ANALYSIS & PDE

Volume 7 No. 4 2014

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ANALYSIS AND PDE
Vol. 7, No. 4, 2014
dx.doi.org/10.2140/apde.2014.7.903

DISPERSION FOR THE SCHRÖDINGER EQUATION ON THE LINE WITH MULTIPLE DIRAC DELTA POTENTIALS AND ON DELTA TREES

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We consider the time-dependent one-dimensional Schrödinger equation with multiple Dirac delta potentials of different strengths. We prove that the classical dispersion property holds under some restrictions on the strengths and on the lengths of the finite intervals. The result is obtained in a more general setting of a Laplace operator on a tree with δ-coupling conditions at the vertices. The proof relies on a careful analysis of the properties of the resolvent of the associated Hamiltonian. With respect to our earlier analysis for Kirchhoff conditions [*J. Math. Phys.* **52**:8 (2011), #083703], here the resolvent is no longer in the framework of Wiener algebra of almost periodic functions, and its expression is harder to analyse.

1. Introduction

In this paper we are concerned with the dispersive properties of the Schrödinger equation with multiple Dirac delta potentials and more generally for the Schrödinger equation on a tree with δ-coupling conditions at the vertices.

Let us first recall that the linear Schrödinger equation on the line,

$$\begin{cases} iu_t(t,x) + u_{xx}(t,x) = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

conserves the L^2-norm

$$\|e^{it\Delta}u_0\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})} \tag{2}$$

and enjoys the dispersive estimate

$$\|e^{it\Delta}u_0\|_{L^\infty(\mathbb{R})} \le \frac{C}{\sqrt{|t|}}\|u_0\|_{L^1(\mathbb{R})}, \quad t \neq 0. \tag{3}$$

It is classical to obtain from these two inequalities the well-known space-time Strichartz estimates [Strichartz 1977; Ginibre and Velo 1985], for $r \ge 2$,

$$\|e^{it\Delta}u_0\|_{L_t^{\frac{4r}{r-2}}(\mathbb{R},\, L_x^r(\mathbb{R}))} \le C\|u_0\|_{L^2(\mathbb{R})}. \tag{4}$$

Banica is partially supported by the French ANR projects “R.A.S.” ANR-08-JCJC-0124-01 and “SchEq” ANR-12-JS01-0005-01. Ignat is partially supported by PN-II-ID-PCE-2011-3-0075 of CNCSIS–UEFISCDI Romania, MTM2011-29306-C02-00, MICINN, Spain, and ERC Advanced Grant FP7-246775 NUMERIWAVES.
MSC2010: 35B45, 35J10, 35R05, 35CXX, 35R02.
Keywords: Schrödinger equation on metric graphs, with 1-D delta potentials, representation of solutions, dispersion and Strichartz estimates.

These dispersive estimates have been successfully applied to obtain results for the nonlinear Schrödinger equation (see, for example [Cazenave 2003; Tao 2006] and the references therein).

Our general framework in this paper refers to the Dirac delta Hamiltonian on a tree with a finite number of vertices, with the external edges (those that have only one internal vertex as an endpoint) formed by infinite strips. The particular case of a tree with all the internal vertices having degree two will give us a result for the Schrödinger equation on the line with several Dirac potentials. Although the latter is a corollary of the former, we shall start our presentation with the case of the line. This is motivated by the fact that historically dispersive properties have been studied first in this case (only with one or with two delta Dirac potentials) and that the previous results on graphs concern only star-shaped graphs (with only one vertex), where the proofs are in the same spirit as on the line with one Dirac delta potential.

So we first consider the semigroup $\exp(-itH_\alpha)$, where H_α is a perturbation of the Laplace operator with n Dirac delta potentials with real strengths $\{\alpha_j\}_{j=1}^{p}$,

$$H_\alpha = -\Delta + \sum_{j=1}^{p} \alpha_j \delta(x - x_j). \tag{5}$$

The spectral properties of the Laplacian with multiple Dirac delta potentials on $\mathbb{R}^n$ have been extensively studied. Operator H_α has at most p eigenvalues, which are all negative and simple, and there are no eigenvalues in the case of positive strengths $\alpha_i > 0$. The remaining part of the spectrum is absolutely continuous and $\sigma_{ac}(H_\alpha) = [0, \infty)$. We will denote by P_e the L^2 projection onto the subspace of the eigenfunctions and by P the projection outside the discrete spectrum. Regarding the spectral properties of H_α we refer to [Albeverio et al. 2005, § II.2] and to the references within. The time-dependent propagator of the linear Schrödinger equation has also been considered in the case of one Dirac delta potential [Gaveau and Schulman 1986; Manoukian 1989; Adami and Sacchetti 2005; Datchev and Holmer 2009], or one point interactions [Albeverio et al. 1994; Adami and Noja 2009; Fukuizumi et al. 2008], or two symmetric Dirac delta potentials [Kovařík and Sacchetti 2010]. In particular, in the case of the line with one delta interaction, without sign condition on the strength, dispersive estimates has been proved but for $e^{-itH_\alpha}P$ [Adami and Sacchetti 2005; Datchev and Holmer 2009]. A similar result was proved to hold in the case of two-point interactions, under a condition on the delta-strength and on the distance between the location of the point interactions [Kovařík and Sacchetti 2010]; see also [Angulo Pava and Ferreira 2013]. Also in [Duchêne et al. 2011] the problem of dispersion for several-delta potentials has been considered, as well as wave operator bounds from which dispersive estimates can be obtained as a consequence. Here Jost and distorted plane functions are used in spectral formulae. A weighted weaker than classical dispersion estimate is obtained for a class of potentials with singularities.

Concerning the nonlinear Schrödinger equation with a Dirac delta potential, standing wave and bound states have been analysed [Fukuizumi and Jeanjean 2008; Fukuizumi et al. 2008; Le Coz et al. 2008], as well as the time dynamics of solitons [Holmer and Zworski 2007; Holmer et al. 2007a; 2007b].

For stating our first result concerning the case of several Dirac potentials, we need to introduce the following functions. With the notations in Lemma 3.1 in the case when $n_j = 2$, we define $f_p = \det D_p$

and $g_p = \dfrac{\det \widetilde{D}_p}{\det D_p}$, defined by recursion as follows:

$$f_1(\omega) = \frac{2\,\omega + \alpha_1}{\omega + \alpha_1}, \quad f_p(\omega) = \frac{2\,\omega + \alpha_p}{\omega + \alpha_p} e^{\omega a_{p-1}} f_{p-1}(\omega)\left(1 - \frac{\alpha_p}{2\,\omega + \alpha_p}\, e^{-2\omega a_{p-1}} g_{p-1}(\omega)\right),$$

where

$$g_1(\omega) = \frac{\alpha_1}{n_1\,\omega + \alpha_1}, \quad g_p(\omega) = \frac{\dfrac{\alpha_p}{n_p\,\omega + \alpha_p} - \dfrac{-2\,\omega + \alpha_p}{2\,\omega + \alpha_p}\, e^{-2\omega a_{p-1}} g_{p-1}(\omega)}{1 - \dfrac{\alpha_p}{2\omega + \alpha_p}\, e^{-2\omega a_{p-1}} g_{p-1}(\omega)}.$$

These functions will appear naturally when computing the resolvent of H_α.

Theorem 1.1. *For any $\{\alpha_j\}_{j=1}^p$ and $\{x_j\}_{j=1}^p$ such that*

$$\partial_\omega^{p-1} f_p\big|_{\omega=0} \neq 0, \tag{6}$$

the solution of the linear Schrödinger equation on the line with multiple delta interactions of strength α_j located at x_j satisfies the dispersion inequality

$$\|e^{-itH_\alpha} P u_0\|_{L^\infty(\mathbb{R})} \le \frac{C}{\sqrt{|t|}} \|u_0\|_{L^1(\mathbb{R})} \quad \textit{for all } t \neq 0. \tag{7}$$

Moreover, in the case of positive strengths $\alpha_j > 0$, condition (6) *is fulfilled and we have*

$$\|e^{-itH_\alpha} u_0\|_{L^\infty(\mathbb{R})} \le \frac{C}{\sqrt{|t|}} \|u_0\|_{L^1(\mathbb{R})} \quad \textit{for all } t \neq 0. \tag{8}$$

We first notice that, in view of the definition of $f_p(\omega)$, condition (6) is not fulfilled only in a few explicit situations. For instance, if $p = 2$, the situations to be avoided are when $x_2 - x_1 + \dfrac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} = 0$, already used in [Kovařík and Sacchetti 2010].

In the previous works on dispersive estimates for one or two delta Dirac potentials, given the particular structure of the operator H_α, the authors obtain explicit representations of the resolvent and then of e^{-itH_α}. However in the general case of multiple delta interactions an explicit representation is not easy to obtain; even in [Albeverio et al. 1984; 2005, § II.2] the resolvent is obtained in terms of the inverse of some matrix D_n that depends on $\{\alpha_j\}_{j=1}^p$ and on the lengths of the finite segments $\{x_j - x_{j-1}\}_{j=2}^p$.

The line setting might be seen as the special case of the equation posed on a simple graph with n vertices, with only two edges starting from any vertex and with delta connection conditions at each vertex ($x_0 = -\infty$, $x_{p+1} = \infty$):

$$\begin{cases} i u_t(t, x) + u_{xx}(t, x) = 0, & x \in (x_{j-1}, x_j),\ j = 1, \dots, p, \\ u_x(t, x_j^+) - u_x(t, x_j^-) = \alpha_j u(x_j), & t > 0,\ j = 1, \dots, p. \end{cases} \tag{9}$$

Our second framework refers to the Dirac delta Hamiltonian H_α^Γ on a tree $\Gamma = (V, E)$ with a finite number of vertices V, with the external edges (those that have only one internal vertex as an endpoint) formed by infinite strips. We consider the linear Schrödinger equation in the case of a tree Γ, with delta conditions of not necessarily equal strength at the vertices

$$\begin{cases} i\boldsymbol{u}_t(t,x) = H_\alpha^\Gamma \boldsymbol{u}(t,x), & (t,x) \in \mathbb{R}\times\Gamma, \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x), & x \in \Gamma. \end{cases} \tag{10}$$

The presentation of the operator H_α^Γ will be given in full detail in Section 2. Let us just say here that H_α^Γ acts on a function $\boldsymbol{u}$ on a graph as $-\partial_{xx}$ on each restriction of $\boldsymbol{u}$ to an edge of the tree and that its domain consists of those functions $\boldsymbol{u}$ for which δ-coupling conditions must be fulfilled. The δ-coupling conditions are a continuity condition for the function $\boldsymbol{u}$ and a δ-transmission condition at the level of its first derivative at all internal vertices v:

$$\sum_{e\in E_v} \partial_n \boldsymbol{u}(v) = \alpha(v)\boldsymbol{u}(v).$$

The operator H_α^Γ shares the same properties of H_α above: only a finite number of negative eigenvalues, and no eigenvalues for positive strengths, and $\sigma_{ac}(H_\alpha^\Gamma) = [0,\infty)$. These properties follow as in [Albeverio et al. 2005, § II.2].

