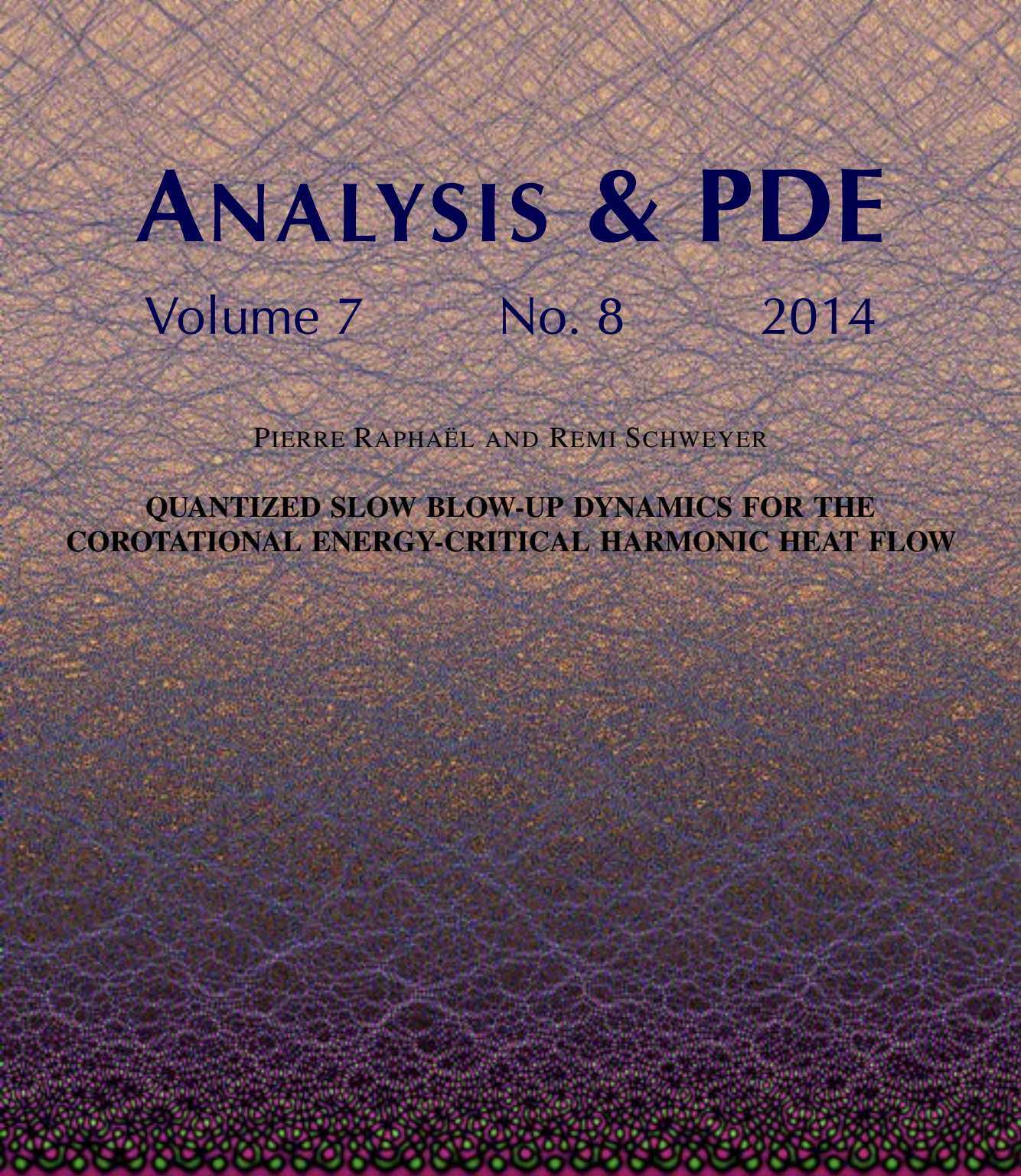
ANALYSIS & PDE

Volume 7 No. 8 2014

PIERRE RAPHAËL AND REMI SCHWEYER

QUANTIZED SLOW BLOW-UP DYNAMICS FOR THE COROTATIONAL ENERGY-CRITICAL HARMONIC HEAT FLOW



msp

ANALYSIS AND PDE
Vol. 7, No. 8, 2014
dx.doi.org/10.2140/apde.2014.7.1713

QUANTIZED SLOW BLOW-UP DYNAMICS FOR THE COROTATIONAL ENERGY-CRITICAL HARMONIC HEAT FLOW

PIERRE RAPHAËL AND REMI SCHWEYER

We consider the energy-critical harmonic heat flow from $\mathbb{R}^2$ into a smooth compact revolution surface of $\mathbb{R}^3$. For initial data with corotational symmetry, the evolution reduces to the semilinear radially symmetric parabolic problem

$$\partial_t u - \partial_r^2 u - \frac{\partial_r u}{r} + \frac{f(u)}{r^2} = 0$$

for a suitable class of functions f. Given an integer $L \in \mathbb{N}^*$, we exhibit a set of initial data arbitrarily close to the least energy harmonic map Q in the energy-critical topology such that the corresponding solution blows up in finite time by concentrating its energy

$$\nabla u(t,r) - \nabla Q\left(\frac{r}{\lambda(t)}\right) \to u^* \quad \text{in } L^2$$

at a speed given by the *quantized* rates

$$\lambda(t) = c(u_0)(1+o(1))\frac{(T-t)^L}{|\log(T-t)|^{2L/(2L-1)}},$$

in accordance with the formal predictions of van den Berg et al. (2003). The case $L = 1$ corresponds to the stable regime exhibited in our previous work (*CPAM*, 2013), and the data for $L \geq 2$ leave on a manifold of codimension $L-1$ in some weak sense. Our analysis is a continuation of work by Merle, Rodnianski, and the authors (in various combinations) and it further exhibits the mechanism for the existence of the excited slow blow-up rates and the associated instability of these threshold dynamics.

1. Introduction

The parabolic heat flow. The harmonic heat flow between two embedded Riemannian manifolds (N, g_N), (M, g_M) is the gradient flow associated to the Dirichlet energy of maps from $N \to M$:

$$\begin{cases} \partial_t v = \mathbb{P}_{T_v M}(\Delta_{g_N} v), \\ v_{|t=0} = v_0, \end{cases} \qquad (t,x) \in \mathbb{R} \times N, \ v(t,x) \in M, \tag{1-1}$$

where $\mathbb{P}_{T_v M}$ is the projection onto the tangent space to M at v. The special case $N = \mathbb{R}^2$, $M = \mathbb{S}^2$ corresponds to the harmonic heat flow to the 2-sphere

$$\partial_t v = \Delta v + |\nabla v|^2 v, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \ v(t,x) \in \mathbb{S}^2, \tag{1-2}$$

and is related to the Landau–Lifschitz equation of ferromagnetism; we refer to [van den Berg et al. 2003;

MSC2010: 35K58.
Keywords: blow-up heat flow.

Angenent et al. 2009; Guan et al. 2009; Gustafson et al. 2010] for a complete introduction to this class of problems. We shall from now on restrict our discussion to the case

$$N = \mathbb{R}^2.$$

Smooth initial data yield unique local-in-time smooth solutions which dissipate the Dirichlet energy

$$\frac{d}{dt}\left\{\int_{\mathbb{R}^2} |\nabla v|^2\right\} = -2\int_{\mathbb{R}^2} |\partial_t v|^2.$$

An essential feature of the problem is that the scaling symmetry

$$u \mapsto u_\lambda(t,x) = u(\lambda^2 t, \lambda x), \quad \lambda > 0,$$

leaves the Dirichlet energy unchanged, and hence the problem is *energy-critical*.

Corotational flows. We restrict our attention in this paper to flows with so-called corotational symmetry. More precisely, let us consider a smooth closed curve in the plane parametrized by arclength

$$u \in [-\pi, \pi] \mapsto \begin{vmatrix} g(u) \\ z(u), \end{vmatrix} \quad (g')^2 + (z')^2 = 1,$$

where

$$(H) \quad \begin{cases} g \in \mathscr{C}^\infty(\mathbb{R}) \text{ is odd and } 2\pi \text{ periodic}, \\ g(0) = g(\pi) = 0,\ g(u) > 0 \quad \text{for } 0 < u < \pi, \\ g'(0) = 1,\ g'(\pi) = -1. \end{cases} \tag{1-3}$$

Then the revolution surface M with parametrization

$$(\theta, u) \in [0, 2\pi] \times [0, \pi] \mapsto \begin{vmatrix} g(u)\cos\theta \\ g(u)\sin\theta \\ z(u) \end{vmatrix}$$

is a smooth [1] compact revolution surface of $\mathbb{R}^3$ with metric $(du)^2 + g^2(u)(d\theta)^2$. Given a homotopy degree $k \in \mathbb{Z}^*$, the k-corotational reduction to (1-1) corresponds to solutions of the form

$$v(t,r) = \begin{vmatrix} g(u(t,r))\cos(k\theta) \\ g(u(t,r))\sin(k\theta) \\ z(u(t,r)), \end{vmatrix} \tag{1-4}$$

which leads to the semilinear parabolic equation

$$\begin{cases} \partial_t u - \partial_r^2 u - \dfrac{\partial_r u}{r} + k^2 \dfrac{f(u)}{r^2} = 0, \\ u_{t=0} = u_0, \end{cases} \qquad f = gg'. \tag{1-5}$$

The k-corotational Dirichlet energy becomes

$$E(u) = \int_0^{+\infty} \left[|\partial_r u|^2 + k^2 \frac{(g(u))^2}{r^2} \right] r\, dr \tag{1-6}$$

[1] See [Gallot et al. 2004], for example.

and is minimized among maps with boundary conditions

$$u(0) = 0, \qquad \lim_{r \to +\infty} u(r) = \pi \tag{1-7}$$

onto the least energy harmonic map Q_k, which is the unique, up-to-scaling solution to

$$r\,\partial_r Q_k = k g(Q_k) \tag{1-8}$$

satisfying (1-7); see for example [Côte 2005]. In the case of $\mathbb{S}^2$ target $g(u) = \sin u$, the harmonic map is explicitly given by

$$Q_k(r) = 2\tan^{-1}(r^k). \tag{1-9}$$

The blow-up problem. The question of the existence of blow-up solutions and the description of the associated concentration of energy scenario has attracted considerable attention for the past thirty years. In the pioneering works of Struwe [1985], Ding and Tian [1995], and Qing and Tian [1997] (see [Topping 2004] for a complete history of the problem), it was shown that if occurring, the concentration of energy implies the bubbling off of a nontrivial harmonic map at a finite number of blow-up points

$$v(t_i, a_i + \lambda(t_i)x) \to Q_i, \qquad \lambda(t_i) \to 0 \tag{1-10}$$

locally in space. In particular, this shows the existence of a global in time flow on negatively curved targets where no nontrivial harmonic map exists.

For corotational data and homotopy number $k \geq 2$, Guan, Gustaffson, Nakanishi, and Tsai [Guan et al. 2009; Gustafson et al. 2010] proved that the flow is globally defined near the ground state harmonic map. In fact, Q_k is asymptotically stable for $k \geq 3$, and in particular no blow-up will occur. Eternally oscillating solutions and infinite time grow up solutions are exhibited for $k = 2$.

In contrast, for $k = 1$, the existence of finite time blow-up solutions has been proved in various geometrical settings strongly using the maximum principle; see in particular the work of Chang, Ding, and Ye [Chang et al. 1992], Coron and Ghidaglia [1989], Qing and Tian [1997], and Topping [2004]. Despite some serious efforts and the use of the maximum principle (see in particular [Angenent et al. 2009]), very little was known until recently about the description of the blow-up bubble and the derivation of the blow-up speed, in particular due to the critical nature of the problem.

For the rest of the paper, we focus on the degree

$$k = 1$$

case, which generates the least energy, nontrivial harmonic map $Q \equiv Q_1$. For $\mathbb{D}^2$ initial manifold and $\mathbb{S}^2$ target, van den Berg, Hulshof, and King [van den Berg et al. 2003], in continuation of [Herrero and Velázquez 1994], implemented a formal analysis based on the matched asymptotics techniques and predicted the existence of blow-up solutions of the form

$$u(t, r) \sim Q\left(\frac{r}{\lambda(t)}\right) \tag{1-11}$$

with blow-up speed governed by the quantized rates

$$\lambda(t) \sim \frac{(T-t)^L}{|\log(T-t)|^{2L/(2L-1)}}, \quad L \in \mathbb{N}^*.$$

We will further discuss the presence of quantized rates which is reminiscent of the classification of type II blow-up for the supercritical nonlinear heat equation [Mizoguchi 2007].

We completely revisited the blow-up analysis in [Raphaël and Schweyer 2013] by adapting the strategy developed in [Raphaël and Rodnianski 2012; Merle et al. 2011] for the study of wave and Schrödinger maps, with two main new approaches:

- We completely avoid the formal matched asymptotics approach and replace it by an elementary derivation of an explicit and universal system of ODE's which drives the blow-up speed. A similar simplification further occurred in related critical settings; see in particular [Raphaël and Schweyer 2014].
- We designed a robust universal *energy method* to control the solution in the blow-up regime, which applies both to parabolic and dispersive problems. In particular, we aim to make no use of the maximum principle.

These techniques led to [Raphaël and Schweyer 2013] the construction of an *open set* of corotational initial data arbitrarily close to the ground state harmonic map in the energy-critical topology such that the corresponding solution to (1-5) bubbles off a harmonic map according to (1-11) at the speed

$$\lambda(t) \sim \frac{T-t}{|\log(T-t)|^2}, \quad \text{that is, } L = 1.$$

This is the *stable*[2] blow-up regime.

Statement of the result. Our main claim in this paper is that the analysis in [Raphaël and Schweyer 2013] can be further extended to exhibit the unstable modes which are responsible for *a discrete sequence of quantized slow blow-up rates*.

Theorem 1.1 (excited slow blow-up dynamics for the 1-corotational heat flow). *Let $k = 1$ and g satisfy (1-3). Let Q be the least energy harmonic map. Let $L \in \mathbb{N}^*$. Then there exists a smooth corotational initial data $u_0(r)$ such that the corresponding solution to (1-5) blows up in finite time $T = T(u_0) > 0$ by bubbling off a harmonic map*

$$\nabla u(t,r) - \nabla Q\left(\frac{r}{\lambda(t)}\right) \to \nabla u^* \quad \text{in } L^2 \quad \text{as } t \to T \tag{1-12}$$

at the excited rate

$$\lambda(t) = c(u_0)(1 + o_{t\to T}(1))\frac{(T-t)^L}{|\log(T-t)|^{2L/(2L-1)}}, \quad c(u_0) > 0. \tag{1-13}$$

Moreover, u_0 can be taken arbitrarily close to Q in the energy-critical topology.

[2] In the presence of corotational symmetry, blow-up dynamics are expected to be unstable by rotation under general perturbations; see [Merle et al. 2011].

Comments on the result. 1. *Regularity of the asymptotic profile*. Arguing as in [Raphaël and Schweyer 2013] and using the estimates of Proposition 3.1, one can directly relate the rate of blow-up (1-13) to the regularity of the remaining excess of energy, in the sense that u^* exhibits an H^{L+1} regularity is some suitable Sobolev sense; see Remark 4.1. See also [Merle and Raphaël 2005b] for a related phenomenon in the dispersive setting.

2. *Stable and excited blow-up rates*. The case $L = 1$ is treated in [Raphaël and Schweyer 2013] and corresponds to stable blow-up. For $L \geq 2$, the set of initial data leading to (1-13) is of codimension $(L-1)$ in the following sense: there exist fixed directions $(\psi_i)_{2\leq i\leq L}$ such that, for any suitable perturbation ε_0 of Q, there exist $(a_i(\varepsilon_0))_{2\leq i\leq L} \in \mathbb{R}^{L-1}$ such that the solution to (1-5) with data

$$Q + \varepsilon_0 + \sum_{i=2}^{L} a_i(\varepsilon_0)\psi_i$$

blows up in finite time with the blow-up speed (1-13). Building a smooth manifold would require proving local uniqueness and smoothness of the flow $\varepsilon_0 \mapsto a_i(\varepsilon_0))_{2\leq i\leq L}$, which is a separate problem; see, for example, [Krieger and Schlag 2009] for an introduction to this kind of issue. The control of the unstable modes relies on a classical soft and powerful Brouwer type topological argument in continuation of [Côte et al. 2011; Côte and Zaag 2013; Hillairet and Raphaël 2012].

3. *On quantized blow-up rates*. There is an important formal and rigorous literature on the existence of quantized blow-up rates for parabolic problems. In the pioneering works [Herrero and Velázquez 1994; Filippas et al. 2000], the authors predicted the existence of a sequence of quantized blow-up rates for the supercritical power nonlinearity heat equation

$$\partial_t u = \Delta u + u^p, \quad x \in \mathbb{R}^d,\ p > p(d),\ d < 7,$$

and this sequence is in one to one correspondence with the spectrum of the linearized operator close to the explicit singular self similar solution. After this formal work, and using the a priori bounds on radial type II blow-up solutions of Matano and Merle [2009; 2004], Mizogushi completely classified the radial data type II blow-up according to these quantized rates. Note that Mizogushi finishes the classification using the Matano–Merle a priori estimates on threshold dynamics, which heavily rely on the maximum principle, but the argument is not constructive. One of the main points of our work is to revisit the formal derivation of the sequence of blow-up rates and to relate it not to a spectral problem, but to the structure of the resonances of the linearized operator H close to Q *and of its iterates*, that is, the growing solutions to

$$H^k T_k = 0, \quad k \in \mathbb{N}^*.$$

In particular, we show how the *dynamics of tails* as initiated in [Raphaël and Rodnianski 2012; Merle et al. 2011] lead to a universal dynamical system driving the blow-up speed, which admits unstable solutions (1-13) corresponding to a codimension $(L-1)$ set of initial data. Another by-product of this analysis is the first explicit construction of type II blow-up for the energy-critical nonlinear heat equation [Schweyer 2012].

4. *Classification of the flow near Q.* The question of the classification of the flow near the harmonic map, and more generally near the ground state solitary wave in nonlinear evolution problems, has attracted considerable attention recently; see, for example, [Raphaël 2013]. This program has been concluded for the mass-critical (gKdV) equation in [Martel et al. 2012a; 2012b; 2012c], where it is shown that, provided the data is taken close enough to the ground state *in a suitable topology* which is strictly smaller than the energy norm, the blow-up dynamics are completely classified. In contrast, arbitrarily slow blow-up can be achieved for large deformations of the ground state in this restricted sense. The existence of such slow blow-up regimes remains however open in many important instances, in particular for the mass-critical NLS equation; see [Merle et al. 2013] for a further introduction to this delicate problem. For energy-critical problems like wave or Schrödinger maps, Krieger et al. [2008] showed that arbitrarily slow blow-up can be achieved, but the known examples so far are never $\mathscr{C}^\infty$ smooth. The structure of the flow near Q is also somewhat mysterious, and various new kinds of global dynamics have been constructed; see [Donninger and Krieger 2013; Bejenaru and Tataru 2014]. One of the new results of our analysis in this paper is to show the essential role played by the control of higher order Sobolev norms, which provide a new topology to measure the distance to the solitary wave which is sharp enough to see all the blow-up regimes (1-13). The control of these norms acts in the energy method as a replacement of the counting of the number of intersections of the solution with the ground state, which, in the parabolic setting, plays an essential role for the classification of the blow-up dynamics [Mizoguchi 2007], but relies in an essential way on maximum principle techniques. We believe that the blow-up solutions we construct in this paper are the building blocks to classify the blow-up dynamics near the ground state in a suitable topology.

5. *Extension to dispersive problems.* We treat in this paper the parabolic problem, but the robustness of our approach has been shown in [Raphaël and Rodnianski 2012; Merle et al. 2011], which treat the dispersive wave and Schrödinger maps with $\mathbb{S}^2$ target. We expect that similar constructions can be performed there as well to produce arbitrarily slow $\mathscr{C}^\infty$ blow-up solutions with quantized rate, hence completing the analysis of these excited regimes, which started in the seminal work [Krieger et al. 2008].

Notations. We introduce the differential operator

$$\Lambda f = y \cdot \nabla f \quad \text{(energy-critical scaling)}.$$

Given a parameter $\lambda > 0$, we let

$$u_\lambda(r) = u(y) \quad \text{with } y = \frac{r}{\lambda}.$$

Given a positive number $b_1 > 0$, we let

$$B_0 = \frac{1}{\sqrt{b_1}}, \quad B_1 = \frac{|\log b_1|}{\sqrt{b_1}}. \tag{1-14}$$

We let χ be a positive nonincreasing smooth cut-off function with

$$\chi(y) = \begin{cases} 1 & \text{for } y \le 1, \\ 0 & \text{for } y \ge 2. \end{cases}$$

Given a parameter $B > 0$, we denote

$$\chi_B(y) = \chi\left(\frac{y}{B}\right). \tag{1-15}$$

We shall systematically omit the measure in all radial two dimensional integrals and note that

$$\int f = \int_0^{+\infty} f(r) r\, dr.$$

Given a p-uplet $J = (j_1, \dots, j_p) \in \mathbb{N}^p$, we introduce the norms

$$|J|_1 = \sum_{k=1}^{p} j_k, \quad |J|_2 = \sum_{k=1}^{p} k j_k. \tag{1-16}$$

We note that

$$\mathcal{B}_d(R) = \left\{ x \in \mathbb{R}^d,\ |x| = \left(\sum_{i=1}^{d} x_i^2 \right)^{\frac{1}{2}} \le R \right\}.$$

Strategy of the proof. Let us give brief insight into the strategy of the proof of Theorem 1.1.

(i). *Renormalized flow and iterated resonances.* Let us look for a modulated solution $u(t, r)$ of (1-5) in renormalized form

$$u(t, r) = v(s, y), \quad y = \frac{r}{\lambda(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda^2(t)}, \tag{1-17}$$

which leads to the self-similar equation

$$\partial_s v - \Delta v + b_1 \Lambda v + \frac{f(v)}{y^2} = 0, \quad b_1 = -\frac{\lambda_s}{\lambda}. \tag{1-18}$$

We know from theoretical ground that if blow-up occurs, $v(s, y) = Q(y) + \varepsilon(s, y)$ for some small $\varepsilon(s, y)$, and hence the linear part of the ε flow is governed by the Schrödinger operator

$$H = -\Delta + \frac{f'(Q)}{y^2}.$$

The energy-critical structure of the problem induces an explicit resonance

$$H(\Lambda Q) = 0,$$

where from explicit computation,

$$\Lambda Q \sim \frac{2}{y} \quad \text{as } y \to \infty. \tag{1-19}$$

More generally, the iterates of the kernel of H computed iteratively through the scheme

$$H T_{k+1} = -T_k, \quad T_0 = \Lambda Q, \tag{1-20}$$

display a nontrivial tail at infinity:

$$T_k(y) \sim y^{2k-1}(c_k \log y + d_k) \quad \text{for } y \gg 1. \tag{1-21}$$

(ii). *Tail dynamics*. We now generalize the approach developed in [Raphaël and Rodnianski 2012; Merle et al. 2011] and claim that $(T_k)_{k\geq 1}$ correspond to unstable directions which can be excited in a universal way. To see this, let us look for a slowly modulated solution to (1-18) of the form $v(s, y) = Q_{b(s)}(y)$ with

$$b = (b_1, \dots, b_L), \quad Q_b = Q(y) + \sum_{i=1}^{L} b_i T_i(y) + \sum_{i=2}^{L+2} S_i(y) \tag{1-22}$$

and with a priori bounds

$$b_i \sim b_1^i, \quad |S_i(y)| \lesssim b_1^i y^{C_i},$$

so that S_i is in some sense homogeneous of degree i in b_1. Our strategy is the following: choose the universal dynamical system driving the modes $(b_i)_{1\leq i\leq L}$ which generates the *least growing in space solution* S_i. Let us illustrate the procedure.

$\underline{O(b_1)}$. We do not adjust the law of b_1 for the first term.[3] We therefore obtain from (1-18) the equation

$$b_1(HT_1 + \Lambda Q) = 0.$$

$\underline{O(b_1^2, b_2)}$. We obtain

$$(b_1)_s T_1 + b_1^2 \Lambda T_1 + b_2 H T_2 + H S_2 = b_1^2 NL(T_1, Q),$$

where $NL(T_1, Q)$ corresponds to nonlinear interaction terms. When considering the far away tail (1-21), we have, for y large,

$$\Lambda T_1 \sim T_1, \quad HT_2 = -T_1,$$

and thus

$$(b_1)_s T_1 + b_1^2 \Lambda T_1 + b_2 H T_2 \sim ((b_1)_s + b_1^2 - b_2) T_1.$$

Hence the leading order growth is canceled by the choice

$$(b_1)_s + b_1^2 - b_2 = 0. \tag{1-23}$$

We then solve for

$$HS_2 = -b_1^2(\Lambda T_1 - T_1) + NL(T_1, Q)$$

and check that $S_2 \ll b_1^2 T_1$ for y large.

$\underline{O(b_1^{k+1}, b_{k+1})}$. At the k-th iteration, we obtain an elliptic equation of the form

$$(b_k)_s T_k + b_1 b_k \Lambda T_k + b_{k+1} H T_{k+1} + H S_1 = b_1^{k+1} NL_k(T_1, \dots, T_k, Q).$$

From (1-21), we have, for tails,

$$\Lambda T_k \sim (2k-1) T_k,$$

and therefore

$$(b_k)_s T_k + b_1 b_k \Lambda T_k + b_{k+1} H T_{k+1} \sim ((b_k)_s + (2k-1) b_1 b_k - b_{k+1}) T_k.$$

[3] If $(b_1)_s = -c_1 b_1$, then $-\lambda_s/\lambda \sim b_1 \sim e^{-c_1 s}$, and hence after integration in time, $|\log \lambda| \lesssim 1$ and there is no blow-up.

The cancellation of the leading order growth occurs for

$$(b_k)_s + (2k-1)b_1 b_k - b_{k+1} = 0.$$

We then solve for the remaining S_{k+1} term and check that $S_{k+1} \ll b_1^{k+1} T_{k+1}$ for y large.

(iii). *The universal system of ODE's.* The above approach leads to the universal system of ODE's which we stop after the L-th iterate:

$$(b_k)_s + (2k-1)b_1 b_k - b_{k+1} = 0, \quad 1 \le k \le L, \quad b_{L+1} \equiv 0, \quad -\frac{\lambda_s}{\lambda} = b_1. \tag{1-24}$$

It turns out, and this is classical for critical problems, that an additional logarithmic gain related to the growth (1-21) can be captured, and this turns out to be essential for the analysis.[4] This leads to the sharp dynamical system

$$\begin{cases} (b_k)_s + \left(2k-1+\dfrac{2}{|\log b_1|}\right) b_1 b_k - b_{k+1} = 0, \quad 1 \le k \le L,\ b_{L+1} \equiv 0, \\ -\dfrac{\lambda_s}{\lambda} = b_1, \\ \dfrac{ds}{dt} = \dfrac{1}{\lambda^2}. \end{cases} \tag{1-25}$$

It is easily seen (see Lemma 2.14) that (1-25) rewritten in the original t time variable admits solutions such that $\lambda(t)$ touches 0 in finite time T with the asymptotic (1-13). Equivalently in renormalized variables,

$$\lambda(s) \sim \frac{(\log s)^{|d_1|}}{s^{c_1}}, \quad b(s) \sim \frac{c_1}{s} \quad \text{with } c_1 = \frac{L}{2L-1},\ d_1 = \frac{-2L}{(2L-1)^2}. \tag{1-26}$$

Moreover (see Lemma 2.15), the corresponding solution is stable for $L = 1$. This is the stable blow-up regime, and unstable with $(L-1)$ directions of instabilities for $L \ge 2$.

(iv). *Decomposition of the flow and modulation equations.* Let the approximate solution Q_b be given by (1-22), which by construction generates an approximate solution to the renormalized flow (1-18):

$$\Psi_b = \partial_s Q_b - \Delta Q_b + b \Lambda Q_b + \frac{f(Q_b)}{y^2} = \mathrm{Mod}(t) + O(b^{2L+2}),$$

where, roughly,

$$\mathrm{Mod}(t) = \sum_{i=1}^{L} \left[(b_i)_s + \left(2i-1+\frac{2}{|\log b_1|}\right) b_1 b_i - b_{i+1} \right] T_i.$$

We localize Q_b in the zone $y \le B_1$ to avoid the irrelevant growing tails for $y \gg 1/\sqrt{b_1}$. We then pick an initial data of the form

$$u_0(y) = Q_b(y) + \varepsilon_0(y), \quad |\varepsilon_0(y)| \ll 1$$

[4] See, for example, [Raphaël and Rodnianski 2012] for further discussion.

in some suitable sense where $b(0)$ is chosen initially close to the exact excited solution to (1-24). From standard modulation argument, we dynamically introduce a modulated decomposition of the flow

$$u(t,r) = (Q_{b(t)} + \varepsilon)\left(t, \frac{r}{\lambda(t)}\right) = (Q_{b(t)})\left(t, \frac{r}{\lambda(t)}\right) + w(t,r), \tag{1-27}$$

where the $L+1$ modulation parameters $(b(t), \lambda(t))$ are chosen in order to manufacture the orthogonality conditions

$$(\varepsilon(t), H^k \Phi_M) = 0, \quad 0 \le k \le M. \tag{1-28}$$

Here $\Phi_M(y)$ is some fixed direction depending on some large constant M which generates an approximation of the kernel of the iterates of H; see (3-7). This orthogonal decomposition, which, for each fixed time t, directly follows from the implicit function theorem, now allows us to compute the modulation equations governing the parameters $(b(t), \lambda(t))$. The Q_b construction is precisely manufactured to produce the expected ODE's:[5]

$$\left|\frac{\lambda_s}{\lambda} + b_1\right| + \sum_{i=1}^{L} \left|(b_i)_s + \left(2i - 1 + \frac{2}{|\log b_1|}\right) b_1 b_i - b_{i+1}\right| \lesssim \|\varepsilon\|_{\mathrm{loc}} + b_1^{L+\frac{3}{2}}, \tag{1-29}$$

where $\|\varepsilon\|_{\mathrm{loc}}$ measures a *local-in-space* interaction with the harmonic map.

(v). *Control of the radiation and monotonicity formula.* According to (1-29), the core of our analysis is now to show that local norms of ε are under control and do not perturb the dynamical system (1-24). This is achieved using high order Sobolev norms adapted to the linear flow, and in particular we claim that the orthogonality conditions (1-28) ensure the Hardy type coercivity of the iterated operator

$$\mathscr{E}_{2k+2} = \int |H^{k+1}\varepsilon|^2 \gtrsim \int \frac{|\varepsilon|^2}{(1+y^{4k+4})(1+|\log y|^2)}, \quad 0 \le k \le L.$$

We now claim the we can control theses norms thanks to an energy estimate *seen on the linearized equation in original variables*, that is, by working with w in (1-27) and not ε, as initiated in [Raphaël and Rodnianski 2012; Merle et al. 2011]. Here the parabolic structure of the problem simplifies the analysis and displays a repulsive property of the renormalized linearized operator; see the proof of (3-48). The outcome is an estimate of the form

$$\frac{d}{ds}\left\{\frac{\mathscr{E}_{2k+2}}{\lambda^{4k+2}}\right\} \lesssim \frac{b_1^{2k+3}}{\lambda^{4k+2}} |\log b_1|^{c_k}, \tag{1-30}$$

where the right hand side is controlled by the size of the error in the construction of the approximate blow-up profile. Integrating this in time yields two contributions, one from data and one from the error:

$$\mathscr{E}_{2k+2}(s) \lesssim \lambda^{4k+2}(s)\mathscr{E}_{2k+2}(0) + \lambda^{4k+2}(s) \int_{s_0}^{s} \frac{b_1^{2k+3}}{\lambda^{4k+2}} |\log b_1|^{c_k}\, d\sigma.$$

[5] See Lemma 3.3.

The second contribution is estimated in the regime (1-26) using the fundamental algebra

$$(2k+3)-c_1(4k+2)=1+\frac{2(L-k-1)}{2L-1}\begin{cases}\geq 1 & \text{for } k\leq L-1,\\ <1 & \text{for } k=L.\end{cases} \tag{1-31}$$

Hence data dominates for $k\leq L-1$ up to a logarithmic error

$$\lambda^{4k+2}(s)\int_{s_0}^{s}\frac{b_1^{2k+3}}{\lambda^{4k+2}}|\log b_1|^{c_k}\,d\sigma\sim\lambda^{4k+2}(\log s)^C\int_{s_0}^{s}\frac{d\sigma}{\sigma^{2k+3-c_1(4k+2)}}\sim\lambda^{4k+2}(\log s)^C,$$

which yields the bound

$$\mathscr{E}_{2k+2}\lesssim\lambda^{4k+2}|\log s|^C,\quad 0\leq k\leq L-1, \tag{1-32}$$

which simply expresses the boundedness up to a log of w in some Sobolev type H^{k+1} norm. On the other hand, for $k=L$, we can first derive a *sharp logarithmic gain* in (1-30),

$$\frac{d}{ds}\left\{\frac{\mathscr{E}_{2L+2}}{\lambda^{4k+2}}\right\}\lesssim\frac{b_1^{2L+3}}{\lambda^{4L+2}|\log b_1|^2}, \tag{1-33}$$

and then the integral diverges from (1-31) and

$$\lambda^{4k+2}(s)\int_{s_0}^{s}\frac{b_1^{2L+3}}{\lambda^{4L+2}|\log b_1|^2}\,d\sigma\sim\lambda^{4k+2}(s)\int_{s_0}^{s}\frac{1}{\sigma}\frac{b_1^{2L+2}}{\lambda^{4L+2}|\log b_1|^2}\,d\sigma\sim\frac{b_1^{2L+2}}{|\log b_1|^2}\gg\lambda^{4k+2}.$$

We therefore obtain

$$\mathscr{E}_{2L+2}\lesssim\frac{b_1^{2L+2}}{|\log b_1|^2}. \tag{1-34}$$

The difference between the controls (1-32) for $0\leq k\leq L-1$ and the sharp control (1-34) is an essential feature of the analysis and explains the introduction of an exactly order $L+1$ Sobolev energy.

We can now reinject this bound into (1-29) and, thanks to the logarithmic gain in (1-33), show that ε does not perturb the system (1-25), modulo the control of the associated unstable $L-1$ modes by a further adjusted choice of the initial data. This concludes the proof of Theorem 1.1.

This paper is organized as follows. In Section 2, we construct the approximate self-similar solutions Q_b and obtain sharp estimates on the error term Ψ_b. We also exhibit an explicit solution to the dynamical system (1-25) and show that it displays $(L-1)$ directions of instability. In Section 3, we set up the bootstrap argument in Proposition 3.1 and derive the fundamental monotonicity of the Sobolev-type norm $\|H^{L+1}\varepsilon\|_{L^2}^2$ in Proposition 3.6, which is the heart of the analysis. In Section 4, we close the bootstrap bounds, which easily imply the blow-up statement of Theorem 1.1.

2. Construction of the approximate profile

This section is devoted to the construction of the approximate Q_b blow-up profile and the study of the associated dynamical system for $b=(b_1,\dots,b_L)$.

The linearized Hamiltonian. Let us start by recalling the structure of the harmonic map Q, which is the unique up-to-scaling solution to

$$\Lambda Q = g(Q), \quad Q(0) = 0, \quad \lim_{r\to+\infty} Q(r) = \pi. \tag{2-1}$$

This equation can be integrated explicitly.[6] Q is smooth $Q \in \mathscr{C}^\infty([0,+\infty),[0,\pi))$ and using (1-3) admits a Taylor expansion[7] to all order at the origin,

$$Q(y) = \sum_{i=0}^{p} c_i y^{2i+1} + O(y^{2p+3}) \quad \text{as } y \to 0, \tag{2-2}$$

and at infinity,

$$Q(y) = \pi - \frac{2}{y} - \sum_{i=1}^{p} \frac{d_i}{y^{2i+1}} + O\left(\frac{1}{y^{2p+3}}\right) \quad \text{as } y \to +\infty. \tag{2-3}$$

The linearized operator close to Q displays a remarkable structure. Indeed, let the potentials

$$Z = g'(Q), \quad V = Z^2 + \Lambda Z = f'(Q), \quad \widetilde{V} = (1+Z)^2 - \Lambda Z, \tag{2-4}$$

which, from (2-2),(2-3), satisfy the following behavior at $0, +\infty$:

$$Z(y) = \begin{cases} 1 + \sum_{i=1}^{p} c_i y^{2i} + O(y^{2p+2}) & \text{as } y \to 0, \\ -1 + \sum_{i=1}^{p} \dfrac{c_i}{y^{2i}} + O\left(\dfrac{1}{y^{2p+2}}\right) & \text{as } y \to +\infty, \end{cases} \tag{2-5}$$

$$V(y) = \begin{cases} 1 + \sum_{i=1}^{p} c_i y^{2i} + O(y^{2p+2}) & \text{as } y \to 0, \\ 1 + \sum_{i=1}^{p} \dfrac{c_i}{y^{2i}} + O\left(\dfrac{1}{y^{2p+2}}\right) & \text{as } y \to +\infty, \end{cases} \tag{2-6}$$

$$\widetilde{V}(y) = \begin{cases} 4 + \sum_{i=1}^{p} c_i y^{2i} + O(y^{2p+2}) & \text{as } y \to 0, \\ \sum_{i=1}^{p} \dfrac{c_i}{y^{2i}} + O\left(\dfrac{1}{y^{2p+2}}\right) & \text{as } y \to +\infty, \end{cases} \tag{2-7}$$

where $(c_i)_{i\geq 1}$ stands for some generic sequence of constants which depend on the Taylor expansion of g at $(0,\pi)$. The linearized operator close to Q is the Schrödinger operator

$$H = -\Delta + \frac{V}{y^2}. \tag{2-8}$$

and admits the factorization

$$H = A^* A \tag{2-9}$$

with

$$A = -\partial_y + \frac{Z}{y}, \quad A^* = \partial_y + \frac{1+Z}{y}, \quad Z(y) = g'(Q).$$

[6] See [Raphaël and Schweyer 2013] for more details.

[7] up to scaling

Observe that, equivalently,

$$Au = -\Lambda Q \frac{\partial}{\partial y}\left(\frac{u}{\Lambda Q}\right), \quad A^* u = \frac{1}{y\Lambda Q}\frac{\partial}{\partial y}(uy\Lambda Q), \tag{2-10}$$

and thus the kernels of A and A^* on $\mathbb{R}_+^*$ are explicit:

$$Au = 0 \quad \text{if and only if} \quad u \in \operatorname{Span}(\Lambda Q), \qquad A^* u = 0 \quad \text{if and only if} \quad u \in \operatorname{Span}\left(\frac{1}{y\Lambda Q}\right). \tag{2-11}$$

Hence the kernel of H on $\mathbb{R}_+^*$ is

$$Hu = 0 \quad \text{if and only if} \quad u \in \operatorname{Span}(\Lambda Q, \Gamma) \tag{2-12}$$

with

$$\Gamma(y) = \Lambda\phi \int_1^y \frac{dx}{x(\Lambda\phi(x))^2} = \begin{cases} O\left(\frac{1}{y}\right) & \text{as } y \to 0, \\ \frac{y}{4} + O\left(\frac{\log y}{y}\right) & \text{as } y \to +\infty. \end{cases} \tag{2-13}$$

In particular, H is a positive operator on $\dot{H}^1_{\mathrm{rad}}$ with a *resonance* ΛQ at the origin induced by the energy-critical scaling invariance. We also introduce the conjugate Hamiltonian

$$\widetilde{H} = AA^* = -\Delta + \frac{\widetilde{V}}{y^2}, \tag{2-14}$$

which is definite positive by construction and (2-11); see Lemma B.2.

Admissible functions. Explicit knowledge of the Green's functions allows us to introduce the formal inverse

$$H^{-1} f = -\Gamma(y) \int_0^y f \Lambda Q x \, dx + \Lambda Q(y) \int_0^y f \Gamma x \, dx. \tag{2-15}$$

Given a function f, we introduce the suitable derivatives of f by considering the sequence

$$f_0 = f, \quad f_{k+1} = \begin{cases} A^* f_k & \text{for k odd,} \\ A f_k & \text{for k even,} \end{cases} \quad k \geq 0. \tag{2-16}$$

We shall introduce the formal notation

$$f_k = \mathscr{A}^k f.$$

We define a first class of admissible functions which display a suitable behavior both at the origin and infinity.

