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# WAVE AND KLEIN–GORDON EQUATIONS ON HYPERBOLIC SPACES

JEAN-PHILIPPE ANKER AND VITTORIA PIERFELICE

We consider the Klein–Gordon equation associated with the Laplace–Beltrami operator $\Delta$ on real hyperbolic spaces of dimension $n \geq 2$; as $\Delta$ has a spectral gap, the wave equation is a particular case of our study. After a careful kernel analysis, we obtain dispersive and Strichartz estimates for a large family of admissible couples. As an application, we prove global well-posedness results for the corresponding semilinear equation with low regularity data.

## 1. Introduction

Dispersive properties of the wave and other evolution equations have been proved to be very useful in the study of nonlinear problems. The theory is well-established for the Euclidean wave equation in dimension $n \geq 3$:

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = F(t,x), \\ u(0,x) = f(x), \quad \partial_t|_{t=0} u(t,x) = g(x). \end{cases} \tag{1}$$

The Strichartz estimates

$$\|\nabla_{\mathbb{R}\times\mathbb{R}^n} u\|_{L^p(I;\dot H^{-\sigma,q}(\mathbb{R}^n))} \lesssim \|f\|_{\dot H^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^{\tilde p'}(I;\dot H^{\tilde\sigma,\tilde q'}(\mathbb{R}^n))}$$

hold for solutions $u$ to the Cauchy problem (1) on any (possibly unbounded) time interval $I \subseteq \mathbb{R}$ under the assumptions that

$$\sigma \geq \frac{n+1}{2}\Big(\frac{1}{2}-\frac{1}{q}\Big) \quad \text{and} \quad \tilde\sigma \geq \frac{n+1}{2}\Big(\frac{1}{2}-\frac{1}{\tilde q}\Big)$$

and the couples $(p,q), (\tilde p, \tilde q) \in [2,\infty] \times [2,\infty)$ satisfy

$$\frac{2}{p}+\frac{n-1}{q}=\frac{n-1}{2} \quad \text{and} \quad \frac{2}{\tilde p}+\frac{n-1}{\tilde q}=\frac{n-1}{2}. \tag{2}$$

We refer to [Ginibre and Velo 1995; Keel and Tao 1998] for more details.

These estimates serve as a tool for several existence results about the nonlinear wave equation in the Euclidean setting. The problem of finding minimal regularity conditions on the initial data ensuring local well-posedness for semilinear wave equations was addressed in [Kapitanski 1994] and then almost completely answered in [Lindblad and Sogge 1995; Keel and Tao 1998] (see Figure 5 in Section 6). In general, local solutions cannot be extended to global ones unless further assumptions are made on the

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nonlinearity or on the initial data. A successful machinery was developed towards the global existence of small solutions to the semilinear wave equation

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = F(u), \\ u(0,x) = f(x), \quad \partial_t|_{t=0} u(t,x) = g(x) \end{cases} \tag{3}$$

with power-like nonlinearities

$$F(u) \sim |u|^\gamma \quad (\gamma > 1). \tag{4}$$

The results depend on the space dimension $n$. After the pioneer work of John [1979] in dimension $n = 3$, Strauss [1989] conjectured that the problem (3) is globally well posed in dimension $n \geq 2$ for small initial data provided that

$$\gamma > \gamma_0 = \frac{1}{2} + \frac{1}{n-1} + \sqrt{\Big(\frac{1}{2} + \frac{1}{n-1}\Big)^2 + \frac{2}{n-1}}. \tag{5}$$

On one hand, the negative part of the conjecture was established by Sideris [1984], who proved blow-up for nonlinearities $F(u) = |u|^\gamma$ with $1 < \gamma < \gamma_0$ and for rather general initial data. On the other hand, the positive part of the conjecture was proved for any dimension in several steps (see, e.g., [Klainerman and Ponce 1983; Georgiev et al. 1997; D'Ancona et al. 2001] and [Georgiev 2000] for a comprehensive survey).

Analogous results hold for the Klein–Gordon equation

$$\partial_t^2 u(t,x) - \Delta_x u(t,x) + u(t,x) = F(t,x)$$

though its study has not been carried out as thoroughly as for the wave equation; in particular, the sharpness of several well-posedness results is yet unknown (see [Bahouri and Gérard 1999; Ginibre and Velo 1985; Machihara et al. 2004; Nakanishi 1999] and the references therein).

In view of the rich Euclidean theory, it is natural to look at the corresponding equations on more general manifolds. Here we consider real hyperbolic spaces $\mathbb{H}^n$, which are the most simple examples of noncompact Riemannian manifolds with negative curvature. For geometric reasons, we expect better dispersive properties and hence stronger results than in the Euclidean setting.

Consider the wave equation associated to the Laplace–Beltrami operator $\Delta = \Delta_{\mathbb{H}^n}$ on $\mathbb{H}^n$:

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = F(t,x), \\ u(0,x) = f(x), \quad \partial_t|_{t=0} u(t,x) = g(x). \end{cases} \tag{6}$$

The operator $-\Delta$ is positive on $L^2(\mathbb{H}^n)$, and its $L^2$-spectrum is the half-line $[\rho^2, +\infty)$, where $\rho = (n-1)/2$. Thus, (6) may be considered as a special case of the family of Klein–Gordon equations

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) + cu(t,x) = F(t,x), \\ u(0,x) = f(x), \quad \partial_t|_{t=0} u(t,x) = g(x), \end{cases} \tag{7}$$

where

$$c \geq -\rho^2 = -\frac{(n-1)^2}{4} \tag{8}$$

is a constant. In the limit case $c = -\rho^2$, (7) is called the *shifted* wave equation.

Pierfelice [2008] obtained Strichartz estimates for the nonshifted wave equation (6) with radial data on a class of Riemannian manifolds containing all hyperbolic spaces. The wave equation (6) was also investigated on the 3-dimensional hyperbolic space by Metcalfe and Taylor [2011; 2012], who proved dispersive and Strichartz estimates with applications to small-data global well-posedness for the semilinear wave equation. In his recent thesis, Hassani [2011a; 2011b] obtains a first set of results on noncompact Riemannian symmetric spaces of higher rank.

To our knowledge, the shifted wave equation (7) in the limit case $c = -\rho^2$ was first considered by Fontaine [1994; 1997] in low dimensions $n = 3$ and $n = 2$. Tataru [2001] obtained dispersive estimates for the operators $\sin\bigl(t\sqrt{\Delta+\rho^2}\bigr)/\sqrt{\Delta+\rho^2}$ and $\cos\bigl(t\sqrt{\Delta+\rho^2}\bigr)$ acting on inhomogeneous Sobolev spaces on $\mathbb{H}^n$ and then transferred them to $\mathbb{R}^n$ in order to get well-posedness results for the Euclidean semilinear wave equation (see also [Georgiev 2000]). Complementary results were obtained by Ionescu [2000], who investigated $L^q \to L^q$ Sobolev estimates for the above operators on all hyperbolic spaces.

A more detailed analysis of the shifted wave equation was carried out in [Anker et al. 2012]. There Strichartz estimates were obtained for a wider range of couples than in the Euclidean setting, and consequently stronger well-posedness results were shown to hold for the nonlinear equations. Corresponding results for the Schrödinger equation were obtained in [Anker and Pierfelice 2009; Anker et al. 2011; Ionescu and Staffilani 2009].

In the present paper, we study the family of equations (7) in the remaining range $c > -\rho^2$ and in dimension $n \geq 2$, which includes the particular case $c = 0$ and $n = 3$ considered in [Metcalfe and Taylor 2011; 2012]. In order to state and describe our results, it is convenient to rewrite the constant (8) as

$$c = \kappa^2 - \rho^2 \quad \text{with } \kappa > 0 \tag{9}$$

and to introduce the operator

$$D = \sqrt{-\Delta - \rho^2 + \kappa^2} \tag{10}$$

as well as

$$\widetilde{D} = \sqrt{-\Delta - \rho^2 + \tilde{\kappa}^2}, \tag{11}$$

where $\tilde{\kappa} > \rho$ is another fixed constant. Thus, our family of equations (7) becomes

$$\begin{cases} \partial_t^2 u(t,x) + D_x^2 u(t,x) = F(t,x), \\ u(0,x) = f(x), \quad \partial_t|_{t=0} u(t,x) = g(x), \end{cases} \tag{12}$$

the wave equation (6) corresponding to the choice $\kappa = \rho$ and the shifted wave equation to the limit case $\kappa = 0$.

Let us now describe the content of this paper and present our main results, which we state for simplicity in dimension $n \geq 3$. In Section 2, we recall the basic tools of spherical Fourier analysis on real hyperbolic spaces $\mathbb{H}^n$. After analyzing carefully the integral kernel of the half-wave operator

$$W_t^\sigma = \widetilde{D}^{-\sigma} e^{itD}$$

in Section 3, we prove in Section 4 the following dispersive estimates, which combine the small time estimates [Anker et al. 2012] for the shifted wave equation and the large time estimates [Anker and

Pierfelice 2009] for the Schrödinger equation:

$$\|W_t^\sigma\|_{L^q\to L^{q'}} \lesssim \begin{cases} |t|^{-(n-1)(1/2-1/q)} & \text{if } 0<|t|<1, \\ |t|^{-3/2} & \text{if } |t|\geq 1, \end{cases}$$

where $2<q<\infty$ and $\sigma\geq(n+1)(1/2-1/q)$. Notice that we don't deal with the limit case $q=\infty$, where Metcalfe and Taylor [2011] have obtained an $H^1\to\mathrm{BMO}$ estimate in dimension $n=3$.

In Section 5, we deduce the Strichartz estimates

$$\|\nabla_{\mathbb{R}\times\mathbb{H}^n}u\|_{L^p(I;H^{-\sigma,q}(\mathbb{H}^n))} \lesssim \|f\|_{H^1(\mathbb{H}^n)}+\|g\|_{L^2(\mathbb{H}^n)}+\|F\|_{L^{\tilde{p}'}(I;H^{\tilde{\sigma},\tilde{q}'}(\mathbb{H}^n))}$$

for solutions $u$ to (12). Here $I\subseteq\mathbb{R}$ is any time interval,

$$\sigma\geq\frac{n+1}{2}\Big(\frac{1}{2}-\frac{1}{q}\Big) \quad\text{and}\quad \tilde{\sigma}\geq\frac{n+1}{2}\Big(\frac{1}{2}-\frac{1}{\tilde{q}}\Big),$$

and the couples $(1/p,1/q)$ and $(1/\tilde{p},1/\tilde{q})$ belong to the triangle

$$\Big\{\Big(\frac{1}{p},\frac{1}{q}\Big)\in\Big(0,\frac{1}{2}\Big]\times\Big(0,\frac{1}{2}\Big)\;\Big|\;\frac{1}{p}\geq\frac{n-1}{2}\Big(\frac{1}{2}-\frac{1}{q}\Big)\Big\}\cup\Big\{\Big(0,\frac{1}{2}\Big)\Big\}. \tag{13}$$

These estimates are similar to those obtained in [Anker et al. 2012] for the shifted wave equation except that they involve standard Sobolev spaces and no exotic ones. Notice that the range (13) of admissible couples for $\mathbb{H}^n$ is substantially wider than the range (2) for $\mathbb{R}^n$, which corresponds to the lower edge of the triangle (13).

In Section 6, we apply the results of the previous sections to the problem of global existence with small data for the corresponding semilinear equations. In contrast with the Euclidean case, where the range of admissible nonlinearities $F(u)\sim|u|^\gamma$ is restricted to $\gamma>\gamma_0$, we prove global well-posedness for powers $\gamma>1$ arbitrarily close to 1. This result improves in particular [Metcalfe and Taylor 2011], where global well-posedness for (6) was obtained in the case $n=3$ and $\kappa=\rho$ under the assumption $\gamma>\frac{5}{3}$.

As already observed for the Schrödinger equation [Anker and Pierfelice 2009; Anker et al. 2011] and for the shifted wave equation [Anker et al. 2012; 2014], the fact that better results hold for $\mathbb{H}^n$ than for $\mathbb{R}^n$ may be regarded as a consequence of the stronger dispersion properties in negative curvature. The final section is the Appendix, where we estimate some oscillatory integrals occurring in the kernel analysis carried out in Section 3.

## 2. Essentials about real hyperbolic spaces

In this paper, we consider the simplest class of Riemannian symmetric spaces of the noncompact type, namely real hyperbolic spaces $\mathbb{H}^n$ of dimension $n\geq 2$. We refer to Helgason's books [2001; 2000; 1994] and to Koornwinder's survey [1984] for their algebraic structure and geometric properties as well as for harmonic analysis on these spaces, and we shall be content with the following information. $\mathbb{H}^n$ can be realized as the symmetric space $G/K$, where $G=\mathrm{SO}(1,n)_0$ and $K=\mathrm{SO}(n)$. In geodesic polar coordinates, the Laplace–Beltrami operator on $\mathbb{H}^n$ writes

$$\Delta_{\mathbb{H}^n}=\partial_r^2+(n-1)\coth r\,\partial_r+\sinh^{-2}r\,\Delta_{\mathbb{S}^{n-1}}.$$

The spherical functions $\varphi_\lambda$ on $\mathbb{H}^n$ are normalized radial eigenfunctions of $\Delta = \Delta_{\mathbb{H}^n}$:

$$\begin{cases} \Delta\varphi_\lambda = -(\lambda^2+\rho^2)\varphi_\lambda, \\ \varphi_\lambda(0) = 1, \end{cases}$$

where $\lambda \in \mathbb{C}$ and $\rho = (n-1)/2$. They can be expressed in terms of special functions:

$$\varphi_\lambda(r) = \phi_\lambda^{(n/2-1,-1/2)}(r) = {}_2F_1\big(\tfrac{\rho}{2} + i\tfrac{\lambda}{2}, \tfrac{\rho}{2} - i\tfrac{\lambda}{2}; \tfrac{n}{2}; -\sinh^2 r\big),$$

where $\phi_\lambda^{(\alpha,\beta)}$ denotes the Jacobi functions and ${}_2F_1$ the Gauss hypergeometric function. In the sequel, we shall use the Harish-Chandra formula

$$\varphi_\lambda(r) = \int_K dk\, e^{-(\rho+i\lambda)\mathrm{H}(a_{-r}k)} \tag{14}$$

and the basic estimate

$$|\varphi_\lambda(r)| \le \varphi_0(r) \lesssim (1+r)e^{-\rho r} \quad \forall \lambda \in \mathbb{R},\ r \ge 0. \tag{15}$$

We shall also use the Harish-Chandra expansion

$$\varphi_\lambda(r) = \boldsymbol{c}(\lambda)\Phi_\lambda(r) + \boldsymbol{c}(-\lambda)\Phi_{-\lambda}(r) \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{Z},\ r > 0, \tag{16}$$

where the Harish-Chandra $\boldsymbol{c}$-function is given by

$$\boldsymbol{c}(\lambda) = \frac{\Gamma(2\rho)}{\Gamma(\rho)} \frac{\Gamma(i\lambda)}{\Gamma(i\lambda+\rho)} \tag{17}$$

and

$$\begin{aligned} \Phi_\lambda(r) &= (2\sinh r)^{i\lambda-\rho}{}_2F_1\big(\tfrac{\rho}{2} - i\tfrac{\lambda}{2}, -\tfrac{\rho-1}{2} - i\tfrac{\lambda}{2}; 1 - i\lambda; -\sinh^{-2} r\big) \\ &= (2\sinh r)^{-\rho} e^{i\lambda r} \sum_{k=0}^{+\infty} \Gamma_k(\lambda) e^{-2kr} \\ &\sim e^{(i\lambda-\rho)r} \quad \text{as } r \to +\infty. \end{aligned} \tag{18}$$

The coefficients $\Gamma_k(\lambda)$ in the expansion (18) are rational functions of $\lambda \in \mathbb{C}$ that satisfy the recurrence formula

$$\begin{cases} \Gamma_0(\lambda) = 1, \\ \Gamma_k(\lambda) = \dfrac{\rho(\rho-1)}{k(k-i\lambda)} \displaystyle\sum_{j=0}^{k-1} (k-j)\Gamma_j(\lambda). \end{cases}$$

Their classical estimates were improved as follows in [Anker et al. 2011, Lemma 2.1].

**Lemma 2.1.** *Let* $0 < \varepsilon < 1$ *and* $\Omega_\varepsilon = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \le \varepsilon|\lambda|,\ \operatorname{Im}\lambda \le -1+\varepsilon\}$. *Then there exist* $\nu \ge 0$ *and, for every* $\ell \in \mathbb{N}$, $C_\ell \ge 0$ *such that*

$$|\partial_\lambda^\ell \Gamma_k(\lambda)| \le C_\ell k^\nu (1+|\lambda|)^{-\ell-1} \quad \forall k \in \mathbb{N}^*,\ \lambda \in \mathbb{C} \setminus \Omega_\varepsilon. \tag{19}$$

Under suitable assumptions, the spherical Fourier transform of a bi-$K$-invariant function $f$ on $G$ is defined by

$$\mathcal{H}f(\lambda) = \int_G dg\, f(g)\varphi_\lambda(g),$$

and the following inversion formula holds:

$$f(x) = \text{const} \int_0^{+\infty} d\lambda\, |\boldsymbol{c}(\lambda)|^{-2} \mathcal{H} f(\lambda) \varphi_\lambda(x).$$

Here is a well-known estimate of the Plancherel density:

$$|\boldsymbol{c}(\lambda)|^{-2} \lesssim |\lambda|^2 (1 + |\lambda|)^{n-3} \quad \forall \lambda \in \mathbb{R}. \tag{20}$$

Via the spherical Fourier transform, the Laplace–Beltrami operator $\Delta$ corresponds to

$$-\lambda^2 - \rho^2$$

and hence the operators $D = \sqrt{-\Delta - \rho^2 + \kappa^2}$ and $\widetilde{D} = \sqrt{-\Delta - \rho^2 + \tilde{\kappa}^2}$ to

$$\sqrt{\lambda^2 + \kappa^2} \quad \text{and} \quad \sqrt{\lambda^2 + \tilde{\kappa}^2}.$$

Recall eventually the definition of Sobolev spaces on $\mathbb{H}^n$ and the Sobolev embedding theorem. We refer to [Triebel 1992] for more details about function spaces on Riemannian manifolds. Let $\sigma \in \mathbb{R}$ and $1 < q < \infty$. Then $H^{\sigma,q}(\mathbb{H}^n)$ denotes the image of $L^q(\mathbb{H}^n)$ under $(-\Delta)^{-\sigma/2}$ (in the space of distributions on $\mathbb{H}^n$) equipped with the norm

$$\|f\|_{H^{\sigma,q}} = \|(-\Delta)^{\sigma/2} f\|_{L^q}.$$

In this definition, we may replace $(-\Delta)^{-\sigma/2}$ by $D^{-\sigma} = (-\Delta - \rho^2 + \kappa^2)^{-\sigma/2}$ as long as $\kappa > 2|1/2 - 1/q|\rho$ and in particular by $\widetilde{D}^{-\sigma} = (-\Delta - \rho^2 + \tilde{\kappa}^2)^{-\sigma/2}$ since $\tilde{\kappa} > \rho$. If $\sigma = N$ is a nonnegative integer, then $H^{\sigma,q}(\mathbb{H}^n)$ coincides with the Sobolev space

$$W^{N,q}(\mathbb{H}^n) = \{ f \in L^q(\mathbb{H}^n) \mid \nabla^j f \in L^q(\mathbb{H}^n) \ \ \forall j,\ 1 \le j \le N \}$$

defined in terms of covariant derivatives. In the $L^2$ setting, we write $H^\sigma(\mathbb{H}^n)$ instead of $H^{\sigma,2}(\mathbb{H}^n)$.