The dispersion inequality for (10) was proved in [Banica and Ignat 2011] for the case of Kirchhoff's connection condition on trees, that is $\alpha(v) = 0$ for all internal vertices of the tree (see also [Ignat 2010]). The case of δ- and δ'-coupling on a star-shaped tree (i.e., only one vertex) has been considered in [Adami et al. 2011], where the main result concerns the time evolution of a fast soliton for the nonlinear equation, in the spirit of [Holmer et al. 2007a]. Finally, we mention that for the stationary nonlinear equation, the study of bound states on a star-shaped tree with delta conditions has been analysed in a series of papers [Adami et al. 2012a; 2012b; 2012c; 2012d].

The main result of this paper is the following, involving the expression of a determinant function $\det D_{\Gamma_p}(\omega)$ defined by recursion in Lemma 3.1.

Theorem 1.2. *Let us consider a tree $\Gamma = (V,E)$ with p vertices. If the strengths at the vertices and the lengths of the finite edges are such that*

$$\partial_\omega^{(p-1)} \det D_{\Gamma_p}\big|_{\omega=0} \neq 0, \tag{11}$$

then the solution of the linear Schrödinger equation on a tree with delta connection conditions satisfies the dispersion inequality

$$\|e^{-itH_\alpha^\Gamma} P\boldsymbol{u}_0\|_{L^\infty(\Gamma)} \leq \frac{C}{\sqrt{|t|}} \|\boldsymbol{u}_0\|_{L^1(\Gamma)} \quad \textit{for all } t \neq 0. \tag{12}$$

Moreover, in the case of positive strengths $\alpha_j > 0$, condition (11) is fulfilled and we have

$$\|e^{-itH_\alpha^\Gamma} \boldsymbol{u}_0\|_{L^\infty(\Gamma)} \leq \frac{C}{\sqrt{|t|}} \|\boldsymbol{u}_0\|_{L^1(\Gamma)} \quad \textit{for all } t \neq 0. \tag{13}$$

The proof of Theorem 1.2 uses elements from [Banica 2003; Banica and Ignat 2011; Gavrus 2012] in an appropriate way related to the delta connection conditions on the tree. The starting point consists of writing the solution in terms of the resolvent of the Laplacian, which in turn is determined by recursion on the number of vertices. With respect to the previous works with Kirchhoff conditions, the novelty here is that we are no longer in the framework of the almost periodic Wiener algebra of functions, and the expression of the resolvent is harder to analyse.

The linear solution $e^{-itH_\alpha^\Gamma}\boldsymbol{u}_0$ will be shown to be a combination of oscillatory integrals, that becomes more and more involved as the number of vertices of the tree grows. We do not have any more that $e^{-itH_\alpha^\Gamma}\boldsymbol{u}_0$ is a summable superposition of solutions of the linear Schrödinger equation on the line, as for Kirchhoff conditions in [Banica and Ignat 2011].

Theorem 1.1 follows from Theorem 1.2 by considering the particular case of a tree Γ with all the internal vertices having degree two.

As classically noticed [Rauch 1978; Jensen and Kato 1979; Journé et al. 1991; Rodnianski and Schlag 2004; Goldberg and Schlag 2004], one can expect dispersion in the absence of eigenvalues and of zero resonances. In the δ-coupling case the nongeneric condition (6) for $p=2$ is precisely in link with the presence of a zero resonance (see formula (2.1.29) in Chapter II of [Albeverio et al. 2005]), so one might expect that in the absence of eigenvalues the dispersion holds generically, even for more general coupling. We shall give in Appendix C some sufficient conditions to obtain dispersion for general couplings.

Finally, we note that in the presence of eigenfunctions, the dispersion estimate cannot be valid globally in time. Denoting by H either H_α or H_α^Γ, the general classical TT^* argument and the Christ–Kiselev lemma allow one to infer global in time Strichartz estimates as on $\mathbb{R}$ for $e^{-itH}P$, the dispersive part of e^{-itH} (see for instance the short proof of Theorem 2.3 in [Tao 2006]). This together with the regularity of the eigenfunctions of the operator H give us the following result:

Theorem 1.3. *Let $T>0$ and let (q,r) and (q',r') be two 1-admissible couples, in the sense that $4\le q\le\infty$, $2\le r\le\infty$ and $\frac{2}{q}+\frac{1}{r}=\frac{1}{2}$. For any $\alpha\ge 1$, there exists a constant $C>0$ such that the homogeneous Strichartz estimates*

$$\|e^{-itH}\boldsymbol{u}_0\|_{L^q((0,T),L^r(\Gamma))}\le C\big(\|\boldsymbol{u}_0\|_{L^2(\Gamma)}+T^{1/q}\|\boldsymbol{u}_0\|_{L^\alpha(\Gamma)}\big),$$

and the inhomogeneous Strichartz estimates

$$\left\|\int_0^t e^{-i(t-s)H}\boldsymbol{F}(s)\,ds\right\|_{L^q((0,T),L^r(\Gamma))}\le C\big(\|\boldsymbol{F}\|_{L^{\tilde{q}'}((0,T),L^{\tilde{r}'}(\Gamma))}+T^{1/q}\|\boldsymbol{F}\|_{L^1((0,T),L^\alpha(\Gamma))}\big),$$

hold. Here x' stands for the conjugate of x, defined by $\dfrac{1}{x}+\dfrac{1}{x'}=1$.

We shall give in Appendix B a proof inspired by [Datchev and Holmer 2009]. As a typical result for the nonlinear Schrödinger equation based on the Strichartz estimates, one obtains the global in time well-posedness for subcritical $L^2(\Gamma)$ solutions:

Theorem 1.4. *Let $p\in(0,4)$. For any $\boldsymbol{u}_0\in L^2(\Gamma)$ there exists a unique solution*

$$\boldsymbol{u}\in C(\mathbb{R},L^2(\Gamma))\cap\bigcap_{(q,r)\ 1\text{-adm.}}L^q(\mathbb{R},L^r(\Gamma))$$

of the nonlinear Schrödinger equation

$$\begin{cases} i\boldsymbol{u}_t+H\boldsymbol{u}\pm|\boldsymbol{u}|^p\boldsymbol{u}=0, & t\neq 0,\\ \boldsymbol{u}(0)=\boldsymbol{u}_0, & t=0.\end{cases}\tag{14}$$

Moreover, the $L^2(\Gamma)$*-norm of* $\boldsymbol{u}$ *is conserved along the time*: $\|\boldsymbol{u}(t)\|_{L^2(\Gamma)} = \|\boldsymbol{u}_0\|_{L^2(\Gamma)}$.

Local in time existence with lifespan depending on the L^2 size of the initial data follows from a classical fixed point argument as on $\mathbb{R}$ (see for instance Proposition 3.15 in [Tao 2006]). The extension to global solutions is obtained from the conservation of the $L^2(\Gamma)$-norm that in turn follows by taking the imaginary part of (14) multiplied by $\bar{\boldsymbol{u}}$ and integrating on Γ.

The paper is organised as follows. In the next section we introduce the framework of the Laplacian analysis on a graph. In Section 3 we give the proof of Theorem 1.2. In Appendix A we show how the conditions of the theorems are fulfilled for positive strengths of interactions. Appendix B contains the proof of Theorem 1.3. In Appendix C we shall describe the approach for general coupling conditions.

2. Preliminaries on graphs and δ-coupling

In this section we present some generalities about metric graphs and introduce the Dirac delta Hamiltonian H_α^Γ on such structure. More general types of self-adjoint operators, $\Delta(A, B)$, have been considered in [Kostrykin and Schrader 2006; 1999]. We collect here some basic facts on metric graphs and on some operators that could be defined on such structures [Kuchment 2008; 2004; 2005; Kostrykin and Schrader 2006; Gnutzmann and Smilansky 2006; Exner 2011].

Let $\Gamma = (V, E)$ be a graph where V is the set of vertices and E the set of edges. For each $v \in V$ we denote by $E_v = \{e \in E : v \in e\}$ the set of edges branching from v. We assume that V is connected and the degree of each vertex v of Γ is finite: $d(v) = |E_v| < \infty$. The edges could be of finite length and then their ends are vertices of V, or they could have infinite length and then we assume that each infinite edge is a ray with a single vertex belonging to V (see [Kuchment 2008] for more details on graphs with infinite edges). The vertices are called internal if $d(v) \geq 2$ or external if $d(v) = 1$. In this paper we will assume that there are no external vertices.

We fix an orientation of Γ and for each oriented edge e, we denote by $I(e)$ the initial vertex and by $T(e)$ the terminal one. Of course in the case of infinite edges we have only initial vertices.

We identify every edge e of Γ with an interval I_e, where $I_e = [0, l_e]$ if the edge is finite and $I_e = [0, \infty)$ if the edge is infinite. This identification introduces a coordinate x_e along the edge e. In this way Γ is a metric space and is often called a metric graph [Kuchment 2008].

Let v be a vertex of V and e be an edge in E_v. We set, for finite edges e,

$$j(v, e) = \begin{cases} 0 & \text{if } v = I(e), \\ l_e & \text{if } v = T(e), \end{cases}$$

and, for infinite edges,

$$j(v, e) = 0 \quad \text{if } v = I(e).$$

We identify any function $\boldsymbol{u}$ on Γ with a collection $\{u^e\}_{e \in E}$ of functions u^e defined on the edges e of Γ. Each u^e can be considered as a function on the interval I_e. In fact, we use the same notation u^e for both the function on the edge e and the function on the interval I_e identified with e. For a function $\boldsymbol{u} : \Gamma \to \mathbb{C}$, $\boldsymbol{u} = \{u^e\}_{e \in E}$, we denote by $f(\boldsymbol{u}) : \Gamma \to \mathbb{C}$ the family $\{f(u^e)\}_{e \in E}$, where $f(u^e) : e \to \mathbb{C}$.

