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CONCENTRATION OF SMALL WILLMORE SPHERES IN RIEMANNIAN 3-MANIFOLDS

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Given a three-dimensional Riemannian manifold (M, g), we prove that, if (Φ_k) is a sequence of Willmore spheres (or more generally area-constrained Willmore spheres) having Willmore energy bounded above uniformly strictly by 8π and Hausdorff converging to a point $\bar{p} \in M$, then $\mathrm{Scal}(\bar{p}) = 0$ and $\nabla \mathrm{Scal}(\bar{p}) = 0$ (respectively, $\nabla \mathrm{Scal}(\bar{p}) = 0$). Moreover, a suitably rescaled sequence smoothly converges, up to subsequences and reparametrizations, to a round sphere in the euclidean three-dimensional space. This generalizes previous results of Lamm and Metzger. An application to the Hawking mass is also established.

1. Introduction

Let Σ be a closed two-dimensional surface and (M, g) a three-dimensional Riemannian manifold. Given a smooth immersion $\Phi : \Sigma \hookrightarrow M$, $W(\Phi)$ denotes the Willmore energy of Φ defined by

$$W(\Phi) := \int_\Sigma H^2 \, d\mathrm{vol}_{\bar{g}}, \tag{1}$$

where $\bar{g} := \Phi^*(g)$ is the pullback metric on Σ (i.e., the metric induced by the immersion), $d\mathrm{vol}_{\bar{g}}$ is the associated volume form, and H is the mean curvature of the immersion Φ (we adopt the convention that $H = \frac{1}{2}\bar{g}^{ij} A_{ij}$, where A_{ij} is the second fundamental form; or in other words, H is the arithmetic mean of the two principal curvatures).

In case the ambient manifold is the euclidean three-dimensional space, the topic is classical and goes back to the works of Blaschke and Thomsen in 1920–1930, who were looking for a conformal invariant theory that included minimal surfaces; the functional was later rediscovered by Willmore [1993] in the 1960s, and from that moment, there has been a flourishing of results (let us mention the fundamental paper of Simon [1993], the work of Kuwert and Schätzle [2001; 2004; 2007], the more recent approach by Rivière [2008; 2014; 2013], etc.) culminating in the recent proof of the Willmore conjecture by Marques and Neves [2014] by min–max techniques (let us mention that partial results towards the Willmore conjecture were previously obtained by Li and Yau [1982], Montiel and Ros [1986], Ros [1999], Topping [2000], etc., and that a crucial role in the proof of the conjecture is played by a result of Urbano [1990]).

On the other hand, the investigation of the Willmore functional in nonconstantly curved Riemannian manifolds is a much more recent topic started in [Mondino 2010] (see also [Mondino 2013] and the

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more recent joint work [Carlotto and Mondino 2014]), where the second author studied existence and nonexistence of Willmore surfaces in a perturbative setting.

Smooth minimizers of the L^2-norm of the second fundamental form among spheres in compact Riemannian 3-manifolds were obtained in collaboration with Kuwert and Schygulla in [Kuwert et al. 2014], where the full regularity theory for minimizers was settled, taking inspiration from the approach of Simon [1993] (see also [Mondino and Schygulla 2014] for minimization in noncompact Riemannian manifolds).

Let us finally mention the work in collaboration with Rivière [Mondino and Rivière 2014; 2013], where using a "parametric approach" inspired by the euclidean theory of [Rivière 2008; 2014; 2013], the necessary tools for studying the calculus of variations of the Willmore functional in Riemannian manifolds (i.e., the definition of the weak objects and related compactness and regularity issues) are settled together with applications; in particular, the existence and regularity of Willmore spheres in homotopy classes is established.

Since — as usual in the calculus of variations — the existence results are obtained by quite general techniques and do not describe the minimizing object, the purpose of the present paper is to investigate the geometric properties of the critical points of W.

More precisely, we investigate the following natural questions. Let $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ be a sequence of smooth critical points of the Willmore functional W (or more generally we will also consider critical points under area constraint) converging to a point $\bar{p} \in M$ in Hausdorff distance sense; what can we say about Φ_k? Are they becoming more and more round? Does the limit point $\bar{p}$ have some special geometric property?

These questions have already been addressed in recent articles — below the main known results are recalled for the reader's convenience — but in the present paper we are going to obtain the sharp answers.

Before describing the known and the new results in this direction, let us recall that a critical point of the Willmore functional is called a *Willmore surface* and it satisfies

$$\Delta_{\bar{g}} H + H|A^\circ|^2 + H \operatorname{Ric}(\vec{n}, \vec{n}) = 0, \tag{2}$$

where $\Delta_{\bar{g}}$ is the Laplace–Beltrami operator corresponding to the metric $\bar{g}$, $(A^\circ)_{ij} := A_{ij} - H\bar{g}_{ij}$ is the trace-free second fundamental form, $\vec{n}$ is a normal unit vector to Φ, and Ric is the Ricci tensor of the ambient manifold (M, g). Notice that (2) is a fourth-order nonlinear elliptic PDE in the parametrization map Φ.

Throughout the paper, we will consider more generally *area-constrained Willmore surfaces*, i.e., critical points of the Willmore functional under area constraint; the immersion Φ is an area-constrained Willmore surface if and only if it satisfies

$$\Delta_{\bar{g}} H + H|A^\circ|^2 + H \operatorname{Ric}(\vec{n}, \vec{n}) = \lambda H \tag{3}$$

for some $\lambda \in \mathbb{R}$ playing the role of Lagrange multiplier.

The first result in the direction of the above questions was achieved in the master degree thesis of Mondino [2010], where it was proved that, if (Φ_k) is a sequence of Willmore surfaces obtained as normal graphs over shrinking geodesic spheres centered at a point $\bar{p}$, then the scalar curvature at $\bar{p}$ must vanish: $\mathrm{Scal}(\bar{p}) = 0$.

In subsequent papers, Lamm and Metzger [2010; 2013] proved that, if $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ is a sequence of area-constrained Willmore surfaces converging to a point $\bar{p}$ in Hausdorff distance sense and such that[1]

$$W(\Phi_k) \leq 4\pi + \varepsilon \quad \text{for some } \varepsilon > 0 \text{ small enough,} \tag{4}$$

then $\nabla \mathrm{Scal}(\bar{p}) = 0$ and, up to subsequences, Φ_k is $W^{2,2}$-asymptotic to a geodesic sphere centered at $\bar{p}$. Moreover in [Lamm and Metzger 2013], using the regularity theory developed in [Kuwert et al. 2014], they showed that, if (M, g) is any compact Riemannian 3-manifold and a_k is any sequence of positive real numbers such that $a_k \downarrow 0$, then there exists a smooth minimizer Φ_k of W under the area constraint $\mathrm{Area}(\Phi_k) = a_k$; moreover, such a sequence (Φ_k) satisfies (4) and therefore $W^{2,2}$-converges to a round critical point of the scalar curvature. Let us mention that the existence of area-constrained Willmore spheres was generalized in [Mondino and Rivière 2013] to any value of the area.

The goal of this paper is multiple. The main achievement is the improvement of the perturbative bound (4) above to the global bound

$$\limsup_k W(\Phi_k) < 8\pi. \tag{5}$$

Secondly, we improve the $W^{2,2}$-convergence above to *smooth* convergence towards a *round* critical point of the scalar curvature; i.e., we show that, if we rescale (M, g) around $\bar{p}$ in such a way that the sequence of surfaces has fixed area equal to 1 (for more details, see Section 2), then the sequence converges smoothly, up to subsequences, to a round sphere centered at $\bar{p}$ and $\bar{p}$ is a critical point of the scalar curvature of (M, g).

Finally we give an application of these results to the Hawking mass.

We believe that the bound (5) is sharp in order to have smooth convergence to a *round* point (in the sense specified above); indeed, if (5) is violated, then the sequence (Φ_k) may degenerate to a couple of bubbles, each one costing almost 4π in terms of Willmore energy.

Now let us state the main results of the present article. The first theorem below concerns the case of a sequence of Willmore immersions and is a consequence of the second more general theorem about area-constrained Willmore immersions.

Theorem 1.1. *Let* (M, g) *be a three-dimensional Riemannian manifold, and let* $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ *be a sequence of Willmore surfaces satisfying the energy bound* (5) *and Hausdorff converging to a point* $\bar{p} \in M$.

Then $\mathrm{Scal}(\bar{p}) = 0$ *and* $\nabla \mathrm{Scal}(\bar{p}) = 0$; *moreover, if we rescale* (M, g) *around* $\bar{p}$ *in such a way that the rescaled immersions* $\widetilde{\Phi}_k$ *have fixed area equal to* 1, *then* $\widetilde{\Phi}_k$ *converges smoothly, up to subsequences and up to reparametrizations, to a round sphere in the three-dimensional euclidean space.*

Actually, we prove the following more general result about sequences of area-constrained Willmore immersions:

[1]The normalization of the Willmore functional used in [Lamm and Metzger 2010; 2013] differs from our convention by a factor of 2.

Theorem 1.2. *Let (M, g) be a three-dimensional Riemannian manifold, and let $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ be a sequence of area-constrained Willmore surfaces satisfying the energy bound* (5) *and Hausdorff converging to a point $\bar{p} \in M$.*

Then $\nabla \operatorname{Scal}(\bar{p}) = 0$; moreover, if we rescale (M, g) around $\bar{p}$ in such a way that the rescaled immersions $\widetilde{\Phi}_k$ have fixed area equal to 1, *then $\widetilde{\Phi}_k$ converges smoothly, up to subsequences and up to reparametrizations, to a round sphere in the three-dimensional euclidean space.*

Of course, Theorem 1.2 implies Theorem 1.1 except the property $\operatorname{Scal}(\bar{p}) = 0$. This fact follows by the aforementioned [Mondino 2010, Theorem 1.3] holding for Willmore graphs over geodesic spheres together with the smooth convergence to a round point ensured by Theorem 1.2.

Now we pass to discuss an application to the Hawking mass m_H, defined for an immersed sphere $\Phi : \mathbb{S}^2 \hookrightarrow (M, g)$ by

$$m_H(\Phi) = \frac{\operatorname{Area}_g(\Phi)}{16\pi^{3/2}}(4\pi - W(\Phi)). \tag{6}$$

Of course, the critical points of the Hawking mass under area constraint are exactly the area-constrained Willmore spheres (see [Lamm et al. 2011] and the references therein for more material about the Hawking mass); moreover, it is clear that the inequality $m_H(\Phi) \geq 0$ implies that $W(\Phi) \leq 4\pi$.