**Proposition 2.2.** *Let $1 < q_1, q_2 < \infty$ and $\sigma_1, \sigma_2 \in \mathbb{R}$ such that $\sigma_1 - \sigma_2 \ge n/q_1 - n/q_2 \ge 0$. Then*

$$H^{\sigma_1,q_1}(\mathbb{H}^n) \subset H^{\sigma_2,q_2}(\mathbb{H}^n).$$

*By this inclusion, we mean that there exists a constant $C \ge 0$ such that*

$$\|f\|_{H^{\sigma_2,q_2}} \le C \|f\|_{H^{\sigma_1,q_1}} \quad \forall f \in C_c^\infty(\mathbb{H}^n).$$

## 3. Kernel estimates

In this section, we derive pointwise estimates for the radial convolution kernel $w_t^\sigma$ of the operator $W_t^\sigma = \widetilde{D}^{-\sigma} e^{itD}$ for suitable exponents $\sigma \in \mathbb{R}$. By the inversion formula of the spherical Fourier transform,

$$w_t^\sigma(r) = \text{const} \int_{-\infty}^{+\infty} d\lambda\, |\boldsymbol{c}(\lambda)|^{-2} (\lambda^2 + \tilde{\kappa}^2)^{-\sigma/2} \varphi_\lambda(r) e^{it\sqrt{\lambda^2+\kappa^2}}.$$

Contrarily to the Euclidean case, this kernel has different behaviors depending on whether $t$ is small or large, and therefore, we cannot use any rescaling. Let us split up

$$\begin{aligned} w_t^\sigma(r) &= w_t^{\sigma,0}(r) + w_t^{\sigma,\infty}(r) \\ &= \text{const} \int_{-\infty}^{+\infty} d\lambda\, \chi_0(\lambda) |\boldsymbol{c}(\lambda)|^{-2} (\lambda^2+\tilde{\kappa}^2)^{-\sigma/2} \varphi_\lambda(r) e^{it\sqrt{\lambda^2+\kappa^2}} \\ &\quad + \text{const} \int_{-\infty}^{+\infty} d\lambda\, \chi_\infty(\lambda) |\boldsymbol{c}(\lambda)|^{-2} (\lambda^2+\tilde{\kappa}^2)^{-\sigma/2} \varphi_\lambda(r) e^{it\sqrt{\lambda^2+\kappa^2}} \end{aligned}$$

using smooth, even cut-off functions $\chi_0$ and $\chi_\infty$ on $\mathbb{R}$ such that

$$\chi_0(\lambda) + \chi_\infty(\lambda) = 1 \quad \text{and} \quad \begin{cases} \chi_0(\lambda) = 1 & \forall |\lambda| \le \kappa, \\ \chi_\infty(\lambda) = 1 & \forall |\lambda| \ge \kappa + 1. \end{cases}$$

We shall first estimate $w_t^{\sigma,0}$ and next a variant of $w_t^{\sigma,\infty}$. The kernel $w_t^{\sigma,\infty}$ has indeed a logarithmic singularity on the sphere $r = t$ when $\sigma = (n+1)/2$. We bypass this problem by considering the analytic family of operators

$$\widetilde{W}_t^{\sigma,\infty} = \frac{e^{\sigma^2}}{\Gamma((n+1)/2-\sigma)} \chi_\infty(D) \widetilde{D}^{-\sigma} e^{itD}$$

in the vertical strip $0 \le \operatorname{Re}\sigma \le (n+1)/2$ and the corresponding kernels

$$\widetilde{w}_t^{\sigma,\infty}(r) = \text{const} \frac{e^{\sigma^2}}{\Gamma((n+1)/2-\sigma)} \int_{-\infty}^{+\infty} d\lambda\, \chi_\infty(\lambda) |\boldsymbol{c}(\lambda)|^{-2} (\lambda^2+\tilde{\kappa}^2)^{-\sigma/2} \varphi_\lambda(r) e^{it\sqrt{\lambda^2+\kappa^2}}. \tag{21}$$

Notice that the gamma function (which occurs naturally in the theory of Riesz distributions) will allow us to deal with the boundary point $\sigma = (n+1)/2$ while the exponential function yields boundedness at infinity in the vertical strip.

### 3A. *Estimate of $w_t^0 = w_t^{\sigma,0}$.*

**Theorem 3.1.** *Let $\sigma \in \mathbb{R}$. The following pointwise estimates hold for the kernel $w_t^0$:*

(i) *For every $t \in \mathbb{R}$ and $r \ge 0$, we have*

$$|w_t^0(r)| \lesssim \varphi_0(r).$$

(ii) *Assume that $|t| \ge 2$. Then for every $0 \le r \le |t|/2$, we have*

$$|w_t^0(r)| \lesssim |t|^{-3/2}(1+r)\varphi_0(r).$$

*Proof.* Recall that

$$w_t^0(r) = \text{const} \int_{-\kappa-1}^{\kappa+1} d\lambda\, \chi_0(\lambda) |\boldsymbol{c}(\lambda)|^{-2} (\lambda^2+\tilde{\kappa}^2)^{-\sigma/2} \varphi_\lambda(r) e^{it\sqrt{\lambda^2+\kappa^2}}. \tag{22}$$

By symmetry, we may assume that $t > 0$.

It follows from the estimates (15) and (20) that

$$|w_t^0(r)| \lesssim \int_{-\kappa-1}^{\kappa+1} d\lambda\, \lambda^2 \varphi_0(r) \lesssim \varphi_0(r),$$

proving (i). We prove (ii) by substituting in (22) the first integral representation of $\varphi_\lambda$ in (14) and by reducing in this way to Fourier analysis on $\mathbb{R}$. Specifically,

$$w_t^0(r) = \int_K dk\, e^{-\rho \mathrm{H}(a_{-r}k)} \int_{-\infty}^{+\infty} d\lambda\, a(\lambda) e^{it(\sqrt{\lambda^2+\kappa^2}-\mathrm{H}(a_{-r}k)\lambda/t)},$$

where $a(\lambda) = \text{const}\, \chi_0(\lambda)|\boldsymbol{c}(\lambda)|^{-2}(\lambda^2+\tilde{\kappa}^2)^{-\sigma/2}$. Since

$$\int_K dk\, e^{-\rho \mathrm{H}(a_{-r}k)} = \varphi_0(r)$$

and $|\mathrm{H}(a_{-r}k)| \le r$, it remains for us to estimate the oscillatory integral

$$I(t,x) = \int_{-\infty}^{+\infty} d\lambda\, a(\lambda) e^{it(\sqrt{\lambda^2+\kappa^2}-x\lambda/t)}$$

by $|t|^{-3/2}(1+|x|)$. This is obtained by the method of stationary phase. More precisely, we apply Lemma A.1 in the Appendix, whose assumption (A-1) is fulfilled according to (20). □

### 3B. *Estimate of* $\tilde{w}_t^\infty = \tilde{w}_t^{\sigma,\infty}$.

**Theorem 3.2.** *The following pointwise estimates hold for the kernel* $\tilde{w}_t^\infty$, *uniformly in* $\sigma \in \mathbb{C}$ *with* $\operatorname{Re}\sigma = (n+1)/2$:

(i) *Assume that* $|t| \ge 2$. *Then for every* $r \ge 0$, *we have*

$$|\tilde{w}_t^\infty(r)| \lesssim |t|^{-\infty}.$$

(ii) *Assume that* $0 < |t| \le 2$.

(a) *If* $r \ge 3$, *then* $\tilde{w}_t^\infty(r) = \mathrm{O}(r^{-1}e^{-\rho r})$.

(b) *If* $0 \le r \le 3$, *then* $|\tilde{w}_t^\infty(r)| \lesssim \begin{cases} |t|^{-(n-1)/2} & \text{if } n \ge 3, \\ |t|^{-1/2}(1-\log|t|) & \text{if } n = 2. \end{cases}$

Throughout the proof of Theorem 3.2, we may assume again by symmetry that $t > 0$.

*Proof of Theorem 3.2(i).* By evenness, we have

$$\tilde{w}_t^\infty(r) = 2\,\text{const} \frac{e^{\sigma^2}}{\Gamma((n+1)/2-\sigma)} \int_0^{+\infty} d\lambda\, \chi_\infty(\lambda)|\boldsymbol{c}(\lambda)|^{-2}(\lambda^2+\tilde{\kappa}^2)^{-\sigma/2}\varphi_\lambda(r) e^{it\sqrt{\lambda^2+\kappa^2}}. \tag{23}$$

If $0 \le r \le t/2$, we resume the proof of Theorem 3.1(ii), using Lemma A.2 instead of Lemma A.1, and conclude that

$$|\tilde{w}_t^\infty(r)| \lesssim t^{-\infty}\varphi_0(r). \tag{24}$$

If $r \ge t/2$, we substitute in (23) the Harish-Chandra expansion (16) of $\varphi_\lambda(r)$ and reduce this way again to Fourier analysis on $\mathbb{R}$. Specifically, our task consists in estimating the expansion

$$\tilde{w}_t^\infty(r) = (\sinh r)^{-\rho} \sum_{k=0}^{+\infty} e^{-2kr}\{I_k^{+,\infty}(t,r) + I_k^{-,\infty}(t,r)\} \tag{25}$$

involving oscillatory integrals

$$I_k^{\pm,\infty}(t,r)=\int_0^{+\infty}d\lambda\,a_k^{\pm}(\lambda)e^{i(t\sqrt{\lambda^2+\kappa^2}\pm r\lambda)}$$

with amplitudes

$$a_k^{\pm}(\lambda)=2\,\text{const}\frac{e^{\sigma^2}}{\Gamma((n+1)/2-\sigma)}\chi_\infty(\lambda)\boldsymbol{c}(\mp\lambda)^{-1}(\lambda^2+\tilde{\kappa}^2)^{-\sigma/2}\Gamma_k(\pm\lambda).$$

Notice that $a_k^{\pm}(\lambda)$ is a symbol of order

$$d=\begin{cases}-1 & \text{if } k=0,\\ -2 & \text{if } k\in\mathbb{N}^*\end{cases}$$

uniformly in $\sigma\in\mathbb{C}$ with $\operatorname{Re}\sigma=(n+1)/2$. This follows indeed from the expression (17) of the $\boldsymbol{c}$-function and from the estimate (19) of the coefficients $\Gamma_k$. Consequently, the integrals

$$I_k^{\pm,\infty}(t,r)=\mathrm{O}(k^{\nu})\tag{26}$$

are easy to estimate when $k>0$ while $I_0^{+,\infty}(t,r)$ and especially $I_0^{-,\infty}(t,r)$ require more work. As far as the penultimate integral is concerned, we integrate by parts

$$I_0^{+,\infty}(t,r)=\int_0^{+\infty}d\lambda\,a_0^{+}(\lambda)e^{it\phi(\lambda)},$$

using $e^{it\phi(\lambda)}=(it\phi'(\lambda))^{-1}\frac{\partial}{\partial\lambda}e^{it\phi(\lambda)}$ and the following properties of $\phi(\lambda)=\sqrt{\lambda^2+\kappa^2}+(r/t)\lambda$:

- $\phi'(\lambda)=\dfrac{\lambda}{\sqrt{\lambda^2+\kappa^2}}+\dfrac{r}{t}\geq\dfrac{r}{t}\geq\dfrac{1}{2}$,
- $\phi''(\lambda)=\kappa^2(\lambda^2+\kappa^2)^{-3/2}$ is a symbol of order $-3$.

We obtain this way

$$I_0^{+,\infty}(t,r)=\mathrm{O}(r^{-1})\tag{27}$$

and actually

$$I_0^{+,\infty}(t,r)=\mathrm{O}(r^{-\infty})$$

by repeated integrations by parts. Let us turn to the last integral, which we rewrite as follows:

$$I_0^{-,\infty}(t,r)=\int_0^{+\infty}d\lambda\,a_0^{-}(\lambda)e^{it(\sqrt{\lambda^2+\kappa^2}-\lambda)}e^{i(t-r)\lambda}.$$

After performing an integration by parts based on $e^{i(t-r)\lambda}=-i(t-r)^{-1}\frac{\partial}{\partial\lambda}e^{i(t-r)\lambda}$ and by using the fact that

$$\psi(\lambda)=\sqrt{\lambda^2+\kappa^2}-\lambda=\frac{\kappa^2}{\sqrt{\lambda^2+\kappa^2}+\lambda}\tag{28}$$

is a symbol of order $-1$, we obtain

$$I_0^{-,\infty}(t,r)=\mathrm{O}\Big(\frac{t}{|r-t|}\Big).\tag{29}$$

This estimate is enough for our purpose as long as $r$ stays away from $t$. If $|r-t|\leq 1$, let us split up

$$e^{it\psi(\lambda)}=1+\mathrm{O}(t\psi(\lambda))$$

and

$$I_0^{-,\infty}(t,r)=\int_0^{+\infty}d\lambda\, a_0^-(\lambda)e^{i(t-r)\lambda}+\mathrm{O}(t) \tag{30}$$

accordingly. The remaining integral was estimated in [Anker et al. 2011] at the end of the proof of Theorem 4.2(ii):

$$\int_0^{+\infty}d\lambda\, a_0^-(\lambda)e^{i(t-r)\lambda}=\mathrm{O}(1). \tag{31}$$

By combining the estimates (26), (27), (29), (30), and (31), we deduce from (25) that

$$|\widetilde{w}_t^{\infty}(r)|\lesssim e^{-\rho r}t\lesssim t^{-\infty}\quad \forall r\geq \frac{t}{2}\geq 1$$

uniformly in $\sigma\in\mathbb{C}$ with $\operatorname{Re}\sigma=(n+1)/2$. This concludes the proof of Theorem 3.2(i). □

Let us turn to the small time estimates in Theorem 3.2.

*Proof of Theorem 3.2(ii)(a).* Since $0<t\leq 2$ and $r\geq 3$, we can resume the proof of Theorem 3.2(i) in the case $r\geq t+1\geq t/2$. By using the expansion (25) and the estimates (26), (27), and (29), we obtain

$$|\widetilde{w}_t^{\infty}(r)|\lesssim r^{-1}e^{-\rho r}$$

uniformly in $\sigma\in\mathbb{C}$ with $\operatorname{Re}\sigma=(n+1)/2$. This concludes the proof of Theorem 3.2(ii)(a). □

*Proof of Theorem 3.2(ii)(b).* Let us rewrite and expand (23) as follows:

$$\widetilde{w}_t^{\infty}(r)=2\,\mathrm{const}\frac{e^{\sigma^2}}{\Gamma((n+1)/2-\sigma)}\int_0^{+\infty}d\lambda\,\chi_\infty(\lambda)|\boldsymbol{c}(\lambda)|^{-2}(\lambda^2+\tilde{\kappa}^2)^{-\sigma/2}e^{it\psi(\lambda)}e^{it\lambda}\varphi_\lambda(r) \tag{32}$$

$$=\int_0^{+\infty}d\lambda\, a(\lambda)e^{it\lambda}\varphi_\lambda(r)+\int_0^{+\infty}d\lambda\, b(\lambda)e^{it\lambda}\varphi_\lambda(r), \tag{33}$$

where $\psi$ is given by (28),

$$a(\lambda)=2\,\mathrm{const}\frac{e^{\sigma^2}}{\Gamma((n+1)/2-\sigma)}\chi_\infty(\lambda)|\boldsymbol{c}(\lambda)|^{-2}(\lambda^2+\tilde{\kappa}^2)^{-\sigma/2}$$

is a symbol of order $(n-3)/2$, uniformly in $\sigma\in\mathbb{C}$ with $\operatorname{Re}\sigma=(n+1)/2$, and

$$b(\lambda)=2\,\mathrm{const}\frac{e^{\sigma^2}}{\Gamma((n+1)/2-\sigma)}\chi_\infty(\lambda)|\boldsymbol{c}(\lambda)|^{-2}(\lambda^2+\tilde{\kappa}^2)^{-\sigma/2}\{e^{it\psi(\lambda)}-1\}$$

is a symbol of $(n-5)/2$, uniformly in $0<t\leq 2$ and $\sigma\in\mathbb{C}$ with $\operatorname{Re}\sigma=(n+1)/2$. The first integral in (33) was analyzed in [Anker et al. 2011, Appendix C] and estimated there by

$$C\begin{cases} t^{-(n-1)/2} & \text{if } n\geq 3,\\ t^{-1/2}(1-\log|t|) & \text{if } n=2.\end{cases}$$

The second integral is easier to estimate for instance by $Ct^{-(n-2)/2}$. This concludes the proof of Theorem 3.2(ii)(b). □

**Remark 3.3.** As far as local estimates of wave kernels are concerned, we might have used the *Hadamard parametrix* [Hörmander 2007, §17.4] instead of spherical analysis.

**Remark 3.4.** The kernel analysis carried out in this section still holds for the operators $D^{-\sigma}\widetilde{D}^{-\widetilde{\sigma}}e^{itD}$, provided that we assume $\operatorname{Re}\sigma+\operatorname{Re}\widetilde{\sigma}=(n+1)/2$ in Theorem 3.2.

