsilon(1+t+|q|)^{-1}, \tag{12.2.7a}$$

$$(1+|q|)^{-1}|h| + |\nabla h| + |\nabla \mathscr{F}| \le C\varepsilon(1+t+|q|)^{-1/2}(1+|q|)^{-1/2}(1+q_-)^{-\mu}, \tag{12.2.7b}$$

*where $q_- = 0$ if $q \ge 0$ and $q_- = |q|$ if $q < 0$. Then there exists a constant $C_1 > 0$ such that, if $0 < \varepsilon \le \mu/C_1$, then the following integral inequality holds for $t \in [0, T)$:*

$$\int_{\Sigma_t} |\dot{\mathscr{F}}|^2 w(q)\, d^3x + \int_0^t \int_{\Sigma_\tau} \left( |\dot{\mathscr{F}}|^2_{\mathscr{L}\mathscr{N}} + |\dot{\mathscr{F}}|^2_{\mathscr{T}\mathscr{T}} \right) w'(q)\, d^3x\, d\tau$$

$$\lesssim \int_{\Sigma_0} |\dot{\mathscr{F}}|^2 w(q)\, d^3x + \varepsilon \int_0^t \int_{\Sigma_\tau} \frac{|\dot{\mathscr{F}}|^2}{1+\tau} w(q)\, d^3x\, d\tau + \int_0^t \int_{\Sigma_\tau} |\dot{\mathscr{F}}_{0\kappa} \dot{\mathfrak{F}}^\kappa| w(q)\, d^3x\, d\tau. \tag{12.2.8}$$

**Remark 12.4.** Proposition 12.3 will not be used until the proof of Theorem 16.1, where it plays a key role; see Section 16.2. We also remark that the hypotheses of the proposition are implied by the hypotheses of the theorem; see Section 2.14 and Remark 16.2.

*Proof.* Inequality (12.2.6) was proved as Proposition 6.2 of [Lindblad and Rodnianski 2010]. The proof was based on using Lemma 11.11 to estimate the inhomogeneous terms on the right-hand side of (12.2.4). Rather than reproving this inequality, we only give the proof of (12.2.8), which is based on (12.2.1) and uses related ideas.

We commence with the proof of (12.2.8), our goal being to deduce suitable pointwise bounds for some of the terms appearing on the right-hand side of (12.2.1). For the cubic terms, we use Lemma 11.12, the hypotheses of the proposition, and the inequality $|ab| \lesssim a^2 + b^2$ to conclude that

$$\left| (\nabla_\mu h^{\mu\kappa}) \dot{\mathscr{F}}_{\kappa\zeta} \dot{\mathscr{F}}_0{}^\zeta - (\nabla_\mu h^{\kappa\lambda}) \dot{\mathscr{F}}^\mu{}_\kappa \dot{\mathscr{F}}_{0\lambda} + \tfrac{1}{2} (\nabla_t h^{\kappa\lambda}) \dot{\mathscr{F}}_{\kappa\eta} \dot{\mathscr{F}}_\lambda{}^\eta \right|$$

$$\lesssim (|\nabla h|_{\mathscr{L}\mathscr{L}} + |\overline{\nabla} h|)|\dot{\mathscr{F}}|^2 + |\nabla h||\dot{\mathscr{F}}|(|\dot{\mathscr{F}}|_{\mathscr{L}\mathscr{N}} + |\dot{\mathscr{F}}|_{\mathscr{T}\mathscr{T}})$$

$$\lesssim \varepsilon(1+t+|q|)^{-1}|\dot{\mathscr{F}}|^2 + \varepsilon(1+|q|)^{-1}(1+q_-)^{-2\mu} \left( |\dot{\mathscr{F}}|^2_{\mathscr{L}\mathscr{N}} + |\dot{\mathscr{F}}|^2_{\mathscr{T}\mathscr{T}} \right) \tag{12.2.9}$$

and

$$\left| L_\mu h^{\mu\kappa} \dot{\mathscr{F}}_{\kappa\zeta} \dot{\mathscr{F}}_0{}^\zeta + L_\mu h^{\kappa\lambda} \dot{\mathscr{F}}^\mu{}_\kappa \dot{\mathscr{F}}_{0\lambda} + \tfrac{1}{2} h^{\kappa\lambda} \dot{\mathscr{F}}_{\kappa\eta} \dot{\mathscr{F}}_\lambda{}^\eta \right|$$

$$\lesssim |h|_{\mathscr{L}\mathscr{L}} |\dot{\mathscr{F}}|^2 + |h||\dot{\mathscr{F}}|(|\dot{\mathscr{F}}|_{\mathscr{L}\mathcal{N}} + |\dot{\mathscr{F}}|_{\mathcal{T}\mathcal{T}})$$

$$\lesssim \varepsilon(1+|q|)(1+t+|q|)^{-1}|\dot{\mathscr{F}}|^2 + \varepsilon(1+q_-)^{-2\mu}\big(|\dot{\mathscr{F}}|^2_{\mathscr{L}\mathcal{N}} + |\dot{\mathscr{F}}|^2_{\mathcal{T}\mathcal{T}}\big). \quad (12.2.10)$$

For the higher-order terms, we use (3.7.2h), the hypotheses of the proposition, and the inequality $|ab| \lesssim a^2 + b^2$ to deduce that

$$\left| (\nabla_\mu N_\triangle^{\#\mu\zeta\kappa\lambda}) \dot{\mathscr{F}}_{\kappa\lambda} \dot{\mathscr{F}}_{0\zeta} - \tfrac{1}{4} (\nabla_t N_\triangle^{\#\zeta\eta\kappa\lambda}) \dot{\mathscr{F}}_{\zeta\eta} \dot{\mathscr{F}}_{\kappa\lambda} \right| \lesssim \big( |(h,\mathscr{F})||(\nabla h, \nabla\mathscr{F})| \big) |\dot{\mathscr{F}}|^2$$

$$\lesssim \varepsilon(1+t+|q|)^{-1}|\dot{\mathscr{F}}|^2 \quad (12.2.11)$$

and

$$\left| L_\mu N_\triangle^{\#\mu\zeta\kappa\lambda} \dot{\mathscr{F}}_{\kappa\lambda} \dot{\mathscr{F}}_{0\zeta} + \tfrac{1}{4} N_\triangle^{\#\zeta\eta\kappa\lambda} \dot{\mathscr{F}}_{\zeta\eta} \dot{\mathscr{F}}_{\kappa\lambda} \right| \lesssim |(h,\mathscr{F})|^2 |\dot{\mathscr{F}}|^2$$

$$\lesssim \varepsilon(1+|q|)(1+t+|q|)^{-1}|\dot{\mathscr{F}}|^2. \quad (12.2.12)$$

Inserting (12.2.9)–(12.2.12) into the right-hand side of (12.2.1) and using (12.1.2), we have that

$$\int_{\Sigma_t} |\dot{\mathscr{F}}|^2 w(q)\, d^3x + \int_0^t \int_{\Sigma_\tau} \big( |\dot{\mathscr{F}}|^2_{\mathscr{L}\mathcal{N}} + |\dot{\mathscr{F}}|^2_{\mathcal{T}\mathcal{T}} \big) w'(q)\, d^3x\, d\tau$$

$$\leq C \int_{\Sigma_0} |\dot{\mathscr{F}}|^2 w(q)\, d^3x + C_1 \varepsilon \int_0^t \int_{\Sigma_\tau} \Big( \frac{|\dot{\mathscr{F}}|^2}{1+\tau} w(q) + \big( |\dot{\mathscr{F}}|^2_{\mathscr{L}\mathcal{N}} + |\dot{\mathscr{F}}|^2_{\mathcal{T}\mathcal{T}} \big) \frac{w'(q)}{\mu} \Big) d^3x\, d\tau$$

$$+ C \int_0^t \int_{\Sigma_\tau} |\dot{\mathscr{F}}_{0\kappa} \dot{\mathfrak{F}}^\kappa| w(q)\, d^3x\, d\tau. \quad (12.2.13)$$

Now if $C_1\varepsilon/\mu$ is sufficiently small, we can absorb the $C_1\varepsilon \int_0^t \int_{\Sigma_\tau} [(|\dot{\mathscr{F}}|^2_{\mathscr{L}\mathcal{N}} + |\dot{\mathscr{F}}|^2_{\mathcal{T}\mathcal{T}}) w'(q)/\mu]\, d^3x\, d\tau$ term on the right-hand side of (12.2.13) into the second term on the left-hand side at the expense of increasing the constants $C$. Inequality (12.2.8) thus follows. $\qquad\square$

## 13. Pointwise decay estimates for wave equations in a curved spacetime

In this section, we state a lemma and a corollary proved in [Lindblad and Rodnianski 2010]. They allow one to deduce pointwise decay estimates for solutions to inhomogeneous wave equations (e.g., for the $h_{\mu\nu}$). The main advantage of these estimates is that, if one has good control over the inhomogeneous terms, then the pointwise decay estimates provided by the lemma and its corollary are *improvements over what can be deduced from the weighted Klainerman–Sobolev inequalities* of Proposition B.1. In particular, the lemma and its corollary play a fundamental role in the proofs of Propositions 15.6 and 15.7. See the beginning of Section 15 for additional details regarding this improvement.

**Remark 13.1.** The Faraday tensor analogs of Lemma 13.2 and Corollary 13.3 are contained in the estimates of Proposition 11.5. More specifically, the analogous inequalities would arise from integrating (in the direction of the first-order vector field differential operators on the left-hand sides of the inequalities) the inequalities in the proposition. We will carry out these integrations in Section 15, which will allow us to derive improved pointwise decay estimates for the lower-order Lie derivatives of the Faraday

tensor (improved relative to what can be deduced from the weighted Klainerman–Sobolev inequalities of Proposition B.1).

**13.1. _The decay estimate weight function_ $\varpi(q)$.** As in [Lindblad and Rodnianski 2010], our decay estimates will involve the following weight function $\varpi(q)$, which is chosen to complement the energy-estimate weight function $w(q)$ defined in (12.1.1):

$$\varpi = \varpi(q) = \begin{cases} (1+|q|)^{1+\gamma'} & \text{if } q > 0, \\ (1+|q|)^{1/2-\mu'} & \text{if } q < 0, \end{cases} \tag{13.1.1}$$

where $0 < \delta < \mu' < \frac{1}{2} - \mu$ and $0 < \gamma' < \gamma - \delta$ are fixed constants. Its complementary role will become apparent in Section 15.

**13.2. _Pointwise decay estimates._** We now state the lemma concerning pointwise decay estimates for solutions to inhomogeneous quasilinear wave equations.

**Lemma 13.2** (Pointwise decay estimates for solutions to a scalar wave equation [Lindblad and Rodnianski 2010, Lemma 7.1]). _Let $\phi$ be a solution of the scalar wave equation_ (13.2.1)

$$\widetilde{\Box}_g \phi = \mathfrak{I} \tag{13.2.1}$$

_on a curved background with metric $g_{\mu\nu}$. Assume that the tensor $H^{\mu\nu} \overset{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$ obeys the following estimates_:

$$|H| \le \varepsilon', \qquad \int_0^\infty (1+t)^{-1} \|H(t, \cdot)\|_{L^\infty(D_t)} \, dt \le \tfrac{1}{4}, \quad \text{and} \quad |H|_{\mathscr{L}\mathscr{T}} \le \varepsilon'(1+t+|x|)^{-1}(1+|q|) \tag{13.2.2}$$

_in the region_

$$D_t \overset{\text{def}}{=} \{x \mid t/2 < |x| < 2t\} \tag{13.2.3}$$

_for $t \in [0, T)$. Then with $\alpha \overset{\text{def}}{=} \max(1 + \gamma', \frac{1}{2} - \mu')$, the following pointwise estimate holds for $(t, x) \in [0, T) \times \mathbb{R}^3$:_

$$(1+t+|q|)\varpi(q)|\nabla\phi| \lesssim \sup_{0 \le \tau \le t} \sum_{|I| \le 1} \|\varpi(q)\nabla_{\mathscr{L}}^I \phi(\tau, \cdot)\|_{L^\infty} + \int_{\tau=0}^t \varepsilon'\alpha \|\varpi(q)\nabla\phi(\tau, \cdot)\|_{L^\infty} \, d\tau$$

$$+ \int_{\tau=0}^t (1+\tau)\|\varpi(q)\mathfrak{I}(\tau, \cdot)\|_{L^\infty(D_\tau)} \, d\tau$$

$$+ \int_{\tau=0}^t \sum_{|I| \le 2} (1+\tau)^{-1}\|\varpi(q)\nabla_{\mathscr{L}}^I \phi(\tau, \cdot)\|_{L^\infty(D_\tau)} \, d\tau. \tag{13.2.4}$$

We now state the following corollary, which provides similar decay estimates for the null components of tensorial systems of wave equations:

**Corollary 13.3** (Pointwise decay estimates for solutions to a system of tensorial wave equations [Lindblad and Rodnianski 2010, Corollary 7.2]). _Let $\phi_{\mu\nu}$ be a solution of the system_

$$\widetilde{\Box}_g \phi_{\mu\nu} = \mathfrak{I}_{\mu\nu} \tag{13.2.5}$$

*on a curved background with a metric $g_{\mu\nu}$. Assume that the tensor $H^{\mu\nu} \overset{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$ obeys the following estimates*:

$$|H| \leq \frac{\varepsilon'}{4}, \qquad \int_0^\infty (1+t)^{-1} \|H(t, \cdot)\|_{L_\infty(D_t)} \, dt \leq \varepsilon', \quad \text{and} \quad |H|_{\mathscr{L}\mathscr{T}} \leq \frac{\varepsilon'}{4}(1+t+|q|)^{-1}(1+|q|) \quad (13.2.6)$$

*in the region*

$$D_t \overset{\text{def}}{=} \{x \mid t/2 < |x| < 2t\} \tag{13.2.7}$$

*for $t \in [0, T)$. Then for any $\mathscr{U}, \mathscr{V} \in \{\mathscr{L}, \mathscr{T}, \mathscr{N}\}$ and with $\alpha \overset{\text{def}}{=} \max(1+\gamma', \frac{1}{2} - \mu')$, the following pointwise estimate holds for $(t, x) \in [0, T) \times \mathbb{R}^3$:*

$$(1+t+|q|)\varpi(q)|\nabla\phi|_{\mathscr{U}\mathscr{V}} \lesssim \sup_{0 \leq \tau \leq t} \sum_{|I| \leq 1} \|\varpi(q)\nabla_{\mathscr{L}}^I\phi(\tau, \cdot)\|_{L^\infty} + \int_{\tau=0}^t \varepsilon'\alpha\|\varpi(q)|\nabla\phi(\tau, \cdot)|_{\mathscr{U}\mathscr{V}}\|_{L^\infty} \, d\tau$$

$$+ \int_{\tau=0}^t (1+\tau)\|\varpi(q)|\mathfrak{I}(\tau, \cdot)|_{\mathscr{U}\mathscr{V}}\|_{L^\infty(D_\tau)} \, d\tau$$

$$+ \sum_{|I| \leq 2} \int_{\tau=0}^t (1+\tau)^{-1}\|\varpi(q)\nabla_{\mathscr{L}}^I\phi(\tau, \cdot)\|_{L^\infty(D_\tau)} \, d\tau. \tag{13.2.8}$$

## 14. Local well-posedness and the continuation principle for the reduced equations

In this short section, we state for convenience a standard proposition concerning local well-posedness and a continuation principle for the reduced equations (3.7.1a)–(3.7.1c). The continuation principle shows that a suitable a priori bound on the energy of the solution implies global existence. It therefore plays a fundamental role in our global stability argument of Section 16.

**Proposition 14.1** (Local well-posedness and the continuation principle). *Let $(h_{\mu\nu}^{(1)}|_{t=0}, \partial_t h_{\mu\nu}^{(1)}|_{t=0}, \mathscr{F}_{\mu\nu}|_{t=0})$ $(\mu, \nu = 0, 1, 2, 3)$ be initial data for the reduced equations (3.7.1a)–(3.7.1c) constructed from abstract initial data $(\mathring{\underline{h}}_{jk}^{(1)}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ $(j, k = 1, 2, 3)$ on the manifold $\mathbb{R}^3$ satisfying the constraints (4.1.1a)– (4.1.2b) as described in Section 4.2. Assume that the data are asymptotically flat in the sense of (1.0.4a)– (1.0.4f). Let $\ell \geq 4$ be an integer, and let $\gamma > 0$ and $\mu > 0$ be constants satisfying the restrictions stated in Section 2.14. Assume that $E_{\ell;\gamma}(0) < \varepsilon$, where $E_{\ell;\gamma}(0)$ is the norm of the abstract data defined in (10.0.3). Then if $\varepsilon$ is sufficiently small,[29] these data launch a unique classical solution to the reduced equations existing on a nontrivial maximal spacetime slab $[0, T_{\max}) \times \mathbb{R}^3$. The energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$ of the solution, which is defined in (1.2.7), satisfies $\mathscr{E}_{\ell;\gamma;\mu}(0) \lesssim \varepsilon$ and is continuous on $[0, T_{\max})$. Furthermore, either $T_{\max} = \infty$, or one of the following two "breakdown" scenarios must occur*:

  (i) $\lim_{t \uparrow T_{\max}} \mathscr{E}_{\ell;\gamma;\mu}(t) = \infty$.

 (ii) *The solution escapes the regime of hyperbolicity of the reduced equations.*

**Remark 14.2.** The classification of the two breakdown scenarios is known as a *continuation principle*.

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[29] This smallness assumption ensures that the reduced data lie within the regime of hyperbolicity of the reduced equations.

**Remark 14.3.** Note that, in order to deduce global existence, Proposition 14.1 shows that it suffices to derive an a priori bound on $\mathscr{E}_{4;\gamma;\mu}(t)$ together with a bound ensuring that the solution remains in the regime of hyperbolicity. However, our methods do not allow us to derive an a priori bound for $\mathscr{E}_{4;\gamma;\mu}(t)$ alone; our derivation of upgraded pointwise estimates (see Section 15), which are essential for our derivation of an a priori energy estimate, requires that we work with $\mathscr{E}_{\ell;\gamma;\mu}(t)$ for $\ell \geq 10$.

The main ingredients in the proof of Proposition 14.1 are Lemmas 12.1 and 12.2, which provide weighted energy estimates for linearized versions of the reduced equations. Based on the availability of these estimates, the proof is rather standard, and we omit the details. Readers may consult, e.g., [Hörmander 1997, Chapter VI; Majda 1984, Chapter 2; Shatah and Struwe 1998, Chapter 5; Sogge 2008, Chapter 1; Speck 2009b; Taylor 1996, Chapter 16] for details concerning local existence and, e.g., [Hörmander 1997, Chapter VI; Sogge 2008, Chapter 1; Speck 2009a] for the ideas behind the continuation principle.

## 15. The fundamental energy bootstrap assumption and pointwise decay estimates for the reduced equations

In this section, we introduce our fundamental bootstrap assumption (15.0.1) for the energy of a solution to the reduced equations. Under this assumption, we derive a collection of pointwise decay estimates that will play a crucial role in the proof of Theorem 16.1. In particular, these decay estimates are used to deduce the factors $(1+\tau)^{-1}$ and $(1+\tau)^{-1+C\varepsilon}$ in (16.2.10), which are essential for deriving the a priori energy bound (16.1.8). The decay estimates can be roughly divided into two classes: the weak pointwise decay estimates and the upgraded pointwise decay estimates. The weak decay estimates are consequences of the weighted Klainerman–Sobolev inequality (1.2.10). These estimates inherit a loss of approximately $(1+t)^{\delta}$ relative to what is needed to prove our main result. We remark that $\delta$ is a fixed small constant that is independent of the data while $\varepsilon$ is connected to the size of the data. The loss comes from the loss we allow in our energy bootstrap assumption. Roughly speaking, if one tried to prove global stability using only the weak estimates, then the factors $(1+\tau)^{-1}$ and $(1+\tau)^{-1+C\varepsilon}$ in (16.2.10) would have to be replaced with $(1+\tau)^{-1+\delta}$; this loss of approximately $(1+t)^{\delta}$ would completely destroy the viability of our approach. The purpose of the upgraded pointwise decay estimates is precisely to eliminate some of this loss for the lower-order derivatives of the solution. The upgraded estimates are derived using the weak estimates and the special structure of the equations in wave coordinates; that is, many of the estimates we derive in this section rely upon the wave-coordinate condition.

We recall that the spacetime metric $g_{\mu\nu}$ is split into the pieces $g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}$ and that the energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$ (see (1.2.7)) is a functional of $(h^{(1)}, \mathscr{F})$. Our main bootstrap assumption for the energy is

$$\mathscr{E}_{\ell;\gamma;\mu}(t) \leq \varepsilon(1+t)^{\delta}, \tag{15.0.1}$$

where $\ell \geq 10$ is an integer, $0 < \gamma < \frac{1}{2}$ is a fixed constant, $\delta$ is a fixed constant satisfying both $0 < \delta < \frac{1}{4}$ and $0 < \delta < \gamma$, $0 < \mu < \frac{1}{2}$ is a fixed constant (all of which will be chosen during the proof of Theorem 16.3), and $\varepsilon$ is a small positive number whose required smallness is adjusted (as many times as necessary) during the derivation of our inequalities. With the help of (6.5.22), inequality (15.0.1) implies the following

more explicit consequence of the energy bootstrap assumption:

$$\sum_{|I|\leq \ell} \left( \left\| w^{1/2}(q)\nabla\nabla_{\mathscr{L}}^I h^{(1)} \right\|_{L^2} + \left\| w^{1/2}(q)\mathscr{L}_{\mathscr{L}}^I \mathscr{F} \right\|_{L^2} \right) \leq C\varepsilon(1+t)^{\delta}. \tag{15.0.2}$$

In the remaining estimates in this article, *we will also often make the following smallness assumption on the ADM mass*:

$$M \leq \varepsilon. \tag{15.0.3}$$

**15.1. *Preliminary (weak) pointwise decay estimates.*** In this section, we provide some preliminary pointwise decay estimates that are essentially a consequence of the weighted Klainerman–Sobolev inequalities of Appendix B. Unlike the upgraded pointwise decay estimates of the next section, these estimates do not take into account the special structure of the reduced equations under the wave-coordinate condition.

We begin with a simple lemma concerning pointwise decay estimates for the Schwarzschild tail of the metric and its derivatives.

**Lemma 15.1** (Decay estimates for $h^{(0)}$). *Let $h^{(0)}$ be as in* (1.2.1c), *and let $I$ be any $\nabla$-multi-index. Then the following pointwise estimate holds for $(t, x) \in [0, \infty) \times \mathbb{R}^3$:*

$$|\nabla^I h^{(0)}| \leq CM(1+t+|q|)^{-(1+|I|)}, \tag{15.1.1a}$$

*where $M$ is the ADM mass.*

*Furthermore, if $I$ is any $\nabla$-multi-index and $J$ is any $\mathscr{L}$-multi-index, then the following pointwise estimate holds for $(t, x) \in [0, \infty) \times \mathbb{R}^3$:*

$$|\nabla^I \nabla_{\mathscr{L}}^J h^{(0)}| + |\nabla_{\mathscr{L}}^J \nabla^I h^{(0)}| \leq CM(1+t+|q|)^{-(1+|I|)}. \tag{15.1.1b}$$

**Remark 15.2.** Since $H_{(0)\mu\nu} = -h_{\mu\nu}^{(0)}$ (where $H_{(0)}^{\mu\nu}$ is defined in (11.1.2)), the above estimates also hold if we replace $h^{(0)}$ with $H_{(0)}$.

*Proof.* The lemma follows from simple computations, the definition (4.2.1) of the cut-off function $\chi$, the definition of $h^{(0)}$, and the definitions of the vector fields $Z \in \mathscr{L}$. □

**Corollary 15.3** (Weak pointwise decay estimates; slight extension of [Lindblad and Rodnianski 2010, Corollary 9.4]). *Let $\ell \geq 10$ be an integer. Assume that the abstract initial data are asymptotically flat in the sense of* (1.0.4a)–(1.0.4f), *that the ADM mass smallness condition* (15.0.3) *holds, that the constraints* (4.1.1a)–(4.1.2b) *are satisfied, and that the initial data for the reduced system are constructed from the abstract initial data as described in Section 4.2. Let $(g_{\mu\nu} \stackrel{\text{def}}{=} m_{\mu\nu} + h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}, \mathscr{F}_{\mu\nu})$ be the corresponding solution to the reduced system* (3.7.1a)–(3.7.1c) *existing on a slab $(t, x) \in [0, T) \times \mathbb{R}^3$, where $h^{(1)}$ is defined in* (1.2.1b). *In particular, by Proposition 4.2, the wave-coordinate condition* (3.1.1a) *holds for $(t, x) \in [0, T) \times \mathbb{R}^3$. Assume in addition that the pair $(h^{(1)}, \mathscr{F})$ satisfies the energy bootstrap assumption* (15.0.1) *on the interval $[0, T)$. Then if $\varepsilon$ is sufficiently small, the following pointwise estimates hold for $(t, x) \in [0, T) \times \mathbb{R}^3$:*

$$|\nabla \nabla_{\mathscr{L}}^I h^{(1)}| |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}| \leq \begin{cases} C\varepsilon(1+t+|q|)^{-1}(1+t)^{\delta}(1+|q|)^{-1-\gamma} & \text{if } q > 0, \\ C\varepsilon(1+t+|q|)^{-1}(1+t)^{\delta}(1+|q|)^{-1/2} & \text{if } q < 0 \end{cases} \quad (|I| \leq \ell - 3), \quad (15.1.2a)$$

$$|\nabla_{\mathscr{L}}^I h^{(1)}| \leq \begin{cases} C\varepsilon(1+t+|q|)^{-1+\delta}(1+|q|)^{-\gamma} & \text{if } q > 0, \\ C\varepsilon(1+t+|q|)^{-1+\delta}(1+|q|)^{1/2} & \text{if } q < 0 \end{cases} \quad (|I| \leq \ell - 3), \quad (15.1.2b)$$

$$|\overline{\nabla} \nabla_{\mathscr{L}}^I h^{(1)}| + (1+|q|)|\overline{\nabla} \mathscr{L}_{\mathscr{L}}^I \mathscr{F}|$$
$$\leq \begin{cases} C\varepsilon(1+t+|q|)^{-2+\delta}(1+|q|)^{-\gamma} & \text{if } q > 0, \\ C\varepsilon(1+t+|q|)^{-2+\delta}(1+|q|)^{1/2} & \text{if } q < 0 \end{cases} \quad (|I| \leq \ell - 4). \quad (15.1.2c)$$

*In addition, the tensor field $H_{(1)}^{\mu\nu}$ defined in (11.1.2) satisfies the same estimates as $h_{\mu\nu}^{(1)}$. Furthermore, if we make the substitution $\gamma \to \delta$ in the above inequalities, then the same estimates hold for the tensor fields $h_{\mu\nu}^{(0)}$, $h_{\mu\nu} \stackrel{\text{def}}{=} h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}$, $H_{(0)\mu\nu} \stackrel{\text{def}}{=} -h_{\mu\nu}^{(0)}$, $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$, and $H_{(1)}^{\mu\nu} \stackrel{\text{def}}{=} H^{\mu\nu} - H_{(0)}^{\mu\nu}$.*

*Proof.* This corollary is a slight extension of Corollary 9.4 of [Lindblad and Rodnianski 2010], in which estimates for $h^{(0)} = -H_{(0)}$, $h^{(1)}$, and $h$ were proved. The main idea in the proof is to use the weighted Klainerman–Sobolev estimates of Proposition B.1 under the assumption (15.0.2) together with the decay (1.0.4c)–(1.0.4f) of the initial data at spatial infinity and Lemma 15.1. The estimates for $\mathscr{F}$ follow in a straightforward fashion from the arguments of [Lindblad and Rodnianski 2010, Corollary 9.4] while the estimates for $H_{(1)}$ and $H$ follow from those for $h^{(1)}$ and $h$ together with (3.3.11a). $\qquad\square$

In the next lemma, we use the weak decay estimates to derive pointwise estimates for the Schwarzschild tail term $\nabla_{\mathscr{L}}^I \widetilde{\Box}_g h^{(0)}$ appearing on the right-hand side of (7.0.1).

**Lemma 15.4** (Pointwise decay estimates for $\nabla_{\mathscr{L}}^I \widetilde{\Box}_g h^{(0)}$ [Lindblad and Rodnianski 2010, Lemma 9.9]). *Let $h^{(0)}$ be the Schwarzschild part of $h$ as defined in (1.2.1c), and assume the hypotheses/conclusions of Corollary 15.3. Let $I$ be a $\mathscr{L}$-multi-index subject to the restrictions stated below. Then if $\varepsilon$ is sufficiently small, the following pointwise estimates hold for $(t, x) \in [0, T) \times \mathbb{R}^3$, where $M$ is the ADM mass:*

$$|\nabla_{\mathscr{L}}^I \widetilde{\Box}_g h^{(0)}| \leq \begin{cases} CM\varepsilon(1+t+|q|)^{-4+\delta}(1+|q|)^{-\delta} & \text{if } q > 0, \\ CM(1+t+|q|)^{-3} & \text{if } q < 0 \end{cases} \quad (|I| \leq \ell - 3). \quad (15.1.3a)$$

*Furthermore, the following pointwise estimates also hold for $(t, x) \in [0, T) \times \mathbb{R}^3$:*

$$|\nabla_{\mathscr{L}}^I \widetilde{\Box}_g h^{(0)}| \leq CM \sum_{|J| \leq |I|} (1+t+|q|)^{-3}|\nabla_{\mathscr{L}}^J h^{(1)}| + \begin{cases} CM\varepsilon(1+t+|q|)^{-4} & \text{if } q > 0, \\ CM(1+t+|q|)^{-3} & \text{if } q < 0 \end{cases} \quad (|I| \leq \ell). \quad (15.1.3b)$$

*Proof.* We first observe that $\widetilde{\Box}_g h^{(0)} = \Box_m h^{(0)} + H^{\kappa\lambda} \nabla_\kappa \nabla_\lambda h^{(0)}$, where $\Box_m \stackrel{\text{def}}{=} (m^{-1})^{\kappa\lambda} \nabla_\kappa \nabla_\lambda$ is the Minkowski wave operator. From (15.1.1b), the definition of $h^{(0)}$, the Leibniz rule, and the fact that $\Box_m(1/r) = 0$ for $r > 0$, it follows that

$$|\nabla_{\mathscr{L}}^I \Box_m h^{(0)}| \lesssim M(1+t+|q|)^{-3} \chi_0\left(\frac{1}{2} \leq \frac{r}{t} \leq \frac{3}{4}\right), \quad (15.1.4)$$

$$\left|\nabla_{\mathscr{L}}^I \left(H^{\kappa\lambda} \nabla_\kappa \nabla_\lambda h^{(0)}\right)\right| \lesssim M(1+t+|q|)^{-3} \sum_{|J| \leq |I|} |\nabla_{\mathscr{L}}^J H|, \quad (15.1.5)$$

where $\chi_0\left(\frac{1}{2} \leq z \leq \frac{3}{4}\right)$ is the characteristic function of the interval $\left[\frac{1}{2}, \frac{3}{4}\right]$. Furthermore, using $H = -h^{(0)} - h^{(1)} + O^\infty(|h^{(0)} + h^{(1)}|^2)$, we deduce that

$$\sum_{|J| \leq |I|} |\nabla^J_{\mathscr{L}} H| \lesssim \varepsilon(1 + t + |q|)^{-1} + \sum_{|J| \leq |I|} |\nabla^J_{\mathscr{L}} h^{(1)}|. \tag{15.1.6}$$

Using (15.1.5), (15.1.6), and the estimate (15.1.2b), we have that

$$\left|\nabla^I_{\mathscr{L}}\left(H^{\kappa\lambda} \nabla_\kappa \nabla_\lambda h^{(0)}\right)\right| \lesssim \begin{cases} M\varepsilon(1 + t + |q|)^{-4+\delta}(1 + |q|)^{-\delta} & \text{if } q > 0, \\ M\varepsilon(1 + t + |q|)^{-4+\delta}(1 + |q|)^{1/2} & \text{if } q < 0 \end{cases} \quad (|I| \leq \ell - 3) \tag{15.1.7}$$

and

$$\left|\nabla^I_{\mathscr{L}}\left(H^{\kappa\lambda} \nabla_\kappa \nabla_\lambda h^{(0)}\right)\right| \lesssim M\varepsilon(1 + t + |q|)^{-4} + M\varepsilon(1 + t + |q|)^{-3} \sum_{|J| \leq |I|} |\nabla^J_{\mathscr{L}} h^{(1)}| \quad (|I| \leq \ell). \tag{15.1.8}$$

Inequalities (15.1.3a) and (15.1.3b) now easily follow from the above estimates. $\qquad\square$

**15.2. Initial upgraded pointwise decay estimates for $|\mathscr{L}^I_{\mathscr{L}} \mathscr{F}|_{\mathscr{L}\mathscr{N}}$ and $|\mathscr{L}^I_{\mathscr{L}} \mathscr{F}|_{\mathscr{T}\mathscr{T}}$.** In this section, we prove some upgraded pointwise decay estimates for the "favorable" components of the lower-order Lie derivatives of $\mathscr{F}$. Our estimates take into account the special structure revealed by our null decomposition of the electromagnetic equations of variations, a structure that was captured by Proposition 11.5 and that depends in part upon the wave-coordinate condition. We remark that in Section 15.3 some of these decay estimates will be further improved (hence our use of the terminology "initial upgraded" here).

**Proposition 15.5** (Initial upgraded pointwise decay estimates for $|\mathscr{L}^I_{\mathscr{L}} \mathscr{F}|_{\mathscr{L}\mathscr{N}}$ and $|\mathscr{L}^I_{\mathscr{L}} \mathscr{F}|_{\mathscr{T}\mathscr{T}}$). *Assume the hypotheses/conclusions of Corollary 15.3. Then if $\varepsilon$ is sufficiently small, the following pointwise estimates hold for $(t, x) \in [0, T) \times \mathbb{R}^3$:*

$$|\mathscr{L}^I_{\mathscr{L}} \mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}^I_{\mathscr{L}} \mathscr{F}|_{\mathscr{T}\mathscr{T}} \leq \begin{cases} C\varepsilon(1 + t + |q|)^{-2+2\delta}(1 + |q|)^{-\gamma-\delta} & \text{if } q > 0, \\ C\varepsilon(1 + t + |q|)^{-2+2\delta}(1 + |q|)^{1/2-\delta} & \text{if } q < 0 \end{cases} \quad (|I| \leq \ell - 4). \tag{15.2.1}$$

*Proof.* Since $|\mathscr{L}^I_{\mathscr{L}} \mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}^I_{\mathscr{L}} \mathscr{F}|_{\mathscr{T}\mathscr{T}} \approx |\alpha[\mathscr{L}^I_{\mathscr{L}} \mathscr{F}]| + |\rho[\mathscr{L}^I_{\mathscr{L}} \mathscr{F}]| + |\sigma[\mathscr{L}^I_{\mathscr{L}} \mathscr{F}]|$, it suffices to prove the desired decay estimates for $|\alpha[\mathscr{L}^I_{\mathscr{L}} \mathscr{F}]|$, $|\rho[\mathscr{L}^I_{\mathscr{L}} \mathscr{F}]|$, and $|\sigma[\mathscr{L}^I_{\mathscr{L}} \mathscr{F}]|$ separately. We provide proof for the null component $\alpha[\mathscr{L}^I_{\mathscr{L}} \mathscr{F}]$. The proofs for the components $\rho[\mathscr{L}^I_{\mathscr{L}} \mathscr{F}]$ and $\sigma[\mathscr{L}^I_{\mathscr{L}} \mathscr{F}]$ are similar, and we leave these details to the reader. Let $\mathscr{W} \stackrel{\text{def}}{=} \{(t, x) \mid |x| \geq 1 + t/2\} \cap \{(t, x) \mid |x| \leq 2t - 1\}$ denote the "wave-zone" region. Then for $(t, x) \notin \mathscr{W}$, we have that $1 + |q| \approx (1 + t + |q|)$. Using this fact, for $(t, x) \notin \mathscr{W}$, we can bound $|\alpha[\mathscr{L}^I_{\mathscr{L}} \mathscr{F}]|$ by the right-hand side of (15.2.1) by using the weak decay estimate (15.1.2a).

We now consider the case $(t, x) \in \mathscr{W}$. Let $f \stackrel{\text{def}}{=} r\alpha[\mathscr{L}^I_{\mathscr{L}} \mathscr{F}]$. Then from (11.1.13b), the fact that $r \approx (1 + t + |q|) \approx (1 + s + |q|)$ on $\mathscr{W}$, and the weak decay estimates of Corollary 15.3, it follows that (with $\partial_q$ defined in Section 2.7)

$$|\partial_q f(t, x)| \lesssim \begin{cases} \varepsilon(1 + s + |q|)^{-1+2\delta}(1 + |q|)^{-1-\gamma-\delta} & \text{if } q > 0, \\ \varepsilon(1 + s + |q|)^{-1+2\delta}(1 + |q|)^{-1/2-\delta} & \text{if } q < 0. \end{cases} \tag{15.2.2}$$

Let $(\tau(q'), y(q'))$ be the $q'$-parametrized line segment of constant $s$ and angular values that begins at $(t, x)$ and terminates at the point $(t_0, x_0)$ lying to the *past* of $(t, x)$ and on the boundary of $\mathscr{W}$. Let $q$ and $s$ be the null coordinates corresponding to $(t, x)$. Then the null coordinates corresponding to $(t_0, x_0)$

are $q_0 = \frac{s}{3} - \frac{2}{3}$ and $s_0 = s$. Integrating the inequality (15.2.2) along this line segment (i.e., integrating $dq'$), we have that

$$|f(t,x)| \lesssim |f(t_0, x_0)| + \int_{q'=q}^{q'=s/3-2/3} \begin{cases} \varepsilon(1+s+|q'|)^{-1+2\delta}(1+|q'|)^{-1-\gamma-\delta} & \text{if } q' > 0, \\ \varepsilon(1+s+|q'|)^{-1+\delta}(1+|q'|)^{-1/2-\delta} & \text{if } q' < 0 \end{cases} \right\} dq'$$

$$\lesssim |f(t_0, x_0)| + \begin{cases} \varepsilon(1+s)^{-1+2\delta}(1+|q|)^{-\gamma-\delta} & \text{if } q > 0, \\ \varepsilon(1+s)^{-1+\delta}(1+|q|)^{1/2-\delta} & \text{if } q < 0. \end{cases} \tag{15.2.3}$$

From the facts that $r_0 \approx 1 + |q_0| \approx 1 + t_0 + |q_0| \approx 1 + s_0 + |q_0| \approx 1 + s$, together with the weak decay estimate (15.1.2a), it follows that

$$|f(t_0, x_0)| \lesssim \varepsilon(1+s)^{-1-\gamma+\delta}. \tag{15.2.4}$$

Combining (15.2.3) and (15.2.4), and using the fact that $1+s \approx 1+t+|q|$, we deduce that $|\alpha[\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}(t, x)]|$ is bounded from above by the right-hand side of (15.2.1). This completes our proof of (15.2.1) for the $\alpha[\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}]$ component. $\qquad \square$

## 15.3. Upgraded pointwise decay estimates for $|\nabla_{\mathscr{Z}}^I h|$ and $|\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|$ and fully upgraded pointwise decay estimates for $|\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|_{\mathscr{L}\mathscr{N}}$ and $|\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|_{\mathscr{T}\mathscr{T}}$.

In this section, we state two propositions that strengthen some of the pointwise decay estimates proved in Sections 15.1 and 15.2. Their proofs, which are provided in Sections 15.4 and 15.5, are based on a careful analysis of the special structure of the reduced equations and in particular rely upon the decompositions performed in Section 11, which in turn rely in part upon the wave-coordinate condition. These estimates play a central role in our derivation of the "strong" a priori energy estimate (16.1.8), which is the main step in the proof of our stability theorem.

**Proposition 15.6** (Upgraded pointwise decay estimates for $\mathscr{F}$ and certain components of $h$, $\nabla h$, and $\nabla_Z h$; extension of [Lindblad and Rodnianski 2010, Proposition 10.1]). *Assume the hypotheses/conclusions of Corollary 15.3. In particular, by Proposition 4.2, the wave-coordinate condition (3.1.1a) holds for $(t, x) \in [0, T) \times \mathbb{R}^3$. Then if $\varepsilon$ is sufficiently small, for every vector field $Z \in \mathscr{Z}$, the following pointwise estimates hold for $(t, x) \in [0, T) \times \mathbb{R}^3$:*

$$|\nabla h|_{\mathscr{L}\mathscr{T}} + |\nabla \nabla_Z h|_{\mathscr{L}\mathscr{L}} \le \begin{cases} C\varepsilon(1+t+|q|)^{-2+\delta}(1+|q|)^{-\delta} & \text{if } q > 0, \\ C\varepsilon(1+t+|q|)^{-2+\delta}(1+|q|)^{1/2} & \text{if } q < 0, \end{cases} \tag{15.3.1a}$$

$$|h|_{\mathscr{L}\mathscr{T}} + |\nabla_Z h|_{\mathscr{L}\mathscr{L}} \le \begin{cases} C\varepsilon(1+t+|q|)^{-1} & \text{if } q > 0, \\ C\varepsilon(1+t+|q|)^{-1}(1+|q|)^{1/2+\delta} & \text{if } q < 0, \end{cases} \tag{15.3.1b}$$

$$|\nabla h|_{\mathscr{T}\mathscr{N}} \le C\varepsilon(1+t+|q|)^{-1}, \tag{15.3.2a}$$

$$|\nabla h| \le C\varepsilon(1+t+|q|)^{-1}(1+\ln(1+t)), \tag{15.3.2b}$$

$$|\mathscr{F}| \le C\varepsilon(1+t+|q|)^{-1}. \tag{15.3.3}$$

*Furthermore, the same estimates hold for the tensor fields $h_{\mu\nu}^{(0)}$, $h_{\mu\nu}^{(1)}$, $H^{\mu\nu} \overset{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$, $H_{(0)}^{\mu\nu}$, and $H_{(1)}^{\mu\nu}$.*

**Proposition 15.7** (Upgraded pointwise decay estimates for the lower-order derivatives of $h$ and $\mathscr{F}$; extension of [Lindblad and Rodnianski 2010, Proposition 10.2]). *Under the assumptions of Proposition 15.6,*

*let $0 < \gamma' < \gamma - \delta$ and $0 < \delta < \mu' < \frac{1}{2}$ be fixed constants. Let $I$ be any $\mathscr{L}$-multi-index subject to the restrictions stated below. Then there exist constants $M_k$ and $C_k$ depending on $\gamma'$, $\mu'$, and $\delta$ such that, if $\varepsilon$ is sufficiently small, then the following pointwise estimates hold for $(t, x) \in [0, T) \times \mathbb{R}^3$:*

$$
|\nabla \nabla_{\mathscr{L}}^I h^{(1)}| + |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|
$$
$$
\leq \begin{cases} C_k \varepsilon (1 + t + |q|)^{-1 + M_k \varepsilon} (1 + |q|)^{-1-\gamma'} & \text{if } q > 0, \\ C_k \varepsilon (1 + t + |q|)^{-1 + M_k \varepsilon} (1 + |q|)^{-1/2+\mu'} & \text{if } q < 0 \end{cases} \quad (|I| = k \leq \ell - 5), \qquad (15.3.4a)
$$

$$
|\nabla_{\mathscr{L}}^I h^{(1)}| \leq \begin{cases} C_k \varepsilon (1 + t + |q|)^{-1 + M_k \varepsilon} (1 + |q|)^{-\gamma'} & \text{if } q > 0, \\ C_k \varepsilon (1 + t + |q|)^{-1 + M_k \varepsilon} (1 + |q|)^{1/2+\mu'} & \text{if } q < 0 \end{cases} \quad (|I| = k \leq \ell - 5), \qquad (15.3.4b)
$$

$$
|\overline{\nabla} \nabla_{\mathscr{L}}^I h^{(1)}| + (1 + |q|)|\overline{\nabla} \mathscr{L}_{\mathscr{L}}^I \mathscr{F}| + |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{L} \mathscr{N}} + |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{T} \mathscr{T}}
$$
$$
\leq \begin{cases} C_k \varepsilon (1 + t + |q|)^{-2 + M_k \varepsilon} (1 + |q|)^{-\gamma'} & \text{if } q > 0, \\ C_k \varepsilon (1 + t + |q|)^{-2 + M_k \varepsilon} (1 + |q|)^{1/2+\mu'} & \text{if } q < 0 \end{cases} \quad (|I| = k \leq \ell - 6). \qquad (15.3.4c)
$$

*Furthermore, the same estimates hold for $h_{\mu\nu} \overset{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$ and $H^{\mu\nu} \overset{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$ if we replace $\gamma'$ with $M_k \varepsilon$.*

**15.4. *Proof of Proposition 15.6.*** We only prove the estimates for $h_{\mu\nu}$ and $\mathscr{F}_{\mu\nu}$. The estimates for $h_{\mu\nu}^{(0)}$, $h_{\mu\nu}^{(1)}$, $H^{\mu\nu}$, $H_{(0)}^{\mu\nu}$, and $H_{(1)}^{\mu\nu}$ follow easily from those for $h_{\mu\nu}$, (3.3.11a), and Lemma 15.1.

**15.4.1.** *Proofs of* (15.3.1a) *and* (15.3.1b). We will argue as in Lemma 10.4 of [Lindblad and Rodnianski 2010]; we first provide a lemma that establishes a more general version of the desired estimates.

**Lemma 15.8** (Pointwise estimates for $|\nabla \nabla_{\mathscr{L}}^I h|_{\mathscr{L} \mathscr{L}}$, $|\nabla_{\mathscr{L}}^I h|_{\mathscr{L} \mathscr{L}}$, $|\nabla \nabla_{\mathscr{L}}^I h|_{\mathscr{L} \mathscr{T}}$, and $|\nabla_{\mathscr{L}}^I h|_{\mathscr{L} \mathscr{T}}$ [Lindblad and Rodnianski 2010, Lemma 10.4]). *Under the hypotheses of Proposition 15.6, if $k \leq \ell - 4$ and $\varepsilon$ is sufficiently small, then the following pointwise estimates hold for $(t, x) \in [0, T) \times \mathbb{R}^3$:*

$$
\sum_{|I| \leq k} |\nabla \nabla_{\mathscr{L}}^I h|_{\mathscr{L} \mathscr{L}} + \underbrace{\sum_{|J| \leq k-1} |\nabla \nabla_{\mathscr{L}}^J h|_{\mathscr{L} \mathscr{T}}}_{\text{absent if } k = 0} \lesssim \underbrace{\sum_{|K| \leq k-2} |\nabla \nabla_{\mathscr{L}}^K h|}_{\text{absent if } k \leq 1}
$$
$$
+ \begin{cases} \varepsilon (1 + t + |q|)^{-2+2\delta} (1 + |q|)^{-2\delta} & \text{if } q > 0, \\ \varepsilon (1 + t + |q|)^{-2+2\delta} (1 + |q|)^{1/2-\delta} & \text{if } q < 0, \end{cases} \qquad (15.4.1)
$$

$$
\sum_{|I| \leq k} |\nabla_{\mathscr{L}}^I h|_{\mathscr{L} \mathscr{L}} + \underbrace{\sum_{|J| \leq k-1} |\nabla_{\mathscr{L}}^J h|_{\mathscr{L} \mathscr{T}}}_{\text{absent if } k = 0} \lesssim \underbrace{\sum_{|K| \leq k-2} \int_{\varrho = |x|}^{\varrho = |x|+t} |\nabla \nabla_{\mathscr{L}}^K h|(t + |q| - \varrho, \varrho x / |x|) \, d\varrho}_{\text{absent if } k \leq 1}
$$
$$
+ \begin{cases} \varepsilon (1 + t + |q|)^{-1} & \text{if } q > 0, \\ \varepsilon (1 + t + |q|)^{-1} (1 + |q|)^{1/2+\delta} & \text{if } q < 0. \end{cases} \qquad (15.4.2)
$$

*Furthermore, the same estimates hold for the tensor $H^{\mu\nu} \overset{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$.*

*Proof.* By Proposition 11.1, we have that

$$
\sum_{|I| \leq k} |\nabla \nabla_{\mathscr{L}}^I h|_{\mathscr{L} \mathscr{L}} + \underbrace{\sum_{|J| \leq k-1} |\nabla \nabla_{\mathscr{L}}^J h|_{\mathscr{L} \mathscr{T}}}_{\text{absent if } k = 0} \lesssim \underbrace{\sum_{|K| \leq k-2} |\nabla \nabla_{\mathscr{L}}^J h|}_{\text{absent if } k \leq 1} + \sum_{|J| \leq k} |\overline{\nabla} \nabla_{\mathscr{L}}^J h| + \sum_{|I_1| + |I_2| \leq k} |\nabla_{\mathscr{L}}^{I_1} h| |\nabla \nabla_{\mathscr{L}}^{I_2} h|. \quad (15.4.3)
$$

By Corollary 15.3, we have that

$$
\sum_{|J| \leq k} |\overline{\nabla} \nabla_{\mathcal{L}}^J h| + \sum_{|I_1|+|I_2| \leq k} |\nabla \nabla_{\mathcal{L}}^{I_1} h| |\nabla \nabla_{\mathcal{L}}^{I_2} h|
$$
$$
\lesssim \begin{cases} \varepsilon (1+t+|q|)^{-2+2\delta}(1+|q|)^{-2\delta} & \text{if } q > 0, \\ \varepsilon (1+t+|q|)^{-2+2\delta}(1+|q|)^{1/2-\delta} & \text{if } q < 0 \end{cases} \quad (k \leq \ell - 4). \quad (15.4.4)
$$

Combining (15.4.3) and (15.4.4), we deduce (15.4.1). Inequality (15.4.2) follows from integrating inequality (15.4.1) for $|\partial_q \nabla_{\mathcal{L}}^I h| \lesssim |\nabla \nabla_{\mathcal{L}}^I h|$, $q \overset{\text{def}}{=} |x| - t$, along the lines along which the angle $\omega \overset{\text{def}}{=} x/|x|$ and the null coordinate $s = |x| + t$ are constant (i.e., integrating $dq$) and using (15.1.2b) at $t = 0$.

The proofs of the estimates for $H^{\mu\nu}$ follow from the estimates for $h_{\mu\nu}$, (3.3.11a), and Corollary 15.3. This concludes our proof of the lemma. $\qquad \square$

Having proved the lemma, inequalities (15.3.1a) and (15.3.1b) now follow from inequalities (15.4.1) and (15.4.2) and the weak decay estimates of Corollary 15.3.

**15.4.2.** *Proof of* (15.3.3). Let $\mathcal{W} \overset{\text{def}}{=} \{(t, x) \mid |x| \geq 1 + t/2\} \cap \{(t, x) \mid |x| \leq 2t - 1\}$ denote the "wave-zone" region. Note that $r \approx 1 + t + |q| \approx 1 + t + s$ for $(t, x) \in \mathcal{W}$. Now as in the proof of Proposition 15.5, inequality (15.3.3) follows from the weak decay estimates of Corollary 15.3 if $(t, x) \notin \mathcal{W}$. Furthermore, we have that $|\mathcal{F}| \approx |\underline{\alpha}[\mathcal{F}]| + |\alpha[\mathcal{F}]| + |\rho[\mathcal{F}]| + |\sigma[\mathcal{F}]|$, and by Proposition 15.5, inequality (15.3.3) has already been shown to hold for $|\alpha[\mathcal{F}]| + |\rho[\mathcal{F}]| + |\sigma[\mathcal{F}]| \approx |\mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{F}|_{\mathcal{T}\mathcal{T}}$.

It remains to prove the desired estimate for $|\underline{\alpha}[\mathcal{F}(t, x)]|$ under the assumption that $(t, x) \in \mathcal{W}$. To this end, we use (11.1.12), the weak decay estimates of Corollary 15.3, and Proposition 15.5 to deduce that if $(t, x) \in \mathcal{W}$ then

$$
\left| \nabla_\Lambda (r \underline{\alpha}[\mathcal{F}]) \right| \lesssim \varepsilon (1+t+|q|)^{-3/2+\delta} \left| r \underline{\alpha}[\mathcal{F}] \right| + \varepsilon (1+t+|q|)^{-2+3\delta}, \quad (15.4.5)
$$

where $\Lambda \overset{\text{def}}{=} L + \frac{1}{4} h_{LL} \underline{L}$. Let $(\tau(\lambda), y(\lambda))$ be the integral curve[30] of the vector field $\Lambda$ passing through the point $(t, x) = (\tau(\lambda_1), y(\lambda_1)) \in \mathcal{W}$. By the already-proved smallness estimate (15.3.1b) for $h_{LL}$, every such integral curve must intersect the boundary of $\mathcal{W}$ at a point $(t_0, x_0) = (\tau(\lambda_0), y(\lambda_0))$ to the *past* of $(t, x)$. Furthermore, by (15.3.1b) again, we have that $\frac{d\tau}{d\lambda} \approx 1$ along the integral curves, and for all $(\tau, y) \in \mathcal{W}$, we have that $|y| \approx \tau \approx 1 + |\tau| \approx 1 + |\tau| + ||y| - \tau|$. We now set $f(\lambda) \overset{\text{def}}{=} \left| |y(\lambda)| \underline{\alpha}[\mathcal{F}(\tau(\lambda), y(\lambda))] \right|$, integrate inequality (15.4.5) along the integral curve (which is contained in $\mathcal{W}$), use the assumption $0 < \delta < \frac{1}{4}$, and change variables so that $\tau$ is the integration variable to obtain

$$
\overbrace{\left| r \underline{\alpha}[\mathcal{F}](t, x) \right|}^{f(\lambda(t))} \leq \overbrace{\left| r_0 \underline{\alpha}[\mathcal{F}(t_0, x_0)] \right|}^{f(\lambda_0)} + C\varepsilon \int_{\lambda=\lambda_0}^{\lambda=\lambda_1} [1+\tau(\lambda)]^{-2+3\delta} \, d\lambda + C\varepsilon \int_{\lambda=\lambda_0}^{\lambda=\lambda_1} [1+\tau(\lambda)]^{-3/2+\delta} f(\lambda) \, d\lambda
$$
$$
\leq C\varepsilon + C\varepsilon \int_{\tau=t_0}^{\tau=t} (1+\tau)^{-2+3\delta} \, d\tau + C\varepsilon \int_{\tau=t_0}^{\tau=t} (1+\tau)^{-3/2+\delta} f(\lambda \circ \tau) \, d\tau
$$
$$
\leq C\varepsilon + C\varepsilon \int_{\tau=t_0}^{\tau=t} (1+\tau)^{-3/2+\delta} f(\lambda \circ \tau) \, d\tau, \quad (15.4.6)
$$

---

[30]By integral curve, we mean the solution to the ODE system $\frac{d\tau}{d\lambda} = \Lambda^0(\tau, y)$ and $\frac{dy^j}{d\lambda} = \Lambda^j(\tau, y)$ $(j = 1, 2, 3)$ passing through the point $(t, x)$ at parameter value $\lambda = \lambda_1$.

where we have used (15.1.2a) to obtain the bound $|r_0\underline{\alpha}[\mathcal{F}(t_0, x_0)]| \le C\varepsilon$ for the point $(t_0, x_0)$ lying on the boundary of $\mathcal{W}$. Applying Gronwall's lemma to (15.4.6), we deduce that

$$\left| r\underline{\alpha}[\mathcal{F}(t, x)] \right| \le C\varepsilon \exp\left( C\varepsilon \int_{\tau=t_0}^{\tau=t} (1+\tau)^{-3/2+\delta} \, d\tau \right) \le C\varepsilon, \tag{15.4.7}$$

from which it trivially follows that

$$\left| \underline{\alpha}[\mathcal{F}(t, x)] \right| \le C\varepsilon r^{-1} \le C\varepsilon(1+t+|q|)^{-1} \tag{15.4.8}$$

as desired.

**15.4.3.** *Proofs of* (15.3.2a) *and* (15.3.2b). In the next two lemmas, we will use the fact that the tensor field $h_{\mu\nu} \stackrel{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$ is a solution to the system

$$\widetilde{\Box}_g h_{\mu\nu} = \mathfrak{H}_{\mu\nu}, \tag{15.4.9}$$

where the inhomogeneous term $\mathfrak{H}_{\mu\nu}$ is defined in (3.7.2a).

**Lemma 15.9** (Pointwise estimates for the $\mathfrak{H}_{\mu\nu}$ inhomogeneities; extension of [Lindblad and Rodnianski 2010, Lemma 10.5]). *Suppose that the assumptions of Proposition 15.6 hold. Then if $\varepsilon$ is sufficiently small, the following pointwise estimates hold for $(t, x) \in [0, T) \times \mathbb{R}^3$:*

$$|\mathfrak{H}|_{\mathcal{T}\mathcal{N}} \le C\varepsilon(1+t+|q|)^{-3/2+\delta}|\nabla h| + C\varepsilon(1+t+|q|)^{-5/2+\delta}, \tag{15.4.10a}$$

$$|\mathfrak{H}| \le C\varepsilon(1+t+|q|)^{-3/2+\delta}|\nabla h| + C|\nabla h|_{\mathcal{T}\mathcal{N}}^2 + C\varepsilon^2(1+t+|q|)^{-2}. \tag{15.4.10b}$$

*Proof.* Lemma 15.9 follows from Proposition 11.3, Corollary 15.3, Proposition 15.5, the *already-proved* estimate (15.3.3), and the assumption $0 < \delta < \frac{1}{4}$. □

**Lemma 15.10** (Integral inequalities for $|\nabla h|_{\mathcal{T}\mathcal{N}}$ and $|\nabla h|$; extension of [Lindblad and Rodnianski 2010, Lemma 10.6]). *Suppose that the assumptions of Proposition 15.6 hold. Then if $\varepsilon$ is sufficiently small, the following integral inequalities hold for $t \in [0, T)$:*

$$(1+t)\||\nabla h|_{\mathcal{T}\mathcal{N}}(t, \cdot)\|_{L^\infty} \le C\varepsilon + C\varepsilon \int_0^t (1+\tau)^{-1/2+\delta}\|\nabla h(\tau, \cdot)\|_{L^\infty} \, d\tau, \tag{15.4.11a}$$

$$(1+t)\|\nabla h(t, \cdot)\|_{L^\infty} \le C\varepsilon + C\varepsilon^2 \ln(1+t) + C\varepsilon \int_0^t (1+\tau)^{-1/2+\delta}\|\nabla h(\tau, \cdot)\|_{L^\infty} \, d\tau$$

$$+ C\varepsilon \int_0^t (1+\tau)\||\nabla h|_{\mathcal{T}\mathcal{N}}^2(\tau, \cdot)\|_{L^\infty} \, d\tau. \tag{15.4.11b}$$

*Proof.* We first observe that (15.1.2b) and (15.3.1b) (the version for the tensor $H$) imply that the hypotheses of Lemma 13.2 and Corollary 13.3 hold. Therefore, using the lemma and the corollary with $\varpi(q) \stackrel{\text{def}}{=} 1$ and $\alpha \stackrel{\text{def}}{=} 0$, and noting that $h_{\mu\nu}$ satisfies the system (15.4.9), we have that

$$(1+t)|\nabla h|_{\mathcal{T}\mathcal{N}} \lesssim \sup_{0\le\tau\le t} \sum_{|I|\le 1} \|\nabla_{\mathscr{L}}^I h(\tau, \cdot)\|_{L^\infty} + \int_{\tau=0}^t (1+\tau)\||\mathfrak{H}|_{\mathcal{T}\mathcal{N}}\|_{L^\infty(D_\tau)} \, d\tau$$

$$+ \sum_{|I|\le 2} \int_{\tau=0}^t (1+\tau)^{-1}\|\nabla_{\mathscr{L}}^I h\|_{L^\infty(D_\tau)} \, d\tau. \tag{15.4.12}$$

Using (15.1.2b) (the version for the tensor $h$), we estimate the first and third terms on the right-hand side of (15.4.12) as follows:

$$\sup_{0 \leq \tau \leq t} \sum_{|I| \leq 1} \|\nabla_{\mathscr{L}}^I h(\tau, \cdot)\|_{L^\infty} \leq C\varepsilon, \tag{15.4.13}$$

$$\sum_{|I| \leq 2} \int_{\tau=0}^t (1+\tau)^{-1} \|\nabla_{\mathscr{L}}^I h\|_{L^\infty(D_\tau)} \, d\tau \leq C\varepsilon \int_{\tau=0}^\infty (1+\tau)^{-3/2+\delta} \, d\tau \leq C\varepsilon. \tag{15.4.14}$$

To estimate the second term, we use (15.4.10a) to conclude that for $x \in D_t$ we have that

$$(1+t)|\mathfrak{H}|_{\mathscr{T}\mathscr{N}} \leq C\varepsilon(1+t)^{-1/2+\delta}|\nabla h| + C\varepsilon(1+t)^{-3/2+\delta}. \tag{15.4.15}$$

Inequality (15.4.11a) now follows from (15.4.12)–(15.4.15) and the fact that $C\varepsilon \int_0^t (1+\tau)^{-3/2+\delta} \, d\tau \leq C\varepsilon$. Inequality (15.4.11b) can be obtained in a similar fashion by using (15.4.10b). $\qquad\square$

To finish the proof of Proposition 15.6, we will apply the following Gronwall-type inequality:

**Lemma 15.11** (Gronwall-type inequality; slight modification of [Lindblad and Rodnianski 2010, Lemma 10.7]). *Assume that the continuous functions $b(t) \geq 0$ and $c(t) \geq 0$ satisfy*

$$b(t) \leq C\varepsilon + C\varepsilon \int_0^t (1+\tau)^{-1-a} c(\tau) \, d\tau, \tag{15.4.16a}$$

$$c(t) \leq C\varepsilon + C\varepsilon^2 \ln(1+t) + C\varepsilon \int_0^t (1+\tau)^{-1-a} c(\tau) \, d\tau + C \int_0^t (1+\tau)^{-1} b^2(\tau) \, d\tau \tag{15.4.16b}$$

*for some positive constants $a$ and $C$ such that $\varepsilon < a/4C$ and $\varepsilon < 2a/(1+4C^2)$. Then*

$$b(t) \leq 2C\varepsilon, \tag{15.4.17a}$$

$$c(t) \leq 2C\varepsilon(1 + a\ln(1+t)). \tag{15.4.17b}$$

*Proof.* We slightly modify the proof of [Lindblad and Rodnianski 2010, Lemma 10.7]. Let $T$ be the largest time such that the bounds (15.4.17a)–(15.4.17b) hold. Then inserting these bounds into the inequalities (15.4.16a)–(15.4.16b) and using the bound (and the change of variables $z \overset{\text{def}}{=} a\ln(1+\tau)$)

$$\int_{\tau=0}^\infty (1+\tau)^{-1-a}(1 + a\ln(1+\tau)) \, d\tau = a^{-1} \int_{z=0}^\infty e^{-z}(1+z) \, dz = 2a^{-1}, \tag{15.4.18}$$

we deduce that the following inequalities hold for $t \in [0, T]$:

$$b(t) \leq C\varepsilon(1 + 4C\varepsilon a^{-1}) < 2C\varepsilon, \tag{15.4.19}$$

$$c(t) \leq C\varepsilon(1 + 4C\varepsilon a^{-1} + (1+4C^2)\varepsilon\ln(1+t)) < 2C\varepsilon(1 + a\ln(1+t)). \tag{15.4.20}$$

Since the above inequalities are a strict improvement of the assumed bounds (15.4.17a)–(15.4.17b), we thus conclude that $T = \infty$. $\qquad\square$

To complete the proofs of (15.3.2a) and (15.3.2b), we apply Lemmas 15.10 and 15.11 with $b(t) \overset{\text{def}}{=} (1+t)\||\nabla h|_{\mathscr{T}\mathscr{N}}(t, \cdot)\|_{L^\infty}$ and $c(t) \overset{\text{def}}{=} (1+t)\|\nabla h(t, \cdot)\|_{L^\infty}$. This implies (15.3.2a) and (15.3.2b) with

$(1 + t)$ in place of $(1 + t + |q|)$. The additional decay in $|q|$ in (15.3.2a) and (15.3.2b) follows directly from (15.1.2a) (the version for the tensor $h$). $\qquad\square$

**15.5. *Proof of Proposition 15.7.*** We will prove the proposition using a series of inductive steps. We only prove the estimates for $h_{\mu\nu}^{(1)}$ and $\mathscr{F}_{\mu\nu}$. The estimates for $h_{\mu\nu}$ and $H^{\mu\nu}$ follow easily from those for $h_{\mu\nu}^{(1)}$, (3.3.11a), and Lemma 15.1. We first prove a technical lemma that will be used during the proof of the proposition.

**Lemma 15.12** (Pointwise estimates for the $|\nabla_{\mathscr{L}}^I \mathfrak{H}|$ inhomogeneities). *Suppose that the hypotheses of Proposition 15.6 hold, and let $\mathfrak{H}_{\mu\nu}$ be the inhomogeneous term on the right-hand side of the reduced equation* (3.7.1a). *Then if $I$ is any $\mathscr{L}$-multi-index with $|I| \leq \ell$, the following pointwise estimates hold for $(t, x) \in [0, T) \times \mathbb{R}^3$:*

$$
|\nabla_{\mathscr{L}}^I \mathfrak{H}| \leq C\varepsilon \sum_{|J| \leq |I|} (1 + t + |q|)^{-1} \big(|\nabla \nabla_{\mathscr{L}}^J h^{(1)}| + |\nabla_{\mathscr{L}}^J \mathscr{F}|\big)
$$

$$
+ C \sum_{\substack{|I_1| + |I_2| \leq |I| \\ |I_1|, |I_2| \leq |I| - 1}} \big(|\nabla \nabla_{\mathscr{L}}^{I_1} h^{(1)}| + |\mathscr{L}_{\mathscr{L}}^{I_1} \mathscr{F}|\big)\big(|\nabla \nabla_{\mathscr{L}}^{I_2} h^{(1)}| + |\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}|\big) + C\varepsilon^2 (1 + t + |q|)^{-4}. \quad (15.5.1)
$$

*Proof.* Lemma 15.12 follows from (11.1.5c), Lemma 15.1, the weak decay estimates of Corollary 15.3, (15.3.2a), (15.3.3), and the assumption that $0 < \delta < \frac{1}{4}$. We remark that the $C\varepsilon^2(1 + t + |q|)^{-4}$ term arises from the estimate $|\nabla \nabla_{\mathscr{L}}^{I_1} h^{(0)}||\nabla \nabla_{\mathscr{L}}^{I_2} h^{(0)}| \leq C\varepsilon^2 (1 + t + |q|)^{-4}$. $\qquad\square$

We are now ready for the proof of Proposition 15.7. To prove (15.3.4a)–(15.3.4c), we will argue inductively, using the inequalities in the case $|I| \leq k$ to deduce that they hold in the case $|I| = k + 1$. We also remark that the base case $k = 0$ is covered by our argument.

*Induction Step 1: Upgraded pointwise decay estimates for $|\nabla_{\mathscr{L}}^I h|_{\mathscr{L}\mathscr{L}}$ for $|I| = k + 1$ and $|\nabla_{\mathscr{L}}^J h|_{\mathscr{L}\mathscr{T}}$ for $|J| = k$.* As a first step, we will use the wave-coordinate condition to upgrade the estimates for $|\nabla_{\mathscr{L}}^I h|_{\mathscr{L}\mathscr{L}}$ for $|I| = k + 1$ and $|\nabla_{\mathscr{L}}^J h|_{\mathscr{L}\mathscr{T}}$ for $|J| = k$. To this end, we appeal to inequality (15.4.2), using inequality (15.3.4a) for $h$ under the induction hypothesis to bound the integrand and thereby concluding that

$$
\sum_{|I| = k+1} |\nabla_{\mathscr{L}}^I h|_{\mathscr{L}\mathscr{L}} + \sum_{|J| = k} |\nabla_{\mathscr{L}}^J h|_{\mathscr{L}\mathscr{T}} \lesssim \begin{cases} \varepsilon(1 + t + |q|)^{-1 + M_{k-1}\varepsilon}(1 + |q|)^{-M_{k-1}\varepsilon} & \text{if } q > 0, \\ \varepsilon(1 + t + |q|)^{-1 + M_{k-1}\varepsilon}(1 + |q|)^{1/2 + \mu'} & \text{if } q < 0. \end{cases} \quad (15.5.2)
$$

In the above estimates, the constant $\mu'$ is subject to the restrictions stated in the hypotheses of Proposition 15.7. Furthermore, since $H^{\mu\nu} = -h^{\mu\nu} + O^\infty(|h|^2)$, (15.1.2b) implies that the same estimates hold for the tensor $H$.

*Induction Step 2: Upgraded pointwise decay estimates for $|\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|$ and $|I| = k + 1$.* Let $\mathscr{W} \overset{\text{def}}{=} \{(t, x) \mid |x| \geq 1 + t/2\} \cap \{(t, x) \mid |x| \leq 2t - 1\}$ denote the "wave-zone" region. Then for $(t, x) \notin \mathscr{W}$, we have that $1 + |q| \approx 1 + t + |q|$. Using this fact, we see that for $(t, x) \notin \mathscr{W}$ the weak decay estimate (15.1.2a) implies that inequality (15.3.4a) holds for $|\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|$ in the case $|I| = k + 1$. Furthermore, by Proposition 15.5, the inequality (15.3.4a) holds for the null components $|\alpha[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}]|$, $|\rho[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}]|$, and $|\sigma[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}]|$ when $|I| = k + 1$.

It remains to consider $|\underline{\alpha}[\mathcal{L}_{\mathcal{L}}^I \mathcal{F}(t, x)]|$ in the case $(t, x) \in \mathcal{W}$. Note that $r \approx 1 + t + |q| \approx 1 + t + s$ for $(t, x) \in \mathcal{W}$. We will make use of the weight $\varpi(q)$ defined in (13.1.1). From (11.1.13a), Corollary 15.3 (the version for the tensor field $h$), Proposition 15.5, (15.3.1b), (15.3.3), the induction hypothesis, and (15.5.2), it follows that

$$\sum_{|I| \leq k+1} \left| \nabla_\Lambda \left( r\varpi(q)\underline{\alpha}[\mathcal{L}_{\mathcal{L}}^I \mathcal{F}] \right) \right| \leq C\varepsilon(1 + t + |q|)^{-1} \sum_{|I| \leq k+1} \left| r\varpi(q)\underline{\alpha}[\mathcal{L}_{\mathcal{L}}^I \mathcal{F}] \right|$$

$$+ C\varepsilon(1 + t + |q|)^{-(1+a)} + C\varepsilon^2(1 + t + |q|)^{-1+C\varepsilon}, \quad (15.5.3)$$

where $0 < a < \min\{\mu' - \delta, \gamma - \delta - \gamma'\}$ is a fixed constant and $\Lambda \overset{\text{def}}{=} L + \frac{1}{4}h_{LL}\underline{L}$. Note the importance of the independent estimate (15.3.1b) for bounding the second, fourth, and fifth sums on the right-hand side of (11.1.13a) and of the independent estimate (15.5.2) (in the case $|I| = k + 1$) for bounding the third sum on the right-hand side of (11.1.13a).

Let $(\tau(\lambda), y(\lambda))$ be the integral curve (as defined in Section 15.4.2) of the vector field $\Lambda$ passing through the point $(t, x) = (\tau(\lambda_1), y(\lambda_1)) \in \mathcal{W}$. By the inequality (15.3.1b) for $h_{LL}$, every such integral curve must intersect the boundary of $\mathcal{W}$ at a point $(t_0, x_0) = (\tau(\lambda_0), y(\lambda_0))$ lying to the *past* of $(t, x)$. Using (15.3.1b) again, we have that $\frac{dt}{d\lambda} \approx 1$ along the integral curves, and in the entire region $\mathcal{W}$, we have that $|y| \approx \tau \approx 1 + |\tau| \approx 1 + |\tau| + ||y| - \tau|$. We define $f(\lambda) \overset{\text{def}}{=} \sum_{|I| \leq k+1} ||y(\lambda)|\varpi(q(\lambda))\underline{\alpha}[\mathcal{L}_{\mathcal{L}}^I \mathcal{F}(\tau(\lambda), y(\lambda))]|$, where $q(\lambda) \overset{\text{def}}{=} |y(\lambda)| - \tau(\lambda)$. Note that $f(\lambda_1) = \sum_{|I| \leq k+1} |r\varpi(q)\underline{\alpha}[\mathcal{L}_{\mathcal{L}}^I \mathcal{F}]|$, where $q \overset{\text{def}}{=} q(\lambda_1) = |x| - t$ while the weak decay estimate (15.1.2a) implies that $f(\lambda_0) \leq C\varepsilon$. Integrating inequality (15.5.3) and changing variables so that $\tau$ is the integration variable, we have that

$$\underbrace{f(\lambda_1)}_{f(\lambda(t))} \leq f(\lambda_0) + C\varepsilon \int_{\lambda=\lambda_0}^{\lambda=\lambda_1} [1 + \tau(\lambda)]^{-1} f(\lambda) \, d\lambda$$

$$+ C\varepsilon \int_{\lambda=\lambda_0}^{\lambda=\lambda_1} [1 + \tau(\lambda)]^{-(1+a)} \, d\lambda + C\varepsilon^2 \int_{\lambda=\lambda_0}^{\lambda=\lambda_1} [1 + \tau(\lambda)]^{-1+C\varepsilon} \, d\lambda$$

$$\leq C\varepsilon(1 + t)^{C\varepsilon} + C\varepsilon \int_{\tau=t_0}^{\tau=t} (1 + \tau)^{-1} f(\lambda \circ \tau) \, d\tau. \quad (15.5.4)$$

Applying Gronwall's inequality to (15.5.4), we have that

$$f(\underbrace{\lambda \circ t}_{\lambda_1}) \leq C\varepsilon(1 + t)^{C\varepsilon} \exp\left( C\varepsilon \int_{\tau=t_0}^{\tau=t} (1 + \tau)^{-1} \, d\tau \right)$$

$$\leq C\varepsilon(1 + t)^{2C\varepsilon}, \quad (15.5.5)$$

from which it easily follows that for $(t, x) \in \mathcal{W}$ we have that

$$\sum_{|I| \leq k+1} \left| \underline{\alpha}[\mathcal{L}_{\mathcal{L}}^I \mathcal{F}] \right| \leq C\varepsilon(1 + t)^{-1+2C\varepsilon}\varpi^{-1}(q). \quad (15.5.6)$$

Combining (15.5.6) and the previous arguments covering $(t, x) \notin \mathcal{W}$ and the other null components of $\mathcal{L}_{\mathcal{L}}^I \mathcal{F}$, we have shown that the estimate (15.3.4a) holds for $|\mathcal{L}_{\mathcal{L}}^I \mathcal{F}|$ in the case $|I| = k + 1$.

*Final Induction Step: Upgraded pointwise decay estimates for* $|\nabla \nabla_{\mathscr{L}}^I h|$ *and* $|\nabla_{\mathscr{L}}^I h|$ $(|I| = k+1)$. Our first goal is to prove the following estimate in the case $|I| = k+1$:

$$|\widetilde{\Box}_g \nabla_{\mathscr{L}}^I h^{(1)}| \lesssim \varepsilon \sum_{|K| \leq |I|} (1+t+|q|)^{-1} |\nabla \nabla_{\mathscr{L}}^K h^{(1)}| + \begin{cases} \varepsilon^2 (1+t+|q|)^{-4+\delta} (1+|q|)^{-\delta} & \text{if } q > 0, \\ \varepsilon (1+t+|q|)^{-3} & \text{if } q < 0 \end{cases}$$

$$+ \begin{cases} \varepsilon^2 (1+t+|q|)^{-2+2M_k \varepsilon} (1+|q|)^{-1-\gamma'} & \text{if } q > 0, \\ \varepsilon^2 (1+t+|q|)^{-2+2M_k \varepsilon} (1+|q|)^{-1/2+\mu'} & \text{if } q < 0. \end{cases} \quad (15.5.7)$$

To prove (15.5.7), we first recall Corollary 11.7, which states that

$$|\widetilde{\Box}_g \nabla_{\mathscr{L}}^I h^{(1)}| \lesssim |\widehat{\nabla}_{\mathscr{L}}^I \mathfrak{H}| + |\widehat{\nabla}_{\mathscr{L}}^I \widetilde{\Box}_g h^{(0)}| + (1+t+|q|)^{-1} \sum_{|K| \leq |I|} \sum_{|J|+(|K|-1)_+ \leq |I|} |\nabla \nabla_{\mathscr{L}}^K h^{(1)}| |\nabla_{\mathscr{L}}^J H|$$

$$+ (1+|q|)^{-1} \sum_{|K| \leq |I|} \sum_{|J|+(|K|-1)_+ \leq |I|} |\nabla \nabla_{\mathscr{L}}^K h^{(1)}| |\nabla_{\mathscr{L}}^J H|_{\mathscr{L}\mathscr{L}} \quad (15.5.8)$$

$$+ (1+|q|)^{-1} \sum_{|K| \leq |I|} \sum_{|J'|+(|K|-1)_+ \leq |I|-1} |\nabla \nabla_{\mathscr{L}}^K h^{(1)}| |\nabla_{\mathscr{L}}^{J'} H|_{\mathscr{L}\mathscr{T}}$$

$$+ (1+|q|)^{-1} \underbrace{\sum_{|K| \leq |I|} \sum_{|J''|+(|K|-1)_+ \leq |I|-2} |\nabla \nabla_{\mathscr{L}}^K h^{(1)}| |\nabla_{\mathscr{L}}^{J''} H|}_{\text{absent if } |I| \leq 1 \text{ or } |K| = |I|}, \quad (15.5.9)$$

where $(|K|-1)_+ \overset{\text{def}}{=} 0$ if $|K| = 0$ and $(|K|-1)_+ \overset{\text{def}}{=} |K|-1$ if $|K| \geq 1$. We first bound the terms from line (15.5.8) onwards, considering separately the cases $|K| < |I|$ and $|K| = |I| = k+1$. For $|K| < |I| = k+1$, we use (15.5.2) (for the tensor field $H$) and (15.3.4b) (for the tensor field $H$) under the induction hypotheses to conclude that

$$(1+|q|)^{-1} \sum_{\substack{|J| \leq k+1 \\ |J'| \leq k \\ |J''| \leq k-1}} \left( |\nabla_{\mathscr{L}}^J H|_{\mathscr{L}\mathscr{L}} + |\nabla_{\mathscr{L}}^{J'} H|_{\mathscr{L}\mathscr{T}} + |\nabla_{\mathscr{L}}^{J''} H| \right)$$
$$\lesssim \begin{cases} \varepsilon (1+t+|q|)^{-1+M_k \varepsilon} (1+|q|)^{-1-M_k \varepsilon} & \text{if } q > 0, \\ \varepsilon (1+t+|q|)^{-1+M_k \varepsilon} (1+|q|)^{-1/2+\mu'} & \text{if } q < 0. \end{cases} \quad (15.5.10)$$

Also using (15.3.4a) under the induction hypotheses to bound $|\nabla \nabla_{\mathscr{L}}^K h^{(1)}|$, we deduce that all of the terms from line (15.5.8) onwards in the case $|K| < |I|$ can be bounded by the last term on the right-hand side of (15.5.7).