Therefore, combining this easy observations with Theorem 1.2, we obtain the following corollary:

Corollary 1.3. *Let (M, g) be a three-dimensional Riemannian manifold, and let $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ be a sequence of critical points of m_H under area constraint having nonnegative Hawking mass and Hausdorff converging to a point $\bar{p} \in M$.*

Then $\nabla \operatorname{Scal}(\bar{p}) = 0$; moreover, if we rescale (M, g) around $\bar{p}$ in such a way that the rescaled immersions $\widetilde{\Phi}_k$ have fixed area equal to 1, *then $\widetilde{\Phi}_k$ converges smoothly, up to subsequences and up to reparametrizations, to a round sphere in the three-dimensional euclidean space.*

First of all, let us mention that Corollary 1.3 also follows by the analysis performed in [Lamm and Metzger 2010] with the only difference that here we improved the $W^{2,2}$ convergence to the smooth one. Now let us briefly comment on the relevance of Corollary 1.3 despite the triviality of its proof. Recall that, from the note of Christodoulou and Yau [1988], if (M, g) has nonnegative scalar curvature then isoperimetric spheres (and more generally stable CMC spheres) have positive Hawking mass; on the other hand, it is known (see for instance [Druet 2002] or [Nardulli 2009]) that, if M is compact, then small isoperimetric regions converge to geodesic spheres centered at a maximum point of the scalar curvature as the enclosed volume converges to 0 (see also [Mondino and Nardulli 2012] for the noncompact case). Therefore, a link between regions with positive Hawking mass and critical points of the scalar curvature was already present in literature, but Corollary 1.3 expresses this link precisely.

We end the introduction by outlying the structure of the paper and the main ideas of the proof. First of all, as already noticed, it is enough to prove Theorem 1.2 in order to get all the stated results. To prove it, we adopt the blow-up technique taking inspiration from [Laurain 2012], where the first author analyzed the corresponding questions in the context of CMC-surfaces; such technique was introduced in the analysis of the Yamabe problem, which is a second-order scalar problem (for a detailed overview of the method including applications see [Druet et al. 2004]). The technical novelty of [Laurain 2012] was that a second-order *vectorial* problem was considered; the technical originality of the present paper from the point of view of the blow-up method is that we study a *fourth-order vectorial problem*.

More precisely, in Section 2, we consider normal coordinates centered at the limit point $\bar{p}$ and we rescale appropriately the metric g such that the rescaled surfaces all have diameter 1 (or thanks to the monotonicity formula, it is equivalent to fix the area of the rescaled surfaces equal to 1); notice that the rescaled ambient metrics g_k are becoming more and more euclidean.

In Section 2A, by exploiting the divergence form of the Willmore equation established in [Mondino and Rivière 2013], we give a decay estimate on the Lagrange multipliers as k goes to infinity.

Section 3 is devoted to the proof of Theorem 1.2; we start in Section 3A by establishing a fundamental technical result that, under the above working assumptions, the sequence (Φ_k) converges smoothly to a round sphere, up to subsequences and reparametrizations. Let us remark that in the proof we exploit in a crucial way the assumption (5); otherwise, it may be possible for the sequence to degenerate to a couple of bubbles. Once we have smooth convergence to a round sphere ω, we study the remainder given by the difference between Φ_k and ω: in Section 3C, we use the linearized Willmore operator (recalled in the Appendix) in order to give precise asymptotics of such a remainder term, and in the final Section 3D, we refine these estimates and conclude the proof.

2. Notation and preliminaries

Throughout the paper, (M, g) is a Riemannian 3-manifold and $\mathbb{S}^2$ is the round 2-sphere of unit radius in $\mathbb{R}^3$. The Greek indexes α, β, γ, μ, and ν will run from 1 to 3 and will denote quantities in M; Latin indexes will run from 1 to 2 and will denote quantities on $\Phi_k(\mathbb{S}^2)$; we will always use Einstein notation on summation over indexes. Given a smooth immersion $\Phi : \mathbb{S}^2 \hookrightarrow (M, g)$, we call $\bar{g} = \Phi^*(g)$ the pullback metric, $d\mathrm{vol}_{\bar{g}}$ the induced area form, and $H_{g,\Phi}$ the mean curvature and

$$W_g(\Phi) := \int_{\mathbb{S}^2} |H_{g,\Phi}|^2 \, d\mathrm{vol}_{\bar{g}}$$

is the Willmore functional.

Now let (Φ_k) be a sequence of smooth immersions from $\mathbb{S}^2$ into M. Under our working assumptions, where $\mathrm{diam}_g(\Omega)$ is the diameter of the subset Ω of M with respect to the metric g, we will always have

$$\varepsilon_k := \mathrm{diam}_g(\Phi_k(\mathbb{S}^2)) \to 0, \tag{7}$$

$$W_g(\Phi_k) := \int_{\mathbb{S}^2} |H_{g,\Phi_k}|^2 \, d\mathrm{vol}_{\bar{g}_k} \le 8\pi - 2\delta \quad \text{for some } \delta > 0 \text{ independent of } k, \tag{8}$$

where $d\mathrm{vol}_{\bar{g}_k}$ is the area form on $\mathbb{S}^2$ associated to the pullback metric $\bar{g}_k = \Phi_k^*(g)$ and H_{g,Φ_k} is the mean curvature of Φ_k.

Notice that in case M is compact then (7) is sufficient to ensure that, up to subsequences, $\Phi_k(\mathbb{S}^2)$ converges to a point $\bar{p} \in M$ in Hausdorff distance sense; but since there is no further reason to restrict to a compact ambient manifold, we assume the convergence to $\bar{p}$ in the hypothesis of our main results instead of a compactness assumption on M.

In order to efficiently handle the geometric quantities, we need good coordinates; let us now introduce them. Take coordinates (x^μ), $\mu = 1, 2, 3$, around $\bar{p}$, and let $p_k = (p_k^1, p_k^2, p_k^3)$ be the center of mass of $\Phi_k(\mathbb{S}^2)$:

$$p_k^\mu = \frac{1}{\mathrm{Area}_g(\Phi_k)} \int_{\mathbb{S}^2} \Phi_k^\mu \, d\mathrm{vol}_{\bar{g}_k}, \quad \mu = 1, 2, 3,$$

where $\mathrm{Area}_g(\Phi_k) = \int_{\mathbb{S}^2} d\mathrm{vol}_{\bar{g}_k}$ is the area of $\Phi_k(\mathbb{S}^2)$. Clearly, up to subsequences, $p_k \to \bar{p}$.

For every $k \in \mathbb{N}$, consider the exponential normal coordinates centered in p_k and rescale this chart by a factor $1/\varepsilon_k$ with respect to the center of these coordinates. Hence, we get a new sequence of immersions $\widetilde{\Phi}_k : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_{\varepsilon_k})$, in the following simply denoted by Φ_k, where the metric g_{ε_k} is defined by

$$g_{\varepsilon_k}(y)(u, v) := g(\varepsilon_k y)(\varepsilon_k^{-1} u, \varepsilon_k^{-1} v). \tag{9}$$

Notice that now we have

$$W_{g_{\varepsilon_k}}(\Phi_k) \le 8\pi - 2\delta, \quad \mathrm{diam}_{g_{\varepsilon_k}}(\Phi_k(\mathbb{S}^2)) = 1, \quad \text{and} \quad \Phi_k(\mathbb{S}^2) \subset B_{g_{\varepsilon_k}}\big(0, \tfrac{3}{2}\big), \tag{10}$$

where the first inequality is a consequence of the invariance under rescaling of the Willmore functional and $B_{g_{\varepsilon_k}}\big(0, \frac{3}{2}\big)$ is the metric ball in $(\mathbb{R}^3, g_{\varepsilon_k})$ of center 0 and radius $\frac{3}{2}$. By the classical expression of the metric in normal coordinates, we get that (see Appendix B in [Laurain 2012])

$$(g_{\varepsilon_k})_{\mu\nu}(y) = \delta_{\mu\nu} + \tfrac{1}{3}\varepsilon_k^2 R_{\alpha\mu\nu\beta}(p_k) y^\alpha y^\beta + \tfrac{1}{6}\varepsilon_k^3 R_{\alpha\mu\nu\beta,\gamma}(p_k) y^\alpha y^\beta y^\gamma + o(\varepsilon_k^3), \tag{11}$$

the inverse metric is

$$(g_{\varepsilon_k})^{\mu\nu}(y) = \delta_{\mu\nu} - \tfrac{1}{3}\varepsilon_k^2 R_{\alpha\mu\nu\beta}(p_k) y^\alpha y^\beta - \tfrac{1}{6}\varepsilon_k^3 R_{\alpha\mu\nu\beta,\gamma}(p_k) y^\alpha y^\beta y^\gamma + o(\varepsilon_k^3), \tag{12}$$

the volume form of g_{ε_k} can be written as

$$\sqrt{|g_{\varepsilon_k}|}(y) = 1 - \tfrac{1}{6}\varepsilon_k^2 \,\mathrm{Ric}_{\alpha\beta}(p_k) y^\alpha y^\beta - \tfrac{1}{12}\varepsilon_k^3 \,\mathrm{Ric}_{\alpha\beta,\gamma}(p_k) y^\alpha y^\beta y^\gamma + o(\varepsilon_k^3), \tag{13}$$

and the Christoffel symbols of g_{ε_k} can be expanded as

$$(\Gamma_{\varepsilon_k})^\gamma_{\alpha\beta}(y) = A_{\alpha\beta\gamma\mu}(p_k) y^\mu \varepsilon_k^2 + B_{\alpha\beta\gamma\mu\nu}(p_k) y^\mu y^\nu \varepsilon_k^3 + o(\varepsilon_k^3), \tag{14}$$

where $A_{\alpha\beta\gamma\mu}(p_k) = \frac{1}{3}(R_{\beta\mu\alpha\gamma}(p_k) + R_{\alpha\mu\beta\gamma}(p_k))$ and $B_{\alpha\beta\gamma\mu\nu}(p_k) = \frac{1}{12}(2R_{\beta\mu\alpha\gamma,\nu}(p_k) + 2R_{\alpha\mu\beta\gamma,\nu}(p_k) + R_{\beta\mu\nu\gamma,\alpha}(p_k) + R_{\alpha\mu\nu\gamma,\beta}(p_k) - R_{\alpha\mu\nu\beta,\gamma}(p_k))$.