Definition 2.1 (admissible functions). We say a smooth function $f \in \mathscr{C}^\infty(\mathbb{R}_+, \mathbb{R})$ is admissible of degree $(p_1, p_2) \in \mathbb{N} \times \mathbb{Z}$ if

(i) f admits a Taylor expansion at the origin to all order

$$f(y) = \sum_{k=p_1}^{p} c_k y^{2k+1} + O(y^{2p+3}); \tag{2-17}$$

(ii) f and its suitable derivatives admit a bound, for $y \geq 2$,

$$\text{for all } k \geq 0, \quad |f_k(y)| \lesssim \begin{cases} y^{2p_2-k-1}(1+|\log y|) & \text{for } 2p_2-k \geq 1, \\ y^{2p_2-k-1} & \text{for } 2p_2-k \leq 0. \end{cases} \tag{2-18}$$

H naturally acts on the class of admissible functions in the following way.

Lemma 2.2 (action of H and H^{-1} on admissible functions). *Let f be an admissible function of degree (p_1, p_2). Then*

(i) *for all $l \geq 1$, $H^l f$ is admissible of degree*

$$(\max(p_1 - l, 0),\ p_2 - l), \tag{2-19}$$

(ii) *for all l, $p_2 \geq 0$, $H^{-l} f$ is admissible of degree*

$$(p_1 + l,\ p_2 + l). \tag{2-20}$$

Proof of Lemma 2.2. This a simple consequence of the expansions (2-2), (2-3).

Let us first show that Hf is admissible of degree at least $(\max(p_1 - 1, 1),\ p_2 - 1)$, which yields (2-19) by induction. We inject the Taylor expansions (2-17), (2-18) into (2-8). Near the origin, the claim directly follows from the Taylor expansion (2-6) and the cancellation $H(y) = cy + O(y^3)$ at the origin. The claim at infinity directly follows from the relation $u_k = f_{k+2}$ by definition.

Now let $p_2 \geq 0$ and $u = H^{-1} f$ be given by (2-15), and let us show that u is admissible of degree at least $(p_1 + 1,\ p_2 + 1)$, which yields (2-20) by induction. From the relation $u_k = f_{k-2}$ for $k \geq 2$, we need only consider $k = 0, 1$. We first observe from the Wronskian relation $\Gamma'(\Lambda Q) - (\Lambda Q)'\Gamma = 1/y$ that

$$A\Gamma = -\Gamma' + \frac{Z}{y}\Gamma = -\Gamma' + \frac{(\Lambda Q)'}{\Lambda Q}\Gamma = -\frac{1}{y\Lambda Q}.$$

Thus, using the cancellation $A\Lambda Q = 0$, we compute

$$Au = -A\Gamma \int_0^y f \Lambda Q x\, dx = \frac{1}{y\Lambda Q} \int_0^y f \Lambda Q x\, dx. \tag{2-21}$$

Moreover, we may invert A using (2-10) and the boundary condition $u = O(y^3)$ from (2-15), which yields

$$u = -\Lambda Q \int_0^y \frac{Au}{\Lambda Q}\, dx = -\Lambda Q(y) \int_0^y \frac{dx}{x(\Lambda Q(x))^2} \int_0^x f(z) \Lambda Q(z) z\, dz. \tag{2-22}$$

Using (2-2), this yields the Taylor expansion near the origin:

$$Au = \sum_{k=p_1}^{p} c_k^{(1)} y^{2k+2} + O(y^{2p+4}), \quad u = -\Lambda Q \int_0^y \frac{Au}{\Lambda Q}\, dx = \sum_{k=p_1}^{p} c_k^{(2)} y^{2k+3} + O(y^{2p+5}),$$

and hence u is of degree at least p_1+1 near the origin. For $y \geq 1$, from (2-21), (2-22), (2-18), we estimate by brute force, for $p_2 \geq 1$,

$$|Au| = |u_1| \lesssim \int_0^y \tau^{2p_2-1}(1+|\log\tau|)\,d\tau \lesssim y^{2p_2}(1+|\log y|),$$
$$|u| \lesssim \frac{1}{y}\int_0^y \tau^{2p_2}(1+|\log\tau|)\tau\,d\tau \lesssim y^{2p_2+1}(1+|\log y|),$$

and, for $p_2 = 0$,

$$|Au| = |u_1| \lesssim \int_1^y \tau^{-1}\,d\tau \lesssim 1+|\log y|,$$
$$|u| \lesssim \frac{1}{y}\int_0^y (1+|\log\tau|)\tau\,d\tau \lesssim y(1+|\log y|).$$

Hence u satisfies (2-18) with $p_2 \to p_2+1$ and $k = 0, 1$. □

Let us give an explicit example of admissible functions which will be essential for the analysis. From (2-2) and the cancellation $A\Lambda Q = 0$, ΛQ is admissible of degree $(0, 0)$, and hence Lemma 2.2 ensures the following.

Lemma 2.3 (generators of the kernel of H^i). *Let the sequence of profiles for $i \geq 1$ be*

$$T_i = (-1)^i H^{-i}\Lambda Q. \tag{2-23}$$

Then T_i is admissible of degree (i, i).

b_1-admissible functions. We will need an extended notion of admissible functions for the construction of the blow-up profile. In the sequel, we consider a small enough $0 < b_1 \ll 1$ and let B_0, χ_{B_0} be given by (1-14), (1-15). Given $l \in \mathbb{Z}$, we let

$$g_l(b_1, y) = \begin{cases} \dfrac{1+|\log(\sqrt{b_1}y)|}{|\log b_1|}\mathbf{1}_{y\leq 3B_0} & \text{for } l \geq 1, \\ \dfrac{\mathbf{1}_{y\leq 3B_0}}{|\log b_1|} & \text{for } l \leq 0, \end{cases} \tag{2-24}$$

and, similarly,

$$\tilde{g}_l(b_1, y) = \begin{cases} \dfrac{1+|\log y|}{|\log b_1|}\mathbf{1}_{y\leq 3B_0} & \text{for } l \geq 1, \\ \dfrac{\mathbf{1}_{y\leq 3B_0}}{|\log b_1|} & \text{for } l \leq 0. \end{cases} \tag{2-25}$$

We then define the extended class of b_1-admissible functions.

Definition 2.4 (b_1-admissible functions). We say a smooth function $f \in \mathscr{C}^\infty(\mathbb{R}_+^* \times \mathbb{R}_+, \mathbb{R})$ is b_1-admissible of degree $(p_1, p_2) \in \mathbb{N}\times\mathbb{Z}$ if the following hold:

(i) For $y \leq 1$, f admits a representation

$$f(b_1, y) = \sum_{j=1}^{J} h_j(b_1)\tilde{f}_j(y) \tag{2-26}$$

for some finite order $J \in \mathbb{N}^*$, some smooth functions $\tilde{f}_j(y)$ with a Taylor expansion at the origin to all order, for all $y \le 1$,

$$\tilde{f}_j(y) = \sum_{k=p_1}^{p} c_{k,j} y^{2k+1} + O(y^{2p+3}), \tag{2-27}$$

and some smooth functions $h_j(b_1)$ away from the origin with

$$\text{for all } l \ge 0, \quad \left| \frac{\partial^l h_j}{\partial b_1^l} \right| \lesssim \frac{1}{b_1^l}. \tag{2-28}$$

(ii) The function f and its suitable derivatives (2-16) satisfy a uniform bound for some constant $c_{p_2} > 0$: for all $y \ge 2$ and all $k \ge 0$,

$$|f_k(b_1, y)| \lesssim y^{2p_2-k-1} g_{2p_2-k}(b_1, y) + y^{2p_2-k-3} |\log y|^{c_{p_2}} + F_{p_2,k,0}(b_1) y^{2p_2-k-3} \mathbf{1}_{y \ge 3B_0}, \tag{2-29}$$

and, for all $l \ge 1$,

$$\begin{aligned} &\left| \frac{\partial^l}{\partial b_1^l} f_k(b_1, y) \right| \\ &\quad \lesssim \frac{1}{b_1^l |\log b_1|} \left\{ y^{2p_2-k-1} \tilde{g}_{2p_2-k}(b_1, y) + y^{2p_2-k-3} |\log y|^{c_{p_2}} \right\} + F_{p_2,k,l}(b_1) y^{2p_2-k-3} \mathbf{1}_{y \ge 3B_0}, \end{aligned} \tag{2-30}$$

where, for all $l \ge 0$,

$$F_{p_2,k,l}(b_1) = \begin{cases} 0 & \text{for } 2p_2 - k - 3 \le -1, \\ 1/(b_1^{l+1} |\log b_1|) & \text{for } 2p_2 - k - 3 \ge 0. \end{cases} \tag{2-31}$$

Remark 2.5. *Let us consider the solution T_1 to*

$$H T_1 = -\Lambda Q.$$

An explicit computation reveals the growth for y large

$$\Lambda Q \sim \frac{1}{y}, \quad T_1(y) \sim y \log y.$$

The b_1-admissibility corresponds to a $\log b_1$ gain on the growth at ∞, which is an essential feature of the slowly growing tails in the construction of the modulated blow-up profile in Proposition 2.12. Observe for example that (2-29), (2-31) imply the rough bound

$$|f_k| \lesssim (1+y)^{2p_2-1-k}, \quad \left| \frac{\partial^l f_k}{\partial b_1^l} \right| \lesssim \frac{(1+y)^{2p_2-1-k}}{|\log b_1|}, \quad k \ge 0,\ l \ge 1, \tag{2-32}$$

and hence a logarithmic improvement with respect to (2-18). This gain will be measured in a sharp way through the computation of suitable weighted Sobolev bounds; see Lemma 2.8.

We claim that H, H^{-1} and the scaling operators naturally act on the class of b_1-admissible functions in the following way.

Lemma 2.6 (action of H, H^{-1} and scaling operators on b_1-admissible functions). *Let f be a b_1-admissible function of degree (p_1, p_2). Then we get*:

(i) *For all $l \geq 1$, $H^l f$ is b_1-admissible of degree*

$$(\max(p_1 - l, 0),\ p_2 - l). \tag{2-33}$$

(ii) *For all l, $p_2 \geq 1$, $H^{-l} f$ is b_1-admissible of degree*

$$(p_1 + l,\ p_2 + l). \tag{2-34}$$

(iii) $\Lambda f = y\partial_y f$ *is admissible of degree* (p_1, p_2).

(iv) $b_1 \partial f/(\partial b_1)$ *is admissible of degree* (p_1, p_2).

Proof of Lemma 2.6. *Proof of* (i). We show that $u = Hf$ is b_1-admissible of degree $(\max(p_1-1, 0),\ p_2-1)$, which yields (2-33) by induction. Near the origin, the claim directly follows from the Taylor expansion (2-27) with (2-26) and the cancellation $H(y) = cy + O(y^3)$ at the origin. For $y \geq 1$, H is independent of b_1 so that, by definition,

$$\text{for all } l \geq 0, \quad \frac{\partial^l u_k}{\partial b_1^l} = \frac{\partial^l f_{k+2}}{\partial b_1^l},$$

which satisfies (2-29), (2-30), (2-31) with $p_2 \to p_2 - 1$ and $F_{p_2-1,k,l}(b_1) = F_{p_2,k+2,l}(b_1)$. Equation (2-33) follows.

Proof of (ii). Now let $p_2 \geq 1$ and let us show that $u = H^{-1} f$ is admissible of degree $(p_1 + 1,\ p_2 + 1)$, which yields (2-34) by induction. Observe that, for $k \geq 2$ and all $l \geq 0$,

$$\frac{\partial^l u_k}{\partial b_1^l} = \frac{\partial^l f_{k-2}}{\partial b_1^l},$$

which satisfies (2-29), (2-30), (2-31) with $p_2 \to p_2 + 1$ and $F_{p_2+1,k,l}(b_1) = F_{p_2,k-2,l}(b_1)$. It thus only remains to estimate u, Au, and their derivatives in b_1.

Estimate for u near the origin. The inversion formulas (2-21), (2-22) ensure the decomposition of variables near the origin

$$u(b_1, y) = \sum_{j=1}^{J} h_j(b_1)\tilde{u}_j(y),$$

where, using (2-2) the Taylor expansion near the origin,

$$A\tilde{u}_j = \sum_{k=p_1}^{p} c_{k,j}^{(1)} y^{2k+2} + O(y^{2p+4}), \quad \tilde{u}_j = -\Lambda Q \int_0^y \frac{A\tilde{u}_j}{\Lambda Q}\, dx = \sum_{k=p_1}^{p} c_{k,j}^{(2)} y^{2k+3} + O(y^{2p+5}).$$

Hence u is of degree at least $p_1 + 1$ near the origin.

Estimate for $u_1 = Au$ for $y \geq 1$. We use the formula (2-21) and the assumption $p_2 \geq 1$ to estimate, for $1 \leq y \leq 3B_0$,

$$\begin{aligned}
|Au| &\lesssim \int_0^y |f|\,d\tau \lesssim \int_0^y [\tau^{2p_2-1} g_{2p_2-1}(b_1,\tau) + \tau^{2p_2-3}|\log\tau|^{c_{p_2}}]\,d\tau \\
&\lesssim \frac{1}{b_1^{p_2}|\log b_1|}\int_0^{\sqrt{b_1}y} \sigma^{2p_2-1}(1+|\log\sigma|)\,d\sigma + O(y^{2p_2-2}|\log y|^{1+c_{p_2}}) \\
&\lesssim y^{2p_2}\frac{1+|\log(\sqrt{b_1}y)|}{|\log b_1|} + y^{2p_2-2}|\log y|^{1+c_{p_2}} \\
&= y^{2(p_2+1)-2} g_{2(p_2+1)-1}(b_1,y) + y^{2(p_2+1)-4}|\log y|^{1+c_{p_2}},
\end{aligned}$$

and, for $y \geq 3B_0$,

$$\begin{aligned}
|Au| &\lesssim \int_0^y |f|\,d\tau \lesssim \int_0^{3B_0} \tau^{2p_2-1} g_1(b_1,\tau)\,d\tau + \int_{3B_0}^y F_{p_2,0,0}(b_1)\tau^{2p_2-3}\,d\tau + O(y^{2p_2-2}|\log y|^{1+c_{p_2}}) \\
&\lesssim \frac{1}{b_1^{p_2}|\log b_1|} + \int_{3B_0}^y F_{p_2,0,0}(b_1)\tau^{2p_2-3}\,d\tau + O(y^{2p_2-2}|\log y|^{1+c_{p_2}}).
\end{aligned}$$

If $p_2 = 1$, which is the borderline case $2p_2 - 3 = -1$, then $F_{p_2,0,0} = 0$, and we thus get the bound, for all $p_2 \geq 1$, $y \geq 3B_0$,

$$\begin{aligned}
|Au| &\lesssim y^{2p_2-2}\left(\frac{1}{b_1|\log b_1|} + F_{p_2,0,0}(b_1)\right) + y^{2p_2-2}|\log y|^{1+c_{p_2}} \\
&\lesssim \frac{1}{b_1|\log b_1|} y^{2p_2-2} + y^{2(p_2+1)-4}|\log y|^{1+c_{p_2}},
\end{aligned}$$

and (2-31) is satisfied for $(p_2 \to p_2+1, k=1)$ thanks to $2(p_2+1)-1-3 \geq 0$.

We now pick $l \geq 1$. H is independent of b_1, so

$$H\left(\frac{\partial^l u}{\partial b_1^l}\right) = \frac{\partial^l f}{\partial b_1^l},$$

and therefore, from (2-21), we compute

$$\frac{\partial^l u_1}{\partial b_1^l} = \frac{1}{y\Lambda Q}\int_0^y \Lambda Q \frac{\partial^l f}{\partial b_1^l} x\,dx.$$

This yields the bound, for $|y| \leq 3B_0$,

$$\begin{aligned}
\left|\frac{\partial^l u_1}{\partial b_1^l}\right| &\lesssim \int_0^y \frac{1}{b_1^l|\log b_1|}\{y^{2p_2-1}\tilde{g}_{2p_2-1}(b_1,y) + y^{2p_2-3}|\log y|^{c_{p_2}}\}\,dy \\
&\lesssim \frac{1}{b_1^l|\log b_1|}[y^{2p_2}\tilde{g}_1(b_1,y) + y^{2p_2-2}|\log y|^{c_{p_2}+1}] \\
&= \frac{1}{b_1^l|\log b_1|}[y^{2(p_2+1)-2}\tilde{g}_{2(p_2+1)-1}(b_1,y) + y^{2(p_2+1)-4}|\log y|^{c_{p_2}+1}],
\end{aligned}$$

and, for $|y| \geq 3B_0$,

$$\left|\frac{\partial^l u_1}{\partial b_1^l}\right| \lesssim \frac{1}{b_1^l|\log b_1|}\left[\frac{1}{b_1^{p_2}} + y^{2p_2-2}|\log y|^{c_{p_2}+1}\right] + \int_{3B_0}^{y} F_{p_2,0,l}(b_1) y^{2p_2-3}\, dy.$$

Again, if $p_2 = 1$, then $F_{p_2,0,l} = 0$, and we therefore obtain the bound, for all $p_2 \geq 1$,

$$\begin{aligned}\left|\frac{\partial^l u_1}{\partial b_1^l}\right| &\lesssim y^{2p_2-2}\left[\frac{1}{b_1^{l+1}|\log b_1|} + F_{p_2,0,l}(b_1)\right] + \frac{y^{2p_2-2}|\log y|^{c_{p_2}+1}}{b_1^l|\log b_1|} \\ &\lesssim \frac{y^{2(p_2+1)-1-3}}{b_1^{l+1}|\log b_1|} + \frac{1}{b_1^l|\log b_1|} y^{2(p_2+1)-1-3}|\log y|^{c_{p_2}+1},\end{aligned}$$

and (2-31) is satisfied for $(p_2 \to p_2+1, k=1)$ thanks to $2(p_2+1)-1-3 \geq 0$.

Estimate for u. Now, from the above bounds and (2-22), for $1 \leq y \leq 3B_0$ we estimate

$$\begin{aligned}|u| &\lesssim \frac{1}{y}\int_0^y |Au|\tau\, d\tau \lesssim \frac{1}{y}\int_0^y [\tau^{2p_2+1} g_{2p_2+1}(b_1,\tau) + \tau^{2p_2-1}|\log\tau|^{1+c_{p_2}}]\, d\tau \\ &\lesssim y^{2p_2+1}\frac{1+|\log(\sqrt{b_1}y)|}{|\log b_1|} + y^{2p_2-1}|\log y|^{2+c_{p_2}} \\ &= y^{2(p_2+1)-1} g_{2(p_2+1)}(b_1, y) + y^{2(p_2+1)-3}|\log y|^{2+c_{p_2}}\end{aligned}$$

and for $y \geq 3B_0$ we estimate

$$\begin{aligned}|u| &\lesssim \frac{1}{y}\left[\int_0^{3B_0} \tau^{2p_2+1} g(b_1,\tau)\, d\tau + \int_{3B_0}^{y} F_{p_2+1,1,0}(b_1)\tau^{2p_2-1}\, d\tau\right] + y^{2(p_2+1)-3}|\log y|^{2+c_{p_2}} \\ &\lesssim y^{2p_2-1}\left[\frac{1}{b_1|\log b_1|}\right] + y^{2(p_2+1)-3}|\log y|^{2+c_{p_2}},\end{aligned}$$

which satisfies (2-29) for $(p_2 \to p_2+1, k=0)$ thanks to $2(p_2+1)-3-1 \geq 0$. Finally, for $l \geq 1$, $1 \leq y \leq 3B_0$,

$$\begin{aligned}\left|\frac{\partial^l u}{\partial b_1^l}\right| &\lesssim \frac{1}{y}\int_0^y \left|\frac{\partial^l u_1}{\partial b_1^l}\right|\tau\, d\tau \lesssim \frac{1}{y b_1^l|\log b_1|}\int_0^y [\tau^{2p_2+1}\tilde{g}_{2p_2+1}(b_1,\tau) + \tau^{2p_2-1}|\log\tau|^{1+c_{p_2}}]\, d\tau \\ &\lesssim \frac{1}{b_1^l|\log b_1|}[y^{2(p_2+1)-1}\tilde{g}_{2(p_2+1)}(b_1, y) + y^{2(p_2+1)-3}|\log y|^{c_{p_2}+1}],\end{aligned}$$

and, for $y \geq 3B_0$,

$$\begin{aligned}\left|\frac{\partial^l u}{\partial b_1^l}\right| &\lesssim \frac{1}{y}\left[\int_0^{3B_0} \frac{\tau^{2p_2+1}\tilde{g}_1(b_1,\tau)}{b_1^l|\log b_1|}\, d\tau + \int_{3B_0}^{y} F_{p_2+1,1,l}(b_1)\tau^{2p_2-1}\, d\tau\right] + \frac{y^{2(p_2+1)-3}|\log y|^{2+c_{p_2}}}{b_1^l|\log b_1|} \\ &\lesssim \frac{y^{2(p_2+1)-3}}{b_1^{l+1}|\log b_1|} + \frac{y^{2(p_2+1)-3}|\log y|^{2+c_{p_2}}}{b_1^l|\log b_1|}.\end{aligned}$$

Hence u is b_1-admissible of degree (p_1+1, p_2+1).

Proof of (iii) *and* (iv). The property (iv) is a direct consequence of the definition of b_1 admissible functions (Definition 2.4) and the trivial bound

$$\frac{\tilde{g}_l(b_1, y)}{|\log b_1|} \lesssim g_l(b_1, y).$$

We now turn to the proof of (iii). First we rewrite the scaling operator as

$$\Lambda = y\partial_y = -Id - yA + (1+Z).$$

Near the origin, the existence of the decomposition (2-26) follows directly from the even parity of the Taylor expansion of Z at the origin (2-5). Far out, let

$$\Lambda f = -f - yf_1 + (1+Z)f.$$

A simple induction argument similar to Lemma D.1 yields the expansion for $k \geq 1$,

$$(yf_1)_k = c_{k+1} y f_{k+1} + c_{k+2} f_k + \sum_{i=1}^{k} P_{k,i}(y) f_i, \tag{2-35}$$

with the improved decay

$$|\partial_y^l P_{k,i}(y)| \lesssim \frac{1}{1+y^{2+l+k-i}} \quad \text{for all } l \geq 0,\ y \geq 1. \tag{2-36}$$

We therefore obtain from (2-32), (2-35), (2-36), (2-5) the bound

$$\begin{aligned}|(\Lambda f)_k| &\lesssim |yf_{k+1}| + |f_k| + \sum_{i=0}^{k} \frac{1}{y^{2+k-i}} y^{2p_2-i-1} \\ &\lesssim y^{2p_2-k-1}(g_{2p_2-(k+1)} + g_{2p_2-k}) + y^{2p_2-k-3}|\log y|^{c_{p_2}} + (F_{p_2,k,0} + F_{p_2,k+1,0}) y^{2p_2-k-3}\mathbf{1}_{y\geq 3B_0}.\end{aligned}$$

We now observe the monotonicity $g_{2p_2-k-1} \lesssim g_{2p_2-k}$ from (2-24) and $F_{p_2,k+1,0} \lesssim F_{p_2,k,0}$ from (2-31), and thus $(\Lambda f)_k$ satisfies (2-30), (2-31) for $l = 0$. Similarly, for $k \geq 0$, $l \geq 1$, we use the bound, for $y \gtrsim B_0$,

$$y^{2p_2-k-5}|F_{p_2,i,l}(b_1)| \lesssim \frac{y^{2p_2-k-5}}{b_1^{l+1}|\log b_1|} \lesssim \frac{y^{2p_2-k-3}}{b_1^l |\log b_1|},$$

to estimate

$$\begin{aligned}\left|\frac{\partial^l(\Lambda f)_k}{\partial b_1^l}\right| &\lesssim \left|y\frac{\partial^l f_{k+1}}{\partial b_1^l}\right| + \left|\frac{\partial^l f_k}{\partial b_1^l}\right| \\ &\quad + \sum_{i=0}^{k} \frac{1}{y^{k-i+2}} \left\{ \frac{1}{b_1^l|\log b_1|}[y^{2p_2-i-1} + y^{2p_2-i-3}|\log y|^{c_{p_2}}] + F_{p_2,i,l}(b_1) y^{2p_2-i-3}\mathbf{1}_{y\geq 3B_0} \right\} \\ &\lesssim \frac{1}{b_1^l|\log b_1|}[y^{2p_2-k-1}\tilde{g}_{2p_2-(k+1)} + \tilde{g}_{2p_2-k}) + y^{2p_2-k-3}|\log y|^{c_{p_2}}] \\ &\qquad + (F_{p_2,k,l} + F_{p_2,k+1,l}) y^{2p_2-k-3}\mathbf{1}_{y\geq 3B_0},\end{aligned}$$

and the bounds $\tilde{g}_{2p_2-k-1} \lesssim \tilde{g}_{2p_2-k}$, $F_{p_2,k+1,l} \lesssim F_{p_2,k,l}$ now ensure (2-30), (2-31) for $l \geq 1$. □

Slowly growing tails. Let us give an example of admissible profiles which will be central in the construction of the leading order slowly modulated blow-up profile. Given $b_1 > 0$ small enough, we let the radiation be

$$\Sigma_{b_1} = H^{-1}\{-c_{b_1}\chi_{B_0/4}\Lambda Q + d_{b_1} H[(1-\chi_{B_0})\Lambda Q]\} \tag{2-37}$$

with

$$c_{b_1} = \frac{4}{\int \chi_{B_0/4}(\Lambda Q)^2}, \quad d_{b_1} = c_{b_1}\int_0^{B_0} \chi_{B_0/4}\Lambda Q \Gamma y\, dy. \tag{2-38}$$

Lemma 2.7 (slowly growing tails). *Let $(T_i)_{i\geq 1}$ be given by (2-23). Then the sequence of profiles for $i \geq 1$*

$$\Theta_i = \Lambda T_i - (2i-1)T_i - (-1)^{i+1} H^{-i+1}\Sigma_{b_1} \tag{2-39}$$

is b_1-admissible of degree (i, i).

Proof of Lemma 2.7. *Step* 1: *Structure of* T_1. Let us consider $T_1 = -H^{-1}\Lambda Q$, which is admissible of degree (1, 1) from Lemma 2.3. For $y \geq 1$, explicit computation using the expansion (2-3) into (2-15) yields

$$T_1(y) = y\log y + e_0 y + O\left(\frac{|\log y|^2}{y}\right), \quad \Lambda T_1 = y\log y + (1+e_0)y + O\left(\frac{|\log y|^2}{y}\right) \tag{2-40}$$

for some universal constant e_0. Hence we get the essential cancellation

$$\Lambda T_1 - T_1 = y + O\left(\frac{|\log y|^2}{y}\right). \tag{2-41}$$

We now prove that Θ_i is of order (i, i) by induction on i.

Step 2: $i = 1$. By definition,

$$\Sigma_{b_1} = \Gamma(y)\int_0^y c_{b_1}\chi_{B_0/4}(\Lambda Q)^2 x\, dx - \Lambda Q(y)\int_0^y c_{b_1}\chi_{B_0/4}\Gamma\Lambda Q x\, dx + d_{b_1}(1-\chi_{B_0})\Lambda Q(y), \tag{2-42}$$

and thus, by the definition of c_{b_1}, d_{b_1} in (2-38),

$$\Sigma_{b_1} = \begin{cases} c_{b_1} T_1 & \text{for } y \leq B_0/4, \\ 4\Gamma & \text{for } y \geq 3B_0. \end{cases} \tag{2-43}$$

In particular, Σ_{b_1} admits a representation (2-26) near the origin with $J = 1$, $h_1(b_1) = c_{b_1}$, and $\tilde{f}_1(y) = T_1(y)$, and thus an expansion (2-27) of order $p_1 = 1$ from the first step. A direct computation on the formula (2-38) yields the bounds

$$c_{b_1} = \frac{2}{|\log b_1|}\left[1 + O\left(\frac{1}{|\log b_1|}\right)\right], \quad |d_{b_1}| \lesssim \frac{1}{b_1|\log b_1|}, \tag{2-44}$$

and

$$\left|\frac{\partial^l c_{b_1}}{\partial b_1^l}\right| \lesssim \frac{1}{b_1^l|\log b_1|^2}, \quad \left|\frac{\partial^l d_{b_1}}{\partial b_1^l}\right| \lesssim \frac{1}{b_1^{l+1}|\log b_1|} \qquad \text{for all } l \geq 1, \tag{2-45}$$

which imply (2-28).

For $y \geq 3B_0$, from (2-13), (2-43), we estimate

$$\Sigma_{b_1}(y) = y + O\left(\frac{\log y}{y}\right), \tag{2-46}$$

and, for $2 \leq y \leq 3B_0$,

$$\begin{aligned}\Sigma_{b_1}(y) &= c_{b_1}\left(\frac{y}{4} + O\left(\frac{\log y}{y}\right)\right)\left[\int_0^y \chi_{B_0/4}(\Lambda Q)^2 x\,dx\right] - c_{b_1}\Lambda Q(y)\int_1^y O(1)x\,dx \\ &= y\frac{\int_0^y \chi_{B_0/4}(\Lambda Q)^2}{\int \chi_{B_0/4}(\Lambda Q)^2} + O\left(\frac{1+y}{|\log b_1|}\right).\end{aligned} \tag{2-47}$$

We thus conclude from (2-41), (2-47) that, for $y \leq 3B_0$,

$$\Theta_1(y) = y - y\frac{\int_0^y \chi_{B_0/4}(\Lambda Q)^2}{\int \chi_{B_0/4}(\Lambda Q)^2} + O\left(\frac{1+y}{|\log b_1|}\right) + O\left(\frac{|\log y|^2}{1+y}\right) = O\left(\frac{1+y}{|\log b_1|}(1+|\log(y\sqrt{b_1})|)\right),$$

which, together with the bounds (2-40), (2-46) for $y \geq 3B_0$, yields the bound, for $y \geq 2$,

$$|\Theta_1(y)| \lesssim y g_2(b_1, y) + O\left(\frac{|\log y|^2}{y}\right). \tag{2-48}$$

Now, from (2-21), (2-37), we compute

$$A\Sigma_{b_1} = \frac{1}{y\Lambda Q}\int_0^y \Lambda Q[-c_{b_1}\chi_{B_0/4}\Lambda Q + d_{b_1}H[(1-\chi_{B_0})\Lambda Q]]x\,dx,$$

and from (2-44) we estimate, for $y \leq 3B_0$,

$$A\Sigma_{b_1} = -\frac{4}{y\Lambda Q} + \frac{c_{b_1}}{y\Lambda Q}\int_y^{B_0}(\Lambda Q)^2 x\,dx + O\left(\frac{d_{b_1}}{B_0^2}\mathbf{1}_{B_0\leq y\leq 3B_0}\right) = -2 + O(g_1(b_1, y)) \tag{2-49}$$

and, for $y \geq 3B_0$,

$$A\Sigma_{b_1} = -\frac{4}{y\Lambda Q} = -2 + O\left(\frac{1}{y^2}\right). \tag{2-50}$$

Moreover, a simple rescaling argument yields the formula

$$A(\Lambda u) = Au + \Lambda Au - \frac{\Lambda Z}{y}u$$

and thus, using (2-40), (2-5),

$$A(\Lambda T_1 - T_1) = \Lambda A T_1 - \frac{\Lambda Z}{y}T_1 = \Lambda A T_1 + O\left(\frac{\log y}{y^2}\right).$$

Now, from (2-21), (2-3), we estimate

$$AT_1 = -\left[\frac{1}{y\Lambda Q}\int_0^y (\Lambda Q)^2 x\,dx\right] = -2\log y + O\left(\frac{\log y}{y^2}\right),$$

and, similarly,

$$\Lambda A T_1 = -2 + O\left(\frac{\log y}{y^2}\right),$$

from which

$$A(\Lambda T_1 - T_1) = -2 + O\left(\frac{\log y}{y^2}\right). \tag{2-51}$$

We thus conclude from (2-48), (2-49), (2-50), (2-51) that

$$|A\Theta_1| \lesssim g_1(b_1, y) + O\left(\frac{\log y}{y^2}\right).$$

We now turn to the control of $H\Theta_1$. First, from a simple rescaling argument, we compute

$$H(\Lambda u) = 2Hu + \Lambda Hu - \frac{\Lambda V}{y^2}u, \tag{2-52}$$

which implies

$$H(\Lambda T_1 - T_1) = -\Lambda Q - \Lambda^2 Q + O\left(\frac{\log y}{y^3}\right) = O\left(\frac{\log y}{y^3}\right).$$

Hence, according to (2-24), we get the desired cancellation

$$|H\Theta_1| \lesssim |H(\Lambda T_1 - T_1)| + |H\Sigma_{b_1}| \lesssim \frac{1}{(1+y)|\log b_1|}\mathbf{1}_{y\le 3B_0} + O\left(\frac{\log y}{y^3}\right).$$

The control of higher order suitable derivatives in y now follows by iteration using (2-3), (2-6). Hence Θ_1 satisfies the bound (2-29) with $p_2 = 1$, $l = 0$.

We now take derivatives in b_1, in which case from (2-42), for $l \ge 1$,

$$\frac{\partial^l \Theta_1}{\partial^l b_1} = -\frac{\partial^l \Sigma_{b_1}}{\partial b_1^l} = \Gamma(y)\int_0^y \frac{\partial^l}{\partial b_1^l}\{c_{b_1}\chi_{B_0/4}\}(\Lambda Q)^2 x\,dx$$
$$- \Lambda Q(y)\int_0^y \frac{\partial^l}{\partial b_1^l}\{c_{b_1}\chi_{B_0/4}\}\Gamma\Lambda Q x\,dx + \frac{\partial^l}{\partial b_1^l}\{d_{b_1}(1-\chi_{B_0})\}\Lambda Q(y),$$

and from (2-43),

$$\frac{\partial^l \Theta_1}{\partial^l b_1} = -\frac{\partial^l \Sigma_{b_1}}{\partial b_1^l}(y) = 0 \quad \text{for } y \ge 3B_0.$$

From (1-14), we estimate by brute force

$$\left|\frac{\partial^l \chi_{B_0}}{\partial b_1^l}\right| \lesssim \frac{\mathbf{1}_{B_0\le y\le 2B_0}}{b_1^l}$$

and thus, from the Leibniz rule and (2-45), for $y \le 3B_0$, we obtain

$$\left|\frac{\partial^l \Theta_1}{\partial b_1^l}\right| \lesssim \frac{y}{b_1^l|\log b_1|^2}(1+|\log y|) + \left[\sum_{k=1}^{l}\frac{1}{b_1^{l-k}b_1^k|\log b_1|^2}\right] y\mathbf{1}_{B_0/2\le y\le 3B_0}$$
$$+\left[\sum_{k=0}^{l}\frac{1}{b_1^{l-k}b_1^{k+1}|\log b_1|}\right]\frac{\mathbf{1}_{B_0/2\le y\le 3B_0}}{y}$$
$$\lesssim \frac{y(1+|\log y|)}{b_1^l|\log b_1|^2} \lesssim \frac{y\tilde{g}_1}{b_1^l|\log b_1|}.$$

The control of higher suitable derivatives $(\partial^l \mathcal{A}^k \Theta_1/(\partial b_1^l))_{l,k\geq 1}$ follows similarly using the explicit formula (2-37). This concludes the proof of the estimate (2-30) with $p_2 = 1$, and thus Θ_1 is b_1-admissible of degree $(1, 1)$.

Step 3: $i \to i+1$. We assume the claim for Θ_i and prove it for Θ_{i+1}. From (2-23), (2-39), (2-52),

$$\begin{aligned} H\Theta_{i+1} &= H(\Lambda T_{i+1}) - (2i+1)HT_{i+1} - (-1)^i H^{-i+1}\Sigma_{b_1} \\ &= \Lambda H T_{i+1} - (2i-1)HT_{i+1} + (-1)^{i+1}H^{-i+1}\Sigma_{b_1} - \frac{\Lambda V}{y^2}T_{i+1} \\ &= -[(\Lambda T_i - (2i-1)T_i - (-1)^{i+1}H^{-i+1}\Sigma_{b_1}] - \frac{\Lambda V}{y^2}T_{i+1} \\ &= -\Theta_i - \frac{\Lambda V}{y^2}T_{i+1}. \end{aligned}$$

The induction hypothesis ensures that Θ_i is b_1-admissible of order (i, i). Moreover, near the origin, T_{i+1} is from Lemma 2.3 of degree $i+1$ and hence the development (2-6) ensures that $(\Lambda V/y^2)T_{i+1}$ is of degree $i+1$ near the origin. For $y \geq 1$, (2-6) ensures the improved bound

$$\left|\frac{\partial^p}{\partial y^p}\left(\frac{\Lambda V}{y^2}\right)\right| \lesssim \frac{1}{y^{p+4}}, \quad p \geq 0,$$

and since T_{i+1} is of degree $i+1$, we obtain from the Leibniz rule the rough bound, for all $k \geq 0$,

$$\left|\mathcal{A}^k\left[\frac{\Lambda V}{y^2}T_{i+1}\right]\right| \lesssim \sum_{p=0}^{k} \frac{1}{y^{k-p+4}} y^{2(i+1)-p-1}|\log y|^{c_i} \lesssim y^{2i-k-3}|\log y|^{c_i}.$$

Hence $(\Lambda V/y^2)T_{i+1}$, which is independent of b_1, satisfies (2-29) and is b_1-admissible of degree (i, i). We conclude from Lemma 2.6 that Θ_{i+1} is admissible of order $(i+1, i+1)$. □

Sobolev bounds on b_1-admissible functions. The property of b_1-admissibility leads to simple Sobolev bounds with sharp logarithmic gains. We let B_1 be given by (1-14).

Lemma 2.8 (estimate of b_1-admissible function). *Let $i \geq 1$ and f be a b_1-admissible function of degree (i, i). Then*

$$\int_{y\leq 2B_1} |H^k f|^2 \lesssim \frac{|\log b_1|^{4(i-k-1)}}{b_1^{2(i-k)}|\log b_1|^2} \quad \text{for } 0 \leq k \leq i-1, \tag{2-53}$$

$$\int_{y\leq 2B_1} |H^k f|^2 \lesssim 1 \quad \text{for } k \geq i, \tag{2-54}$$

and

$$\int_{y\leq 2B_1} \frac{1+|\log y|^2}{1+y^4}|H^k f|^2 + \int_{y\leq 2B_1} \frac{1+|\log y|^2}{1+y^2}|AH^k f|^2 \lesssim |\log b_1|^3 \quad \text{for } k \geq i-1. \tag{2-55}$$

Remark 2.9. *The boundedness of the Sobolev norm (2-54) in the borderline case $k = i$ is a consequence of the definition (2-24). Indeed,*

$$\int_{y\le 3B_0}\left|\frac{1+|\log\sqrt{b_1}y|}{(1+y)|\log b_1|}\right|^2 \sim |\log b_1|,$$

but

$$\int_{y\le 3B_0}\left|\frac{1}{(1+y)|\log b_1|}\right|^2 \lesssim 1. \tag{2-56}$$

Proof of Lemma 2.8. Let $k \ge 0$. Near the origin, the cancellation $A(y) = y^2 + O(y)$ and the Taylor expansion (2-27) ensure that $H^k f$ is bounded uniformly in $y \le 1$, $|b_1| \le \frac{1}{2}$. For $y \ge 1$, from (2-29) we estimate

$$\int_{y\le 2B_1}|H^k f|^2 = \int |f_{2k}|^2 \lesssim \int_{3B_0\le y\le 2B_1}|F_{i,2k,0}(b_1)y^{2i-2k-3}\mathbf{1}_{y\ge 3B_0}|^2 + \int_{1\le y\le 2B_1}|y^{2i-2k-1}g_{2i-2k}(b,y) + y^{2i-2k-3}|\log y|^{c_i}|^2.$$

For $k \ge i$, $F_{i,2k,0} = 0$ and from (2-24) we estimate (2-56)

$$\int_{y\le 2B_1}|H^k f|^2 \lesssim 1 + \int_{1\le y\le 2B_1}\left|y^{2(i-k)-1}\frac{\mathbf{1}_{y\le 3B_0}}{|\log b_1|} + y^{2i-2k-3}|\log y|^{c_i}\right|^2 \lesssim 1.$$

For $k \le i-1$, the growth can be controlled in a sharp way. Indeed, using $F_{i,2k,0} = 0$ for $k = i-1$ precisely to avoid an additional logarithmic error, we estimate

$$\begin{aligned}\int_{y\le 2B_1}|H^k f|^2 &\lesssim \frac{B_1^{4i-4k-4}}{b_1^2|\log b_1|^2} + \frac{1}{|\log b_1|^2}\int_{y\le 3B_0} y^{4(i-k)-2}(1+|\log\sqrt{b_1}y|^2) + B_1^{4(i-k)-4}|\log b_1|^{2c_{p_2}+1}\\ &\lesssim 1 + \frac{B_0^{4(i-k)}}{|\log b_1|^2} + \frac{|\log b_1|^{4(i-k-1)}}{b_1^{2(i-k)}|\log b_1|^2} \lesssim \frac{|\log b_1|^{4(i-k-1)}}{b_1^{2(i-k)}|\log b_1|^2}.\end{aligned}$$

Finally, for $k \ge i-1$, using the rough bound (2-32), we estimate

$$\begin{aligned}\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^4}|H^k f|^2 + \int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^2}|AH^k f|^2 &\lesssim \int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^4}|(1+y)^{2i-2k-1}|^2 + \int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^2}|(1+y)^{2i-2(k+1)-1}|^2\\ &\lesssim |\log b_1|^3.\end{aligned}$$

□

Slowly modulated blow-up profiles. In this section we construct the approximate modulated blow-up profile. Let us start by introducing the notion of homogeneous admissible functions.

