ANALYSIS & PDE

Volume 7 No. 4 2014

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ANALYSIS AND PDE
Vol. 7, No. 4, 2014
dx.doi.org/10.2140/apde.2014.7.997

PROBABILISTIC GLOBAL WELL-POSEDNESS FOR THE SUPERCRITICAL NONLINEAR HARMONIC OSCILLATOR

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Thanks to an approach inspired by Burq and Lebeau [*Ann. Sci. Éc. Norm. Supér.* (4) **6**:6 (2013)], we prove stochastic versions of Strichartz estimates for Schrödinger with harmonic potential. As a consequence, we show that the nonlinear Schrödinger equation with quadratic potential and any polynomial nonlinearity is almost surely locally well-posed in $L^2(\mathbb{R}^d)$ for any $d \geq 2$. Then, we show that we can combine this result with the high-low frequency decomposition method of Bourgain to prove a.s. global well-posedness results for the cubic equation: when $d=2$, we prove global well-posedness in $\mathcal{H}^s(\mathbb{R}^2)$ for any $s>0$, and when $d=3$ we prove global well-posedness in $\mathcal{H}^s(\mathbb{R}^3)$ for any $s>\frac{1}{6}$, which is a supercritical regime.

Furthermore, we also obtain almost sure global well-posedness results with scattering for NLS on $\mathbb{R}^d$ without potential. We prove scattering results for L^2-supercritical equations and L^2-subcritical equations with initial conditions in L^2 without additional decay or regularity assumption.

1. Introduction and results

1A. ***Introduction.*** It is known from several works that a probabilistic approach can help to give insight into the dynamics of dispersive nonlinear PDEs, even for low Sobolev regularity. This point of view was initiated by Lebowitz, Rose and Speer [1988], developed by Bourgain [1994; 1996] and Zhidkov [2001], and enhanced by Tzvetkov [2006; 2008; 2010], Burq and Tzvetkov [2008a; 2008b], Oh [2009/10; 2009], Colliander and Oh [2012] and others. In this paper we study the Cauchy problem for the nonlinear Schrödinger–Gross–Pitaevskii equation

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u - |x|^2 u = \pm |u|^{p-1}u, & (t,x)\in\mathbb{R}\times\mathbb{R}^d, \\ u(0)=u_0, \end{cases} \tag{1-1}$$

with $d \geq 2$, $p \geq 3$ an odd integer and where u_0 is a random initial condition.

Much work has been done on dispersive PDEs with random initial conditions since the papers of Burq and Tzvetkov [2008a; 2008b]. In these articles, the authors showed that, thanks to a randomisation of the initial condition, one can prove well-posedness results even for data with supercritical Sobolev regularity. We also refer to [Burq and Tzvetkov 2014; Thomann 2009; Burq et al. 2010; Poiret 2012a; 2012b;

Robert was partly supported by the grant “NOSEVOL” ANR-2011-BS01019 01. Thomann was partly supported by the grant “HANDDY” ANR-10-JCJC 0109 and by the grant “ANAÉ” ANR-13-BS01-0010-03.
MSC2010: 35P05, 35Q55, 35R60.
Keywords: harmonic oscillator, supercritical nonlinear Schrödinger equation, random initial conditions, scattering, global solutions.

de Suzzoni 2013; Nahmod and Staffilani 2013] for strong solutions in a probabilistic sense. Concerning weak solutions, see [Burq et al. 2012; $\geq$ 2014; Nahmod et al. 2013].

More recently, Burq and Lebeau [2013] considered a different randomisation method, and thanks to fine spectral estimates they obtained better stochastic bounds, which enabled them to improve the previous known results for the supercritical wave equation on a compact manifold. In [Poiret et al. 2013] we extended the results of [Burq and Lebeau 2013] to the harmonic oscillator in $\mathbb{R}^d$. This approach enables us to prove a stochastic version of the usual Strichartz estimates with a gain of $d/2$ derivatives, which we will use here to apply to the nonlinear problem. These estimates (the result of Proposition 2.1) can be seen as a consequence of [Poiret et al. 2013, Inequality (1.6)], but we give here an alternative proof suggested by Nicolas Burq.

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{g_n\}_{n\geq 0}$ be a sequence of real random variables, which we will assume to be independent and identically distributed. We assume that the common law ν of g_n satisfies, for some $c > 0$, the bound

$$\int_{-\infty}^{+\infty} e^{\gamma x}\,\mathrm{d}\nu \leq e^{c\gamma^2} \quad \text{for all } \gamma \in \mathbb{R}. \tag{1-2}$$

This condition implies in particular that the g_n are centred variables. It is easy to check that (1-2) is satisfied for centred Gauss laws and for any centred law with bounded support. Under condition (1-2), we can prove the Khinchin inequality (Lemma 2.3), which we will use in the sequel.

Let $d \geq 2$. We denote by

$$H = -\Delta + |x|^2$$

the harmonic oscillator and by $\{\varphi_j \mid j \geq 1\}$ an orthonormal basis of $L^2(\mathbb{R}^d)$ of eigenvectors of H (the Hermite functions). The eigenvalues of H are the $\{2(\ell_1 + \cdots + \ell_d) + d \mid \ell \in \mathbb{N}^d\}$, and we can order them in a non-decreasing sequence $\{\lambda_j \mid j \geq 1\}$, repeated according to their multiplicities, and so that $H\varphi_j = \lambda_j \varphi_j$.

We define the harmonic Sobolev spaces for $s \geq 0$, $p \geq 1$ by

$$\mathcal{W}^{s,p} = \mathcal{W}^{s,p}(\mathbb{R}^d) = \{u \in L^p(\mathbb{R}^d) \mid H^{s/2}u \in L^p(\mathbb{R}^d)\},$$
$$\mathcal{H}^s = \mathcal{H}^s(\mathbb{R}^d) = \mathcal{W}^{s,2}.$$

The natural norms are denoted by $\|u\|_{\mathcal{W}^{s,p}}$ and up to equivalence of norms, for $1 < p < +\infty$, we have [Yajima and Zhang 2004, Lemma 2.4]

$$\|u\|_{\mathcal{W}^{s,p}} = \|H^{s/2}u\|_{L^p} \equiv \|(-\Delta)^{s/2}u\|_{L^p} + \|\langle x\rangle^s u\|_{L^p}.$$

For $j \geq 1$, let

$$I(j) = \{n \in \mathbb{N} \mid 2j \leq \lambda_n < 2(j+1)\}.$$

Observe that, for all $j \geq d/2$, $I(j) \neq \varnothing$ and that $\#I(j) \sim c_d\, j^{d-1}$ when $j \to +\infty$.

Let $s \in \mathbb{R}$. Any $u \in \mathcal{H}^s(\mathbb{R}^d)$ can be written in a unique fashion as

$$u = \sum_{j=1}^{+\infty} \sum_{n \in I(j)} c_n \varphi_n.$$

Following a suggestion of Nicolas Burq, we introduce the condition

$$|c_k|^2 \le \frac{C}{\#I(j)} \sum_{n\in I(j)} |c_n|^2 \quad \text{for all } \ j \ge 1 \text{ and } k \in I(j), \tag{1-3}$$

which means that the coefficients have almost the same size on each level of energy $I(j)$. Observe that this condition is always satisfied in dimension $d = 1$. We define the set $\mathcal{A}_s \subset \mathcal{H}^s(\mathbb{R}^d)$ by

$$\mathcal{A}_s = \left\{ u = \sum_{j=1}^{+\infty} \sum_{n\in I(j)} c_n \varphi_n \in \mathcal{H}^s(\mathbb{R}^d) \;\middle|\; \text{condition (1-3) holds for some } C > 0 \right\}.$$

It is easy to check the following properties:

- If $u \in \mathcal{A}_s$, then for all $c \in \mathbb{C}$, $cu \in \mathcal{A}_s$.
- The set $\mathcal{A}_s$ is neither closed nor open in $\mathcal{H}^s$.
- The set $\mathcal{A}_s$ is invariant under the linear Schrödinger flow e^{-itH}.
- The set $\mathcal{A}_s$ depends on the choice of the orthonormal basis $(\varphi_n)_{n\ge 1}$. Indeed, given $u \in \mathcal{H}^s$, it is easy to see that there exists a Hilbertian basis $(\widetilde{\varphi}_n)_{n\ge 1}$ such that $u \in \widetilde{\mathcal{A}}_s$, where $\widetilde{\mathcal{A}}_s$ is the space based on $(\widetilde{\varphi}_n)_{n\ge 1}$.

Let $\gamma \in \mathcal{A}_s$. We define the probability measure μ_γ on $\mathcal{H}^s$ via the map

$$\begin{aligned} &\Omega \to \mathcal{H}^s(\mathbb{R}^d), \\ &\omega \mapsto \gamma^\omega = \sum_{j=1}^{+\infty} \sum_{n\in I(j)} c_n g_n(\omega) \varphi_n. \end{aligned}$$

In other words, μ_γ is defined by the condition, that for all measurable $F : \mathcal{H}^s \to \mathbb{R}$,

$$\int_{\mathcal{H}^s(\mathbb{R}^d)} F(v)\, d\mu_\gamma(v) = \int_\Omega F(\gamma^\omega)\, d\mathbb{P}(\omega).$$

In particular, we can check that μ_γ satisfies:

- If $\gamma \in \mathcal{H}^s \backslash \mathcal{H}^{s+\varepsilon}$, then $\mu_\gamma(\mathcal{H}^{s+\varepsilon}) = 0$.
- Assume that for all $j \ge 1$ such that $I(j) \ne \varnothing$ we have $c_j \ne 0$. Then for all nonempty open subsets $B \subset \mathcal{H}^s$, $\mu_\gamma(B) > 0$.

Finally, we denote by $\mathcal{M}^s$ the set of all such measures, $\mathcal{M}^s = \bigcup_{\gamma\in\mathcal{A}_s} \{\mu_\gamma\}$.

1B. ***Main results.*** Before we state our results, let us recall some facts concerning the deterministic study of the nonlinear Schrödinger equation (1-1). We say that (1-1) is locally well-posed in $\mathcal{H}^s(\mathbb{R}^d)$ if, for any initial condition $u_0 \in \mathcal{H}^s(\mathbb{R}^d)$, there exists a unique local in time solution $u \in \mathcal{C}([-T, T]; \mathcal{H}^s(\mathbb{R}^d))$, and if the flow-map is uniformly continuous. We denote by

$$s_c = \frac{d}{2} - \frac{2}{p-1}$$

the critical Sobolev index. Then one can show that NLS is well-posed in $\mathcal{H}^s(\mathbb{R}^d)$ when $s > \max(s_c, 0)$, and ill-posed when $s < s_c$. We refer to the introduction of [Thomann 2009] for more details on this topic.

1B1. *Local existence results.* We are now able to state our first result on the local well-posedness of (1-1).

Theorem 1.1. *Let $d \geq 2$, let $p \geq 3$ be an odd integer, and fix $\mu = \mu_\gamma \in \mathcal{M}^0$. Then there exists $\Sigma \subset L^2(\mathbb{R}^d)$ with $\mu(\Sigma) = 1$ and such that*:

(i) *For all $u_0 \in \Sigma$ there exist $T > 0$ and a unique local solution u to* (1-1) *with initial data u_0 satisfying*

$$u(t) - e^{-itH}u_0 \in \mathscr{C}([-T, T]; \mathcal{H}^s(\mathbb{R}^d)), \tag{1-4}$$

for some s such that $\frac{d}{2} - \frac{2}{p-1} < s < \frac{d}{2}$.

(ii) *More precisely, for all $T > 0$, there exists $\Sigma_T \subset \Sigma$ with*

$$\mu(\Sigma_T) \geq 1 - C\exp\bigl(-cT^{-\delta}\|\gamma\|^{-2}_{L^2(\mathbb{R}^2)}\bigr), \quad C, c, \delta > 0,$$

and such that for all $u_0 \in \Sigma_T$ the lifespan of u is larger than T.

Let $\gamma = \sum_{n=0}^{+\infty} c_n\varphi_n(x)$. Then

$$u_0^\omega := \sum_{n=0}^{+\infty} g_n(\omega)c_n\varphi_n(x)$$

is a typical element in the support of μ_γ. Another way to state Theorem 1.1 is: for any $T > 0$, there exists an event $\Omega_T \subset \Omega$ such that

$$\mathbb{P}(\Omega_T) \geq 1 - C\exp\bigl(-cT^{-\delta}\|\gamma\|^{-2}_{L^2(\mathbb{R}^d)}\bigr), \quad C, c, \delta > 0,$$

and that for all $\omega \in \Omega_T$, there exists a unique solution of the form (1-4) to (1-1) with initial data u_0^ω.

We will see in Proposition 2.1 that the stochastic approach yields a gain of $d/2$ derivatives compared to the deterministic theory. To prove Theorem 1.1 we only have to gain $s_c = d/2 - 2/(p-1)$ derivatives. The solution is constructed by a fixed point argument in a Strichartz space $X_T^s \subset \mathscr{C}([-T, T]; \mathcal{H}^s(\mathbb{R}^d))$ with continuous embedding, and uniqueness holds in the class X_T^s.

The deterministic Cauchy problem for (1-1) was studied by Oh [1989] (see also [Cazenave 2003, Chapter 9] for more references). Thomann [2009] has proven an almost sure local existence result for (1-1) in the supercritical regime (with a gain of $\frac{1}{4}$ of a derivative), for any $d \geq 1$. This local existence result was improved by [Burq et al. 2010] when $d = 1$ (gain of $\frac{1}{2}$ a derivative), by [Deng 2012] when $d = 2$, and by Poiret [2012a; 2012b] in any dimension.

Remark 1.2. The results of Theorem 1.1 also hold true for any quadratic potential

$$V(x) = \sum_{1 \leq j \leq d} \alpha_j x_j^2, \quad \alpha_j > 0, \quad 1 \leq j \leq d,$$

and for more general potentials such that $V(x) \approx \langle x \rangle^2$.

1B2. *Global existence and scattering results for NLS.* As an application of the results of the previous part, we are able to construct global solutions to the nonlinear Schrödinger equation without potential, which scatter when $t \to \pm\infty$. Consider the equation

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u = \pm |u|^{p-1}u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d. \\ u(0) = u_0. \end{cases} \tag{1-5}$$

The well-posedness indexes for this equation are the same as for (1-1). Namely, (1-5) is well-posed in $H^s(\mathbb{R}^d)$ when $s > \max(s_c, 0)$, and ill-posed when $s < s_c$.