We now consider the case $|K| = |I| = k+1$. Since $|J| \leq 1$ and $|J'| = 0$ in this case, we can use (15.3.1b) (for the tensor field $H$) to deduce the bound

$$(1+|q|)^{-1} \sum_{|K|=|I|} \left( |\nabla \nabla_{\mathscr{L}}^K h^{(1)}| \left( \sum_{|J|+(|K|-1)_+ \leq |I|} |\nabla_{\mathscr{L}}^J H|_{\mathscr{L}\mathscr{L}} + \sum_{|J'|+(|K|-1)_+ \leq |I|-1} |\nabla_{\mathscr{L}}^{J'} H|_{\mathscr{L}\mathscr{T}} \right) \right)$$

$$\lesssim \varepsilon \sum_{|K|=|I|} (1+t+|q|)^{-1} |\nabla \nabla_{\mathscr{L}}^K h^{(1)}|. \quad (15.5.11)$$

Thus, all of the terms from line (15.5.8) onwards in the case $|K| = |I| = k+1$ can be bounded by the first term on the right-hand side of (15.5.7).

With the help of Corollary 15.3 (the version for the tensor field $H$), the

$$(1+t+|q|)^{-1} \sum_{|K|\leq|I|} \sum_{|J|+(|K|-1)_+\leq|I|} |\nabla_{\mathscr{L}}^J H||\nabla\nabla_{\mathscr{L}}^K h^{(1)}|$$

sum on the right-hand side of (15.5.9) can be bounded by the first sum on the right-hand side of (15.5.7).

For the $|\widehat{\nabla}_{\mathscr{L}}^I \widetilde{\square}_g h^{(0)}|$ term from the right-hand side of (15.5.9), we simply use Lemma 15.4, which shows that $|\widehat{\nabla}_{\mathscr{L}}^I \widetilde{\square}_g h^{(0)}|$ is bounded by the next-to-last term on the right-hand side of (15.5.7).

To bound the $|\nabla_{\mathscr{L}}^I \mathfrak{H}|$ term from the right-hand side of (15.5.9), we apply Lemma 15.12. Using the already-proved upgraded estimates for $|\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|$ ($|I| \leq k+1$), we see that the first and third sums from the right-hand side of (15.5.1) are bounded by the right-hand side of (15.5.7). The second sum

$$\sum_{\substack{|J|+|K|\leq|I| \\ |J|\leq|K|<|I|}} \left(|\nabla\nabla_{\mathscr{L}}^J h^{(1)}| + |\mathscr{L}_{\mathscr{L}}^J \mathscr{F}|\right)\left(|\nabla\nabla_{\mathscr{L}}^K h^{(1)}| + |\mathscr{L}_{\mathscr{L}}^K \mathscr{F}|\right)$$

from the right-hand side of (15.5.1) can be bounded by the last term on the right-hand side of (15.5.7) by using the induction hypotheses since $|J| \leq |K| \leq k$. This completes the proof of (15.5.7) in the case of $|I| = k+1$.

To obtain the desired upgraded pointwise estimate for $|\nabla\nabla_{\mathscr{L}}^I h^{(1)}|$, we will estimate the quantity

$$n_{k+1}(t) \overset{\text{def}}{=} (1+t) \sum_{|I|\leq k+1} \left\| \varpi(q)\nabla\nabla_{\mathscr{L}}^I h^{(1)}(t,\cdot) \right\|_{L^\infty}, \tag{15.5.12}$$

where $\varpi(q)$ is the weight defined in (13.1.1). Our goal is to use Lemma 13.2 with $\phi \overset{\text{def}}{=} \nabla_{\mathscr{L}}^I h_{\mu\nu}^{(1)}$ to obtain an integral inequality for $n_{k+1}(t)$ that is amenable to Gronwall's inequality. We begin by estimating the terms on the right-hand side of (13.2.8). First, with $a \overset{\text{def}}{=} \min(\mu'-\delta, \gamma-\delta-\gamma') > 0$, by the weak decay estimate (15.1.2b), we have that

$$\varpi(q)|\nabla_{\mathscr{L}}^I h^{(1)}| \lesssim \begin{cases} \varepsilon(1+t+|q|)^{-1+\delta}(1+|q|)^{1+\gamma'-\gamma} & \text{if } q>0, \\ \varepsilon(1+t+|q|)^{-1+\delta}(1+|q|)^{1-\mu'} & \text{if } q<0 \end{cases} \lesssim \varepsilon(1+t)^{-a} \quad (|I|\leq\ell-3). \tag{15.5.13}$$

This will serve as a suitable bound for estimating the first and fourth sums on the right-hand side of (13.2.8).

Next, using (15.5.7) and the definition (15.5.12), we deduce the following pointwise estimate:

$$\varpi(q)|\widetilde{\square}_g \nabla_{\mathscr{L}}^I h^{(1)}| \lesssim (1+t)^{-2}\left(\varepsilon n_{k+1} + \varepsilon^2(1+t)^{2M_k\varepsilon} + \varepsilon(1+t)^{-1/2-\mu'}\right). \tag{15.5.14}$$

This will serve as a suitable bound for estimating the third sum on the right-hand side of (13.2.8).

We now apply Lemma 13.2, using (15.5.13), (15.5.14), and the assumption $k+1 \leq \ell-5$ to deduce that

$$
\begin{aligned}
n_{k+1}(t) \leq C \sup_{0 \leq \tau \leq t} \sum_{|I| \leq k+2} & \left\| \varpi(q) \nabla_{\mathscr{L}}^I h^{(1)}(\tau, \cdot) \right\|_{L^\infty} \\
& + C \sum_{|I| \leq k+1} \int_0^t \varepsilon \left\| \varpi(q) \nabla \nabla_{\mathscr{L}}^I h^{(1)}(\tau, \cdot) \right\|_{L^\infty} d\tau \\
& + C \sum_{|I| \leq k+1} \int_0^t (1+\tau) \left\| \varpi(q) |\widetilde{\Box}_g \nabla_{\mathscr{L}}^I h^{(1)}|(\tau, \cdot) \right\|_{L^\infty(D_\tau)} d\tau \\
& + C \sum_{|I| \leq k+3} \int_0^t (1+\tau)^{-1} \left\| \varpi(q) \nabla_{\mathscr{L}}^I h^{(1)}(\tau, \cdot) \right\|_{L^\infty(D_\tau)} d\tau \\
\leq C\varepsilon(1+t)^{-a} + C & \int_0^t (1+\tau)^{-1} \varepsilon n_{k+1}(\tau) \, d\tau \\
& + C \int_0^t (1+\tau)^{-1} \left\{ \varepsilon^2 (1+\tau)^{C\varepsilon} + \varepsilon(1+\tau)^{-1/2-\mu'} + \varepsilon(1+\tau)^{-a} \right\} d\tau \\
\leq C\varepsilon + C\varepsilon(1+t)^{C\varepsilon} & + C\varepsilon \int_0^t (1+\tau)^{-1} n_{k+1}(\tau) \, d\tau. \tag{15.5.15}
\end{aligned}
$$

From (15.5.15) and Gronwall's inequality, we conclude that $n_{k+1}(t) \leq 2C\varepsilon(1+t)^{2C\varepsilon}$, which proves (15.3.4a) in the case $|I| = k+1$. As in our proof of Lemma 15.8, the estimate (15.3.4b) follows from integrating the bound for $|\partial_q \nabla_{\mathscr{L}}^I h^{(1)}|$ implied by (15.3.4a) along the line $\omega \overset{\text{def}}{=} x/|x| = \text{constant}$ and $t + |x| = \text{constant}$, from the hyperplane $t = 0$, and using (15.1.2b) at $t = 0$. This closes the induction argument. We have completed the proof of Proposition 15.7 with the exception of showing that inequality (15.3.4c) holds for $|\overline{\nabla} \nabla_{\mathscr{L}}^I h^{(1)}|$, $|\overline{\nabla} \mathscr{L}_{\mathscr{L}}^I \mathscr{F}|$, $|\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{L}\mathscr{N}}$, and $|\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{T}\mathscr{T}}$, where $|I| \leq \ell-6$. In the next paragraph, we address these inequalities using an argument that is not part of the induction process.

*Upgraded pointwise decay estimates for* $|\overline{\nabla} \nabla_{\mathscr{L}}^I h^{(1)}|$, $|\overline{\nabla} \mathscr{L}_{\mathscr{L}}^I \mathscr{F}|$, $|\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{L}\mathscr{N}}$, *and* $|\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{T}\mathscr{T}}$ ($|I| \leq \ell - 6$). We first note that inequality (15.3.4c) for $|\overline{\nabla} \nabla_{\mathscr{L}}^I h^{(1)}|$ and $|\overline{\nabla} \mathscr{L}_{\mathscr{L}}^I \mathscr{F}|$ follows from Lemma 6.16, (6.5.22), (15.3.4a), and (15.3.4b).

We now focus on proving the estimate (15.3.4c) for $|\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{L}\mathscr{N}}$ and $|\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{T}\mathscr{T}}$ in (15.3.4c); all of the other estimates of Proposition 15.7 have already been proved. Recall that $|\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{T}\mathscr{T}} \approx |\alpha[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}]| + |\rho[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}]| + |\sigma[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}]|$. We will prove the desired estimate for $|\alpha[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}]|$ in detail; the proofs for $|\rho[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}]|$ and $|\sigma[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}]|$ are similar.

Our proof mirrors the proof of Proposition 15.5 except that we now are able to use the already-proved upgraded estimates of Proposition 15.7 in place of the weak decay estimates of Corollary 15.3. We will use the notation defined in the proof of Proposition 15.5. With the help of the upgraded pointwise decay estimates (15.3.4a) and (15.3.4b) (including the versions for the tensor field $h = h^{(0)} + h^{(1)}$), inequality (15.2.2) for $f(t, x) \overset{\text{def}}{=} r\alpha[\mathscr{L}_{\mathscr{L}}^I \mathscr{F}(t, x)]$ can be upgraded to

$$
|\partial_q f(t, x)| \leq \begin{cases} C_k \varepsilon(1+s)^{-1+C\varepsilon}(1+|q|)^{-1-\gamma'} & \text{if } q > 0, \\ C_k \varepsilon(1+s)^{-1+C\varepsilon}(1+|q|)^{-1/2+\mu'} & \text{if } q < 0 \end{cases} \quad (|I| \leq \ell - 6). \tag{15.5.16}
$$

Arguing as in the proof of Proposition 15.5, and using in particular (15.2.4), we deduce from (15.5.16) that

$$
\left| r\alpha[\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}(t, x)] \right| \leq C\varepsilon (1+s)^{-1-\overbrace{(\gamma-\delta)}^{>0}}
$$

$$
+ \begin{cases} C_k \varepsilon (1+s)^{-1+C\varepsilon}(1+|q'|)^{-\gamma'} & \text{if } q' > 0, \\ C_k \varepsilon (1+s)^{-1+C\varepsilon}(1+|q'|)^{1/2+\mu'} & \text{if } q' < 0 \end{cases} \quad (|I| \leq \ell - 6), \quad (15.5.17)
$$

from which it easily follows that

$$
\left| \alpha[\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}(t, x)] \right| \leq \begin{cases} C_k \varepsilon (1+t+|q|)^{-2+C\varepsilon}(1+|q|)^{-\gamma'} & \text{if } q > 0, \\ C_k \varepsilon (1+t+|q|)^{-2+C\varepsilon}(1+|q|)^{1/2+\mu'} & \text{if } q < 0 \end{cases} \quad (|I| \leq \ell - 6). \quad (15.5.18)
$$

We have thus obtained the desired bound (15.3.4c) for $|\alpha[\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}]|$. $\qquad\square$

## 16. Global existence and stability

In this section, we prove our main stability results. We separate our results into two theorems. The main conclusions are proved in Theorem 16.3, which is an easy consequence of Theorem 16.1. Theorem 16.1, which concerns the reduced equations (3.7.1a)–(3.7.1c), contains the crux of our bootstrap argument. In this theorem, we make certain assumptions concerning the smallness of the abstract initial data and various pointwise decay estimates for the solution on a local interval of existence $[0, T)$. We then use these assumptions to derive a "strong" a priori estimate for the energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$ of the reduced solution on the same interval $[0, T)$. Furthermore, in Section 15, the pointwise decay assumptions of Theorem 16.1 were shown to be *automatic consequences* of the smallness assumptions on the data and the "weak" bootstrap assumption (15.0.1) for $\mathscr{E}_{\ell;\gamma;\mu}(t)$ as long as $\ell \geq 10$. Consequently, in our proof of Theorem 16.3, we will be able to appeal to the continuation principle of Proposition 14.1 to conclude that the solution to the reduced equation exists globally in time. Furthermore, this line of reasoning leads to an estimate on the size of $\mathscr{E}_{\ell;\gamma;\mu}(t)$, which can be used to deduce various decay estimates for the global solution. The wave-coordinate condition plays a central role in many of the estimates in this section.

### 16.1. *Statement of the strong-a priori-energy-estimate theorem and proof of the global stability theorem.* We begin by recalling that the norm $E_{\ell;\gamma}(0) \geq 0$ of the abstract initial data is

$$
E_{\ell;\gamma}^2(0) \overset{\text{def}}{=} \|\underline{\nabla}\underline{\mathring{h}}^{(1)}\|_{H_{1/2+\gamma}^\ell}^2 + \|\mathring{K}\|_{H_{1/2+\gamma}^\ell}^2 + \|\mathring{\mathfrak{D}}\|_{H_{1/2+\gamma}^\ell}^2 + \|\mathring{\mathfrak{B}}\|_{H_{1/2+\gamma}^\ell}^2. \tag{16.1.1}
$$

We furthermore recall that the energy $\mathscr{E}_{\ell;\gamma;\mu}(t) \geq 0$ of the reduced solution is

$$
\mathscr{E}_{\ell;\gamma;\mu}^2(t) \overset{\text{def}}{=} \sup_{0 \leq \tau \leq t} \sum_{|I| \leq \ell} \int_{\Sigma_\tau} \left( |\nabla \nabla_{\mathscr{Z}}^I h^{(1)}|^2 + |\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|^2 \right) w(q)\, d^3x. \tag{16.1.2}
$$

In the above expressions, the weight function $w(q)$ and its derivative $w'(q)$ are

$$w = w(q) \overset{\text{def}}{=} \begin{cases} 1 + (1 + |q|)^{1+2\gamma} & \text{if } q > 0, \\ 1 + (1 + |q|)^{-2\mu} & \text{if } q < 0, \end{cases} \tag{16.1.3a}$$

$$w'(q) = \begin{cases} (1 + 2\gamma)(1 + |q|)^{2\gamma} & \text{if } q > 0, \\ 2\mu(1 + |q|)^{-2\mu-1} & \text{if } q < 0. \end{cases} \tag{16.1.3b}$$

The constants $\mu$ and $\gamma$ are subject to the restrictions summarized in Section 2.14. The spacetime metric is split into the three pieces

$$g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}, \tag{16.1.4a}$$

$$h_{\mu\nu}^{(0)} = \chi\left(\frac{r}{t}\right)\chi(r)\frac{2M}{r}\delta_{\mu\nu}, \tag{16.1.4b}$$

where the cut-off function $\chi$ is defined in (4.2.1). Furthermore, by Proposition 10.4, if $\varepsilon$ is sufficiently small and $E_{\ell;\gamma}(0) + M \leq \varepsilon$, then the initial energy for the reduced solution satisfies

$$\mathscr{E}_{\ell;\gamma;\mu}(0) \lesssim E_{\ell;\gamma}(0) + M \lesssim \varepsilon. \tag{16.1.5}$$

We now state our technical theorem concerning the derivation of a "strong" a priori energy estimate. The proof will be provided in Section 16.2.

**Theorem 16.1** (Derivation of a strong a priori energy estimate). *Let* $(g_{\mu\nu} \overset{\text{def}}{=} m_{\mu\nu} + \overbrace{h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}}^{h_{\mu\nu}}, \mathscr{F}_{\mu\nu})$ *be a local-in-time solution of the reduced equations* (3.7.1a)–(3.7.1c) *satisfying the wave-coordinate condition* (3.1.1a) *for* $(t, x) \in [0, T) \times \mathbb{R}^3$. *Let* $\ell \geq 0$ *be an integer. Suppose also that, for some constants* $\mu'$ *and* $\gamma$ *satisfying* $0 < \mu' < \frac{1}{2}$ *and* $0 < \gamma < \frac{1}{2}$, *for all vector fields* $Z \in \mathscr{Z}$, *for all* $\mathscr{Z}$-*multi-indices* $I$ *subject to the restrictions stated below, and for the sets* $\mathscr{L} = \{L\}$, $\mathscr{T} = \{L, e_1, e_2\}$, *and* $\mathscr{N} = \{\underline{L}, L, e_1, e_2\}$, *the following pointwise decay estimates hold for* $(t, x) \in [0, T) \times \mathbb{R}^3$:

$$(1 + |q|)^{-1}|h|_{\mathscr{L}\mathscr{T}} + (1 + |q|)^{-1}|\nabla_Z h|_{\mathscr{L}\mathscr{L}} + |\nabla h|_{\mathscr{T}\mathscr{N}} + |\mathscr{F}| \leq C\varepsilon(1 + t + |q|)^{-1}, \tag{16.1.6a}$$

$$(1 + |q|)^{-1}|\nabla_{\mathscr{Z}}^I h| + |\nabla\nabla_{\mathscr{Z}}^I h| + |\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|$$
$$\leq \begin{cases} C\varepsilon(1 + t + |q|)^{-1+C\varepsilon}(1 + |q|)^{-1-C\varepsilon} & \text{if } q > 0, \\ C\varepsilon(1 + t + |q|)^{-1+C\varepsilon}(1 + |q|)^{-1/2+\mu'} & \text{if } q < 0 \end{cases} \quad (|I| \leq \lfloor \ell/2 \rfloor), \tag{16.1.6b}$$

$$|\overline{\nabla}\nabla_{\mathscr{Z}}^I h| + (1 + |q|)|\overline{\nabla}\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}| + |\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|_{\mathscr{T}\mathscr{T}}$$
$$\leq \begin{cases} C\varepsilon(1 + t + |q|)^{-2+C\varepsilon}(1 + |q|)^{-C\varepsilon} & \text{if } q > 0, \\ C\varepsilon(1 + t + |q|)^{-2+C\varepsilon}(1 + |q|)^{1/2+\mu'} & \text{if } q < 0 \end{cases} \quad (|I| \leq \lfloor \ell/2 \rfloor). \tag{16.1.6c}$$

*In addition, assume that the following smallness conditions on the abstract initial data and ADM mass hold*:

$$E_{\ell;\gamma}(0) + M \leq \mathring{\varepsilon}. \tag{16.1.7}$$

*Then for any constant* $\mu$ *satisfying* $0 < \mu < \frac{1}{2} - \mu'$, *there exist positive constants* $\varepsilon_\ell$, $c_\ell$, *and* $\tilde{c}_\ell$ *depending on* $\ell$, $\mu$, $\mu'$, *and* $\gamma$ *such that, if* $\mathring{\varepsilon} \leq \varepsilon \leq \varepsilon_\ell$, *then the following energy inequality holds for* $t \in [0, T)$:

$$\mathscr{E}_{\ell;\gamma;\mu}(t) \leq c_\ell(\mathring{\varepsilon} + \varepsilon^{3/2})(1 + t)^{\tilde{c}_\ell \varepsilon}. \tag{16.1.8}$$

**Remark 16.2.** By Lemma 15.1, the decompositions $h = h^{(0)} + h^{(1)}$ and $H = H_{(0)} + H_{(1)}$ (where $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$), and the fact that $H^{\mu\nu}_{(1)} = -h^{(1)\mu\nu} + O^{\infty}(|h^{(0)} + h^{(1)}|^2)$, it follows that the estimates stated in the assumptions of the theorem also hold if we replace $h$ with $h^{(0)}$, $H_{(0)}$, $h^{(1)}$, or $H_{(1)}$.

We now state and (using the results of Theorem 16.1) prove our main global stability theorem.

**Theorem 16.3** (Global stability of the Minkowski spacetime solution). *Let* $(\mathring{\underline{g}}_{jk} = \delta_{jk} + \mathring{\underline{h}}^{(0)}_{jk} + \mathring{\underline{h}}^{(1)}_{jk}$, $\mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ $(j, k = 1, 2, 3)$ *be abstract initial data on the manifold* $\mathbb{R}^3$ *for the Einstein-nonlinear electromagnetic system* (1.0.1a)–(1.0.1c) *that satisfy the constraints* (4.1.1a)–(4.1.2b), *and let* $(g_{\mu\nu}|_{t=0} = m_{\mu\nu} + h^{(0)}_{\mu\nu}|_{t=0} + h^{(1)}_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0} = \partial_t h^{(0)}_{\mu\nu}|_{t=0} + \partial_t h^{(1)}_{\mu\nu}|_{t=0}, \mathscr{F}_{\mu\nu}|_{t=0})$ $(\mu, \nu = 0, 1, 2, 3)$ *be the corresponding initial data for the reduced system* (3.7.1a)–(3.7.1c) *as defined in Section 4.2. Assume that the abstract initial data are asymptotically flat in the sense that* (1.0.4a)–(1.0.4f) *hold. Let* $\ell \geq 10$ *be an integer, and let* $0 < \gamma < \frac{1}{2}$ *be a fixed constant. Let* $E_{\ell;\gamma}(0)$ *be the norm of the abstract data given in* (16.1.1), *and let* $M$ *be the ADM mass corresponding to the abstract data. Then there exists a constant* $\varepsilon_\ell > 0$ *depending on* $\gamma$ *and* $\ell$ *such that, if* $\varepsilon \leq \varepsilon_\ell$ *and if*

$$E_{\ell;\gamma}(0) + M \leq \varepsilon, \tag{16.1.9}$$

*then the reduced data launch a unique, classical solution* $(g_{\mu\nu} \stackrel{\text{def}}{=} m_{\mu\nu} + h^{(0)}_{\mu\nu} + h^{(1)}_{\mu\nu}, \mathscr{F}_{\mu\nu})$ *that exists for* $(t, x) \in (-\infty \times \infty) \times \mathbb{R}^3$. *The solution satisfies* both[31] *the reduced system* (3.7.1a)–(3.7.1c) *and the Einstein-nonlinear electromagnetic system* (1.0.1a)–(1.0.1c), *and the spacetime* $(\mathbb{R}^{1+3}, g_{\mu\nu})$ *is geodesically complete. In addition, the coordinates* $(t, x)$ *form a global system of wave coordinates. Furthermore, there exists a constant* $0 < \mu < \frac{1}{2}$ *(see Remark 1.2), and constants* $c_\ell > 0$ *and* $\tilde{c}_\ell > 0$ *depending on* $\gamma$ *and* $\ell$, *such that the solution's energy* (16.1.2) *satisfies the following bound for all* $t \in (-\infty, \infty)$:

$$\mathscr{E}_{\ell;\gamma;\mu}(t) \leq c_\ell \varepsilon (1 + |t|)^{\tilde{c}_\ell \varepsilon}. \tag{16.1.10}$$

*In addition, there exists a constant* $C_\ell > 0$ *depending on* $\gamma$ *and* $\ell$ *such that the following pointwise decay estimates hold for all* $(t, x) \in (-\infty, \infty) \times \mathbb{R}^3$:

$$(1 + |t| + |q|)^{1-\tilde{c}_\ell \varepsilon}(1 + |q|)^{-3/2}|h^{(1)}|_{\mathscr{L}\mathscr{T}} + (1 + |t| + |q|)^{1-\tilde{c}_\ell \varepsilon}(1 + |q|)^{-3/2}|\nabla_Z h^{(1)}|_{\mathscr{L}\mathscr{L}}$$
$$+ (1 + |t| + |q|)^{1-\tilde{c}_\ell \varepsilon}(1 + |q|)^{-1/2}|\nabla h^{(1)}|_{\mathscr{L}\mathscr{T}} + (1 + |t| + |q|)^{1-\tilde{c}_\ell \varepsilon}(1 + |q|)^{-1/2}|\nabla\nabla_Z h^{(1)}|_{\mathscr{L}\mathscr{L}}$$
$$+ |\nabla h^{(1)}|_{\mathscr{T}\mathscr{N}} + \{1 + \ln(1 + |t|)\}^{-1}|\nabla h^{(1)}| + |\mathscr{F}|$$
$$\leq C_\ell \varepsilon (1 + |t| + |q|)^{-1}, \tag{16.1.11a}$$

$$(1 + |q|)^{-1}|\nabla^I_{\mathscr{Z}} h^{(1)}| + |\nabla\nabla^I_{\mathscr{Z}} h^{(1)}| + |\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}|$$
$$\leq \begin{cases} C_\ell \varepsilon (1 + |t| + |q|)^{-1+\tilde{c}_\ell \varepsilon}(1 + |q|)^{-1-\gamma} & \text{if } q > 0, \\ C_\ell \varepsilon (1 + |t| + |q|)^{-1+\tilde{c}_\ell \varepsilon}(1 + |q|)^{-1/2} & \text{if } q < 0 \end{cases} \quad (|I| \leq \ell - 3), \tag{16.1.11b}$$

$$|\bar{\nabla}\nabla^I_{\mathscr{Z}} h^{(1)}| + (1 + |q|)|\bar{\nabla}\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}| + |\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}|_{\mathscr{T}\mathscr{T}}$$
$$\leq \begin{cases} C_\ell \varepsilon (1 + |t| + |q|)^{-2+\tilde{c}_\ell \varepsilon}(1 + |q|)^{-\gamma} & \text{if } q > 0, \\ C_\ell \varepsilon (1 + |t| + |q|)^{-2+\tilde{c}_\ell \varepsilon}(1 + |q|)^{1/2} & \text{if } q < 0 \end{cases} \quad (|I| \leq \ell - 4). \tag{16.1.11c}$$

---

[31] Of course, we technically mean here that the pair $(h^{(1)}_{\mu\nu}, \mathscr{F}_{\mu\nu})$ is a solution to the version (3.7.1a)–(3.7.1c) of the reduced equations while the pair $(g_{\mu\nu}, \mathscr{F}_{\mu\nu})$ is a solution to (1.0.1a)–(1.0.1c).

**Remark 16.4.** Some of the $(1+|q|)$-decay estimates in inequalities (16.1.11a)–(16.1.11c) are not optimal and can be improved with additional work. For example, in [Lindblad and Rodnianski 2010, Section 16], with the help of the fundamental solution of the Minkowski wave operator $\Box_m$, the $(1+|q|)$-decay estimates (16.1.11b)–(16.1.11c) for the tensor field $h^{(1)}$ are strengthened by a power of $\frac{1}{2}$ in the interior region $\{q < 0\}$.

**Remark 16.5.** Proposition 4.2 shows that the wave-coordinate condition (3.1.1a) holds in the domain of classical existence of the solution to the reduced equations; this is why the reduced solution also satisfies the Einstein-nonlinear electromagnetic equations (1.0.1a)–(1.0.1c).

**Remark 16.6.** A global stability result for the reduced equations under the wave-coordinate assumption, without regard for the abstract initial data, can be deduced from the smallness of $\mathscr{E}_{\ell;\gamma;\mu}(0) + |M|$ (we could even allow for negative $M$!) together with the assumption $\liminf_{|x|\to\infty}|h^{(1)}(0, x)| = 0$; this latter assumption, which is needed to deduce the inequalities (15.1.2b) at $t = 0$, is automatically implied by the assumptions of Theorem 16.3.

*Proof.* We only discuss the region of spacetime in which $t \geq 0$; the argument for $t \leq 0$ is similar. We define $E_{\ell;\gamma}(0) + M \overset{\text{def}}{=} \mathring{\varepsilon}$. By Proposition 14.1, we can choose constants $\gamma'$, $\mu$, $\mu'$, and $\delta$ subject to the restrictions described in Section 2.14 (in particular, these constants depend on $\gamma$) and a constant $A_\ell > 0$ such that, if $\varepsilon \overset{\text{def}}{=} A_\ell \mathring{\varepsilon}$, $A_\ell$ is sufficiently large, and $\mathring{\varepsilon}$ is sufficiently small, then there exists a nontrivial spacetime slab $[0, T) \times \mathbb{R}^3$ upon which the solution to the reduced equations exists and satisfies the energy bound $\mathscr{E}_{\ell;\gamma;\mu}(t) \leq \varepsilon(1+t)^\delta$ for $t \in [0, T)$. We then define

$$T_* \overset{\text{def}}{=} \sup\big\{T \;\big|\; \text{the solution exists classically and remains in the regime}$$

$$\text{of hyperbolicity of the reduced equations, and } \mathscr{E}_{\ell;\gamma;\mu}(t) \leq \varepsilon(1+t)^\delta \text{ for } t \in [0, T)\big\}.$$

Note that, under the above assumptions, we have that $T_* > 0$.

We now observe that the main energy bootstrap assumption (15.0.1) is satisfied on $[0, T_*)$. Thus, if $\varepsilon$ is sufficiently small, then by Propositions 15.6 and 15.7, all of the hypotheses of Theorem 16.1 are necessarily satisfied on $[0, T_*)$. Here, we are using the fact that $\lfloor \ell/2 \rfloor \leq \ell - 5$, which holds if $\ell \geq 10$. Consequently, the conclusion of that theorem (i.e., estimate (16.1.8)) allows us to deduce that the following energy estimate holds for $t \in [0, T_*)$:

$$\mathscr{E}_{\ell;\gamma;\mu}(t) \leq c_\ell(\mathring{\varepsilon} + \varepsilon^{3/2})(1+t)^{\tilde{c}_\ell \varepsilon} = c_\ell\Big(\frac{\varepsilon}{A_\ell} + \varepsilon^{3/2}\Big)(1+t)^{\tilde{c}_\ell \varepsilon}. \tag{16.1.12}$$

Now if $A_\ell > 3c_\ell$ and $\mathring{\varepsilon}$ is sufficiently small, then (16.1.12) implies that

$$\mathscr{E}_{\ell;\gamma;\mu}(t) < \tfrac{1}{2}A_\ell \mathring{\varepsilon}(1+t)^{A_\ell \tilde{c}_\ell \mathring{\varepsilon}} = \tfrac{1}{2}\varepsilon(1+t)^{\tilde{c}_\ell \varepsilon}, \tag{16.1.13}$$

which is a strict improvement over the bootstrap assumption (15.0.1). Thus, by (16.1.13), the weighted Klainerman–Sobolev inequality (B.4) (which, together with (6.5.22) and the smallness of $\mathscr{E}_{\ell;\gamma;\mu}(t)$, implies that the solution remains within the regime of hyperbolicity of the reduced equations), the continuation principle of Proposition 14.1, and the continuity of $\mathscr{E}_{\ell;\gamma;\mu}(t)$, it follows that, if $A_\ell$ is sufficiently large and $\mathring{\varepsilon}$ is sufficiently small, then $T_* = \infty$. Furthermore, under these assumptions, it is an obvious consequence

of this reasoning that (16.1.13) holds for $t \in [0, \infty)$. After renaming the constants in (16.1.13), we arrive at (16.1.10).

The inequalities (16.1.11b) follow as in the proof of Corollary 15.3 but with the strong energy estimate (16.1.10) in place of the energy bootstrap assumption (15.0.1). Similarly, the inequalities (16.1.11a) follow as in our proof of Proposition 15.6 but with the strong energy estimate (16.1.10) in place of the energy bootstrap assumption (15.0.1). The inequalities (16.1.11c) for $|\overline{\nabla} \nabla^I_{\mathscr{L}} h^{(1)}|$ and $|\overline{\nabla} \mathscr{L}^I_{\mathscr{L}} \mathscr{F}|$ follow from Lemma 6.16, (6.5.22), and (16.1.11b). The inequalities (16.1.11c) for $|\mathscr{L}^I_{\mathscr{L}} \mathscr{F}|_{\mathscr{L}\mathscr{N}}$ and $|\mathscr{L}^I_{\mathscr{L}} \mathscr{F}|_{\mathscr{T}\mathscr{T}}$ follow as in our proof of (15.2.1) but with the strong energy estimate (16.1.10) in place of the energy bootstrap assumption (15.0.1).

Based on these pointwise decay estimates, the geodesic completeness of the spacetime $(\mathbb{R}^{1+3}, g_{\mu\nu} \overset{\text{def}}{=} m_{\mu\nu} + h^{(0)}_{\mu\nu} + h^{(1)}_{\mu\nu})$ follows as in [Lindblad and Rodnianski 2005, Section 16; Loizelet 2008, Section 9]. $\square$

### 16.2. *The main argument in the proof of Theorem 16.1.*

Our goal is to use *only* the assumptions of Theorem 16.1 to deduce (for all sufficiently small nonnegative $\varepsilon$ and for sufficiently large fixed constants $c_\ell$ and $\tilde{c}_\ell$) the "strong" a priori energy estimate (16.1.8), which we restate for convenience:

$$\mathscr{E}_{\ell;\gamma;\mu}(t) \leq c_\ell(\overset{\circ}{\varepsilon} + \varepsilon^{3/2})(1+t)^{\tilde{c}_\ell \varepsilon}. \tag{16.2.1}$$

The proof of (16.2.1) is based on a hierarchy of Gronwall-amenable inequalities for $\mathscr{E}_{k;\gamma;\mu}(t)$ $(0 \leq k \leq \ell)$. We derive this hierarchy by carefully analyzing the integrals of Proposition 12.3 involving the inhomogeneous terms $\mathfrak{H}^{(1;I)}_{\mu\nu}$ and $\mathfrak{F}^{\nu}_{(I)}$. We recall that the structure of these inhomogeneous terms is captured by Propositions 7.1 and 8.1, which state that $\nabla^I_{\mathscr{L}} h^{(1)}_{\mu\nu}$ and $\mathscr{L}^I_{\mathscr{L}} \mathscr{F}_{\mu\nu}$ are solutions to the following system of equations:

$$\widetilde{\square}_g \nabla^I_{\mathscr{L}} h^{(1)}_{\mu\nu} = \mathfrak{H}^{(1;I)}_{\mu\nu} \qquad (\mu, \nu = 0, 1, 2, 3), \tag{16.2.2a}$$

$$\nabla_\lambda \mathscr{L}^I_{\mathscr{L}} \mathscr{F}_{\mu\nu} + \nabla_\mu \mathscr{L}^I_{\mathscr{L}} \mathscr{F}_{\nu\lambda} + \nabla_\nu \mathscr{L}^I_{\mathscr{L}} \mathscr{F}_{\lambda\mu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{16.2.2b}$$

$$N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{L}^I_{\mathscr{L}} \mathscr{F}_{\kappa\lambda} = \mathfrak{F}^{\nu}_{(I)} \qquad (\nu = 0, 1, 2, 3). \tag{16.2.2c}$$

Most of the work goes into obtaining suitable estimates for the integrals involving $\mathfrak{H}^{(1;I)}_{\mu\nu}$ and $\mathfrak{F}^{\nu}_{(I)}$. In order to avoid impeding the flow of the proof, we prove most of the desired inequalities later in this section after the main argument. For the main part of the argument, we simply quote Corollaries 16.12 and 16.18, which are the key estimates that allow us to apply a suitable version of Gronwall's inequality. We will then return to the proofs of Corollaries 16.12 and 16.18, which follow from a large collection of lemmas, each of which involves the analysis of one of the constituent pieces of the integrals involving $\mathfrak{H}^{(1;I)}_{\mu\nu}$ and $\mathfrak{F}^{\nu}_{(I)}$.

We now proceed to the main argument. We first note that the hypotheses of Proposition 12.3 are implied by the hypotheses of Theorem 16.1. Therefore, we can use Proposition 12.3 (with $\dot{\mathscr{F}} \overset{\text{def}}{=} \mathscr{L}^I_{\mathscr{L}} \mathscr{F}$ in the proposition) and Corollaries 16.12 and 16.18 to deduce that

$$\sum_{|I|\le k}\int_{\Sigma_t}\left|\binom{\nabla\nabla_{\mathscr{L}}^I h^{(1)}}{\mathscr{L}_{\mathscr{L}}^I\mathscr{F}}\right|^2 w(q)\,d^3x + \sum_{|I|\le k}\int_0^t\int_{\Sigma_\tau}\left(|\overline{\nabla}\nabla_{\mathscr{L}}^I h^{(1)}|^2 + |\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|_{\mathscr{L}\mathscr{N}}^2 + |\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|_{\mathscr{T}\mathscr{T}}^2\right)w'(q)\,d^3x\,d\tau$$

$$\le C\sum_{|I|\le k}\int_{\Sigma_0}\left|\binom{\nabla\nabla_{\mathscr{L}}^I h^{(1)}}{\mathscr{L}_{\mathscr{L}}^I\mathscr{F}}\right|^2 w(q)\,d^3x + C\varepsilon\sum_{|I|\le k}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1}\left|\binom{\nabla\nabla_{\mathscr{L}}^I h^{(1)}}{\mathscr{L}_{\mathscr{L}}^I\mathscr{F}}\right|^2 w(q)\,d^3x\,d\tau$$

$$+C\sum_{|I|\le k}\int_0^t\int_{\Sigma_\tau}\left(|\mathfrak{H}^{(1;I)}||\nabla\nabla_{\mathscr{L}}^I h^{(1)}| + |(\mathscr{L}_{\mathscr{L}}^I\mathscr{F}_{0\nu})\mathfrak{F}_{(I)}^\nu|\right)w(q)\,d^3x\,d\tau$$

$$\le C\sum_{|I|\le k}\int_{\Sigma_0}\left|\binom{\nabla\nabla_{\mathscr{L}}^I h^{(1)}}{\mathscr{L}_{\mathscr{L}}^I\mathscr{F}}\right|^2 w(q)\,d^3x + CM\sum_{|I|\le k}\int_0^t\left((1+\tau)^{-3/2}\sqrt{\int_{\Sigma_\tau}|\nabla\nabla_{\mathscr{L}}^I h^{(1)}|^2 w(q)\,d^3x}\right)d\tau$$

$$+C\varepsilon\sum_{|I|\le k}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1}\left|\binom{\nabla\nabla_{\mathscr{L}}^I h^{(1)}}{\mathscr{L}_{\mathscr{L}}^I\mathscr{F}}\right|^2 w(q)\,d^3x\,d\tau$$

$$+C\varepsilon\sum_{|I|\le k}\int_0^t\int_{\Sigma_\tau}\left(|\overline{\nabla}\nabla_{\mathscr{L}}^I h^{(1)}|^2 + |\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|_{\mathscr{L}\mathscr{N}}^2 + |\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|_{\mathscr{T}\mathscr{T}}^2\right)w'(q)\,d^3x\,d\tau$$

$$\tag{16.2.3}$$

$$+C\varepsilon\underbrace{\sum_{|J|\le k-1}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1+C\varepsilon}\left|\binom{\nabla\nabla_{\mathscr{L}}^J h^{(1)}}{\mathscr{L}_{\mathscr{L}}^J\mathscr{F}}\right|^2 w(q)\,d^3x\,d\tau}_{\text{absent if }k=0} + C\varepsilon^3.$$

Recalling the definition (where the dependence on $\mu$ and $\gamma$ is through $w(q)$)

$$\mathscr{E}_{k;\gamma;\mu}^2(t) \overset{\text{def}}{=} \sup_{0\le\tau\le t}\sum_{|I|\le k}\int_{\Sigma_\tau}\left(|\nabla\nabla_{\mathscr{L}}^I h^{(1)}|^2 + |\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|^2\right)w(q)\,d^3x$$

and introducing the quantity $\mathscr{S}_{k;\gamma;\mu}(t)\ge 0$, which is defined by

$$\mathscr{S}_{k;\gamma;\mu}^2(t) \overset{\text{def}}{=} \sum_{|I|\le k}\int_0^t\int_{\Sigma_\tau}\left(|\overline{\nabla}\nabla_{\mathscr{L}}^I h^{(1)}|^2 + |\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|_{\mathscr{L}\mathscr{N}}^2 + |\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|_{\mathscr{T}\mathscr{T}}^2\right)w'(q)\,d^3x\,d\tau, \tag{16.2.4}$$

it therefore follows from the final inequality of (16.2.3) that

$$\mathscr{E}_{k;\gamma;\mu}^2(t) + \mathscr{S}_{k;\gamma;\mu}^2(t) \le C\mathscr{E}_{k;\gamma;\mu}^2(0) + CM\int_0^t(1+\tau)^{-3/2}\mathscr{E}_{k;\gamma;\mu}(\tau)\,d\tau + C\varepsilon\int_0^t(1+\tau)^{-1}\mathscr{E}_{k;\gamma;\mu}^2(\tau)\,d\tau$$

$$+\underbrace{C\varepsilon\mathscr{S}_{k;\gamma;\mu}^2(t)}_{\text{absorb into left-hand side}} + C\varepsilon\int_0^t(1+\tau)^{-1+C\varepsilon}\mathscr{E}_{k-1;\gamma;\mu}^2(\tau)\,d\tau + C\varepsilon^3. \tag{16.2.5}$$

For $\varepsilon$ sufficiently small, we may absorb the $C\varepsilon\mathscr{S}_{k;\gamma;\mu}^2(t)$ term from (16.2.5) into the left-hand side at the expense of increasing all of the constants. We can similarly absorb the term $CM\int_0^t(1+\tau)^{-3/2}\mathscr{E}_{k;\gamma;\mu}(\tau)\,d\tau$ by using the inequality $CM\int_0^t(1+\tau)^{-3/2}\mathscr{E}_{k;\gamma;\mu}(\tau)\,d\tau \le \frac{1}{2}\mathscr{E}_{k;\gamma;\mu}^2(t) + C^2M^2$, which follows from the algebraic estimate $CM\mathscr{E}_{k;\gamma;\mu}(\tau) \le \frac{1}{4}\mathscr{E}_{k;\gamma;\mu}^2(\tau) + C^2M^2$, the integral inequality $\int_0^t(1+\tau)^{-3/2}\,d\tau \le 2$, and the fact that $\mathscr{E}_{k;\gamma;\mu}^2(\tau)$ is increasing. If we also use the fact that $\mathscr{E}_{k;\gamma;\mu}^2(0) \le C(E_{\ell;\gamma}^2(0) + M^2) \le C\mathring{\varepsilon}^2$ (i.e, Proposition 10.4) and the inequality $M \le \mathring{\varepsilon}$, then we arrive at the following inequality, valid for all small $\varepsilon$:

$$\mathscr{E}^2_{k;\gamma;\mu}(t) + \mathscr{S}^2_{k;\gamma;\mu}(t) \le C(\mathring{\varepsilon}^2 + \varepsilon^3) + C\varepsilon \int_0^t (1+\tau)^{-1}\mathscr{E}^2_{k;\gamma;\mu}(\tau)\,d\tau$$

$$+ C\varepsilon \underbrace{\int_0^t (1+\tau)^{-1+C\varepsilon}\mathscr{E}^2_{k-1;\gamma;\mu}(\tau)\,d\tau}_{\text{absent if } k=0}. \quad (16.2.6)$$

For $k=0$, (16.2.6) implies that

$$\mathscr{E}^2_{0;\gamma;\mu}(t) \le C(\mathring{\varepsilon}^2 + \varepsilon^3) + c_0\varepsilon \int_0^t (1+\tau)^{-1}\mathscr{E}^2_{0;\gamma;\mu}(\tau)\,d\tau. \quad (16.2.7)$$

From (16.2.7) and Gronwall's inequality, we deduce that

$$\mathscr{E}^2_{0;\gamma;\mu}(t) \le C(\mathring{\varepsilon}^2 + \varepsilon^3)(1+t)^{c_0\varepsilon}. \quad (16.2.8)$$

Using (16.2.6) and the base case (16.2.8), we will argue inductively to derive the following estimate for $k \ge 1$:

$$\mathscr{E}^2_{k;\gamma;\mu}(t) \le C(\mathring{\varepsilon}^2 + \varepsilon^3)(1+t)^{c_k\varepsilon}. \quad (16.2.9)$$

Assuming that (16.2.9) holds for the case $k-1$, we insert inequality (16.2.9) for $\mathscr{E}^2_{k-1;\gamma;\mu}(t)$ into the right-hand side of (16.2.6) and deduce that

$$\mathscr{E}^2_{k;\gamma;\mu}(t) + \mathscr{S}^2_{k;\gamma;\mu}(t) \le C(\mathring{\varepsilon}^2 + \varepsilon^3) + C\varepsilon \int_0^t (1+\tau)^{-1}\mathscr{E}^2_{k;\gamma;\mu}(\tau)\,d\tau + C\varepsilon(\mathring{\varepsilon}^2 + \varepsilon^3)\int_0^t (1+\tau)^{-1+C\varepsilon}\,d\tau$$

$$\le C(\mathring{\varepsilon}^2 + \varepsilon^3)(1+t)^{C\varepsilon} + C\varepsilon \int_0^t (1+\tau)^{-1}\mathscr{E}^2_{k;\gamma;\mu}(\tau)\,d\tau. \quad (16.2.10)$$

Finally, from (16.2.10) and Gronwall's lemma, we conclude that, if $\varepsilon$ is sufficiently small, then

$$\mathscr{E}^2_{k;\gamma;\mu}(t) \le C(\mathring{\varepsilon}^2 + \varepsilon^3)(1+t)^{c_k\varepsilon}. \quad (16.2.11)$$

We have therefore closed the induction and shown (16.1.8). This concludes the proof of Theorem 16.1.

**16.3.** *Integral inequalities for the* $\nabla^I_{\mathscr{L}} h^{(1)}_{\mu\nu}$ *inhomogeneities.* In this section, we analyze the integrals in Proposition 12.3 corresponding to the inhomogeneous terms $\mathfrak{H}^{(1;I)}_{\mu\nu}$ in (16.2.2a). The main goal is to arrive at Corollary 16.12. The main point is that right-hand sides of the inequalities in the corollary can be bounded in terms of time integrals of the energies $\mathscr{E}_{k;\gamma;\mu}(t)$ (this was carried out in inequality (16.2.5)). As opposed to the estimates proved in Section 16.4, most of the estimates proved in this section are a straightforward generalization of the ones proved in [Lindblad and Rodnianski 2010]; i.e., the estimates involve a similar analysis but with additional terms arising from the presence of the $\mathscr{F}$ terms appearing on the right-hand side of the reduced equation (3.7.1a). The additional terms result in the presence of the $\mathscr{L}^J_{\mathscr{L}}\mathscr{F}$ component of the first term on the right-hand side of inequality (16.3.2) and the $\mathscr{L}^{J'}_{\mathscr{L}}\mathscr{F}$ component of the next-to-last term of the same inequality. The most important aspect of our analysis is showing that these additional terms respectively appear with the factors $\varepsilon(1+t)^{-1}$ and $\varepsilon(1+t)^{-1+C\varepsilon}$.

We begin with a lemma that follows easily from algebraic estimates of the form $|ab| \lesssim a^2 + b^2$:

**Lemma 16.7** (Arithmetic-geometric mean inequality). *Let*

$$\mathfrak{H}^{(1;I)}_{\mu\nu} = \widehat{\nabla}^I_{\mathscr{L}}\mathfrak{H}_{\mu\nu} - \widehat{\nabla}^I_{\mathscr{L}}\widetilde{\square}h^{(0)}_{\mu\nu} - (\widehat{\nabla}^I_{\mathscr{L}}\widetilde{\square}_g h^{(1)}_{\mu\nu} - \widetilde{\square}_g\nabla^I_{\mathscr{L}}h^{(1)}_{\mu\nu})$$

*be the inhomogeneous term on the right-hand side of* (7.0.1). *Then the following algebraic inequality holds*:

$$|\mathfrak{H}^{(1;I)}||\nabla\nabla^I_{\mathscr{L}}h^{(1)}| \le \varepsilon^{-1}(1+t)|\widehat{\nabla}^I_{\mathscr{L}}\mathfrak{H}|^2 + \varepsilon^{-1}(1+t)|\widehat{\nabla}^I_{\mathscr{L}}\widetilde{\Box}_g h^{(1)}_{\mu\nu} - \widetilde{\Box}_g \nabla^I_{\mathscr{L}}h^{(1)}_{\mu\nu}|^2$$

$$+ \varepsilon(1+t)^{-1}|\nabla\nabla^I_{\mathscr{L}}h^{(1)}|^2 + |\widehat{\nabla}^I_{\mathscr{L}}\widetilde{\Box}_g h^{(0)}||\nabla\nabla^I_{\mathscr{L}}h^{(1)}|. \quad (16.3.1)$$

$$\square$$

The next lemma provides a preliminary pointwise estimate for the $|\widehat{\nabla}^I_{\mathscr{L}}\mathfrak{H}|$ term on the right-hand side of (16.3.1).

**Lemma 16.8** (Pointwise estimates for the $|\nabla^I_{\mathscr{L}}\mathfrak{H}|$ inhomogeneities; extension of [Lindblad and Rodnianski 2010, Lemma 11.2]). *Under the assumptions of Theorem 16.1, if $I$ is any $\mathscr{L}$-multi-index with $|I| \le \ell$ and if $\varepsilon$ is sufficiently small, then the following pointwise estimates hold for $(t, x) \in [0, T) \times \mathbb{R}^3$:*

$$|\nabla^I_{\mathscr{L}}\mathfrak{H}| \lesssim \varepsilon \sum_{|J|\le|I|}(1+t)^{-1}\left|\begin{pmatrix}\nabla\nabla^J_{\mathscr{L}}h^{(1)}\\ \mathscr{L}^J_{\mathscr{L}}\mathscr{F}\end{pmatrix}\right| + \varepsilon \sum_{|J|\le|I|}(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-1/2+\mu'}|\overline{\nabla}\nabla^J_{\mathscr{L}}h^{(1)}|$$

$$+ \varepsilon^2 \sum_{|J|\le|I|}(1+t+|q|)^{-1}(1+|q|)^{-1}|\nabla^J_{\mathscr{L}}h^{(1)}|$$

$$+ \varepsilon \underbrace{\sum_{|J'|\le|I|-1}(1+t)^{-1+C\varepsilon}\left|\begin{pmatrix}\nabla\nabla^{J'}_{\mathscr{L}}h^{(1)}\\ \mathscr{L}^{J'}_{\mathscr{L}}\mathscr{F}\end{pmatrix}\right|}_{\text{absent if }|I|=0} + \varepsilon^2(1+t+|q|)^{-4}. \quad (16.3.2)$$

*Proof.* By Proposition 11.3, we have that

$$|\nabla^I_{\mathscr{L}}\mathfrak{H}| \lesssim |\text{(i)}| + |\text{(ii)}| + |\text{(iii)}|, \quad (16.3.3)$$

where

$$|\text{(i)}| = \sum_{|J|+|K|\le|I|}|\nabla\nabla^J_{\mathscr{L}}h|_{\mathscr{T}\mathscr{N}}|\nabla\nabla^K_{\mathscr{L}}h|_{\mathscr{T}\mathscr{N}} + |\overline{\nabla}\nabla^J_{\mathscr{L}}h||\nabla\nabla^K_{\mathscr{L}}h|$$

$$+ \underbrace{\sum_{|J''|+|K''|\le|I|-2}|\nabla\nabla^{J''}_{\mathscr{L}}h||\nabla\nabla^{K''}_{\mathscr{L}}h|}_{\text{absent if }|I|\le 1}, \quad (16.3.4)$$

$$|\text{(ii)}| = \sum_{|J|+|K|\le|I|}|\mathscr{L}^J_{\mathscr{L}}\mathscr{F}||\mathscr{L}^K_{\mathscr{L}}\mathscr{F}|, \quad (16.3.5)$$

$$|\text{(iii)}| = \sum_{|I_1|+|I_2|+|I_3|\le|I|}|\nabla^{I_1}_{\mathscr{L}}h||\nabla\nabla^{I_2}_{\mathscr{L}}h||\nabla\nabla^{I_3}_{\mathscr{L}}h| + \sum_{|I_1|+|I_2|+|I_3|\le|I|}|\nabla^{I_1}_{\mathscr{L}}h||\mathscr{L}^{I_2}_{\mathscr{L}}\mathscr{F}||\mathscr{L}^{I_3}_{\mathscr{L}}\mathscr{F}|$$

$$+ \sum_{|I_1|+|I_2|+|I_3|\le|I|}|\mathscr{L}^{I_1}_{\mathscr{L}}\mathscr{F}||\mathscr{L}^{I_2}_{\mathscr{L}}\mathscr{F}||\mathscr{L}^{I_3}_{\mathscr{L}}\mathscr{F}|. \quad (16.3.6)$$

The desired bound for $|\text{(i)}|$ was proved in Lemma 11.2 of [Lindblad and Rodnianski 2010] by using the decomposition $h = h^{(1)} + h^{(0)}$ and by combining Lemma 15.1 and inequalities (16.1.6a)–(16.1.6c). The term $|\text{(ii)}|$ is the main contribution to $|\nabla^I_{\mathscr{L}}\mathfrak{H}|$ arising from the presence of nonzero electromagnetic fields. To bound $|\text{(ii)}|$ by the right-hand side of (16.3.2), we consider the cases $(|J| = \ell, |K| = 0)$, $(|J| = 0, |K| = \ell)$, $(|J| \le \ell - 1, |K| \le \lfloor \ell/2 \rfloor)$, and $(|J| \le \lfloor \ell/2 \rfloor, |K| \le \ell - 1)$; clearly this exhausts

all possible cases. In the first two cases, we use (16.1.6a) to achieve the desired bound while in the last two we use (16.1.6b). The cubic terms from case (iii) can be similarly bounded by using the decomposition $h = h^{(1)} + h^{(0)}$ and by combining Lemma 15.1 and inequality (16.1.6b).    □

Using the previous lemma, we now derive the desired integral inequalities corresponding to the $\varepsilon^{-1}(1+t)|\widehat{\nabla}_{\mathscr{L}}^{I}\mathfrak{H}|^2$ term on the right-hand side of (16.3.1).

**Lemma 16.9** (Integral estimates for $\varepsilon^{-1}(1+t)|\widehat{\nabla}_{\mathscr{L}}^{I}\mathfrak{H}|^2 w(q)$; extension of [Lindblad and Rodnianski 2010, Lemma 11.3]). *Under the assumptions of Theorem 16.1, if $I$ is any $\mathscr{L}$-multi-index with $|I| \leq \ell$ and if $\varepsilon$ is sufficiently small, then the following integral estimate holds for $t \in [0, T)$:*

$$
\varepsilon^{-1} \int_0^t \int_{\Sigma_\tau} (1+\tau)|\widehat{\nabla}_{\mathscr{L}}^{I}\mathfrak{H}|^2 w(q) \, d^3x \, d\tau
$$
$$
\lesssim \varepsilon \sum_{|J| \leq |I|} \int_0^t \int_{\Sigma_\tau} \left( (1+\tau)^{-1} \left| \binom{\nabla \nabla_{\mathscr{L}}^{J} h^{(1)}}{\mathscr{L}_{\mathscr{L}}^{J}\mathscr{F}} \right|^2 w(q) + |\overline{\nabla}\nabla_{\mathscr{L}}^{J} h^{(1)}|^2 w'(q) \right) d^3x \, d\tau
$$
$$
+ \varepsilon \underbrace{\sum_{|J'| \leq |I|-1} \int_0^t \int_{\Sigma_\tau} (1+\tau)^{-1+C\varepsilon} \left| \binom{\nabla \nabla_{\mathscr{L}}^{J'} h^{(1)}}{\mathscr{L}_{\mathscr{L}}^{J'}\mathscr{F}} \right|^2 w(q) \, d^3x \, d\tau + \varepsilon^3.}_{\text{absent if } |I| = 0} \quad (16.3.7)
$$

*Proof.* After squaring both sides of (16.3.2), multiplying by $\varepsilon^{-1}(1+t)w(q)$, using the inequality $(1+|q|)^{-1}(1+q_-)^{-2\mu}w(q) \lesssim w'(q)$ (i.e., inequality (12.1.2)) and the fact that $\mu+\mu' < \frac{1}{2}$, and integrating, we see that the only terms that are not manifestly bounded by the right-hand side of (16.3.7) are

$$
\varepsilon^3 \sum_{|J| \leq |I|} \int_0^t \int_{\Sigma_t} (1+\tau)^{-1}(1+|q|)^{-2}|\nabla_{\mathscr{L}}^{J} h^{(1)}|^2 w(q) \, d^3x \, d\tau. \quad (16.3.8)
$$

The desired bound for these terms can be achieved with the help of the Hardy inequalities of Proposition C.1, which imply that

$$
\int_{\Sigma_t} (1+\tau)^{-1}(1+|q|)^{-2}|\nabla_{\mathscr{L}}^{J} h^{(1)}|^2 w(q) \, d^3x \lesssim \int_{\Sigma_t} (1+\tau)^{-1}|\nabla \nabla_{\mathscr{L}}^{J} h^{(1)}|^2 w(q) \, d^3x. \quad (16.3.9)
$$

This concludes the proof.    □

We now derive the desired integral inequalities corresponding to the $|\widehat{\nabla}_{\mathscr{L}}^{I}\widetilde{\Box}_g h^{(0)}||\nabla \nabla_{\mathscr{L}}^{I} h^{(1)}|$ term on the right-hand side of (16.3.1).

**Lemma 16.10** (Integral estimates for $|\widehat{\nabla}_{\mathscr{L}}^{I}\widetilde{\Box}_g h^{(0)}||\nabla \nabla_{\mathscr{L}}^{I} h^{(1)}|w(q)$ [Lindblad and Rodnianski 2010, Lemma 11.4]). *Let M be the ADM mass. Under the assumptions of Theorem 16.1, if $I$ is a $\mathscr{L}$-multi-index satisfying $|I| \leq \ell$ and if $\varepsilon$ is sufficiently small, then the following integral inequality holds for $t \in [0, T)$:*

$$\int_0^t \int_{\Sigma_\tau} |\widehat{\nabla}_{\mathscr{L}}^I \widetilde{\Box}_g h^{(0)}| |\nabla \nabla_{\mathscr{L}}^I h^{(1)}| w(q) \, d^3x \, d\tau$$

$$\lesssim M \sum_{|J| \leq |I|} \int_0^t \int_{\Sigma_\tau} (1+\tau)^{-2} |\nabla \nabla_{\mathscr{L}}^I h^{(1)}|^2 w(q) \, d^3x \, d\tau$$

$$+ M \sum_{|J| \leq |I|} \int_0^t \left( (1+\tau)^{-3/2} \sqrt{\int_{\Sigma_\tau} |\nabla \nabla_{\mathscr{L}}^I h^{(1)}|^2 w(q) \, d^3x} \right) d\tau. \quad (16.3.10)$$

*Proof.* We first use the Cauchy–Schwarz inequality for integrals to obtain

$$\int_0^t \int_{\Sigma_\tau} |\widehat{\nabla}_{\mathscr{L}}^I \widetilde{\Box}_g h^{(0)}| |\nabla \nabla_{\mathscr{L}}^I h^{(1)}| w(q) \, d^3x \, d\tau$$

$$\leq \int_0^t \left[ \left( \int_{\Sigma_\tau} |\widehat{\nabla}_{\mathscr{L}}^I \widetilde{\Box}_g h^{(0)}|^2 w(q) \, d^3x \right)^{1/2} \times \left( \int_{\Sigma_\tau} |\nabla \nabla_{\mathscr{L}}^I h^{(1)}|^2 w(q) \, d^3x \right)^{1/2} \right] d\tau. \quad (16.3.11)$$

Furthermore, under the present assumptions, the previous proof of inequality (15.1.3b) remains valid. Thus, from (15.1.3b) and the Hardy inequalities of Proposition C.1, it follows that

$$\int_{\Sigma_t} |\widehat{\nabla}_{\mathscr{L}}^I \widetilde{\Box}_g h^{(0)}|^2 w(q) \, d^3x \lesssim M^2 (1+t)^{-3} + M^2 (1+t)^{-4} \sum_{|J| \leq |I|} \int_{\Sigma_t} |\nabla \nabla_{\mathscr{L}}^J h^{(1)}|^2 w(q) \, d^3x. \quad (16.3.12)$$

The estimate (16.3.10) now follows from (16.3.11), (16.3.12), and the inequalities $\sqrt{|a|+|b|} \lesssim \sqrt{|a|} + \sqrt{|b|}$ and $|ab| \lesssim a^2 + b^2$. $\square$

The following integral estimate for the commutator term $\varepsilon^{-1}(1+t)|\widehat{\nabla}_{\mathscr{L}}^I \widetilde{\Box}_g h_{\mu\nu}^{(1)} - \widetilde{\Box}_g (\nabla_{\mathscr{L}}^I h_{\mu\nu}^{(1)})|^2$ on the right-hand side of (16.3.1) was proved in [Lindblad and Rodnianski 2010]. Its lengthy proof is similar to our proof of Lemma 16.17 below, and we do not repeat it here.

**Lemma 16.11** (Integral estimates for $\varepsilon^{-1}|\widehat{\nabla}_{\mathscr{L}}^I \widetilde{\Box}_g h_{\mu\nu}^{(1)} - \widetilde{\Box}_g \nabla_{\mathscr{L}}^I h_{\mu\nu}^{(1)}|^2 w(q)$ [Lindblad and Rodnianski 2010, Lemma 11.5]). *Under the assumptions of Theorem 16.1, if $I$ is a $\mathscr{L}$-multi-index satisfying $1 \leq |I| \leq \ell$ and if $\varepsilon$ is sufficiently small, then the following integral inequality holds for $t \in [0, T)$:*

$$\varepsilon^{-1} \int_0^t \int_{\Sigma_\tau} (1+\tau) \left| \widehat{\nabla}_{\mathscr{L}}^I \widetilde{\Box}_g h_{\mu\nu}^{(1)} - \widetilde{\Box}_g \nabla_{\mathscr{L}}^I h_{\mu\nu}^{(1)} \right|^2 w(q) \, d^3x \, d\tau$$

$$\lesssim \varepsilon \sum_{|J| \leq |I|} \int_0^t \int_{\Sigma_\tau} \left( (1+\tau)^{-1} |\nabla \nabla_{\mathscr{L}}^J h^{(1)}|^2 w(q) + |\bar{\nabla} \nabla_{\mathscr{L}}^J h^{(1)}|^2 w'(q) \right) d^3x \, d\tau$$

$$+ \varepsilon \sum_{|J'| \leq |I|-1} \int_0^t \int_{\Sigma_\tau} (1+\tau)^{-1+C\varepsilon} |\nabla \nabla_{\mathscr{L}}^{J'} h^{(1)}|^2 w(q) \, d^3x \, d\tau + \varepsilon^3. \quad (16.3.13)$$

Combining Lemmas 16.7, 16.9, 16.10, and 16.11, we arrive at the following corollary:

**Corollary 16.12** (Estimates for the energy integrals corresponding to the $h^{(1)}$ inhomogeneities). *Under the assumptions of Theorem 16.1, if $0 \leq k \leq \ell$ and $\varepsilon$ is sufficiently small, then the following integral inequality holds for $t \in [0, T)$:*

$$\sum_{|I| \leq k} \int_0^t \int_{\Sigma_\tau} |\mathfrak{H}^{(1;I)}| |\nabla \nabla_{\mathscr{L}}^I h^{(1)}| \, d^3 x \, d\tau \lesssim M \sum_{|I| \leq k} \int_0^t \left( (1+\tau)^{-3/2} \sqrt{\int_{\Sigma_\tau} |\nabla \nabla_{\mathscr{L}}^I h^{(1)}|^2 w(q) \, d^3 x} \right) d\tau$$

$$+ \varepsilon \sum_{|I| \leq k} \int_0^t \int_{\Sigma_\tau} (1+\tau)^{-1} \left| \begin{pmatrix} \nabla \nabla_{\mathscr{L}}^I h^{(1)} \\ \mathscr{L}_{\mathscr{L}}^I \mathscr{F} \end{pmatrix} \right|^2 w(q) \, d^3 x \, d\tau$$

$$+ \varepsilon \sum_{|I| \leq k} \int_0^t \int_{\Sigma_\tau} |\bar{\nabla} \nabla_{\mathscr{L}}^I h^{(1)}|^2 w'(q) \, d^3 x \, d\tau$$

$$+ \varepsilon \underbrace{\sum_{|J| \leq k-1} \int_0^t \int_{\Sigma_\tau} (1+\tau)^{-1+C\varepsilon} \left| \begin{pmatrix} \nabla \nabla_{\mathscr{L}}^J h^{(1)} \\ \mathscr{L}_{\mathscr{L}}^J \mathscr{F} \end{pmatrix} \right|^2 w(q) \, d^3 x \, d\tau}_{\text{absent if } k = 0}$$

$$+ \varepsilon^3. \tag{16.3.14}$$

This completes our analysis of the integral inequalities for the $h_{\mu\nu}^{(1)}$ inhomogeneities.

**16.4. Integral inequalities for the $\mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{\mu\nu}$ inhomogeneities.** In this section, we estimate the integrals corresponding to the inhomogeneous terms in the $\mathscr{L}_{\mathscr{L}}^I$-commuted electromagnetic equations. More precisely, we analyze the integrals in Proposition 12.3 corresponding to the inhomogeneous terms $\mathfrak{F}_{(I)}^\nu$ in (16.2.2c). The main goal is to arrive at Corollary 16.18. As was the case for Corollary 16.12, the main point is that right-hand sides of the inequalities in Corollary 16.18 can be bounded in terms of time integrals of the energies $\mathscr{E}_{k;\gamma;\mu}(t)$ (this was carried out in inequality (16.2.5)).

We begin with the following lemma, which provides pointwise estimates for the wave-coordinate-controlled quantities $|\nabla \nabla_{\mathscr{L}}^I h^{(1)}|_{\mathscr{L}\mathscr{L}}$ and $|\nabla \nabla_{\mathscr{L}}^J h^{(1)}|_{\mathscr{L}\mathscr{T}}$ for $|I| \leq \ell$ and $|J| \leq \ell - 1$. These pointwise estimates will be used to help to derive suitable integral estimates later in this section.

**Lemma 16.13** (Pointwise estimates for $\sum_{|I| \leq k} |\nabla \nabla_{\mathscr{L}}^I h^{(1)}|_{\mathscr{L}\mathscr{L}} + \sum_{|J| \leq k-1} |\nabla \nabla_{\mathscr{L}}^J h^{(1)}|_{\mathscr{L}\mathscr{T}}$). *Under the assumptions of Theorem 16.1, if $0 \leq k \leq \ell$ and $\varepsilon$ is sufficiently small, then the following pointwise inequality holds for $(t, x) \in [0, T) \times \mathbb{R}^3$:*

$$\sum_{|I| \leq k} |\nabla \nabla_{\mathscr{L}}^I h^{(1)}|_{\mathscr{L}\mathscr{L}} + \overbrace{\sum_{|J| \leq k-1} |\nabla \nabla_{\mathscr{L}}^J h^{(1)}|_{\mathscr{L}\mathscr{T}}}^{\text{absent if } k = 0}$$

$$\lesssim \sum_{|I| \leq k} |\bar{\nabla} \nabla_{\mathscr{L}}^I h^{(1)}| + \varepsilon (1 + t + |q|)^{-2} \chi_0(1/2 \leq r/t \leq 3/4) + \varepsilon^2 (1 + t + |q|)^{-3}$$

$$+ \varepsilon \sum_{|I| \leq k} (1 + t + |q|)^{-1+C\varepsilon} (1 + |q|)^{1/2+\mu'} |\nabla \nabla_{\mathscr{L}}^I h^{(1)}|$$

$$+ \varepsilon \sum_{|I| \leq k} (1 + t + |q|)^{-1+C\varepsilon} (1 + |q|)^{-1/2+\mu'} |\nabla_{\mathscr{L}}^I h^{(1)}| + \overbrace{\sum_{|J'| \leq k-2} |\nabla \nabla_{\mathscr{L}}^{J'} h^{(1)}|}^{\text{absent if } k \leq 1}, \quad (16.4.1)$$

*where $\chi_0\left(\frac{1}{2} \leq z \leq \frac{3}{4}\right)$ is the characteristic function of the interval $\left[\frac{1}{2}, \frac{3}{4}\right]$.*

*Proof.* Lemma 16.13 follows from Lemma 11.2 (for the tensor field $h_{\mu\nu}^{(1)}$) and the pointwise decay assumptions (16.1.6b) for $h_{\mu\nu}^{(1)}$.                                                                              □

In the next lemma, we derive pointwise estimates for the term $|(\mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{0\nu})\widehat{\mathscr{L}}_{\mathscr{L}}^I \mathfrak{F}^\nu|$. This term appears in the second spacetime integral on the right-hand side of (12.2.1), which is our basic energy inequality for the Faraday tensor and its Lie derivatives. The pointwise estimates are preliminary estimates that will be used in the subsequent lemma to estimate the corresponding spacetime integral.

**Lemma 16.14** (Pointwise estimates for $|(\mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{0\nu})\widehat{\mathscr{L}}_{\mathscr{L}}^I \mathfrak{F}^\nu|$). *Let* $\mathfrak{F}_{(I)}^\nu = \widehat{\mathscr{L}}_{\mathscr{L}}^I \mathfrak{F}^\nu + [N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathscr{L}}^I (N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda})]$ *be the inhomogeneous term* (8.1.2b) *in the equations of variation* (8.1.1b) *satisfied by* $\dot{\mathscr{F}} \overset{\text{def}}{=} \mathscr{L}_{\mathscr{L}}^I \mathscr{F}$. *Under the assumptions of Theorem 16.1, if* $0 \leq k \leq \ell$ *and* $\varepsilon$ *is sufficiently small, then the following pointwise inequality holds for* $(t, x) \in [0, T) \times \mathbb{R}^3$:

$$\sum_{|I| \leq k} |(\mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{0\nu})\widehat{\mathscr{L}}_{\mathscr{L}}^I \mathfrak{F}^\nu| \lesssim \varepsilon \sum_{|I| \leq k} (1+t+|q|)^{-1}\left(|\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|^2 + |\nabla \nabla_{\mathscr{L}}^I h^{(1)}|^2\right)$$

$$+ \varepsilon \sum_{|I| \leq k} (1+|q|)^{-1}(1+q_-)^{-2\mu}|\overline{\nabla} \nabla_{\mathscr{L}}^I h^{(1)}|^2$$

$$+ \varepsilon \sum_{|I| \leq |k|} (1+|q|)^{-1}(1+q_-)^{-2\mu}\left(|\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{L}\mathscr{N}}^2 + |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{T}\mathscr{T}}^2\right)$$

$$+ \varepsilon \sum_{|I| \leq k} (1+t+|q|)^{-1}(1+|q|)^{-2}|\nabla_{\mathscr{L}}^I h^{(1)}|. \tag{16.4.2}$$

*Proof.* From (11.1.11a) with $X_\nu \overset{\text{def}}{=} \mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{0\nu}$, the pointwise decay assumptions of Theorem 16.1, the decomposition $h = h^{(0)} + h^{(1)}$, and the $h^{(0)}$-decay estimates of Lemma 15.1, it follows that

$$\sum_{|I| \leq k} |(\mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{0\nu})\widehat{\mathscr{L}}_{\mathscr{L}}^I \mathfrak{F}^\nu| \lesssim \sum_{\substack{|I| \leq k \\ |I_1|+|I_2| \leq |I|}} |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}||\overline{\nabla} \nabla_{\mathscr{L}}^{I_1} h^{(1)}||\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}|$$

$$+ \sum_{\substack{|I| \leq k \\ |I_1|+|I_2| \leq |I|}} |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}||\nabla \nabla_{\mathscr{L}}^{I_1} h^{(1)}|\left(|\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}|_{\mathscr{T}\mathscr{T}}\right)$$

$$+ \sum_{\substack{|I| \leq k \\ |I_1|+|I_2|+|I_3| \leq |I|}} |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}||\nabla \nabla_{\mathscr{L}}^{I_1} h^{(1)}||\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}||\mathscr{L}_{\mathscr{L}}^{I_3} \mathscr{F}|$$

$$+ \sum_{\substack{|I| \leq k \\ |I_1|+|I_2|+|I_3| \leq |I|}} |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}||\nabla_{\mathscr{L}}^{I_1} h^{(1)}||\nabla \nabla_{\mathscr{L}}^{I_2} h^{(1)}||\mathscr{L}_{\mathscr{L}}^{I_3} \mathscr{F}|$$

$$+ \varepsilon \sum_{|I| \leq k} (1+t+|q|)^{-1}|\nabla \nabla_{\mathscr{L}}^I h^{(1)}|^2$$

$$+ \varepsilon \sum_{|I| \leq k} (1+t+|q|)^{-1}(1+|q|)^{-2}|\nabla_{\mathscr{L}}^I h^{(1)}|^2$$

$$+ \varepsilon \sum_{|I| \leq k} (1+t+|q|)^{-1}|\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|^2. \tag{16.4.3}$$

Inequality (16.4.2) now follows from the assumptions of Theorem 16.1, (16.4.3), and repeated application of algebraic inequalities of the form $|ab| \lesssim \varsigma a^2 + \varsigma^{-1}b^2$. As an example, we consider the

term $|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}||\nabla\nabla_{\mathcal{Z}}^{I_1} h^{(1)}||\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}|_{\mathcal{L}\mathcal{N}}$ in the case that $|I_1| \leq |I| \leq \lfloor \ell/2 \rfloor$ (such an inequality must be satisfied by either $|I_1|$ or $|I_2|$). Then with the help of (16.1.6b) and the fact that $\mu + \mu' < \frac{1}{2}$, it follows that, if $\varepsilon$ is sufficiently small, then

$$
|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}||\nabla\nabla_{\mathcal{Z}}^{I_1} h^{(1)}||\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}|_{\mathcal{L}\mathcal{N}} \lesssim \varepsilon(1+t+|q|)^{-1}|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|^2 + \varepsilon^{-1}(1+t+|q|)|\nabla\nabla_{\mathcal{Z}}^{I_1} h^{(1)}|^2 |\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}|_{\mathcal{L}\mathcal{N}}^2
$$
$$
\lesssim \varepsilon(1+t+|q|)^{-1}|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|^2 + \varepsilon(1+|q|)^{-1}(1+q_-)^{-2\mu}|\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}|_{\mathcal{L}\mathcal{N}}^2. \quad (16.4.4)
$$

We now observe that the right-hand side of the above inequality is manifestly bounded by the right-hand side of (16.4.2). $\qquad\qquad\square$

We now use the pointwise estimates of the previous lemma to estimate part of the second space-time integral on the right-hand side of (12.2.1). These estimates are easier than the corresponding estimates involving the commutator term $N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathcal{L}_{\mathcal{Z}}\mathcal{F}_{\kappa\lambda} - \widehat{\mathcal{L}}_{\mathcal{Z}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathcal{F}_{\kappa\lambda})$, which are derived in Lemma 16.17.

**Lemma 16.15** (Integral estimates for $|(\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}_{0\nu})\widehat{\mathcal{L}}_{\mathcal{Z}}^I \mathfrak{F}^\nu|w(q))$. *Under the assumptions of Lemma 16.14, if $0 \leq k \leq \ell$ and $\varepsilon$ is sufficiently small, then the following integral inequality holds for $t \in [0, T]$:*

$$
\sum_{|I| \leq k} \int_0^t \!\!\int_{\Sigma_\tau} \left| (\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}_{0\nu})\widehat{\mathcal{L}}_{\mathcal{Z}}^I \mathfrak{F}^\nu \right| w(q)\, d^3x\, d\tau
$$
$$
\lesssim \varepsilon \sum_{|I| \leq k} \int_0^t \!\!\int_{\Sigma_\tau} (1+\tau)^{-1}|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|^2 w(q)\, d^3x\, d\tau + \varepsilon \sum_{|I| \leq k} \int_0^t \!\!\int_{\Sigma_\tau} (1+\tau)^{-1}|\nabla\nabla_{\mathcal{Z}}^I h^{(1)}|^2 w(q)\, d^3x\, d\tau
$$
$$
+ \varepsilon \sum_{|I| \leq k} \int_0^t \!\!\int_{\Sigma_\tau} |\overline{\nabla}\nabla_{\mathcal{Z}}^I h^{(1)}|^2 w'(q)\, d^3x\, d\tau
$$
$$
+ \varepsilon \sum_{|I| \leq k} \int_0^t \!\!\int_{\Sigma_\tau} \left( |\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{L}\mathcal{N}}^2 + |\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{T}\mathcal{T}}^2 \right) w'(q)\, d^3x\, d\tau. \quad (16.4.5)
$$

*Proof.* Inequality (16.4.5) follows from multiplying inequality (16.4.2) by $w(q)$, integrating $\int_0^t \int_{\Sigma_\tau} d^3x\, d\tau$, using the fact that $(1+|q|)^{-1}(1+q_-)^{-2\mu}w(q) \lesssim w'(q)$, and using the Hardy estimate (16.3.9) to bound the integral corresponding to the last sum on the right-hand side of (16.4.2) by the second sum on the right-hand side of (16.4.5). $\qquad\square$

The next lemma is a companion to Lemma 16.14. In the lemma, we derive pointwise estimates for the term $|(\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}_{0\nu})[N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathcal{L}_{\mathcal{Z}}^I \mathcal{F}_{\kappa\lambda} - \widehat{\mathcal{L}}_{\mathcal{Z}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathcal{F}_{\kappa\lambda})]|$. This term appears in the second spacetime integral on the right-hand side of (12.2.1), which is our basic energy inequality for the Faraday tensor and its Lie derivatives. As before, these pointwise estimates are preliminary estimates that will be used in the subsequent lemma to estimate the corresponding spacetime integral.

