ANALYSIS & PDE

Volume 7 No. 6 2014

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ANALYSIS AND PDE
Vol. 7, No. 6, 2014
dx.doi.org/10.2140/apde.2014.7.1317

OPTIMAL CONTROL OF SINGULAR FOURIER MULTIPLIERS BY MAXIMAL OPERATORS

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Dedicated to the memory of Adela Moyua, 1956–2013.

We control a broad class of singular (or "rough") Fourier multipliers by geometrically defined maximal operators via general weighted $L^2(\mathbb{R})$ norm inequalities. The multipliers involved are related to those of Coifman, Rubio de Francia and Semmes, satisfying certain weak Marcinkiewicz-type conditions that permit highly oscillatory factors of the form $e^{i|\xi|^\alpha}$ for both α positive and negative. The maximal functions that arise are of some independent interest, involving fractional averages associated with tangential approach regions (related to those of Nagel and Stein), and more novel "improper fractional averages" associated with "escape" regions. Some applications are given to the theory of L^p–L^q multipliers, oscillatory integrals and dispersive PDE, along with natural extensions to higher dimensions.

1. Introduction and statements of results

Given a Fourier multiplier m, with corresponding convolution operator T_m, there has been considerable interest in identifying, where possible, "geometrically defined" maximal operators $\mathcal{M}$ for which a weighted L^2-norm inequality of the form

$$\int_{\mathbb{R}^n} |T_m f|^2 w \le \int_{\mathbb{R}^n} |f|^2 \mathcal{M} w \tag{1}$$

holds for all admissible input functions f and weight functions w. This very general Fourier multiplier problem was made particularly explicit in the 1970s in work of A. Córdoba and C. Fefferman [1976], following the emergence of fundamental connections between the theory of Fourier multipliers and elementary geometric notions such as curvature (see [Fefferman 1971; Córdoba 1977; Stein 1979], in particular). Such control of a multiplier m by a maximal operator $\mathcal{M}$, combined with an elementary duality argument, reveals that, for $p, q \ge 2$,

$$\|m\|_{p,q} := \|T_m\|_{L^p - L^q} \le \|\mathcal{M}\|^{1/2}_{L^{(q/2)'} - L^{(p/2)'}}. \tag{2}$$

Thus it is of particular interest to identify an "optimal" maximal operator $\mathcal{M}$ for which (1) holds, in the sense that (2) permits optimal L^p–L^q bounds for $\mathcal{M}$ to be transferred to optimal bounds for T_m.

There are a variety of results of this nature, although often formulated in terms of the convolution kernel rather than the multiplier. For example, if T denotes a Calderón–Zygmund singular integral operator

Supported by ERC Starting Grant 307617.
MSC2010: primary 42B15, 42B25, 42B20; secondary 42B37.
Keywords: Fourier multipliers, maximal operators, weighted inequalities.

on $\mathbb{R}^n$, such as the Hilbert transform on the line, Córdoba and Fefferman [1976] (see also [Hunt et al. 1973]) showed that for each $s > 1$ there is a constant $C_s < \infty$ for which

$$\int_{\mathbb{R}^n} |Tf|^2 w \le C_s \int_{\mathbb{R}^n} |f|^2 (Mw^s)^{1/s} \tag{3}$$

holds, where M denotes the classical Hardy–Littlewood maximal operator. This result extends to weighted L^p estimates for $1 < p < \infty$; see [Córdoba and Fefferman 1976]. The inequality (3) may be viewed as a consequence of the classical theory of Muckenhoupt A_p weights through the fundamental fact that if $(Mw^s)^{1/s} < \infty$ a.e. and $s > 1$ then $(Mw^s)^{1/s} \in A_1 \subset A_2$; see [Stein 1993] and the references there. Of course, for any fixed $s > 1$ the maximal operator $w \mapsto (Mw^s)^{1/s}$ in (3) is not optimal, since it fails to be L^p-bounded in the range $1 < p \le s$, while T is bounded on L^p for all $1 < p < \infty$. More recently this was remedied by Wilson [1989], who showed that[1]

$$\int_{\mathbb{R}^n} |Tf|^2 w \lesssim \int_{\mathbb{R}^n} |f|^2 M^3 w, \tag{4}$$

where $M^3 = M \circ M \circ M$ denotes the 3-fold composition of M with itself. As with (3), this useful result extends to weighted L^p norms for $1 < p < \infty$; see [Wilson 1989; Pérez 1994; Reguera and Thiele 2012]. There are numerous further results belonging to the considerable theory surrounding the A_p weights; see for example [García-Cuerva and Rubio de Francia 1985; Pérez 1995; Hytönen 2012; Lacey et al. 2014; Hytönen et al. 2013; Lerner 2013].

In the setting of *oscillatory integrals* the controlling maximal operators appear to acquire a much more interesting geometric nature, well beyond the scope of the classical A_p theory. This is illustrated well by a compelling and seemingly very deep conjecture concerning the classical *Bochner–Riesz multipliers*,

$$m_\delta(\xi) = (1 - |\xi|^2)_+^\delta,$$

where $\xi \in \mathbb{R}^n$ and $\delta \ge 0$. Of course, m_0 is simply the characteristic function of the unit ball in $\mathbb{R}^n$, allowing us to interpret m_δ for $\delta > 0$ as a certain regularisation of this characteristic function. The classical Bochner–Riesz conjecture concerns the range of exponents p for which m_δ is an L^p-multiplier. In the 1970s, A. Córdoba [1977] and E. M. Stein [1979] raised the possibility that a weighted inequality of the form (1) holds where $\mathcal{M}$ is some suitable variant of the Nikodym maximal operator

$$\mathcal{N}_\delta w(x) := \sup_{T \ni x} \frac{1}{|T|} \int_T w;$$

see also [Fefferman 1971; 1973]. Here the supremum is taken over all cylindrical tubes of eccentricity less than $1/\delta$ that contain the point x. This maximal operator $\mathcal{M}$ should be geometrically defined (very much like $\mathcal{N}_\delta$) and its known/conjectured bounds should be similar to those of $\mathcal{N}_\delta$, thus essentially implying the full Bochner–Riesz conjecture via (2).[2] Such a result is rather straightforward for $n = 1$ as it reduces to

[1] Throughout this paper we shall write $A \lesssim B$ if there exists a constant c such that $A \le cB$. In particular, this constant will always be independent of the input function f and weight function w. The relations $A \gtrsim B$ and $A \sim B$ are defined similarly.

[2] Similar weighted inequalities relating the Fourier restriction and Kakeya conjectures have also received some attention in the literature; see [Bennett et al. 2006] for further discussion.

the aforementioned inequality for the Hilbert transform. In higher dimensions this question is far from having a satisfactory answer already for $n = 2$ (see [Bourgain 1991; Christ 1985; Carbery et al. 1992; Carbery and Soria 1997a; 1997b; Carbery and Seeger 2000; Bennett et al. 2006; Duoandikoetxea et al. 2008; Lee et al. 2012; Córdoba and Rogers 2014] for some related results). The associated convolution kernel

$$K_\delta(x) := \mathscr{F}^{-1} m_\delta(x) = \frac{c J_{n/2+\delta}(2\pi|x|)}{|x|^{n/2+\delta}} = c\frac{e^{2\pi i|\xi|} + e^{-2\pi i|\xi|} + o(1)}{|\xi|^{(n+1)/2+\delta}},$$

unlike the Hilbert kernel, is (for δ sufficiently small) very far from being Lebesgue integrable. Here J_λ denotes the Bessel function of order λ, making K_δ highly oscillatory.

In [Bennett and Harrison 2012], using arguments from [Bennett et al. 2006] in the setting of Fourier extension operators, we gave nontrivial examples of such "optimal" control of oscillatory kernels on the line by geometrically defined maximal operators. In particular, for integers $\ell \geq 3$, we showed that

$$\int_{\mathbb{R}} \left|e^{i(\cdot)^\ell} * f\right|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^2 \mathcal{M} M^4 w, \tag{5}$$

where

$$\mathcal{M}w(x) := \sup_{(y,r)\in\Gamma(x)} \frac{1}{r^{1/(\ell-1)}} \int_{y-r}^{y+r} w$$

and

$$\Gamma(x) = \left\{(y,r) : 0 < r \leq 1,\ |x-y| \leq r^{-1/(\ell-1)}\right\}. \tag{6}$$

The maximal operator $\mathcal{M}$ here may be interpreted as a fractional Hardy–Littlewood maximal operator associated with an approach region $\Gamma(x)$. This maximal operator is similar in spirit to those studied by Nagel and Stein [1984], although here tangential approach to infinite order is permitted. It is shown in [Bennett and Harrison 2012] that $\mathcal{M}$ has a sharp bound on $L^{(\ell/2)'}$, which may be reconciled via (5) with a sharp L^ℓ bound for convolution with e^{ix^ℓ}. We note in passing that the factors of the Hardy–Littlewood maximal operator appearing in (5) are of secondary importance as $\mathcal{M}$ and $M^2\mathcal{M}M^4$ share the same L^p–L^q mapping properties. This follows from the L^p-boundedness of M for $1 < p \leq \infty$.