For the next result, we will need an additional condition on the law ν. We assume that

$$\mathbb{P}(|g_n| < \rho) > 0 \quad \text{for all } \rho > 0, \tag{1-6}$$

which ensures that the random variable can take arbitrarily small values. Then we can prove:

Theorem 1.3. *Let $d \geq 2$, let $p \geq 3$ be an odd integer, and fix $\mu = \mu_\gamma \in \mathcal{M}^0$. Assume that* (1-6) *holds. Then there exists $\Sigma \subset L^2(\mathbb{R}^d)$ with $\mu(\Sigma) > 0$ and such that*:

(i) *For all $u_0 \in \Sigma$ there exists a unique global solution u to* (1-5) *with initial data u_0 satisfying*

$$u(t) - e^{it\Delta}u_0 \in \mathscr{C}(\mathbb{R}; \mathscr{H}^s(\mathbb{R}^d)),$$

for some s such that $\frac{d}{2} - \frac{2}{p-1} < s < \frac{d}{2}$.

(ii) *For all $u_0 \in \Sigma$ there exist states $f_+, f_- \in \mathscr{H}^s(\mathbb{R}^d)$ such that when $t \to \pm\infty$,*

$$\|u(t) - e^{it\Delta}(u_0 + f_\pm)\|_{H^s(\mathbb{R}^d)} \to 0.$$

(iii) *If we assume that the distribution of ν is symmetric, then*

$$\mu\big(u_0 \in L^2(\mathbb{R}^d) : \textit{assertion (ii) holds true} \;\big|\; \|u_0\|_{L^2(\mathbb{R}^d)} \leq \eta\big) \to 1,$$

when $\eta \to 0$.

We can show [Poiret 2012a, Théorème 20] that for all $s > 0$, if $u_0 \notin \mathscr{H}^\sigma(\mathbb{R}^d)$ then $\mu(H^\sigma(\mathbb{R}^d)) = 0$. This shows that the randomisation does not yield a gain of derivative in the Sobolev scale; thus Theorem 1.3 gives results for initial conditions not covered by the deterministic theory.

There is a large literature on the deterministic local and global theory with scattering for (1-5). We refer to [Banica et al. 2008; Nakanishi and Ozawa 2002; Carles 2009] for such results and more references.

We do not give here the details of the proof of Theorem 1.3, since one can follow the main lines of the argument of Poiret [2012a; 2012b] but with different constants (see, e.g., [Poiret 2012b, Théorème 4]). The proof of (i) and (ii) is based on the use of an explicit transform, called the lens transform $\mathscr{L}$, which links the solutions of (1-5) to solutions of NLS with harmonic potential. The transform $\mathscr{L}$ has been used in different contexts; see [Carles 2009] for scattering results and more references. More precisely, for $u(t,x) : \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\times \mathbb{R}^d \to \mathbb{C}$ we define

$$v(t,x) = \mathscr{L}u(t,x) = \left(\frac{1}{\sqrt{1+4t^2}}\right)^{d/2} u\left(\frac{\arctan(2t)}{2}, \frac{x}{\sqrt{1+4t^2}}\right) e^{i|x|^2 t/(1+4t^2)},$$

then u is a solution to

$$i\frac{\partial u}{\partial t} - Hu = \lambda \cos(2t)^{\frac{1}{2}d(p-1)-2}|u|^{p-1}u$$

if and only if v satisfies $i\,\partial v/\partial t + \Delta v = \lambda|v|^{p-1}v$. Theorem 1.1 provides solutions with lifespan larger than $\pi/4$ for large probabilities, provided that the initial conditions are small enough.

Part (iii) is stated in [Poiret 2012a, Théorème 9], and can be understood as a small data result.

In Theorem 1.3 we assumed that $d \geq 2$ and that $p \geq 3$ was an odd integer, so we had $p \geq 1 + 4/d$, or, in other words, we were in an L^2-supercritical setting. Our approach also allows to get results in an L^2-subcritical context, i.e., when $1 + 2/d < p < 1 + 4/d$.

Theorem 1.4. *Let $d = 2$ and $2 < p < 3$. Assume that* (1-6) *holds and fix $\mu = \mu_\gamma \in \mathcal{M}^0$. Then there exists $\Sigma \subset L^2(\mathbb{R}^2)$ with $\mu(\Sigma) > 0$ and such that for all* $0 < \varepsilon < 1$:

(i) *For all $u_0 \in \Sigma$ there exists a unique global solution u to* (1-5) *with initial data u_0 satisfying*

$$u(t) - e^{it\Delta}u_0 \in \mathscr{C}(\mathbb{R}; \mathscr{H}^{1-\varepsilon}(\mathbb{R}^2)).$$

(ii) *For all $u_0 \in \Sigma$ there exist states $f_+, f_- \in \mathscr{H}^{1-\varepsilon}(\mathbb{R}^2)$ such that when $t \to \pm\infty$,*

$$\|u(t) - e^{it\Delta}(u_0 + f_\pm)\|_{H^{1-\varepsilon}(\mathbb{R}^2)} \to 0.$$

(iii) *If we assume that the distribution of ν is symmetric, then*

$$\mu\big(u_0 \in L^2(\mathbb{R}^2) : \textit{assertion (ii) holds true} \;\big|\; \|u_0\|_{L^2(\mathbb{R}^2)} \leq \eta\big) \to 1,$$

when $\eta \to 0$.

In the case $p \leq 1+2/d$, Barab [1984] showed that a nontrivial solution to (1-5) never scatters; therefore even with a stochastic approach one can not have scattering in this case. When $d = 2$, the condition $p > 2$ in Theorem 1.4 is therefore optimal. Usually, deterministic scattering results in L^2-subcritical contexts are obtained in the space $H^1 \cap \mathscr{F}(H^1)$. Here we assume $u_0 \in L^2$, and thus we relax both the regularity and the decay assumptions (this latter point is the most striking in this context). Again we refer to [Banica et al. 2008] for an overview of scattering theory for NLS.

When $\mu \in \mathcal{M}^\sigma$ for some $0 < \sigma < 1$ we are able to prove the same result with $\varepsilon = 0$. Since the proof is much easier, we give it before the case $\sigma = 0$ (see Section 3B).

Finally, we point out that in Theorem 1.4 we are only able to consider the case $d = 2$ because of the lack of regularity of the nonlinear term $|u|^{p-1}u$.

1B3. *Global existence results for NLS with quadratic potential.* We also get global existence results for defocusing Schrödinger equation with harmonic potential. For $d = 2$ or $d = 3$, consider the equation

$$\begin{cases} i\dfrac{\partial u}{\partial t} - Hu = |u|^2u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0) = u_0, \end{cases} \tag{1-7}$$

and denote by E the energy of (1-7), namely

$$E(u) = \|u\|^2_{\mathscr{H}^1(\mathbb{R}^d)} + \tfrac{1}{2}\|u\|^4_{L^4(\mathbb{R}^d)}.$$

Deterministic global existence for (1-7) has been studied by Zhang [2005] and by Carles [2011] in the case of time-dependent potentials.

When $d = 3$, our global existence result for (1-7) is the following:

Theorem 1.5. *Let $d = 3$, $\frac{1}{6} < s < 1$ and fix $\mu = \mu_\gamma \in \mathcal{M}^s$. Then there exists a set $\Sigma \subset \mathcal{H}^s(\mathbb{R}^3)$ such that $\mu(\Sigma) = 1$ and that the following holds true*:

(i) *For all $u_0 \in \Sigma$, there exists a unique global solution to* (1-7), *which reads*

$$u(t) = e^{-itH} u_0 + w(t), \quad w \in \mathcal{C}(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^3)).$$

(ii) *The previous line defines a global flow Φ, which leaves the set Σ invariant*:

$$\Phi(t)(\Sigma) = \Sigma, \quad \text{for all } t \in \mathbb{R}.$$

(iii) *There exist $C, c_s > 0$ such that, for all $t \in \mathbb{R}$,*

$$E(w(t)) \leq C(M + |t|)^{c_s+},$$

where M is a positive random variable such that

$$\mu(u_0 \in \mathcal{H}^s(\mathbb{R}^3) : M > K) \leq Ce^{-cK^\delta/\|\gamma\|^2_{\mathcal{H}^s(\mathbb{R}^3)}}.$$

Here the critical Sobolev space is $\mathcal{H}^{1/2}(\mathbb{R}^3)$; thus the local deterministic theory, combined with the conservation of the energy, immediately gives global well-posedness in $\mathcal{H}^1(\mathbb{R}^3)$. Using a kind of interpolation method due to Bourgain, one may obtain deterministic global well-posedness in $\mathcal{H}^s(\mathbb{R}^3)$ for some $1/2 < s < 1$. Instead, for the proof of Theorem 1.5, we will rely on the almost well-posedness result of Theorem 1.1, and this gives global well-posedness in a supercritical context.

The constant $c_s > 0$ can be computed explicitly (see (4-16)), and we do not think that we have obtained the optimal rate. By reversibility of the equation, it is enough to consider only positive times.

With a similar approach, in dimension $d = 2$, we can prove:

Theorem 1.6. *Let $d = 2$, $0 < s < 1$ and fix $\mu = \mu_\gamma \in \mathcal{M}^s$. Then there exists a set $\Sigma \subset \mathcal{H}^s(\mathbb{R}^2)$ such that $\mu(\Sigma) = 1$ and that, for all $u_0 \in \Sigma$, there exists a unique global solution to* (1-7),

$$u(t) = e^{-itH} u_0 + w(t), \quad w \in \mathcal{C}(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^2)).$$

In addition, statements (ii) *and* (iii) *of Theorem 1.5 are also satisfied with $c_s = \dfrac{1-s}{s}$.*

Here the critical Sobolev space is $L^2(\mathbb{R}^2)$; thus Theorem 1.6 shows global well-posedness for any subcritical cubic nonlinear Schrödinger equations in dimension two.

Using the smoothing effect, which yields a gain of $\frac{1}{2}$ a derivative, a global well-posedness result for (1-1), in the defocusing case, was given in [Burq et al. 2010] in the case $d = 1$, for any $p \geq 3$. The global existence is proved for a typical initial condition on the support of a Gibbs measure, which is $\bigcap_{\sigma>0} \mathcal{H}^{-\sigma}(\mathbb{R})$. This result was extended by Deng [2012] in dimension $d = 2$ for radial functions. However, this approach has the drawback that it relies on the invariance of a Gibbs measure, which is a rigid object, and is supported in rough Sobolev spaces. Therefore it seems difficult to adapt this strategy in higher dimensions.

Here instead we obtain the results of Theorems 1.5 and 1.6 as a combination of Theorem 1.1 with the high-low frequency decomposition method of [Bourgain 1999, p. 84]. This approach has been successful in different contexts, and has been first used together with probabilistic arguments by Colliander and Oh [2012] for the cubic Schrödinger below $L^2(\mathbb{S}^1)$ and later on by Burq and Tzvetkov [2014] for the wave equation.

1C. ***Notations and plan of the paper.*** In this paper $c, C > 0$ denote constants, the value of which may change from line to line. These constants will always be universal, or uniformly bounded with respect to the other parameters.

We let $L^p_T = L^p_{[-T,T]} = L^p(-T,T)$ for $T > 0$ and we write $L^p_x = L^p(\mathbb{R}^d)$. We denote the harmonic oscillator on $\mathbb{R}^d$ by $H = -\Delta + |x|^2 = \sum_{j=1}^d (-\partial_j^2 + x_j^2)$, and for $s \geq 0$ we define the Sobolev space $\mathcal{H}^s$ by the norm $\|u\|_{\mathcal{H}^s} = \|H^{s/2}u\|_{L^2(\mathbb{R}^d)}$. More generally, we define the spaces $\mathcal{W}^{s,p}$ by the norm $\|u\|_{\mathcal{W}^{s,p}} = \|H^{s/2}u\|_{L^p(\mathbb{R}^d)}$. If E is a Banach space and μ is a measure on E, we write $L^p_\mu = L^p(d\mu)$ and $\|u\|_{L^p_\mu E} = \big\| \|u\|_E \big\|_{L^p_\mu}$.

The rest of the paper is organised as follows. In Section 2 we recall some deterministic results on the spectral function, and prove stochastic Strichartz estimates. Section 3 is devoted to the proof of Theorem 1.1 and of the scattering results for NLS without potential. Finally, in Section 4 we study the global existence for the Schrödinger–Gross–Pitaevskii equation (1-1).

2. Stochastic Strichartz estimates

The main result of this section is the following probabilistic improvement of the Strichartz estimates.

Proposition 2.1. *Let $s \in \mathbb{R}$ and $\mu = \mu_\gamma \in \mathcal{M}^s$. Let $1 \leq q < +\infty$, $2 \leq r \leq +\infty$, and set $\alpha = d\left(\frac{1}{2} - \frac{1}{r}\right)$ if $r < +\infty$ and $\alpha < d/2$ if $r = +\infty$. Then there exist $c, C > 0$ such that, for all $\tau \in \mathbb{R}$,*

$$\mu\Big(u \in \mathcal{H}^s(\mathbb{R}^d) : \|e^{-i(t+\tau)H}u\|_{L^q_{[0,T]}\mathcal{W}^{s+\alpha,r}(\mathbb{R}^d)} > K\Big) \leq C e^{-cK^2/T^{2/q}\|\gamma\|^2_{\mathcal{H}^s(\mathbb{R}^d)}}.$$

When $r = +\infty$, this result expresses a gain μ-a.s. of $d/2$ derivatives in space compared to the deterministic Strichartz estimates (see the bound (3-2)).

Proposition 2.1 is a consequence of [Poiret et al. 2013, Inequality (1.6)], but we give here a self-contained proof suggested by Nicolas Burq.

There are two key ingredients in the proof of Proposition 2.1. The first one is a deterministic estimate on the spectral function given in Lemma 2.2, and the second is the Khinchin inequality stated in Lemma 2.3.

2A. ***Deterministic estimates of the spectral function.*** We define the spectral function π_H for the harmonic oscillator by

$$\pi_H(\lambda; x, y) = \sum_{\lambda_j \leq \lambda} \varphi_j(x)\overline{\varphi_j(y)},$$

and this definition does not depend on the choice of $\{\varphi_j \mid j \in \mathbb{N}\}$.