Since by (11) the metric g_{ε_k} is close to the euclidean metric in the C^∞-norm on $B_{g_0}(0, 2)$, where $B_{g_0}(0, 2)$ is the euclidean ball in $\mathbb{R}^3$ of center 0 and radius 2, recalling (10), we get the following lemma:

Lemma 2.1. *Let g_{ε_k} be the metric defined in* (9) *having the form* (11); *let $\Phi_k : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_{\varepsilon_k})$ be smooth immersions with $\Phi_k(\mathbb{S}^2) \subset B_{g_{\varepsilon_k}}(0, 2)$ satisfying*

$$W_{g_{\varepsilon_k}}(\Phi_k) \le 8\pi - 2\delta \quad \textit{for some } \delta > 0.$$

Then, for k large enough, we have

$$W_{g_0}(\Phi_k) \le 8\pi - \delta, \quad \tfrac{1}{2} \le \operatorname{diam}_{g_0}(\Phi_k(\mathbb{S}^2)) \le 2, \quad \textit{and} \quad \Phi_k(\mathbb{S}^2) \subset B_{g_0}(0, 2), \tag{15}$$

where g_0 *is the euclidean metric on* $\mathbb{R}^3$, W_{g_0} *is the euclidean Willmore functional, and* $B_{g_0}(0, 2)$ *is the euclidean ball of center* 0 *and radius* 2 *in* $\mathbb{R}^3$. *It follows that, for large k,* $\Phi_k : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_{\varepsilon_k})$ *is a smooth embedding and that there exist constants* $C_1, C_2 > 0$ *such that*

$$0 < \frac{1}{C_1} \le \frac{1}{C_2} \operatorname{Area}_{g_0}(\Phi_k) \le \operatorname{Area}_{g_{\varepsilon_k}}(\Phi_k) \le C_2 \operatorname{Area}_{g_0}(\Phi_k) \le C_1 < \infty. \tag{16}$$

Proof. The properties expressed in (15) follow from (10) by a direct estimate of the remainders given by the curvature terms of the metric g_{ε_k}; for such estimates, we refer to Lemmas 2.1–2.4 in [Mondino and Schygulla 2014].

It is classically known that, if the Willmore functional of an immersed closed surface in $(\mathbb{R}^3, g_0)$ is strictly below 8π, then the immersion is actually an embedding (see [Li and Yau 1982] or [Simon 1993]), so our second statement follows.

In order to prove (16), let us recall Lemma 1.1 in [Simon 1993] stating that

$$\sqrt{\frac{\operatorname{Area}_{g_0}(\Phi_k)}{W_{g_0}(\Phi_k)}} \le \operatorname{diam}_{g_0} \Phi_k(\mathbb{S}^2) \le C\sqrt{\operatorname{Area}_{g_0}(\Phi_k) W_{g_0}(\Phi_k)} \quad \text{for some universal } C > 0,$$

which, combined with the bound on $\operatorname{diam}_{g_0}(\Phi_k(\mathbb{S}^2))$ and $W_{g_0}(\Phi_k)$ expressed in (15), gives that there exists a constant $C_0 > 0$ such that

$$0 < \frac{1}{C_0} \le \operatorname{Area}_{g_0}(\Phi_k) \le C_0 < \infty;$$

the desired chain of inequalities (16) follows then by estimating the remainders as in Lemma 2.2 in [Mondino and Schygulla 2014]. □

2A. ***The area-constrained Willmore equation and an estimate of the Lagrange multiplier.*** In the rest of the paper, we will work with area-constrained Willmore immersions, i.e., critical points of the Willmore functional under the constraint that the area is fixed. If $\Phi : \mathbb{S}^2 \hookrightarrow (M, g)$ is a smooth area-constraint Willmore immersion, then it satisfies the following PDE (see for instance Section 3 in [Lamm et al. 2011] for the derivation of the equation)

$$\triangle_{\bar{g}} H_{g,\Phi} + H_{g,\Phi} |A^\circ_{g,\Phi}|^2_{\bar{g}} + H_{g,\Phi} \operatorname{Ric}_g(\vec{n}_{g,\Phi}, \vec{n}_{g,\Phi}) = \lambda H_{g,\Phi} \tag{17}$$

for some $\lambda \in \mathbb{R}$, where $\vec{n}_{g,\Phi}$ is a normal unit vector to $\Phi(\mathbb{S}^2) \subset (M, g)$, $(A^\circ_{g,\Phi})_{ij}$ is the traceless second fundamental form $(A^\circ_{g,\Phi})_{ij} = (A_{g,\Phi})_{ij} - \bar{g}_{ij} H_{g,\Phi}$ (of course $(A_{g,\Phi})_{ij}$ is the second fundamental form of Φ in (M, g)), and $|A^\circ_{g,\Phi}|^2_{\bar{g}} = \bar{g}^{ik}\bar{g}^{jl}(A^\circ_{g,\Phi})_{ij}(A^\circ_{g,\Phi})_{kl}$ is its norm with respect to the metric $\bar{g} = \Phi^* g$.

Now let (Φ_k) be a sequence of smooth area-constrained Willmore immersions of $\mathbb{S}^2$ into (M, g) satisfying (7)–(8); perform the rescaling procedure described above, and obtain the immersions $(\widetilde{\Phi}_k)$ of $\mathbb{S}^2$ into $(\mathbb{R}^3, g_{\varepsilon_k})$ (for simplicity denoted again with Φ_k from now on), where g_{ε_k} is defined in (9), satisfying (10). Since the Willmore functional is scale invariant, the rescaled surfaces are still area-constrained Willmore surfaces, so they satisfy the equation

$$\triangle_{\bar{g}_{\varepsilon_k}} H_{g_{\varepsilon_k},\Phi_k} + H_{g_{\varepsilon_k},\Phi_k} |A^{\circ}_{g_{\varepsilon_k},\Phi_k}|^2_{\bar{g}_{\varepsilon_k}} + H_{g_{\varepsilon_k},\Phi_k} \operatorname{Ric}_{g_{\varepsilon_k}}(\vec{n}_{g_{\varepsilon_k},\Phi_k}, \vec{n}_{g_{\varepsilon_k},\Phi_k}) = \lambda_k H_{g_{\varepsilon_k},\Phi_k}. \tag{18}$$

The first step in our arguments is to show that the Lagrange multipliers λ_k are controlled by ε_k^2. Let us mention that this was already proved in [Lamm and Metzger 2013], the idea being to use the invariance under rescaling of the Willmore functional. Here we slightly modify the proof in [Lamm and Metzger 2013] by exploiting the divergence structure of the Willmore equation in Riemannian manifolds discovered in [Mondino and Rivière 2013] (let us stress that the divergence structure of the Willmore equation in euclidean setting was a breakthrough by Rivière [2008]).

Lemma 2.2. *Let* (Φ_k) *be a sequence of smooth area-constrained Willmore immersions of* $\mathbb{S}^2$ *into* $(\mathbb{R}^3, g_{\varepsilon_k})$, *where* g_{ε_k} *has the form* (11) *with* $\varepsilon_k \to 0$ *and* $\Phi_k(\mathbb{S}^2) \subset B_{g_0}(0,2)$, *the euclidean ball of center* 0 *and radius* 2.

Then the Lagrange multipliers λ_k *appearing in* (18) *satisfy*

$$\sup_{k\in\mathbb{N}} \frac{|\lambda_k|}{\varepsilon_k^2} < \infty. \tag{19}$$

Proof. Since (Φ_k) are area-constrained Willmore immersions, for every variation vector field $\vec{X}$ on $\mathbb{R}^3$, we have that

$$\delta_{\vec{X}} W_{g_{\varepsilon_k}}(\Phi_k) = \lambda_k \delta_{\vec{X}} \operatorname{Area}_{g_{\varepsilon_k}}(\Phi_k), \tag{20}$$

where $\delta_{\vec{X}} W$ and $\delta_{\vec{X}}$ Area are the first variations of the Willmore and the Area functionals corresponding to the vector field $\vec{X}$. Observe that the vector field corresponding to the dilations in $\mathbb{R}^3$ is the position vector field $\vec{x}$, so the first variation of the euclidean Willmore functional in $\mathbb{R}^3$ with respect to $\vec{x}$ is null: $\delta_{\vec{x}} W_{g_0} = 0$; on the other hand, the first variation of euclidean area with respect to the $\vec{x}$ variation is easy to compute using the tangential divergence formula:

$$\delta_{\vec{x}} \operatorname{Area}_{g_0}(\Phi) = -2\int_{\mathbb{S}^2} \langle \vec{H}, \vec{x}\rangle_{g_0}\, d\mathrm{vol}_{\bar{g}_0} = \int_{\mathbb{S}^2} \operatorname{div}_{\Phi,g_0} \vec{x}\, d\mathrm{vol}_{\bar{g}_0} = 2\operatorname{Area}_{g_0}(\Phi),$$

where $\operatorname{div}_{\Phi,g_0}$ is the tangential divergence on $\Phi(\mathbb{S}^2)$ with respect to the euclidean metric. The two euclidean formulas give the well known fact that every area-constraint Willmore surface is actually a Willmore surface.

In the present framework, the ambient metric g_{ε_k} is a perturbation of order ε_k^2 of the euclidean metric g_0, so it is natural to expect that the Lagrange multiplier maybe does not vanish but at least is of order ε_k^2. Let us prove it. First of all, by the expansion of the Christoffel symbols (14), it follows that the covariant derivative in metric g_{ε_k} of the position vector field $\vec{x}$ has the form

$$\nabla^{g_{\varepsilon_k}} \vec{x} = \mathrm{Id} + O(\varepsilon_k^2). \tag{21}$$

It follows that the tangential divergence of $\vec{x}$ on $\Phi_k(\mathbb{S}^2)$ with respect to the metric $\bar{g}_k$ is $\mathrm{div}_{\Phi, g_{\varepsilon_k}} \vec{x} = 2 + O(\varepsilon_k^2)$, and by the tangential divergence formula, we obtain as before

$$\delta_{\vec{x}} \operatorname{Area}_{g_{\varepsilon_k}}(\Phi) = -2 \int_{\mathbb{S}^2} \langle \vec{H}_{\Phi_k, g_{\varepsilon_k}}, \vec{x} \rangle_{g_{\varepsilon_k}} \, d\mathrm{vol}_{\bar{g}_{\varepsilon_k}} = \int_{\mathbb{S}^2} \mathrm{div}_{\Phi_k, g_{\varepsilon_k}} \vec{x} \, d\mathrm{vol}_{\bar{g}_{\varepsilon_k}} = [2 + O(\varepsilon_k^2)] \operatorname{Area}_{g_{\varepsilon_k}}(\Phi_k);$$

recalling the uniform area bound given in (16), we get that there exists $C > 0$ such that

$$0 \leq \frac{1}{C} \leq \delta_{\vec{x}} \operatorname{Area}_{g_{\varepsilon_k}}(\Phi) \leq C < \infty. \tag{22}$$