**Lemma 16.16** (Pointwise estimates for $|(\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}_{0\nu})[N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathcal{L}_{\mathcal{Z}}^I \mathcal{F}_{\kappa\lambda} - \widehat{\mathcal{L}}_{\mathcal{Z}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathcal{F}_{\kappa\lambda})]|$). *Let $N^{\#\mu\nu\kappa\lambda}\cdot \nabla_\mu \mathcal{L}_{\mathcal{Z}}^I \mathcal{F}_{\kappa\lambda} - \widehat{\mathcal{L}}_{\mathcal{Z}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathcal{F}_{\kappa\lambda})$ be the inhomogeneous commutator term (8.1.3b) in the equations of variation (8.1.1b) satisfied by $\dot{\mathfrak{F}}_{\mu\nu} \stackrel{\text{def}}{=} \mathcal{L}_{\mathcal{Z}}^I \mathcal{F}_{\mu\nu}$. Under the assumptions of Theorem 16.1, if $1 \leq k \leq \ell$ and $\varepsilon$ is sufficiently small, then the following pointwise inequality holds for $(t, x) \in [0, T] \times \mathbb{R}^3$:*

$$\sum_{|I|\leq k}\left|(\mathscr{L}_{\mathscr{L}}^I\mathscr{F}_{0\nu})\big(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{L}_{\mathscr{L}}^I\mathscr{F}_{\kappa\lambda}-\widehat{\mathscr{L}}_{\mathscr{L}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{F}_{\kappa\lambda})\big)\right|$$

$$\lesssim\varepsilon\sum_{|I|\leq|k|}(1+t+|q|)^{-1}|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|^2+\varepsilon\sum_{|I|\leq|k|}(1+t+|q|)^{-1}(1+|q|)^{-2}|\nabla_{\mathscr{L}}^I h^{(1)}|^2$$

$$+\varepsilon\sum_{|I|\leq|k|}(1+|q|)^{-1}(1+q_-)^{-2\mu}\big(|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|^2_{\mathscr{L}\mathscr{N}}+|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|^2_{\mathscr{T}\mathscr{T}}\big)$$

$$+\varepsilon\sum_{|I|\leq k}(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-(2+C\varepsilon)}(1+q_-)^{-2\mu}|\nabla_{\mathscr{L}}^I h^{(1)}|^2_{\mathscr{L}\mathscr{L}}$$

$$+\varepsilon\sum_{|J|\leq k-1}(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-(2+C\varepsilon)}(1+q_-)^{-2\mu}|\nabla_{\mathscr{L}}^J h^{(1)}|^2_{\mathscr{L}\mathscr{T}}$$

$$+\varepsilon\underbrace{\sum_{|J'|\leq k-2}(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-2}|\nabla_{\mathscr{L}}^{J'} h^{(1)}|^2}_{\text{absent if }k=1}$$

$$+\varepsilon\sum_{|J|\leq k-1}(1+t+|q|)^{-1+C\varepsilon}|\mathscr{L}_{\mathscr{L}}^J\mathscr{F}|^2. \tag{16.4.6}$$

*Proof.* From inequality (11.1.11b) with $X_\nu\overset{\text{def}}{=}\mathscr{L}_{\mathscr{L}}^I\mathscr{F}_{0\nu}$, the pointwise decay assumptions of Theorem 16.1, together with the decomposition $h=h^{(0)}+h^{(1)}$ and the $h^{(0)}$ decay estimates of Lemma 15.1, it follows that

$$\sum_{|I|\leq k}\left|(\mathscr{L}_{\mathscr{L}}^I\mathscr{F}_{0\nu})\big(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{L}_{\mathscr{L}}^I\mathscr{F}_{\kappa\lambda}-\widehat{\mathscr{L}}_{\mathscr{L}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{F}_{\kappa\lambda})\big)\right|$$

$$\lesssim\sum_{\substack{|I|\leq k,\ |I'|\leq k\\|J|\leq 1}}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}||\nabla_{\mathscr{L}}^{I'} h^{(1)}|_{\mathscr{L}\mathscr{L}}|\mathscr{L}_{\mathscr{L}}^J\mathscr{F}|+\sum_{\substack{|I|\leq k,\ |I'|\leq k\\|J|\leq 1}}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}||\nabla_{\mathscr{L}}^J h^{(1)}|_{\mathscr{L}\mathscr{L}}|\mathscr{L}_{\mathscr{L}}^{I'}\mathscr{F}|$$

$$+\sum_{|I|\leq k}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|^2|h|_{\mathscr{L}\mathscr{T}}+\sum_{\substack{|I|\leq k\\|I_1|+|I_2|\leq k+1\\|I_1|,|I_2|\leq k}}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}||\nabla_{\mathscr{L}}^{I_1} h^{(1)}|\big(|\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|_{\mathscr{L}\mathscr{N}}+|\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|_{\mathscr{T}\mathscr{T}}\big)$$

$$+\sum_{\substack{|I|\leq k\\|I_1|+|I_2|\leq k+1\\|I_1|,|I_2|\leq k}}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|_{\mathscr{L}\mathscr{N}}|\nabla_{\mathscr{L}}^{I_1} h^{(1)}||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|$$

$$+\sum_{\substack{|I|\leq k\\|I_1|+|I_2|\leq k+1\\|I_1|,|I_2|\leq k}}(1+t+|q|)^{-1}|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}||\nabla_{\mathscr{L}}^{I_1} h^{(1)}||\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|$$

$$+\varepsilon\sum_{|I|\leq k}(1+t+|q|)^{-1}|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}|^2+\varepsilon\sum_{|I|\leq k}(1+t+|q|)^{-1}(1+|q|)^{-2}|\nabla_{\mathscr{L}}^I h^{(1)}|^2$$

$$+\sum_{\substack{|I|\leq k\\|I_1|+|I_2|\leq k+1\\|I_1|\leq k-1,\ |I_2|\leq k-1}}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}||\nabla_{\mathscr{L}}^{I_1} h^{(1)}|_{\mathscr{L}\mathscr{L}}|\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|$$

$$+\sum_{\substack{|I|\leq k\\|I_1|+|I_2|\leq k\\|I_1|\leq k-1,\ |I_2|\leq k-1}}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{L}}^I\mathscr{F}||\nabla_{\mathscr{L}}^{I_1} h^{(1)}|_{\mathscr{L}\mathscr{T}}|\mathscr{L}_{\mathscr{L}}^{I_2}\mathscr{F}|$$

$$+ \sum_{\substack{|I| \leq k \\ |I_1|+|I_2| \leq k-1 \\ |I_1| \leq k-2,\, |I_2| \leq k-1}} (1+|q|)^{-1} |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}| |\nabla_{\mathscr{L}}^{I_1} h^{(1)}| |\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}|$$

<center>absent if $k=1$</center>

$$+ \sum_{\substack{|I| \leq k \\ |I_1|+|I_2|+|I_3| \leq k+1 \\ |I_1|,|I_2|,|I_3| \leq k}} (1+|q|)^{-1} |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}| |\nabla_{\mathscr{L}}^{I_1} h^{(1)}| |\nabla_{\mathscr{L}}^{I_2} h^{(1)}| |\mathscr{L}_{\mathscr{L}}^{I_3} \mathscr{F}|$$

$$+ \sum_{\substack{|I| \leq k \\ |I_1|+|I_2|+|I_3| \leq k+1 \\ |I_1|,|I_2|,|I_3| \leq k}} (1+|q|)^{-1} |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}| |\nabla_{\mathscr{L}}^{I_1} h^{(1)}| |\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}| |\mathscr{L}_{\mathscr{L}}^{I_3} \mathscr{F}|$$

$$+ \sum_{\substack{|I| \leq k \\ |I_1|+|I_2|+|I_3| \leq k+1 \\ |I_1|,|I_2|,|I_3| \leq k}} (1+|q|)^{-1} |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}| |\mathscr{L}_{\mathscr{L}}^{I_1} \mathscr{F}| |\mathscr{L}_{\mathscr{L}}^{I_2} \mathscr{F}| |\mathscr{L}_{\mathscr{L}}^{I_3} \mathscr{F}|. \tag{16.4.7}$$

We remark that the $\varepsilon \sum_{|I| \leq k} (1+t+|q|)^{-1} |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|^2$ and $\varepsilon \sum_{|I| \leq k} (1+t+|q|)^{-1} (1+|q|)^{-2} |\nabla_{\mathscr{L}}^I h^{(1)}|^2$ sums on the right-hand side of (16.4.7) account for all of the terms containing a factor $\nabla_{\mathscr{L}}^J h^{(0)}$ for some $J$. Inequality (16.4.6) now follows from (16.4.7), the pointwise decay assumptions of Theorem 16.1 (including the implied estimates for $h^{(1)}$), and simple algebraic estimates of the form $|ab| \lesssim \varsigma a^2 + \varsigma^{-1} b^2$ (as in (16.4.4)). $\quad\square$

The next lemma is a companion to Lemma 16.15. In the lemma, we use the pointwise estimates of the previous lemma to estimate the part of the second spacetime integral on the right-hand side of (12.2.1) that was not addressed by Lemma 16.15.

**Lemma 16.17** (Integral estimates for $|(\mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{0\nu})[N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathscr{L}}^I (N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{F}_{\kappa\lambda})]|$). *Under the assumptions of Lemma 16.14, if $1 \leq k \leq \ell$ and $\varepsilon$ is sufficiently small, then the following integral inequality holds for $t \in [0, T)$:*

$$\sum_{|I| \leq k} \int_0^t \int_{\Sigma_\tau} \left| (\mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{0\nu}) \left( N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathscr{L}}^I (N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{F}_{\kappa\lambda}) \right) \right| w(q) \, d^3x \, d\tau$$

$$\lesssim \varepsilon \sum_{|I| \leq k} \int_0^t \int_{\Sigma_\tau} (1+\tau)^{-1} |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|^2 w(q) \, d^3x \, d\tau + \varepsilon \sum_{|I| \leq k} \int_0^t \int_{\Sigma_\tau} (1+\tau)^{-1} |\nabla \nabla_{\mathscr{L}}^I h^{(1)}|^2 w(q) \, d^3x \, d\tau$$

$$+ \varepsilon \sum_{|I| \leq k} \int_0^t \int_{\Sigma_\tau} \left( |\overline{\nabla} \nabla_{\mathscr{L}}^I h^{(1)}|^2 + |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{L}\mathcal{N}}^2 + |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{T}\mathscr{T}}^2 \right) w'(q) \, d^3x \, d\tau$$

$$+ \varepsilon \sum_{|J'| \leq k-2} \int_0^t \int_{\Sigma_\tau} (1+\tau+|q|)^{-1+C\varepsilon} |\nabla \nabla_{\mathscr{L}}^{J'} h^{(1)}|^2 w(q) \, d^3x \, d\tau$$

<center>absent if $k=1$</center>

$$+ \varepsilon \sum_{|J| \leq k-1} \int_0^t \int_{\Sigma_\tau} (1+\tau+|q|)^{-1+C\varepsilon} |\mathscr{L}_{\mathscr{L}}^J \mathscr{F}|^2 w(q) \, d^3x \, d\tau + \varepsilon^3. \tag{16.4.8}$$

*Proof.* We begin by multiplying both sides of (16.4.6) by $w(q)$ and integrating $\int_0^t \int_{\Sigma_\tau} d^3x \, d\tau$. The integrals corresponding to the first and last sums on the right-hand side of (16.4.6) are manifestly bounded by the first and next-to-last terms on the right-hand side of (16.4.8). Using also the fact that $(1+|q|)^{-1}(1+q_-)^{-2\mu}w(q) \lesssim w'(q)$, we deduce that the integral corresponding to the third sum on the right-hand side of (16.4.6) is bounded by the third sum on the right-hand side of (16.4.8).

To bound the integral corresponding to the second sum on the right-hand side of (16.4.6), we simply use the Hardy inequalities of Proposition C.1 to derive the inequality

$$
\sum_{|I| \le k} \int_0^t \int_{\Sigma_\tau} (1+\tau+|q|)^{-1}(1+|q|)^{-2} |\nabla_{\mathscr{L}}^I h^{(1)}|^2 w(q) \, d^3x \, d\tau
$$
$$
\lesssim \sum_{|I| \le k} \int_0^t \int_{\Sigma_\tau} (1+\tau+|q|)^{-1} |\nabla \nabla_{\mathscr{L}}^I h^{(1)}|^2 w(q) \, d^3x \, d\tau. \quad (16.4.9)
$$

After multiplication by $\varepsilon$, we see that the right-hand side of the above inequality is manifestly bounded by the second sum on the right-hand side of (16.4.8). Using the same reasoning, we obtain the following bound for the integral corresponding to the next-to-last sum on the right-hand side of (16.4.6):

$$
\sum_{|J'| \le k-2} \int_0^t \int_{\Sigma_\tau} (1+\tau+|q|)^{-1+C\varepsilon}(1+|q|)^{-2} |\nabla_{\mathscr{L}}^{J'} h^{(1)}|^2 w(q) \, d^3x \, d\tau
$$
$$
\lesssim \sum_{|J'| \le k-2} \int_0^t \int_{\Sigma_\tau} (1+\tau+|q|)^{-1+C\varepsilon} |\nabla \nabla_{\mathscr{L}}^{J'} h^{(1)}|^2 w(q) \, d^3x \, d\tau. \quad (16.4.10)
$$

We then multiply (16.4.10) by $\varepsilon$ and observe that the right-hand side of the resulting inequality is manifestly bounded by the right-hand side of (16.4.8).

To estimate the integrals corresponding to the fourth and fifth sums on the right-hand side of (16.4.6), we will make use of the weight $\widetilde{w}(q)$, which is defined by

$$
\widetilde{w}(q) \stackrel{\text{def}}{=} \min\{w'(q), (1+t+|q|)^{-1+C\varepsilon} w(q)\}. \quad (16.4.11)
$$

We note that by (12.1.2) the following inequality is satisfied:

$$
\widetilde{w}(q) \lesssim (1+|q|)^{-1} w(q). \quad (16.4.12)
$$

With the help of Lemma 16.13, (16.4.12), and the Hardy inequalities of Proposition C.1, we estimate the integral corresponding to the fourth sum on the right-hand side of (16.4.6) as follows:

$$
\sum_{|I| \le k} \int_0^t \int_{\Sigma_\tau} (1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-(2+C\varepsilon)}(1+q_-)^{-2\mu} |\nabla_{\mathscr{L}}^I h^{(1)}|_{\mathscr{L}\mathscr{L}}^2 w(q) \, d^3x \, d\tau
$$
$$
\lesssim \sum_{|I| \le k} \int_0^t \int_{\Sigma_\tau} |\nabla \nabla_{\mathscr{L}}^I h^{(1)}|_{\mathscr{L}\mathscr{L}}^2 \widetilde{w}(q) \, d^3x \, d\tau
$$

$$\lesssim \sum_{|I| \leq k} \int_0^t \int_{\Sigma_\tau} |\overline{\nabla} \nabla_{\mathscr{L}}^I h^{(1)}|^2 w'(q) \, d^3x \, d\tau$$

$$+ \varepsilon^2 \int_0^t \int_{\Sigma_\tau} (1 + \tau + |q|)^{-4} \chi_0^2 \left( \frac{1}{2} < \frac{r}{t} < \frac{3}{4} \right) w'(q) \, d^3x \, d\tau$$

$$+ \varepsilon^4 \int_0^t \int_{\Sigma_\tau} (1 + \tau + |q|)^{-6} w'(q) \, d^3x \, d\tau$$

$$+ \varepsilon^2 \sum_{|I| \leq k} \int_0^t \int_{\Sigma_\tau} (1 + \tau)^{-1} |\nabla \nabla_{\mathscr{L}}^I h^{(1)}|^2 w(q) \, d^3x \, d\tau$$

$$+ \varepsilon^2 \sum_{|I| \leq k} \int_0^t \int_{\Sigma_\tau} (1 + \tau + |q|)^{-1} (1 + |q|)^{-2} |\nabla_{\mathscr{L}}^I h^{(1)}|^2 w(q) \, d^3x \, d\tau$$

$$+ \underbrace{\sum_{|J'| \leq k-2} \int_0^t \int_{\Sigma_\tau} (1 + \tau + |q|)^{-1+C\varepsilon} |\nabla \nabla_{\mathscr{L}}^{J'} h^{(1)}|^2 w(q) \, d^3x \, d\tau}_{\text{absent if } k = 1}$$

$$\lesssim \sum_{|I| \leq k} \int_0^t \int_{\Sigma_\tau} |\overline{\nabla} \nabla_{\mathscr{L}}^I h^{(1)}|^2 w'(q) \, d^3x \, d\tau$$

$$+ \sum_{|I| \leq k} \int_0^t \int_{\Sigma_\tau} (1 + \tau)^{-1} |\nabla \nabla_{\mathscr{L}}^I h^{(1)}|^2 w(q) \, d^3x \, d\tau$$

$$+ \underbrace{\sum_{|J'| \leq k-2} \int_0^t \int_{\Sigma_\tau} (1 + \tau)^{-1+C\varepsilon} |\nabla \nabla_{\mathscr{L}}^{J'} h^{(1)}|^2 w(q) \, d^3x \, d\tau}_{\text{absent if } k = 1} + \varepsilon^2, \quad (16.4.13)$$

where to pass to the final inequality we have again used Proposition C.1 to estimate

$$\sum_{|I| \leq k} \int_0^t \int_{\Sigma_\tau} (1 + \tau + |q|)^{-1} (1 + |q|)^{-2} |\nabla_{\mathscr{L}}^I h^{(1)}|^2 w(q) \, d^3x \, d\tau$$

$$\lesssim \sum_{|I| \leq k} \int_0^t \int_{\Sigma_\tau} (1 + \tau)^{-1} |\nabla \nabla_{\mathscr{L}}^I h^{(1)}|^2 w(q) \, d^3x \, d\tau.$$

After multiplying both sides of (16.4.13) by $\varepsilon$, we see that the resulting right-hand side is manifestly bounded by the right-hand side of (16.4.8) as desired. The integral corresponding to the fifth sum on the right-hand side of (16.4.6) can be bounded via the same reasoning. □

Combining Lemmas 16.15 and 16.17, we arrive at the following corollary:

**Corollary 16.18** (Estimates for the energy integrals corresponding to the $\mathscr{F}$ inhomogeneities). *Let*

$$\mathfrak{F}_{(I)}^\nu = \widehat{\mathscr{L}}_{\mathscr{L}}^I \mathfrak{F}^\nu + [N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathscr{L}}^I (N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{F}_{\kappa\lambda})]$$

*be the inhomogeneous term* (8.1.3b) *in the equations of variation* (8.1.1b) *satisfied by* $\dot{\mathscr{F}}_{\mu\nu} \overset{\text{def}}{=} \mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{\mu\nu}$. *Under the assumptions of Theorem 16.1, if* $0 \leq k \leq \ell$ *and* $\varepsilon$ *is sufficiently small, then the following integral*

*inequality holds for $t \in [0, T]$:*

$$\sum_{|I| \le k} \int_0^t \int_{\Sigma_\tau} |(\mathscr{L}_{\mathscr{L}}^I \mathscr{F}_{0\nu}) \mathfrak{F}_{(I)}^\nu| w(q) \, d^3x \, d\tau$$

$$\lesssim \varepsilon \sum_{|I| \le k} \int_0^t \int_{\Sigma_\tau} (1 + \tau)^{-1} \left| \begin{pmatrix} \nabla \nabla_{\mathscr{L}}^I h^{(1)} \\ \mathscr{L}_{\mathscr{L}}^I \mathscr{F} \end{pmatrix} \right|^2 w(q) \, d^3x \, d\tau$$

$$+ \varepsilon \sum_{|I| \le k} \int_0^t \int_{\Sigma_\tau} \left( |\overline{\nabla} \nabla_{\mathscr{L}}^I h^{(1)}|^2 + |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathscr{L}\mathcal{N}}^2 + |\mathscr{L}_{\mathscr{L}}^I \mathscr{F}|_{\mathcal{T}\mathcal{T}}^2 \right) w'(q) \, d^3x \, d\tau$$

$$\underbrace{+ \varepsilon \sum_{|J| \le k-1} \int_0^t \int_{\Sigma_\tau} (1 + \tau)^{-1+C\varepsilon} \left| \begin{pmatrix} \nabla \nabla_{\mathscr{L}}^J h^{(1)} \\ \mathscr{L}_{\mathscr{L}}^J \mathscr{F} \end{pmatrix} \right|^2 w(q) \, d^3x \, d\tau}_{\text{absent if } k = 0} + \varepsilon^3. \qquad (16.4.14)$$

## Appendix A: Weighted Sobolev–Moser inequalities

The propositions and corollaries stated in this section were used in Section 10 to relate the smallness condition on the abstract initial data to a smallness condition on the initial energy of the corresponding solution to the reduced equations. The lemmas we state are slight extensions of Lemmas 2.4 and 2.5 of [Choquet-Bruhat and Christodoulou 1981] while the corollaries are easy (and nonoptimal) consequences of the lemmas. Throughout the appendix, we use the abbreviations

$$C_\eta^\ell \overset{\text{def}}{=} C_\eta^\ell(\mathbb{R}^3), \qquad H_\eta^\ell \overset{\text{def}}{=} H_\eta^\ell(\mathbb{R}^3),$$

and so on (see Definitions 10.1 and 10.2). Furthermore, $(x^1, x^2, x^3)$ denotes the standard Euclidean coordinate system on $\mathbb{R}^3$ and $|x| \overset{\text{def}}{=} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$.

**Proposition A.1** (Weighted Sobolev embedding [Choquet-Bruhat and Christodoulou 1981, Lemma 2.4]). *Let $\ell$ and $\ell'$ be integers, and let $\eta$ and $\eta'$ be real numbers subject to the constraints $\ell' < \ell - \frac{3}{2}$ and $\eta' < \eta + \frac{3}{2}$. Assume that $v \in H_\eta^\ell$. Then $v \in C_{\eta'}^{\ell'}$, and*

$$\|v\|_{C_{\eta'}^{\ell'}} \lesssim \|v\|_{H_\eta^\ell}. \qquad (A.1)$$

**Proposition A.2** (Weighted Sobolev multiplication properties [Choquet-Bruhat and Christodoulou 1981, Lemma 2.5]). *Let $\ell_1, \ldots, \ell_p \ge 0$ be integers, and let $\eta_1, \ldots, \eta_p$ be real numbers. Suppose that $v_j \in H_{\eta_j}^{\ell_j}$ for $j = 1, \ldots, p$. Assume that the integer $\ell$ satisfies $0 \le \ell \le \min\{\ell_1, \ldots, \ell_p\}$ and $\ell \le \sum_{j=1}^p \ell_j - (p-1)\frac{3}{2}$ and that $\eta < \sum_{j=1}^p \eta_j + (p-1)\frac{3}{2}$. Then*

$$\prod_{j=1}^p v_j \in H_\eta^\ell, \qquad (A.2)$$

*and the multiplication map*

$$H_{\eta_1}^{\ell_1} \times \cdots \times H_{\eta_p}^{\ell_p} \to H_\ell^\eta, \qquad (v_1, \ldots, v_p) \to \prod_{j=1}^p v_j \qquad (A.3)$$

*is continuous.*

**Corollary A.3.** *Let $\ell \geq 2$ be an integer, and let $\eta \geq 0$. Assume that $v_j \in H_\eta^\ell$ for $j = 1, \ldots, p$ and that $I_1, \ldots, I_p$ are $\underline{\nabla}$-multi-indices satisfying $\sum_{j=1}^p |I_j| \leq \ell$. Then*

$$(1 + |x|^2)^{(\eta + \sum_{j=1}^p |I_j|)/2} \prod_{i=1}^p \underline{\nabla}^{I_i} v_i \in L^2 \tag{A.4}$$

*and*

$$\left\| (1 + |x|^2)^{(\eta + \sum_{j=1}^p |I_j|)/2} \prod_{i=1}^p \underline{\nabla}^{I_i} v_i \right\|_{L^2} \lesssim \prod_{i=1}^p \|v_i\|_{H_\eta^\ell}. \tag{A.5}$$

**Corollary A.4.** *Let $\ell \geq 2$ be an integer, let $\mathfrak{K}$ be a compact set, and let $F(\cdot) \in C^\ell(\mathfrak{K})$ be a function. Assume that $v_1$ is a function on $\mathbb{R}^3$ such that $v_1(\mathbb{R}^3) \subset \mathfrak{K}$. Furthermore, assume that $\underline{\nabla} v_1, v_2 \in H_\eta^\ell$. Then $(F \circ v_1)v_2 \in H_\eta^\ell$, and*

$$\|(F \circ v_1)v_2\|_{H_\eta^\ell} \lesssim \|v_2\|_{H_\eta^\ell} |F|_{\mathfrak{K}} + \|(1 + |x|)v_2\|_{L^\infty} \|\underline{\nabla} v_1\|_{H_\eta^{\ell-1}} \sum_{j=1}^\ell |F^{(j)}|_{\mathfrak{K}} \|v_1\|_{L^\infty}^{j-1}, \tag{A.6}$$

*where $F^{(j)}$ denotes the array of all $j$-th order partial derivatives of $F$ with respect to its arguments and $|F^{(j)}|_{\mathfrak{K}} \stackrel{\mathrm{def}}{=} \sup_{v \in \mathfrak{K}} |F^{(j)}(v)|$.*

### Appendix B: Weighted Klainerman–Sobolev inequalities

In this section, we recall the weighted Klainerman–Sobolev inequalities that were proved in [Lindblad and Rodnianski 2010]. Throughout this section, the weight function $w(q)$ is defined by

$$w \stackrel{\mathrm{def}}{=} w(q) \stackrel{\mathrm{def}}{=} \begin{cases} 1 + (1 + |q|)^{1+2\gamma} & \text{if } q > 0, \\ 1 + (1 + |q|)^{-2\mu} & \text{if } q < 0. \end{cases} \tag{B.1}$$

In this section, we assume that $\gamma$ and $\mu$ are fixed constants satisfying $0 < \gamma < 1$ and $0 < \mu < \frac{1}{2}$. It easily follows from (B.1) that

$$w' \stackrel{\mathrm{def}}{=} w'(q) = \begin{cases} (1 + 2\gamma)(1 + |q|)^{2\gamma} & \text{if } q > 0, \\ 2\mu(1 + |q|)^{-1-2\mu} & \text{if } q < 0, \end{cases} \tag{B.2}$$

and

$$w' \leq 4(1 + |q|)^{-1} w \leq 16\mu^{-1}(1 + q_-)^{2\mu} w'. \tag{B.3}$$

**Proposition B.1** (Weighted Klainerman–Sobolev inequality [Lindblad and Rodnianski 2010, Proposition 14.1]). *There exists a $C > 0$ such that, for all $\phi(t, \cdot) \in C_0^\infty(\mathbb{R}^3)$, the following inequality holds:*

$$(1 + t + |x|)[(1 + |q|)w(q)]^{1/2} |\phi(t, x)| \leq C \sum_{|I| \leq 3} \left\| w^{1/2} \nabla_{\mathcal{L}}^I \phi(t, \cdot) \right\|_{L^2}, \qquad q \stackrel{\mathrm{def}}{=} |x| - t. \tag{B.4}$$

*Furthermore,*

$$(1 + t + |x|)[(1 + |q|)w(q)]^{1/2} |\nabla \phi(t, x)| \leq C \sum_{|I| \leq 3} \left\| w^{1/2} \nabla \nabla_{\mathcal{L}}^I \phi(t, \cdot) \right\|_{L^2}, \qquad q \stackrel{\mathrm{def}}{=} |x| - t. \tag{B.5}$$

*Proof.* Equation (B.4) was proved in the paper cited; (B.5) follows from Lemma 6.11 and (B.4). □

## Appendix C: Hardy-type inequalities

In this section, we recall the weighted Hardy-type inequalities proved in [Lindblad and Rodnianski 2010].

**Proposition C.1** (Hardy inequalities [Lindblad and Rodnianski 2010, Corollary 13.3]). *Let $\gamma > 0$ and $\mu > 0$, $q \overset{\text{def}}{=} |x| - t$, and let $w(q)$ and $w'(q)$ be as defined in* (B.1) *and* (B.2), *respectively. Then for any $-1 \le a \le 1$, there exists a $C > 0$ such that, for all $\phi \in C_0^\infty(\mathbb{R}^3)$, we have the integral inequality*

$$\int_{\mathbb{R}^3} (1+t+|q|)^{-1+a}(1+|q|)^{-2}|\phi|^2 w(q)\, d^3x \le C \int_{\mathbb{R}^3} (1+t+|q|)^{-1+a}|\partial_r\phi|^2 w(q)\, d^3x, \qquad \text{(C.1)}$$

*where $\partial_r = \omega^b \partial_b$, $\omega^j \overset{\text{def}}{=} x^j/r$, denotes the radial vector field.*

*If in addition $a < 2\min\{\gamma, \mu\}$, then with*

$$\widetilde{w}(q) \overset{\text{def}}{=} \min\{w'(q), (1+t+|q|)^{-1+a}w(q)\}, \qquad \text{(C.2)}$$

*there exists a constant $C > 0$ such that the integral inequality*

$$\int_{\mathbb{R}^3} (1+t+|q|)^{-1+a}(1+|q|)^{-(a+2)}(1+q_-)^{-2\mu}|\phi|^2 w(q)\, d^3x \le C \int_{\mathbb{R}^3} |\partial_r\phi|^2 \widetilde{w}(q)\, d^3x, \qquad \text{(C.3)}$$

*holds, where $q_- \overset{\text{def}}{=} |q|$ if $q \le 0$ and $q_- = 0$ if $q > 0$.*

**Corollary C.2.** *Assume the hypotheses of Proposition C.1, and let $P_{\mu\nu}$ be a type-$\binom{0}{2}$ tensor field. Let $\mathcal{V}$ and $\mathcal{W}$ be any two of the subsets of null frame-field vectors defined in* (5.1.12). *Then the same conclusions of Proposition C.1 hold if we replace $|\phi|$ and $|\partial_r\phi|$ with the contraction seminorms $|P|_{\mathcal{V}\mathcal{W}}$ and $|\nabla P|_{\mathcal{V}\mathcal{W}}$, respectively, where the contraction seminorms are defined in Definition 5.8.*

*Proof.* Let $\not{h}$ be the first fundamental form of the $S_{r,t}$ defined in (5.1.4b), and recall that the tensor $\not{h}_\mu{}^\nu$ projects $m$-orthogonally onto the $S_{r,t}$. Since $\partial_r = \frac{1}{2}(L - \underline{L})$, it follows from (5.1.9a), (5.1.9b), and (5.1.10) that

$$\partial_r(\underline{L}^\kappa \underline{L}^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}\underline{L}^\kappa \underline{L}^\lambda(\nabla_L - \nabla_{\underline{L}})P_{\kappa\lambda}, \qquad \text{(C.4)}$$

$$\partial_r(\underline{L}^\kappa L^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}\underline{L}^\kappa L^\lambda(\nabla_L - \nabla_{\underline{L}})P_{\kappa\lambda}, \qquad \text{(C.5)}$$

$$\partial_r(L^\kappa \underline{L}^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}L^\kappa \underline{L}^\lambda(\nabla_L - \nabla_{\underline{L}})P_{\kappa\lambda}, \qquad \text{(C.6)}$$

$$\partial_r(L^\kappa L^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}L^\kappa L^\lambda(\nabla_L - \nabla_{\underline{L}})P_{\kappa\lambda} \qquad (\mu = 0,1,2,3), \qquad \text{(C.7)}$$

$$\partial_r(\not{h}_\mu{}^\kappa \underline{L}^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}\not{h}_\mu{}^\kappa \underline{L}^\lambda(\nabla_L - \nabla_{\underline{L}})P_{\kappa\lambda} \qquad (\mu = 0,1,2,3), \qquad \text{(C.8)}$$

$$\partial_r(\underline{L}^\kappa \not{h}_\mu{}^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}\underline{L}^\kappa \not{h}_\mu{}^\lambda(\nabla_L - \nabla_{\underline{L}})P_{\kappa\lambda} \qquad (\mu = 0,1,2,3), \qquad \text{(C.9)}$$

$$\partial_r(\not{h}_\mu{}^\kappa L^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}\not{h}_\mu{}^\kappa L^\lambda(\nabla_L - \nabla_{\underline{L}})P_{\kappa\lambda} \qquad (\mu = 0,1,2,3), \qquad \text{(C.10)}$$

$$\partial_r(L^\kappa \not{h}_\mu{}^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}L^\kappa \not{h}_\mu{}^\lambda(\nabla_L - \nabla_{\underline{L}})P_{\kappa\lambda} \qquad (\mu = 0,1,2,3), \qquad \text{(C.11)}$$

$$\partial_r(\not{h}_\mu{}^\kappa \not{h}_\nu{}^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}\not{h}_\mu{}^\kappa \not{h}_\nu{}^\lambda(\nabla_L - \nabla_{\underline{L}})P_{\kappa\lambda} \qquad (\mu, \nu = 0,1,2,3). \qquad \text{(C.12)}$$

That is to say, $\partial_r$ commutes with the null decomposition of $P$. The conclusion of the corollary now easily follows from applying the proposition with $\phi$ equal to the scalar-valued functions $\underline{L}^\kappa \underline{L}^\lambda P_{\kappa\lambda}$, $\underline{L}^\kappa L^\lambda P_{\kappa\lambda}$, $\dots$, $\not{h}_\mu{}^\kappa \not{h}_\nu{}^\lambda P_{\kappa\lambda}$, respectively. $\qquad \square$

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# DISPERSION FOR THE SCHRÖDINGER EQUATION ON THE LINE WITH MULTIPLE DIRAC DELTA POTENTIALS AND ON DELTA TREES

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We consider the time-dependent one-dimensional Schrödinger equation with multiple Dirac delta potentials of different strengths. We prove that the classical dispersion property holds under some restrictions on the strengths and on the lengths of the finite intervals. The result is obtained in a more general setting of a Laplace operator on a tree with $\delta$-coupling conditions at the vertices. The proof relies on a careful analysis of the properties of the resolvent of the associated Hamiltonian. With respect to our earlier analysis for Kirchhoff conditions [*J. Math. Phys.* **52**:8 (2011), #083703], here the resolvent is no longer in the framework of Wiener algebra of almost periodic functions, and its expression is harder to analyse.

## 1. Introduction

In this paper we are concerned with the dispersive properties of the Schrödinger equation with multiple Dirac delta potentials and more generally for the Schrödinger equation on a tree with $\delta$-coupling conditions at the vertices.

Let us first recall that the linear Schrödinger equation on the line,

$$\begin{cases} i u_t(t, x) + u_{xx}(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

conserves the $L^2$-norm

$$\|e^{it\Delta} u_0\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})} \tag{2}$$

and enjoys the dispersive estimate

$$\|e^{it\Delta} u_0\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{|t|}} \|u_0\|_{L^1(\mathbb{R})}, \quad t \neq 0. \tag{3}$$

It is classical to obtain from these two inequalities the well-known space-time Strichartz estimates [Strichartz 1977; Ginibre and Velo 1985], for $r \geq 2$,

$$\|e^{it\Delta} u_0\|_{L_t^{\frac{4r}{r-2}}(\mathbb{R}, L_x^r(\mathbb{R}))} \leq C \|u_0\|_{L^2(\mathbb{R})}. \tag{4}$$

These dispersive estimates have been successfully applied to obtain results for the nonlinear Schrödinger equation (see, for example [Cazenave 2003; Tao 2006] and the references therein).

Our general framework in this paper refers to the Dirac delta Hamiltonian on a tree with a finite number of vertices, with the external edges (those that have only one internal vertex as an endpoint) formed by infinite strips. The particular case of a tree with all the internal vertices having degree two will give us a result for the Schrödinger equation on the line with several Dirac potentials. Although the latter is a corollary of the former, we shall start our presentation with the case of the line. This is motivated by the fact that historically dispersive properties have been studied first in this case (only with one or with two delta Dirac potentials) and that the previous results on graphs concern only star-shaped graphs (with only one vertex), where the proofs are in the same spirit as on the line with one Dirac delta potential.

So we first consider the semigroup $\exp(-itH_\alpha)$, where $H_\alpha$ is a perturbation of the Laplace operator with $n$ Dirac delta potentials with real strengths $\{\alpha_j\}_{j=1}^{p}$,

$$H_\alpha = -\Delta + \sum_{j=1}^{p} \alpha_j \delta(x - x_j). \tag{5}$$

The spectral properties of the Laplacian with multiple Dirac delta potentials on $\mathbb{R}^n$ have been extensively studied. Operator $H_\alpha$ has at most $p$ eigenvalues, which are all negative and simple, and there are no eigenvalues in the case of positive strengths $\alpha_i > 0$. The remaining part of the spectrum is absolutely continuous and $\sigma_{ac}(H_\alpha) = [0, \infty)$. We will denote by $P_e$ the $L^2$ projection onto the subspace of the eigenfunctions and by $P$ the projection outside the discrete spectrum. Regarding the spectral properties of $H_\alpha$ we refer to [Albeverio et al. 2005, §II.2] and to the references within. The time-dependent propagator of the linear Schrödinger equation has also been considered in the case of one Dirac delta potential [Gaveau and Schulman 1986; Manoukian 1989; Adami and Sacchetti 2005; Datchev and Holmer 2009], or one point interactions [Albeverio et al. 1994; Adami and Noja 2009; Fukuizumi et al. 2008], or two symmetric Dirac delta potentials [Kovařík and Sacchetti 2010]. In particular, in the case of the line with one delta interaction, without sign condition on the strength, dispersive estimates has been proved but for $e^{-itH_\alpha}P$ [Adami and Sacchetti 2005; Datchev and Holmer 2009]. A similar result was proved to hold in the case of two-point interactions, under a condition on the delta-strength and on the distance between the location of the point interactions [Kovařík and Sacchetti 2010]; see also [Angulo Pava and Ferreira 2013]. Also in [Duchêne et al. 2011] the problem of dispersion for several-delta potentials has been considered, as well as wave operator bounds from which dispersive estimates can be obtained as a consequence. Here Jost and distorted plane functions are used in spectral formulae. A weighted weaker than classical dispersion estimate is obtained for a class of potentials with singularities.

Concerning the nonlinear Schrödinger equation with a Dirac delta potential, standing wave and bound states have been analysed [Fukuizumi and Jeanjean 2008; Fukuizumi et al. 2008; Le Coz et al. 2008], as well as the time dynamics of solitons [Holmer and Zworski 2007; Holmer et al. 2007a; 2007b].

For stating our first result concerning the case of several Dirac potentials, we need to introduce the following functions. With the notations in Lemma 3.1 in the case when $n_j = 2$, we define $f_p = \det D_p$

and $g_p = \dfrac{\det \widetilde{D}_p}{\det D_p}$, defined by recursion as follows:

$$f_1(\omega) = \frac{2\omega + \alpha_1}{\omega + \alpha_1}, \quad f_p(\omega) = \frac{2\omega + \alpha_p}{\omega + \alpha_p} e^{\omega a_{p-1}} f_{p-1}(\omega) \left( 1 - \frac{\alpha_p}{2\omega + \alpha_p} e^{-2\omega a_{p-1}} g_{p-1}(\omega) \right),$$

where

$$g_1(\omega) = \frac{\alpha_1}{n_1 \omega + \alpha_1}, \quad g_p(\omega) = \frac{\dfrac{\alpha_p}{n_p \omega + \alpha_p} - \dfrac{-2\omega + \alpha_p}{2\omega + \alpha_p} e^{-2\omega a_{p-1}} g_{p-1}(\omega)}{1 - \dfrac{\alpha_p}{2\omega + \alpha_p} e^{-2\omega a_{p-1}} g_{p-1}(\omega)}.$$

These functions will appear naturally when computing the resolvent of $H_\alpha$.

**Theorem 1.1.** *For any $\{\alpha_j\}_{j=1}^p$ and $\{x_j\}_{j=1}^p$ such that*

$$\partial_\omega^{p-1} f_p \big|_{\omega=0} \neq 0, \tag{6}$$

*the solution of the linear Schrödinger equation on the line with multiple delta interactions of strength $\alpha_j$ located at $x_j$ satisfies the dispersion inequality*

$$\| e^{-itH_\alpha} P u_0 \|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{|t|}} \| u_0 \|_{L^1(\mathbb{R})} \quad \text{for all } t \neq 0. \tag{7}$$

*Moreover, in the case of positive strengths $\alpha_j > 0$, condition* (6) *is fulfilled and we have*

$$\| e^{-itH_\alpha} u_0 \|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{|t|}} \| u_0 \|_{L^1(\mathbb{R})} \quad \text{for all } t \neq 0. \tag{8}$$

We first notice that, in view of the definition of $f_p(\omega)$, condition (6) is not fulfilled only in a few explicit situations. For instance, if $p = 2$, the situations to be avoided are when $x_2 - x_1 + \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} = 0$, already used in [Kovařík and Sacchetti 2010].

In the previous works on dispersive estimates for one or two delta Dirac potentials, given the particular structure of the operator $H_\alpha$, the authors obtain explicit representations of the resolvent and then of $e^{-itH_\alpha}$. However in the general case of multiple delta interactions an explicit representation is not easy to obtain; even in [Albeverio et al. 1984; 2005, §II.2] the resolvent is obtained in terms of the inverse of some matrix $D_n$ that depends on $\{\alpha_j\}_{j=1}^p$ and on the lengths of the finite segments $\{x_j - x_{j-1}\}_{j=2}^p$.

The line setting might be seen as the special case of the equation posed on a simple graph with $n$ vertices, with only two edges starting from any vertex and with delta connection conditions at each vertex ($x_0 = -\infty$, $x_{p+1} = \infty$):

$$\begin{cases} i u_t(t, x) + u_{xx}(t, x) = 0, & x \in (x_{j-1}, x_j), \ j = 1, \ldots, p, \\ u_x(t, x_j^+) - u_x(t, x_j^-) = \alpha_j u(x_j), & t > 0, \ j = 1, \ldots, p. \end{cases} \tag{9}$$

Our second framework refers to the Dirac delta Hamiltonian $H_\alpha^\Gamma$ on a tree $\Gamma = (V, E)$ with a finite number of vertices $V$, with the external edges (those that have only one internal vertex as an endpoint) formed by infinite strips. We consider the linear Schrödinger equation in the case of a tree $\Gamma$, with delta conditions of not necessarily equal strength at the vertices

$$\begin{cases} i\boldsymbol{u}_t(t,x) = H_\alpha^\Gamma \boldsymbol{u}(t,x), & (t,x) \in \mathbb{R} \times \Gamma, \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x), & x \in \Gamma. \end{cases} \tag{10}$$

The presentation of the operator $H_\alpha^\Gamma$ will be given in full detail in Section 2. Let us just say here that $H_\alpha^\Gamma$ acts on a function $\boldsymbol{u}$ on a graph as $-\partial_{xx}$ on each restriction of $\boldsymbol{u}$ to an edge of the tree and that its domain consists of those functions $\boldsymbol{u}$ for which $\delta$-coupling conditions must be fulfilled. The $\delta$-coupling conditions are a continuity condition for the function $\boldsymbol{u}$ and a $\delta$-transmission condition at the level of its first derivative at all internal vertices $v$:

$$\sum_{e \in E_v} \partial_n \boldsymbol{u}(v) = \alpha(v)\boldsymbol{u}(v).$$

The operator $H_\alpha^\Gamma$ shares the same properties of $H_\alpha$ above: only a finite number of negative eigenvalues, and no eigenvalues for positive strengths, and $\sigma_{ac}(H_\alpha^\Gamma) = [0,\infty)$. These properties follow as in [Albeverio et al. 2005, §II.2].

The dispersion inequality for (10) was proved in [Banica and Ignat 2011] for the case of Kirchhoff's connection condition on trees, that is $\alpha(v) = 0$ for all internal vertices of the tree (see also [Ignat 2010]). The case of $\delta$- and $\delta'$-coupling on a star-shaped tree (i.e., only one vertex) has been considered in [Adami et al. 2011], where the main result concerns the time evolution of a fast soliton for the nonlinear equation, in the spirit of [Holmer et al. 2007a]. Finally, we mention that for the stationary nonlinear equation, the study of bound states on a star-shaped tree with delta conditions has been analysed in a series of papers [Adami et al. 2012a; 2012b; 2012c; 2012d].

The main result of this paper is the following, involving the expression of a determinant function $\det D_{\Gamma_p}(\omega)$ defined by recursion in Lemma 3.1.

**Theorem 1.2.** *Let us consider a tree $\Gamma = (V, E)$ with $p$ vertices. If the strengths at the vertices and the lengths of the finite edges are such that*

$$\partial_\omega^{(p-1)} \det D_{\Gamma_p}\big|_{\omega=0} \neq 0, \tag{11}$$

*then the solution of the linear Schrödinger equation on a tree with delta connection conditions satisfies the dispersion inequality*

$$\|e^{-itH_\alpha^\Gamma} P\boldsymbol{u}_0\|_{L^\infty(\Gamma)} \leq \frac{C}{\sqrt{|t|}} \|\boldsymbol{u}_0\|_{L^1(\Gamma)} \quad \text{for all } t \neq 0. \tag{12}$$

*Moreover, in the case of positive strengths $\alpha_j > 0$, condition* (11) *is fulfilled and we have*

$$\|e^{-itH_\alpha^\Gamma} \boldsymbol{u}_0\|_{L^\infty(\Gamma)} \leq \frac{C}{\sqrt{|t|}} \|\boldsymbol{u}_0\|_{L^1(\Gamma)} \quad \text{for all } t \neq 0. \tag{13}$$

The proof of Theorem 1.2 uses elements from [Banica 2003; Banica and Ignat 2011; Gavrus 2012] in an appropriate way related to the delta connection conditions on the tree. The starting point consists of writing the solution in terms of the resolvent of the Laplacian, which in turn is determined by recursion on the number of vertices. With respect to the previous works with Kirchhoff conditions, the novelty here

is that we are no longer in the framework of the almost periodic Wiener algebra of functions, and the expression of the resolvent is harder to analyse.

The linear solution $e^{-itH_\alpha^\Gamma} \boldsymbol{u}_0$ will be shown to be a combination of oscillatory integrals, that becomes more and more involved as the number of vertices of the tree grows. We do not have any more that $e^{-itH_\alpha^\Gamma} \boldsymbol{u}_0$ is a summable superposition of solutions of the linear Schrödinger equation on the line, as for Kirchhoff conditions in [Banica and Ignat 2011].

Theorem 1.1 follows from Theorem 1.2 by considering the particular case of a tree $\Gamma$ with all the internal vertices having degree two.

As classically noticed [Rauch 1978; Jensen and Kato 1979; Journé et al. 1991; Rodnianski and Schlag 2004; Goldberg and Schlag 2004], one can expect dispersion in the absence of eigenvalues and of zero resonances. In the $\delta$-coupling case the nongeneric condition (6) for $p = 2$ is precisely in link with the presence of a zero resonance (see formula (2.1.29) in Chapter II of [Albeverio et al. 2005]), so one might expect that in the absence of eigenvalues the dispersion holds generically, even for more general coupling. We shall give in Appendix C some sufficient conditions to obtain dispersion for general couplings.

Finally, we note that in the presence of eigenfunctions, the dispersion estimate cannot be valid globally in time. Denoting by $H$ either $H_\alpha$ or $H_\alpha^\Gamma$, the general classical $TT^*$ argument and the Christ–Kiselev lemma allow one to infer global in time Strichartz estimates as on $\mathbb{R}$ for $e^{-itH} P$, the dispersive part of $e^{-itH}$ (see for instance the short proof of Theorem 2.3 in [Tao 2006]). This together with the regularity of the eigenfunctions of the operator $H$ give us the following result:

**Theorem 1.3.** *Let $T > 0$ and let $(q, r)$ and $(q', r')$ be two $1$-admissible couples, in the sense that $4 \le q \le \infty$, $2 \le r \le \infty$ and $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$. For any $\alpha \ge 1$, there exists a constant $C > 0$ such that the homogeneous Strichartz estimates*

$$\|e^{-itH} \boldsymbol{u}_0\|_{L^q((0,T), L^r(\Gamma))} \le C \big( \|\boldsymbol{u}_0\|_{L^2(\Gamma)} + T^{1/q} \|\boldsymbol{u}_0\|_{L^\alpha(\Gamma)} \big),$$

*and the inhomogeneous Strichartz estimates*

$$\left\| \int_0^t e^{-i(t-s)H} \boldsymbol{F}(s) \, ds \right\|_{L^q((0,T), L^r(\Gamma))} \le C \big( \|\boldsymbol{F}\|_{L^{\tilde{q}'}((0,T), L^{\tilde{r}'}(\Gamma))} + T^{1/q} \|\boldsymbol{F}\|_{L^1((0,T), L^\alpha(\Gamma))} \big),$$

*hold. Here $x'$ stands for the conjugate of $x$, defined by $\frac{1}{x} + \frac{1}{x'} = 1$.*

We shall give in Appendix B a proof inspired by [Datchev and Holmer 2009]. As a typical result for the nonlinear Schrödinger equation based on the Strichartz estimates, one obtains the global in time well-posedness for subcritical $L^2(\Gamma)$ solutions:

**Theorem 1.4.** *Let $p \in (0, 4)$. For any $\boldsymbol{u}_0 \in L^2(\Gamma)$ there exists a unique solution*

$$\boldsymbol{u} \in C(\mathbb{R}, L^2(\Gamma)) \cap \bigcap_{(q,r) \ 1\text{-adm.}} L^q(\mathbb{R}, L^r(\Gamma))$$

*of the nonlinear Schrödinger equation*

$$\begin{cases} i\boldsymbol{u}_t + H\boldsymbol{u} \pm |\boldsymbol{u}|^p \boldsymbol{u} = 0, & t \ne 0, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, & t = 0. \end{cases} \tag{14}$$

*Moreover, the $L^2(\Gamma)$-norm of $\boldsymbol{u}$ is conserved along the time*: $\|\boldsymbol{u}(t)\|_{L^2(\Gamma)} = \|\boldsymbol{u}_0\|_{L^2(\Gamma)}$.

Local in time existence with lifespan depending on the $L^2$ size of the initial data follows from a classical fixed point argument as on $\mathbb{R}$ (see for instance Proposition 3.15 in [Tao 2006]). The extension to global solutions is obtained from the conservation of the $L^2(\Gamma)$-norm that in turn follows by taking the imaginary part of (14) multiplied by $\bar{\boldsymbol{u}}$ and integrating on $\Gamma$.

The paper is organised as follows. In the next section we introduce the framework of the Laplacian analysis on a graph. In Section 3 we give the proof of Theorem 1.2. In Appendix A we show how the conditions of the theorems are fulfilled for positive strengths of interactions. Appendix B contains the proof of Theorem 1.3. In Appendix C we shall describe the approach for general coupling conditions.

## 2. Preliminaries on graphs and $\delta$-coupling

In this section we present some generalities about metric graphs and introduce the Dirac delta Hamiltonian $H_\alpha^\Gamma$ on such structure. More general types of self-adjoint operators, $\Delta(A, B)$, have been considered in [Kostrykin and Schrader 2006; 1999]. We collect here some basic facts on metric graphs and on some operators that could be defined on such structures [Kuchment 2008; 2004; 2005; Kostrykin and Schrader 2006; Gnutzmann and Smilansky 2006; Exner 2011].

Let $\Gamma = (V, E)$ be a graph where $V$ is the set of vertices and $E$ the set of edges. For each $v \in V$ we denote by $E_v = \{e \in E : v \in e\}$ the set of edges branching from $v$. We assume that $V$ is connected and the degree of each vertex $v$ of $\Gamma$ is finite: $d(v) = |E_v| < \infty$. The edges could be of finite length and then their ends are vertices of $V$, or they could have infinite length and then we assume that each infinite edge is a ray with a single vertex belonging to $V$ (see [Kuchment 2008] for more details on graphs with infinite edges). The vertices are called internal if $d(v) \geq 2$ or external if $d(v) = 1$. In this paper we will assume that there are no external vertices.

We fix an orientation of $\Gamma$ and for each oriented edge $e$, we denote by $I(e)$ the initial vertex and by $T(e)$ the terminal one. Of course in the case of infinite edges we have only initial vertices.

We identify every edge $e$ of $\Gamma$ with an interval $I_e$, where $I_e = [0, l_e]$ if the edge is finite and $I_e = [0, \infty)$ if the edge is infinite. This identification introduces a coordinate $x_e$ along the edge $e$. In this way $\Gamma$ is a metric space and is often called a metric graph [Kuchment 2008].

Let $v$ be a vertex of $V$ and $e$ be an edge in $E_v$. We set, for finite edges $e$,

$$j(v, e) = \begin{cases} 0 & \text{if } v = I(e), \\ l_e & \text{if } v = T(e), \end{cases}$$

and, for infinite edges,

$$j(v, e) = 0 \quad \text{if } v = I(e).$$

We identify any function $\boldsymbol{u}$ on $\Gamma$ with a collection $\{u^e\}_{e \in E}$ of functions $u^e$ defined on the edges $e$ of $\Gamma$. Each $u^e$ can be considered as a function on the interval $I_e$. In fact, we use the same notation $u^e$ for both the function on the edge $e$ and the function on the interval $I_e$ identified with $e$. For a function $\boldsymbol{u} : \Gamma \to \mathbb{C}$, $\boldsymbol{u} = \{u^e\}_{e \in E}$, we denote by $f(\boldsymbol{u}) : \Gamma \to \mathbb{C}$ the family $\{f(u^e)\}_{e \in E}$, where $f(u^e) : e \to \mathbb{C}$.

A function $\boldsymbol{u} = \{u^e\}_{e \in E}$ is continuous if and only if $u^e$ is continuous on $I_e$ for every $e \in E$ and, moreover, is continuous at the vertices of $\Gamma$:

$$u^e(j(v,e)) = u^{e'}(j(v,e')) \quad \text{for all } e, e' \in E_v \text{ and } v \in V.$$

The space $L^p(\Gamma)$, $1 \leq p < \infty$ consists of all functions $\boldsymbol{u} = \{u_e\}_{e \in E}$ on $\Gamma$ that belong to $L^p(I_e)$ for each edge $e \in E$ and

$$\|\boldsymbol{u}\|_{L^p(\Gamma)}^p = \sum_{e \in E} \|u^e\|_{L^p(I_e)}^p < \infty.$$

Similarly, the space $L^\infty(\Gamma)$ consists of all functions that belong to $L^\infty(I_e)$ for each edge $e \in E$ and

$$\|\boldsymbol{u}\|_{L^\infty(\Gamma)} = \sup_{e \in E} \|u^e\|_{L^\infty(I_e)} < \infty.$$

The Sobolev space $H^m(\Gamma)$, for an integer $m \geq 1$, consists of all continuous functions on $\Gamma$ that belong to $H^m(I_e)$ for each $e \in E$ and

$$\|\boldsymbol{u}\|_{H^m(\Gamma)}^2 = \sum_{e \in E} \|u^e\|_{H^m(e)}^2 < \infty.$$

The above spaces are Hilbert spaces with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_{L^2(\Gamma)} = \sum_{e \in E}(u^e, v^e)_{L^2(I_e)} = \sum_{e \in E} \int_{I_e} u^e(x)\overline{v^e}(x)\,dx$$

and

$$(\boldsymbol{u}, \boldsymbol{v})_{H^m(\Gamma)} = \sum_{e \in E}(u^e, v^e)_{H^m(I_e)} = \sum_{e \in E}\sum_{k=0}^{m} \int_{I_e} \frac{d^k u^e}{dx^k}\,\overline{\frac{d^k v^e}{dx^k}}\,dx.$$

We now define the normal exterior derivative of a function $\boldsymbol{u} = \{u^e\}_{e \in E}$ at the endpoints of the edges. For each $e \in E$ and $v$ an endpoint of $e$ we consider the normal derivative of the restriction of $\boldsymbol{u}$ to the edge $e$ of $E_v$ evaluated at $j(v,e)$, to be defined by

$$\frac{\partial u^e}{\partial n_e}(j(v,e)) = \begin{cases} -u_x^e(0^+) & \text{if } j(v,e) = 0, \\ u_x^e(l_e^-) & \text{if } j(v,e) = l_e. \end{cases}$$

We now introduce $H_\alpha^\Gamma$. It generalises the classical Dirac delta interactions with strength parameters (5). The Dirac delta Hamiltonian is defined on the domain

$$D(H_\alpha^\Gamma) = \left\{ \boldsymbol{u} \in H^2(\Gamma) : \sum_{e \in E_v} \frac{\partial u^e}{\partial n_e}(j(v,e)) = \alpha(v)\boldsymbol{u}(v), \ \forall v \in V \right\}. \tag{15}$$

For any $\boldsymbol{u} = \{u^e\}_{e \in E}$, the operator $H_\alpha^\Gamma$ acts by

$$(H_\alpha^\Gamma \boldsymbol{u})(x) = -u_{xx}^e(x), \quad x \in I_e, e \in E.$$

The quadratic form associated to $H_\alpha^\Gamma$ is defined on $H^1(\Gamma)$ and it is given by

$$\mathscr{E}_\alpha^\Gamma(u) = \sum_{e \in E} \int_{I_e} |u_x^e(x)|^2\,dx + \sum_{v \in V} \alpha(v)|\boldsymbol{u}(v)|^2.$$

When all strengths vanish this corresponds to the Kirchhoff coupling analysed in [Banica and Ignat 2011].

Finally, let us mention that there are other coupling conditions (see [Kostrykin and Schrader 1999]), which allow one to define a "Laplace" operator on a metric graph. To be more precise, let us consider an operator that acts on functions on the graph $\Gamma$ as the second derivative $d^2/dx^2$, and whose domain consists of all functions $\boldsymbol{u}$ that belong to the Sobolev space $H^2(e)$ on each edge $e$ of $\Gamma$ and satisfy the following boundary condition at the vertices:

$$A(v)\boldsymbol{u}(v) + B(v)\boldsymbol{u}'(v) = 0 \quad \text{for each vertex } v. \tag{16}$$

Here $\boldsymbol{u}(v)$ and $\boldsymbol{u}'(v)$ are correspondingly the vector of values of $\boldsymbol{u}$ at $v$ attained from directions of different edges converging at $v$ and the vector of derivatives at $v$ in the outgoing directions. For each vertex $v$ of the tree we assume that matrices $A(v)$ and $B(v)$ are of size $d(v)$ and satisfy the following two conditions:

(1) the joint matrix $(A(v), B(v))$ has maximal rank $d(v)$,

(2) $A(v)B(v)^T = B(v)A(v)^T$.

Under those assumptions it was proved in [Kostrykin and Schrader 1999] that the operator under consideration, denoted by $\Delta(A, B)$, is self-adjoint. The case considered in this paper, of $\delta$-coupling, corresponds to the matrices

$$A(v) = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -\alpha(v) \end{pmatrix}, \quad B(v) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

More examples of matrices satisfying the above conditions are given in [Kostrykin and Schrader 1999; Kostrykin et al. 2008].

## 3. Proof of Theorem 1.2

We shall use a description of the solution of the linear Schrödinger equation in terms of the resolvent. For $\omega > 0$ such that $-\omega^2$ is not an eigenvalue, let $R_\omega$ be the resolvent of the Laplacian on a tree

$$R_\omega \boldsymbol{u}_0 = (H_\alpha^\Gamma + \omega^2 I)^{-1} \boldsymbol{u}_0.$$

Before starting let us choose an orientation on the tree $\Gamma$. Choose an internal vertex $\mathbb{0}$ to be the root of the tree and the initial vertex for all the edges that branch from it. This procedure introduces an orientation for all the edges starting from $\mathbb{0}$. For the other endpoints of the edges belonging to $E_\mathbb{0}$ we repeat the above procedure and inductively we construct an orientation on $\Gamma$.

**3A.** *The structure of the resolvent.* In order to obtain the expression of the resolvent, second-order equations

$$(R_\omega \boldsymbol{u}_0)'' = \omega^2 R_\omega \boldsymbol{u}_0 - \boldsymbol{u}_0$$

must be solved on each edge of the tree together with coupling conditions at each vertex. Then, on each edge parametrised by $I_e$, for $x \in I_e$, since $\omega \neq 0$,

$$R_\omega \boldsymbol{u}_0(x) = c_e \, e^{\omega x} + \tilde{c}_e \, e^{-\omega x} + \frac{t_e(x, \omega)}{\omega}, \tag{17}$$

with

$$t_e(x, \omega) = \frac{1}{2} \int_{I_e} \boldsymbol{u}_0(y) \, e^{-\omega|x-y|} \, dy.$$

Since $R_\omega \boldsymbol{u}_0$ belongs to $L^2(\Gamma)$, the coefficients $c_e$ and $\tilde{c}_e$ are zero on the infinite edges $e \in \mathcal{E}$, parametrised by $[0, \infty)$. If we denote by $\mathcal{I}$ the set of internal edges, we have $2|\mathcal{I}| + |\mathcal{E}|$ coefficients. The delta conditions of continuity of $R_\omega \boldsymbol{u}_0$ and of transmission of $(R_\omega \boldsymbol{u}_0)'$ at the vertices of the tree give the system of equations on the coefficients. We have the same number of equations as of unknowns. We denote by $D_{\Gamma_p}(\omega)$ the matrix of the system, where $p$ is the number of vertices of the tree, and $T_{\Gamma_p}(\omega)$ is the column of the free terms in the system.

Therefore the resolvent $R_\omega \boldsymbol{u}_0(x)$ on an edge $I_e$ is

$$R_\omega \boldsymbol{u}_0(x) = \frac{\det M_{\Gamma_p}^{c_e}(\omega)}{\det D_{\Gamma_p}(\omega)} e^{\omega x} + \frac{\det M_{\Gamma_p}^{\tilde{c}_e}(\omega)}{\det D_{\Gamma_p}(\omega)} e^{-\omega x} + \frac{t_e(x, \omega)}{\omega}, \tag{18}$$

where $M_{\Gamma_p}^{c_e}(\omega)$ and $M_{\Gamma_p}^{\tilde{c}_e}(\omega)$ are obtained from $D_{\Gamma_p}(\omega)$ by replacing the column corresponding to the unknown $c_e$ and $\tilde{c}_e$, respectively, by the column of the free terms $T_{\Gamma_p}(\omega)$.

**3B. *The expression of* $\det D_{\Gamma_p}(\omega)$.** In view of the form (17) of the resolvent, we obtain on an edge $I_e$

$$R_\omega \boldsymbol{u}_0(0) = c_e + \tilde{c}_e + \frac{t_e(0, \omega)}{\omega},$$
$$(R_\omega \boldsymbol{u}_0)'(0) = c_e \omega - \tilde{c}_e \omega + t_e(0, \omega), \tag{19}$$

and, if $I_e$ is parametrised by $[0, a]$ with $a < \infty$,

$$R_\omega \boldsymbol{u}_0(a) = c_e \, e^{\omega a} + \tilde{c}_e \, e^{-\omega a} + \frac{t_e(0, \omega)}{\omega},$$
$$(R_\omega \boldsymbol{u}_0)'(a) = c_e \, \omega \, e^{\omega a} - \tilde{c}_e \, \omega \, e^{-\omega a} - t_e(a, \omega). \tag{20}$$

**3B1. *The star-shaped tree case.*** In the case of a single vertex and $n_1 \geq 2$ edges $I_j$, $1 \leq j \leq n_1$, parametrised by $[0, \infty)$ we have only the coefficient $\tilde{c}_j$ on each edge $I_j$ since each $c_j$ vanishes. The delta conditions are continuity of the resolvent at the vertex, and the fact that the sum of the first derivatives must be equal to $\alpha$ times the value of the resolvent at the vertex:

$$(R_\omega \boldsymbol{u}_0)_j(0) = (R_\omega \boldsymbol{u}_0)_1(0), \quad \sum_{1 \leq j \leq n_1} (R_\omega \boldsymbol{u}_0)_j'(0) = \alpha_1 \, (R_\omega \boldsymbol{u}_0)_1(0).$$

From (19) we obtain as the matrix for the system of $\tilde{c}$'s

$$
D_{\Gamma_1}(\omega) =
\begin{pmatrix}
1 & -1 & & & & & \\
 & 1 & -1 & & & & \\
 & & & \ddots & & & \\
 & & & & 1 & -1 & \\
 & & & & & 1 & -1 \\
1 & \dfrac{\omega}{\omega+\alpha_1} & \dfrac{\omega}{\omega+\alpha_1} & \cdots & \dfrac{\omega}{\omega+\alpha_1} & \dfrac{\omega}{\omega+\alpha_1} & \dfrac{\omega}{\omega+\alpha_1}
\end{pmatrix},
$$

and as a free term column

$$
T_{\Gamma_1}(\omega) =
\begin{pmatrix}
\dfrac{t_2(0,\omega)-t_1(0,\omega)}{\omega} \\[2mm]
\vdots \\[2mm]
\dfrac{t_{n_1}(0,\omega)-t_{n_1-1}(0,\omega)}{\omega} \\[2mm]
\dfrac{\omega-\alpha_1}{\omega+\alpha_1}\dfrac{t_1(0,\omega)}{\omega} + \dfrac{\omega}{\omega+\alpha_1} \displaystyle\sum_{2\le j\le n_1} \dfrac{t_j(0,\omega)}{\omega}.
\end{pmatrix}
$$

By developing $\det D_{\Gamma_1}(\omega)$ with respect to its last column, we obtain by recursion that

$$
\det D_{\Gamma_1}(\omega) = \frac{n_1\omega+\alpha_1}{\omega+\alpha_1}.
$$

Thus $\det D_{\Gamma_1}$ does not vanish on the imaginary axis and $\omega R_\omega u_0$ can be analytically continued in a region containing the imaginary axis.

We introduce here the matrix $\tilde{D}_{\Gamma_1}(\omega)$, which is the matrix of the coefficients of the resolvent if on the last edge $I_{n_1}$ we should have $c_{n_1}e^{\omega x}$ instead of $\tilde{c}_{n_1}e^{-\omega x}$. This changes only the $(n_1,n_1)$-entry of $D_{\Gamma_1}(\omega)$, which is $-\dfrac{\omega}{\omega+\alpha_1}$ instead of $\dfrac{\omega}{\omega+\alpha_1}$:

$$
\tilde{D}_{\Gamma_1}(\omega) =
\begin{pmatrix}
1 & -1 & & & & & \\
 & 1 & -1 & & & & \\
 & & & \ddots & & & \\
 & & & & 1 & -1 & \\
 & & & & & 1 & -1 \\
1 & \dfrac{\omega}{\omega+\alpha_1} & \dfrac{\omega}{\omega+\alpha_1} & \cdots & \dfrac{\omega}{\omega+\alpha_1} & \dfrac{\omega}{\omega+\alpha_1} & -\dfrac{\omega}{\omega+\alpha_1}
\end{pmatrix}.
$$

Moreover, the free term column remains the same for this new system. Again, by recursion, we have

$$
\det \tilde{D}_{\Gamma_1}(\omega) = \frac{(n_1-2)\omega+\alpha_1}{\omega+\alpha_1}.
$$

**3B2.** *The general tree case.* Any tree $\Gamma_p$ with $p$ vertices, $p \geq 2$, can be seen as a tree $\Gamma_{p-1}$ with $p-1$ vertices, to which we add a new vertex on one of its infinite edges, and $n_p - 1$ new infinite edges from it. Let us denote by $N$ the number of edges of $\Gamma_{p-1}$. By this transformation $I_N$ becomes an internal edge, parametrised by $[0, a_{p-1}]$, and we have in addition external edges $I_{N+j}$, for $1 \leq j \leq n_p - 1$. We denote by $\alpha_p$ the strength of the delta condition in the new $p$-th vertex. The matrix of the new system (the unknowns of the $\Gamma_{p-1}$ system, together with an extra unknown on the new internal line $I_N$ and $n_p - 1$ unknowns on the new $n_p - 1$ external edges) is denoted by $D_{\Gamma_p}(\omega)$. Notice that if we write the system of unknowns of $\Gamma_p$ by changing the order of the unknowns (i.e., permuting columns) or the order of the conditions at vertices (i.e., permuting lines), then the determinant remains unchanged or it changes sign, and the ratio $\det \widetilde{D}_{\Gamma_p}(\omega)/\det D_{\Gamma_p}(\omega)$ remains unchanged.

For $\Gamma_p$, by writing the delta conditions at the end of $I_N$, together with the two conditions involving the coefficients on $I_N$ at the beginning of $I_N$, we obtain the matrix $D_{\Gamma_p}(\omega)$ as

$$
\begin{pmatrix}
\begin{array}{c|ccccccc}
& -1 & & & & & & \\
D_{\Gamma_{p-1}}(\omega) & -\dfrac{\omega}{\omega+\alpha_{p-1}} & & & & & & \\
\hline
e^{-\omega a_{p-1}} & e^{\omega a_{p-1}} & -1 & & & & & \\
& & 1 & -1 & & & & \\
& & & & \ddots & & & \\
& & & & & 1 & -1 & \\
& & & & & & 1 & -1 \\
\dfrac{-\omega+\alpha_p}{\omega+\alpha_p}e^{-\omega a_{p-1}} & e^{\omega a_{p-1}} & \dfrac{\omega}{\omega+\alpha_p} & \dfrac{\omega}{\omega+\alpha_p} & \cdots & \dfrac{\omega}{\omega+\alpha_p} & \dfrac{\omega}{\omega+\alpha_p} & \dfrac{\omega}{\omega+\alpha_p}
\end{array}
\end{pmatrix}
$$

and the free term column as

$$
T_{\Gamma_p}(\omega) = 
\begin{pmatrix}
T_{\Gamma_{p-1}}(\omega) \\
\dfrac{t_{N+1}(0,\omega)-t_N(a_{p-1},\omega)}{\omega} \\
\vdots \\
\dfrac{t_{N+n_p-1}(0,\omega)-t_{N+n_p-2}(0,\omega)}{\omega} \\
\dfrac{\omega-\alpha_p}{\omega+\alpha_p}\dfrac{t_N(a_{p-1},\omega)}{\omega} + \dfrac{\omega}{\omega+\alpha_p}\displaystyle\sum_{1\leq j\leq n_p-1}\dfrac{t_{N+j}(0,\omega)}{\omega}
\end{pmatrix}.
$$

We point out that $D_{\Gamma_p}$ has $p-1$ pairs of columns that are equal at $\omega = 0$. This implies that $\omega = 0$ is a zero of order at least $p-1$ for $D_{\Gamma_p}$. The assumption imposed in Theorem 1.1 guarantees that the order of $\omega = 0$ is exactly $p-1$. This will avoid the existence of zero resonances for the resolvent $R_\omega$. In the case when all the strengths $\{\alpha_k\}_{k=1}^n$ are positive the condition in Theorem 1.1 is fulfilled; this will be proved in Appendix A.

**Lemma 3.1.** *We have the recursion formulae*

$$\det D_{\Gamma_1}(\omega) = \frac{n_1\omega + \alpha_1}{\omega + \alpha_1}, \quad \frac{\det \widetilde{D}_{\Gamma_1}(\omega)}{\det D_{\Gamma_1}(\omega)} = \frac{(n_1 - 2)\omega + \alpha_1}{n_1\omega + \alpha_1},$$

$$\det D_{\Gamma_p}(\omega) = \frac{n_p\omega + \alpha_p}{\omega + \alpha_p} e^{\omega a_{p-1}} \det D_{\Gamma_{p-1}}(\omega) \left(1 - \frac{(n_p - 2)\omega + \alpha_p}{n_p\omega + \alpha_p} e^{-2\omega a_{p-1}} \frac{\det \widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}\right),$$

$$\frac{\det \widetilde{D}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)} = \frac{\dfrac{(n_p - 2)\omega + \alpha_p}{n_p\omega + \alpha_p} - \dfrac{(n_p - 4)\omega + \alpha_p}{n_p\omega + \alpha_p} e^{-2\omega a_{p-1}} \dfrac{\det \widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}}{1 - \dfrac{(n_p - 2)\omega + \alpha_p}{n_p\omega + \alpha_p} e^{-2\omega a_{p-1}} \dfrac{\det \widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}}. \tag{21}$$

*Proof.* The part about $\Gamma_1$ was proved in Section 3B1.

By developing $\det D_{\Gamma_p}$ with respect to the last $n_p$ lines, we obtain an alternated sum of determinants of $n_p \times n_p$ minors composed of the last $n_p$ lines of $D_{\Gamma_p}$ times the determinant of the matrix $D_{\Gamma_p}$, without the lines and columns the minor is made of. On the last $n_p$ lines, there are only $n_p + 1$ columns that do not identically vanish. The only possible way to obtain a $n_p \times n_p$ minor composed from the last $n_p$ lines of $D_{\Gamma_p}$ with determinant different from zero is to choose all of the last $n_p - 1$ columns together with a previous one. This follows from the fact that if we eliminate from $\det D_{\Gamma_n}$ both previous columns together with $n_p - 2$ columns among the last $n_p$ columns, we obtain a block-diagonal type matrix, with first diagonal block $D_{\Gamma_{p-1}}$ with its last column replaced by zeros, so its determinant vanishes. Therefore

$$\det D_{\Gamma_p} = \det D_{\Gamma_{p-1}} \det A^{n_p} - \det \widetilde{D}_{\Gamma_{p-1}} \det B^{n_p},$$

where for $m \geq 1$, $A^m$ and $B^m$ are the $m \times m$ matrices

$$A^m = \begin{pmatrix} e^{\omega a_{p-1}} & -1 & & & & & \\ & 1 & -1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & -1 & \\ & & & & & 1 & -1 \\ e^{\omega a_{p-1}} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \cdots & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} \end{pmatrix},$$

$$B^m = \begin{pmatrix} e^{-\omega a_{p-1}} & -1 & & & & & \\ & 1 & -1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & -1 & \\ & & & & & 1 & -1 \\ \frac{-\omega+\alpha_p}{\omega+\alpha_p} e^{-\omega a_{p-1}} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \cdots & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} \end{pmatrix}.$$

We have

$$\det A^2 = \frac{2\omega + \alpha_p}{\omega + \alpha_p} e^{\omega a_{p-1}},$$

and by developing $A^m$ with respect to the first last column we obtain the recursion formula

$$\det A^m = \frac{\omega}{\omega + \alpha_p} e^{\omega a_{p-1}} + \det A^{n_p - 1},$$

so

$$\det A^m = \frac{m\omega + \alpha_p}{\omega + \alpha_p} e^{\omega a_{p-1}}.$$

Similarly we obtain

$$\det B^m = \frac{(m-2)\,\omega + \alpha_p}{\omega + \alpha_p} e^{-\omega a_{p-1}}.$$

Therefore we find, indeed,

$$\det D_{\Gamma_p}(\omega) = \frac{n_p \omega + \alpha_p}{\omega + \alpha_p} e^{\omega a_{p-1}} \det D_{\Gamma_{p-1}}(\omega) \left( 1 - \frac{(n_p - 2)\omega + \alpha_p}{n_p \omega + \alpha_p} e^{-2\omega a_{p-1}} \frac{\det \widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)} \right).$$

In a similar way we get

$$\det \widetilde{D}_{\Gamma_p}(\omega) = \frac{(n_p - 2)\omega + \alpha_p}{\omega + \alpha_p} e^{\omega a_{p-1}} \det D_{\Gamma_{p-1}}(\omega) - \frac{(n_p - 4)\omega + \alpha_p}{\omega + \alpha_p} e^{-\omega a_{p-1}} \det \widetilde{D}_{\Gamma_{p-1}}(\omega),$$

which leads directly to (21), completing the proof of the lemma. $\qquad\square$

### 3C. *A lower bound for* $\det D_{\Gamma_p}(i\tau)$ *away from* 0.

**Lemma 3.2.** *The function* $\det D_{\Gamma_p}(\omega)$ *is bounded away from zero by a positive constant on a strip containing the imaginary axis*:

$$\forall \delta > 0, \exists c_{\Gamma_p}, \epsilon_{\Gamma_p} > 0, \exists 0 < r_{\Gamma_p} < 1 \text{ such that } |\det D_{\Gamma_p}(\omega)| > c_{\Gamma_p}, \quad \left| \frac{\det \widetilde{D}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)} \right| < r_\Gamma,$$

*for all* $\omega \in \mathbb{C}$ *with* $|\Re\omega| < \epsilon_{\Gamma_p}$ *and* $|\Im\omega| > \delta$.