## 4. Dispersive estimates

In this section, we obtain $L^{q'}\to L^q$ estimates for the operator $W_t^\sigma=\widetilde{D}^{-\sigma}e^{itD}$, which will be crucial for our Strichartz estimates in next section. Let us split up its kernel $w_t^\sigma=w_t^{\sigma,0}+w_t^{\sigma,\infty}$ as before. We will handle the contribution of $w_t^{\sigma,0}$, using the pointwise estimates obtained in Section 3A and the following criterion (see for instance [Anker et al. 2011, Theorem 3.4]) based on the Kunze–Stein phenomenon:

**Lemma 4.1.** *There exists a constant $C>0$ such that, for every radial measurable function $\kappa$ on $\mathbb{H}^n$ and for every $2\le q<\infty$ and $f\in L^{q'}(\mathbb{H}^n)$,*

$$\|f*\kappa\|_{L^q}\le C_q\|f\|_{L^{q'}}\left\{\int_0^{+\infty}dr\,(\sinh r)^{n-1}|\kappa(r)|^{q/2}\varphi_0(r)\right\}^{2/q}.$$

For the second part $w_t^{\sigma,\infty}$, we resume the Euclidean approach, which consists of interpolating analytically between $L^2\to L^2$ and $L^1\to L^\infty$ estimates for the family of operators

$$\widetilde{W}_t^{\sigma,\infty}=\frac{e^{\sigma^2}}{\Gamma((n+1)/2-\sigma)}\chi_\infty(D)\widetilde{D}^{-\sigma}e^{itD}\tag{34}$$

in the vertical strip $0\le\operatorname{Re}\sigma\le(n+1)/2$.

### 4A. *Small-time dispersive estimates.*

**Theorem 4.2.** *Assume that $0<|t|\le 2$, $2<q<\infty$, and $\sigma\ge(n+1)(1/2-1/q)$. Then*

$$\|\widetilde{D}^{-\sigma}e^{itD}\|_{L^{q'}\to L^q}\lesssim\begin{cases}|t|^{-(n-1)(1/2-1/q)} & \text{if } n\ge 3,\\ |t|^{-(1/2-1/q)}(1-\log|t|)^{1-2/q} & \text{if } n=2.\end{cases}$$

*Proof.* We divide the proof into two parts, corresponding to the kernel decomposition $w_t=w_t^0+w_t^\infty$. By applying Lemma 4.1 and using the pointwise estimates in Theorem 3.1(i), we obtain on one hand

$$\begin{aligned}\|f*w_t^0\|_{L^q}&\lesssim\left\{\int_0^{+\infty}dr\,(\sinh r)^{n-1}\varphi_0(r)|w_t^0(r)|^{q/2}\right\}^{2/q}\|f\|_{L^{q'}}\\&\lesssim\left\{\int_0^{+\infty}dr\,(1+r)^{q/2+1}e^{-(q/2-1)\rho r}\right\}^{2/q}\|f\|_{L^{q'}}\\&\lesssim\|f\|_{L^{q'}}\quad\forall f\in L^{q'}.\end{aligned}$$

On the other hand, in order to estimate the $L^{q'}\to L^q$ norm of $f\mapsto f*w_t^\infty$, we proceed by interpolation for the analytic family (34). If $\operatorname{Re}\sigma=0$, then

$$\|f*\widetilde{w}_t^\infty\|_{L^2}\lesssim\|f\|_{L^2}\quad\forall f\in L^2.$$

If $\operatorname{Re}\sigma=(n+1)/2$, we deduce from the pointwise estimates in Theorem 3.2(ii) that

$$\|f*\widetilde{w}_t^{\infty}\|_{L^\infty}\lesssim|t|^{-(n-1)/2}\|f\|_{L^1}\quad\forall f\in L^1.$$

By interpolation, we conclude for $\sigma=(n+1)(1/2-1/q)$ that

$$\|f*w_t^{\infty}\|_{L^q}\lesssim|t|^{-(n-1)(1/2-1/q)}\|f\|_{L^{q'}}\quad\forall f\in L^{q'}.$$

□

**4B.** ***Large-time dispersive estimate.***

**Theorem 4.3.** *Assume that* $|t|\geq 2$, $2<q<\infty$, *and* $\sigma\geq(n+1)(1/2-1/q)$. *Then*

$$\|\widetilde{D}^{-\sigma}e^{itD}\|_{L^{q'}\to L^q}\lesssim|t|^{-3/2}.$$

*Proof.* We divide the proof into three parts, corresponding to the kernel decomposition

$$w_t=\mathbb{1}_{B(0,|t|/2)}w_t^0+\mathbb{1}_{\mathbb{H}^n\setminus B(0,|t|/2)}w_t^0+w_t^{\infty}.$$

*Estimate 1.* By applying Lemma 4.1 and using the pointwise estimate in Theorem 3.1(ii), we obtain

$$\begin{aligned}\|f*\{\mathbb{1}_{B(0,|t|/2)}w_t^0\}\|_{L^q}&\lesssim\left\{\int_0^{|t|/2}dr\,(\sinh r)^{n-1}\varphi_0(r)|w_t^0(r)|^{q/2}\right\}^{2/q}\|f\|_{L^{q'}}\\&\lesssim\underbrace{\left\{\int_0^{|t|/2}dr\,(1+r)^{q+1}e^{-(q/2-1)\rho r}\right\}^{2/q}}_{<+\infty}|t|^{-3/2}\|f\|_{L^{q'}}\quad\forall f\in L^{q'}.\end{aligned}$$

*Estimate 2.* By applying Lemma 4.1 and using the pointwise estimate in Theorem 3.1(i), we obtain

$$\begin{aligned}\|f*\{\mathbb{1}_{\mathbb{H}^n\setminus B(0,|t|/2)}w_t^0\}\|_{L^q}&\lesssim\left\{\int_{|t|/2}^{+\infty}dr\,(\sinh r)^{n-1}\varphi_0(r)|w_t^0(r)|^{q/2}\right\}^{2/q}\|f\|_{L^{q'}}\\&\lesssim\underbrace{\left\{\int_{|t|/2}^{+\infty}dr\,r^{q/2+1}e^{-(q/2-1)\rho r}\right\}^{2/q}}_{\lesssim|t|^{-\infty}}\|f\|_{L^{q'}}\quad\forall f\in L^{q'}.\end{aligned}$$

*Estimate 3.* We proceed by interpolation for the analytic family (34). If $\operatorname{Re}\sigma=0$, then

$$\|f*\widetilde{w}_t^{\infty}\|_{L^2}\lesssim\|f\|_{L^2}\quad\forall f\in L^2.$$

If $\operatorname{Re}\sigma=(n+1)/2$, we deduce from Theorem 3.2(i) that

$$\|f*\widetilde{w}_t^{\infty}\|_{L^\infty}\lesssim|t|^{-\infty}\|f\|_{L^1}\quad\forall f\in L^1.$$

By interpolation, we obtain for $\sigma=(n+1)(1/2-1/q)$ that

$$\|f*w_t^{\infty}\|_{L^q}\lesssim|t|^{-\infty}\|f\|_{L^{q'}}\quad\forall f\in L^{q'}.$$

We conclude the proof of Theorem 4.3 by summing up the previous estimates. □

**4C.** ***Global dispersive estimates.*** As noticed in Remark 3.4, similar results hold for the operators $D^{-\sigma}\widetilde{D}^{-\widetilde{\sigma}}e^{itD}$.

**Corollary 4.4.** *Let $2 < q < \infty$ and $\sigma, \widetilde{\sigma} \in \mathbb{R}$ such that $\sigma + \widetilde{\sigma} \geq (n+1)(1/2 - 1/q)$. Then*

$$\|D^{-\sigma}\widetilde{D}^{-\widetilde{\sigma}}e^{itD}\|_{L^{q'}\to L^q} \lesssim \begin{cases} |t|^{-(n-1)(1/2-1/q)} & \text{if } 0 < |t| \leq 1, \\ |t|^{-3/2} & \text{if } |t| \geq 1. \end{cases} \tag{35}$$

*In particular, if $2 < q < \infty$ and $\sigma \geq (n+1)(1/2 - 1/q)$, then*

$$\|\widetilde{D}^{-\sigma}e^{itD}\|_{L^{q'}\to L^q} + \Big\|\widetilde{D}^{1-\sigma}\frac{e^{itD}}{D}\Big\|_{L^{q'}\to L^q} \lesssim \begin{cases} |t|^{-(n-1)(1/2-1/q)} & \text{if } 0 < |t| \leq 1, \\ |t|^{-3/2} & \text{if } |t| \geq 1. \end{cases} \tag{36}$$

*These results hold in dimension $n \geq 3$. In dimension $n = 2$, there is an additional logarithmic factor in the small time bound, which becomes $|t|^{-(1/2-1/q)}(1 - \log|t|)^{1-2/q}$.*

**Remark 4.5.** On $L^2(\mathbb{H}^n)$, we know by spectral theory that

- $e^{itD}$ is a 1-parameter group of unitary operators,
- $D^{-\sigma}\widetilde{D}^{-\widetilde{\sigma}}$ is a bounded operator if $\sigma + \widetilde{\sigma} \geq 0$.

**Remark 4.6.** Let us specialize our results for the wave equation (6). In this case, we have $D = \sqrt{-\Delta}$, and we may take $\widetilde{D} = D$. Let $2 < q < \infty$ and $\sigma \geq (n+1)(1/2 - 1/q)$. Then

$$\|D^{-\sigma}e^{itD}\|_{L^{q'}\to L^q} \lesssim \begin{cases} |t|^{-(n-1)(1/2-1/q)} & \text{if } 0 < |t| \leq 1, \\ |t|^{-3/2} & \text{if } |t| \geq 1 \end{cases} \tag{37}$$

in dimension $n \geq 3$ and

$$\|D^{-\sigma}e^{itD}\|_{L^{q'}\to L^q} \lesssim \begin{cases} |t|^{-(1/2-1/q)}(1 - \log|t|)^{1-2/q} & \text{if } 0 < |t| \leq 1, \\ |t|^{-3/2} & \text{if } |t| \geq 1 \end{cases}$$

in dimension $n = 2$. Let us compare (37) with the dispersive estimates by Metcalfe and Taylor [2011; 2012] in dimension $n = 3$. Actually, the weaker bound $|t|^{-6(1/2-1/q)}$, obtained in [Metcalfe and Taylor 2011, §3] when $|t|$ is large and $2 < q < 4$, was improved in [Metcalfe and Taylor 2012] after the release of a preprint version of the present paper. On the other hand, these authors are able to deal with the endpoint case $q = \infty$, using local Hardy and BMO spaces on $\mathbb{H}^n$.

## 5. Strichartz estimates

We shall assume $n \geq 4$ throughout this section and discuss the dimensions $n = 3$ and $n = 2$ in the final remarks. Consider the linear equation (12) on $\mathbb{H}^n$, whose solution is given by Duhamel's formula:

$$u(t,x) = \underbrace{(\cos tD_x)f(x) + \frac{\sin tD_x}{D_x}g(x)}_{u_{\text{hom}}(t,x)} + \underbrace{\int_0^t ds\,\frac{\sin(t-s)D_x}{D_x}F(s,x)}_{u_{\text{inhom}}(t,x)}.$$

**Definition 5.1.** A couple $(p, q)$ will be called admissible (see Figure 1) if $(1/p, 1/q)$ belongs to the triangle

$$\Big\{\Big(\frac{1}{p},\frac{1}{q}\Big) \in \Big(0,\frac{1}{2}\Big] \times \Big(0,\frac{1}{2}\Big) \,\Big|\, \frac{1}{p} \geq \frac{n-1}{2}\Big(\frac{1}{2}-\frac{1}{q}\Big)\Big\} \cup \Big\{\Big(0,\frac{1}{2}\Big)\Big\}. \tag{38}$$

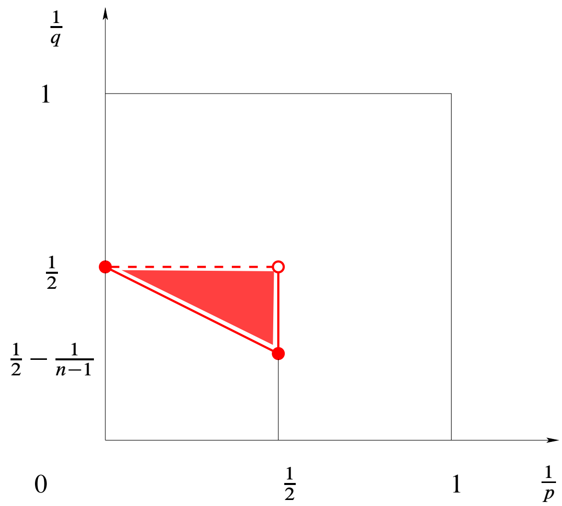


**Figure 1.** Admissibility in dimension $n \geq 4$.

**Theorem 5.2.** *Let $(p,q)$ and $(\tilde{p},\tilde{q})$ be two admissible couples, and let*

$$\sigma \geq \frac{n+1}{2}\Big(\frac{1}{2}-\frac{1}{q}\Big) \quad \text{and} \quad \tilde{\sigma} \geq \frac{n+1}{2}\Big(\frac{1}{2}-\frac{1}{\tilde{q}}\Big). \tag{39}$$

*Then the following Strichartz estimate holds for solutions to the Cauchy problem* (12):

$$\|\nabla_{\mathbb{R}\times\mathbb{H}^n} u\|_{L^p H^{-\sigma,q}} \lesssim \|f\|_{H^1} + \|g\|_{L^{p'}L^2} + \|F\|_{L^{\tilde{p}'}H^{\tilde{\sigma},\tilde{q}'}}. \tag{40}$$

*Proof.* We shall prove the following estimate, which amounts to (40):

$$\begin{aligned}\|\widetilde{D}_x^{-\sigma+1/2}u(t,x)\|_{L_t^pL_x^q} &+ \|\widetilde{D}_x^{-\sigma-1/2}\partial_t u(t,x)\|_{L_t^pL_x^q}\\ &\lesssim \|D_x^{1/2}f(x)\|_{L_x^2} + \|D_x^{-1/2}g(x)\|_{L_x^2} + \|\widetilde{D}_x^{\tilde{\sigma}-1/2}F(t,x)\|_{L_t^{\tilde{p}'}L_x^{\tilde{q}'}}.\end{aligned} \tag{41}$$

Consider the operator

$$Tf(t,x) = \widetilde{D}_x^{-\sigma+1/2}\frac{e^{\pm itD_x}}{\sqrt{D_x}}f(x),$$

initially defined from $L^2(\mathbb{H}^n)$ into $L^\infty(\mathbb{R};H^\sigma(\mathbb{H}^n))$, and its formal adjoint

$$T^*F(x) = \int_{-\infty}^{+\infty} ds\, \widetilde{D}_x^{-\sigma+1/2}\frac{e^{\mp isD_x}}{\sqrt{D_x}}F(s,x),$$

initially defined from $L^1(\mathbb{R};H^{-\sigma}(\mathbb{H}^n))$ into $L^2(\mathbb{H}^n)$. The $TT^*$ method consists in proving first the $L^{p'}(\mathbb{R};L^{q'}(\mathbb{H}^n)) \to L^p(\mathbb{R};L^q(\mathbb{H}^n))$ boundedness of the operator

$$TT^*F(t,x) = \int_{-\infty}^{+\infty} ds\, \widetilde{D}_x^{-2\sigma+1}\frac{e^{\pm i(t-s)D_x}}{D_x}F(s,x)$$

and of its truncated version

$$\mathcal{T}F(t,x)=\int_{-\infty}^{t}ds\,\widetilde{D}_x^{-2\sigma+1}\frac{e^{\pm i(t-s)D_x}}{D_x}F(s,x),$$

for every admissible couple $(p,q)$ and for every $\sigma\geq(n+1)(1/2-1/q)/2$, and in decoupling next the indices.

We may disregard the endpoint case $(p,q)=(\infty,2)$, which is easily dealt with, using the boundedness on $L^2(\mathbb{H}^n)$ of $e^{itD}$ ($t\in\mathbb{R}$) and $\widetilde{D}^{-\sigma+1/2}D^{-1/2}$ ($\sigma\geq 0$). Thus, assume that $(p,q)$ is an admissible couple that is different from the endpoints $(\infty,2)$ and $(2,2(n-1)/(n-3))$. It follows from (36) that the norms $\|TT^*F(t,x)\|_{L_t^pL_x^q}$ and $\|\mathcal{T}F(t,x)\|_{L_t^pL_x^q}$ are bounded above by

$$\biggl\|\int_{0<|t-s|<1}ds\,|t-s|^{-\alpha}\|F(s,x)\|_{L_x^{q'}}\biggr\|_{L_t^p}+\biggl\|\int_{|t-s|\geq 1}ds\,|t-s|^{-3/2}\|F(s,x)\|_{L_x^{q'}}\biggr\|_{L_t^p},\tag{42}$$

where $\alpha=(n-1)(1/2-1/q)\in(0,1)$. On one hand, the convolution kernel $|t-s|^{-3/2}\mathbb{1}_{\{|t-s|\geq 1\}}$ defines obviously a bounded operator from $L^{p_1}(\mathbb{R})$ to $L^{p_2}(\mathbb{R})$ for all $1\leq p_1\leq p_2\leq\infty$ in particular from $L^{p'}(\mathbb{R})$ to $L^p(\mathbb{R})$ since $p\geq 2$. On the other hand, the convolution kernel $|t-s|^{-\alpha}\mathbb{1}_{\{0<|t-s|\leq 1\}}$ with $0<\alpha<1$ defines a bounded operator from $L^{p_1}(\mathbb{R})$ to $L^{p_2}(\mathbb{R})$ for all $1<p_1,p_2<\infty$ such that $0\leq 1/p_1-1/p_2\leq 1-\alpha$ in particular from $L^{p'}(\mathbb{R})$ to $L^p(\mathbb{R})$ since $p\geq 2$ and $2/p\geq\alpha$.

At the endpoint $(p,q)=(2,2(n-1)/(n-3))$, we have $\alpha=1$. Thus, the previous argument breaks down and is replaced by the refined analysis carried out in [Keel and Tao 1998]. Notice that the problem lies only in the first part of (42) and not in the second one, which involves an integrable convolution kernel on $\mathbb{R}$.