Now let us compute the variation of the Willmore functional with respect to the variation $\vec{x}$:

$$\delta_{\vec{x}} W_{g_{\varepsilon_k}}(\Phi_k) = \int_{\mathbb{S}^2} \langle \vec{x}, \vec{n} \rangle_{g_{\varepsilon_k}} (\triangle_{\bar{g}_{\varepsilon_k}} H + H|A^\circ|^2 + H \operatorname{Ric}(\vec{n}, \vec{n})) \, d\mathrm{vol}_{\bar{g}_{\varepsilon_k}}, \tag{23}$$

where of course all the quantities are computed on Φ_k and with respect to the metric g_{ε_k}. In order to continue the computations, it is useful to rewrite the first variation of W in divergence form. Up to a reparametrization, we can assume that Φ_k are conformal so that the following identity holds (see Theorem 2.1 in [Mondino and Rivière 2013]):

$$[\triangle_{\bar{g}_{\varepsilon_k}} H\vec{n} + \vec{H}|A^\circ|^2 - R^\perp_\Phi(T\Phi)] \, d\mathrm{vol}_{\bar{g}_{\varepsilon_k}} = D^*\big[\nabla H\vec{n} - \tfrac{1}{2} H D\vec{n} + \tfrac{1}{2} H \star_{g_{\varepsilon_k}} (\vec{n} \wedge D^\perp \vec{n})\big], \tag{24}$$

where $\vec{H} = H\vec{n}$ is the mean curvature vector of the immersion Φ_k, $\star_{g_{\varepsilon_k}}$ is the Hodge operator associated to metric g_{ε_k}, $D\,\cdot := (\nabla_{\partial_{x_1}\Phi_k} \cdot\,, \nabla_{\partial_{x_2}\Phi_k} \cdot)$ and $D^\perp \cdot := (-\nabla_{\partial_{x_2}\Phi_k} \cdot\,, \nabla_{\partial_{x_1}\Phi_k} \cdot)$, and D^* is an operator acting on couples of vector fields $(\vec{V}_1, \vec{V}_2)$ along $(\Phi_k)_*(T\mathbb{S}^2)$ defined as

$$D^*(\vec{V}_1, \vec{V}_2) := \nabla_{\partial_{x_1}\Phi_k} \vec{V}_1 + \nabla_{\partial_{x_2}\Phi_k} \vec{V}_2.$$

Finally $R^\perp_{\Phi_k}(T\Phi_k) := (\operatorname{Riem}(\vec{e}_1, \vec{e}_2)\vec{H})^\perp = \star_{g_{\epsilon_k}}(\vec{n} \wedge \operatorname{Riem}^h(\vec{e}_1, \vec{e}_2)\vec{H})$, where $\vec{e}_i = \partial_{x_i}\Phi / |\partial_{x_i}\Phi|$ for $i = 1, 2$.

Plugging (24) into (23) and integrating by parts, we obtain

$$\delta_{\vec{x}} W_{g_{\varepsilon_k}}(\Phi_k) = \int_{\mathbb{S}^2} \big\langle -D\vec{x}, \nabla H\vec{n} - \tfrac{1}{2} H D\vec{n} + \tfrac{1}{2} H \star_{g_{\varepsilon_k}} (\vec{n} \wedge D^\perp \vec{n}) \big\rangle_{g_{\varepsilon_k}} d\mathrm{vol}_{\mathbb{S}^2} + \int_{\mathbb{S}^2} \langle \vec{x}, R^\perp_\Phi(T\Phi_k) + \vec{H} \operatorname{Ric}(\vec{n}, \vec{n}) \rangle_{g_{\varepsilon_k}} d\mathrm{vol}_{\bar{g}_{\varepsilon_k}}. \tag{25}$$

Since the Riemannian curvature tensor of the metric g_{ε_k} is of order $O(\varepsilon_k^2)$ and both the curvature terms are linear in H, using Schwartz inequality, the integral in the second line can be estimated as

$$\int_{\mathbb{S}^2} \langle \vec{x}, R^\perp_{\Phi_k}(T\Phi_k) + \vec{H} \operatorname{Ric}(\vec{n}, \vec{n}) \rangle_{g_{\varepsilon_k}} d\mathrm{vol}_{\bar{g}_{\varepsilon_k}} = O(\varepsilon_k^2)(W_{g_{\varepsilon_k}}(\Phi_k) \operatorname{Area}_{g_{\varepsilon_k}}(\Phi_k))^{1/2} = O(\varepsilon_k^2). \tag{26}$$

The first line of the right hand side of (23) can be written explicitly as

$$\int_{\mathbb{S}^2} \big\langle -\partial_{x^1}\Phi_k - \vec{\Gamma}^{g_{\varepsilon_k}}_{\alpha\beta} (\partial_{x^1}\Phi_k^\alpha)\Phi^\beta, (\partial_{x^1} H)\vec{n} + \tfrac{1}{2} H A_1^j (\partial_{x^j}\Phi_k) + \tfrac{1}{2} H A_2^j \star_{g_{\varepsilon_k}} (\vec{n} \wedge \partial_{x^j}\Phi_k) \big\rangle_{g_{\varepsilon_k}} d\mathrm{vol}_{\mathbb{S}^2}$$
$$+ \int_{\mathbb{S}^2} \big\langle -\partial_{x^2}\Phi_k - \vec{\Gamma}^{g_{\varepsilon_k}}_{\alpha\beta} (\partial_{x^2}\Phi_k^\alpha)\Phi^\beta, (\partial_{x^2} H)\vec{n} + \tfrac{1}{2} H A_2^j (\partial_{x^j}\Phi_k) - \tfrac{1}{2} H A_1^j \star_{g_{\varepsilon_k}} (\vec{n} \wedge \partial_{x^j}\Phi_k) \big\rangle_{g_{\varepsilon_k}} d\mathrm{vol}_{\mathbb{S}^2}. \tag{27}$$

Recalling that $\star_{g_{\varepsilon_k}}(\vec{n}\wedge\partial_{x^1}\Phi_k)=\partial_{x^2}\Phi_k$ and $\star_{g_{\varepsilon_k}}(\vec{n}\wedge\partial_{x^2}\Phi_k)=-\partial_{x^1}\Phi_k$, we obtain that all terms obtained doing the scalar product with $-\partial_{x^1}\Phi_k$ in the first line and with $-\partial_{x^2}\Phi_k$ in the second line simplify and just the terms containing the Christoffel symbols remain; since $\Phi_k\subset B_{\gamma_{\varepsilon_k}}(0,2)$ and the Christoffel symbols are of order $O(\varepsilon_k^2)$ by (14), (27) can be written as

$$\int_{\mathbb{S}^2}-\sum_{i=1}^{2}\langle\vec{\Gamma}^{g_{\varepsilon_k}}_{\alpha\beta}(\partial_{x^i}\Phi_k^\alpha)\Phi^\beta,(\partial_{x^i}H)\vec{n}\rangle\,d\mathrm{vol}_{\mathbb{S}^2}+O(\varepsilon_k^2)\int_{\mathbb{S}^2}|H_{\Phi_k,g_{\varepsilon_k}}||A_{\Phi_k,g_{\varepsilon_k}}|\,d\mathrm{vol}_{\bar{g}_{\varepsilon_k}};\tag{28}$$

using Schwartz inequality, of course, the second summand can be bounded by

$$O(\varepsilon_k^2)\biggl(\int_{\mathbb{S}^2}|H_{\Phi_k,g_{\varepsilon_k}}|^2\,d\mathrm{vol}_{\bar{g}_{\varepsilon_k}}\biggr)^{1/2}\biggl(\int_{\mathbb{S}^2}|A_{\Phi_k,g_{\varepsilon_k}}|^2\,d\mathrm{vol}_{\bar{g}_{\varepsilon_k}}\biggr)^{1/2}=O(\varepsilon_k^2),\tag{29}$$

where we used the Gauss equations, Gauss–Bonnet theorem, and area bound (16) to infer that

$$\int_{\mathbb{S}^2}|A_{\Phi_k,g_{\varepsilon_k}}|^2\,d\mathrm{vol}_{\bar{g}_{\varepsilon_k}}\le C(W_{g_{e_k}}(\Phi_k)+1)\le C_1.$$

In order to estimate the first integral of (28), we integrate by parts the derivative on H and we recall (14), obtaining

$$\begin{aligned}\int_{\mathbb{S}^2}-\sum_{i=1}^{2}\langle\vec{\Gamma}^{g_{\varepsilon_k}}_{\alpha\beta}(\partial_{x^i}\Phi_k^\alpha)\Phi^\beta,(\partial_{x^i}H)\vec{n}\rangle\,d\mathrm{vol}_{\mathbb{S}^2}&=O(\varepsilon_k^2)\int_{\mathbb{S}^2}(|H_{\Phi_k,g_{\varepsilon_k}}|+|H_{\Phi_k,g_{\varepsilon_k}}||A_{\Phi_k,g_{\varepsilon_k}}|)\,d\mathrm{vol}_{\bar{g}_{\varepsilon_k}}\\&=O(\varepsilon_k^2)(W_{g_{\varepsilon_k}}(\Phi_k))^{1/2}\biggl[(\mathrm{Area}_{g_{\varepsilon_k}}(\Phi_k))^{1/2}+\biggl(\int_{\mathbb{S}^2}|A_{\Phi_k,g_{\varepsilon_k}}|^2\,d\mathrm{vol}_{\bar{g}_{\varepsilon_k}}\biggr)^{1/2}\biggr]=O(\varepsilon_k^2).\end{aligned}\tag{30}$$

Collecting (25)–(30), we obtain that

$$\delta_{\vec{x}}W_{g_{\varepsilon_k}}(\Phi_k)=O(\varepsilon_k^2).$$

Combining the last equation with (22) and (20), we obtain that $\lambda_k=O(\varepsilon_k^2)$ as desired. ☐

3. The blow-up analysis and the proof of the main theorem

3A. *Existence of just one bubble and convergence.*

Lemma 3.1. *Let g_{ε_k} be the metrics on $\mathbb{R}^3$ defined in* (9) *having the expression* (11), *and let* (Φ_k) *be area-constrained Willmore immersions of $\mathbb{S}^2$ into* $(\mathbb{R}^3,g_{\varepsilon_k})$ *satisfying* (10); *without loss of generality, we can assume Φ_k to be conformal with respect to the euclidean metric g_0. Up to a rotation in the domain, we can also assume that, for every $k\in\mathbb{N}$, the north pole $N\in\mathbb{S}^2$ is the maximum point of the quantity* $|\nabla\Phi_k|^2+|\nabla^2\Phi_k|$:

$$\mu_k:=|\nabla\Phi_k|_h^2(N)+|\nabla^2\Phi_k|_h(N)=\max_{\mathbb{S}^2}|\nabla\Phi_k|_h^2+|\nabla^2\Phi_k|_h,$$

where h is the standard round metric of $\mathbb{S}^2$ of constant Gauss curvature equal to 1 *and $|\nabla\Phi_k|_h$ and $|\nabla^2\Phi_k|_h$ are the norms evaluated in the h metric.*

With $S \in \mathbb{S}^2$ the south pole and $P : \mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2$ the stereographic projection, consider the new parametrizations $\widetilde{\Phi}_k$, in the following simply denoted with Φ_k, defined by

$$\widetilde{\Phi}_k(P^{-1}(z)) := \Phi_k\left(P^{-1}\left(\frac{z}{\mu_k^{1/2}}\right)\right) \quad \textit{for all } z \in \mathbb{R}^2.$$

Then $\widetilde{\Phi}_k$, a priori just defined on $\mathbb{S}^2 \setminus \{S\}$, extend to smooth conformal immersions of $\mathbb{S}^2$ into $(\mathbb{R}^3, g_0)$ and converge to a conformal parametrization of a round sphere in the $C^l(\mathbb{S}^2, h)$-norm for every $l \in \mathbb{N}$.