Definition 2.10 (homogeneous functions). Given parameters $b=(b_k)_{1\le k\le L}$ and $(p_1,p_2,p_3)\in\mathbb{N}\times\mathbb{Z}\times\mathbb{N}$, we say a function $S(b,y)$ is homogeneous of degree (p_1,p_2,p_3) if it is of the form

$$S(b,y)=\sum_{J=(j_1,\dots,j_L),\,|J|_2=p_3}\left[c_J\left(\prod_{k=1}^{L}b_k^{j_k}\right)\widetilde{S}_J(b_1,y)\right],$$

where

$$J=(j_1,\dots,j_L)\in\mathbb{Z}\times\mathbb{N}^{L-1},\quad |J|_2=\sum_{k=1}^{L}kj_k,$$

and for some b_1-admissible profiles $\widetilde{S}_J$ of degree (p_1,p_2) in the sense of Definition 2.4. We note that

$$\deg(S)=(p_1,p_2,p_3).$$

Remark 2.11. *We allow for negative powers of b_1 only in the above definition. This ensures from Lemma 2.6 that the space of homogeneous functions of a given degree is stable by application of the operator $b_1\partial/(\partial b_1)$. It is also stable by multiplication by c_{b_1} from (2-44), (2-45).*

We may now proceed to the construction of the slowly modulated blow-up profiles.

Proposition 2.12 (construction of the approximate profile). *Let $M>0$ be a large enough universal constant. Then there exists a small enough universal constant $b^*(M)>0$ such that the following holds true. Let there be a $\mathcal{C}^1$ map*

$$b=(b_k)_{1\le k\le L}:[s_0,s_1]\mapsto(-b^*(M),b^*(M))^L$$

with a priori bounds on $[s_0,s_1]$

$$0<b_1<b^*(M),\quad |b_k|\lesssim b_1^k\quad \text{for } 2\le k\le L. \tag{2-57}$$

Let B_1 be given by (1-14) and $(T_i)_{1\le i\le L}$ be given by (2-23). Then there exist homogeneous profiles

$$\begin{cases} S_i(b,y), & 2\le i\le L+2,\\ S_1=0 \end{cases}$$

with

$$\begin{cases} \deg(S_i)=(i,i,i),\\ \dfrac{\partial S_i}{\partial b_j}=0 \qquad\qquad \text{for } 2\le i\le j\le L \end{cases} \tag{2-58}$$

such that

$$Q_{b(s)}(y)=Q(y)+\alpha_{b(s)}(y),\quad \alpha_b(y)=\sum_{i=1}^{L}b_iT_i(y)+\sum_{i=2}^{L+2}S_i(y) \tag{2-59}$$

generates an approximate solution to the renormalized flow

$$\partial_s Q_b-\Delta Q_b+b_1\Lambda Q_b+\frac{f(Q_b)}{y^2}=\Psi_b+\mathrm{Mod}(t) \tag{2-60}$$

with

$$\mathrm{Mod}(t) = \sum_{i=1}^{L}[(b_i)_s + (2i-1+c_{b_1})b_1 b_i - b_{i+1}]\Big[T_i + \sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}\Big]. \tag{2-61}$$

Here we used the convention

$$b_{L+1} = 0, \quad T_0 = \Lambda Q,$$

and Ψ_b *satisfies*

(i) *the global weighted bounds*

$$\text{for all } 1 \le k \le L, \quad \int_{y\le 2B_1} |H^k \Psi_b|^2 \lesssim b_1^{2k+2}|\log b_1|^C, \tag{2-62}$$

$$\int_{y\le 2B_1} \frac{1+|\log y|^2}{1+y^4}|H^L\Psi_b|^2 + \int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^2}|AH^L\Psi_b|^2 \lesssim \frac{b_1^{2L+3}}{|\log b_1|^2}, \tag{2-63}$$

$$\int_{y\le 2B_1} |H^{L+1}\Psi_b|^2 \lesssim \frac{b_1^{2L+4}}{|\log b_1|^2}; \tag{2-64}$$

and

(ii) $$\text{for all } 0 \le k \le L+1, \quad \int_{y\le 2M} |H^k\Psi_b|^2 \lesssim M^C b_1^{2L+6} \tag{2-65}$$

for some universal constant $C = C(L) > 0$ (*improved local control*).

Proof of Proposition 2.12. *Step* 1: *Computation of the error*. From (2-59), (2-60) we compute

$$\partial_s Q_b - \Delta Q_b + b_1 \Lambda Q_b + \frac{f(Q_b)}{y^2} = A_1 + A_2$$

with

$$A_1 = b_1\Lambda Q + \sum_{i=1}^{L}[(b_i)_s T_i + b_i H T_i + b_1 b_i \Lambda T_i] + \sum_{i=2}^{L+2}[\partial_s S_i + H S_i + b_1 \Lambda S_i],$$

$$A_2 = \frac{1}{y^2}[f(Q+\alpha_b) - f(Q) - f'(Q)\alpha_b].$$

Let us rearrange the first sum using the definition (2-23):

$$\begin{aligned} A_1 &= b_1\Lambda Q + \partial_s S_{L+2} + b_1\Lambda S_{L+2} + \sum_{i=1}^{L}[(b_i)_s T_i - b_i T_{i-1} + b_1 b_i \Lambda T_i] + \sum_{i=2}^{L+1}[\partial_s S_i + b_1 \Lambda S_i] + \sum_{i=1}^{L+1} H S_{i+1} \\ &= [\partial_s S_{L+2} + b_1 \Lambda S_{L+2}] + [H S_{L+2} + \partial_s S_{L+1} + b_1 \Lambda S_{L+1}] + \sum_{i=1}^{L}[(b_i)_s + (2i-1+c_{b_1})b_1 b_i - b_{i+1}]T_i \\ &\quad + \sum_{i=1}^{L}[H S_{i+1} + \partial_s S_i + b_1 b_i(\Lambda T_i - (2i-1+c_{b_1})T_i) + b_1 \Lambda S_i], \end{aligned}$$

where c_{b_1} is given by (2-38). We now treat the time dependence using the anticipated approximate

modulation equation

$$\partial_s S_i = \sum_{j=1}^{L} (b_j)_s \frac{\partial S_i}{\partial b_j} = \sum_{j=1}^{L} ((b_j)_s + (2j-1+c_{b_1})b_1 b_j - b_{j+1}) \frac{\partial S_i}{\partial b_j} - \sum_{j=1}^{L} ((2j-1+c_{b_1})b_1 b_j - b_{j+1}) \frac{\partial S_i}{\partial b_j},$$

and thus, using (2-58),

$$\begin{aligned} A_1 = & \left\{ b_1 \Lambda S_{L+2} - \sum_{i=1}^{L} ((2i-1+c_{b_1})b_1 b_i - b_{i+1}) \frac{\partial S_{L+2}}{\partial b_i} \right\} \\ & + \left\{ H S_{L+2} + b_1 \Lambda S_{L+1} - \sum_{i=1}^{L} ((2i-1+c_{b_1})b_1 b_i - b_{i+1}) \frac{\partial S_{L+1}}{\partial b_i} \right\} \\ & + \sum_{i=1}^{L} \left[H S_{i+1} + b_1 b_i (\Lambda T_i - (2i-1+c_{b_1}) T_i) + b_1 \Lambda S_i - \sum_{j=1}^{i-1} ((2j-1+c_{b_1})b_1 b_j - b_{j+1}) \frac{\partial S_i}{\partial b_j} \right] \\ & + \sum_{i=1}^{L} [(b_i)_s + (2i-1+c_{b_1})b_1 b_i - b_{i+1}] \left[T_i + \sum_{j=i+1}^{L+2} \frac{\partial S_j}{\partial b_i} \right]. \end{aligned}$$

We now expand A_2 using a Taylor expansion:

$$A_2 = \frac{1}{y^2} \left\{ \sum_{j=2}^{L+2} \frac{f^{(j)}(Q)}{j!} \alpha_b^j + R_2 \right\}$$

with

$$R_2 = \frac{\alpha_b^{L+3}}{(L+2)!} \int_0^1 (1-\tau)^{L+2} f^{(L+3)}(Q + \tau \alpha_b)\, d\tau. \tag{2-66}$$

Using the notation (1-16) for the $2L+1$ uplet $J = (i_1, \dots, i_L, j_2, \dots, j_{L+2}) \in \mathbb{N}^{2L+1}$, we sort the Taylor polynomial[8]

$$|J|_1 = \sum_{k=1}^{L} i_k + \sum_{k=2}^{L+2} j_k, \quad |J|_2 = \sum_{k=1}^{L} k i_k + \sum_{k=2}^{L+2} k j_k,$$

and thus

$$\sum_{j=2}^{L+2} \frac{f^{(j)}(Q)}{j!} \alpha_b^j = \sum_{j=2}^{L+2} \frac{f^{(j)}(Q)}{j!} \sum_{|J|_1 = j} c_J \prod_{k=1}^{L} b_k^{i_k} T_k^{i_k} \prod_{k=2}^{L+2} S_k^{j_k} = \sum_{i=2}^{L+2} P_i + R_1,$$

where

$$P_i = \sum_{j=2}^{L+2} \frac{f^{(j)}(Q)}{j!} \sum_{|J|_1 = j, |J|_2 = i} c_J \prod_{k=1}^{L} b_k^{i_k} T_k^{i_k} \prod_{k=2}^{L+2} S_k^{j_k}, \tag{2-67}$$

$$R_1 = \sum_{j=2}^{L+2} \frac{f^{(j)}(Q)}{j!} \sum_{|J|_1 = j, |J|_2 \geq L+3} c_J \prod_{k=1}^{L} b_k^{i_k} T_k^{i_k} \prod_{k=2}^{L+2} S_k^{j_k}. \tag{2-68}$$

[8]Remember that b_i is order b_1^i from (2-57).

We finally use the definitions (2-23), (2-39), (2-61) to rewrite

$$\Lambda T_i - (2i-1+c_{b_1})T_i = \Theta_i + (-1)^{i+1}H^{-i+1}\Sigma_{b_1} - c_{b_1}T_i = \Theta_i + (-1)^{i+1}H^{-i+1}(\Sigma_{b_1} - c_{b_1}T_1),$$

which, together with (2-60), yields the following expression for the error:

$$\begin{aligned}\Psi_b &= \sum_{i=1}^{L}(-1)^{i+1}b_1 b_i H^{-i+1}(\Sigma_{b_1} - c_{b_1}T_1)\\ &\quad + \left\{b_1\Lambda S_{L+2} - \sum_{i=1}^{L}((2i-1+c_{b_1})b_1 b_i - b_{i+1})\frac{\partial S_{L+2}}{\partial b_i} + \frac{1}{y^2}[R_1+R_2]\right\}\\ &\quad + \left\{HS_{L+2} + b_1\Lambda S_{L+1} + \frac{P_{L+2}}{y^2} - \sum_{i=1}^{L}((2i-1+c_{b_1})b_1 b_i - b_{i+1})\frac{\partial S_{L+1}}{\partial b_i}\right\}\\ &\quad + \sum_{i=1}^{L}\left[HS_{i+1} + b_1 b_i\Theta_i + b_1\Lambda S_i + \frac{P_{i+1}}{y^2} - \sum_{j=1}^{i-1}((2j-1+c_{b_1})b_1 b_j - b_{j+1})\frac{\partial S_i}{\partial b_j}\right]. \end{aligned} \tag{2-69}$$

We now construct iteratively the sequence of profiles $(S_i)_{1\le i\le L+2}$ through the scheme

$$\begin{cases} S_1 = 0,\\ S_i = -H^{-1}\Phi_i, \quad 2\le i\le L+2, \end{cases} \tag{2-70}$$

where, for $1\le i\le L$,

$$\Phi_{i+1} = b_1 b_i\Theta_i + b_1\Lambda S_i + \frac{P_{i+1}}{y^2} - \sum_{j=1}^{i-1}((2j-1+c_{b_1})b_1 b_j - b_{j+1})\frac{\partial S_i}{\partial b_j}, \tag{2-71}$$

$$\Phi_{L+2} = b_1\Lambda S_{L+1} + \frac{P_{L+2}}{y^2} - \sum_{i=1}^{L}((2i-1+c_{b_1})b_1 b_i - b_{i+1})\frac{\partial S_{L+1}}{\partial b_i}. \tag{2-72}$$

Step 2: *Control of* Φ_i, S_i. We claim by induction on i that Φ_i is homogeneous with

$$\deg(\Phi_i) = (i-1, i-1, i) \quad \text{for } 2\le i\le L+2 \tag{2-73}$$

and

$$\frac{\partial\Phi_i}{\partial b_j} = 0 \quad \text{for } 2\le i\le j\le L+2. \tag{2-74}$$

This implies from Lemma 2.6 that S_i given by (2-70) is homogeneous and satisfies (2-58) for $2\le i\le L+2$.

Case 1: $i=1$. We compute explicitly

$$\Phi_2 = b_1^2\Theta_1 + b_1^2\frac{f''(Q)}{2y^2}T_1^2,$$

which satisfies (2-74). Recall from (1-3) that $f = gg'$ is odd and π periodic so that the expansions (2-2), (2-3) yield, at the origin,

$$\frac{f^{(j)}(Q)}{y^2} = \begin{cases} \sum_{k=-1}^{p} y^{2k+1} + O(y^{2p+3}) & \text{for j even},\\ \sum_{k=-1}^{p} y^{2k} + O(y^{2p+2}) & \text{for j odd} \end{cases} \tag{2-75}$$

and, at infinity,

$$\frac{f^{(j)}(Q)}{y^2} = \begin{cases} \sum_{k=1}^{p} y^{-2k-1} + O(y^{-2p-3}) & \text{for j even,} \\ \sum_{k=1}^{p} y^{-2k} + O(y^{-2p-2}) & \text{for j odd.} \end{cases} \tag{2-76}$$

From Lemmas 2.3 and 2.7, T_1 and Θ_1 are respectively admissible and b_1-admissible of order $(1, 1)$. In particular, we have the Taylor expansion near the origin

$$\frac{f''(Q)}{2y^2} T_1^2 = \sum_{k=1}^{p} c_k y^{2k+1} + O(y^{2p+3}), \quad p \geq 1,$$

and the bound at infinity

$$\left| \mathscr{A}^k \left(\frac{f''(Q)}{2y^2} T_1^2 \right) \right| \lesssim \frac{1}{y^{3+k}} y^2 |\log y|^2 \lesssim y^{2-k-3} |\log y|^2, \quad k \geq 0.$$

Hence $(f''(Q)/2y^2)T_1^2$ is b_1-admissible of degree $(1, 1)$. We conclude that Φ_2 is homogeneous with

$$\deg(\Phi_2) = (1, 1, 2).$$

Case 2: $i \to i+1$. We estimate all terms in (2-71). Equation (2-74) holds by direct inspection. From Lemma 2.7, $b_1 b_i \Theta_i$ is homogeneous of degree $(i, i, i+1)$. From Lemma 2.6, $b_1 \Lambda S_i$ is homogeneous of degree $(i, i, i+1)$ by induction. For $j \geq 2$, by definition and induction we have that

$$((2j - 1 + c_{b_1}) b_1 b_j - b_{j+1}) \frac{\partial S_i}{\partial b_j}$$

is homogeneous of degree $(i, i, i+1)$. For $j = 1$, we rewrite the term

$$((1 + c_{b_1}) b_1^2 - b_2) \frac{\partial S_i}{\partial b_1} = \left((1 + c_{b_1}) b_1 - \frac{b_2}{b_1} \right) \left(b_1 \frac{\partial S_i}{\partial b_1} \right)$$

and, recalling Remark 2.11, conclude that this term is also homogeneous of degree $(i, i, i+1)$. It thus remains to estimate the nonlinear term P_{i+1}/y^2 in (2-71), which, from (2-67), is a linear combination of monomials of the form[9]

$$M_J(y) = \frac{f^{(j)}(Q)}{y^2} \prod_{k=1}^{i} b_k^{i_k} T_k^{i_k} \prod_{k=2}^{i} S_k^{j_k}, \quad |J|_1 = j, \ |J|_2 = i+1, \ 2 \leq j \leq i+1.$$

Using (2-75), (2-76), we conclude that M_J is admissible with the following development at the origin: for $j = 2l$,

$$\begin{aligned} M_J(y) &= y^{-1} y^{\sum_{k \geq 1} i_k (2k+1) + j_k (2k+1)} (c_0 + c_2 y^2 + \cdots + c_p y^{2p} + o(y^{2p+1})) \\ &= y^{2|J|_2 + j - 1} (c_0 + c_2 y^2 + \cdots + c_p y^{2p} + o(y^{2p+1})) \\ &= y^{2(i+l)+1} (c_0 + c_2 y^2 + \cdots + c_p y^{2p} + o(y^{2p+1})), \end{aligned}$$

[9]Observe that terms involving $k \geq i+1$ are indeed forbidden in the last product from the constraint $|J|_1 \geq 2$, $|J|_2 = i+1$.

and, for $j = 2l+1$,

$$\begin{aligned} M_J(y) &= y^{-2} y^{\sum_{k\geq 1} i_k(2k+1)+j_k(2k+1)}(c_0 + c_2 y^2 + \ldots + c_p y^{2p} + o(y^{2p+1})) \\ &= y^{2|J|_2+j-2}(c_0 + c_2 y^2 + \ldots + c_p y^{2p} + o(y^{2p+1})) \\ &= y^{2(i+l)+1}(c_0 + c_2 y^2 + \ldots + c_p y^{2p} + o(y^{2p+1})). \end{aligned}$$

Now $j \geq 2$ ensures $l \geq 1$, and hence M_J admits a Taylor expansion (2-27) at the origin with $p_1 = i+1$. For $y \geq 1$, the rough bound (2-32) and (2-18) imply

$$|S_j| \lesssim b_1^j y^{2j-1}, \quad |T_j| \lesssim y^{2j-1}|\log y|^C,$$

which, together with (2-76), yields the control

$$M_J(y) \lesssim |\log y|^C \begin{cases} y^{2|J|_2-j-3} = y^{2(i-l)-1} & \text{for } j = 2l \geq 2, \\ y^{2|J|_2-j-2} = y^{2(i-l)-1} & \text{for } j = 2l+1 \geq 3 \end{cases} \lesssim y^{2i-3}|\log y|^C, \tag{2-77}$$

which is compatible with the degree i control at infinity (2-29). The control of further derivatives in (y, b_1) follows from (2-32) and the Leibniz rule. This concludes the proof of (2-73).

Step 3: *Estimate on the error*. From (2-69), we compute

$$\Psi_b = \Psi_b^{(0)} + \Psi_b^{(1)}, \tag{2-78}$$

$$\Psi_b^{(0)} = \sum_{i=1}^{L} (-1)^{i+1} b_1 b_i H^{-i+1} \widetilde{\Sigma}_{b_1} \quad \text{with } \widetilde{\Sigma}_{b_1} = \Sigma_{b_1} - c_{b_1} T_1, \tag{2-79}$$

$$\Psi_b^{(1)} = b_1 \Lambda S_{L+2} - \sum_{i=1}^{L} ((2i-1+c_{b_1}) b_1 b_i - b_{i+1}) \frac{\partial S_{L+2}}{\partial b_i} + \frac{1}{y^2}[R_1 + R_2]. \tag{2-80}$$

Estimates for $\Psi_b^{(0)}$. First observe from (2-43), (2-79) that

$$\operatorname{Supp} \widetilde{\Sigma}_{b_1} \subset \left\{ y \geq \frac{B_0}{4} \right\}. \tag{2-81}$$

We extract from (2-42) the rough bound for $k \geq 0$ and $B_0/4 \leq y \leq 2B_1$

$$|H^{-k} \widetilde{\Sigma}_{b_1}| \lesssim 1 + y^{2k+1}.$$

Thus

$$\int_{y \leq 2B_1} |H^{-k} \widetilde{\Sigma}_{b_1}|^2 \lesssim b_1^{-2k-2} |\log b_1|^C, \quad 0 \leq k \leq L.$$

On the other hand, from (2-37) and the cancellation $H \Lambda Q = 0$, we have

$$|H \widetilde{\Sigma}_{b_1}| \lesssim \frac{1}{|\log b_1|} \left(\frac{1}{1+y} \right) \mathbf{1}_{y \geq B_0/4}, \tag{2-82}$$

$$|H^k \widetilde{\Sigma}_{b_1}| \lesssim \frac{1}{B_0^{2(k-1)} |\log b_1|} \left(\frac{1}{1+y} \right) \mathbf{1}_{B_0 \leq y \leq 3B_0} \quad \text{for } k \geq 2. \tag{2-83}$$

This leads to the bound

$$\int_{y\le 2B_1} |H\widetilde{\Sigma}_{b_1}|^2 \lesssim \frac{1}{|\log b_1|}, \quad \int |H^k\widetilde{\Sigma}_{b_1}|^2 \lesssim \frac{b_1^{2k-2}}{|\log b_1|^2} \quad \text{for } k\ge 2.$$

Thus from (2-57), for $0\le k\le L$, we estimate

$$\int_{y\le 2B_1} |H^k\Psi_b^{(0)}|^2 \lesssim |\log b_1|^C \sum_{i=1}^{L} b_1^{2+2i} b_1^{2(k-i+1)-2} \lesssim b_1^{2k+2}|\log b_1|^C$$

and the sharp logarithmic gain

$$\int |H^{L+1}\Psi_b^{(0)}|^2 \lesssim \sum_{i=1}^{L} b_1^{2+2i}\|H^{L+2-i}\widetilde{\Sigma}_{b_1}\|_{L^2}^2 \lesssim \frac{1}{|\log b_1|^2}\sum_{i=1}^{L} b_1^{2+2i} b_1^{2(L+1-i+1)-2} \lesssim \frac{b_1^{2L+4}}{|\log b_1|^2}.$$

Similarly, using (2-82), (2-83),

$$\begin{aligned}\int_{y\le 2B_1} \frac{1+|\log y|^2}{1+y^4}|H^L\Psi_b^{(0)}|^2 &\lesssim \sum_{i=1}^{L} b_1^{2+2i}\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^4}|H^{L-i+1}\widetilde{\Sigma}_{b_1}|^2\\ &\lesssim \sum_{i=1}^{L} b_1^{2+2i}\int_{y\ge B_0/4}\frac{1+|\log y|^2}{1+y^4}\frac{b_1^{2(L-i+1-1)}}{|\log b_1|^2(1+y^2)} \lesssim b_1^{2L+4},\end{aligned}$$

and

$$\begin{aligned}\int_{y\le 2B_1} \frac{1+|\log y|^2}{1+y^2}|AH^L\Psi_b^{(0)}|^2 &\lesssim \sum_{i=1}^{L} b_1^{2+2i}\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^2}|AH^{L-i+1}\widetilde{\Sigma}_{b_1}|^2\\ &\lesssim \sum_{i=1}^{L} b_1^{2+2i} b_1^{2(L-i)}\int_{y\ge B_0/4}\frac{1+|\log y|^2}{y^4(1+y^2)}\\ &\lesssim b_1^{2L+4}|\log b_1|^C.\end{aligned}$$

Estimates for $\Psi_b^{(1)}$. By construction, S_{L+2} is homogeneous of degree $(L+2, L+2, L+2)$ and thus so is ΛS_{L+2}. We therefore estimate from (2-53), (2-57), for all $0\le k\le L+1$,

$$\int_{y\le 2B_1} |b_1 H^k\Lambda S_{L+2}|^2 \lesssim \frac{b_1^2 b_1^{2L+4}|\log b_1|^{4(L+2-k-1)}}{b_1^{2(L+2-k)}|\log b_1|^2} = \frac{b_1^{2k+2}}{|\log b_1|^2}|\log b_1|^{4(L+1-k)},$$

and, using the rough bound (2-32),

$$\begin{aligned}&\int_{y\le 2B_1}(1+|\log y|^2)\left[\frac{|b_1 H^L\Lambda S_{L+2}|^2}{1+y^4}+\frac{|b_1 AH^L\Lambda S_{L+2}|^2}{1+y^2}\right]\\ &\qquad\lesssim b_1^2 b_1^{2L+4}\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^4}(1+y^2)^{2(L+2)-1-2L} \lesssim b_1^{2L+4}|\log b_1|^C.\end{aligned}$$

We now turn to the control of R_1, which, from (2-68), is a linear combination of terms of the form

$$\widetilde{M}_J = \frac{f^{(j)}(Q)}{y^2}\prod_{k=1}^{L} b_k^{i_k} T_k^{i_k} \prod_{k=2}^{L+2} S_k^{j_k}, \quad |J|_1 = j,\ |J|_2 \geq L+3,\ 2 \leq j \leq L+2.$$

At the origin, the homogeneity of S_i and the admissibility of T_i ensure the bound, for $y \leq 1$,

$$|\widetilde{M}_J(y)| \lesssim b_1^{L+3} \begin{cases} y^{2|J|_2+j-1} = y^{2(|J|_2+l)-1} & \text{for } j = 2l, \\ y^{2|J|_2+j-2} = y^{2(|J|_2+l)-1} & \text{for } j = 2l+1 \end{cases} \lesssim b_1^{L+3} y^{2L+6},$$

and similarly for (2-77), for $1 \leq y \leq 2B_1$,

$$|\widetilde{M}_J(y)| \lesssim b_1^{|J|_2} |\log b_1|^C \begin{cases} y^{2|J|_2-j-3} = y^{2(|J|_2-l)-3} & \text{for } j = 2l \\ y^{2|J|_2-j-2} = y^{2(|J|_2-l)-3} & \text{for } j = 2l+1 \end{cases} \lesssim b_1^{|J|_2} y^{2|J|_2-5} |\log b_1|^C,$$

where we used $j \geq 2$, and similarly for higher derivatives. This ensures the control at the origin

$$|H^k \widetilde{M}_J(y)| \lesssim b_1^{L+3} \quad \text{for } 0 \leq k \leq L+1,\ y \leq 1$$

and, for $y \geq 1$,

$$|H^k \widetilde{M}_J(y)| \lesssim b_1^{|J|_2} y^{2(|J|_2-k)-5}, \quad 0 \leq k \leq L+1.$$

Thus, for all $0 \leq k \leq L+1$,

$$\begin{aligned} \int_{y \leq 2B_1} |H^k \widetilde{M}_J|^2 &\lesssim b_1^{2L+6} + b_1^{2|J|_2} |\log b_1|^C \int_{y \leq 2B_1} y^{4(|J|_2-k)-10} \lesssim b_1^{2L+6} + b_1^{2|J|_2} B_1^{4(|J|_2-k)-8} |\log b_1|^C \\ &\lesssim b_1^{2L+6} + b_1^{2k+4} |\log b_1|^C \lesssim b_1^{2k+3}. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{y \leq 2B_1} (1+|\log y|^2) \left[\frac{|H^L \widetilde{M}_J|^2}{1+y^4} + \frac{|AH^L \widetilde{M}_J|^2}{1+y^2} \right] &\lesssim b_1^{2L+6} + |\log b_1|^C \int_{1 \leq y \leq 2B_1} \frac{1+|\log y|^2}{1+y^4} |b_1^{|J|_2} y^{2(|J|_2-L)-5}|^2 \\ &\lesssim b_1^{2L+6} + b_1^{2|J|_2} B_1^{4(|J|_2-L)-12} |\log b_1|^C \lesssim b_1^{2L+6} |\log b_1|^C. \end{aligned}$$

It remains to estimate the R_2 term given by (2-66). Near the origin $y \leq 1$, by construction, we have $|\alpha_b| \lesssim b_1 y^3$, and thus, for $0 \leq k \leq L+1$, $y \leq 1$,

$$\left| H^k \left(\frac{R_2}{y^2} \right) \right| \lesssim b_1^{L+3} y^{3(L+3)-2-2k} \lesssim b_1^{L+3}.$$

For $y \geq 1$, we use the rough bound by construction, for $1 \leq y \leq 2B_1$,

$$|\alpha_b| \lesssim b_1 y |\log y|^C,$$

which yields the bound, for $0 \leq k \leq L+1$, $1 \leq y \leq 2B_1$,

$$\left| H^k \left(\frac{R_2}{y^2} \right) \right| \lesssim b_1^{L+3} |\log b_1|^C y^{L+3-2-2k},$$

from which, for $0 \le k \le L+1$,

$$\begin{aligned}\int_{y\le 2B_1} |H^k\Big(\frac{R_2}{y^2}\Big)|^2 &\lesssim b_1^{2L+6} + b_1^{2L+6}|\log b_1|^C \int_{1\le y\le 2B_1} y^{2L+2-4k} \\ &\lesssim \begin{cases} b_1^{2L+6} & \text{for } 2L+2-4k<-1, \\ b_1^{2L+6} B_1^{2L+4-4k}|\log b_1|^C = b_1^{2k+L+4}|\log b_1|^C \end{cases} \\ &\lesssim b_1^{2k+5}|\log b_1|^C.\end{aligned}$$

Similarly,

$$\int_{y\le 2B_1} (1+|\log y|^2)\Big[\frac{1}{1+y^4}|H^L\Big(\frac{R_2}{y^2}\Big)|^2 + \frac{1}{1+y^2}|AH^L\Big(\frac{R_2}{y^2}\Big)|^2\Big] \lesssim b_1^{2L+5}.$$

The collection of above estimates yields (2-62), (2-64).

Finally, the local control (2-65) is a simple consequence of the support localization (2-81) and the fact that $\Psi_b^{(1)}$ given by (2-80) satisfies by construction a bound on compact sets:

$$|H^k\Psi_b^{(1)}(y)| \lesssim M^C b_1^{L+3} \quad \text{for all } y \le 2M \ll B_0 \text{ and all } 0 \le k \le L+1.$$

This concludes the proof of Proposition 2.12. □

Localization of the profile. We now proceed to a simple localization procedure of the profile Q_b to avoid some irrelevant growth in the region $y \ge 2B_1$.

Proposition 2.13 (localization). *Under the assumptions of Proposition 2.12, assume the a priori bound*

$$|(b_1)_s| \lesssim b_1^2. \tag{2-84}$$

Consider the localized profile

$$\widetilde{Q}_{b(s)}(y) = Q(y) + \tilde{\alpha}_{b(s)}(y), \quad \tilde{\alpha}_b(y) = \sum_{i=1}^{L} b_i \widetilde{T}_i(y) + \sum_{i=2}^{L+2} \widetilde{S}_i(y), \tag{2-85}$$

with

$$\widetilde{T}_i = \chi_{B_1} T_i, \quad \widetilde{S}_i = \chi_{B_1} S_i. \tag{2-86}$$

Then

$$\begin{aligned}\partial_s \widetilde{Q}_b - \Delta \widetilde{Q}_b + b_1 \Lambda \widetilde{Q}_b + \frac{f(\widetilde{Q}_b)}{y^2} \\ = \widetilde{\Psi}_b + \sum_{i=1}^{L} [(b_i)_s + (2i-1+c_{b_1})b_1 b_i - b_{i+1}]\Big[\widetilde{T}_i + \chi_{B_1} \sum_{j=i+1}^{L+2} \frac{\partial S_j}{\partial b_i}\Big],\end{aligned} \tag{2-87}$$

where $\widetilde{\Psi}_b$ satisfies

(i) *the weighted bounds*

$$\text{for all } 1 \le k \le L, \quad \int |H^k \widetilde{\Psi}_b|^2 \lesssim b_1^{2k+2}|\log b_1|^C, \tag{2-88}$$

$$\int_{y\le 2B_1} \frac{1+|\log y|^2}{1+y^4}|H^L\Psi_b|^2 + \int \frac{1+|\log y|^2}{1+y^2}|AH^L\widetilde{\Psi}_b|^2 \lesssim \frac{b_1^{2L+3}}{|\log b_1|^2}, \tag{2-89}$$

$$\int |H^{L+1}\widetilde{\Psi}_b|^2 \lesssim \frac{b_1^{2L+4}}{|\log b_1|^2}, \tag{2-90}$$

and

$$\text{(ii)} \qquad \text{for all } 0 \le k \le L+1, \quad \int_{y\le 2M} |H^k \widetilde{\Psi}_b|^2 \lesssim M^C b_1^{2L+6} \tag{2-91}$$

for some universal constant $C = C(L) > 0$ (improved local control).

Proof of Proposition 2.13. From localization we compute

$$\begin{aligned}\partial_s \widetilde{Q}_b - \Delta \widetilde{Q}_b + b_1 \Lambda \widetilde{Q}_b + \frac{f(\widetilde{Q}_b)}{y^2} \\ = \chi_{B_1}\left\{\partial_s Q_b - \Delta Q_b + b_1 \Lambda Q_b + \frac{f(Q_b)}{y^2}\right\} + (\partial_s \chi_{B_1})\alpha_b - 2\partial_y \chi_{B_1}\partial_y \alpha_b - \alpha_b \Delta \chi_{B_1} + b_1 \alpha_b \Lambda \chi_{B_1} \\ + b_1(1-\chi_{B_1})\Lambda Q + \frac{1}{y^2}\{f(\widetilde{Q}_b) - f(Q) - \chi_{B_1}(f(Q_b) - f(Q))\}\end{aligned}$$

so that

$$\widetilde{\Psi}_b = \chi_{B_1}\Psi_b + \widetilde{\Psi}_b^{(0)}$$

with

$$\begin{aligned}\widetilde{\Psi}_b^{(0)} = \frac{1}{y^2}\{f(\widetilde{Q}_b) - f(Q) - \chi_{B_1}(f(Q_b) - f(Q))\} \\ + (\partial_s \chi_{B_1})\alpha_b - 2\partial_y \chi_{B_1}\partial_y \alpha_b - \alpha_b \Delta \chi_{B_1} + b_1 \alpha_b \Lambda \chi_{B_1} + b_1(1-\chi_{B_1})\Lambda Q. \end{aligned} \tag{2-92}$$

Note that all terms on the right hand side above are localized in $B_1 \le y \le 2B_1$ except the last one, for which $\mathrm{Supp}((1-\chi_{B_1})\Lambda Q) \subset \{y \ge B_1\}$. Hence (2-65) implies (2-91). The bounds (2-88), (2-89), (2-90) for $\chi_{B_1}\Psi_b$ follow verbatim as in the proof of (2-62), (2-63), (2-64).

To estimate the second error induced by localization in (2-92), first observe from (2-84) the bound

$$|\partial_s \chi_{B_1}| \lesssim \frac{|(b_1)_s|}{b_1}|y\chi'_{B_1}| \lesssim b_1 \mathbf{1}_{B_1 \le y \le 2B_1}.$$

Moreover, from the admissibility of T_i and the b_1-admissibility of S_i, T_i terms dominate for $y \sim B_1$ in α_b, and we estimate from (2-18), for all $k \ge 0$ and $B_1 \le y \le 2B_1$,

$$\left|\frac{\partial^k}{\partial y^k}\alpha_b\right| \lesssim \sum_{i=1}^{L} b_1^i y^{2i-k-1}(1 + |\log b_1|) \lesssim \frac{|\log b_1|}{B_1^{k+1}}. \tag{2-93}$$

This yields, for all $1 \le k \le L$,

$$\begin{aligned}\int \left| H^k\left((\partial_s \chi_{B_1})\alpha_b - 2\partial_y \chi_{B_1}\partial_y \alpha_b - \alpha_b \Delta \chi_{B_1} + b_1 \alpha_b \Lambda \chi_{B_1}\right)\right|^2 \\ \lesssim \int_{B_1 \le y \le 2B_1} \left|\frac{b_1|\log b_1|}{B_1^{2k+1}} + \frac{|\log b_1|}{B_1^{2k+1+2}}\right|^2 \lesssim \frac{b_1^2}{B_1^{4k}}|\log b_1|^C \\ \lesssim b_1^{2k+2}|\log b_1|^C \end{aligned} \tag{2-94}$$

and

$$\int \left| H^{L+1}\left((\partial_s \chi_{B_1})\alpha_b - 2\partial_y \chi_{B_1}\partial_y \alpha_b - \alpha_b \Delta \chi_{B_1} + b_1 \alpha_b \Lambda \chi_{B_1} \right) \right|^2 \lesssim \int_{B_1 \le y \le 2B_1} \left| \frac{b_1 |\log b_1|}{B_1^{2(L+1)+1}} + \frac{|\log b_1|}{B_1^{2(L+1)+1+2}} \right|^2 \lesssim \frac{b_1^{2L+4}}{|\log b_1|^2}. \quad (2\text{-}95)$$

We next estimate by brute force

$$\left| \frac{d^k}{dy^k}[(1-\chi_{B_1})\Lambda Q] \right| \lesssim \frac{1}{y^{k+1}} \mathbf{1}_{y \ge B_1},$$

from which, for all $1 \le k \le L+1$,

$$\int |H^k(b_1(1-\chi_{B_1})\Lambda Q)|^2 \lesssim b_1^2 \int_{B_1 \le y \le 2B_1} \frac{1}{y^{4k+2}} \lesssim \frac{b_1^{2k+2}}{|\log b_1|^{4k}}.$$

It remains to estimate the nonlinear term, for which, using (2-93) and $|f'| \lesssim 1$, we estimate

$$\left| \frac{\partial^k}{\partial y^k} \left\{ \frac{1}{y^2}[f(\widetilde{Q}_b) - f(Q) - \chi_{B_1}(f(Q_b) - f(Q))] \right\} \right| \lesssim \frac{|\log b_1|}{B_1^{k+1}}.$$

The corresponding terms are estimated as in (2-94), (2-95). □

Study of the dynamical system for $b = (b_1, \dots, b_L)$. The essence of the construction of the Q_b profile is to generate according to (2-61) the finite dimensional dynamical system (1-25) for $b = (b_1, \dots, b_L)$:

$$(b_k)_s + \left(2k - 1 + \frac{2}{\log s}\right) b_1 b_k - b_{k+1} = 0, \quad 1 \le k \le L,\ b_{L+1} \equiv 0. \quad (2\text{-}96)$$

We show in this section that (2-96) admits exceptional solutions, and that the linearized operator close to these solutions is explicit.

Lemma 2.14 (approximate solution for the b system). *Let $L \ge 2$ and let $s_0 \gg 1$ be a large enough universal constant. We write the sequences*

$$\begin{cases} c_1 = \dfrac{L}{2L-1}, \\ c_{k+1} = -\dfrac{L-k}{2L-1} c_k, \quad 1 \le k \le L-1, \end{cases} \quad (2\text{-}97)$$

$$\begin{cases} d_1 = -\dfrac{2L}{(2L-1)^2}, \\ d_{k+1} = -\dfrac{L-k}{2L-1} d_k + \dfrac{4L(L-k)}{(2L-1)^2} c_k, \quad 1 \le k \le L-1. \end{cases} \quad (2\text{-}98)$$

Then the explicit choice

$$b_k^e(s) = \frac{c_k}{s^k} + \frac{d_k}{s^k \log s}, \quad 1 \le k \le L,\ b_{L+1}^e \equiv 0 \quad (2\text{-}99)$$

generates an approximate solution to (2-96) in the sense that

$$(b_k^e)_s + \left(2k-1+\frac{2}{\log s}\right)b_1^e b_k^e - b_{k+1}^e = O\left(\frac{1}{s^{k+1}(\log s)^2}\right), \quad 1 \le k \le L. \tag{2-100}$$

The proof of Lemma 2.14 is an explicit computation which is left to the reader. We now claim that this solution corresponds to a codimension $(L-1)$ exceptional manifold.