Thus, $TT^*$ and $\mathcal{T}$ are bounded from $L^{p'}(\mathbb{R};L^{q'}(\mathbb{H}^n))$ to $L^p(\mathbb{R};L^q(\mathbb{H}^n))$ for every admissible couple $(p,q)$. As a consequence, $T^*$ is bounded from $L^{p'}(\mathbb{R};L^{q'}(\mathbb{H}^n))$ to $L^2(\mathbb{H}^n)$ and $T$ is bounded from $L^2(\mathbb{H}^n)$ to $L^p(\mathbb{R};L^q(\mathbb{H}^n))$. We deduce in particular that

$$\|\widetilde{D}_x^{-\sigma+1/2}(\cos tD_x)f(x)\|_{L_t^pL_x^q}\lesssim\|\widetilde{D}_{x\sigma+1/2}e^{\pm itD_x}f(x)\|_{L_t^pL_x^q}\lesssim\|D_x^{1/2}f(x)\|_{L_x^2}$$

and

$$\Bigl\|\widetilde{D}_x^{-\sigma+1/2}\frac{\sin tD_x}{D_x}g(x)\Bigr\|_{L_t^pL_x^q}\lesssim\|\widetilde{D}_x^{-\sigma+1/2}D_x^{-1}e^{\pm itD_x}g(x)\|_{L_t^pL_x^q}\lesssim\|D_x^{-1/2}g(x)\|_{L_x^2}.$$

In summary,

$$\|\widetilde{D}_x^{-\sigma+1/2}u_{\mathrm{hom}}(t,x)\|_{L_t^pL_x^q}\lesssim\|D_x^{1/2}f(x)\|_{L_x^2}+\|D_x^{-1/2}g(x)\|_{L_x^2}.\tag{43}$$

We next decouple the indices in the $L^{p'}L^{q'}\to L^qL^q$ estimate of $TT^*$ and $\mathcal{T}$. Let $(p,q)\neq(\widetilde{p},\widetilde{q})$ be two admissible couples, and let $\sigma\geq(n+1)(1/2-1/q)/2$ and $\widetilde{\sigma}\geq(n+1)(1/2-1/\widetilde{q})/2$. Since $T$ and $T^*$ are separately continuous, the operator

$$TT^*F(t,x)=\int_{-\infty}^{+\infty}ds\,\widetilde{D}_x^{-\sigma-\widetilde{\sigma}+1}\frac{e^{\pm i(t-s)D_x}}{D_x}F(s,x)$$

is bounded from $L^{\widetilde{p}'}(\mathbb{R};L^{\widetilde{q}'}(\mathbb{H}^n))$ to $L^p(\mathbb{R};L^q(\mathbb{H}^n))$. According to [Christ and Kiselev 2001], this result remains true for the truncated operator

$$\mathcal{T}F(t,x)=\int_{-\infty}^{t}ds\,\widetilde{D}_x^{-\sigma-\widetilde{\sigma}+1}\frac{e^{\pm i(t-s)D_x}}{D_x}F(s,x)$$

and hence for

$$\widetilde{\mathcal{T}}F(t,x)=\int_{0}^{t}ds\,\widetilde{D}_x^{-\sigma-\widetilde{\sigma}+1}\frac{\sin(t-s)D_x}{D_x}F(s,x)$$

as long as $p$ and $\widetilde{p}$ are not both equal to 2. For the remaining case, where $p=\widetilde{p}=2$ and $2<q\neq\widetilde{q}\leq 2(n-1)/(n-3)$, we argue as in the proof of [Anker et al. 2011, Theorem 6.3] by resuming part of the bilinear approach in [Keel and Tao 1998]. Hence,

$$\|\widetilde{D}_x^{-\sigma+1/2}u_{\mathrm{inhom}}(t,x)\|_{L_t^pL_x^q}\lesssim\|\widetilde{D}_x^{\widetilde{\sigma}-1/2}F(t,x)\|_{L_t^{\widetilde{p}'}L_x^{\widetilde{q}'}}\tag{44}$$

for all admissible couples $(p,q)$ and $(\widetilde{p},\widetilde{q})$.

The Strichartz estimate

$$\|\widetilde{D}_x^{-\sigma+1/2}u(t,x)\|_{L_t^pL_x^q}\lesssim\|D_x^{1/2}f(x)\|_{L_x^2}+\|D_x^{-1/2}g(x)\|_{L_x^2}+\|\widetilde{D}_x^{\widetilde{\sigma}-1/2}F(t,x)\|_{L_t^{\widetilde{p}'}L_x^{\widetilde{q}'}}$$

is obtained by summing up the homogeneous estimate (43) and the inhomogeneous estimate (44). As far as it is concerned, the Strichartz estimate of

$$\partial_t u(t,x)=-(\sin tD_x)D_xf(x)+(\cos tD_x)g(x)+\int_0^t ds\,[\cos(t-s)D_x]F(s,x)$$

is obtained in the same way and is actually easier. More precisely, we consider this time the operator

$$\widetilde{T}f(t,x)=\widetilde{D}_x^{-\sigma}e^{\pm itD_x}f(x)$$

and its adjoint

$$\widetilde{T}^*F(x)=\int_{-\infty}^{+\infty}ds\,\widetilde{D}_x^{-\sigma}e^{\mp isD_x}F(s,x). \qquad \square$$

By using the Sobolev embedding theorem, Theorem 5.2 can be extended to all couples $(1/p,1/q)$ and $(1/\widetilde{p},1/\widetilde{q})$ in the square

$$\left[0,\tfrac{1}{2}\right]\times\left(0,\tfrac{1}{2}\right)\cup\left\{\left(0,\tfrac{1}{2}\right)\right\}.\tag{45}$$

**Corollary 5.3.** *Let $(p,q)$ and $(\widetilde{p},\widetilde{q})$ be two couples corresponding to the square* (45), *and let $\sigma,\widetilde{\sigma}\in\mathbb{R}$. Assume that $\sigma\geq\sigma(p,q)$, where*

$$\sigma(p,q)=\frac{n+1}{2}\Big(\frac{1}{2}-\frac{1}{q}\Big)+\max\Big\{0,\frac{n-1}{2}\Big(\frac{1}{2}-\frac{1}{q}\Big)-\frac{1}{p}\Big\}=\begin{cases}\dfrac{n+1}{2}\Big(\dfrac{1}{2}-\dfrac{1}{q}\Big) & \text{if } \dfrac{1}{p}\geq\dfrac{n-1}{2}\Big(\dfrac{1}{2}-\dfrac{1}{q}\Big),\\ n\Big(\dfrac{1}{2}-\dfrac{1}{q}\Big)-\dfrac{1}{p} & \text{if } \dfrac{1}{p}\leq\dfrac{n-1}{2}\Big(\dfrac{1}{2}-\dfrac{1}{q}\Big),\end{cases}$$

*and similarly, $\widetilde{\sigma}\geq\sigma(\widetilde{p},\widetilde{q})$ (see Figure 2). Then the conclusion of Theorem 5.2 holds for solutions to the Cauchy problem* (12). *More precisely, we have again the Strichartz estimate*

$$\|\nabla_{\mathbb{R}\times\mathbb{H}^n}u\|_{L^pH^{-\sigma,q}}\lesssim\|f\|_{H^1}+\|g\|_{L^2}+\|F\|_{L^{\widetilde{p}'}H^{\widetilde{\sigma},\widetilde{q}'}},\tag{40}$$

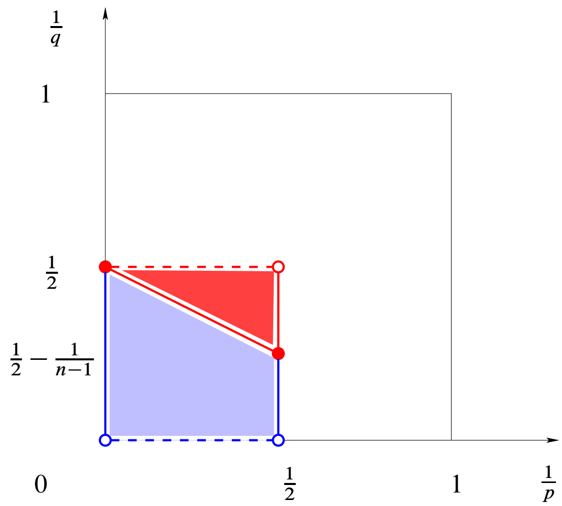


**Figure 2.** Case $n \geq 4$.

*which amounts to*

$$\|\widetilde{D}_x^{-\sigma+1/2}u(t,x)\|_{L_t^pL_x^q}+\|\widetilde{D}_x^{-\sigma-1/2}\partial_t u(t,x)\|_{L_t^pL_x^q} \lesssim \|D_x^{1/2}f(x)\|_{L_x^2}+\|D_x^{-1/2}g(x)\|_{L_x^2}+\|\widetilde{D}_x^{\tilde{\sigma}-1/2}F(t,x)\|_{L_t^{\tilde{p}'}L_x^{\tilde{q}'}}. \tag{41}$$

*Proof.* We may restrict to the limit cases $\sigma=\sigma(p,q)$ and $\tilde{\sigma}=\sigma(\tilde{p},\tilde{q})$. Define $Q$ by

$$\frac{1}{Q}=\begin{cases}\frac{1}{q} & \text{if } \frac{1}{p}\geq\frac{n-1}{2}\Big(\frac{1}{2}-\frac{1}{q}\Big),\\ \frac{1}{2}-\frac{2}{n-1}\frac{1}{p} & \text{if } \frac{1}{p}\leq\frac{n-1}{2}\Big(\frac{1}{2}-\frac{1}{q}\Big)\end{cases}$$

and $\widetilde{Q}$ similarly. Since $(p,Q)$ and $(\tilde{p},\widetilde{Q})$ are admissible couples, it follows from Theorem 5.2 and more precisely from (41) that

$$\|\widetilde{D}_x^{-\Sigma+1/2}u(t,x)\|_{L_t^pL_x^Q}+\|\widetilde{D}_x^{-\Sigma-1/2}\partial_t u(t,x)\|_{L_t^pL_x^Q} \lesssim \|D_x^{1/2}f(x)\|_{L_x^2}+\|D_x^{-1/2}g(x)\|_{L_x^2}+\|\widetilde{D}_x^{\widetilde{\Sigma}-1/2}F(t,x)\|_{L_t^{\tilde{p}'}L_x^{\widetilde{Q}'}}, \tag{46}$$

where $\Sigma=(n+1)(1/2-1/Q)/2$ and $\widetilde{\Sigma}=(n+1)(1/2-1/\widetilde{Q})/2$. Since $\sigma-\Sigma=n(1/Q-1/q)$, we have

$$\|\widetilde{D}_x^{-\sigma+1/2}u(t,x)\|_{L_t^pL_x^q}\lesssim\|\widetilde{D}_x^{-\Sigma+1/2}u(t,x)\|_{L_t^pL_x^Q} \tag{47}$$

according to the Sobolev embedding theorem (Proposition 2.2). Similarly,

$$\|\widetilde{D}_x^{\widetilde{\Sigma}-1/2}F(t,x)\|_{L_t^{\tilde{p}'}L_x^{\widetilde{Q}'}}\lesssim\|\widetilde{D}_x^{\tilde{\sigma}-1/2}F(t,x)\|_{L_t^{\tilde{p}'}L_x^{\tilde{q}'}}. \tag{48}$$

We conclude by combining (46), (47), and (48). □

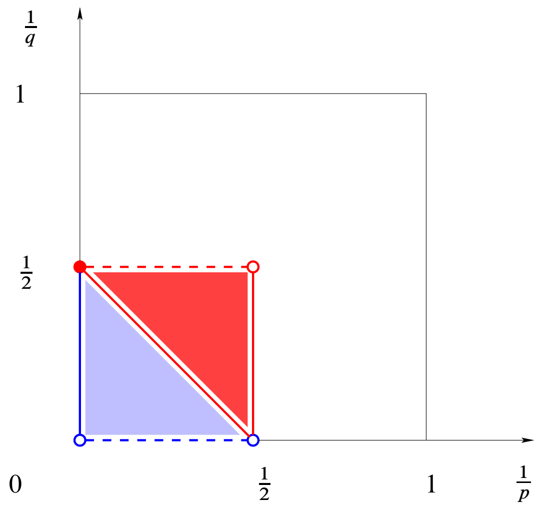


**Figure 3.** Case $n = 3$.

**Remark 5.4.** Theorem 5.2 and Corollary 5.3 hold true in dimension $n = 3$ with the same proofs. Notice that the endpoint $(p, q) = (2, \infty)$ is excluded (see Figure 3). These results hold in particular for the 3-dimensional wave equation (6) and include the Strichartz estimates obtained by Metcalfe and Taylor [2011, §4] in the smaller region

$$\left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in \left[0, \frac{1}{2}\right] \times \left(0, \frac{1}{2}\right] \;\middle|\; \frac{1}{p} \leq 3\left(\frac{1}{2} - \frac{1}{q}\right)\right\} \setminus \left\{\left(\frac{1}{2}, \frac{1}{3}\right)\right\}.$$

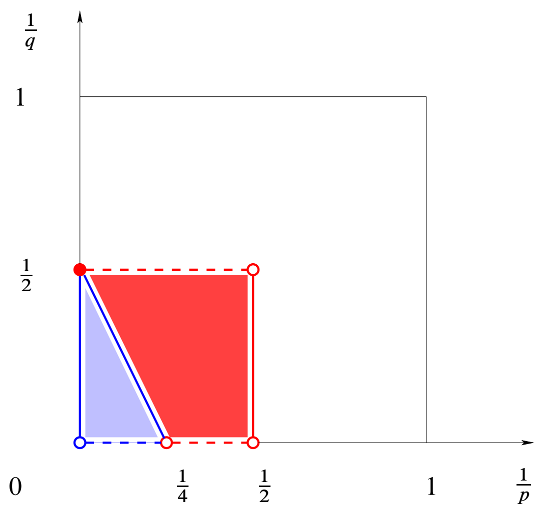


**Figure 4.** Case $n = 2$.

**Remark 5.5.** The analysis carried out in this section still holds in dimension $n=2$ except for the first convolution kernel in (42), which becomes

$$|t-s|^{-\alpha}(1-\log|t-s|)^{\beta}\mathbb{1}_{\{0<|t-s|<1\}}$$

with $\alpha=1/2-1/q$ and $\beta=2(1/2-1/q)$. Consequently, the admissibility region in Theorem 5.2 becomes

$$\Big\{\Big(\frac{1}{p},\frac{1}{q}\Big)\in\Big(0,\frac{1}{2}\Big]\times\Big(0,\frac{1}{2}\Big)\;\Big|\;\frac{1}{p}>\frac{1}{2}\Big(\frac{1}{2}-\frac{1}{q}\Big)\Big\}\cup\Big\{\Big(0,\frac{1}{2}\Big)\Big\}$$

and the inequalities $\sigma\geq\sigma(p,q)$, $\widetilde{\sigma}\geq\sigma(\widetilde{p},\widetilde{q})$ in Corollary 5.3 (see Figure 4) become strict in the triangle

$$\Big\{\Big(\frac{1}{p},\frac{1}{q}\Big)\in\Big(0,\frac{1}{4}\Big)\times\Big(0,\frac{1}{2}\Big)\;\Big|\;\frac{1}{p}\leq\frac{1}{2}\Big(\frac{1}{2}-\frac{1}{q}\Big)\Big\}.$$

## 6. Global well-posedness in $L^p(\mathbb{R},L^q(\mathbb{H}^n))$

In this section, following the classical fixed-point scheme, we use the Strichartz estimates obtained in Section 5 to prove global well-posedness for the semilinear equation

$$\begin{cases}\partial_t^2u(t,x)+D_x^2u(t,x)=F(u(t,x))\\ u(0,x)=f(x),\quad \partial_t|_{t=0}u(t,x)=g(x)\end{cases}\tag{49}$$

on $\mathbb{H}^n$ with power-like nonlinearities

$$F(u)\sim|u|^{\gamma}\quad(\gamma>1)$$

and small initial data $f$ and $g$. We assume $n\geq3$ throughout the section and discuss the 2-dimensional case in the final remark. The statement and proof of our result involve the powers

$$\gamma_1=1+\frac{3}{n},\qquad \gamma_2=1+\frac{2}{\frac{n-1}{2}+\frac{2}{n-1}},\qquad \gamma_{\text{conf}}=1+\frac{4}{n-1},$$

$$\gamma_3=\begin{cases}\dfrac{1}{n}\left(\dfrac{n+6}{2}+\dfrac{2}{n-1}+\sqrt{4n+\Big(\dfrac{6-n}{2}+\dfrac{2}{n-1}\Big)^2}\right) & \text{if } n\leq5,\\[2ex] 1+\dfrac{2}{\frac{n-1}{2}-\frac{1}{n-1}} & \text{if } n\geq6,\end{cases}\tag{50}$$

$$\gamma_4=\begin{cases}1+\dfrac{4}{n-2} & \text{if } n\leq5,\\[2ex] \dfrac{n-1}{2}+\dfrac{3}{n+1}-\sqrt{\Big(\dfrac{n-3}{2}+\dfrac{3}{n+1}\Big)^2-4\dfrac{n-1}{n+1}} & \text{if } n\geq6,\end{cases}$$

which are computed in Table 1, and the curves

$$\sigma_1(\gamma)=\frac{n+1}{4}-\frac{(n+1)(n+5)}{8n}\frac{1}{\gamma-\frac{n+1}{2n}},\qquad \sigma_2(\gamma)=\frac{n+1}{4}-\frac{1}{\gamma-1},\quad\text{and}\quad \sigma_3(\gamma)=\frac{n}{2}-\frac{2}{\gamma-1}.\tag{51}$$

The powers $\gamma_1$, $\gamma_2$, and $\gamma_{\text{conf}}$ and the curves $C_1$, $C_2$, and $C_3$ parametrized by $\sigma_1$, $\sigma_2$, and $\sigma_3$ occur already in the Euclidean setting. More precisely, they are involved in the conditions, illustrated in Figure 5,

| $n$ | $\gamma_1$ | $\gamma_2$ | $\gamma_{\text{conf}}$ | $\gamma_3$ | $\gamma_4$ |
|---|---|---|---|---|---|
| 3 | 2 | 2 | 3 | $\frac{11+\sqrt{73}}{6} \simeq 3.26$ | 5 |
| 4 | $\frac{7}{4} = 1.75$ | $\frac{25}{13} \simeq 1.92$ | $\frac{7}{3} \simeq 2.33$ | $\frac{5}{2} \simeq 2.5$ | 3 |
| 5 | $\frac{8}{5} \simeq 1.6$ | $\frac{9}{5} \simeq 1.8$ | 2 | $\frac{6+\sqrt{21}}{5} \simeq 2.12$ | $\frac{7}{3} \simeq 2.33$ |
| 6 | $\frac{3}{2} = 1.5$ | $\frac{49}{29} \simeq 1.69$ | $\frac{9}{5} = 1.8$ | $\frac{43}{23} \simeq 1.87$ | 2 |
| $\geq 7$ | $< \gamma_2$ | $< \gamma_{\text{conf}}$ | $< \gamma_3$ | $< \gamma_4$ | $< 2$ |

**Table 1.** Critical powers.

of minimal regularity $\sigma$ on the initial data $f$ and $g$ that are needed in order to ensure local well-posedness of (49). We refer again to [Kapitanski 1994; Lindblad and Sogge 1995; Keel and Tao 1998] for more details. Notice that, in dimension $n = 3$, $\gamma_1$ coincides with $\gamma_2$ and there is no curve $C_1$.