Proof. Step a. There exists a smooth conformal parametrization $\Phi_\infty : \mathbb{S}^2 \to (\mathbb{R}^3, g_0)$ of a round sphere in $\mathbb{R}^3$ endowed with the euclidean metric g_0 such that, up to subsequences, $\widetilde{\Phi}_k \to \Phi_\infty$ in the $C^l_{\text{loc}}(\mathbb{S}^2 \setminus \{S\})$-norm for every $l \in \mathbb{N}$.

Denote by u_k the conformal factor associated to $\widetilde{\Phi}_k$, i.e.,

$$\widetilde{\Phi}_k^*(g_0) = e^{2u_k} h,$$

where g_0 is the euclidean metric in $\mathbb{R}^3$. Observe that, by construction, for any compact subset of the form

$$K := \mathbb{S}^2 \setminus B^h_\delta(S) \quad \text{for some } \delta > 0,$$

there holds

$$\sup_{k\in\mathbb{N}} \sup_K (|\nabla\widetilde{\Phi}_k|_h^2 + |\nabla^2\widetilde{\Phi}_k|_h) < \infty. \tag{31}$$

Then for every compact subset, there exists a constant C_K depending just on K such that, for every $x_0 \in K$ and every $\rho \in (0, \operatorname{dist}(K, S)/2)$,

$$\sup_{k\in\mathbb{N}} \sup_{B^h_\rho(x_0)} |\nabla^2\widetilde{\Phi}_k|^2 \le C_K,$$

where $B^h_\rho(x_0)$ is the ball of center x_0 and radius ρ in the metric h. By the conformal invariance of the Dirichlet energy, with $\pi_{\tilde{\vec{n}}_k}$ the projection on the normal space to $\widetilde{\Phi}_k$, we infer that for every $\varepsilon_0 > 0$ there exists $\rho_{\varepsilon_0,K} > 0$ (small enough) depending just on K and on ε_0 but not on $k \in \mathbb{N}$ such that, for every $\rho \in (0, \rho_{\varepsilon_0,K})$ and $x_0 \in K$,

$$\begin{aligned}\int_{B^h_\rho(x_0)} |\nabla\tilde{\vec{n}}_k|^2_{\widetilde{\Phi}_k^*(g_0)}\, d\mathrm{vol}_{\widetilde{\Phi}_k^*(g_0)} &= \int_{B^h_\rho(x_0)} |\nabla\tilde{\vec{n}}_k|_h^2\, d\mathrm{vol}_h = \int_{B^h_\rho(x_0)} |\pi_{\tilde{\vec{n}}_k}(\nabla^2\widetilde{\Phi}_k)|_h^2\, d\mathrm{vol}_h \\ &\le \int_{B^h_\rho(x_0)} |\nabla^2\widetilde{\Phi}_k|_h^2\, d\mathrm{vol}_h \le C_K\rho^2 \le \varepsilon_0. \end{aligned} \tag{32}$$

Taking $\varepsilon_0 \le \frac{8}{3}\pi$, for any $x_0 \in K$ and $\rho < \rho_{\varepsilon_0,K}$, we can apply the Hélein moving frame method based on Chern construction of conformal coordinates (for more details, see [Rivière 2013, Section 3]) and infer that, up to a reparametrization of $\widetilde{\Phi}_k$ on $B_\rho(x_0)$, with $\bar{u}_k$ the mean value of u_k on $B^h_\rho(x_0)$,

$$\|u_k - \bar{u}_k\|_{L^\infty(B^h_\rho(x_0))} \le \widetilde{C}$$

for some $\widetilde{C} > 0$ independent of $k \in \mathbb{N}$. Covering K by finitely many balls as above, the connectedness of K implies that any two balls of the finite covering are connected by a chain of balls of the same covering and therefore there exists constants $c_{k,K} \in \mathbb{R}$ and $k \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \|u_k - c_{k,K}\|_{L^\infty(K)} < \infty. \tag{33}$$

Observe that $\sup_{k\in\mathbb{N}} c_{k,K} < +\infty$; indeed, if $\limsup_k c_{k,K} = +\infty$, then $\limsup_k \operatorname{Area}(\widetilde{\Phi}_k(K)) = +\infty$, contradicting the area bound (16) (here we use that K has positive h-volume). Now let us consider separately the cases $\sup_k |c_{k,K}| < \infty$ and $\liminf_k c_{k,K} = -\infty$.

Case 1: $\sup_k |c_{k,K}| < \infty$. Estimate (33) yields a uniform bound on the conformal factors u_k on the subset K. Since by assumption the immersions $\widetilde{\Phi}_k$ are area-constrained Willmore immersions satisfying (32) with arbitrarily small Lagrange multipliers thanks to Lemma 2.2, then by ε-regularity,[2] we infer that for every $l \in \mathbb{N}$ there exists C_l such that

$$|e^{-lu_k}\nabla^l \widetilde{\Phi}_k|_{L^\infty(B^h_{\rho/2}(x_0))} \le C_l \left(\int_{B^h_\rho(x_0)} |\nabla \tilde{\vec{n}}_k|^2_h \, d\mathrm{vol}_h + 1 \right)^{1/2} \le \widehat{C}_l,$$

and therefore, by the assumed uniform bound on $|u_k|$ and by covering K by finitely many balls, we get

$$\sup_{k\in\mathbb{N}} |\nabla^l \widetilde{\Phi}_k|_{L^\infty(K)} < \infty \quad \text{for all } l \in \mathbb{N}. \tag{34}$$

By the Arzelà–Ascoli theorem and by the estimate on the Lagrange multipliers given in Lemma 2.2, up to subsequences, the maps $\widetilde{\Phi}_k$ converge in the $C^l(K)$-norm, for every $l \in \mathbb{N}$, to a limit Willmore immersion $\widetilde{\Phi}_\infty$ of K into $(\mathbb{R}^3, g_0)$; repeating the above argument to $K = \mathbb{S}^2 \setminus B^h_\delta(S)$, for every $\delta > 0$, we get that, up to subsequences, the maps $\widetilde{\Phi}_k$ converge in the $C^l_{\mathrm{loc}}(\mathbb{S}^2 \setminus \{S\})$-norm, for every $l \in \mathbb{N}$, to a limit Willmore immersion $\Phi_\infty : \mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^3$, a smooth Willmore conformal immersion with finite area and L^2-bounded second fundamental form; therefore, by Lemma A.5 in [Rivière 2014] (let us mention that this result was already present in [Müller and Šverák 1995]; see also [Kuwert and Li 2012]), the map Φ_∞ can be extended up to the south pole S to a possibly branched immersion; i.e., the south pole S is a possible branch point for Φ_∞ and the following expansion around S holds:

$$(C - o(1))|z|^{n-1} \le \left| \frac{\partial \Phi_\infty}{\partial z} \right| \le (C + o(1))|z|^{n-1}, \tag{35}$$

where z is a complex coordinate around the south pole and $n-1$ is the branching order. We claim that the branching order is 0 or in other words that Φ_∞ is unbranched; indeed, by the strong convergence of $\widetilde{\Phi}_k$ to Φ_∞ and the smooth convergence of g_{ϵ_k} to the euclidean metric g_0, we have that

$$W_{g_0}(\Phi_\infty) \le \liminf_k W_{g_{\epsilon_k}}(\widetilde{\Phi}_k) < 8\pi; \tag{36}$$

[2] Note that ε-regularity for Willmore immersions was first proved by Kuwert and Schätzle [2001]. Here we use the ε-regularity theorem proved by Rivière (see Theorem I.5 in [Rivière 2008]; see also Theorem I.1 in [Bernard and Rivière 2014]); to this aim, observe that the ε-regularity theorem was stated for *Willmore immersions*, but the proof can be repeated verbatim to *area-constrained Willmore immersions in metric* g_{ϵ_k}: indeed the Lagrange multiplier $\lambda \vec{H}$ and the Riemannian terms are lower-order terms that can be absorbed in the already present error terms $\vec{g}_1$ and $\vec{g}_2$ in the proof of Theorem I.5 at pp. 24–26 in [Rivière 2008]. Of course, ε-regularity is a consequence of the ellipticity of the equation.

therefore, by the Li–Yau inequality [1982], we get that $n-1=0$, i.e., Φ_∞ is an immersion also at the south pole S. Since Φ_∞ is a smooth Willmore immersion of $\mathbb{S}^2$ into $\mathbb{R}^3$ with energy less than 8π, by the classification of Willmore spheres by Bryant [1984], Φ_∞ is a smooth conformal parametrization of a round sphere in $\mathbb{R}^3$.