Let us recall some results of π_H, which were essentially obtained by Thangavelu [1993, Lemma 3.2.2, p. 70] (see also [Karadzhov 1995] and [Poiret et al. 2013, Section 3] for more details).

Thanks to the Mehler formula, we can prove

$$\pi_H(\lambda; x, x) \le C\lambda^{d/2} \exp\left(-c\frac{|x|^2}{\lambda}\right) \quad \text{for all } x \in \mathbb{R}^d \text{ and } \lambda \ge 1. \tag{2-1}$$

One also has the following more subtle bound, which is the heart of [Karadzhov 1995]:

$$|\pi_H(\lambda+\mu; x, x) - \pi_H(\lambda; x, x)| \le C(1+|\mu|)\lambda^{d/2-1} \quad \text{for } \lambda \ge 1,\ |\mu| \le C_0\lambda. \tag{2-2}$$

This inequality gives a bound on π_H in energy interval of size ~ 1, which is the finest one can obtain.

Then we can prove (see [Poiret et al. 2013, Lemma 3.5]):

Lemma 2.2. *Let $d \ge 2$ and assume that $|\mu| \le c_0$, $r \ge 1$ and $\theta \ge 0$. Then there exists $C > 0$ such that for all $\lambda \ge 1$*

$$\|\pi_H(\lambda+\mu; x, x) - \pi_H(\lambda; x, x)\|_{L^r(\mathbb{R}^d)} \le C\lambda^{\frac{1}{2}d(1+1/r)-1}.$$

2B. *Proof of Proposition 2.1.* To begin with, recall the Khinchin inequality, which shows a smoothing property of the random series in the L^k spaces for $k \ge 2$; for example, see [Burq and Tzvetkov 2008a, Lemma 4.2].

Lemma 2.3. *There exists $C > 0$ such that for all real $k \ge 2$ and $(c_n) \in \ell^2(\mathbb{N})$*

$$\left\|\sum_{n\ge 1} g_n(\omega)\, c_n\right\|_{L^k_{\mathbb{P}}} \le C\sqrt{k}\left(\sum_{n\ge 1} |c_n|^2\right)^{\frac{1}{2}}.$$

Now we fix $\gamma = \sum_{n=0}^{+\infty} c_n\varphi_n \in \mathcal{A}_s$ and let $\gamma^\omega = \sum_{n=0}^{+\infty} g_n(\omega) c_n \varphi_n$.

Firstly, we treat the case $r < +\infty$. Set $\alpha = d\left(\frac{1}{2} - \frac{1}{r}\right)$ and set $\sigma = s + \alpha$. Observe that it suffices to prove the estimation for $K \gg \|\gamma\|_{\mathcal{H}^s(\mathbb{R}^d)}$.

Let $k \ge 1$. By definition,

$$\begin{aligned}\int_{\mathcal{H}^s(\mathbb{R}^d)} \left\|e^{-i(t+\tau)H} u\right\|^k_{L^q_{[0,T]}\mathcal{W}^{\sigma,r}(\mathbb{R}^d)} d\mu(u) &= \int_\Omega \left\|e^{-i(t+\tau)H}\gamma^\omega\right\|^k_{L^q_{[0,T]}\mathcal{W}^{\sigma,r}(\mathbb{R}^d)} d\mathbb{P}(\omega)\\ &= \int_\Omega \left\|e^{-i(t+\tau)H} H^{\sigma/2}\gamma^\omega\right\|^k_{L^q_{[0,T]}L^r(\mathbb{R}^d)} d\mathbb{P}(\omega).\end{aligned} \tag{2-3}$$

Since $e^{-i(t+\tau)H} H^{\sigma/2}\gamma^\omega(x) = \sum_{n=0}^{+\infty} g_n(\omega) c_n \lambda_n^{\sigma/2} e^{-i(t+\tau)\lambda_n}\varphi_n(x)$, by Lemma 2.3 we get

$$\left\|e^{-i(t+\tau)H} H^{\sigma/2}\gamma^\omega(x)\right\|_{L^k_{\mathbb{P}}} \le C\sqrt{k}\left\|e^{-i(t+\tau)H} H^{\sigma/2}\gamma^\omega(x)\right\|_{L^2_{\mathbb{P}}} = C\sqrt{k}\left(\sum_{n=0}^{+\infty} \lambda_n^\sigma |c_n|^2 |\varphi_n(x)|^2\right)^{\frac{1}{2}}.$$

Assume that $k \geq r$. By the integral Minkowski inequality, the previous line and the triangle inequality we get

$$
\begin{aligned}
\big\| e^{-i(t+\tau)H} H^{\sigma/2} \gamma^{\omega} \big\|_{L^k_{\mathbb{P}} L^r_x} &\leq \big\| e^{-i(t+\tau)H} H^{\sigma/2} \gamma^{\omega} \big\|_{L^r_x L^k_{\mathbb{P}}} \\
&\leq C\sqrt{k} \Big\| \sum_{k=0}^{+\infty} \lambda_k^{\sigma} |c_k|^2 |\varphi_k|^2 \Big\|^{\frac{1}{2}}_{L^{r/2}(\mathbb{R}^d)} \\
&\leq C\sqrt{k} \Big(\sum_{j=1}^{+\infty} \Big\| \sum_{k \in I(j)} \lambda_k^{\sigma} |c_k|^2 |\varphi_k|^2 \Big\|_{L^{r/2}(\mathbb{R}^d)} \Big)^{\frac{1}{2}}.
\end{aligned}
\tag{2-4}
$$

Condition (1-3) implies that for all $x \in \mathbb{R}^d$ and $k \in I(j) = \{n \in \mathbb{N} \mid 2j \leq \lambda_n < 2(j+1)\}$

$$
\lambda_k^{\sigma} |c_k|^2 |\varphi_k(x)|^2 \leq C j^{\sigma} \sum_{n \in I(j)} |c_n|^2 \frac{|\varphi_k(x)|^2}{\# I(j)},
$$

and thus, by Lemma 2.2 and the fact that $\# I(j) \sim c j^{d-1}$,

$$
\begin{aligned}
\Big\| \sum_{k \in I(j)} \lambda_k^{\sigma} |c_k|^2 |\varphi_k(x)|^2 \Big\|_{L^{r/2}(\mathbb{R}^d)} &\leq C j^{\sigma} \sum_{n \in I(j)} |c_n|^2 \frac{\big\| \sum_{k \in I(j)} |\varphi_k(x)|^2 \big\|_{L^{r/2}(\mathbb{R}^d)}}{\# I(j)} \\
&\leq C j^{\sigma + d(1/r - 1/2)} \sum_{n \in I(j)} |c_n|^2 \\
&= C j^{s} \sum_{n \in I(j)} |c_n|^2.
\end{aligned}
$$

The latter inequality together with (2-4) gives

$$
\big\| e^{-i(t+\tau)H} H^{\sigma/2} \gamma^{\omega} \big\|_{L^k_{\mathbb{P}} L^r_x} \leq C\sqrt{k} \|\gamma\|_{\mathcal{H}^s(\mathbb{R}^d)},
$$

and for $k \geq r$, by Minkowski,

$$
\big\| e^{-i(t+\tau)H} H^{\sigma/2} \gamma^{\omega} \big\|_{L^k_{\mathbb{P}} L^q_{[0,T]} L^r_x} \leq C\sqrt{k}\, T^{1/q} \|\gamma\|_{\mathcal{H}^s(\mathbb{R}^d)}.
$$

Then, using (2-3) and the Bienaymé–Chebishev inequality, we obtain

$$
\begin{aligned}
\mu\big(u \in \mathcal{H}^s : \|e^{-i(t+\tau)H} u\|_{L^q_{[0,T]} \mathcal{W}^{\sigma,r}(\mathbb{R}^d)} > K\big) &\leq \big(K^{-1} \|e^{-i(t+\tau)H} H^{\sigma/2} \gamma^{\omega}\|_{L^k_{\mathbb{P}} L^q_{[0,T]} L^r_x}\big)^k \\
&\leq \big(C K^{-1} \sqrt{k}\, T^{1/q} \|\gamma\|_{\mathcal{H}^s(\mathbb{R}^d)}\big)^k.
\end{aligned}
$$

Finally, if $K \gg \|\gamma\|_{\mathcal{H}^s(\mathbb{R}^d)}$, we can choose $k = K^2 \big/ 2CT^{2/q} \|\gamma\|^2_{\mathcal{H}^s(\mathbb{R}^d)} \geq r$, which yields the result.

Now assume $r = +\infty$. We use the Sobolev inequality to get $\|u\|_{\mathcal{W}^{s,\infty}} \leq C \|u\|_{\mathcal{W}^{\tilde{s},\tilde{r}}}$ with $\tilde{s} = s + 2d/\tilde{r}$ for $\tilde{r} \geq 1$ large enough; hence we can apply the previous result for $r < +\infty$.

Remark 2.4. A similar result to Proposition 2.1 holds, with the same gain of derivatives, when $I(\lambda)$ is replaced with the dyadic interval $J(j) = \{n \in \mathbb{N} \mid 2^j \leq \lambda_n < 2^{j+1}\}$. Then the condition (1-3) becomes

$$|c_k|^2 \leq \frac{C}{\#J(j)} \sum_{n \in J(j)} |c_n|^2 \quad \text{for all } \ j \geq 1 \text{ and } k \in J(j), \tag{2-5}$$

which seems more restrictive. Indeed neither condition imply the other.

Observe that if we want to prove the result under condition (2-5), the subtle estimate (2-2) is not needed; (2-1) is enough.

Remark 2.5. For $d = 1$, condition (1-3) is always satisfied but condition (2-2) is not. Instead we can use that $\|\varphi_k\|_p \leq C\lambda_k^{-\theta(p)}$ with $\theta(p) > 0$ for $p > 2$ [Koch and Tataru 2005]. For example if $p > 4$ we have $\theta(p) = \frac{1}{4} - (p-1)/6p$. Thus we get the Proposition 2.1 with $s = p\theta(p)/4$ (see [Thomann 2009; Burq et al. 2010], where this is used).

Remark 2.6. Another approach could have been to exploit the particular basis $(\varphi_n)_{n\geq 1}$, which satisfies the good L^∞ estimates given in [Poiret et al. 2013, Theorem 1.3], and to construct the measures μ as the image measures of random series of the form

$$\gamma^\omega(x) = \sum_{n\geq 1} c_n g_n(\omega)\varphi_n(x),$$

with $c_n \in \ell^2(\mathbb{N})$ not necessarily satisfying (1-3). A direct application of the Khinchin inequality (as in [Thomann 2009, Proposition 2.3]) then gives the same bounds as in Proposition 2.1. Observe that condition (1-3) is also needed in this approach, but it directly intervenes in the construction of the φ_n.

We believe that the strategy we adopted here is slightly more general, since it seems to work even in cases where we do not have a basis of eigenfunctions that satisfy bounds analogous to [Poiret et al. 2013, Theorem 1.3], as for example in the case of the operator $-\Delta + |x|^4$.

3. Application to the local theory of the supercritical Schrödinger equation

3A. ***Almost sure local well-posedness.*** This subsection, devoted to the proof of Theorem 1.1, follows the argument of [Poiret 2012b].

Let $u_0 \in L^2(\mathbb{R}^d)$. We look for a solution to (1-1) of the form $u = e^{-itH}u_0 + v$, where v is some fluctuation term more regular than the linear profile $e^{-itH}u_0$. By the Duhamel formula, the unknown v has to be a fixed point of the operator

$$L(v) := \mp i \int_0^t e^{-i(t-s)H} |e^{-isH}u_0 + v(s)|^{p-1}(e^{-isH}u_0 + v(s))\, ds, \tag{3-1}$$

in some adequate functional space, which is a Strichartz space.

To begin with, we recall the Strichartz estimates for the harmonic oscillator. A couple $(q, r) \in [2, +\infty]^2$ is called admissible if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \quad \text{and} \quad (d, q, r) \neq (2, 2, +\infty),$$

and if one defines

$$X_T^s := \bigcap_{\substack{(q,r)\\ \text{admissible}}} L^q([-T,T];\mathcal{W}^{s,r}(\mathbb{R}^d)),$$

then for all $T>0$ there exists $C_T>0$ such that for all $u_0\in\mathcal{H}^s(\mathbb{R}^d)$ we have

$$\|e^{-itH}u_0\|_{X_T^s}\le C_T\|u_0\|_{\mathcal{H}^s(\mathbb{R}^d)}. \tag{3-2}$$

We will also need the inhomogeneous version of Strichartz: For all $T>0$, there exists $C_T>0$ such that for all admissible couples (q,r) and functions $F\in L^{q'}([T,T];\mathcal{W}^{s,r'}(\mathbb{R}^d))$,

$$\Big\|\int_0^t e^{-i(t-s)H}F(s)\,ds\Big\|_{X_T^s}\le C_T\|F\|_{L^{q'}([-T,T],\mathcal{W}^{s,r'}(\mathbb{R}^d))}, \tag{3-3}$$

where q' and r' are the Hölder conjugates of q and r. We refer to [Poiret 2012b] for a proof.

The next result is a direct application of the Sobolev embeddings and Hölder.

Lemma 3.1. *Let $(q,r)\in[2,\infty[\times[2,\infty]$, and let $s,s_0\ge 0$ be such that $s-s_0>\dfrac{d}{2}-\dfrac{2}{q}-\dfrac{d}{r}$. Then there exist $\kappa,C>0$ such that for any $T\ge 0$ and $u\in X_T^s$,*

$$\|u\|_{L^q([-T,T],\mathcal{W}^{s_0,r}(\mathbb{R}^d))}\le CT^\kappa\|u\|_{X_T^s}.$$

We now introduce the appropriate sets in which we can profit from the stochastic estimates of the previous section. Fix $\mu=\mu_\gamma\in\mathcal{M}^0$ and, for $K\ge 0$ and $\varepsilon>0$, define the set $G_d(K)$ as

$$G_d(K)=\big\{w\in L^2(\mathbb{R}^d)\mid \|w\|_{L^2(\mathbb{R}^d)}\le K \text{ and } \|e^{-itH}w\|_{L^{1/\varepsilon}_{[-2\pi,2\pi]}\mathcal{W}^{d/2-\varepsilon,\infty}(\mathbb{R}^d)}\le K\big\}.$$

Then by Proposition 2.1,

$$\mu(G_d(K)^c)\le\mu\big(\|w\|_{L^2(\mathbb{R}^d)}>K\big)+\mu\big(\|e^{-itH}w\|_{L^{1/\varepsilon}_{[-2\pi,2\pi]}\mathcal{W}^{d/2-\varepsilon,\infty}(\mathbb{R}^d)}>K\big)\le Ce^{-cK^2/\|\gamma\|^2_{L^2}}. \tag{3-4}$$

We want to perform a fixed point argument on L with initial condition $u_0\in G_d(K)$ for some $K>0$ and $\varepsilon>0$ small enough. We begin by establishing some estimates.