In this paper we seek an understanding of the "map" $m \mapsto \mathcal{M}$, from Fourier multiplier to optimal controlling maximal operator, for which (1) holds. As we shall see, an inequality of the form (1) does indeed hold for a wide class of multipliers m and a surprisingly rich family of geometrically defined maximal operators $\mathcal{M}$. This class of multipliers is sufficiently singular to apply to a variety of highly oscillatory convolution kernels, placing (5) in a much broader context. The maximal operators turn out to be fractional Hardy–Littlewood maximal operators associated with a diverse family of approach and "escape" regions in the half-space. While such operators corresponding to approach regions have arisen before [Nagel and Stein 1984; Bennett et al. 2006; Bennett and Harrison 2012], those associated with "escape" regions appear to be quite novel, involving improper fractional averages.

As is well known, at least in one dimension, the *variation* of a multiplier can play a decisive role in determining its behaviour as an operator. For example, if a multiplier m is of bounded variation on the line, then it often satisfies the same norm inequalities as the Hilbert transform. This is a straightforward

consequence of the elementary identity

$$T_m = \lim_{t\to-\infty} m(t)I + \frac{1}{2}\int_{\mathbb{R}} (I + iM_{-t}HM_t)\,dm(t). \tag{7}$$

Here I denotes the identity operator on $\mathbb{R}^d$, the modulation operator M_t is given by $M_t f(x) = e^{-2\pi i x t} f(x)$, and $dm(t)$ denotes the Lebesgue–Stieltjes measure (which we identify with $|m'(t)|\,dt$ throughout). In particular, combining this with (4) quickly leads to the inequality

$$\int_{\mathbb{R}} |T_m f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^3 w. \tag{8}$$

Invoking classical weighted Littlewood–Paley theory for dyadic decompositions of the line (see [Wilson 2007] and [Bennett and Harrison 2012] for further discussion) leads to the following weighted version of the Marcinkiewicz multiplier theorem (cf. [Kurtz 1980]).

Theorem 1. *If $m : \mathbb{R} \to \mathbb{C}$ is a bounded function which is uniformly of bounded variation on dyadic intervals, that is,*

$$\sup_{R>0} \int_{R\le|\xi|\le 2R} |m'(\xi)|\,d\xi < \infty, \tag{9}$$

then

$$\int_{\mathbb{R}} |T_m f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^7 w.$$

The control of m here by a power of the Hardy–Littlewood maximal operator is optimal in the sense that Theorem 1, combined with the Hardy–Littlewood maximal theorem, implies the classical Marcinkiewicz multiplier theorem via (2). It would seem unlikely that the particular power of M that features here is best possible; here and throughout this paper we do not concern ourselves with such finer points.

Our goal is to establish versions of Theorem 1 which apply to much more singular (or "rougher") multipliers. A natural class of singular multipliers on the line, defined in terms of the so-called "r-variation" was introduced by Coifman, Rubio de Francia and Semmes in [Coifman et al. 1988]. For a function m on an interval $[a, b]$ we define the r-variation of m to be the supremum of the quantity

$$\left(\sum_{j=0}^{N-1} |m(x_{j+1}) - m(x_j)|^r\right)^{1/r}$$

over all partitions $a = x_0 < x_1 < \cdots < x_N = b$ of $[a, b]$. We say that m is a V_r *multiplier* if it has uniformly bounded r-variation on each dyadic interval. (Of course, if $r = 1$ this class reduces to the classical Marcinkiewicz multipliers.) In [Coifman et al. 1988] it is shown that if m is a V_r multiplier then m is an $L^p(\mathbb{R})$ multiplier for $|1/p - 1/2| < 1/r$, considerably generalising the classical Marcinkiewicz multiplier theorem on the line. With the possible exception of the endpoint, this result is sharp, as may be seen from the specific multipliers

$$m_{\alpha,\beta}(\xi) := \frac{e^{i|\xi|^\alpha}}{(1+|\xi|^2)^{\beta/2}}, \quad \alpha, \beta \ge 0, \tag{10}$$

first studied by Hirschman [1959] (see [Stein 1970] for further discussion). Indeed $m_{\alpha,\beta}$ is a V_r multiplier if $\beta r = \alpha$, while being an L^p multiplier if and only if $\alpha|1/p - 1/2| \le \beta$; see [Hirschman 1959; Miyachi 1981]. The endpoint case $|1/p - 1/2| = 1/r$ remains open in general for V_r multipliers; see [Tao and Wright 2001] for further discussion and related results.

For the purposes of identifying *optimal* controlling maximal operators we will confine attention to a subclass of the V_r multipliers that retains some of the structure of the specific example (10). Before we describe this subclass let us discuss some motivating examples.

The multiplier corresponding to the convolution kernel e^{ix^ℓ} appearing in (5) coincides with the (generalised) Airy function

$$Ai^{(\ell)}(\xi) = \int_{-\infty}^{\infty} e^{i(x^\ell + x\xi)}\, dx = c_0 \frac{e^{ic_1|\xi|^{\ell/(\ell-1)}} + o(1)}{|\xi|^{(\ell-2)/(2(\ell-1))}}$$

as $|\xi| \to \infty$; here c_0 and c_1 are appropriate constants. As standard Airy function asymptotics reveal, the variation of this multiplier on dyadic intervals is unbounded. This multiplier, with its highly oscillatory behaviour as $|\xi| \to \infty$, belongs to a more general class of multipliers satisfying

$$m(\xi) = O\big(|\xi|^{-\beta}\big), \quad m'(\xi) = O\big(|\xi|^{-\beta+\alpha-1}\big) \tag{11}$$

as $|\xi| \to \infty$. Here $\alpha, \beta \ge 0$, and of course the specific multiplier in (10) is a model example. In addition to multipliers whose derivatives can have strong singularities at *infinity*, it is also natural to consider those which are singular at a *point*. In particular, we might hope to control multipliers satisfying (11) as $|\xi| \to 0$ for $\alpha, \beta \le 0$. Such singular multipliers, which were studied by Miyachi [1980; 1981], arise frequently in the study of oscillatory and oscillatory-singular integrals; see for example [Stein 1993; Miyachi 1981; Sjölin 1981; Chanillo et al. 1986]. See also [Miyachi 1980; 1981] for a general $L^p(\mathbb{R})$ (and Hardy space $H^p(\mathbb{R})$) multiplier theorem under the specific hypothesis (11). The following class of multipliers, which we denote $\mathscr{C}(\alpha, \beta)$, involves a Marcinkiewicz-type variation condition specifically designed to capture these Miyachi-type examples.

The class of multipliers. For each $\alpha, \beta \in \mathbb{R}$, let $\mathscr{C}(\alpha, \beta)$ be the class of functions $m : \mathbb{R} \to \mathbb{C}$ for which

$$\operatorname{supp}(m) \subseteq \{\xi : |\xi|^\alpha \ge 1\}, \tag{12}$$

$$\sup_{\xi} |\xi|^\beta |m(\xi)| < \infty, \tag{13}$$

$$\sup_{R^\alpha \ge 1} \sup_{\substack{I \subseteq [R,2R] \\ \ell(I) = R^{-\alpha}R}} R^\beta \int_{\pm I} |m'(\xi)|\, d\xi < \infty. \tag{14}$$

Here the supremum is taken over all subintervals I of $[R, 2R]$ of length $\ell(I) = R^{-\alpha}R$.

Remarks.

(i) The support condition (12) has no content for $\alpha = 0$. For $\alpha > 0$ and $\alpha < 0$ it reduces to, respectively, $\operatorname{supp}(m) \subseteq \{|\xi| \ge 1\}$ and $\operatorname{supp}(m) \subseteq \{|\xi| \le 1\}$. A similar interpretation applies to the outermost supremum in (14).

(ii) The case $\alpha = 0$ is of course somewhat degenerate. As is easily verified, the class $\mathscr{C}(\alpha, \beta)$ reduces to the classical Marcinkiewicz multipliers when $\alpha = \beta = 0$. Further, the fractional integration multiplier $\xi \mapsto |\xi|^{-\beta}$ lies in $\mathscr{C}(0, \beta)$.

(iii) The model behaviour of a multiplier in $\mathscr{C}(\alpha, \beta)$ in the nondegenerate case $\alpha \neq 0$ is that of the Miyachi multipliers (11) as $|\xi|^{\alpha} \to \infty$.

(iv) An elementary calculation reveals that if m lies in $\mathscr{C}(\alpha, \beta)$ then m is a V_r multiplier provided $\beta r = \alpha$. We also note that the additional structure of the class $\mathscr{C}(\alpha, \beta)$ yields L^p–L^q estimates for certain $q \neq p$ — see the forthcoming Corollary 5.