Lemma 2.15 (linearization). 1. Computation of the linearized system. *Let*

$$b_k(s) = b_k^e(s) + \frac{U_k(s)}{s^k(\log s)^{5/4}}, \quad 1 \le k \le L,\ b_{L+1} = U_{L+1} \equiv 0, \tag{2-101}$$

and note $U = (U_1, \dots, U_L)$. *Then*

$$\begin{aligned}(b_k)_s + \left(2k-1+\frac{2}{\log s}\right)b_1 b_k - b_{k+1} \\ = \frac{1}{s^{k+1}(\log s)^{5/4}}\left[s(U_k)_s - (A_L U)_k + O\left(\frac{1}{\sqrt{\log s}} + \frac{|U|+|U|^2}{\log s}\right)\right],\end{aligned} \tag{2-102}$$

where

$$A_L = (a_{i,j})_{1\le i,j,\le L} \quad \textit{with} \quad \begin{cases} a_{11} = -1/(2L-1), & \\ a_{i,i+1} = 1, & 1 \le i \le L-1, \\ a_{1,i} = -(2i-1)c_i, & 2 \le i \le L, \\ a_{i,i} = (L-i)/(2L-1), & 2 \le i \le L, \\ a_{i,j} = 0 & \textit{otherwise}. \end{cases} \tag{2-103}$$

2. Diagonalization of the linearized matrix. A_L *is diagonalizable*:

$$A_L = P_L^{-1} D_L P_L, \quad D_L = \operatorname{diag}\left\{-1, \frac{2}{2L-1}, \frac{3}{2L-1}, \dots, \frac{L}{2L-1}\right\}. \tag{2-104}$$

Proof of Lemma 2.15. *Step* 1: *Linearization*. A simple computation from (2-99) ensures

$$\begin{aligned}(b_k)_s + \left(2k-1+\frac{2}{\log s}\right)b_1 b_k - b_{k+1} \\ = \frac{1}{s^{k+1}(\log s)^{5/4}}\left[s(U_k)_s - kU_k + O\left(\frac{|U|}{\log s}\right)\right] + O\left(\frac{1}{s^{k+1}(\log s)^2}\right) \\ + \frac{1}{s^{k+1}(\log s)^{5/4}}\left[(2k-1)c_k U_1 + (2k-1)c_1 U_k - U_{k+1} + O\left(\frac{|U|+|U|^2}{\log s}\right)\right],\end{aligned}$$

and then the relation

$$(2k-1)c_1 - k = \frac{(2k-1)L}{2L-1} - k = -\frac{L-k}{2L-1}$$

ensures

$$\begin{aligned}(b_k)_s + \left(2k-1+\frac{2}{\log s}\right)b_1 b_k - b_{k+1} \\ = \frac{1}{s^{k+1}(\log s)^{5/4}}\left[s(U_k)_s + (2k-1)c_k U_1 - \frac{L-k}{2L-1}U_k - U_{k+1} + O\left(\frac{1}{\sqrt{\log s}} + \frac{|U|+|U|^2}{\log s}\right)\right],\end{aligned}$$

which is equivalent to (2-102), (2-103).

Step 2 Diagonalization. The proof follows by computing the characteristic polynomial. The cases $L = 2, 3$ are done by direct inspection. Let us assume $L \geq 4$. We compute

$$P_L(X) = \det(A_L - X\,\mathrm{Id})$$

by developing on the last row. This yields

$$P_L(x) = (-1)^{L+1}(-1)(2L-1)c_L$$
$$+(-X)\left\{(-1)^L(-1)(2L-3)c_{L-1}+\left(\frac{1}{2L-1}-X\right)\left[(-1)^{L-1}(-1)(2L-5)c_{L-2}+\left(\frac{2}{2L-1}-X\right)\cdots\right]\right\}.$$

We use the recurrence relation (2-97) to compute explicitly

$$(-1)^{L+1}(-1)(2L-1)c_L$$
$$+(-X)\left\{(-1)^L(-1)(2L-3)c_{L-1}+\left(\frac{1}{2L-1}-X\right)[(-1)^{L-1}(-1)(2L-5)c_{L-2}]\right\}$$
$$=(-1)^L\left\{(2L-3)c_{L-1}\left(X-\frac{1}{2L-3}\right)+(2L-5)c_{L-2}\left(X-\frac{1}{2L-1}\right)X\right\}.$$

We now compute from (2-97), for $1 \leq k \leq L-2$,

$$(2L-(2k+1))c_{L-k}\left(X-\frac{1}{2L-(2k+1)}\right)+(2L-(2k+3))c_{L-(k+1)}X\left(X-\frac{1}{2L-1}\right)$$
$$=(2L-(2k+3))c_{L-(k+1)}\left[X\left(X-\frac{1}{2L-1}\right)-\frac{2L-(2k+1)}{2L-(2k+3)}\frac{k+1}{2L-1}\left(X-\frac{1}{2L-(2k+1)}\right)\right]$$
$$=(2L-(2k+3))c_{L-(k+1)}\left(X-\frac{k+1}{2L-1}\right)\left(X-\frac{1}{2L-(2k+3)}\right). \tag{2-105}$$

We therefore obtain inductively

$$P_L(X)=(-1)^L\left\{(2L-3)c_{L-1}\left(X-\frac{1}{2L-3}\right)+(2L-5)c_{L-2}\left(X-\frac{1}{2L-1}\right)X\right\}$$
$$+(-X)\left(\frac{1}{2L-1}-X\right)\left(\frac{2}{2L-1}-X\right)\left[(-1)^{L-2}(-1)(2L-7)c_{L-3}+\left(\frac{3}{2L-1}-X\right)\cdots\right]$$
$$=(-1)^L\left(X-\frac{2}{2L-1}\right)\left\{(2L-5)c_{L-2}\left(X-\frac{1}{2L-5}\right)+(2L-7)c_{L-3}X\left(X-\frac{1}{2L-1}\right)\right\}$$
$$+(-X)\left(\frac{1}{2L-1}-X\right)\left(\frac{2}{2L-1}-X\right)\left(\frac{3}{2L-1}-X\right)[(-1)^{L-3}(-1)(2L-9)c_{L-4}\cdots]$$
$$=(-1)^L\left(X-\frac{2}{2L-1}\right)\cdots\left(X-\frac{L-2}{2L-1}\right)$$
$$\times\left\{3c_2\left(X-\frac{1}{3}\right)+X\left(X-\frac{1}{2L-1}\right)\left(c_1+X-\frac{L-1}{2L-1}\right)\right\}.$$

We use (2-105) with $k = L-2$ to compute the last polynomial:

$$\begin{aligned}
3c_2\Big(X-\frac{1}{3}\Big)+X\Big(X-\frac{1}{2L-1}\Big)\Big(c_1+X-\frac{L-1}{2L-1}\Big)&\\
&=\Big\{3c_2\Big(X-\frac{1}{3}\Big)+c_1X\Big(X-\frac{1}{2L-1}\Big)\Big\}+X\Big(X-\frac{1}{2L-1}\Big)\Big(X-\frac{L-1}{2L-1}\Big)\\
&=c_1\Big(X-\frac{L-1}{2L-1}\Big)(X-1)+X\Big(X-\frac{1}{2L-1}\Big)\Big(X-\frac{L-1}{2L-1}\Big)\\
&=\Big(X-\frac{L-1}{2L-1}\Big)\Big[\frac{L}{2L-1}(X-1)+X\Big(X-\frac{1}{2L-1}\Big)\Big]\\
&=\Big(X-\frac{L-1}{2L-1}\Big)\Big(X-\frac{L}{2L-1}\Big)\Big(X+1\Big).
\end{aligned}$$

We have therefore computed

$$P_L(x)=(-1)^L\Big(X-\frac{2}{2L-1}\Big)\cdots\Big(X-\frac{L-2}{2L-1}\Big)\Big(X-\frac{L-1}{2L-1}\Big)\Big(X-\frac{L}{2L-1}\Big)\Big(X+1\Big)$$

and (2-104) is proved. □

3. The trapped regime

In this section, we introduce the main dynamical tools at the heart of the proof of Theorem 1.1. We start with describing the bootstrap regime in which the blow-up solutions of Theorem 1.1 will be trapped. We then exhibit the Lyapounov type control of H^k norms, which is the heart of our analysis.

Modulation. We describe in this section the set of initial data leading to the blow-up scenario of Theorem 1.1. Let there be a smooth 1-corotational initial data

$$v(0,x)=\begin{vmatrix} g(u_0(r))\cos\theta\\ g(u_0(r))\sin\theta\\ z(u_0(r))\end{vmatrix}\quad\text{with}\quad \|\nabla u_0-\nabla Q\|_{L^2}\ll 1, \tag{3-1}$$

and let $v(t,x)$ be the corresponding smooth solution to (1-1) with life time $0<T<+\infty$. From (A-1), we may decompose on a small time interval

$$v(t,x)=\begin{vmatrix} g(u(t,r))\cos\theta\\ g(u(t,r))\sin\theta\\ z(u(t,r)),\end{vmatrix} \tag{3-2}$$

where

$$\tilde{\varepsilon}(t,r)=u(t,r)-Q(r)\quad\text{satisfies (A-4).} \tag{3-3}$$

Moreover, from a standard argument,

$$T<+\infty\quad\text{implies}\ \|\Delta v(t)\|_{L^2}\to+\infty\quad\text{as } t\to T. \tag{3-4}$$

We now modulate the solution and introduce from a standard argument[10] using the initial smallness (3-1) the unique decomposition of the flow defined on a small time $t \in [0, t_1]$:

$$u(t,r) = (\widetilde{Q}_{b(t)} + \varepsilon(t,r))_{\lambda(t)}, \quad \lambda(t) > 0,\ b = (b_1, \dots, b_L), \tag{3-5}$$

where $\varepsilon(t)$ satisfies the $L+1$ orthogonality conditions

$$(\varepsilon, H^k \Phi_M) = 0, \quad 0 \le k \le L \tag{3-6}$$

and the smallness

$$\|\nabla \varepsilon(t)\|_{L^2} + \left\| \frac{\varepsilon(t)}{y} \right\|_{L^2} + |b(t)| \ll 1.$$

Here, given $M > 0$ large enough, we define

$$\Phi_M = \sum_{p=0}^{L} c_{p,M} H^p(\chi_M \Lambda Q), \tag{3-7}$$

where

$$c_{0,M} = 1, \quad c_{k,M} = (-1)^{k+1} \frac{\sum_{p=0}^{k-1} c_{p,M} (\chi_M H^p(\chi_M \Lambda Q), T_k)}{(\chi_M \Lambda Q, \Lambda Q)}, \quad 1 \le k \le L,$$

is manufactured to ensure the nondegeneracy

$$(\Phi_M, \Lambda Q) = (\chi_M \Lambda Q, \Lambda Q) = 4 \log M (1 + o(1)) \quad \text{as } M \to +\infty \tag{3-8}$$

and the cancellation, for all $1 \le k \le L$,

$$(\Phi_M, T_k) = \sum_{p=0}^{k-1} c_{p,M} (H^p(\chi_M \Lambda Q), T_k) + c_{k,M} (-1)^k (\chi_M \Lambda Q, \Lambda Q) = 0. \tag{3-9}$$

In particular,

$$(H^i T_j, \Phi_M) = (-1)^j (\chi_M \Lambda Q, \Lambda Q) \delta_{i,j}, \quad 0 \le i, j \le L. \tag{3-10}$$

Observe also by induction that

$$\text{for all } 1 \le p \le L, \quad |c_{p,M}| \lesssim M^{2p}, \tag{3-11}$$

from which

$$\int |\Phi_M|^2 \lesssim \int |\chi_M \Lambda Q|^2 + \sum_{p=1}^{L} c_{p,M}^2 \int |H^p(\chi_M \Lambda Q)|^2 \lesssim \log M. \tag{3-12}$$

The existence of the decomposition (3-5) is a standard consequence of the implicit function theorem and the explicit relations

$$\left| \left(\frac{\partial}{\partial \lambda} (\widetilde{Q}_b)_\lambda, \frac{\partial}{\partial b_1} (\widetilde{Q}_b)_\lambda, \dots, \frac{\partial}{\partial b_L} (\widetilde{Q}_b)_\lambda \right) \right|_{\lambda=1,\, b=0} = (\Lambda Q, T_1, \dots, T_L),$$

[10] See, for example, [Martel and Merle 2000; Merle and Raphaël 2005a; Raphaël and Rodnianski 2012] for a further introduction to modulation.

which, using (3-9), imply the nondegeneracy of the Jacobian

$$\left|\left(\frac{\partial}{\partial(\lambda, b_j)}(\widetilde{Q}_b)_\lambda, H^i\Phi_M\right)_{1\le j\le L,\,0\le i\le L}\right|_{\lambda=1,\,b=0} = (\chi_M\Lambda Q, \Lambda Q)^{L+1} \neq 0.$$

The decomposition (3-5) exists as long as $t < T$ and $\varepsilon(t, r)$ remains small in the energy topology. Observe also from (3-3), (3-5), and the explicit structure of $\widetilde{Q}_b$ that ε satisfies (A-4), and in particular Lemma B.5 applies. In other words, we may measure the regularity of the map through the following coercive norms of ε: the energy norm

$$\|\varepsilon\|_{\mathcal{H}}^2 = \int |\partial_y\varepsilon|^2 + \int \frac{|\varepsilon|^2}{y^2}, \tag{3-13}$$

and higher order Sobolev norms adapted to the linearized operator

$$\mathcal{E}_{2k} = \int |H^k\varepsilon|^2, \quad 1 \le k \le L+1. \tag{3-14}$$

Setting up the bootstrap. We now choose our set of initial data in a more restricted way. More precisely, we pick a large enough time $s_0 \gg 1$ and rewrite the decomposition (3-5) as

$$u(t, r) = (\widetilde{Q}_{b(s)} + \varepsilon)(s, y), \tag{3-15}$$

where we introduce the renormalized variables

$$y = \frac{r}{\lambda(t)}, \quad s(t) = s_0 + \int_0^t \frac{d\tau}{\lambda^2(\tau)} \tag{3-16}$$

and measure time in s, which will be proved to be a global time. We introduce a decomposition (2-101):

$$b_k = b_k^e + \frac{U_k}{s^k(\log s)^{5/4}}, \quad 1 \le k \le L,\ b_{k+1} = U_{k+1} \equiv 0. \tag{3-17}$$

We consider the variable

$$V = P_L U, \tag{3-18}$$

where P_L refers to the diagonalization (2-104) of A_L. We assume that initially

$$|V_1(0)| \le 1, \quad (V_2(0), \dots, V_L(0)) \in \mathcal{B}_{L-1}(2). \tag{3-19}$$

We also assume the explicit initial smallness of the data:

$$\int |\nabla\varepsilon(0)|^2 + \int \left|\frac{\varepsilon(0)}{y}\right|^2 \le b_1^2(0), \tag{3-20}$$

$$|\mathcal{E}_{2k}(0)| \le [b_1(0)]^{10L+4}, \quad 1 \le k \le L+1. \tag{3-21}$$

Note also that, up to a fixed rescaling, we may always assume

$$\lambda(0) = 1. \tag{3-22}$$

Proposition 3.1 (bootstrap). *There exists*

$$(V_2(0), \dots, V_L(0)) \in \mathcal{B}_{L-1}(2)$$

such that the following bounds hold for all $s \geq s_0$:

- *Control of the radiation*:

$$\int |\nabla \varepsilon(s)|^2 + \int \left|\frac{\varepsilon(s)}{y}\right|^2 \leq 10(b_1(0))^{\frac{1}{4}}, \tag{3-23}$$

$$|\mathscr{E}_{2k}(s)| \leq b_1^{(2k-1)2L/(2L-1)}(s)|\log b_1(s)|^K, \quad 1 \leq k \leq L, \tag{3-24}$$

$$|\mathscr{E}_{2L+2}(s)| \leq K\frac{b_1^{2L+2}(s)}{|\log b_1(s)|^2}. \tag{3-25}$$

- *Control of the unstable modes*:

$$|V_1(s)| \leq 2, \quad (V_2(s), \dots, V_L(s)) \in \mathcal{B}_{L-1}(2). \tag{3-26}$$

Remark 3.2. *Note that the bounds (3-24) easily imply*[11] *the control of the H^2 norm of the full map (3-2)*

$$\int |\Delta v(s)|^2 < C(s) < +\infty, \quad s < s^*,$$

and therefore the blow-up criterion (3-4) ensures that the map is well defined on $[s, s^*)$.

Equivalently, given $(\varepsilon(0), V(0))$ as above, we introduce the time

$$s^* = s^*(\varepsilon(0), V(0)) = \sup\{s \geq s_0 \text{ such that (3-23), (3-24), (3-25), (3-26) hold on } [s_0, s]\}.$$

Observe that the continuity of the flow and the initial smallness (3-20), (3-21) ensure that $s^* > 0$. We then assume by contradiction that

$$\text{for all } (V_2(0), \dots, V_L(0)) \in \mathcal{B}_{L-1}(2), \quad s^* < +\infty, \tag{3-27}$$

and look for a contradiction. Our main claim is that the a priori control of the unstable modes (3-26) is enough to improve the bounds (3-23), (3-24), (3-25), and then the claim follows from the $(L-1)$ codimensional instability (2-104) of the system (2-96) near the exceptional solution b^e through a standard topological argument à la Brouwer.

The rest of this section is devoted to the derivation of the key lemmas for the proof of Proposition 3.1. We will make a systematic implicit use of the interpolation bounds of Lemma C.1, which are a consequence of the coercivity of the $\mathscr{E}_{2k+2}$ energy given by Lemma B.5.

Equation for the radiation. Recall the decomposition of the flow

$$u(t, r) = (\widetilde{Q}_{b(t)} + \varepsilon)(s, y) = (Q + \tilde{\alpha}_{b(t)})_{\lambda(s)} + w(t, r).$$

We use the rescaling formulas

$$u(t, r) = v(s, y), \quad y = \frac{r}{\lambda(t)}, \quad \partial_t u = \frac{1}{\lambda^2(t)}\left(\partial_s v - \frac{\lambda_s}{\lambda}\Lambda v\right)_\lambda$$

[11] See [Raphaël and Schweyer 2013] for the full computation.

to derive the equation for ε in renormalized variables,

$$\partial_s \varepsilon - \frac{\lambda_s}{\lambda}\Lambda\varepsilon + H\varepsilon = F - \widetilde{\mathrm{Mod}} = \mathscr{F}. \tag{3-28}$$

Here H is the linearized operator given by (2-8), $\widetilde{\mathrm{Mod}}(t)$ is given by

$$\widetilde{\mathrm{Mod}}(t) = -\left(\frac{\lambda_s}{\lambda} + b_1\right)\Lambda\widetilde{Q}_b + \sum_{i=1}^{L}[(b_i)_s + (2i - 1 + c_{b_1})b_1 b_i - b_{i+1}]\left[\widetilde{T}_i + \chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}\right], \tag{3-29}$$

and

$$F = -\widetilde{\Psi}_b + L(\varepsilon) - N(\varepsilon), \tag{3-30}$$

where L is the linear operator corresponding to the error in the linearized operator from Q to $\widetilde{Q}_b$

$$L(\varepsilon) = \frac{f'(Q) - f'(\widetilde{Q}_b)}{y^2}\varepsilon, \tag{3-31}$$

and the remainder term is the purely nonlinear term

$$N(\varepsilon) = \frac{f(\widetilde{Q}_b + \varepsilon) - f(\widetilde{Q}_b) - \varepsilon f'(\widetilde{Q}_b)}{y^2}. \tag{3-32}$$

We also need to write the flow (3-28) in original variables. For this we use the rescaled operators

$$\begin{aligned} A_\lambda &= -\partial_r + \frac{Z_\lambda}{r}, & A_\lambda^* &= \partial_r + \frac{1 + Z_\lambda}{r}, \\ H_\lambda &= A_\lambda^* A_\lambda = -\Delta + \frac{V_\lambda}{r^2}, & \widetilde{H}_\lambda &= A_\lambda A_\lambda^* = -\Delta + \frac{\widetilde{V}_\lambda}{r^2}, \end{aligned} \tag{3-33}$$

and the renormalized function

$$w(t, r) = \varepsilon(s, y).$$

Then (3-28) becomes

$$\partial_t w + H_\lambda w = \frac{1}{\lambda^2}\mathscr{F}_\lambda. \tag{3-34}$$

Observe from (2-99) that, for $s < s^*$,

$$|b_k| \lesssim b_1^k, \quad 0 < b_1 \ll 1, \tag{3-35}$$

and hence the a priori bound (2-57) holds.

Modulation equations. Let us now compute the modulation equations for (b, λ) as a consequence of the choice of orthogonality conditions (3-6).

Lemma 3.3 (modulation equations). *We have the bound on the modulation parameters*

$$\left|\frac{\lambda_s}{\lambda}+b_1\right|+\sum_{k=1}^{L-1}|(b_k)_s+(2k-1+c_{b_1})b_1b_k-b_{k+1}|\lesssim b_1^{L+\frac{3}{2}}, \tag{3-36}$$

$$\left|(b_L)_s+(2L-1+c_{b_1})b_1b_L\right|\lesssim\frac{1}{\sqrt{\log M}}\left(\sqrt{\mathscr{E}_{2L+2}}+\frac{b_1^{L+1}}{|\log b_1|}\right). \tag{3-37}$$

Remark 3.4. *Note that this implies in the bootstrap the rough bound*

$$|(b_1)_s|\le 2b_1^2, \tag{3-38}$$

and, in particular, (2-84) *holds.*

Proof of Lemma 3.3. *Step* 1: *Law for* b_L. Let

$$D(t)=\left|\frac{\lambda_s}{\lambda}+b_1\right|+\sum_{k=1}^{L}|(b_k)_s+(2k-1+c_{b_1})b_1b_k-b_{k+1}|. \tag{3-39}$$

We take the inner product of (3-28) with $H^L\Phi_M$ and, using the orthogonality (3-6), obtain

$$(\widetilde{\mathrm{Mod}}(t),H^L\Phi_M)=-(\widetilde{\Psi}_b,H^L\Phi_M)-(H^L\varepsilon,H\Phi_M)-\left(-\frac{\lambda_s}{\lambda}\Lambda\varepsilon-L(\varepsilon)+N(\varepsilon),H^L\Phi_M\right). \tag{3-40}$$

First, from the construction of the profile, (3-29), the localization $\mathrm{Supp}(\Phi_M)\subset[0,2M]$ from (3-7), and the identities (3-8), (3-9), (3-10), we compute

$$\begin{aligned}&(H^L(\widetilde{\mathrm{Mod}}(t)),\Phi_M)\\&\quad=-\left(b_1+\frac{\lambda_s}{\lambda}\right)(H^L\Lambda\widetilde{Q}_b,\Phi_M)+\sum_{i=1}^{L}[(b_i)_s+(2i-1+c_{b_1})b_1b_i-b_{i+1}]\left(\widetilde{T}_i+\chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i},H^L\Phi_M\right)\\&\quad=(-1)^L(\Lambda Q,\Phi_M)((b_L)_s+(2L-1+c_{b_1})b_1b_L)+O(M^Cb_1|D(t)|).\end{aligned}$$

The linear term in (3-40) is estimated[12] from (3-24), (3-12):

$$|(H^L\varepsilon,H\Phi_M)|\lesssim\|H^{L+1}\varepsilon\|_{L^2}\sqrt{\log M}=\sqrt{\log M\mathscr{E}_{2L+2}}.$$

The remaining nonlinear term is estimated using the Hardy bounds of Appendix A:

$$\left|\left(-\frac{\lambda_s}{\lambda}\Lambda\varepsilon+L(\varepsilon)+N(\varepsilon),H^L\Phi_M\right)\right|\lesssim M^Cb_1(\sqrt{\mathscr{E}_{2L+2}}+|D(t)|).$$

[12] Observe that we do not use the interpolated bounds of Lemma C.1, but directly the definition (3-14) of $\mathscr{E}_{2L+2}$, and hence the dependence of the constant in M is explicit. This will be crucial for the analysis.

We inject these estimates into (3-40) and conclude from (3-8) and the local estimate (2-91) that

$$|(b_L)_s + (2L-1+c_{b_1})b_1 b_L| = \frac{\sqrt{\log M \mathcal{E}_{2L+2}}}{\log M} + M^C b_1 |D(t)| + M^C b_1^{L+\frac{3}{2}}$$
$$\lesssim \frac{1}{\sqrt{\log M}}\left(\sqrt{\mathcal{E}_{2L+2}} + \frac{b_1^{L+1}}{|\log b_1|}\right) + M^C b_1 |D(t)|. \tag{3-41}$$

Step 2: Degeneracy of the law for λ and $(b_k)_{1\le k\le L-1}$. We now take the inner product of (3-28) with $H^k\Phi_M$, $0 \le k \le L-1$ and obtain

$$(\widetilde{\mathrm{Mod}}(t), H^k\Phi_M) = -(\widetilde{\Psi}_b, H^k\Phi_M) - (H^{k+1}\varepsilon, H\Phi_M) - \left(-\frac{\lambda_s}{\lambda}\Lambda\varepsilon - L(\varepsilon) + N(\varepsilon), H^k\Phi_M\right). \tag{3-42}$$

Note first that the choice of orthogonality conditions (3-6) gets rid of the linear term in ε:

$$\text{for all } 0 \le k \le L-1, \quad (H^{k+1}\varepsilon, \Phi_M) = 0.$$

Next, from (3-29), the localization $\mathrm{Supp}(\Phi_M) \subset [0, 2M]$ from (3-7), and the identities (3-8), (3-9), (3-10), we compute

$$(H^k(\widetilde{\mathrm{Mod}}(t)), \Phi_M)$$
$$= -\left(b + \frac{\lambda_s}{\lambda}\right)(H^k\Lambda\widetilde{Q}_b, \Phi_M) + \sum_{i=1}^{L}[(b_i)_s + (2i-1+c_{b_1})b_1 b_i - b_{i+1}]\left(\widetilde{T}_i + \chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}, H^k\Phi_M\right)$$
$$= (\Lambda Q, \Phi_M)\begin{cases} -(\lambda_s/\lambda + b_1) & \text{for } k=0, \\ (-1)^k((b_k)_s + (2k-1+c_{b_1})b_1 b_k - b_{k+1}) & \text{for } 1 \le k \le L-1 \end{cases} + O(M^C b_1|D(t)|).$$

Nonlinear terms are easily estimated using the Hardy bounds

$$\left|\left(-\frac{\lambda_s}{\lambda}\Lambda\varepsilon + L(\varepsilon) + N(\varepsilon), H^k\Phi_M\right)\right| \lesssim M^C b_1(\sqrt{\mathcal{E}_{2L+2}} + |D(t)|) \lesssim b_1^{L+\frac{3}{2}} + b_1 M^C |D(t)|.$$

Injecting this bound into (3-42) together with the local bound (2-91) yields the first bound,

$$D(t) \lesssim b_1^{L+\frac{3}{2}}, \tag{3-43}$$

and (3-36) is proved. Injecting this bound into (3-41) yields (3-37). □

Improved modulation equation for b_L. Observe that (3-37), (3-25) yield the pointwise bound

$$|(b_L)_s + (2L-1+c_{b_1})b_1 b_L| \lesssim \frac{1}{\sqrt{\log M}}\left(\sqrt{\mathcal{E}_{2L+2}} + \frac{b_1^{L+1}}{|\log b_1|}\right) \lesssim \frac{b_1^{L+1}}{|\log b_1|},$$

which is worse than (3-36) and critical to close (3-26). We claim that a $|\log b_1|$ is easily gained up to an oscillation in time.

Lemma 3.5 (improved control of b_L). *Let $B_\delta = B_0^\delta$ and*

$$\tilde{b}_L = b_L + \frac{(-1)^L(H^L\varepsilon, \chi_{B_\delta}\Lambda Q)}{4\delta|\log b_1|}. \tag{3-44}$$

Then

$$|\tilde{b}_L - b_L| \lesssim b_1^{L+\frac{1}{2}} \tag{3-45}$$

and $\tilde{b}_L$ satisfies the pointwise differential equation

$$|(\tilde{b}_L)_s + (2L-1+c_{b_1})b_1\tilde{b}_L| \lesssim \frac{C(M)}{\sqrt{|\log b_1|}}\left[\sqrt{\mathscr{E}_{2L+2}} + \frac{b_1^{L+1}}{|\log b_1|}\right]. \tag{3-46}$$

Proof of Lemma 3.5. We commute (3-28) with H^L and take the scalar product with $\chi_{B_\delta}\Lambda Q$ for some small enough universal constant $0 < \delta \ll 1$. This yields

$$\frac{d}{ds}\{(H^L\varepsilon, \chi_{B^\delta}\Lambda Q)\} - (H^L\varepsilon, \Lambda Q\partial_s(\chi_{B^\delta})) = -(H^{L+1}\varepsilon, \chi_{B^\delta}\Lambda Q) + \frac{\lambda_s}{\lambda}(H^L\Lambda\varepsilon, \chi_{B_\delta}\Lambda Q) + (F - \widetilde{\mathrm{Mod}}, H^L\chi_{B_\delta}\Lambda Q).$$

The linear term is estimated by Cauchy–Schwarz:

$$|(H^{L+1}\varepsilon, \chi_{B^\delta}\Lambda Q)| \lesssim C(M)\sqrt{|\log b_1|}\sqrt{\mathscr{E}_{2L+2}}.$$

Using (3-36), we similarly estimate

$$|(H^L\varepsilon, \Lambda Q\partial_s(\chi_{B^\delta}))| + \left|\frac{\lambda_s}{\lambda}(H^L\Lambda\varepsilon, \chi_{B_\delta}\Lambda Q)\right| \lesssim C(M)\frac{|(b_1)_s|}{b_1}\frac{1}{b_1^{C\delta}}\sqrt{\mathscr{E}_{2L+2}} + \frac{b_1}{b_1^{C\delta}}C(M)\sqrt{\mathscr{E}_{2L+2}} \lesssim \sqrt{|\log b_1|}\sqrt{\mathscr{E}_{2L+2}}.$$

The estimate on the error terms easily follows from the Hardy bounds

$$|(L(\varepsilon), H^L\chi_{B^\delta}\Lambda Q)| + (N(\varepsilon), H^L\chi_{B^\delta}\Lambda Q)| \lesssim \frac{b_1}{b_1^{C\delta}}C(M)\sqrt{\mathscr{E}_{2L+2}} \lesssim \sqrt{|\log b_1|}\sqrt{\mathscr{E}_{2L+2}}.$$

From (2-91) we further estimate

$$|(H^L\varepsilon, \tilde{\Psi}_b)| \lesssim \frac{b_1^{L+3}}{b_1^{C\delta}}C(M)\sqrt{\mathscr{E}_{2L+2}} \lesssim \sqrt{|\log b_1|}\sqrt{\mathscr{E}_{2L+2}}.$$

From (3-36), (3-29), we now compute

$$\begin{aligned}
-(\widetilde{\mathrm{Mod}}, H^L\chi_{B_\delta}\Lambda Q) &= O\left(\frac{b_1^{L+3/2}}{b_1^{C\delta}}\right) + [(b_L)_s + (2L-1+c_{b_1})b_1b_L]\left(H^L\widetilde{T}_L + \sum_{j=L+1}^{L+2} H^L\left[\chi_{B_1}\frac{\partial S_j}{\partial_{b_L}}\right], \chi_{B_\delta}\Lambda Q\right)\\
&= (-1)^L[(b_L)_s + (2L-1+c_{b_1})b_1b_L]\left[(\Lambda Q, \chi_{B_\delta}\Lambda Q) + O(b_1^{1-C\delta})\right] + O\left(\frac{b_1^{L+3/2}}{b_1^{C\delta}}\right)\\
&= (-1)^L[(b_L)_s + (2L-1+c_{b_1})b_1b_L]4\delta|\log b_1| + O(\sqrt{|\log b_1|}\sqrt{\mathscr{E}_{2L+2}} + b_1^{L+1}).
\end{aligned}$$

The collection of above bounds yields the preliminary estimate

$$\left|\frac{d}{ds}\{(H^L\varepsilon,\chi_{B^\delta}\Lambda Q)\}+(-1)^L\big[((b_L)_s+(2L-1+c_{b_1}))b_1b_L\big]4\delta|\log b_1|\right| \lesssim C(M)\sqrt{|\log b_1|}\left[\sqrt{\mathscr{E}_{2L+2}}+\frac{b_1^{L+1}}{|\log b_1|}\right] \tag{3-47}$$

By brute force, from (3-44) we estimate

$$|\tilde{b}_L-b_L|\lesssim|\log b_1|^C b_1^{L+1-C\delta}\lesssim b_1^{L+\frac12}$$

and we therefore rewrite (3-47) using (3-38) as

$$|(\tilde{b}_L)_s+(2L-1+c_{b_1})b_1\tilde{b}_L|\lesssim|(H^L\varepsilon,\chi_{B_\delta}\Lambda Q)|\left|\frac{d}{ds}\left\{\frac{1}{4\delta\log b_1}\right\}\right|+\frac{C(M)\sqrt{|\log b_1|}}{|\log b_1|}\left[\sqrt{\mathscr{E}_{2L+2}}+\frac{b_1^{L+1}}{|\log b_1|}\right]$$
$$\lesssim b_1^{1-C\delta}\sqrt{\mathscr{E}_{2L+2}}+\frac{C(M)}{\sqrt{|\log b_1|}}\left[\sqrt{\mathscr{E}_{2L+2}}+\frac{b_1^{L+1}}{|\log b_1|}\right]$$

and (3-46) is proved. □

The Lyapounov monotonicity. We now turn to the core of the argument which is the derivation of a suitable Lyapounov functional for the $\mathscr{E}_{2L+2}$ energy.

Proposition 3.6 (Lyapounov monotonicity). *We have*

$$\frac{d}{dt}\left\{\frac{1}{\lambda^{4L+2}}\left[\mathscr{E}_{2L+2}+O\left(b_1^{\frac45}\frac{b_1^{2L+2}}{|\log b_1|^2}\right)\right]\right\}\le C\frac{b_1}{\lambda^{4L+4}}\left[\frac{\mathscr{E}_{2L+2}}{\sqrt{\log M}}+\frac{b_1^{2L+2}}{|\log b_1|^2}+\frac{b_1^{L+1}\sqrt{\mathscr{E}_{2L+2}}}{|\log b_1|}\right] \tag{3-48}$$

for some universal constant $C>0$ independent of M and of the bootstrap constant K in (3-23), (3-24).

Proof of Proposition 3.6. *Step* 1: *Suitable derivatives.* We define the derivatives of w associated with the linearized Hamiltonian H_λ by

$$w_1=A_\lambda w,\quad w_{k+1}=\begin{cases}A_\lambda^* w_k & \text{for k odd,}\\ A_\lambda w_k & \text{for k even,}\end{cases}\qquad 1\le k\le 2L+1$$

and we define its renormalized version by

$$\varepsilon_1=A\varepsilon,\quad \varepsilon_{k+1}=\begin{cases}A^*\varepsilon_k & \text{for k odd,}\\ A\varepsilon_k & \text{for k even,}\end{cases}\qquad 1\le k\le 2L+1.$$

From (3-34), we compute

$$\partial_t w_{2L}+H_\lambda w_{2L}=[\partial_t,H_\lambda^L]w+H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right) \tag{3-49}$$

$$\partial_t w_{2L+1}+\widetilde{H}_\lambda w_{2L+1}=\frac{\partial_t Z_\lambda}{r}w_{2L}+A_\lambda([\partial_t,H_\lambda^L]w)+A_\lambda H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right). \tag{3-50}$$

We recall the action of time derivatives on rescaling:

$$\partial_t v_\lambda=\frac{1}{\lambda^2}\left(\partial_s v-\frac{\lambda_s}{\lambda}\Lambda v\right)_\lambda. \tag{3-51}$$

Step 2: *Modified energy identity*. We compute the energy identity on (3-50) using (3-51):

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathscr{E}_{2L+2} &= \frac{1}{2}\frac{d}{dt}\left\{\int \widetilde{H}_\lambda w_{2L+1} w_{2L+1}\right\}\\ &= \int \widetilde{H}_\lambda w_{2L+1}\partial_t w_{2L+1} + \int \frac{\partial_t \widetilde{V}_\lambda}{2r^2} w_{2L+1}^2\\ &= -\int (\widetilde{H}_\lambda w_{2L+1})^2 + b_1 \int \frac{(\Lambda \widetilde{V})_\lambda}{2\lambda^2 r^2} w_{2L+1}^2 - \left(\frac{\lambda_s}{\lambda} + b_1\right)\int \frac{(\Lambda \widetilde{V})_\lambda}{2\lambda^2 r^2} w_{2L+1}^2\\ &\qquad + \int \widetilde{H}_\lambda w_{2L+1}\left[\frac{\partial_t Z_\lambda}{r} w_{2L} + A_\lambda([\partial_t, H_\lambda^L]w) + A_\lambda H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right]. \quad (3\text{-}52)\end{aligned}$$

From (3-49), (3-50) we further compute

$$\begin{aligned}\frac{d}{dt}\left\{\int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L+1} w_{2L}\right\} &= \int \frac{d}{dt}\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}\right) w_{2L+1} w_{2L}\\ &\quad + \int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L}\left[-\widetilde{H}_\lambda w_{2L+1} + \frac{\partial_t Z_\lambda}{r} w_{2L} + A_\lambda([\partial_t, H_\lambda^L]w) + A_\lambda H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right]\\ &\quad + \int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L+1}\left[-A_\lambda^* w_{2L+1} + [\partial_t, H_\lambda^L]w + H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right].\end{aligned}$$

We now integrate by parts using (2-4) to compute

$$\begin{aligned}\int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L+1} A_\lambda^* w_{2L+1} &= \frac{b_1}{\lambda^{4L+4}}\int \frac{\Lambda Z}{y}\varepsilon_{2L+1} A^*\varepsilon_{2L+1}\\ &= \frac{b_1}{\lambda^{4L+4}}\int \frac{2(1+Z)\Lambda Z - \Lambda^2 Z}{2y^2}\varepsilon_{2L+1}^2\\ &= \frac{b_1}{\lambda^{4L+4}}\int \frac{\Lambda \widetilde{V}}{2y^2}\varepsilon_{2L+1}^2 = b_1\int \frac{(\Lambda \widetilde{V})_\lambda}{2\lambda^2 r^2} w_{2L+1}^2.\end{aligned}$$

Injecting this into the energy identity (3-52) yields the modified energy identity

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\mathscr{E}_{2L+2} + 2\int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L+1} w_{2L}\right\}\\ &\quad = -\int (\widetilde{H}_\lambda w_{2L+1})^2 - \left(\frac{\lambda_s}{\lambda} + b_1\right)\int \frac{(\Lambda \widetilde{V})_\lambda}{2\lambda^2 r^2} w_{2L+1}^2 + \int \frac{d}{dt}\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}\right) w_{2L+1} w_{2L}\\ &\qquad + \int \widetilde{H}_\lambda w_{2L+1}\left[\frac{\partial_t Z_\lambda}{r} w_{2L} + A_\lambda([\partial_t, H_\lambda^L]w) + A_\lambda H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right]\\ &\qquad + \int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L}\left[-\widetilde{H}_\lambda w_{2L+1} + \frac{\partial_t Z_\lambda}{r} w_{2L} + A_\lambda([\partial_t, H_\lambda^L]w) + A_\lambda H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right]\\ &\qquad + \int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L+1}\left[[\partial_t, H_\lambda^L]w + H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right]. \quad (3\text{-}53)\end{aligned}$$

We now aim at estimating all terms in the right hand side of (3-53). Throughout the proof, we shall make implicit use of the coercivity estimates of Lemma B.2 and Lemma C.1.