*Proof.* We shall prove this lemma by recursion on $p$. For $p = 1$, Lemma 3.1 ensures that

$$\det D_{\Gamma_1}(\omega) = \frac{n_1 \omega + \alpha_1}{\omega + \alpha_1}, \quad \frac{\det \widetilde{D}_{\Gamma_1}(\omega)}{\det D_{\Gamma_1}(\omega)} = \frac{(n_1 - 2)\omega + \alpha_1}{n_1 \omega + \alpha_1}.$$

We obtain a positive lower bound for $|\det D_{\Gamma_1}(\omega)|$ if we show that it does not approach zero. Therefore the existence of $c_{\Gamma_1} > 0$ is obtained by considering $\epsilon_{\Gamma_1} \leq \frac{|\alpha_1|}{2n_1}$. Next, we have

$$\left| \frac{(n_1 - 2)\omega + \alpha_1}{n_1 \omega + \alpha_1} \right| < 1 \iff 0 < \alpha_1 \Re\omega + (n_1 - 1)|\omega|^2,$$

so for any $\delta > 0$ we get an appropriate $0 < r_{\Gamma_1} < 1$ by choosing

$$\epsilon_{\Gamma_1} \leq \frac{(n_1 - 1)\delta^2}{2|\alpha_1|}.$$

Assume that we have proved this lemma for $p - 1$. We shall show now that it also holds for $p$. Now, from the ratio information part in this lemma for $\Gamma_{p-1}$ we can choose $\epsilon_{\Gamma_p}$ small enough to have, for

$|\Re\omega| < \epsilon_{\Gamma_p}$ and $|\Im\omega| > \delta$,

$$\left| 1 - \frac{(n_p - 2)\omega + \alpha_p}{n_p\omega + \alpha_p} e^{-2\omega a_{p-1}} \frac{\det \widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)} \right| > c_0 > 0.$$

Also from this lemma for $\Gamma_{p-1}$ we have the existence of two positive constants $c_{\Gamma_{p-1}}$ and $\epsilon_{\Gamma_{p-1}}$ such that $|\det D_{\Gamma_{p-1}}(\omega)| > c_{\Gamma_{p-1}}$, for all $\omega \in \mathbb{C}$ with $|\Re\omega| < \epsilon_{\Gamma_{p-1}}$ and $|\Im\omega| > \delta$. Finally, $(n_p\omega + \alpha_p)/(\omega + \alpha_p)$ is bounded below by a positive constant for small enough $\Re\omega$, so eventually we get

$$\exists c_{\Gamma_p}, \epsilon_{\Gamma_p} > 0 \text{ such that } |\det D_{\Gamma_p}(\omega)| > c_{\Gamma_p} \text{ for all } \omega \in \mathbb{C} \text{ with } |\Re\omega| < \epsilon_{\Gamma_p}, |\Im\omega| > \delta.$$

We are left with showing that the ratio $\det \widetilde{D}_{\Gamma_p}(\omega)/\det D_{\Gamma_p}(\omega)$ is of modulus less than one. In view of the recursion formula on the ratio from Lemma 3.1, we first impose as a condition on $\epsilon_{\Gamma_p}$ that

$$\tilde{r}_{\Gamma_{p-1}} := e^{2\epsilon_{\Gamma_p} a_{p-1}} r_{\Gamma_{p-1}} < 1,$$

and then we have to show that for $|z| < \tilde{r}_{\Gamma_{p-1}}$,

$$\left| \frac{(n_p - 2)\omega + \alpha_p - ((n_p - 4)\omega + \alpha_p)z}{n_p\omega + \alpha_p - ((n_p - 2)\omega + \alpha_p)z} \right| < r_{\Gamma_p},$$

for all complex $\omega$ with $|\Re\omega| < \epsilon_{\Gamma_p}$ and $|\Im\omega| > \delta$, for $\epsilon_{\Gamma_p}$ to be chosen and $r_{\Gamma_p} < 1$. By letting $q = (n_p - 2)\omega + \alpha_p$, the above inequality becomes

$$|q - (q - 2\omega)z| < |(q + 2\omega) - qz| \iff |q(1-z) + 2\omega z| < |q(1-z) + 2\omega|.$$

Expanding this last inequality we find that we have to prove that

$$0 < |\omega|^2(1 - |z|^2) + |1 - z|^2((n_p - 2)|\omega|^2 + \alpha_p\Re(\omega)).$$

Since $n_p \geq 2$ and $|z| < \tilde{r}_{\Gamma_{p-1}} < 1$, it is enough to have

$$0 < |\omega|^2(1 - |z|^2) + |1 - z|^2\alpha_p\Re(\omega).$$

Also, $|\Re z| < \tilde{r}_{\Gamma_{p-1}} < 1$, so by choosing $\epsilon_{\Gamma_p} \leq \dfrac{(1 - \tilde{r}_{\Gamma_{p-1}}^2)\delta^2}{2|\alpha_p|(1 - \tilde{r}_{\Gamma_{p-1}})^2}$ we get the existence of $r_{\Gamma_p} < 1$. $\square$

## 3D. *Vanishing of the numerator at $\tau = 0$.*

Recall that we have denoted by $M_{\Gamma_p}^{c_e}(\omega)$ the matrix $D_{\Gamma_p}(\omega)$ with $D_{\Gamma_p}^e(\omega)$, the column corresponding to the unknown $c_e$, replaced by the free terms column $T_{\Gamma_p}(\omega)$. In particular $\omega \det M_{\Gamma_p}^{c_e}(\omega)$ is the determinant of the matrix $D_{\Gamma_p}(\omega)$ with the column corresponding to the unknown $c_e$ replaced by $\omega T_{\Gamma_p}(\omega)$. The same holds for $M_{\Gamma_p}^{\tilde{c}_e}(\omega)$ with the appropriate substitutions.

**Lemma 3.3.** $\qquad -(\omega T_{\Gamma_p}(\omega))(0) = \displaystyle\sum_{e \in \mathscr{E}} t_e(0, 0) D_{\Gamma_p}^e(0) + \sum_{e \in \mathscr{I}} t_e(0, 0) D_{\Gamma_p}^{\tilde{e}}(0)$

**Remark 3.4.** From the shape of $D_{\Gamma_p}(\omega)$ displayed in the proof of Lemma 3.1 we notice that the two junction columns with $D_{\Gamma_{p-1}}(\omega)$, corresponding to the coefficients of the resolvent on the connecting

edge $I_N$, are

$$D_{\Gamma_p}^{I_N}(\omega) = \left(0, \ldots, 0, -1, \frac{\omega}{\omega+\alpha_{p-1}}, e^{-\omega a_{p-1}}, 0, \ldots, 0, \frac{-\omega+\alpha_p}{\omega+\alpha_p} e^{-\omega a_{p-1}}\right)^T$$

and

$$D_{\Gamma_p}^{\tilde{I}_N}(\omega) = \left(0, \ldots, 0, -1, -\frac{\omega}{\omega+\alpha_{p-1}}, e^{\omega a_{p-1}}, 0, \ldots, 0, e^{\omega a_{p-1}}\right)^T.$$

In particular, these two columns are the same at $\omega = 0$. Moreover, $D_{\Gamma_p}(\omega)$ contains $p-1$ such pairs of columns: $D_{\Gamma_p}^e(0) = D_{\Gamma_p}^{\tilde{e}}(0)$ for all $e \in \mathcal{I}$. Thus, the last term in the right side of Lemma 3.3 could be either $D_{\Gamma_p}^e(0)$ or $D_{\Gamma_p}^{\tilde{e}}(0)$, for $e \in \mathcal{I}$.

*Proof.* We will prove this identity inductively. For $p = 1$, $(\omega T_{\Gamma_1})$ is given in Section 3B1. We choose $X_1 = (t_1(0,0), t_2(0,0), \ldots, t_{n_1}(0,0))^T$, then $D_{\Gamma_1}(0)X_1 = -(\omega T_{\Gamma_1})(0)$, which proves the lemma for $p = 1$.

Given now $X_{p-1}$ such that $D_{\Gamma_{p-1}}(0)X_{p-1} = -(\omega T_{\Gamma_{p-1}}(\omega))(0)$, we construct $X_p$ as follows:

$$X_p^T = (X_{p-1}^T, 0, t_{N+1}(0,0), \ldots, t_{N+n_p-1}(0,0)).$$

Using the recursion between $D_{\Gamma_p}$ and $D_{\Gamma_{p-1}}$ used in the proof of Lemma 3.1, the identity

$$\omega T_{\Gamma_p}(\omega) = \begin{pmatrix} \omega T_{\Gamma_{p-1}}(\omega) \\ t_{N+1}(0,\omega) - t_N(a_{p-1},\omega) \\ \vdots \\ t_{N+n_p-1}(0,\omega) - t_{N+n_p-2}(0,\omega) \\ \frac{\omega-\alpha_p}{\omega+\alpha_p} t_N(a_{p-1},\omega) + \frac{\omega}{\omega+\alpha_p} \sum_{1\leq j\leq n_p-1} t_{N+j}(0,\omega) \end{pmatrix},$$

and the fact that $t_e(0,0) = t_e(a_e, 0)$ for all $e \in \mathcal{I}$, we obtain that $X_p$ satisfies $D_{\Gamma_p}(0)X_p = -(\omega T_{\Gamma_p}(\omega))(0)$. Writing this identity in terms of the columns of the matrix $D_{\Gamma_p}(0)$ we obtain the desired identity. $\square$

**Lemma 3.5.** $\omega = 0$ *is a root of order at least $p-1$ of $\omega \det M_{\Gamma_p}^{c_e}(\omega)$ and of $\omega \det M_{\Gamma_p}^{\tilde{c}_e}(\omega)$ for all edges $e$.*

*Proof.* We shall give the proof for $\omega \det M_{\Gamma_p}^{c_e}(\omega)$; the result for $\omega \det M_{\Gamma_p}^{\tilde{c}_e}(\omega)$ will be the same. From the shape of $D_{\Gamma_p}(\omega)$ displayed in the proof of Lemma 3.1 and Remark 3.4 we have $p-1$ pairs of columns that are equal at $\omega = 0$. Moreover, by Lemma 3.3, $(\omega T_{\Gamma_p})(0)$ is a linear combination of these columns evaluated at $\omega = 0$.

The derivative of a determinant is the sum of the determinants of the matrices obtained by differentiating one column. When $T_{\Gamma_p}$ does not replace any of these $2(p-1)$ columns it follows that the lemma holds since there are always two identical columns. Then by the above argument we have

$$\partial_\omega^k(\omega \det M_{\Gamma_p}^{c_e})(0) = 0 \quad \text{if } 0 \leq k \leq p-3. \tag{22}$$

Assume now that $T_{\Gamma_p}$ replaces one of these $2(p-1)$ columns. To finish the proof we must show that

$$\partial_\omega^{p-2}(\omega \det M_{\Gamma_p}^{c_e})(0) = 0. \tag{23}$$

Using again the fact that $D_{\Gamma_p}(\omega)$ contains $p-1$ pairs of columns that match at $\omega = 0$, we only need to show that $\det A_{\Gamma_p}(0) = 0$, where $A_{\Gamma_p}(\omega)$ is $D_{\Gamma_p}(\omega)$ with the column $\omega T_{\Gamma_p}(\omega)$ replacing one column of one pair, and one column of each of the remaining $p-2$ pairs of columns is differentiated. In particular $A_{\Gamma_p}(0)$ contains one unchanged column of each of the $p-1$ pairs. By Lemma 3.3 we know that $(\omega T_{\Gamma_p}(\omega))(0)$ is a linear combination of the columns corresponding to external edges and of the internal ones (each one from the $p-1$ pairs), so the new determinant vanishes and the proof is finished. $\quad\square$

**Lemma 3.6.** *For all edge indices $\lambda$ and $e$, $\omega = 0$ is a root of order at least $p-2$ of the coefficient $f_{\lambda,e}(\omega)$ of $t_\lambda(0,\omega)$ in $\omega \det M_{\Gamma_p}^{c_e}(\omega)$, and the same holds for the coefficient $\tilde{f}_{\lambda,e}(\omega)$ of $t_\lambda(0,\omega)$ in $\omega \det M_{\Gamma_p}^{\tilde{c}_e}(\omega)$.*

*Proof.* This result follows from the discussion that led to (22): the matrix $\omega M_{\Gamma_p}^{c_e}(\omega)$ has $p-2$ pairs of columns that are identical at $\omega = 0$. $\quad\square$

**Lemma 3.7.** *For each edge index $e$ and each external edge index $\lambda$, $\omega = 0$ is a root of order at least $p-1$ of the coefficient $f_{\lambda,e}(\omega)$ of $t_\lambda(0,\omega)$ in $\omega \det M_{\Gamma_p}^{c_e}(\omega)$, and the same holds for the coefficient $\tilde{f}_{\lambda,e}(\omega)$ of $t_\lambda(0,\omega)$ in $\omega \det M_{\Gamma_p}^{\tilde{c}_e}(\omega)$.*

*Proof.* The statement corresponds to the particular case of Lemma 3.5 where all the components of $T_{\Gamma_p}$ are taken to be 0 except $t_\lambda(0,\omega)$, which is replaced by 1. $\quad\square$

**Lemma 3.8.** *For each edge index $e$ and each internal edge index $\lambda$, $\omega = 0$ is a root of order at least $p-1$ of $f_{\lambda,e}^1(\omega) + f_{\lambda,e}^2(\omega)$, where $f_{\lambda,e}^1(\omega)$ and $f_{\lambda,e}^2(\omega)$ are respectively the coefficients of $t_\lambda(0,\omega)$ and $t_\lambda(a_\lambda,\omega)$ in $\omega \det M_{\Gamma_p}^{c_e}(\omega)$. The same holds for $\tilde{f}_{\lambda,e}^1(\omega) + \tilde{f}_{\lambda,e}^2(\omega)$, where $\tilde{f}_{\lambda,e}^1(\omega)$ and $\tilde{f}_{\lambda,e}^2(\omega)$ are respectively the coefficients of $t_\lambda(0,\omega)$ and $t_\lambda(a_\lambda,\omega)$ in $\omega \det M_{\Gamma_p}^{\tilde{c}_e}(\omega)$.*

*Proof.* The proof goes the same as for Lemma 3.7. $\quad\square$

**3E.** ***The end of the proof.*** Now we shall use the theorem hypothesis, $\partial_\omega^{(p-1)} \det D_{\Gamma_p}\big|_{\omega=0} \neq 0$. We obtain that $\omega = 0$ is a root of order $p-1$ of $\det D_{\Gamma_p}$. From the previous subsections we conclude the following:

**Lemma 3.9.** *$\omega R_\omega f(x)$ can be analytically continued in a region containing the imaginary axis.*

*Proof.* The proof is an immediate consequence of decomposition (18) for $x \in I_e$, together with Lemma 3.2, Lemma 3.5 and the fact that $\omega = 0$ is a root of order $p-1$ of $\det D_{\Gamma_p}$. $\quad\square$

*Proof of Theorem 1.2.* As a consequence of Lemma 3.9 we can use a spectral calculus argument to write the solution of the Schrödinger equation with initial data $\boldsymbol{u}_0$ as

$$e^{-itH_\alpha^\Gamma} P\boldsymbol{u}_0(x) = \frac{1}{i\pi} \int_{-\infty}^\infty e^{-it\tau^2} \tau R_{i\tau}\boldsymbol{u}_0(x)\,d\tau. \tag{24}$$

In view of the definition of $t_e$ and with the notations from Lemmas 3.7 and 3.8 we can also write the

decomposition (18) as

$$\tau R_{i\tau} \boldsymbol{u}_0(x) = \frac{1}{2} \int_{I_e} \boldsymbol{u}_0 \, e^{-i\tau|x-y|} \, dy \, i + \sum_{\lambda \in \mathscr{E}} \frac{f_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} \int_{I_\lambda} \boldsymbol{u}_0(y) e^{i\tau y} \, dy \, e^{i\tau x}$$

$$+ \sum_{\lambda \in \mathscr{E}} \frac{\tilde{f}_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} \int_{I_\lambda} \boldsymbol{u}_0(y) e^{i\tau y} \, dy \, e^{-i\tau x}$$

$$+ \sum_{\lambda \in \mathscr{I}} \int_{I_\lambda} \boldsymbol{u}_0(y) \left( e^{i\tau y} \frac{f^1_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} + e^{i\tau(a_\lambda - y)} \frac{f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} \right) dy \, e^{i\tau x}$$

$$+ \sum_{\lambda \in \mathscr{I}} \int_{I_\lambda} \boldsymbol{u}_0(y) \left( e^{i\tau y} \frac{\tilde{f}^1_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} + e^{i\tau(a_\lambda - y)} \frac{\tilde{f}^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} \right) dy \, e^{-i\tau x}. \quad (25)$$

Moreover, in view of the results in Lemma 3.8 and Lemma 3.7 we gather the terms as follows:

$$\tau R_{i\tau} \boldsymbol{u}_0(x) = \frac{1}{2} \int_{I_e} \boldsymbol{i} \boldsymbol{u}_0, e^{-i\tau|x-y|} \, dy + \sum_{\lambda \in \mathscr{E}} \int_{I_\lambda} \boldsymbol{u}_0(y) \frac{f_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} e^{i\tau(x+y)} \, dy$$

$$+ \sum_{\lambda \in \mathscr{E}} \int_{I_\lambda} \boldsymbol{u}_0(y) \frac{\tilde{f}_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} e^{i\tau(y-x)} \, dy + \sum_{\lambda \in \mathscr{I}} \int_{I_\lambda} \boldsymbol{u}_0(y) \frac{f^1_{\lambda,e}(i\tau) + f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} e^{i\tau(x+y)} \, dy$$

$$+ \sum_{\lambda \in \mathscr{I}} \int_{I_\lambda} \boldsymbol{u}_0(y) \frac{\tilde{f}^1_{\lambda,e}(i\tau) + \tilde{f}^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} e^{i\tau(y-x)} \, dy$$

$$+ \sum_{\lambda \in \mathscr{I}} \int_{I_\lambda} \boldsymbol{u}_0(y) \frac{(e^{i\tau(a_\lambda - y)} - e^{i\tau y}) f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} e^{i\tau x} \, dy$$

$$+ \sum_{\lambda \in \mathscr{I}} \int_{I_\lambda} \boldsymbol{u}_0(y) \frac{(e^{i\tau(a_\lambda - y)} - e^{i\tau y}) \tilde{f}^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} e^{-i\tau x} \, dy. \quad (26)$$

Let $e$ be an external edge. In view of Lemma 3.7 and the fact that $\omega = 0$ is a root of order $p - 1$ of $\det D_{\Gamma_p}$, we obtain that the fraction $f_{\lambda,e}(i\tau)/\det D_{\Gamma_p}(i\tau)$ is upper bounded near $\tau = 0$. Outside a neighbourhood of $\tau = 0$ we use Lemma 3.2 to infer that $|\det D_{\Gamma_p}(i\tau)|$ is positively bounded below outside neighbourhoods of $\tau = 0$. Moreover, in view of the explicit entries of $M^{c_e}_{\Gamma_p}(i\tau)$, we see that $f_{\lambda,e}(i\tau)$ is upper bounded for any $\tau \in \mathbb{R}$ since all the entries of matrix $D_{\Gamma_p}(i\tau)$ as well as the coefficients of $t_\lambda$ in $T_{\Gamma_p(i\tau)}$ have absolute value less than one. Summarising, we have obtained that

$$\frac{f_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} \in L^\infty(\mathbb{R}).$$

The derivative of this fraction is upper bounded near $\tau = 0$ by limited development at $\tau = 0$. Outside neighbourhoods of $\tau = 0$ we have that $\partial_\tau f_{\lambda,e}(i\tau)$ and $\partial_\tau \det D_{\Gamma_p}(i\tau)$ have upper bounds of type $1/\tau^2$. This is because each term of $\partial_\tau f_{\lambda,e}(i\tau)$ and $\partial_\tau \det D_{\Gamma_p}(i\tau)$ contains a derivative of an element of the line given by the $\delta$-condition involving the derivatives in the root vertex $\mathbb{O}$. This vertex is the one which

is an initial vertex for all $n$ edges emerging from it: $I(e) = \mathbb{O}$, for all $e \in E$, $\mathbb{O} \in e$. If $\alpha$ denotes the strength of the $\delta$-condition in $\mathbb{O}$, then this line of the matrix $D_{\Gamma_p}(i\tau)$ is composed of $0, 1$ and $\pm\frac{i\tau}{i\tau+\alpha}$, where the minus sign appears only on the finite edges that star from $\mathbb{O}$, and this line for the column matrix $i\tau T_{\Gamma_p}(i\tau)$ is

$$\left( \frac{i\tau-\alpha}{i\tau+\alpha} t_1(0,i\tau) + \frac{i\tau}{i\tau+\alpha} \sum_{2 \leq j \leq n} t_j(0,i\tau) \right).$$

Finally, as above, $f_{\lambda,e}(i\tau)$ and $\det D_{\Gamma_p}(i\tau)$ are upper bounded and from Lemma 3.2 we have that $|\det D_{\Gamma_p}(i\tau)|$ is positively bounded below outside neighbourhoods of $\tau = 0$. In conclusion we infer that

$$\partial_\tau \frac{f_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} \in L^1(\mathbb{R}).$$

The same argument using Lemmas 3.7, 3.8 and 3.6 can be performed to obtain that

$$\frac{\tilde{f}_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}, \quad \frac{f^1_{\lambda,e}(i\tau) + f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}, \quad \frac{\tilde{f}^1_{\lambda,e}(i\tau) + \tilde{f}^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)},$$

$$\frac{(e^{i\tau(a_\lambda-y)} - e^{i\tau y}) f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}, \quad \frac{(e^{i\tau(a_\lambda-y)} - e^{i\tau y}) \tilde{f}^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}$$

are in $L^\infty$ with derivative in $L^1$. Notice that when $\lambda$ belongs to an internal edge $I_\lambda$ it follows that the interval $I_\lambda$ has finite length. Therefore for the last fractions we use that $(e^{i\tau(a_\lambda-y)} - e^{i\tau y}) f^2_{\lambda,e}(i\tau)$ vanishes with order $p-1$ at $\tau = 0$ and repeat the argument used above. The only difference from the previous cases is that we will obtain bounds in terms of the parameter $y$. Since $y$ is now on an internal edge $I_\lambda$ of finite length we obtain uniform bounds. Therefore the dispersion estimate (12) of Theorem 1.2 follows from (24) by using (26) and the classical oscillatory integral estimate

$$\left| \int_{-\infty}^{\infty} e^{-it\tau^2} e^{i\tau a} g(\tau)\, d\tau \right| \leq \frac{C}{\sqrt{|t|}} (\|g\|_{L^\infty} + \|g'\|_{L^1}). \qquad \square$$

## Appendix A: The multiplicity of the root $\omega = 0$ of $\det D_{\Gamma_p}(\omega)$

Here we prove that the condition (11) is fulfilled in the case of positive strengths. We shall show first the following double property.

**Lemma A.1.** *For all $p \geq 1$ we have the following properties*:

$$(\mathscr{P}^1_p) : \frac{\det \tilde{D}_{\Gamma_p}}{\det D_{\Gamma_p}}(0) = 1, \quad (\mathscr{P}^2_p) : \partial_\omega \left( \frac{\det \tilde{D}_{\Gamma_p}}{\det D_{\Gamma_p}} \right)(0) < 0.$$

*Proof.* Lemma 3.1 ensures that $\frac{\det \tilde{D}_{\Gamma_1}}{\det D_{\Gamma_1}}(\omega) = \frac{(n_1-2)\omega+\alpha_1}{n_1\omega+\alpha_1}$, and in particular

$$\partial_\omega \left( \frac{\det \tilde{D}_{\Gamma_1}}{\det D_{\Gamma_1}} \right)(\omega) = -\frac{2\alpha_1}{(n_1\omega+\alpha_1)^2}, \tag{27}$$

and the lemma follows for $p = 1$, since $\alpha_1 > 0$. We shall show the general case by recursion. Let us denote by $P_p(\omega)$ and $Q_p(\omega)$ the numerator and respectively the denominator in the recursion formula of the ratio from Lemma 3.1:

$$P_p(\omega) = \frac{(n_p - 2)\omega + \alpha_p}{n_p \omega + \alpha_p} - \frac{(n_p - 4)\omega + \alpha_p}{n_p \omega + \alpha_p} e^{-2\omega a_{p-1}} \frac{\det \widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)},$$

$$Q_p(\omega) = 1 - \frac{(n_p - 2)\omega + \alpha_p}{n_p \omega + \alpha_p} e^{-2\omega a_{p-1}} \frac{\det \widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}.$$

We have $P_p(0) = Q_p(0) = 0$, and in view of $(\mathscr{P}^1_{p-1})$ we compute

$$\partial_\omega P_p(0) = \partial_\omega Q_p(0) = \frac{2}{\alpha_p} + 2a_{p-1} - \partial_\omega \left( \frac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}} \right)(0).$$

Therefore $(\mathscr{P}^2_{p-1})$ ensures that $\partial_\omega P_p(0) = \partial_\omega Q_p(0) \neq 0$ and we apply l'Hôpital's rule to conclude $(\mathscr{P}^1_p)$.
    Since

$$P_p(\omega) - Q_p(\omega) = -\frac{2\omega}{n_p \omega + \alpha_p} \left( 1 - e^{-2a_{p-1}\omega} \frac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}}(\omega) \right),$$

we define $\widetilde{P}_p(\omega)$ and $\widetilde{Q}_p(\omega)$ by

$$P_p(\omega) = \frac{2\omega}{n_p \omega + \alpha_p} \widetilde{P}_p(\omega), \quad Q_p(\omega) = \frac{2\omega}{n_p \omega + \alpha_p} \widetilde{Q}_p(\omega).$$

In particular

$$\frac{\det \widetilde{D}_{\Gamma_p}}{\det D_{\Gamma_p}}(\omega) = \frac{\widetilde{P}_p}{\widetilde{Q}_p}(\omega), \quad \widetilde{P}_p(\omega) - \widetilde{Q}_p(\omega) = -\left( 1 - e^{-2a_{p-1}\omega} \frac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}}(\omega) \right).$$

By using $(\mathscr{P}^1_{p-1})$ and $(\mathscr{P}^2_{p-1})$, we obtain

$$\partial_\omega(\widetilde{P}_p - \widetilde{Q}_p)(0) = -2a_{p-1} + \partial_\omega \left( \frac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}} \right)(0).$$

Moreover, $\widetilde{P}_p(0) = \widetilde{Q}_p(0) = \frac{1}{2}\alpha_p \partial_\omega P_p(0) = \frac{1}{2}\alpha_p \partial_\omega Q_p(0) \neq 0$, and we can compute

$$\partial_\omega \left( \frac{\det \widetilde{D}_{\Gamma_p}}{\det D_{\Gamma_p}} \right)(0) = \frac{\partial_\omega \widetilde{P}_p(0) \widetilde{Q}_p(0) - \widetilde{P}_p(0) \partial_\omega \widetilde{Q}_p(0)}{(\widetilde{Q}_p(0))^2} = \frac{\partial_\omega(\widetilde{P}_p - \widetilde{Q}_p)(0)}{\widetilde{Q}_p(0)}$$

$$= -\frac{2a_{p-1} - \partial_\omega \left( \dfrac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}} \right)(0)}{\dfrac{\alpha_p}{2} \left( \dfrac{2}{\alpha_p} + 2a_{p-1} - \partial_\omega \left( \dfrac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}} \right)(0) \right)}.$$

By using again $(\mathscr{P}^2_{p-1})$, we obtain $(\mathscr{P}^2_p)$.                                                $\square$

**Lemma A.2.** $\omega = 0$ *is a root of order $p - 1$ of* $\det D_{\Gamma_p}(\omega)$. *In particular, condition* (11) *is fulfilled.*

*Proof.* From Lemma 3.1 we have $\det D_{\Gamma_1}(\omega) = n_1\omega + \alpha$, so $\det D_{\Gamma_1}(0) \neq 0$. Lemma 3.1 also gives us

$$\det D_{\Gamma_p}(\omega) = \frac{n_p\omega + \alpha_p}{\omega + \alpha_p} e^{\omega a_{p-1}} \det D_{\Gamma_{p-1}}(\omega) \left( 1 - \frac{(n_p - 2)\omega + \alpha_p}{n_p\omega + \alpha_p} e^{-2\omega a_{p-1}} \frac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}}(\omega) \right),$$

so by recursion it is enough to show that $\omega = 0$ is a simple root for

$$1 - \frac{(n_p - 2)\,\omega + \alpha_p}{n_p\,\omega + \alpha_p} e^{-2\omega a_{p-1}} \frac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}}(\omega).$$

This is precisely $Q_p(\omega)$ from the proof of Lemma A.1, where it was proved that $\partial_\omega Q_p(0) \neq 0$.  $\square$

## Appendix B: Strichartz estimates

We prove Theorem 1.3. Let us first remark that by the definition of $P_e$ we have

$$P_e\phi = \sum_{k=1}^{m} \langle \phi, \varphi_k \rangle \varphi_k,$$

where $\{\varphi_k\}_{k=1}^m$ are eigenfunctions of the operator $H$. Since $\varphi_k \in L^2(\Gamma)$ we have that $\varphi_k \in L^1(\Gamma) \cap L^\infty(\Gamma)$. Indeed, on the infinite edges the eigenfunctions corresponding to an eigenvalue $\lambda < 0$ are of type $C \exp(-\sqrt{-\lambda})x$. This means that they belong to $L^1(e) \cap L^\infty(e)$ for any external edge $e$. On the internal edges this property trivially holds.

Then $P_e$ is defined for any $\phi \in L^r(\Gamma)$, $1 \leq r \leq \infty$, and for any $1 \leq r_1, r_2 \leq \infty$ we have, by Hölder's inequality,

$$\|P_e\phi\|_{L^{r_2}(\Gamma)} \leq \sum_{k=1}^{m} |\langle \phi, \varphi_k \rangle| \|\varphi_k\|_{L^{r_2}(\Gamma)} \leq \|\phi\|_{L^{r_1}(\Gamma)} \sum_{k=1}^{m} \|\varphi_k\|_{L^{r_1'}(\Gamma)} \|\varphi_k\|_{L^{r_2}(\Gamma)}$$

$$\leq C(\Gamma, r_1, r_2) \|\phi\|_{L^{r_1}(\Gamma)}.$$

*Proof of Theorem 1.3.* Using the dispersive estimate (12) and the mass conservation

$$\|e^{-itH} \boldsymbol{u}_0\|_{L^2(\Gamma)} = \|\boldsymbol{u}_0\|_{L^2(\Gamma)}$$

we obtain, by applying the classical $TT^*$ argument and Christ–Kiselev lemma [2001], the estimates

$$\|e^{-itH} P\boldsymbol{u}_0\|_{L^q(\mathbb{R}, L^r(\Gamma))} \leq C \|\boldsymbol{u}_0\|_{L^2(\Gamma)}, \tag{28}$$

and

$$\left\| \int_0^t e^{-i(t-s)} P\boldsymbol{F}(s)ds \right\|_{L^q((0,T), L^r(\Gamma))} \leq C \|\boldsymbol{F}\|_{L^{\tilde{r}'}((0,T), L^{\tilde{r}'}(\Gamma))}. \tag{29}$$

Now using Stone's theorem we obtain

$$e^{-itH}\phi = e^{-itH} P\phi + e^{-itH} P_e\phi = e^{-itH} P\phi + \sum_{k=1}^{m} e^{it\lambda_k^2} \langle \phi, \varphi_k \rangle \varphi_k,$$

where by $\lambda_k$ we denote the eigenvalue of the eigenfunction $\varphi_k$. We claim that, for all $\alpha \geq 1$,

$$\|e^{-itH} P_e\boldsymbol{u}_0\|_{L^q((0,T), L^r(\Gamma))} \leq C T^{1/q} \|\boldsymbol{u}_0\|_{L^\alpha(\Gamma)} \tag{30}$$

and

$$\left\| \int_0^t e^{-i(t-s)} P_e \boldsymbol{F}(s) ds \right\|_{L^q((0,T),L^r(\Gamma))} \le C T^{1/q} \|\boldsymbol{F}\|_{L^1((0,T),L^\alpha(\Gamma))}. \tag{31}$$

Putting together estimates (28), (29), (30) and (31) we obtain the desired result. We now prove estimates (30) and (31).

In the case of estimate (30), using the fact that

$$e^{-itH} P_e \boldsymbol{u}_0 = \sum_{k=1}^m e^{it\lambda_k^2} \langle \boldsymbol{u}_0, \varphi_k \rangle \varphi_k$$

we obtain by Hölder's inequality that, for any $\alpha \ge 1$,

$$\|e^{-itH} P_e \boldsymbol{u}_0\|_{L^r(\Gamma)} \le \sum_{k=1}^m |\langle \boldsymbol{u}_0, \varphi_k \rangle| \|\varphi_k\|_{L^r(\Gamma)} \le \|\boldsymbol{u}_0\|_{L^\alpha(\Gamma)} \sum_{k=1}^m \|\varphi_k\|_{L^{\alpha'}(\Gamma)} \|\varphi_k\|_{L^r(\Gamma)} \le C \|\boldsymbol{u}_0\|_{L^\alpha(\Gamma)}.$$

Taking the $L^q$-norm on the time interval $(0, T)$ we obtain estimate (30).

In a similar way we have

$$\|e^{-i(t-s)H} P_e \boldsymbol{F}(s)\|_{L^r(\Gamma)} \le C \|\boldsymbol{F}(s)\|_{L^\alpha(\Gamma)}.$$

Using Minkowski's inequality we obtain that

$$\left\| \int_0^t e^{-i(t-s)H} P_e \boldsymbol{F}(s) ds \right\|_{L^q((0,T),L^r(\Gamma))} \le \left\| \int_0^t \|e^{-i(t-s)H} P_e \boldsymbol{F}(s)\|_{L^r(\Gamma)} ds \right\|_{L^q(0,T)}$$

$$\le T^{1/q} \int_0^T \|\boldsymbol{F}(s)\|_{L^\alpha(\Gamma)} ds,$$

which proves estimate (31). $\qquad\square$

## Appendix C: General couplings

We consider general coupling conditions at each vertex $v$ (see (16) in Section 2),

$$A^v \boldsymbol{u}(v) + B^v \boldsymbol{u}'(v) = 0.$$

Using the notations introduced in this article, we shall give the recursion formulae for obtaining $\det D_{\Gamma_p}$ for general couplings. As a consequence, we shall give a sufficient condition for obtaining the dispersion.

We follow the approach in Section 3A for computing the resolvent. For a star-shaped graph with $n_1$ edges $I_j$ parametrised by $x \in [0, \infty)$, with coupling conditions $(A^1, B^1)$, the resolvent on each edge $I_j$ is

$$R_\omega \boldsymbol{u}_0(x) = \tilde{c}_j e^{-\omega x} + \frac{1}{2\omega} \int_0^\infty \boldsymbol{u}_0(y) e^{-\omega|x-y|} dy.$$

The coupling conditions yield as a system for the $\tilde{c}$'s

$$(A^1 + \omega B^1)\begin{pmatrix} \tilde{c}_1 \\ \vdots \\ \tilde{c}_{n_1} \end{pmatrix} = \begin{pmatrix} \sum_{1 \le j \le n_1} \dfrac{t_j(0, \omega)}{\omega}(b_{1,j}\omega - a_{1,j}) \\ \vdots \\ \sum_{1 \le j \le n_1} \dfrac{t_j(0, \omega)}{\omega}(b_{n_1,j}\omega - a_{n_1,j}) \end{pmatrix}.$$

We denote by $D_{\Gamma_1}(\omega)$ the matrix of the system. We define $\tilde{D}_{\Gamma_1}(\omega)$ to be the matrix

$$\big((A^1 + \omega B^1)_1, (A^1 + \omega B^1)_2, \ldots, (A^1 - \omega B^1)_{n_1}\big),$$

where by $(A^1 + \omega B^1)_j$ we mean the $j$-th column of $A^1 + \omega B^1$.

The case of a general tree with $p$ vertices can again be seen as constructed by adding a new vertex $v^p$ to a $(p-1)$-vertex tree, with coupling conditions $(A^p, B^p)$, from which emerge new $n_p - 1$ infinite edges. Similarly to Lemma 3.1 we derive the recursion formulae

$$\det D_{\Gamma_1}(\omega) = \det(A^1 + \omega B^1),$$

$$\frac{\det \tilde{D}_{\Gamma_1}(\omega)}{\det D_{\Gamma_1}(\omega)} = \frac{\det\big((A^1 + \omega B^1)_1, (A^1 + \omega B^1)_2, \ldots, (A^1 - \omega B^1)_{n_1}\big)}{\det(A^1 + \omega B^1)},$$

$$\det D_{\Gamma_p}(\omega) = \det(A^p + \omega B^p)e^{\omega a_{p-1}}\det D_{\Gamma_{p-1}}(\omega)$$

$$\times \left(1 - \frac{\det\big((A^p - \omega B^p)_1, (A^p + \omega B^p)_2, \ldots, (A^p + \omega B^p)_{n_p}\big)}{\det(A^p + \omega B^p)}e^{-2\omega a_{p-1}}\frac{\det \tilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}\right),$$

$$\frac{\det \tilde{D}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)} = \left(\frac{\det\big((A^p + \omega B^p)_1, (A^p + \omega B^p)_2, \ldots, (A^p - \omega B^p)_{n_p}\big)}{\det(A^p + \omega B^p)}\right.$$

$$- \frac{\det\big((A^p - \omega B^p)_1, (A^p + \omega B^p)_2, \ldots, (A^p - \omega B^p)_{n_p}\big)}{\det(A^p + \omega B^p)}e^{-2\omega a_{p-1}}\frac{\det \tilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}\right)$$

$$\times \left(1 - \frac{\det\big((A^p - \omega B^p)_1, (A^p + \omega B^p)_2, \ldots, (A^p + \omega B^p)_{n_p}\big)}{\det(A^p + \omega B^p)}e^{-2\omega a_{p-1}}\frac{\det \tilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}\right)^{-1}.$$

A sufficient condition for using the spectral formula as in Section 3E and then for getting the dispersion as the following constraint, depending only on the entries of $(A^j, B^j)_{1 \le j \le p}$, is

$$|\det D_{\Gamma_p}(i\omega)| \ne 0 \quad \text{for all } \omega \in \mathbb{R}. \tag{32}$$

This is the way the Kirchhoff coupling case was ruled in [Banica and Ignat 2011] and this might be used in other cases. In the $\delta$-coupling case presented in this article, and probably in many other cases, such an estimate is not valid. Then an analysis around the zeros of $\det D_{\Gamma_p}(\omega)$ has to be done starting from the above recursion formulae.

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# THE CUNTZ SEMIGROUP AND STABILITY OF CLOSE $C^*$-ALGEBRAS

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We prove that separable $C^*$-algebras which are completely close in a natural uniform sense have isomorphic Cuntz semigroups, continuing a line of research developed by Kadison–Kastler, Christensen, and Khoshkam. This result has several applications: we are able to prove that the property of stability is preserved by close $C^*$-algebras provided that one algebra has stable rank one; close $C^*$-algebras must have affinely homeomorphic spaces of lower-semicontinuous quasitraces; strict comparison is preserved by sufficient closeness of $C^*$-algebras. We also examine $C^*$-algebras which have a positive answer to Kadison's Similarity Problem, as these algebras are completely close whenever they are close. A sample consequence is that sufficiently close $C^*$-algebras have isomorphic Cuntz semigroups when one algebra absorbs the Jiang–Su algebra tensorially.

## 1. Introduction

Kadison and Kastler [1972] introduced a metric $d$ on the $C^*$-subalgebras of a given $C^*$-algebra by equipping the unit balls of the subalgebras with the Hausdorff metric (in norm). They conjectured that sufficiently close $C^*$-subalgebras of $\mathcal{B}(\mathcal{H})$ should be isomorphic, and this conjecture was recently established by Christensen et al. [2012] when one $C^*$-algebra is separable and nuclear. The one-sided version of this result—that a sufficiently close near inclusion of a nuclear separable $C^*$-algebra into another $C^*$-algebra gives rise to a true inclusion—was later proved by Hirshberg, Kirchberg, and White [Hirshberg et al. 2012]. These results and others (see [Christensen et al. 2010; Cameron et al. 2012]) have given new momentum to the perturbation theory of operator algebras.

The foundational paper [Kadison and Kastler 1972] was concerned with structural properties of close algebras, showing that the type decomposition of a von Neumann algebra transfers to nearby algebras. We continue this theme here asking "Which properties or invariants of $C^*$-algebras are preserved by small perturbations?" With the proof of the Kadison–Kastler conjecture, the answer for nuclear separable $C^*$-algebras is "all of them". Here we consider general separable $C^*$-algebras where already, there are some results. Sufficiently close $C^*$-algebras have isomorphic lattices of ideals [Phillips 1973/74] and algebras whose stabilisations are sufficiently close have isomorphic K-theories [Kirchberg 1996]. This was extended to the Elliott invariant consisting of K-theory, traces, and their natural pairing, in [Christensen et al. 2010]. A natural next step is to consider the Cuntz semigroup of (equivalence classes of) positive

elements (in the stabilisation) of a $C^*$-algebra, due both to its exceptional sensitivity in determining non-isomorphism [Toms 2008], classification results using the semigroup [Robert 2012] and the host of $C^*$-algebraic properties that can be formulated as order-theoretic properties of the semigroup; for example there is strong evidence to suggest that the behaviour of the Cuntz semigroup characterises important algebraic regularity properties of simple separable nuclear $C^*$-algebras [Matui and Sato 2012; Winter 2010; 2012]. We prove that algebras whose stabilisations are sufficiently close do indeed have isomorphic Cuntz semigroups, a surprising fact given the sensitivity of a Cuntz class to perturbations of its representing positive element. This is in stark contrast with the case of Murray–von Neumann equivalence classes of projections, where classes are stable under perturbations of the representing projection of size strictly less than one. The bridge between these two situations is that we can arrange for the representing positive element of a Cuntz class to be almost a projection in trace. We exploit this fact through the introduction of what we call *very rapidly increasing sequences* of positive contractions, increasing sequences where each element almost acts as a unit on its predecessor.

The Kadison–Kastler metric $d$ is equivalent to a complete version $d_{\text{cb}}$ (given by applying $d$ to the stabilisations) if and only if Kadison's Similarity Problem has a positive solution [Christensen et al. 2010; Cameron et al. 2013]; the latter is known to hold in considerable generality, for instance in the case of $\mathcal{Z}$-stable algebras [Johanesová and Winter 2012]. We show how this result, and a number of other similarity results for $C^*$-algebras, can be put in a common framework using Christensen's [1977] property $D_k$ and, building on [Christensen et al. 2010], make a more careful study of automatic complete closeness and its relation to property $D_k$. We prove that if an algebra $A$ has $D_k$ for some $k$, then $d(A \otimes \mathcal{K}, B \otimes \mathcal{K}) \leq C(k)d(A, B)$, where $C(k)$ is a constant independent of $A$ and $B$; as a consequence, sufficiently close $C^*$-algebras have isomorphic Cuntz semigroups provided one algebra is $\mathcal{Z}$-stable.

Stability is perhaps the most basic property one could study in perturbation theory, yet proving its permanence under small perturbations has seen very little progress. We take a significant step here by proving that stability is indeed preserved if one of the algebras considered has stable rank one. The proof is an application of our permanence result for the Cuntz semigroup. Another application is our proof that stably close $C^*$-algebras have affinely homeomorphic spaces of lower semicontinuous 2-quasitraces. This extends and improves results from [Christensen et al. 2010], showing that the affine isomorphism between the trace spaces of stably close $C^*$-algebras obtained there is weak*-weak*-continuous.

The paper is organized as follows: Section 2 contains the preliminaries on the Cuntz semigroup and the Kadison–Kastler metric; Section 3 establishes the permanence of the Cuntz semigroup under complete closeness; Section 4 discusses property $D_k$ and proves our permanence result for stability; Section 5 proves permanence for quasitraces.

## 2. Preliminaries

Throughout the paper we write $A^+$ for the positive elements of a $C^*$-algebra $A$, $A_1$ for the unit ball of $A$ and $A_1^+$ for the positive contractions in $A$.

In the next two subsections we review the definition and basic properties of the Cuntz semigroup. A complete account can be found in the survey [Ara et al. 2011].

***The Cuntz semigroup.*** Let $A$ be a $C^*$-algebra. Let us consider on $(A \otimes \mathcal{K})^+$ the relation $a \precsim b$ if $v_n b v_n^* \to a$ for some sequence $(v_n)$ in $A \otimes \mathcal{K}$. Let us write $a \sim b$ if $a \precsim b$ and $b \precsim a$. In this case we say that $a$ is *Cuntz equivalent* to $b$. Let $\mathrm{Cu}(A)$ denote the set $(A \otimes \mathcal{K})^+/\sim$ of Cuntz equivalence classes. We use $\langle a \rangle$ to denote the class of $a$ in $\mathrm{Cu}(A)$. It is clear that $\langle a \rangle \leq \langle b \rangle \Leftrightarrow a \precsim b$ defines an order on $\mathrm{Cu}(A)$. We also endow $\mathrm{Cu}(A)$ with an addition operation by setting $\langle a \rangle + \langle b \rangle := \langle a' + b' \rangle$, where $a'$ and $b'$ are orthogonal and Cuntz equivalent to $a$ and $b$ respectively (the choice of $a'$ and $b'$ does not affect the Cuntz class of their sum). The semigroup $W(A)$ is then the subsemigroup of $\mathrm{Cu}(A)$ of Cuntz classes with a representative in $\bigcup_n M_n(A)_+$.

Alternatively, $\mathrm{Cu}(A)$ can be defined to consist of equivalence classes of countably generated Hilbert modules over $A$ [Coward et al. 2008]. The equivalence relation boils down to isomorphism in the case that $A$ has stable rank one, but is rather more complicated in general and as we do not require the precise definition of this relation in the sequel, we omit it. We note, however, that the identification of these two approaches to $\mathrm{Cu}(A)$ is achieved by associating the element $\langle a \rangle$ to the class of the Hilbert module $\overline{a\ell_2(A)}$.

***The category* Cu.** The semigroup $\mathrm{Cu}(A)$ is an object in a category of ordered Abelian monoids denoted by **Cu** introduced in [Coward et al. 2008] with additional properties. Before stating them, we require the notion of order-theoretic compact containment. Let $T$ be a preordered set with $x, y \in T$. We say that $x$ is *compactly contained* in $y$, denoted by $x \ll y$, if for any increasing sequence $(y_n)$ in $T$ with supremum $y$, we have $x \leq y_{n_0}$ for some $n_0 \in \mathbb{N}$. An object $S$ of **Cu** enjoys the following properties (see [Coward et al. 2008; Ara et al. 2011]), which we use repeatedly in the sequel. In particular the existence of suprema in property P3 is a crucial in our construction of a map between the Cuntz semigroups of stably close $C^*$-algebras.

P1. $S$ contains a zero element.

P2. The order on $S$ is compatible with addition: $x_1 + x_2 \leq y_1 + y_2$ whenever $x_i \leq y_i$, $i \in \{1, 2\}$.

P3. Every countable upward directed set in $S$ has a supremum.

P4. For each $x \in S$, the set $x_{\ll} = \{y \in S \mid y \ll x\}$ is upward directed with respect to both $\leq$ and $\ll$, and contains a sequence $(x_n)$ such that $x_n \ll x_{n+1}$ for every $n \in \mathbb{N}$ and $\sup_n x_n = x$.

P5. The operation of passing to the supremum of a countable upward directed set and the relation $\ll$ are compatible with addition: if $S_1$ and $S_2$ are countable upward directed sets in $S$, then $S_1 + S_2$ is upward directed and $\sup(S_1 + S_2) = \sup S_1 + \sup S_2$, and if $x_i \ll y_i$ for $i \in \{1, 2\}$, then $x_1 + x_2 \ll y_1 + y_2$.

We say that a sequence $(x_n)$ in $S \in$ **Cu** is *rapidly increasing* if $x_n \ll x_{n+1}$ for all $n$. We take the scale $\Sigma(\mathrm{Cu}(A))$ to be the subset of $\mathrm{Cu}(A)$ obtained as supremums of increasing sequences from $A^+$.

For objects $S$ and $T$ from **Cu**, the map $\phi \colon S \to T$ is a morphism in the category **Cu** if

M1. $\phi$ is order-preserving;

M2. $\phi$ is additive and maps 0 to 0;

M3. $\phi$ preserves the suprema of increasing sequences;

M4. $\phi$ preserves the relation $\ll$.

*The Kadison–Kastler metric.* Let us recall the definition of the metric on the collection of all $C^*$-subalgebras of a $C^*$-algebra introduced in [Kadison and Kastler 1972].

**Definition 2.1.** Let $A$, $B$ be $C^*$-subalgebras of a $C^*$-algebra $C$. Define a metric $d$ on all such pairs as follows: $d(A, B) < \gamma$ if and only if for each $x$ in the unit ball of $A$ or $B$, there is $y$ in the unit ball of the other algebra such that $\|x - y\| < \gamma$.

In this definition, we typically take $C = \mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. The complete, or stabilised, version of the Kadison–Kastler metric is defined by $d_{\mathrm{cb}}(A, B) = d(A \otimes \mathcal{K}, B \otimes \mathcal{K})$ inside $C \otimes \mathcal{K}$ (here $\mathcal{K}$ is the $C^*$-algebra of compact operators on $\ell^2(\mathbb{N})$); the notion $d_{\mathrm{cb}}$ is used for this metric as $d_{\mathrm{cb}}(A, B) \leq \gamma$ is equivalent to the condition that $d(\mathrm{M}_n(A), \mathrm{M}_n(B)) \leq \gamma$ for every $n$.

We repeatedly use the standard fact that if $d(A, B) < \gamma$, then given a positive contraction $a \in A_1^+$, there exists a positive contraction $b \in B_1^+$ with $\|a - b\| < 2\alpha$. One way of seeing this is to use the hypothesis $d(A, B) < \gamma$ to approximate $a^{1/2}$ by some $c \in B_1$ with $\|a^{1/2} - c\| < \gamma$. Then take $b = cc^*$ so that

$$\|a - b\| \leq \|a^{1/2}(a^{1/2} - c)\| + \|(a^{1/2} - c^*)c\| < 2\gamma.$$

There is also a one-sided version of closeness introduced by Christensen [1980], which is referred to as a *$\gamma$-near inclusion*:

**Definition 2.2.** Let $A$, $B$ be $C^*$-subalgebras of a $C^*$-algebra $C$ and let $\gamma > 0$. Write $A \subseteq_\gamma B$ if for every $x$ in the unit ball of $B$, there is $y \in B$ such that $\|x - y\| \leq \gamma$ (note that $y$ need not be in the unit ball of $B$). Write $A \subset_\gamma B$ if there exists $\gamma' < \gamma$ with $A \subseteq_{\gamma'} B$. As with the Kadison–Kastler metric, we also use complete, or stabilised, near inclusions: write $A \subseteq_{\mathrm{cb}, \gamma} B$ when $A \otimes \mathrm{M}_n \subseteq_\gamma B \otimes \mathrm{M}_n$ for all $n$, and $A \subset_{\mathrm{cb}, \gamma} B$ when there exists $\gamma' < \gamma$ with $A \subseteq_{\mathrm{cb}, \gamma'} B$.

### 3. Very rapidly increasing sequences and the Cuntz semigroup

We start by noting that, for close $C^*$-algebras of real rank zero, an isomorphism between their Cuntz semigroups can be deduced from existing results in the literature. For a $C^*$-algebra $A$, let $V(A)$ be the Murray and von Neumann semigroup of equivalence classes of projections in $\bigcup_{n=1}^\infty A \otimes \mathrm{M}_n$ and write $\Sigma(V(A)) = \{[p] \in V(A) \mid p = p^2 = p^* \in A\}$. This is a local semigroup in the sense that if $p, q, p'$ and $q'$ are projections in $A$ with $p'q' = 0$ and $p \sim p'$, $q \sim q'$, then $[p] + [q] = [p' + q'] \in \Sigma(V(A))$. Recall that, if $A$ has real rank zero, then the work of Zhang [1990] shows that $V(A)$ has the Riesz refinement property. By definition, this means that whenever $x_1, \ldots, x_n, y_1, \ldots, y_m \in V(A)$ satisfy $\sum_i x_i = \sum_j y_j$, then there exist $z_{i,j} \in V(A)$ with $\sum_j z_{i,j} = x_i$ and $\sum_i z_{i,j} = y_j$ for each $i, j$. The case $m = n = 2$ of this can be found as [Ara and Pardo 1996, Lemma 2.3], and the same proof works in general.

The Cuntz semigroup of a $C^*$-algebra of real rank zero is completely determined by its semigroup of projections (see [Perera 1997] when $A$ additionally has stable rank one and [Antoine et al. 2011] for the general case). We briefly recall how this is done. An *interval* in $V(A)$ is a nonempty, order hereditary and upward directed subset $I$ of $V(A)$, which is said to be countably generated provided there is an increasing sequence $(x_n)$ in $V(A)$ such that $I = \{x \in V(A) \mid x \leq x_n \text{ for some } n\}$. The set of countably generated intervals is denoted by $\Lambda_\sigma(V(A))$, and it has a natural semigroup structure. Namely, if $I$ and $J$ have

generating sequences $(x_n)$ and $(y_n)$ respectively, then $I + J$ is the interval generated by $(x_n + y_n)$. Given a positive element $a$ in $A \otimes \mathcal{K}$ in a $\sigma$-unital C*-algebra of real rank zero $A$, put $I(a) = \{[p] \in V(A) \mid p \precsim a\}$. The correspondence $[a] \mapsto I(a)$ defines an ordered semigroup isomorphism $\mathrm{Cu}(A) \cong \Lambda_\sigma(V(A))$.

**Theorem 3.1.** *Let $A$ and $B$ be $\sigma$-unital C*-subalgebras of a C*-algebra $C$, with $d(A, B) < 1/8$. If $A$ has real rank zero, then $B$ also has real rank zero and $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$.*

*Proof.* That $B$ has real rank zero follows from [Christensen et al. 2010, Theorem 6.3]. We know from [Phillips and Raeburn 1979, Theorem 2.6] that there is an isomorphism of local semigroups $\Phi_1 \colon \Sigma(V(A)) \to \Sigma(V(B))$ (with inverse, say, $\Psi_1$). This is defined as $\Phi_1[p] = [q]$, where $q$ is a projection in $B$ such that $\|p - q\| < 1/8$. Given $p \in M_n(A)$, by [Zhang 1990, Theorem 3.2] we can find projections $\{p_i\}_{i=1,\dots,n}$ in $A$ such that $[p] = \sum_i [p_i]$. Now extend $\Phi_1$ to $\Phi \colon V(A) \to V(B)$ by $\Phi([p]) = \sum_i \Phi_1([p_i])$. Let us check that $\Phi$ is well defined. If $[p] = \sum_i [p_i] = \sum_j [q_j]$ for projections $p_i$ and $q_j$ in $A$, then use refinement to find elements $a_{ij} \in V(A)$ such that $[p_i] = \sum_j a_{ij}$ and $[q_j] = \sum_i a_{ij}$. We may also clearly choose projections $z_{ij}, z'_{ij} \in A$ such that $a_{ij} = [z_{ij}] = [z'_{ij}]$, and such that $z_{ij} \perp z_{ik}$ if $j \ne k$, and $z'_{ij} \perp z'_{lj}$ if $i \ne l$. Then

$$\sum \Phi_1([p_i]) = \sum_i \sum_j \Phi_1([z_{ij}]) = \sum_i \sum_j \Phi_1([z'_{ij}]) = \sum_j \sum_i \Phi_1([z'_{ij}]) = \sum_j \Phi_1([q_j]).$$

Clearly $\Phi$ is additive and $\Phi|_{\Sigma(V(A))} = \Phi_1$. Using $\Psi_1$, we construct an additive map $\Psi \colon V(B) \to V(A)$, with $\Psi|_{\Sigma(V(B))} = \Psi_1$. Since $\Psi_1 \circ \Phi_1 = \mathrm{id}_{\Sigma(V(A))}$, it follows that $\Psi \circ \Phi = \mathrm{id}_{V(A)}$. Similarly $\Phi \circ \Psi = \mathrm{id}_{V(B)}$.

Since $\mathrm{Cu}(A) \cong \Lambda_\sigma(V(A))$ and $\mathrm{Cu}(B) \cong \Lambda_\sigma(V(B))$, it follows that $\mathrm{Cu}(A)$ is isomorphic to $\mathrm{Cu}(B)$. $\square$

We turn now to very rapidly increasing sequences. These provide the key tool we use to transfer information between close algebras at the level of the Cuntz semigroup.

**Definition 3.2.** Let $A$ be a C*-algebra. We say that a rapidly increasing sequence $(a_n)_{n=1}^\infty$ in $A_1^+$ is *very rapidly increasing* if given $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $m_0 \in \mathbb{N}$ such that for $m \ge m_0$, there exists $v \in A_1$ with $\|(v a_m v^*) a_n - a_n\| < \varepsilon$. Say that a very rapidly increasing sequence $(a_n)_{n=1}^\infty$ in $(A \otimes \mathcal{K})_+^1$ *represents* $x \in \mathrm{Cu}(A)$ if $\sup_n \langle a_n \rangle = x$.

The following two functions are used in the sequel to manipulate very rapidly increasing sequences. Given $a \in A_+$ and $\varepsilon > 0$, write $(a - \varepsilon)_+$ for $h_\varepsilon(a)$, where $h_\varepsilon$ is the continuous function $h_\varepsilon(t) = \max(0, t - \varepsilon)$. For $0 \le \beta < \gamma$, let $g_{\beta, \gamma}$ be the piecewise linear function on $\mathbb{R}$ given by

$$g_{\beta, \gamma}(t) = \begin{cases} 0 & \text{if } t \le \beta, \\ \dfrac{t - \beta}{\gamma - \beta} & \text{if } \beta < t < \gamma, \\ 1 & \text{if } t \ge \gamma. \end{cases} \tag{3-1}$$

With this notation, the standard example of a very rapidly increasing sequence is given by

$$(g_{2^{-(n+1)}, 2^{-n}}(a))_{n=1}^\infty \quad \text{for } a \in A_1^+.$$

This sequence represents $\langle a \rangle$. In this way every element of the Cuntz semigroup of $A$ is represented by a very rapidly increasing sequence from $(A \otimes \mathcal{K})_1^+$. In the next few lemmas we develop properties of very rapidly increasing sequences, starting with a technical observation.

**Lemma 3.3.** *Let $A$ be a $C^*$-algebra and let $a, b \in A_1^+$ and $v \in A_1$ satisfy $\|v^*bva - a\| \leq \delta$ for some $\delta > 0$. Suppose that $0 < \beta < 1$ and $\gamma \geq 0$ satisfy $\gamma + \delta\beta^{-1} < 1$, then $\langle (a - \beta)_+ \rangle \leq \langle (b - \gamma)_+ \rangle$ in $\mathrm{Cu}(A)$.*

*Proof.* Let $p \in A^{**}$ denote the spectral projection of $a$ for the interval $[\beta, 1]$. When $p = 0$, then $(a - \beta)_+ = 0$ and the result is trivial, so we may assume that $p \neq 0$. Then $ap$ is invertible in $pA^{**}p$ with inverse $x$ satisfying $\|x\| \leq \beta^{-1}$. Compressing $(v^*bva - a)$ by $p$ and multiplying by $x$, we have $\|pv^*bvp - p\| \leq \delta\beta^{-1}$. Thus

$$\|pv^*(b - \gamma)_+vp - p\| \leq \|(b - \gamma)_+ - b\| + \|pv^*bvp - p\| \leq \gamma + \delta\beta^{-1},$$

and so

$$pv^*(b - \gamma)_+vp \geq (1 - (\gamma + \delta\beta^{-1}))p.$$

As $p$ acts as a unit on $(a - \beta)_+$, we have

$$\begin{aligned}
(a - \beta)_+ &= (a - \beta)_+^{1/2} p (a - \beta)_+^{1/2} \\
&\leq (1 - (\gamma + \delta\beta^{-1}))^{-1}(a - \beta)_+^{1/2} pv^*(b - \gamma)_+vp(a - \beta)_+^{1/2} \\
&= (1 - (\gamma + \delta\beta^{-1}))^{-1}(a - \beta)_+^{1/2} v^*(b - \gamma)_+v(a - \beta)_+^{1/2}.
\end{aligned}$$

Thus $(a - \beta)_+ \precsim (b - \gamma)_+$. $\qquad\square$

The next lemma encapsulates the fact that the element of the Cuntz semigroup represented by a very rapidly increasing sequence $(a_n)_{n=1}^\infty$ of contractions depends only on the behaviour of parts of the $a_n$ with spectrum near 1.

**Lemma 3.4.** *Let $(a_n)_{n=1}^\infty$ be a very rapidly increasing sequence in $A_1^+$. Then for each $\lambda < 1$, the sequence $(\langle (a_n - \lambda)_+ \rangle)_{n=1}^\infty$ has the property that for each $n \in \mathbb{N}$, there is $m_0 \in \mathbb{N}$ such that, for $m \geq m_0$, we have $\langle (a_n - \lambda)_+ \rangle \ll \langle (a_m - \lambda)_+ \rangle$. Furthermore,*

$$\sup_n \langle (a_n - \lambda)_+ \rangle = \sup_n \langle a_n \rangle. \tag{3-2}$$

*Proof.* Fix $n \in \mathbb{N}$ and $0 < \varepsilon < \lambda$ and take $0 < \delta$ small enough that $\lambda + \varepsilon^{-1}\delta < 1$. As $(a_n)_{n=1}^\infty$ is very rapidly increasing, there exists $m_0$ such that for $m \geq m_0$, there exists $v \in A_1$ with $\|(v^*a_mv)a_n - a_n\| < \delta$. Lemma 3.3 gives

$$\langle (a_n - \varepsilon)_+ \rangle \leq \langle (a_m - \lambda)_+ \rangle,$$

so that $\langle (a_n - \lambda)_+ \rangle \ll \langle (a_m - \lambda)_+ \rangle$, as $\varepsilon < \lambda$. This shows that $(\langle (a_r - \lambda)_+ \rangle)_{r=1}^\infty$ is upward directed and that

$$\langle (a_n - \delta)_+ \rangle \leq \sup_r \langle (a_r - \lambda)_+ \rangle,$$

for all $n$ and all $\varepsilon > 0$, from which (3-2) follows. $\qquad\square$

We can modify elements sufficiently far down a very rapidly increasing sequences with contractions so that they almost act as units for positive contractions dominated in the Cuntz semigroup by the sequence.

**Lemma 3.5.** *Let $A$ be a $C^*$-algebra.*

(1) *Suppose that $a, b \in A_1^+$ satisfy $a \precsim b$. Then for all $\varepsilon > 0$, there exists $v \in A$ with $\|v^*bva - a\| \leq \varepsilon$ and $\|v^*bv\| \leq 1$.*

(2) *Let $(a_n)_{n=1}^\infty$ be a very rapidly increasing sequence in $A_1^+$ and suppose $a \in A_1^+$ satisfies $\langle a \rangle \ll \sup\langle a_n \rangle$. Then, for every $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that, for $m \geq m_0$, there exists $v \in A_1$ with $\|(v^*a_m v)a - a\| < \varepsilon$.*

*Proof.* (1). Fix $\varepsilon > 0$ and find $r > 0$ so that $\|a^{1+r} - a\| \leq \varepsilon/2$. Now $a^r \precsim b$, so there exists $w \in A$ with $\|a^r - w^*bw\| \leq \varepsilon/4$. Thus $\|w^*bw\| \leq 1 + \varepsilon/4$, and so, writing $v = (1 + \varepsilon/4)^{-1/2}w$, we have $\|v^*bv\| \leq 1$ and $\|w^*bw - v^*bv\| \leq \varepsilon/4$. As such $\|a^r - v^*bv\| \leq \varepsilon/2$ and so

$$\|v^*bva - a\| \leq \|v^*bv - a^r\|\|a\| + \|a^{1+r} - a\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as claimed.

(2) As $\langle a \rangle \ll \sup\langle a_n \rangle$, there exists some $m_1 \in \mathbb{N}$ with $a \precsim a_{m_1} \sim a_{m_1}^2$. Fix $\varepsilon > 0$ and by part (1), find $w \in A$ with $\|(w^*a_{m_1}^2 w)a - a\| < \varepsilon/2$ and $\|w^*a_{m_1}^2 w\| \leq 1$. Now set $\varepsilon' = \varepsilon/(2\|w\|)$ and, as $(a_n)_{n=1}^\infty$ is very rapidly increasing, find some $m_0 > m_1$ such that for $m \geq m_0$ there exists $t \in A_1$ with $\|(t^*a_m t)a_{m_1} - a_{m_1}\| \leq \varepsilon'$. Given such $m$ and $t$, we have

$$\|(w^*a_{m_1}t^*a_m ta_{m_1}w)a - a\| \leq \|w^*a_{m_1}\|\|(t^*a_m t)a_{m_1} - a_{m_1}\|\|w\|\|a\| + \|(w^*a_{m_1}^2 w)a - a\|$$

$$\leq \|w\|\varepsilon' + \tfrac{1}{2}\varepsilon = \varepsilon,$$

as $\|w^*a_{m_1}\| \leq 1$. Thus we can take $v = ta_{m_1}w \in A_1$.  $\square$

It follows immediately from part (2) above that two very rapidly increasing sequences representing the same element of the Cuntz semigroup can be intertwined to a single very rapidly increasing sequence.

**Proposition 3.6.** *Let $(a_n)_{n=1}^\infty$, $(a_n')_{n=1}^\infty$ be very rapidly increasing sequences in a $C^*$-algebra $A$ representing the same element $x \in \mathrm{Cu}(A)$. Then these sequences can be intertwined after telescoping to form a very rapidly increasing sequence which also represents $x$, i.e., there exists $m_1 < m_2 < \cdots$ and $n_1 < n_2 < \cdots$ such that $(a_{m_1}, a_{n_1}', a_{m_2}, a_{n_2}', \ldots)$ is a very rapidly increasing sequence.*

Given a rapidly increasing sequence in $A_1^+$, we can use the functions $g_{\beta,\gamma}$ from (3-1) to push the spectrum of the elements of the sequence out to 1 and extract a very rapidly increasing sequence representing the same element of the Cuntz semigroup.

**Lemma 3.7.** *Let $A$ be a $C^*$-algebra and $(a_n)_{n=1}^\infty$ be a rapidly increasing sequence in $A_1^+$. There exists a sequence $(m_n)_{n=1}^\infty$ in $\mathbb{N}$ such that the sequence $(g_{2^{-(m_n+1)}, 2^{-m_n}}(a_n))_{n=1}^\infty$ is very rapidly increasing and*

$$\sup_n \langle g_{2^{-(m_n+1)}, 2^{-m_n}}(a_n) \rangle = \sup_n \langle a_n \rangle.$$

*In particular, every element of the scale $\Sigma(\mathrm{Cu}(A))$ can be expressed as a very rapidly increasing sequence of elements from $A_1^+$.*

*Proof.* We will construct the $m_n$ so that $a_{n-1} \precsim g_{2^{-(m_n+1)}, 2^{-m_n}}(a_n)$ and for each $1 \leq r < n$, there exists $v \in A_1$ with

$$\|(v^* g_{2^{-(m_n+1)}, 2^{-m_n}}(a_n)v) g_{2^{-(m_r+1)}, 2^{-m_r}}(a_r) - g_{2^{-(m_r+1)}, 2^{-m_r}}(a_r)\| < 2^{-n}.$$

Fix $n \in \mathbb{N}$ and suppose $m_1, \ldots, m_{n-1}$ have been constructed with these properties. As $(g_{2^{-(m+1)}, 2^{-m}}(a_n))_{m=1}^\infty$ is a very rapidly increasing sequence representing $\langle a_n \rangle$, and $\langle a_{n-1} \rangle \ll \langle a_n \rangle$, there exists $\widetilde{m}_n$ such that $\langle a_{n-1} \rangle \ll \langle (g_{2^{-(m+1)}, 2^{-m}}(a_n)) \rangle$ for $m \geq \widetilde{m}_n$. Further, for $1 \leq r < n$,

$$\langle g_{2^{-(m_r+1)}, 2^{-m_r}}(a_r) \rangle \ll \langle a_r \rangle \ll \sup_m \langle (g_{2^{-(m+1)}, 2^{-m}}(a_n)) \rangle$$

and so the required $m_n$ can be found using part (2) of Lemma 3.5.

The resulting sequence $(g_{2^{-(m_n+1)}, 2^{-m_n}}(a_n))_{n=1}^\infty$ is very rapidly increasing by construction. Since $a_{n-1} \precsim g_{2^{-(m_n+1)}, 2^{-m_n}}(a_n) \precsim a_n$ for all $n$, we have $\sup_n \langle g_{2^{-(m_n+1)}, 2^{-m_n}}(a_n) \rangle = \sup_n \langle a_n \rangle$. $\qquad\square$

We now consider the situation where we have two close $C^*$-algebras acting on the same Hilbert space. The following lemma ensures that we can produce a well defined map between the Cuntz semigroups.

**Lemma 3.8.** *Let $A$, $B$ be $C^*$-algebras acting on the same Hilbert space and suppose that $a \in A_1^+$ and $b \in B_1^+$ satisfy $\|a - b\| < 2\alpha$ for some $\alpha < 1/27$. Suppose that $(a_n)_{n=1}^\infty$ is a very rapidly increasing sequence in $A_1^+$ with $\langle a \rangle \ll \sup\langle a_n \rangle$. Then, there exists $n_0 \in \mathbb{N}$ with the property that for $n \geq n_0$ and $b_n \in B_1^+$ with $\|b_n - a_n\| < 2\alpha$, we have*

$$\langle (b - 18\alpha)_+ \rangle \ll \langle (b_n - \gamma)_+ \rangle \ll \langle (b_n - 18\alpha)_+ \rangle$$

*in $\mathrm{Cu}(B)$, for all $\gamma$ with $18\alpha < \gamma < 2/3$.*

*Proof.* Fix $\gamma$ with $2/3 > \gamma > 18\alpha$ (which is possible as $\alpha < 1/27$). By taking $\varepsilon = 2\alpha - \|a - b\|$ in Lemma 3.5(2), there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, there exists $v \in A_1$ with

$$\|(v^* a_n v)a - a\| < 2\alpha - \|a - b\|.$$

Fix such an $n \geq n_0$ and $v \in A_1$, and take $b_n \in B_1^+$ with $\|a_n - b_n\| < 2\alpha$ and choose some $w \in B_1$ with $\|w - v\| < \alpha$. We have

$$\|w^* b_n w - v^* a_n v\| \leq 2\|w - v\| + \|b_n - a_n\| < 4\alpha$$

so that

$$\begin{aligned}
\|(w^* b_n w)b - b\| &\leq \|((w^* b_n w) - 1)(b - a)\| + \|(w^* b_n w - v^* a_n v)a\| + \|(v^* a_n v)a - a\| \\
&\leq \|b - a\| + 4\alpha + \|(v^* a_n v)a - a\| \\
&\leq 6\alpha.
\end{aligned}$$

Taking $\delta = 6\alpha$, $\beta = 18\alpha$ and $2/3 > \gamma' > \gamma > 18\alpha$ so that $\gamma' + \delta\beta^{-1} < 1$, Lemma 3.3 gives

$$\langle (b - 18\alpha)_+ \rangle \leq \langle (b_n - \gamma')_+ \rangle \ll \langle (b_n - \gamma)_+ \rangle \ll \langle (b_n - 18\alpha)_+ \rangle. \qquad\square$$

**Proposition 3.9.** Let $A$, $B$ be $C^*$-algebras acting on the same Hilbert space with the property that there exists $\alpha < 1/27$ such that for each $a \in A_1$ there exists $b \in B_1$ with $\|a - b\| < \alpha$. Then there is a well defined, order-preserving map $\Phi : \Sigma(\mathrm{Cu}(A)) \to \Sigma(\mathrm{Cu}(B))$ given by

$$\Phi(\sup\langle a_n \rangle) = \sup\langle (b_n - 18\alpha)_+ \rangle,$$

whenever $(a_n)_{n=1}^{\infty}$ is a very rapidly increasing sequence in $A_1^+$ and $b_n \in B_1^+$ have $\|a_n - b_n\| < 2\alpha$ for all $n \in \mathbb{N}$. Moreover, if $d(A, B) < \alpha$ for $\alpha < 1/42$, then $\Phi$ is a bijection with inverse $\Psi : \Sigma(\mathrm{Cu}(B)) \to \Sigma(\mathrm{Cu}(A))$ obtained from interchanging the roles of $A$ and $B$ in the definition of $\Phi$.

*Proof.* Suppose first that $\alpha < 1/27$. To see that $\Phi$ is well defined, we apply Lemma 3.8 repeatedly. Firstly, given a very rapidly increasing sequence $(a_n)_{n=1}^{\infty}$ in $A_1^+$ representing an element $x \in \Sigma(\mathrm{Cu}(A))$ and a sequence $(b_n)_{n=1}^{\infty}$ in $B_1^+$ with $\|a_n - b_n\| < 2\alpha$ for all $n$, Lemma 3.8 shows that the sequence $\left(\langle (b_n - 18\alpha)_+ \rangle\right)_{n=1}^{\infty}$ is upward directed. Indeed, for each $m$, take $a = a_m$ and $b = b_m$ in Lemma 3.8 so that $\langle (b_m - 18\alpha)_+ \rangle \ll \langle (b_n - 18\alpha)_+ \rangle$ for all sufficiently large $n$. As such, $\sup_n \langle (b_n - 18\alpha)_+ \rangle$ exists in $\Sigma(\mathrm{Cu}(B))$.

Secondly, this supremum does not depend on the choice of $(b_n)_{n=1}^{\infty}$. Consider two sequences $(b_n)_{n=1}^{\infty}$ and $(b'_n)_{n=1}^{\infty}$ satisfying $\|b_n - a_n\| < 2\alpha$ and $\|b'_n - a_n\| < 2\alpha$ for all $n$. For each $n$, Lemma 3.8 shows that there exists $m_0$ such that, for $m \geq m_0$, we have

$$\langle (b_n - 18\alpha)_+ \rangle \ll \langle (b'_m - 18\alpha)_+ \rangle, \quad \text{and} \quad \langle (b'_n - 18\alpha)_+ \rangle \ll \langle (b_m - 18\alpha)_+ \rangle.$$

Thus $\sup_n \langle (b_n - 18\alpha)_+ \rangle = \sup_n \langle (b'_n - 18\alpha)_+ \rangle$.

Thirdly, for two very rapidly increasing sequences $(a'_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$ in $A_1^+$ with $\sup_n \langle a'_n \rangle \leq \sup_n \langle a_n \rangle$, and sequences $(b'_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ in $B_1^+$ with $\|b'_n - a'_n\|$, $\|b_n - a_n\| < 2\alpha$ for all $n$, Lemma 3.8 gives $\sup_n \langle (b'_n - 18\alpha)_+ \rangle \leq \sup_n \langle (b_n - 18\alpha)_+ \rangle$. In particular, when $(a'_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$ represent the same element of $\Sigma(\mathrm{Cu}(A))$, this shows that the map $\Phi$ given in the proposition is well defined. In general, this third observation also shows that $\Phi$ is order-preserving.

Now suppose that $d(A, B) < \alpha < 1/42$ and let $\Psi : \Sigma(\mathrm{Cu}(B)) \to \Sigma(\mathrm{Cu}(A))$ be the order-preserving map obtained by interchanging the roles of $A$ and $B$ above. Take $x \in \Sigma(\mathrm{Cu}(A))$ and fix a very rapidly increasing sequence $(a_n)_{n=1}^{\infty}$ in $A_1^+$ representing $x$. Fix a sequence $(b_n)_{n=1}^{\infty}$ in $B_1^+$ with $\|a_n - b_n\| < 2\alpha$ for all $n$. For each $n$, Lemma 3.8 gives $m > n$ with

$$\langle (b_n - 18\alpha)_+ \rangle \ll \langle (b_m - \gamma)_+ \rangle \ll \langle (b_m - 18\alpha)_+ \rangle,$$

for any $\gamma$ with $18\alpha < \gamma < 2/3$. Passing to a subsequence if necessary, we can assume this holds for $m = n + 1$ and hence $((b_n - 18\alpha)_+)_{n=1}^{\infty}$ is a rapidly increasing sequence. By Lemma 3.7, there exists a sequence $(m_n)_{n=1}^{\infty}$ in $\mathbb{N}$ so that, defining $b'_n = g_{2^{-(m_n+1)}, 2^{-m_n}}((b_n - 18\alpha)_+)$, we have a very rapidly increasing sequence $(b'_n)_{n=1}^{\infty}$ in $B_1^+$ with

$$\sup_n \langle b'_n \rangle = \sup_n \langle (b_n - 18\alpha)_+ \rangle = \Phi(x).$$

Choose a sequence $(c_n)_{n=1}^{\infty}$ in $A_1^+$ with $\|c_n - b'_n\| < 2\alpha$ for each $n$ so that the definition of $\Psi$ gives $\Psi(\Phi(x)) = \sup\langle (c_n - 18\alpha)_+ \rangle$. We now show that $x \leq \Psi(\Phi(x)) \leq x$.

Fix $0 < \beta < 1$ with $\alpha(18 + 24\beta^{-1}) < 1$. This choice can be made as $\alpha < 1/42$. Fix $n \in \mathbb{N}$. As $\langle (b_n - 18\alpha)_+ \rangle \ll \sup_r \langle b'_r \rangle$, Lemma 3.5 (2) provides $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, there exists $w \in B_1$ with

$$\|(w^* b'_m w)(b_n - 18\alpha)_+ - (b_n - 18\alpha)_+ \| < 2\alpha - \|a_n - b_n\|. \tag{3-3}$$

Take $v \in A_1$ with $\|v - w\| < \alpha$. Then

$$\|(v^* c_m v - w^* b'_m w)\| \leq \|c_m - b'_m\| + 2\|v - w\| < 4\alpha. \tag{3-4}$$

Combining the estimates (3-3), (3-4) and noting that $\|w^* b'_m w - 1\| \leq 1$ as $w$ is a contraction, gives

$$\|(v^* c_m v) a_n - a_n\| \leq \|((v^* c_m v) - 1)(a_n - b_n)\| + \|(v^* c_m v - w^* b'_m w) b_n\|$$
$$+ \|(w^* b'_m w - 1)(b_n - (b_n - 18\alpha)_+)\| + \|(w^* b'_m w)(b_n - 18\alpha)_+ - (b_n - 18\alpha)_+\|$$
$$< \|a_n - b_n\| + 4\alpha + 18\alpha + (2\alpha - \|a_n - b_n\|) = 24\alpha.$$

Taking $\gamma = 18\alpha$, $\delta = 24\alpha$, Lemma 3.3 gives $\langle (a_n - \beta)_+ \rangle \leq \langle (c_m - 18\alpha)_+ \rangle \leq \Psi(\Phi(x))$. As $n$ was arbitrary, $\sup_n \langle (a_n - \beta)_+ \rangle \leq \Psi(\Phi(x))$. As $\beta < 1$, Lemma 3.4 gives $\sup_n \langle (a_n - \beta)_+ \rangle = \sup_n \langle a_n \rangle = x$, so that $x \leq \Psi(\Phi(x))$.