(v) An elementary change of variables argument reveals that a multiplier $m \in \mathscr{C}(\alpha, \beta)$ if and only if $\widetilde{m} \in \mathscr{C}(-\alpha, -\beta)$, where $\widetilde{m}(\xi) := m(1/\xi)$. The main point is that the diffeomorphism $\mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$, $\xi \mapsto 1/\xi$ preserves dyadic intervals and (essentially) any lattice structure within them.

(vi) Unlike the V_r multipliers, if $\alpha \neq 0$ the class $\mathscr{C}(\alpha, \beta)$ is not dilation-invariant due to the distinguished role of the unit scale $R = 1$. See the forthcoming Theorem 3 for a natural dilation-invariant formulation.

We now introduce the family of maximal operators that will control these multipliers via (1).

The controlling maximal operators. For $\alpha, \beta \in \mathbb{R}$ we define the maximal operator $\mathcal{M}_{\alpha,\beta}$ by

$$\mathcal{M}_{\alpha,\beta} f(x) = \sup_{(r,y)\in\Gamma_\alpha(x)} \frac{r^{2\beta}}{r} \int_{|y-z|\leq r} f(z)\, dz, \tag{15}$$

where

$$\Gamma_\alpha(x) = \big\{(r, y) : 0 < r^{\alpha} \leq 1 \text{ and } |y - x| \leq r^{1-\alpha}\big\}. \tag{16}$$

This family of maximal operators is of some independent interest. When $\alpha = 0$ the approach region $\Gamma_\alpha(x)$ is simply a cone with vertex x, and the associated maximal operator $\mathcal{M}_{\alpha,\beta}$ is equivalent to the classical fractional Hardy–Littlewood maximal operator

$$M_{2\beta} w(x) := \sup_{r>0} \frac{r^{2\beta}}{r} \int_{x-r}^{x+r} w. \tag{17}$$

When $0 < \alpha < 1$ the maximal operators $\mathcal{M}_{\alpha,\beta}$ have also been considered before and originate in work of Nagel and Stein [1984] on fractional maximal operators associated with more general nontangential approach regions. However, as we have already mentioned, the above definitions also permit $\alpha \geq 1$ and $\alpha < 0$, where one sees dramatic transitions in the nature of the region Γ_α. In particular if $\alpha \geq 1$ then the situation is similar to that in (6), where tangential approach to infinite order is permitted; see [Bennett et al. 2006] for the origins of such regions. Furthermore, for $\alpha < 0$ we have

$$\Gamma_\alpha(x) = \big\{(r, y) : r \geq 1 \text{ and } |y - x| \leq r^{1-\alpha}\big\},$$

which may be viewed as an "escape", rather than "approach", region. Notice also that if $\beta < 0$ we interpret $\mathcal{M}_{\alpha,\beta}$ as an *improper* fractional maximal operator.

The maximal operators $\mathcal{M}_{\alpha,\beta}$ are significant improvements on the controlling maximal operators $w \mapsto (Mw^s)^{1/s}$ that typically arise via classical A_p-weighted inequalities. Crudely estimating $\mathcal{M}_{\alpha,\beta} w$ pointwise using Hölder's inequality reveals that

$$\mathcal{M}_{\alpha,\beta} w \le (Mw^s)^{1/s} \quad \text{when } 2s\beta = \alpha. \tag{18}$$

This allows the forthcoming Theorem 2 to be reconciled with certain A_p-weighted inequalities established by Chanillo, Kurtz and Sampson in [Chanillo et al. 1983; 1986]. In Section 2 we provide necessary and sufficient conditions for $\mathcal{M}_{\alpha,\beta}$ to be bounded from L^p to L^q. In particular, we see that $\mathcal{M}_{\alpha,\beta}$ is bounded on L^s when $2s\beta = \alpha$, a property that does not follow from (18).

The main result of this paper is the following.

Theorem 2. *Let $\alpha, \beta \in \mathbb{R}$. If $m \in \mathscr{C}(\alpha, \beta)$ then*

$$\int_{\mathbb{R}} |T_m f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^6 \mathcal{M}_{\alpha,\beta} M^4 w. \tag{19}$$

It is interesting to contrast this result with the recent weighted variational Carleson theorem of Do and Lacey [2012]; see also [Oberlin et al. 2012; Lacey 2007].

As may be expected, the factors of the Hardy–Littlewood maximal operator M arising in Theorem 2 are of secondary importance, and to some extent occur for technical reasons. Since M is bounded on L^p for all $1 < p \le \infty$, the maximal operators $M^6 \mathcal{M}_{\alpha,\beta} M^4$ and $\mathcal{M}_{\alpha,\beta}$ share the same L^p–L^q bounds. The forthcoming Theorem 4 clarifies the L^p–L^q behaviour of these operators.

It is perhaps helpful to make some further remarks about the nonsingular case $\alpha = 0$ of Theorem 2. As is immediately verified, the class of multipliers $\mathscr{C}(0, \beta)$ is precisely those satisfying the conditions

$$\sup_{\xi \in \mathbb{R}} |\xi|^\beta |m(\xi)| < \infty, \tag{20}$$

and

$$\sup_{R>0} R^\beta \int_{R \le |\xi| \le 2R} |m'(\xi)|\, d\xi < \infty. \tag{21}$$

For such "classical" multipliers, Theorem 2 reduces to the weighted inequality

$$\int_{\mathbb{R}} |T_m f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^6 M_{2\beta} M^4 w, \tag{22}$$

where $M_{2\beta}$ is the fractional Hardy–Littlewood maximal operator given by (17). When $\beta = 0$, the conditions (20) and (21) become those of the classical Marcinkiewicz multiplier theorem, and the resulting inequality (22) reduces — up to factors of M — to the classical Theorem 1. Noting that the multiplier $\xi \mapsto |\xi|^{-\beta}$ lies in $\mathscr{C}(0, \beta)$, again up to factors of M we recover the one-dimensional case of Pérez's [1995] result.

Of course the class $\mathscr{C}(\alpha, \beta)$ is neither scale-invariant nor facilitates quantification of the implicit constants in Theorem 2. Our arguments, along with elementary scaling considerations, reveal the following.

Theorem 3. *Let $\alpha, \beta \in \mathbb{R}$ and $\lambda, C > 0$. If $m : \mathbb{R} \to \mathbb{C}$ is such that*

$$\operatorname{supp}(m) \subseteq \{\xi : |\xi|^\alpha \geq \lambda^\alpha\}, \tag{23}$$

$$\sup_\xi |\xi|^\beta |m(\xi)| \leq C, \tag{24}$$

$$\sup_{R^\alpha \geq \lambda^\alpha} \sup_{\substack{I \subseteq [R,2R] \\ \ell(I) = (R/\lambda)^{-\alpha} R}} R^\beta \int_{\pm I} |m'(\xi)|\, d\xi \leq C, \tag{25}$$

then there exists an absolute constant $c > 0$ such that

$$\int_{\mathbb{R}} |T_m f|^2 w \leq cC^2 \int_{\mathbb{R}} |f|^2 M^6 \mathcal{M}^\lambda_{\alpha,\beta} M^4 w, \tag{26}$$

where

$$\mathcal{M}^\lambda_{\alpha,\beta} w(x) = \sup_{(y,r) \in \Gamma^\lambda_\alpha(x)} \frac{r^{2\beta}}{r} \int_{y-r}^{y+r} w \quad \text{and} \quad \Gamma^\lambda_\alpha(x) = \big\{(y,r) : 0 < r^\alpha \leq \lambda^{-\alpha},\ |x-y| \leq \lambda^{-\alpha} r^{1-\alpha}\big\}.$$

The hypotheses of Theorem 3 are scale-invariant. More precisely, if m satisfies (23)–(25) with parameter $\lambda = \eta$, then $\eta^\beta m(\eta\,\cdot)$ satisfies (23)–(25) with parameter $\lambda = 1$.

Organisation of the paper. Our proof of Theorem 2 rests crucially on a certain Littlewood–Paley type square function estimate. This is presented in Section 3. Section 4 contains the proof of Theorem 2, Section 5 concerns extensions to higher dimensions, and finally Section 6 is devoted to the L^p–L^q boundedness properties of the maximal operators $\mathcal{M}_{\alpha,\beta}$. We begin by presenting some applications and interpretations of Theorem 2.

2. Applications and interpretations

Here we present three distinct applications (or interpretations) of Theorem 2.

2.1. *L^p–L^q multipliers.* Our first application of Theorem 2 is to the theory of L^p–L^q multipliers on the line. Such a multiplier theorem will follow from Theorem 2 via (2) once we have suitable bounds on the maximal operators $\mathcal{M}_{\alpha,\beta}$.