Step 3: *Lower order quadratic terms*. We treat the lower order quadratic terms in (3-53) using dissipation. Indeed, from (2-5), (2-6), (3-38), we have the bounds

$$|\partial_t Z_\lambda| + |\partial_t V_\lambda| \lesssim \frac{b_1}{\lambda^2}(|\Lambda Z| + |\Lambda V|)_\lambda \lesssim \frac{b_1}{\lambda^2}\frac{y^2}{1+y^4}. \tag{3-54}$$

We moreover claim the bound

$$\int \frac{([\partial_t, H_\lambda^L]w)^2}{\lambda^2(1+y^2)} + \int |A_\lambda([\partial_t, H_\lambda^L]w)|^2 \lesssim C(M)\frac{b_1^2}{\lambda^{4L+4}}\mathscr{E}_{2L+2}, \tag{3-55}$$

which is proved in Appendix E. From Cauchy–Schwartz, the rough bound (3-38), and Lemma C.1, we conclude

$$\begin{aligned}\int \left|\widetilde{H}_\lambda w_{2L+1}\left[\frac{\partial_t Z_\lambda}{r}w_{2L} + \int A_\lambda([\partial_t, H_\lambda^L]w)\right]\right| + \int |\widetilde{H}_\lambda w_{2L+1}|\left|\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_{2L}\right| \\ \le \frac{1}{2}\int |\widetilde{H}_\lambda w_{2L+1}|^2 + \frac{b_1^2}{\lambda^{4L+4}}\left[\int \frac{\varepsilon_{2L}^2}{1+y^6} + C(M)\mathscr{E}_{2L+2}\right] \\ \le \frac{1}{2}\int |\widetilde{H}_\lambda w_{2L+1}|^2 + \frac{b_1}{\lambda^{4L+4}}C(M)b_1\mathscr{E}_{2L+2}.\end{aligned}$$

All other quadratic terms are lower order by a factor b_1 again using (3-38), (3-55), (3-36), and Lemma C.1:

$$\begin{aligned}\left|\frac{\lambda_s}{\lambda} + b_1\right|\left|\int \left|\frac{(\Lambda\widetilde{V})_\lambda}{2\lambda^2 r^2}w_{2L+1}^2\right| + \int \left|\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2L}\left[\frac{\partial_t Z_\lambda}{r}w_{2L} + A_\lambda([\partial_t, H_\lambda^L]w)\right]\right| \\ + \int \left|\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2L+1}[\partial_t, H_\lambda^L]w\right| + \left|\int \frac{d}{dt}\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}\right)w_{2L+1}w_{2L}\right| \\ \lesssim \frac{b_1^2}{\lambda^{4L+4}}\left[\int \frac{\varepsilon_{2L+1}^2}{1+y^4} + \int \frac{\varepsilon_{2L}^2}{1+y^6} + C(M)\mathscr{E}_{2L+2}\right] \lesssim \frac{b_1}{\lambda^{4L+4}}C(M)b_1\mathscr{E}_{2L+2}.\end{aligned}$$

We similarly estimate the boundary term in time using (C-10):

$$\left|\int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2L+1}w_{2L}\right| \lesssim \frac{b_1}{\lambda^{4L+2}}\left[\int \frac{\varepsilon_{2L+1}^2}{1+y^2} + \int \frac{\varepsilon_{2L}^2}{1+y^4}\right] \lesssim \frac{b_1}{\lambda^{4L+2}}|\log b_1|^C b_1^{2L+2}.$$

We inject these estimates into (3-53) to derive the preliminary bound

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\{\frac{1}{\lambda^{4L+2}}\left[\mathscr{E}_{2L+2} + O\left(b_1^{\frac{4}{5}}\frac{b^{2L+2}}{|\log b|^2}\right)\right]\right\} \\ \le -\frac{1}{2}\int (\widetilde{H}_\lambda w_{2L+1})^2 + \int \widetilde{H}_\lambda w_{2L+1}A_\lambda H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right) \\ + \int H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\left[\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2L+1} + A_\lambda^*\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2L}\right)\right] + \frac{b_1}{\lambda^{4L+4}}\sqrt{b_1}b_1^{2L+2}\end{aligned} \tag{3-56}$$

with constants independent of M for $|b| < b^*(M)$ small enough.

We now estimate all terms in the right hand side of (3-56).

Step 4: *Further use of dissipation*. Let us introduce the decomposition from (3-28), (3-30),

$$\mathscr{F} = \mathscr{F}_0 + \mathscr{F}_1, \quad \mathscr{F}_0 = -\widetilde{\Psi}_b - \widetilde{\mathrm{Mod}}(t), \quad \mathscr{F}_1 = L(\varepsilon) - N(\varepsilon). \tag{3-57}$$

The first term in the right hand side of (3-56) is estimated after an integration by parts:

$$\begin{aligned}\left|\int \widetilde{H}_\lambda w_{2L+1} A_\lambda H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right| &\le \frac{C}{\lambda^{4L+4}}\|A^*\varepsilon_{2L+1}\|_{L^2}\|H^{L+1}\mathscr{F}_0\|_{L^2} + \frac{1}{4}\int |\widetilde{H}_\lambda w_{2L+1}|^2 + \frac{C}{\lambda^{4L+4}}\int |AH^L\mathscr{F}_1|^2 \\ &\le \frac{C}{\lambda^{4L+4}}\big[\|H^{L+1}\mathscr{F}_0\|_{L^2}\sqrt{\mathscr{E}_{2L+2}} + \|AH^L\mathscr{F}_1\|_{L^2}^2\big] + \frac{1}{4}\int |\widetilde{H}_\lambda w_{2L+1}|^2 \end{aligned} \tag{3-58}$$

for some universal constant $C > 0$ independent of M.

The last two terms in (3-56) can be estimated by brute force from Cauchy–Schwarz:

$$\begin{aligned}\left|\int H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L+1}\right| &\lesssim \frac{b_1}{\lambda^{4L+4}}\left(\int \frac{1+|\log y|^2}{1+y^4}|H^L\mathscr{F}|^2\right)^{\frac12}\left(\int \frac{\varepsilon_{2L+1}^2}{y^2(1+|\log y|^2)}\right)^{\frac12} \\ &\lesssim \frac{b_1}{\lambda^{4L+4}}\sqrt{\mathscr{E}_{2L+2}}\left(\int \frac{1+|\log y|^2}{1+y^4}|H^L\mathscr{F}|^2\right)^{\frac12}, \end{aligned} \tag{3-59}$$

where constants are independent of M thanks to the estimate (B-2) for ε_{2L+1}. Similarly,

$$\begin{aligned}\left|\int H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)A_\lambda^*\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L}\right)\right| &\lesssim \frac{b_1}{\lambda^{4L+4}}\left(\int \frac{1+|\log y|^2}{1+y^2}|AH^L\mathscr{F}|^2\right)^{\frac12}\left(\int \frac{\varepsilon_{2L}^2}{(1+y^4)(1+|\log y|^2)}\right)^{\frac12} \\ &\lesssim \frac{b_1}{\lambda^{4L+4}}C(M)\sqrt{\mathscr{E}_{2L+2}}\left(\int \frac{1+|\log y|^2}{1+y^2}|AH^L\mathscr{F}_0|^2 + \int |AH^L\mathscr{F}_1|^2\right)^{\frac12}. \end{aligned} \tag{3-60}$$

We now claim the bounds

$$\int \frac{1+|\log y|^2}{1+y^4}|H^L\mathscr{F}|^2 \lesssim \frac{b_1^{2L+2}}{|\log b_1|^2} + \frac{\mathscr{E}_{2L+2}}{\log M}, \tag{3-61}$$

$$\int \frac{1+|\log y|^2}{1+y^2}|AH^L\mathscr{F}_0|^2 \lesssim \delta(\alpha^*)\left[\frac{b_1^{2L+2}}{|\log b_1|^2} + \mathscr{E}_{2L+2}\right], \tag{3-62}$$

$$\int |H^{L+1}\mathscr{F}_0|^2 \lesssim b_1^2\left[\frac{b_1^{2L+2}}{|\log b_1|^2} + \frac{\mathscr{E}_{2L+2}}{\log M}\right], \tag{3-63}$$

$$\int |AH^L\mathscr{F}_1|^2 \lesssim b_1\left[\frac{b_1^{2L+2}}{|\log b_1|^2} + \frac{\mathscr{E}_{2L+2}}{\log M}\right] \tag{3-64}$$

with all $\lesssim$ constants independent of M for $|b| < \alpha^*(M)$ small enough, and where

$$\delta(\alpha^*) \to 0 \quad \text{as } \alpha^*(M) \to 0.$$

Injecting these bounds together with (3-58), (3-59), (3-60) into (3-56) concludes the proof of (3-48). We now turn to the proof of (3-61), (3-62), (3-63), (3-64).

Step 5: $\widetilde{\Psi}_b$ *terms*. The contribution of $\widetilde{\Psi}_b$ terms to (3-61), (3-62), (3-63) is estimated from (2-89), (2-90), which are at the heart of the construction of $\widetilde{Q}_b$ and yield the desired bounds.

Step 6: $\widetilde{\mathrm{Mod}(t)}$ *terms*. Recall (3-29),

$$\widetilde{\mathrm{Mod}}(t) = -\left(\frac{\lambda_s}{\lambda} + b_1\right)\Lambda\widetilde{Q}_b + \sum_{i=1}^{L}[(b_i)_s + (2i-1+c_{b_1})b_1 b_i - b_{i+1}]\left[\widetilde{T}_i + \chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}\right],$$

and the notation (3-39).

Proof of (3-63) *for* $\widetilde{\mathrm{Mod}}$. We recall that

$$|b_k| \lesssim b_1^k,$$

and, from Lemma 2.8, we estimate

$$\begin{aligned}\int |H^{L+1}\Lambda\widetilde{Q}_b|^2 &\lesssim \sum_{i=1}^{L}\int |H^{L+1}b_i\Lambda\widetilde{T}_i|^2 + \sum_{i=2}^{L+2}\int |H^{L+1}\Lambda\widetilde{S}_i|^2\\ &\lesssim \sum_{i=1}^{L} b_1^{2i}\int_{y\le 2B_1}\left|\frac{(1+|\log y|^C)y^{2i-1}}{1+y^{2L+2}}\right|^2 + \sum_{i=2}^{L+1} b_1^{2i} + \frac{b_1^{2L+4}}{b_1^2|\log b_1|^2} \lesssim b_1^2.\end{aligned}$$

We then use the cancellation $H^{L+1}T_i = 0$ for $1 \le i \le L$ to estimate

$$\sum_{i=1}^{L}\int |H^{L+1}\widetilde{T}_i|^2 \lesssim \sum_{i=1}^{L}\int_{B_1\le y\le 2B_1}\left|\frac{y^{2i-1}}{y^{2L+2}}\right|^2 \lesssim b_1^2.$$

Then, using Lemma 2.8 again,[13] for $1 \le i \le L$,

$$\sum_{j=i+1}^{L+2}\int\left|H^{L+1}\left[\chi_{B_1}\frac{\partial S_j}{\partial b_i}\right]\right|^2 \lesssim \sum_{j=i+1}^{L+1} b_1^{2(j-i)} + \frac{b_1^{2(L+2-i)}}{b_1^2|\log b_1|^2} \lesssim b_1^2.$$

We thus obtain from Lemma 3.3 the expected bound:

$$\int |H^{L+1}\widetilde{\mathrm{Mod}}|^2 \lesssim b_1^2|D(t)|^2 \lesssim b_1^2\left[\frac{\mathscr{E}_{2L+2}}{|\log M|} + \frac{b_1^{2L+2}}{|\log b_1|^2}\right]. \qquad \square$$

[13] This is where we used the logarithmic gain (2-54) induced by (2-24).

Proof of (3-61) for $\widetilde{\mathrm{Mod}}$. We use Lemma 2.8 to derive the rough bound

$$\begin{aligned}
&\int \frac{1+|\log y|^2}{1+y^4}|H^L\Lambda\widetilde{Q}_b|^2\\
&\quad\lesssim \sum_{i=1}^{L}\int \frac{1+|\log y|^2}{1+y^4}|H^L b_i\Lambda\widetilde{T}_i|^2+\sum_{i=2}^{L+2}\int \frac{1+|\log y|^2}{1+y^4}|H^L\Lambda\widetilde{S}_i|^2\\
&\quad\lesssim \sum_{i=1}^{L} b_1^{2i}\int_{y\le 2B_1}\frac{1+|\log y|^C}{1+y^4}\left|\frac{y^{2i-1}}{1+y^{2L}}\right|^2+\sum_{i=2}^{L+1} b_1^{2i}|\log b_1|^3+b_1^{2L+4}\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^4}\left|\frac{1+y^{2(L+2)-1}}{1+y^{2L}}\right|^2\\
&\quad\lesssim 1.
\end{aligned}$$

Next,

$$\sum_{j=1}^{L}\frac{1+|\log y|^2}{1+y^4}|H^L\widetilde{T}_i|^2\lesssim\sum_{j=1}^{L}\int_{y\le 2B_1}\frac{1+|\log y|^C}{1+y^4}\left|\frac{y^{2i-1}}{1+y^{2L}}\right|^2\lesssim 1,$$

and finally, again using Lemma 2.8, for $1\le i\le L$,

$$\begin{aligned}
&\sum_{j=i+1}^{L+2}\int\frac{1+|\log y|^2}{1+y^4}\left|H^L\left[\chi_{B_1}\frac{\partial S_j}{\partial b_i}\right]\right|^2\\
&\qquad\lesssim\sum_{j=i+1}^{L+1} b_1^{2(j-i)}|\log b_1|^2+b_1^{2(L-i)+4}\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^4}\left|\frac{1+y^{2(L+2)-1}}{1+y^{2L}}\right|^2\lesssim 1.
\end{aligned}$$

We thus obtain from Lemma 3.3 the expected bound:

$$\int\frac{1+|\log y|^2}{1+y^4}|H^L\widetilde{\mathrm{Mod}}|^2\lesssim|D(t)|^2\lesssim\frac{\mathscr{E}_{2L+2}}{|\log M|}+\frac{b_1^{2L+2}}{|\log b_1|^2}.$$ □

Proof of (3-62) for $\widetilde{\mathrm{Mod}}$. We use Lemma 2.8 to estimate

$$\begin{aligned}
&\int\frac{1+|\log y|^2}{1+y^2}|AH^L\Lambda\widetilde{Q}_b|^2\\
&\quad\lesssim\sum_{i=1}^{L}\int\frac{1+|\log y|^2}{1+y^2}|H^L b_i\Lambda\widetilde{T}_i|^2+\sum_{i=2}^{L+2}\int\frac{1+|\log y|^2}{1+y^2}|AH^L\Lambda\widetilde{S}_i|^2\\
&\quad\lesssim\sum_{i=1}^{L} b_1^{2i}\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^2}\left|\frac{y^{2i-1}}{1+y^{2L}}\right|^2+\sum_{i=2}^{L+1} b_1^{2i}|\log b_1|^3+b_1^{2L+4}\int_{B_1\le y\le 2B_1}\frac{1+|\log y|^2}{1+y^2}\left|\frac{1+y^{2(L+2)-1}}{1+y^{2L+1}}\right|^2\\
&\quad\lesssim b_1^2.
\end{aligned}$$

Next, using the cancellation $AH^L T_i=0$, $1\le i\le L$,

$$\sum_{j=1}^{L}\frac{1+|\log y|^2}{1+y^2}|AH^L\widetilde{T}_i|^2\lesssim\sum_{j=1}^{L}\int_{B_1\le y\le 2B_1}\frac{1+|\log y|^C}{1+y^2}\left|\frac{y^{2i-1}}{1+y^{2L}}\right|^2\lesssim b_1|\log b_1|^C,$$

and finally using Lemma 2.8 again, for $1 \le i \le L$, we have

$$\sum_{j=i+1}^{L+2} \int \frac{1+|\log y|^2}{1+y^2} \left| AH^L \left[\chi_{B_1} \frac{\partial S_j}{\partial b_i} \right] \right|^2 \lesssim \sum_{j=i+1}^{L+1} b_1^{2(j-i)} |\log b_1|^C + b_1^{2(L-i)+4} \int_{y \le 2B_1} \frac{1+|\log y|^2}{1+y^2} \left| \frac{1+y^{2(L+2)-1}}{1+y^{2L+1}} \right|^2 \lesssim b_1.$$

We thus obtain from Lemma 3.3 the desired bound:

$$\int \frac{1+|\log y|^2}{1+y^2} |AH^L \widetilde{\mathrm{Mod}}|^2 \le \sqrt{b_1} |D(t)|^2 \lesssim \delta(\alpha^*) \left[\mathscr{E}_{2L+2} + \frac{b_1^{2L+2}}{|\log b_1|^2} \right]. \qquad \square$$

Step 7: *Nonlinear term* $N(\varepsilon)$. *Control near the origin* $y \le 1$. From (3-32) and a Taylor Lagrange formula, we rewrite

$$N(\varepsilon) = zN_0(\varepsilon), \quad z = y\left(\frac{\varepsilon}{y}\right)^2, \quad N_0(\varepsilon) = \frac{1}{y} \int_0^1 (1-\tau) f''(\widetilde{Q}_b + \tau\varepsilon)\, d\tau. \tag{3-65}$$

First observe from (C-2) and the Taylor expansion at the origin of T_i given by (2-39) that

$$z = \frac{1}{y} \left[\sum_{i=1}^{L+1} c_i T_{L+1-i} + r_\varepsilon \right]^2 = \sum_{i=0}^{L} \tilde{c}_i y^{2i+1} + \tilde{r}_\varepsilon, \tag{3-66}$$

where, from (C-3), (C-4),

$$\begin{aligned} |\tilde{c}_i| &\lesssim C(M) \mathscr{E}_{2L+2}, \\ |\partial_y^k \tilde{r}_\varepsilon| &\lesssim y^{2L+1-k} |\log y| C(M) \mathscr{E}_{2L+2}, \quad 0 \le k \le 2L+1. \end{aligned} \tag{3-67}$$

We now let $\tau \in [0,1]$ and

$$v_\tau = \widetilde{Q}_b + \tau\varepsilon,$$

and obtain from Proposition 2.12 and (C-2) the Taylor expansion at the origin

$$v_\tau = \sum_{i=0}^{L} \hat{c}_i y^{2i+1} + \hat{r}_\varepsilon \tag{3-68}$$

with

$$|\hat{c}_i| \lesssim 1, \quad |\partial_y^k \hat{r}_\varepsilon| \lesssim y^{2L+1-k} |\log y|, \quad 0 \le k \le 2L+1. \tag{3-69}$$

Recall that $f \in \mathscr{C}^\infty$ with $f^{2k}(0) = 0$, $k \ge 0$. We therefore obtain a Taylor expansion

$$f''(v_\tau) = \sum_{i=1}^{L+1} \frac{f^{(2i+1)}(0)}{i!} v_\tau^{2i-1} + \frac{v^{2L+2}}{(2L+1)!} \int_0^1 (1-\sigma)^{2L+1} f^{(2L+4)}(\sigma v_\tau)\, d\sigma,$$

which, together with (3-68), ensures an expansion

$$N_0(\varepsilon) = \sum_{i=0}^{L} \hat{\tilde{c}}_i y^{2i} + \hat{\tilde{r}}_\varepsilon, \quad |\hat{\tilde{c}}_i| \lesssim 1, \quad |\partial_y^k \hat{\tilde{r}}_\varepsilon| \lesssim y^{2L-k} |\log y|, \quad 0 \le k \le 2L+1.$$

Combining this with (3-66) ensures the expansion

$$N(\varepsilon) = zN_0(\varepsilon) = \sum_{i=0}^{L} \tilde{\tilde{c}}_i y^{2i+1} + \tilde{\tilde{r}}_\varepsilon \tag{3-70}$$

with

$$|\tilde{\tilde{c}}_i| \lesssim C(M)\mathscr{E}_{2L+2}, \quad |\partial_y^k \tilde{\tilde{r}}_\varepsilon| \lesssim y^{2L+1-k}|\log y| C(M)\mathscr{E}_{2L+2}, \quad 0 \le k \le 2L+1.$$

Observe that from a direct check this implies the bound

$$|\mathscr{A}^k \tilde{\tilde{r}}_\varepsilon| \lesssim \sum_{i=0}^{k} \frac{\partial_y^i \tilde{\tilde{r}}_\varepsilon}{y^{k-i}} \lesssim C(M)\mathscr{E}_{2L+2} \sum_{i=0}^{k} \frac{|\log y| y^{2L+1-i}}{y^{k-i}} \lesssim y^{2L+1-k}|\log y| C(M)\mathscr{E}_{2L+2}, \quad 0 \le k \le 2L+1.$$

We now compute using a simple induction based on the expansions (2-5), (2-6) and the cancellation $A(y) = O(y^2)$ that, for $y \le 1$,

$$\begin{cases} \mathscr{A}^{2k+1}\left(\sum_{i=0}^{L} \tilde{\tilde{c}}_i y^{2i+1}\right) = \sum_{i=k+1}^{L} c_{i,2k+1} y^{2(i-k)} + O(y^{2(L-k)+2}), \\ \mathscr{A}^{2k+2}\left(\sum_{i=0}^{L} \tilde{\tilde{c}}_i y^{2i+1}\right) = \sum_{i=k+1}^{L} c_{i,2k+2} y^{2(i-k)-1} + O(y^{2(L-k)+1}). \end{cases} \tag{3-71}$$

From (3-70), we conclude

$$\|\mathscr{A}^k N(\varepsilon)\|_{L^\infty(y\le 1)} \lesssim C(M)\mathscr{E}_{2L+2}, \quad 0 \le k \le 2L+1, \tag{3-72}$$

and thus, in particular, we get the control near the origin

$$\int_{y\le 1} \frac{1+|\log y|^2}{1+y^4} |H^L N(\varepsilon)|^2 + \int_{y\le 1} |AH^L \mathcal{N}(\varepsilon)|^2 \lesssim C(M)(\mathscr{E}_{2L+2})^2 \lesssim b_1^2 b_1^{2L+2}.$$

Control for $y \ge 1$. We give a detailed proof of (3-64). The proof of (3-61) follows the exact same lines (with more room in fact) and is left to the reader. Let

$$\zeta = \frac{\varepsilon}{y}, \quad N_1(\varepsilon) = \int_0^1 (1-\tau) f''(\widetilde{Q}_b + \tau\varepsilon)\, d\tau \quad \text{so that } N(\varepsilon) = \zeta^2 N_1. \tag{3-73}$$

We first estimate from (C-14): for $(i, j) \in \mathbb{N} \times \mathbb{N}$ with $1 \le i + j \le 2L+1$,

$$\left\| \frac{\partial_y^i \zeta}{y^{j-1}} \right\|_{L^\infty(y\ge 1)}^2 \lesssim \sum_{k=0}^{i} \left\| \frac{\partial_y^k \varepsilon}{y^{j+i-k}} \right\|_{L^\infty(y\ge 1)}^2 \lesssim |\log b_1|^C \begin{cases} b_1^{(i+j)2L/(2L-1)} & \text{for } 1 \le i+j \le 2L-1, \\ b_1^{2L+1} & \text{for } i+j = 2L, \\ b_1^{2L+2} & \text{for } i+j = 2L+1. \end{cases} \tag{3-74}$$

Similarly, from (C-12), for $(i, j) \in \mathbb{N} \times \mathbb{N}^*$ with $2 \le i + j \le 2L+2$,

$$\begin{aligned} \int_{y\ge 1} \frac{1+|\log y|^C}{1+y^{2j-2}} |\partial_y^i \zeta|^2 &\lesssim \sum_{k=0}^{i} \int_{y\ge 1} \frac{1+|\log y|^C}{1+y^{2j+2(i-k)}} |\partial_y^k \varepsilon|^2 \\ &\lesssim |\log b_1|^C \begin{cases} b_1^{(i+j-1)2L/(2L-1)} & \text{for } 2 \le i+j \le 2L, \\ b_1^{2L+1} & \text{for } i+j = 2L+1, \\ b_1^{2L+2} & \text{for } i+j = 2L+2. \end{cases} \end{aligned} \tag{3-75}$$

Moreover, from the energy bound (3-23),

$$\int_{y\geq 1} |\zeta|^2 \lesssim 1. \tag{3-76}$$

We now claim the pointwise bound, for $y \geq 1$,

$$\text{for all } 1 \leq k \leq 2L+1, \quad |\partial_y^k N_1(\varepsilon)| \lesssim |\log b_1|^C \left[\frac{1}{y^{k+1}} + b_1^{a_k/2}\right], \tag{3-77}$$

with

$$a_k = \begin{cases} k2L/(2L-1) & \text{for } 1 \leq k \leq 2L-1 \\ 2L+1 & \text{for } k = 2L, \\ 2L+2 & \text{for } k = 2L+1, \end{cases} \tag{3-78}$$

which is proved below. For $k = 0$, we simply need the obvious bound

$$\|N_1(\varepsilon)\|_{L^\infty(y\geq 1)} \lesssim 1. \tag{3-79}$$

Then, by brute force, from (3-73), (3-77), (3-79), we estimate

$$\begin{aligned} |AH^L N(\varepsilon)| &\lesssim \sum_{k=0}^{2L+1} \frac{|\partial_y^k N(\varepsilon)|}{y^{2L+1-k}} \lesssim \sum_{k=0}^{2L+1} \frac{1}{y^{2L+1-k}} \sum_{i=0}^{k} |\partial_y^i \zeta^2| |\partial_y^{k-i} N_1(\varepsilon)| \\ &\lesssim \sum_{k=0}^{2L+1} \frac{|\partial_y^k \zeta^2|}{y^{2L+1-k}} + \sum_{k=1}^{2L+1} \frac{1}{y^{2L+1-k}} \sum_{i=0}^{k-1} |\partial_y^i \zeta^2| |\log b_1|^C \left[\frac{1}{y^{k-i+1}} + b_1^{(a_{k-i})/2}\right] \\ &\lesssim \sum_{k=0}^{2L+1} \frac{|\partial_y^k \zeta^2|}{y^{2L+1-k}} + |\log b_1|^C \sum_{i=0}^{2L} \frac{|\partial_y^i \zeta^2|}{y^{2L+2-i}} + |\log b_1|^C \sum_{k=1}^{2L+1} \sum_{i=0}^{k-1} b_1^{(a_{k-i})/2} \frac{|\partial_y^i \zeta^2|}{y^{2L+1-k}} \\ &\lesssim |\log b_1|^C \left[\sum_{k=0}^{2L+1} \frac{|\partial_y^k \zeta^2|}{y^{2L+1-k}} + \sum_{k=1}^{2L+1} \sum_{i=0}^{k-1} b_1^{(a_{k-i})/2} \frac{|\partial_y^i \zeta^2|}{y^{2L+1-k}}\right], \end{aligned}$$

and hence

$$\begin{aligned} &\int_{y\geq 1} |AH^L N(\varepsilon)|^2 \\ &\quad \lesssim |\log b_1|^C \sum_{k=0}^{2L+1} \sum_{i=0}^{k} \int_{y\geq 1} \frac{|\partial_y^i \zeta|^2 |\partial_y^{k-i} \zeta|^2}{y^{4L+2-2k}} + |\log b_1|^C \sum_{k=1}^{2L+1} \sum_{i=0}^{k-1} \sum_{j=0}^{i} b_1^{a_{k-i}} \int_{y\geq 1} \frac{|\partial_y^j \zeta|^2 |\partial_y^{i-j} \zeta|^2}{y^{4L+2-2k}}. \end{aligned}$$

We now claim the bounds

$$\sum_{k=0}^{2L+1} \sum_{i=0}^{k} \int_{y\geq 1} \frac{|\partial_y^i \zeta|^2 |\partial_y^{k-i} \zeta|^2}{y^{4L+2-2k}} \lesssim |\log b_1|^C b_1^{\delta(L)} b_1^{2L+3}, \tag{3-80}$$

$$|\log b_1|^C \sum_{k=1}^{2L+1} \sum_{i=0}^{k-1} \sum_{j=0}^{i} b_1^{a_{k-i}} \int_{y\geq 1} \frac{|\partial_y^j \zeta|^2 |\partial_y^{i-j} \zeta|^2}{y^{4L+2-2k}} \lesssim |\log b_1|^C b_1^{\delta(L)} b_1^{2L+3} \tag{3-81}$$

for some $\delta(L) > 0$, and this concludes the proof of (3-64) for $N(\varepsilon)$.

Proof of (3-77). We first extract from Proposition 2.12 the rough bound

$$|\partial_y^k \widetilde{Q}_b| \lesssim |\log b_1|^C \left[\frac{1}{y^{k+1}} + \sum_{i=1}^{2L+2} b_1^i y^{2i-1-k} \mathbf{1}_{y\le 2B_1} \right] \lesssim \frac{|\log b_1|^C}{y^{k+1}}. \tag{3-82}$$

Then let $\tau \in [0,1]$ and $v_\tau = \widetilde{Q}_b + \tau\varepsilon$. From (3-82), (C-14), (3-78), we conclude

$$|\partial_y^k v_\tau| \lesssim |\log b_1|^C \left[\frac{1}{y^{k+1}} + b_1^{a_k/2} \right], \quad 1 \le k \le 2L+1,\ y \ge 1. \tag{3-83}$$

We therefore estimate N_1 through the formula (3-73) using the rough bound $|\partial_v^i f| \lesssim 1$ and the Faa di Bruno formula: for $1 \le k \le 2L+1$,

$$\begin{aligned} |\partial_y^k N_1(\varepsilon)| &\lesssim \int_0^1 \sum_{m_1+2m_2+\cdots+km_k=k} |\partial_v^{m_1+\cdots+m_k} f(v_\tau)| \prod_{i=1}^k |\partial_y^i v_\tau|^{m_i}\, d\tau \\ &\lesssim |\log b_1|^C \sum_{m_1+2m_2+\cdots+km_k=k} |\prod_{i=1}^k \left[\frac{1}{y^{i+1}} + b_1^{a_i/2} \right]^{m_i} \\ &\lesssim |\log b_1|^C \left[\frac{1}{y^{k+1}} + b_1^{\alpha_k/2} \right]. \end{aligned}$$

To estimate α_k from the definition (3-78), we observe that for $k \le 2L-1$, $i \le 2L-1$, and thus

$$\alpha_k \ge \sum_{i=0}^k \frac{2iL}{L-1} m_i = \frac{2kL}{L-1} = a_k.$$

For $k = 2L$, we have to treat the boundary term $i = k$, $(m_1, \dots, m_{k-1}, m_k) = (0, \dots, 0, 1) = 1$, which yields

$$\alpha_{2L} \ge \min\left\{ \frac{2L(2L)}{2L-1}; 2L+1 \right\} = 2L+1.$$

For $k = 2L+1$, we have the two boundary terms $(m_1, m_2, \dots, m_{k-2}, m_{k-1}, m_k) = (1, 0, \dots, 0, 1, 0)$, $(m_1, \dots, , m_{k-1}, m_k) = (0, \dots, 0, 1)$, which yield

$$\alpha_{2L+1} \ge \min\left\{ \frac{2L(2L+1)}{2L-1}; 2L+1+\frac{2L}{2L-1}; 2L+2 \right\} = 2L+2,$$

and (3-77) is proved. □

Proof of (3-80). Let $0 \le k \le 2L+1$, $0 \le i \le k$. Let $I_1 = k-i$, $I_2 = i$. Then we can pick $J_2 \in \mathbb{N}^*$ such that

$$\max\{1; 2-i\} \le J_2 \le \min\{2L+3-k; 2L+2-i\}$$

and define

$$J_1 = 2L+3-k-J_2.$$

Then, from direct inspection,

$$(I_1, J_1, I_2, J_2) \in \mathbb{N}^3 \times \mathbb{N}^*, \quad \begin{cases} 1 \le I_1 + J_1 \le 2L+1, \quad 2 \le I_2 + J_2 \le 2L+2, \\ I_1 + I_2 + J_1 + J_2 = 2L+3. \end{cases}$$

Thus

$$A_i = \int_{y\ge 1} \frac{|\partial_y^i \zeta|^2 |\partial_y^{k-i}\zeta|^2}{y^{4L+2-2k}} = \int_{y\ge 1} \frac{|\partial_y^{I_1}\zeta|^2 |\partial_y^{I_2}\zeta|^2}{y^{2J_1-2+2J_2-2}} \lesssim \left\| \frac{\partial_y^{I_1}\zeta}{y^{J_1-1}} \right\|^2_{L^\infty(y\ge 1)} \int_{y\ge 1} \frac{|\partial_y^{I_2}\zeta|^2}{y^{2J_2-2}} \lesssim |\log b_1|^C b_1^{d_{i,k}},$$

where we now compute the exponent $d_{i,k}$ using (3-74), (3-75):

- for $I_1 + J_1 \le 2L-1$, $I_2 + J_2 \le 2L$,
$$d_{i,k} = \frac{2L}{2L-1}(I_1 + J_1 + I_2 + J_2 - 1) = \frac{2L(2L+2)}{2L-1} > 2L+3;$$
- for $I_1 + J_1 = 2L$, $I_2 + J_2 = 3$,
$$d_{i,k} = 2L + 1 + \frac{2L}{2L-1}(3-1) > 2L+3;$$
- for $I_1 + J_1 = 2L+1$, $I_2 + J_2 = 2$,
$$d_{i,k} = 2L + 2 + \frac{2L}{2L-1} > 2L+3;$$
- for $I_2 + J_2 = 2L+1$, $I_1 + J_1 = 2$,
$$d_{i,k} = \frac{2(2L)}{2L-1} + 2L + 1 > 2L+3;$$
- for $I_2 + J_2 = 2L+2$, $I_1 + J_1 = 1$,
$$d_{i,k} = 2L + 2 + \frac{2L}{2L-1} > 2L+3;$$

and (3-80) is proved. □

Proof of (3-81). Let $1 \le k \le 2L+1$, $0 \le j \le i \le k-1$. For $k = 2L+1$ and $0 \le i = j \le 2L$, we use the energy bound (3-76) to estimate

$$b_1^{a_{k-i}} \int_{y\ge 1} \frac{|\partial_y^j \zeta|^2 |\partial_y^{i-j}\zeta|^2}{y^{4L+2-2k}} = b_1^{a_{2L+1-i}} \int_{y\ge 1} |\partial_y^i \zeta|^2 |\zeta|^2 \lesssim b_1^{a_{2L+1-i}} \|\zeta\|^2_{L^\infty(y\ge 1)} \int_{y\ge 1} |\partial_y^i \zeta|^2 \lesssim b_1^{d_{i,2L+1}}$$

with

$$d_{i,2L+1} = \begin{cases} \dfrac{2L}{2L-1} + 2L + 2 & \text{for } i = 0, \\ \dfrac{2L}{2L-1} + 2L + 2 + \dfrac{2L}{2L-1} & \text{for } i = 1, \\ \dfrac{2L}{2L-1} + \dfrac{2L}{2L-1}(i+1-1) + \dfrac{2L}{2L-1}(2L+1-i) & \text{for } 2 \le i \le 2L \end{cases}$$
$$> 2L+3.$$

This exceptional case being treated, we let $I_1 = j$, $I_2 = i - j$, and pick $J_2 \in \mathbb{N}^*$ with

$$\max\{1; 2-(i-j); 2-(k-j)\} \le J_2 \le \min\{2L+3-k; 2L+2-(k-j); 2L+2-(i-j)\}.$$

Let

$$J_1 = 2L+3-k-J_2.$$

Then we can directly check that

$$(I_1, J_1, I_2, J_2) \in \mathbb{N}^3 \times \mathbb{N}^*, \quad \begin{cases} 1 \le I_1 + J_1 \le 2L+1, \quad 2 \le I_2 + J_2 \le 2L+2, \\ I_1 + I_2 + J_1 + J_2 = 2L+3-(k-i). \end{cases}$$

Hence

$$b_1^{a_{k-i}} \int_{y\ge 1} \frac{|\partial_y^j \zeta|^2 |\partial_y^{i-j}\zeta|^2}{y^{4L+2-2k}} = b_1^{a_{k-i}} \int_{y\ge 1} \frac{|\partial_y^{I_1}\zeta|^2 |\partial_y^{I_2}\zeta|^2}{y^{2J_2-2+2J_1-2}} \lesssim b_1^{a_{k-i}} \left\| \frac{\partial_y^{I_1}\zeta}{y^{J_1-1}} \right\|_{L^\infty(y\ge 1)}^2 \int_{y\ge 1} \frac{|\partial_y^{I_2}\zeta|^2}{y^{2J_2-2}} \lesssim |\log b_1|^C b_1^{d_{i,j,k}},$$

where we now compute the exponent $d_{i,k}$ using (3-74), (3-75), (3-78):

- for $I_1 + J_1 \le 2L-1$, $I_2 + J_2 \le 2L$, $k - i \le 2L-1$,
$$d_{i,j,k} = (k-i)\frac{2L}{2L-1} + (2L+3-(k-i)-1)\frac{2L}{2L-1} = \frac{2L(2L+2)}{2L-1} > 2L+3;$$
- for $I_1 + J_1 \le 2L-1$, $I_2 + J_2 \le 2L$, $k - i = 2L$,
$$d_{i,j,k} = 2L+1+(2L+3-2L-1)\frac{2L}{2L-1} > 2L+3;$$
- for $I_1 + J_1 = 2L$, $I_2 + J_2 = 3-(k-i) \ge 2$ and thus $k-i = 1$, $I_2 + J_2 = 2$,
$$d_{i,j,k} = \frac{2L}{2L-1} + 2L + 1 + \frac{2L}{2L_1} > 2L+3;$$
- for $I_2 + J_2 = 2L+1$, $I_1 + J_1 = 2-(k-i) \ge 1$ and thus $k-i = 1$. $I_1 + J_1 = 1$,
$$d_{i,j,k} = \frac{2L}{2L-1} + 2L + 1 + \frac{2L}{2L-1} > 2L+3.$$

This concludes the proof of (3-81). □

Step 8: *small linear term* $L(\varepsilon)$. Let us rewrite from a Taylor expansion

$$L(\varepsilon) = -\varepsilon N_2(\tilde{\alpha}_b), \quad N_2(\tilde{\alpha}_b) = \frac{f'(Q+\tilde{\alpha}_b) - f'(Q)}{y^2} = \frac{\tilde{\alpha}_b}{y^2}\int_0^1 f''(Q+\tau\tilde{\alpha}_b)\,d\tau. \tag{3-84}$$

Control for $y \le 1$. We use a Taylor expansion with the cancellation $f^{2k}(0) = 0$, $k \ge 0$, and Proposition 2.12 to ensure, for $y \le 1$, a decomposition

$$N_2(\tilde{\alpha}_b) = b_1\left[\sum_{i=0}^{L} \tilde{c}_i y^{2i} + r\right], \quad |\tilde{c}_i| \lesssim 1, \quad |\partial_y^k r| \lesssim y^{2L+2-k}, \quad 0 \le k \le 2L+1.$$

We combine this with (C-2) and obtain the representation, for $y \le 1$,

$$L(\varepsilon) = \Big[\sum_{i=1}^{L+1} c_i T_{L+1-i} + r_\varepsilon\Big] b_1 \Big[\sum_{i=0}^{L} c_i y^{2i} + r\Big] = b_1 \Big[\sum_{i=1}^{L} \hat{c}_i y^{2i-1} + \hat{r}_\varepsilon\Big] \tag{3-85}$$

with bounds

$$|\hat{c}_i| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}}, \tag{3-86}$$

$$|\partial_y^k \hat{r}_\varepsilon| \lesssim y^{2L+1-k}|\log y| C(M)\sqrt{\mathscr{E}_{2L+2}}, \quad 0 \le k \le 2L+1,\ y \le 1. \tag{3-87}$$

We now apply $(\mathscr{A}^k)_{0\le k\le 2L+1}$ to (3-85) and conclude using (3-71) that

$$\|\mathscr{A}^k L(\varepsilon)\|_{L^\infty(y\le 1)} \lesssim b_1 C(M)\sqrt{\mathscr{E}_{2L+2}}, \tag{3-88}$$

from which

$$\int_{y\le 1} \frac{1+|\log y|^2}{1+y^4}|H^L L(\varepsilon)|^2 + \int_{y\le 1} |AH^L L(\varepsilon)|^2 \lesssim C(M) b_1^2 \mathscr{E}_{2L+2} \lesssim C(M) b_1^2 b_1^{2L+2}.$$

Control for $y \ge 1$. We give a detailed proof of (3-64). The proof of (3-61) follows the exact same lines and is left to the reader. We claim the pointwise bound, for $y \ge 1$,

$$\text{for all } 0 \le k \le 2L+1, \quad |\partial_y^k N_2(\tilde{\alpha}_b)| \lesssim \frac{b_1|\log b_1|^C}{y^{k+1}}, \tag{3-89}$$

which is proved below. From the Leibniz rule, this yields

$$|\partial_y^k L(\varepsilon)| \lesssim \sum_{i=0}^{k} \frac{b_1|\log b_1|^C |\partial_y^i \varepsilon|}{y^{k-i+1}}, \tag{3-90}$$

and thus

$$\begin{aligned} |AH^L L(\varepsilon)| &\lesssim \sum_{k=0}^{2L+1} \frac{|\partial_y^k L(\varepsilon)|}{y^{2L+1-k}} \lesssim \sum_{k=0}^{2L+1} \frac{1}{y^{2L+1-k}} \sum_{i=0}^{k} \frac{b_1|\log b_1|^C |\partial_y^i \varepsilon|}{y^{k-i+1}} \\ &\lesssim b_1|\log b_1|^C \sum_{i=0}^{2L+1} \frac{|\partial_y^i \varepsilon|}{y^{2L+2-i}}. \end{aligned}$$

Therefore, from (C-11) with $k = L$, we conclude

$$\int_{y\ge 1} |AH^L L(\varepsilon)|^2 \lesssim b_1^2 |\log b_1|^C \sum_{i=0}^{2L+1} \int_{y\ge 1} \frac{|\partial_y^i \varepsilon|^2}{y^{4L+4-2i}} \lesssim |\log b_1|^C b_1^{2L+4},$$

and (3-64) is proved.