Lemma 3.2. *Let $s\in\big]\frac{d}{2}-\frac{2}{p-1},\frac{d}{2}\big[$. For $\varepsilon>0$ small enough there exist $C>0$ and $\kappa>0$ such that for any $0<T\le 1$, $u_0\in G_d(K)$, $v\in X_T^s$ and $f_i=v$ or $f_i=e^{-itH}u_0$,*

$$\Big\|H^{s/2}(v)\prod_{i=2}^p f_i\Big\|_{L^1([-T,T],L^2(\mathbb{R}^d))}\le CT^\kappa\big(K^p+\|v\|^p_{X_T^s}\big), \tag{3-5}$$

and

$$\Big\|H^{s/2}(e^{-itH}u_0)\prod_{i=2}^p f_i\Big\|_{L^1([-T,T],L^2(\mathbb{R}^d))}\le CT^\kappa\big(K^p+\|v\|^p_{X_T^s}\big). \tag{3-6}$$

Proof. First we prove (3-5). Thanks to the Hölder inequality,

$$\Big\||\nabla|^s(v)\prod_{i=2}^p f_i\Big\|_{L^1([-T,T],L^2(\mathbb{R}^d))}\le\||\nabla|^s(v)\|_{L^\infty([-T,T],L^2(\mathbb{R}^d))}\prod_{i=2}^p\|f_i\|_{L^{p-1}([-T,T],L^\infty(\mathbb{R}^d))}$$

and

$$\Big\|\langle x\rangle^s v\prod_{i=2}^{p} f_i\Big\|_{L^1([-T,T],L^2(\mathbb{R}^d))} \le \|\langle x\rangle^s v\|_{L^\infty([-T,T],L^2(\mathbb{R}^d))}\prod_{i=2}^{p}\|f_i\|_{L^{p-1}([-T,T],L^\infty(\mathbb{R}^d))}$$
$$\le \|v\|_{L^\infty([-T,T],H^s(\mathbb{R}^d))}\prod_{i=2}^{p}\|f_i\|_{L^{p-1}([-T,T],L^\infty(\mathbb{R}^d))}.$$

If $f_i=v$, then as $s>\dfrac{d}{2}-\dfrac{2}{p-1}$, we can use Lemma 3.1 to obtain

$$\|v\|_{L^{p-1}([-T,T],L^\infty(\mathbb{R}^d))}\le CT^\kappa\|v\|_{X_T^s}.$$

If $f_i=e^{-itH}u_0$, then by definition of $G_d(K)$ we have, for $\varepsilon>0$ small enough,

$$\|e^{-itH}u_0\|_{L^{p-1}([-T,T],L^\infty(\mathbb{R}^d))}\le T^\kappa\|e^{-itH}u_0\|_{L^{1/\varepsilon}([-2\pi,2\pi],\mathcal{W}^{d/2-\varepsilon,\infty}(\mathbb{R}^d))}\le T^\kappa K.$$

We now turn to (3-6). Thanks to the Hölder inequality, we have

$$\Big\||\nabla|^s(e^{-itH}u_0)\prod_{i=2}^{p} f_i\Big\|_{L^1([-T,T],L^2(\mathbb{R}^d))}$$
$$\le \||\nabla|^s(e^{-itH}u_0)\|_{L^p([-T,T],L^{2dp}(\mathbb{R}^d))}\prod_{i=2}^{p}\|f_i\|_{L^p([-T,T],L^{2dp(p-1)/(dp-1)}(\mathbb{R}^d))}$$
$$\le \|e^{-itH}u_0\|_{L^p([-T,T],\mathcal{W}^{s,2dp}(\mathbb{R}^d))}\prod_{i=2}^{p}\|f_i\|_{L^p([-T,T],L^{2dp(p-1)/(dp-1)}(\mathbb{R}^d))}.$$

If $f_i=e^{-itH}u_0$, by interpolation we obtain, for some $0\le\theta\le 1$,

$$\|e^{-itH}u_0\|_{L^p([-T,T],L^{2dp(p-1)/(dp-1)}(\mathbb{R}^d))}\le CT^\kappa\|u_0\|^{1-\theta}_{L^2(\mathbb{R}^d)}\|e^{-itH}u_0\|^{\theta}_{L^{1/\varepsilon}([-T,T],L^\infty(\mathbb{R}^d))}\le CT^\kappa K.$$

If $f_i=v$, as $s>\dfrac{d}{2}-\dfrac{2}{p-1}>\dfrac{d}{2}-\dfrac{2}{p}-\dfrac{d(dp-1)}{2dp(p-1)}$ (because $p\ge 3$ and $d\ge 2$), then thanks to Lemma 3.1 we find

$$\|v\|_{L^p([-T,T],L^{2dp(p-1)/(dp-1)}(\mathbb{R}^d))}\le CT^\kappa\|v\|_{X_T^s}. \qquad \square$$

We are now able to establish the estimates that will be useful in the application of a fixed point theorem.

Proposition 3.3. *Let* $s\in\big]\frac{d}{2}-\frac{2}{p-1},\frac{d}{2}\big[$. *Then for* $\varepsilon>0$ *small enough, there exist* $C>0$ *and* $\kappa>0$ *such that if* $u_0\in G_d(K)$ *for some* $K>0$. *For any* $v,v_1,v_2\in X_T^s$ *and* $0<T\le 1$,

$$\Big\|\int_0^t e^{-i(t-s)H}|e^{-isH}u_0+v|^{p-1}(e^{-isH}u_0+v)\,ds\Big\|_{X_T^s}\le CT^\kappa(K^p+\|v\|^p_{X_T^s}),$$

and

$$\Big\|\int_0^t e^{-i(t-s)H}|e^{-isH}u_0+v_1|^{p-1}(e^{-isH}u_0+v_1)\,ds$$
$$-\int_0^t e^{-i(t-s)H}|e^{-isH}u_0+v_2|^{p-1}(e^{-isH}u_0+v_2)\,ds\Big\|_{X_T^s}$$
$$\le CT^\kappa\|v_1-v_2\|_{X_T^s}\big(K^{p-1}+\|v_1\|^{p-1}_{X_T^s}+\|v_2\|^{p-1}_{X_T^s}\big).$$

Proof. We only prove the first claim, since the proof of the second is similar. Using the Strichartz inequalities (3-3), we obtain

$$\Big\| \int_0^t e^{-i(t-s)H} |e^{-isH}u_0 + v|^{p-1}(e^{-isH}u_0 + v)\,ds \Big\|_{X_T^s} \\ \le C \big\| |e^{-isH}u_0 + v|^{p-1}(e^{-isH}u_0 + v) \big\|_{L^1_{[-T,T]}\mathcal{H}^s(\mathbb{R}^d)}.$$

Then, using Lemma 3.2, we obtain the existence of $\kappa > 0$ such that for any $u_0 \in G_d(K)$, $0 < T \le 1$ and $v \in X_T^s$,

$$\big\| H^{s/2}\big(|e^{-isH}u_0 + v|^{p-1}(e^{-isH}u_0 + v)\big) \big\|_{L^1([-T,T],L^2(\mathbb{R}^d))} \le CT^\kappa \big(K^p + \|v\|^p_{X_T^s}\big). \qquad \square$$

Proof of Theorem 1.1. We now complete the contraction argument on L defined in (3-1) with some $u_0 \in G_d(K)$. According to Proposition 3.3, there exist $C > 0$ and $\kappa > 0$ such that

$$\|L(v)\|_{X_T^s} \le CT^\kappa \big(K^p + \|v\|^p_{X_T^s}\big)$$
$$\|L(v_1) - L(v_2)\|_{X_T^s} \le CT^\kappa \|v_1 - v_2\|_{X_T^s} \big(K^{p-1} + \|v_1\|^{p-1}_{X_T^s} + \|v_2\|^{p-1}_{X_T^s}\big).$$

Hence, if we choose $T > 0$ such that $K = (8CT^\kappa)^{-1/(p-1)}$, then L is a contraction in the space $B_{X_T^s}(0, K)$ (the ball of radius K in X_T^s). Thus if we set $\Sigma_T = G_d(K)$, with the previous choice of K, the result follows from (3-4). $\square$

Proof of Theorem 1.3. We introduce

$$\begin{cases} i\dfrac{\partial w}{\partial t} - Hw = \pm \cos(2t)^{\frac12 d(p-1)-2}|w|^{p-1}w, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ v(0) = u_0, \end{cases} \tag{3-7}$$

and let $s \in \big]\frac{d}{2} - \frac{2}{p-1}, \frac{d}{2}\big[$, $T = \frac{\pi}{4}$ and $1 \gg \varepsilon > 0$. Thanks to Proposition 3.3, there exist $C > 0$ and $\kappa > 0$ such that if $u_0 \in G_d(K)$ for some $K > 0$ then, for all v,

$$\Big\| \int_0^t e^{-i(t-s)H}\big(\cos(2s)^{\frac{d}{2}(p-1)-2}|e^{-isH}u_0 + v|^{p-1}(e^{-isH}u_0 + v)\big)\,ds \Big\|_{X_T^s} \le CT^\kappa\big(K^p + \|v\|^p_{X_T^s}\big). \tag{3-8}$$

As in Theorem 1.1, we can choose $K = (8CT^\kappa)^{-1/(p-1)}$ to obtain, for $u_0 \in G_d(K)$, a unique local solution $w = e^{-itH}u_0 + v$ in time interval $\big]-\frac{\pi}{4}, \frac{\pi}{4}\big[$ to (3-7) with $v \in X_T^1$.

We set $u = \mathscr{L}v$. Then u is a global solution to (1-5). Thanks to [Poiret 2012b, Propositions 20 and 22], we obtain that $u = e^{it\Delta}u_0 + v'$ with $v' \in X_T^1$.

Moreover, thanks to (3-8), we have that

$$\int_0^t e^{-i(t-s)H}\big(\cos(2s)^{\frac{d}{2}(p-1)-2}|e^{-isH}u_0 + v|^{p-1}(e^{-isH}u_0 + v)\big)\,ds \in \mathscr{C}^0\big([-T,T], \mathcal{H}^s(\mathbb{R}^d)\big).$$

Then there exist $L \in \mathcal{H}^s$ such that

$$\lim_{t\to T} \Big\| e^{-itH}\int_0^t e^{-isH}\big(\cos(2s)^{\frac{d}{2}(p-1)-2}|e^{-isH}u_0 + v|^{p-1}(e^{-isH}u_0 + v)\big)\,ds - L \Big\|_{\mathcal{H}^s(\mathbb{R}^d)} = 0.$$

Using [Poiret 2012b, Lemma 70], we obtain that

$$\lim_{t\to T}\left\|u(t)-e^{it\Delta}u_0-e^{it\Delta}(-ie^{-iTH}L)\right\|_{H^s(\mathbb{R}^d)}=0.$$

Finally, to establish Theorem 1.3, it suffices to set $\Sigma=G_d(K)$ and to prove that $\mu(u_0\in G_d(K))>0$. We can write

$$u_0=\chi\left(\frac{H}{N}\right)u_0+(1-\chi)\left(\frac{H}{N}\right)u_0:=[u_0]_N+[u_0]^N,$$

where χ is a truncation function. Using the triangle inequality and independence, we obtain that

$$\mu(u_0\in G_d(K))\geq\mu\big([u_0]_N\in G_d(K/2)\big)\mu\big([u_0]^N\in G_d(K/2)\big).$$

For all N, $\mu\big([u_0]_N\in G_d(K/2)\big)>0$ because the hypothesis (1-6) is satisfied and thanks to Proposition 2.1 we have

$$\mu\big([u_0]^N\in G_d(K/2)\big)\geq 1-Ce^{-cK^2/\|[u_0]^N\|^2_{L^2}}\to 1\quad\text{as } N\to\infty,$$

and there exists N such that $\mu\big([u_0]^N\in G_d(K/2)\big)>0$. □

3B. ***Almost sure local well-posedness of the time dependent equation and scattering for NLS.*** This section is devoted to the proof of Theorem 1.4. The strategy is similar to the proof of Theorem 1.3: we solve the equation which is mapped by $\mathscr{L}$ to (1-5) up to time $T=\pi/4$ and we conclude as previously. The difference here is that the nonlinear term of the equation we have to solve is singular at time $T=\pi/4$. More precisely, we consider the equation

$$\begin{cases} i\dfrac{\partial u}{\partial t}-Hu=\pm\cos(2t)^{p-3}|u|^{p-1}u, & (t,x)\in\mathbb{R}\times\mathbb{R}^2,\\ u(0)=u_0, \end{cases}\tag{3-9}$$

when $2<p<3$.

Let us first consider the easier case $\sigma>0$.

3B1. *Proof of Theorem 1.4 in the case $\sigma>0$.* Let $\sigma>0$ and $\mu=\mu_\gamma\in\mathcal{M}^\sigma$, and for $K\geq 0$ and $\varepsilon>0$ define the set $F_\sigma(K)$ as

$$F_\sigma(K)=\big\{w\in\mathscr{H}^\sigma(\mathbb{R}^2)\mid \|w\|_{\mathscr{H}^\sigma(\mathbb{R}^2)}\leq K \text{ and } \|e^{-itH}w\|_{L^{1/\varepsilon}_{[0,2\pi]}\mathscr{W}^{1+\sigma-\varepsilon,\infty}(\mathbb{R}^2)}\leq K\big\}.$$

The parameter $\varepsilon>0$ will be chosen small enough that we can apply Proposition 2.1 and get

$$\mu(F_\sigma(K)^c)\leq\mu(\|w\|_{\mathscr{H}^\sigma}>K)+\mu\Big(\|e^{-itH}w\|_{L^{1/\varepsilon}_{[0,2\pi]}\mathscr{W}^{1+\sigma-\varepsilon,\infty}}>K\Big)\leq Ce^{-cK^2/\|\gamma\|^2_{\mathscr{H}^\sigma}}.$$

The next proposition is the key in the proof of Theorem 1.4 when $\sigma>0$.