Theorem 4. *Let $1 < p \leq q \leq \infty$ and $\alpha, \beta \in \mathbb{R}$. If $\alpha > 0$ then $\mathcal{M}_{\alpha,\beta}$ is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ if and only if*

$$\beta \geq \frac{\alpha}{2q} + \frac{1}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big). \tag{27}$$

If $\alpha = 0$ then $\mathcal{M}_{\alpha,\beta}$ is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ if and only if

$$\beta = \frac{1}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big). \tag{28}$$

If $\alpha < 0$ then $\mathcal{M}_{\alpha,\beta}$ is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ if and only if

$$\beta \leq \frac{\alpha}{2q} + \frac{1}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big). \tag{29}$$

Remarks. When $\alpha = 0$ Theorem 4 of course reduces to the well-known L^p–L^q boundedness properties of the classical fractional Hardy–Littlewood maximal operator in one dimension; see [Muckenhoupt and Wheeden 1974]. For $0 \le \alpha < 1$ (the case of nontangential approach regions) and $p = q$, this result follows from the work of [Nagel and Stein 1984]. Certain particular cases of Theorem 4 in the region $\alpha > 1$ are established in [Bennett and Harrison 2012], following arguments in [Bennett et al. 2006]. Our proof, which extends further the arguments in [Bennett et al. 2006], follows by establishing a corresponding endpoint Hardy space result when $p = 1$; see Section 6.

Combining Theorems 2 and 4 yields the following unweighted Marcinkiewicz-type multiplier theorem.

Corollary 5. *Let $2 \le p \le q < \infty$, $\alpha, \beta \in \mathbb{R}$ and suppose $m \in \mathscr{C}(\alpha, \beta)$. If one of*

$$\begin{aligned} &\alpha > 0 \quad \text{and} \quad \beta \ge \alpha\Big(\frac{1}{2} - \frac{1}{p}\Big) + \frac{1}{p} - \frac{1}{q}, \\ &\alpha = 0 \quad \text{and} \quad \beta = \frac{1}{p} - \frac{1}{q}, \\ \text{or} \quad &\alpha < 0 \quad \text{and} \quad \beta \le \alpha\Big(\frac{1}{2} - \frac{1}{p}\Big) + \frac{1}{p} - \frac{1}{q} \end{aligned}$$

holds, then m is an $L^p(\mathbb{R})$–$L^q(\mathbb{R})$ multiplier.

Remarks. Corollary 5, which modestly generalises a number of well-known results, is optimal subject to the (inevitable) constraint $p, q \ge 2$; see [Miyachi 1980; 1981]. However, as the examples in those papers also suggest, unless $p = q$, Corollary 5 is unlikely to lead to optimal results in the full range $1 \le p, q \le \infty$. If $\alpha \ne 0$ then, by duality and interpolation, we may conclude that m is an $L^p(\mathbb{R})$ multiplier for all $1 < p < \infty$ satisfying the familiar condition $|1/2 - 1/p| \le \beta/\alpha$. This generalises the L^p (as opposed to H^p) multiplier results of [Miyachi 1980] in dimension $n = 1$. If $\alpha = 0$ then Corollary 5 reduces to the classical one-dimensional Marcinkiewicz multiplier theorem on setting $p = q$, since m is a Marcinkiewicz multiplier if and only if $m \in \mathscr{C}(0, 0)$. The special case $\alpha = 0$ also generalises the classical Hardy–Littlewood–Sobolev theorem on fractional integration since the multiplier $|\xi|^{-\beta}$ belongs to $\mathscr{C}(0, \beta)$.

2.2. ***Oscillatory convolution kernels on the line.*** The method of stationary phase permits Theorem 2 to be applied to a variety of explicit oscillatory convolution operators on the line. For example, for $a > 0$ with $a \ne 1$ and $1 - a/2 \le b < 1$, consider the convolution kernel $K_{a,b} : \mathbb{R}\backslash\{0\} \to \mathbb{C}$ given by

$$K_{a,b}(x) = \frac{e^{i|x|^a}}{(1 + |x|)^b}.$$

The corresponding convolution operator is well understood on L^p, with

$$\|K_{a,b} * f\|_p \lesssim \|f\|_p \quad \Longleftrightarrow \quad p_0 \le p \le p_0', \tag{30}$$

where $p_0 = a/(a + b - 1)$; see [Sjölin 1981; Jurkat and Sampson 1981]. As we shall see, an application of Theorem 2 quickly leads to the following.

Theorem 6. *If $a > 0$ with $a \neq 1$ and $1 - a/2 \leq b < 1$, then*

$$\int_{\mathbb{R}} |K_{a,b} * f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^6 \mathcal{M}_{\alpha,\beta} M^4 w, \tag{31}$$

where $\alpha = \dfrac{a}{a-1}$ *and* $\beta = \dfrac{a/2+b-1}{a-1}$.

This theorem is optimal in the sense that it allows us to recover (30) (and indeed more general L^p–L^q estimates) from Theorem 4 via (2). Notice that if $0 < a < 1$ then $\alpha := a/(a-1) < 0$ and so the controlling maximal operator $\mathcal{M}_{\alpha,\beta}$ corresponds to an *escape* region. Similarly, if $a > 1$ then $\alpha > 0$ and so $\mathcal{M}_{\alpha,\beta}$ corresponds to an *approach* region. Theorem 6 may of course be viewed as a generalisation (modulo factors of M) of the inequality (5).

In order to deduce Theorem 6 from Theorem 2 we simply observe that, up to a couple of well-behaved "error" terms, the multiplier $\widehat{K}_{a,b}$ belongs to $\mathscr{C}(\alpha, \beta)$. Let us begin by handling the portion of $K_{a,b}$ in a neighbourhood of the origin (where the kernel lacks smoothness). Let $\eta \in C_c^\infty(\mathbb{R})$ be an even function satisfying $\eta(x) = 1$ for $|x| \leq 1$, and write $K_{a,b} = K_{a,b,0} + K_{a,b,\infty}$, where $K_{a,b,0} = \eta K_{a,b}$. Since $K_{a,b,0}$ is rapidly decreasing, by the Cauchy–Schwarz inequality we have

$$\int_{\mathbb{R}} |K_{a,b,0} * f|^2 w \leq \|K_{a,b,0}\|_1 \int_{\mathbb{R}} |f|^2 |K_{a,b,0}| * w \lesssim \int_{\mathbb{R}} |f|^2 M^1 w,$$

where

$$M^1 w(x) := \sup_{r \geq 1} \frac{1}{2r} \int_{x-r}^{x+r} w.$$

The claimed inequality (31) for the portion of the kernel $K_{a,b,0}$ now follows from the elementary pointwise bound

$$M^1 w \lesssim A M^1 w \leq \mathcal{M}_{\alpha,\beta} M^1 w \leq \mathcal{M}_{\alpha,\beta} M w \leq M^6 \mathcal{M}_{\alpha,\beta} M^4 w,$$

where the averaging operator A is given by

$$A w(x) = \frac{1}{2} \int_{x-1}^{x+1} w.$$

It thus remains to prove (31) for the portion $K_{a,b,\infty}$. In order to force the support hypothesis (12) we introduce a function $\psi \in C^\infty(\mathbb{R})$ such that $\psi(\xi) = 0$ when $|\xi|^\alpha \leq 1$ and $\psi(\xi) = 1$ when $|\xi|^\alpha \geq 2$. Writing $m_0 = (1 - \psi)\widehat{K}_{a,b,\infty}$ and $m_1 = \psi \widehat{K}_{a,b,\infty}$, it suffices to show that

$$\int_{\mathbb{R}} |T_{m_j} f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^6 \mathcal{M}_{\alpha,\beta} M^4 w \tag{32}$$

for $j = 0, 1$. A standard stationary phase argument (see [Sjölin 1981] for explicit details) reveals that m_1 satisfies the Miyachi-type bounds (11) as $|\xi|^\alpha \to \infty$. Hence $m_1 \in \mathscr{C}(\alpha, \beta)$, yielding (32) for $j = 1$ by Theorem 2. The multiplier m_0 is less interesting, being the Fourier transform of a rapidly decreasing function (again, see [Sjölin 1981] for further details). Arguing as we did for the portion $K_{a,b,0}$ establishes (32) for $j = 0$, completing the proof.

For a more far-reaching discussion relating to the asymptotics of Fourier transforms of oscillatory kernels, see [Stein 1993], and what Stein refers to as the "duality of phases".