Proof of (3-89). Let

$$N_3 = \int_0^1 f''(Q + \tau\tilde{\alpha}_b)\, d\tau.$$

Letting $\tilde{v}_\tau = Q + \tau\tilde{\alpha}_b$, $0 \le \tau \le 1$, from Proposition 2.12 we estimate

$$|\partial_y^k \tilde{v}_\tau| \lesssim \frac{|\log b_1|^C}{y^{k+1}}, \quad 1 \le k \le 2L+1,\ y \ge 1,$$

and hence, using the Faa di Bruno formula,

$$\begin{aligned}|\partial_y^k N_3(\tilde{\alpha}_b)| &\lesssim \int_0^1 \sum_{m_1+2m_2+\cdots+km_k=k} |\partial_v^{m_1+\cdots+m_k} f(\tilde{v}_\tau)| \prod_{i=1}^k |\partial_y^i \tilde{v}_\tau|^{m_i}\, d\tau \\ &\lesssim |\log b_1|^C \sum_{m_1+2m_2+\cdots+km_k=k} \prod_{i=1}^k \left[\frac{1}{y^{i+1}}\right]^{m_i} \lesssim \frac{|\log b_1|^C}{y^{k+1}}.\end{aligned}$$

This yields in particular the rough bound

$$|\partial_y^k N_3(\tilde{\alpha}_b)| \lesssim \frac{|\log b_1|^C}{y^k}, \quad y \ge 1,\ 0 \le k \le 2L+1,$$

and hence, from the Leibniz rule,

$$\left|\partial_y^k \left(\frac{N_3(\tilde{\alpha}_b)}{y^2}\right)\right| \lesssim \frac{|\log b_1|^C}{y^{k+2}}, \quad y \ge 1,\ 0 \le k \le 2L+1. \tag{3-91}$$

From Proposition 2.12, we extract the rough bound

$$|\partial_y^k \tilde{\alpha}_b| \lesssim \frac{|\log b_1|^C b_1}{y^{k-1}}, \quad 0 \le k \le 2L+1$$

and, from the Leibniz rule, we conclude

$$|\partial_y^k N_2| \lesssim \sum_{i=0}^k |\log b_1|^C \frac{b_1}{y^{i+2} y^{k-i-1}} \lesssim \frac{b_1 |\log b_1|^C}{y^{k+1}},$$

which proves (3-89). □

This concludes the proofs of (3-61), (3-62), (3-63), (3-64), and thus of Proposition 3.6. □

4. Closing the bootstrap and proof of Theorem 1.1

We are now in position to close the bootstrap bounds of Proposition 3.1. The proof of Theorem 1.1 will easily follow.

Proof of Proposition 3.1.

Proof. Our aim is first to show that for $s < s^*$, the a priori bounds (3-23), (3-24), (3-25) can be improved, and then the unstable modes $(U_k)_{2\le k\le L}$ will be controlled through a standard topological argument.

Step 1: *improved $\dot{H}^1$ bound*. First observe from (3-17) and the a priori bound on U_k for $s < s^*$ that

$$|b_k(s)| \lesssim |b_k(0)|. \tag{4-1}$$

The energy bound (3-23) is now a straightforward consequence of the dissipation of energy and the bounds (4-1), (3-26). Indeed, let

$$\tilde{\varepsilon} = \varepsilon + \hat{\alpha}.$$

Then

$$E_0 = \int |\partial_y (Q+\tilde{\varepsilon})|^2 + \int \frac{g^2(Q+\tilde{\varepsilon})}{y^2} = E(Q) + (H\tilde{\varepsilon}, \tilde{\varepsilon}) + \int \frac{1}{y^2}[g^2(Q+\tilde{\varepsilon}) - 2f(Q)\tilde{\varepsilon} - f'(Q)\tilde{\varepsilon}^2]. \quad (4\text{-}2)$$

We first use the bound on the profile which is easily extracted from Proposition 2.12

$$\int |\partial_y \hat{\alpha}|^2 + \int \frac{|\hat{\alpha}|^2}{y^2} \lesssim b_1 |\log b_1|^C \leq \sqrt{b_1(0)}$$

using (4-1). Using Lemma B.1 ensures the coercivity

$$(H\tilde{\varepsilon}, \tilde{\varepsilon}) \geq c(M)\left[\int |\partial_y \tilde{\varepsilon}|^2 + \int \frac{|\tilde{\varepsilon}|^2}{y^2}\right] - \frac{1}{c(M)}(\tilde{\varepsilon}, \Phi_M)^2 \geq c(M)\left[\int |\partial_y \varepsilon|^2 + \int \frac{|\varepsilon|^2}{y^2}\right] - \sqrt{b_1(0)}.$$

The nonlinear term is estimated from a Taylor expansion:

$$\left|\int \frac{1}{y^2}[g^2(Q+\tilde{\varepsilon}) - 2f(Q)\tilde{\varepsilon} - f'(Q)\tilde{\varepsilon}^2]\right| \lesssim \int \frac{|\tilde{\varepsilon}|^3}{y^2} \lesssim \left(\int |\partial_y \tilde{\varepsilon}|^2 + \int \frac{|\tilde{\varepsilon}|^2}{y^2}\right)^{\frac{3}{2}},$$

where we used the Sobolev bound

$$\|\tilde{\varepsilon}\|_{L^\infty}^2 \lesssim \|\partial_y \tilde{\varepsilon}\|_{L^2} \left\|\frac{\tilde{\varepsilon}}{y}\right\|_{L^2}.$$

We inject these bounds into the dissipation of energy (4-2) together with the initial bound (3-20) to estimate

$$\int |\partial_y \varepsilon|^2 + \int \frac{|\varepsilon|^2}{y^2} \lesssim \int |\partial_y \tilde{\varepsilon}|^2 + \int \frac{|\tilde{\varepsilon}|^2}{y^2} + b_1 |\log b_1|^C \leq c(M)\sqrt{b_1(0)} \leq (b_1(0))^{\frac{1}{4}} \quad (4\text{-}3)$$

for $|b_1(0)| \leq b_1^*(M)$ small enough.

Step 2: *integration of the scaling law*. Let us compute explicitly the scaling parameter for $s < s^*$. From (3-36), (3-26), (2-101), (2-99), we have the rough bound

$$-\frac{\lambda_s}{\lambda} = \frac{c_1}{s} - \frac{|d_1|}{\log s} + O\left(\frac{1}{s(\log s)^{5/4}}\right),$$

which we rewrite as

$$\left|\frac{d}{ds}\left\{\frac{s^{c_1}\lambda(s)}{(\log s)^{|d_1|}}\right\}\right| \lesssim \frac{1}{s(\log s)^{5/4}}. \quad (4\text{-}4)$$

We integrate this using the initial value $\lambda(0) = 1$ and conclude

$$\frac{s^{c_1}\lambda(s)}{(\log s)^{|d_1|}} = \frac{s_0^{c_1}}{(\log s_0)^{|d_1|}} + O\left(\frac{1}{(\log s_0)^{1/4}}\right). \quad (4\text{-}5)$$

Together with the law for b_1 given by (3-26), (2-101), (2-99), this implies

$$b_1(0)^{c_1}|\log b_1(0)|^{|d_1|} \lesssim \frac{b_1^{c_1}(s)|\log b_1|^{|d_1|}}{\lambda(s)} \lesssim b_1(0)^{c_1}|\log b_1(0)|^{|d_1|}. \tag{4-6}$$

Step 3: *improved control of* $\mathcal{E}_{2L+2}$. We now improve the control of the high order $\mathcal{E}_{2L+2}$ energy (3-25) by reintegrating the Lyapounov monotonicity (3-48) in the regime governed by (4-5), (2-101). Indeed, we inject the bootstrap bound (3-25) into the monotonicity formula (3-48) and integrate in time s: for all $s \in [s_0, s^*)$,

$$\begin{aligned}\mathcal{E}_{2L+2}(s) \le 2\left(\frac{\lambda(s)}{\lambda(0)}\right)^{4L+2}\left[\mathcal{E}_{2L+2}(0) + Cb_1^{\frac{4}{5}}(0)\frac{b_1^{2L+2}(0)}{|\log b_1(0)|^2}\right] + \frac{b_1^{2L+2}(s)}{|\log b_1(s)|^2} \\ + C\left[1 + \frac{K}{\log M} + \sqrt{K}\right]\lambda^{2L+4}(s)\int_{s_0}^{s}\frac{b_1}{\lambda^{4L+2}}\frac{b_1^{2L+2}}{|\log b_1|^2}\,d\sigma\end{aligned} \tag{4-7}$$

for some universal constant $C > 0$ independent of M. We now observe from (4-6) that the integral in the right hand side of (4-7) is divergent, since

$$\frac{b_1}{\lambda^{4L+2}}\frac{b_1^{2L+2}}{|\log b_1|^2} \gtrsim C(b_0)\frac{b_1^{2L+3}}{b_1^{(4L+2)c_1}|\log b_1|^C} \gtrsim \frac{C(b_0)}{(\log s)^C s^{2L+3-(4L+2)L/(2L-1)}} = \frac{C(b_0)}{(\log s)^C s^{(2L-3)/(2L-1)}},$$

and therefore, from (4-6) and $1/s \lesssim b_1 \lesssim 1/s$,

$$\lambda^{4L+2}(s)\int_{s_0}^{s}\frac{b_1}{\lambda^{4L+2}}\frac{b_1^{2L+2}}{|\log b_1|^2}\,d\sigma \lesssim \frac{b_1^{2L+2}(s)}{|\log b_1(s)|^2}.$$

We now estimate the contribution of the initial data using (4-6) and the initial bounds (3-21), (3-22):

$$\begin{aligned}&\left(\frac{\lambda(s)}{\lambda(0)}\right)^{4L+2}\left[\mathcal{E}_{2L+2}(0) + Cb_1^{\frac{4}{5}}(0)\frac{b_1^{2L+2}(0)}{|\log b_1(0)|^2}\right] \\ &\quad\lesssim \lambda^{4L+2}(s)b_1^{\frac{4}{5}}(0)\frac{b_1^{2L+2}(0)}{|\log b_1(0)|^2} \\ &\quad\lesssim (b_1(s))^{(4L+2)L/(2L-1)}|\log b_1(s)|^C(b_1(0))^{\frac{4}{5}+2L+2-(4L+2)L/(2L-1)}|\log b_1(0)|^C \lesssim \frac{b_1^{2L+2}(s)}{|\log b_1(s)|^2},\end{aligned}$$

where we used the algebra, for $L \ge 2$,

$$0 < \frac{L(4L+2)}{2L-1} - (2L+2) = \frac{2}{2L-1} < \frac{4}{5}.$$

Injecting these bounds into (4-7) yields

$$\mathcal{E}_{2L+2}(s) \lesssim \frac{b_1^{2L+2}(s)}{|\log b_1(s)|^2}\left[1 + \frac{K}{\log M} + \sqrt{K}\right] \le \frac{K}{2}\frac{b_1^{2L+2}(s)}{|\log b_1(s)|^2} \tag{4-8}$$

for K large enough independent of M.

Step 4: *Improved control of* $\mathscr{E}_{2k+2}$, $0 \le k \le L-1$. We now claim the improved bound on the intermediate energies

$$\mathscr{E}_{2k+2} \le b_1^{(2k+1)2L/(2L-1)} |\log b_1|^{C+\sqrt{K}}. \tag{4-9}$$

This follows from the monotonicity formula, for $0 \le k \le L-1$,

$$\frac{d}{dt}\left\{\frac{1}{\lambda^{4k+2}}\left[\mathscr{E}_{2k+2} + O\left(b_1^{\frac{1}{2}} b_1^{(4k+2)2L/(2L-1)}\right)\right]\right\} \lesssim \frac{|\log b_1|^C}{\lambda^{4k+4}}\left[b_1^{2k+3} + b_1^{1+\delta+(2k+1)2L/(2L-1)} + \sqrt{b_1^{2k+4}\mathscr{E}_{2k+2}}\right] \tag{4-10}$$

for some universal constants $C, \delta > 0$ independent of the bootstrap constant K. The proof is similar to that of (4-8) and in fact simpler since we allow for logarithmic losses; details are given in Appendix F.

Using (4-5), we now estimate

$$\lambda^{4k+2}(s) \int_{s_0}^{s} \frac{b_1^{2k+3}}{\lambda^{4k+2}} |\log b_1|^C \lesssim \frac{(\log s)^{|d_1|}}{s^{(4k+2)c_1}} \int_{s_0}^{s} \frac{(\log \sigma)^C}{\sigma^{2k+3-c_1(4k+2)}}\, d\sigma.$$

From (2-97), we compute

$$(2k+3) - c_1(4k+2) = 1 + \frac{2(L-k-1)}{2L-1}, \tag{4-11}$$

and hence

$$\lambda^{4k+2}(s) \int_{s_0}^{s} \frac{b_1^{2k+3}}{\lambda^{4k+2}} |\log b_1|^C \lesssim \frac{(\log s)^{|d_1|+C}}{s^{(4k+2)c_1}} \lesssim b_1^{(4k+2)c_1} |\log b_1|^C.$$

Similarly, from (4-6),

$$\lambda^{4k+2}(s) \int_{s_0}^{s} \frac{b_1^{1+\delta+(2k+1)2L/(2L-1)}}{\lambda^{4k+2}} |\log b_1|^C\, d\sigma \lesssim \frac{(\log s)^{|d_1|+C}}{s^{(4k+2)c_1}} \int_{s_0}^{s} \frac{(\log \sigma)^C}{\sigma^{1+\delta}}\, d\sigma \lesssim b_1^{(4k+2)c_1} |\log b_1|^C,$$

and, using (4-11), (3-24),

$$\begin{aligned}\lambda^{4k+2}(s) \int_{s_0}^{s} \frac{|\log b_1|^C}{\lambda^{4k+2}} \sqrt{b_1^{2k+4}\mathscr{E}_{2k+2}}\, d\sigma &\lesssim \frac{(\log s)^{|d_1|+C}}{s^{(4k+2)c_1}} \int_{s_0}^{s} \frac{(\log \sigma)^{C+\sqrt{K}}}{\sqrt{s^{2k+4-(2k+1)2L/(2L-1)}}}\, d\sigma \\ &\lesssim |\log b_1|^{C+\sqrt{K}} b_1^{(4k+2)c_1} \int_{s_0}^{s} \frac{d\sigma}{\sigma^{1+(L-k-1)/(2L-1)}} \lesssim |\log b_1|^{C+\sqrt{K}} b_1^{(4k+2)c_1}.\end{aligned}$$

Using the initial smallness (3-21) and (4-6), the time integration of (4-10) from $s = s_0$ to s therefore yields

$$\mathscr{E}_{2k+2}(s) \lesssim \lambda^{4k+2}(s) b_1(0)^{10L+4} + |\log b_1(s)|^{C+\sqrt{K}} b_1^{4k+2}(s) \lesssim |\log b_1(s)|^{C+\sqrt{K}} b_1^{(4k+2)c_1}(s),$$

and (4-9) is proved.

Remark 4.1. *For* $0 \le k \le L-2$, *the above argument shows the bound*

$$\mathscr{E}_{2k+2} \lesssim \lambda^{4k+2},$$

which equivalently corresponds to a uniform high order Sobolev control w. The logarithmic loss for $k = L-1$ could be gained as well with a little more work; see [Raphaël and Schweyer 2013] for the case $L = 1$. This shows that the limiting excess of energy u^* in (1-12) enjoys some suitable high order Sobolev regularity.

Step 5 contradiction through a topological argument. Let us consider

$$\tilde{b}_k = b_k \quad \text{for } 1 \le k \le L,\ \tilde{b}_L \text{ given by (3-44)}$$

and the associated variables

$$\tilde{b}_k = b_k^e + \frac{\widetilde{U}_k}{s^k(\log s)^{5/4}}, \quad 1 \le k \le L, \quad \tilde{b}_{k+1} = \widetilde{U}_{k+1} \equiv 0, \quad \widetilde{V} = P_L \widetilde{U}.$$

From (3-45),

$$|V - \widetilde{V}| \lesssim s^L |\log s|^C b_1^{L+\frac{1}{2}} \lesssim \frac{1}{s^{1/4}}. \tag{4-12}$$

Let the associated control of the unstable models be

$$|\widetilde{V}_1(s)| \le 2, \quad (\widetilde{V}_2(s), \dots, \widetilde{V}_L(s)) \in \mathcal{B}_{L-1}\big(\tfrac{1}{2}\big), \tag{4-13}$$

and the slightly modified exit time

$$\tilde{s}^* = \sup\{s \ge s_0 \text{ such that (3-23), (3-24), (3-25), (4-13) hold on } [s_0, s]\}.$$

Then (4-12) and the assumption (3-27) imply

$$\text{for all } (\widetilde{V}_2(0), \dots, \widetilde{V}_L(0)) \in \mathcal{B}_{L-1}\big(\tfrac{1}{2}\big), \quad \tilde{s}^* < +\infty. \tag{4-14}$$

We claim that this contradicts Brouwer's fixed point theorem.

Indeed, first, from (2-102), we estimate

$$(\tilde{b}_k)_s + \left(2k - 1 + \frac{2}{\log s}\right)\tilde{b}_1\tilde{b}_k - \tilde{b}_{k+1} = \frac{1}{s^{k+1}(\log s)^{5/4}}\left[s(\widetilde{U}_k)_s - (A_L\widetilde{U})_k + O\left(\frac{1}{\sqrt{\log s}}\right)\right],$$

and thus, from (3-37), (3-46), (4-8), and (2-44),

$$\begin{aligned}
|s(\widetilde{U}_k)_s - (A_L\widetilde{U})_k| &\lesssim \frac{1}{\sqrt{\log s}} + s^{k+1}(\log s)^{\frac{5}{4}}\left|(\tilde{b}_k)_s + \left(2k-1+\frac{2}{\log s}\right)\tilde{b}_1\tilde{b}_k - \tilde{b}_{k+1}\right| \\
&\lesssim \frac{1}{\sqrt{\log s}} + s^{k+1}(\log s)^{\frac{5}{4}}\left[b_1^{L+\frac{3}{2}} + \frac{1}{s^{k+1}(\log s)^2} + \frac{b_1^{L+1}}{|\log b_1|^{3/2}}\right] \\
&\lesssim \frac{1}{(\log s)^{1/4}}.
\end{aligned}$$

Hence, using the diagonalization (2-104),

$$s(\tilde{V})_s = D_L\widetilde{V}_s + O\left(\frac{1}{(\log s)^{1/4}}\right). \tag{4-15}$$

This first implies the control of the stable mode $\widetilde{V}_1$ from (2-104),

$$|(s\widetilde{V}_1)_s| \lesssim \frac{1}{(\log s)^{1/4}},$$

and thus, from (3-19),

$$|\widetilde{V}_1(s)| \lesssim \frac{1}{s} + \frac{1}{s}\int_{s_0}^{s} \frac{d\tau}{(\log \tau)^{1/4}} \leq \tfrac{1}{10}. \tag{4-16}$$

Now, from (4-3), (4-8), (4-9), (4-16), and a standard continuity argument,

$$\sum_{i=2}^{L} \widetilde{V}_i^2(s^*) = \tfrac{1}{4}. \tag{4-17}$$

We then compute from (4-15) the fundamental strict outgoing condition at the exit time $\tilde{s}^*$ defined by (4-17):

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\left\{\sum_{i=2}^{L}\widetilde{V}_i^2\right\}_{|s=\tilde{s}^*} &= \sum_{i=2}^{L}(\widetilde{V}_i)_s\widetilde{V}_i = \frac{1}{\tilde{s}^*}\left[\sum_{i=2}^{L}\frac{i}{2L-1}\widetilde{V}_i^2(s^*) + O\left(\frac{1}{(\log \tilde{s}^*)^{1/4}}\right)\right]\\ &\geq \frac{1}{s^*}\left[\frac{2}{2L-1}\frac{1}{4} + O\left(\frac{1}{(\log \tilde{s}^*)^{1/4}}\right)\right] > 0.\end{aligned}$$

This implies from a standard argument the continuity of the map

$$(\widetilde{V}_i)_{2\leq i\leq L} \in \mathcal{B}_{L-1}\left(\tfrac{1}{2}\right) \mapsto \tilde{s}^*[(\widetilde{V}_i)_{2\leq i\leq L}],$$

and hence the continuous map

$$\begin{aligned}&\mathcal{B}_{L-1}\left(\tfrac{1}{2}\right) \to \mathcal{B}_{L-1}\left(\tfrac{1}{2}\right),\\ &(\widetilde{V}_i)_{2\leq i\leq L} \mapsto \{\widetilde{V}_i[\tilde{s}^*((\widetilde{V}_i)_{2\leq i\leq L})]\}\end{aligned}$$

is the identity on the boundary sphere $\mathbb{S}_{L-1}\left(\frac{1}{2}\right)$, a contradiction to Brouwer's fixed point theorem. This concludes the proof of Proposition 3.1. □

Proof of Theorem 1.1.

Proof. We pick initial data satisfying the conclusions of Proposition 3.1. In particular, (4-4) implies the existence of $c(u_0) > 0$ such that

$$\lambda(s) = c(u_0)\frac{(\log s)^{|d_1|}}{s^{c_1}}\left[1 + O\left(\frac{1}{(\log s)^{1/4}}\right)\right],$$

and then, from (3-36), (2-101),

$$-\lambda\lambda_t = -\frac{\lambda_s}{\lambda} = b_1 + O\left(\frac{1}{s^L}\right) = \frac{c_1}{s}\left[1 + O\left(\frac{1}{\log s}\right)\right] = \frac{c(u_0)\lambda^{1/c_1}}{|\log\lambda|^{|d_1|/c_1}}\left[1 + O\left(\frac{1}{(\log s)^{1/4}}\right)\right].$$

Hence we get the pointwise differential equation

$$-\lambda^{(-(L-1))/L}|\log\lambda|^{2/(2L-1)}\lambda_t = c(u_0)(1 + o(1)).$$

We easily conclude that λ touches zero at some finite time $T = T(u_0) < +\infty$ with near blow-up time

$$\lambda(t) = c(u_0)(1+o(1))\frac{(T-t)^L}{|\log(T-t)|^{2L/(2L-1)}}(1+o(1)).$$

The strong convergence (1-12) now follows as in [Raphaël and Schweyer 2013]. This concludes the proof of Theorem 1.1. □

Appendix A: Regularity in corotational symmetry

We detail in this appendix the regularity of smooth maps with 1-corotational symmetry.

Lemma A.1 (regularity in corotational symmetry). *Let v be a smooth 1-corotational map*

$$v(y,\theta) = \begin{vmatrix} g(u(y))\cos\theta \\ g(u(y))\sin\theta \\ z(u(y)) \end{vmatrix} \tag{A-1}$$

with

$$v(0) = e_z, \qquad \lim_{y\to+\infty} v(x) \to -e_z. \tag{A-2}$$

Assume that v is smooth in the Sobolev sense:

$$\sum_{i=1}^{N}\int |(-\Delta)^{i/2}v|^2 < +\infty$$

for some $N \gg L$. Then:

(*i*) *u is a smooth function of y with a Taylor expansion at the origin for $p \le 10L$*:

$$u(y) = \sum_{k=0}^{p} c_k y^{2k+1} + O(y^{2p+3}). \tag{A-3}$$

(*ii*) *Assume that $u(y) = Q(y) + \varepsilon(y)$ with*

$$\|\nabla\varepsilon\|_{L^2} + \left\|\frac{\varepsilon}{y}\right\|_{L^2} \ll 1, \tag{A-4}$$

and consider the sequence of suitable derivatives $\varepsilon_k = \mathscr{A}^k\varepsilon$. Then, for all $1 \le k \le L$,

$$\int |\varepsilon_{2k+2}|^2 + \int \frac{|\varepsilon_{2k+1}|^2}{y^2(1+y^2)} + \sum_{p=0}^{k}\int\left[\frac{|\varepsilon_{2p-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-p)})} + \frac{|\varepsilon_{2p}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-p)})}\right] < +\infty. \tag{A-5}$$

Proof of Lemma A.1. Let us consider the rotation matrix

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A-6}$$

and rewrite (A-1) as

$$v(r,\theta) = e^{\theta R} w \quad \text{with } w(r) = \begin{vmatrix} w_1 = g(u) \\ 0 \\ w_3 = z(u). \end{vmatrix} \tag{A-7}$$

Step 1: *Control at the origin*. We compute the energy density

$$|\nabla v|^2 = |\partial_y w|^2 + \frac{|w|^2}{y^2}, \tag{A-8}$$

which is bounded from the smoothness of v, from which

$$\left|\frac{w_1}{y}\right| + |\partial_y w_1| \lesssim 1. \tag{A-9}$$

Similarly,

$$\Delta v = e^{\theta R}\left(\Delta w + R^2 \frac{w}{y^2}\right) = e^{\theta R} \begin{vmatrix} -\mathbb{H} w_1 \\ 0 \\ \Delta w_3, \end{vmatrix} \tag{A-10}$$

where

$$\mathbb{H} w_1 = -\Delta w_1 + \frac{w_1}{y^2} = \mathbb{A}^* \mathbb{A} w_1$$

with

$$\mathbb{A} = -\partial_y + \frac{1}{y}, \quad \mathbb{A}^* = \partial_y + \frac{2}{y}.$$

The regularity of v implies

$$|\mathbb{H} w_1| \lesssim 1$$

near the origin, which, together with (A-9), yields

$$\mathbb{A} w_1(y) = \frac{1}{y^2} \int_0^r (\mathbb{H} w_1) \tau^2 \, d\tau = O(y). \tag{A-11}$$

We now observe that

$$\mathbb{H} w_1 = -\partial_{yy} w_1 + \frac{1}{y} \mathbb{A} w_1$$

and conclude

$$|\partial_{yy}^2 w_1| \lesssim 1.$$

We now iterate this argument once on (A-10). Indeed, at the origin,

$$|\partial_y \mathbb{H} w_1|^2 + \frac{|\mathbb{H} w_1|^2}{y^2} \lesssim |\nabla \Delta v|^2 \lesssim 1, \quad |\mathbb{H}^2 w_1| \lesssim |\Delta^2 v| \lesssim 1,$$

and hence

$$|\mathbb{H} w_1| \lesssim y, \quad |\mathbb{A} \mathbb{H} w_1| \lesssim y, \quad |\mathbb{H}^2 w_1| \lesssim 1.$$

This yields the $\mathscr{C}^3$-regularity of w_1 at the origin and, from (A-11), the improved bound

$$\mathbb{A} w_1(y) = \frac{1}{y^2} \int_0^y (\mathbb{H} w_1) \tau^2 \, d\tau = O(y^2).$$

A simple induction now yields for all $k \geq 1$ the $\mathscr{C}^k$-regularity of w_1, and that the sequence

$$(w_1)_0 = w_1, \quad (w_1)_{k+1} = \begin{cases} \mathbb{A}^*(w_1)_k & \text{for k odd,} \\ \mathbb{A}(w_1)_k & \text{for k even,} \end{cases} \qquad k \geq 1$$

satisfies the bound

$$|(w_1)_k| \lesssim \begin{cases} y & \text{for k even,} \\ y^2 & \text{for k odd.} \end{cases} \tag{A-12}$$

We therefore let a Taylor expansion at the origin

$$w_1(y) = \sum_{i=1}^{p} c_i y^i + O(y^{p+1})$$

apply successively the operator $\mathbb{A}$, $\mathbb{A}^*$ and conclude from the relations

$$\mathbb{A}(r^k) = -(k-1)y^{k-1}, \quad \mathbb{A}^*(y^k) = (k+2)y^{k-1}$$

and (A-12) that

$$c_{2k} = 0 \quad \text{for all } k \geq 1.$$

We now recall from (A-7) that $w_1 = g(u)$ and the Taylor expansion (A-3) now follows from the odd parity of g at the origin.

We now claim that this implies the bound (A-5) at the origin. Indeed, ε admits a Taylor expansion (A-3) from (2-2) to which we apply successively the operators A, A^*. We observe from (2-5) the cancellation

$$A(y) = cy^2 + O(y^3),$$

which ensures the bound near the origin

$$|\varepsilon_{2k}| \lesssim y, \quad |\varepsilon_{2k+1}| \lesssim y^2, \tag{A-13}$$

and hence the finiteness of the norms (A-5) at the origin.

Step 2: *control for* $r \geq 1$. We first claim

$$\int \frac{\varepsilon^2}{y^2} + \sum_{k=1}^{L+2} \int (\partial_y^k \varepsilon)^2 < +\infty. \tag{A-14}$$

Indeed, from (A-4),

$$\|\varepsilon\|_{L^\infty} \lesssim \|\nabla \varepsilon\|_{L^2} + \left\| \frac{\varepsilon}{y} \right\|_{L^2} \ll 1. \tag{A-15}$$

From (A-8),

$$\int |\Delta g(u)|^2 \lesssim \int |\Delta v|^2 < +\infty.$$

Now

$$\Delta g(u) = g'(u)\Delta u + (\partial_y u)^2 g'(u) g''(u)$$

and, using Sobolev and the L^∞ control (A-15), we estimate

$$\int ((\partial_y \varepsilon)^2 g'(u) g''(u))^2 \lesssim \int |\Delta \varepsilon|^2 \int |\nabla \varepsilon|^2 \ll \int |\Delta \varepsilon|^2.$$

Moreover, from the smallness (A-15) and the structure of Q,

$$|g'(u)| \gtrsim 1 \quad \text{as } r \to +\infty,$$

from which

$$1 + \int |\Delta v|^2 \gtrsim \int |\Delta \varepsilon|^2.$$

The control of higher order Sobolev norms (A-14) now follows similarly by induction using the Faa di Bruno formula for the computation of $\partial_y^k g(u)$. This is left to the reader. Now (A-14) easily implies

$$\sum_{i=1}^{L+2} \int_{y \geq 1} |\varepsilon_k|^2 < +\infty,$$

and the bound (A-5) is proved far out. □

Appendix B: Coercivity bounds

Given $M \geq 1$, we let Φ_M be given by (3-7). Let us recall the coercivity of the operator H, which is a standard consequence of the knowledge of the kernel of H and the nondegeneracy (3-8).

Lemma B.1 (coercivity of H). *Let $M \geq 1$ be large enough. Then there exists $C(M) > 0$ such that, for all radially symmetric u with*

$$\int \left[|\partial_y u|^2 + \frac{|u|^2}{y^2} \right] < +\infty$$

satisfying

$$(u, \Phi_M) = 0,$$

we have

$$(Hu, u) \geq C(M) \left[\int |\partial_y u|^2 + \frac{|u|^2}{y^2} \right].$$

We now recall the coercivity of $\widetilde{H}$, which is a simple consequence of (2-11) and is proved in [Raphaël and Rodnianski 2012].

Lemma B.2 (coercivity of $\widetilde{H}$). *Let u be such that*

$$\int |\partial_y u|^2 + \int \frac{|u|^2}{y^2} < +\infty. \tag{B-1}$$

Then

$$(\widetilde{H} u, u) = \|A^* u\|_{L^2}^2 \geq c_0 \left[\int |\partial_y u|^2 + \int \frac{|u|^2}{y^2 (1 + |\log y|^2)} \right] \tag{B-2}$$

for some universal constant $c_0 > 0$.

We now claim the following weighted coercivity bound on H.

Lemma B.3 (coercivity of $\mathscr{E}_2$). *There exists $C(M) > 0$ such that, for all radially symmetric u with*

$$\int \frac{|u|^2}{y^4(1+|\log y|^2)} + \int \frac{|Au|^2}{y^2(1+y^2)} < +\infty \tag{B-3}$$

and

$$(u, \Phi_M) = 0,$$

we have

$$\int |Hu|^2 \geq C(M)\left[\int \frac{|u|^2}{y^4(1+|\log y|^2)} + \int \frac{|Au|^2}{y^2(1+|\log y|^2)}\right]. \tag{B-4}$$

Proof of Lemma B.3. This lemma is proved in [Raphaël and Rodnianski 2012] in the case of the sphere target. Let us briefly recall the argument for the sake of completeness.

Step 1: *conclusion using a subcoercivity lower bound*. For any u satisfying (B-10), we claim the subcoercivity lower bound

$$\begin{aligned}\int |Hu|^2 \\ \gtrsim \int \frac{|\partial_y^2 u|^2}{(1+|\log y|^2)} + \int \frac{(\partial_y u)^2}{y^2(1+|\log y|^2)} + \int \frac{|u|^2}{y^4(1+|\log y|^2)} - C\left[\int \frac{(\partial_y u)^2}{1+y^3} + \int \frac{|u|^2}{1+y^5}\right].\end{aligned} \tag{B-5}$$

Let us assume (B-5) and conclude the proof of (B-4). By contradiction, let $M > 0$ be fixed and consider a normalized sequence u_n,

$$\int \frac{|u_n|^2}{y^4(1+|\log y|^2)} + \int \frac{|Au_n|^2}{y^2(1+|\log y|^2)} = 1, \tag{B-6}$$

satisfying the orthogonality condition

$$(u_n, \Phi_M) = 0 \tag{B-7}$$

and the smallness

$$\int |Hu_n|^2 \leq \frac{1}{n}. \tag{B-8}$$

Note that the normalization condition implies

$$\int \frac{|u_n|^2}{y^4(1+|\log y|^2)} + \int \frac{|\partial_y u_n|^2}{y^2(1+|\log y|^2)} \lesssim 1,$$

and thus, from (B-5), the sequence u_n is bounded in H^2_{loc}. Hence there exists $u_\infty \in H^2_{\mathrm{loc}}$ such that, up to a subsequence and for any smooth cut-off function ζ vanishing in a neighborhood of $y = 0$, the sequence ζu_n is uniformly bounded in H^2_{loc} and converges to ζu_∞ in H^1_{loc}. Moreover, (B-8) implies

$$Hu_\infty = 0,$$

and by lower semicontinuity of the norm and (B-6),

$$\int \frac{|u_\infty|^2}{y^4(1+|\log y|^2)} < +\infty,$$

which implies from (2-12) that

$$u = \alpha \Lambda Q \quad \text{for some } \alpha \in \mathbb{R}.$$

We may moreover pass to the limit in (B-7) from (B-6) and the local compactness of Sobolev embeddings, and thus

$$(u_\infty, \Lambda Q) = 0, \quad \text{from which } \alpha = 0,$$

where we used the nondegeneracy (3-8). Hence $u_\infty = 0$. Now the subcoercivity lower bound (B-5) together with (B-6), (B-8), and the H^2_{loc} uniform bound imply the existence of $\varepsilon, c > 0$ such that

$$\int_{\varepsilon \le y \le 1/\varepsilon} \left[\frac{|\partial_y u_\infty|^2}{1+y^3} + \frac{|u_\infty|^2}{1+y^5} \right] \ge c > 0,$$

which contradicts the established identity $u_\infty = 0$. Thus (B-4) is proved.

Step 2: *proof of* (B-5). Let us first apply Lemma B.2 to Au, which satisfies (B-1) by assumption, and estimate

$$\int (Hu)^2 = (\widetilde{H} Au, Au) \gtrsim \int |\partial_y (Au)|^2 + \int \frac{|Au|^2}{y^2(1+y^2)}. \tag{B-9}$$

Near the origin, we now recall the logarithmic Hardy inequality

$$\int_{y \le 1} \frac{|v|^2}{y^2(1+|\log y|^2)} \lesssim \int_{1 \le y \le 2} |v|^2 + \int_{y \le 1} |\partial_y v|^2,$$

and thus, using (2-10),

$$\int \frac{|Au|^2}{y^2} = \int \frac{1}{y^2} \left| \Lambda Q \partial_y \left(\frac{u}{\Lambda Q} \right) \right|^2 \gtrsim \int_{y \le 1} \left| \frac{u}{\Lambda Q} \right|^2 \frac{dy}{y^2(1+|\log y|^2)} - \int_{y \le 1} (|\partial_y u|^2 + |u|^2),$$

which, together with (B-9), yields

$$\int |Hu|^2 \gtrsim \int_{y \le 1} \left[\frac{(\partial_y u)^2}{y^2(1+|\log y|^2)} + \int \frac{|u|^2}{y^4(1+|\log y|^2)} \right] - C \int_{y \le 1} (|\partial_y u|^2 + |u|^2).$$

To control the second derivative, we rewrite near the origin

$$Hu = -\partial_y^2 u + \frac{1}{y} \left(-\partial_y u + \frac{u}{y} \right) + \frac{V-1}{y^2} u = -\partial_y^2 u + \frac{Au}{y} + \frac{(V-1)+(1-Z)}{y^2} u$$

and (B-5) follows near the origin.

Away from the origin, let $\zeta(y)$ be a smooth cut-off function with support in $y \ge \frac{1}{2}$ and equal to 1 for $y \ge 1$. We use the logarithmic Hardy inequality

$$\int_{y \ge 1} \frac{|u|^2}{y^2(1+|\log y|^2)} \lesssim \int_{1 \le y \le 2} |u|^2 + \int |\partial_y u|^2$$

to conclude from (B-9) that

$$\int (Hu)^2 \gtrsim \int \zeta \frac{|Au|^2}{y^2(1+|\log y|^2)} - C \int_{1 \le y \le 2} (|u|^2 + |\partial_y u|^2).$$

Now, from (2-5), we estimate

$$\int \zeta \frac{|Au|^2}{y^2(1+|\log y|^2)} = \int \zeta \frac{|-\partial_y u - \frac{u}{y}|^2}{y^2(1+|\log y|^2)} - C\int \zeta \frac{|u|^2}{y^6(1+|\log y|^2)}$$
$$\gtrsim \int \frac{\zeta}{y^2(1+|\log y|^2)}\left[|\partial_y u|^2 + \frac{|u|^2}{y^2}\right] - C\int\left[\frac{|u|^2}{y^5} + \frac{|\partial_y u|^2}{y^3}\right],$$

where we integrated by parts in the last step. The control of the second derivative follows from the explicit expression of H. This concludes the proof of (B-5). □

We now aim at generalizing the coercivity of the $\mathscr{E}_2$ energy of Lemma B.3 to higher order energies. This first requires a generalization of the weighted estimate (B-4).