Proposition 3.4. *Let $\sigma>0$. There exist $C>0$ and $\kappa>0$ such that if $u_0\in F_\sigma(K)$ for some $K>0$ then for any $v,v_1,v_2\in X^1_T$ and $0<T\leq 1$,*

$$\left\|\int_0^t e^{-i(t-\tau)H}\big(\cos(2\tau)^{p-3}|e^{-i\tau H}u_0+v|^{p-1}(e^{-i\tau H}u_0+v)\big)\,d\tau\right\|_{X^1_T}\leq CT^\kappa(K^p+\|v\|^p_{X^1_T})\tag{3-10}$$

and

$$\begin{aligned}\Big\| \int_0^t e^{-i(t-\tau)H} \big(\cos(2\tau)^{p-3} |e^{-i\tau H}u_0+v_1|^{p-1}(e^{-i\tau H}u_0+v_1)\big)\,d\tau \\ - \int_0^t e^{-i(t-\tau)H} \big(\cos(2\tau)^{p-3} |e^{-i\tau H}u_0+v_2|^{p-1}(e^{-i\tau H}u_0+v_2)\big)\,d\tau \Big\|_{X_T^1} \\ \leq CT^\kappa \|v_1-v_2\|_{X_T^1}\big(K^{p-1}+\|v_1\|_{X_T^1}^{p-1}+\|v_2\|_{X_T^1}^{p-1}\big).\end{aligned}\tag{3-11}$$

Proof. We first prove (3-10). Using the Strichartz inequalities (3-3), we obtain

$$\begin{aligned}\Big\| \int_0^t e^{-i(t-\tau)H} \big(\cos(2\tau)^{p-3} |e^{-i\tau H}u_0+v|^{p-1}(e^{-i\tau H}u_0+v)\big)\,d\tau \Big\|_{X_T^1} \\ \leq C\big\|\cos(2\tau)^{p-3} |e^{-i\tau H}u_0+v|^{p-1}(e^{-i\tau H}u_0+v)\big\|_{L^1_{[-T,T]}\mathcal{H}^1(\mathbb{R}^2)}.\end{aligned}$$

We use the formula

$$\nabla(|u|^{p-1}u)=\frac{p+1}{2}|u|^{p-1}\nabla u+\frac{p-1}{2}|u|^{p-3}u^2\nabla\bar{u}.\tag{3-12}$$

We let $f=e^{-isH}u_0$, then

$$\begin{aligned}\big\|\nabla(|f+v|^{p-1}(f+v))\big\|_{L^2(\mathbb{R}^2)} &\leq C\big\||f+v|^{p-1}\nabla(f+v)\big\|_{L^2(\mathbb{R}^2)}+C\big\||f+v|^{p-3}(f+v)^2\nabla(f+v)\big\|_{L^2(\mathbb{R}^2)} \\ &\leq C\|f+v\|_{L^\infty(\mathbb{R}^2)}^{p-1}\|\nabla v\|_{L^2(\mathbb{R}^2)}+C\|f+v\|_{L^{2(p-1)}(\mathbb{R}^2)}^{p-1}\|\nabla f+v\|_{L^\infty(\mathbb{R}^2)}.\end{aligned}$$

Therefore

$$\begin{aligned}&\big\||f+v|^{p-1}(f+v)\big\|_{\mathcal{H}^1(\mathbb{R}^2)} \\ &\quad\leq C\big(\|f\|_{L^\infty(\mathbb{R}^2)}^{p-1}+\|v\|_{L^\infty(\mathbb{R}^2)}^{p-1}\big)\|v\|_{\mathcal{H}^1(\mathbb{R}^2)}+C\big(\|f\|_{L^{2(p-1)}(\mathbb{R}^2)}^{p-1}+\|v\|_{L^{2(p-1)}(\mathbb{R}^2)}^{p-1}\big)\|f\|_{\mathcal{W}^{1,\infty}(\mathbb{R}^2)}.\end{aligned}$$

Now observe that $\|v\|_{L^\infty_{[-T,T]}L^{2(p-1)}}\leq\|v\|_{X_T^1}$ as well as, for all $r<+\infty$, $\|v\|_{L^r_{[-T,T]}L^\infty}\leq\|v\|_{X_T^1}$. Then, for all $q>1$,

$$\begin{aligned}\big\||f+v|^{p-1}(f+v)\big\|_{L^q_{[-T,T]}\mathcal{H}^1(\mathbb{R}^2)} &\leq CT^\kappa\Big(\big(\|f\|_{L^{\infty-}_{[-T,T]}L^\infty(\mathbb{R}^2)}^{p-1}+\|v\|_{X_T^1}^{p-1}\big)\|v\|_{X_T^1} \\ &\qquad+\big(\|f\|_{L^{\infty-}_{[-T,T]}L^{2(p-1)}(\mathbb{R}^2)}^{p-1}+\|v\|_{X_T^1}^{p-1}\big)\|f\|_{L^{\infty-}_{[-T,T]}\mathcal{W}^{1,\infty}(\mathbb{R}^2)}\Big) \\ &\leq CT^\kappa\big(K^p+\|v\|_{X_T^1}^p\big).\end{aligned}\tag{3-13}$$

Choose $q>1$ so that $q'(3-p)<1$. We have $\|\cos(2\tau)^{p-3}\|_{L^{q'}_{[-T,T]}}<\infty$; thus from (3-13) and Hölder, we infer

$$\begin{aligned}&\big\|\cos(2\tau)^{p-3}|f+v|^{p-1}(f+v)\big\|_{L^1([-T,T],\mathcal{H}^1(\mathbb{R}^2))} \\ &\qquad\leq C\|\cos(2\tau)^{p-3}\|_{L^{q'}_{[-T,T]}}\big\||f+v|^{p-1}(f+v)\big\|_{L^q_{[-T,T]}\mathcal{H}^1(\mathbb{R}^2)} \\ &\qquad\leq CT^\kappa(K^p+\|v\|_{X_T^1}^p).\end{aligned}$$

For the proof of (3-11) we can proceed similarly. Namely, we use the estimates

$$\big||z_1|^{p-1}-|z_2|^{p-1}\big| \le C(|z_1|^{p-2}+|z_2|^{p-2})|z_1-z_2| \tag{3-14}$$

and

$$\big||z_1|^{p-3}z_1^2-|z_2|^{p-3}z_2^2\big| \le C(|z_1|^{p-2}+|z_2|^{p-2})|z_1-z_2|,$$

which are proven in [Cazenave et al. 2011, Remark 2.3] together with (3-12). □

3B2. *Proof of Theorem 1.4 in the case $\sigma=0$.* The strategy of the proof in this case is similar, at the price of some technicalities, since the Leibniz rule (3-12) does not hold true for non-integer derivatives. Actually, when $\sigma=0$, we will have to work in X^s_T for $s<1$ because the probabilistic term $e^{-itH}u_0 \notin \mathcal{W}^{1,\infty}(\mathbb{R}^2)$.

Moreover, we are not able to obtain a contraction estimate in X^s_T. Therefore, we will do a fixed point in the space $\{\|v\|_{X^s_T}\le K\}$ endowed with the weaker metric induced by X^0_T. We can check that this space is complete. Actually, by the Banach–Alaoglu theorem, the closed balls of each component space of X^s_T are compact for the weak* topology.

For $0<s<1$, we use the following characterisation of the usual $H^s(\mathbb{R}^2)$ norm:

$$\|g\|_{H^s(\mathbb{R}^2)} = \bigg(\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|g(x)-g(y)|^2}{|x-y|^{2s+2}}\,dx\,dy\bigg)^{1/2}. \tag{3-15}$$

For $\mu=\mu_\gamma\in\mathcal{M}^0$, $K\ge 0$ and $\varepsilon>0$, define the set $\widetilde{F}_0(K)$ as

$$\widetilde{F}_0(K) = \big\{w\in L^2(\mathbb{R}^2) \mid \|w\|_{L^2(\mathbb{R}^2)}\le K,\ \|e^{-itH}w\|_{L^{1/\varepsilon}_{[0,2\pi]}\mathcal{W}^{1-\varepsilon,\infty}(\mathbb{R}^2)}\le K \\ \text{and } \big\|(e^{-itH}w)(x)-(e^{-itH}w)(y)\big\|_{L^\infty_{t\in[0,2\pi]}} \le K|x-y|^{1-\varepsilon}\big\}.$$

The next result states that $\widetilde{F}_0(K)$ is a set with large measure.

Lemma 3.5. *If $\varepsilon>0$ is small enough,*

$$\mu\big((\widetilde{F}_0(K))^c\big) \le Ce^{-cK^2/\|\gamma\|^2_{L^2(\mathbb{R}^2)}}.$$

Proof. We only have to study the contribution of the Lipschitz term in $\widetilde{F}_0(K)$, since the others are controlled by Proposition 2.1.

We fix $\gamma=\sum_{n=0}^{+\infty} c_n\varphi_n\in\mathcal{A}_0$ and set $\gamma^\omega=\sum_{n=0}^{+\infty} g_n(\omega)c_n\varphi_n$. Let $k\ge 1$. By definition,

$$\int_{L^2(\mathbb{R}^2)} \big\|e^{-itH}u(x)-e^{-itH}u(y)\big\|^k_{L^\infty_{[0,2\pi]}}\,d\mu(u) \\ = \int_\Omega \big\|e^{-itH}\gamma^\omega(x)-e^{-itH}\gamma^\omega(y)\big\|^k_{L^\infty_{[0,2\pi]}}\,d\mathbb{P}(\omega). \tag{3-16}$$

We have $e^{-itH}\gamma^\omega(x)-e^{-itH}\gamma^\omega(y)=\sum_{n=0}^{+\infty} g_n(\omega)c_n e^{-it\lambda_n}(\varphi_n(x)-\varphi_n(y))$. By Khinchin (Lemma 2.3) we get

$$\begin{aligned}\left\|e^{-itH}\gamma^\omega(x)-e^{-itH}\gamma^\omega(y)\right\|_{L^k_{\mathbb{P}}} &\le C\sqrt{k}\Big(\sum_{n=0}^{+\infty}|c_n|^2|\varphi_n(x)-\varphi_n(y)|^2\Big)^{1/2}\\ &= C\sqrt{k}\Big(\sum_{j=1}^{+\infty}\sum_{n\in I(j)}|c_n|^2|\varphi_n(x)-\varphi_n(y)|^2\Big)^{1/2},\end{aligned}$$

Recall that $k\in I(j)=\{n\in\mathbb{N}\mid 2j\le\lambda_n<2(j+1)\}$ and that $\#I(j)\sim cj$. Next, by condition (1-3), we deduce that

$$\left\|e^{-itH}\gamma^\omega(x)-e^{-itH}\gamma^\omega(y)\right\|_{L^k_{\mathbb{P}}}\le C\sqrt{k}\Big(\sum_{j=1}^{+\infty}j^{-1}\Big(\sum_{\ell\in I(j)}|c_\ell|^2\Big)\sum_{n\in I(j)}|\varphi_n(x)-\varphi_n(y)|^2\Big)^{1/2}$$

Now we need the following estimate, proven in [Imekraz et al. 2014, Lemma 6.1]:

$$\sum_{n\in I(j)}|\varphi_n(y)-\varphi_n(x)|^2\le C|y-x|^2 j.$$

Therefore, we obtain

$$\left\|e^{-itH}\gamma^\omega(x)-e^{-itH}\gamma^\omega(y)\right\|_{L^k_{\mathbb{P}}}\le C\sqrt{k}\,|x-y|\,\|\gamma\|_{L^2(\mathbb{R}^2)},$$

and for $k\ge q$ an integration in time and Minkowski yield

$$\left\|e^{-itH}\gamma^\omega(x)-e^{-itH}\gamma^\omega(y)\right\|_{L^k_{\mathbb{P}}L^q_{[0,2\pi]}}\le C\sqrt{k}\,|x-y|\,\|\gamma\|_{L^2(\mathbb{R}^2)}.$$

However, since the case $q=+\infty$ is forbidden, the previous estimate is not enough to have a control on the $L^\infty_{[0,2\pi]}$-norm. To tackle this issue, we claim that for $k\ge q$ we have

$$\left\|e^{-itH}\gamma^\omega(x)-e^{-itH}\gamma^\omega(y)\right\|_{L^k_{\mathbb{P}}W^{1,q}_{[0,2\pi]}}\le C\sqrt{k}\,\|\gamma\|_{L^2(\mathbb{R}^2)}. \tag{3-17}$$

Then by a usual Sobolev embedding argument we get (by taking $q\gg 1$ large enough) that for all $\varepsilon>0$

$$\left\|e^{-itH}\gamma^\omega(x)-e^{-itH}\gamma^\omega(y)\right\|_{L^k_{\mathbb{P}}L^\infty_{[0,2\pi]}}\le C\sqrt{k}\,|x-y|^{1-\varepsilon}\|\gamma\|_{L^2(\mathbb{R}^2)},$$

which in turn by (3-16) implies that

$$\mu\Big(u\in L^2(\mathbb{R}^2)\mid \left\|e^{-itH}u(x)-e^{-itH}u(y)\right\|_{L^\infty_{[0,2\pi]}}>K|x-y|^{1-\varepsilon}\Big)\le Ce^{-cK^2/\|\gamma\|^2_{L^2}},$$

as we did in the end of the proof of Proposition 2.1.

Let us now prove (3-17). We have

$$\partial_t(e^{-itH}\gamma^\omega(x)-e^{-itH}\gamma^\omega(y))=-i\sum_{n=0}^{+\infty}g_n(\omega)\lambda_n c_n e^{-it\lambda_n}(\varphi_n(x)-\varphi_n(y)),$$

and with the previous arguments we get

$$\left\|\partial_t(e^{-itH}\gamma^\omega(x)-e^{-itH}\gamma^\omega(y))\right\|_{L^k_{\mathbb{P}}}\le C\sqrt{k}\biggl(\sum_{j=1}^{+\infty}\biggl(\sum_{\ell\in I(j)}|c_\ell|^2\biggr)\sum_{n\in I(j)}|\varphi_n(x)-\varphi_n(y)|^2\biggr)^{1/2}$$
$$\le C\sqrt{k}\|\gamma\|_{L^2(\mathbb{R}^2)},$$

where here we have used the Thangavelu–Karadzhov estimate (see [Poiret et al. 2013, Lemma 3.5])

$$\sup_{x\in\mathbb{R}^2}\sum_{n\in I(j)}|\varphi_n(x)|^2\le C.$$

We conclude the proof of (3-17) by integrating in time and using Minkowski. □

We will also need the following technical result.