Case 2: $\liminf_k c_{k,K} = -\infty$. This cannot happen. In this case, up to subsequences, we have that $\widetilde{\Phi}_k(K) \to \bar{x} \in M$ in Hausdorff distance sense. Consider then the rescaled immersions

$$\widehat{\Phi}_k := e^{-c_{k,K}} \widetilde{\Phi}_k \tag{37}$$

of K, and observe that by construction $\sup_k |\hat{u}_{k,K}| < \infty$, where $\hat{u}_{k,K}$ is the conformal factor of $\widehat{\Phi}_k$. Moreover, since the integrals appearing in (32) are invariant under rescaling, estimate (32) holds for $\widehat{\Phi}_k$ as well. Therefore, up to a diagonal extraction, $\widehat{\Phi}_k \to \Phi_\infty$ in the $C^l_{\mathrm{loc}}(\mathbb{S}^2 \setminus \{S\})$-norm. In particular, $\widetilde{\Phi}_k \to 0$ in the $C^2_{\mathrm{loc}}(\mathbb{S}^2 \setminus \{S\})$-norm, which contradicts the fact that

$$|\nabla \widetilde{\Phi}_k|^2_h(N) + |\nabla^2 \widetilde{\Phi}_k|_h(N) = 1.$$

Step b. $\widetilde{\Phi}_k \to \Phi_\infty$ in $C^l(\mathbb{S}^2)$ for every $l \in \mathbb{N}$; namely, the convergence of *Step a* is *on the whole* $\mathbb{S}^2$.

Observe that, if there exists $\bar{\rho} > 0$ such that $\sup_k \sup_{B^h_{\bar{\rho}}(S)} |\nabla \widetilde{\Phi}_k|^2 + |\nabla^2 \widetilde{\Phi}_k| < \infty$, then in *Step a*, we can choose as compact subset K the whole $\mathbb{S}^2$ and the claim of *Step b* follows by the same arguments as *Step a*. So assume by contradiction that there exists a sequence $\rho_k \downarrow 0$ such that, for

$$\bar{\mu}_k := \sup_{B^h_{\rho_k}(\bar{x})} |\nabla \widetilde{\Phi}_k|^2 + |\nabla^2 \widetilde{\Phi}_k|,$$

one has

$$\limsup_k \bar{\mu}_k = +\infty.$$

By a small rotation in the domain $\mathbb{S}^2$, we can assume that, for every $k \in \mathbb{N}$, the maximum of $|\nabla \widetilde{\Phi}_k|^2 + |\nabla^2 \widetilde{\Phi}_k|$ on $B^h_{\rho_k}(S)$ is attained at the south pole S and that, up to subsequences in k,

$$\lim_k \bar{\mu}_k := \lim_k |\nabla \widetilde{\Phi}_k|^2(S) + |\nabla^2 \widetilde{\Phi}_k|(S) = +\infty. \tag{38}$$

Analogously to the above, with $P_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$ the stereographic projection centered at the north pole N, we consider the reparametrized immersions

$$\overline{\Phi}_k(P_N^{-1}(z)) := \widetilde{\Phi}_k\left(P_N^{-1}\left(\frac{z}{\bar{\mu}_k^{1/2}}\right)\right).$$

Observe that, in this way, the compact subsets K considered above are shrinking towards the north pole N and, by the arguments above, their $\overline{\Phi}_k$-images are converging to a round sphere; repeating the arguments above to compact subsets this time containing the south pole S and avoiding the north pole N, we infer that, up to subsequences, $\overline{\Phi}_k$ (or a further rescaling of it) converges smoothly, away the north pole N, to a round sphere, namely a second bubble. Combining the bubble formed in *Step a* and this second bubble, since each bubble contributes 4π of Willmore energy, we infer that

$$\limsup_k W_{g_{\epsilon_k}}(\Phi_k) \geq 8\pi, \tag{39}$$

contradicting the assumption (10). This concludes the proof of the *Step b* and of the lemma. $\square$

3B. ***Expansion of the equation.*** Recalling that $\Phi_k : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_{\varepsilon_k})$ is a smooth immersion satisfying the area-constrained Willmore equation in metric g_{ε_k} and that g_{ε_k} smoothly converge to the euclidean metric g_0, in the present section, we expand this differential equation with respect to ε_k. Without loss of generality, we can assume that Φ_k is conformal with respect to the metric g_{ε_k}. We will see that curvature terms appear at ε_k^2 order while the derivatives of the curvature appear at ε_k^3 order.

From now on, in order to make the notation a bit lighter, we replace ε_k by ε.

Recall that the area-constrained Willmore equation in metric g_ε has the form

$$\triangle_{\bar{g}_\varepsilon} H_\varepsilon + H_\varepsilon |A^\circ_\varepsilon|^2_{\bar{g}_\varepsilon} + \mathrm{Ric}_{g_\varepsilon}(\vec{n}_\varepsilon, \vec{n}_\varepsilon) H_\varepsilon = \lambda_\varepsilon H_\varepsilon. \tag{40}$$

Since $\triangle_{\bar{g}_\varepsilon} = (2/|\nabla\Phi_\varepsilon|^2_{g_\varepsilon})\Delta$, where Δ is the flat laplacian in $\mathbb{R}^2$, multiplying (40) by $|\nabla\Phi_\varepsilon|^2_{g_\varepsilon}/2$, we get

$$\Delta H_\varepsilon + \tfrac{1}{2}|\nabla\Phi_\varepsilon|^2_{g_\varepsilon} H_\varepsilon |A^\circ_\varepsilon|^2_{\bar{g}_\varepsilon} + \tfrac{1}{2}|\nabla\Phi_\varepsilon|^2_{g_\varepsilon} H_\varepsilon \,\mathrm{Ric}_{g_\varepsilon}(\vec{n}_\varepsilon, \vec{n}_\varepsilon) = \tfrac{1}{2}\lambda_\varepsilon |\nabla\Phi_\varepsilon|^2_{g_\varepsilon} H_\varepsilon. \tag{41}$$

First of all, recalling that $H_\varepsilon = g_\varepsilon(\triangle_{\bar{g}_\varepsilon}\Phi_\varepsilon, \vec{n}_\varepsilon)/2$, we expand H_ε as

$$H_\varepsilon = \frac{1}{|\nabla\Phi_\varepsilon|^2_{g_\varepsilon}}(g_\varepsilon)_{\alpha\beta}\triangle\Phi^\alpha_\varepsilon \sqrt{|g_\varepsilon|}\, g^{\beta\gamma}_\varepsilon (\vec{\nu}_\varepsilon)_\gamma = \frac{\sqrt{|g_\varepsilon|}}{|\nabla\Phi_\varepsilon|^2_{g_\varepsilon}}\triangle\Phi^\alpha_\varepsilon \vec{\nu}_{\varepsilon\alpha}, \tag{42}$$

where $\vec{\nu}_\varepsilon$ is the inward-pointing unit normal with respect to g_0. Using (11) and (13), we get

$$|\nabla\Phi_\varepsilon|^2_{g_\varepsilon} = |\nabla\Phi_\varepsilon|^2 + \tfrac{1}{3}\varepsilon^2 R_{\alpha\beta\gamma\eta}(p_k)\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon\langle\nabla\Phi^\alpha_\varepsilon, \nabla\Phi^\eta_\varepsilon\rangle + \tfrac{1}{6}\varepsilon^3 R_{\alpha\beta\gamma\eta,\mu}(p_k)\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon\Phi^\mu_\varepsilon\langle\nabla\Phi^\alpha_\varepsilon, \nabla\Phi^\eta_\varepsilon\rangle + O(\varepsilon^4)$$

so that

$$\frac{1}{|\nabla\Phi_\varepsilon|^2_{g_\varepsilon}} = \frac{1}{|\nabla\Phi_\varepsilon|^2}\Big(1 - \frac{\varepsilon^2}{3|\nabla\Phi_\varepsilon|^2} R_{\alpha\beta\gamma\eta}(p_k)\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon\langle\nabla\Phi^\alpha_\varepsilon, \nabla\Phi^\eta_\varepsilon\rangle - \frac{\varepsilon^3}{6|\nabla\Phi_\varepsilon|^2} R_{\alpha\beta\gamma\eta,\mu}(p_k)\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon\Phi^\mu_\varepsilon\langle\nabla\Phi^\alpha_\varepsilon, \nabla\Phi^\eta_\varepsilon\rangle + O(\varepsilon^4)\Big); \tag{43}$$

moreover,

$$\sqrt{|g_\varepsilon|} = 1 - \tfrac{1}{6}\varepsilon^2 \,\mathrm{Ric}_{\alpha\beta}(p_k)\Phi^\alpha_\varepsilon\Phi^\beta_\varepsilon - \tfrac{1}{6}\varepsilon^3 \,\mathrm{Ric}_{\alpha\beta,\gamma}(p_k)\Phi^\alpha_\varepsilon\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon + O(\varepsilon^4). \tag{44}$$

Combining (42) with (43) and (44), we can write

$$H_\varepsilon = \frac{\triangle\Phi^\alpha_\varepsilon\vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2}(1 + \varepsilon^2 S_\varepsilon + \varepsilon^3 T_\varepsilon + O(\varepsilon^4)), \tag{45}$$

where

$$S_\varepsilon := -\frac{1}{3|\nabla\Phi_\varepsilon|^2} R_{\alpha\beta\gamma\eta}(p_k)\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon\langle\nabla\Phi^\alpha_\varepsilon, \nabla\Phi^\eta_\varepsilon\rangle - \tfrac{1}{6}\,\mathrm{Ric}_{\alpha\beta}(p_k)\Phi^\alpha_\varepsilon\Phi^\beta_\varepsilon$$

and

$$T_\varepsilon := -\frac{1}{6|\nabla\Phi_\varepsilon|^2} R_{\alpha\beta\gamma\eta,\mu}(p_k)\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon\Phi^\mu_\varepsilon\langle\nabla\Phi^\alpha_\varepsilon, \nabla\Phi^\eta_\varepsilon\rangle - \tfrac{1}{6}\,\mathrm{Ric}_{\alpha\beta,\gamma}(p_k)\Phi^\alpha_\varepsilon\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon.$$

The combination of (44) and (45) gives

$$\mathrm{Ric}_{g_\varepsilon}(\vec{n}_\varepsilon,\vec{n}_\varepsilon)H_\varepsilon=\varepsilon^2\frac{\Delta\Phi^\alpha_\varepsilon\vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2}\,\mathrm{Ric}_g(p_k)(\vec{\nu}_\varepsilon,\vec{\nu}_\varepsilon)+O(\varepsilon^4). \tag{46}$$