2.3. ***Spatial regularity of solutions of dispersive equations.*** Theorem 2 has an interesting interpretation in the context of spatial regularity of solutions to dispersive equations. For example, applying[3] Theorem 2 to the multiplier $m_{2,\beta}$ given by (10) yields

$$\int_{\mathbb{R}} \left|e^{i\partial^2} f\right|^2 w \lesssim \int_{\mathbb{R}} \left|(I-\partial^2)^{\beta/2} f\right|^2 M^6 \mathcal{M}_{2,\beta} M^4 w$$

for all $\beta \geq 0$. Using the scale-invariant inequality (26) with $\lambda = t^{-1/2}$, a similar statement may be made for the operator $e^{it\partial^2}$, namely

$$\int_{\mathbb{R}} \left|e^{it\partial^2} f\right|^2 w \lesssim \int_{\mathbb{R}} \left|(t^{-1}I-\partial^2)^{\beta/2} f\right|^2 M^6 \mathcal{M}_{2,\beta}^{t^{-1/2}} M^4 w,$$

with implicit constant independent of $t > 0$. It is perhaps more natural to rewrite this as

$$\int_{\mathbb{R}} \left|e^{it\partial^2} f\right|^2 w \lesssim \int_{\mathbb{R}} \left|(I-t\partial^2)^{\beta/2} f\right|^2 M^6 \mathfrak{M}_t M^4 w,$$

where

$$\mathfrak{M}_t w(x) := \sup_{(y,r)\in\Lambda_t(x)} r^{2\beta} \frac{1}{t^{1/2} r} \int_{y-t^{1/2}r}^{y+t^{1/2}r} w \quad \text{and} \quad \Lambda_t(x) = \left\{(y,r) : 0 < r \leq 1,\ |x-y| \leq t^{1/2}/r\right\},$$

so that the degeneracy as $t \to 0$ is more apparent. The resulting L^p multiplier theorem at $t = 1$ (see Corollary 5 in the case $q = p$) is the inequality

$$\left\|e^{i\partial^2} f\right\|_{L^p(\mathbb{R})} \lesssim \|f\|_{W^{\beta,p}}$$

for $\beta \geq 2|1/2 - 1/p|$. Here $W^{\beta,p}$ denotes the classical inhomogeneous L^p Sobolev space. This optimal Sobolev inequality, which goes back to Miyachi [1981], describes the regularity loss in $L^p(\mathbb{R})$ for a solution to the Schrödinger equation with initial data in $L^p(\mathbb{R})$. Naturally this interpretation applies equally well to the wave, Airy and more general (pseudo)differential dispersive equations. Similar conclusions, at least for the Schrödinger equation, may be reached in higher dimensions using the results of Section 5; see also [Miyachi 1981].

3. Weighted inequalities for a lattice square function

In this section we present the forward and reverse weighted Littlewood–Paley square function estimates that underpin our proof of Theorem 2. We formulate our results in $\mathbb{R}^n$ in anticipation of higher-dimensional applications in Section 5.

Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\operatorname{supp}(\widehat{\Psi}) \subseteq [-1,1]^n$ and

$$\sum_{k\in\mathbb{Z}^n} \widehat{\Psi}(\xi - k) = 1$$

for all $\xi \in \mathbb{R}^n$. Such a function may of course be constructed by defining $\widehat{\Psi} = \chi_{[-1/2,1/2]^n} * \phi$, for a function $\phi \in C_c^\infty(\mathbb{R}^n)$ of suitably small support and integral 1.

[3] Strictly speaking we are applying Theorem 2 to a portion of the multiplier supported away from the origin, and dealing with the portion near the origin by other (elementary) means. See Section 2.2 for further details.

For each $t \in (0,\infty)^n$ we define the $n \times n$ dilation matrix $\delta(t) := \operatorname{diag}(t_1, \dots, t_n)$, and the rectangular box $B(t) := \delta(t)^{-1}[-1,1]^n = [-1/t_1, 1/t_1] \times \cdots \times [-1/t_n, 1/t_n]$.

Now let $R' \in (0,\infty)^n$ and decompose $\mathbb{R}^n$ into a lattice of rectangles $\{\rho_k\}$ as follows. For each $k \in \mathbb{Z}^n$ let

$$\rho_k = \delta(R')\big(\{k\} + \big[-\tfrac{1}{2}, \tfrac{1}{2}\big]^n\big),$$

making ρ_k the axis-parallel rectangular cell centred at $\delta(R')k = (R'_1 k_1, \dots, R'_n k_n)$ with j-th side length R'_j. Defining $\Psi_k \in \mathcal{S}(\mathbb{R}^n)$ by

$$\widehat{\Psi}_k(\xi) = \widehat{\Psi}(\delta(R')^{-1}\xi - k),$$

we have

$$\sum_{k\in\mathbb{Z}^n} \widehat{\Psi}_k \equiv 1, \tag{33}$$
$$\operatorname{supp}(\widehat{\Psi}_k) \subseteq \tilde{\rho}_k,$$

for each $k \in \mathbb{Z}^n$. Here $\tilde{\rho}_k$ denotes the concentric double of ρ_k. Finally, let the operator S_k be given by $\widehat{S_k f} = \widehat{\Psi}_k \hat{f}$.

For the operators S_k we have the following essentially standard square function estimate. Very similar results may be found in several places in the literature, including [Córdoba 1982; Rubio de Francia 1985; Bennett et al. 2006].

Proposition 7.
$$\int_{\mathbb{R}^n} \sum_k |S_k f|^2 w \lesssim \int_{\mathbb{R}^n} |f|^2 M_S w \tag{34}$$

uniformly in R', where M_S denotes the strong maximal function.

A reverse weighted inequality, where the function f is controlled by the square function $\big(\sum_k |S_k f|^2\big)^{1/2}$, is rather more subtle, and is the main content of this section.

Theorem 8. *Suppose $R \in (0,\infty)^n$ is such that $R_j \geq R'_j$ for each $1 \leq j \leq n$, and let ρ be an axis-parallel rectangle in $\mathbb{R}^n$ of j-th side length R_j. If $\operatorname{supp}(\hat{f}) \subseteq \rho$ then*

$$\int_{\mathbb{R}^n} |f|^2 w \lesssim \int_{\mathbb{R}^n} \sum_k |S_k f|^2 M_S A_{R,R'} M_S w,$$

where the operator $A_{R,R'}$ is given by

$$A_{R,R'} w(x) = \sup_{y \in \{x\} + B(R')} \frac{1}{|B(R)|} \int_{\{y\}+B(R)} w.$$

Remark. As the following proof reveals, Theorem 8 continues to hold if the operators S_k are replaced by the genuine frequency-projection operators defined by $\widehat{S_k f} = \chi_{\rho_k} \hat{f}$.

Proof of Theorem 8. We begin by exploiting the Fourier support hypothesis on f to mollify the weight w. Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ be an even function satisfying $\widehat{\Phi} = 1$ on $[-1,1]^n$. Observe that if we define $\Phi_R \in \mathcal{S}(\mathbb{R}^n)$ by $\widehat{\Phi}_R(\xi) = \widehat{\Phi}(\delta(R)^{-1}\xi) = \widehat{\Phi}(\xi_1/R_1, \dots, \xi_n/R_n)$, then $f = f * (M_{\xi_\rho} \Phi_R)$. Here

$M_{\xi_\rho}\Phi_R(x) = e^{-2\pi i x\cdot\xi_\rho}\Phi_R(x)$ and ξ_ρ denotes the centre of ρ. A standard application of the Cauchy–Schwarz inequality and Fubini's theorem reveals that

$$\int_{\mathbb{R}^n} |f|^2 w = \int_{\mathbb{R}^n} |f * (M_{\xi_\rho}\Phi_R)|^2 w \le \|\Phi_R\|_1 \int_{\mathbb{R}^n} |f|^2 |\Phi_R| * w \lesssim \int_{\mathbb{R}^n} |f|^2 w_1, \tag{35}$$

where $w_1 := |\Phi_R| * w$. The final inequality here follows since the functions Φ_R are normalised in L^1.

Now, by (33) we have

$$f = \sum_k S_k f.$$

This raises issues of orthogonality for the operators S_k on $L^2(w_1)$. Although the weight w_1 is smooth, in order for us to have any (almost) orthogonality we should expect to need an improved smoothness consistent with a mollification by $|\Phi_{R'}|$ rather than $|\Phi_R|$. We thus seek an efficient way of dominating w_1 by such an improved weight.[4] This ingredient, which is based on an argument in [Bennett et al. 2006], comes in two simple steps. First define the weight function w_2 by

$$w_2(x) = \sup_{y\in\{x\}+B(R')} w_1(y).$$

Certainly w_2 dominates w_1 pointwise, although w_2 will not in general be sufficiently smooth for our purposes. Let $\Theta \in \mathcal{S}(\mathbb{R}^n)$ be a nonnegative function whose Fourier transform is nonnegative and compactly supported, and let

$$w_3 = \Theta_{R'} * w_2,$$

where $\Theta_{R'}$ is defined by $\widehat{\Theta}_{R'}(\xi) = \widehat{\Theta}(\delta(R')^{-1}\xi) = \widehat{\Theta}(\xi_1/R_1', \dots, \xi_n/R_n')$. By construction w_3 has Fourier support in $\{\xi : |\xi_j| \lesssim R_j',\ 1 \le j \le n\}$ and so by Parseval's theorem we have the desired almost orthogonality:

$$\langle S_k f, S_{k'} f\rangle_{L^2(w_3)} = 0 \quad \text{if } |k-k'| \gtrsim 1. \tag{36}$$

Despite its improved smoothness, this new weight w_3 continues to dominate w_1.