Lemma 3.6. *Let $u_0\in\widetilde{F}_0(K)$ and $f(t,x)=e^{-itH}u_0(x)$. Let $2\le q<+\infty$ and $g\in L^q([-T,T];L^2(\mathbb{R}^2))$. Then, if $\varepsilon>0$ is small enough in the definition of $\widetilde{F}_0(K)$,*

$$\left\|\biggl(\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|f(t,x)-f(t,y)|^2|g(t,x)|^2}{|x-y|^{2s+2}}\,dx\,dy\biggr)^{1/2}\right\|_{L^q_{[-T,T]}}\le CK\|g\|_{L^q_{[-T,T]}L^2(\mathbb{R}^2)}. \tag{3-18}$$

Proof. We consider such f,g, and we split the integral. On the one hand, we use that f is Lipschitz:

$$\int_{|x-y|\le 1}\frac{|f(t,x)-f(t,y)|^2|g(t,x)|^2}{|x-y|^{2s+2}}\,dx\,dy\le K^2\int_{x\in\mathbb{R}^2}|g(t,x)|^2\biggl(\int_{y:|x-y|\le 1}\frac{dy}{|x-y|^{2s+2\varepsilon}}\biggr)dx$$
$$\le CK^2\|g(t,\cdot)\|^2_{L^2(\mathbb{R}^2)},$$

provided that $s+\varepsilon<1$. We take the $L^q_{[-T,T]}$-norm, and we see that this contribution is bounded by the right side of (3-18).

On the other hand

$$\int_{|x-y|\ge 1}\frac{|f(t,x)-f(t,y)|^2|g(t,x)|^2}{|x-y|^{2s+2}}\,dx\,dy$$
$$\le C\|f(t,\cdot)\|^2_{L^\infty(\mathbb{R}^2)}\int_{x\in\mathbb{R}^2}|g(t,x)|^2\biggl(\int_{y:|x-y|\ge 1}\frac{dy}{|x-y|^{2s+2}}\biggr)dx$$
$$\le C\|f(t,\cdot)\|^2_{L^\infty(\mathbb{R}^2)}\|g(t,\cdot)\|^2_{L^2(\mathbb{R}^2)}$$

if $s>0$. Now we take the $L^q_{[-T,T]}$-norm, and use the fact that $\|f\|_{L^q_{[0,2\pi]}L^\infty(\mathbb{R}^2)}\le K$ if $\varepsilon<1/q$. □

We now state the main estimates of this section.

Proposition 3.7. *There exist $C>0$ and $\kappa>0$ such that if $u_0\in\widetilde{F}_0(K)$ for some $K>0$ then for any $v,v_1,v_2\in X^s_T$ and $0<T\le 1$,*

$$\left\|\int_0^t e^{-i(t-\tau)H}\bigl(\cos(2\tau)^{p-3}|e^{-i\tau H}u_0+v|^{p-1}(e^{-i\tau H}u_0+v)\bigr)\,d\tau\right\|_{X^s_T}\le CT^\kappa\bigl(K^p+\|v\|^p_{X^s_T}\bigr), \tag{3-19}$$

and

$$\begin{aligned}\Big\| \int_0^t & e^{-i(t-\tau)H}\big(\cos(2\tau)^{p-3}|e^{-i\tau H}u_0+v_1|^{p-1}(e^{-i\tau H}u_0+v_1)\big)\,d\tau \\ &- \int_0^t e^{-i(t-\tau)H}\big(\cos(2\tau)^{p-3}|e^{-i\tau H}u_0+v_2|^{p-1}(e^{-i\tau H}u_0+v_2)\big)\,d\tau\Big\|_{X_T^0} \\ &\leq CT^{\kappa}\|v_1-v_2\|_{X_T^0}\big(K^{p-1}+\|v_1\|_{X_T^s}^{p-1}+\|v_2\|_{X_T^s}^{p-1}\big). \qquad (3\text{-}20)\end{aligned}$$

Proof. Let $u_0 \in \widetilde{F}_0(K)$ and set $f = e^{-isH}u_0$. Let $2 < p < 3$, then there exists $q \gg 1$ such that $q'(3-p) < 1$, which in turn implies $\|\cos(2s)^{p-3}\|_{L^{q'}_{[-T,T]}} \leq CT^{\kappa}$. Next, if $s < 1$ is large enough we have, by Sobolev,

$$\|v\|_{L^{\infty}_{[-T,T]}L^{2(p-1)}(\mathbb{R}^2)} \leq \|v\|_{X_T^s} \quad \text{and} \quad \|v\|_{L^{q(p-1)}_{[-T,T]}L^{\infty}(\mathbb{R}^2)} \leq \|v\|_{X_T^s}. \qquad (3\text{-}21)$$

First we prove (3-19). From Strichartz and Hölder, we get

$$\begin{aligned}\Big\|\int_0^t e^{-i(t-s)H}\big(\cos(2s)^{p-3}|f+v|^{p-1}(f+v)\big)\,ds\Big\|_{X_T^s} &\leq C\big\|\cos(2s)^{p-3}|f+v|^{p-1}(f+v)\big\|_{L^1_{[-T,T]}\mathcal{H}^s(\mathbb{R}^2)} \\ &\leq C\|\cos(2s)^{p-3}\|_{L^{q'}_{[-T,T]}}\big\||f+v|^{p-1}(f+v)\big\|_{L^q_{[-T,T]}\mathcal{H}^s(\mathbb{R}^2)} \\ &\leq CT^{\kappa}\big\||f+v|^{p-1}(f+v)\big\|_{L^q_{[-T,T]}\mathcal{H}^s(\mathbb{R}^2)}. \qquad (3\text{-}22)\end{aligned}$$

By using the characterization (3-15), we will prove that

$$\big\||f+v|^{p-1}(f+v)\big\|_{L^q_{[-T,T]}\mathcal{H}^s(\mathbb{R}^2)} \leq C\big(K^p+\|v\|_{X_T^s}^p\big). \qquad (3\text{-}23)$$

The term $\|\langle x\rangle^s|f+v|^{p-1}(f+v)\|_{L^q_{[-T,T]}L^2(\mathbb{R}^2)}$ is easily controlled; thus we only detail the contribution of the H^s norm. With (3-14), it is easy to check that, for all $x, y \in \mathbb{R}^2$,

$$\begin{aligned}\big||f+v|^{p-1}(f+v)(x)-|f+v|^{p-1}(f+v)(y)\big| &\leq C|v(x)-v(y)|\big(|v(x)|^{p-1}+|v(y)|^{p-1}+|f(x)|^{p-1}+|f(y)|^{p-1}\big) \\ &\quad + C|f(x)-f(y)|\big(|v(x)|^{p-1}+|v(y)|^{p-1}+|f(x)|^{p-1}+|f(y)|^{p-1}\big).\end{aligned}$$

By (3-21) the contribution in $L^q_{[-T,T]}H^s(\mathbb{R}^2)$ of the first term in the previous expression is at most

$$C\big(\|f\|^{p-1}_{L^{q(p-1)}_{[-T,T]}L^{\infty}(\mathbb{R}^2)}+\|v\|^{p-1}_{L^{q(p-1)}_{[-T,T]}L^{\infty}(\mathbb{R}^2)}\big)\|v\|_{X_T^s} \leq C\big(K^{p-1}+\|v\|^{p-1}_{X_T^s}\big)\|v\|_{X_T^s}.$$

To bound the second term, we apply Lemma 3.6, which gives a contribution of at most

$$\big(\|f\|^{p-1}_{L^{q(p-1)}_{[-T,T]}L^{2(p-1)}(\mathbb{R}^2)}+\|v\|^{p-1}_{L^{q(p-1)}_{[-T,T]}L^{2(p-1)}(\mathbb{R}^2)}\big)K \leq C\big(K^{p-1}+\|v\|^{p-1}_{X_T^s}\big)K,$$

which concludes the proof of (3-23).

The proof of (3-20) is in the same spirit, and even easier. We do not write the details. □

Thanks to the estimates of Proposition 3.7, for $K > 0$ small enough (see the proof of Theorem 1.3 for more details) we are able to construct a unique solution $v \in \mathcal{C}([-\pi/4, \pi/4]; L^2(\mathbb{R}^2))$ such that $v \in L^\infty([-\pi/4, \pi/4]; \mathcal{H}^s(\mathbb{R}^2))$. By interpolation we deduce that $v \in \mathcal{C}([-\pi/4, \pi/4]; \mathcal{H}^{s'}(\mathbb{R}^2))$ for all $s' < s$. The end of the proof of Theorem 1.4 is similar to the proof of Theorem 1.3, using here Lemma 3.5.

4. Global well-posedness for the cubic equation

4A. ***The case of dimension $d = 3$.*** We now turn to the proof of Theorem 1.5, which is obtained thanks to the high-low frequency decomposition method of [Bourgain 1999, p. 84].

Let $0 \le s < 1$ and fix $\mu = \mu_\gamma \in \mathcal{M}^s$. For $K \ge 0$ define the set $F_s(K)$ as

$$F_s(K) = \big\{ w \in \mathcal{H}^s(\mathbb{R}^3) \mid \|w\|_{\mathcal{H}^s(\mathbb{R}^3)} \le K,\ \|w\|_{L^4(\mathbb{R}^3)} \le K \text{ and } \|e^{-itH} w\|_{L^{1/\varepsilon}_{[0,2\pi]} \mathcal{W}^{3/2+s-\varepsilon,\infty}(\mathbb{R}^3)} \le K \big\}.$$

Then, by Proposition 2.1,

$$\begin{aligned} &\mu\big((F_s(K))^c\big) \\ &\quad \le \mu(\|w\|_{\mathcal{H}^s} > K) + \mu(\|w\|_{L^4} > K) + \mu\big(\|e^{-itH} w\|_{L^{1/\varepsilon}_{[0,2\pi]} \mathcal{W}^{3/2+s-\varepsilon,\infty}} > K\big) \le C e^{-cK^2/\|\gamma\|^2_{\mathcal{H}^s}}. \end{aligned} \tag{4-1}$$

Now we define a smooth version of the usual spectral projector. Let $\chi \in \mathcal{C}_0^\infty(-1, 1)$, so that $0 \le \chi \le 1$, with $\chi = 1$ on $\big[-\frac{1}{2}, \frac{1}{2}\big]$. We define the operators $S_N = \chi\big(\frac{H}{N^2}\big)$ as

$$S_N\Big(\sum_{n=0}^{+\infty} c_n \varphi_n\Big) = \sum_{n=0}^{+\infty} \chi\Big(\frac{\lambda_n}{N^2}\Big) c_n \varphi_n,$$

and we write

$$v_N = S_N v, \quad v^N = (1 - S_N) v.$$

It is clear that for any $\sigma \ge 0$ we have $\|S_N\|_{\mathcal{H}^\sigma \to \mathcal{H}^\sigma} = 1$. Moreover, by [Burq et al. 2010, Proposition 4.1], for all $1 \le r \le +\infty$, $\|S_N\|_{L^r \to L^r} \le C$, uniformly in $N \ge 1$.

It is straightforward to check that

$$\|v_N\|_{\mathcal{H}^1} \le N^{1-s} \|v\|_{\mathcal{H}^s}, \quad \|v^N\|_{L^2} \le N^{-s} \|v\|_{\mathcal{H}^s}. \tag{4-2}$$

Next, let $u_0 \in F_s(N^\varepsilon)$. By the definition of $F_s(N^\varepsilon)$ and (4-2), $\|u_{0,N}\|_{\mathcal{H}^1} \le N^{1-s} \|u_0\|_{\mathcal{H}^s} \le N^{1-s+\varepsilon}$. The nonlinear term of the energy can be controlled by the quadratic term. Indeed

$$\|u_{0,N}\|^4_{L^4} \le C N^\varepsilon \le N^{2(1-s+\varepsilon)},$$

and thus

$$E(u_{0,N}) \le 2 N^{2(1-s+\varepsilon)}. \tag{4-3}$$

We also have

$$\|u_{0,N}\|_{L^2} \le \|u_0\|_{\mathcal{H}^s} \le N^\varepsilon.$$

For a nice description of the stochastic version of the low-high frequency decomposition method we use here, we refer to the introduction of [Colliander and Oh 2012]. To begin with, we look for a solution u to (1-7) of the form $u = u^1 + v^1$, where u^1 is the solution to

$$\begin{cases} i\dfrac{\partial u^1}{\partial t} - Hu^1 = |u^1|^2 u^1, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^3, \\ u^1(0) = u_{0,N}, \end{cases} \tag{4-4}$$

and where $v^1 = e^{-itH}u_0^N + w^1$ satisfies

$$\begin{cases} i\dfrac{\partial w^1}{\partial t} - Hw^1 = |w^1 + e^{-itH}u_0^N + u^1|^2(w^1 + e^{-itH}u_0^N + u^1) - |u^1|^2u^1, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^3, \\ w^1(0) = 0. \end{cases} \tag{4-5}$$

Since (4-4) is $\mathcal{H}^1$-subcritical, by the usual deterministic arguments there exists a unique global solution $u^1 \in \mathscr{C}(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^3))$.

We now turn to (4-5), for which we have the next local existence result.