For the reverse inequality, fix $k \in \mathbb{N}$ and apply Lemma 3.8 (with the roles of $A$ and $B$ reversed, $b'_k$ playing the role of $a$, $(b'_n)_{n=1}^\infty$ the role of $(a_n)$) and $\gamma = 1/2$ (so $18\alpha < \gamma < 2/3$) to find some $n \in \mathbb{N}$ such that $\langle (c_k - 18\alpha)_+ \rangle \leq \langle (c_n - 1/2)_+ \rangle$. Now, just as in the proof of Lemma 3.8, there is $z \in B_1$ with $\|(z^* b_{n+1} z) b_n - b_n\| \leq 6\alpha$. Let $p \in B^{**}$ be the spectral projection of $b_n$ for $[18\alpha, 1]$, so that, just as in the proof of Lemma 3.3, $\|z^* b_{n+1} z p - p\| \leq 1/3$. Fix $y \in A_1$ with $\|y - z\| \leq \alpha$. Since $p$ is a unit for $(b_n - 18\alpha)_+$, it is a unit for $b'_n = g_{2^{-(m_n+1)}, 2^{-m_n}}((b_n - 18\alpha)_+)$, giving the estimate

$$\|y^* a_{n+1} y c_n - c_n\| \leq \|y^* a_{n+1} y c_n - z^* b_{n+1} z c_n\| + \|(z^* b_{n+1} z - 1)(c_n - b'_n)\| + \|(z^* b_{n+1} z) b'_n - b'_n\|$$
$$\leq 4\alpha + 2\alpha + \tfrac{1}{3} = 6\alpha + \tfrac{1}{3}.$$

Take $\delta = 6\alpha + 1/3$, $\beta = 1/2$ and $\gamma = 0$, so that $\gamma + \beta^{-1}\delta = 2/3 + 12\alpha < 1$. Thus Lemma 3.3 gives

$$\langle (c_n - 1/2)_+ \rangle \leq \langle a_{n+1} \rangle,$$

and hence

$$\langle (c_k - 18\alpha)_+ \rangle \leq \langle a_{n+1} \rangle \leq x.$$

Taking the supremum over $k$ gives $\Psi(\Phi(x)) \leq x$. $\qquad\qquad\square$

**Theorem 3.10.** *Let $A$ and $B$ be $C^*$-algebras acting on the same Hilbert space with $d_{cb}(A, B) < \alpha < 1/42$. Then $(\mathrm{Cu}(A), \Sigma(\mathrm{Cu}(A)))$ is isomorphic to $(\mathrm{Cu}(B), \Sigma(\mathrm{Cu}(B)))$. Moreover, an order-preserving isomorphism $\Phi : \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ can be defined by $\Phi(\sup\langle a_n \rangle) = \sup\langle (b_n - 18\alpha)_+ \rangle$, whenever $(a_n)_{n=1}^\infty$ is a very rapidly increasing sequence in $(A \otimes \mathcal{K})_1^+$ and $(b_n)_{n=1}^\infty$ is a sequence in $(B \otimes \mathcal{K})_1^+$ with $\|a_n - b_n\| < 2\alpha$ for all $n \in \mathbb{N}$.*

*Proof.* We have $d(A \otimes \mathcal{K}, B \otimes \mathcal{K}) < \alpha < 1/42$. By definition,

$$\Sigma(\mathrm{Cu}(A \otimes \mathcal{K})) = \mathrm{Cu}(A) \quad \text{and} \quad \Sigma(\mathrm{Cu}(A \otimes \mathcal{K})) = \mathrm{Cu}(B).$$

By applying Proposition 3.9 to $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$, we obtain mutually inverse order-preserving bijections $\Phi : \mathrm{Cu}(A \otimes \mathcal{K}) \to \mathrm{Cu}(B \otimes \mathcal{K})$ and $\Psi : \mathrm{Cu}(B \otimes \mathcal{K}) \to \mathrm{Cu}(A \otimes \mathcal{K})$, given by $\Phi(\sup\langle a_n \rangle) = \sup\langle (b_n - 18\alpha)_+ \rangle$, whenever $(a_n)_{n=1}^{\infty}$ is a very rapidly increasing sequence in $(A \otimes \mathcal{K})_1^+$ and $(b_n)_{n=1}^{\infty}$ is a sequence in $(B \otimes \mathcal{K})_1^+$ with $\|a_n - b_n\| < 2\alpha$ for all $n \in \mathbb{N}$. Given a very rapidly increasing sequence $(a_n)_{n=1}^{\infty}$ in $A_1^+$ representing an element $x \in \Sigma(\mathrm{Cu}(A))$, we can find a sequence $(b_n)_{n=1}^{\infty}$ in $B_1^+$ with $\|a_n - b_n\| < 2\alpha$, so that $\Phi(x) = \sup\langle (b_n - 18\alpha)_+ \rangle \in \Sigma(\mathrm{Cu}(B))$. Since $\Phi$ and $\Phi^{-1}$ are order-preserving bijections, they also preserve the relation $\ll$ of compact containment and suprema of countable upward directed sets, as these notions are determined by the order relation $\leq$. Further, taking $a_n = b_n = 0$ for all $n$ shows that $\Phi(0_{\mathrm{Cu}(A)}) = 0_{\mathrm{Cu}(B)}$. Finally, note that $\Phi$ preserves addition: given very rapidly increasing sequences $(a_n)_{n=1}^{\infty}$ and $(a_n')_{n=1}^{\infty}$ in $(A \otimes \mathcal{K})_1^+$ representing $x$ and $y$ in $\mathrm{Cu}(A)$, the sequence $(a_n \oplus a_n')$ is very rapidly increasing in $M_2(A \otimes \mathcal{K}) \cong A \otimes \mathcal{K}$. If $(b_n)_{n=1}^{\infty}, (b_n')_{n=1}^{\infty}$ have $\|a_n - b_n\|, \|a_n' - b_n'\| < 2\alpha$ for all $n$, then

$$\|(a_n \oplus a_n') - (b_n \oplus b_n')\| < 2\alpha,$$

and has

$$((b_n \oplus b_n') - 18\alpha)_+ = (b_n - 18\alpha)_+ \oplus (b_n' - 18\alpha)_+.$$

In this way we see that $\Phi(x + y) = \Phi(x) + \Phi(y)$. $\qquad\square$

In particular properties of a $C^*$-algebra which are determined by its Cuntz semigroup transfer to completely close $C^*$-algebras. One of the most notable of these properties is that of strict comparison.

**Corollary 3.11.** *Let $A$ and $B$ be $C^*$-algebras acting on the same Hilbert space with $d_{\mathrm{cb}}(A, B) < 1/42$ and suppose that $A$ has strict comparison. Then so too does $B$.*

## 4. $\mathscr{L}$-stability and automatic complete closeness

Given a $C^*$-algebra $A \subset \mathcal{B}(\mathcal{H})$, [Cameron et al. 2013] shows that the metrics $d(A, \cdot)$ and $d_{\mathrm{cb}}(A, \cdot)$ are equivalent if and only if $A$ has a positive answer to Kadison's [1955] similarity problem. The most useful reformulation of the similarity property for working with close $C^*$-algebras is due to Christensen [1982, Theorem 3.1] and Kirchberg [1996]. Combining these, it follows that a $C^*$-algebra $A$ has a positive answer to the similarity problem if and only if $A$ has Christensen's [1980] property $D_k$ for some $k$.

**Definition 4.1.** Given an operator $T \in \mathcal{B}(\mathcal{H})$, we write $\mathrm{ad}(T)$ for the derivation $\mathrm{ad}(T)(x) = xT - Tx$. A $C^*$-algebra $A$ has *property $D_k$* for some $k > 0$ if, for every nondegenerate representation $\pi : A \to \mathcal{B}(\mathcal{H})$, the inequality

$$d(T, \pi(A)') \leq k\|\mathrm{ad}(T)|_{\pi(A)}\| \tag{4-1}$$

holds for all $T \in \mathcal{B}(\mathcal{H})$. A von Neumann algebra $A$ is said to have the property $D_k^*$ if the inequality (4-1) holds for all unital normal representations $\pi$ on $\mathcal{H}$ and all $T \in \mathcal{B}(\mathcal{H})$.

By taking weak*-limit points, it follows that if $A$ is a weak*-dense $C^*$-subalgebra of a von Neumann algebra $\mathcal{M}$ and $A$ has property $D_k$, then $\mathcal{M}$ has property $D_k^*$.

That property $D_k$ converts near containments to completely bounded near containments originates in [Christensen 1980, Theorem 3.1]. The version we give below improves on the bounds $\gamma' = 6k\gamma$ from there and $\gamma' = (1 + \gamma)^{2k} - 1$ from [Christensen et al. 2010, Corollary 2.12].

**Proposition 4.2.** *Suppose that $A$ has property $D_k$ for some $k > 0$. Then for $\gamma > 0$, every near inclusion $A \subseteq_\gamma B$ (or $A \subset_\gamma B$) with $A$ and $B$ acting nondegenerately on the same Hilbert space, gives rise to a completely bounded near inclusion $A \subseteq_{\mathrm{cb}, \gamma'} B$ (or $A \subset_{\mathrm{cb}, \gamma'}$), where $\gamma' = 2k\gamma$.*

*Proof.* Suppose $A \subseteq_\gamma B$ is a near inclusion of $C^*$-algebras acting nondegenerately on $\mathcal{H}$ and fix $n \in \mathbb{N}$. Let $C = C^*(A, B)$ and let $\pi : C \to \mathcal{B}(\mathcal{H})$ be the universal representation of $C$. Then $\pi(A)''$ has property $D_k^*$ so that $\pi(A)'' \subseteq_{\mathrm{cb}, 2k\gamma} \pi(B)''$, by [Cameron et al. $\geq$ 2014, Proposition 2.2.4]. By definition, for $n \in \mathbb{N}$ we have $\pi(A)'' \otimes M_n \subseteq_{2k\gamma} \pi(B)'' \otimes M_n$. As $\pi$ is the universal representation of $C$, the Hahn–Banach argument used to deduce [Christensen 1980, equation (3)] from [ibid., equation (2)] gives $A \otimes M_n \subseteq_{2k\gamma} B \otimes M_n$, as required. The result when we work with strict near inclusions $A \subset_\gamma B$ follows immediately. $\square$

$C^*$-algebras with no bounded traces (such as stable algebras) were shown to have the similarity property in [Haagerup 1983]. Using the property $D_k$ version of this fact, the previous proposition gives automatic complete closeness when one algebra has no bounded traces. The argument below which transfers the absence of bounded traces to a nearby $C^*$-algebra essentially goes back to [Kadison and Kastler 1972, Lemma 9]. We use more recent results in order to get better estimates.

**Corollary 4.3.** *Suppose that $A$ and $B$ are $C^*$-algebras which act nondegenerately on the same Hilbert space and satisfy $d(A, B) < \gamma$ for $\gamma < (2 + 6\sqrt{2})^{-1}$. Suppose that $A$ has no bounded traces (for example if $A$ is stable). Then $B$ has no bounded traces, and therefore $A \subset_{\mathrm{cb}, 3\gamma} B$, $B \subset_{\mathrm{cb}, 3\gamma} A$ and $d_{\mathrm{cb}}(A, B) < 6\gamma$.*

*Proof.* Suppose $d(A, B) < (2 + 6\sqrt{2})^{-1}$ and $\tau : B \to \mathbb{C}$ is a bounded trace. Let $\pi : B \to \mathcal{B}(\mathcal{H})$ be the GNS-representation of $B$ corresponding to $\tau$. Then there is a larger Hilbert space $\tilde{\mathcal{H}}$ and a representation $\tilde{\pi} : C^*(A, B) \to \mathcal{B}(\tilde{\mathcal{H}})$ such that $\pi$ is a direct summand of $\tilde{\pi}|_B$. That is, the projection $p$ from $\tilde{\mathcal{H}}$ onto $\mathcal{H}$ is central in $\tilde{\pi}(B)$ and $\pi(b) = p\tilde{\pi}(b)p$ for all $b \in B$. Then, by [Kadison and Kastler 1972, Lemma 5], we have $d(\tilde{\pi}(A)'', \tilde{\pi}(B)'') \leq d(A, B)$, and hence there is a projection $q \in \tilde{\pi}(A)''$ with $\|p - q\| \leq \gamma/\sqrt{2}$ by [Khoshkam 1984, Lemma 1.10(ii)]. If $q$ is an infinite projection in $\tilde{\pi}(A)''$, then as $d(A, B) < (2 + 6\sqrt{2})^{-1}$, one can follow the argument of [Christensen et al. 2010, Lemma 6.1] (using the estimate $\|p - q\| < \gamma/\sqrt{2}$ in place of $\|p - q\| < 2\gamma$) to see that $p$ is infinite in $\tilde{\pi}(B)''$, giving a contradiction. If $q$ is finite, then $q\tilde{\pi}(A)''q$ has a finite trace $\rho$ and $\rho \circ \tilde{\pi}|_A$ defines a bounded trace on $A$, and again we have a contradiction. Thus $B$ has no bounded traces.

Theorem 2.4 of [Christensen 1977] shows that a properly infinite von Neumann algebra has property $D_{3/2}^*$. As such, every $C^*$-algebra with no bounded traces has property $D_{3/2}$. Since $A$ and $B$ both have property $D_{3/2}$, Proposition 4.2 gives $A \subset_{\mathrm{cb}, 3\gamma} B$ and $B \subset_{\mathrm{cb}, 3\gamma} A$, whence $d_{\mathrm{cb}}(A, B) < 6\gamma$. $\square$

**Corollary 4.4.** *Suppose that $A$ and $B$ are $C^*$-algebras which act nondegenerately on the same Hilbert space and satisfy $d(A, B) < 1/252$ and suppose that $A$ has no bounded traces (for example, when $A$ is stable). Then*

$$\big(\mathrm{Cu}(A), \Sigma(\mathrm{Cu}(A))\big) \cong \big(\mathrm{Cu}(B), \Sigma(\mathrm{Cu}(B))\big).$$

*Proof.* Combine Corollary 4.3 with Theorem 3.10, noting that $6d(A, B) < 1/42$. ☐

We can use the Cuntz semigroup to show that stability transfers to close $C^*$-algebras provided one algebra has stable rank one. To detect stability for a $\sigma$-unital $C^*$-algebra we use the following criterion from [Ortega et al. 2012, Lemma 5.4] which reformulates the earlier characterisation from [Hjelmborg and Rørdam 1998].

**Lemma 4.5.** *Let $A$ be a $\sigma$-unital $C^*$-algebra and let $c \in A$ be a strictly positive element. Then, $A$ is stable if and only if for every $\epsilon > 0$, there is $b \in A^+$ such that $(c - \epsilon)_+ \perp b$ and $(c - \epsilon)_+ \precsim b$.*

Following [Robert and Santiago 2010], we say that a $C^*$-algebra $A$ has *weak cancellation* provided $\mathrm{Cu}(A)$ satisfies the property that $x + z \ll y + z$ implies $x \leq y$. It was proved in [Rørdam and Winter 2010, Theorem 4.3] that if $A$ has stable rank one, then $W(A)$ has the property defining weak cancellation. When $A$ has stable rank one, so too does $A \otimes \mathcal{K}$ [Rieffel 1983, Theorem 3.6], and so $A$ has weak cancellation.

**Lemma 4.6.** *Let $A$ be a $\sigma$-unital $C^*$-algebra with weak cancellation. Then $A$ is stable if and only if $\mathrm{Cu}(A) = \Sigma(\mathrm{Cu}(A))$.*

*Proof.* If $A$ is stable, we have $\Sigma(\mathrm{Cu}(A)) = \mathrm{Cu}(A)$. Indeed, given $n \in \mathbb{N}$, choose an automorphism $\theta_n : \mathcal{K} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ with $\theta_n(\mathcal{K} \otimes e_{11}) = K \otimes M_n$ and let $\psi : A \to A \otimes \mathcal{K}$ be an isomorphism. Then $(\psi^{-1} \otimes \mathrm{id}_{\mathcal{K}})(\mathrm{id}_A \otimes \theta_n)(\psi \otimes \mathrm{id}_{\mathcal{K}})$ is an automorphism of $A \otimes \mathcal{K}$ which maps $A \otimes e_{11}$ onto $A \otimes M_n$. In this way the class of a positive element in $A \otimes M_n$ lies in the scale $\Sigma(\mathrm{Cu}(A))$. For $x \in (A \otimes \mathcal{K})_+$ and $\varepsilon > 0$, we have $(x - \varepsilon)_+ \in \bigcup_{n=1}^{\infty}(A \otimes M_n)$, and hence $\langle(x - \varepsilon)_+\rangle \in \Sigma(\mathrm{Cu}(A))$. Since the scale is defined to be closed under suprema, it follows that $\mathrm{Cu}(A) = \Sigma(\mathrm{Cu}(A))$.

Conversely, let $c \in A$ be a strictly positive element so that $\Sigma(\mathrm{Cu}(A)) = \{x \in \mathrm{Cu}(A) : x \leq \langle c \rangle\}$ and let $\epsilon > 0$ be given. The hypothesis ensures that $2\langle c \rangle \leq \langle c \rangle$, and so we can find $\delta > 0$ such that $2\langle(c - \epsilon/4)_+\rangle \ll \langle(c - \delta)_+\rangle$. Now write

$$(c - \delta)_+ = (c - \delta)_+ g_{\epsilon/2, \epsilon}(c) + (c - \delta)_+(1_{M(A)} - g_{\epsilon/2, \epsilon}(c)),$$

and observe that

$$\langle(c - \delta)_+ g_{\epsilon/2, \epsilon}(c)\rangle \leq \langle g_{\epsilon/2, \epsilon}(c)\rangle = \langle(c - \epsilon/2)_+\rangle \ll \langle(c - \epsilon/4)_+\rangle.$$

We now have that

$$2\langle(c - \epsilon/4)_+\rangle \ll \langle(c - \delta)_+\rangle \leq \langle(c - \epsilon/2)_+\rangle + \langle(c - \delta)_+(1_{M(A)} - g_{\epsilon/2, \epsilon}(c))\rangle$$

and so weak cancellation enables us to conclude that

$$\langle(c - \epsilon/4)_+\rangle \leq \langle(c - \delta)_+(1 - g_{\epsilon/2, \epsilon}(c))\rangle.$$

Let $b = (c - \delta)_+(1 - g_{\epsilon/2, \epsilon}(c))$. It is clear that $b \perp (c - \epsilon)_+$ and that $(c - \epsilon)_+ \leq (c - \epsilon/4)_+ \precsim b$. Thus we may invoke Lemma 4.5 to conclude that $A$ is stable. ☐

**Theorem 4.7.** *Let $A$ and $B$ be $\sigma$-unital $C^*$-algebras with $A$ stable and $d(A, B) < 1/252$ and suppose either $A$ or $B$ has stable rank one. Then $B$ is stable.*

*Proof.* By Corollary 4.4, we have an isomorphism

$$(\mathrm{Cu}(A), \Sigma(\mathrm{Cu}(A)) \cong (\mathrm{Cu}(B), \Sigma(\mathrm{Cu}(B)).$$

Since $A$ is stable $\mathrm{Cu}(A) = \Sigma(\mathrm{Cu}(A))$. Our isomorphism condition now tells us that $\mathrm{Cu}(B) = \Sigma(\mathrm{Cu}(B))$. If $B$ has stable rank one, then it has weak cancellation, whereas if $A$ has stable rank one, $A$ has weak cancellation and, as weak cancellation is a property of the Cuntz semigroup, so too does $B$. The result now follows from Lemma 4.6. $\qquad\square$

We now turn to the situation in which one $C^*$-algebra is $\mathcal{Z}$-stable. Christensen [1977] shows that McDuff II$_1$ factors have property $D_{5/2}$, and hence via the estimates of [Pisier 1998], have similarity length at most 5. (In fact, McDuff factors, and more generally II$_1$ factors with Murray and von Neumann's property $\Gamma$ have length 3 [Christensen 2001], but at present we do not know how to use this fact to obtain better estimates for automatic complete closeness of close factors with property $\Gamma$.)

Analogous results have been established in a $C^*$-setting, in particular, $\mathcal{Z}$-stable $C^*$-algebras [Johanesová and Winter 2012] and $C^*$-algebras of the form $A \otimes B$, where $B$ is nuclear and has arbitrarily large unital matrix subalgebras [Pop 2004] have similarity degree (and hence length) at most 5. See also [Li and Shen 2008]. Here we show how to use the original von Neumann techniques from [Christensen 1977] to show that a class of algebras generalising both these examples have property $D_{5/2}$ (recapturing the upper bound 5 on the length). A similar result has been obtained independently by Hadwin and Li [2014, Corollary 1] working in terms of the similarity degree as opposed to property $D_k$. Once we have this $D_k$ estimate, Proposition 4.2 applies. In particular we obtain uniform estimates on the cb-distance $d_{\mathrm{cb}}(A, B)$ in terms of $d(A, B)$ when $A$ is $\mathcal{Z}$-stable.

Given a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{B}(\mathcal{H})$, write $\overline{\mathrm{co}}_{\mathcal{M}}^w(x)$ for the weak*-closed convex hull of $\{uxu^* : u \in \mathcal{U}(\mathcal{M})\}$. If $\mathcal{M}$ is injective, then by Schwartz's property P, $\overline{\mathrm{co}}_{\mathcal{M}}^w(x) \cap \mathcal{M}'$ is nonempty for all $x \in \mathcal{B}(\mathcal{H})$. Note that for a nondegenerately represented $C^*$-algebra $A \subset \mathcal{B}(\mathcal{H})$, we have $\|\mathrm{ad}(T)|_A\| = \|\mathrm{ad}(T)|_{A''}\|$. We say that an inclusion $A \subset C$ of $C^*$-algebras is nondegenerate if the inclusion map is nondegenerate.

**Proposition 4.8.** Let $C$ be a $C^*$-algebra and $A, B \subset C$ be commuting nondegenerate $C^*$-subalgebras which generate $C$. Suppose $B$ is nuclear and has no nonzero finite-dimensional representations. Then $C$ has property $D_{5/2}$, and hence similarity length at most 5.

*Proof.* Suppose $C$ is nondegenerately represented on $\mathcal{H}$ and fix $x \in \mathcal{B}(\mathcal{H})$. The nondegeneracy assumption ensures that $A$ and $B$ are nondegenerately represented on $\mathcal{H}$. Note that $C''$ has no finite type I part as $B$ has no nonzero finite-dimensional representations. Let $p$ be the central projection in $C''$ so that $C''p$ is type II$_1$ and $C''(1-p)$ is properly infinite. Fix a unital type I$_\infty$ subalgebra $\mathcal{M}_0 \subset (1-p)C''(1-p)$ and let $\mathcal{M} = (\mathcal{M}_0 \cup pB)''$ which is injective. By Schwartz's property P, there exists $y \in \overline{\mathrm{co}}_{\mathcal{M}}^w(x) \cap (\mathcal{M} \cup \{p\})'$. As in Theorems 2.3 and 2.4 of [Christensen 1977], $\|y - x\| \le \|\mathrm{ad}(x)|_{C''}\|$ and $\|\mathrm{ad}(y)|_{C''}\| \le \|\mathrm{ad}(x)|_{C''}\|$. Write $y_1 = yp$ and $y_2 = y(1-p)$. If $p \ne 1$, then the properly infinite algebra $\mathcal{M}_0$ lies in $C''(1-p) \cap \{y_2, y_2^*\}'$ and so by Corollary 2.2 of the same reference, $\|\mathrm{ad}(y_2)|_{C''(1-p)}\| = 2d(y_2, C'(1-p))$. Take $x_2 \in C''(1-p)$ with $\|x_2 - y_2\| = \|\mathrm{ad}(y_2)|_{C''(1-p)}\|/2 \le \|\mathrm{ad}(x)|_{C''}\|/2$.

If $p \neq 0$, then we argue exactly as in the proof of [Christensen 1977, Proposition 2.8] to produce first $z_1 \in A'p$ with $\|y_1 - z_1\| \leq \|\mathrm{ad}(y_1)|_{C''p}\|/2 \leq \|\mathrm{ad}(x)|_{C''}\|/2$. Continuing with Christensen's proof, as $B''p$ and $A''p$ commute, $\overline{\mathrm{co}}^w_{B''p}(z_1)$ is contained in $A'p$ and hence there exists $x_1 \in \overline{\mathrm{co}}^w_{B''p}(z_1) \cap B'p$ with

$$\|x_1 - z_1\| \leq \|\mathrm{ad}(z_1)|_{B''p}\| \leq \|\mathrm{ad}(z_1 - y_1)|_{B''p}\| \leq 2\|z_1 - y_1\| \leq \|\mathrm{ad}(x)|_{C''}\|.$$

Then

$$\|y_1 - x_1\| \leq \|y_1 - z_1\| + \|z_1 - x_1\| \leq \tfrac{3}{2}\|\mathrm{ad}(x)|_{C''}\|.$$

If $p = 0$, take $x_1 = 0$ and the same inequality holds. The element $x_1 + x_2 \in C'$ has

$$\|x - (x_1 + x_2)\| \leq \|x - y\| + \|(y_1 - x_1) + (y_2 - x_2)\|$$
$$\leq \|\mathrm{ad}(x)|_{C''}\| + \max(\|y_1 - x_2\|, \|y_2 - x_2\|) \leq \tfrac{5}{2}\|\mathrm{ad}(x)_{C''}\|.$$

Therefore $C$ has property $D_{5/2}$, and so by [Pisier 1998, Remark 4.7] has length at most 5. □

**Corollary 4.9.** *Let $A$ be a $\mathscr{Z}$-stable $C^*$-algebra. Then $A$ has property $D_{5/2}$ and length at most 5.*

The main result of [Christensen et al. 2010]—a reference we will abbreviate to [CSSW] for the remainder of this section—is that the similarity property transfers to close $C^*$-algebras. This work is carried out with estimates depending on the length and length constant of $A$, but it is equally possible to carry out this work entirely in terms of property $D_k$ so it can be applied to $\mathscr{Z}$-stable algebras. Our objective is to obtain a version of [CSSW, Corollary 4.6] replacing the hypothesis that $A$ has length at most $\ell$ and length constant at most $K$ with the formally weaker hypothesis that $A$ has property $D_k$ (if $A$ has the specified length and length constants, then it has property $D_k$ for $k = K\ell/2$, conversely if $A$ has property $D_k$, then it has length at most $\lfloor 2k \rfloor$, but a length constant estimate is not known in this case; see [Pisier 1998, Remark 4.7]). This enables us to use Corollary 4.9 obtain an isomorphism between the Cuntz semigroups of sufficiently close $C^*$-algebras when one algebra is $\mathscr{Z}$-stable. To achieve a $D_k$ version of [CSSW, Section 4], we adjust the hypotheses in Lemma 4.1, Theorem 4.2 and Theorem 4.4 of that reference in turn, starting with Lemma 4.1. We begin by isolating a technical observation.

**Lemma 4.10.** *Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful tracial state acting in standard form on $\mathscr{H}$ and let $J$ be the conjugate linear modular conjugation operator inducing an isometric antisomorphism $x \mapsto JxJ$ of $\mathscr{M}$ onto $\mathscr{M}' \cong \mathscr{M}^{\mathrm{op}}$. Suppose that $\mathscr{S}$ is another von Neumann algebra acting nondegenerately on $\mathscr{H}$ with $\mathscr{M}' \subset_\gamma \mathscr{S}$. If $\mathscr{M}$ has property $D_k^*$, then $\mathscr{M}' \subset_{\mathrm{cb},2k\gamma} \mathscr{S}$.*

*Proof.* As $J$ is isometric, $\mathscr{M} \subset_\gamma J\mathscr{S}J$, so that $\mathscr{M} \subset_{\mathrm{cb},2k\gamma} J\mathscr{S}J$ by Proposition 4.2. Now, for each $n \in \mathbb{N}$, let $J_n$ denote the isometric conjugate linear operator of component-wise complex conjugation on $\mathbb{C}^n$ so that $J \otimes J_n$ is a conjugate linear isometry on $\mathscr{H} \otimes \mathbb{C}^n$. We can conjugate the near inclusion $\mathscr{M} \otimes M_n \subset_{2k\gamma} J\mathscr{S}J \otimes M_n$ by $J \otimes J_n$ to obtain $\mathscr{M}' \otimes M_n \subset_{2k\gamma} \mathscr{S} \otimes M_n$, as required. □

The next lemma is the modification of [CSSW, Lemma 4.1]. The expression for $\beta$ below is a slight improvement over that of the original.

**Lemma 4.11.** *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras of type II$_1$ faithfully and nondegenerately represented on $\mathcal{H}$ with common centre Z which admits a faithful state. Suppose $d(\mathcal{M}, \mathcal{N}) = \alpha$ and $\mathcal{M}$ has property $D_k^*$. If $\alpha$ satisfies*

$$24(12\sqrt{2}k + 4k + 1)\alpha < \tfrac{1}{200},$$

*then $d(\mathcal{M}', \mathcal{N}') < 2\beta + 1200k\alpha(1 + \beta)$, where $\beta = 96k\alpha(600k + 1)$.*

*Proof.* This amounts to showing that the hypothesis in [CSSW, Lemma 4.1] that $\mathcal{M}$ contains an weak*-dense $C^*$-algebra $A$ of length at most $\ell$ and length constant at most $K$ can be replaced by the statement that $\mathcal{M}$ has property $D_k^*$ (and that the specified expressions on $\beta$ are valid). The hypothesis that $\mathcal{M}$ has such a weak*-dense $C^*$-algebra is initially used to see that $\mathcal{M}$ has property $D_k$ at the beginning of the lemma and then applied to a unital normal representation to obtain [CSSW, equation (4.5)]. As such property $D_k^*$ suffices for this estimate.

The other use of this hypothesis comes on p. 385 in the last paragraph of the lemma, to obtain [CSSW, equation (4.28)]. Using the notation of this paragraph, the von Neumann algebra $T_{\mathcal{M}}$ is a cutdown of $\mathcal{M}$ acting as $\mathcal{M} \otimes I_{\mathcal{G}}$ on $\mathcal{H} \otimes \mathcal{G}$ by the projection $e_{i_0,i_0}$ from the commutant of $\mathcal{M}$ on this space. Since $e_{i_0,i_0}$ is unitarily equivalent in this commutant to a projection of the form $e \otimes g_0$, where $e$ is a projection from the commutant of $\mathcal{M}$ on $\mathcal{H}$ of full central support and $g_0$ is a minimal projection in $\mathcal{B}(\mathcal{G})$, it follows that $e_{i_0,i_0}$ has full central support in the commutant of $\mathcal{M}$ on $\mathcal{H} \otimes \mathcal{G}$. As such $T_{\mathcal{M}}$ is isomorphic to $\mathcal{M}$, so has property $D_k^*$. Thus Lemma 4.10 can be applied to the near inclusion $T_{\mathcal{M}}' \subset_{48(600k\alpha+\alpha)} T_{\mathcal{N}_2}'$ from [CSSW, equation (4.25)] giving

$$T_{\mathcal{M}}' \subset_{\mathrm{cb}, 96k(600k\alpha+\alpha)} T_{\mathcal{N}_2}'.$$

It then follows that

$$T_{\mathcal{M}} \,\overline{\otimes}\, \mathcal{B}(\ell^2(\Lambda)) \subset_{96k(600k\alpha+\alpha)} T_{\mathcal{N}} \,\overline{\otimes}\, \mathcal{B}(\ell^2(\Lambda)),$$

which is precisely [CSSW, equation (4.28)] with our new estimate for $\beta$ replacing that of the original. We then deduce that $d(\mathcal{M}', \mathcal{N}') \le 2\beta + 1200k\alpha(1 + \beta)$ in just the same way that [CSSW, equation (4.30)] is obtained from [CSSW, equation (4.28)]. $\square$

Now we adjust Theorem 4.2 of [CSSW]. The resulting constant $\beta$ is obtained by taking $\alpha = 11\gamma$ in the previous lemma. Note that there is an unfortunate omission in the value of $\beta$ in Theorem 4.2 of [CSSW], which should be given by taking $\alpha = 11\gamma$ in Lemma 4.1 of [CSSW], so should be $K((1 + 316800k\gamma + 528\gamma)^\ell - 1)$; this has no knock-on consequences to Theorem 4.4 of [CSSW], where the correct value of $\beta$ is used.

**Lemma 4.12.** *Let A and B be $C^*$-algebras acting on a Hilbert space and suppose that $d(A, B) = \gamma$. Suppose A has property $D_k$ and $24(12\sqrt{2}k + 4k + 1)\gamma < 1/2200$. Then*

$$d(A', B') \le 10\gamma + 2\beta + 13200k\gamma(1 + \beta),$$

*where $\beta = 1056k(600k\gamma + \gamma)$.*

*Proof.* This amounts to replacing the hypothesis that $A$ has length at most $\ell$ and length constant at most $K$ with the condition that $A$ has property $D_k$ in Theorem 4.2 of [CSSW]. The length hypothesis on $A$

is used to show that certain $\text{II}_1$ von Neumann closures of $A$ satisfy [CSSW, Lemma 4.1], but since the weak*-closure of a $C^*$-algebra with property $D_k$ has property $D_k^*$, Lemma 4.11 can be used in place of that lemma. Note that in the proof of [CSSW, Theorem 4.2] the reference to injective von Neumann algebras having property $D_1$ is incorrect (it is an open question whether $\prod_{n=1}^{\infty} M_n$ has the similarity property). The correct statement is that these algebras have property $D_1^*$, which is all that is used. □

Finally we can convert Theorem 4.4 of [CSSW]. Note the typo in the statement of this theorem; the definition of $\tilde{k}$ should be $k/(1 - 2\eta - 2k\gamma)$ rather than $k/(1 - 2\eta - k\gamma)$. The same change should be made in Corollary 4.6 of [CSSW].

**Proposition 4.13.** Let $A$ and $B$ be $C^*$-subalgebras of some $C^*$-algebra $C$ with $d(A, B) < \gamma$ and suppose $A$ has property $D_k$. Write $\beta = 1056(600k\gamma + \gamma)$ and $\eta = 10\gamma + 2\beta + 13200k\gamma(1 + \beta)$ and suppose that

$$24(12\sqrt{2}k + 4k + 1)\gamma < \frac{1}{2200}, \quad 2\eta + 2k\gamma < 1. \tag{4-2}$$

Then $d_{\text{cb}}(A, B) \le 4\tilde{k}\gamma$, where

$$\tilde{k} = \frac{k}{1 - 2\eta - 2k\gamma}.$$

*Proof.* We check that $B$ has property $D_{\tilde{k}}$. This amounts to weakening the hypothesis of [CSSW, Theorem 4.4] in just the same way as the preceding lemmas. Applying Lemma 4.12 in place of Theorem 4.2 of [CSSW] in the proof of their Theorem 4.4 shows that under the hypotheses of this proposition $B$ has property $D_{\tilde{k}}$, where

$$\tilde{k} = \frac{k}{1 - 2\eta - 2k\gamma}.$$

This is valid as property $D_k$ descends to quotients so, following the proof of [CSSW, Theorem 4.4], the algebra $\rho(A)$ inherits property $D_k$ allowing the use of Lemma 4.12 above in place of [CSSW, Theorem 4.2]. Note that one should take care with issues of degeneracy here. In particular, the representation $\pi$ of $B$ in the proof of Theorem 4.4 of [CSSW] should be assumed nondegenerate.

Proposition 4.2 now shows that $B \subset_{\text{cb}, 2\tilde{k}} A$ and $A \subset_{\text{cb}, 2k} B$. Therefore

$$d_{\text{cb}}(A, B) \le 2\max(2\tilde{k}\gamma, 2k\gamma) = 4\tilde{k}\gamma. \qquad \square$$

**Corollary 4.14.** *Let $A$ be a $C^*$-algebra generated by two commuting nondegenerate $C^*$-subalgebras one of which is nuclear and has no finite-dimensional irreducible representations. Suppose that $A \subset \mathcal{B}(\mathcal{H})$ and $B$ is another $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ with $d(A, B) < \gamma$ for $\gamma < 1/6422957$. Then $d_{\text{cb}}(A, B) < 1/42$ and $(\text{Cu}(A), \Sigma(\text{Cu}(A))) \cong (\text{Cu}(B), \Sigma(\text{Cu}(B)))$.*

*Proof.* By Proposition 4.8, $A$ has property $D_k$ for $k = 5/2$ so in Proposition 4.13, $\beta = 1585056\gamma$ and $\eta = 3203122\gamma + 52306848000\gamma^2$, so that $2\eta + 2k\gamma < 10^{11}\gamma < 1$ for $\gamma < 10^{-11}$. The bound on $\gamma$ ensures that (4-2) holds so that Proposition 4.13 applies. Further this bound gives

$$\frac{4k\gamma}{1 - 2\eta - 2k\gamma} < \frac{1}{42},$$

and so the result follows from Proposition 4.13 and Theorem 3.10. □

In particular, $C^*$-algebras sufficiently close to $\mathfrak{Z}$-stable algebras are automatically completely close and have the Cuntz semigroup of a $\mathfrak{Z}$-stable algebra. The question of whether the property of $\mathfrak{Z}$-stability transfers to sufficiently close subalgebras raised in [Christensen et al. 2012] remains open.

**Corollary 4.15.** *Let $A$ be a $\mathfrak{Z}$-stable $C^*$-algebra and suppose that $B$ is another $C^*$-algebra acting on the same Hilbert space as $A$ with $d(A, B) < 1/6422957$. Then $d_{\mathrm{cb}}(A, B) < 1/42$, and $(\mathrm{Cu}(A), \Sigma(\mathrm{Cu}(A)))$ is isomorphic to $(\mathrm{Cu}(B), \Sigma(\mathrm{Cu}(B)))$. In particular, $B$ has the Cuntz semigroup of a $\mathfrak{Z}$-stable algebra.*

## 5. Quasitraces

In this section we use our isomorphism between the Cuntz semigroups of completely close $C^*$-algebras to give an affine homeomorphism between the lower semicontinuous quasitraces on such algebras. This isomorphism is compatible with the affine isomorphism of the trace spaces of close $C^*$-algebras constructed in [Christensen et al. 2010, Section 5].

Given a $C^*$-algebra $A$, write $T(A)$ for the cone of lower semicontinuous traces on $A$ and $QT_2(A)$ for the cone of lower semicontinuous 2-quasitraces on $A$. Precisely, a trace $\tau$ on $A$ is a linear function $\tau : A_+ \to [0, \infty]$ vanishing at 0 and satisfying the trace identity $\tau(xx^*) = \tau(x^*x)$ for all $x \in A$. A 2-quasitrace is a function $\tau : A_+ \to [0, \infty]$ vanishing at 0 which satisfies the trace identity and which is linear on commuting elements of $A_+$. Write $T_s(A)$ for the simplex of tracial states on $A$ and $QT_{2,s}(A)$ for the bounded 2-quasitraces on $A$ of norm one. Lower semicontinuous traces and 2-quasitraces on $A$ extend uniquely to lower semicontinuous traces and 2-quasitraces respectively on $A \otimes \mathcal{K}$; see [Blanchard and Kirchberg 2004, Remark 2.27(viii)].

In [Elliott et al. 2011, Section 4], Elliott, Robert and Santiago extend earlier work of Blackadar and Handelman, setting out how functionals on $\mathrm{Cu}(A)$ arise from elements of $QT_2(A)$. Precisely, a functional on $\mathrm{Cu}(A)$ is a map $f : \mathrm{Cu}(A) \to [0, \infty]$ which is additive, order-preserving, has $f(0) = 0$ and preserves the suprema of increasing sequences. Given $\tau \in QT_2(A)$, the expression $d_\tau(\langle a \rangle) = \lim_{n \to \infty} \tau(a^{1/n})$ gives a well defined functional on $\mathrm{Cu}(A)$, where we abuse notation by using $\tau$ to denote the extension of the original lower semicontinuous 2-quasitrace to $A \otimes \mathcal{K}$. Alternatively, one can define $d_\tau$ by $d_\tau(\langle a \rangle) = \lim_{n \to \infty} \tau(a_n)$, where $(a_n)_{n=1}^\infty$ is any very rapidly increasing sequence from $(A \otimes \mathcal{K})_1^+$ representing $\langle a \rangle$. Conversely, given a functional $f$ on $\mathrm{Cu}(A)$, a lower semicontinuous 2-quasitrace on $A \otimes \mathcal{K}$ (and hence on $A$) is given by $\tau_f(a) = \int_0^\infty f\big(\langle (a - t)_+ \rangle\big) \, dt$. With this notation, the assignments $\tau \mapsto d_\tau$ and $f \mapsto \tau_f$ are mutually inverse (see [ibid., Proposition 4.2]).

The topology on $QT_2(A)$ is specified by saying that a net $(\tau_i)$ in $QT_2(A)$ converges to $\tau \in QT_2(A)$ if and only if

$$\limsup_i \tau_i((a - \varepsilon)_+) \le \tau(a) \le \liminf_i \tau_i(a)$$

for all $a \in A_+$ and $\varepsilon > 0$. With this topology $QT_2(A)$ is a compact Hausdorff space [ibid., Theorem 4.4] and $T(A)$ is compact in the induced topology [ibid., Theorem 3.7]. In a similar fashion, the cone of functionals on $\mathrm{Cu}(A)$ is topologised by defining $\lambda_i \to \lambda$ if and only if

$$\limsup_i \lambda_i\big(\langle (a - \varepsilon)_+ \rangle\big) \le \lambda(\langle a \rangle) \le \liminf_i \lambda_i(\langle a \rangle)$$

for all $a \in (A \otimes \mathcal{K})_+$ and $\varepsilon > 0$. Theorem 4.4 of [ibid.] shows that the affine map $\tau \mapsto d_\tau$ is a homeomorphism between the cone $QT_2(A)$ and the cone of functionals on the Cuntz semigroup.

**Theorem 5.1.** (1) *Let $A$, $B$ be $C^*$-algebras acting nondegenerately on a Hilbert space, with $d_{\mathrm{cb}}(A, B) <$ $1/42$. The isomorphism $\Phi : \big(\mathrm{Cu}(A), \Sigma(\mathrm{Cu}(A))\big) \to \big(\mathrm{Cu}(B), \Sigma(\mathrm{Cu}(B))\big)$ given by Theorem 3.10 induces an affine homeomorphism*

$$\widehat{\Phi} : QT_2(B) \to QT_2(A)$$

*satisfying*

$$d_{\widehat{\Phi}(\tau)}(x) = d_\tau(\Phi(x)) \tag{5-1}$$

*for all $x \in \mathrm{Cu}(A)$ and $\tau \in QT_2(B)$.*

(2) *Suppose additionally that $A$ and $B$ are unital and $d_{\mathrm{cb}}(A, B) < \gamma < 1/2200$. Then $\widehat{\Phi}$ is compatible with the map $\Psi : T_s(B) \to T_s(A)$ given in Lemma 5.4 of [Christensen et al. 2010]. Precisely, for $\tau \in T_s(B)$, we have $\widehat{\Phi}(\tau) \in T_s(A) \subset QT_2(A)$ and $\widehat{\Phi}(\tau) = \Psi(\tau)$.*

*Proof.* Part (1) of the theorem is a consequence of Theorem 3.10 and [Elliott et al. 2011, Proposition 4.2]: given $\tau \in QT_2(B)$, define $\widehat{\Phi}(\tau)$ to be the lower semicontinuous 2-quasitrace induced by the functional $d_\tau \circ \Phi$ on $\mathrm{Cu}(A)$. It is immediate from the construction that the map $\widehat{\Phi}$ is affine, bijective and the identity (5-1) holds.

To show that $\widehat{\Phi}$ is continuous, we use the homeomorphism between the cone of lower semicontinuous quasitraces and functionals on the Cuntz semigroup in [Elliott et al. 2011, Theorem 4.4]. Consider a net $(\tau_i)$ in $QT_2(B)$ with $\tau_i \to \tau$. Fix $a \in A_+$, then

$$d_\tau\big(\Phi(\langle a \rangle)\big) \leq \liminf_i d_{\tau_i}\big(\Phi(\langle a \rangle)\big),$$

as $d_{\tau_i} \to d_\tau$. Now take $\varepsilon > 0$ and fix a contraction $b \in (B \otimes K)_+$ with $\Phi(\langle a \rangle) = \langle b \rangle$. As $\big(\langle (b - 1/n)_+ \rangle\big)_{n=1}^\infty$ is very rapidly increasing with supremum $\langle b \rangle$, there exists $n \in \mathbb{N}$ with $\Phi\big(\langle (a - \varepsilon)_+ \rangle\big) \leq \langle (b - 1/n)_+ \rangle$. As

$$\limsup_i d_{\tau_i}(\langle (b - \tfrac{1}{n})_+ \rangle) \leq d_\tau(\langle b \rangle),$$

it follows that

$$\limsup_i d_{\widehat{\Phi}(\tau_i)}\big(\langle (a - \varepsilon)_+ \rangle\big) \leq d_{\widehat{\Phi}(\tau)}(\langle a \rangle) \leq \liminf_i d_{\widehat{\Phi}(\tau)}(\langle a \rangle).$$

Thus $d_{\widehat{\Phi}(\tau_i)} \to d_{\widehat{\Phi}(\tau)}$ and so, using the homeomorphism between $QT_2(A)$ and functionals on $\mathrm{Cu}(A)$, we have $\widehat{\Phi}(\tau_i) \to \widehat{\Phi}(\tau)$. Therefore $\widehat{\Phi}$ is continuous, and hence a homeomorphism between $QT_2(B)$ and $QT_2(A)$.

For the second part we first need to review the construction of the map $\Psi$ from [Christensen et al. 2010], which we again abbreviate [CSSW]. Suppose $d_{\mathrm{cb}}(A, B) < \gamma < 1/2200$. Write $C = C^*(A, B)$ and let $C \subset \mathcal{B}(\mathcal{H})$ be the universal representation of $C$ so that $\mathcal{M} = A''$ and $\mathcal{N} = B''$ are isometrically isomorphic to $A^{**}$ and $B^{**}$ respectively. Note that the Kaplansky density argument of [Kadison and Kastler 1972, Lemma 5] gives $d_{\mathrm{cb}}(\mathcal{M}, \mathcal{N}) \leq d_{\mathrm{cb}}(A, B)$. Following the proof of [CSSW, Lemma 5.4] we can find a unitary $u \in (Z(\mathcal{M}) \cup Z(\mathcal{N}))''$ such that $Z(u\mathcal{M}u^*) = Z(\mathcal{N})$ and $\|u - 1_C\| \leq 5\gamma$. We write

$A_1 = uAu^*$ and $\mathcal{M}_1 = u\mathcal{M}u^*$. There is now a projection $z_{\text{fin}} \in Z(\mathcal{M}_1) = Z(\mathcal{N})$ which simultaneously decomposes $\mathcal{M}_1 = \mathcal{M}_1 z_{\text{fin}} \oplus \mathcal{M}_1(1 - z_{\text{fin}})$ and $\mathcal{N} = \mathcal{N}z_{\text{fin}} \oplus \mathcal{N}(1 - z_{\text{fin}})$ into the finite and properly infinite parts respectively (see [CSSW, Lemma 3.5] or [Kadison and Kastler 1972]). Given a tracial state $\tau$ on $B$, there is a unique extension $\tau''$ to $\mathcal{N}$, which then factors uniquely through the centre valued trace $\text{Tr}_{\mathcal{N}z_{\text{fin}}}$ on $\mathcal{N}z_{\text{fin}}$. That is, $\tau''(x) = (\phi_\tau \circ \text{Tr}_{\mathcal{N}z_{\text{fin}}})(xz_{\text{fin}})$ for some state $\phi_\tau$ on $\mathcal{N}z_{\text{fin}}$. The map $\Psi$ in [CSSW] is then given by defining $\Psi(\tau)(y) = (\phi_\tau \circ \text{Tr}_{\mathcal{M}_1 z_{\text{fin}}})(uyu^*z_{\text{fin}})$ for $y \in A$.

Now fix $\tau \in T_s(B)$. For $m \in \mathbb{N}$ and $a \in (A \otimes M_m)_1^+$, consider the standard very rapidly increasing sequence $(g_{2^{-(n+1)},2^{-n}}(a))_{n=1}^\infty$ which represents $\langle a \rangle$. Let $p_n \in \mathcal{M} \otimes M_m$ be the spectral projection for $a$ for $[2^{-(n+1)}, 1]$, so that the alternating sequence

$$g_{2^{-2},2^{-1}}(a), p_1, g_{2^{-3},2^{-2}}(a), p_2, g_{2^{-4},2^{-3}}(a), p_3, \ldots$$

is very rapidly increasing. Then

$$d_{\Psi(\tau)}(\langle a \rangle) = \sup_n (\Psi(\tau))(g_{2^{-n},2^{-(n+1)}}(a)) = \sup_n \Psi(\tau)''(p_n). \tag{5-2}$$

Choose $b_n \in (B \otimes M_m)_1^+$ with $\|g_{2^{-(n+1)},2^{-n}}(a) - b_n\| \leq 2\gamma$ and projections $q_n \in \mathcal{N} \otimes M_m$ with $\|p_n - q_n\| \leq 2\gamma$ by a standard functional calculus argument [Christensen 1974/75, Lemma 2.1]. Note that $d_{\text{cb}}(\mathcal{M}_1, \mathcal{N}) \leq 11\gamma$ and the algebras $(\mathcal{M}_1 \otimes M_m)(z_{\text{fin}} \otimes 1_m)$ and $(\mathcal{N}_1 \otimes M_m)(z_{\text{fin}} \otimes 1_m)$ have the same centre. Since

$$\left\| (u \otimes 1_m) p (u \otimes 1_m)^*(z_{\text{fin}} \otimes 1_m) - q(z_{\text{fin}} \otimes 1_m) \right\| < \tfrac{1}{2},$$

Lemma 3.6 of [CSSW] applies to show that

$$(\text{Tr}_{\mathcal{M}_1 z_{\text{fin}}} \otimes \text{tr}_m)\big((u \otimes 1_m) p_n (u \otimes 1_m)^*(z_{\text{fin}} \otimes 1_m)\big) = (\text{Tr}_{\mathcal{N}z_{\text{fin}}} \otimes \text{tr}_m)(q(z_{\text{fin}} \otimes 1_m)).$$

This ensures that $\Psi(\tau)''(p_n) = \tau''(q_n)$ for all $n$.

As each $(q_n - 18\gamma)_+ = q_n$, the sequence

$$(b_1 - 18\gamma)_+, q_1, (b_2 - 18\gamma)_+, q_2, (b_3 - 18\gamma)_+, q_3, \ldots$$

is upwards directed by Lemma 3.8 and the supremum of this sequence defines $\Phi(\langle a \rangle)$. We then have

$$d_\tau(\Phi(\langle a \rangle)) = \sup_n \tau''(q_n). \tag{5-3}$$

Indeed, $d_\tau(\Phi(\langle a \rangle))$ is given by $\sup \tau(c_n)$, where $(c_n)_{n=1}^\infty$ is any very rapidly increasing sequence in $(B \otimes \mathcal{K})_+$ representing $\Phi(\langle a \rangle)$. But, working in $\text{Cu}(\mathcal{N})$, Proposition 3.6 shows that any such very rapidly increasing sequence $(c_n)_{n=1}^\infty$ can be intertwined with the very rapidly increasing sequence $(q_n)_{n=1}^\infty$ after telescoping, and this establishes (5-3). Combining (5-2) and (5-3), we have

$$d_{\Psi(\tau)}(\langle a \rangle) = d_{\widehat{\Phi}(\tau)}(\langle a \rangle) \tag{5-4}$$

for all $m \in \mathbb{N}$ and $a \in (A \otimes M_m)_+$. As functionals on the Cuntz semigroup preserve suprema, (5-4) holds for all $a \in (A \otimes \mathcal{K})_+$, whence $\Psi(\tau) = \widehat{\Phi}(\tau)$.    $\square$

The homeomorphism between the lower semicontinuous quasitraces can be used to establish the weak*-continuity of the map between the tracial state spaces of close unital $C^*$-algebras from [CSSW, Section 5] resolving a point left open there. In particular this shows that the map defined in [CSSW] provides an isomorphism between the Elliott invariants of completely close algebras.

For any closed two-sided ideal $I \trianglelefteq A$, the subcone $T_I(A)$ of $T(A)$ consists of those $\tau \in T(A)$ such that the closed two-sided ideal generated by $\{x \in A_+ : \tau(x) < \infty\}$ is $I$. Proposition 3.11 of [Elliott et al. 2011] shows that the relative topology on $T_I(A)$ is the topology of pointwise convergence on the positive elements of the Pedersen ideal of $I$. In particular, $T_s(A) \subset T_A(A)$. In particular, the induced topology on $T_s(A)$ is just the weak*-topology.

**Corollary 5.2.** *Suppose that $A$ and $B$ are unital $C^*$-algebras acting nondegenerately on a Hilbert space with $d_{\mathrm{cb}}(A, B) < 1/42$ and $d(A, B) < 1/2200$. Then the affine isomorphism $\Psi : T_s(B) \to T_s(A)$ between tracial state spaces in* [Christensen et al. 2010, Section 5] *is a homeomorphism with respect to the weak*-topologies.*

We end with two further corollaries of Theorem 5.1.

**Corollary 5.3.** *Let $A$ and $B$ be unital $C^*$-algebras acting nondegenerately on the same Hilbert space with $d_{\mathrm{cb}}(A, B) < 1/2200$. Suppose every bounded 2-quasitrace on $A$ is a trace, then the same property holds for $B$.*

*Proof.* Given $\tau \in QT_{2,s}(B)$, its image $\widehat{\Phi}(\tau)$ lies in $QT_{2,s}(A) = T_s(A)$. By Theorem 5.1 (2) (applied with $A$ and $B$ interchanged)

$$\tau = \widehat{\Phi}^{-1}(\widehat{\Phi}(\tau)) = \Psi^{-1}(\widehat{\Phi}(\tau)) \in T_s(B),$$

as claimed. $\square$

The question of whether exactness transfers to (completely) close $C^*$-algebras raised in [Christensen et al. 2010] remains open, but we do at least obtain the following corollary.

**Corollary 5.4.** *Let $A$ and $B$ be unital $C^*$-algebras acting nondegenerately on the same Hilbert space with $d_{\mathrm{cb}}(A, B) < 1/2200$ and suppose $A$ is exact. Then every bounded 2-quasitrace on $B$ is a trace.*

*Proof.* This is immediate from Haagerup's result [1991] that bounded 2-quasitraces on exact $C^*$-algebras are traces and the previous corollary. $\square$

We end by noting that the isomorphism between the Cuntz semigroups of completely close algebras in Theorem 3.10 can also be used to directly recapture an isomorphism between the Elliott invariants in significant cases. Let $\mathrm{Cu}_{\mathbb{T}}$ be the functor $A \mapsto \mathrm{Cu}(A \otimes C(\mathbb{T}))$ mapping the category of $C^*$-algebras into the category **Cu** introduced in [Coward et al. 2008] and let Ell be the Elliott invariant functor taking values in the category **Inv** whose objects are the 4-tuples arising from the Elliott invariant. Let $\mathscr{C}$ be the subcategory of separable, unital, simple finite and $\mathscr{Z}$-stable algebras $A$ with $QT_2(A) = T(A)$ (for example if $A$ is exact). Then, building on work from [Brown et al. 2008; Brown and Toms 2007], Theorem 4.2 of [Antoine et al. 2014] provides functors $F : \mathbf{Inv} \to \mathbf{Cu}$ and $G : \mathbf{Cu} \to \mathbf{Inv}$ such that there are natural equivalences of functors $F \circ \mathrm{Ell}|_{\mathscr{C}} \cong \mathrm{Cu}_{\mathbb{T}}|_{\mathscr{C}}$ and $G \circ \mathrm{Cu}_{\mathbb{T}}|_{\mathscr{C}} \cong \mathrm{Ell}|_{\mathscr{C}}$ (a similar result for simple unital ASH algebras which are not type I and have slow dimension growth can be found in [Tikuisis 2011]). Note that

in Theorem 4.2 of [Antoine et al. 2014] there is an implicit nuclearity hypothesis, which is only actually used in order to see $QT_2(A) = T(A)$; therefore the result holds in the generality stated. Thus if $A$ and $B$ are $\mathscr{Z}$-stable $C^*$-algebras with $d_{cb}(A, B)$ sufficiently small, and $A$ is simple, separable, unital finite and has $QT_2(A) = T(A)$, then $B$ enjoys all these properties. Further, since tensoring by an abelian algebra does not increase the complete distance between $A$ and $B$ (see [Christensen 1980, Theorem 3.2] for this result in the context of near inclusions—the same proof works for the metric $d_{cb}$), $\mathrm{Cu}_{\mathbb{T}}(A) \cong \mathrm{Cu}_{\mathbb{T}}(B)$ by Theorem 3.10. Thus $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

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# WAVE AND KLEIN–GORDON EQUATIONS ON HYPERBOLIC SPACES

JEAN-PHILIPPE ANKER AND VITTORIA PIERFELICE

We consider the Klein–Gordon equation associated with the Laplace–Beltrami operator $\Delta$ on real hyperbolic spaces of dimension $n \geq 2$; as $\Delta$ has a spectral gap, the wave equation is a particular case of our study. After a careful kernel analysis, we obtain dispersive and Strichartz estimates for a large family of admissible couples. As an application, we prove global well-posedness results for the corresponding semilinear equation with low regularity data.

## 1. Introduction

Dispersive properties of the wave and other evolution equations have been proved to be very useful in the study of nonlinear problems. The theory is well-established for the Euclidean wave equation in dimension $n \geq 3$:

$$\begin{cases} \partial_t^2 u(t, x) - \Delta_x u(t, x) = F(t, x), \\ u(0, x) = f(x), \quad \partial_t|_{t=0} u(t, x) = g(x). \end{cases} \tag{1}$$

The Strichartz estimates

$$\|\nabla_{\mathbb{R} \times \mathbb{R}^n} u\|_{L^p(I; \dot{H}^{-\sigma, q}(\mathbb{R}^n))} \lesssim \|f\|_{\dot{H}^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^{\tilde{p}'}(I; \dot{H}^{\tilde{\sigma}, \tilde{q}'}(\mathbb{R}^n))}$$

hold for solutions $u$ to the Cauchy problem (1) on any (possibly unbounded) time interval $I \subseteq \mathbb{R}$ under the assumptions that

$$\sigma \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) \quad \text{and} \quad \tilde{\sigma} \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{\tilde{q}}\right)$$

and the couples $(p, q), (\tilde{p}, \tilde{q}) \in [2, \infty] \times [2, \infty)$ satisfy

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2} \quad \text{and} \quad \frac{2}{\tilde{p}} + \frac{n-1}{\tilde{q}} = \frac{n-1}{2}. \tag{2}$$

We refer to [Ginibre and Velo 1995; Keel and Tao 1998] for more details.

These estimates serve as a tool for several existence results about the nonlinear wave equation in the Euclidean setting. The problem of finding minimal regularity conditions on the initial data ensuring local well-posedness for semilinear wave equations was addressed in [Kapitanski 1994] and then almost completely answered in [Lindblad and Sogge 1995; Keel and Tao 1998] (see Figure 5 in Section 6). In general, local solutions cannot be extended to global ones unless further assumptions are made on the

nonlinearity or on the initial data. A successful machinery was developed towards the global existence of small solutions to the semilinear wave equation

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = F(u), \\ u(0,x) = f(x), \quad \partial_t|_{t=0} u(t,x) = g(x) \end{cases} \tag{3}$$

with power-like nonlinearities

$$F(u) \sim |u|^\gamma \quad (\gamma > 1). \tag{4}$$

The results depend on the space dimension $n$. After the pioneer work of John [1979] in dimension $n = 3$, Strauss [1989] conjectured that the problem (3) is globally well posed in dimension $n \geq 2$ for small initial data provided that

$$\gamma > \gamma_0 = \frac{1}{2} + \frac{1}{n-1} + \sqrt{\left(\frac{1}{2} + \frac{1}{n-1}\right)^2 + \frac{2}{n-1}}. \tag{5}$$

On one hand, the negative part of the conjecture was established by Sideris [1984], who proved blow-up for nonlinearities $F(u) = |u|^\gamma$ with $1 < \gamma < \gamma_0$ and for rather general initial data. On the other hand, the positive part of the conjecture was proved for any dimension in several steps (see, e.g., [Klainerman and Ponce 1983; Georgiev et al. 1997; D'Ancona et al. 2001] and [Georgiev 2000] for a comprehensive survey).

Analogous results hold for the Klein–Gordon equation

$$\partial_t^2 u(t,x) - \Delta_x u(t,x) + u(t,x) = F(t,x)$$

though its study has not been carried out as thoroughly as for the wave equation; in particular, the sharpness of several well-posedness results is yet unknown (see [Bahouri and Gérard 1999; Ginibre and Velo 1985; Machihara et al. 2004; Nakanishi 1999] and the references therein).

In view of the rich Euclidean theory, it is natural to look at the corresponding equations on more general manifolds. Here we consider real hyperbolic spaces $\mathbb{H}^n$, which are the most simple examples of noncompact Riemannian manifolds with negative curvature. For geometric reasons, we expect better dispersive properties and hence stronger results than in the Euclidean setting.

Consider the wave equation associated to the Laplace–Beltrami operator $\Delta = \Delta_{\mathbb{H}^n}$ on $\mathbb{H}^n$:

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = F(t,x), \\ u(0,x) = f(x), \quad \partial_t|_{t=0} u(t,x) = g(x). \end{cases} \tag{6}$$

The operator $-\Delta$ is positive on $L^2(\mathbb{H}^n)$, and its $L^2$-spectrum is the half-line $[\rho^2, +\infty)$, where $\rho = (n-1)/2$. Thus, (6) may be considered as a special case of the family of Klein–Gordon equations

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) + c u(t,x) = F(t,x), \\ u(0,x) = f(x), \quad \partial_t|_{t=0} u(t,x) = g(x), \end{cases} \tag{7}$$

where

$$c \geq -\rho^2 = -\frac{(n-1)^2}{4} \tag{8}$$

is a constant. In the limit case $c = -\rho^2$, (7) is called the *shifted* wave equation.

Pierfelice [2008] obtained Strichartz estimates for the nonshifted wave equation (6) with radial data on a class of Riemannian manifolds containing all hyperbolic spaces. The wave equation (6) was also investigated on the 3-dimensional hyperbolic space by Metcalfe and Taylor [2011; 2012], who proved dispersive and Strichartz estimates with applications to small-data global well-posedness for the semilinear wave equation. In his recent thesis, Hassani [2011a; 2011b] obtains a first set of results on noncompact Riemannian symmetric spaces of higher rank.

To our knowledge, the shifted wave equation (7) in the limit case $c = -\rho^2$ was first considered by Fontaine [1994; 1997] in low dimensions $n = 3$ and $n = 2$. Tataru [2001] obtained dispersive estimates for the operators $\sin\!\big(t\sqrt{\Delta + \rho^2}\big)/\sqrt{\Delta + \rho^2}$ and $\cos\!\big(t\sqrt{\Delta + \rho^2}\big)$ acting on inhomogeneous Sobolev spaces on $\mathbb{H}^n$ and then transferred them to $\mathbb{R}^n$ in order to get well-posedness results for the Euclidean semilinear wave equation (see also [Georgiev 2000]). Complementary results were obtained by Ionescu [2000], who investigated $L^q \to L^q$ Sobolev estimates for the above operators on all hyperbolic spaces.

A more detailed analysis of the shifted wave equation was carried out in [Anker et al. 2012]. There Strichartz estimates were obtained for a wider range of couples than in the Euclidean setting, and consequently stronger well-posedness results were shown to hold for the nonlinear equations. Corresponding results for the Schrödinger equation were obtained in [Anker and Pierfelice 2009; Anker et al. 2011; Ionescu and Staffilani 2009].

In the present paper, we study the family of equations (7) in the remaining range $c > -\rho^2$ and in dimension $n \geq 2$, which includes the particular case $c = 0$ and $n = 3$ considered in [Metcalfe and Taylor 2011; 2012]. In order to state and describe our results, it is convenient to rewrite the constant (8) as

$$c = \kappa^2 - \rho^2 \quad \text{with } \kappa > 0 \tag{9}$$

and to introduce the operator

$$D = \sqrt{-\Delta - \rho^2 + \kappa^2} \tag{10}$$

as well as

$$\widetilde{D} = \sqrt{-\Delta - \rho^2 + \tilde{\kappa}^2}, \tag{11}$$

where $\tilde{\kappa} > \rho$ is another fixed constant. Thus, our family of equations (7) becomes

$$\begin{cases} \partial_t^2 u(t,x) + D_x^2 u(t,x) = F(t,x), \\ u(0,x) = f(x), \quad \partial_t|_{t=0} u(t,x) = g(x), \end{cases} \tag{12}$$

the wave equation (6) corresponding to the choice $\kappa = \rho$ and the shifted wave equation to the limit case $\kappa = 0$.

Let us now describe the content of this paper and present our main results, which we state for simplicity in dimension $n \geq 3$. In Section 2, we recall the basic tools of spherical Fourier analysis on real hyperbolic spaces $\mathbb{H}^n$. After analyzing carefully the integral kernel of the half-wave operator

$$W_t^\sigma = \widetilde{D}^{-\sigma} e^{itD}$$

in Section 3, we prove in Section 4 the following dispersive estimates, which combine the small time estimates [Anker et al. 2012] for the shifted wave equation and the large time estimates [Anker and

Pierfelice 2009] for the Schrödinger equation:

$$\|W_t^\sigma\|_{L^q \to L^{q'}} \lesssim \begin{cases} |t|^{-(n-1)(1/2-1/q)} & \text{if } 0 < |t| < 1, \\ |t|^{-3/2} & \text{if } |t| \geq 1, \end{cases}$$

where $2 < q < \infty$ and $\sigma \geq (n+1)(1/2-1/q)$. Notice that we don't deal with the limit case $q = \infty$, where Metcalfe and Taylor [2011] have obtained an $H^1 \to \mathrm{BMO}$ estimate in dimension $n = 3$.

In Section 5, we deduce the Strichartz estimates

$$\|\nabla_{\mathbb{R} \times \mathbb{H}^n} u\|_{L^p(I; H^{-\sigma,q}(\mathbb{H}^n))} \lesssim \|f\|_{H^1(\mathbb{H}^n)} + \|g\|_{L^2(\mathbb{H}^n)} + \|F\|_{L^{\tilde{p}'}(I; H^{\tilde{\sigma}, \tilde{q}'}(\mathbb{H}^n))}$$

for solutions $u$ to (12). Here $I \subseteq \mathbb{R}$ is any time interval,

$$\sigma \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) \quad \text{and} \quad \tilde{\sigma} \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{\tilde{q}}\right),$$

and the couples $(1/p, 1/q)$ and $(1/\tilde{p}, 1/\tilde{q})$ belong to the triangle

$$\left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in \left(0, \frac{1}{2}\right] \times \left(0, \frac{1}{2}\right) \,\Big|\, \frac{1}{p} \geq \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)\right\} \cup \left\{\left(0, \frac{1}{2}\right)\right\}. \tag{13}$$

These estimates are similar to those obtained in [Anker et al. 2012] for the shifted wave equation except that they involve standard Sobolev spaces and no exotic ones. Notice that the range (13) of admissible couples for $\mathbb{H}^n$ is substantially wider than the range (2) for $\mathbb{R}^n$, which corresponds to the lower edge of the triangle (13).

In Section 6, we apply the results of the previous sections to the problem of global existence with small data for the corresponding semilinear equations. In contrast with the Euclidean case, where the range of admissible nonlinearities $F(u) \sim |u|^\gamma$ is restricted to $\gamma > \gamma_0$, we prove global well-posedness for powers $\gamma > 1$ arbitrarily close to 1. This result improves in particular [Metcalfe and Taylor 2011], where global well-posedness for (6) was obtained in the case $n = 3$ and $\kappa = \rho$ under the assumption $\gamma > \frac{5}{3}$.

As already observed for the Schrödinger equation [Anker and Pierfelice 2009; Anker et al. 2011] and for the shifted wave equation [Anker et al. 2012; 2014], the fact that better results hold for $\mathbb{H}^n$ than for $\mathbb{R}^n$ may be regarded as a consequence of the stronger dispersion properties in negative curvature. The final section is the Appendix, where we estimate some oscillatory integrals occurring in the kernel analysis carried out in Section 3.

## 2. Essentials about real hyperbolic spaces

In this paper, we consider the simplest class of Riemannian symmetric spaces of the noncompact type, namely real hyperbolic spaces $\mathbb{H}^n$ of dimension $n \geq 2$. We refer to Helgason's books [2001; 2000; 1994] and to Koornwinder's survey [1984] for their algebraic structure and geometric properties as well as for harmonic analysis on these spaces, and we shall be content with the following information. $\mathbb{H}^n$ can be realized as the symmetric space $G/K$, where $G = \mathrm{SO}(1, n)_0$ and $K = \mathrm{SO}(n)$. In geodesic polar coordinates, the Laplace–Beltrami operator on $\mathbb{H}^n$ writes

$$\Delta_{\mathbb{H}^n} = \partial_r^2 + (n-1)\coth r\, \partial_r + \sinh^{-2} r\, \Delta_{\mathbb{S}^{n-1}}.$$

The spherical functions $\varphi_\lambda$ on $\mathbb{H}^n$ are normalized radial eigenfunctions of $\Delta = \Delta_{\mathbb{H}^n}$:

$$\begin{cases} \Delta\varphi_\lambda = -(\lambda^2 + \rho^2)\varphi_\lambda, \\ \varphi_\lambda(0) = 1, \end{cases}$$

where $\lambda \in \mathbb{C}$ and $\rho = (n-1)/2$. They can be expressed in terms of special functions:

$$\varphi_\lambda(r) = \phi_\lambda^{(n/2-1,-1/2)}(r) = {}_2F_1\left(\tfrac{\rho}{2} + i\tfrac{\lambda}{2}, \tfrac{\rho}{2} - i\tfrac{\lambda}{2}; \tfrac{n}{2}; -\sinh^2 r\right),$$

where $\phi_\lambda^{(\alpha,\beta)}$ denotes the Jacobi functions and ${}_2F_1$ the Gauss hypergeometric function. In the sequel, we shall use the Harish-Chandra formula

$$\varphi_\lambda(r) = \int_K dk\, e^{-(\rho+i\lambda)\mathrm{H}(a_{-r}k)} \tag{14}$$

and the basic estimate

$$|\varphi_\lambda(r)| \leq \varphi_0(r) \lesssim (1+r)e^{-\rho r} \quad \forall\lambda \in \mathbb{R},\ r \geq 0. \tag{15}$$

We shall also use the Harish-Chandra expansion

$$\varphi_\lambda(r) = c(\lambda)\Phi_\lambda(r) + c(-\lambda)\Phi_{-\lambda}(r) \quad \forall\lambda \in \mathbb{C}\setminus\mathbb{Z},\ r > 0, \tag{16}$$

where the Harish-Chandra $c$-function is given by

$$c(\lambda) = \frac{\Gamma(2\rho)}{\Gamma(\rho)}\frac{\Gamma(i\lambda)}{\Gamma(i\lambda + \rho)} \tag{17}$$

and

$$\begin{aligned}
\Phi_\lambda(r) &= (2\sinh r)^{i\lambda-\rho}{}_2F_1\left(\tfrac{\rho}{2} - i\tfrac{\lambda}{2}, -\tfrac{\rho-1}{2} - i\tfrac{\lambda}{2}; 1 - i\lambda; -\sinh^{-2} r\right) \\
&= (2\sinh r)^{-\rho}e^{i\lambda r}\sum_{k=0}^{+\infty}\Gamma_k(\lambda)e^{-2kr} \\
&\sim e^{(i\lambda-\rho)r} \quad \text{as } r \to +\infty.
\end{aligned} \tag{18}$$

The coefficients $\Gamma_k(\lambda)$ in the expansion (18) are rational functions of $\lambda \in \mathbb{C}$ that satisfy the recurrence formula

$$\begin{cases} \Gamma_0(\lambda) = 1, \\ \Gamma_k(\lambda) = \dfrac{\rho(\rho-1)}{k(k-i\lambda)}\sum_{j=0}^{k-1}(k-j)\Gamma_j(\lambda). \end{cases}$$

Their classical estimates were improved as follows in [Anker et al. 2011, Lemma 2.1].

**Lemma 2.1.** *Let $0 < \varepsilon < 1$ and $\Omega_\varepsilon = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq \varepsilon|\lambda|,\ \operatorname{Im}\lambda \leq -1 + \varepsilon\}$. Then there exist $\nu \geq 0$ and, for every $\ell \in \mathbb{N}$, $C_\ell \geq 0$ such that*

$$|\partial_\lambda^\ell \Gamma_k(\lambda)| \leq C_\ell k^\nu (1 + |\lambda|)^{-\ell-1} \quad \forall k \in \mathbb{N}^*,\ \lambda \in \mathbb{C}\setminus\Omega_\varepsilon. \tag{19}$$

Under suitable assumptions, the spherical Fourier transform of a bi-$K$-invariant function $f$ on $G$ is defined by

$$\mathscr{H}f(\lambda) = \int_G dg\, f(g)\varphi_\lambda(g),$$

and the following inversion formula holds:

$$f(x) = \text{const} \int_0^{+\infty} d\lambda \, |c(\lambda)|^{-2} \mathscr{H} f(\lambda) \varphi_\lambda(x).$$

Here is a well-known estimate of the Plancherel density:

$$|c(\lambda)|^{-2} \lesssim |\lambda|^2 (1 + |\lambda|)^{n-3} \quad \forall \lambda \in \mathbb{R}. \tag{20}$$

Via the spherical Fourier transform, the Laplace–Beltrami operator $\Delta$ corresponds to

$$-\lambda^2 - \rho^2$$

and hence the operators $D = \sqrt{-\Delta - \rho^2 + \kappa^2}$ and $\widetilde{D} = \sqrt{-\Delta - \rho^2 + \tilde{\kappa}^2}$ to

$$\sqrt{\lambda^2 + \kappa^2} \quad \text{and} \quad \sqrt{\lambda^2 + \tilde{\kappa}^2}.$$

Recall eventually the definition of Sobolev spaces on $\mathbb{H}^n$ and the Sobolev embedding theorem. We refer to [Triebel 1992] for more details about function spaces on Riemannian manifolds. Let $\sigma \in \mathbb{R}$ and $1 < q < \infty$. Then $H^{\sigma,q}(\mathbb{H}^n)$ denotes the image of $L^q(\mathbb{H}^n)$ under $(-\Delta)^{-\sigma/2}$ (in the space of distributions on $\mathbb{H}^n$) equipped with the norm

$$\|f\|_{H^{\sigma,q}} = \|(-\Delta)^{\sigma/2} f\|_{L^q}.$$

In this definition, we may replace $(-\Delta)^{-\sigma/2}$ by $D^{-\sigma} = (-\Delta - \rho^2 + \kappa^2)^{-\sigma/2}$ as long as $\kappa > 2|1/2 - 1/q|\rho$ and in particular by $\widetilde{D}^{-\sigma} = (-\Delta - \rho^2 + \tilde{\kappa}^2)^{-\sigma/2}$ since $\tilde{\kappa} > \rho$. If $\sigma = N$ is a nonnegative integer, then $H^{\sigma,q}(\mathbb{H}^n)$ coincides with the Sobolev space

$$W^{N,q}(\mathbb{H}^n) = \{ f \in L^q(\mathbb{H}^n) \mid \nabla^j f \in L^q(\mathbb{H}^n) \ \forall j, \ 1 \le j \le N \}$$

defined in terms of covariant derivatives. In the $L^2$ setting, we write $H^\sigma(\mathbb{H}^n)$ instead of $H^{\sigma,2}(\mathbb{H}^n)$.

**Proposition 2.2.** *Let $1 < q_1, q_2 < \infty$ and $\sigma_1, \sigma_2 \in \mathbb{R}$ such that $\sigma_1 - \sigma_2 \ge n/q_1 - n/q_2 \ge 0$. Then*

$$H^{\sigma_1, q_1}(\mathbb{H}^n) \subset H^{\sigma_2, q_2}(\mathbb{H}^n).$$

*By this inclusion, we mean that there exists a constant $C \ge 0$ such that*

$$\|f\|_{H^{\sigma_2, q_2}} \le C \|f\|_{H^{\sigma_1, q_1}} \quad \forall f \in C_c^\infty(\mathbb{H}^n).$$

## 3. Kernel estimates

In this section, we derive pointwise estimates for the radial convolution kernel $w_t^\sigma$ of the operator $W_t^\sigma = \widetilde{D}^{-\sigma} e^{itD}$ for suitable exponents $\sigma \in \mathbb{R}$. By the inversion formula of the spherical Fourier transform,

$$w_t^\sigma(r) = \text{const} \int_{-\infty}^{+\infty} d\lambda \, |c(\lambda)|^{-2} (\lambda^2 + \tilde{\kappa}^2)^{-\sigma/2} \varphi_\lambda(r) e^{it\sqrt{\lambda^2 + \kappa^2}}.$$

Contrarily to the Euclidean case, this kernel has different behaviors depending on whether $t$ is small or large, and therefore, we cannot use any rescaling. Let us split up

$$w_t^\sigma(r) = w_t^{\sigma,0}(r) + w_t^{\sigma,\infty}(r)$$

$$= \text{const} \int_{-\infty}^{+\infty} d\lambda \, \chi_0(\lambda) |c(\lambda)|^{-2} (\lambda^2 + \tilde{\kappa}^2)^{-\sigma/2} \varphi_\lambda(r) e^{it\sqrt{\lambda^2 + \kappa^2}}$$

$$+ \text{const} \int_{-\infty}^{+\infty} d\lambda \, \chi_\infty(\lambda) |c(\lambda)|^{-2} (\lambda^2 + \tilde{\kappa}^2)^{-\sigma/2} \varphi_\lambda(r) e^{it\sqrt{\lambda^2 + \kappa^2}}$$

using smooth, even cut-off functions $\chi_0$ and $\chi_\infty$ on $\mathbb{R}$ such that

$$\chi_0(\lambda) + \chi_\infty(\lambda) = 1 \quad \text{and} \quad \begin{cases} \chi_0(\lambda) = 1 & \forall |\lambda| \leq \kappa, \\ \chi_\infty(\lambda) = 1 & \forall |\lambda| \geq \kappa + 1. \end{cases}$$

We shall first estimate $w_t^{\sigma,0}$ and next a variant of $w_t^{\sigma,\infty}$. The kernel $w_t^{\sigma,\infty}$ has indeed a logarithmic singularity on the sphere $r = t$ when $\sigma = (n+1)/2$. We bypass this problem by considering the analytic family of operators

$$\widetilde{W}_t^{\sigma,\infty} = \frac{e^{\sigma^2}}{\Gamma((n+1)/2 - \sigma)} \chi_\infty(D) \widetilde{D}^{-\sigma} e^{itD}$$

in the vertical strip $0 \leq \text{Re}\,\sigma \leq (n+1)/2$ and the corresponding kernels

$$\widetilde{w}_t^{\sigma,\infty}(r) = \text{const} \frac{e^{\sigma^2}}{\Gamma((n+1)/2 - \sigma)} \int_{-\infty}^{+\infty} d\lambda \, \chi_\infty(\lambda) |c(\lambda)|^{-2} (\lambda^2 + \tilde{\kappa}^2)^{-\sigma/2} \varphi_\lambda(r) e^{it\sqrt{\lambda^2 + \kappa^2}}. \quad (21)$$

Notice that the gamma function (which occurs naturally in the theory of Riesz distributions) will allow us to deal with the boundary point $\sigma = (n+1)/2$ while the exponential function yields boundedness at infinity in the vertical strip.