Lemma 9. $w_2 \lesssim w_3$.

Proof. By dilating Θ by an absolute constant if necessary, we may assume that $\Theta \gtrsim 1$ on $[-1,1]^n$. Consequently,

$$w_3(0) \gtrsim \frac{1}{|B(R')|}\int_{B(R')} w_2(x)\,dx.$$

Now let $B_1, B_2, \dots, B_{2^n}$ be the intersections of $B(R')$ with the 2^n coordinate hyperoctants of $\mathbb{R}^n$. It will suffice to show that there exists $\ell \in \{1, 2, \dots, 2^n\}$ such that $w_2(x) \ge w_2(0)$ for all $x \in B_\ell$. To see this we suppose, for a contradiction, that there exist $x_\ell \in B_\ell$ such that $w_2(x_\ell) < w_2(0)$ for each $1 \le \ell \le 2^n$. Thus, by the definition of w_2 we have

$$\sup_{x\in\{x_\ell\}+B(R')} w_1(x) < w_2(0) \quad \text{for } 1 \le \ell \le 2^n.$$

[4] This idea is somewhat reminiscent of the classical fact that if $(Mw^s)^{1/s} < \infty$ a.e. and $s > 1$ then $w \le (Mw^s)^{1/s} \in A_1 \subset A_2$; see the discussion following (3).

However, since

$$B(R') \subseteq \bigcup_{\ell=1}^{2^n} (\{x_\ell\} + B(R')),$$

$\sup_{x\in B(R')} w_1(x) < w_2(0)$, contradicting the definition of $w_2(0)$. □

Combining (35), Lemma 9 and the orthogonality property (36) we obtain

$$\int_{\mathbb{R}^n} |f|^2 w \lesssim \int_{\mathbb{R}^n} \sum_k |S_k f|^2 w_3. \tag{37}$$

In order to complete the proof of Theorem 8 it remains to show that $w_3(x) \lesssim M_S A_{R,R'} M_S w(x)$ uniformly in x and R, R'. Since $w_3(x) \lesssim M_S w_2(x)$ it suffices to show that $w_2(x) \lesssim A_{R,R'} M_S w(x)$. Further, by translation invariance, it is enough to deal with the case $x = 0$. To see this we define the maximal operator $M_S^{(R)}$ by

$$M_S^{(R)} w(y) = \sup_{r\geq 1} \frac{1}{|rB(R)|} \int_{\{y\}+rB(R)} w.$$

Notice that $M_S^{(R)} w \leq M_S w$. Using the rapid decay of Φ and elementary considerations we have

$$w_1(y) = |\Phi_R| * w(y) \lesssim M_S^{(R)} w(y) \lesssim \frac{1}{|B(R)|} \int_{\{y\}+B(R)} M_S^{(R)} w$$

and so

$$w_2(0) \lesssim \sup_{y\in B(R')} \frac{1}{|B(R)|} \int_{\{y\}+B(R)} M_S w = A_{R,R'} M_S w(0)$$

uniformly in R, R', as required. □

4. The proof of Theorem 2

The proof we present combines the essential ingredients of the standard proof of the Marcinkiewicz multiplier theorem (see [Stein 1970; Duoandikoetxea 2001], for example) and the square function estimates from Section 3.

By standard weighted Littlewood–Paley theory (see [Bennett and Harrison 2012] for further details) it suffices to prove that

$$\int_{\mathbb{R}} |T_m f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^5 \mathcal{M}_{\alpha,\beta} M w \tag{38}$$

holds for functions f with Fourier support in the dyadic interval $\pm[R, 2R]$, with bounds uniform in $R^\alpha \geq 1$.

Suppose that $\operatorname{supp}(\hat{f}) \subseteq \pm[R, 2R]$ for some $R^\alpha \geq 1$. We begin by applying Theorem 8 with $n = 1$, $R' = R^{-\alpha} R$ and $\rho = \pm[R, 2R]$. For each $k \in \mathbb{Z}$ let ρ_k, $\tilde{\rho}_k$, Ψ_k and S_k be as in Section 3. By Theorem 8 we have

$$\int_{\mathbb{R}} |T_m f|^2 w \lesssim \int_{\mathbb{R}} \sum_k |S_k T_m f|^2 M A_{R,R'} M w \tag{39}$$

uniformly in $R^\alpha \geq 1$. Of course the case $R = 1$ (as with the case $\alpha = 0$) is somewhat degenerate here, although we note that the conclusion (39) does retain some content.

Next we invoke the standard representation formula

$$S_k T_m f(x) = m(a_k) S_k f(x) + \int_{\tilde{\rho}_k} U_\xi S_k f(x) m'(\xi)\, d\xi, \tag{40}$$

where $a_k = \inf \tilde{\rho}_k$ and U_ξ is defined by

$$\widehat{U_\xi f} = \chi_{[\xi,\infty)} \hat{f}. \tag{41}$$

In order to see (40), which is a minor variant of (7), we use the Fourier inversion formula to write

$$\begin{aligned} S_k T_m f(x) &= \int_{\tilde{\rho}_k} e^{ix\xi} \widehat{\Psi}_k(\xi) m(\xi) \hat{f}(\xi)\, d\xi \\ &= -\int_{\tilde{\rho}_k} \frac{\partial}{\partial \xi} \left(\int_\xi^\infty \widehat{\Psi}_k(t) \hat{f}(t) e^{ixt} dt \right) m(\xi)\, d\xi \\ &= m(a_k) S_k f(x) + \int_{\tilde{\rho}_k} \left(\int_{\mathbb{R}} \chi_{[\xi,\infty)}(t) \widehat{\Psi}_k(t) \hat{f}(t) e^{ixt}\, dt \right) m'(\xi)\, d\xi \\ &= m(a_k) S_k f(x) + \int_{\tilde{\rho}_k} U_\xi S_k f(x) m'(\xi)\, d\xi. \end{aligned}$$

Applying Minkowski's inequality to (40), we obtain

$$\begin{aligned} &\left(\int_{\mathbb{R}} |S_k T_m f|^2 M A_{R,R'} M w \right)^{1/2} \\ &\qquad \le |m(a_k)| \left(\int_{\mathbb{R}} |S_k f|^2 M A_{R,R'} M w \right)^{1/2} + \int_{\tilde{\rho}_k} \left(\int_{\mathbb{R}} |U_\xi S_k f|^2 M A_{R,R'} M w \right)^{1/2} |m'(\xi)|\, d\xi. \end{aligned}$$

Since $U_\xi = \frac{1}{2}(I + i M_{-\xi} H M_\xi)$, where $M_\xi f(x) := e^{-2\pi i x\xi} f(x)$ and H is the Hilbert transform, an application of (4) yields

$$\int_{\mathbb{R}} |U_\xi S_k f|^2 M A_{R,R'} M w \lesssim \int_{\mathbb{R}} |S_k f|^2 M^4 A_{R,R'} M w$$

uniformly in ξ, k and R. Using this along with the hypotheses (13) and (14) yields

$$\int_{\mathbb{R}} |S_k T_m f|^2 M A_{R,R'} M w \lesssim R^{-2\beta} \int_{\mathbb{R}} |S_k f|^2 M^4 A_{R,R'} M w$$

uniformly in k and R. Here we have used the fact that $|a_k| \sim R$. Thus by (39) and Proposition 7 we have

$$\int_{\mathbb{R}} |T_m f|^2 w \lesssim R^{-2\beta} \int_{\mathbb{R}} |f|^2 M^5 A_{R,R'} M w$$

uniformly in $R^\alpha \ge 1$. Inequality (38) now follows from the elementary observation that

$$R^{-2\beta} A_{R,R^{-\alpha}R} w(x) \lesssim \mathcal{M}_{\alpha,\beta} w(x)$$

uniformly in x and $R^\alpha \ge 1$.

5. Extensions to higher dimensions

Theorem 2 has a natural generalisation to higher dimensions. It should be pointed out that this generalisation, being of Marcinkiewicz type in formulation, is not motivated by multipliers of the form (11), but rather by tensor products of such one-dimensional multipliers. For the sake of simplicity we confine our attention to two dimensions. Just as with the classical Marcinkiewicz multiplier theorem, this is already typical of the general situation.