A function $\boldsymbol{u} = \{u^e\}_{e\in E}$ is continuous if and only if u^e is continuous on I_e for every $e \in E$ and, moreover, is continuous at the vertices of Γ:

$$u^e(j(v,e)) = u^{e'}(j(v,e')) \quad \text{for all } e, e' \in E_v \text{ and } v \in V.$$

The space $L^p(\Gamma)$, $1 \le p < \infty$ consists of all functions $\boldsymbol{u} = \{u_e\}_{e\in E}$ on Γ that belong to $L^p(I_e)$ for each edge $e \in E$ and

$$\|\boldsymbol{u}\|^p_{L^p(\Gamma)} = \sum_{e\in E} \|u^e\|^p_{L^p(I_e)} < \infty.$$

Similarly, the space $L^\infty(\Gamma)$ consists of all functions that belong to $L^\infty(I_e)$ for each edge $e \in E$ and

$$\|\boldsymbol{u}\|_{L^\infty(\Gamma)} = \sup_{e\in E} \|u^e\|_{L^\infty(I_e)} < \infty.$$

The Sobolev space $H^m(\Gamma)$, for an integer $m \ge 1$, consists of all continuous functions on Γ that belong to $H^m(I_e)$ for each $e \in E$ and

$$\|\boldsymbol{u}\|^2_{H^m(\Gamma)} = \sum_{e\in E} \|u^e\|^2_{H^m(e)} < \infty.$$

The above spaces are Hilbert spaces with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_{L^2(\Gamma)} = \sum_{e\in E} (u^e, v^e)_{L^2(I_e)} = \sum_{e\in E} \int_{I_e} u^e(x)\overline{v^e}(x)\, dx$$

and

$$(\boldsymbol{u}, \boldsymbol{v})_{H^m(\Gamma)} = \sum_{e\in E} (u^e, v^e)_{H^m(I_e)} = \sum_{e\in E} \sum_{k=0}^{m} \int_{I_e} \frac{d^k u^e}{dx^k} \overline{\frac{d^k v^e}{dx^k}}\, dx.$$

We now define the normal exterior derivative of a function $\boldsymbol{u} = \{u^e\}_{e\in E}$ at the endpoints of the edges. For each $e \in E$ and v an endpoint of e we consider the normal derivative of the restriction of $\boldsymbol{u}$ to the edge e of E_v evaluated at $j(v,e)$, to be defined by

$$\frac{\partial u^e}{\partial n_e}(j(v,e)) = \begin{cases} -u^e_x(0^+) & \text{if } j(v,e) = 0, \\ u^e_x(l_e^-) & \text{if } j(v,e) = l_e. \end{cases}$$

We now introduce H^Γ_α. It generalises the classical Dirac delta interactions with strength parameters (5). The Dirac delta Hamiltonian is defined on the domain

$$D(H^\Gamma_\alpha) = \left\{ \boldsymbol{u} \in H^2(\Gamma) : \sum_{e\in E_v} \frac{\partial u^e}{\partial n_e}(j(v,e)) = \alpha(v)\boldsymbol{u}(v),\ \forall v \in V \right\}. \tag{15}$$

For any $\boldsymbol{u} = \{u^e\}_{e\in E}$, the operator H^Γ_α acts by

$$(H^\Gamma_\alpha \boldsymbol{u})(x) = -u^e_{xx}(x), \quad x \in I_e, e \in E.$$

The quadratic form associated to H^Γ_α is defined on $H^1(\Gamma)$ and it is given by

$$\mathscr{E}^\Gamma_\alpha(u) = \sum_{e\in E} \int_{I_e} |u^e_x(x)|^2\, dx + \sum_{v\in V} \alpha(v)|\boldsymbol{u}(v)|^2.$$

When all strengths vanish this corresponds to the Kirchhoff coupling analysed in [Banica and Ignat 2011].

Finally, let us mention that there are other coupling conditions (see [Kostrykin and Schrader 1999]), which allow one to define a "Laplace" operator on a metric graph. To be more precise, let us consider an operator that acts on functions on the graph Γ as the second derivative d^2/dx^2, and whose domain consists of all functions $\boldsymbol{u}$ that belong to the Sobolev space $H^2(e)$ on each edge e of Γ and satisfy the following boundary condition at the vertices:

$$A(v)\boldsymbol{u}(v)+B(v)\boldsymbol{u}'(v)=0 \quad \text{for each vertex } v. \tag{16}$$

Here $\boldsymbol{u}(v)$ and $\boldsymbol{u}'(v)$ are correspondingly the vector of values of $\boldsymbol{u}$ at v attained from directions of different edges converging at v and the vector of derivatives at v in the outgoing directions. For each vertex v of the tree we assume that matrices $A(v)$ and $B(v)$ are of size $d(v)$ and satisfy the following two conditions:

(1) the joint matrix $(A(v), B(v))$ has maximal rank $d(v)$,

(2) $A(v)B(v)^T = B(v)A(v)^T$.

Under those assumptions it was proved in [Kostrykin and Schrader 1999] that the operator under consideration, denoted by $\Delta(A, B)$, is self-adjoint. The case considered in this paper, of δ-coupling, corresponds to the matrices

$$A(v)=\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0\\ 0 & 1 & -1 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & -1\\ 0 & 0 & 0 & \cdots & 0 & -\alpha(v)\end{pmatrix}, \quad B(v)=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0\\ 1 & 1 & 1 & \cdots & 1 & 1\end{pmatrix}.$$

More examples of matrices satisfying the above conditions are given in [Kostrykin and Schrader 1999; Kostrykin et al. 2008].

3. Proof of Theorem 1.2

We shall use a description of the solution of the linear Schrödinger equation in terms of the resolvent. For $\omega > 0$ such that $-\omega^2$ is not an eigenvalue, let R_ω be the resolvent of the Laplacian on a tree

$$R_\omega \boldsymbol{u}_0 = (H_\alpha^\Gamma + \omega^2 I)^{-1}\boldsymbol{u}_0.$$

Before starting let us choose an orientation on the tree Γ. Choose an internal vertex $\mathbb{O}$ to be the root of the tree and the initial vertex for all the edges that branch from it. This procedure introduces an orientation for all the edges starting from $\mathbb{O}$. For the other endpoints of the edges belonging to $E_{\mathbb{O}}$ we repeat the above procedure and inductively we construct an orientation on Γ.

3A. ***The structure of the resolvent.*** In order to obtain the expression of the resolvent, second-order equations

$$(R_\omega \boldsymbol{u}_0)'' = \omega^2 R_\omega \boldsymbol{u}_0 - \boldsymbol{u}_0$$

must be solved on each edge of the tree together with coupling conditions at each vertex. Then, on each edge parametrised by I_e, for $x \in I_e$, since $\omega \neq 0$,

$$R_\omega \boldsymbol{u}_0(x) = c_e\, e^{\omega x} + \tilde{c}_e\, e^{-\omega x} + \frac{t_e(x,\omega)}{\omega}, \tag{17}$$

with

$$t_e(x,\omega) = \frac{1}{2}\int_{I_e} \boldsymbol{u}_0(y)\, e^{-\omega|x-y|}\, dy.$$

Since $R_\omega \boldsymbol{u}_0$ belongs to $L^2(\Gamma)$, the coefficients c_e and $\tilde{c}_e$ are zero on the infinite edges $e \in \mathscr{E}$, parametrised by $[0,\infty)$. If we denote by $\mathscr{I}$ the set of internal edges, we have $2|\mathscr{I}| + |\mathscr{E}|$ coefficients. The delta conditions of continuity of $R_\omega \boldsymbol{u}_0$ and of transmission of $(R_\omega \boldsymbol{u}_0)'$ at the vertices of the tree give the system of equations on the coefficients. We have the same number of equations as of unknowns. We denote by $D_{\Gamma_p}(\omega)$ the matrix of the system, where p is the number of vertices of the tree, and $T_{\Gamma_p}(\omega)$ is the column of the free terms in the system.

Therefore the resolvent $R_\omega \boldsymbol{u}_0(x)$ on an edge I_e is

$$R_\omega \boldsymbol{u}_0(x) = \frac{\det M_{\Gamma_p}^{c_e}(\omega)}{\det D_{\Gamma_p}(\omega)}\, e^{\omega x} + \frac{\det M_{\Gamma_p}^{\tilde{c}_e}(\omega)}{\det D_{\Gamma_p}(\omega)}\, e^{-\omega x} + \frac{t_e(x,\omega)}{\omega}, \tag{18}$$

where $M_{\Gamma_p}^{c_e}(\omega)$ and $M_{\Gamma_p}^{\tilde{c}_e}(\omega)$ are obtained from $D_{\Gamma_p}(\omega)$ by replacing the column corresponding to the unknown c_e and $\tilde{c}_e$, respectively, by the column of the free terms $T_{\Gamma_p}(\omega)$.