## 3A. *Estimate of $w_t^0 = w_t^{\sigma,0}$.*

**Theorem 3.1.** *Let $\sigma \in \mathbb{R}$. The following pointwise estimates hold for the kernel $w_t^0$:*

(i) *For every $t \in \mathbb{R}$ and $r \geq 0$, we have*

$$|w_t^0(r)| \lesssim \varphi_0(r).$$

(ii) *Assume that $|t| \geq 2$. Then for every $0 \leq r \leq |t|/2$, we have*

$$|w_t^0(r)| \lesssim |t|^{-3/2} (1 + r) \varphi_0(r).$$

*Proof.* Recall that

$$w_t^0(r) = \text{const} \int_{-\kappa-1}^{\kappa+1} d\lambda \, \chi_0(\lambda) |c(\lambda)|^{-2} (\lambda^2 + \tilde{\kappa}^2)^{-\sigma/2} \varphi_\lambda(r) e^{it\sqrt{\lambda^2 + \kappa^2}}. \quad (22)$$

By symmetry, we may assume that $t > 0$.

It follows from the estimates (15) and (20) that

$$|w_t^0(r)| \lesssim \int_{-\kappa-1}^{\kappa+1} d\lambda \, \lambda^2 \varphi_0(r) \lesssim \varphi_0(r),$$

proving (i). We prove (ii) by substituting in (22) the first integral representation of $\varphi_\lambda$ in (14) and by reducing in this way to Fourier analysis on $\mathbb{R}$. Specifically,

$$w_t^0(r) = \int_K dk \, e^{-\rho H(a_{-r}k)} \int_{-\infty}^{+\infty} d\lambda \, a(\lambda) e^{it(\sqrt{\lambda^2+\kappa^2}-H(a_{-r}k)\lambda/t)},$$

where $a(\lambda) = \text{const} \, \chi_0(\lambda)|c(\lambda)|^{-2}(\lambda^2 + \tilde{\kappa}^2)^{-\sigma/2}$. Since

$$\int_K dk \, e^{-\rho H(a_{-r}k)} = \varphi_0(r)$$

and $|H(a_{-r}k)| \leq r$, it remains for us to estimate the oscillatory integral

$$I(t,x) = \int_{-\infty}^{+\infty} d\lambda \, a(\lambda) e^{it(\sqrt{\lambda^2+\kappa^2}-x\lambda/t)}$$

by $|t|^{-3/2}(1 + |x|)$. This is obtained by the method of stationary phase. More precisely, we apply Lemma A.1 in the Appendix, whose assumption (A-1) is fulfilled according to (20). $\qquad\square$

## 3B. *Estimate of $\widetilde{w}_t^\infty = \widetilde{w}_t^{\sigma,\infty}$.*

**Theorem 3.2.** *The following pointwise estimates hold for the kernel $\widetilde{w}_t^\infty$, uniformly in $\sigma \in \mathbb{C}$ with $\text{Re}\,\sigma = (n+1)/2$:*

(i) *Assume that $|t| \geq 2$. Then for every $r \geq 0$, we have*

$$|\widetilde{w}_t^\infty(r)| \lesssim |t|^{-\infty}.$$

(ii) *Assume that $0 < |t| \leq 2$.*

    (a) *If $r \geq 3$, then $\widetilde{w}_t^\infty(r) = O(r^{-1}e^{-\rho r})$.*

    (b) *If $0 \leq r \leq 3$, then $|\widetilde{w}_t^\infty(r)| \lesssim \begin{cases} |t|^{-(n-1)/2} & \text{if } n \geq 3, \\ |t|^{-1/2}(1 - \log|t|) & \text{if } n = 2. \end{cases}$*

  Throughout the proof of Theorem 3.2, we may assume again by symmetry that $t > 0$.

*Proof of Theorem 3.2(i).* By evenness, we have

$$\widetilde{w}_t^\infty(r) = 2\,\text{const}\frac{e^{\sigma^2}}{\Gamma((n+1)/2-\sigma)} \int_0^{+\infty} d\lambda \, \chi_\infty(\lambda)|c(\lambda)|^{-2}(\lambda^2 + \tilde{\kappa}^2)^{-\sigma/2}\varphi_\lambda(r)e^{it\sqrt{\lambda^2+\kappa^2}}. \quad (23)$$

If $0 \leq r \leq t/2$, we resume the proof of Theorem 3.1(ii), using Lemma A.2 instead of Lemma A.1, and conclude that

$$|\widetilde{w}_t^\infty(r)| \lesssim t^{-\infty}\varphi_0(r). \quad (24)$$

If $r \geq t/2$, we substitute in (23) the Harish-Chandra expansion (16) of $\varphi_\lambda(r)$ and reduce this way again to Fourier analysis on $\mathbb{R}$. Specifically, our task consists in estimating the expansion

$$\widetilde{w}_t^\infty(r) = (\sinh r)^{-\rho} \sum_{k=0}^{+\infty} e^{-2kr}\{I_k^{+,\infty}(t,r) + I_k^{-,\infty}(t,r)\} \quad (25)$$

involving oscillatory integrals

$$I_k^{\pm,\infty}(t,r) = \int_0^{+\infty} d\lambda \, a_k^\pm(\lambda) e^{i(t\sqrt{\lambda^2+\kappa^2}\pm r\lambda)}$$

with amplitudes

$$a_k^\pm(\lambda) = 2\,\mathrm{const}\frac{e^{\sigma^2}}{\Gamma((n+1)/2-\sigma)}\chi_\infty(\lambda)\boldsymbol{c}(\mp\lambda)^{-1}(\lambda^2+\tilde{\kappa}^2)^{-\sigma/2}\Gamma_k(\pm\lambda).$$

Notice that $a_k^\pm(\lambda)$ is a symbol of order

$$d = \begin{cases} -1 & \text{if } k = 0, \\ -2 & \text{if } k \in \mathbb{N}^* \end{cases}$$

uniformly in $\sigma \in \mathbb{C}$ with $\mathrm{Re}\,\sigma = (n+1)/2$. This follows indeed from the expression (17) of the $\boldsymbol{c}$-function and from the estimate (19) of the coefficients $\Gamma_k$. Consequently, the integrals

$$I_k^{\pm,\infty}(t,r) = \mathrm{O}(k^\nu) \tag{26}$$

are easy to estimate when $k > 0$ while $I_0^{+,\infty}(t,r)$ and especially $I_0^{-,\infty}(t,r)$ require more work. As far as the penultimate integral is concerned, we integrate by parts

$$I_0^{+,\infty}(t,r) = \int_0^{+\infty} d\lambda \, a_0^+(\lambda) e^{it\phi(\lambda)},$$

using $e^{it\phi(\lambda)} = (it\phi'(\lambda))^{-1}\frac{\partial}{\partial\lambda}e^{it\phi(\lambda)}$ and the following properties of $\phi(\lambda) = \sqrt{\lambda^2+\kappa^2} + (r/t)\lambda$:

- $\phi'(\lambda) = \dfrac{\lambda}{\sqrt{\lambda^2+\kappa^2}} + \dfrac{r}{t} \geq \dfrac{r}{t} \geq \dfrac{1}{2}$,
- $\phi''(\lambda) = \kappa^2(\lambda^2+\kappa^2)^{-3/2}$ is a symbol of order $-3$.

We obtain this way

$$I_0^{+,\infty}(t,r) = \mathrm{O}(r^{-1}) \tag{27}$$

and actually

$$I_0^{+,\infty}(t,r) = \mathrm{O}(r^{-\infty})$$

by repeated integrations by parts. Let us turn to the last integral, which we rewrite as follows:

$$I_0^{-,\infty}(t,r) = \int_0^{+\infty} d\lambda \, a_0^-(\lambda) e^{it(\sqrt{\lambda^2+\kappa^2}-\lambda)} e^{i(t-r)\lambda}.$$

After performing an integration by parts based on $e^{i(t-r)\lambda} = -i(t-r)^{-1}\frac{\partial}{\partial\lambda}e^{i(t-r)\lambda}$ and by using the fact that

$$\psi(\lambda) = \sqrt{\lambda^2+\kappa^2} - \lambda = \frac{\kappa^2}{\sqrt{\lambda^2+\kappa^2}+\lambda} \tag{28}$$

is a symbol of order $-1$, we obtain

$$I_0^{-,\infty}(t,r) = \mathrm{O}\Big(\frac{t}{|r-t|}\Big). \tag{29}$$

This estimate is enough for our purpose as long as $r$ stays away from $t$. If $|r - t| \leq 1$, let us split up

$$e^{it\psi(\lambda)} = 1 + \mathrm{O}(t\psi(\lambda))$$

and

$$I_0^{-,\infty}(t, r) = \int_0^{+\infty} d\lambda \, a_0^-(\lambda) e^{i(t-r)\lambda} + \mathrm{O}(t) \tag{30}$$

accordingly. The remaining integral was estimated in [Anker et al. 2011] at the end of the proof of Theorem 4.2(ii):

$$\int_0^{+\infty} d\lambda \, a_0^-(\lambda) e^{i(t-r)\lambda} = \mathrm{O}(1). \tag{31}$$

By combining the estimates (26), (27), (29), (30), and (31), we deduce from (25) that

$$|\widetilde{w}_t^\infty(r)| \lesssim e^{-\rho r} t \lesssim t^{-\infty} \quad \forall r \geq \frac{t}{2} \geq 1$$

uniformly in $\sigma \in \mathbb{C}$ with $\mathrm{Re}\,\sigma = (n+1)/2$. This concludes the proof of Theorem 3.2(i).     □

Let us turn to the small time estimates in Theorem 3.2.

*Proof of Theorem 3.2(ii)(a).* Since $0 < t \leq 2$ and $r \geq 3$, we can resume the proof of Theorem 3.2(i) in the case $r \geq t + 1 \geq t/2$. By using the expansion (25) and the estimates (26), (27), and (29), we obtain

$$|\widetilde{w}_t^\infty(r)| \lesssim r^{-1} e^{-\rho r}$$

uniformly in $\sigma \in \mathbb{C}$ with $\mathrm{Re}\,\sigma = (n+1)/2$. This concludes the proof of Theorem 3.2(ii)(a).     □

*Proof of Theorem 3.2(ii)(b).* Let us rewrite and expand (23) as follows:

$$\widetilde{w}_t^\infty(r) = 2\,\mathrm{const}\,\frac{e^{\sigma^2}}{\Gamma((n+1)/2-\sigma)} \int_0^{+\infty} d\lambda \, \chi_\infty(\lambda) |c(\lambda)|^{-2} (\lambda^2 + \tilde{\kappa}^2)^{-\sigma/2} e^{it\psi(\lambda)} e^{it\lambda} \varphi_\lambda(r) \tag{32}$$

$$= \int_0^{+\infty} d\lambda \, a(\lambda) e^{it\lambda} \varphi_\lambda(r) + \int_0^{+\infty} d\lambda \, b(\lambda) e^{it\lambda} \varphi_\lambda(r), \tag{33}$$

where $\psi$ is given by (28),

$$a(\lambda) = 2\,\mathrm{const}\,\frac{e^{\sigma^2}}{\Gamma((n+1)/2-\sigma)} \chi_\infty(\lambda) |c(\lambda)|^{-2} (\lambda^2 + \tilde{\kappa}^2)^{-\sigma/2}$$

is a symbol of order $(n-3)/2$, uniformly in $\sigma \in \mathbb{C}$ with $\mathrm{Re}\,\sigma = (n+1)/2$, and

$$b(\lambda) = 2\,\mathrm{const}\,\frac{e^{\sigma^2}}{\Gamma((n+1)/2-\sigma)} \chi_\infty(\lambda) |c(\lambda)|^{-2} (\lambda^2 + \tilde{\kappa}^2)^{-\sigma/2} \{e^{it\psi(\lambda)} - 1\}$$

is a symbol of $(n-5)/2$, uniformly in $0 < t \leq 2$ and $\sigma \in \mathbb{C}$ with $\mathrm{Re}\,\sigma = (n+1)/2$. The first integral in (33) was analyzed in [Anker et al. 2011, Appendix C] and estimated there by

$$C \begin{cases} t^{-(n-1)/2} & \text{if } n \geq 3, \\ t^{-1/2}(1 - \log|t|) & \text{if } n = 2. \end{cases}$$

The second integral is easier to estimate for instance by $Ct^{-(n-2)/2}$. This concludes the proof of Theorem 3.2(ii)(b).     □

**Remark 3.3.** As far as local estimates of wave kernels are concerned, we might have used the *Hadamard parametrix* [Hörmander 2007, §17.4] instead of spherical analysis.

**Remark 3.4.** The kernel analysis carried out in this section still holds for the operators $D^{-\sigma}\widetilde{D}^{-\widetilde{\sigma}}e^{itD}$, provided that we assume $\operatorname{Re}\sigma + \operatorname{Re}\widetilde{\sigma} = (n+1)/2$ in Theorem 3.2.

## 4. Dispersive estimates

In this section, we obtain $L^{q'}\to L^q$ estimates for the operator $W_t^\sigma = \widetilde{D}^{-\sigma}e^{itD}$, which will be crucial for our Strichartz estimates in next section. Let us split up its kernel $w_t^\sigma = w_t^{\sigma,0} + w_t^{\sigma,\infty}$ as before. We will handle the contribution of $w_t^{\sigma,0}$, using the pointwise estimates obtained in Section 3A and the following criterion (see for instance [Anker et al. 2011, Theorem 3.4]) based on the Kunze–Stein phenomenon:

**Lemma 4.1.** *There exists a constant $C > 0$ such that, for every radial measurable function $\kappa$ on $\mathbb{H}^n$ and for every $2 \leq q < \infty$ and $f \in L^{q'}(\mathbb{H}^n)$,*

$$\|f * \kappa\|_{L^q} \leq C_q \|f\|_{L^{q'}} \left\{ \int_0^{+\infty} dr \, (\sinh r)^{n-1}|\kappa(r)|^{q/2}\varphi_0(r) \right\}^{2/q}.$$

For the second part $w_t^{\sigma,\infty}$, we resume the Euclidean approach, which consists of interpolating analytically between $L^2 \to L^2$ and $L^1 \to L^\infty$ estimates for the family of operators

$$\widetilde{W}_t^{\sigma,\infty} = \frac{e^{\sigma^2}}{\Gamma((n+1)/2-\sigma)}\chi_\infty(D)\widetilde{D}^{-\sigma}e^{itD} \tag{34}$$

in the vertical strip $0 \leq \operatorname{Re}\sigma \leq (n+1)/2$.

### 4A. *Small-time dispersive estimates.*

**Theorem 4.2.** *Assume that $0 < |t| \leq 2$, $2 < q < \infty$, and $\sigma \geq (n+1)(1/2-1/q)$. Then*

$$\|\widetilde{D}^{-\sigma}e^{itD}\|_{L^{q'}\to L^q} \lesssim \begin{cases} |t|^{-(n-1)(1/2-1/q)} & \text{if } n \geq 3, \\ |t|^{-(1/2-1/q)}(1-\log|t|)^{1-2/q} & \text{if } n = 2. \end{cases}$$

*Proof.* We divide the proof into two parts, corresponding to the kernel decomposition $w_t = w_t^0 + w_t^\infty$. By applying Lemma 4.1 and using the pointwise estimates in Theorem 3.1(i), we obtain on one hand

$$\|f * w_t^0\|_{L^q} \lesssim \left\{ \int_0^{+\infty} dr \, (\sinh r)^{n-1}\varphi_0(r)|w_t^0(r)|^{q/2} \right\}^{2/q}\|f\|_{L^{q'}}$$

$$\lesssim \left\{ \int_0^{+\infty} dr \, (1+r)^{q/2+1}e^{-(q/2-1)\rho r} \right\}^{2/q}\|f\|_{L^{q'}}$$

$$\lesssim \|f\|_{L^{q'}} \quad \forall f \in L^{q'}.$$

On the other hand, in order to estimate the $L^{q'} \to L^q$ norm of $f \mapsto f * w_t^\infty$, we proceed by interpolation for the analytic family (34). If $\operatorname{Re}\sigma = 0$, then

$$\|f * \widetilde{w}_t^\infty\|_{L^2} \lesssim \|f\|_{L^2} \quad \forall f \in L^2.$$

If $\operatorname{Re}\sigma = (n+1)/2$, we deduce from the pointwise estimates in Theorem 3.2(ii) that

$$\|f * \widetilde{w}_t^\infty\|_{L^\infty} \lesssim |t|^{-(n-1)/2} \|f\|_{L^1} \quad \forall f \in L^1.$$

By interpolation, we conclude for $\sigma = (n+1)(1/2 - 1/q)$ that

$$\|f * w_t^\infty\|_{L^q} \lesssim |t|^{-(n-1)(1/2-1/q)} \|f\|_{L^{q'}} \quad \forall f \in L^{q'}. \qquad \square$$

## 4B. *Large-time dispersive estimate.*

**Theorem 4.3.** *Assume that* $|t| \geq 2$, $2 < q < \infty$, *and* $\sigma \geq (n+1)(1/2 - 1/q)$. *Then*

$$\|\widetilde{D}^{-\sigma} e^{itD}\|_{L^{q'} \to L^q} \lesssim |t|^{-3/2}.$$

*Proof.* We divide the proof into three parts, corresponding to the kernel decomposition

$$w_t = \mathbb{1}_{B(0,|t|/2)} w_t^0 + \mathbb{1}_{\mathbb{H}^n \setminus B(0,|t|/2)} w_t^0 + w_t^\infty.$$

*Estimate 1.* By applying Lemma 4.1 and using the pointwise estimate in Theorem 3.1(ii), we obtain

$$\|f * \{\mathbb{1}_{B(0,|t|/2)} w_t^0\}\|_{L^q} \lesssim \left\{ \int_0^{|t|/2} dr\, (\sinh r)^{n-1} \varphi_0(r) |w_t^0(r)|^{q/2} \right\}^{2/q} \|f\|_{L^{q'}}$$

$$\lesssim \underbrace{\left\{ \int_0^{|t|/2} dr\, (1+r)^{q+1} e^{-(q/2-1)\rho r} \right\}^{2/q}}_{< +\infty} |t|^{-3/2} \|f\|_{L^{q'}} \quad \forall f \in L^{q'}.$$

*Estimate 2.* By applying Lemma 4.1 and using the pointwise estimate in Theorem 3.1(i), we obtain

$$\|f * \{\mathbb{1}_{\mathbb{H}^n \setminus B(0,|t|/2)} w_t^0\}\|_{L^q} \lesssim \left\{ \int_{|t|/2}^{+\infty} dr\, (\sinh r)^{n-1} \varphi_0(r) |w_t^0(r)|^{q/2} \right\}^{2/q} \|f\|_{L^{q'}}$$

$$\lesssim \underbrace{\left\{ \int_{|t|/2}^{+\infty} dr\, r^{q/2+1} e^{-(q/2-1)\rho r} \right\}^{2/q}}_{\lesssim |t|^{-\infty}} \|f\|_{L^{q'}} \quad \forall f \in L^{q'}.$$

*Estimate 3.* We proceed by interpolation for the analytic family (34). If $\operatorname{Re}\sigma = 0$, then

$$\|f * \widetilde{w}_t^\infty\|_{L^2} \lesssim \|f\|_{L^2} \quad \forall f \in L^2.$$

If $\operatorname{Re}\sigma = (n+1)/2$, we deduce from Theorem 3.2(i) that

$$\|f * \widetilde{w}_t^\infty\|_{L^\infty} \lesssim |t|^{-\infty} \|f\|_{L^1} \quad \forall f \in L^1.$$

By interpolation, we obtain for $\sigma = (n+1)(1/2 - 1/q)$ that

$$\|f * w_t^\infty\|_{L^q} \lesssim |t|^{-\infty} \|f\|_{L^{q'}} \quad \forall f \in L^{q'}.$$

We conclude the proof of Theorem 4.3 by summing up the previous estimates. $\qquad \square$

**4C.** *Global dispersive estimates.* As noticed in Remark 3.4, similar results hold for the operators $D^{-\sigma} \tilde{D}^{-\tilde{\sigma}} e^{itD}$.

**Corollary 4.4.** *Let* $2 < q < \infty$ *and* $\sigma, \tilde{\sigma} \in \mathbb{R}$ *such that* $\sigma + \tilde{\sigma} \geq (n+1)(1/2 - 1/q)$. *Then*

$$\|D^{-\sigma} \tilde{D}^{-\tilde{\sigma}} e^{itD}\|_{L^{q'} \to L^q} \lesssim \begin{cases} |t|^{-(n-1)(1/2-1/q)} & \text{if } 0 < |t| \leq 1, \\ |t|^{-3/2} & \text{if } |t| \geq 1. \end{cases} \tag{35}$$

*In particular, if* $2 < q < \infty$ *and* $\sigma \geq (n+1)(1/2 - 1/q)$, *then*

$$\|\tilde{D}^{-\sigma} e^{itD}\|_{L^{q'} \to L^q} + \left\| \tilde{D}^{1-\sigma} \frac{e^{itD}}{D} \right\|_{L^{q'} \to L^q} \lesssim \begin{cases} |t|^{-(n-1)(1/2-1/q)} & \text{if } 0 < |t| \leq 1, \\ |t|^{-3/2} & \text{if } |t| \geq 1. \end{cases} \tag{36}$$

*These results hold in dimension* $n \geq 3$. *In dimension* $n = 2$, *there is an additional logarithmic factor in the small time bound, which becomes* $|t|^{-(1/2-1/q)} (1 - \log|t|)^{1-2/q}$.

**Remark 4.5.** On $L^2(\mathbb{H}^n)$, we know by spectral theory that

- $e^{itD}$ is a 1-parameter group of unitary operators,
- $D^{-\sigma} \tilde{D}^{-\tilde{\sigma}}$ is a bounded operator if $\sigma + \tilde{\sigma} \geq 0$.

**Remark 4.6.** Let us specialize our results for the wave equation (6). In this case, we have $D = \sqrt{-\Delta}$, and we may take $\tilde{D} = D$. Let $2 < q < \infty$ and $\sigma \geq (n+1)(1/2 - 1/q)$. Then

$$\|D^{-\sigma} e^{itD}\|_{L^{q'} \to L^q} \lesssim \begin{cases} |t|^{-(n-1)(1/2-1/q)} & \text{if } 0 < |t| \leq 1, \\ |t|^{-3/2} & \text{if } |t| \geq 1 \end{cases} \tag{37}$$

in dimension $n \geq 3$ and

$$\|D^{-\sigma} e^{itD}\|_{L^{q'} \to L^q} \lesssim \begin{cases} |t|^{-(1/2-1/q)} (1 - \log|t|)^{1-2/q} & \text{if } 0 < |t| \leq 1, \\ |t|^{-3/2} & \text{if } |t| \geq 1 \end{cases}$$

in dimension $n = 2$. Let us compare (37) with the dispersive estimates by Metcalfe and Taylor [2011; 2012] in dimension $n = 3$. Actually, the weaker bound $|t|^{-6(1/2-1/q)}$, obtained in [Metcalfe and Taylor 2011, §3] when $|t|$ is large and $2 < q < 4$, was improved in [Metcalfe and Taylor 2012] after the release of a preprint version of the present paper. On the other hand, these authors are able to deal with the endpoint case $q = \infty$, using local Hardy and BMO spaces on $\mathbb{H}^n$.

## 5. Strichartz estimates

We shall assume $n \geq 4$ throughout this section and discuss the dimensions $n = 3$ and $n = 2$ in the final remarks. Consider the linear equation (12) on $\mathbb{H}^n$, whose solution is given by Duhamel's formula:

$$u(t, x) = \underbrace{(\cos t D_x) f(x) + \frac{\sin t D_x}{D_x} g(x)}_{u_{\text{hom}}(t,x)} + \underbrace{\int_0^t ds \, \frac{\sin(t-s) D_x}{D_x} F(s, x)}_{u_{\text{inhom}}(t,x)}.$$

**Definition 5.1.** A couple $(p, q)$ will be called admissible (see Figure 1) if $(1/p, 1/q)$ belongs to the triangle

$$\left\{ \left( \frac{1}{p}, \frac{1}{q} \right) \in \left( 0, \frac{1}{2} \right] \times \left( 0, \frac{1}{2} \right) \,\Big|\, \frac{1}{p} \geq \frac{n-1}{2} \left( \frac{1}{2} - \frac{1}{q} \right) \right\} \cup \left\{ \left( 0, \frac{1}{2} \right) \right\}. \tag{38}$$
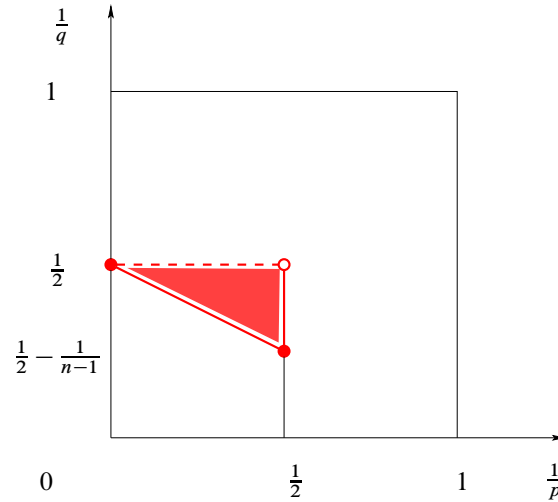
**Figure 1.** Admissibility in dimension $n \geq 4$.

**Theorem 5.2.** *Let* $(p, q)$ *and* $(\widetilde{p}, \widetilde{q})$ *be two admissible couples, and let*

$$\sigma \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) \quad and \quad \widetilde{\sigma} \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{\widetilde{q}}\right). \tag{39}$$

*Then the following Strichartz estimate holds for solutions to the Cauchy problem* (12):

$$\|\nabla_{\mathbb{R}\times\mathbb{H}^n} u\|_{L^p H^{-\sigma,q}} \lesssim \|f\|_{H^1} + \|g\|_{L^{p'}L^2} + \|F\|_{L^{\widetilde{p}'} H^{\widetilde{\sigma},\widetilde{q}'}}. \tag{40}$$

*Proof.* We shall prove the following estimate, which amounts to (40):

$$\|\widetilde{D}_x^{-\sigma+1/2} u(t,x)\|_{L_t^p L_x^q} + \|\widetilde{D}_x^{-\sigma-1/2} \partial_t u(t,x)\|_{L_t^p L_x^q}$$
$$\lesssim \|D_x^{1/2} f(x)\|_{L_x^2} + \|D_x^{-1/2} g(x)\|_{L_x^2} + \|\widetilde{D}_x^{\widetilde{\sigma}-1/2} F(t,x)\|_{L_t^{\widetilde{p}'} L_x^{\widetilde{q}'}}. \tag{41}$$

Consider the operator

$$Tf(t,x) = \widetilde{D}_x^{-\sigma+1/2} \frac{e^{\pm itD_x}}{\sqrt{D_x}} f(x),$$

initially defined from $L^2(\mathbb{H}^n)$ into $L^\infty(\mathbb{R}; H^\sigma(\mathbb{H}^n))$, and its formal adjoint

$$T^* F(x) = \int_{-\infty}^{+\infty} ds \, \widetilde{D}_x^{-\sigma+1/2} \frac{e^{\mp isD_x}}{\sqrt{D_x}} F(s,x),$$

initially defined from $L^1(\mathbb{R}; H^{-\sigma}(\mathbb{H}^n))$ into $L^2(\mathbb{H}^n)$. The $TT^*$ method consists in proving first the $L^{p'}(\mathbb{R}; L^{q'}(\mathbb{H}^n)) \to L^p(\mathbb{R}; L^q(\mathbb{H}^n))$ boundedness of the operator

$$TT^* F(t,x) = \int_{-\infty}^{+\infty} ds \, \widetilde{D}_x^{-2\sigma+1} \frac{e^{\pm i(t-s)D_x}}{D_x} F(s,x)$$

and of its truncated version

$$\mathcal{T} F(t, x) = \int_{-\infty}^{t} ds \, \widetilde{D}_x^{-2\sigma+1} \frac{e^{\pm i(t-s)D_x}}{D_x} F(s, x),$$

for every admissible couple $(p, q)$ and for every $\sigma \geq (n+1)(1/2 - 1/q)/2$, and in decoupling next the indices.

We may disregard the endpoint case $(p, q) = (\infty, 2)$, which is easily dealt with, using the boundedness on $L^2(\mathbb{H}^n)$ of $e^{itD}$ ($t \in \mathbb{R}$) and $\widetilde{D}^{-\sigma+1/2} D^{-1/2}$ ($\sigma \geq 0$). Thus, assume that $(p, q)$ is an admissible couple that is different from the endpoints $(\infty, 2)$ and $(2, 2(n-1)/(n-3))$. It follows from (36) that the norms $\|T T^* F(t, x)\|_{L_t^p L_x^q}$ and $\|\mathcal{T} F(t, x)\|_{L_t^p L_x^q}$ are bounded above by

$$\left\| \int_{0 < |t-s| < 1} ds \, |t-s|^{-\alpha} \|F(s, x)\|_{L_x^{q'}} \right\|_{L_t^p} + \left\| \int_{|t-s| \geq 1} ds \, |t-s|^{-3/2} \|F(s, x)\|_{L_x^{q'}} \right\|_{L_t^p}, \qquad (42)$$

where $\alpha = (n-1)(1/2 - 1/q) \in (0, 1)$. On one hand, the convolution kernel $|t - s|^{-3/2} \mathbb{1}_{\{|t-s| \geq 1\}}$ defines obviously a bounded operator from $L^{p_1}(\mathbb{R})$ to $L^{p_2}(\mathbb{R})$ for all $1 \leq p_1 \leq p_2 \leq \infty$ in particular from $L^{p'}(\mathbb{R})$ to $L^p(\mathbb{R})$ since $p \geq 2$. On the other hand, the convolution kernel $|t - s|^{-\alpha} \mathbb{1}_{\{0 < |t-s| \leq 1\}}$ with $0 < \alpha < 1$ defines a bounded operator from $L^{p_1}(\mathbb{R})$ to $L^{p_2}(\mathbb{R})$ for all $1 < p_1, p_2 < \infty$ such that $0 \leq 1/p_1 - 1/p_2 \leq 1 - \alpha$ in particular from $L^{p'}(\mathbb{R})$ to $L^p(\mathbb{R})$ since $p \geq 2$ and $2/p \geq \alpha$.

At the endpoint $(p, q) = (2, 2(n-1)/(n-3))$, we have $\alpha = 1$. Thus, the previous argument breaks down and is replaced by the refined analysis carried out in [Keel and Tao 1998]. Notice that the problem lies only in the first part of (42) and not in the second one, which involves an integrable convolution kernel on $\mathbb{R}$.

Thus, $T T^*$ and $\mathcal{T}$ are bounded from $L^{p'}(\mathbb{R}; L^{q'}(\mathbb{H}^n))$ to $L^p(\mathbb{R}; L^q(\mathbb{H}^n))$ for every admissible couple $(p, q)$. As a consequence, $T^*$ is bounded from $L^{p'}(\mathbb{R}; L^{q'}(\mathbb{H}^n))$ to $L^2(\mathbb{H}^n)$ and $T$ is bounded from $L^2(\mathbb{H}^n)$ to $L^p(\mathbb{R}; L^q(\mathbb{H}^n))$. We deduce in particular that

$$\|\widetilde{D}_x^{-\sigma+1/2}(\cos t D_x) f(x)\|_{L_t^p L_x^q} \lesssim \|\widetilde{D}_{x\sigma+1/2} e^{\pm it D_x} f(x)\|_{L_t^p L_x^q} \lesssim \|D_x^{1/2} f(x)\|_{L_x^2}$$

and

$$\left\| \widetilde{D}_x^{-\sigma+1/2} \frac{\sin t D_x}{D_x} g(x) \right\|_{L_t^p L_x^q} \lesssim \|\widetilde{D}_x^{-\sigma+1/2} D_x^{-1} e^{\pm it D_x} g(x)\|_{L_t^p L_x^q} \lesssim \|D_x^{-1/2} g(x)\|_{L_x^2}.$$

In summary,

$$\|\widetilde{D}_x^{-\sigma+1/2} u_{\text{hom}}(t, x)\|_{L_t^p L_x^q} \lesssim \|D_x^{1/2} f(x)\|_{L_x^2} + \|D_x^{-1/2} g(x)\|_{L_x^2}. \qquad (43)$$

We next decouple the indices in the $L^{p'} L^{q'} \to L^q L^q$ estimate of $T T^*$ and $\mathcal{T}$. Let $(p, q) \neq (\widetilde{p}, \widetilde{q})$ be two admissible couples, and let $\sigma \geq (n+1)(1/2 - 1/q)/2$ and $\widetilde{\sigma} \geq (n+1)(1/2 - 1/\widetilde{q})/2$. Since $T$ and $T^*$ are separately continuous, the operator

$$T T^* F(t, x) = \int_{-\infty}^{+\infty} ds \, \widetilde{D}_x^{-\sigma-\widetilde{\sigma}+1} \frac{e^{\pm i(t-s)D_x}}{D_x} F(s, x)$$

is bounded from $L^{\widetilde{p}'}(\mathbb{R}; L^{\widetilde{q}'}(\mathbb{H}^n))$ to $L^p(\mathbb{R}; L^q(\mathbb{H}^n))$. According to [Christ and Kiselev 2001], this result remains true for the truncated operator

$$\mathscr{T} F(t, x) = \int_{-\infty}^t ds \, \widetilde{D}_x^{-\sigma-\widetilde{\sigma}+1} \frac{e^{\pm i(t-s)D_x}}{D_x} F(s, x)$$

and hence for

$$\widetilde{\mathscr{T}} F(t, x) = \int_0^t ds \, \widetilde{D}_x^{-\sigma-\widetilde{\sigma}+1} \frac{\sin(t-s)D_x}{D_x} F(s, x)$$

as long as $p$ and $\widetilde{p}$ are not both equal to 2. For the remaining case, where $p = \widetilde{p} = 2$ and $2 < q \neq \widetilde{q} \leq 2(n-1)/(n-3)$, we argue as in the proof of [Anker et al. 2011, Theorem 6.3] by resuming part of the bilinear approach in [Keel and Tao 1998]. Hence,

$$\|\widetilde{D}_x^{-\sigma+1/2} u_{\text{inhom}}(t, x)\|_{L_t^p L_x^q} \lesssim \|\widetilde{D}_x^{\widetilde{\sigma}-1/2} F(t, x)\|_{L_t^{\widetilde{p}'} L_x^{\widetilde{q}'}} \tag{44}$$

for all admissible couples $(p, q)$ and $(\widetilde{p}, \widetilde{q})$.

The Strichartz estimate

$$\|\widetilde{D}_x^{-\sigma+1/2} u(t, x)\|_{L_t^p L_x^q} \lesssim \|D_x^{1/2} f(x)\|_{L_x^2} + \|D_x^{-1/2} g(x)\|_{L_x^2} + \|\widetilde{D}_x^{\widetilde{\sigma}-1/2} F(t, x)\|_{L_t^{\widetilde{p}'} L_x^{\widetilde{q}'}}$$

is obtained by summing up the homogeneous estimate (43) and the inhomogeneous estimate (44). As far as it is concerned, the Strichartz estimate of

$$\partial_t u(t, x) = -(\sin t D_x) D_x f(x) + (\cos t D_x) g(x) + \int_0^t ds \, [\cos(t-s) D_x] F(s, x)$$

is obtained in the same way and is actually easier. More precisely, we consider this time the operator

$$\widetilde{T} f(t, x) = \widetilde{D}_x^{-\sigma} e^{\pm it D_x} f(x)$$

and its adjoint

$$\widetilde{T}^* F(x) = \int_{-\infty}^{+\infty} ds \, \widetilde{D}_x^{-\sigma} e^{\mp is D_x} F(s, x). \hspace{2cm} \square$$

By using the Sobolev embedding theorem, Theorem 5.2 can be extended to all couples $(1/p, 1/q)$ and $(1/\widetilde{p}, 1/\widetilde{q})$ in the square

$$[0, \tfrac{1}{2}] \times (0, \tfrac{1}{2}) \cup \{(0, \tfrac{1}{2})\}. \tag{45}$$

**Corollary 5.3.** *Let $(p, q)$ and $(\widetilde{p}, \widetilde{q})$ be two couples corresponding to the square* (45), *and let $\sigma, \widetilde{\sigma} \in \mathbb{R}$. Assume that $\sigma \geq \sigma(p, q)$, where*

$$\sigma(p, q) = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) + \max\left\{0, \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{p}\right\} = \begin{cases} \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) & \text{if } \frac{1}{p} \geq \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{q}\right), \\ n\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{p} & \text{if } \frac{1}{p} \leq \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{q}\right), \end{cases}$$

*and similarly, $\widetilde{\sigma} \geq \sigma(\widetilde{p}, \widetilde{q})$ (see Figure 2). Then the conclusion of Theorem 5.2 holds for solutions to the Cauchy problem* (12). *More precisely, we have again the Strichartz estimate*

$$\|\nabla_{\mathbb{R} \times \mathbb{H}^n} u\|_{L^p H^{-\sigma, q}} \lesssim \|f\|_{H^1} + \|g\|_{L^2} + \|F\|_{L^{\widetilde{p}'} H^{\widetilde{\sigma}, \widetilde{q}'}}, \tag{40}$$
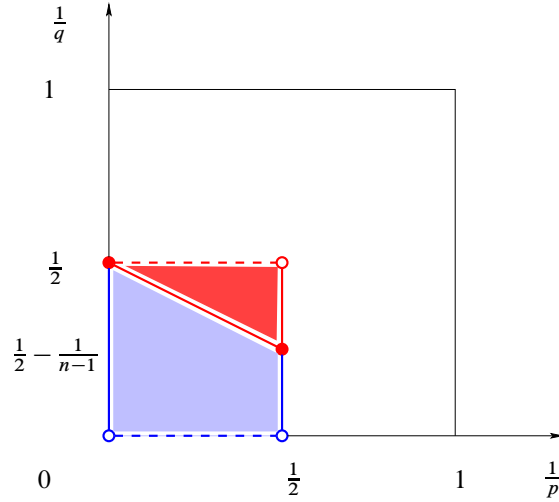
**Figure 2.** Case $n \geq 4$.

*which amounts to*

$$\|\tilde{D}_x^{-\sigma+1/2} u(t,x)\|_{L_t^p L_x^q} + \|\tilde{D}_x^{-\sigma-1/2} \partial_t u(t,x)\|_{L_t^p L_x^q}$$
$$\lesssim \|D_x^{1/2} f(x)\|_{L_x^2} + \|D_x^{-1/2} g(x)\|_{L_x^2} + \|\tilde{D}_x^{\tilde{\sigma}-1/2} F(t,x)\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}. \quad (41)$$

*Proof.* We may restrict to the limit cases $\sigma = \sigma(p,q)$ and $\tilde{\sigma} = \sigma(\tilde{p}, \tilde{q})$. Define $Q$ by

$$\frac{1}{Q} = \begin{cases} \dfrac{1}{q} & \text{if } \dfrac{1}{p} \geq \dfrac{n-1}{2}\left(\dfrac{1}{2} - \dfrac{1}{q}\right), \\ \dfrac{1}{2} - \dfrac{2}{n-1}\dfrac{1}{p} & \text{if } \dfrac{1}{p} \leq \dfrac{n-1}{2}\left(\dfrac{1}{2} - \dfrac{1}{q}\right) \end{cases}$$

and $\tilde{Q}$ similarly. Since $(p, Q)$ and $(\tilde{p}, \tilde{Q})$ are admissible couples, it follows from Theorem 5.2 and more precisely from (41) that

$$\|\tilde{D}_x^{-\Sigma+1/2} u(t,x)\|_{L_t^p L_x^Q} + \|\tilde{D}_x^{-\tilde{\Sigma}-1/2} \partial_t u(t,x)\|_{L_t^p L_x^Q}$$
$$\lesssim \|D_x^{1/2} f(x)\|_{L_x^2} + \|D_x^{-1/2} g(x)\|_{L_x^2} + \|\tilde{D}_x^{\tilde{\Sigma}-1/2} F(t,x)\|_{L_t^{\tilde{p}'} L_x^{\tilde{Q}'}}, \quad (46)$$

where $\Sigma = (n+1)(1/2 - 1/Q)/2$ and $\tilde{\Sigma} = (n+1)(1/2 - 1/\tilde{Q})/2$. Since $\sigma - \Sigma = n(1/Q - 1/q)$, we have

$$\|\tilde{D}_x^{-\sigma+1/2} u(t,x)\|_{L_t^p L_x^q} \lesssim \|\tilde{D}_x^{-\Sigma+1/2} u(t,x)\|_{L_t^p L_x^Q} \quad (47)$$

according to the Sobolev embedding theorem (Proposition 2.2). Similarly,

$$\|\tilde{D}_x^{\tilde{\Sigma}-1/2} F(t,x)\|_{L_t^{\tilde{p}'} L_x^{\tilde{Q}'}} \lesssim \|\tilde{D}_x^{\tilde{\sigma}-1/2} F(t,x)\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}. \quad (48)$$

We conclude by combining (46), (47), and (48). $\qquad\square$
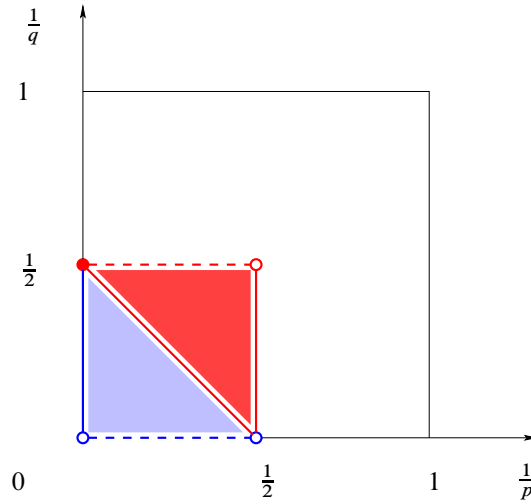
**Figure 3.** Case $n = 3$.

**Remark 5.4.** Theorem 5.2 and Corollary 5.3 hold true in dimension $n = 3$ with the same proofs. Notice that the endpoint $(p, q) = (2, \infty)$ is excluded (see Figure 3). These results hold in particular for the 3-dimensional wave equation (6) and include the Strichartz estimates obtained by Metcalfe and Taylor [2011, §4] in the smaller region

$$\left\{ \left( \frac{1}{p}, \frac{1}{q} \right) \in \left[ 0, \frac{1}{2} \right] \times \left( 0, \frac{1}{2} \right] \middle| \frac{1}{p} \leq 3 \left( \frac{1}{2} - \frac{1}{q} \right) \right\} \setminus \left\{ \left( \frac{1}{2}, \frac{1}{3} \right) \right\}.$$
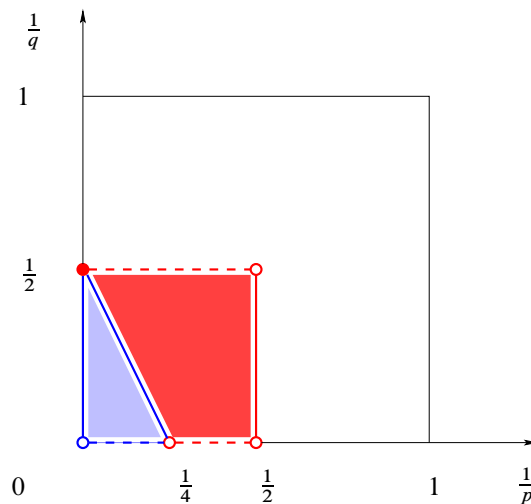


**Figure 4.** Case $n = 2$.

**Remark 5.5.** The analysis carried out in this section still holds in dimension $n = 2$ except for the first convolution kernel in (42), which becomes

$$|t - s|^{-\alpha}(1 - \log|t - s|)^{\beta}\mathbb{1}_{\{0 < |t-s| < 1\}}$$

with $\alpha = 1/2 - 1/q$ and $\beta = 2(1/2 - 1/q)$. Consequently, the admissibility region in Theorem 5.2 becomes

$$\left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in \left(0, \frac{1}{2}\right] \times \left(0, \frac{1}{2}\right) \Big| \frac{1}{p} > \frac{1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)\right\} \cup \left\{\left(0, \frac{1}{2}\right)\right\}$$

and the inequalities $\sigma \geq \sigma(p, q)$, $\tilde{\sigma} \geq \sigma(\tilde{p}, \tilde{q})$ in Corollary 5.3 (see Figure 4) become strict in the triangle

$$\left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in \left(0, \frac{1}{4}\right) \times \left(0, \frac{1}{2}\right) \Big| \frac{1}{p} \leq \frac{1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)\right\}.$$

## 6. Global well-posedness in $L^p(\mathbb{R}, L^q(\mathbb{H}^n))$

In this section, following the classical fixed-point scheme, we use the Strichartz estimates obtained in Section 5 to prove global well-posedness for the semilinear equation

$$\begin{cases} \partial_t^2 u(t, x) + D_x^2 u(t, x) = F(u(t, x)) \\ u(0, x) = f(x), \quad \partial_t|_{t=0} u(t, x) = g(x) \end{cases} \tag{49}$$

on $\mathbb{H}^n$ with power-like nonlinearities

$$F(u) \sim |u|^{\gamma} \quad (\gamma > 1)$$

and small initial data $f$ and $g$. We assume $n \geq 3$ throughout the section and discuss the 2-dimensional case in the final remark. The statement and proof of our result involve the powers

$$\gamma_1 = 1 + \frac{3}{n}, \qquad \gamma_2 = 1 + \frac{2}{\frac{n-1}{2} + \frac{2}{n-1}}, \qquad \gamma_{\text{conf}} = 1 + \frac{4}{n-1},$$

$$\gamma_3 = \begin{cases} \dfrac{1}{n}\left(\dfrac{n+6}{2} + \dfrac{2}{n-1} + \sqrt{4n + \left(\dfrac{6-n}{2} + \dfrac{2}{n-1}\right)^2}\right) & \text{if } n \leq 5, \\ 1 + \dfrac{2}{\frac{n-1}{2} - \frac{1}{n-1}} & \text{if } n \geq 6, \end{cases} \tag{50}$$

$$\gamma_4 = \begin{cases} 1 + \dfrac{4}{n-2} & \text{if } n \leq 5, \\ \dfrac{n-1}{2} + \dfrac{3}{n+1} - \sqrt{\left(\dfrac{n-3}{2} + \dfrac{3}{n+1}\right)^2 - 4\dfrac{n-1}{n+1}} & \text{if } n \geq 6, \end{cases}$$

which are computed in Table 1, and the curves

$$\sigma_1(\gamma) = \frac{n+1}{4} - \frac{(n+1)(n+5)}{8n}\frac{1}{\gamma - \frac{n+1}{2n}}, \qquad \sigma_2(\gamma) = \frac{n+1}{4} - \frac{1}{\gamma - 1}, \quad \text{and} \quad \sigma_3(\gamma) = \frac{n}{2} - \frac{2}{\gamma - 1}. \tag{51}$$

The powers $\gamma_1$, $\gamma_2$, and $\gamma_{\text{conf}}$ and the curves $C_1$, $C_2$, and $C_3$ parametrized by $\sigma_1$, $\sigma_2$, and $\sigma_3$ occur already in the Euclidean setting. More precisely, they are involved in the conditions, illustrated in Figure 5,

| $n$ | $\gamma_1$ | $\gamma_2$ | $\gamma_{\text{conf}}$ | $\gamma_3$ | $\gamma_4$ |
|-----|------------|------------|------------------------|------------|------------|
| 3 | 2 | 2 | 3 | $\frac{11+\sqrt{73}}{6} \simeq 3.26$ | 5 |
| 4 | $\frac{7}{4} = 1.75$ | $\frac{25}{13} \simeq 1.92$ | $\frac{7}{3} \simeq 2.33$ | $\frac{5}{2} \simeq 2.5$ | 3 |
| 5 | $\frac{8}{5} \simeq 1.6$ | $\frac{9}{5} \simeq 1.8$ | 2 | $\frac{6+\sqrt{21}}{5} \simeq 2.12$ | $\frac{7}{3} \simeq 2.33$ |
| 6 | $\frac{3}{2} = 1.5$ | $\frac{49}{29} \simeq 1.69$ | $\frac{9}{5} = 1.8$ | $\frac{43}{23} \simeq 1.87$ | 2 |
| $\geq 7$ | $< \gamma_2$ | $< \gamma_{\text{conf}}$ | $< \gamma_3$ | $< \gamma_4$ | $< 2$ |

**Table 1.** Critical powers.

of minimal regularity $\sigma$ on the initial data $f$ and $g$ that are needed in order to ensure local well-posedness of (49). We refer again to [Kapitanski 1994; Lindblad and Sogge 1995; Keel and Tao 1998] for more details. Notice that, in dimension $n = 3$, $\gamma_1$ coincides with $\gamma_2$ and there is no curve $C_1$.

As mentioned in the introduction, global well-posedness of (49) on $\mathbb{R}^n$ requires additional conditions. Recall that smooth solutions with small-amplitude blow up or not depending on whether $\gamma$ is smaller or larger than the critical power $\gamma_0$ defined in (5).

In Section 5, we have obtained Strichartz estimates on $\mathbb{H}^n$ for a range of admissible couples that is wider than on $\mathbb{R}^n$. As a consequence, we deduce in this section stronger well-posedness results for (49). In particular, we prove global well-posedness for small initial data in $H^\sigma(\mathbb{H}^n) \times H^{\sigma-1}(\mathbb{H}^n)$ if $1 < \gamma < \gamma_1$ and $\sigma > 0$ is small. Thus, there is no blow-up for small powers $\gamma > 1$ on $\mathbb{H}^n$ in sharp contrast with $\mathbb{R}^n$.
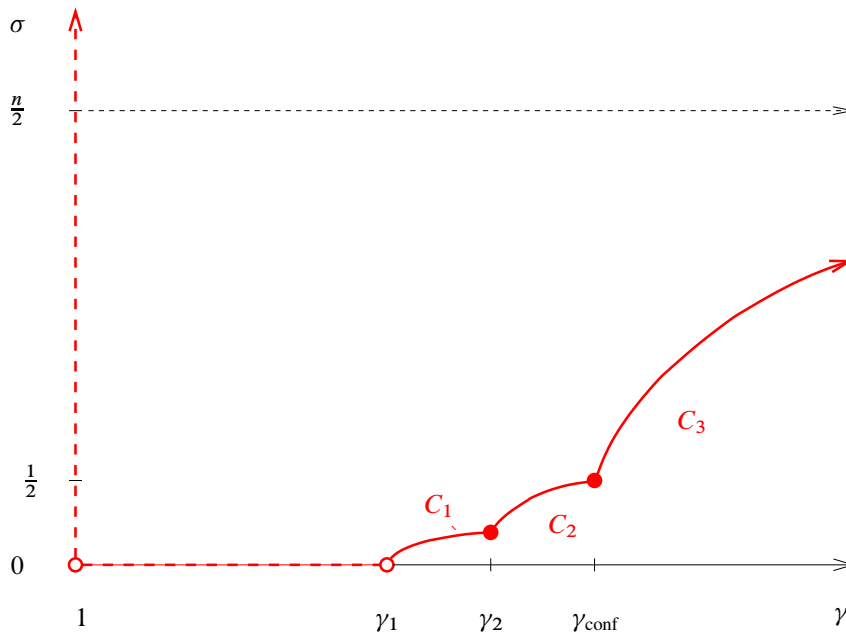


**Figure 5.** Regularity for local well-posedness on $\mathbb{R}^n$ in dimension $n \geq 3$.
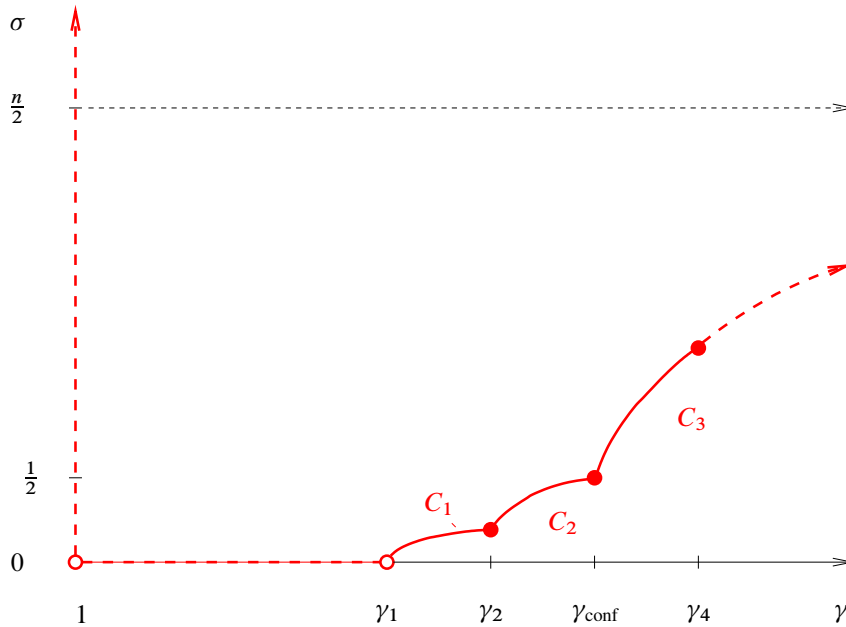
**Figure 6.** Regularity for global well-posedness on $\mathbb{H}^n$ in dimension $n \geq 3$.

**Theorem 6.1.** *Assume that the nonlinearity $F$ satisfies*

$$|F(u)| \leq C|u|^{\gamma} \quad and \quad |F(u) - F(v)| \leq C(|u|^{\gamma-1} + |v|^{\gamma-1})|u - v|. \tag{52}$$

*Then in dimension $n \geq 3$, (49) is globally well posed for small initial data in $H^{\sigma}(\mathbb{H}^n) \times H^{\sigma-1}(\mathbb{H}^n)$ provided that*

$$\begin{cases} \sigma = 0^+ & \text{if } 1 < \gamma \leq \gamma_1, \\ \sigma = \sigma_1(\gamma) & \text{if } \gamma_1 < \gamma \leq \gamma_2, \\ \sigma = \sigma_2(\gamma) & \text{if } \gamma_2 \leq \gamma \leq \gamma_{\text{conf}}, \\ \sigma = \sigma_3(\gamma) & \text{if } \gamma_{\text{conf}} \leq \gamma \leq \gamma_4, \end{cases} \tag{53}$$

*where $\sigma = 0^+$ stands for any $\sigma > 0$ sufficiently close to 0 (see Figure 6). More precisely, in each case, there exist $2 \leq p, q < \infty$ and $\delta, \varepsilon > 0$ such that, for any initial data $(f, g) \in H^{\sigma}(\mathbb{H}^n) \times H^{\sigma-1}(\mathbb{H}^n)$ with norm $\leq \delta$, the Cauchy problem (49) has a unique solution $u$ with norm $\leq \varepsilon$ in the Banach space*

$$X = C(\mathbb{R}; H^{\sigma}(\mathbb{H}^n)) \cap C^1(\mathbb{R}; H^{\sigma-1}(\mathbb{H}^n)) \cap L^p(\mathbb{R}; L^q(\mathbb{H}^n)).$$

**Remark 6.2.** In dimension $n = 3$, $\gamma_1$ coincides with $\gamma_2$, the second and third conditions in (53) boil down to

$$\sigma \geq \sigma_2(\gamma) \quad \text{if } \gamma_1 = \gamma_2 < \gamma \leq \gamma_{\text{conf}},$$

and there is no curve $C_1$ in Figure 6.

*Proof of Theorem 6.1 for* $1 < \gamma \leq \gamma_{\text{conf}}$. We resume the fixed-point method based on Strichartz estimates. Define $u = \Phi(v)$ as the solution to the Cauchy problem

$$\begin{cases} \partial_t^2 u(t, x) + D_x^2 u(t, x) = F(v(t, x)), \\ u(0, x) = f(x), \quad \partial_t|_{t=0} u(t, x) = g(x), \end{cases} \tag{54}$$

which is given by Duhamel's formula:

$$u(t, x) = (\cos t D_x) f(x) + \frac{\sin t D_x}{D_x} g(x) + \int_0^t ds \, \frac{\sin(t - s) D_x}{D_x} F(s, x).$$

On one hand, according to Theorem 5.2, the Strichartz estimate

$$\|u(t, x)\|_{L_t^\infty H_x^\sigma} + \|\partial_t u(t, x)\|_{L_t^\infty H_x^{\sigma - 1}} + \|u(t, x)\|_{L_t^p L_x^q}$$
$$\lesssim \|f(x)\|_{H_x^\sigma} + \|g(x)\|_{H_x^{\sigma - 1}} + \|F(v(t, x))\|_{L_t^{\tilde{p}'} H_x^{\sigma + \tilde{\sigma} - 1, \tilde{q}'}}$$

holds whenever

$$\begin{cases} (p, q) \text{ and } (\tilde{p}, \tilde{q}) \text{ are admissible couples,} \\ \sigma \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{\tilde{q}}\right) \text{ and } \tilde{\sigma} \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{\tilde{q}}\right). \end{cases}$$

On the other hand, by our nonlinear assumption (52) and by the Sobolev embedding theorem (Proposition 2.2), we have

$$\|F(v(t, x))\|_{L_t^{\tilde{p}'} H_x^{\sigma + \tilde{\sigma} - 1, \tilde{q}'}} \lesssim \||v(t, x)|^\gamma\|_{L_t^{\tilde{p}'} H_x^{\sigma + \tilde{\sigma} - 1, \tilde{q}'}} \lesssim \||v(t, x)|^\gamma\|_{L_t^{\tilde{p}'} L_x^{\tilde{Q}'}} \lesssim \|v(t, x)\|_{L_t^{\gamma \tilde{p}'} L_x^{\gamma \tilde{Q}'}}^\gamma,$$

provided that

$$\sigma + \tilde{\sigma} \leq 1, \qquad 1 < \tilde{Q}' \leq \tilde{q}' < \infty, \quad \text{and} \quad \frac{n}{\tilde{Q}'} - \frac{n}{\tilde{q}'} \leq 1 - \sigma - \tilde{\sigma}. \tag{55}$$

In order to remain within the same function space, we require in addition that

$$\gamma \tilde{p}' = p \quad \text{and} \quad \gamma \tilde{Q}' = q.$$

In summary,

$$\|u(t, x)\|_{L_t^\infty H_x^\sigma} + \|\partial_t u(t, x)\|_{L_t^\infty H_x^{\sigma - 1}} + \|u(t, x)\|_{L_t^p L_x^q}$$
$$\leq C\left\{\|f(x)\|_{H_x^\sigma} + \|g(x)\|_{H_x^{\sigma - 1}} + \|v\|_{L_t^p L_x^q}^\gamma\right\} \tag{56}$$

if the following set of conditions is satisfied:

$$\begin{cases} \text{(a)} & (p, q) \text{ and } (\tilde{p}, \tilde{q}) \text{ are admissible couples,} \\ \text{(b)} & \sigma \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right), \tilde{\sigma} \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{\tilde{q}}\right), \text{ and } \sigma + \tilde{\sigma} \leq 1, \\ \text{(c)} & \frac{\gamma}{p} + \frac{1}{\tilde{p}} = 1, \\ \text{(d)} & 1 \leq \frac{\gamma}{q} + \frac{1}{\tilde{q}} \leq 1 + \frac{1 - \sigma - \tilde{\sigma}}{n}, \\ \text{(e)} & q > \gamma. \end{cases} \tag{57}$$

For such a choice, $\Phi$ maps the Banach space

$$X = C(\mathbb{R}; H^\sigma(\mathbb{H}^n)) \cap C^1(\mathbb{R}; H^{\sigma-1}(\mathbb{H}^n)) \cap L^p(\mathbb{R}; L^q(\mathbb{H}^n)),$$

equipped with the norm

$$\|u\|_X = \|u(t,x)\|_{L_t^\infty H_x^\sigma} + \|\partial_t u(t,x)\|_{L_t^\infty H_x^{\sigma-1}} + \|u\|_{L_t^p L_x^q},$$

into itself. Let us show that $\Phi$ is a contraction on the ball

$$X_\varepsilon = \{u \in X \mid \|u\|_X \le \varepsilon\},$$

provided that $\varepsilon > 0$ and $\|f\|_{H^\sigma} + \|g\|_{H^{\sigma-1}}$ are sufficiently small. Let $v, \tilde{v} \in X$, $u = \Phi(v)$, and $\tilde{u} = \Phi(\tilde{v})$. By resuming the arguments leading to (56) and by using in addition Hölder's inequality, we obtain the estimate

$$\begin{aligned}
\|u - \tilde{u}\|_X &\le C \|F(v) - F(\tilde{v})\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} \\
&\le C \|\{|v|^{\gamma-1} + |\tilde{v}|^{\gamma-1}\} |v - \tilde{v}|\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} \\
&\le C \{\|v\|_{L_t^p L_x^q}^{\gamma-1} + \|\tilde{v}\|_{L_t^p L_x^q}^{\gamma-1}\} \|v - \tilde{v}\|_{L_t^p L_x^q} \\
&\le C \{\|v\|_X^{\gamma-1} + \|\tilde{v}\|_X^{\gamma-1}\} \|v - \tilde{v}\|_X. \tag{58}
\end{aligned}$$

Thus, if we assume $\|v\|_X \le \varepsilon$, $\|\tilde{v}\|_X \le \varepsilon$, and $\|f\|_{H^\sigma} + \|g\|_{H^{\sigma-1}} \le \delta$, then (56) and (58) yield

$$\|u\|_X \le C\delta + C\varepsilon^\gamma, \qquad \|\tilde{u}\|_X \le C\delta + C\varepsilon^\gamma, \quad \text{and} \quad \|u - \tilde{u}\|_X \le 2C\varepsilon^{\gamma-1} \|v - \tilde{v}\|_X.$$

Hence,

$$\|u\|_X \le \varepsilon, \qquad \|\tilde{u}\|_X \le \varepsilon, \quad \text{and} \quad \|u - \tilde{u}\|_X \le \tfrac{1}{2} \|v - \tilde{v}\|_X$$

if $C\varepsilon^{\gamma-1} \le \tfrac{1}{4}$ and $C\delta \le \tfrac{3}{4}\varepsilon$. One concludes by applying the fixed-point theorem in the complete metric space $X_\varepsilon$.

It remains for us to check that the set of conditions (57) can be fulfilled in the various cases (53). Notice that we may assume the following equalities in (57)(b):

$$\sigma = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) \quad \text{and} \quad \tilde{\sigma} = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{\tilde{q}}\right).$$

Thus, (57) reduces to the set of conditions:

$$\begin{cases}
\text{(a)} & (p, q) \text{ and } (\tilde{p}, \tilde{q}) \text{ are admissible couples,} \\
\text{(b)} & \dfrac{1}{q} + \dfrac{1}{\tilde{q}} \ge \dfrac{n-1}{n+1}, \\
\text{(c)} & \dfrac{\gamma}{p} + \dfrac{1}{\tilde{p}} = 1, \\
\text{(d)(i)} & \dfrac{\gamma}{q} + \dfrac{1}{\tilde{q}} \ge 1, \\
\text{(d)(ii)} & \left(\dfrac{2n}{n-1}\gamma - \dfrac{n+1}{n-1}\right)\dfrac{1}{q} + \dfrac{1}{\tilde{q}} \le \dfrac{n+1}{n-1}, \\
\text{(e)} & q > \gamma.
\end{cases} \tag{59}$$

We shall discuss these conditions first in high dimensions and next in low dimensions.

▶ Assume that $n \geq 6$.

Firstly notice that $\gamma_{\mathrm{conf}} < 2$. As $\gamma \leq \gamma_{\mathrm{conf}}$ and $q > 2$, (59)(e) is trivially satisfied. Secondly, we claim that (59)(a) and (59)(c) reduce to the single condition

$$\frac{\gamma}{q} + \frac{1}{\widetilde{q}} \geq \frac{\gamma+1}{2} - \frac{2}{n-1} \tag{60}$$

in the square

$$R = \left[\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2}\right) \times \left[\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2}\right). \tag{61}$$

More precisely, if $(p, q)$ and $(\widetilde{p}, \widetilde{q})$ are admissible couples satisfying (59)(c), then $(1/q, 1/\widetilde{q})$ is a point in the square $R$ satisfying (60). Conversely, if $(1/q, 1/\widetilde{q}) \in R$ satisfies (60), then there exists a 1-parameter family of admissible couples $(p, q)$ and $(\widetilde{p}, \widetilde{q})$ satisfying (59)(c). All these claims can be deduced from Figure 7.
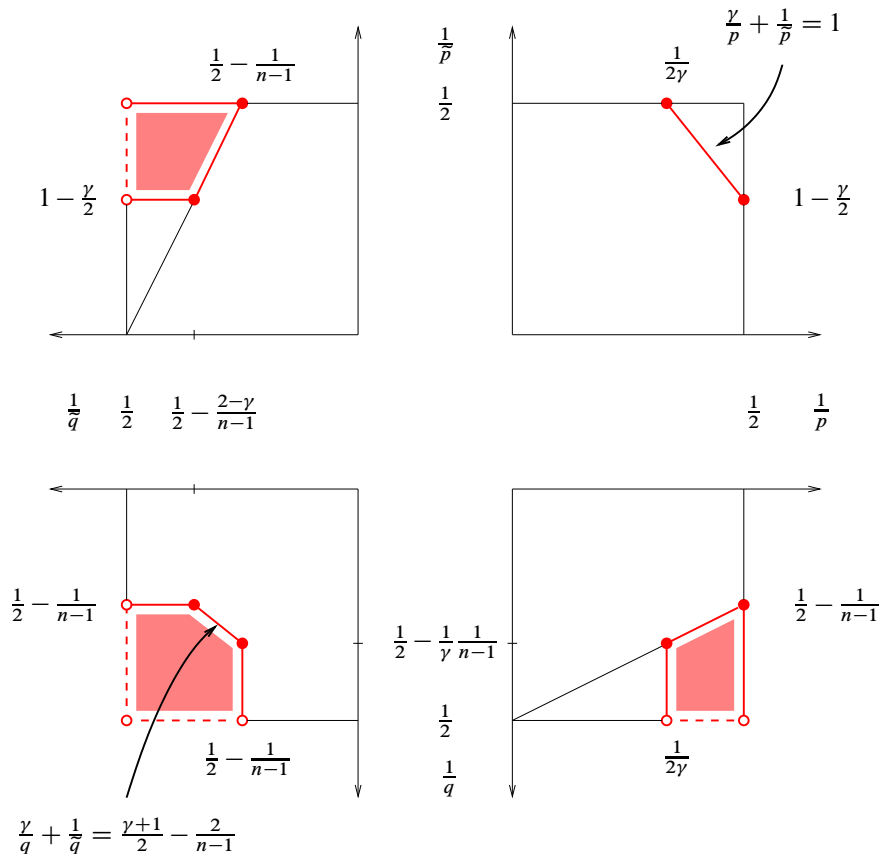


**Figure 7.** Case $\gamma < 2$.
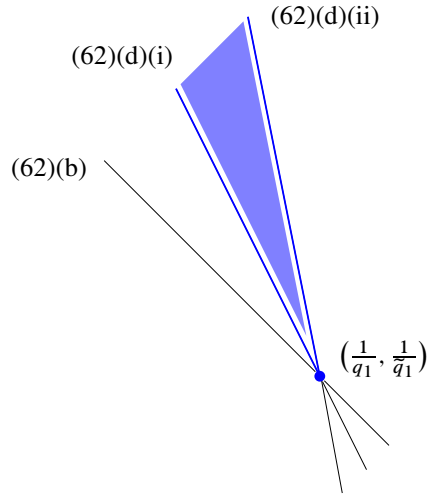
**Figure 8.** Sector $S$.

Thirdly, as $\gamma \leq \gamma_{\text{conf}}$, (60) follows actually from (59)(d)(i). Fourthly, we claim that (59)(b) follows from (59)(d)(i) and (59)(d)(ii). Consider indeed the three lines

$$
\begin{cases}
\text{(b)} & \dfrac{1}{q} + \dfrac{1}{\tilde{q}} = \dfrac{n-1}{n+1}, \\[2mm]
\text{(d)(i)} & \dfrac{\gamma}{q} + \dfrac{1}{\tilde{q}} = 1, \\[2mm]
\text{(d)(ii)} & \left(\dfrac{2n}{n-1}\gamma - \dfrac{n+1}{n-1}\right)\dfrac{1}{q} + \dfrac{1}{\tilde{q}} = \dfrac{n+1}{n-1}
\end{cases}
\tag{62}
$$
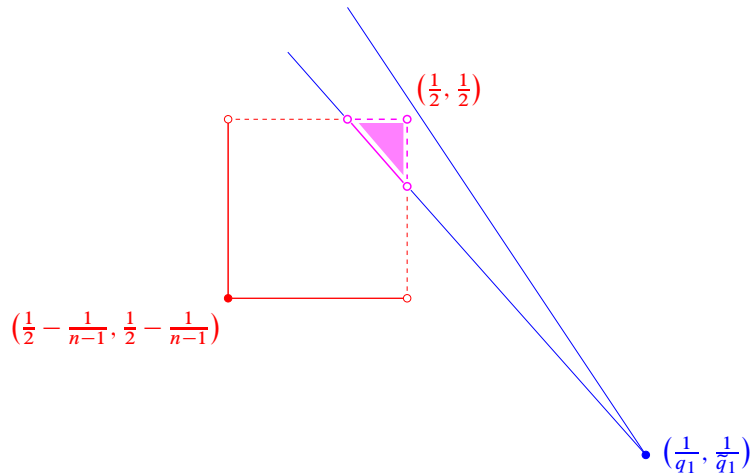


**Figure 9.** Case $1 < \gamma \leq 1 + \dfrac{2}{n}$.
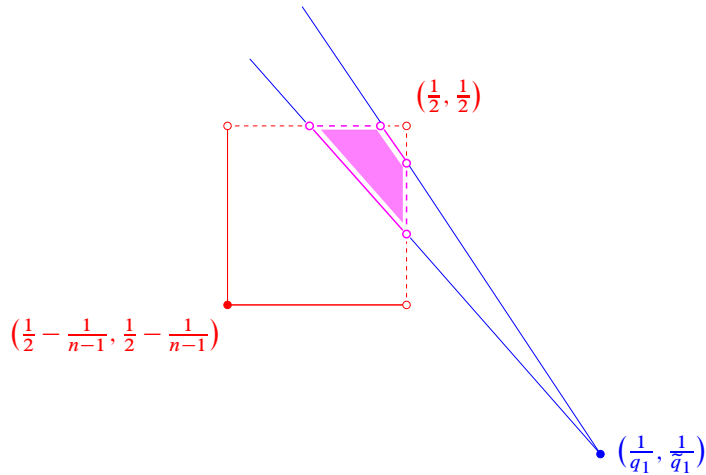
**Figure 10.** Case $1 + \dfrac{2}{n} \leq \gamma \leq 1 + \dfrac{2}{n-1}$.

in the plane with coordinates $(1/q, 1/\widetilde{q})$. On one hand, they meet at the same point, whose coordinates are

$$\begin{cases} \dfrac{1}{q_1} = \dfrac{2}{n+1}\dfrac{1}{\gamma-1}, \\ \dfrac{1}{\widetilde{q}_1} = \dfrac{n-1}{n+1} - \dfrac{2}{n+1}\dfrac{1}{\gamma-1}. \end{cases} \tag{63}$$

On the other hand, the coefficients of $1/q$ occur in increasing order in (62):

$$1 < \gamma < \frac{2n}{n-1}\gamma - \frac{n+1}{n-1}.$$

Hence, (59)(b) follows from (59)(d)(i) and (59)(d)(ii), which define the sector $S$ with vertex $(1/q_1, 1/\widetilde{q}_1)$ and edges (62)(d)(i) and (62)(d)(ii) depicted in Figure 8.

In summary, the set of conditions (59) reduce to the three conditions (59)(d)(i), (59)(d)(ii), and (61) in the plane with coordinates $(1/q, 1/\widetilde{q})$. In order to conclude, we examine the possible intersections of the
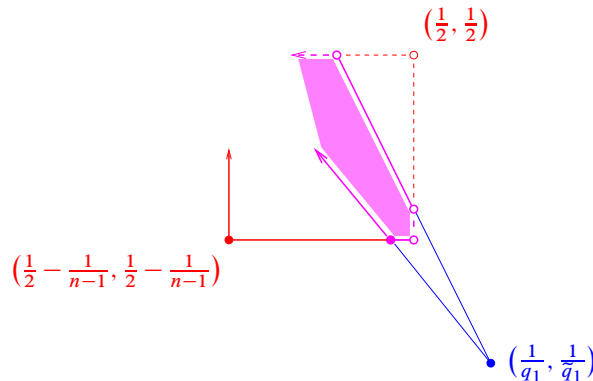


**Figure 11.** Case $1 + \dfrac{2}{n-1} \leq \gamma \leq \gamma_1$.
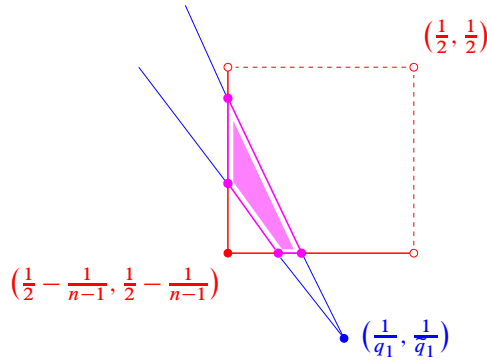
**Figure 12.** Case $\gamma_1 < \gamma \le \gamma_2$.

sector $S$ defined by (59)(d)(i) and (59)(d)(ii) with the square $R$ defined by (61), and we determine in each case the minimal regularity $\sigma = (n+1)(1/2 - 1/q)/2$.

● *Case 1*: $1 < \gamma \le \gamma_1$.

In the following three subcases, the minimal regularity condition is $\sigma > 0$ as $1/q > 1/2$ can be chosen arbitrarily close to $\frac{1}{2}$:

○ *Subcase 1.1*: $1 < \gamma \le 1 + \dfrac{2}{n}$ (see Figure 9).

○ *Subcase 1.2*: $1 + \dfrac{2}{n} \le \gamma \le 1 + \dfrac{2}{n-1}$ (see Figure 10).

○ *Subcase 1.3*: $1 + \dfrac{2}{n-1} \le \gamma \le \gamma_1$ (see Figure 11).

● *Case 2*: $\gamma_1 < \gamma \le \gamma_2$ (see Figure 12).