Finally, using (45), (46), and (19), we expand (41) up to ε^2 order (the term $H_\varepsilon|A^\circ_\varepsilon|^2_{\bar{g}_\varepsilon}$ will be expanded in the next subsection) as

$$\begin{aligned}\Delta H_\varepsilon&+\tfrac12|\nabla\Phi_\varepsilon|^2_{g_\varepsilon}H_\varepsilon|A^\circ_\varepsilon|^2_{\bar{g}_\varepsilon}+\tfrac12|\nabla\Phi_\varepsilon|^2_{g_\varepsilon}H_\varepsilon\,\mathrm{Ric}_{g_\varepsilon}(\vec{n}_\varepsilon,\vec{n}_\varepsilon)-\lambda_\varepsilon H_\varepsilon\tfrac12|\nabla\Phi_\varepsilon|^2_{g_\varepsilon}\\
&=\Delta\Big(\frac{\Delta\Phi^\alpha_\varepsilon\vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2}\Big)+\varepsilon^2\Big(\Delta\Big(\frac{\Delta\Phi^\alpha_\varepsilon\vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2}\Big)S_\varepsilon+2\Big\langle\nabla\Big(\frac{\Delta\Phi^\alpha_\varepsilon\vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2}\Big),\nabla S_\varepsilon\Big\rangle+\frac{\Delta\Phi^\alpha_\varepsilon\vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2}\Delta S_\varepsilon\Big)\\
&\qquad+\tfrac12|\nabla\Phi_\varepsilon|^2_{g_\varepsilon}H_\varepsilon|A^\circ_\varepsilon|^2_{\bar{g}_\varepsilon}+\tfrac12\varepsilon^2\Delta\Phi^\alpha_\varepsilon\vec{\nu}_{\varepsilon\alpha}\,\mathrm{Ric}_g(p)(\vec{\nu}_\varepsilon,\vec{\nu}_\varepsilon)-\tfrac12\lambda_\varepsilon\Delta\Phi^\alpha_\varepsilon\vec{\nu}_{\varepsilon\alpha}+o(\varepsilon^2).\end{aligned} \tag{47}$$

3C. ***Approximated solutions to the area-constrained Willmore equation.*** In this section, we solve (47) up to the ε^2 order. For this, let ω be the inverse of the stereographic projection with respect to the north pole and notice that ω is a solution of the equation when $\varepsilon=0$. We make the ansatz of looking for a solution up to the order ε^2 of the form $\omega+\varepsilon^2\rho$ for some function ρ. Since $|A^\circ|^2=0$ for ω, it is clear that

$$H_\varepsilon|A^\circ_\varepsilon|^2_{\bar{g}_\varepsilon}=O(\varepsilon^4); \tag{48}$$

in particular, since for our arguments it is enough to expand the equation up to ε^3 order, this term will never play a role and therefore will be neglected.

Observing that $\Delta\omega^\alpha\omega_\alpha/|\nabla\omega|^2\equiv-1$, (47) implies that ρ must solve

$$\begin{aligned}L_\omega(\rho)=\Delta\Big(\frac{1}{3|\nabla\omega|^2}R_{\alpha\beta\gamma\mu}(p_k)\omega^\beta\omega^\gamma\langle\nabla\omega^\alpha,\nabla\omega^\mu\rangle+\tfrac16\,\mathrm{Ric}_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta\Big)&\\
-\tfrac12|\nabla\omega|^2\,\mathrm{Ric}_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta+\frac{\lambda_\varepsilon}{2\varepsilon^2}|\nabla\omega|^2,&\end{aligned} \tag{49}$$

where L_ω is the linearized Willmore operator at ω; see the Appendix for more details. Using the identity

$$\langle\nabla\omega^\alpha,\nabla\omega^\beta\rangle=(\delta_{\alpha\beta}-\omega^\alpha\omega^\beta)\tfrac12|\nabla\omega|^2, \tag{50}$$

(49) reduces to

$$\begin{aligned}L_\omega(\rho)&=\tfrac13\Delta(\mathrm{Ric}_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta)-\tfrac12|\nabla\omega|^2\,\mathrm{Ric}_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta+\frac{\lambda_\varepsilon}{2\varepsilon^2}|\nabla\omega|^2\\
&=\Big(-\mathrm{Ric}_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta+\Big(\frac{\lambda_\varepsilon}{2\varepsilon^2}+\tfrac13\,\mathrm{Scal}(p_k)\Big)\Big)|\nabla\omega|^2.\end{aligned} \tag{51}$$

Hence, we easily check that

$$\rho_\varepsilon=\tfrac13\,\mathrm{Ric}_{\alpha\beta}(p_k)\omega^\beta+\frac{\lambda_\varepsilon}{\varepsilon^2}f(r)\omega \tag{52}$$

with

$$f(r)=\frac{r^2\ln(r^2/(1+r^2))-1-\ln(1+r^2)}{1+r^2},$$

where $r^2 = x^2 + y^2$, is the desired function. Moreover, it is not difficult to check that this perturbed ω satisfies the conformal conditions up to ε^2 order, that is to say

$$\begin{cases} g_\varepsilon((\omega+\varepsilon^2\rho_\varepsilon)_x, (\omega+\varepsilon^2\rho_\varepsilon)_x) - g_\varepsilon((\omega+\varepsilon^2\rho_\varepsilon)_y, (\omega+\varepsilon^2\rho_\varepsilon)_y) = O(\varepsilon^3), \\ g_\varepsilon((\omega+\varepsilon^2\rho_\varepsilon)_x, (\omega+\varepsilon^2\rho_\varepsilon)_y) = O(\varepsilon^3); \end{cases} \tag{53}$$

a way to prove it is to use the expansion of the metric with the fact that in dimension 3 one has

$$R_{\alpha\beta\gamma\mu} = (g_{\alpha\gamma}\operatorname{Ric}_{\beta\mu} - g_{\alpha\mu}\operatorname{Ric}_{\beta\gamma} + g_{\beta\mu}\operatorname{Ric}_{\alpha\gamma} - g_{\beta\gamma}\operatorname{Ric}_{\alpha\mu}) + \tfrac{1}{2}\operatorname{Scal}(g_{\alpha\mu}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\mu}).$$

3D. ***Proof of Theorem 1.2.*** Let us briefly recall the setting. Let $\Phi_k : \mathbb{S}^2 \hookrightarrow (M, g)$ be conformal Willmore immersions satisfying

$$\varepsilon := \operatorname{diam}_g(\Phi_k(\mathbb{S}^2)) \to 0, \tag{54}$$

$$W_g(\Phi_k) := \int_{\mathbb{S}^2} |H_{g,\Phi_k}|^2 \, d\mathrm{vol}_{\bar{g}_k} \le 8\pi - 2\delta \quad \text{for some } \delta > 0 \text{ independent of } k. \tag{55}$$

Thanks to Lemma 2.2, we associate to Φ_k the new immersion $\Phi^\varepsilon : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_\varepsilon)$, where $g_\varepsilon(y)(u, v) := g(\varepsilon y)(\varepsilon^{-1}u, \varepsilon^{-1}v)$, which satisfies the area-constrained Willmore equation

$$\triangle_{\bar{g}_\varepsilon} H_{g_\varepsilon,\Phi^\varepsilon} + H_{g_\varepsilon,\Phi^\varepsilon}|A^\circ_{g_\varepsilon,\Phi^\varepsilon}|^2_{\bar{g}_\varepsilon} + H_{g_\varepsilon,\Phi^\varepsilon}\operatorname{Ric}_{g_\varepsilon}(\vec{n}_{g_\varepsilon,\Phi^\varepsilon}, \vec{n}_{g_\varepsilon,\Phi^\varepsilon}) = \lambda_\varepsilon H_{g_\varepsilon,\Phi^\varepsilon} \tag{56}$$

with $\lambda_\varepsilon = O(\varepsilon^2)$. Moreover, by Lemma 3.1, we know that, up to conformal reparametrizations and up to subsequences, we have

$$\Phi^\varepsilon \to \Phi \text{ in } C^2(\mathbb{S}^2),$$

where Φ is a conformal diffeomorphism of $\mathbb{S}^2$. Clearly, up to reparametrizing our sequence, we can assume that $\Phi = \mathrm{Id}$. In the following, we perform all the computations in the chart given by the stereographic projection (which is conformal); we denote by ω the inverse of the stereographic projection.

Before proceeding with the proof, we need to make a small adjustment to the immersions. We claim that there exist $a^\varepsilon \in \mathbb{R}^2$, $b^\varepsilon \in \mathbb{R}^2$, $R^\varepsilon \in SO(3)$, and $z^\varepsilon \in \mathbb{C}$ satisfying

$$a^\varepsilon = o(1), \quad b^\varepsilon = o(1), \quad |\mathrm{Id} - R^\varepsilon| = o(1), \quad \text{and} \quad z^\varepsilon = o(1) \tag{57}$$

such that, up to replacing Φ^ε by $\Phi^\varepsilon(a^\varepsilon + z^\varepsilon\,\cdot\,)$ and $\Omega^\varepsilon = \omega^\varepsilon + \varepsilon^2\rho^\varepsilon$, where ρ^ε is given by (52), by $R^\varepsilon[\omega(\,\cdot\, + b^\varepsilon) + \varepsilon^2\rho^\varepsilon(\,\cdot\, + b_\varepsilon)]$, we get

$$|\nabla\Phi^\varepsilon| \text{ and } |\nabla\Omega^\varepsilon| \text{ are maximal at } 0, \quad \operatorname{Vect}\{\Phi^\varepsilon_x(0), \Phi^\varepsilon_y(0)\} = \operatorname{Vect}\{\Omega^\varepsilon_x(0), \Omega^\varepsilon_y(0)\}, \text{ and } \Phi^\varepsilon_x(0) = \Omega^\varepsilon_x(0). \tag{58}$$

This is a simple consequence of the $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ convergence of Φ^ε to ω. Indeed, we first choose a^ε and b^ε such that $|\nabla\Phi^\varepsilon|$ and $|\nabla\Omega^\varepsilon|$ are maximal at 0 and then R^ε such that the tangent plane of Φ^ε and $R^\varepsilon\Omega^\varepsilon$ coincide at 0, and finally we find z_ε in order to adjust the first derivatives.

Therefore, from now on, we will assume that (58) is satisfied.