3B. ***The expression of* det $\boldsymbol{D_{\Gamma_p}(\omega)}$.** In view of the form (17) of the resolvent, we obtain on an edge I_e

$$\begin{aligned} R_\omega \boldsymbol{u}_0(0) &= c_e + \tilde{c}_e + \frac{t_e(0,\omega)}{\omega}, \\ (R_\omega \boldsymbol{u}_0)'(0) &= c_e\omega - \tilde{c}_e\omega + t_e(0,\omega), \end{aligned} \tag{19}$$

and, if I_e is parametrised by $[0,a]$ with $a < \infty$,

$$\begin{aligned} R_\omega \boldsymbol{u}_0(a) &= c_e\, e^{\omega a} + \tilde{c}_e\, e^{-\omega a} + \frac{t_e(0,\omega)}{\omega}, \\ (R_\omega \boldsymbol{u}_0)'(a) &= c_e\, \omega\, e^{\omega a} - \tilde{c}_e\, \omega\, e^{-\omega a} - t_e(a,\omega). \end{aligned} \tag{20}$$

3B1. *The star-shaped tree case.* In the case of a single vertex and $n_1 \geq 2$ edges I_j, $1 \leq j \leq n_1$, parametrised by $[0,\infty)$ we have only the coefficient $\tilde{c}_j$ on each edge I_j since each c_j vanishes. The delta conditions are continuity of the resolvent at the vertex, and the fact that the sum of the first derivatives must be equal to α times the value of the resolvent at the vertex:

$$(R_\omega \boldsymbol{u}_0)_j(0) = (R_\omega \boldsymbol{u}_0)_1(0), \quad \sum_{1\leq j\leq n_1} (R_\omega \boldsymbol{u}_0)_j'(0) = \alpha_1\, (R_\omega \boldsymbol{u}_0)_1(0).$$

From (19) we obtain as the matrix for the system of $\tilde{c}$'s

$$D_{\Gamma_1}(\omega)=\begin{pmatrix} 1 & -1 & & & & & \\ & 1 & -1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & -1 & \\ & & & & & 1 & -1 \\ 1 & \frac{\omega}{\omega+\alpha_1} & \frac{\omega}{\omega+\alpha_1} & \cdots & \frac{\omega}{\omega+\alpha_1} & \frac{\omega}{\omega+\alpha_1} & \frac{\omega}{\omega+\alpha_1} \end{pmatrix},$$

and as a free term column

$$T_{\Gamma_1}(\omega)=\begin{pmatrix} \frac{t_2(0,\omega)-t_1(0,\omega)}{\omega} \\ \vdots \\ \frac{t_{n_1}(0,\omega)-t_{n_1-1}(0,\omega)}{\omega} \\ \frac{\omega-\alpha_1}{\omega+\alpha_1}\frac{t_1(0,\omega)}{\omega}+\frac{\omega}{\omega+\alpha_1}\sum_{2\le j\le n_1}\frac{t_j(0,\omega)}{\omega}. \end{pmatrix}$$

By developing $\det D_{\Gamma_1}(\omega)$ with respect to its last column, we obtain by recursion that

$$\det D_{\Gamma_1}(\omega)=\frac{n_1\omega+\alpha_1}{\omega+\alpha_1}.$$

Thus $\det D_{\Gamma_1}$ does not vanish on the imaginary axis and $\omega R_\omega \boldsymbol{u}_0$ can be analytically continued in a region containing the imaginary axis.

We introduce here the matrix $\widetilde{D}_{\Gamma_1}(\omega)$, which is the matrix of the coefficients of the resolvent if on the last edge I_{n_1} we should have $c_{n_1}e^{\omega x}$ instead of $\tilde{c}_{n_1}e^{-\omega x}$. This changes only the (n_1,n_1)-entry of $D_{\Gamma_1}(\omega)$, which is $-\frac{\omega}{\omega+\alpha_1}$ instead of $\frac{\omega}{\omega+\alpha_1}$:

$$\widetilde{D}_{\Gamma_1}(\omega)=\begin{pmatrix} 1 & -1 & & & & & \\ & 1 & -1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & -1 & \\ & & & & & 1 & -1 \\ 1 & \frac{\omega}{\omega+\alpha_1} & \frac{\omega}{\omega+\alpha_1} & \cdots & \frac{\omega}{\omega+\alpha_1} & \frac{\omega}{\omega+\alpha_1} & -\frac{\omega}{\omega+\alpha_1} \end{pmatrix}.$$

Moreover, the free term column remains the same for this new system. Again, by recursion, we have

$$\det \widetilde{D}_{\Gamma_1}(\omega)=\frac{(n_1-2)\omega+\alpha_1}{\omega+\alpha_1}.$$

3B2. *The general tree case.* Any tree Γ_p with p vertices, $p \geq 2$, can be seen as a tree Γ_{p-1} with $p-1$ vertices, to which we add a new vertex on one of its infinite edges, and $n_p - 1$ new infinite edges from it. Let us denote by N the number of edges of Γ_{p-1}. By this transformation I_N becomes an internal edge, parametrised by $[0, a_{p-1}]$, and we have in addition external edges I_{N+j}, for $1 \leq j \leq n_p - 1$. We denote by α_p the strength of the delta condition in the new p-th vertex. The matrix of the new system (the unknowns of the Γ_{p-1} system, together with an extra unknown on the new internal line I_N and $n_p - 1$ unknowns on the new $n_p - 1$ external edges) is denoted by $D_{\Gamma_p}(\omega)$. Notice that if we write the system of unknowns of Γ_p by changing the order of the unknowns (i.e., permuting columns) or the order of the conditions at vertices (i.e., permuting lines), then the determinant remains unchanged or it changes sign, and the ratio det $\widetilde{D}_{\Gamma_p}(\omega)/\det D_{\Gamma_p}(\omega)$ remains unchanged.

For Γ_p, by writing the delta conditions at the end of I_N, together with the two conditions involving the coefficients on I_N at the beginning of I_N, we obtain the matrix $D_{\Gamma_p}(\omega)$ as

$$\left(\begin{array}{c|ccccccc}
D_{\Gamma_{p-1}}(\omega) & -1 & & & & & & \\
 & -\frac{\omega}{\omega+\alpha_{p-1}} & & & & & & \\ \hline
e^{-\omega a_{p-1}} & e^{\omega a_{p-1}} & -1 & & & & & \\
 & & 1 & -1 & & & & \\
 & & & & \ddots & & & \\
 & & & & & 1 & -1 & \\
 & & & & & & 1 & -1 \\
\frac{-\omega+\alpha_p}{\omega+\alpha_p}e^{-\omega a_{p-1}} & e^{\omega a_{p-1}} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \cdots & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p}
\end{array}\right)$$

and the free term column as

$$T_{\Gamma_p}(\omega) = \begin{pmatrix} T_{\Gamma_{p-1}}(\omega) \\ \frac{t_{N+1}(0,\omega)-t_N(a_{p-1},\omega)}{\omega} \\ \vdots \\ \frac{t_{N+n_p-1}(0,\omega)-t_{N+n_p-2}(0,\omega)}{\omega} \\ \frac{\omega-\alpha_p}{\omega+\alpha_p}\frac{t_N(a_{p-1},\omega)}{\omega} + \frac{\omega}{\omega+\alpha_p}\sum_{1\leq j\leq n_p-1}\frac{t_{N+j}(0,\omega)}{\omega} \end{pmatrix}.$$

We point out that D_{Γ_p} has $p-1$ pairs of columns that are equal at $\omega = 0$. This implies that $\omega = 0$ is a zero of order at least $p-1$ for D_{Γ_p}. The assumption imposed in Theorem 1.1 guarantees that the order of $\omega = 0$ is exactly $p-1$. This will avoid the existence of zero resonances for the resolvent R_ω. In the case when all the strengths $\{\alpha_k\}_{k=1}^n$ are positive the condition in Theorem 1.1 is fulfilled; this will be proved in Appendix A.

Lemma 3.1. *We have the recursion formulae*

$$\det D_{\Gamma_1}(\omega)=\frac{n_1\omega+\alpha_1}{\omega+\alpha_1},\quad \frac{\det \widetilde{D}_{\Gamma_1}(\omega)}{\det D_{\Gamma_1}(\omega)}=\frac{(n_1-2)\omega+\alpha_1}{n_1\omega+\alpha_1},$$

$$\det D_{\Gamma_p}(\omega)=\frac{n_p\omega+\alpha_p}{\omega+\alpha_p}e^{\omega a_{p-1}}\det D_{\Gamma_{p-1}}(\omega)\left(1-\frac{(n_p-2)\omega+\alpha_p}{n_p\omega+\alpha_p}\,e^{-2\omega a_{p-1}}\frac{\det \widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}\right),$$

$$\frac{\det \widetilde{D}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)}=\frac{\dfrac{(n_p-2)\omega+\alpha_p}{n_p\omega+\alpha_p}-\dfrac{(n_p-4)\omega+\alpha_p}{n_p\omega+\alpha_p}e^{-2\omega a_{p-1}}\dfrac{\det \widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}}{1-\dfrac{(n_p-2)\omega+\alpha_p}{n_p\omega+\alpha_p}e^{-2\omega a_{p-1}}\dfrac{\det \widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}}. \tag{21}$$

Proof. The part about Γ_1 was proved in Section 3B1.

By developing $\det D_{\Gamma_p}$ with respect to the last n_p lines, we obtain an alternated sum of determinants of $n_p\times n_p$ minors composed of the last n_p lines of D_{Γ_p} times the determinant of the matrix D_{Γ_p}, without the lines and columns the minor is made of. On the last n_p lines, there are only n_p+1 columns that do not identically vanish. The only possible way to obtain a $n_p\times n_p$ minor composed from the last n_p lines of D_{Γ_p} with determinant different from zero is to choose all of the last n_p-1 columns together with a previous one. This follows from the fact that if we eliminate from $\det D_{\Gamma_n}$ both previous columns together with n_p-2 columns among the last n_p columns, we obtain a block-diagonal type matrix, with first diagonal block $D_{\Gamma_{p-1}}$ with its last column replaced by zeros, so its determinant vanishes. Therefore

$$\det D_{\Gamma_p}=\det D_{\Gamma_{p-1}}\det A^{n_p}-\det \widetilde{D}_{\Gamma_{p-1}}\det B^{n_p},$$

where for $m\ge 1$, A^m and B^m are the $m\times m$ matrices

$$A^m=\begin{pmatrix} e^{\omega a_{p-1}} & -1 & & & & \\ & 1 & -1 & & & \\ & & & \ddots & & \\ & & & 1 & -1 & \\ & & & & 1 & -1 \\ e^{\omega a_{p-1}} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \cdots & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p}\end{pmatrix},$$

$$B^m=\begin{pmatrix} e^{-\omega a_{p-1}} & -1 & & & & \\ & 1 & -1 & & & \\ & & & \ddots & & \\ & & & 1 & -1 & \\ & & & & 1 & -1 \\ \frac{-\omega+\alpha_p}{\omega+\alpha_p}e^{-\omega a_{p-1}} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \cdots & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p} & \frac{\omega}{\omega+\alpha_p}\end{pmatrix}.$$

We have

$$\det A^2=\frac{2\omega+\alpha_p}{\omega+\alpha_p}e^{\omega a_{p-1}},$$

and by developing A^m with respect to the first last column we obtain the recursion formula

$$\det A^m = \frac{\omega}{\omega+\alpha_p} e^{\omega a_{p-1}} + \det A^{n_p-1},$$

so

$$\det A^m = \frac{m\omega+\alpha_p}{\omega+\alpha_p} e^{\omega a_{p-1}}.$$

Similarly we obtain

$$\det B^m = \frac{(m-2)\,\omega+\alpha_p}{\omega+\alpha_p} e^{-\omega a_{p-1}}.$$

Therefore we find, indeed,

$$\det D_{\Gamma_p}(\omega) = \frac{n_p\omega+\alpha_p}{\omega+\alpha_p} e^{\omega a_{p-1}} \det D_{\Gamma_{p-1}}(\omega)\left(1 - \frac{(n_p-2)\omega+\alpha_p}{n_p\omega+\alpha_p}\, e^{-2\omega a_{p-1}} \frac{\det \widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}\right).$$

In a similar way we get

$$\det \widetilde{D}_{\Gamma_p}(\omega) = \frac{(n_p-2)\omega+\alpha_p}{\omega+\alpha_p} e^{\omega a_{p-1}} \det D_{\Gamma_{p-1}}(\omega) - \frac{(n_p-4)\omega+\alpha_p}{\omega+\alpha_p} e^{-\omega a_{p-1}} \det \widetilde{D}_{\Gamma_{p-1}}(\omega),$$

which leads directly to (21), completing the proof of the lemma. □

3C. *A lower bound for* $\det D_{\Gamma_p}(i\tau)$ *away from* 0.