Lemma B.4 (weighted coercivity bound). *Let $L \geq 1$, $0 \leq k \leq L$, and $M \geq M(L)$ be large enough. Then there exists $C(M) > 0$ such that, for all radially symmetric u with*

$$\int \frac{|u|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})} + \int \frac{|Au|^2}{y^6(1+|\log y|^2)(1+y^{4k+4})} < +\infty \tag{B-10}$$

and

$$(u, \Phi_M) = 0,$$

we have

$$\int \frac{|Hu|^2}{y^4(1+|\log y|^2)(1+y^{4k})} \geq C(M)\left\{\int \frac{|u|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})} + \int \frac{|Au|^2}{y^6(1+|\log y|^2)(1+y^{4k})}\right\}. \tag{B-11}$$

Proof of Lemma B.4. *Step* 1: *subcoercivity lower bound.* For any u satisfying (B-10), we claim the subcoercivity lower bound

$$\int \frac{|Hu|^2}{y^4(1+|\log y|)^2(1+y^{4k})} \gtrsim \int \frac{|\partial_y^2 u|^2}{y^4(1+|\log y|^2)(1+y^{4k})} + \int \frac{(\partial_y u)^2}{y^2(1+|\log y|)^2(1+y^{4k+4})}$$
$$+ \int \frac{|u|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})} - C\left[\int \frac{(\partial_y u)^2}{1+y^{4k+8}} + \int \frac{|u|^2}{1+y^{4k+10}}\right]. \tag{B-12}$$

Control near the origin. Recall from the finiteness of the norm (B-10) and the formula (2-21) that

$$Au(y) = \frac{1}{y\Lambda Q(y)} \int_0^y \tau \Lambda Q(\tau) Hu(\tau)\, d\tau.$$

Then from Cauchy–Schwarz and Fubini we estimate

$$\int_{y\le 1}\frac{|Au|^2}{y^5(1+|\log y|^2)}\,dy \lesssim \int_{0\le y\le 1}\int_{0\le\tau\le y}\frac{y^5}{y^9(1+|\log y|^2)}|Hu(\tau)|^2\,dy\,d\tau$$
$$\lesssim \int_{0\le\tau\le 1}|Hu(\tau)|^2\left[\int_{\tau\le y\le 1}\frac{dy}{y^4(1+|\log y|^2)}\right]d\tau \lesssim \int_{\tau\le 1}\frac{|Hu(\tau)|^2}{\tau^3(1+|\log\tau|^2)}\,d\tau,$$

and thus

$$\int_{y\le 1}\frac{|Au|^2}{y^6(1+|\log y|^2)} \lesssim \int_{y\le 1}\frac{|Hu|^2}{y^4(1+|\log y|^2)}. \tag{B-13}$$

We now invert A and get from (2-10) the existence of $c(u)$ such that

$$u(y) = c(u)\Lambda Q(y) - \Lambda Q(y)\int_0^y \frac{Au(\tau)}{\Lambda Q(\tau)}\,d\tau.$$

We estimate from Cauchy–Schwarz and (B-13), for $1\le y\le 1$,

$$\left|\int_0^y \frac{Au(\tau)}{\Lambda Q(\tau)}\,d\tau\right|^2 \lesssim y^4(1+|\log y|^2)\int_0^y\frac{|Au|^2}{\tau^5(1+|\log\tau|^2)}\,d\tau \lesssim y^3\int_{y\le 1}\frac{|Hu|^2}{y^4(1+|\log y|^2)},$$

from which

$$|c(u)|^2 \lesssim \int_{y\le 1}|u|^2 + \int_{y\le 1}\frac{|Hu|^2}{y^4(1+|\log y|^2)}$$

and

$$\int_{y\le 1}\frac{|u|^2}{y^4(1+|\log y|^2)} \lesssim \int_{y\le 1}\frac{|Hu|^2}{y^4(1+|\log y|^2)} + \int_{1\le y\le 2}|u|^2. \tag{B-14}$$

The control of the first derivative follows from (B-13), (B-14), and the definition of A:

$$\int_{y\le 1}\frac{|\partial_y u|^2}{y^2(1+|\log y|^2)} \lesssim \int_{y\le 1}\frac{|Au|^2}{y^2(1+|\log y|^2)} + \int_{y\le 1}\frac{|u|^2}{y^4(1+|\log y|^2)}$$
$$\lesssim \int_{y\le 1}\frac{|Hu|^2}{y^4(1+|\log y|^2)} + \int_{1\le y\le 2}|u|^2.$$

To control the second derivative, we rewrite near the origin

$$Hu = -\partial_y^2 u + \frac{1}{y}\left(-\partial_y u + \frac{u}{y}\right) + \frac{V-1}{y^2}u = -\partial_y^2 u + \frac{Au}{y} + \frac{(V-1)+(1-Z)}{y^2}u,$$

which using (B-13), (B-14) and (2-5), (2-6) implies

$$\int_{y\le 1}\frac{|\partial_y^2 u|^2}{y^4(1+|\log y|^2)} \lesssim \int_{y\le 1}\frac{|Hu|^2}{y^4(1+|\log y|^2)} + \int_{1\le y\le 2}|u|^2.$$

This concludes the proof of (B-12) near the origin.

Control away from the origin. Let $\zeta(y)$ be a smooth cut-off function with support in $y \geq \frac{1}{2}$ and equal to 1 for $y \geq 1$. We compute

$$\begin{aligned}\int \zeta \frac{|Hu|^2}{y^{4k+4}(1+|\log y|)^2} &= \int \zeta \frac{|-\partial_y(y\partial_y u)+\frac{V}{y}u|^2}{y^{4k+6}(1+|\log y|)^2} \\ &= \int \zeta \frac{|\partial_y(y\partial_y u)|^2}{y^{4k+6}(1+|\log y|)^2} - 2\int \zeta \frac{\partial_y(y\partial_y u)\cdot Vu}{y^{4k+7}(1+|\log y|)^2} + \int \zeta \frac{V^2|u|^2}{y^{4k+8}(1+|\log y|)^2} \\ &= \int \zeta \frac{|\partial_y(y\partial_y u)|^2}{y^{4k+6}(1+|\log y|)^2} + 2\int \zeta \frac{V(\partial_y u)^2}{y^{4k+6}(1+|\log y|)^2} + \int \zeta \frac{V^2|u|^2}{y^{4k+8}(1+|\log y|)^2} \\ &\quad - \int |u|^2 \Delta\left(\frac{\zeta V}{y^{4k+6}(1+|\log y|)^2}\right). \end{aligned} \tag{B-15}$$

We now use the two dimensional logarithmic Hardy inequality with best constant:[14] for all $\gamma > 0$,

$$\frac{\gamma^2}{4}\int_{y\geq 1} \frac{|v|^2}{y^{2+\gamma}(1+|\log y|)^2} \leq C_\gamma \int_{1\leq y\leq 2} |v|^2 + \int_{y\geq 1} \frac{|\partial_y v|^2}{y^\gamma (1+|\log y|)^2} \tag{B-16}$$

with $\gamma = 4k+6$. We estimate

$$\begin{aligned}\int \zeta \frac{|\partial_y(y\partial_y u)|^2}{y^{4k+6}(1+|\log y|)^2} &\geq \frac{(4k+6)^2}{4}\int_{y\geq 1} \frac{|\partial_y u|^2}{y^{4k+6}(1+|\log y|)^2} - C_k \int_{1\leq y\leq 2} |\partial_y u|^2 \\ &\geq \frac{(4k+6)^4}{16}\int_{y\geq 1} \frac{|u|^2}{y^{4k+8}(1+|\log y|)^2} - C_k \int_{1\leq y\leq 2} [|\partial_y u|^2 + |u|^2].\end{aligned}$$

We now observe that, for $k \geq 0$ and $y \geq 1$,

$$\partial_y^k V(y) = \partial_y^k(1) + O(y^{-2-k}).$$

We compute

$$\Delta\left(\frac{1}{y^{4k+6}}\right) = \frac{(4k+6)^2}{y^{4k+8}}.$$

Injecting these bounds into (B-15) yields the lower bound

$$\begin{aligned}\int \zeta \frac{|Hu|^2}{y^{4k+4}(1+|\log y|)^2} &\geq \left[\frac{(4k+6)^4}{16} - (4k+6)^2\right]\int_{y\geq 1} \frac{|u|^2}{y^{4k+8}(1+|\log y|)^2} - C_k \int \left[\frac{|\partial_y u|^2}{1+y^{4k+8}} + \frac{|u|^2}{1+y^{4k+10}}\right].\end{aligned}$$

Note that we can always keep the control of the first two derivatives in these estimates, and the control (B-12) follows away from the origin.

[14] which can be obtained by a simple integration by parts; see [Merle et al. 2011].

Step 2: *proof of* (B-11). By contradiction, let $M > 0$ be fixed and consider a normalized sequence u_n,

$$\int \frac{|u_n|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})} + \int \frac{|Au_n|^2}{y^6(1+|\log y|^2)(1+y^{4k})} = 1, \tag{B-17}$$

satisfying the orthogonality condition

$$(u_n, \Phi_M) = 0 \tag{B-18}$$

and the smallness

$$\int \frac{|Hu_n|^2}{y^4(1+|\log y|^2)(1+y^{4k})} \le \frac{1}{n}. \tag{B-19}$$

Note that the normalization condition implies

$$\int \frac{|u_n|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})} + \int \frac{|\partial_y u_n|^2}{y^2(1+|\log y|^2)(1+y^{4k+4})} \lesssim 1, \tag{B-20}$$

and thus, from (B-12), the sequence u_n is bounded in H^2_{loc}. Hence, there exists $u_\infty \in H^2_{\mathrm{loc}}$ such that, up to a subsequence and for any smooth cut-off function ζ vanishing in a neighborhood of $y = 0$, the sequence ζu_n is uniformly bounded in H^2_{loc} and converges to ζu_∞ in H^1_{loc}. Moreover, (B-19) implies

$$Hu_\infty = 0,$$

and, by lower semicontinuity of the norm and (B-17),

$$\int \frac{|u_\infty|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})} < +\infty,$$

which implies from (2-12) that

$$u = \alpha \Lambda Q \quad \text{for some } \alpha \in \mathbb{R}.$$

We may moreover pass to the limit in (B-18) from (B-17) and the local compactness embedding, and thus

$$(u_\infty, \Lambda Q) = 0, \quad \text{from which } \alpha = 0,$$

where we used the nondegeneracy (3-8). Hence $u_\infty = 0$.

Now from (B-13), (B-14), (B-19), and (B-17),

$$\int_{y\ge 1} \frac{|u_n|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})} + \int_{y\ge 1} \frac{|\partial_y u_n|^2}{y^6(1+|\log y|^2)(1+y^{4k})} \gtrsim 1,$$

and hence, from (B-12), (B-19),

$$\frac{|\partial_y u_n|^2}{1+y^{4k+8}} + \frac{|u_n|^2}{1+y^{4k+10}} \gtrsim 1,$$

which, from the local compactness of Sobolev embeddings and the a priori bound (B-20), ensures

$$\int \frac{|\partial_y u_\infty|^2}{1+y^{4k+8}} + \int \frac{|u_\infty|^2}{1+y^{4k+10}} \gtrsim 1.$$

This contradicts the established identity $u_\infty = 0$. □

We are now in a position to prove the coercivity of the higher order $(\mathscr{E}_{2k+2})_{0\le k\le L}$ energies under suitable orthogonality conditions. Given a radially symmetric function ε, we recall the definition of suitable derivatives:

$$\varepsilon_{-1}=0,\quad \varepsilon_0=\varepsilon,\quad \varepsilon_{k+1}=\begin{cases} A^*\varepsilon_k & \text{for k odd,}\\ A\varepsilon_k & \text{for k even,}\end{cases}\qquad 0\le k\le 2L+1.$$

Lemma B.5 (coercivity of $\mathscr{E}_{2k+2}$). *Let $L\ge 1$, $0\le k\le L$, and $M\ge M(L)$ be large enough. Then there exists $C(M)>0$ such that, for all ε with*

$$\int |\varepsilon_{2k+2}|^2+\int \frac{|\varepsilon_{2k+1}|^2}{y^2(1+y^2)} +\sum_{p=0}^{k}\int\left[\frac{|\varepsilon_{2p-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-p)})}+\frac{|\varepsilon_{2p}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-p)})}\right]<+\infty \tag{B-21}$$

satisfying

$$(\varepsilon, H^p\Phi_M)=0,\quad 0\le p\le k, \tag{B-22}$$

we have

$$\mathscr{E}_{2k+2}(\varepsilon)=\int (H^{k+1}\varepsilon)^2\ge C(M)\left\{\int \frac{|\varepsilon_{2k+1}|^2}{y^2(1+|\log y|^2)} +\sum_{p=0}^{k}\int\left[\frac{|\varepsilon_{2p-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-p)})}+\frac{|\varepsilon_{2p}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-p)})}\right]\right\}. \tag{B-23}$$

Proof of Lemma B.1. We argue by induction on k. The case $k=0$ is Lemma B.3. We assume the claim for k and prove it for $1\le k+1\le L$. Indeed, let $v=H\varepsilon$. Then $v_p=\varepsilon_{p+2}$, and thus v satisfies (B-21) and[15]

$$\text{for all } 0\le p\le k,\quad (v,H^p\Phi_M)=(\varepsilon,H^{p+1}\Phi_M)=0.$$

We may thus apply the induction claim for k to v and estimate

$$\begin{aligned}
&\int (H^{k+2}\varepsilon)^2\\
&=\int (H^{k+1}v)^2\\
&\ge C(M)\left\{\int \frac{|\varepsilon_{2k+3}|^2}{y^2(1+|\log y|^2)}+\sum_{p=0}^{k}\int\left[\frac{|\varepsilon_{2p+1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-p)})}+\frac{|\varepsilon_{2p+2}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-p)})}\right]\right\}\\
&\ge C(M)\left\{\int \frac{|\varepsilon_{2k+3}|^2}{y^2(1+|\log y|^2)} +\sum_{p=1}^{k+1}\int\left[\frac{|\varepsilon_{2p-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k+1-p)})}+\frac{|\varepsilon_{2p}|^2}{y^4(1+|\log y|^2)(1+y^{4(k+1-p)})}\right]\right\}.
\end{aligned} \tag{B-24}$$

[15] from $k\le L+1$

The orthogonality condition $(\varepsilon, \Phi_M) = 0$ and (B-21) allow us to use Lemma B.4 and to deduce from the weighted bound (B-11) the control

$$\int \frac{|\varepsilon_2|^2}{y^4(1+|\log y|^2)(1+y^{4k})} \gtrsim \int \frac{|\varepsilon|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})},$$

which together with (B-24) concludes the proof of Lemma B.1. □

Appendix C: Interpolation bounds

We derive in this section interpolation bounds on ε in the setting of the bootstrap proposition 3.1, and which are a consequence of the coercivity property of Lemma B.5.

Lemma C.1 (interpolation bounds). (i). Weighted Sobolev bounds for ε_k. *For* $0 \le k \le L$,

$$\sum_{i=0}^{2k+1} \int \frac{|\varepsilon_i|^2}{y^2(1+y^{4k-2i+2})(1+|\log y|^2)} + \int |\varepsilon_{2k+2}|^2 \le C(M)\mathscr{E}_{2k+2}. \tag{C-1}$$

(ii). Development near the origin. *ε admits a Taylor–Lagrange-like expansion*

$$\varepsilon = \sum_{i=1}^{L+1} c_i T_{L+1-i} + r_\varepsilon \tag{C-2}$$

with bounds

$$|c_i| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}}, \tag{C-3}$$

$$|\partial_y^k r_\varepsilon| \lesssim y^{2L+1-k}|\log y| C(M)\sqrt{\mathscr{E}_{2L+2}}, \quad 0 \le k \le 2L+1,\ y \le 1. \tag{C-4}$$

(iii). Bounds near the origin for ε_k. *For* $|y| \le \frac{1}{2}$,

$$|\varepsilon_{2k}| \lesssim C(M) y|\log y|\sqrt{\mathscr{E}_{2L+2}}, \quad 0 \le k \le L, \tag{C-5}$$

$$|\varepsilon_{2k-1}| \lesssim C(M) y^2|\log y|\sqrt{\mathscr{E}_{2L+2}}, \quad 1 \le k \le L, \tag{C-6}$$

$$|\varepsilon_{2L+1}| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}}. \tag{C-7}$$

(iv). Bounds near the origin for $\partial_y^k \varepsilon$. *For* $|y| \le \frac{1}{2}$,

$$|\partial_y^{2k}\varepsilon| \lesssim C(M) y|\log y|\sqrt{\mathscr{E}_{2L+2}}, \quad 0 \le k \le L, \tag{C-8}$$

$$|\partial_y^{2k-1}\varepsilon| \lesssim C(M)|\log y|\sqrt{\mathscr{E}_{2L+2}}, \quad 1 \le k \le L+1. \tag{C-9}$$

(v). Lossy bound.

$$\sum_{i=0}^{2k+1} \int \frac{1+|\log y|^C}{1+y^{4k-2i+4}}|\varepsilon_i|^2 \lesssim |\log b_1|^C \begin{cases} b_1^{(4k+2)L/(2L-1)}, & 0 \le k \le L-1, \\ b_1^{2L+2} & \text{for } k = L, \end{cases} \tag{C-10}$$

$$\sum_{i=0}^{2k+1} \int \frac{1+|\log y|^C}{1+y^{4k-2i+4}}|\partial_y^i\varepsilon|^2 \lesssim |\log b_1|^C \begin{cases} b_1^{(4k+2)L/(2L-1)}, & 0 \le k \le L-1, \\ b_1^{2L+2} & \text{for } k = L. \end{cases} \tag{C-11}$$

(vi). Generalized lossy bound. *Let* $(i,j)\in\mathbb{N}\times\mathbb{N}^*$ *with* $2\le i+j\le 2L+2$. *Then*

$$\int \frac{1+|\log y|^C}{1+y^{2j}}|\partial_y^i\varepsilon|^2 \lesssim |\log b_1|^C \begin{cases} b_1^{(i+j-1)2L/(2L-1)} & \text{for } 2\le i+j\le 2L,\\ b_1^{2L+1} & \text{for } i+j=2L+1,\\ b_1^{2L+2} & \text{for } i+j=2L+2. \end{cases} \tag{C-12}$$

Moreover,

$$\int \frac{|\partial_y^i\varepsilon|^2}{1+|\log y|^2} \lesssim |\log b_1|^C \begin{cases} b_1^{(i-1)2L/(2L-1)}, & 2\le i\le 2L+1,\\ b_1^{2L+2} & \text{for } i=2L+2, \end{cases} \tag{C-13}$$

(vii). Pointwise bound far away. *Let* $(i,j)\in\mathbb{N}\times\mathbb{N}$ *with* $1\le i+j\le 2L+1$. *Then*

$$\left\|\frac{\partial_y^i\varepsilon}{y^j}\right\|^2_{L^\infty(y\ge 1)} \lesssim |\log b_1|^C \begin{cases} b_1^{(i+j)2L/(2L-1)} & \text{for } 1\le i+j\le 2L-1,\\ b_1^{2L+1} & \text{for } i+j=2L,\\ b_1^{2L+2} & \text{for } i+j=2L+1. \end{cases} \tag{C-14}$$

Proof. *Step* 1: *proof of* (i). The estimate (C-1) follows from (B-23) with $0\le k\le L$.

Step 2: *adapted Taylor expansion*. *Initialization*. Recall the boundary condition origin at the origin (A-13), which implies $|\varepsilon_{2L+1}(y)|\le C_{\varepsilon_{2L+1}}y^2$ as $y\to 0$. Together with (2-10) and the behavior $\Lambda Q\sim y$ near the origin, this implies

$$r_1=\varepsilon_{2L+1}(y)=\frac{1}{y\Lambda Q}\int_0^y \varepsilon_{2L+2}\Lambda Q x\,dx, \tag{C-15}$$

and this yields the pointwise bound, for $y\le 1$,

$$|r_1(y)|\lesssim \frac{1}{y^2}\left(\int_{y\le 1}|\varepsilon_{2L+2}|^2x\,dx\right)^{\frac12}\left(\int_0^y x^2x\,dx\right)^{\frac12}\lesssim C(M)\sqrt{\mathscr{E}_{2L+2}}. \tag{C-16}$$

We now remark that there exists $\frac12<a<2$ such that

$$|\varepsilon_{2L+1}(a)|^2\lesssim\int_{|y|\le 1}|\varepsilon_{2L+1}|^2\lesssim C(M)\mathscr{E}_{2L+2}$$

from (C-1). We then define

$$r_2=-\Lambda Q\int_a^y\frac{r_1}{\Lambda Q}\,dx$$

and obtain from (C-16) the pointwise bound, for $y\le 1$,

$$|r_2|\lesssim y|\log y|C(M)\sqrt{\mathscr{E}_{2L+2}}. \tag{C-17}$$

Now observe that, by construction, using (2-10),

$$Ar_2=r_1=\varepsilon_{2L+1},\quad Hr_2=A^*\varepsilon_{2L+1}=\varepsilon_{2L+2}=H\varepsilon_{2L}. \tag{C-18}$$

Now, from (B-24),

$$\int_{y\le 1}\frac{|\varepsilon_{2L}|^2}{y^4(1+|\log y|^2)}y\,dy<+\infty,$$

and hence $|\varepsilon_{2L}(y_n)| < +\infty$ on some sequence $y_n \to 0$, and from (C-17), (C-18), the explicit knowledge of the kernel of H, and the singular behavior (2-13), we conclude that there exists $c_2 \in \mathbb{R}$ such that

$$\varepsilon_{2L} = c_2 \Lambda Q + r_2. \tag{C-19}$$

Moreover, there exists $\frac{1}{2} < a < 2$ such that

$$|\varepsilon_{2L}(a)|^2 \lesssim \int_{|y|\le 1} |\varepsilon_{2L}|^2 \lesssim C(M)\mathscr{E}_{2L+2}$$

from (C-1), and thus, from (C-17), (C-19),

$$|c_2| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}}, \quad |\varepsilon_{2L}| \lesssim y|\log y| C(M)\sqrt{\mathscr{E}_{2L+2}}. \tag{C-20}$$

Induction. We now build by induction the sequence

$$r_{2k+1} = \frac{1}{y\Lambda Q}\int_0^y r_{2k}\Lambda Q x\,dx, \quad r_{2k+2} = -\Lambda Q \int_0^y \frac{r_{2k+1}}{\Lambda Q}\,dx, \quad 1 \le k \le L.$$

We claim by induction that, for all $1 \le k \le L+1$, $\varepsilon_{2L+2-2k}$ admits a Taylor expansion at the origin

$$\varepsilon_{2L+2-2k} = \sum_{i=1}^{k} c_{i,k} T_{k-i} + r_{2k}, \quad 1 \le k \le L+1, \tag{C-21}$$

with the bounds, for $|y| \le 1$,

$$|c_{i,k}| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}}, \tag{C-22}$$

$$|\mathscr{A}^i r_{2k}| \lesssim |\log y| y^{2k-1-i} C(M)\sqrt{\mathscr{E}_{2L+2}}, \quad 0 \le i \le 2k-1. \tag{C-23}$$

This follows from (C-19), (C-20), (C-17), (C-18) for $k = 1$. We now let $1 \le k \le L$, assume the claim for k, and prove it for $k+1$.

By construction, using (2-10),

$$Ar_{2k+2} = r_{2k+1}, \quad Hr_{2k+2} = r_{2k}, \tag{C-24}$$

and thus $\mathscr{A}^i r_{2k} = r_{2k-i}$. In particular, for $i \ge 2$, $\mathscr{A}^{i-2} r_{2k+2} = r_{2k-i}$, and therefore the bounds (C-23) for $k+1$ and $2 \le i \le 2k+1$ follow from the induction claim. We now estimate by definition and induction, for $|y| \le 1$,

$$|Ar_{2k+2}| = |r_{2k+1}(y)| = \left|\frac{1}{y\Lambda Q}\int_0^y r_{2k}\Lambda Q x\,dx\right| \lesssim \frac{C(M)\sqrt{\mathscr{E}_{2L+2}}}{y^2} y^{3+2k-1}$$

$$|r_{2k+2}| = \left|\Lambda Q \int_0^y \frac{r_{2k+1}}{\Lambda Q}\,dx\right| \lesssim y y^{2k} C(M)\sqrt{\mathscr{E}_{2L+2}},$$

and (C-23) is proved for $k = 1$ and $i = 0, 1$. From the regularity at the origin (A-13), (C-24), the relation

$$H\varepsilon_{2L+2-2(k+1)} = \varepsilon_{2L+2-2k} = \sum_{i=1}^{k} c_{i,k} T_{k-i} + r_{2k},$$

and the bound (C-23), there exists c_{2k+2} such that

$$\varepsilon_{2L+2-2(k+1)} = \sum_{i=1}^{k} c_{i,k} T_{k+1-i} + c_{2k+2} \Lambda Q + r_{2k+2}.$$

We now observe that there exists $\frac{1}{2} < a < 2$ such that

$$|\varepsilon_{2L-2k}(a)|^2 \lesssim \int_{|y|\le 1} |\varepsilon_{2L-2k}|^2 \lesssim C(M)\mathscr{E}_{2L+2}$$

from (C-1), and thus, using (C-23),

$$|c_{2k+2}| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}}.$$

This completes the induction claim.

Step 3: *proof of* (ii), (iii), and (iv). We obtain from (C-21), (C-3) with $k = L+1$ the Taylor expansion

$$\varepsilon = \sum_{i=1}^{L+1} c_{i,k} T_{k-i} + r_\varepsilon, \quad r_\varepsilon = r_{2L+2}, \quad |c_{i,k}| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}},$$

where, from (C-23),

$$|\mathscr{A}^i r_\varepsilon| \lesssim |\log y| y^{2L+1-i} C(M)\sqrt{\mathscr{E}_{2L+2}}, \quad 0 \le i \le 2L+1.$$

A brute force computation using the expansions (2-5), (2-6) near the origin ensure that, for any function f,

$$\partial_y^k f = \sum_{i=0}^{k} P_{i,k} \mathscr{A}^i f, \quad |P_{i,k}| \lesssim \frac{1}{y^{k-i}}, \tag{C-25}$$

and we therefore estimate, for $0 \le k \le 2L+1$,

$$|\partial_y^k r_\varepsilon| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}} \sum_{i=0}^{k} \frac{|\log y| y^{2L+1-i}}{y^{k-i}} \lesssim y^{2L+1-k}|\log y| C(M)\sqrt{\mathscr{E}_{2L+2}}.$$

This concludes the proof of (ii). The estimates of (iii), (iv) now directly follow from (ii) using the Taylor expansion of T_i at the origin given by Lemma 2.3, and (C-16) for (C-7).

Step 4: *proof of* (v). We first claim that, for $0 \le k \le L$,

$$\sum_{i=0}^{2k+2} \int \frac{|\partial_y^i \varepsilon|^2}{(1+|\log y|^2)(1+y^{4k-2i+4})} \lesssim C(M)(\mathscr{E}_{2k+2} + \mathscr{E}_{2L+2}). \tag{C-26}$$

Observe that this implies (C-13) by taking $i = 2k+2$.

Indeed, from (C-8), (C-9), we estimate

$$\sum_{i=0}^{2k+1} \int_{y\le 1} \frac{1+|\log y|^C}{1+y^{4k-2i+4}} |\partial_y^i \varepsilon|^2 \lesssim C(M)\mathscr{E}_{2L+2}. \tag{C-27}$$

For $y \geq 1$, we recall from the brute force computation (C-25) that

$$|\partial_y^k \varepsilon| \lesssim \sum_{i=0}^{k} \frac{|\varepsilon_i|}{y^{k-i}}, \tag{C-28}$$

and thus, using (C-1), for $0 \leq k \leq L$,

$$\begin{aligned}\sum_{i=0}^{2k+2} \int_{y\geq 1} \frac{|\partial_y^i \varepsilon|^2}{(1+|\log y|^2)(1+y^{4k-2i+4})} &\lesssim \sum_{i=0}^{2k+2} \sum_{j=0}^{i} \int_{y\geq 1} \frac{|\varepsilon_j|^2}{(1+|\log y|^2)(1+y^{4k-2i+4+2i-2j})} \\ &\lesssim \sum_{j=0}^{2k+2} \int \frac{|\varepsilon_j|^2}{(1+|\log y|^2)(1+y^{4k+4-2j})} \lesssim C(M)\mathscr{E}_{2k+2},\end{aligned}$$

and (C-26) is proved. In particular, together with the energy bound (3-23), this yields the rough Sobolev bound

$$\int \frac{|\varepsilon|^2}{y^2} + \sum_{k=1}^{2L+2} \int \frac{|\partial_y^k \varepsilon|^2}{1+|\log y|^2} \lesssim 1.$$

Therefore, again using (C-26), we estimate

$$\begin{aligned}\sum_{i=0}^{2k+1} \int \frac{1+|\log y|^C}{1+y^{4k-2i+4}} |\partial_y^i \varepsilon|^2 &\lesssim \sum_{i=0}^{2k+1} \left[\int_{y\leq B_0^{100L}} \frac{1+|\log y|^C}{1+y^{4k-2i+4}} |\partial_y^i \varepsilon|^2 + \int_{y\geq B_0^{100L}} \frac{1+|\log y|^C}{y^2} |\partial_y^i \varepsilon|^2 \right] \\ &\lesssim |\log b_1|^C \mathscr{E}_{2k+2} + \frac{1}{B_0^{10L}}\end{aligned} \tag{C-29}$$

and (C-11) follows. The estimate (C-10) now follows from (C-5), (C-6), (C-7) for $y \leq 1$ with also (1-31), and (C-11) for $y \geq 1$.

Step 5: *proof of* (vi). Let $i \geq 0$, $j \geq 1$ with $2 \leq i + j \leq 2L + 2$.

Case 1: $i + j$ *even*. We have

$$i + j = 2(k+1), \quad 0 \leq k \leq L.$$

For $k \leq L-1$, from (C-11) and $0 \leq i = 2k+2-j \leq 2k+1$, we estimate

$$\int \frac{1+|\log y|^C}{1+y^{2j}} |\partial_y^i \varepsilon|^2 = \int \frac{1+|\log y|^C}{1+y^{4k+4-2i}} |\partial_y^i \varepsilon|^2 \lesssim b_1^{(4k+2)L/(2L-1)} |\log b_1|^C \lesssim b_1^{(i+j-1)2L/(2L-1)} |\log b_1|^C.$$

For $k = L$, from (C-11), we have

$$\int \frac{1+|\log y|^C}{1+y^{2j}} |\partial_y^i \varepsilon|^2 = \int \frac{1+|\log y|^C}{1+y^{4k+4-2i}} |\partial_y^i \varepsilon|^2 \lesssim b_1^{2L+2} |\log b_1|^C.$$

Case 2: $i+j$ *odd*. We have $i+j=2k+1$, $1\le k\le L$. Assume $k\le L-1$. If $j\ge 2$, then $i\le 2k+1-j\le 2(k-1)+1$, and thus, from (C-11),

$$\begin{aligned}\int \frac{1+|\log y|^C}{1+y^{2j}}|\partial_y^i\varepsilon|^2 &= \int \frac{1+|\log y|^C}{1+y^{4k+2-2i}}|\partial_y^i\varepsilon|^2\\ &\lesssim \left(\int \frac{1+|\log y|^C}{1+y^{4k+4-2i}}|\partial_y^i\varepsilon|^2\right)^{\frac12}\left(\int \frac{1+|\log y|^C}{1+y^{4(k-1)+4-2i}}|\partial_y^i\varepsilon|^2\right)^{\frac12}\\ &\lesssim |\log b_1|^C b_1^{L/(2(2L-1))(4k+2+4(k-1)+2)} = b_1^{(i+j-1)2L/(2L-1)}|\log b_1|^C.\end{aligned}$$

For the extremal case $j=1$, $i=2k$, $1\le k\le L-1$, we estimate, from (C-10), (C-26),

$$\begin{aligned}\int \frac{1+|\log y|^C}{1+y^2}|\partial_y^{2k}\varepsilon|^2 &\lesssim \left(\int \frac{1+|\log y|^C}{1+y^4}|\partial_y^{2k}\varepsilon|^2\right)^{\frac12}\left(\int \frac{|\partial_y^{2k}\varepsilon|^2}{1+|\log y|^2}\right)^{\frac12}\\ &\lesssim |\log b_1|^C b_1^{L/(2(2L-1))(4k+2+4(k-1)+2)} = b_1^{(i+j-1)2L/(2L-1)}|\log b_1|^C.\end{aligned}$$

If $k=L$, then for $j\ge 2$, we have $i\le 2k+1-j\le 2(k-1)+1$, and thus, from (C-11),

$$\begin{aligned}\int \frac{1+|\log y|^C}{1+y^{2j}}|\partial_y^i\varepsilon|^2 &= \int \frac{1+|\log y|^C}{1+y^{4k+2-2i}}|\partial_y^i\varepsilon|^2\\ &\lesssim \left(\int \frac{1+|\log y|^C}{1+y^{4k+4-2i}}|\partial_y^i\varepsilon|^2\right)^{\frac12}\left(\int \frac{1+|\log y|^C}{1+y^{4(k-1)+4-2i}}|\partial_y^i\varepsilon|^2\right)^{\frac12}\\ &\lesssim |\log b_1|^C b_1^{\frac12\left(2L+2+(4(k-1)+2)L/(2L-1)\right)} = b_1^{2L+1}|\log b_1|^C,\end{aligned}$$

and for $j=1$, $i=2L$, from (C-10), (C-13),

$$\begin{aligned}\int \frac{1+|\log y|^C}{1+y^2}|\partial_y^{2L}\varepsilon|^2 &\lesssim \left(\int \frac{1+|\log y|^C}{1+y^4}|\partial_y^{2L}\varepsilon|^2\right)^{\frac12}\left(\int \frac{|\partial_y^{2L}\varepsilon|^2}{1+|\log y|^2}\right)^{\frac12}\\ &\lesssim |\log b_1|^C b_1^{\frac12\left(2L+2+(4(L-1)+2)L/(2L-1)\right)} = b_1^{2L+1}|\log b_1|^C.\end{aligned}$$

Step 6: *proof of* (vii). From Cauchy–Schwarz we estimate

$$\left\|\frac{\varepsilon}{y}\right\|_{L^\infty(y\ge1)}^2 \lesssim \int_{y\ge1}|\varepsilon\partial_y\varepsilon|\,dy \lesssim \int \frac{(1+|\log y|^2)|\varepsilon|^2}{y^2}+\int\frac{|\partial_y\varepsilon|^2}{1+|\log y|^2}.$$

Let $i,j\ge 0$ with $1\le i+j\le 2L+1$. Then $2\le i+j+1\le 2L$, and we conclude from (C-12), (C-13) that

$$\begin{aligned}\left\|\frac{\partial_y^i\varepsilon}{y^j}\right\|_{L^\infty(y\ge1)}^2 &\lesssim \int_{y\ge1}\frac{(1+|\log y|^2)|\partial_y^i\varepsilon|^2}{y^{2j+2}}+\int_{y\ge1}\frac{|\partial_y^{i+1}\varepsilon|^2}{y^{2j}(1+|\log y|^2)}\\ &\lesssim |\log b_1|^C\begin{cases} b_1^{(i+j)2L/(2L-1)} & \text{for } 2\le i+j+1\le 2L,\\ b_1^{2L+1} & \text{for } i+j+1=2L+1,\\ b_1^{2L+2} & \text{for } i+j+1=2L+2.\end{cases}\end{aligned}$$

□

Appendix D: Leibniz rule for H^k

Given a smooth function Φ, we prove the following Leibniz rule.

Lemma D.1 (Leibniz rule for H^k). *Let $k \geq 1$. Then*

$$\begin{aligned} \mathscr{A}^{2k-1}(\Phi\varepsilon) &= \sum_{i=0}^{k-1} \Phi_{2i,2k-1}\varepsilon_{2i} + \sum_{i=1}^{k} \Phi_{2i-1,2k-1}\varepsilon_{2i-1}, \\ \mathscr{A}^{2k}(\Phi\varepsilon) &= \sum_{i=0}^{k} \Phi_{2i,2k}\varepsilon_{2i} + \sum_{i=1}^{k} \Phi_{2i-1,2k}\varepsilon_{2i-1}, \end{aligned} \tag{D-1}$$

where $\Phi_{i,k}$ is computed through the recurrence relation

$$\Phi_{0,1} = -\partial_y\Phi, \quad \Phi_{1,1} = \Phi\Phi_{0,2} = -\partial_{yy}\Phi - \frac{1+2Z}{y}\partial_y\Phi, \quad \Phi_{1,2} = 2\partial_y\Phi, \quad \Phi_{2,2} = \Phi, \tag{D-2}$$

$$\begin{cases} \Phi_{2k+2,2k+2} = \Phi_{2k+1,2k+1}, \\ \Phi_{2i,2k+2} = \Phi_{2i-1,2k+1} + \partial_y\Phi_{2i,2k+1} + ((1+2Z)/y)\Phi_{2i,2k+1} & 1 \leq i \leq k, \\ \Phi_{0,2k+2} = \partial_y\Phi_{0,2k+1} + (1+2Z)/y\Phi_{0,2k+1}, \\ \Phi_{2i-1,2k+2} = -\Phi_{2i-2,2k+1} + \partial_y\Phi_{2i-1,2k+1}, & 1 \leq i \leq k+1, \end{cases} \tag{D-3}$$

$$\begin{cases} \Phi_{2k+1,2k+1} = \Phi_{2k,2k}, \\ \Phi_{2i-1,2k+1} = \Phi_{2i-2,2k} + ((1+2Z)/y)\Phi_{2i-1,2k} - \partial_y\Phi_{2i-1,2k}, & 1 \leq i \leq k, \\ \Phi_{2i,2k+1} = -\partial_y\Phi_{2i,2k} - \Phi_{2i-1,2k}, & 1 \leq i \leq k, \\ \Phi_{0,2k+1} = -\partial_y\Phi_{0,2k}. \end{cases} \tag{D-4}$$

Proof. We compute

$$\begin{aligned} A(\Phi\varepsilon) &= \Phi\varepsilon_1 - (\partial_y\Phi)\varepsilon, \\ H(\Phi\varepsilon) = A^*A\varepsilon &= \Phi\varepsilon_2 + \partial_y\Phi\varepsilon_1 - \left(-A + \frac{1+2Z}{y}\right)(\partial_y\Phi\varepsilon) \\ &= \Phi\varepsilon_2 + 2\partial_y\Phi\varepsilon_1 + \left[-\partial_{yy}\Phi - \frac{1+2Z}{y}\partial_y\Phi\right]\varepsilon. \end{aligned}$$

and

$$\begin{aligned} &\mathscr{A}^{2k+1}(\Phi\varepsilon) \\ &= \sum_{i=0}^{k} A[\Phi_{2i,2k}\varepsilon_{2i}] + \sum_{i=1}^{k}\left(-A^* + \frac{1+2Z}{y}\right)\Phi_{2i-1,2k}\varepsilon_{2i-1} \\ &= \sum_{i=0}^{k}\{\Phi_{2i,2k}\varepsilon_{2i+1} - \partial_y\Phi_{2i,2k}\varepsilon_{2i}\} + \sum_{i=1}^{k}\left\{-\Phi_{2i-1,2k}\varepsilon_{2i} + \left[\frac{1+2Z}{y}\Phi_{2i-1,2k} - \partial_y\Phi_{2i-1,2k}\right]\varepsilon_{2i-1}\right\} \\ &= -\partial_y\Phi_{0,2k}\varepsilon + \sum_{i=1}^{k}(-\partial_y\Phi_{2i,2k} - \Phi_{2i-1,2k})\varepsilon_{2i} + \sum_{i=1}^{k}\left\{\Phi_{2i-2,2k} + \frac{1+2Z}{y}\Phi_{2i-1,2k} - \partial_y\Phi_{2i-1,2k}\right\}\varepsilon_{2i-1}, \\ &\qquad + \Phi_{2k,2k}\varepsilon_{2k+1}, \end{aligned}$$

which is (D-4). We then compute

$$
\begin{aligned}
&\mathscr{A}^{2k+2}(\Phi\varepsilon)\\
&=\sum_{i=0}^{k}\left[-A+\frac{1+2Z}{y}\right]\{\Phi_{2i,2k+1}\varepsilon_{2i}\}+\sum_{1=i}^{k+1}A^*(\Phi_{2i-1,2k+1}\varepsilon_{2i-1})\\
&=\sum_{i=0}^{k}\left\{-\Phi_{2i,2k+1}\varepsilon_{2i+1}+\left[\partial_y\Phi_{2i,2k+1}+\frac{1+2Z}{y}\Phi_{2i,2k+1}\right]\varepsilon_{2i}\right\}+\sum_{i=1}^{k+1}\{\Phi_{2i-1,2k+1}\varepsilon_{2i}+\partial_y\Phi_{2i-1,2k+1}\varepsilon_{2i-1}\}\\
&=\left[\partial_y\Phi_{0,2k+1}+\frac{1+2Z}{y}\Phi_{0,2k+1}\right]\varepsilon+\Phi_{2k+1,2k+1}\varepsilon_{2k+2}\\
&\qquad+\sum_{i=1}^{k}\left[\Phi_{2i-1,2k+1}+\partial_y\Phi_{2i,2k+1}+\frac{1+2Z}{y}\Phi_{2i,2k+1}\right]\varepsilon_{2i}+\sum_{i=1}^{k+1}\left[-\Phi_{2i-2,2k+1}+\partial_y\Phi_{2i-1,2k+1}\right]\varepsilon_{2i-1},
\end{aligned}
$$

which is (D-3). □

Appendix E: Proof of (3-55)

A simple induction argument ensures the formula

$$
[\partial_t, H_\lambda^L]w=\sum_{k=0}^{L-1}H_\lambda^k[\partial_t, H_\lambda]H_\lambda^{L-(k+1)}w.
$$

We therefore renormalize and explicitly compute

$$
[\partial_t, H_\lambda^L]w=\frac{1}{\lambda^{2L+2}}\sum_{k=0}^{L-1}H^k\left(-\frac{\lambda_s}{\lambda}\frac{\Lambda V}{y^2}H^{L-(k+1)}\varepsilon\right). \tag{E-1}
$$

We now apply the Leibniz rule Lemma D.1 with $\Phi=\Lambda V/y^2$. In view of the expansion (2-6) and the recurrence formula (D-3), we have an expansion at the origin to all orders, for even $k\geq 2$,

$$
\begin{cases}
\Phi_{2i,2k}(y)=\displaystyle\sum_{p=0}^{N}c_{i,k,p}y^{2p}+O(y^{2N+2}), & 0\leq i\leq k,\\
\Phi_{2i+1,2k}(y)=\displaystyle\sum_{p=0}^{N}c_{i,k,p}y^{2p+1}+O(y^{2N+3}), & 1\leq i\leq k-1.
\end{cases}
$$

and, for odd $k\geq 1$,

$$
\begin{cases}
\Phi_{2i-1,2k+1}(y)=\displaystyle\sum_{p=0}^{N}c_{i,k,p}y^{2p}+O(y^{2N+2}), & 1\leq i\leq k+1,\\
\Phi_{2i,2k+1}(y)=\displaystyle\sum_{p=0}^{N}c_{i,k,p}y^{2p+1}+O(y^{2N+3}), & 1\leq i\leq k-1.
\end{cases}
$$

We also have a bound, for $y \geq 1$,

$$|\Phi_{i,k}| \lesssim \frac{1}{1+y^{4+(2k-i)}}, \quad 0 \leq i \leq 2k.$$

Therefore, from (D-1), we estimate

$$\text{for all } k \geq 0, \quad \left| H^k\left(\frac{\Lambda V}{y^2}\varepsilon\right)\right| \leq \sum_{i=0}^{2k} c_{i,k}\frac{|\varepsilon_i|}{1+y^{4+(2k-i)}}. \tag{E-2}$$

Similarly,

$$\left|AH^k\left(\frac{\Lambda V}{y^2}\varepsilon\right)\right| \lesssim \sum_{i=0}^{2k} c_{i,k}\frac{1}{1+y^{4+(2k-i)}}\left[|\partial_y\varepsilon_i| + \frac{|\varepsilon_i|}{y}\right] \lesssim \sum_{i=0}^{2k+1} c_{i,k}\frac{|\varepsilon_i|}{y(1+y^{4+(2k-i)})}.$$

We now inject (E-2) into (E-1) and obtain using (3-36) the pointwise bound on the commutator

$$|[\partial_t, H_\lambda^L]w| \lesssim \frac{|b_1|}{\lambda^{2L+2}}\sum_{k=0}^{L-1}\sum_{i=0}^{2k} c_{i,k}\frac{|\varepsilon_{2(L-k-1)+i}|}{1+y^{4+(2k-i)}} \lesssim \frac{|b_1|}{\lambda^{2L+2}}\sum_{m=0}^{2L-2}\frac{|\varepsilon_{2L-2-m}|}{1+y^{4+m}} = \frac{|b_1|}{\lambda^{2L+2}}\sum_{m=0}^{2L-2}\frac{|\varepsilon_m|}{1+y^{2+2L-m}}.$$

Hence, after a change of variables in the integral, and using (C-1), we have

$$\int \frac{|[\partial_t, H_\lambda^L]w|^2}{\lambda^2(1+y^2)} \lesssim \frac{|b_1|^2}{\lambda^{4L+4}}\sum_{m=0}^{2L-2}\int \frac{\varepsilon_m^2}{(1+y^2)(1+y^{4+4L-2m})} \lesssim \frac{C(M)b_1^2}{\lambda^{4L+4}}\mathscr{E}_{2L+2},$$

and, similarly,

$$\int |A_\lambda[\partial_t, H_\lambda^L]w|^2 \lesssim \frac{|b_1|^2}{\lambda^{4L+4}}\sum_{m=0}^{2L-1}\int \frac{\varepsilon_m^2}{y^2(1+y^{4+4L-2m})} \lesssim \frac{C(M)b_1^2}{\lambda^{4L+4}}\mathscr{E}_{2L+2},$$

which is (3-55).