Now we prove Theorem 1.2. We set

$$\Phi^\varepsilon = \Omega^\varepsilon + r^\varepsilon$$

for some function r^ε, and thanks to the computations of Section 3C, we see that r^ε satisfies

$$L_\omega(r^\varepsilon) = O(\varepsilon^3) + o(|\nabla r^\varepsilon| + |\nabla^2 r^\varepsilon| + |\nabla^3 r^\varepsilon| + |\nabla^4 r^\varepsilon|). \tag{59}$$

Moreover, combining (53) and (58), we get that

$$g^\varepsilon(\nabla r^\varepsilon, \nabla r^\varepsilon)(0) = O(\varepsilon^6). \tag{60}$$

Indeed, the error terms of $r_x^\varepsilon(0)$ and $r_y^\varepsilon(0)$ lie in the plane generated by $\Omega_x^\varepsilon(0)$ and $\Omega_y^\varepsilon(0)$. So it suffices to estimate their projection against $\Omega_x^\varepsilon(0)$ and $\Omega_y^\varepsilon(0)$. But this one vanishes up to the ε^3 order thanks to (53). Observe that we also have

$$g^\varepsilon(\nabla^2 r^\varepsilon, \nabla\omega^\varepsilon)(0) = O(\varepsilon^3). \tag{61}$$

Claim. $$\sup_{\mathbb{R}^2} |\nabla r^\varepsilon| + |\nabla^2 r^\varepsilon| + |\nabla^3 r^\varepsilon| + |\nabla^4 r^\varepsilon| = O(\varepsilon^3).$$

Proof of the claim. Let us denote $\mu_\varepsilon := |\nabla r^\varepsilon| + |\nabla^2 r^\varepsilon| + |\nabla^3 r^\varepsilon| + |\nabla^4 r^\varepsilon|$, and assume by contradiction that $\lim \varepsilon^3/\mu_\varepsilon = 0$. Up to a reparametrization, we can assume that this sup is achieved at some point z_ε that is confined in a fixed compact subset of $\mathbb{R}^2$. In fact, we can do a reparametrization in order to make this requirement satisfied before performing the adjustments of the previous page. Then we set

$$\tilde r_\varepsilon = \frac{r_\varepsilon - r_\varepsilon(0)}{\mu^\varepsilon}.$$

By construction, $\tilde r^\varepsilon$ is bounded in the C^4-norm on every compact subset of $\mathbb{R}^2$, and therefore, by the Arzelà–Ascoli theorem, it converges up to subsequences to a limit function $\tilde r$ in C^3_{loc}-topology. Thanks to (59), $\tilde r$ is a solution of the linearized equation (A-1) and, recalling (60)-(61), satisfies (A-2) with $\nabla \tilde r(0) = 0$ and $\langle \nabla^2 \tilde r, \nabla\omega\rangle(0) = 0$. Then, applying Lemma A.1, we get that $\nabla\tilde r \equiv 0$, which is in contradiction with the fact that $|\nabla \tilde r| + |\nabla^2 \tilde r| + |\nabla^3 \tilde r| + |\nabla^4 \tilde r| = 1$ at some point at finite distance. This proves the claim. □

Mimicking the proof of the claim above, one can prove that by setting

$$\tilde r_\varepsilon = \frac{r_\varepsilon - r_\varepsilon(0)}{\varepsilon^3}$$

then, up to subsequences, $\tilde r_\varepsilon$ converges to a function $\tilde r$ in $C^3_{\mathrm{loc}}(\mathbb{R}^2)$ that, using (41), (45), and (46), satisfies the linearized Willmore equation

$$L_\omega(\tilde r) = \Delta\Big(\frac{1}{6|\nabla\omega|^2} R_{\alpha\beta\gamma\mu,\nu}(p_k)\omega^\beta\omega^\gamma\omega^\nu\langle\nabla\omega^\alpha, \nabla\omega^\mu\rangle + \tfrac{1}{6}\operatorname{Ric}_{\alpha\beta,\gamma}(p_k)\omega^\alpha\omega^\beta\omega^\gamma\Big).$$

Recalling identity (50), the last equation can be rewritten as

$$L_\omega(\tilde r) = \Delta\big(\tfrac{1}{12}\operatorname{Ric}_{\alpha\beta,\gamma}(p_k)\omega^\alpha\omega^\beta\omega^\gamma\big).$$

Finally, integrating this relation against the ω^α, for $\alpha = 1, \dots, 3$, which are solutions of the linearized equation, we get

$$\int_{\mathbb{R}^2} \Delta\omega\big(\tfrac{1}{12}\operatorname{Ric}_{\alpha\beta,\gamma}(p_k)\omega^\alpha\omega^\beta\omega^\gamma\big)\,dz = 0.$$

Let us note that the integration by parts above has been possible thanks to the decay of ω and its derivatives at infinity. The last identity gives

$$\int_{\mathbb{R}^2} (\mathrm{Ric}_{\alpha\beta,\gamma}(p_k)\omega^\alpha\omega^\beta\omega^\gamma)\tfrac{1}{2}\omega|\nabla\omega|^2\,dz = 0.$$

Then by a change of variable, we get

$$\int_{\mathbb{S}^2} (\mathrm{Ric}_{\alpha\beta,\gamma}(p_k)(p_k)y^\alpha y^\beta y^\gamma)y\,d\mathrm{vol}_h = 0,$$

where h is the standard metric on $\mathbb{S}^2$ and y^α are the position coordinates of $\mathbb{S}^2$ in $\mathbb{R}^3$. Finally, using the relation

$$\int_{S^2} y^\alpha y^\beta y^\gamma y^\mu\,d\mathrm{vol}_h = \tfrac{4}{15}\pi(\delta^{\alpha\beta}\delta^{\mu\gamma} + \delta^{\alpha\mu}\delta^{\beta\gamma} + \delta^{\alpha\gamma}\delta^{\beta\mu})$$

and the second Bianchi identity, we obtain

$$\nabla\,\mathrm{Scal}(\bar{p}) = 0,$$

which proves the theorem. □

Appendix A: The linearized Willmore operator

The aim of this appendix is to derive the linearized Willmore equation and to classify its solution.

The Willmore equation for a conformal immersion Φ into $\mathbb{R}^3$ can be written as

$$W'(\Phi) = \Delta_{\bar{g}}(H) + H|A^\circ|^2_{\bar{g}} = 0,$$

where $\Delta_{\bar{g}} = (2/|\nabla\Phi|^2)\Delta$, H is the mean curvature, and A° is the traceless second fundamental form.

Equivalently, one has

$$H = \tfrac{1}{2}\langle\Delta_{\bar{g}}\Phi, \vec{\nu}\rangle,$$

where $\vec{\nu}$ is the inward-pointing unit normal of the immersion Φ. Hence, by multiplying the first equation by $|\nabla\Phi|^2/2$, we can consider the equivalent equation

$$\widetilde{W}'(\Phi) = \Delta H + \langle\Delta\Phi, \vec{\nu}\rangle\tfrac{1}{2}|A^\circ|^2_{\bar{g}} = 0.$$

Of course, any conformal parametrization, ω, of a round sphere is a solution. Then expanding $\widetilde{W}'(\omega + t\rho)$ for some function ρ and using the fact that $A^\circ \equiv 0$ for a round sphere, we get

$$L_\omega(\rho) := \delta\widetilde{W}_\omega(\rho) = -\Delta\Big(\frac{\langle\Delta\rho, \omega\rangle + 2\langle\nabla\omega, \nabla\rho\rangle}{|\nabla\omega|^2}\Big) = 0. \tag{A-1}$$

Also consider the linearization of the conformality condition, which gives

$$\begin{cases}\langle\omega_x, \rho_x\rangle - \langle\omega_y, \rho_y\rangle = 0,\\ \langle\omega_x, \rho_y\rangle + \langle\omega_y, \rho_x\rangle = 0.\end{cases} \tag{A-2}$$

In the following lemma, we classify the solutions of the linearized operator following the previous work [Laurain 2012] concerning the linearized operator for the constant mean curvature equation:[3]

Lemma A.1. *Let* $\rho \in \mathring{H}^2(\mathbb{R}^2, \mathbb{R}^3)$ *be a solution of the linearized equation* (A-1) *that satisfies* (A-2) *and the additional normalizing conditions*

$$\nabla\rho(0) = 0 \quad \textit{and} \quad \langle \nabla^2\rho, \nabla\omega\rangle(0) = 0.$$

Then $\nabla\rho \equiv 0$.

Proof. First we remark that, thanks to the definition of $\mathring{H}^2(\mathbb{R}^2, \mathbb{R}^3)$, we have

$$\frac{\langle \Delta\rho, \omega\rangle + 2\langle \nabla\omega, \nabla\rho\rangle}{|\nabla\omega|^2} \in L^2(\mathbb{R}^2).$$

Hence, using Liouville's theorem, we get that

$$\langle \Delta\rho, \omega\rangle + 2\langle \nabla\omega, \nabla\rho\rangle = 0. \tag{A-3}$$

Then thanks to the fact that $(\omega_x, \omega_y, \omega)$ is a basis of $\mathbb{R}^3$ and (A-2), there exist $a, b, c, d : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\begin{cases} \rho_x = a\omega_x + b\omega_y + c\omega, \\ \rho_y = -b\omega_x + a\omega_y + d\omega. \end{cases} \tag{A-4}$$

Then plugging (A-4) into (A-3) and using the relation $\rho_{xy} = \rho_{yx}$, we see that a, b, c, and d satisfy the equations

$$a_y + b_x = d, \tag{A-5}$$

$$b_y - a_x = -c, \tag{A-6}$$

$$c_y - d_x = b|\nabla\omega|^2,$$

$$c_x + d_y = -a|\nabla\omega|^2.$$

These equations imply that a and b satisfy

$$\Delta a = -a|\nabla\omega|^2 \quad \text{and} \quad \Delta b = -b|\nabla\omega|^2.$$

Since $\rho \in \mathring{H}^1(\mathbb{R}^2, \mathbb{R}^3)$, then a and b can be seen as functions in $H^1(S^2)$ satisfying $\Delta\alpha = 2\alpha$; therefore, a and b are linear combinations of the first nonvanishing eigenfunctions of $\Delta_{\mathbb{S}^2}$ (see also Lemma C.1 of [Laurain 2012]); that is to say

$$a = \sum_{i=0}^{2} a_i\psi_i \quad \text{and} \quad b = \sum_{i=0}^{2} b_i\psi_i,$$

where

$$\psi_i(x) = \frac{x_i}{(1+|x|^2)} \quad \text{for } i = 1, 2 \text{ and } \psi_0(x) = \frac{1-|x|^2}{1+|x|^2}.$$

[3] In this statement, $\mathring{H}^2(\mathbb{R}^2, \mathbb{R}^3)$ is the pushforward of $H^2(S^2)$ on $\mathbb{R}^2$ via stereographic projection.

Finally using the facts that $\nabla\rho(0) = 0$ and $\langle\nabla^2\rho, \nabla\omega\rangle(0) = 0$, (A-5), and (A-6), we can conclude that $a \equiv b \equiv c \equiv d \equiv 0$, which proves the lemma. □

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