Lemma 3.2. *The function* $\det D_{\Gamma_p}(\omega)$ *is bounded away from zero by a positive constant on a strip containing the imaginary axis*:

$$\forall \delta > 0, \exists c_{\Gamma_p}, \epsilon_{\Gamma_p} > 0, \exists 0 < r_{\Gamma_p} < 1 \textit{ such that } |\det D_{\Gamma_p}(\omega)| > c_{\Gamma_p}, \left|\frac{\det \widetilde{D}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)}\right| < r_\Gamma,$$

for all $\omega \in \mathbb{C}$ *with* $|\Re\omega| < \epsilon_{\Gamma_p}$ *and* $|\Im\omega| > \delta$.

Proof. We shall prove this lemma by recursion on p. For $p = 1$, Lemma 3.1 ensures that

$$\det D_{\Gamma_1}(\omega) = \frac{n_1\omega+\alpha_1}{\omega+\alpha_1}, \quad \frac{\det \widetilde{D}_{\Gamma_1}(\omega)}{\det D_{\Gamma_1}(\omega)} = \frac{(n_1-2)\omega+\alpha_1}{n_1\omega+\alpha_1}.$$

We obtain a positive lower bound for $|\det D_{\Gamma_1}(\omega)|$ if we show that it does not approach zero. Therefore the existence of $c_{\Gamma_1} > 0$ is obtained by considering $\epsilon_{\Gamma_1} \le \frac{|\alpha_1|}{2n_1}$. Next, we have

$$\left|\frac{(n_1-2)\omega+\alpha_1}{n_1\omega+\alpha_1}\right| < 1 \quad \Longleftrightarrow \quad 0 < \alpha_1\Re\omega + (n_1-1)|\omega|^2,$$

so for any $\delta > 0$ we get an appropriate $0 < r_{\Gamma_1} < 1$ by choosing

$$\epsilon_{\Gamma_1} \le \frac{(n_1-1)\delta^2}{2|\alpha_1|}.$$

Assume that we have proved this lemma for $p-1$. We shall show now that it also holds for p. Now, from the ratio information part in this lemma for Γ_{p-1} we can choose ϵ_{Γ_p} small enough to have, for

$|\Re\omega| < \epsilon_{\Gamma_p}$ and $|\Im\omega| > \delta$,

$$\left|1 - \frac{(n_p-2)\omega+\alpha_p}{n_p\omega+\alpha_p} e^{-2\omega a_{p-1}} \frac{\det \widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}\right| > c_0 > 0.$$

Also from this lemma for Γ_{p-1} we have the existence of two positive constants $c_{\Gamma_{p-1}}$ and $\epsilon_{\Gamma_{p-1}}$ such that $|\det D_{\Gamma_{p-1}}(\omega)| > c_{\Gamma_{p-1}}$, for all $\omega \in \mathbb{C}$ with $|\Re\omega| < \epsilon_{\Gamma_{p-1}}$ and $|\Im\omega| > \delta$. Finally, $(n_p\omega+\alpha_p)/(\omega+\alpha_p)$ is bounded below by a positive constant for small enough $\Re\omega$, so eventually we get

$$\exists c_{\Gamma_p}, \epsilon_{\Gamma_p} > 0 \text{ such that } |\det D_{\Gamma_p}(\omega)| > c_{\Gamma_p} \text{ for all } \omega \in \mathbb{C} \text{ with } |\Re\omega| < \epsilon_{\Gamma_p}, |\Im\omega| > \delta.$$

We are left with showing that the ratio $\det \widetilde{D}_{\Gamma_p}(\omega)/\det D_{\Gamma_p}(\omega)$ is of modulus less than one. In view of the recursion formula on the ratio from Lemma 3.1, we first impose as a condition on ϵ_{Γ_p} that

$$\tilde{r}_{\Gamma_{p-1}} := e^{2\epsilon_{\Gamma_p} a_{p-1}} r_{\Gamma_{p-1}} < 1,$$

and then we have to show that for $|z| < \tilde{r}_{\Gamma_{p-1}}$,

$$\left|\frac{(n_p-2)\omega+\alpha_p-((n_p-4)\omega+\alpha_p)z}{n_p\omega+\alpha_p-((n_p-2)\omega+\alpha_p)z}\right| < r_{\Gamma_p},$$

for all complex ω with $|\Re\omega| < \epsilon_{\Gamma_p}$ and $|\Im\omega| > \delta$, for ϵ_{Γ_p} to be chosen and $r_{\Gamma_p} < 1$. By letting $q = (n_p-2)\omega+\alpha_p$, the above inequality becomes

$$|q-(q-2\omega)z| < |(q+2\omega)-qz| \quad \iff \quad |q(1-z)+2\omega z| < |q(1-z)+2\omega|.$$

Expanding this last inequality we find that we have to prove that

$$0 < |\omega|^2(1-|z|^2) + |1-z|^2((n_p-2)|\omega|^2 + \alpha_p\Re(\omega)).$$

Since $n_p \geq 2$ and $|z| < \tilde{r}_{\Gamma_{p-1}} < 1$, it is enough to have

$$0 < |\omega|^2(1-|z|^2) + |1-z|^2\alpha_p\Re(\omega).$$

Also, $|\Re z| < \tilde{r}_{\Gamma_{p-1}} < 1$, so by choosing $\epsilon_{\Gamma_p} \leq \dfrac{(1-\tilde{r}^2_{\Gamma_{p-1}})\delta^2}{2|\alpha_p|(1-\tilde{r}_{\Gamma_{p-1}})^2}$ we get the existence of $r_{\Gamma_p} < 1$. □

3D. ***Vanishing of the numerator at $\tau = 0$.*** Recall that we have denoted by $M^{c_e}_{\Gamma_p}(\omega)$ the matrix $D_{\Gamma_p}(\omega)$ with $D^e_{\Gamma_p}(\omega)$, the column corresponding to the unknown c_e, replaced by the free terms column $T_{\Gamma_p}(\omega)$. In particular $\omega \det M^{c_e}_{\Gamma_p}(\omega)$ is the determinant of the matrix $D_{\Gamma_p}(\omega)$ with the column corresponding to the unknown c_e replaced by $\omega T_{\Gamma_p}(\omega)$. The same holds for $M^{\tilde{c}_e}_{\Gamma_p}(\omega)$ with the appropriate substitutions.

Lemma 3.3. $$-(\omega T_{\Gamma_p}(\omega))(0) = \sum_{e\in\mathscr{E}} t_e(0,0)D^e_{\Gamma_p}(0) + \sum_{e\in\mathscr{I}} t_e(0,0)D^{\tilde{e}}_{\Gamma_p}(0)$$

Remark 3.4. From the shape of $D_{\Gamma_p}(\omega)$ displayed in the proof of Lemma 3.1 we notice that the two junction columns with $D_{\Gamma_{p-1}}(\omega)$, corresponding to the coefficients of the resolvent on the connecting edge I_N, are

$$D^{I_N}_{\Gamma_p}(\omega)=\left(0,\dots,0,-1,\frac{\omega}{\omega+\alpha_{p-1}},e^{-\omega a_{p-1}},0,\dots,0,\frac{-\omega+\alpha_p}{\omega+\alpha_p}e^{-\omega a_{p-1}}\right)^T$$

and

$$D^{\tilde{I}_N}_{\Gamma_p}(\omega)=\left(0,\dots,0,-1,-\frac{\omega}{\omega+\alpha_{p-1}},e^{\omega a_{p-1}},0,\dots,0,e^{\omega a_{p-1}}\right)^T.$$

In particular, these two columns are the same at $\omega=0$. Moreover, $D_{\Gamma_p}(\omega)$ contains $p-1$ such pairs of columns: $D^e_{\Gamma_p}(0)=D^{\tilde{e}}_{\Gamma_p}(0)$ for all $e\in\mathcal{I}$. Thus, the last term in the right side of Lemma 3.3 could be either $D^e_{\Gamma_p}(0)$ or $D^{\tilde{e}}_{\Gamma_p}(0)$, for $e\in\mathcal{I}$.

Proof. We will prove this identity inductively. For $p=1$, (ωT_{Γ_1}) is given in Section 3B1. We choose $X_1=(t_1(0,0),t_2(0,0),\dots,t_{n_1}(0,0))^T$, then $D_{\Gamma_1}(0)X_1=-(\omega T_{\Gamma_1})(0)$, which proves the lemma for $p=1$.