Proposition 4.1. *Let $0 < s < 1$ and $\mu = \mu_\gamma \in \mathcal{M}^s$. Set $T = N^{-4(1-s)-\varepsilon}$ with $\varepsilon > 0$. Assume that $E(u^1) \le 4N^{2(1-s+\varepsilon)}$ and $\|u^1\|_{L^\infty_{[0,T]}L^2} \le 2N^\varepsilon$. Then:*

(i) *There exists a set $\Sigma_T^1 \subset \mathcal{H}^s$, which only depends on T, so that*

$$\mu(\Sigma_T^1) \ge 1 - C\exp\bigl(-cT^{-\delta}\|\gamma\|^{-2}_{\mathcal{H}^s(\mathbb{R}^3)}\bigr), \quad \textit{with some } \delta > 0.$$

(ii) *For all $u_0 \in \Sigma_T^1$ there exists a unique solution $w^1 \in \mathscr{C}([0,T], \mathcal{H}^1(\mathbb{R}^3))$ to* (4-5), *which satisfies the bounds*

$$\|w^1\|_{L^\infty_{[0,T]}\mathcal{H}^1} \le CN^{\beta(s)+c\varepsilon}, \tag{4-6}$$

with

$$\beta(s) = \begin{cases} -\frac{5}{2} & \textit{if } 0 \le s \le \frac12, \\ 2s - \frac72 & \textit{if } \frac12 \le s \le 1, \end{cases} \tag{4-7}$$

and

$$\|w^1\|_{L^\infty_{[0,T]}L^2} \le CN^{-9/2+2s+c\varepsilon}. \tag{4-8}$$

Proof. In the next lines, we write $C^{a+} = C^{a+b\varepsilon}$, for some absolute quantity $b > 0$. Since $d = 3$, for $T > 0$ we define the space $X_T^1 = L^\infty([0,T]; \mathcal{H}^1(\mathbb{R}^3)) \cap L^2([0,T]; \mathcal{W}^{1,6}(\mathbb{R}^3))$. Let $\varepsilon > 0$, and define $\Sigma_T^1 = F_s(N^\varepsilon)$. By (4-1) and the choice $T = N^{-4(1-s)-\varepsilon}$, the set Σ_T^1 satisfies (i).

Let $u_0 \in \Sigma_T^1$. To simplify the notations in the proof, we write $w = w^1$, $u = u^1$ and $f = e^{-itH}u_0^N$. We define the map

$$L(w) = \mp i \int_0^t e^{-i(t-s)H}\bigl(|f+u+w|^2(f+u+w) - |u|^2u\bigr)(s)\,ds. \tag{4-9}$$

First we prove (4-6). By Strichartz (3-3),

$$\|L(w)\|_{X_T^1} \le C\bigl\||f+u+w|^2(f+u+w) - |u|^2u\bigr\|_{L^1_T\mathcal{H}^1 + L^2\mathcal{W}^{1,6/5}}. \tag{4-10}$$

By estimating the contribution of every term, we now prove that

$$\|L(w)\|_{X_T^1} \le CN^{\beta(s)+} + N^{0-}\|w\|_{X_T^1} + N^{-2(1-s)+}\|w\|^3_{X_T^1}, \tag{4-11}$$

where $\beta(s) < (1-s)$ is as in the statement. It is enough to prove that L maps a ball of size $CN^{\beta(s)+}$ into itself for times $T = N^{-4(1-s)-\varepsilon}$. With similar arguments one can show that L is a contraction (we do not write the details) and get w satisfying (4-6).

Observe that the complex conjugation is harmless with respect to the norms considered; thus we can forget it. By the definition of $\Sigma_T^1 = F_s(N^\varepsilon)$ and (4-2) we have the estimates used in the sequel: for all $\sigma < \frac{3}{2}$,

$$\|f\|_{L_T^\infty L^2} \le CN^{-s+\varepsilon} \quad \text{and} \quad \|H^{\sigma/2} f\|_{L_T^{\infty-} L^\infty} \le CN^{\sigma-3/2-s+2\varepsilon}. \tag{4-12}$$

Let us prove the second estimate in detail:

$$\begin{aligned}
\|H^{\sigma/2} f\|_{L_T^{\infty-} L^\infty} &= N^\sigma \left\| \left(\frac{H}{N^2}\right)^{\sigma/2} \left(1-\chi\left(\frac{H}{N^2}\right)\right) e^{-itH} u_0 \right\|_{L_T^{\infty-} L^\infty} \\
&\le CN^\sigma \left\| \left(\frac{H}{N^2}\right)^{(3/2+s-\varepsilon)/2} \left(1-\chi\left(\frac{H}{N^2}\right)\right) e^{-itH} u_0 \right\|_{L_T^{\infty-} L^\infty} \\
&\le CN^{\sigma-3/2-s+\varepsilon} \|e^{-itH} u_0\|_{L_T^{\infty-} \mathcal{W}^{3/2+s-\varepsilon,\infty}} \\
&\le CN^{\sigma-3/2-s+2\varepsilon},
\end{aligned}$$

where we have used that $x^{\sigma/2}(1-\chi(x)) \le Cx^{(3/2+s-\varepsilon)/2}(1-\chi(x))$.

Observe also that by assumption

$$\|u\|_{L_T^\infty L^2} \le CN^\varepsilon, \quad \|u\|_{L_T^\infty \mathcal{H}^1} \le CN^{1-s+\varepsilon}, \quad \text{and} \quad \|u\|_{L_T^\infty L^4} \le CN^{(1-s+\varepsilon)/2}.$$

We now estimate each term in the right side of (4-10):

- Source terms: Observe that $L_T^{4/3}\mathcal{W}^{1,3/2} \subset L_T^1 \mathcal{H}^1 + L^2 \mathcal{W}^{1,6/5}$. By Hölder and (4-12),

$$\begin{aligned}
\|fu^2\|_{L_T^1 \mathcal{H}^1 + L^2 \mathcal{W}^{1,6/5}} &\le C\|fuH^{1/2}u\|_{L_T^{4/3} L^{3/2}} + C\|u^2 H^{1/2} f\|_{L_T^1 L^2} \\
&\le CT^{3/4-} \|u\|_{L_T^\infty \mathcal{H}^1} \|u\|_{L_T^\infty L^6} \|f\|_{L_T^{\infty-} L^\infty} + CT^{1-} \|u\|_{L_T^\infty L^4}^2 \|H^{1/2} f\|_{L_T^{\infty-} L^\infty} \\
&\le CN^{-5/2+} + CN^{-7/2+2s+} \le CN^{\beta(s)+},
\end{aligned}$$

where we have set $\beta(s) = \max\left(-\frac{5}{2}, -\frac{7}{2}+2s\right)$, which is precisely (4-7). Similarly,

$$\begin{aligned}
\|f^2 u\|_{L_T^1 \mathcal{H}^1} &\le C\|f^2 H^{1/2} u\|_{L_T^1 L^2} + C\|ufH^{1/2} f\|_{L_T^1 L^2} \\
&\le CT^{1-} \|u\|_{L_T^\infty \mathcal{H}^1} \|f\|_{L_T^{\infty-} L^\infty}^2 + CT^{1-} \|u\|_{L_T^\infty L^2} \|f\|_{L_T^{\infty-} L^\infty} \|H^{1/2} f\|_{L_T^{\infty-} L^\infty} \\
&\le CT^{1-} N^{-2-3s+} + CT^{1-} N^{-2-2s+} \le CN^{-6+2s+} \le CN^{\beta(s)+}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\|f^3\|_{L_T^1 \mathcal{H}^1} \le C\|f^2 H^{1/2} f\|_{L_T^1 L^2} &\le CT^{1-} \|H^{1/2} f\|_{L_T^{\infty-} L^\infty} \|f\|_{L_T^{\infty-} L^\infty} \|f\|_{L_T^\infty L^2} \\
&\le CT^{1-} N^{-1/2-s+} N^{-3/2-s+} N^{-s+} \le CN^{-6+s+} \le CN^{\beta(s)+}.
\end{aligned}$$

• Linear terms in w:

$$\begin{aligned}\|wf^2\|_{L^1_T\mathcal{H}^1} &\le C\|f^2H^{1/2}w\|_{L^1_TL^2}+C\|wfH^{1/2}f\|_{L^1_TL^2}\\ &\le CT^{1-}\|f\|^2_{L^{\infty-}_TL^\infty}\|w\|_{L^\infty_T\mathcal{H}^1}+CT^{1-}\|w\|_{L^\infty_TL^2}\|f\|_{L^{\infty-}_TL^\infty}\|H^{1/2}f\|_{L^{\infty-}_TL^\infty}\\ &\le CN^{-6+2s+}\|w\|_{X^1_T}\le CN^{0-}\|w\|_{X^1_T}.\end{aligned}$$

Using that $\|w\|_{L^{4/3+}_TL^{\infty-}}\le CT^{1/2-}\|w\|_{L^4_TL^{\infty-}}\le CT^{1/2-}\|w\|_{L^4_T\mathcal{W}^{1,3}}$ and $X^1_T\subset L^4([0,T];\mathcal{W}^{1,3})$,

$$\begin{aligned}\|wu^2\|_{L^1_T\mathcal{H}^1+L^2\mathcal{W}^{1,6/5}} &\le C\|u^2H^{1/2}w\|_{L^1_TL^2}+C\|wuH^{1/2}u\|_{L^{4/3+}_TL^{3/2-}}\\ &\le C\|u\|^2_{L^4_TL^6}\|w\|_{L^2_T\mathcal{W}^{1,6}}+C\|w\|_{L^{4/3+}_TL^{\infty-}}\|u\|_{L^\infty_TL^6}\|u\|_{L^\infty_T\mathcal{H}^1}\\ &\le CT^{1/2-}\|u\|^2_{L^\infty_T\mathcal{H}^1}\|w\|_{X^1_T}\le CN^{0-}\|w\|_{X^1_T}.\end{aligned}$$

• The cubic term in w: by Sobolev and $X^1_T\subset L^{4-}([0,T];\mathcal{W}^{1,3+})\subset L^{4-}([0,T];L^\infty)$, we have

$$\begin{aligned}\|w^3\|_{L^1_T\mathcal{H}^1} &\le C\|w^2H^{1/2}w\|_{L^1_TL^2}\le C\|w\|_{L^\infty_T\mathcal{H}^1}\|w\|^2_{L^2_TL^\infty}\\ &\le CT^{1/2-}\|w\|^3_{X^1_T}\le CN^{-2(1-s)+}\|w\|^3_{X^1_T}.\end{aligned}$$

• Quadratic terms in w: with similar arguments, we check that they are controlled by the previous ones.

This completes the proof of (4-11). Hence, for all $u_0\in\Sigma^1_T$, L has a unique fixed point w.

Let $w\in X^1_T$ be defined this way, and let us prove that $\|w\|_{X^0_T}\le CN^{-9/2+2s+}$, which will imply (4-8). By the Strichartz inequality (3-3),

$$\|w\|_{X^0_T}\le C\big\||f+u+w|^2(f+u+w)-|u|^2u\big\|_{L^1_TL^2+L^2L^{6/5}}.$$

As previously, the main contribution in the source term is

$$\|fu^2\|_{L^1_TL^2}\le T^{1-}\|u\|^2_{L^\infty_TL^4}\|f\|_{L^{\infty-}_TL^\infty}\le CN^{-4(1-s)+1-s-3/2-s+}=CN^{-9/2+2s+}.$$

For the cubic term we write

$$\begin{aligned}\|w^3\|_{L^1_TL^2} &\le \|w\|_{L^\infty_TL^2}\|w\|^2_{L^2_TL^\infty}\le CT^{1/2-}\|w\|_{L^\infty_TL^2}\|w\|^2_{X^1_T}\\ &\le CN^{-2(1-s)+\beta(s)+}\|w\|_{L^\infty_TL^2}\le CN^{0-}\|w\|_{X^0_T},\end{aligned}$$

which gives a control by the linear term.

The other terms are controlled with similar arguments, and we leave the details to the reader. This finishes the proof of Proposition 4.1. □

Lemma 4.2. *Under the assumptions of Proposition 4.1, for all $u_0\in\Sigma^1_T$ we have*

$$|E(u^1(T)+w^1(T))-E(u^1(T))|\le CN^{1-s+\beta(s)+}.$$

Proof. Write $u = u^1$ and $w = w^1$. A direct expansion and Hölder give

$$|E(u(T)+w(T)) - E(u(T))| \\ \leq 2\|u\|_{L_T^\infty \mathcal{H}^1}\|w\|_{L_T^\infty \mathcal{H}^1} + \|w\|^2_{L_T^\infty \mathcal{H}^1} + C\|w\|_{L_T^\infty L^4}\|u\|^3_{L_T^\infty L^4} + C\|w\|^4_{L_T^\infty L^4}.$$

Since $\beta(s) \leq (1-s)$, we directly have

$$2\|u\|_{L_T^\infty \mathcal{H}^1}\|w\|_{L_T^\infty \mathcal{H}^1} + \|w\|^2_{L_T^\infty \mathcal{H}^1} \leq CN^{1-s+\beta(s)+}.$$

By Sobolev and Proposition 4.1,

$$\|w\|_{L_T^\infty L^4} \leq C\|w\|_{L_T^\infty \mathcal{H}^{3/4}} \leq C\|w\|^{1/4}_{L_T^\infty L^2}\|w\|^{3/4}_{L_T^\infty \mathcal{H}^1} \leq CN^{\eta(s)+}, \tag{4-13}$$

with $\eta(s) = \max(-3+s/2, -15/4+2s) \leq (1-s+\beta(s))/3$. Hence,

$$\|w\|^3_{L_T^\infty L^4} \leq CN^{1-s+\beta(s)+}.$$

From the bounds $\|u\|_{L_T^\infty L^4} \leq CN^{(1-s)/2}$ and (4-13), we infer

$$\|w\|_{L_T^\infty L^4}\|u\|^3_{L_T^\infty L^4} \leq CN^{\delta(s)+},$$

where $\delta(s) = \max(-3+s/2, -15/4+2s) \leq 1-s+\beta(s)$ (with equality when $0 < s \leq \frac{1}{2}$). This completes the proof. $\square$

With the results of Proposition 4.1 and Lemma 4.2, we are able to iterate the argument. At time $t = T$, write $u = u^2 + v^2$ where u^2 is the solution to

$$\begin{cases} i\dfrac{\partial u^2}{\partial t} - Hu^2 = |u^2|^2 u^2, & (t,x) \in \mathbb{R}\times\mathbb{R}^3, \\ u^2(T) = u^1(T) + w^1(T) \in \mathcal{H}^1(\mathbb{R}^3), \end{cases} \tag{4-14}$$

and where $v^2 = e^{-itH}u_0^N + w^2$ satisfies

$$\begin{cases} i\dfrac{\partial w^2}{\partial t} - Hw^2 = \left|w^2 + e^{-itH}u_0^N + u^2\right|^2(w^2 + e^{-itH}u_0^N + u^2) - |u^2|^2u^2, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^3, \\ w^2(T) = 0. \end{cases}$$

By Proposition 4.1, $w^1(T) \in \mathcal{H}^1(\mathbb{R}^3)$; thus (4-14) is globally well-posed. Then, thanks to Lemma 4.2, by the conservation of the energy,

$$E(u^2) = E(u^1(T) + w^1(T)) \leq 4N^{2(1-s+\varepsilon)},$$

and, by the conservation of the mass,

$$\|u^2\|_{L_T^\infty L^2} = \|u^1(T) + w^1(T)\|_{L^2} \leq 2N^\varepsilon.$$

Therefore there exists a set $\Sigma_T^2 \subset \mathcal{H}^s$ with

$$\mu(\Sigma_T^2) \geq 1 - C\exp(-cT^{-\delta}\|\gamma\|^{-2}_{\mathcal{H}^s}),$$

and such that for all $u_0 \in \Sigma_T^2$ there exists a unique $w^2 \in \mathscr{C}([T, 2T], \mathscr{H}^1(\mathbb{R}^3))$ that satisfies the result of Proposition 4.1, with the same $T > 0$. Here we use crucially that the large deviation bounds of Proposition 2.1 are invariant under time shift τ.