As mentioned in the introduction, global well-posedness of (49) on $\mathbb{R}^n$ requires additional conditions. Recall that smooth solutions with small-amplitude blow up or not depending on whether $\gamma$ is smaller or larger than the critical power $\gamma_0$ defined in (5).

In Section 5, we have obtained Strichartz estimates on $\mathbb{H}^n$ for a range of admissible couples that is wider than on $\mathbb{R}^n$. As a consequence, we deduce in this section stronger well-posedness results for (49). In particular, we prove global well-posedness for small initial data in $H^\sigma(\mathbb{H}^n) \times H^{\sigma-1}(\mathbb{H}^n)$ if $1 < \gamma < \gamma_1$ and $\sigma > 0$ is small. Thus, there is no blow-up for small powers $\gamma > 1$ on $\mathbb{H}^n$ in sharp contrast with $\mathbb{R}^n$.

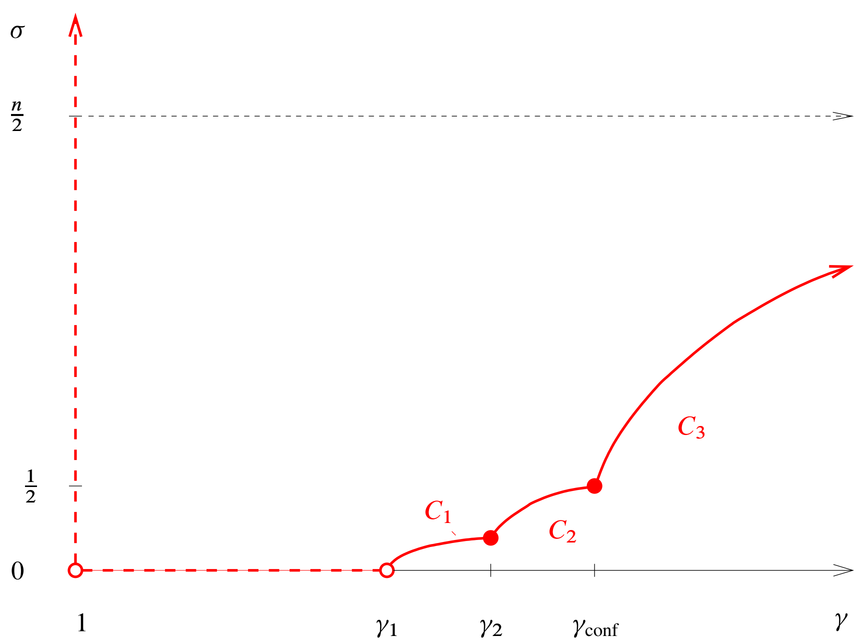


**Figure 5.** Regularity for local well-posedness on $\mathbb{R}^n$ in dimension $n \geq 3$.

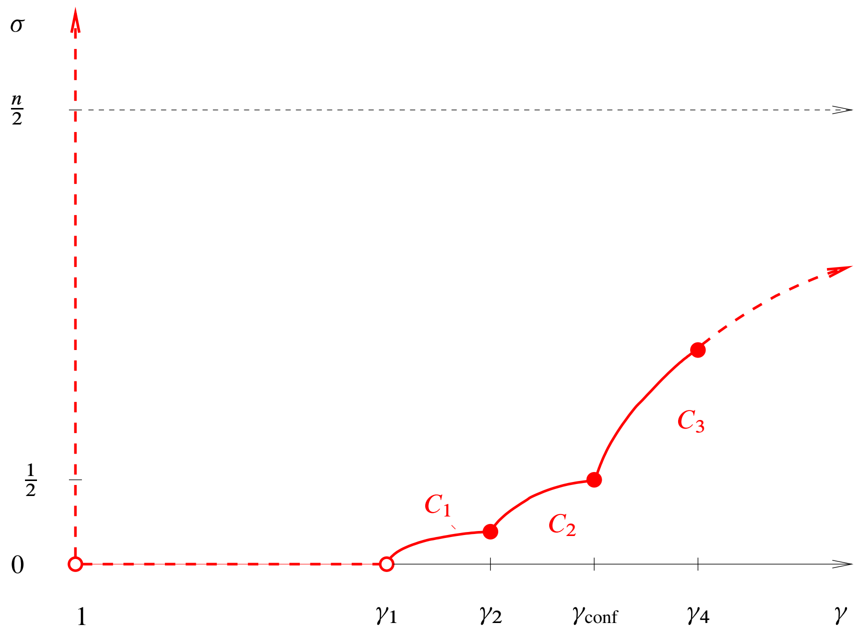


**Figure 6.** Regularity for global well-posedness on $\mathbb{H}^n$ in dimension $n \geq 3$.

**Theorem 6.1.** *Assume that the nonlinearity $F$ satisfies*

$$|F(u)| \leq C|u|^{\gamma} \quad \textit{and} \quad |F(u)-F(v)| \leq C(|u|^{\gamma-1}+|v|^{\gamma-1})|u-v|. \tag{52}$$

*Then in dimension $n \geq 3$,* (49) *is globally well posed for small initial data in $H^{\sigma}(\mathbb{H}^n) \times H^{\sigma-1}(\mathbb{H}^n)$ provided that*

$$\begin{cases} \sigma = 0^+ & \textit{if } 1 < \gamma \leq \gamma_1, \\ \sigma = \sigma_1(\gamma) & \textit{if } \gamma_1 < \gamma \leq \gamma_2, \\ \sigma = \sigma_2(\gamma) & \textit{if } \gamma_2 \leq \gamma \leq \gamma_{\text{conf}}, \\ \sigma = \sigma_3(\gamma) & \textit{if } \gamma_{\text{conf}} \leq \gamma \leq \gamma_4, \end{cases} \tag{53}$$

*where $\sigma = 0^+$ stands for any $\sigma > 0$ sufficiently close to* 0 *(see Figure 6). More precisely, in each case, there exist $2 \leq p, q < \infty$ and $\delta, \varepsilon > 0$ such that, for any initial data $(f, g) \in H^{\sigma}(\mathbb{H}^n) \times H^{\sigma-1}(\mathbb{H}^n)$ with norm $\leq \delta$, the Cauchy problem* (49) *has a unique solution $u$ with norm $\leq \varepsilon$ in the Banach space*

$$X = C(\mathbb{R}; H^{\sigma}(\mathbb{H}^n)) \cap C^1(\mathbb{R}; H^{\sigma-1}(\mathbb{H}^n)) \cap L^p(\mathbb{R}; L^q(\mathbb{H}^n)).$$

**Remark 6.2.** In dimension $n=3$, $\gamma_1$ coincides with $\gamma_2$, the second and third conditions in (53) boil down to

$$\sigma \geq \sigma_2(\gamma) \quad \text{if } \gamma_1 = \gamma_2 < \gamma \leq \gamma_{\text{conf}},$$

and there is no curve $C_1$ in Figure 6.

*Proof of Theorem 6.1 for* $1 < \gamma \le \gamma_{\mathrm{conf}}$. We resume the fixed-point method based on Strichartz estimates. Define $u = \Phi(v)$ as the solution to the Cauchy problem

$$\begin{cases} \partial_t^2 u(t,x) + D_x^2 u(t,x) = F(v(t,x)), \\ u(0,x) = f(x), \quad \partial_t|_{t=0} u(t,x) = g(x), \end{cases} \tag{54}$$

which is given by Duhamel's formula:

$$u(t,x) = (\cos t D_x) f(x) + \frac{\sin t D_x}{D_x} g(x) + \int_0^t ds\, \frac{\sin(t-s) D_x}{D_x} F(s,x).$$

On one hand, according to Theorem 5.2, the Strichartz estimate

$$\begin{aligned} &\|u(t,x)\|_{L_t^\infty H_x^\sigma} + \|\partial_t u(t,x)\|_{L_t^\infty H_x^{\sigma-1}} + \|u(t,x)\|_{L_t^p L_x^q} \\ &\qquad\qquad \lesssim \|f(x)\|_{H_x^\sigma} + \|g(x)\|_{H_x^{\sigma-1}} + \|F(v(t,x))\|_{L_t^{\tilde{p}'} H_x^{\sigma+\tilde{\sigma}-1,\tilde{q}'}} \end{aligned}$$

holds whenever

$$\begin{cases} (p,q) \text{ and } (\tilde{p},\tilde{q}) \text{ are admissible couples,} \\ \sigma \ge \frac{n+1}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big) \text{ and } \tilde{\sigma} \ge \frac{n+1}{2}\Big(\frac{1}{2} - \frac{1}{\tilde{q}}\Big). \end{cases}$$

On the other hand, by our nonlinear assumption (52) and by the Sobolev embedding theorem (Proposition 2.2), we have

$$\|F(v(t,x))\|_{L_t^{\tilde{p}'} H_x^{\sigma+\tilde{\sigma}-1,\tilde{q}'}} \lesssim \||v(t,x)|^\gamma\|_{L_t^{\tilde{p}'} H_x^{\sigma+\tilde{\sigma}-1,\tilde{q}'}} \lesssim \||v(t,x)|^\gamma\|_{L_t^{\tilde{p}'} L_x^{\tilde{Q}'}} \lesssim \|v(t,x)\|^\gamma_{L_t^{\gamma\tilde{p}'} L_x^{\gamma\tilde{Q}'}},$$

provided that

$$\sigma + \tilde{\sigma} \le 1, \qquad 1 < \tilde{Q}' \le \tilde{q}' < \infty, \quad \text{and} \quad \frac{n}{\tilde{Q}'} - \frac{n}{\tilde{q}'} \le 1 - \sigma - \tilde{\sigma}. \tag{55}$$

In order to remain within the same function space, we require in addition that

$$\gamma \tilde{p}' = p \quad \text{and} \quad \gamma \tilde{Q}' = q.$$

In summary,

$$\begin{aligned} &\|u(t,x)\|_{L_t^\infty H_x^\sigma} + \|\partial_t u(t,x)\|_{L_t^\infty H_x^{\sigma-1}} + \|u(t,x)\|_{L_t^p L_x^q} \\ &\qquad\qquad \le C\big\{\|f(x)\|_{H_x^\sigma} + \|g(x)\|_{H_x^{\sigma-1}} + \|v\|^\gamma_{L_t^p L_x^q}\big\} \end{aligned} \tag{56}$$

if the following set of conditions is satisfied:

$$\begin{cases} \text{(a)} \quad (p,q) \text{ and } (\tilde{p},\tilde{q}) \text{ are admissible couples,} \\ \text{(b)} \quad \sigma \ge \frac{n+1}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big),\ \tilde{\sigma} \ge \frac{n+1}{2}\Big(\frac{1}{2} - \frac{1}{\tilde{q}}\Big), \text{ and } \sigma + \tilde{\sigma} \le 1, \\ \text{(c)} \quad \frac{\gamma}{p} + \frac{1}{\tilde{p}} = 1, \\ \text{(d)} \quad 1 \le \frac{\gamma}{q} + \frac{1}{\tilde{q}} \le 1 + \frac{1-\sigma-\tilde{\sigma}}{n}, \\ \text{(e)} \quad q > \gamma. \end{cases} \tag{57}$$

For such a choice, $\Phi$ maps the Banach space

$$X = C(\mathbb{R}; H^{\sigma}(\mathbb{H}^n)) \cap C^1(\mathbb{R}; H^{\sigma-1}(\mathbb{H}^n)) \cap L^p(\mathbb{R}; L^q(\mathbb{H}^n)),$$

equipped with the norm

$$\|u\|_X = \|u(t,x)\|_{L^\infty_t H^\sigma_x} + \|\partial_t u(t,x)\|_{L^\infty_t H^{\sigma-1}_x} + \|u\|_{L^p_t L^q_x},$$

into itself. Let us show that $\Phi$ is a contraction on the ball

$$X_\varepsilon = \{u \in X \mid \|u\|_X \le \varepsilon\},$$

provided that $\varepsilon > 0$ and $\|f\|_{H^\sigma} + \|g\|_{H^{\sigma-1}}$ are sufficiently small. Let $v, \tilde{v} \in X$, $u = \Phi(v)$, and $\tilde{u} = \Phi(\tilde{v})$. By resuming the arguments leading to (56) and by using in addition Hölder's inequality, we obtain the estimate

$$\begin{aligned}
\|u - \tilde{u}\|_X &\le C\,\|F(v) - F(\tilde{v})\|_{L^{\tilde{p}'}_t L^{\tilde{Q}'}_x} \\
&\le C\,\|\{|v|^{\gamma-1} + |\tilde{v}|^{\gamma-1}\}|v - \tilde{v}|\|_{L^{\tilde{p}'}_t L^{\tilde{Q}'}_x} \\
&\le C\{\|v\|^{\gamma-1}_{L^p_t L^q_x} + \|\tilde{v}\|^{\gamma-1}_{L^p_t L^q_x}\}\|v - \tilde{v}\|_{L^p_t L^q_x} \\
&\le C\{\|v\|^{\gamma-1}_X + \|\tilde{v}\|^{\gamma-1}_X\}\|v - \tilde{v}\|_X.
\end{aligned} \tag{58}$$

Thus, if we assume $\|v\|_X \le \varepsilon$, $\|\tilde{v}\|_X \le \varepsilon$, and $\|f\|_{H^\sigma} + \|g\|_{H^{\sigma-1}} \le \delta$, then (56) and (58) yield

$$\|u\|_X \le C\delta + C\varepsilon^\gamma, \qquad \|\tilde{u}\|_X \le C\delta + C\varepsilon^\gamma, \quad \text{and} \quad \|u - \tilde{u}\|_X \le 2C\varepsilon^{\gamma-1}\|v - \tilde{v}\|_X.$$

Hence,

$$\|u\|_X \le \varepsilon, \qquad \|\tilde{u}\|_X \le \varepsilon, \quad \text{and} \quad \|u - \tilde{u}\|_X \le \tfrac{1}{2}\|v - \tilde{v}\|_X$$

if $C\varepsilon^{\gamma-1} \le \frac{1}{4}$ and $C\delta \le \frac{3}{4}\varepsilon$. One concludes by applying the fixed-point theorem in the complete metric space $X_\varepsilon$.

It remains for us to check that the set of conditions (57) can be fulfilled in the various cases (53). Notice that we may assume the following equalities in (57)(b):

$$\sigma = \frac{n+1}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big) \quad \text{and} \quad \tilde{\sigma} = \frac{n+1}{2}\Big(\frac{1}{2} - \frac{1}{\tilde{q}}\Big).$$

Thus, (57) reduces to the set of conditions:

$$\begin{cases}
\text{(a)} & (p,q) \text{ and } (\tilde{p},\tilde{q}) \text{ are admissible couples,} \\
\text{(b)} & \dfrac{1}{q} + \dfrac{1}{\tilde{q}} \ge \dfrac{n-1}{n+1}, \\
\text{(c)} & \dfrac{\gamma}{p} + \dfrac{1}{\tilde{p}} = 1, \\
\text{(d)(i)} & \dfrac{\gamma}{q} + \dfrac{1}{\tilde{q}} \ge 1, \\
\text{(d)(ii)} & \Big(\dfrac{2n}{n-1}\gamma - \dfrac{n+1}{n-1}\Big)\dfrac{1}{q} + \dfrac{1}{\tilde{q}} \le \dfrac{n+1}{n-1}, \\
\text{(e)} & q > \gamma.
\end{cases} \tag{59}$$

We shall discuss these conditions first in high dimensions and next in low dimensions.

► Assume that $n \geq 6$.

Firstly notice that $\gamma_{\text{conf}} < 2$. As $\gamma \leq \gamma_{\text{conf}}$ and $q > 2$, (59)(e) is trivially satisfied. Secondly, we claim that (59)(a) and (59)(c) reduce to the single condition

$$\frac{\gamma}{q} + \frac{1}{\widetilde{q}} \geq \frac{\gamma+1}{2} - \frac{2}{n-1} \tag{60}$$

in the square

$$R = \Big[\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2}\Big) \times \Big[\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2}\Big). \tag{61}$$

More precisely, if $(p,q)$ and $(\widetilde{p},\widetilde{q})$ are admissible couples satisfying (59)(c), then $(1/q, 1/\widetilde{q})$ is a point in the square $R$ satisfying (60). Conversely, if $(1/q, 1/\widetilde{q}) \in R$ satisfies (60), then there exists a 1-parameter family of admissible couples $(p,q)$ and $(\widetilde{p},\widetilde{q})$ satisfying (59)(c). All these claims can be deduced from Figure 7.