Given now X_{p-1} such that $D_{\Gamma_{p-1}}(0)X_{p-1}=-(\omega T_{\Gamma_{p-1}}(\omega))(0)$, we construct X_p as follows:

$$X_p^T=(X_{p-1}^T,0,t_{N+1}(0,0),\dots,t_{N+n_p-1}(0,0)).$$

Using the recursion between D_{Γ_p} and $D_{\Gamma_{p-1}}$ used in the proof of Lemma 3.1, the identity

$$\omega T_{\Gamma_p}(\omega)=\begin{pmatrix}\omega T_{\Gamma_{p-1}}(\omega)\\ t_{N+1}(0,\omega)-t_N(a_{p-1},\omega)\\ \vdots\\ t_{N+n_p-1}(0,\omega)-t_{N+n_p-2}(0,\omega)\\ \dfrac{\omega-\alpha_p}{\omega+\alpha_p}t_N(a_{p-1},\omega)+\dfrac{\omega}{\omega+\alpha_p}\displaystyle\sum_{1\le j\le n_p-1}t_{N+j}(0,\omega)\end{pmatrix},$$

and the fact that $t_e(0,0)=t_e(a_e,0)$ for all $e\in\mathcal{I}$, we obtain that X_p satisfies $D_{\Gamma_p}(0)X_p=-(\omega T_{\Gamma_p}(\omega))(0)$. Writing this identity in terms of the columns of the matrix $D_{\Gamma_p}(0)$ we obtain the desired identity. □

Lemma 3.5. *$\omega=0$ is a root of order at least $p-1$ of $\omega\det M^{c_e}_{\Gamma_p}(\omega)$ and of $\omega\det M^{\tilde{c}_e}_{\Gamma_p}(\omega)$ for all edges e.*

Proof. We shall give the proof for $\omega\det M^{c_e}_{\Gamma_p}(\omega)$; the result for $\omega\det M^{\tilde{c}_e}_{\Gamma_p}(\omega)$ will be the same. From the shape of $D_{\Gamma_p}(\omega)$ displayed in the proof of Lemma 3.1 and Remark 3.4 we have $p-1$ pairs of columns that are equal at $\omega=0$. Moreover, by Lemma 3.3, $(\omega T_{\Gamma_p})(0)$ is a linear combination of these columns evaluated at $\omega=0$.

The derivative of a determinant is the sum of the determinants of the matrices obtained by differentiating one column. When T_{Γ_p} does not replace any of these $2(p-1)$ columns it follows that the lemma holds since there are always two identical columns. Then by the above argument we have

$$\partial^k_\omega(\omega\det M^{c_e}_{\Gamma_p})(0)=0\quad\text{if }0\le k\le p-3.\tag{22}$$

Assume now that T_{Γ_p} replaces one of these $2(p-1)$ columns. To finish the proof we must show that

$$\partial^{p-2}_\omega\big(\omega\det M^{c_e}_{\Gamma_p}\big)(0)=0.\tag{23}$$

Using again the fact that $D_{\Gamma_p}(\omega)$ contains $p-1$ pairs of columns that match at $\omega = 0$, we only need to show that $\det A_{\Gamma_p}(0) = 0$, where $A_{\Gamma_p}(\omega)$ is $D_{\Gamma_p}(\omega)$ with the column $\omega T_{\Gamma_p}(\omega)$ replacing one column of one pair, and one column of each of the remaining $p-2$ pairs of columns is differentiated. In particular $A_{\Gamma_p}(0)$ contains one unchanged column of each of the $p-1$ pairs. By Lemma 3.3 we know that $(\omega T_{\Gamma_p}(\omega))(0)$ is a linear combination of the columns corresponding to external edges and of the internal ones (each one from the $p-1$ pairs), so the new determinant vanishes and the proof is finished. □

Lemma 3.6. *For all edge indices λ and e, $\omega = 0$ is a root of order at least $p-2$ of the coefficient $f_{\lambda,e}(\omega)$ of $t_\lambda(0,\omega)$ in $\omega \det M^{c_e}_{\Gamma_p}(\omega)$, and the same holds for the coefficient $\tilde{f}_{\lambda,e}(\omega)$ of $t_\lambda(0,\omega)$ in $\omega \det M^{\tilde{c}_e}_{\Gamma_p}(\omega)$.*

Proof. This result follows from the discussion that led to (22): the matrix $\omega M^{c_e}_{\Gamma_p}(\omega)$ has $p-2$ pairs of columns that are identical at $\omega = 0$. □

Lemma 3.7. *For each edge index e and each external edge index λ, $\omega = 0$ is a root of order at least $p-1$ of the coefficient $f_{\lambda,e}(\omega)$ of $t_\lambda(0,\omega)$ in $\omega \det M^{c_e}_{\Gamma_p}(\omega)$, and the same holds for the coefficient $\tilde{f}_{\lambda,e}(\omega)$ of $t_\lambda(0,\omega)$ in $\omega \det M^{\tilde{c}_e}_{\Gamma_p}(\omega)$.*

Proof. The statement corresponds to the particular case of Lemma 3.5 where all the components of T_{Γ_p} are taken to be 0 except $t_\lambda(0,\omega)$, which is replaced by 1. □

Lemma 3.8. *For each edge index e and each internal edge index λ, $\omega = 0$ is a root of order at least $p-1$ of $f^1_{\lambda,e}(\omega) + f^2_{\lambda,e}(\omega)$, where $f^1_{\lambda,e}(\omega)$ and $f^2_{\lambda,e}(\omega)$ are respectively the coefficients of $t_\lambda(0,\omega)$ and $t_\lambda(a_\lambda,\omega)$ in $\omega \det M^{c_e}_{\Gamma_p}(\omega)$. The same holds for $\tilde{f}^1_{\lambda,e}(\omega) + \tilde{f}^2_{\lambda,e}(\omega)$, where $\tilde{f}^1_{\lambda,e}(\omega)$ and $\tilde{f}^2_{\lambda,e}(\omega)$ are respectively the coefficients of $t_\lambda(0,\omega)$ and $t_\lambda(a_\lambda,\omega)$ in $\omega \det M^{\tilde{c}_e}_{\Gamma_p}(\omega)$.*

Proof. The proof goes the same as for Lemma 3.7. □

3E. ***The end of the proof.*** Now we shall use the theorem hypothesis, $\partial^{(p-1)}_\omega \det D_{\Gamma_p}\big|_{\omega=0} \neq 0$. We obtain that $\omega = 0$ is a root of order $p-1$ of $\det D_{\Gamma_p}$. From the previous subsections we conclude the following:

Lemma 3.9. *$\omega R_\omega \boldsymbol{f}(x)$ can be analytically continued in a region containing the imaginary axis.*

Proof. The proof is an immediate consequence of decomposition (18) for $x \in I_e$, together with Lemma 3.2, Lemma 3.5 and the fact that $\omega = 0$ is a root of order $p-1$ of $\det D_{\Gamma_p}$. □

Proof of Theorem 1.2. As a consequence of Lemma 3.9 we can use a spectral calculus argument to write the solution of the Schrödinger equation with initial data $\boldsymbol{u}_0$ as

$$e^{-itH^\Gamma_\alpha} P\boldsymbol{u}_0(x) = \frac{1}{i\pi}\int_{-\infty}^{\infty} e^{-it\tau^2}\tau R_{i\tau}\boldsymbol{u}_0(x)\,d\tau. \tag{24}$$

In view of the definition of t_e and with the notations from Lemmas 3.7 and 3.8 we can also write the decomposition (18) as

$$
\begin{aligned}
\tau R_{i\tau}\boldsymbol{u}_0(x) = \frac{1}{2}\int_{I_e} \boldsymbol{u}_0\, e^{-i\tau|x-y|}\,dy\, i + \sum_{\lambda\in\mathscr{E}} \frac{f_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} \int_{I_\lambda} \boldsymbol{u}_0(y) e^{i\tau y}\,dy\, e^{i\tau x} \\
+ \sum_{\lambda\in\mathscr{E}} \frac{\tilde{f}_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} \int_{I_\lambda} \boldsymbol{u}_0(y) e^{i\tau y}\,dy\, e^{-i\tau x} \\
+ \sum_{\lambda\in\mathscr{I}} \int_{I_\lambda} \boldsymbol{u}_0(y)\Big(e^{i\tau y}\frac{f^1_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} + e^{i\tau(a_\lambda - y)}\frac{f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}\Big)\,dy\, e^{i\tau x} \\
+ \sum_{\lambda\in\mathscr{I}} \int_{I_\lambda} \boldsymbol{u}_0(y)\Big(e^{i\tau y}\frac{\tilde{f}^1_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} + e^{i\tau(a_\lambda - y)}\frac{\tilde{f}^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}\Big)\,dy\, e^{-i\tau x}. \quad (25)
\end{aligned}
$$

Moreover, in view of the results in Lemma 3.8 and Lemma 3.7 we gather the terms as follows:

$$
\begin{aligned}
\tau R_{i\tau}\boldsymbol{u}_0(x) = \frac{1}{2}\int_{I_e} \dot{\boldsymbol{u}}_0, e^{-i\tau|x-y|}\,dy + \sum_{\lambda\in\mathscr{E}} \int_{I_\lambda} \boldsymbol{u}_0(y) \frac{f_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} e^{i\tau(x+y)}\,dy \\
+ \sum_{\lambda\in\mathscr{E}} \int_{I_\lambda} \boldsymbol{u}_0(y) \frac{\tilde{f}_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} e^{i\tau(y-x)}\,dy + \sum_{\lambda\in\mathscr{I}} \int_{I_\lambda} \boldsymbol{u}_0(y) \frac{f^1_{\lambda,e}(i\tau) + f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} e^{i\tau(x+y)}\,dy \\
+ \sum_{\lambda\in\mathscr{I}} \int_{I_\lambda} \boldsymbol{u}_0(y) \frac{\tilde{f}^1_{\lambda,e}(i\tau) + \tilde{f}^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} e^{i\tau(y-x)}\,dy \\
+ \sum_{\lambda\in\mathscr{I}} \int_{I_\lambda} \boldsymbol{u}_0(y) \frac{(e^{i\tau(a_\lambda-y)} - e^{i\tau y}) f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}\, e^{i\tau x}\,dy \\
+ \sum_{\lambda\in\mathscr{I}} \int_{I_\lambda} \boldsymbol{u}_0(y) \frac{(e^{i\tau(a_\lambda-y)} - e^{i\tau y}) \tilde{f}^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} e^{-i\tau x}\,dy. \quad (26)
\end{aligned}
$$