Fix a time $A > 0$. We can iterate the previous argument and construct u^j, v^j and w^j for $1 \le j \le \lfloor A/T \rfloor$ such that the function u^j is the solution to (4-14) with initial condition

$$u^j(t = (j-1)T) = u^{j-1}((j-1)T) + w^{j-1}((j-1)T),$$

then we set $v^j(t) = e^{-itH} u_0^N + w^j(t)$, where the function w^j is the solution to

$$\begin{cases} i\dfrac{\partial w^j}{\partial t} - Hw^j = |w^j + e^{-itH}u_0^N + u^j|^2(w^j + e^{-itH}u_0^N + u^j) - |u^j|^2 u^j, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ w^j((j-1)T) = 0. \end{cases}$$

This enables us to define a unique solution u to the initial problem (1-7) defined by $u(t) = u^j(t) + v^j(t)$ for $t \in [(j-1)T, jT]$, with $1 \le j \le \lfloor A/T \rfloor$ provided that $u_0 \in \Gamma_T^A$, where $\Gamma_T^A := \bigcap_{j=1}^{\lfloor A/T \rfloor} \Sigma_T^j$.

Thanks to the exponential bounds, we have $\mu((\Gamma_T^A)^c) \le C \exp(-cT^{-\delta/2} \|\gamma\|_{\mathscr{H}^s}^{-2})$, with $T = N^{-4(1-s)-\varepsilon}$.

For uniform bounds on the energy and the mass, it remains to check whether $E(u^j) \le 4N^{2(1-s+\varepsilon)}$ and $\|u^j\|_{L^2(\mathbb{R}^3)} \le 2N^{\varepsilon}$ for all $1 \le j \le \lfloor A/T \rfloor$. By Lemma 4.2, for $T = N^{-4(1-s)-}$,

$$E(u^j) \le E(u_{0,N}) + CAT^{-1}N^{1-s+\beta(s)+} \le 2N^{2(1-s+\varepsilon)} + CAN^{\beta(s)+5(1-s)+}, \tag{4-15}$$

which satisfies the prescribed bound if and only if $3(1-s) + \beta(s) < 0$.

- If $\frac{1}{2} \le s \le 1$, the condition is $3(1-s) + 2s - \frac{7}{2} < 0$, or equivalently $s > -\frac{1}{2}$, which is satisfied.
- If $0 \le s \le \frac{1}{2}$, the condition is $3(1-s) - \frac{5}{2} < 0$, or equivalently $s > \frac{1}{6}$. The same argument applies to control $\|u^j\|_{L^2}$.

If $\frac{1}{6} < s < 1$, we optimise in (4-15) with the choice of $N \ge 1$ so that $A \sim cN^{-3(1-s)-\beta(s)}$, and get that, for $1 \le j \le \lfloor A/T \rfloor$,

$$E(u^j) \le CA^{c_s+},$$

with

$$c_s = \begin{cases} \dfrac{2(1-s)}{6s-1} & \text{if } \frac{1}{6} < s \le \frac{1}{2}, \\ \dfrac{2(1-s)}{2s+1} & \text{if } \frac{1}{2} \le s \le 1. \end{cases} \tag{4-16}$$

Denote by $\Gamma^A = \Gamma_T^A$ the set defined with the previous choice of N and $T = N^{-4(1-s)-\varepsilon}$.

Lemma 4.3. *Let $\frac{1}{6} < s < 1$. Then for all $A \in \mathbb{N}$ and all $u_0 \in \Gamma^A$ there exists a unique solution to* (1-7) *on* $[0, A]$, *which reads*

$$u(t) = e^{-itH}u_0 + w(t), \quad \textit{with } w \in \mathscr{C}\big([0, A], \mathscr{H}^1(\mathbb{R}^3)\big), \ \sup_{t \in [0,A]} E\big(w(t)\big) \le CA^{c_s+}.$$

Proof. On the time interval $[(j-1)T, jT]$ we have $u = u^j + v^j$ where $v^j = e^{-itH}u_0^N + w^j$ and $u^j = e^{-itH}u_{0,N} + z^j$, for some $z_j \in \mathscr{C}([0, +\infty[, \mathscr{H}^1(\mathbb{R}^3))$. Therefore, if we define $w \in \mathscr{C}([0, A], \mathscr{H}^1(\mathbb{R}^3))$

by $w(t) = z^j(t) + w^j(t)$ for $t \in [(j-1)T, jT]$ and $1 \le j \le \lfloor A/T \rfloor$, we get $u(t) = e^{-itH}u_0 + w(t)$ for all $t \in [0, A]$. Next, for $t \in [(j-1)T, jT]$,

$$E(w(t)) \le CE(z^j) + CE(w^j) \le CE(u^j) + CE(e^{-itH}u_{0,N}) + CE(w^j) \le CA^{c_s+},$$

which was the claim. □

We are now able to complete the proof of Theorem 1.5. Set

$$\Theta = \bigcap_{k=1}^{+\infty} \bigcup_{A \ge k} \Gamma^A \quad \text{and} \quad \Sigma = \Theta + \mathcal{H}^1.$$

We have $\mu(\Theta) = \lim_{k\to\infty} \mu\big(\bigcup_{A\ge k} \Gamma^A\big)$ and $\mu\big(\bigcup_{A\ge k} \Gamma^A\big) \ge 1 - c\exp(-k^\delta \|\gamma\|_{\mathcal{H}^s}^{-2})$. So $\mu(\Theta) = 1$, and thus $\mu(\Sigma) = 1$.

By definition, for all $u_0 \in \Theta$, there exists a unique global solution to (1-7), which reads

$$u(t) = e^{-itH}u_0 + w(t), \quad w \in \mathscr{C}(\mathbb{R}, \mathcal{H}^1(\mathbb{R}^3)).$$

Then by Lemma 4.3 for all $u_0 \in \Theta$, there exists a unique $w \in \mathscr{C}([0, +\infty[, \mathcal{H}^1(\mathbb{R}^3))$, which satisfies, for all N, the bound

$$\sup_{t\in[0,N]} E(w(t)) \le CN^{c_s+}.$$

Now, if $U_0 \in \Sigma$ then $U_0 = u_0 + v$ with $u_0 \in \Theta$, $v \in \mathcal{H}^1$ and we can use the method of Proposition 4.1, Lemma 4.2 and Lemma 4.3 with $U_{0,N}$ replaced by $u_{0,N} + v$. And the set Σ satisfies properties (i) and (ii).

Coming back to the definition of Σ_T^j, we have $e^{-itH}(\Sigma_T^j) = \Sigma_T^j$ for all $t \in \mathbb{R}$; thus $e^{-itH}(\Theta) = \Theta$. Finally, thanks to property (i), the set Σ is invariant under the dynamics and property (iii) is satisfied.

4B. ***The case of dimension $d = 2$.*** In this section, we prove Theorem 1.6. The proof is analogous to Theorem 1.5 in a simpler context; that is why we only explain the key estimates.

According to Proposition 2.1, we set

$$F_s(K) = \big\{ w \in \mathcal{H}^s(\mathbb{R}^2) \mid \|w\|_{\mathcal{H}^s(\mathbb{R}^2)} \le K, \|w\|_{L^4(\mathbb{R}^2)} \le K \text{ and } \|e^{-itH}w\|_{L^{1/\varepsilon}_{[0,2\pi]}\mathcal{W}^{1+s-\varepsilon,\infty}(\mathbb{R}^2)} \le K \big\},$$

and we fix $u_0 \in F_s(N^\varepsilon)$.

Then, if $f = e^{-itH}u_0^N$, we have

$$\|f\|_{L^\infty_{[0,2\pi]}L^2} \le CN^{-s+\varepsilon} \quad \text{and} \quad \|H^{\sigma/2} f\|_{L^{\infty-}_{[0,2\pi]},L^\infty} \le CN^{\sigma-1-s+\varepsilon}.$$

In Proposition 4.1 we can choose $T = N^{-2(1-s)-\varepsilon}$ to have

$$\|u^1\|_{L^\infty_T L^2} \le CN^\varepsilon \quad \text{and} \quad \|u^1\|_{L^\infty_T \mathcal{H}^1} \le CN^{1-s+\varepsilon}.$$

Moreover, as $u_0 \in F_s(N^\varepsilon)$, we obtain

$$\|u_{0,N}\|_{L^4} \le CN^\varepsilon.$$

Hence, we establish

$$E(u^1) = \|u^1\|^2_{\mathcal{H}^1(\mathbb{R}^2)} + \tfrac{1}{2}\|u^1\|^4_{L^4(\mathbb{R}^2)} = \|u_{0,N}\|^2_{\mathcal{H}^1(\mathbb{R}^2)} + \tfrac{1}{2}\|u_{0,N}\|^4_{L^4(\mathbb{R}^2)} \le N^{2(1-s+\varepsilon)} + CN^{4\varepsilon} \le 4N^{2(1-s+\varepsilon)},$$

and

$$\|u^1\|_{L^\infty_T L^4} \le CN^{(1-s+\varepsilon)/2}.$$

In Proposition 4.1, we obtain $\|w^1\|_{L^\infty_{[0,T]}\mathcal{H}^1} \le CN^{-1+}$ and $\|w^1\|_{L^\infty_{[0,T]}L^2} \le CN^{-2+}$. The proof is essentially the same. We define the map L as in (4-9). For the first estimate, we prove that

$$\|L(w)\|_{X^1_T} \le CN^{-1+} + N^{0-}\|w\|_{X^1_T} + N^{-2(1-s)+}\|w\|^3_{X^1_T}.$$

We only give details for the source terms. First,

$$\begin{aligned}\|fu^2\|_{L^{1+}_T\mathcal{W}^{1,2-}+L^1_T\mathcal{H}^1} &\le C\|fuH^{1/2}u\|_{L^{1+}_T L^{2-}} + C\|u^2H^{1/2}f\|_{L^1_T L^2}\\ &\le CT^{1-}\|u\|_{L^\infty_T\mathcal{H}^1}\|u\|_{L^\infty_T L^{\infty-}}\|f\|_{L^{\infty-}_T L^\infty} + CT^{1-}\|u\|^2_{L^\infty_T L^4}\|H^{1/2}f\|_{L^{\infty-}_T L^\infty}\\ &\le CT^{1-}\max(N^{1-3s+}, N^{1-2s+}) \le CT^{1-}N^{1-2s+} \le CN^{-1+}.\end{aligned}$$

Similarly,

$$\begin{aligned}\|f^2u\|_{L^1_T\mathcal{H}^1} &\le C\|f^2H^{1/2}u\|_{L^1_T L^2} + C\|ufH^{1/2}f\|_{L^1_T L^2}\\ &\le CT^{1-}\|u\|_{L^\infty_T\mathcal{H}^1}\|f\|^2_{L^{\infty-}_T L^\infty} + CT^{1-}\|u\|_{L^\infty_T L^2}\|f\|_{L^{\infty-}_T L^\infty}\|H^{1/2}f\|_{L^{\infty-}_T L^\infty}\\ &\le CT^{1-}\max(N^{-1-3s+}, N^{-1-2s+}) \le CT^{1-}N^{-1-2s+} \le CN^{-3+} \le CN^{-1+}.\end{aligned}$$

Finally,

$$\begin{aligned}\|f^3\|_{L^1_T\mathcal{H}^1} &\le C\|f^2H^{1/2}f\|_{L^1_T L^2} \le CT^{1-}\|H^{1/2}f\|_{L^{\infty-}_T L^\infty}\|f\|_{L^{\infty-}_T L^\infty}\|f\|_{L^\infty_T L^2}\\ &\le CT^{1-}N^{-s+}N^{-1-s+}N^{-s+} \le CN^{-3-s+} \le CN^{-1+}.\end{aligned}$$

Analogously to Lemma 4.2, we obtain $|E(u^1(T)+w^1(T)) - E(u^1(T))| \le CN^{-s+}$, because here $\beta(s) = 1-$ and the estimates on u^1 are the same as in dimension $d = 3$.

Finally, the globalisation argument holds if (4-15) is satisfied, that is to say

$$CAT^{-1}N^{-s+} \le 4N^{2(1-s)+},$$

which is equivalent to $2(1-s)-s < 2(1-s)$, hence $s > 0$. In this case, we set $A \sim cN^s$ and we get that, for $0 \le t \le A$,

$$E(w(t)) \le CA^{c_s+}, \text{ with } c_s = \frac{1-s}{s}.$$

Theorem 1.6 follows.

Acknowledgements

We are grateful to Nicolas Burq for discussions on this subject. We thank Rémi Carles for discussions on scattering theory which led us toward Theorem 1.4.

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Received 20 Sep 2013. Revised 9 Apr 2014. Accepted 8 May 2014.

AURÉLIEN POIRET: aurelien.poiret@math.u-psud.fr
Laboratoire de Mathématiques, UMR 8628 du CNRS, Université Paris Sud, 91405 Orsay Cedex, France

DIDIER ROBERT: didier.robert@univ-nantes.fr
Laboratoire de Mathématiques J. Leray, UMR 6629 du CNRS, Université de Nantes, 2, rue de la Houssinière, 44322 Nantes Cedex 03, France

LAURENT THOMANN: laurent.thomann@univ-nantes.fr
Laboratoire de Mathématiques J. Leray, UMR 6629 du CNRS, Université de Nantes, 2, rue de la Houssinière, 44322 Nantes Cedex 03, France

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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Analysis & PDE

Volume 7 No. 4 2014

2157-5045(2014)7:4;1-C