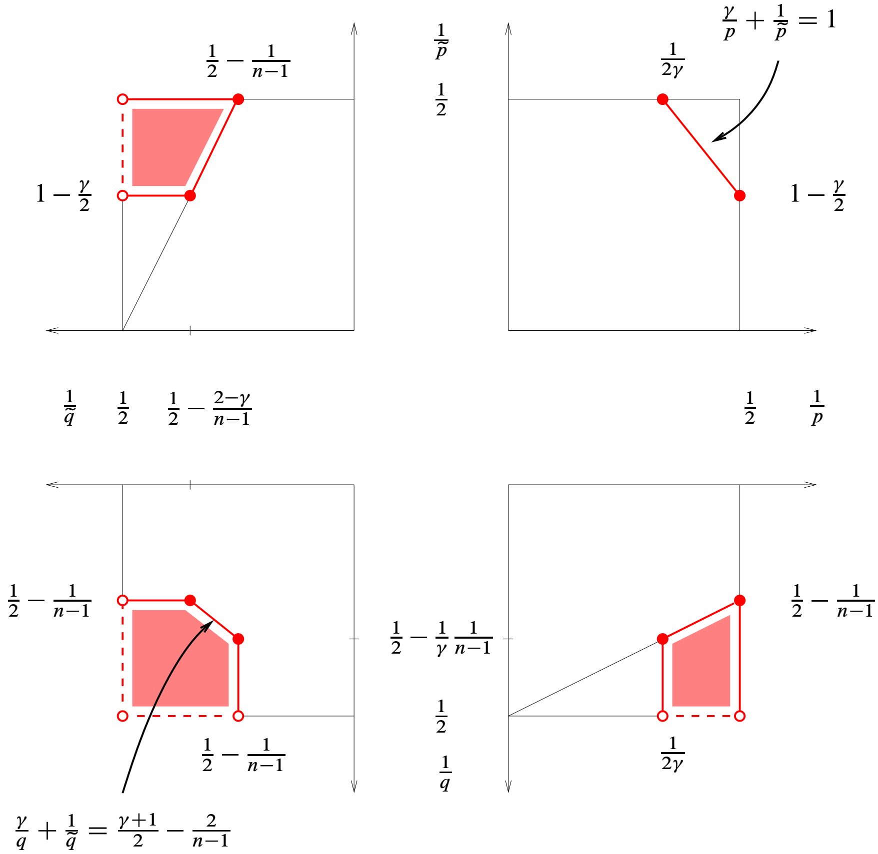


**Figure 7.** Case $\gamma < 2$.

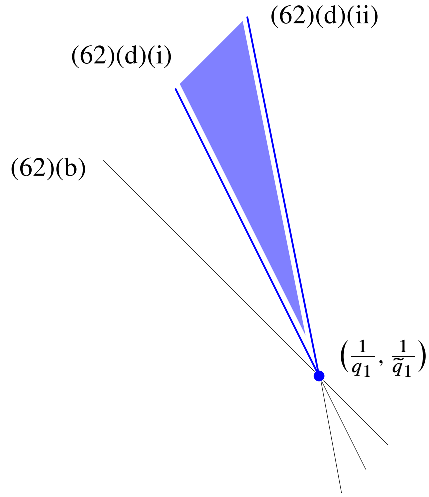


**Figure 8.** Sector $S$.

Thirdly, as $\gamma \leq \gamma_{\text{conf}}$, (60) follows actually from (59)(d)(i). Fourthly, we claim that (59)(b) follows from (59)(d)(i) and (59)(d)(ii). Consider indeed the three lines

$$\begin{cases} \text{(b)} & \dfrac{1}{q} + \dfrac{1}{\tilde{q}} = \dfrac{n-1}{n+1}, \\ \text{(d)(i)} & \dfrac{\gamma}{q} + \dfrac{1}{\tilde{q}} = 1, \\ \text{(d)(ii)} & \left(\dfrac{2n}{n-1}\gamma - \dfrac{n+1}{n-1}\right)\dfrac{1}{q} + \dfrac{1}{\tilde{q}} = \dfrac{n+1}{n-1} \end{cases} \tag{62}$$

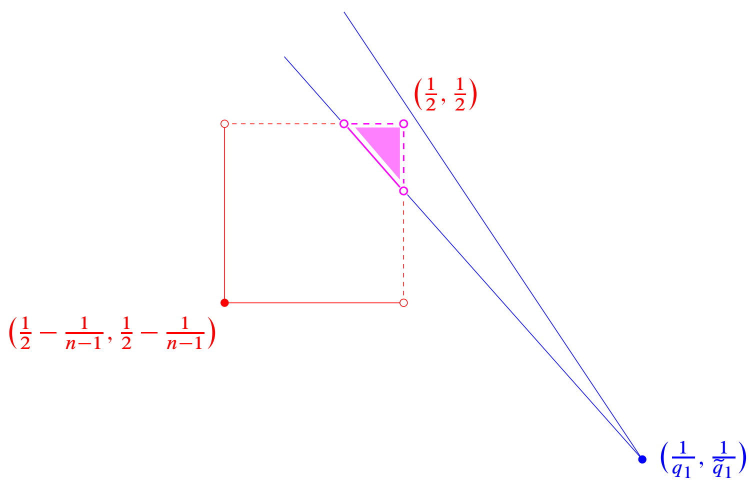


**Figure 9.** Case $1 < \gamma \leq 1 + \dfrac{2}{n}$.

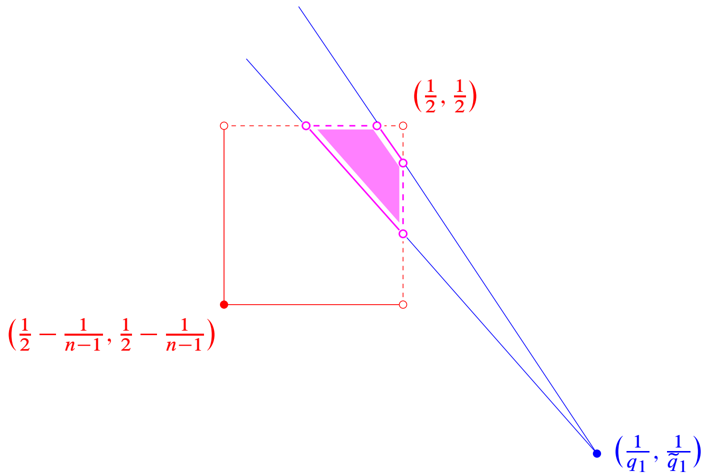


**Figure 10.** Case $1+\dfrac{2}{n} \leq \gamma \leq 1+\dfrac{2}{n-1}$.

in the plane with coordinates $(1/q, 1/\widetilde{q})$. On one hand, they meet at the same point, whose coordinates are

$$\begin{cases} \dfrac{1}{q_1} = \dfrac{2}{n+1}\dfrac{1}{\gamma-1}, \\ \dfrac{1}{\widetilde{q}_1} = \dfrac{n-1}{n+1} - \dfrac{2}{n+1}\dfrac{1}{\gamma-1}. \end{cases} \tag{63}$$

On the other hand, the coefficients of $1/q$ occur in increasing order in (62):

$$1 < \gamma < \frac{2n}{n-1}\gamma - \frac{n+1}{n-1}.$$

Hence, (59)(b) follows from (59)(d)(i) and (59)(d)(ii), which define the sector $S$ with vertex $(1/q_1, 1/\widetilde{q}_1)$ and edges (62)(d)(i) and (62)(d)(ii) depicted in Figure 8.

In summary, the set of conditions (59) reduce to the three conditions (59)(d)(i), (59)(d)(ii), and (61) in the plane with coordinates $(1/q, 1/\widetilde{q})$. In order to conclude, we examine the possible intersections of the

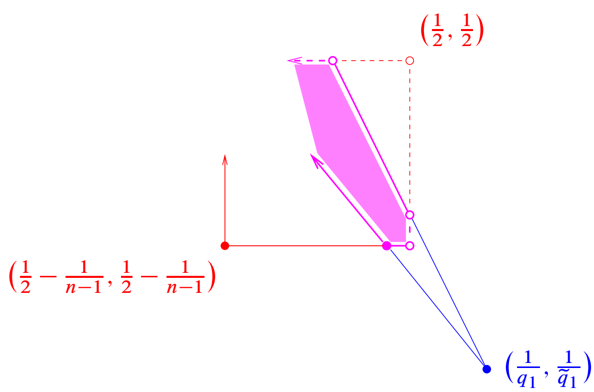


**Figure 11.** Case $1+\dfrac{2}{n-1} \leq \gamma \leq \gamma_1$.

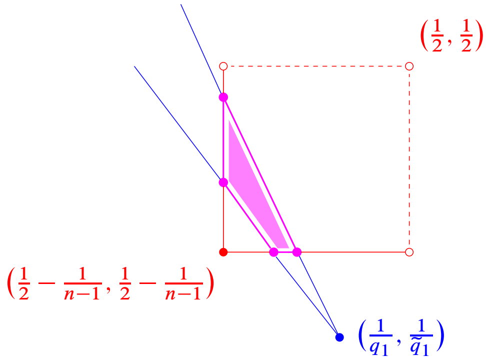


**Figure 12.** Case $\gamma_1 < \gamma \le \gamma_2$.

sector $S$ defined by (59)(d)(i) and (59)(d)(ii) with the square $R$ defined by (61), and we determine in each case the minimal regularity $\sigma = (n+1)(1/2-1/q)/2$.

- *Case 1*: $1 < \gamma \le \gamma_1$.

In the following three subcases, the minimal regularity condition is $\sigma > 0$ as $1/q > 1/2$ can be chosen arbitrarily close to $\frac{1}{2}$:

- ∘ *Subcase 1.1*: $1 < \gamma \le 1 + \dfrac{2}{n}$ (see Figure 9).
- ∘ *Subcase 1.2*: $1 + \dfrac{2}{n} \le \gamma \le 1 + \dfrac{2}{n-1}$ (see Figure 10).
- ∘ *Subcase 1.3*: $1 + \dfrac{2}{n-1} \le \gamma \le \gamma_1$ (see Figure 11).

- *Case 2*: $\gamma_1 < \gamma \le \gamma_2$ (see Figure 12).

The minimal regularity $\sigma = \sigma_1(\gamma)$ is reached at the boundary point

$$\left(\frac{1}{q},\frac{1}{\tilde{q}}\right) = \left(\frac{n+5}{4n}\,\frac{1}{\gamma-(n+1)/2n},\ \frac{1}{2}-\frac{1}{n-1}\right).$$

- *Case 3*: $\gamma_2 \le \gamma \le \gamma_{\text{conf}}$ (see Figure 13).

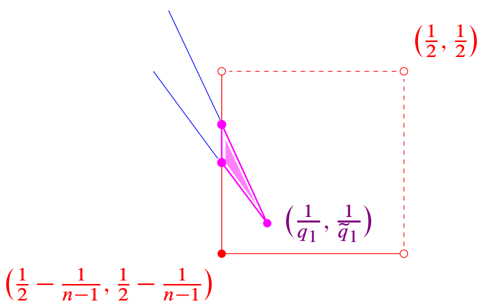


**Figure 13.** Case $\gamma_2 \le \gamma \le \gamma_{\text{conf}}$.

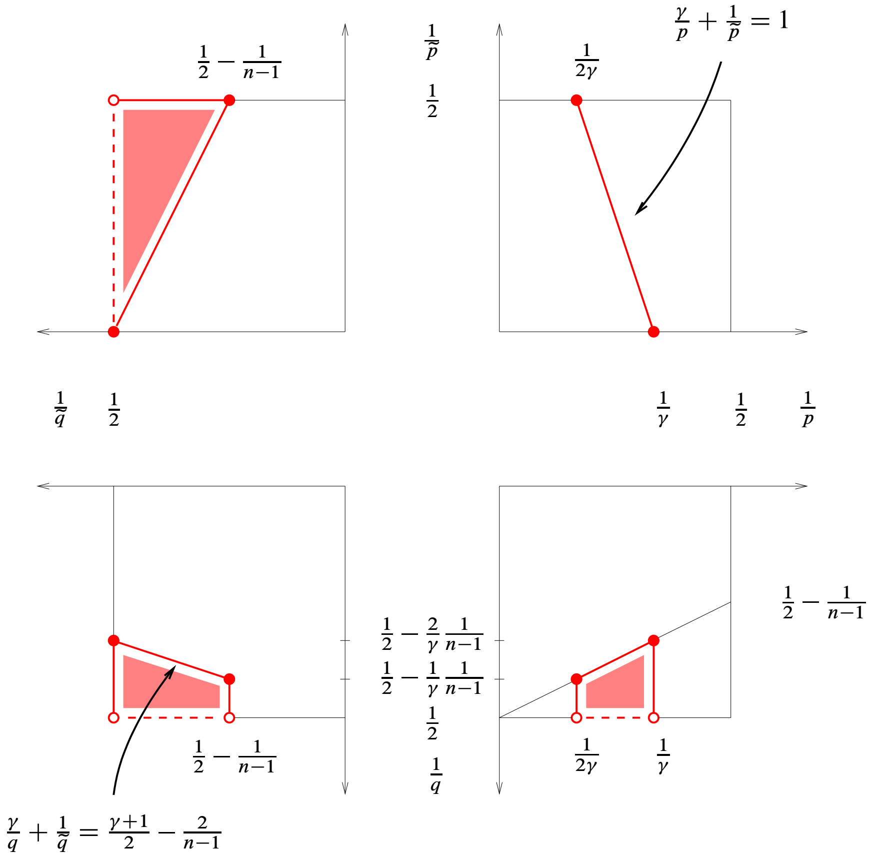


**Figure 14.** Case $\gamma \geq 2$.

The minimal regularity $\sigma=\sigma_2(\gamma)$ is reached at the vertex $(1/q_1, 1/\tilde{q}_1)$. In the limit case $\gamma=\gamma_{\text{conf}}$, notice that all indices $1/q_1$, $1/\tilde{q}_1$, $1/p_1=(n-1)(1/2-1/q_1)/2$, and $1/\tilde{p}_1=(n-1)(1/2-1/\tilde{q}_1)/2$ become equal to the Strichartz index $(n-1)/2(n+1)=1/2-1/(n+1)$.

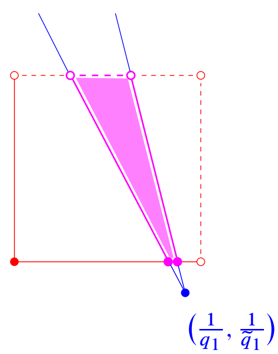


**Figure 15.** Case $\gamma_1 < \gamma \leq \gamma_2$.

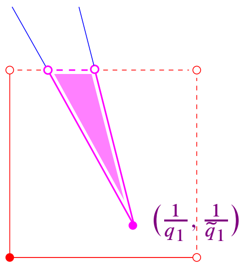


**Figure 16.** Subcase $\gamma_2 \leq \gamma < 2$.

This concludes the proof of Theorem 6.1 for $1 < \gamma \leq \gamma_{\mathrm{conf}}$ and $n \geq 6$.

► Assume that $n = 4$ or 5.

Let us adapt the proof above. If $\gamma \geq 2$, (59)(e) must be checked and (59)(a) and (59)(c) reduce again to (60) but this time in the slightly larger square

$$R = \Big[\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2}\Big) \times \Big[\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2}\Big] \tag{64}$$

(see Figure 14). Thus, (59) reduces to

$$\begin{cases} \text{(59)(d)(i), (59)(d)(ii), and (64)} & \text{if } 1 < \gamma < 2, \\ \text{(59)(d)(i), (59)(d)(ii), (59)(e), and (64)} & \text{if } 2 \leq \gamma \leq \gamma_{\mathrm{conf}}. \end{cases}$$

The case-by-case study of the intersection $S \cap R$ is carried out as above and yields the same results. The only difference lies in the fact that the sector $S$ exits the square $R$ through the top edge instead of the left edge (see Figures 15, 16, and 17 below). Notice that (59)(e) is satisfied as $q_1 > \gamma$ when $2 \leq \gamma \leq \gamma_{\mathrm{conf}}$.

• *Case 2*: $\gamma_1 < \gamma \leq \gamma_2$ (see Figure 15).

• *Case 3*: $\gamma_2 \leq \gamma < \gamma_{\mathrm{conf}}$.

∘ *Subcase 3.1*: $\gamma_1 < \gamma \leq \gamma_2$ (see Figure 16).

∘ *Subcase 3.2*: $\gamma_2 \leq \gamma < 2$ (see Figure 17).

This concludes the proof of Theorem 6.1 for $1 < \gamma \leq \gamma_{\mathrm{conf}}$ and $n = 4, 5$.

► Assume that $n = 3$.

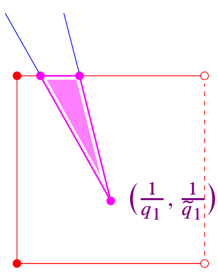


**Figure 17.** Subcase $2 \leq \gamma \leq \gamma_{\mathrm{conf}}$.

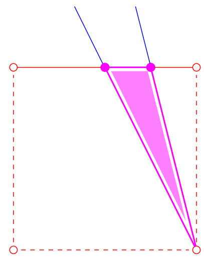

**Figure 18.** Case $\gamma = 2$.

The proof works the same except that the square becomes

$$R = \begin{cases} \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right) & \text{if } 1 < \gamma < 2, \\ \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right] & \text{if } 2 \le \gamma \le \gamma_{\text{conf}} \end{cases} \tag{65}$$

and that $(1/q_1, 1/\tilde{q}_1)$ enters the square $R$ through the vertex $\left(\frac{1}{2}, 0\right)$ instead of the bottom edge. This happens when $\gamma = 2$ (see Figure 18), and in this case, (59)(e) is satisfied. It is further satisfied when $2 < \gamma \le \gamma_{\text{conf}}$ as $q_1 > \gamma$.