Let e be an external edge. In view of Lemma 3.7 and the fact that $\omega = 0$ is a root of order $p-1$ of $\det D_{\Gamma_p}$, we obtain that the fraction $f_{\lambda,e}(i\tau)/\det D_{\Gamma_p}(i\tau)$ is upper bounded near $\tau = 0$. Outside a neighbourhood of $\tau = 0$ we use Lemma 3.2 to infer that $|\det D_{\Gamma_p}(i\tau)|$ is positively bounded below outside neighbourhoods of $\tau = 0$. Moreover, in view of the explicit entries of $M^{c_e}_{\Gamma_p}(i\tau)$, we see that $f_{\lambda,e}(i\tau)$ is upper bounded for any $\tau \in \mathbb{R}$ since all the entries of matrix $D_{\Gamma_p}(i\tau)$ as well as the coefficients of t_λ in $T_{\Gamma_p(i\tau)}$ have absolute value less than one. Summarising, we have obtained that

$$
\frac{f_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)} \in L^\infty(\mathbb{R}).
$$

The derivative of this fraction is upper bounded near $\tau = 0$ by limited development at $\tau = 0$. Outside neighbourhoods of $\tau = 0$ we have that $\partial_\tau f_{\lambda,e}(i\tau)$ and $\partial_\tau \det D_{\Gamma_p}(i\tau)$ have upper bounds of type $1/\tau^2$. This is because each term of $\partial_\tau f_{\lambda,e}(i\tau)$ and $\partial_\tau \det D_{\Gamma_p}(i\tau)$ contains a derivative of an element of the line given by the δ-condition involving the derivatives in the root vertex $\mathbb{O}$. This vertex is the one which is an initial vertex for all n edges emerging from it: $I(e)=\mathbb{O}$, for all $e\in E$, $\mathbb{O}\in e$. If α denotes the strength of the δ-condition in $\mathbb{O}$, then this line of the matrix $D_{\Gamma_p}(i\tau)$ is composed of 0, 1 and $\pm\frac{i\tau}{i\tau+\alpha}$, where the minus sign appears only on the finite edges that star from $\mathbb{O}$, and this line for the column matrix $i\tau T_{\Gamma_p}(i\tau)$ is

$$\left(\frac{i\tau-\alpha}{i\tau+\alpha}t_1(0,i\tau)+\frac{i\tau}{i\tau+\alpha}\sum_{2\le j\le n}t_j(0,i\tau)\right).$$

Finally, as above, $f_{\lambda,e}(i\tau)$ and $\det D_{\Gamma_p}(i\tau)$ are upper bounded and from Lemma 3.2 we have that $|\det D_{\Gamma_p}(i\tau)|$ is positively bounded below outside neighbourhoods of $\tau=0$. In conclusion we infer that

$$\partial_\tau\frac{f_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}\in L^1(\mathbb{R}).$$

The same argument using Lemmas 3.7, 3.8 and 3.6 can be performed to obtain that

$$\frac{\tilde f_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)},\quad \frac{f^1_{\lambda,e}(i\tau)+f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)},\quad \frac{\tilde f^1_{\lambda,e}(i\tau)+\tilde f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)},$$
$$\frac{(e^{i\tau(a_\lambda-y)}-e^{i\tau y})f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)},\quad \frac{(e^{i\tau(a_\lambda-y)}-e^{i\tau y})\tilde f^2_{\lambda,e}(i\tau)}{\det D_{\Gamma_p}(i\tau)}$$

are in L^∞ with derivative in L^1. Notice that when λ belongs to an internal edge I_λ it follows that the interval I_λ has finite length. Therefore for the last fractions we use that $(e^{i\tau(a_\lambda-y)}-e^{i\tau y})f^2_{\lambda,e}(i\tau)$ vanishes with order $p-1$ at $\tau=0$ and repeat the argument used above. The only difference from the previous cases is that we will obtain bounds in terms of the parameter y. Since y is now on an internal edge I_λ of finite length we obtain uniform bounds. Therefore the dispersion estimate (12) of Theorem 1.2 follows from (24) by using (26) and the classical oscillatory integral estimate

$$\left|\int_{-\infty}^{\infty}e^{-it\tau^2}e^{i\tau a}g(\tau)\,d\tau\right|\le\frac{C}{\sqrt{|t|}}(\|g\|_{L^\infty}+\|g'\|_{L^1}).\qquad\square$$

Appendix A: The multiplicity of the root $\omega=0$ of $\det D_{\Gamma_p}(\omega)$

Here we prove that the condition (11) is fulfilled in the case of positive strengths. We shall show first the following double property.

Lemma A.1. *For all $p\ge1$ we have the following properties*:

$$(\mathcal{P}^1_p):\ \frac{\det\widetilde D_{\Gamma_p}}{\det D_{\Gamma_p}}(0)=1,\quad (\mathcal{P}^2_p):\ \partial_\omega\left(\frac{\det\widetilde D_{\Gamma_p}}{\det D_{\Gamma_p}}\right)(0)<0.$$

Proof. Lemma 3.1 ensures that $\dfrac{\det\widetilde D_{\Gamma_1}}{\det D_{\Gamma_1}}(\omega)=\dfrac{(n_1-2)\omega+\alpha_1}{n_1\,\omega+\alpha_1}$, and in particular

$$\partial_\omega\left(\frac{\det\widetilde D_{\Gamma_1}}{\det D_{\Gamma_1}}\right)(\omega)=-\frac{2\alpha_1}{(n_1\omega+\alpha_1)^2},\tag{27}$$

and the lemma follows for $p = 1$, since $\alpha_1 > 0$. We shall show the general case by recursion. Let us denote by $P_p(\omega)$ and $Q_p(\omega)$ the numerator and respectively the denominator in the recursion formula of the ratio from Lemma 3.1:

$$P_p(\omega) = \frac{(n_p-2)\omega+\alpha_p}{n_p\omega+\alpha_p} - \frac{(n_p-4)\omega+\alpha_p}{n_p\omega+\alpha_p} e^{-2\omega a_{p-1}} \frac{\det \widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)},$$

$$Q_p(\omega) = 1 - \frac{(n_p-2)\omega+\alpha_p}{n_p\omega+\alpha_p} e^{-2\omega a_{p-1}} \frac{\det \widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}.$$

We have $P_p(0) = Q_p(0) = 0$, and in view of $(\mathcal{P}^1_{p-1})$ we compute

$$\partial_\omega P_p(0) = \partial_\omega Q_p(0) = \frac{2}{\alpha_p} + 2a_{p-1} - \partial_\omega \left(\frac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}} \right)(0).$$

Therefore $(\mathcal{P}^2_{p-1})$ ensures that $\partial_\omega P_p(0) = \partial_\omega Q_p(0) \neq 0$ and we apply l'Hôpital's rule to conclude $(\mathcal{P}^1_p)$.

Since

$$P_p(\omega) - Q_p(\omega) = -\frac{2\omega}{n_p\omega+\alpha_p} \left(1 - e^{-2a_{p-1}\omega} \frac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}}(\omega) \right),$$

we define $\widetilde{P}_p(\omega)$ and $\widetilde{Q}_p(\omega)$ by

$$P_p(\omega) = \frac{2\omega}{n_p\omega+\alpha_p} \widetilde{P}_p(\omega), \qquad Q_p(\omega) = \frac{2\omega}{n_p\omega+\alpha_p} \widetilde{Q}_p(\omega).$$

In particular

$$\frac{\det \widetilde{D}_{\Gamma_p}}{\det D_{\Gamma_p}}(\omega) = \frac{\widetilde{P}_p}{\widetilde{Q}_p}(\omega), \qquad \widetilde{P}_p(\omega) - \widetilde{Q}_p(\omega) = -\left(1 - e^{-2a_{p-1}\omega} \frac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}}(\omega) \right).$$

By using $(\mathcal{P}^1_{p-1})$ and $(\mathcal{P}^2_{p-1})$, we obtain

$$\partial_\omega(\widetilde{P}_p - \widetilde{Q}_p)(0) = -2a_{p-1} + \partial_\omega \left(\frac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}} \right)(0).$$

Moreover, $\widetilde{P}_p(0) = \widetilde{Q}_p(0) = \frac{1}{2}\alpha_p \partial_\omega P_p(0) = \frac{1}{2}\alpha_p \partial_\omega Q_p(0) \neq 0$, and we can compute

$$\partial_\omega \left(\frac{\det \widetilde{D}_{\Gamma_p}}{\det D_{\Gamma_p}} \right)(0) = \frac{\partial_\omega \widetilde{P}_p(0)\widetilde{Q}_p(0) - \widetilde{P}_p(0)\partial_\omega \widetilde{Q}_p(0)}{(\widetilde{Q}_p(0))^2} = \frac{\partial_\omega(\widetilde{P}_p - \widetilde{Q}_p)(0)}{\widetilde{Q}_p(0)}$$

$$= -\frac{2a_{p-1} - \partial_\omega \left(\frac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}} \right)(0)}{\frac{\alpha_p}{2} \left(\frac{2}{\alpha_p} + 2a_{p-1} - \partial_\omega \left(\frac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}} \right)(0) \right)}.$$

By using again $(\mathcal{P}^2_{p-1})$, we obtain $(\mathcal{P}^2_p)$. $\square$

Lemma A.2. *$\omega = 0$ is a root of order $p-1$ of* $\det D_{\Gamma_p}(\omega)$. *In particular, condition* (11) *is fulfilled.*

Proof. From Lemma 3.1 we have $\det D_{\Gamma_1}(\omega) = n_1\omega + \alpha$, so $\det D_{\Gamma_1}(0) \neq 0$. Lemma 3.1 also gives us

$$\det D_{\Gamma_p}(\omega) = \frac{n_p\omega + \alpha_p}{\omega + \alpha_p} e^{\omega a_{p-1}} \det D_{\Gamma_{p-1}}(\omega)\left(1 - \frac{(n_p-2)\omega + \alpha_p}{n_p\omega + \alpha_p} e^{-2\omega a_{p-1}} \frac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}}(\omega)\right),$$

so by recursion it is enough to show that $\omega = 0$ is a simple root for

$$1 - \frac{(n_p-2)\,\omega + \alpha_p}{n_p\,\omega + \alpha_p}\, e^{-2\omega a_{p-1}} \frac{\det \widetilde{D}_{\Gamma_{p-1}}}{\det D_{\Gamma_{p-1}}}(\omega).$$

This is precisely $Q_p(\omega)$ from the proof of Lemma A.1, where it was proved that $\partial_\omega Q_p(0) \neq 0$. □

Appendix B: Strichartz estimates

We prove Theorem 1.3. Let us first remark that by the definition of P_e we have

$$P_e\phi = \sum_{k=1}^{m} \langle \phi, \varphi_k\rangle \varphi_k,$$

where $\{\varphi_k\}_{k=1}^m$ are eigenfunctions of the operator H. Since $\varphi_k \in L^2(\Gamma)$ we have that $\varphi_k \in L^1(\Gamma) \cap L^\infty(\Gamma)$. Indeed, on the infinite edges the eigenfunctions corresponding to an eigenvalue $\lambda < 0$ are of type $C \exp(-\sqrt{-\lambda})x$. This means that they belong to $L^1(e) \cap L^\infty(e)$ for any external edge e. On the internal edges this property trivially holds.