For $\alpha, \beta \in \mathbb{R}^2$ let $\mathscr{C}(\alpha, \beta)$ denote the class of functions $m : \mathbb{R}^2 \to \mathbb{C}$ for which

$$\operatorname{supp}(m) \subseteq \{\xi \in \mathbb{R}^2 : |\xi_1|^{\alpha_1} \geq 1,\ |\xi_2|^{\alpha_2} \geq 1\}, \tag{42}$$

$$\sup_{\xi_2} \sup_{\xi_1} |\xi_2|^{\beta_2} |\xi_1|^{\beta_1} |m(\xi_1, \xi_2)| < \infty, \tag{43}$$

$$\sup_{\xi_2} |\xi_2|^{\beta_2} \left\{ \sup_{R_1^{\alpha_1} \geq 1} \sup_{\substack{I_1 \subseteq [R_1, 2R_1] \\ \ell(I_1) = R_1^{-\alpha_1} R_1}} R_1^{\beta_1} \int_{\pm I_1} \left| \frac{\partial m}{\partial \xi_1} \right| d\xi_1 \right\} < \infty, \tag{44}$$

$$\sup_{\xi_1} |\xi_1|^{\beta_1} \left\{ \sup_{R_2^{\alpha_2} \geq 1} \sup_{\substack{I_2 \subseteq [R_2, 2R_2] \\ \ell(I_2) = R_2^{-\alpha_2} R_2}} R_2^{\beta_2} \int_{\pm I_2} \left| \frac{\partial m}{\partial \xi_2} \right| d\xi_2 \right\} < \infty, \tag{45}$$

$$\sup_{R_2^{\alpha_2} \geq 1} \sup_{\substack{I_2 \subseteq [R_2, 2R_2] \\ \ell(I_2) = R_2^{-\alpha_2} R_2}} \sup_{R_1^{\alpha_1} \geq 1} \sup_{\substack{I_1 \subseteq [R_1, 2R_1] \\ \ell(I_1) = R_1^{-\alpha_1} R_1}} R_2^{\beta_2} R_1^{\beta_1} \int_{\pm I_2} \int_{\pm I_1} \left| \frac{\partial^2 m}{\partial \xi_1 \partial \xi_2} \right| d\xi_1 d\xi_2 < \infty. \tag{46}$$

Although these conditions might appear rather complicated, it is straightforward to verify that the tensor product $\mathscr{C}(\alpha_1, \beta_1) \otimes \mathscr{C}(\alpha_2, \beta_2)$ is contained in $\mathscr{C}(\alpha, \beta)$, and that $\mathscr{C}(0, 0)$ is precisely the classical Marcinkiewicz multipliers on $\mathbb{R}^2$.

Theorem 10. *If $m \in \mathscr{C}(\alpha, \beta)$ then*

$$\int_{\mathbb{R}^2} |T_m f|^2 w \lesssim \int_{\mathbb{R}^2} |f|^2 M_S^9 \mathcal{M}_{\alpha,\beta} M_S^7 w,$$

where

$$\mathcal{M}_{\alpha,\beta} w(x) = \sup_{(r_1, y_1) \in \Gamma_{\alpha_1}(x_1)} \sup_{(r_2, y_2) \in \Gamma_{\alpha_2}(x_2)} \frac{r_1^{2\beta_1}}{r_1} \frac{r_2^{2\beta_2}}{r_2} \int_{|y_1 - z_1| \leq r_1} \int_{|y_2 - z_2| \leq r_2} w(z)\, dz$$

and M_S denotes the strong maximal function.

Proof of Theorem 10. The proof we present is very similar to the one-dimensional case. By standard weighted Littlewood–Paley theory (again, see [Bennett and Harrison 2012] for details) it suffices to prove that

$$\int_{\mathbb{R}^2} |T_m f|^2 w \lesssim \int_{\mathbb{R}^2} |f|^2 M_S^8 \mathcal{M}_{\alpha,\beta} M_S w \tag{47}$$

holds for functions f with Fourier support in $(\pm[R_1, 2R_1]) \times (\pm[R_2, 2R_2])$, with bounds uniform in $R_1^{\alpha_1}, R_2^{\alpha_2} \geq 1$.

Assuming such a restriction we can apply Theorem 8 with $n = 2$, $R' = (R_1^{-\alpha_1} R_1, R_2^{-\alpha_2} R_2)$ and

$\rho = (\pm[R_1, 2R_1]) \times (\pm[R_2, 2R_2])$. For each $k \in \mathbb{Z}^2$ let ρ_k, $\tilde{\rho}_k$, Ψ_k and S_k be as in Section 3. By Theorem 8 we have

$$\int_{\mathbb{R}^2} |T_m f|^2 w \lesssim \int_{\mathbb{R}^2} \sum_{k\in\mathbb{Z}^2} |S_k T_m f|^2 M_S A_{R,R'} M_S w \tag{48}$$

uniformly in R.

In what follows $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ denote the coordinate projections $\pi_1 x = x_1$ and $\pi_2 x = x_2$, and for each k we define $a_k \in \mathbb{R}^2$ by $a_k = (\inf \pi_1 \tilde{\rho}_k, \inf \pi_2 \tilde{\rho}_k)$, making a_k the bottom-left vertex of the axis-parallel rectangle $\tilde{\rho}_k$.

Now, taking our cue again from the standard proof of the classical Marcinkiewicz multiplier theorem, we write

$$\begin{aligned} S_k T_m f(x) = m(a_k) S_k f(x) + \int_{\pi_1\tilde{\rho}_k} U^{(1)}_{\xi_1} S_k f(x) \frac{\partial m}{\partial \xi_1}(\xi_1, \pi_2 a_k)\, d\xi_1 + \int_{\pi_2\tilde{\rho}_k} U^{(2)}_{\xi_2} S_k f(x) \frac{\partial m}{\partial \xi_2}(\pi_1 a_k, \xi_2)\, d\xi_2 \\ + \int_{\tilde{\rho}_k} U^{(2)}_{\xi_2} U^{(1)}_{\xi_1} S_k f(x) \frac{\partial^2 m}{\partial \xi_1 \partial \xi_2}(\xi_1, \xi_2)\, d\xi_1 d\xi_2, \end{aligned} \tag{49}$$

where $U^{(j)}_{\xi_j}$ denotes the operator U_{ξ_j}, defined in (41), acting in the j-th variable. Applying Minkowski's inequality we obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^2} |S_k T_m f|^2 M_S A_{R,R'} M_S w\right)^{1/2} &\le |m(a_k)| \left(\int_{\mathbb{R}^2} |S_k f|^2 M_S A_{R,R'} M_S w\right)^{1/2} \\ &+ \int_{\pi_1\tilde{\rho}_k} \left(\int_{\mathbb{R}^2} |U^{(1)}_{\xi_1} S_k f|^2 M_S A_{R,R'} M_S w\right)^{1/2} \left|\frac{\partial m}{\partial \xi_1}(\xi_1, \pi_2 a_k)\right| d\xi_1 \\ &+ \int_{\pi_2\tilde{\rho}_k} \left(\int_{\mathbb{R}^2} |U^{(2)}_{\xi_2} S_k f|^2 M_S A_{R,R'} M_S w\right)^{1/2} \left|\frac{\partial m}{\partial \xi_2}(\pi_1 a_k, \xi_2)\right| d\xi_2 \\ &+ \int_{\tilde{\rho}_k} \left(\int_{\mathbb{R}^2} |U^{(2)}_{\xi_2} U^{(1)}_{\xi_1} S_k f|^2 M_S A_{R,R'} M_S w\right)^{1/2} \left|\frac{\partial^2 m}{\partial \xi_1 \partial \xi_2}(\xi_1, \xi_2)\right| d\xi. \end{aligned}$$

We denote the summands on the right side by I, II, III, IV.

For I we use the facts that $|\pi_1 a_k| \sim R_1$ and $|\pi_2 a_k| \sim R_2$, along with (43) to obtain

$$I \lesssim R_2^{-\beta_2} R_1^{-\beta_1} \left(\int_{\mathbb{R}^2} |S_k f|^2 M_S A_{R,R'} M_S w\right)^{1/2}$$

uniformly in k. For II, following the proof of Theorem 2, we apply (4) in the first variable to obtain

$$II \lesssim \int_{\pi_1\tilde{\rho}_k} \left(\int_{\mathbb{R}^2} |S_k f|^2 M_S^4 A_{R,R'} M_S w\right)^{1/2} \left|\frac{\partial m}{\partial \xi_1}(\xi_1, \pi_2 a_k)\right| d\xi_1,$$

which by (44) yields

$$II \lesssim R_2^{-\beta_2} R_1^{-\beta_1} \left(\int_{\mathbb{R}^2} |S_k f|^2 M_S^4 A_{R,R'} M_S w\right)^{1/2}$$

uniformly in k. By (45) and symmetry, it follows that III satisfies the same bound. The final term IV is potentially the most interesting as it involves using a weighted bound on the double Hilbert transform. By

a twofold application of (4), followed by (46), we obtain

$$IV \lesssim R_2^{-\beta_2} R_1^{-\beta_1} \bigg(\int_{\mathbb{R}^2} |S_k f|^2 M_S^7 A_{R,R'} M_S w \bigg)^{1/2}.$$

Thus, by (48) and Proposition 7 we have

$$\int_{\mathbb{R}^2} |T_m f|^2 w \lesssim R_2^{-2\beta_2} R_1^{-2\beta_1} \int_{\mathbb{R}^2} |S_k f|^2 M_S^8 A_{R,R'} M_S w$$

uniformly in $R_1^{\alpha_1}, R_2^{\alpha_2} \geq 1$. Inequality (47) now follows on observing that

$$R_2^{-2\beta_2} R_1^{-2\beta_1} A_{R,R'} w(x) \lesssim \mathcal{M}_{\alpha,\beta} w(x)$$

uniformly in x and $R_1^{\alpha_1}, R_2^{\alpha_2} \geq 1$. □

Remarks. The above arguments raise certain basic questions about weighted inequalities for various multiparameter operators in harmonic analysis. For instance, for which powers $k \in \mathbb{N}$ do we have

$$\int_{\mathbb{R}^n} |Tf|^2 w \lesssim \int_{\mathbb{R}^n} |f|^2 M_S^k w$$

for classical product Calderón–Zygmund operators T on $\mathbb{R}^n$ with $n \geq 2$? As we have seen, crudely applying the one-dimensional result of Wilson [1989] separately in each variable allows us to take $k = 3n$. Reducing this power would of course lead to a reduction in the number of factors of M_S in the statement of Theorem 10.