This concludes the proof of Theorem 6.1 for $1 < \gamma \le \gamma_{\text{conf}}$. □

*Proof of Theorem 6.1 for* $\gamma_{\text{conf}} \le \gamma \le \gamma_4$. We resume the fixed-point method above, using Corollary 5.3 instead of Theorem 5.2, and obtain in this way the set of conditions

$$\begin{cases} \text{(a)} & 2 \le p \le \infty \text{ and } 2 \le q < \infty \text{ satisfy } \dfrac{1}{p} \le \dfrac{n-1}{2}\left(\dfrac{1}{2} - \dfrac{1}{q}\right), \\ \text{(ã)} & 2 \le \tilde{p} \le \infty \text{ and } 2 \le \tilde{q} < \infty \text{ satisfy } \dfrac{1}{\tilde{p}} \le \dfrac{n-1}{2}\left(\dfrac{1}{2} - \dfrac{1}{\tilde{q}}\right), \\ \text{(b)} & \sigma \ge n\left(\dfrac{1}{2} - \dfrac{1}{q}\right) - \dfrac{1}{p}, \ \tilde{\sigma} \ge n\left(\dfrac{1}{2} - \dfrac{1}{\tilde{q}}\right) - \dfrac{1}{\tilde{p}}, \text{ and } \sigma + \tilde{\sigma} \le 1, \\ \text{(c)} & \dfrac{\gamma}{p} + \dfrac{1}{\tilde{p}} = 1, \\ \text{(d)} & 1 \le \dfrac{\gamma}{q} + \dfrac{1}{\tilde{q}} \le 1 + \dfrac{1-\sigma-\tilde{\sigma}}{n}, \\ \text{(e)} & q > \gamma. \end{cases} \tag{66}$$

We may assume that

$$\sigma = n\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{p} \quad \text{and} \quad \tilde{\sigma} = n\left(\frac{1}{2} - \frac{1}{\tilde{q}}\right) - \frac{1}{\tilde{p}}.$$

With this choice, the conditions

$$\sigma + \tilde{\sigma} \le 1 \quad \text{and} \quad \frac{\gamma}{q} + \frac{1}{\tilde{q}} \le 1 + \frac{1-\sigma-\tilde{\sigma}}{n}$$

become

$$\frac{1}{p} + \frac{1}{\tilde{p}} + 1 \ge n\left(1 - \frac{1}{q} - \frac{1}{\tilde{q}}\right) \tag{67}$$

and

$$\frac{1}{p}+\frac{1}{\tilde{p}}+1 \geq (\gamma-1)\frac{n}{q}. \tag{68}$$

Notice moreover that (67) follows from (68), combined with $\gamma/q+1/\tilde{q}\geq 1$, and that (68) can be rewritten as follows, using (66)(c):

$$\frac{1}{p}+\frac{n}{q}\leq\frac{2}{\gamma-1}.$$

Thus, (66) reduces to the set of conditions

$$\begin{cases} \text{(a)} & 2\leq p\leq\infty \text{ and } 2\leq q<\infty \text{ satisfy } \dfrac{1}{p}\leq\dfrac{n-1}{2}\Big(\dfrac{1}{2}-\dfrac{1}{q}\Big),\\ (\tilde{\text{a}}) & 2\leq \tilde{p}\leq\infty \text{ and } 2\leq \tilde{q}<\infty \text{ satisfy } \dfrac{1}{\tilde{p}}\leq\dfrac{n-1}{2}\Big(\dfrac{1}{2}-\dfrac{1}{\tilde{q}}\Big),\\ \text{(c)} & \dfrac{\gamma}{p}+\dfrac{1}{\tilde{p}}=1,\\ \text{(d)(i)} & \dfrac{\gamma}{q}+\dfrac{1}{\tilde{q}}\geq 1,\\ \text{(d)(ii)} & \dfrac{1}{p}+\dfrac{n}{q}\leq\dfrac{2}{\gamma-1},\\ \text{(e)} & q>\gamma. \end{cases} \tag{69}$$

Among these conditions, consider first (69)(a) and (69)(d)(ii). In the plane with coordinates $(1/p, 1/q)$, the two lines

$$\begin{cases} \text{(a)} & \dfrac{1}{p}+\dfrac{n-1}{2}\dfrac{1}{q}=\dfrac{n-1}{4},\\ \text{(d)(ii)} & \dfrac{1}{p}+\dfrac{n}{q}=\dfrac{2}{\gamma-1} \end{cases} \tag{70}$$

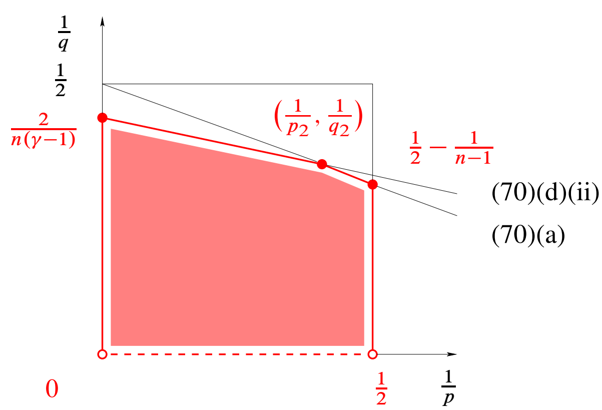


**Figure 19.** *Case 4*: $\gamma_{\text{conf}}\leq\gamma\leq\gamma_3$.

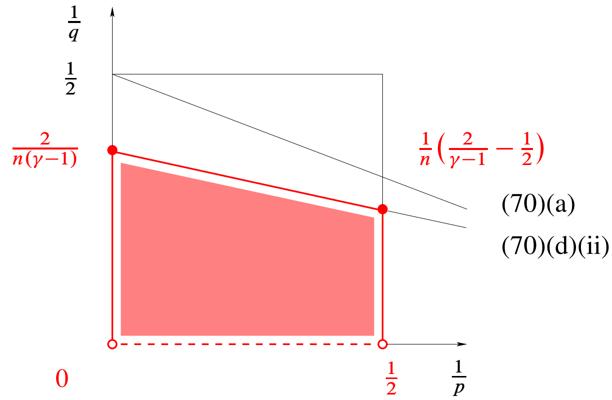


**Figure 20.** *Case 5*: $\gamma_3 \le \gamma \le \gamma_4$.

meet at the point $(1/p_2, 1/q_2)$ given by

$$\begin{cases} \dfrac{1}{p_2} = \dfrac{n-1}{n+1}\Big(\dfrac{n}{2} - \dfrac{2}{\gamma-1}\Big), \\ \dfrac{1}{q_2} = \dfrac{1}{n+1}\Big(\dfrac{4}{\gamma-1} - \dfrac{n-1}{2}\Big). \end{cases} \tag{71}$$

As $\gamma$ varies between $\gamma_{\mathrm{conf}}$ and $\gamma_3$, this point moves on the line (70)(a) between the Strichartz point $(1/2-1/(n+1), 1/2-1/(n+1))$ and the Keel–Tao endpoint $(1/2, 1/2-1/(n-1))$, where it exits the square $\big[0, \frac{1}{2}\big] \times \big(0, \frac{1}{2}\big]$. Thus, (69)(a) and (69)(d)(ii) determine the regions depicted in Figure 19 and in Figure 20. For later use, notice that the minimal regularity

$$\sigma = n\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{1}{p} \ge \sigma_3(\gamma) \tag{72}$$

is reached on the boundary line (70)(d)(ii) and that

$$p_2 < 2\gamma. \tag{73}$$

This inequality holds indeed when $\gamma = \gamma_{\mathrm{conf}}$, and it remains true as $\gamma$ increases while $p_2$ decreases.

Let us next discuss all conditions (69), first in high dimensions and next in low dimensions.

► Assume that $n \ge 6$.

Firstly, notice that (69)(e) is trivially satisfied in this case. On one hand, we have indeed $\gamma \le \gamma_4 \le 2$. On the other hand, it follows from (69)(d)(ii) that

$$\frac{1}{q} \le \frac{2}{n(\gamma-1)} \le \frac{2}{n(\gamma_{\mathrm{conf}}-1)} = \frac{1}{2}\Big(1 - \frac{1}{n}\Big) < \frac{1}{2}.$$

Hence, $\gamma \le 2 < q$.

Secondly, we claim that (69)(a), (69)(ã), (69)(c), and (69)(d)(ii) reduce to the conditions

$$\begin{cases} \text{(a)} & \dfrac{\gamma}{q}+\dfrac{1}{\tilde{q}} \le \dfrac{\gamma+1}{2}-\dfrac{2}{n-1}, \\ \text{(d)(ii)} & \dfrac{\gamma}{q}+\dfrac{n-1}{2n}\dfrac{1}{q} \le \dfrac{n+3}{4n}+\dfrac{2}{n}\dfrac{1}{\gamma-1} \end{cases} \tag{74}$$

in the rectangle

$$R=\Big(0,\frac{1}{n}\Big(\frac{2}{\gamma-1}-\frac{1}{2\gamma}\Big)\Big]\times\Big(0,\frac{1}{2}-\frac{2-\gamma}{n-1}\Big]. \tag{75}$$

Actually, they even reduce to the single condition (74)(d)(ii) if $\gamma \ge \gamma_3$. All these claims are obtained by examining Figures 21 and 22 as we did with Figure 7 in the case $\gamma \le \gamma_{\text{conf}}$.

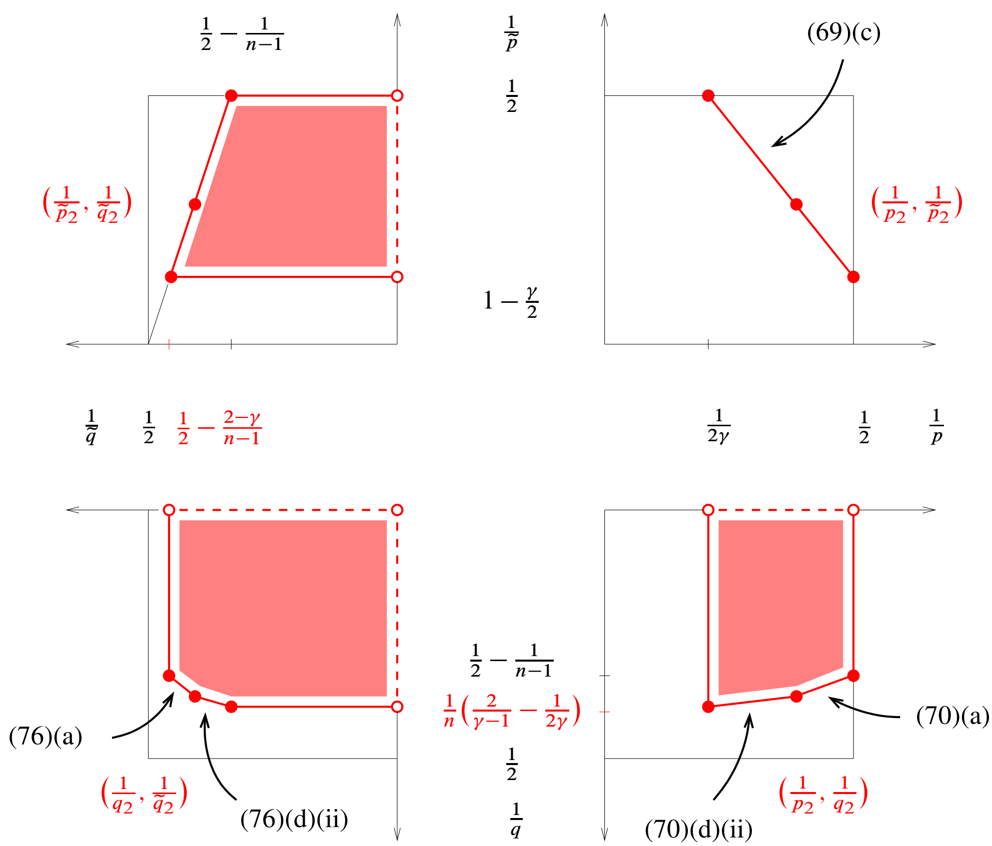


**Figure 21.** *Case 4*: $\gamma_{\text{conf}} \le \gamma \le \gamma_3$.

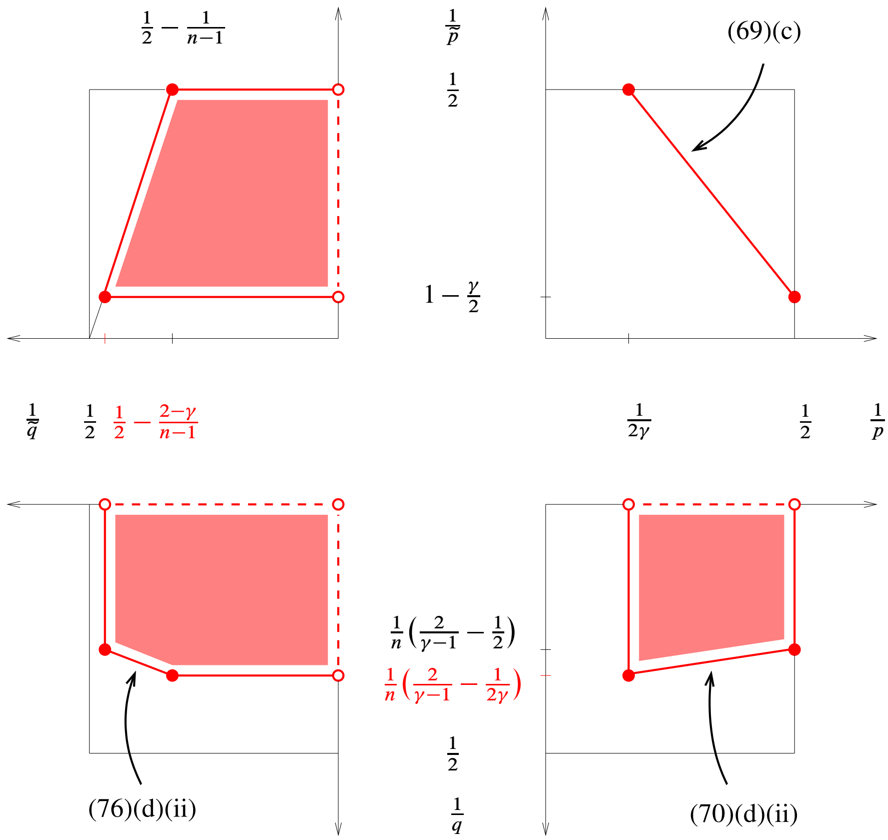


**Figure 22.** *Case 5*: $\gamma_3 \leq \gamma \leq \gamma_4$.

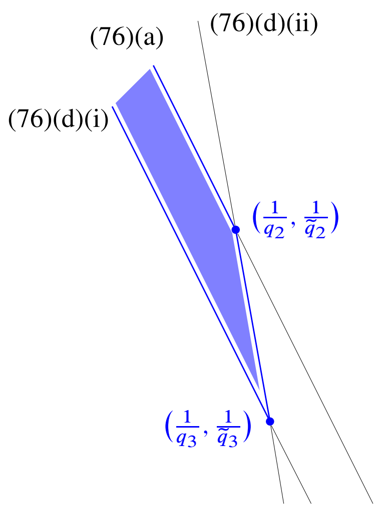


**Figure 23.** Convex region $C$.

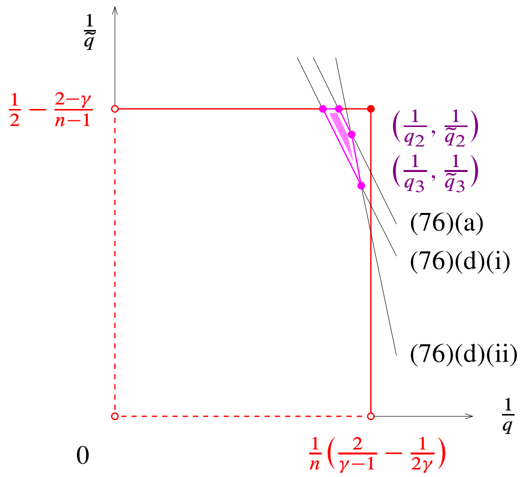


**Figure 24.** *Case 4*: $\gamma_{\mathrm{conf}} \le \gamma \le \gamma_3$.

Thirdly, in the plane with coordinates $(1/q, 1/\tilde{q})$, the conditions (69)(d)(i), (74)(a), and (74)(d)(ii) define the convex region $C$ in Figure 23 with edges

$$\begin{cases} \text{(a)} & \dfrac{\gamma}{q}+\dfrac{1}{\tilde{q}}=\dfrac{\gamma+1}{2}-\dfrac{2}{n-1}, \\ \text{(d)(i)} & \dfrac{\gamma}{q}+\dfrac{1}{\tilde{q}}=1, \\ \text{(d)(ii)} & \dfrac{\gamma}{q}+\dfrac{n-1}{2n}\dfrac{1}{\tilde{q}}=\dfrac{n+3}{4n}+\dfrac{2}{n}\dfrac{1}{\gamma-1} \end{cases} \tag{76}$$

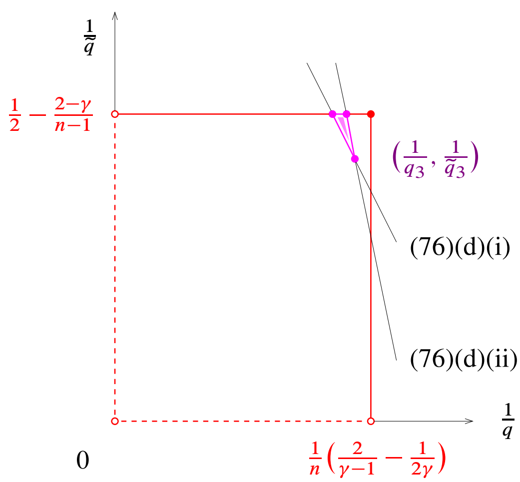


**Figure 25.** *Case 5*: $\gamma_3 \le \gamma \le \gamma_4$.

and with vertices given by

$$\begin{cases} \dfrac{1}{q_2} = \dfrac{4}{n+1}\dfrac{1}{\gamma-1} - \dfrac{1}{2}\dfrac{n-1}{n+1}, & \dfrac{1}{\tilde{q}_2} = \dfrac{n}{n+1}\gamma - \dfrac{4}{n+1}\dfrac{1}{\gamma-1} + \dfrac{1}{2} - \dfrac{2}{n-1} - \dfrac{4}{n+1}, \\ \dfrac{1}{q_3} = \dfrac{4}{n+1}\dfrac{1}{\gamma-1} - \dfrac{1}{2}\dfrac{n+3}{n+1}\dfrac{1}{\gamma}, & \dfrac{1}{\tilde{q}_3} = \dfrac{3}{2}\dfrac{n-1}{n+1} - \dfrac{4}{n+1}\dfrac{1}{\gamma-1}. \end{cases} \tag{77}$$

In order to conclude, it remains for us to determine the possible intersections of the convex region $C$ above with the rectangle $R$ defined by (75) and in each case the minimal regularity $\sigma = n(1/2-1/q)-1/p$.

- *Case 4*: $\gamma_{\text{conf}} \le \gamma \le \gamma_3$ (see Figure 24).
- *Case 5*: $\gamma_3 \le \gamma \le \gamma_4$ (see Figure 25).

In both cases, the minimal regularity $\sigma = \sigma_3(\gamma)$ is reached when $(1/p, 1/q)$ and $(1/q, 1/\tilde{q})$ lie on the edges (70)(d)(ii) and (76)(d)(ii). See Figures 21 and 22. This concludes the proof of Theorem 6.1 for $\gamma_{\text{conf}} < \gamma \le \gamma_4$ and $n \ge 6$.

► Assume that $3 \le n \le 5$.