The minimal regularity $\sigma = \sigma_1(\gamma)$ is reached at the boundary point

$$\left(\frac{1}{q}, \frac{1}{\tilde{q}}\right) = \left(\frac{n+5}{4n} \frac{1}{\gamma - (n+1)/2n}, \frac{1}{2} - \frac{1}{n-1}\right).$$

● *Case 3*: $\gamma_2 \le \gamma \le \gamma_{\mathrm{conf}}$ (see Figure 13).
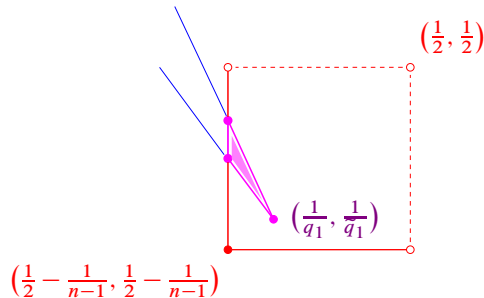


**Figure 13.** Case $\gamma_2 \le \gamma \le \gamma_{\mathrm{conf}}$.
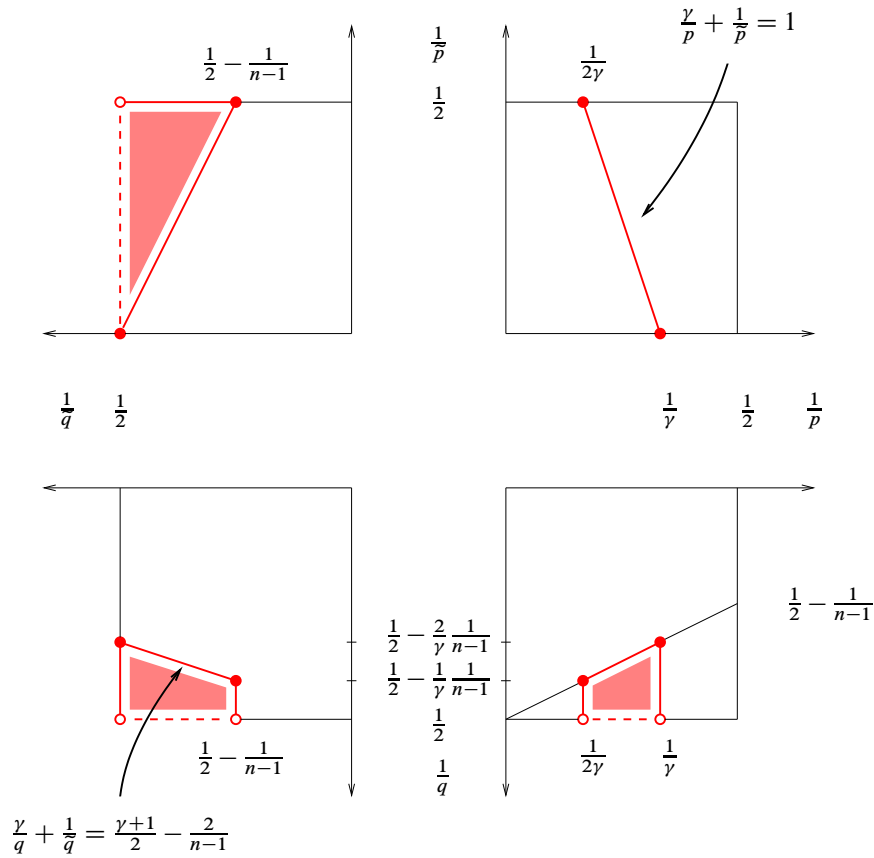
**Figure 14.** Case $\gamma \geq 2$.

The minimal regularity $\sigma = \sigma_2(\gamma)$ is reached at the vertex $(1/q_1, 1/\tilde{q}_1)$. In the limit case $\gamma = \gamma_{\text{conf}}$, notice that all indices $1/q_1$, $1/\tilde{q}_1$, $1/p_1 = (n-1)(1/2 - 1/q_1)/2$, and $1/\tilde{p}_1 = (n-1)(1/2 - 1/\tilde{q}_1)/2$ become equal to the Strichartz index $(n-1)/2(n+1) = 1/2 - 1/(n+1)$.
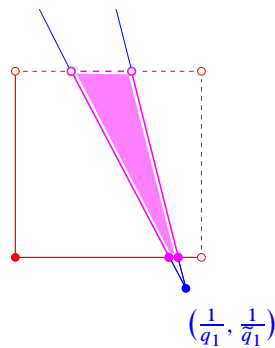


**Figure 15.** Case $\gamma_1 < \gamma \leq \gamma_2$.
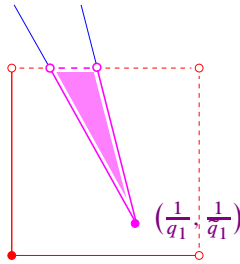
**Figure 16.** Subcase $\gamma_2 \leq \gamma < 2$.

This concludes the proof of Theorem 6.1 for $1 < \gamma \leq \gamma_{\mathrm{conf}}$ and $n \geq 6$.

► Assume that $n = 4$ or $5$.

Let us adapt the proof above. If $\gamma \geq 2$, (59)(e) must be checked and (59)(a) and (59)(c) reduce again to (60) but this time in the slightly larger square

$$R = \left[\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2}\right) \times \left[\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2}\right] \tag{64}$$

(see Figure 14). Thus, (59) reduces to

$$\begin{cases} (59)(d)(i), \ (59)(d)(ii), \text{ and } (64) & \text{if } 1 < \gamma < 2, \\ (59)(d)(i), \ (59)(d)(ii), \ (59)(e), \text{ and } (64) & \text{if } 2 \leq \gamma \leq \gamma_{\mathrm{conf}}. \end{cases}$$

The case-by-case study of the intersection $S \cap R$ is carried out as above and yields the same results. The only difference lies in the fact that the sector $S$ exits the square $R$ through the top edge instead of the left edge (see Figures 15, 16, and 17 below). Notice that (59)(e) is satisfied as $q_1 > \gamma$ when $2 \leq \gamma \leq \gamma_{\mathrm{conf}}$.

• *Case 2*: $\gamma_1 < \gamma \leq \gamma_2$ (see Figure 15).

• *Case 3*: $\gamma_2 \leq \gamma < \gamma_{\mathrm{conf}}$.

○ *Subcase 3.1*: $\gamma_1 < \gamma \leq \gamma_2$ (see Figure 16).

○ *Subcase 3.2*: $\gamma_2 \leq \gamma < 2$ (see Figure 17).

This concludes the proof of Theorem 6.1 for $1 < \gamma \leq \gamma_{\mathrm{conf}}$ and $n = 4, 5$.

► Assume that $n = 3$.
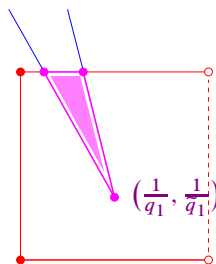


**Figure 17.** Subcase $2 \leq \gamma \leq \gamma_{\mathrm{conf}}$.
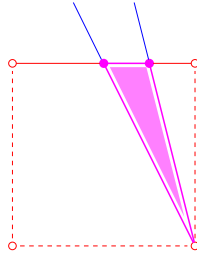
**Figure 18.** Case $\gamma = 2$.

The proof works the same except that the square becomes

$$R = \begin{cases} \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right) & \text{if } 1 < \gamma < 2, \\ \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right] & \text{if } 2 \leq \gamma \leq \gamma_{\text{conf}} \end{cases} \tag{65}$$

and that $(1/q_1, 1/\tilde{q}_1)$ enters the square $R$ through the vertex $\left(\frac{1}{2}, 0\right)$ instead of the bottom edge. This happens when $\gamma = 2$ (see Figure 18), and in this case, (59)(e) is satisfied. It is further satisfied when $2 < \gamma \leq \gamma_{\text{conf}}$ as $q_1 > \gamma$.

This concludes the proof of Theorem 6.1 for $1 < \gamma \leq \gamma_{\text{conf}}$. $\qquad\qquad\qquad\qquad\qquad\square$

*Proof of Theorem 6.1 for $\gamma_{\text{conf}} \leq \gamma \leq \gamma_4$.* We resume the fixed-point method above, using Corollary 5.3 instead of Theorem 5.2, and obtain in this way the set of conditions

$$\begin{cases} \text{(a)} & 2 \leq p \leq \infty \text{ and } 2 \leq q < \infty \text{ satisfy } \dfrac{1}{p} \leq \dfrac{n-1}{2}\left(\dfrac{1}{2} - \dfrac{1}{q}\right), \\[2mm] \text{(ã)} & 2 \leq \tilde{p} \leq \infty \text{ and } 2 \leq \tilde{q} < \infty \text{ satisfy } \dfrac{1}{\tilde{p}} \leq \dfrac{n-1}{2}\left(\dfrac{1}{2} - \dfrac{1}{\tilde{q}}\right), \\[2mm] \text{(b)} & \sigma \geq n\left(\dfrac{1}{2} - \dfrac{1}{q}\right) - \dfrac{1}{p}, \tilde{\sigma} \geq n\left(\dfrac{1}{2} - \dfrac{1}{\tilde{q}}\right) - \dfrac{1}{\tilde{p}}, \text{ and } \sigma + \tilde{\sigma} \leq 1, \\[2mm] \text{(c)} & \dfrac{\gamma}{p} + \dfrac{1}{\tilde{p}} = 1, \\[2mm] \text{(d)} & 1 \leq \dfrac{\gamma}{q} + \dfrac{1}{\tilde{q}} \leq 1 + \dfrac{1-\sigma-\tilde{\sigma}}{n}, \\[2mm] \text{(e)} & q > \gamma. \end{cases} \tag{66}$$

We may assume that

$$\sigma = n\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{p} \quad \text{and} \quad \tilde{\sigma} = n\left(\frac{1}{2} - \frac{1}{\tilde{q}}\right) - \frac{1}{\tilde{p}}.$$

With this choice, the conditions

$$\sigma + \tilde{\sigma} \leq 1 \quad \text{and} \quad \frac{\gamma}{q} + \frac{1}{\tilde{q}} \leq 1 + \frac{1-\sigma-\tilde{\sigma}}{n}$$

become

$$\frac{1}{p} + \frac{1}{\tilde{p}} + 1 \geq n\left(1 - \frac{1}{q} - \frac{1}{\tilde{q}}\right) \tag{67}$$

and

$$\frac{1}{p} + \frac{1}{\tilde{p}} + 1 \geq (\gamma - 1)\frac{n}{q}. \tag{68}$$

Notice moreover that (67) follows from (68), combined with $\gamma/q + 1/\tilde{q} \geq 1$, and that (68) can be rewritten as follows, using (66)(c):

$$\frac{1}{p} + \frac{n}{q} \leq \frac{2}{\gamma - 1}.$$

Thus, (66) reduces to the set of conditions

$$\begin{cases} \text{(a)} & 2 \leq p \leq \infty \text{ and } 2 \leq q < \infty \text{ satisfy } \frac{1}{p} \leq \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{q}\right), \\[2mm] \text{(ã)} & 2 \leq \tilde{p} \leq \infty \text{ and } 2 \leq \tilde{q} < \infty \text{ satisfy } \frac{1}{\tilde{p}} \leq \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{\tilde{q}}\right), \\[2mm] \text{(c)} & \frac{\gamma}{p} + \frac{1}{\tilde{p}} = 1, \\[2mm] \text{(d)(i)} & \frac{\gamma}{q} + \frac{1}{\tilde{q}} \geq 1, \\[2mm] \text{(d)(ii)} & \frac{1}{p} + \frac{n}{q} \leq \frac{2}{\gamma - 1}, \\[2mm] \text{(e)} & q > \gamma. \end{cases} \tag{69}$$

Among these conditions, consider first (69)(a) and (69)(d)(ii). In the plane with coordinates $(1/p, 1/q)$, the two lines

$$\begin{cases} \text{(a)} & \frac{1}{p} + \frac{n-1}{2}\frac{1}{q} = \frac{n-1}{4}, \\[2mm] \text{(d)(ii)} & \frac{1}{p} + \frac{n}{q} = \frac{2}{\gamma - 1} \end{cases} \tag{70}$$
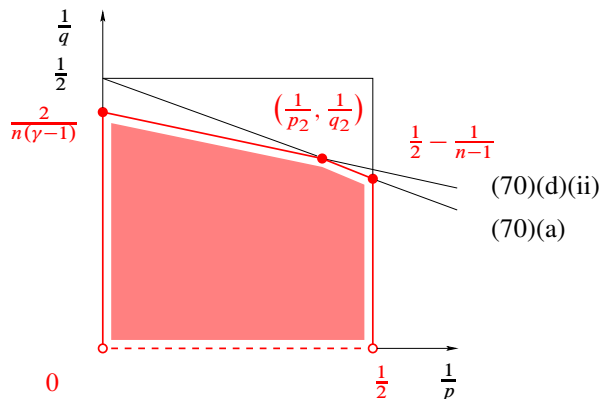


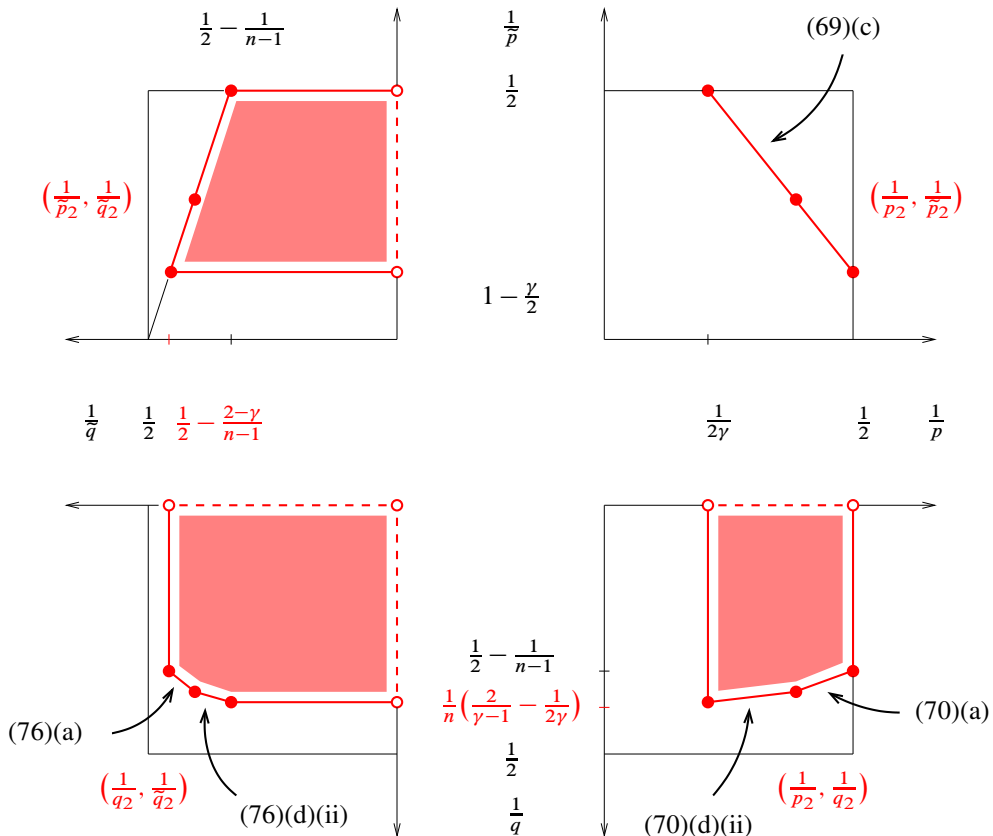**Figure 19.** *Case 4*: $\gamma_{\text{conf}} \leq \gamma \leq \gamma_3$.
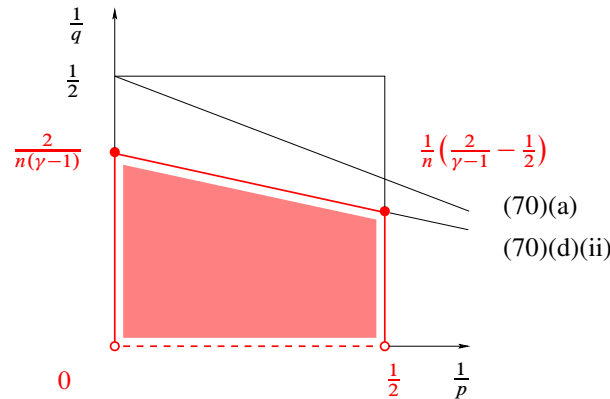
**Figure 20.** *Case 5*: $\gamma_3 \leq \gamma \leq \gamma_4$.

meet at the point $(1/p_2, 1/q_2)$ given by

$$
\begin{cases}
\dfrac{1}{p_2} = \dfrac{n-1}{n+1}\Big(\dfrac{n}{2} - \dfrac{2}{\gamma-1}\Big), \\[2ex]
\dfrac{1}{q_2} = \dfrac{1}{n+1}\Big(\dfrac{4}{\gamma-1} - \dfrac{n-1}{2}\Big).
\end{cases}
\tag{71}
$$

As $\gamma$ varies between $\gamma_{\mathrm{conf}}$ and $\gamma_3$, this point moves on the line (70)(a) between the Strichartz point $(1/2-1/(n+1), 1/2-1/(n+1))$ and the Keel–Tao endpoint $(1/2, 1/2-1/(n-1))$, where it exits the square $\big[0, \frac{1}{2}\big] \times \big(0, \frac{1}{2}\big]$. Thus, (69)(a) and (69)(d)(ii) determine the regions depicted in Figure 19 and in Figure 20. For later use, notice that the minimal regularity

$$
\sigma = n\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{1}{p} \geq \sigma_3(\gamma)
\tag{72}
$$

is reached on the boundary line (70)(d)(ii) and that

$$
p_2 < 2\gamma.
\tag{73}
$$

This inequality holds indeed when $\gamma = \gamma_{\mathrm{conf}}$, and it remains true as $\gamma$ increases while $p_2$ decreases.

Let us next discuss all conditions (69), first in high dimensions and next in low dimensions.

▶ Assume that $n \geq 6$.

Firstly, notice that (69)(e) is trivially satisfied in this case. On one hand, we have indeed $\gamma \leq \gamma_4 \leq 2$. On the other hand, it follows from (69)(d)(ii) that

$$
\frac{1}{q} \leq \frac{2}{n(\gamma-1)} \leq \frac{2}{n(\gamma_{\mathrm{conf}}-1)} = \frac{1}{2}\Big(1 - \frac{1}{n}\Big) < \frac{1}{2}.
$$

Hence, $\gamma \leq 2 < q$.

Secondly, we claim that (69)(a), (69)(ã), (69)(c), and (69)(d)(ii) reduce to the conditions

$$
\begin{cases}
\text{(a)} & \dfrac{\gamma}{q} + \dfrac{1}{\tilde{q}} \le \dfrac{\gamma+1}{2} - \dfrac{2}{n-1}, \\[2mm]
\text{(d)(ii)} & \dfrac{\gamma}{q} + \dfrac{n-1}{2n}\dfrac{1}{q} \le \dfrac{n+3}{4n} + \dfrac{2}{n}\dfrac{1}{\gamma-1}
\end{cases}
\tag{74}
$$

in the rectangle

$$
R = \left(0, \frac{1}{n}\left(\frac{2}{\gamma-1} - \frac{1}{2\gamma}\right)\right] \times \left(0, \frac{1}{2} - \frac{2-\gamma}{n-1}\right].
\tag{75}
$$

Actually, they even reduce to the single condition (74)(d)(ii) if $\gamma \ge \gamma_3$. All these claims are obtained by examining Figures 21 and 22 as we did with Figure 7 in the case $\gamma \le \gamma_{\text{conf}}$.
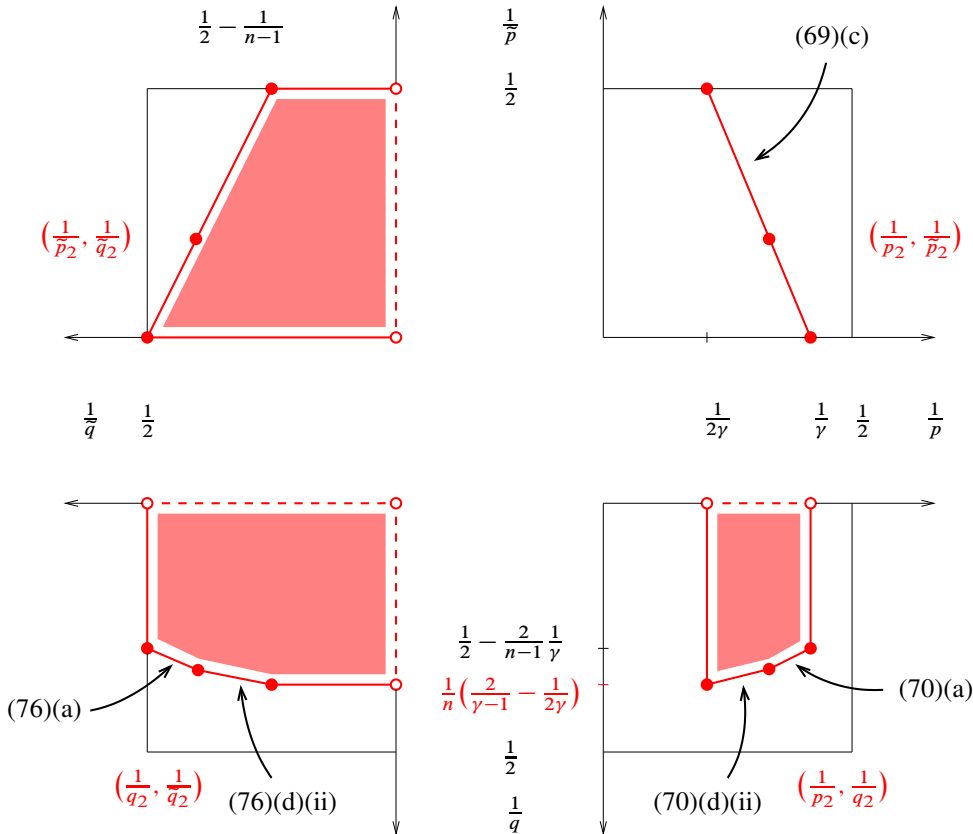


**Figure 21.** *Case 4*: $\gamma_{\text{conf}} \le \gamma \le \gamma_3$.
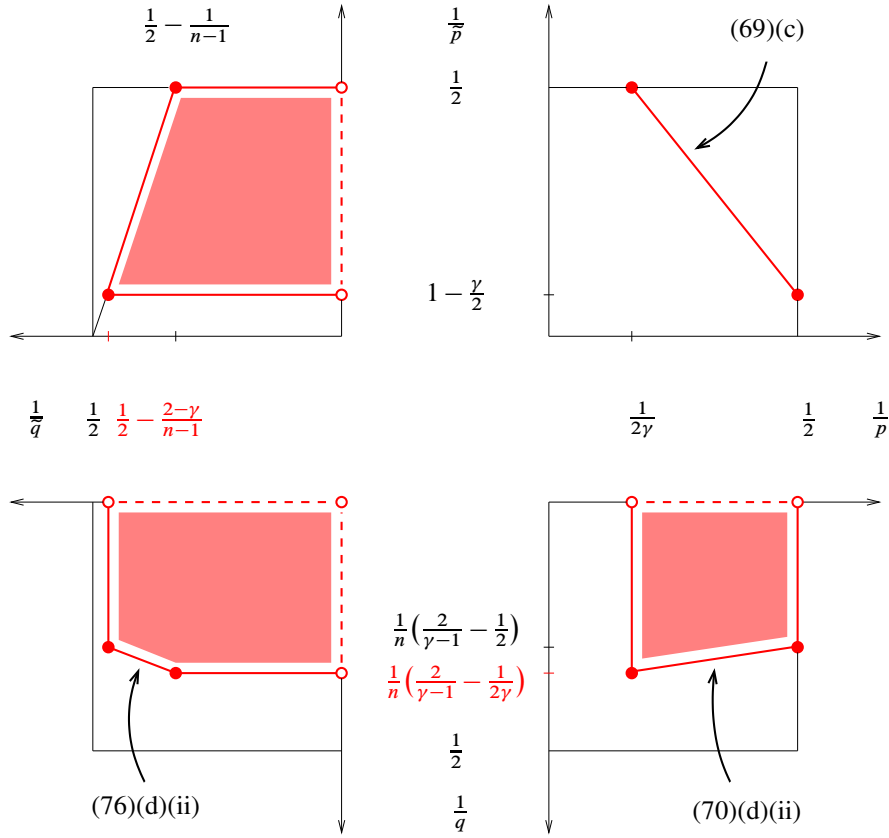
**Figure 22.** *Case 5: $\gamma_3 \le \gamma \le \gamma_4$.*
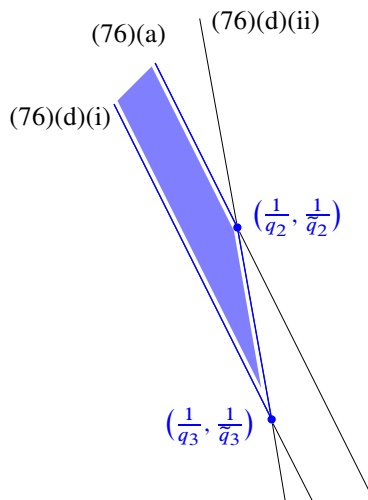


**Figure 23.** Convex region $C$.
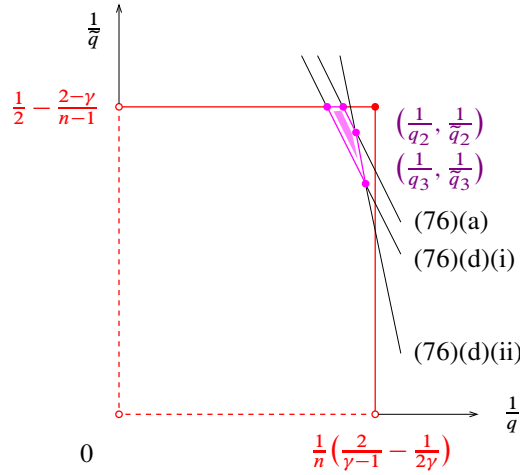
**Figure 24.** *Case 4*: $\gamma_{\mathrm{conf}} \leq \gamma \leq \gamma_3$.

Thirdly, in the plane with coordinates $(1/q, 1/\tilde{q})$, the conditions (69)(d)(i), (74)(a), and (74)(d)(ii) define the convex region $C$ in Figure 23 with edges

$$
\begin{cases}
\text{(a)} & \dfrac{\gamma}{q} + \dfrac{1}{\tilde{q}} = \dfrac{\gamma+1}{2} - \dfrac{2}{n-1}, \\[2mm]
\text{(d)(i)} & \dfrac{\gamma}{q} + \dfrac{1}{\tilde{q}} = 1, \\[2mm]
\text{(d)(ii)} & \dfrac{\gamma}{q} + \dfrac{n-1}{2n}\dfrac{1}{\tilde{q}} = \dfrac{n+3}{4n} + \dfrac{2}{n}\dfrac{1}{\gamma-1}
\end{cases}
\tag{76}
$$
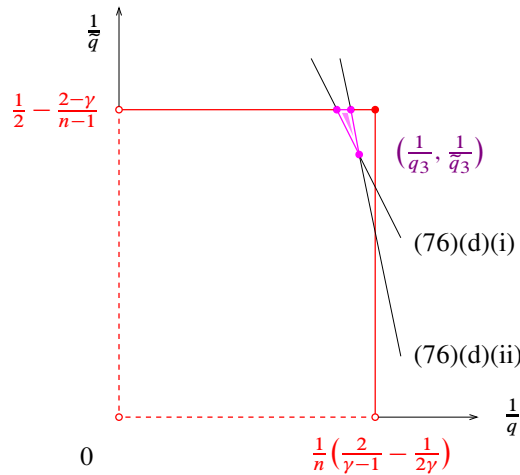


**Figure 25.** *Case 5*: $\gamma_3 \leq \gamma \leq \gamma_4$.

and with vertices given by

$$
\begin{cases}
\dfrac{1}{q_2} = \dfrac{4}{n+1}\dfrac{1}{\gamma-1} - \dfrac{1}{2}\dfrac{n-1}{n+1}, & \dfrac{1}{\widetilde{q}_2} = \dfrac{n}{n+1}\gamma - \dfrac{4}{n+1}\dfrac{1}{\gamma-1} + \dfrac{1}{2} - \dfrac{2}{n-1} - \dfrac{4}{n+1}, \\[2ex]
\dfrac{1}{q_3} = \dfrac{4}{n+1}\dfrac{1}{\gamma-1} - \dfrac{1}{2}\dfrac{n+3}{n+1}\dfrac{1}{\gamma}, & \dfrac{1}{\widetilde{q}_3} = \dfrac{3}{2}\dfrac{n-1}{n+1} - \dfrac{4}{n+1}\dfrac{1}{\gamma-1}.
\end{cases}
\tag{77}
$$

In order to conclude, it remains for us to determine the possible intersections of the convex region $C$ above with the rectangle $R$ defined by (75) and in each case the minimal regularity $\sigma = n(1/2 - 1/q) - 1/p$.

• *Case 4*: $\gamma_{\mathrm{conf}} \le \gamma \le \gamma_3$ (see Figure 24).

• *Case 5*: $\gamma_3 \le \gamma \le \gamma_4$ (see Figure 25).

In both cases, the minimal regularity $\sigma = \sigma_3(\gamma)$ is reached when $(1/p, 1/q)$ and $(1/q, 1/\widetilde{q})$ lie on the edges (70)(d)(ii) and (76)(d)(ii). See Figures 21 and 22. This concludes the proof of Theorem 6.1 for $\gamma_{\mathrm{conf}} < \gamma \le \gamma_4$ and $n \ge 6$.

▶ Assume that $3 \le n \le 5$.

Then $\gamma \ge \gamma_{\mathrm{conf}} \ge 2$, and Figures 21 and 22 become Figures 26 and 27, respectively. Consequently, the four conditions (69)(a), (69)(ã), (69)(c), and (69)(d.ii) reduce again to the two conditions (74)(a)



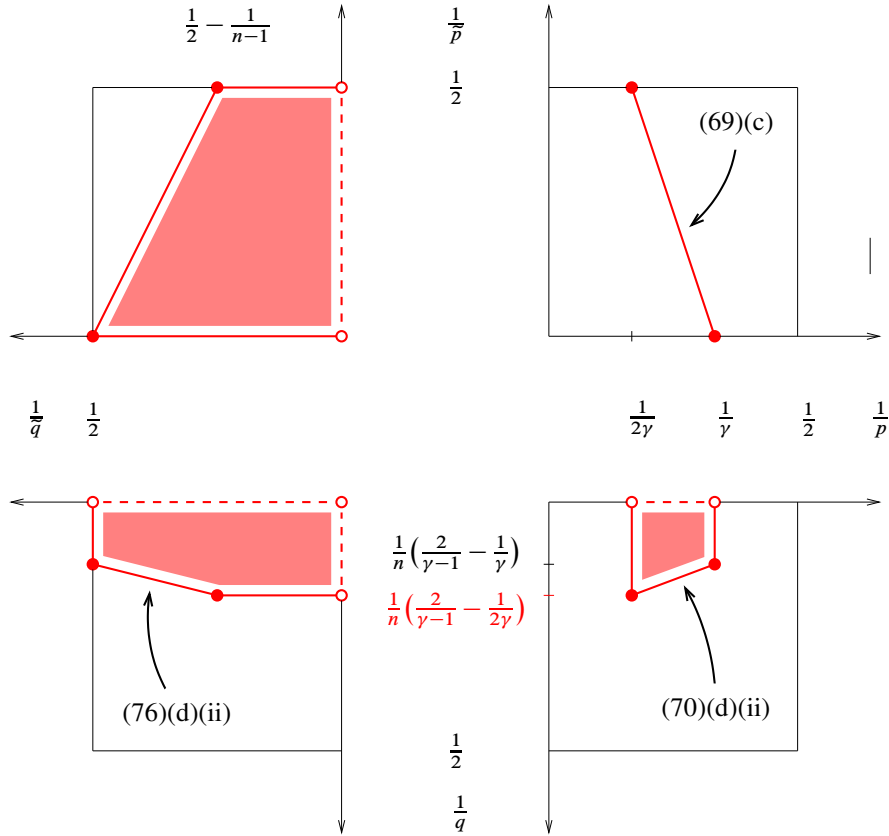**Figure 26.** *Case 4*: $\gamma_{\mathrm{conf}} \le \gamma \le \gamma_3$.

**Figure 27.** *Case 5*: $\gamma_3 \leq \gamma \leq \gamma_4$.
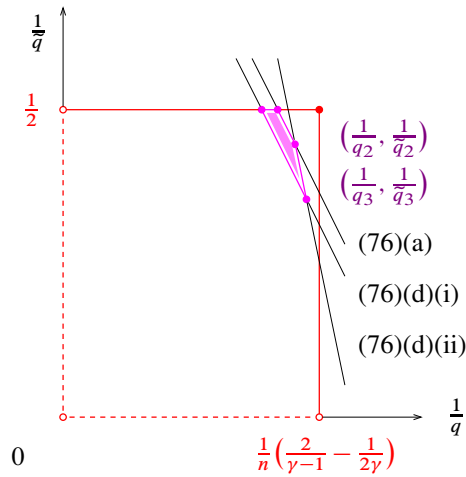


**Figure 28.** *Case 4*: $\gamma_{\mathrm{conf}} \leq \gamma \leq \gamma_3$.
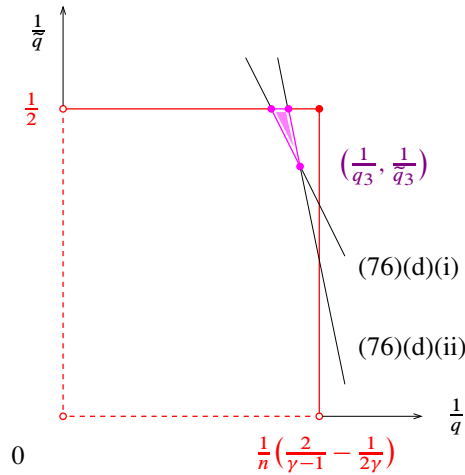
**Figure 29.** *Case 5*: $\gamma_3 \leq \gamma \leq \gamma_4$.

and (74)(d)(ii) if $\gamma_{\mathrm{conf}} \leq \gamma \leq \gamma_3$, and actually to the single condition (74)(d)(ii) if $\gamma_3 \leq \gamma \leq \gamma_4$, but this time in the rectangle

$$R = \left(0, \frac{1}{n}\left(\frac{2}{\gamma-1} - \frac{1}{2\gamma}\right)\right] \times \left(0, \frac{1}{2}\right]. \tag{78}$$

Moreover, (69)(e) is satisfied as $1/q \leq (2/(\gamma - 1) - 1/2\gamma)/n < 1/\gamma$.

We conclude again by examining the possible intersections $C \cap R$ of the convex region defined by (69)(d)(i), (74)(a), and (74)(d)(ii) with the rectangle (78) and by determining in each case the minimal regularity $\sigma = n(1/2 - 1/q) - 1/p$.

- *Case 4*: $\gamma_{\mathrm{conf}} \leq \gamma \leq \gamma_3$ (see Figure 28).

- *Case 5*: $\gamma_3 \leq \gamma \leq \gamma_4$ (see Figure 29).

In both cases, the minimal regularity $\sigma = \sigma_3(\gamma)$ is reached again when $(1/p, 1/q)$ and $(1/q, 1/\tilde{q})$ lie on the edges (70)(d)(ii) and (76)(d)(ii). See Figures 26 and 27. This concludes the proof of Theorem 6.1 for $\gamma_{\mathrm{conf}} < \gamma \leq \gamma_4$ and $3 \leq n \leq 5$. $\qquad\square$
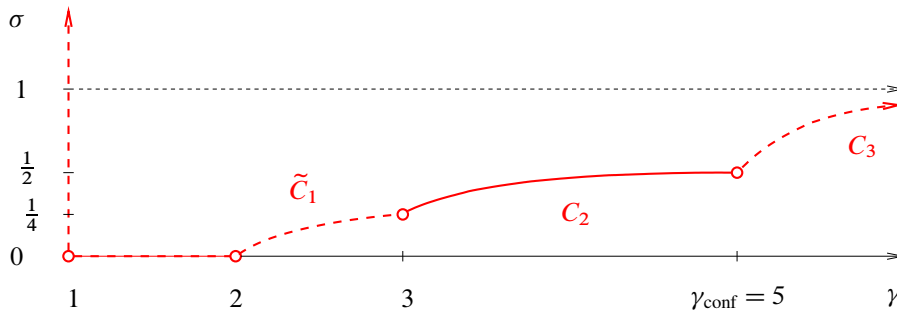


**Figure 30.** Regularity for global well-posedness on $\mathbb{H}^2$.

**Remark 6.3.** In dimension $n = 2$, the statement of Theorem 6.1 holds true with (53) replaced by

$$\begin{cases} \sigma = 0^+ & \text{if } 1 < \gamma \leq 2, \\ \sigma = \tilde{\sigma}_1(\gamma)^+ & \text{if } 2 \leq \gamma \leq 3, \\ \sigma = \sigma_2(\gamma) & \text{if } 3 < \gamma < 5, \\ \sigma = \sigma_3(\gamma)^+ & \text{if } 5 \leq \gamma < \infty, \end{cases} \tag{79}$$

where $\tilde{\sigma}_1(\gamma) = 3/4 - 3/2\gamma$. Notice that the condition $q > \gamma$ is not redundant if $2 < \gamma < 3$ and that it is actually responsible for the curve $\tilde{C}_1$.

**Remark 6.4.** In dimension $n = 3$, Metcalfe and Taylor [2011] obtain a global existence result beyond $\gamma = \gamma_4$. In [Anker and Pierfelice $\geq 2014$], we extend the results of our present paper to Damek–Ricci spaces as we did for the Schrödinger equation in [Anker et al. 2011] and for the shifted wave equation in [Anker et al. 2014], and we also discuss the case $\gamma > \gamma_4$ in this more general setting.

## Appendix A

In this appendix, we collect some lemmas in Fourier analysis on $\mathbb{R}$, which are used in the kernel analysis carried out in Section 3.

**Lemma A.1.** *Consider the oscillatory integral*

$$I(t, x) = \int_{-\infty}^{+\infty} d\lambda \, a(\lambda) e^{it\phi(\lambda)}$$

*where the phase is given by*

$$\phi(\lambda) = \sqrt{\lambda^2 + \kappa^2} - \frac{x\lambda}{t}$$

*(recall that $\kappa$ is a fixed constant $> 0$) and the amplitude $a \in \mathscr{C}_c^\infty(\mathbb{R})$ has the behavior at the origin*

$$a(\lambda) = O(\lambda^2). \tag{A-1}$$

*Then*

$$|I(t, x)| \lesssim \frac{1 + |x|}{(1 + |t|)^{3/2}} \quad \forall |x| \leq \frac{|t|}{2}.$$

*Proof.* Let us compute the first two derivatives

$$\phi'(\lambda) = \frac{\lambda}{\sqrt{\lambda^2 + \kappa^2}} - \frac{x}{t} \quad \text{and} \quad \phi''(\lambda) = \kappa^2(\lambda^2 + \kappa^2)^{-3/2}. \tag{A-2}$$

The phase $\phi$ has a single stationary point:

$$\lambda_0 = \kappa \frac{x}{t} \left(1 - \frac{x^2}{t^2}\right)^{-1/2}, \tag{A-3}$$

which remains bounded under our assumption $|x| \leq |t|/2$:

$$|\lambda_0| \leq \frac{\kappa}{\sqrt{3}} \leq \kappa. \tag{A-4}$$

For later use, let us compute

$$\phi(\lambda_0) = \kappa \left(1 - \frac{x^2}{t^2}\right)^{1/2} \quad \text{and} \quad \phi''(\lambda_0) = \kappa^{-1} \left(1 - \frac{x^2}{t^2}\right)^{3/2}.$$

Since $\phi'' > 0$, we can perform a global change of variables $\lambda \leftrightarrow \mu$ on $\mathbb{R}$ so that

$$\phi(\lambda) - \phi(\lambda_0) = \mu^2.$$

Specifically,

$$\mu = \epsilon(\lambda)(\lambda - \lambda_0),$$

where

$$\epsilon(\lambda) = \left\{ \int_0^1 ds\, (1-s)\phi''\big((1-s)\lambda_0 + s\lambda\big) \right\}^{1/2}.$$

This way, our oscillatory integral becomes

$$I(t, x) = e^{it\phi(\lambda_0)} \int_{\mathbb{R}} d\mu\, \tilde{a}(\mu) e^{(-1+it)\mu^2},$$

where

$$\tilde{a}(\mu) = \frac{d\lambda}{d\mu} a(\lambda(\mu)) e^{\mu^2}$$

is again a smooth function with compact support whose derivatives are controlled uniformly in $t$ and $x$ as long as $|x| \leq |t|/2$. Using Taylor's formula, let us expand

$$\tilde{a}(\mu) = \sum_{j=0}^3 \tilde{a}_j \mu^j + \tilde{a}_4(\mu)\mu^4,$$

where

$$\tilde{a}_0 = \left(\frac{2}{\phi''(\lambda_0)}\right)^{1/2} a(\lambda_0) = \mathrm{O}(\lambda_0^2) = \mathrm{O}\left(\frac{x^2}{t^2}\right),$$

the other constants $\tilde{a}_1$, $\tilde{a}_2$, and $\tilde{a}_3$, and the function $\tilde{a}_4(\mu)$, as well as its derivatives, are bounded uniformly in $t$ and $x$. Let us split up accordingly

$$I(t, x) = \sum_{j=0}^4 I_j(t, x),$$

where

$$I_j(t, x) = \tilde{a}_j e^{it\phi(\lambda_0)} \int_{\mathbb{R}} d\mu\, \mu^j e^{(-1+it)\mu^2} \quad (j = 0, 1, 2, 3)$$

and

$$I_4(t, x) = e^{it\phi(\lambda_0)} \int_{\mathbb{R}} d\mu\, \tilde{a}_4(\mu)\mu^4 e^{(-1+it)\mu^2}.$$

The first and third expressions are handled by elementary complex integration:

$$I_0(t, x) = \tilde{a}_0 \sqrt{\pi} e^{it\phi(\lambda_0)} (1 - it)^{-1/2} = \mathrm{O}\left(\frac{x^2}{t^2(1+|t|)^{1/2}}\right) = \mathrm{O}\left(\frac{1+|x|}{(1+|t|)^{3/2}}\right),$$

$$I_2(t, x) = \tilde{a}_2 \frac{\sqrt{\pi}}{2} e^{it\phi(\lambda_0)} (1 - it)^{-3/2} = \mathrm{O}\big((1+|t|)^{-3/2}\big).$$

The expressions $I_1(t, x)$ and $I_3(t, x)$ vanish by oddness. The expression $I_4(t, x)$ is obviously bounded by the finite integral

$$\int_{\mathbb{R}} d\mu \, \mu^4 e^{-\mu^2}.$$

In order to improve this estimate when $|t|$ is large, let us split up

$$\int_{\mathbb{R}} d\mu = \int_{|\mu| \leq |t|^{-1/2}} d\mu + \int_{|\mu| > |t|^{-1/2}} d\mu.$$

The first integral is easily estimated, using the uniform boundedness of $\tilde{a}_4(\mu)$:

$$\left| \int_{|\mu| \leq |t|^{-1/2}} d\mu \, \tilde{a}_4(\mu) \mu^4 e^{(-1+it)\mu^2} \right| \lesssim \int_{|\mu| \leq |t|^{-1/2}} d\mu \, \mu^4 \lesssim |t|^{-5/2}.$$

After two integration by parts, using $\mu e^{(-1+it)\mu^2} = (2(-1+it))^{-1} \frac{\partial}{\partial \mu} e^{(-1+it)\mu^2}$, the second integral is estimated by

$$|t|^{-5/2} + |t|^{-2} \int_{\mathbb{R}} d\mu \, (1 + |\mu|)^2 e^{-\mu^2}.$$

Altogether,

$$I_4(t, x) = \mathrm{O}((1 + |t|)^{-2}),$$

and this concludes the proof of Lemma A.1. $\qquad\square$

**Lemma A.2.** *Consider the oscillatory integral*

$$J(t, x) = \int_{-\infty}^{+\infty} d\lambda \, a(\lambda) e^{it\phi(\lambda)}$$

*where the phase is given again by*

$$\phi(\lambda) = \sqrt{\lambda^2 + \kappa^2} - \frac{x\lambda}{t}$$

*and the amplitude $a(\lambda)$ is now a symbol (of any order) on $\mathbb{R}$, which vanishes on the interval $[-\kappa, \kappa]$. Then*

$$J(t, x) = \mathrm{O}(|t|^{-\infty}) \quad \forall x, \, 0 \leq |x| \leq \frac{|t|}{2}.$$

*Proof.* According to (A-2), (A-3), and (A-4),

- $\phi$ has a single stationary point $\lambda_0 \in \left[ -\frac{\kappa}{\sqrt{3}}, \frac{\kappa}{\sqrt{3}} \right]$, which remains away from the support of $a$,
- $|\phi'(\lambda)| = \left| \frac{\lambda}{\sqrt{\lambda^2 + \kappa^2}} - \frac{x}{t} \right| \geq \frac{1}{\sqrt{2}} - \frac{1}{2} > 0$ on supp $a$,
- $\phi''$ is a symbol of order $-3$.

These facts allow us to perform several integrations by parts based on

$$e^{it\phi(\lambda)} = \frac{1}{it\phi'(\lambda)} \frac{\partial}{\partial \lambda} e^{it\phi(\lambda)}$$

and to reach the conclusion. $\qquad\square$

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# PROBABILISTIC GLOBAL WELL-POSEDNESS FOR THE SUPERCRITICAL NONLINEAR HARMONIC OSCILLATOR

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Thanks to an approach inspired by Burq and Lebeau [ *Ann. Sci. Éc. Norm. Supér.* (4) **6**:6 (2013)], we prove stochastic versions of Strichartz estimates for Schrödinger with harmonic potential. As a consequence, we show that the nonlinear Schrödinger equation with quadratic potential and any polynomial nonlinearity is almost surely locally well-posed in $L^2(\mathbb{R}^d)$ for any $d \geq 2$. Then, we show that we can combine this result with the high-low frequency decomposition method of Bourgain to prove a.s. global well-posedness results for the cubic equation: when $d = 2$, we prove global well-posedness in $\mathcal{H}^s(\mathbb{R}^2)$ for any $s > 0$, and when $d = 3$ we prove global well-posedness in $\mathcal{H}^s(\mathbb{R}^3)$ for any $s > \frac{1}{6}$, which is a supercritical regime.

Furthermore, we also obtain almost sure global well-posedness results with scattering for NLS on $\mathbb{R}^d$ without potential. We prove scattering results for $L^2$-supercritical equations and $L^2$-subcritical equations with initial conditions in $L^2$ without additional decay or regularity assumption.

## 1. Introduction and results

**1A. *Introduction.*** It is known from several works that a probabilistic approach can help to give insight into the dynamics of dispersive nonlinear PDEs, even for low Sobolev regularity. This point of view was initiated by Lebowitz, Rose and Speer [1988], developed by Bourgain [1994; 1996] and Zhidkov [2001], and enhanced by Tzvetkov [2006; 2008; 2010], Burq and Tzvetkov [2008a; 2008b], Oh [2009/10; 2009], Colliander and Oh [2012] and others. In this paper we study the Cauchy problem for the nonlinear Schrödinger–Gross–Pitaevskii equation

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u - |x|^2 u = \pm |u|^{p-1} u, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0) = u_0, \end{cases} \tag{1-1}$$

with $d \geq 2$, $p \geq 3$ an odd integer and where $u_0$ is a random initial condition.

Much work has been done on dispersive PDEs with random initial conditions since the papers of Burq and Tzvetkov [2008a; 2008b]. In these articles, the authors showed that, thanks to a randomisation of the initial condition, one can prove well-posedness results even for data with supercritical Sobolev regularity. We also refer to [Burq and Tzvetkov 2014; Thomann 2009; Burq et al. 2010; Poiret 2012a; 2012b;

de Suzzoni 2013; Nahmod and Staffilani 2013] for strong solutions in a probabilistic sense. Concerning weak solutions, see [Burq et al. 2012; ≥ 2014; Nahmod et al. 2013].

More recently, Burq and Lebeau [2013] considered a different randomisation method, and thanks to fine spectral estimates they obtained better stochastic bounds, which enabled them to improve the previous known results for the supercritical wave equation on a compact manifold. In [Poiret et al. 2013] we extended the results of [Burq and Lebeau 2013] to the harmonic oscillator in $\mathbb{R}^d$. This approach enables us to prove a stochastic version of the usual Strichartz estimates with a gain of $d/2$ derivatives, which we will use here to apply to the nonlinear problem. These estimates (the result of Proposition 2.1) can be seen as a consequence of [Poiret et al. 2013, Inequality (1.6)], but we give here an alternative proof suggested by Nicolas Burq.

Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and let $\{g_n\}_{n \geq 0}$ be a sequence of real random variables, which we will assume to be independent and identically distributed. We assume that the common law $\nu$ of $g_n$ satisfies, for some $c > 0$, the bound

$$\int_{-\infty}^{+\infty} e^{\gamma x} \, d\nu \leq e^{c\gamma^2} \quad \text{for all } \gamma \in \mathbb{R}. \tag{1-2}$$

This condition implies in particular that the $g_n$ are centred variables. It is easy to check that (1-2) is satisfied for centred Gauss laws and for any centred law with bounded support. Under condition (1-2), we can prove the Khinchin inequality (Lemma 2.3), which we will use in the sequel.

Let $d \geq 2$. We denote by

$$H = -\Delta + |x|^2$$

the harmonic oscillator and by $\{\varphi_j \mid j \geq 1\}$ an orthonormal basis of $L^2(\mathbb{R}^d)$ of eigenvectors of $H$ (the Hermite functions). The eigenvalues of $H$ are the $\{2(\ell_1 + \cdots + \ell_d) + d \mid \ell \in \mathbb{N}^d\}$, and we can order them in a non-decreasing sequence $\{\lambda_j \mid j \geq 1\}$, repeated according to their multiplicities, and so that $H\varphi_j = \lambda_j \varphi_j$.

We define the harmonic Sobolev spaces for $s \geq 0$, $p \geq 1$ by

$$\mathscr{W}^{s,p} = \mathscr{W}^{s,p}(\mathbb{R}^d) = \{u \in L^p(\mathbb{R}^d) \mid H^{s/2}u \in L^p(\mathbb{R}^d)\},$$
$$\mathscr{H}^s = \mathscr{H}^s(\mathbb{R}^d) = \mathscr{W}^{s,2}.$$

The natural norms are denoted by $\|u\|_{\mathscr{W}^{s,p}}$ and up to equivalence of norms, for $1 < p < +\infty$, we have [Yajima and Zhang 2004, Lemma 2.4]

$$\|u\|_{\mathscr{W}^{s,p}} = \|H^{s/2}u\|_{L^p} \equiv \|(-\Delta)^{s/2}u\|_{L^p} + \|\langle x \rangle^s u\|_{L^p}.$$

For $j \geq 1$, let

$$I(j) = \{n \in \mathbb{N} \mid 2j \leq \lambda_n < 2(j+1)\}.$$

Observe that, for all $j \geq d/2$, $I(j) \neq \varnothing$ and that $\#I(j) \sim c_d j^{d-1}$ when $j \to +\infty$.

Let $s \in \mathbb{R}$. Any $u \in \mathscr{H}^s(\mathbb{R}^d)$ can be written in a unique fashion as

$$u = \sum_{j=1}^{+\infty} \sum_{n \in I(j)} c_n \varphi_n.$$

Following a suggestion of Nicolas Burq, we introduce the condition

$$|c_k|^2 \leq \frac{C}{\#I(j)} \sum_{n \in I(j)} |c_n|^2 \quad \text{for all } j \geq 1 \text{ and } k \in I(j), \tag{1-3}$$

which means that the coefficients have almost the same size on each level of energy $I(j)$. Observe that this condition is always satisfied in dimension $d = 1$. We define the set $\mathcal{A}_s \subset \mathcal{H}^s(\mathbb{R}^d)$ by

$$\mathcal{A}_s = \left\{ u = \sum_{j=1}^{+\infty} \sum_{n \in I(j)} c_n \varphi_n \in \mathcal{H}^s(\mathbb{R}^d) \, \middle| \, \text{condition (1-3) holds for some } C > 0 \right\}.$$

It is easy to check the following properties:

- If $u \in \mathcal{A}_s$, then for all $c \in \mathbb{C}$, $cu \in \mathcal{A}_s$.
- The set $\mathcal{A}_s$ is neither closed nor open in $\mathcal{H}^s$.
- The set $\mathcal{A}_s$ is invariant under the linear Schrödinger flow $e^{-itH}$.
- The set $\mathcal{A}_s$ depends on the choice of the orthonormal basis $(\varphi_n)_{n \geq 1}$. Indeed, given $u \in \mathcal{H}^s$, it is easy to see that there exists a Hilbertian basis $(\widetilde{\varphi}_n)_{n \geq 1}$ such that $u \in \widetilde{\mathcal{A}}_s$, where $\widetilde{\mathcal{A}}_s$ is the space based on $(\widetilde{\varphi}_n)_{n \geq 1}$.

Let $\gamma \in \mathcal{A}_s$. We define the probability measure $\mu_\gamma$ on $\mathcal{H}^s$ via the map

$$\Omega \to \mathcal{H}^s(\mathbb{R}^d),$$
$$\omega \mapsto \gamma^\omega = \sum_{j=1}^{+\infty} \sum_{n \in I(j)} c_n g_n(\omega) \varphi_n.$$

In other words, $\mu_\gamma$ is defined by the condition, that for all measurable $F : \mathcal{H}^s \to \mathbb{R}$,

$$\int_{\mathcal{H}^s(\mathbb{R}^d)} F(v) \, d\mu_\gamma(v) = \int_\Omega F(\gamma^\omega) \, d\mathbb{P}(\omega).$$

In particular, we can check that $\mu_\gamma$ satisfies:

- If $\gamma \in \mathcal{H}^s \backslash \mathcal{H}^{s+\varepsilon}$, then $\mu_\gamma(\mathcal{H}^{s+\varepsilon}) = 0$.
- Assume that for all $j \geq 1$ such that $I(j) \neq \varnothing$ we have $c_j \neq 0$. Then for all nonempty open subsets $B \subset \mathcal{H}^s$, $\mu_\gamma(B) > 0$.

Finally, we denote by $\mathcal{M}^s$ the set of all such measures, $\mathcal{M}^s = \bigcup_{\gamma \in \mathcal{A}_s} \{\mu_\gamma\}$.

**1B. *Main results.*** Before we state our results, let us recall some facts concerning the deterministic study of the nonlinear Schrödinger equation (1-1). We say that (1-1) is locally well-posed in $\mathcal{H}^s(\mathbb{R}^d)$ if, for any initial condition $u_0 \in \mathcal{H}^s(\mathbb{R}^d)$, there exists a unique local in time solution $u \in \mathscr{C}([-T, T]; \mathcal{H}^s(\mathbb{R}^d))$, and if the flow-map is uniformly continuous. We denote by

$$s_c = \frac{d}{2} - \frac{2}{p-1}$$

the critical Sobolev index. Then one can show that NLS is well-posed in $\mathcal{H}^s(\mathbb{R}^d)$ when $s > \max(s_c, 0)$, and ill-posed when $s < s_c$. We refer to the introduction of [Thomann 2009] for more details on this topic.

**1B1.** *Local existence results.* We are now able to state our first result on the local well-posedness of (1-1).

**Theorem 1.1.** *Let $d \geq 2$, let $p \geq 3$ be an odd integer, and fix $\mu = \mu_\gamma \in \mathcal{M}^0$. Then there exists $\Sigma \subset L^2(\mathbb{R}^d)$ with $\mu(\Sigma) = 1$ and such that:*

(i) *For all $u_0 \in \Sigma$ there exist $T > 0$ and a unique local solution $u$ to (1-1) with initial data $u_0$ satisfying*

$$u(t) - e^{-itH} u_0 \in \mathscr{C}([-T, T]; \mathcal{H}^s(\mathbb{R}^d)), \tag{1-4}$$

*for some $s$ such that $\frac{d}{2} - \frac{2}{p-1} < s < \frac{d}{2}$.*

(ii) *More precisely, for all $T > 0$, there exists $\Sigma_T \subset \Sigma$ with*

$$\mu(\Sigma_T) \geq 1 - C \exp(-cT^{-\delta} \|\gamma\|_{L^2(\mathbb{R}^2)}^{-2}), \quad C, c, \delta > 0,$$

*and such that for all $u_0 \in \Sigma_T$ the lifespan of $u$ is larger than $T$.*

Let $\gamma = \sum\limits_{n=0}^{+\infty} c_n \varphi_n(x)$. Then

$$u_0^\omega := \sum_{n=0}^{+\infty} g_n(\omega) c_n \varphi_n(x)$$

is a typical element in the support of $\mu_\gamma$. Another way to state Theorem 1.1 is: for any $T > 0$, there exists an event $\Omega_T \subset \Omega$ such that

$$\mathbb{P}(\Omega_T) \geq 1 - C \exp(-cT^{-\delta} \|\gamma\|_{L^2(\mathbb{R}^d)}^{-2}), \quad C, c, \delta > 0,$$

and that for all $\omega \in \Omega_T$, there exists a unique solution of the form (1-4) to (1-1) with initial data $u_0^\omega$.

We will see in Proposition 2.1 that the stochastic approach yields a gain of $d/2$ derivatives compared to the deterministic theory. To prove Theorem 1.1 we only have to gain $s_c = d/2 - 2/(p-1)$ derivatives. The solution is constructed by a fixed point argument in a Strichartz space $X_T^s \subset \mathscr{C}([-T, T]; \mathcal{H}^s(\mathbb{R}^d))$ with continuous embedding, and uniqueness holds in the class $X_T^s$.

The deterministic Cauchy problem for (1-1) was studied by Oh [1989] (see also [Cazenave 2003, Chapter 9] for more references). Thomann [2009] has proven an almost sure local existence result for (1-1) in the supercritical regime (with a gain of $\frac{1}{4}$ of a derivative), for any $d \geq 1$. This local existence result was improved by [Burq et al. 2010] when $d = 1$ (gain of $\frac{1}{2}$ a derivative), by [Deng 2012] when $d = 2$, and by Poiret [2012a; 2012b] in any dimension.

**Remark 1.2.** The results of Theorem 1.1 also hold true for any quadratic potential

$$V(x) = \sum_{1 \leq j \leq d} \alpha_j x_j^2, \quad \alpha_j > 0, \quad 1 \leq j \leq d,$$

and for more general potentials such that $V(x) \approx \langle x \rangle^2$.

**1B2.** *Global existence and scattering results for NLS.* As an application of the results of the previous part, we are able to construct global solutions to the nonlinear Schrödinger equation without potential, which scatter when $t \to \pm\infty$. Consider the equation

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u = \pm|u|^{p-1}u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d. \\ u(0) = u_0. \end{cases} \tag{1-5}$$

The well-posedness indexes for this equation are the same as for (1-1). Namely, (1-5) is well-posed in $H^s(\mathbb{R}^d)$ when $s > \max(s_c, 0)$, and ill-posed when $s < s_c$.

For the next result, we will need an additional condition on the law $\nu$. We assume that

$$\mathbb{P}(|g_n| < \rho) > 0 \quad \text{for all } \rho > 0, \tag{1-6}$$

which ensures that the random variable can take arbitrarily small values. Then we can prove:

**Theorem 1.3.** *Let $d \geq 2$, let $p \geq 3$ be an odd integer, and fix $\mu = \mu_\gamma \in \mathcal{M}^0$. Assume that (1-6) holds. Then there exists $\Sigma \subset L^2(\mathbb{R}^d)$ with $\mu(\Sigma) > 0$ and such that:*

(i) *For all $u_0 \in \Sigma$ there exists a unique global solution $u$ to (1-5) with initial data $u_0$ satisfying*

$$u(t) - e^{it\Delta}u_0 \in \mathscr{C}(\mathbb{R}; \mathscr{H}^s(\mathbb{R}^d)),$$

*for some $s$ such that $\frac{d}{2} - \frac{2}{p-1} < s < \frac{d}{2}$.*

(ii) *For all $u_0 \in \Sigma$ there exist states $f_+, f_- \in \mathscr{H}^s(\mathbb{R}^d)$ such that when $t \to \pm\infty$,*

$$\|u(t) - e^{it\Delta}(u_0 + f_\pm)\|_{H^s(\mathbb{R}^d)} \to 0.$$

(iii) *If we assume that the distribution of $\nu$ is symmetric, then*

$$\mu\big(u_0 \in L^2(\mathbb{R}^d) : \text{assertion (ii) holds true} \mid \|u_0\|_{L^2(\mathbb{R}^d)} \leq \eta\big) \to 1,$$

*when $\eta \to 0$.*

We can show [Poiret 2012a, Théorème 20] that for all $s > 0$, if $u_0 \notin \mathscr{H}^\sigma(\mathbb{R}^d)$ then $\mu(H^\sigma(\mathbb{R}^d)) = 0$. This shows that the randomisation does not yield a gain of derivative in the Sobolev scale; thus Theorem 1.3 gives results for initial conditions not covered by the deterministic theory.

There is a large literature on the deterministic local and global theory with scattering for (1-5). We refer to [Banica et al. 2008; Nakanishi and Ozawa 2002; Carles 2009] for such results and more references.

We do not give here the details of the proof of Theorem 1.3, since one can follow the main lines of the argument of Poiret [2012a; 2012b] but with different constants (see, e.g., [Poiret 2012b, Théorème 4]). The proof of (i) and (ii) is based on the use of an explicit transform, called the lens transform $\mathscr{L}$, which links the solutions of (1-5) to solutions of NLS with harmonic potential. The transform $\mathscr{L}$ has been used in different contexts; see [Carles 2009] for scattering results and more references. More precisely, for $u(t,x) : \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[ \times \mathbb{R}^d \to \mathbb{C}$ we define

$$v(t,x) = \mathscr{L}u(t,x) = \left(\frac{1}{\sqrt{1+4t^2}}\right)^{d/2} u\left(\frac{\arctan(2t)}{2}, \frac{x}{\sqrt{1+4t^2}}\right) e^{i|x|^2 t/(1+4t^2)},$$

then $u$ is a solution to

$$i\frac{\partial u}{\partial t} - Hu = \lambda \cos(2t)^{\frac{1}{2}d(p-1)-2}|u|^{p-1}u$$

if and only if $v$ satisfies $i\,\partial v/\partial t + \Delta v = \lambda |v|^{p-1}v$. Theorem 1.1 provides solutions with lifespan larger than $\pi/4$ for large probabilities, provided that the initial conditions are small enough.

Part (iii) is stated in [Poiret 2012a, Théorème 9], and can be understood as a small data result.

In Theorem 1.3 we assumed that $d \geq 2$ and that $p \geq 3$ was an odd integer, so we had $p \geq 1 + 4/d$, or, in other words, we were in an $L^2$-supercritical setting. Our approach also allows to get results in an $L^2$-subcritical context, i.e., when $1 + 2/d < p < 1 + 4/d$.

**Theorem 1.4.** *Let $d = 2$ and $2 < p < 3$. Assume that (1-6) holds and fix $\mu = \mu_\gamma \in \mathcal{M}^0$. Then there exists $\Sigma \subset L^2(\mathbb{R}^2)$ with $\mu(\Sigma) > 0$ and such that for all $0 < \varepsilon < 1$:*

(i) *For all $u_0 \in \Sigma$ there exists a unique global solution $u$ to (1-5) with initial data $u_0$ satisfying*

$$u(t) - e^{it\Delta}u_0 \in \mathcal{C}(\mathbb{R}; \mathcal{H}^{1-\varepsilon}(\mathbb{R}^2)).$$

(ii) *For all $u_0 \in \Sigma$ there exist states $f_+, f_- \in \mathcal{H}^{1-\varepsilon}(\mathbb{R}^2)$ such that when $t \to \pm\infty$,*

$$\|u(t) - e^{it\Delta}(u_0 + f_\pm)\|_{H^{1-\varepsilon}(\mathbb{R}^2)} \to 0.$$

(iii) *If we assume that the distribution of $v$ is symmetric, then*

$$\mu\big(u_0 \in L^2(\mathbb{R}^2) : \text{assertion (ii) holds true} \mid \|u_0\|_{L^2(\mathbb{R}^2)} \leq \eta\big) \to 1,$$

*when $\eta \to 0$.*

In the case $p \leq 1 + 2/d$, Barab [1984] showed that a nontrivial solution to (1-5) never scatters; therefore even with a stochastic approach one can not have scattering in this case. When $d = 2$, the condition $p > 2$ in Theorem 1.4 is therefore optimal. Usually, deterministic scattering results in $L^2$-subcritical contexts are obtained in the space $H^1 \cap \mathcal{F}(H^1)$. Here we assume $u_0 \in L^2$, and thus we relax both the regularity and the decay assumptions (this latter point is the most striking in this context). Again we refer to [Banica et al. 2008] for an overview of scattering theory for NLS.

When $\mu \in \mathcal{M}^\sigma$ for some $0 < \sigma < 1$ we are able to prove the same result with $\varepsilon = 0$. Since the proof is much easier, we give it before the case $\sigma = 0$ (see Section 3B).

Finally, we point out that in Theorem 1.4 we are only able to consider the case $d = 2$ because of the lack of regularity of the nonlinear term $|u|^{p-1}u$.

**1B3.** *Global existence results for NLS with quadratic potential.* We also get global existence results for defocusing Schrödinger equation with harmonic potential. For $d = 2$ or $d = 3$, consider the equation

$$\begin{cases} i\dfrac{\partial u}{\partial t} - Hu = |u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0) = u_0, \end{cases} \tag{1-7}$$

and denote by $E$ the energy of (1-7), namely

$$E(u) = \|u\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 + \tfrac{1}{2}\|u\|_{L^4(\mathbb{R}^d)}^4.$$

Deterministic global existence for (1-7) has been studied by Zhang [2005] and by Carles [2011] in the case of time-dependent potentials.

When $d = 3$, our global existence result for (1-7) is the following:

**Theorem 1.5.** *Let* $d = 3$, $\frac{1}{6} < s < 1$ *and fix* $\mu = \mu_\gamma \in \mathcal{M}^s$. *Then there exists a set* $\Sigma \subset \mathcal{H}^s(\mathbb{R}^3)$ *such that* $\mu(\Sigma) = 1$ *and that the following holds true*:

  (i) *For all* $u_0 \in \Sigma$, *there exists a unique global solution to* (1-7), *which reads*

$$u(t) = e^{-itH}u_0 + w(t), \quad w \in \mathscr{C}(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^3)).$$

  (ii) *The previous line defines a global flow* $\Phi$, *which leaves the set* $\Sigma$ *invariant*:

$$\Phi(t)(\Sigma) = \Sigma, \quad \text{for all } t \in \mathbb{R}.$$

 (iii) *There exist* $C, c_s > 0$ *such that, for all* $t \in \mathbb{R}$,

$$E(w(t)) \le C(M + |t|)^{c_s+},$$

    *where* $M$ *is a positive random variable such that*

$$\mu(u_0 \in \mathcal{H}^s(\mathbb{R}^3) : M > K) \le Ce^{-cK^\delta/\|\gamma\|^2_{\mathcal{H}^s(\mathbb{R}^3)}}.$$

Here the critical Sobolev space is $\mathcal{H}^{1/2}(\mathbb{R}^3)$; thus the local deterministic theory, combined with the conservation of the energy, immediately gives global well-posedness in $\mathcal{H}^1(\mathbb{R}^3)$. Using a kind of interpolation method due to Bourgain, one may obtain deterministic global well-posedness in $\mathcal{H}^s(\mathbb{R}^3)$ for some $1/2 < s < 1$. Instead, for the proof of Theorem 1.5, we will rely on the almost well-posedness result of Theorem 1.1, and this gives global well-posedness in a supercritical context.

The constant $c_s > 0$ can be computed explicitly (see (4-16)), and we do not think that we have obtained the optimal rate. By reversibility of the equation, it is enough to consider only positive times.

With a similar approach, in dimension $d = 2$, we can prove:

**Theorem 1.6.** *Let* $d = 2$, $0 < s < 1$ *and fix* $\mu = \mu_\gamma \in \mathcal{M}^s$. *Then there exists a set* $\Sigma \subset \mathcal{H}^s(\mathbb{R}^2)$ *such that* $\mu(\Sigma) = 1$ *and that, for all* $u_0 \in \Sigma$, *there exists a unique global solution to* (1-7),

$$u(t) = e^{-itH}u_0 + w(t), \quad w \in \mathscr{C}(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^2)).$$

*In addition, statements* (ii) *and* (iii) *of Theorem 1.5 are also satisfied with* $c_s = \frac{1-s}{s}$.

Here the critical Sobolev space is $L^2(\mathbb{R}^2)$; thus Theorem 1.6 shows global well-posedness for any subcritical cubic nonlinear Schrödinger equations in dimension two.

Using the smoothing effect, which yields a gain of $\frac{1}{2}$ a derivative, a global well-posedness result for (1-1), in the defocusing case, was given in [Burq et al. 2010] in the case $d = 1$, for any $p \ge 3$. The global existence is proved for a typical initial condition on the support of a Gibbs measure, which is $\bigcap_{\sigma>0} \mathcal{H}^{-\sigma}(\mathbb{R})$. This result was extended by Deng [2012] in dimension $d = 2$ for radial functions. However, this approach has the drawback that it relies on the invariance of a Gibbs measure, which is a

rigid object, and is supported in rough Sobolev spaces. Therefore it seems difficult to adapt this strategy in higher dimensions.

Here instead we obtain the results of Theorems 1.5 and 1.6 as a combination of Theorem 1.1 with the high-low frequency decomposition method of [Bourgain 1999, p. 84]. This approach has been successful in different contexts, and has been first used together with probabilistic arguments by Colliander and Oh [2012] for the cubic Schrödinger below $L^2(\mathbb{S}^1)$ and later on by Burq and Tzvetkov [2014] for the wave equation.

**1C.** *Notations and plan of the paper.* In this paper $c, C > 0$ denote constants, the value of which may change from line to line. These constants will always be universal, or uniformly bounded with respect to the other parameters.

We let $L_T^p = L_{[-T,T]}^p = L^p(-T, T)$ for $T > 0$ and we write $L_x^p = L^p(\mathbb{R}^d)$. We denote the harmonic oscillator on $\mathbb{R}^d$ by $H = -\Delta + |x|^2 = \sum_{j=1}^d (-\partial_j^2 + x_j^2)$, and for $s \geq 0$ we define the Sobolev space $\mathcal{H}^s$ by the norm $\|u\|_{\mathcal{H}^s} = \|H^{s/2}u\|_{L^2(\mathbb{R}^d)}$. More generally, we define the spaces $\mathcal{W}^{s,p}$ by the norm $\|u\|_{\mathcal{W}^{s,p}} = \|H^{s/2}u\|_{L^p(\mathbb{R}^d)}$. If $E$ is a Banach space and $\mu$ is a measure on $E$, we write $L_\mu^p = L^p(d\mu)$ and $\|u\|_{L_\mu^p E} = \big\| \|u\|_E \big\|_{L_\mu^p}$.

The rest of the paper is organised as follows. In Section 2 we recall some deterministic results on the spectral function, and prove stochastic Strichartz estimates. Section 3 is devoted to the proof of Theorem 1.1 and of the scattering results for NLS without potential. Finally, in Section 4 we study the global existence for the Schrödinger–Gross–Pitaevskii equation (1-1).

## 2. Stochastic Strichartz estimates

The main result of this section is the following probabilistic improvement of the Strichartz estimates.

**Proposition 2.1.** *Let $s \in \mathbb{R}$ and $\mu = \mu_\gamma \in \mathcal{M}^s$. Let $1 \leq q < +\infty$, $2 \leq r \leq +\infty$, and set $\alpha = d\left(\frac{1}{2} - \frac{1}{r}\right)$ if $r < +\infty$ and $\alpha < d/2$ if $r = +\infty$. Then there exist $c, C > 0$ such that, for all $\tau \in \mathbb{R}$,*

$$\mu\left(u \in \mathcal{H}^s(\mathbb{R}^d) : \big\|e^{-i(t+\tau)H}u\big\|_{L_{[0,T]}^q \mathcal{W}^{s+\alpha,r}(\mathbb{R}^d)} > K\right) \leq C e^{-cK^2/T^{2/q}\|\gamma\|_{\mathcal{H}^s(\mathbb{R}^d)}^2}.$$

When $r = +\infty$, this result expresses a gain $\mu$-a.s. of $d/2$ derivatives in space compared to the deterministic Strichartz estimates (see the bound (3-2)).

Proposition 2.1 is a consequence of [Poiret et al. 2013, Inequality (1.6)], but we give here a self-contained proof suggested by Nicolas Burq.

There are two key ingredients in the proof of Proposition 2.1. The first one is a deterministic estimate on the spectral function given in Lemma 2.2, and the second is the Khinchin inequality stated in Lemma 2.3.

**2A.** *Deterministic estimates of the spectral function.* We define the spectral function $\pi_H$ for the harmonic oscillator by

$$\pi_H(\lambda; x, y) = \sum_{\lambda_j \leq \lambda} \varphi_j(x)\overline{\varphi_j(y)},$$

and this definition does not depend on the choice of $\{\varphi_j \mid j \in \mathbb{N}\}$.

Let us recall some results of $\pi_H$, which were essentially obtained by Thangavelu [1993, Lemma 3.2.2, p. 70] (see also [Karadzhov 1995] and [Poiret et al. 2013, Section 3] for more details).

Thanks to the Mehler formula, we can prove

$$\pi_H(\lambda; x, x) \leq C\lambda^{d/2} \exp\left(-c\frac{|x|^2}{\lambda}\right) \quad \text{for all } x \in \mathbb{R}^d \text{ and } \lambda \geq 1. \tag{2-1}$$

One also has the following more subtle bound, which is the heart of [Karadzhov 1995]:

$$|\pi_H(\lambda + \mu; x, x) - \pi_H(\lambda; x, x)| \leq C(1 + |\mu|)\lambda^{d/2-1} \quad \text{for } \lambda \geq 1, \ |\mu| \leq C_0\lambda. \tag{2-2}$$

This inequality gives a bound on $\pi_H$ in energy interval of size $\sim 1$, which is the finest one can obtain.

Then we can prove (see [Poiret et al. 2013, Lemma 3.5]):

**Lemma 2.2.** *Let $d \geq 2$ and assume that $|\mu| \leq c_0$, $r \geq 1$ and $\theta \geq 0$. Then there exists $C > 0$ such that for all $\lambda \geq 1$*

$$\|\pi_H(\lambda + \mu; x, x) - \pi_H(\lambda; x, x)\|_{L^r(\mathbb{R}^d)} \leq C\lambda^{\frac{1}{2}d(1+1/r)-1}.$$

**2B. *Proof of Proposition 2.1.*** To begin with, recall the Khinchin inequality, which shows a smoothing property of the random series in the $L^k$ spaces for $k \geq 2$; for example, see [Burq and Tzvetkov 2008a, Lemma 4.2].

**Lemma 2.3.** *There exists $C > 0$ such that for all real $k \geq 2$ and $(c_n) \in \ell^2(\mathbb{N})$*

$$\left\|\sum_{n \geq 1} g_n(\omega) c_n\right\|_{L^k_{\mathbb{P}}} \leq C\sqrt{k}\left(\sum_{n \geq 1} |c_n|^2\right)^{\frac{1}{2}}.$$

Now we fix $\gamma = \sum_{n=0}^{+\infty} c_n \varphi_n \in \mathcal{A}_s$ and let $\gamma^\omega = \sum_{n=0}^{+\infty} g_n(\omega) c_n \varphi_n$.

Firstly, we treat the case $r < +\infty$. Set $\alpha = d\left(\frac{1}{2} - \frac{1}{r}\right)$ and set $\sigma = s + \alpha$. Observe that it suffices to prove the estimation for $K \gg \|\gamma\|_{\mathcal{H}^s(\mathbb{R}^d)}$.

Let $k \geq 1$. By definition,

$$\int_{\mathcal{H}^s(\mathbb{R}^d)} \|e^{-i(t+\tau)H}u\|^k_{L^q_{[0,T]}\mathcal{W}^{\sigma,r}(\mathbb{R}^d)}\, d\mu(u) = \int_\Omega \|e^{-i(t+\tau)H}\gamma^\omega\|^k_{L^q_{[0,T]}\mathcal{W}^{\sigma,r}(\mathbb{R}^d)}\, d\mathbb{P}(\omega)$$

$$= \int_\Omega \|e^{-i(t+\tau)H}H^{\sigma/2}\gamma^\omega\|^k_{L^q_{[0,T]}L^r(\mathbb{R}^d)}\, d\mathbb{P}(\omega). \tag{2-3}$$

Since $e^{-i(t+\tau)H}H^{\sigma/2}\gamma^\omega(x) = \sum_{n=0}^{+\infty} g_n(\omega) c_n \lambda_n^{\sigma/2} e^{-i(t+\tau)\lambda_n}\varphi_n(x)$, by Lemma 2.3 we get

$$\|e^{-i(t+\tau)H}H^{\sigma/2}\gamma^\omega(x)\|_{L^k_{\mathbb{P}}} \leq C\sqrt{k}\|e^{-i(t+\tau)H}H^{\sigma/2}\gamma^\omega(x)\|_{L^2_{\mathbb{P}}} = C\sqrt{k}\left(\sum_{n=0}^{+\infty} \lambda_n^\sigma |c_n|^2 |\varphi_n(x)|^2\right)^{\frac{1}{2}}.$$

Assume that $k \geq r$. By the integral Minkowski inequality, the previous line and the triangle inequality we get

$$
\begin{aligned}
\left\| e^{-i(t+\tau)H} H^{\sigma/2} \gamma^\omega \right\|_{L^k_\mathbb{P} L^r_x} &\leq \left\| e^{-i(t+\tau)H} H^{\sigma/2} \gamma^\omega \right\|_{L^r_x L^k_\mathbb{P}} \\
&\leq C \sqrt{k} \left\| \sum_{k=0}^{+\infty} \lambda_k^\sigma |c_k|^2 |\varphi_k|^2 \right\|^{\frac{1}{2}}_{L^{r/2}(\mathbb{R}^d)} \\
&\leq C \sqrt{k} \left( \sum_{j=1}^{+\infty} \left\| \sum_{k \in I(j)} \lambda_k^\sigma |c_k|^2 |\varphi_k|^2 \right\|_{L^{r/2}(\mathbb{R}^d)} \right)^{\frac{1}{2}}.
\end{aligned} \tag{2-4}
$$

Condition (1-3) implies that for all $x \in \mathbb{R}^d$ and $k \in I(j) = \{n \in \mathbb{N} \mid 2j \leq \lambda_n < 2(j+1)\}$

$$
\lambda_k^\sigma |c_k|^2 |\varphi_k(x)|^2 \leq C j^\sigma \sum_{n \in I(j)} |c_n|^2 \frac{|\varphi_k(x)|^2}{\# I(j)},
$$

and thus, by Lemma 2.2 and the fact that $\# I(j) \sim c j^{d-1}$,

$$
\begin{aligned}
\left\| \sum_{k \in I(j)} \lambda_k^\sigma |c_k|^2 |\varphi_k(x)|^2 \right\|_{L^{r/2}(\mathbb{R}^d)} &\leq C j^\sigma \sum_{n \in I(j)} |c_n|^2 \frac{\left\| \sum_{k \in I(j)} |\varphi_k(x)|^2 \right\|_{L^{r/2}(\mathbb{R}^d)}}{\# I(j)} \\
&\leq C j^{\sigma + d(1/r - 1/2)} \sum_{n \in I(j)} |c_n|^2 \\
&= C j^s \sum_{n \in I(j)} |c_n|^2.
\end{aligned}
$$

The latter inequality together with (2-4) gives

$$
\left\| e^{-i(t+\tau)H} H^{\sigma/2} \gamma^\omega \right\|_{L^k_\mathbb{P} L^r_x} \leq C \sqrt{k} \| \gamma \|_{\mathcal{H}^s(\mathbb{R}^d)},
$$

and for $k \geq r$, by Minkowski,

$$
\left\| e^{-i(t+\tau)H} H^{\sigma/2} \gamma^\omega \right\|_{L^k_\mathbb{P} L^q_{[0,T]} L^r_x} \leq C \sqrt{k} \, T^{1/q} \| \gamma \|_{\mathcal{H}^s(\mathbb{R}^d)}.
$$

Then, using (2-3) and the Bienaymé–Chebishev inequality, we obtain

$$
\begin{aligned}
\mu \left( u \in \mathcal{H}^s : \left\| e^{-i(t+\tau)H} u \right\|_{L^q_{[0,T]} W^{\sigma,r}(\mathbb{R}^d)} > K \right) &\leq \left( K^{-1} \left\| e^{-i(t+\tau)H} H^{\sigma/2} \gamma^\omega \right\|_{L^k_\mathbb{P} L^q_{[0,T]} L^r_x} \right)^k \\
&\leq \left( C K^{-1} \sqrt{k} \, T^{1/q} \| \gamma \|_{\mathcal{H}^s(\mathbb{R}^d)} \right)^k.
\end{aligned}
$$

Finally, if $K \gg \| \gamma \|_{\mathcal{H}^s(\mathbb{R}^d)}$, we can choose $k = K^2 / 2 C T^{2/q} \| \gamma \|^2_{\mathcal{H}^s(\mathbb{R}^d)} \geq r$, which yields the result.