As we have already discussed, since Theorem 10 involves Marcinkiewicz-type hypotheses it really belongs to the "multiparameter" theory of multipliers. It is conceivable that a variant may be obtained involving a Hörmander-type hypothesis on sublacunary annuli in $\mathbb{R}^n$; that is, involving hypotheses on quantities of the form

$$\int_{R_j \leq |\xi| < R_{j+1}} \Big| \Big(\frac{\partial}{\partial \xi} \Big)^\gamma m(\xi) \Big|^2 \, d\xi$$

for certain sublacunary sequences (R_j) and multi-indices γ. A very general result of this type (which might permit the radii (R_j) to accumulate away from zero) is likely to be difficult as it would naturally apply to the Bochner–Riesz multipliers. There are of course many other conditions that one might impose, from the above all the way down to the higher-dimensional analogue of the Miyachi condition (11) in [Miyachi 1980; 1981]; see also [Carbery 1985].

6. Proof of Theorem 4

In this section we give a proof of Theorem 4. Our argument is a generalisation of those in [Bennett et al. 2006; Bennett and Harrison 2012]; see also [Nagel and Stein 1984]. As the case $\alpha = 0$ reduces to the L^p–L^q boundedness of the classical fractional Hardy–Littlewood maximal function, we may assume that $\alpha \neq 0$.

The claimed necessity of the conditions (27), (28) and (29) follows from testing the putative L^p–L^q

bound for $\mathcal{M}_{\alpha,\beta}$ on the characteristic function $f_\nu = \chi_{[-\nu,\nu]}$. The necessary conditions follow by taking limits as both $\nu \to 0$ and $\nu \to \infty$. We leave these elementary calculations to the reader.

It will suffice to establish the L^p–L^q boundedness of $\mathcal{M}_{\alpha,\beta}$ for exponents $1 < p \le q \le \infty$ on the sharp line

$$\beta = \frac{\alpha}{2q} + \frac{1}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big). \tag{50}$$

As our proof of Theorem 4 rests on a Hardy space estimate, it is necessary to regularise the averaging in the definition of $\mathcal{M}_{\alpha,\beta}$. To this end let P be a nonnegative, compactly supported bump function which is positive on $[-1, 1]$, let $P_r(x) = r^{-1}P(x/r)$, and define the maximal operator $\tilde{\mathcal{M}}_{\alpha,\beta}$ by

$$\tilde{\mathcal{M}}_{\alpha,\beta} w(x) = \sup_{(y,r)\in\Gamma_\alpha(x)} r^{2\beta} |P_r * w(y)|.$$

Since $\mathcal{M}_{\alpha,\beta} w \lesssim \tilde{\mathcal{M}}_{\alpha,\beta} w$ pointwise uniformly, it suffices to prove that $\tilde{\mathcal{M}}_{\alpha,\beta}$ is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ when (50) holds. Since $\tilde{\mathcal{M}}_{\alpha,0}$ is bounded on $L^\infty(\mathbb{R})$, and $\tilde{\mathcal{M}}_{\alpha,1/2}$ is bounded from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$, by analytic interpolation [Stein 1970] it suffices to prove that $\tilde{\mathcal{M}}_{\alpha,\alpha/2}$ is bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$. We establish this by showing that

$$\|\tilde{\mathcal{M}}_{\alpha,\alpha/2} a\|_1 \lesssim 1 \tag{51}$$

uniformly in H^1-atoms a. By translation invariance we may suppose that the support interval I of a is centred at the origin. Our estimates will be based on the standard and elementary pointwise bound

$$|P_r * a(x)| \lesssim \begin{cases} 1/|I| & \text{if } r \le |I|,\ |x| \le \frac{5}{2}|I|, \\ |I|/r^2 & \text{if } r \ge |I|,\ |x| \le \frac{5}{2} r, \\ 0 & \text{otherwise,} \end{cases}$$

which follows from the smoothness of P and the mean value zero property of a. As the nature of Γ_α is fundamentally different in the cases $\alpha < 0$, $0 < \alpha \le 1$ and $\alpha > 1$, we divide the analysis into three cases. For $\alpha < 0$ and $\alpha > 0$ the interesting situation is, respectively, when $|I| \gtrsim 1$ and $|I| \lesssim 1$.

Case 1: $\alpha < 0$. Elementary considerations reveal that if $|I| \lesssim 1$ then

$$\tilde{\mathcal{M}}_{\alpha,\alpha/2} a(x) \lesssim \begin{cases} |I| & \text{if } |x| \lesssim 1, \\ |I|/|x|^{-(2-\alpha)/(1-\alpha)} & \text{otherwise,} \end{cases}$$

and if $|I| \gtrsim 1$ then

$$\tilde{\mathcal{M}}_{\alpha,\alpha/2} a(x) \lesssim \begin{cases} |I|^{-1}|x|^{\alpha/(1-\alpha)} & \text{if } |x| \lesssim |I|^{1-\alpha}, \\ |I|\,|x|^{-(2-\alpha)/(1-\alpha)} & \text{otherwise.} \end{cases}$$

In both cases (51) follows by direct calculation.

Case 2: $0 < \alpha \le 1$. For technical reasons it is convenient to deal first with the particularly simple case $\alpha = 1$. If $|I| \gtrsim 1$ then arguing similarly we obtain

$$\tilde{\mathcal{M}}_{1,1/2} a(x) \lesssim \begin{cases} |I|^{-1} & \text{if } |x| \lesssim |I|, \\ 0 & \text{otherwise,} \end{cases}$$

and if $|I| \lesssim 1$ then

$$\tilde{\mathcal{M}}_{1,1/2}a(x) \lesssim \begin{cases} 1 & \text{if } |x| \lesssim 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly in both cases (51) follows immediately.

Suppose now that $0 < \alpha < 1$. If $|I| \gtrsim 1$ then

$$\tilde{\mathcal{M}}_{\alpha,\alpha/2}a(x) \lesssim \begin{cases} |I|^{-1} & \text{if } |x| \lesssim |I|, \\ 0 & \text{otherwise,} \end{cases}$$

and if $|I| \lesssim 1$ then

$$\tilde{\mathcal{M}}_{\alpha,\alpha/2}a(x) \lesssim \begin{cases} |I|^{-(1-\alpha)} & \text{if } |x| \lesssim |I|^{1-\alpha}, \\ |I||x|^{-(2-\alpha)/(1-\alpha)} & \text{if } |I|^{1-\alpha} \lesssim |x| \lesssim 1, \\ 0 & \text{otherwise.} \end{cases}$$

Again, in both cases (51) follows directly.

Case 3: $\alpha > 1$. If $|I| \gtrsim 1$ then

$$\tilde{\mathcal{M}}_{\alpha,\alpha/2}a(x) \lesssim \begin{cases} |I|^{-1} & \text{if } |x| \lesssim |I|, \\ |I|^{-1}|x|^{\alpha/(1-\alpha)} & \text{otherwise,} \end{cases}$$

and if $|I| \lesssim 1$ then

$$\tilde{\mathcal{M}}_{\alpha,\alpha/2}a(x) \lesssim \begin{cases} |I|^{-(1-\alpha)} & \text{if } |x| \lesssim |I|^{1-\alpha}, \\ |I|^{-1}|x|^{\alpha/(1-\alpha)} & \text{if } |I|^{1-\alpha} \lesssim |x|. \end{cases}$$

Once again (51) follows.

Acknowledgments

We would like to thank Tony Carbery, Javi Duoandikoetxea, Sam Harrison, Luis Vega and Jim Wright for helpful conversations on aspects of this paper. We also thank David Beltran and Brian Taylor for corrections.

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Received 7 Jun 2013. Accepted 12 Jul 2014.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from MSP.

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Analysis & PDE

Volume 7 No. 6 2014