Then $\gamma \ge \gamma_{\text{conf}} \ge 2$, and Figures 21 and 22 become Figures 26 and 27, respectively. Consequently, the four conditions (69)(a), (69)(ã), (69)(c), and (69)(d.ii) reduce again to the two conditions (74)(a)

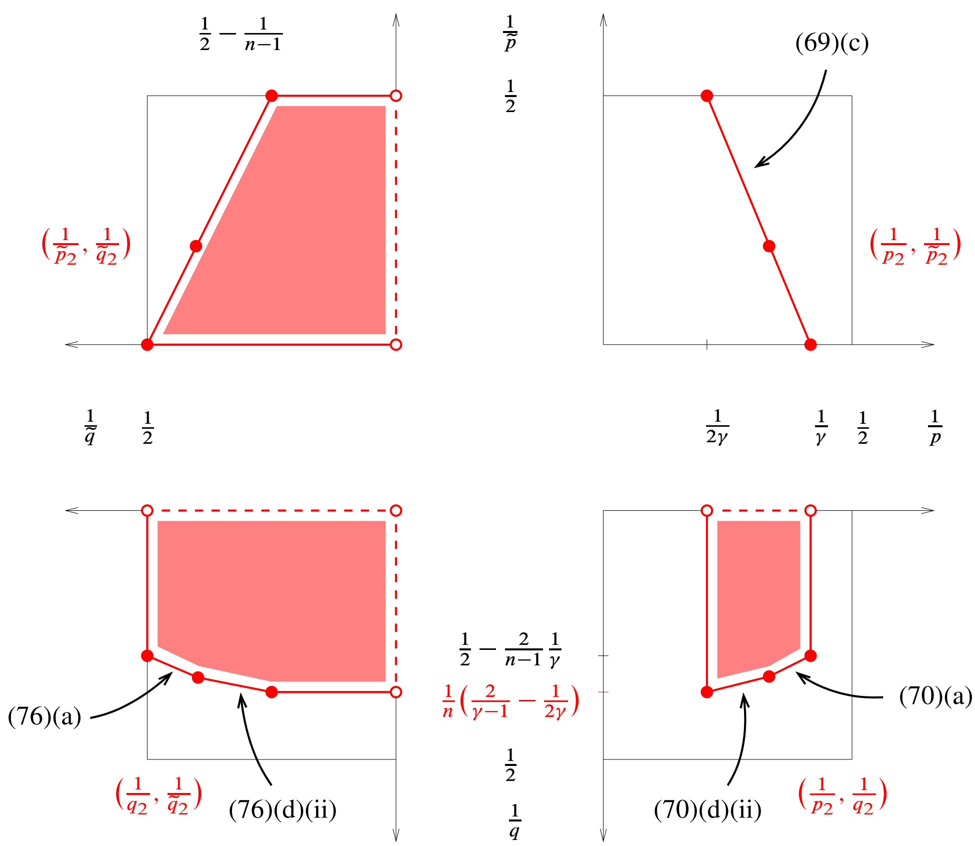


**Figure 26.** *Case 4*: $\gamma_{\text{conf}} \le \gamma \le \gamma_3$.

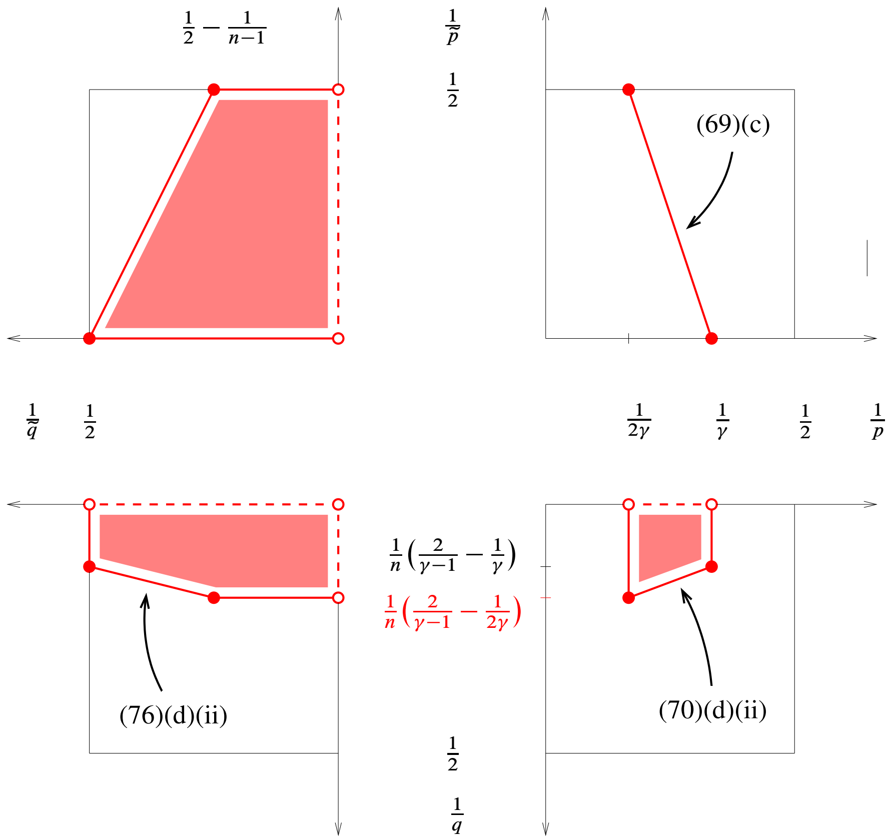


**Figure 27.** *Case 5*: $\gamma_3 \le \gamma \le \gamma_4$.

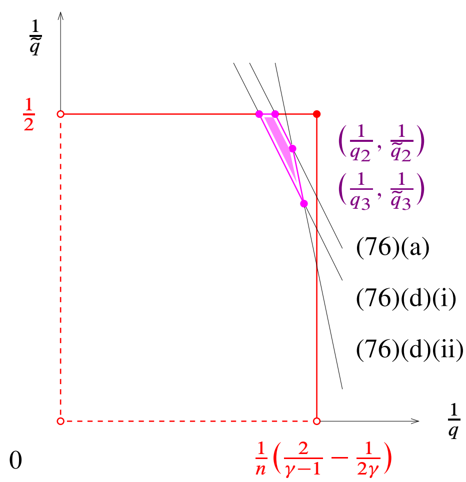


**Figure 28.** *Case 4*: $\gamma_{\text{conf}} \le \gamma \le \gamma_3$.

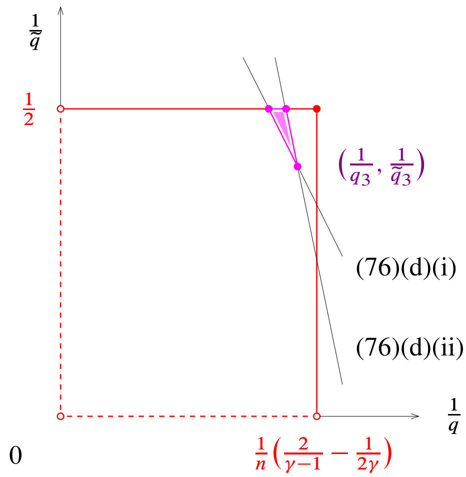


**Figure 29.** *Case 5*: $\gamma_3 \le \gamma \le \gamma_4$.

and (74)(d)(ii) if $\gamma_{\mathrm{conf}} \le \gamma \le \gamma_3$, and actually to the single condition (74)(d)(ii) if $\gamma_3 \le \gamma \le \gamma_4$, but this time in the rectangle

$$R = \left(0, \frac{1}{n}\left(\frac{2}{\gamma-1} - \frac{1}{2\gamma}\right)\right] \times \left(0, \frac{1}{2}\right]. \tag{78}$$

Moreover, (69)(e) is satisfied as $1/q \le (2/(\gamma-1) - 1/2\gamma)/n < 1/\gamma$.

We conclude again by examining the possible intersections $C \cap R$ of the convex region defined by (69)(d)(i), (74)(a), and (74)(d)(ii) with the rectangle (78) and by determining in each case the minimal regularity $\sigma = n(1/2 - 1/q) - 1/p$.

- *Case 4*: $\gamma_{\mathrm{conf}} \le \gamma \le \gamma_3$ (see Figure 28).
- *Case 5*: $\gamma_3 \le \gamma \le \gamma_4$ (see Figure 29).

In both cases, the minimal regularity $\sigma = \sigma_3(\gamma)$ is reached again when $(1/p, 1/q)$ and $(1/q, 1/\tilde{q})$ lie on the edges (70)(d)(ii) and (76)(d)(ii). See Figures 26 and 27. This concludes the proof of Theorem 6.1 for $\gamma_{\mathrm{conf}} < \gamma \le \gamma_4$ and $3 \le n \le 5$. □

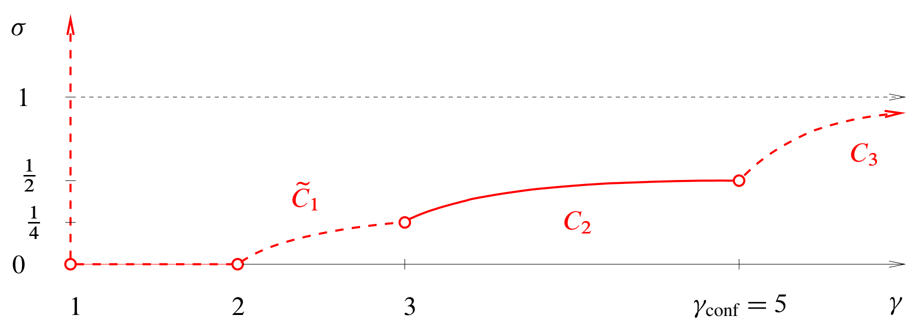


**Figure 30.** Regularity for global well-posedness on $\mathbb{H}^2$.

**Remark 6.3.** In dimension $n=2$, the statement of Theorem 6.1 holds true with (53) replaced by

$$\begin{cases} \sigma = 0^+ & \text{if } 1<\gamma\le 2,\\ \sigma=\widetilde{\sigma}_1(\gamma)^+ & \text{if } 2\le\gamma\le 3,\\ \sigma=\sigma_2(\gamma) & \text{if } 3<\gamma<5,\\ \sigma=\sigma_3(\gamma)^+ & \text{if } 5\le\gamma<\infty,\end{cases} \tag{79}$$

where $\widetilde{\sigma}_1(\gamma)=3/4-3/2\gamma$. Notice that the condition $q>\gamma$ is not redundant if $2<\gamma<3$ and that it is actually responsible for the curve $\widetilde{C}_1$.

**Remark 6.4.** In dimension $n=3$, Metcalfe and Taylor [2011] obtain a global existence result beyond $\gamma=\gamma_4$. In [Anker and Pierfelice $\ge$ 2014], we extend the results of our present paper to Damek–Ricci spaces as we did for the Schrödinger equation in [Anker et al. 2011] and for the shifted wave equation in [Anker et al. 2014], and we also discuss the case $\gamma>\gamma_4$ in this more general setting.

## Appendix A

In this appendix, we collect some lemmas in Fourier analysis on $\mathbb{R}$, which are used in the kernel analysis carried out in Section 3.

**Lemma A.1.** *Consider the oscillatory integral*

$$I(t,x)=\int_{-\infty}^{+\infty} d\lambda\, a(\lambda)e^{it\phi(\lambda)}$$

*where the phase is given by*

$$\phi(\lambda)=\sqrt{\lambda^2+\kappa^2}-\frac{x\lambda}{t}$$

(*recall that $\kappa$ is a fixed constant* $>0$) *and the amplitude $a\in\mathscr{C}_c^\infty(\mathbb{R})$ has the behavior at the origin*

$$a(\lambda)=\mathrm{O}(\lambda^2). \tag{A-1}$$

*Then*

$$|I(t,x)|\lesssim\frac{1+|x|}{(1+|t|)^{3/2}}\quad\forall|x|\le\frac{|t|}{2}.$$

*Proof.* Let us compute the first two derivatives

$$\phi'(\lambda)=\frac{\lambda}{\sqrt{\lambda^2+\kappa^2}}-\frac{x}{t}\quad\text{and}\quad\phi''(\lambda)=\kappa^2(\lambda^2+\kappa^2)^{-3/2}. \tag{A-2}$$

The phase $\phi$ has a single stationary point:

$$\lambda_0=\kappa\frac{x}{t}\Big(1-\frac{x^2}{t^2}\Big)^{-1/2}, \tag{A-3}$$

which remains bounded under our assumption $|x|\le|t|/2$:

$$|\lambda_0|\le\frac{\kappa}{\sqrt{3}}\le\kappa. \tag{A-4}$$

For later use, let us compute

$$\phi(\lambda_0)=\kappa\Big(1-\frac{x^2}{t^2}\Big)^{1/2}\quad\text{and}\quad\phi''(\lambda_0)=\kappa^{-1}\Big(1-\frac{x^2}{t^2}\Big)^{3/2}.$$

Since $\phi''>0$, we can perform a global change of variables $\lambda\leftrightarrow\mu$ on $\mathbb{R}$ so that

$$\phi(\lambda)-\phi(\lambda_0)=\mu^2.$$

Specifically,

$$\mu=\epsilon(\lambda)(\lambda-\lambda_0),$$

where

$$\epsilon(\lambda)=\Big\{\int_0^1 ds\,(1-s)\phi''\big((1-s)\lambda_0+s\lambda\big)\Big\}^{1/2}.$$

This way, our oscillatory integral becomes

$$I(t,x)=e^{it\phi(\lambda_0)}\int_{\mathbb{R}}d\mu\,\tilde a(\mu)e^{(-1+it)\mu^2},$$

where

$$\tilde a(\mu)=\frac{d\lambda}{d\mu}a(\lambda(\mu))e^{\mu^2}$$

is again a smooth function with compact support whose derivatives are controlled uniformly in $t$ and $x$ as long as $|x|\le|t|/2$. Using Taylor's formula, let us expand

$$\tilde a(\mu)=\sum_{j=0}^{3}\tilde a_j\mu^j+\tilde a_4(\mu)\mu^4,$$

where

$$\tilde a_0=\Big(\frac{2}{\phi''(\lambda_0)}\Big)^{1/2}a(\lambda_0)=\mathrm{O}(\lambda_0^2)=\mathrm{O}\Big(\frac{x^2}{t^2}\Big),$$

the other constants $\tilde a_1$, $\tilde a_2$, and $\tilde a_3$, and the function $\tilde a_4(\mu)$, as well as its derivatives, are bounded uniformly in $t$ and $x$. Let us split up accordingly

$$I(t,x)=\sum_{j=0}^{4}I_j(t,x),$$

where

$$I_j(t,x)=\tilde a_j e^{it\phi(\lambda_0)}\int_{\mathbb{R}}d\mu\,\mu^j e^{(-1+it)\mu^2}\quad(j=0,1,2,3)$$

and

$$I_4(t,x)=e^{it\phi(\lambda_0)}\int_{\mathbb{R}}d\mu\,\tilde a_4(\mu)\mu^4e^{(-1+it)\mu^2}.$$

The first and third expressions are handled by elementary complex integration:

$$I_0(t,x)=\tilde a_0\sqrt{\pi}e^{it\phi(\lambda_0)}(1-it)^{-1/2}=\mathrm{O}\Big(\frac{x^2}{t^2(1+|t|)^{1/2}}\Big)=\mathrm{O}\Big(\frac{1+|x|}{(1+|t|)^{3/2}}\Big),$$

$$I_2(t,x)=\tilde a_2\frac{\sqrt{\pi}}{2}e^{it\phi(\lambda_0)}(1-it)^{-3/2}=\mathrm{O}\big((1+|t|)^{-3/2}\big).$$

The expressions $I_1(t,x)$ and $I_3(t,x)$ vanish by oddness. The expression $I_4(t,x)$ is obviously bounded by the finite integral

$$\int_{\mathbb{R}} d\mu\, \mu^4 e^{-\mu^2}.$$

In order to improve this estimate when $|t|$ is large, let us split up

$$\int_{\mathbb{R}} d\mu = \int_{|\mu|\le|t|^{-1/2}} d\mu + \int_{|\mu|>|t|^{-1/2}} d\mu.$$

The first integral is easily estimated, using the uniform boundedness of $\tilde{a}_4(\mu)$:

$$\left|\int_{|\mu|\le|t|^{-1/2}} d\mu\, \tilde{a}_4(\mu)\mu^4 e^{(-1+it)\mu^2}\right| \lesssim \int_{|\mu|\le|t|^{-1/2}} d\mu\, \mu^4 \lesssim |t|^{-5/2}.$$

After two integration by parts, using $\mu e^{(-1+it)\mu^2} = (2(-1+it))^{-1}\frac{\partial}{\partial\mu} e^{(-1+it)\mu^2}$, the second integral is estimated by

$$|t|^{-5/2} + |t|^{-2}\int_{\mathbb{R}} d\mu\, (1+|\mu|)^2 e^{-\mu^2}.$$

Altogether,

$$I_4(t,x) = \mathrm{O}((1+|t|)^{-2}),$$

and this concludes the proof of Lemma A.1. $\square$

**Lemma A.2.** *Consider the oscillatory integral*

$$J(t,x) = \int_{-\infty}^{+\infty} d\lambda\, a(\lambda) e^{it\phi(\lambda)}$$

*where the phase is given again by*

$$\phi(\lambda) = \sqrt{\lambda^2+\kappa^2} - \frac{x\lambda}{t}$$

*and the amplitude $a(\lambda)$ is now a symbol (of any order) on $\mathbb{R}$, which vanishes on the interval $[-\kappa,\kappa]$. Then*

$$J(t,x) = \mathrm{O}(|t|^{-\infty}) \quad \forall x,\ 0 \le |x| \le \frac{|t|}{2}.$$

*Proof.* According to (A-2), (A-3), and (A-4),

- $\phi$ has a single stationary point $\lambda_0 \in \left[-\frac{\kappa}{\sqrt{3}}, \frac{\kappa}{\sqrt{3}}\right]$, which remains away from the support of $a$,
- $|\phi'(\lambda)| = \left|\frac{\lambda}{\sqrt{\lambda^2+\kappa^2}} - \frac{x}{t}\right| \ge \frac{1}{\sqrt{2}} - \frac{1}{2} > 0$ on $\operatorname{supp} a$,
- $\phi''$ is a symbol of order $-3$.

These facts allow us to perform several integrations by parts based on

$$e^{it\phi(\lambda)} = \frac{1}{it\phi'(\lambda)} \frac{\partial}{\partial\lambda} e^{it\phi(\lambda)}$$

and to reach the conclusion. $\square$

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