Appendix F: Proof of (4-10)

We claim the following Lyapounov monotonicity functional for the $\mathscr{E}_{2k+2}$ energy.

Proposition F.1 (Lyapounov monotonicity for $\mathscr{E}_{2k+2}$). *Let $0 \leq k \leq L-1$. Then we have*

$$\frac{d}{dt}\left\{\frac{1}{\lambda^{4k+2}}\left[\mathscr{E}_{2k+2} + O\left(b_1^{\frac{1}{2}}b_1^{(4k+2)2L/(2L-1)}\right)\right]\right\} \lesssim \frac{|\log b_1|^C}{\lambda^{4k+4}}\left[b_1^{2k+3} + b_1^{1+\delta+(2k+1)2L/(2L-1)} + \sqrt{b_1^{2k+4}\mathscr{E}_{2k+2}}\right] \tag{F-1}$$

for some universal constants $C, \delta > 0$ independent of M and of the bootstrap constant K in (3-23), (3-24).

Proof of Proposition F.1. *Step* 1: *modified energy identity*. We follow verbatim the algebra of (3-48) with $L \to k$ and obtain the modified energy identity

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\mathscr{E}_{2k+2} + 2\int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2k+1} w_{2k}\right\} \\
&\quad = -\int (\widetilde{H}_\lambda w_{2k+1})^2 - \left(\frac{\lambda_s}{\lambda} + b_1\right)\int \frac{(\Lambda \widetilde{V})_\lambda}{2\lambda^2 r^2} w_{2k+1}^2 + \int \frac{d}{dt}\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}\right) w_{2k+1} w_{2k} \\
&\qquad + \int \widetilde{H}_\lambda w_{2k+1}\left[\frac{\partial_t Z_\lambda}{r} w_{2k} + A_\lambda([\partial_t, H_\lambda^k]w) + A_\lambda H_\lambda^k\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right] \\
&\qquad + \int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2k}\left[-\widetilde{H}_\lambda w_{2k+1} + \frac{\partial_t Z_\lambda}{r} w_{2k} + A_\lambda([\partial_t, H_\lambda^k]w) + A_\lambda H_\lambda^k\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right] \qquad \text{(F-2)} \\
&\qquad + \int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L+1}\left[[\partial_t, H_\lambda^k]w + H_\lambda^k\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right].
\end{aligned}$$

We now estimate all terms in the right hand side of (F-2).

Step 3: *Lower order quadratic terms*. We treat the lower order quadratic terms in (F-2) using dissipation. The bound

$$\int \frac{([\partial_t, H_\lambda^k]w)^2}{\lambda^2(1+y^2)} + \int |A_\lambda([\partial_t, H_\lambda^k]w)|^2 \lesssim C(M)\frac{b_1^2}{\lambda^{4k+4}}\mathscr{E}_{2k+2} \qquad \text{(F-3)}$$

follows from (3-55) with $L \to k$. From (3-54), the rough bound (3-38), and Lemma C.1, we estimate

$$\begin{aligned}
&\int \left|\widetilde{H}_\lambda w_{2k+1}\left[\frac{\partial_t Z_\lambda}{r} w_{2k} + \int A_\lambda\left([\partial_t, H_\lambda^k]w\right)\right]\right| + \int |\widetilde{H}_\lambda w_{2k+1}|\left|\frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_{2k}\right| \\
&\qquad \le \frac{1}{2}\int |\widetilde{H}_\lambda w_{2k+1}|^2 + \frac{b_1^2}{\lambda^{4k+4}}\left[\int \frac{\varepsilon_{2k}^2}{1+y^6} + C(M)\mathscr{E}_{2k+2}\right] \\
&\qquad \le \frac{1}{2}\int |\widetilde{H}_\lambda w_{2k+1}|^2 + \frac{b_1}{\lambda^{4k+4}} C(M) b_1 \mathscr{E}_{2k+2}.
\end{aligned}$$

All other quadratic terms are lower order by a factor b_1, again using (3-38), (3-55), (3-36), and Lemma C.1:

$$\begin{aligned}
&\left|\frac{\lambda_s}{\lambda} + b_1\right|\left|\int \left|\frac{(\Lambda \widetilde{V})_\lambda}{2\lambda^2 r^2} w_{2k+1}^2\right| + \int \left|\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2k}\left[\frac{\partial_t Z_\lambda}{r} w_{2k} + A_\lambda\left([\partial_t, H_\lambda^k]w\right)\right]\right| \\
&\qquad + \int \left|\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2k+1}[\partial_t, H_\lambda^L]w\right| + \left|\int \frac{d}{dt}\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}\right) w_{2k+1} w_{2k}\right| \\
&\qquad \lesssim \frac{b_1^2}{\lambda^{4k+4}}\left[\int \frac{\varepsilon_{2k+1}^2}{1+y^4} + \int \frac{\varepsilon_{2k}^2}{1+y^6} + C(M)\mathscr{E}_{2k+2}\right] \lesssim \frac{b_1}{\lambda^{4k+4}} C(M) b_1 \mathscr{E}_{2k+2}.
\end{aligned}$$

We similarly estimate the boundary term in time using (C-10):

$$\left|\int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2k+1} w_{2k}\right| \lesssim \frac{b_1}{\lambda^{4k+2}}\left[\int \frac{\varepsilon_{2k+1}^2}{1+y^2} + \int \frac{\varepsilon_{2k}^2}{1+y^4}\right] \lesssim \frac{b_1}{\lambda^{4k+2}} |\log b_1|^C b_1^{(4k+2)2L/(2L-1)}.$$

We inject these estimates into (3-53) to derive the preliminary bound

$$\frac{1}{2}\frac{d}{dt}\left\{\frac{1}{\lambda^{4k+2}}\left[\mathscr{E}_{2k+2}+O\big(b_1^{\frac{1}{2}}b_1^{(4k+2)2L/(2L-1)}\big)\right]\right\}\le-\frac{1}{2}\int(\widetilde{H}_\lambda w_{2k+1})^2+\int\widetilde{H}_\lambda w_{2k+1}A_\lambda H_\lambda^k\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)$$
$$+\int H_\lambda^k\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\left[\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2k+1}+A_\lambda^*\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2k}\right)\right]+\frac{b_1}{\lambda^{4k+4}}b_1^\delta\mathscr{E}_{2k+2}\quad\text{(F-4)}$$

with constants independent of M for $|b|<b^*(M)$ small enough. We now estimate all terms in the right hand side of (F-4).

Step 4: *further use of dissipation*. Recall the decomposition (3-57). The first term in the right hand side of (F-4) is estimated after an integration by parts

$$\left|\int\widetilde{H}_\lambda w_{2k+1}A_\lambda H_\lambda^k\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right|\le\frac{C}{\lambda^{4k+4}}\|A^*\varepsilon_{2k+1}\|_{L^2}\|H^{k+1}\mathscr{F}_0\|_{L^2}+\frac{1}{4}\int|\widetilde{H}_\lambda w_{2k+1}|^2+\frac{C}{\lambda^{4k+4}}\int|AH^k\mathscr{F}_1|^2$$
$$\le\frac{C}{\lambda^{4k+4}}\big[\|H^{k+1}\mathscr{F}_0\|_{L^2}\sqrt{\mathscr{E}_{2k+2}}+\|AH^k\mathscr{F}_1\|_{L^2}^2\big]+\frac{1}{4}\int|\widetilde{H}_\lambda w_{2k+1}|^2\quad\text{(F-5)}$$

for some universal constant $C>0$ independent of M. The last two terms in (F-4) can be estimated by brute force from Cauchy–Schwarz

$$\left|\int H_\lambda^k\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2k+1}\right|\lesssim\frac{b_1}{\lambda^{4k+4}}\left(\int\frac{1+|\log y|^2}{1+y^4}|H^k\mathscr{F}|^2\right)^{\frac{1}{2}}\left(\int\frac{\varepsilon_{2k+1}^2}{y^2(1+|\log y|^2)}\right)^{\frac{1}{2}}$$
$$\lesssim\frac{b_1}{\lambda^{4k+4}}\sqrt{\mathscr{E}_{2k+2}}\left(\int\frac{1+|\log y|^2}{1+y^4}|H^k\mathscr{F}|^2\right)^{\frac{1}{2}},\quad\text{(F-6)}$$

where constants are independent of M thanks to the estimate (B-2) for ε_{2k+1}. Similarly,

$$\left|\int H_\lambda^k\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)A_\lambda^*\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2k}\right)\right|$$
$$\lesssim\frac{b_1}{\lambda^{4k+4}}\left(\int\frac{1+|\log y|^2}{1+y^2}|AH^k\mathscr{F}|^2\right)^{\frac{1}{2}}\left(\int\frac{\varepsilon_{2k}^2}{(1+y^4)(1+|\log y|^2)}\right)^{\frac{1}{2}}$$
$$\lesssim\frac{b_1}{\lambda^{4k+4}}C(M)\sqrt{\mathscr{E}_{2k+2}}\left(\int\frac{1+|\log y|^2}{1+y^2}|AH^k\mathscr{F}_0|^2+\int|AH^k\mathscr{F}_1|^2\right)^{\frac{1}{2}}.\quad\text{(F-7)}$$

We now claim the bounds

$$\int\frac{1+|\log y|^2}{1+y^4}|H^k\mathscr{F}|^2\le b_1^{2k+2}|\log b_1|^C,\quad\text{(F-8)}$$
$$\int\frac{1+|\log y|^2}{1+y^2}|AH^k\mathscr{F}_0|^2\le b_1^{2k+2}|\log b_1|^C,\quad\text{(F-9)}$$
$$\int|H^{k+1}\mathscr{F}_0|^2\le b_1^{2k+4}|\log b_1|^C,\quad\text{(F-10)}$$
$$\int|AH^k\mathscr{F}_1|^2\le b_1^{2k+3}|\log b_1|^C+b_1^{1+\delta+(2k+1)2L/(2L-1)}\quad\text{(F-11)}$$

for some universal constants $\delta, C > 0$ independent of M and of the bootstrap constant K in (3-23), (3-24). Injecting these bounds together with (F-5), (F-6), (F-7) into (F-4) concludes the proof of (F-1). We now turn to the proofs of (F-8), (F-9), (F-10), (F-11).

Step 5: $\widetilde{\Psi}_b$ *terms*. From (2-88) we estimate

$$\int \frac{1+|\log y|^2}{1+y^4}|H^k\widetilde{\Psi}_b|^2 \lesssim \int |H^k\widetilde{\Psi}_b|^2 \lesssim b_1^{2k+2}|\log b_1|^C,$$
$$\int \frac{1+|\log y|^2}{1+y^2}|AH^k\widetilde{\Psi}_b|^2 \lesssim \int |AH^k\widetilde{\Psi}_b|^2 = \int H^k\widetilde{\Psi}_b H^{k+1}\widetilde{\Psi}_b \lesssim b_1^{2k+3}|\log b_1|^C,$$
$$\int |H^{k+1}\widetilde{\Psi}_b|^2 \lesssim b_1^{2(k+1)+2}|\log b_1|^C,$$

and (F-8), (F-9), (F-10) are proved for $\widetilde{\Psi}_b$.

Step 6: $\widetilde{\mathrm{Mod}(t)}$ *terms*. Recall (3-29),

$$\widetilde{\mathrm{Mod}}(t) = -\left(\frac{\lambda_s}{\lambda}+b_1\right)\Lambda\widetilde{Q}_b + \sum_{i=1}^{L}[(b_i)_s + (2i-1+c_{b_1})b_1b_i - b_{i+1}]\left[\widetilde{T}_i + \chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}\right],$$

and the notation (3-39). We will need only the rough bound for b_1-admissible functions (2-32).

Proof of (F-10) *for* $\widetilde{\mathrm{Mod}}$. We estimate from (2-32), for $y \le 2B_1$,

$$|H^{k+1}S_i| + |H^{k+1}\Lambda S_i| + |H^{k+1}b_i\Lambda\widetilde{T}_i| \lesssim b_1^i(1+y)^{2i-1-(2k+2)} \lesssim b_1b_1^{i-1}(1+y)^{2i-2k-3} \lesssim \frac{b_1|\log b_1|^C}{1+y^{2k+1}},$$

and thus, using $H\Lambda Q = 0$,

$$\int |H^{k+1}\Lambda\widetilde{Q}_b|^2 \lesssim \int_{y\le 2B_1}\frac{b_1^2|\log b_1|^C}{1+y^{4k+2}} \lesssim b_1^2|\log b_1|^C.$$

We also have the rough bound, for $1 \le i \le L$, $i+1 \le j \le L_2$, $y \le 2B_1$,

$$|\widetilde{T}_i| + \left|\chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}\right| \lesssim |\log b_1|^C\left[y^{2i-1} + y^{2j-1}b_1^{j-i}|\log b_1|^C\right] \lesssim |\log b_1|^C y^{2i-1}, \tag{F-12}$$

and similarly for suitable derivatives, and hence the bound

$$\sum_{i=1}^{L}\int\left|H^{k+1}\left[\widetilde{T}_i + \chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}\right]\right|^2 \lesssim |\log b_1|^C\int_{y\le 2B_1}|y^{2L-1-(2k+2)}|^2$$
$$\lesssim |\log b_1|^C B_1^{4(L-k)-4} \lesssim \frac{|\log b_1|^C}{b_1^{2(L-k)-2}}.$$

We therefore obtain from Lemma 3.3 the control

$$\int |H^{k+1}\widetilde{\mathrm{Mod}(t)}|^2 \lesssim C(K)|\log b_1|^C b_1^{2L+2}\left[b_1^2 + \frac{1}{b_1^{2(L-k)-2}}\right] \lesssim C(K)b_1^{2k+4}|\log b_1|^C \lesssim |\log b_1|^C b_1^{2k+4}$$

for $b_1 < b_1^*(M)$ small enough.

Proof of (F-8), (F-9). We estimate

$$|H^k S_i| + |H^k \Lambda S_i| + |H^k b_i \Lambda \widetilde{T}_i| \lesssim b_1^i (1+y)^{2i-1-2k} \lesssim \frac{|\log b_1|^C}{1+y^{2k+1}},$$

and thus

$$\int \frac{1+|\log y|^2}{1+y^4} |H^k \Lambda \widetilde{Q}_b|^2 + \int \frac{1+|\log y|^2}{1+y^2} |AH^k \Lambda \widetilde{Q}_b|^2 \lesssim |\log b_1|^C \int \frac{1}{1+y^{4k+2}} \lesssim |\log b_1|^C.$$

Then, from (F-12), we estimate

$$\sum_{i=1}^{L} \int \frac{1+|\log y|^2}{1+y^4} \left| H^k \left[\widetilde{T}_i + \chi_{B_1} \sum_{j=i+1}^{L+2} \frac{\partial S_j}{\partial b_i} \right] \right|^2 + \sum_{i=1}^{L} \int \frac{1+|\log y|^2}{1+y^2} \left| AH^k \left[\widetilde{T}_i + \chi_{B_1} \sum_{j=i+1}^{L+2} \frac{\partial S_j}{\partial b_i} \right] \right|^2$$
$$\lesssim |\log b_1|^C \int_{y \le 2B_1} |y^{2L-1-2k-2}|^2 \lesssim \frac{|\log b_1|^C}{b_1^{2(L-k)-2}},$$

and hence, using Lemma 3.3, we have

$$\int \frac{1+|\log y|^2}{1+y^4} |H^k \widetilde{\mathrm{Mod}}|^2 + \int \frac{1+|\log y|^2}{1+y^2} |AH^k \widetilde{\mathrm{Mod}}|^2 \lesssim |\log b_1|^C C(K) b_1^{2L+2} \left[1 + \frac{1}{b_1^{2(L-k)-2}} \right] \lesssim b_1^{2k+2}.$$

Step 7: *nonlinear term* $N(\varepsilon)$. *Control near the origin* $y \le 1$. The control near the origin follows directly from (3-72).

Control for $y \ge 1$. We detail the proof of the most delicate bound (F-11). The proofs of (F-8) and (F-9) follow similar lines and are left to the reader.

Recall the notations (3-73) and the bounds (3-74), (3-75), (3-76) on ζ. We then have the bounds (3-77), (3-78), (3-79) on $N_1(\varepsilon)$, which yield

$$\begin{aligned}
|AH^k N(\varepsilon)| &\lesssim \sum_{p=0}^{2k+1} \frac{|\partial_y^p N(\varepsilon)|}{y^{2k+1-p}} \lesssim \sum_{p=0}^{2k+1} \frac{1}{y^{2k+1-p}} \sum_{i=0}^{p} |\partial_y^i \zeta^2| |\partial_y^{p-i} N_1(\varepsilon)| \\
&\lesssim \sum_{p=0}^{2k+1} \frac{|\partial_y^k \zeta^2|}{y^{2k+1-p}} + \sum_{p=1}^{2k+1} \frac{1}{y^{2k+1-p}} \sum_{i=0}^{p-1} |\partial_y^i \zeta^2| |\log b_1|^C \left[\frac{1}{y^{p-i+1}} + b_1^{a_{p-i}/2} \right] \\
&\lesssim \sum_{p=0}^{2k+1} \frac{|\partial_y^p \zeta^2|}{y^{2k+1-p}} + |\log b_1|^C \sum_{i=0}^{2k} \frac{|\partial_y^i \zeta^2|}{y^{2k+2-i}} + |\log b_1|^C \sum_{p=1}^{2k+1} \sum_{i=0}^{p-1} b_1^{a_{p-i}/2} \frac{|\partial_y^i \zeta^2|}{y^{2k+1-p}} \\
&\lesssim |\log b_1|^C \left[\sum_{p=0}^{2k+1} \frac{|\partial_y^p \zeta^2|}{y^{2k+1-p}} + \sum_{p=1}^{2k+1} \sum_{i=0}^{p-1} b_1^{a_{p-i}/2} \frac{|\partial_y^i \zeta^2|}{y^{2k+1-p}} \right],
\end{aligned}$$

and hence

$$\int_{y \ge 1} |AH^k N(\varepsilon)|^2 \lesssim |\log b_1|^C \sum_{p=0}^{2k+1} \sum_{i=0}^{p} \int_{y \ge 1} \frac{|\partial_y^i \zeta|^2 |\partial_y^{p-i} \zeta|^2}{y^{4L+2-2p}} + |\log b_1|^C \sum_{p=1}^{2k+1} \sum_{i=0}^{p-1} \sum_{j=0}^{i} b_1^{a_{p-i}} \int_{y \ge 1} \frac{|\partial_y^j \zeta|^2 |\partial_y^{i-j} \zeta|^2}{y^{4L+2-2p}}.$$

We now claim the bounds

$$\sum_{p=0}^{2k+1}\sum_{i=0}^{p}\int_{y\geq 1}\frac{|\partial_y^i\zeta|^2|\partial_y^{p-i}\zeta|^2}{y^{4k+2-2p}}\leq b_1 b_1^{\delta} b_1^{(2k+1)2L/(2L-1)}, \tag{F-13}$$

$$|\log b_1|^C\sum_{p=1}^{2k+1}\sum_{i=0}^{p-1}\sum_{j=0}^{i} b_1^{a_{p-i}}\int_{y\geq 1}\frac{|\partial_y^j\zeta|^2|\partial_y^{i-j}\zeta|^2}{y^{4k+2-2p}}\leq b_1 b_1^{\delta} b_1^{(2k+1)2L/(2L-1)} \tag{F-14}$$

for some $\delta>0$, and this concludes the proof of (F-11) for $N(\varepsilon)$.

Proof of (3-80). Let $0\leq k\leq L-1$, $0\leq p\leq 2k+1$, $0\leq i\leq p$. Let $I_1=p-i$, $I_2=i$. Then we can pick $J_2\in\mathbb{N}^*$ such that

$$\max\{1;2-i\}\leq J_2\leq \min\{2k+3-p;2k+2-i\}$$

and define

$$J_1=2k+3-p-J_2.$$

Then, from direct inspection,

$$(I_1,J_1,I_2,J_2)\in\mathbb{N}^3\times\mathbb{N}^*,\quad \begin{cases}1\leq I_1+J_1\leq 2k+1\leq 2L-1, & 2\leq I_2+J_2\leq 2k+2\leq 2L,\\ I_1+I_2+J_1+J_2=2k+3.\end{cases}$$

Hence, from (3-74), (3-75),

$$\begin{aligned}\int_{y\geq 1}\frac{|\partial_y^i\zeta|^2|\partial_y^{p-i}\zeta|^2}{y^{4k+2-2p}}&\lesssim \left\|\frac{\partial_y^{I_1}\zeta}{y^{J_1-1}}\right\|^2_{L^\infty(y\geq 1)}\int_{y\geq 1}\frac{|\partial_y^{I_2}\zeta|^2}{y^{2J_2-2}}\\ &\lesssim |\log b_1|^{C(K)} b_1^{(I_1+J_1+I_2+J_2-1)2L/(2L-1)}=|\log b_1|^{C(K)} b_1^{(2k+2)2L/(2L-1)}\\ &\leq b_1 b_1^{\delta} b_1^{(2k+1)2L/(2L-1)}.\end{aligned}$$

□

Proof of (3-81). Let $0\leq k\leq L-1$, $1\leq p\leq 2k+1$, $0\leq j\leq i\leq p-1$. For $p=2k+1$ and $0\leq i=j\leq 2k$, we use the energy bound (3-76) to estimate

$$\begin{aligned}b_1^{a_{p-i}}\int_{y\geq 1}\frac{|\partial_y^j\zeta|^2|\partial_y^{i-j}\zeta|^2}{y^{4k+2-2p}}&=b_1^{a_{2k+1-i}}\|\zeta\|^2_{L^\infty(y\geq 1)}\int_{y\geq 1}|\partial_y^i\zeta|^2\\ &\lesssim b_1^{2L/(2L-1)((2k+1-i)+1+i)}|\log b_1|^{C(K)}\leq b_1 b_1^{\delta} b_1^{(2k+1)2L/(2L-1)}.\end{aligned}$$

This exceptional case being treated, we let $I_1=j$, $I_2=i-j$ and pick $J_2\in\mathbb{N}^*$ with

$$\max\{1;2-(i-j);2-(p-j)\}\leq J_2\leq \min\{2k+3-p;2k+2-(p-j);2k+2-(i-j)\}.$$

Let

$$J_1=2k+3-p-J_2.$$

Then we can directly check that

$$(I_1,J_1,I_2,J_2)\in\mathbb{N}^3\times\mathbb{N}^*,\quad \begin{cases}1\leq I_1+J_1\leq 2k+1, & 2\leq I_2+J_2\leq 2k+2,\\ I_1+I_2+J_1+J_2=2k+3-(p-i),\end{cases}$$

and thus

$$b_1^{a_{p-i}}\int_{y\geq 1}\frac{|\partial_y^j\zeta|^2|\partial_y^{i-j}\zeta|^2}{y^{4k+2-2p}}\lesssim b_1^{a_{p-i}}\left\|\frac{\partial_y^{I_1}\zeta}{y^{J_1-1}}\right\|^2_{L^\infty(y\geq 1)}\int_{y\geq 1}\frac{|\partial_y^{I_2}\zeta|^2}{y^{2J_2-2}}$$
$$\lesssim |\log b_1|^{C(K)}b_1^{(p-i+I_1+J_1+I_2+J_2-1)2L/(2L-1)}=|\log b_1|^{C(K)}b_1^{(2k+2)2L/(2L-1)}$$
$$\leq b_1 b_1^{\delta}b_1^{(2k+1)2L/(2L-1)}.$$ □

Step 8: *small linear term* $L(\varepsilon)$. We recall the decomposition (3-84).

Control for $y\leq 1$. The control near the origin directly follows from (3-88).

Control for $y\geq 1$. We give a detailed proof of (F-11) and leave (F-8) to the reader. We recall the bound (3-90):

$$|\partial_y^k L(\varepsilon)|\lesssim\sum_{i=0}^{k}\frac{b_1|\log b_1|^C|\partial_y^i\varepsilon|}{y^{k-i+1}}.$$

This implies

$$|AH^kL(\varepsilon)|\lesssim\sum_{p=0}^{2k+1}\frac{|\partial_y^pL(\varepsilon)|}{y^{2k+1-p}}\lesssim\sum_{p=0}^{2k+1}\frac{1}{y^{2k+1-p}}\sum_{i=0}^{p}\frac{b_1|\log b_1|^C|\partial_y^i\varepsilon|}{y^{p-i+1}}\lesssim b_1|\log b_1|^C\sum_{i=0}^{2k+1}\frac{|\partial_y^i\varepsilon|}{y^{2k+2-i}}.$$

We therefore conclude from (C-11) that

$$\int_{y\geq 1}|AH^kL(\varepsilon)|^2\lesssim b_1^2|\log b_1|^C\sum_{i=0}^{2k+1}\int_{y\geq 1}\frac{|\partial_y^i\varepsilon|^2}{y^{4k+4-2i}}$$
$$\lesssim|\log b_1|^{C(K)}b_1^{2+(2k+1)2L/(2L-1)}\leq b_1^{1+\delta+(2k+1)2L/(2L-1)},$$

and (3-64) is proved.

This concludes the proof of (F-8), (F-9), (F-10), (F-11), and thus of Proposition 3.6. □

Acknowledgments

Raphaël would like to thank Frank Merle and Igor Rodnianski for stimulating discussions on this problem. Both authors are supported by the junior ERC/ANR project SWAP. Raphaël is also partially supported by the senior ERC grant BLOWDISOL.

References

[Angenent et al. 2009] S. B. Angenent, J. Hulshof, and H. Matano, "The radius of vanishing bubbles in equivariant harmonic map flow from D^2 to S^2", *SIAM J. Math. Anal.* **41**:3 (2009), 1121–1137. MR 2010j:53122 Zbl 1223.35198

[Bejenaru and Tataru 2014] I. Bejenaru and D. Tataru, *Near soliton evolution for equivariant Schrödinger maps in two spatial dimensions*, Mem. Amer. Math. Soc. **1069**, Amer. Math. Soc., Providence, RI, 2014. MR 3157482

[van den Berg et al. 2003] J. B. van den Berg, J. Hulshof, and J. R. King, "Formal asymptotics of bubbling in the harmonic map heat flow", *SIAM J. Appl. Math.* **63**:5 (2003), 1682–1717. MR 2004h:35097 Zbl 1037.35023

[Chang et al. 1992] K.-C. Chang, W. Y. Ding, and R. Ye, "Finite-time blow-up of the heat flow of harmonic maps from surfaces", *J. Differential Geom.* **36**:2 (1992), 507–515. MR 93h:58043 Zbl 0765.53026

[Coron and Ghidaglia 1989] J.-M. Coron and J.-M. Ghidaglia, "Explosion en temps fini pour le flot des applications harmoniques", *C. R. Acad. Sci. Paris Sér. I Math.* **308**:12 (1989), 339–344. MR 90g:58026 Zbl 0679.58017

[Côte 2005] R. Côte, "Instability of nonconstant harmonic maps for the $(1+2)$-dimensional equivariant wave map system", *Int. Math. Res. Not.* **2005**:57 (2005), 3525–3549. MR 2006k:58023 Zbl 1101.35055

[Côte and Zaag 2013] R. Côte and H. Zaag, "Construction of a multisoliton blowup solution to the semilinear wave equation in one space dimension", *Comm. Pure Appl. Math.* **66**:10 (2013), 1541–1581. MR 3084698 Zbl 06214311

[Côte et al. 2011] R. Côte, Y. Martel, and F. Merle, "Construction of multi-soliton solutions for the L^2-supercritical gKdV and NLS equations", *Rev. Mat. Iberoam.* **27**:1 (2011), 273–302. MR 2012j:35028 Zbl 1273.35234

[Ding and Tian 1995] W. Ding and G. Tian, "Energy identity for a class of approximate harmonic maps from surfaces", *Comm. Anal. Geom.* **3**:3-4 (1995), 543–554. MR 97e:58055 Zbl 0855.58016

[Donninger and Krieger 2013] R. Donninger and J. Krieger, "Nonscattering solutions and blowup at infinity for the critical wave equation", *Math. Ann.* **357**:1 (2013), 89–163. MR 3084344 Zbl 1280.35135

[Filippas et al. 2000] S. Filippas, M. A. Herrero, and J. J. L. Velázquez, "Fast blow-up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity", *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **456**:2004 (2000), 2957–2982. MR 2002f:35121 Zbl 0988.35032

[Gallot et al. 2004] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, 3rd ed., Springer, Berlin, 2004. MR 2005e:53001 Zbl 1068.53001

[Guan et al. 2009] M. Guan, S. Gustafson, and T.-P. Tsai, "Global existence and blow-up for harmonic map heat flow", *J. Differential Equations* **246**:1 (2009), 1–20. MR 2009i:35154 Zbl 1177.35104

[Gustafson et al. 2010] S. Gustafson, K. Nakanishi, and T.-P. Tsai, "Asymptotic stability, concentration, and oscillation in harmonic map heat-flow, Landau–Lifshitz, and Schrödinger maps on $\mathbb{R}^2$", *Comm. Math. Phys.* **300**:1 (2010), 205–242. MR 2011j:58020 Zbl 1205.35294

[Herrero and Velázquez 1994] M. A. Herrero and J. J. L. Velázquez, "Explosion de solutions d'équations paraboliques semilinéaires supercritiques", *C. R. Acad. Sci. Paris Sér. I Math.* **319**:2 (1994), 141–145. MR 95i:35037 Zbl 0806.35005

[Hillairet and Raphaël 2012] M. Hillairet and P. Raphaël, "Smooth type II blow-up solutions to the four-dimensional energy-critical wave equation", *Anal. PDE* **5**:4 (2012), 777–829. MR 3006642 Zbl 06165684

[Krieger and Schlag 2009] J. Krieger and W. Schlag, "Non-generic blow-up solutions for the critical focusing NLS in 1-D", *J. Eur. Math. Soc.* (*JEMS*) **11**:1 (2009), 1–125. MR 2009m:35488 Zbl 1163.35035

[Krieger et al. 2008] J. Krieger, W. Schlag, and D. Tataru, "Renormalization and blow up for charge one equivariant critical wave maps", *Invent. Math.* **171**:3 (2008), 543–615. MR 2009b:58061 Zbl 1139.35021

[Martel and Merle 2000] Y. Martel and F. Merle, "A Liouville theorem for the critical generalized Korteweg-de Vries equation", *J. Math. Pures Appl.* (9) **79**:4 (2000), 339–425. MR 2001i:37102 Zbl 0963.37058

[Martel et al. 2012a] Y. Martel, F. Merle, and P. Raphaël, "Blow up for the critical gKdV equation I: Dynamics near the solitary wave", preprint, 2012. arXiv 1204.4625

[Martel et al. 2012b] Y. Martel, F. Merle, and P. Raphaël, "Blow up for the critical gKdV equation II: Minimal mass blow up", preprint, 2012. arXiv 1204.4624

[Martel et al. 2012c] Y. Martel, F. Merle, and P. Raphaël, "Blow up for the critical gKdV equation III: Exotic regimes", preprint, 2012. arXiv 1209.2510

[Matano and Merle 2004] H. Matano and F. Merle, "On nonexistence of type II blowup for a supercritical nonlinear heat equation", *Comm. Pure Appl. Math.* **57**:11 (2004), 1494–1541. MR 2005e:35115 Zbl 1112.35098

[Matano and Merle 2009] H. Matano and F. Merle, "Classification of type I and type II behaviors for a supercritical nonlinear heat equation", *J. Funct. Anal.* **256**:4 (2009), 992–1064. MR 2009k:35124 Zbl 1178.35084

[Merle and Raphaël 2005a] F. Merle and P. Raphaël, "The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation", *Ann. of Math.* (2) **161**:1 (2005), 157–222. MR 2006k:35277 Zbl 1185.35263

[Merle and Raphaël 2005b] F. Merle and P. Raphaël, "Profiles and quantization of the blow up mass for critical nonlinear Schrödinger equation", *Comm. Math. Phys.* **253**:3 (2005), 675–704. MR 2006m:35346 Zbl 1062.35137

[Merle et al. 2011] F. Merle, P. Raphaël, and I. Rodnianski, "Blow up dynamics for smooth equivariant solutions to the energy critical Schrödinger map", *C. R. Math. Acad. Sci. Paris* **349**:5-6 (2011), 279–283. MR 2012e:58029 Zbl 1213.35139

[Merle et al. 2013] F. Merle, P. Raphaël, and J. Szeftel, "The instability of Bourgain–Wang solutions for the L^2 critical NLS", *Amer. J. Math.* **135**:4 (2013), 967–1017. MR 3086066 Zbl 1294.35145

[Mizoguchi 2007] N. Mizoguchi, "Rate of type II blowup for a semilinear heat equation", *Math. Ann.* **339**:4 (2007), 839–877. MR 2008k:35246 Zbl 1172.35420

[Qing and Tian 1997] J. Qing and G. Tian, "Bubbling of the heat flows for harmonic maps from surfaces", *Comm. Pure Appl. Math.* **50**:4 (1997), 295–310. MR 98k:58070 Zbl 0879.58017

[Raphaël 2013] P. Raphaël, "On the singularity formation for the nonlinear Schrödinger equation", pp. 269–323 in *Evolution equations*, edited by D. Ellwood et al., Clay Math. Proc. **17**, Amer. Math. Soc., Providence, RI, 2013. MR 3098642 Zbl 06315828

[Raphaël and Rodnianski 2012] P. Raphaël and I. Rodnianski, "Stable blow up dynamics for the critical co-rotational wave maps and equivariant Yang–Mills problems", *Publ. Math. Inst. Hautes Études Sci.* (2012), 1–122. MR 2929728 Zbl 1284.35358

[Raphaël and Schweyer 2013] P. Raphaël and R. Schweyer, "Stable blowup dynamics for the 1-corotational energy critical harmonic heat flow", *Comm. Pure Appl. Math.* **66**:3 (2013), 414–480. MR 3008229 Zbl 1270.35136

[Raphaël and Schweyer 2014] P. Raphaël and R. Schweyer, "On the stability of critical chemotactic aggregation", *Math. Ann.* **359**:1-2 (2014), 267–377. MR 3201901 Zbl 06312293

[Schweyer 2012] R. Schweyer, "Type II blow-up for the four dimensional energy critical semi linear heat equation", *J. Funct. Anal.* **263**:12 (2012), 3922–3983. MR 2990063 Zbl 1270.35137

[Struwe 1985] M. Struwe, "On the evolution of harmonic mappings of Riemannian surfaces", *Comment. Math. Helv.* **60**:4 (1985), 558–581. MR 87e:58056 Zbl 0595.58013

[Topping 2004] P. Topping, "Winding behaviour of finite-time singularities of the harmonic map heat flow", *Math. Z.* **247**:2 (2004), 279–302. MR 2004m:53120 Zbl 1067.53055

Received 10 Jan 2013. Accepted 22 Dec 2013.

PIERRE RAPHAËL: praphael@unice.fr
Laboratoire J. A. Dieudonné, Université de Nice Sophia Antipolis, Institut Universitaire de France, 06000 Nice, France

REMI SCHWEYER: remi.schweyer@math.univ-toulouse.fr
Institut de Mathématiques de Toulouse, Université Paul Sabatier, 31000 Toulouse, France

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2014 is US $180/year for the electronic version, and $355/year (+$50, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Analysis & PDE

Volume 7 No. 8 2014