Now assume $r = +\infty$. We use the Sobolev inequality to get $\| u \|_{W^{s,\infty}} \leq C \| u \|_{W^{\tilde{s},\tilde{r}}}$ with $\tilde{s} = s + 2d/\tilde{r}$ for $\tilde{r} \geq 1$ large enough; hence we can apply the previous result for $r < +\infty$.

**Remark 2.4.** A similar result to Proposition 2.1 holds, with the same gain of derivatives, when $I(\lambda)$ is replaced with the dyadic interval $J(j) = \{n \in \mathbb{N} \mid 2^j \leq \lambda_n < 2^{j+1}\}$. Then the condition (1-3) becomes

$$|c_k|^2 \leq \frac{C}{\#J(j)} \sum_{n \in J(j)} |c_n|^2 \quad \text{for all } j \geq 1 \text{ and } k \in J(j), \tag{2-5}$$

which seems more restrictive. Indeed neither condition imply the other.

Observe that if we want to prove the result under condition (2-5), the subtle estimate (2-2) is not needed; (2-1) is enough.

**Remark 2.5.** For $d = 1$, condition (1-3) is always satisfied but condition (2-2) is not. Instead we can use that $\|\varphi_k\|_p \leq C\lambda_k^{-\theta(p)}$ with $\theta(p) > 0$ for $p > 2$ [Koch and Tataru 2005]. For example if $p > 4$ we have $\theta(p) = \frac{1}{4} - (p-1)/6p$. Thus we get the Proposition 2.1 with $s = p\theta(p)/4$ (see [Thomann 2009; Burq et al. 2010], where this is used).

**Remark 2.6.** Another approach could have been to exploit the particular basis $(\varphi_n)_{n \geq 1}$, which satisfies the good $L^\infty$ estimates given in [Poiret et al. 2013, Theorem 1.3], and to construct the measures $\mu$ as the image measures of random series of the form

$$\gamma^\omega(x) = \sum_{n \geq 1} c_n g_n(\omega)\varphi_n(x),$$

with $c_n \in \ell^2(\mathbb{N})$ not necessarily satisfying (1-3). A direct application of the Khinchin inequality (as in [Thomann 2009, Proposition 2.3]) then gives the same bounds as in Proposition 2.1. Observe that condition (1-3) is also needed in this approach, but it directly intervenes in the construction of the $\varphi_n$.

We believe that the strategy we adopted here is slightly more general, since it seems to work even in cases where we do not have a basis of eigenfunctions that satisfy bounds analogous to [Poiret et al. 2013, Theorem 1.3], as for example in the case of the operator $-\Delta + |x|^4$.

## 3. Application to the local theory of the supercritical Schrödinger equation

**3A.** *Almost sure local well-posedness.* This subsection, devoted to the proof of Theorem 1.1, follows the argument of [Poiret 2012b].

Let $u_0 \in L^2(\mathbb{R}^d)$. We look for a solution to (1-1) of the form $u = e^{-itH}u_0 + v$, where $v$ is some fluctuation term more regular than the linear profile $e^{-itH}u_0$. By the Duhamel formula, the unknown $v$ has to be a fixed point of the operator

$$L(v) := \mp i \int_0^t e^{-i(t-s)H} |e^{-isH}u_0 + v(s)|^{p-1}(e^{-isH}u_0 + v(s)) \, ds, \tag{3-1}$$

in some adequate functional space, which is a Strichartz space.

To begin with, we recall the Strichartz estimates for the harmonic oscillator. A couple $(q, r) \in [2, +\infty]^2$ is called admissible if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \quad \text{and} \quad (d, q, r) \neq (2, 2, +\infty),$$

and if one defines

$$X_T^s := \bigcap_{\substack{(q,r) \\ \text{admissible}}} L^q([-T,T]; \mathcal{W}^{s,r}(\mathbb{R}^d)),$$

then for all $T > 0$ there exists $C_T > 0$ such that for all $u_0 \in \mathcal{H}^s(\mathbb{R}^d)$ we have

$$\|e^{-itH}u_0\|_{X_T^s} \le C_T \|u_0\|_{\mathcal{H}^s(\mathbb{R}^d)}. \tag{3-2}$$

We will also need the inhomogeneous version of Strichartz: For all $T > 0$, there exists $C_T > 0$ such that for all admissible couples $(q,r)$ and functions $F \in L^{q'}([T,T]; \mathcal{W}^{s,r'}(\mathbb{R}^d))$,

$$\left\| \int_0^t e^{-i(t-s)H} F(s)\, ds \right\|_{X_T^s} \le C_T \|F\|_{L^{q'}([-T,T], \mathcal{W}^{s,r'}(\mathbb{R}^d))}, \tag{3-3}$$

where $q'$ and $r'$ are the Hölder conjugates of $q$ and $r$. We refer to [Poiret 2012b] for a proof.

The next result is a direct application of the Sobolev embeddings and Hölder.

**Lemma 3.1.** *Let $(q,r) \in [2, \infty[ \times [2, \infty]$, and let $s, s_0 \ge 0$ be such that $s - s_0 > \dfrac{d}{2} - \dfrac{2}{q} - \dfrac{d}{r}$. Then there exist $\kappa, C > 0$ such that for any $T \ge 0$ and $u \in X_T^s$,*

$$\|u\|_{L^q([-T,T], \mathcal{W}^{s_0,r}(\mathbb{R}^d))} \le C T^\kappa \|u\|_{X_T^s}.$$

We now introduce the appropriate sets in which we can profit from the stochastic estimates of the previous section. Fix $\mu = \mu_\gamma \in \mathcal{M}^0$ and, for $K \ge 0$ and $\varepsilon > 0$, define the set $G_d(K)$ as

$$G_d(K) = \left\{ w \in L^2(\mathbb{R}^d) \mid \|w\|_{L^2(\mathbb{R}^d)} \le K \text{ and } \|e^{-itH}w\|_{L^{1/\varepsilon}_{[-2\pi,2\pi]} \mathcal{W}^{d/2-\varepsilon,\infty}(\mathbb{R}^d)} \le K \right\}.$$

Then by Proposition 2.1,

$$\mu(G_d(K)^c) \le \mu\big(\|w\|_{L^2(\mathbb{R}^d)} > K\big) + \mu\big(\|e^{-itH}w\|_{L^{1/\varepsilon}_{[-2\pi,2\pi]} \mathcal{W}^{d/2-\varepsilon,\infty}(\mathbb{R}^d)} > K\big) \le C e^{-cK^2/\|\gamma\|_{L^2}^2}. \tag{3-4}$$

We want to perform a fixed point argument on $L$ with initial condition $u_0 \in G_d(K)$ for some $K > 0$ and $\varepsilon > 0$ small enough. We begin by establishing some estimates.

**Lemma 3.2.** *Let $s \in \left]\dfrac{d}{2} - \dfrac{2}{p-1}, \dfrac{d}{2}\right[$. For $\varepsilon > 0$ small enough there exist $C > 0$ and $\kappa > 0$ such that for any $0 < T \le 1$, $u_0 \in G_d(K)$, $v \in X_T^s$ and $f_i = v$ or $f_i = e^{-itH}u_0$,*

$$\left\| H^{s/2}(v) \prod_{i=2}^p f_i \right\|_{L^1([-T,T], L^2(\mathbb{R}^d))} \le C T^\kappa \big( K^p + \|v\|_{X_T^s}^p \big), \tag{3-5}$$

*and*

$$\left\| H^{s/2}(e^{-itH}u_0) \prod_{i=2}^p f_i \right\|_{L^1([-T,T], L^2(\mathbb{R}^d))} \le C T^\kappa \big( K^p + \|v\|_{X_T^s}^p \big). \tag{3-6}$$

*Proof.* First we prove (3-5). Thanks to the Hölder inequality,

$$\left\| |\nabla|^s(v) \prod_{i=2}^p f_i \right\|_{L^1([-T,T], L^2(\mathbb{R}^d))} \le \| |\nabla|^s(v) \|_{L^\infty([-T,T], L^2(\mathbb{R}^d))} \prod_{i=2}^p \|f_i\|_{L^{p-1}([-T,T], L^\infty(\mathbb{R}^d))}$$

and

$$\left\| \langle x \rangle^s v \prod_{i=2}^{p} f_i \right\|_{L^1([-T,T],L^2(\mathbb{R}^d))} \leq \|\langle x \rangle^s v\|_{L^\infty([-T,T],L^2(\mathbb{R}^d))} \prod_{i=2}^{p} \|f_i\|_{L^{p-1}([-T,T],L^\infty(\mathbb{R}^d))}$$

$$\leq \|v\|_{L^\infty([-T,T],H^s(\mathbb{R}^d))} \prod_{i=2}^{p} \|f_i\|_{L^{p-1}([-T,T],L^\infty(\mathbb{R}^d))}.$$

If $f_i = v$, then as $s > \dfrac{d}{2} - \dfrac{2}{p-1}$, we can use Lemma 3.1 to obtain

$$\|v\|_{L^{p-1}([-T,T],L^\infty(\mathbb{R}^d))} \leq C T^\kappa \|v\|_{X_T^s}.$$

If $f_i = e^{-itH} u_0$, then by definition of $G_d(K)$ we have, for $\varepsilon > 0$ small enough,

$$\|e^{-itH} u_0\|_{L^{p-1}([-T,T],L^\infty(\mathbb{R}^d))} \leq T^\kappa \|e^{-itH} u_0\|_{L^{1/\varepsilon}([-2\pi,2\pi],\mathcal{W}^{d/2-\varepsilon,\infty}(\mathbb{R}^d))} \leq T^\kappa K.$$

We now turn to (3-6). Thanks to the Hölder inequality, we have

$$\left\| |\nabla|^s (e^{-itH} u_0) \prod_{i=2}^{p} f_i \right\|_{L^1([-T,T],L^2(\mathbb{R}^d))}$$

$$\leq \||\nabla|^s (e^{-itH} u_0)\|_{L^p([-T,T],L^{2dp}(\mathbb{R}^d))} \prod_{i=2}^{p} \|f_i\|_{L^p([-T,T],L^{2dp(p-1)/(dp-1)}(\mathbb{R}^d))}$$

$$\leq \|e^{-itH} u_0\|_{L^p([-T,T],\mathcal{W}^{s,2dp}(\mathbb{R}^d))} \prod_{i=2}^{p} \|f_i\|_{L^p([-T,T],L^{2dp(p-1)/(dp-1)}(\mathbb{R}^d))}.$$

If $f_i = e^{-itH} u_0$, by interpolation we obtain, for some $0 \leq \theta \leq 1$,

$$\|e^{-itH} u_0\|_{L^p([-T,T],L^{2dp(p-1)/(dp-1)}(\mathbb{R}^d))} \leq C T^\kappa \|u_0\|_{L^2(\mathbb{R}^d)}^{1-\theta} \|e^{-itH} u_0\|_{L^{1/\varepsilon}([-T,T],L^\infty(\mathbb{R}^d))}^\theta \leq C T^\kappa K.$$

If $f_i = v$, as $s > \dfrac{d}{2} - \dfrac{2}{p-1} > \dfrac{d}{2} - \dfrac{2}{p} - \dfrac{d(dp-1)}{2dp(p-1)}$ (because $p \geq 3$ and $d \geq 2$), then thanks to Lemma 3.1 we find

$$\|v\|_{L^p([-T,T],L^{2dp(p-1)/(dp-1)}(\mathbb{R}^d))} \leq C T^\kappa \|v\|_{X_T^s}. \qquad \square$$

We are now able to establish the estimates that will be useful in the application of a fixed point theorem.

**Proposition 3.3.** *Let* $s \in \left] \dfrac{d}{2} - \dfrac{2}{p-1}, \dfrac{d}{2} \right[$. *Then for* $\varepsilon > 0$ *small enough, there exist* $C > 0$ *and* $\kappa > 0$ *such that if* $u_0 \in G_d(K)$ *for some* $K > 0$. *For any* $v, v_1, v_2 \in X_T^s$ *and* $0 < T \leq 1$,

$$\left\| \int_0^t e^{-i(t-s)H} |e^{-isH} u_0 + v|^{p-1} (e^{-isH} u_0 + v) \, ds \right\|_{X_T^s} \leq C T^\kappa (K^p + \|v\|_{X_T^s}^p),$$

*and*

$$\left\| \int_0^t e^{-i(t-s)H} |e^{-isH} u_0 + v_1|^{p-1} (e^{-isH} u_0 + v_1) \, ds \right.$$

$$\left. - \int_0^t e^{-i(t-s)H} |e^{-isH} u_0 + v_2|^{p-1} (e^{-isH} u_0 + v_2) \, ds \right\|_{X_T^s}$$

$$\leq C T^\kappa \|v_1 - v_2\|_{X_T^s} \left( K^{p-1} + \|v_1\|_{X_T^s}^{p-1} + \|v_2\|_{X_T^s}^{p-1} \right).$$

*Proof.* We only prove the first claim, since the proof of the second is similar. Using the Strichartz inequalities (3-3), we obtain

$$\left\| \int_0^t e^{-i(t-s)H} |e^{-isH} u_0 + v|^{p-1} (e^{-isH} u_0 + v)\, ds \right\|_{X_T^s}$$
$$\leq C \left\| |e^{-isH} u_0 + v|^{p-1} (e^{-isH} u_0 + v) \right\|_{L_{[-T,T]}^1 \mathcal{H}^s(\mathbb{R}^d)}.$$

Then, using Lemma 3.2, we obtain the existence of $\kappa > 0$ such that for any $u_0 \in G_d(K)$, $0 < T \leq 1$ and $v \in X_T^s$,

$$\left\| H^{s/2} \big( |e^{-isH} u_0 + v|^{p-1} (e^{-isH} u_0 + v) \big) \right\|_{L^1([-T,T], L^2(\mathbb{R}^d))} \leq C T^\kappa \big( K^p + \|v\|_{X_T^s}^p \big). \qquad \square$$

*Proof of Theorem 1.1.* We now complete the contraction argument on $L$ defined in (3-1) with some $u_0 \in G_d(K)$. According to Proposition 3.3, there exist $C > 0$ and $\kappa > 0$ such that

$$\|L(v)\|_{X_T^s} \leq C T^\kappa \big( K^p + \|v\|_{X_T^s}^p \big)$$
$$\|L(v_1) - L(v_2)\|_{X_T^s} \leq C T^\kappa \|v_1 - v_2\|_{X_T^s} \big( K^{p-1} + \|v_1\|_{X_T^s}^{p-1} + \|v_2\|_{X_T^s}^{p-1} \big).$$

Hence, if we choose $T > 0$ such that $K = (8C T^\kappa)^{-1/(p-1)}$, then $L$ is a contraction in the space $B_{X_T^s}(0, K)$ (the ball of radius $K$ in $X_T^s$). Thus if we set $\Sigma_T = G_d(K)$, with the previous choice of $K$, the result follows from (3-4). $\qquad \square$

*Proof of Theorem 1.3.* We introduce

$$\begin{cases} i \dfrac{\partial w}{\partial t} - Hw = \pm \cos(2t)^{\frac{1}{2}d(p-1)-2} |w|^{p-1} w, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ v(0) = u_0, \end{cases} \tag{3-7}$$

and let $s \in \left] \frac{d}{2} - \frac{2}{p-1}, \frac{d}{2} \right[$, $T = \frac{\pi}{4}$ and $1 \gg \varepsilon > 0$. Thanks to Proposition 3.3, there exist $C > 0$ and $\kappa > 0$ such that if $u_0 \in G_d(K)$ for some $K > 0$ then, for all $v$,

$$\left\| \int_0^t e^{-i(t-s)H} \big( \cos(2s)^{\frac{d}{2}(p-1)-2} |e^{-isH} u_0 + v|^{p-1} (e^{-isH} u_0 + v) \big)\, ds \right\|_{X_T^s} \leq C T^\kappa \big( K^p + \|v\|_{X_T^s}^p \big). \tag{3-8}$$

As in Theorem 1.1, we can choose $K = (8C T^\kappa)^{-1/(p-1)}$ to obtain, for $u_0 \in G_d(K)$, a unique local solution $w = e^{-itH} u_0 + v$ in time interval $\left] -\frac{\pi}{4}, \frac{\pi}{4} \right[$ to (3-7) with $v \in X_T^1$.

We set $u = \mathscr{L}v$. Then $u$ is a global solution to (1-5). Thanks to [Poiret 2012b, Propositions 20 and 22], we obtain that $u = e^{it\Delta} u_0 + v'$ with $v' \in X_T^1$.

Moreover, thanks to (3-8), we have that

$$\int_0^t e^{-i(t-s)H} \big( \cos(2s)^{\frac{d}{2}(p-1)-2} |e^{-isH} u_0 + v|^{p-1} (e^{-isH} u_0 + v) \big)\, ds \in \mathscr{C}^0([-T, T], \mathcal{H}^s(\mathbb{R}^d)).$$

Then there exist $L \in \mathcal{H}^s$ such that

$$\lim_{t \to T} \left\| e^{-itH} \int_0^t e^{-isH} \big( \cos(2s)^{\frac{d}{2}(p-1)-2} |e^{-isH} u_0 + v|^{p-1} (e^{-isH} u_0 + v) \big)\, ds - L \right\|_{\mathcal{H}^s(\mathbb{R}^d)} = 0.$$

Using [Poiret 2012b, Lemma 70], we obtain that

$$\lim_{t \to T} \left\| u(t) - e^{it\Delta} u_0 - e^{it\Delta}(-ie^{-iTH} L) \right\|_{H^s(\mathbb{R}^d)} = 0.$$

Finally, to establish Theorem 1.3, it suffices to set $\Sigma = G_d(K)$ and to prove that $\mu(u_0 \in G_d(K)) > 0$. We can write

$$u_0 = \chi\left(\frac{H}{N}\right) u_0 + (1 - \chi)\left(\frac{H}{N}\right) u_0 := [u_0]_N + [u_0]^N,$$

where $\chi$ is a truncation function. Using the triangle inequality and independence, we obtain that

$$\mu(u_0 \in G_d(K)) \geq \mu\big([u_0]_N \in G_d(K/2)\big) \mu\big([u_0]^N \in G_d(K/2)\big).$$

For all $N$, $\mu\big([u_0]_N \in G_d(K/2)\big) > 0$ because the hypothesis (1-6) is satisfied and thanks to Proposition 2.1 we have

$$\mu\big([u_0]^N \in G_d(K/2)\big) \geq 1 - Ce^{-cK^2/\|[u_0]^N\|_{L^2}^2} \to 1 \quad \text{as } N \to \infty,$$

and there exists $N$ such that $\mu\big([u_0]^N \in G_d(K/2)\big) > 0$.                    $\square$

**3B. *Almost sure local well-posedness of the time dependent equation and scattering for NLS.*** This section is devoted to the proof of Theorem 1.4. The strategy is similar to the proof of Theorem 1.3: we solve the equation which is mapped by $\mathscr{L}$ to (1-5) up to time $T = \pi/4$ and we conclude as previously. The difference here is that the nonlinear term of the equation we have to solve is singular at time $T = \pi/4$. More precisely, we consider the equation

$$\begin{cases} i\dfrac{\partial u}{\partial t} - Hu = \pm \cos(2t)^{p-3}|u|^{p-1}u, & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(0) = u_0, \end{cases} \tag{3-9}$$

when $2 < p < 3$.

Let us first consider the easier case $\sigma > 0$.

**3B1. *Proof of Theorem 1.4 in the case $\sigma > 0$.*** Let $\sigma > 0$ and $\mu = \mu_\gamma \in \mathcal{M}^\sigma$, and for $K \geq 0$ and $\varepsilon > 0$ define the set $F_\sigma(K)$ as

$$F_\sigma(K) = \left\{ w \in \mathcal{H}^\sigma(\mathbb{R}^2) \mid \|w\|_{\mathcal{H}^\sigma(\mathbb{R}^2)} \leq K \text{ and } \|e^{-itH} w\|_{L_{[0,2\pi]}^{1/\varepsilon} \mathcal{W}^{1+\sigma-\varepsilon,\infty}(\mathbb{R}^2)} \leq K \right\}.$$

The parameter $\varepsilon > 0$ will be chosen small enough that we can apply Proposition 2.1 and get

$$\mu(F_\sigma(K)^c) \leq \mu(\|w\|_{\mathcal{H}^\sigma} > K) + \mu\big(\|e^{-itH} w\|_{L_{[0,2\pi]}^{1/\varepsilon} \mathcal{W}^{1+\sigma-\varepsilon,\infty}} > K\big) \leq Ce^{-cK^2/\|\gamma\|_{\mathcal{H}^\sigma}^2}.$$

The next proposition is the key in the proof of Theorem 1.4 when $\sigma > 0$.

**Proposition 3.4.** *Let $\sigma > 0$. There exist $C > 0$ and $\kappa > 0$ such that if $u_0 \in F_\sigma(K)$ for some $K > 0$ then for any $v, v_1, v_2 \in X_T^1$ and $0 < T \leq 1$,*

$$\left\| \int_0^t e^{-i(t-\tau)H}\big(\cos(2\tau)^{p-3}|e^{-i\tau H} u_0 + v|^{p-1}(e^{-i\tau H} u_0 + v)\big) \, d\tau \right\|_{X_T^1} \leq CT^\kappa (K^p + \|v\|_{X_T^1}^p) \tag{3-10}$$

*and*

$$\left\| \int_0^t e^{-i(t-\tau)H}\big(\cos(2\tau)^{p-3}|e^{-i\tau H}u_0 + v_1|^{p-1}(e^{-i\tau H}u_0 + v_1)\big)\,d\tau \right.$$

$$\left. - \int_0^t e^{-i(t-\tau)H}\big(\cos(2\tau)^{p-3}|e^{-i\tau H}u_0 + v_2|^{p-1}(e^{-i\tau H}u_0 + v_2)\big)\,d\tau \right\|_{X_T^1}$$

$$\leq CT^\kappa \|v_1 - v_2\|_{X_T^1}\big(K^{p-1} + \|v_1\|_{X_T^1}^{p-1} + \|v_2\|_{X_T^1}^{p-1}\big). \quad (3\text{-}11)$$

*Proof.* We first prove (3-10). Using the Strichartz inequalities (3-3), we obtain

$$\left\| \int_0^t e^{-i(t-\tau)H}\big(\cos(2\tau)^{p-3}|e^{-i\tau H}u_0 + v|^{p-1}(e^{-i\tau H}u_0 + v)\big)\,d\tau \right\|_{X_T^1}$$

$$\leq C \big\|\cos(2\tau)^{p-3}|e^{-i\tau H}u_0 + v|^{p-1}(e^{-i\tau H}u_0 + v)\big\|_{L^1_{[-T,T]}\mathcal{H}^1(\mathbb{R}^2)}.$$

We use the formula

$$\nabla(|u|^{p-1}u) = \frac{p+1}{2}|u|^{p-1}\nabla u + \frac{p-1}{2}|u|^{p-3}u^2\nabla\bar{u}. \quad (3\text{-}12)$$

We let $f = e^{-isH}u_0$, then

$$\big\|\nabla(|f+v|^{p-1}(f+v))\big\|_{L^2(\mathbb{R}^2)} \leq C\big\||f+v|^{p-1}\nabla(f+v)\big\|_{L^2(\mathbb{R}^2)} + C\big\||f+v|^{p-3}(f+v)^2\nabla(f+v)\big\|_{L^2(\mathbb{R}^2)}$$

$$\leq C\|f+v\|_{L^\infty(\mathbb{R}^2)}^{p-1}\|\nabla v\|_{L^2(\mathbb{R}^2)} + C\|f+v\|_{L^{2(p-1)}(\mathbb{R}^2)}^{p-1}\|\nabla f+v\|_{L^\infty(\mathbb{R}^2)}.$$

Therefore

$$\big\||f+v|^{p-1}(f+v)\big\|_{\mathcal{H}^1(\mathbb{R}^2)}$$

$$\leq C\big(\|f\|_{L^\infty(\mathbb{R}^2)}^{p-1} + \|v\|_{L^\infty(\mathbb{R}^2)}^{p-1}\big)\|v\|_{\mathcal{H}^1(\mathbb{R}^2)} + C\big(\|f\|_{L^{2(p-1)}(\mathbb{R}^2)}^{p-1} + \|v\|_{L^{2(p-1)}(\mathbb{R}^2)}^{p-1}\big)\|f\|_{\mathcal{W}^{1,\infty}(\mathbb{R}^2)}.$$

Now observe that $\|v\|_{L^\infty_{[-T,T]}L^{2(p-1)}} \leq \|v\|_{X_T^1}$ as well as, for all $r < +\infty$, $\|v\|_{L^r_{[-T,T]}L^\infty} \leq \|v\|_{X_T^1}$. Then, for all $q > 1$,

$$\big\||f+v|^{p-1}(f+v)\big\|_{L^q_{[-T,T]}\mathcal{H}^1(\mathbb{R}^2)} \leq CT^\kappa \Big(\big(\|f\|_{L^\infty_{[-T,T]}L^\infty(\mathbb{R}^2)}^{p-1} + \|v\|_{X_T^1}^{p-1}\big)\|v\|_{X_T^1}$$

$$+ \big(\|f\|_{L^\infty_{[-T,T]}L^{2(p-1)}(\mathbb{R}^2)}^{p-1} + \|v\|_{X_T^1}^{p-1}\big)\|f\|_{L^\infty_{[-T,T]}\mathcal{W}^{1,\infty}(\mathbb{R}^2)}\Big)$$

$$\leq CT^\kappa\big(K^p + \|v\|_{X_T^1}^p\big). \quad (3\text{-}13)$$

Choose $q > 1$ so that $q'(3-p) < 1$. We have $\|\cos(2\tau)^{p-3}\|_{L^{q'}_{[-T,T]}} < \infty$; thus from (3-13) and Hölder, we infer

$$\big\|\cos(2\tau)^{p-3}|f+v|^{p-1}(f+v)\big\|_{L^1([-T,T],\mathcal{H}^1(\mathbb{R}^2))}$$

$$\leq C\|\cos(2\tau)^{p-3}\|_{L^{q'}_{[-T,T]}}\big\||f+v|^{p-1}(f+v)\big\|_{L^q_{[-T,T]}\mathcal{H}^1(\mathbb{R}^2)}$$

$$\leq CT^\kappa(K^p + \|v\|_{X_T^1}^p).$$

For the proof of (3-11) we can proceed similarly. Namely, we use the estimates

$$\left| |z_1|^{p-1} - |z_2|^{p-1} \right| \leq C(|z_1|^{p-2} + |z_2|^{p-2})|z_1 - z_2| \tag{3-14}$$

and

$$\left| |z_1|^{p-3} z_1^2 - |z_2|^{p-3} z_2^2 \right| \leq C(|z_1|^{p-2} + |z_2|^{p-2})|z_1 - z_2|,$$

which are proven in [Cazenave et al. 2011, Remark 2.3] together with (3-12). $\square$

**3B2.** *Proof of Theorem 1.4 in the case $\sigma = 0$.* The strategy of the proof in this case is similar, at the price of some technicalities, since the Leibniz rule (3-12) does not hold true for non-integer derivatives. Actually, when $\sigma = 0$, we will have to work in $X_T^s$ for $s < 1$ because the probabilistic term $e^{-itH} u_0 \notin \mathcal{W}^{1,\infty}(\mathbb{R}^2)$.

Moreover, we are not able to obtain a contraction estimate in $X_T^s$. Therefore, we will do a fixed point in the space $\{\|v\|_{X_T^s} \leq K\}$ endowed with the weaker metric induced by $X_T^0$. We can check that this space is complete. Actually, by the Banach–Alaoglu theorem, the closed balls of each component space of $X_T^s$ are compact for the weak$^\star$ topology.

For $0 < s < 1$, we use the following characterisation of the usual $H^s(\mathbb{R}^2)$ norm:

$$\|g\|_{H^s(\mathbb{R}^2)} = \left( \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|g(x) - g(y)|^2}{|x - y|^{2s+2}} \, dx \, dy \right)^{1/2}. \tag{3-15}$$

For $\mu = \mu_\gamma \in \mathcal{M}^0$, $K \geq 0$ and $\varepsilon > 0$, define the set $\widetilde{F}_0(K)$ as

$$\widetilde{F}_0(K) = \left\{ w \in L^2(\mathbb{R}^2) \mid \|w\|_{L^2(\mathbb{R}^2)} \leq K, \ \|e^{-itH} w\|_{L_{[0,2\pi]}^{1/\varepsilon} \mathcal{W}^{1-\varepsilon,\infty}(\mathbb{R}^2)} \leq K \right.$$
$$\left. \text{and} \ \|(e^{-itH} w)(x) - (e^{-itH} w)(y)\|_{L_{t \in [0,2\pi]}^\infty} \leq K|x - y|^{1-\varepsilon} \right\}.$$

The next result states that $\widetilde{F}_0(K)$ is a set with large measure.

**Lemma 3.5.** *If $\varepsilon > 0$ is small enough,*

$$\mu\left( (\widetilde{F}_0(K))^c \right) \leq C e^{-cK^2 / \|\gamma\|_{L^2(\mathbb{R}^2)}^2}.$$

*Proof.* We only have to study the contribution of the Lipschitz term in $\widetilde{F}_0(K)$, since the others are controlled by Proposition 2.1.

We fix $\gamma = \sum\limits_{n=0}^{+\infty} c_n \varphi_n \in \mathcal{A}_0$ and set $\gamma^\omega = \sum\limits_{n=0}^{+\infty} g_n(\omega) c_n \varphi_n$. Let $k \geq 1$. By definition,

$$\int_{L^2(\mathbb{R}^2)} \|e^{-itH} u(x) - e^{-itH} u(y)\|_{L_{[0,2\pi]}^\infty}^k \, d\mu(u)$$
$$= \int_\Omega \|e^{-itH} \gamma^\omega(x) - e^{-itH} \gamma^\omega(y)\|_{L_{[0,2\pi]}^\infty}^k \, d\mathbb{P}(\omega). \tag{3-16}$$

We have $e^{-itH}\gamma^{\omega}(x) - e^{-itH}\gamma^{\omega}(y) = \sum_{n=0}^{+\infty} g_n(\omega)c_n e^{-it\lambda_n}(\varphi_n(x) - \varphi_n(y))$. By Khinchin (Lemma 2.3) we get

$$
\begin{aligned}
\left\| e^{-itH}\gamma^{\omega}(x) - e^{-itH}\gamma^{\omega}(y) \right\|_{L_{\mathbb{P}}^k} &\leq C\sqrt{k}\left( \sum_{n=0}^{+\infty} |c_n|^2 |\varphi_n(x) - \varphi_n(y)|^2 \right)^{1/2} \\
&= C\sqrt{k}\left( \sum_{j=1}^{+\infty} \sum_{n\in I(j)} |c_n|^2 |\varphi_n(x) - \varphi_n(y)|^2 \right)^{1/2},
\end{aligned}
$$

Recall that $k \in I(j) = \{n \in \mathbb{N} \mid 2j \leq \lambda_n < 2(j+1)\}$ and that $\#I(j) \sim cj$. Next, by condition (1-3), we deduce that

$$
\left\| e^{-itH}\gamma^{\omega}(x) - e^{-itH}\gamma^{\omega}(y) \right\|_{L_{\mathbb{P}}^k} \leq C\sqrt{k}\left( \sum_{j=1}^{+\infty} j^{-1}\left( \sum_{\ell\in I(j)} |c_\ell|^2 \right) \sum_{n\in I(j)} |\varphi_n(x) - \varphi_n(y)|^2 \right)^{1/2}
$$

Now we need the following estimate, proven in [Imekraz et al. 2014, Lemma 6.1]:

$$
\sum_{n\in I(j)} |\varphi_n(y) - \varphi_n(x)|^2 \leq C|y - x|^2 j.
$$

Therefore, we obtain

$$
\left\| e^{-itH}\gamma^{\omega}(x) - e^{-itH}\gamma^{\omega}(y) \right\|_{L_{\mathbb{P}}^k} \leq C\sqrt{k}\,|x - y|\,\|\gamma\|_{L^2(\mathbb{R}^2)},
$$

and for $k \geq q$ an integration in time and Minkowski yield

$$
\left\| e^{-itH}\gamma^{\omega}(x) - e^{-itH}\gamma^{\omega}(y) \right\|_{L_{\mathbb{P}}^k L_{[0,2\pi]}^q} \leq C\sqrt{k}\,|x - y|\,\|\gamma\|_{L^2(\mathbb{R}^2)}.
$$

However, since the case $q = +\infty$ is forbidden, the previous estimate is not enough to have a control on the $L_{[0,2\pi]}^{\infty}$-norm. To tackle this issue, we claim that for $k \geq q$ we have

$$
\left\| e^{-itH}\gamma^{\omega}(x) - e^{-itH}\gamma^{\omega}(y) \right\|_{L_{\mathbb{P}}^k W_{[0,2\pi]}^{1,q}} \leq C\sqrt{k}\,\|\gamma\|_{L^2(\mathbb{R}^2)}. \tag{3-17}
$$

Then by a usual Sobolev embedding argument we get (by taking $q \gg 1$ large enough) that for all $\varepsilon > 0$

$$
\left\| e^{-itH}\gamma^{\omega}(x) - e^{-itH}\gamma^{\omega}(y) \right\|_{L_{\mathbb{P}}^k L_{[0,2\pi]}^{\infty}} \leq C\sqrt{k}\,|x - y|^{1-\varepsilon}\,\|\gamma\|_{L^2(\mathbb{R}^2)},
$$

which in turn by (3-16) implies that

$$
\mu\left( u \in L^2(\mathbb{R}^2) \mid \left\| e^{-itH}u(x) - e^{-itH}u(y) \right\|_{L_{[0,2\pi]}^{\infty}} > K|x - y|^{1-\varepsilon} \right) \leq C e^{-cK^2/\|\gamma\|_{L^2}^2},
$$

as we did in the end of the proof of Proposition 2.1.

Let us now prove (3-17). We have

$$
\partial_t\left( e^{-itH}\gamma^{\omega}(x) - e^{-itH}\gamma^{\omega}(y) \right) = -i\sum_{n=0}^{+\infty} g_n(\omega)\lambda_n c_n e^{-it\lambda_n}(\varphi_n(x) - \varphi_n(y)),
$$

and with the previous arguments we get

$$\left\| \partial_t \left( e^{-itH} \gamma^\omega(x) - e^{-itH} \gamma^\omega(y) \right) \right\|_{L^k_{\mathbb{P}}} \leq C\sqrt{k} \left( \sum_{j=1}^{+\infty} \left( \sum_{\ell \in I(j)} |c_\ell|^2 \right) \sum_{n \in I(j)} |\varphi_n(x) - \varphi_n(y)|^2 \right)^{1/2}$$

$$\leq C\sqrt{k} \|\gamma\|_{L^2(\mathbb{R}^2)},$$

where here we have used the Thangavelu–Karadzhov estimate (see [Poiret et al. 2013, Lemma 3.5])

$$\sup_{x \in \mathbb{R}^2} \sum_{n \in I(j)} |\varphi_n(x)|^2 \leq C.$$

We conclude the proof of (3-17) by integrating in time and using Minkowski.     □

We will also need the following technical result.

**Lemma 3.6.** *Let $u_0 \in \widetilde{F}_0(K)$ and $f(t, x) = e^{-itH} u_0(x)$. Let $2 \leq q < +\infty$ and $g \in L^q([-T, T]; L^2(\mathbb{R}^2))$. Then, if $\varepsilon > 0$ is small enough in the definition of $\widetilde{F}_0(K)$,*

$$\left\| \left( \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|f(t, x) - f(t, y)|^2 |g(t, x)|^2}{|x - y|^{2s+2}} \, dx \, dy \right)^{1/2} \right\|_{L^q_{[-T, T]}} \leq CK \|g\|_{L^q_{[-T, T]} L^2(\mathbb{R}^2)}. \qquad (3\text{-}18)$$

*Proof.* We consider such $f, g$, and we split the integral. On the one hand, we use that $f$ is Lipschitz:

$$\int_{|x-y| \leq 1} \frac{|f(t, x) - f(t, y)|^2 |g(t, x)|^2}{|x - y|^{2s+2}} \, dx \, dy \leq K^2 \int_{x \in \mathbb{R}^2} |g(t, x)|^2 \left( \int_{y: |x-y| \leq 1} \frac{dy}{|x - y|^{2s+2\varepsilon}} \right) dx$$

$$\leq CK^2 \|g(t, \cdot)\|^2_{L^2(\mathbb{R}^2)},$$

provided that $s + \varepsilon < 1$. We take the $L^q_{[-T, T]}$-norm, and we see that this contribution is bounded by the right side of (3-18).

On the other hand

$$\int_{|x-y| \geq 1} \frac{|f(t, x) - f(t, y)|^2 |g(t, x)|^2}{|x - y|^{2s+2}} \, dx \, dy$$

$$\leq C \|f(t, \cdot)\|^2_{L^\infty(\mathbb{R}^2)} \int_{x \in \mathbb{R}^2} |g(t, x)|^2 \left( \int_{y: |x-y| \geq 1} \frac{dy}{|x - y|^{2s+2}} \right) dx$$

$$\leq C \|f(t, \cdot)\|^2_{L^\infty(\mathbb{R}^2)} \|g(t, \cdot)\|^2_{L^2(\mathbb{R}^2)}$$

if $s > 0$. Now we take the $L^q_{[-T, T]}$-norm, and use the fact that $\|f\|_{L^q_{[0, 2\pi]} L^\infty(\mathbb{R}^2)} \leq K$ if $\varepsilon < 1/q$.     □

We now state the main estimates of this section.

**Proposition 3.7.** *There exist $C > 0$ and $\kappa > 0$ such that if $u_0 \in \widetilde{F}_0(K)$ for some $K > 0$ then for any $v, v_1, v_2 \in X^s_T$ and $0 < T \leq 1$,*

$$\left\| \int_0^t e^{-i(t-\tau)H} \left( \cos(2\tau)^{p-3} |e^{-i\tau H} u_0 + v|^{p-1} (e^{-i\tau H} u_0 + v) \right) d\tau \right\|_{X^s_T} \leq CT^\kappa \left( K^p + \|v\|^p_{X^s_T} \right), \quad (3\text{-}19)$$

*and*

$$\left\| \int_0^t e^{-i(t-\tau)H} \big(\cos(2\tau)^{p-3}|e^{-i\tau H}u_0 + v_1|^{p-1}(e^{-i\tau H}u_0 + v_1)\big)\, d\tau \right.$$
$$\left. - \int_0^t e^{-i(t-\tau)H} \big(\cos(2\tau)^{p-3}|e^{-i\tau H}u_0 + v_2|^{p-1}(e^{-i\tau H}u_0 + v_2)\big)\, d\tau \right\|_{X_T^0}$$
$$\leq CT^\kappa \|v_1 - v_2\|_{X_T^0} \big(K^{p-1} + \|v_1\|_{X_T^s}^{p-1} + \|v_2\|_{X_T^s}^{p-1}\big). \quad (3\text{-}20)$$

*Proof.* Let $u_0 \in \widetilde{F}_0(K)$ and set $f = e^{-isH}u_0$. Let $2 < p < 3$, then there exists $q \gg 1$ such that $q'(3 - p) < 1$, which in turn implies $\| \cos(2s)^{p-3} \|_{L_{[-T,T]}^{q'}} \leq CT^\kappa$. Next, if $s < 1$ is large enough we have, by Sobolev,

$$\|v\|_{L_{[-T,T]}^\infty L^{2(p-1)}(\mathbb{R}^2)} \leq \|v\|_{X_T^s} \quad \text{and} \quad \|v\|_{L_{[-T,T]}^{q(p-1)}L^\infty(\mathbb{R}^2)} \leq \|v\|_{X_T^s}. \quad (3\text{-}21)$$

First we prove (3-19). From Strichartz and Hölder, we get

$$\left\| \int_0^t e^{-i(t-s)H} \big(\cos(2s)^{p-3}|f + v|^{p-1}(f + v)\big)\, ds \right\|_{X_T^s}$$
$$\leq C \big\| \cos(2s)^{p-3}|f + v|^{p-1}(f + v) \big\|_{L_{[-T,T]}^1 \mathcal{H}^s(\mathbb{R}^2)}$$
$$\leq C \big\| \cos(2s)^{p-3} \big\|_{L_{[-T,T]}^{q'}} \big\| |f + v|^{p-1}(f + v) \big\|_{L_{[-T,T]}^q \mathcal{H}^s(\mathbb{R}^2)}$$
$$\leq CT^\kappa \big\| |f + v|^{p-1}(f + v) \big\|_{L_{[-T,T]}^q \mathcal{H}^s(\mathbb{R}^2)}. \quad (3\text{-}22)$$

By using the characterization (3-15), we will prove that

$$\big\| |f + v|^{p-1}(f + v) \big\|_{L_{[-T,T]}^q \mathcal{H}^s(\mathbb{R}^2)} \leq C\big(K^p + \|v\|_{X_T^s}^p\big). \quad (3\text{-}23)$$

The term $\big\| \langle x \rangle^s |f + v|^{p-1}(f + v) \big\|_{L_{[-T,T]}^q L^2(\mathbb{R}^2)}$ is easily controlled; thus we only detail the contribution of the $H^s$ norm. With (3-14), it is easy to check that, for all $x, y \in \mathbb{R}^2$,

$$\big| |f + v|^{p-1}(f + v)(x) - |f + v|^{p-1}(f + v)(y) \big|$$
$$\leq C|v(x) - v(y)|\big(|v(x)|^{p-1} + |v(y)|^{p-1} + |f(x)|^{p-1} + |f(y)|^{p-1}\big)$$
$$+ C|f(x) - f(y)|\big(|v(x)|^{p-1} + |v(y)|^{p-1} + |f(x)|^{p-1} + |f(y)|^{p-1}\big).$$

By (3-21) the contribution in $L_{[-T,T]}^q H^s(\mathbb{R}^2)$ of the first term in the previous expression is at most

$$C\big(\|f\|_{L_{[-T,T]}^{q(p-1)}L^\infty(\mathbb{R}^2)}^{p-1} + \|v\|_{L_{[-T,T]}^{q(p-1)}L^\infty(\mathbb{R}^2)}^{p-1}\big)\|v\|_{X_T^s} \leq C\big(K^{p-1} + \|v\|_{X_T^s}^{p-1}\big)\|v\|_{X_T^s}.$$

To bound the second term, we apply Lemma 3.6, which gives a contribution of at most

$$\big(\|f\|_{L_{[-T,T]}^{q(p-1)}L^{2(p-1)}(\mathbb{R}^2)}^{p-1} + \|v\|_{L_{[-T,T]}^{q(p-1)}L^{2(p-1)}(\mathbb{R}^2)}^{p-1}\big)K \leq C\big(K^{p-1} + \|v\|_{X_T^s}^{p-1}\big)K,$$

which concludes the proof of (3-23).

The proof of (3-20) is in the same spirit, and even easier. We do not write the details. □

Thanks to the estimates of Proposition 3.7, for $K > 0$ small enough (see the proof of Theorem 1.3 for more details) we are able to construct a unique solution $v \in \mathscr{C}([-\pi/4, \pi/4]; L^2(\mathbb{R}^2))$ such that $v \in L^\infty([-\pi/4, \pi/4]; \mathscr{H}^s(\mathbb{R}^2))$. By interpolation we deduce that $v \in \mathscr{C}([-\pi/4, \pi/4]; \mathscr{H}^{s'}(\mathbb{R}^2))$ for all $s' < s$. The end of the proof of Theorem 1.4 is similar to the proof of Theorem 1.3, using here Lemma 3.5.

## 4. Global well-posedness for the cubic equation

**4A.** *The case of dimension $d = 3$.* We now turn to the proof of Theorem 1.5, which is obtained thanks to the high-low frequency decomposition method of [Bourgain 1999, p. 84].

Let $0 \le s < 1$ and fix $\mu = \mu_\gamma \in \mathcal{M}^s$. For $K \ge 0$ define the set $F_s(K)$ as

$$F_s(K) = \big\{w \in \mathscr{H}^s(\mathbb{R}^3) \mid \|w\|_{\mathscr{H}^s(\mathbb{R}^3)} \le K,\ \|w\|_{L^4(\mathbb{R}^3)} \le K \text{ and } \|e^{-itH}w\|_{L^{1/\varepsilon}_{[0,2\pi]}W^{3/2+s-\varepsilon,\infty}(\mathbb{R}^3)} \le K\big\}.$$

Then, by Proposition 2.1,

$$\mu\big((F_s(K))^c\big)$$
$$\le \mu(\|w\|_{\mathscr{H}^s} > K) + \mu(\|w\|_{L^4} > K) + \mu\big(\|e^{-itH}w\|_{L^{1/\varepsilon}_{[0,2\pi]}W^{3/2+s-\varepsilon,\infty}} > K\big) \le Ce^{-cK^2/\|\gamma\|^2_{\mathscr{H}^s}}. \quad (4\text{-}1)$$

Now we define a smooth version of the usual spectral projector. Let $\chi \in \mathscr{C}^\infty_0(-1, 1)$, so that $0 \le \chi \le 1$, with $\chi = 1$ on $\big[-\frac{1}{2}, \frac{1}{2}\big]$. We define the operators $S_N = \chi\big(\frac{H}{N^2}\big)$ as

$$S_N\left(\sum_{n=0}^{+\infty} c_n \varphi_n\right) = \sum_{n=0}^{+\infty} \chi\left(\frac{\lambda_n}{N^2}\right) c_n \varphi_n,$$

and we write

$$v_N = S_N v, \quad v^N = (1 - S_N)v.$$

It is clear that for any $\sigma \ge 0$ we have $\|S_N\|_{\mathscr{H}^\sigma \to \mathscr{H}^\sigma} = 1$. Moreover, by [Burq et al. 2010, Proposition 4.1], for all $1 \le r \le +\infty$, $\|S_N\|_{L^r \to L^r} \le C$, uniformly in $N \ge 1$.

It is straightforward to check that

$$\|v_N\|_{\mathscr{H}^1} \le N^{1-s}\|v\|_{\mathscr{H}^s}, \quad \|v^N\|_{L^2} \le N^{-s}\|v\|_{\mathscr{H}^s}. \quad (4\text{-}2)$$

Next, let $u_0 \in F_s(N^\varepsilon)$. By the definition of $F_s(N^\varepsilon)$ and (4-2), $\|u_{0,N}\|_{\mathscr{H}^1} \le N^{1-s}\|u_0\|_{\mathscr{H}^s} \le N^{1-s+\varepsilon}$. The nonlinear term of the energy can be controlled by the quadratic term. Indeed

$$\|u_{0,N}\|^4_{L^4} \le CN^\varepsilon \le N^{2(1-s+\varepsilon)},$$

and thus

$$E(u_{0,N}) \le 2N^{2(1-s+\varepsilon)}. \quad (4\text{-}3)$$

We also have

$$\|u_{0,N}\|_{L^2} \le \|u_0\|_{\mathscr{H}^s} \le N^\varepsilon.$$

For a nice description of the stochastic version of the low-high frequency decomposition method we use here, we refer to the introduction of [Colliander and Oh 2012]. To begin with, we look for a solution $u$ to (1-7) of the form $u = u^1 + v^1$, where $u^1$ is the solution to

$$\begin{cases} i\dfrac{\partial u^1}{\partial t} - Hu^1 = |u^1|^2 u^1, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u^1(0) = u_{0,N}, \end{cases} \tag{4-4}$$

and where $v^1 = e^{-itH} u_0^N + w^1$ satisfies

$$\begin{cases} i\dfrac{\partial w^1}{\partial t} - Hw^1 = |w^1 + e^{-itH} u_0^N + u^1|^2 (w^1 + e^{-itH} u_0^N + u^1) - |u^1|^2 u^1, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ w^1(0) = 0. \end{cases} \tag{4-5}$$

Since (4-4) is $\mathcal{H}^1$-subcritical, by the usual deterministic arguments there exists a unique global solution $u^1 \in \mathscr{C}(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^3))$.

We now turn to (4-5), for which we have the next local existence result.

**Proposition 4.1.** *Let $0 < s < 1$ and $\mu = \mu_\gamma \in \mathcal{M}^s$. Set $T = N^{-4(1-s)-\varepsilon}$ with $\varepsilon > 0$. Assume that $E(u^1) \leq 4N^{2(1-s+\varepsilon)}$ and $\|u^1\|_{L_{[0,T]}^\infty L^2} \leq 2N^\varepsilon$. Then:*

(i) *There exists a set $\Sigma_T^1 \subset \mathcal{H}^s$, which only depends on $T$, so that*

$$\mu(\Sigma_T^1) \geq 1 - C \exp\bigl(-cT^{-\delta} \|\gamma\|_{\mathcal{H}^s(\mathbb{R}^3)}^{-2}\bigr), \quad \text{with some } \delta > 0.$$

(ii) *For all $u_0 \in \Sigma_T^1$ there exists a unique solution $w^1 \in \mathscr{C}([0,T], \mathcal{H}^1(\mathbb{R}^3))$ to (4-5), which satisfies the bounds*

$$\|w^1\|_{L_{[0,T]}^\infty \mathcal{H}^1} \leq CN^{\beta(s)+c\varepsilon}, \tag{4-6}$$

*with*

$$\beta(s) = \begin{cases} -\dfrac{5}{2} & \text{if } 0 \leq s \leq \dfrac{1}{2}, \\ 2s - \dfrac{7}{2} & \text{if } \dfrac{1}{2} \leq s \leq 1, \end{cases} \tag{4-7}$$

*and*

$$\|w^1\|_{L_{[0,T]}^\infty L^2} \leq CN^{-9/2+2s+c\varepsilon}. \tag{4-8}$$

*Proof.* In the next lines, we write $C^{a+} = C^{a+b\varepsilon}$, for some absolute quantity $b > 0$. Since $d = 3$, for $T > 0$ we define the space $X_T^1 = L^\infty([0,T]; \mathcal{H}^1(\mathbb{R}^3)) \cap L^2([0,T]; \mathcal{W}^{1,6}(\mathbb{R}^3))$. Let $\varepsilon > 0$, and define $\Sigma_T^1 = F_s(N^\varepsilon)$. By (4-1) and the choice $T = N^{-4(1-s)-\varepsilon}$, the set $\Sigma_T^1$ satisfies (i).

Let $u_0 \in \Sigma_T^1$. To simplify the notations in the proof, we write $w = w^1$, $u = u^1$ and $f = e^{-itH} u_0^N$. We define the map

$$L(w) = \mp i \int_0^t e^{-i(t-s)H} \bigl(|f + u + w|^2 (f + u + w) - |u|^2 u\bigr)(s) \, ds. \tag{4-9}$$

First we prove (4-6). By Strichartz (3-3),

$$\|L(w)\|_{X_T^1} \leq C \bigl\| |f + u + w|^2 (f + u + w) - |u|^2 u \bigr\|_{L_T^1 \mathcal{H}^1 + L^2 \mathcal{W}^{1,6/5}}. \tag{4-10}$$

By estimating the contribution of every term, we now prove that

$$\|L(w)\|_{X_T^1} \leq CN^{\beta(s)+} + N^{0-} \|w\|_{X_T^1} + N^{-2(1-s)+} \|w\|_{X_T^1}^3, \tag{4-11}$$

where $\beta(s) < (1-s)$ is as in the statement. It is enough to prove that $L$ maps a ball of size $CN^{\beta(s)+}$ into itself for times $T = N^{-4(1-s)-\varepsilon}$. With similar arguments one can show that $L$ is a contraction (we do not write the details) and get $w$ satisfying (4-6).

Observe that the complex conjugation is harmless with respect to the norms considered; thus we can forget it. By the definition of $\Sigma_T^1 = F_s(N^\varepsilon)$ and (4-2) we have the estimates used in the sequel: for all $\sigma < \frac{3}{2}$,

$$\|f\|_{L_T^\infty L^2} \leq CN^{-s+\varepsilon} \quad \text{and} \quad \|H^{\sigma/2}f\|_{L_T^{\infty-}L^\infty} \leq CN^{\sigma-3/2-s+2\varepsilon}. \tag{4-12}$$

Let us prove the second estimate in detail:

$$\begin{aligned}
\|H^{\sigma/2}f\|_{L_T^{\infty-}L^\infty} &= N^\sigma \left\| \left(\frac{H}{N^2}\right)^{\sigma/2} \left(1 - \chi\left(\frac{H}{N^2}\right)\right) e^{-itH} u_0 \right\|_{L_T^{\infty-}L^\infty} \\
&\leq CN^\sigma \left\| \left(\frac{H}{N^2}\right)^{(3/2+s-\varepsilon)/2} \left(1 - \chi\left(\frac{H}{N^2}\right)\right) e^{-itH} u_0 \right\|_{L_T^{\infty-}L^\infty} \\
&\leq CN^{\sigma-3/2-s+\varepsilon} \|e^{-itH} u_0\|_{L_T^{\infty-}\mathcal{W}^{3/2+s-\varepsilon,\infty}} \\
&\leq CN^{\sigma-3/2-s+2\varepsilon},
\end{aligned}$$

where we have used that $x^{\sigma/2}(1-\chi(x)) \leq Cx^{(3/2+s-\varepsilon)/2}(1-\chi(x))$.

Observe also that by assumption

$$\|u\|_{L_T^\infty L^2} \leq CN^\varepsilon, \quad \|u\|_{L_T^\infty \mathcal{H}^1} \leq CN^{1-s+\varepsilon}, \quad \text{and} \quad \|u\|_{L_T^\infty L^4} \leq CN^{(1-s+\varepsilon)/2}.$$

We now estimate each term in the right side of (4-10):

• Source terms: Observe that $L_T^{4/3}\mathcal{W}^{1,3/2} \subset L_T^1\mathcal{H}^1 + L^2\mathcal{W}^{1,6/5}$. By Hölder and (4-12),

$$\begin{aligned}
\|fu^2\|_{L_T^1\mathcal{H}^1+L^2\mathcal{W}^{1,6/5}} &\leq C\|fuH^{1/2}u\|_{L_T^{4/3}L^{3/2}} + C\|u^2 H^{1/2}f\|_{L_T^1 L^2} \\
&\leq CT^{3/4-}\|u\|_{L_T^\infty \mathcal{H}^1}\|u\|_{L_T^\infty L^6}\|f\|_{L_T^{\infty-}L^\infty} + CT^{1-}\|u\|_{L_T^\infty L^4}^2\|H^{1/2}f\|_{L_T^{\infty-}L^\infty} \\
&\leq CN^{-5/2+} + CN^{-7/2+2s+} \leq CN^{\beta(s)+},
\end{aligned}$$

where we have set $\beta(s) = \max\left(-\frac{5}{2}, -\frac{7}{2} + 2s\right)$, which is precisely (4-7). Similarly,

$$\begin{aligned}
\|f^2 u\|_{L_T^1\mathcal{H}^1} &\leq C\|f^2 H^{1/2}u\|_{L_T^1 L^2} + C\|uf H^{1/2}f\|_{L_T^1 L^2} \\
&\leq CT^{1-}\|u\|_{L_T^\infty \mathcal{H}^1}\|f\|_{L_T^{\infty-}L^\infty}^2 + CT^{1-}\|u\|_{L_T^\infty L^2}\|f\|_{L_T^{\infty-}L^\infty}\|H^{1/2}f\|_{L_T^{\infty-}L^\infty} \\
&\leq CT^{1-}N^{-2-3s+} + CT^{1-}N^{-2-2s+} \leq CN^{-6+2s+} \leq CN^{\beta(s)+}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\|f^3\|_{L_T^1\mathcal{H}^1} &\leq C\|f^2 H^{1/2}f\|_{L_T^1 L^2} \leq CT^{1-}\|H^{1/2}f\|_{L_T^{\infty-}L^\infty}\|f\|_{L_T^{\infty-}L^\infty}\|f\|_{L_T^\infty L^2} \\
&\leq CT^{1-}N^{-1/2-s+}N^{-3/2-s+}N^{-s+} \leq CN^{-6+s+} \leq CN^{\beta(s)+}.
\end{aligned}$$

• Linear terms in $w$:

$$\|wf^2\|_{L^1_T \mathcal{H}^1} \le C\|f^2 H^{1/2} w\|_{L^1_T L^2} + C\|wfH^{1/2}f\|_{L^1_T L^2}$$
$$\le CT^{1-}\|f\|^2_{L^\infty_T - L^\infty}\|w\|_{L^\infty_T \mathcal{H}^1} + CT^{1-}\|w\|_{L^\infty_T L^2}\|f\|_{L^\infty_T - L^\infty}\|H^{1/2}f\|_{L^\infty_T - L^\infty}$$
$$\le CN^{-6+2s+}\|w\|_{X^1_T} \le CN^{0-}\|w\|_{X^1_T}.$$

Using that $\|w\|_{L^{4/3+}_T L^{\infty-}} \le CT^{1/2-}\|w\|_{L^4_T L^{\infty-}} \le CT^{1/2-}\|w\|_{L^4_T \mathcal{W}^{1,3}}$ and $X^1_T \subset L^4([0,T]; \mathcal{W}^{1,3})$,

$$\|wu^2\|_{L^1_T \mathcal{H}^1 + L^2 \mathcal{W}^{1,6/5}} \le C\|u^2 H^{1/2} w\|_{L^1_T L^2} + C\|wuH^{1/2}u\|_{L^{4/3+}_T L^{3/2-}}$$
$$\le C\|u\|^2_{L^4_T L^6}\|w\|_{L^2_T \mathcal{W}^{1,6}} + C\|w\|_{L^{4/3+}_T L^{\infty-}}\|u\|_{L^\infty_T L^6}\|u\|_{L^\infty_T \mathcal{H}^1}$$
$$\le CT^{1/2-}\|u\|^2_{L^\infty_T \mathcal{H}^1}\|w\|_{X^1_T} \le CN^{0-}\|w\|_{X^1_T}.$$

• The cubic term in $w$: by Sobolev and $X^1_T \subset L^{4-}([0,T]; \mathcal{W}^{1,3+}) \subset L^{4-}([0,T]; L^\infty)$, we have

$$\|w^3\|_{L^1_T \mathcal{H}^1} \le C\|w^2 H^{1/2} w\|_{L^1_T L^2} \le C\|w\|_{L^\infty_T \mathcal{H}^1}\|w\|^2_{L^2_T L^\infty}$$
$$\le CT^{1/2-}\|w\|^3_{X^1_T} \le CN^{-2(1-s)+}\|w\|^3_{X^1_T}.$$

• Quadratic terms in $w$: with similar arguments, we check that they are controlled by the previous ones.

This completes the proof of (4-11). Hence, for all $u_0 \in \Sigma^1_T$, $L$ has a unique fixed point $w$.

Let $w \in X^1_T$ be defined this way, and let us prove that $\|w\|_{X^0_T} \le CN^{-9/2+2s+}$, which will imply (4-8). By the Strichartz inequality (3-3),

$$\|w\|_{X^0_T} \le C\big\||f+u+w|^2(f+u+w) - |u|^2 u\big\|_{L^1_T L^2 + L^2 L^{6/5}}.$$

As previously, the main contribution in the source term is

$$\|fu^2\|_{L^1_T L^2} \le T^{1-}\|u\|^2_{L^\infty_T L^4}\|f\|_{L^\infty_T - L^\infty} \le CN^{-4(1-s)+1-s-3/2-s+} = CN^{-9/2+2s+}.$$

For the cubic term we write

$$\|w^3\|_{L^1_T L^2} \le \|w\|_{L^\infty_T L^2}\|w\|^2_{L^2_T L^\infty} \le CT^{1/2-}\|w\|_{L^\infty_T L^2}\|w\|^2_{X^1_T}$$
$$\le CN^{-2(1-s)+\beta(s)+}\|w\|_{L^\infty_T L^2} \le CN^{0-}\|w\|_{X^0_T},$$

which gives a control by the linear term.

The other terms are controlled with similar arguments, and we leave the details to the reader. This finishes the proof of Proposition 4.1. $\qquad\square$

**Lemma 4.2.** *Under the assumptions of Proposition 4.1, for all $u_0 \in \Sigma^1_T$ we have*

$$|E(u^1(T) + w^1(T)) - E(u^1(T))| \le CN^{1-s+\beta(s)+}.$$

*Proof.* Write $u = u^1$ and $w = w^1$. A direct expansion and Hölder give

$$|E(u(T) + w(T)) - E(u(T))|$$
$$\leq 2\|u\|_{L_T^\infty \mathcal{H}^1}\|w\|_{L_T^\infty \mathcal{H}^1} + \|w\|_{L_T^\infty \mathcal{H}^1}^2 + C\|w\|_{L_T^\infty L^4}\|u\|_{L_T^\infty L^4}^3 + C\|w\|_{L_T^\infty L^4}^4.$$

Since $\beta(s) \leq (1-s)$, we directly have

$$2\|u\|_{L_T^\infty \mathcal{H}^1}\|w\|_{L_T^\infty \mathcal{H}^1} + \|w\|_{L_T^\infty \mathcal{H}^1}^2 \leq CN^{1-s+\beta(s)+}.$$

By Sobolev and Proposition 4.1,

$$\|w\|_{L_T^\infty L^4} \leq C\|w\|_{L_T^\infty \mathcal{H}^{3/4}} \leq C\|w\|_{L_T^\infty L^2}^{1/4}\|w\|_{L_T^\infty \mathcal{H}^1}^{3/4} \leq CN^{\eta(s)+}, \tag{4-13}$$

with $\eta(s) = \max(-3 + s/2, -15/4 + 2s) \leq (1 - s + \beta(s))/3$. Hence,

$$\|w\|_{L_T^\infty L^4}^3 \leq CN^{1-s+\beta(s)+}.$$

From the bounds $\|u\|_{L_T^\infty L^4} \leq CN^{(1-s)/2}$ and (4-13), we infer

$$\|w\|_{L_T^\infty L^4}\|u\|_{L_T^\infty L^4}^3 \leq CN^{\delta(s)+},$$

where $\delta(s) = \max(-3 + s/2, -15/4 + 2s) \leq 1 - s + \beta(s)$ (with equality when $0 < s \leq \frac{1}{2}$). This completes the proof. $\square$

With the results of Proposition 4.1 and Lemma 4.2, we are able to iterate the argument. At time $t = T$, write $u = u^2 + v^2$ where $u^2$ is the solution to

$$\begin{cases} i\dfrac{\partial u^2}{\partial t} - Hu^2 = |u^2|^2 u^2, & (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\ u^2(T) = u^1(T) + w^1(T) \in \mathcal{H}^1(\mathbb{R}^3), \end{cases} \tag{4-14}$$

and where $v^2 = e^{-itH}u_0^N + w^2$ satisfies

$$\begin{cases} i\dfrac{\partial w^2}{\partial t} - Hw^2 = |w^2 + e^{-itH}u_0^N + u^2|^2 (w^2 + e^{-itH}u_0^N + u^2) - |u^2|^2 u^2, & (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\ w^2(T) = 0. \end{cases}$$

By Proposition 4.1, $w^1(T) \in \mathcal{H}^1(\mathbb{R}^3)$; thus (4-14) is globally well-posed. Then, thanks to Lemma 4.2, by the conservation of the energy,

$$E(u^2) = E(u^1(T) + w^1(T)) \leq 4N^{2(1-s+\varepsilon)},$$

and, by the conservation of the mass,

$$\|u^2\|_{L_T^\infty L^2} = \|u^1(T) + w^1(T)\|_{L^2} \leq 2N^\varepsilon.$$

Therefore there exists a set $\Sigma_T^2 \subset \mathcal{H}^s$ with

$$\mu(\Sigma_T^2) \geq 1 - C\exp(-cT^{-\delta}\|\gamma\|_{\mathcal{H}^s}^{-2}),$$

and such that for all $u_0 \in \Sigma_T^2$ there exists a unique $w^2 \in \mathscr{C}([T, 2T], \mathscr{H}^1(\mathbb{R}^3))$ that satisfies the result of Proposition 4.1, with the same $T > 0$. Here we use crucially that the large deviation bounds of Proposition 2.1 are invariant under time shift $\tau$.

Fix a time $A > 0$. We can iterate the previous argument and construct $u^j$, $v^j$ and $w^j$ for $1 \le j \le \lfloor A/T \rfloor$ such that the function $u^j$ is the solution to (4-14) with initial condition

$$u^j(t = (j-1)T) = u^{j-1}((j-1)T) + w^{j-1}((j-1)T),$$

then we set $v^j(t) = e^{-itH}u_0^N + w^j(t)$, where the function $w^j$ is the solution to

$$\begin{cases} i\dfrac{\partial w^j}{\partial t} - Hw^j = |w^j + e^{-itH}u_0^N + u^j|^2(w^j + e^{-itH}u_0^N + u^j) - |u^j|^2 u^j, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ w^j((j-1)T) = 0. \end{cases}$$

This enables us to define a unique solution $u$ to the initial problem (1-7) defined by $u(t) = u^j(t) + v^j(t)$ for $t \in [(j-1)T, jT]$, with $1 \le j \le \lfloor A/T \rfloor$ provided that $u_0 \in \Gamma_T^A$, where $\Gamma_T^A := \bigcap_{j=1}^{\lfloor A/T \rfloor} \Sigma_T^j$.

Thanks to the exponential bounds, we have $\mu((\Gamma_T^A)^c) \le C \exp(-cT^{-\delta/2}\|\gamma\|_{\mathscr{H}^s}^{-2})$, with $T = N^{-4(1-s)-\varepsilon}$.

For uniform bounds on the energy and the mass, it remains to check whether $E(u^j) \le 4N^{2(1-s+\varepsilon)}$ and $\|u^j\|_{L^2(\mathbb{R}^3)} \le 2N^\varepsilon$ for all $1 \le j \le \lfloor A/T \rfloor$. By Lemma 4.2, for $T = N^{-4(1-s)-}$,

$$E(u^j) \le E(u_{0,N}) + CAT^{-1}N^{1-s+\beta(s)+} \le 2N^{2(1-s+\varepsilon)} + CAN^{\beta(s)+5(1-s)+}, \tag{4-15}$$

which satisfies the prescribed bound if and only if $3(1-s) + \beta(s) < 0$.

• If $\frac{1}{2} \le s \le 1$, the condition is $3(1-s) + 2s - \frac{7}{2} < 0$, or equivalently $s > -\frac{1}{2}$, which is satisfied.

• If $0 \le s \le \frac{1}{2}$, the condition is $3(1-s) - \frac{5}{2} < 0$, or equivalently $s > \frac{1}{6}$. The same argument applies to control $\|u^j\|_{L^2}$.

If $\frac{1}{6} < s < 1$, we optimise in (4-15) with the choice of $N \ge 1$ so that $A \sim cN^{-3(1-s)-\beta(s)}$, and get that, for $1 \le j \le \lfloor A/T \rfloor$,

$$E(u^j) \le CA^{c_s+},$$

with

$$c_s = \begin{cases} \dfrac{2(1-s)}{6s-1} & \text{if } \frac{1}{6} < s \le \frac{1}{2}, \\ \dfrac{2(1-s)}{2s+1} & \text{if } \frac{1}{2} \le s \le 1. \end{cases} \tag{4-16}$$

Denote by $\Gamma^A = \Gamma_T^A$ the set defined with the previous choice of $N$ and $T = N^{-4(1-s)-\varepsilon}$.

**Lemma 4.3.** *Let $\frac{1}{6} < s < 1$. Then for all $A \in \mathbb{N}$ and all $u_0 \in \Gamma^A$ there exists a unique solution to* (1-7) *on* $[0, A]$, *which reads*

$$u(t) = e^{-itH}u_0 + w(t), \quad \text{with } w \in \mathscr{C}([0, A], \mathscr{H}^1(\mathbb{R}^3)), \quad \sup_{t \in [0,A]} E(w(t)) \le CA^{c_s+}.$$

*Proof.* On the time interval $[(j-1)T, jT]$ we have $u = u^j + v^j$ where $v^j = e^{-itH}u_0^N + w^j$ and $u^j = e^{-itH}u_{0,N} + z^j$, for some $z_j \in \mathscr{C}([0, +\infty[, \mathscr{H}^1(\mathbb{R}^3))$. Therefore, if we define $w \in \mathscr{C}([0, A], \mathscr{H}^1(\mathbb{R}^3))$

by $w(t) = z^j(t) + w^j(t)$ for $t \in [(j-1)T, jT]$ and $1 \le j \le \lfloor A/T \rfloor$, we get $u(t) = e^{-itH}u_0 + w(t)$ for all $t \in [0, A]$. Next, for $t \in [(j-1)T, jT]$,

$$E(w(t)) \le CE(z^j) + CE(w^j) \le CE(u^j) + CE(e^{-itH}u_{0,N}) + CE(w^j) \le CA^{c_s+},$$

which was the claim. $\qquad\square$

We are now able to complete the proof of Theorem 1.5. Set

$$\Theta = \bigcap_{k=1}^{+\infty} \bigcup_{A \ge k} \Gamma^A \quad \text{and} \quad \Sigma = \Theta + \mathcal{H}^1.$$

We have $\mu(\Theta) = \lim_{k \to \infty} \mu\left(\bigcup_{A \ge k} \Gamma^A\right)$ and $\mu\left(\bigcup_{A \ge k} \Gamma^A\right) \ge 1 - c \exp(-k^\delta \|\gamma\|_{\mathcal{H}^s}^{-2})$. So $\mu(\Theta) = 1$, and thus $\mu(\Sigma) = 1$.

By definition, for all $u_0 \in \Theta$, there exists a unique global solution to (1-7), which reads

$$u(t) = e^{-itH}u_0 + w(t), \quad w \in \mathcal{C}(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^3)).$$

Then by Lemma 4.3 for all $u_0 \in \Theta$, there exists a unique $w \in \mathcal{C}([0, +\infty[, \mathcal{H}^1(\mathbb{R}^3))$, which satisfies, for all $N$, the bound

$$\sup_{t \in [0,N]} E(w(t)) \le CN^{c_s+}.$$

Now, if $U_0 \in \Sigma$ then $U_0 = u_0 + v$ with $u_0 \in \Theta$, $v \in \mathcal{H}^1$ and we can use the method of Proposition 4.1, Lemma 4.2 and Lemma 4.3 with $U_{0,N}$ replaced by $u_{0,N} + v$. And the set $\Sigma$ satisfies properties (i) and (ii).

Coming back to the definition of $\Sigma_T^j$, we have $e^{-itH}(\Sigma_T^j) = \Sigma_T^j$ for all $t \in \mathbb{R}$; thus $e^{-itH}(\Theta) = \Theta$. Finally, thanks to property (i), the set $\Sigma$ is invariant under the dynamics and property (iii) is satisfied.

**4B.** *The case of dimension $d = 2$.* In this section, we prove Theorem 1.6. The proof is analogous to Theorem 1.5 in a simpler context; that is why we only explain the key estimates.

According to Proposition 2.1, we set

$$F_s(K) = \left\{w \in \mathcal{H}^s(\mathbb{R}^2) \mid \|w\|_{\mathcal{H}^s(\mathbb{R}^2)} \le K, \|w\|_{L^4(\mathbb{R}^2)} \le K \text{ and } \|e^{-itH}w\|_{L_{[0,2\pi]}^{1/\varepsilon}\mathcal{W}^{1+s-\varepsilon,\infty}(\mathbb{R}^2)} \le K\right\},$$

and we fix $u_0 \in F_s(N^\varepsilon)$.

Then, if $f = e^{-itH}u_0^N$, we have

$$\|f\|_{L_{[0,2\pi]}^\infty L^2} \le CN^{-s+\varepsilon} \quad \text{and} \quad \|H^{\sigma/2}f\|_{L_{[0,2\pi]}^{\infty-},L^\infty} \le CN^{\sigma-1-s+\varepsilon}.$$

In Proposition 4.1 we can choose $T = N^{-2(1-s)-\varepsilon}$ to have

$$\|u^1\|_{L_T^\infty L^2} \le CN^\varepsilon \quad \text{and} \quad \|u^1\|_{L_T^\infty \mathcal{H}^1} \le CN^{1-s+\varepsilon}.$$

Moreover, as $u_0 \in F_s(N^\varepsilon)$, we obtain

$$\|u_{0,N}\|_{L^4} \le CN^\varepsilon.$$

Hence, we establish

$$E(u^1) = \|u^1\|_{\mathscr{H}^1(\mathbb{R}^2)}^2 + \tfrac{1}{2}\|u^1\|_{L^4(\mathbb{R}^2)}^4 = \|u_{0,N}\|_{\mathscr{H}^1(\mathbb{R}^2)}^2 + \tfrac{1}{2}\|u_{0,N}\|_{L^4(\mathbb{R}^2)}^4 \le N^{2(1-s+\varepsilon)} + CN^{4\varepsilon}$$
$$\le 4N^{2(1-s+\varepsilon)},$$

and

$$\|u^1\|_{L_T^\infty L^4} \le CN^{(1-s+\varepsilon)/2}.$$

In Proposition 4.1, we obtain $\|w^1\|_{L_{[0,T]}^\infty \mathscr{H}^1} \le CN^{-1+}$ and $\|w^1\|_{L_{[0,T]}^\infty L^2} \le CN^{-2+}$. The proof is essentially the same. We define the map $L$ as in (4-9). For the first estimate, we prove that

$$\|L(w)\|_{X_T^1} \le CN^{-1+} + N^{0-}\|w\|_{X_T^1} + N^{-2(1-s)+}\|w\|_{X_T^1}^3.$$

We only give details for the source terms. First,

$$\|fu^2\|_{L_T^{1+}\mathscr{W}^{1,2-}+L_T^1\mathscr{H}^1} \le C\|fuH^{1/2}u\|_{L_T^{1+}L^{2-}} + C\|u^2 H^{1/2}f\|_{L_T^1 L^2}$$
$$\le CT^{1-}\|u\|_{L_T^\infty \mathscr{H}^1}\|u\|_{L_T^\infty L^{\infty-}}\|f\|_{L_T^{\infty-}L^\infty} + CT^{1-}\|u\|_{L_T^\infty L^4}^2\|H^{1/2}f\|_{L_T^{\infty-}L^\infty}$$
$$\le CT^{1-}\max(N^{1-3s+}, N^{1-2s+}) \le CT^{1-}N^{1-2s+} \le CN^{-1+}.$$

Similarly,

$$\|f^2 u\|_{L_T^1 \mathscr{H}^1} \le C\|f^2 H^{1/2}u\|_{L_T^1 L^2} + C\|ufH^{1/2}f\|_{L_T^1 L^2}$$
$$\le CT^{1-}\|u\|_{L_T^\infty \mathscr{H}^1}\|f\|_{L_T^{\infty-}L^\infty}^2 + CT^{1-}\|u\|_{L_T^\infty L^2}\|f\|_{L_T^{\infty-}L^\infty}\|H^{1/2}f\|_{L_T^{\infty-}L^\infty}$$
$$\le CT^{1-}\max(N^{-1-3s+}, N^{-1-2s+}) \le CT^{1-}N^{-1-2s+} \le CN^{-3+} \le CN^{-1+}.$$

Finally,

$$\|f^3\|_{L_T^1 \mathscr{H}^1} \le C\|f^2 H^{1/2}f\|_{L_T^1 L^2} \le CT^{1-}\|H^{1/2}f\|_{L_T^{\infty-}L^\infty}\|f\|_{L_T^{\infty-}L^\infty}\|f\|_{L_T^\infty L^2}$$
$$\le CT^{1-}N^{-s+}N^{-1-s+}N^{-s+} \le CN^{-3-s+} \le CN^{-1+}.$$

Analogously to Lemma 4.2, we obtain $|E(u^1(T) + w^1(T)) - E(u^1(T))| \le CN^{-s+}$, because here $\beta(s) = 1-$ and the estimates on $u^1$ are the same as in dimension $d = 3$.

Finally, the globalisation argument holds if (4-15) is satisfied, that is to say

$$CAT^{-1}N^{-s+} \le 4N^{2(1-s)+},$$

which is equivalent to $2(1-s) - s < 2(1-s)$, hence $s > 0$. In this case, we set $A \sim cN^s$ and we get that, for $0 \le t \le A$,

$$E(w(t)) \le CA^{c_s+}, \text{ with } c_s = \frac{1-s}{s}.$$

Theorem 1.6 follows.

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