Then P_e is defined for any $\phi \in L^r(\Gamma)$, $1 \le r \le \infty$, and for any $1 \le r_1, r_2 \le \infty$ we have, by Hölder's inequality,

$$\begin{aligned}\|P_e\phi\|_{L^{r_2}(\Gamma)} \le \sum_{k=1}^{m} |\langle \phi, \varphi_k\rangle| \|\varphi_k\|_{L^{r_2}(\Gamma)} &\le \|\phi\|_{L^{r_1}(\Gamma)} \sum_{k=1}^{m} \|\varphi_k\|_{L^{r_1'}(\Gamma)} \|\varphi_k\|_{L^{r_2}(\Gamma)} \\ &\le C(\Gamma, r_1, r_2)\|\phi\|_{L^{r_1}(\Gamma)}.\end{aligned}$$

Proof of Theorem 1.3. Using the dispersive estimate (12) and the mass conservation

$$\|e^{-itH}\boldsymbol{u}_0\|_{L^2(\Gamma)} = \|\boldsymbol{u}_0\|_{L^2(\Gamma)}$$

we obtain, by applying the classical TT^* argument and Christ–Kiselev lemma [2001], the estimates

$$\|e^{-itH} P\boldsymbol{u}_0\|_{L^q(\mathbb{R}, L^r(\Gamma))} \le C\|\boldsymbol{u}_0\|_{L^2(\Gamma)}, \tag{28}$$

and

$$\left\| \int_0^t e^{-i(t-s)} P\boldsymbol{F}(s)ds \right\|_{L^q((0,T), L^r(\Gamma))} \le C\|\boldsymbol{F}\|_{L^{\tilde{r}'}((0,T), L^{\tilde{r}'}(\Gamma))}. \tag{29}$$

Now using Stone's theorem we obtain

$$e^{-itH}\phi = e^{-itH}P\phi + e^{-itH}P_e\phi = e^{-itH}P\phi + \sum_{k=1}^{m} e^{it\lambda_k^2}\langle \phi, \varphi_k\rangle \varphi_k,$$

where by λ_k we denote the eigenvalue of the eigenfunction φ_k. We claim that, for all $\alpha \ge 1$,

$$\|e^{-itH} P_e\boldsymbol{u}_0\|_{L^q((0,T), L^r(\Gamma))} \le CT^{1/q}\|\boldsymbol{u}_0\|_{L^\alpha(\Gamma)} \tag{30}$$

and

$$\left\| \int_0^t e^{-i(t-s)} P_e \boldsymbol{F}(s) ds \right\|_{L^q((0,T),L^r(\Gamma))} \le C T^{1/q} \|\boldsymbol{F}\|_{L^1((0,T),L^\alpha(\Gamma))}. \tag{31}$$

Putting together estimates (28), (29), (30) and (31) we obtain the desired result. We now prove estimates (30) and (31).

In the case of estimate (30), using the fact that

$$e^{-itH} P_e \boldsymbol{u}_0 = \sum_{k=1}^m e^{it\lambda_k^2} \langle \boldsymbol{u}_0, \varphi_k \rangle \varphi_k$$

we obtain by Hölder's inequality that, for any $\alpha \ge 1$,

$$\|e^{-itH} P_e \boldsymbol{u}_0\|_{L^r(\Gamma)} \le \sum_{k=1}^m |\langle \boldsymbol{u}_0, \varphi_k \rangle| \|\varphi_k\|_{L^r(\Gamma)} \le \|\boldsymbol{u}_0\|_{L^\alpha(\Gamma)} \sum_{k=1}^m \|\varphi_k\|_{L^{\alpha'}(\Gamma)} \|\varphi_k\|_{L^r(\Gamma)} \le C \|\boldsymbol{u}_0\|_{L^\alpha(\Gamma)}.$$

Taking the L^q-norm on the time interval $(0, T)$ we obtain estimate (30).

In a similar way we have

$$\|e^{-i(t-s)H} P_e \boldsymbol{F}(s)\|_{L^r(\Gamma)} \le C \|\boldsymbol{F}(s)\|_{L^\alpha(\Gamma)}.$$

Using Minkowski's inequality we obtain that

$$\begin{aligned} \left\| \int_0^t e^{-i(t-s)H} P_e \boldsymbol{F}(s) ds \right\|_{L^q((0,T),L^r(\Gamma))} &\le \left\| \int_0^t \|e^{-i(t-s)H} P_e \boldsymbol{F}(s)\|_{L^r(\Gamma)} ds \right\|_{L^q(0,T)} \\ &\le T^{1/q} \int_0^T \|\boldsymbol{F}(s)\|_{L^\alpha(\Gamma)} ds, \end{aligned}$$

which proves estimate (31). □

Appendix C: General couplings

We consider general coupling conditions at each vertex v (see (16) in Section 2),

$$A^v \boldsymbol{u}(v) + B^v \boldsymbol{u}'(v) = 0.$$

Using the notations introduced in this article, we shall give the recursion formulae for obtaining $\det D_{\Gamma_p}$ for general couplings. As a consequence, we shall give a sufficient condition for obtaining the dispersion.

We follow the approach in Section 3A for computing the resolvent. For a star-shaped graph with n_1 edges I_j parametrised by $x \in [0, \infty)$, with coupling conditions (A^1, B^1), the resolvent on each edge I_j is

$$R_\omega \boldsymbol{u}_0(x) = \tilde{c}_j\, e^{-\omega x} + \frac{1}{2\omega} \int_0^\infty \boldsymbol{u}_0(y) e^{-\omega|x-y|}\, dy.$$

The coupling conditions yield as a system for the $\tilde{c}$'s

$$(A^1+\omega B^1)\begin{pmatrix}\tilde{c}_1\\ \vdots \\ \tilde{c}_{n_1}\end{pmatrix}=\begin{pmatrix}\displaystyle\sum_{1\le j\le n_1}\frac{t_j(0,\omega)}{\omega}(b_{1,j}\omega-a_{1,j})\\ \vdots \\ \displaystyle\sum_{1\le j\le n_1}\frac{t_j(0,\omega)}{\omega}(b_{n_1,j}\omega-a_{n_1,j})\end{pmatrix}.$$

We denote by $D_{\Gamma_1}(\omega)$ the matrix of the system. We define $\widetilde{D}_{\Gamma_1}(\omega)$ to be the matrix

$$\big((A^1+\omega B^1)_1,(A^1+\omega B^1)_2,\dots,(A^1-\omega B^1)_{n_1}\big),$$

where by $(A^1+\omega B^1)_j$ we mean the j-th column of $A^1+\omega B^1$.

The case of a general tree with p vertices can again be seen as constructed by adding a new vertex v^p to a $(p-1)$-vertex tree, with coupling conditions (A^p, B^p), from which emerge new n_p-1 infinite edges. Similarly to Lemma 3.1 we derive the recursion formulae

$$\begin{aligned}
&\det D_{\Gamma_1}(\omega)=\det(A^1+\omega B^1),\\
&\frac{\det\widetilde{D}_{\Gamma_1}(\omega)}{\det D_{\Gamma_1}(\omega)}=\frac{\det\big((A^1+\omega B^1)_1,(A^1+\omega B^1)_2,\dots,(A^1-\omega B^1)_{n_1}\big)}{\det(A^1+\omega B^1)},\\
&\det D_{\Gamma_p}(\omega)=\det(A^p+\omega B^p)e^{\omega a_{p-1}}\det D_{\Gamma_{p-1}}(\omega)\\
&\qquad\times\left(1-\frac{\det\big((A^p-\omega B^p)_1,(A^p+\omega B^p)_2,\dots,(A^p+\omega B^p)_{n_p}\big)}{\det(A^p+\omega B^p)}e^{-2\omega a_{p-1}}\frac{\det\widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}\right),\\
&\frac{\det\widetilde{D}_{\Gamma_p}(\omega)}{\det D_{\Gamma_p}(\omega)}=\Bigg(\frac{\det\big((A^p+\omega B^p)_1,(A^p+\omega B^p)_2,\dots,(A^p-\omega B^p)_{n_p}\big)}{\det(A^p+\omega B^p)}\\
&\qquad\qquad-\frac{\det\big((A^p-\omega B^p)_1,(A^p+\omega B^p)_2,\dots,(A^p-\omega B^p)_{n_p}\big)}{\det(A^p+\omega B^p)}e^{-2\omega a_{p-1}}\frac{\det\widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}\Bigg)\\
&\qquad\times\left(1-\frac{\det\big((A^p-\omega B^p)_1,(A^p+\omega B^p)_2,\dots,(A^p+\omega B^p)_{n_p}\big)}{\det(A^p+\omega B^p)}e^{-2\omega a_{p-1}}\frac{\det\widetilde{D}_{\Gamma_{p-1}}(\omega)}{\det D_{\Gamma_{p-1}}(\omega)}\right)^{-1}.
\end{aligned}$$

A sufficient condition for using the spectral formula as in Section 3E and then for getting the dispersion as the following constraint, depending only on the entries of $(A^j, B^j)_{1\le j\le p}$, is

$$|\det D_{\Gamma_p}(i\omega)|\neq 0\quad\text{for all }\ \omega\in\mathbb{R}. \tag{32}$$

This is the way the Kirchhoff coupling case was ruled in [Banica and Ignat 2011] and this might be used in other cases. In the δ-coupling case presented in this article, and probably in many other cases, such an estimate is not valid. Then an analysis around the zeros of $\det D_{\Gamma_p}(\omega)$ has to be done starting from the above recursion formulae.

Acknowledgements

The authors are grateful to the referee for the remarks and questions that improved the presentation of this paper.

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Received 3 Dec 2012. Revised 17 Feb 2014. Accepted 1 Apr 2014.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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ANALYSIS & PDE

Volume 7 No. 4 2014

2157-5045(2014)7:4;1-C