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HOLE PROBABILITIES OF SU($m+1$) GAUSSIAN RANDOM POLYNOMIALS

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In this paper, we study hole probabilities $P_{0,m}(r, N)$ of SU($m+1$) Gaussian random polynomials of degree N over a polydisc $(D(0,r))^m$. When $r \geq 1$, we find asymptotic formulas and the decay rate of $\log P_{0,m}(r, N)$. In dimension one, we also consider hole probabilities over some general open sets and compute asymptotic formulas for the generalized hole probabilities $P_{k,1}(r, N)$ over a disc $D(0,r)$.

Introduction

Hole probability is the probability that some random field never vanishes over some set. For Gaussian random entire functions we have this (see also [Zrebiec 2007, Theorem 1.2] for a multivariable result):

Theorem [Sodin and Tsirelson 2005, Theorem 1]. *Let $\psi(z) = \sum_{k=0}^{\infty} c_k z^k/\sqrt{k!}$, where the c_k $(k \geq 0)$ are i.i.d. standard complex Gaussian random variables. Then there exist constants $C_1 \geq C_2 > 0$ such that*

$$\exp\{-C_1 r^4\} \leq \operatorname{Prob}\bigl\{0 \notin \psi(D(0,r))\bigr\} \leq \exp\{-C_2 r^4\}.$$

The case of Gaussian random sections was considered in [Shiffman et al. 2008]: Let M be a compact Kähler manifold with complex dimension m and $(L, h) \to M$ a positive holomorphic line bundle. Let γ_N denote the Gaussian probability measure on $H^0(M, L^N)$ induced by the fiberwise inner product h^N and the polarized volume form $dV_M = \omega_h^m/m! = ((\sqrt{-1}/2\pi)\Theta_h)^m/m!$, where Θ_h is the Chern curvature tensor of (L, h).

Theorem [Shiffman et al. 2008, Theorem 1.4]. *For any nonempty open set $U \subset M$, if there exists s in $H^0(M, L)$ such that s does not vanish on $\overline{U}$, then there exist constants $C_1 \geq C_2 > 0$ such that, for $N \gg 1$,*

$$\exp\{-C_1 N^{m+1}\} \leq \gamma_N\{s_N \in H^0(M, L^N) : 0 \notin s_N(U)\} \leq \exp\{-C_2 N^{m+1}\}.$$

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Therefore, it is natural to ask: can we find sharp constants C_1, C_2 in these two theorems, and is it possible to obtain an asymptotic formula and a decay rate for the hole probability? Using Cauchy's integral estimates, Nishry answered this in the random entire function case as follows (an analogous result for Gaussian random power series is obtained in [Peres and Virág 2005, Corollary 3]).

Theorem [Nishry 2010, Theorem 1]. *Let $\psi(z) = \sum_{k=0}^{\infty} c_k z^k/\sqrt{k!}$, where the c_k $(k \geq 0)$ are i.i.d. standard complex Gaussian random variables. Then*

$$\operatorname{Prob}\{0 \notin \psi(D(0,r))\} = \exp\left\{-\tfrac{1}{2}e^2 r^4 + O(r^{\frac{18}{5}})\right\}.$$

This suggests to us that, for those line bundles with polynomial sections, maybe it is possible to find an asymptotic formula for the hole probability.

If $P_{0,m}(r,N)$ denotes the hole probability of $\mathrm{SU}(m+1)$ Gaussian random polynomials over the polydisc $(D(0,r))^m$, $d_m x$ denotes the Lebesgue measure on $\mathbb{R}^m$ and

$$E_r(x) := 2\sum_{i=1}^{m} x_i \log r - \left[\sum_{i=1}^{m} x_i \log x_i + \left(1-\sum_{i=1}^{m} x_i\right)\log\left(1-\sum_{i=1}^{m} x_i\right)\right]$$

is a continuous function defined over the standard simplex $\Sigma_m := \{x = (x_1,\ldots,x_m) \in \mathbb{R}^{m+} : \sum_{i=1}^{m} x_i \leq 1\}$ (here we adopt the convention that $0\log 0 = 0$), we have the following results:

Theorem 0.1. *For $r \geq 1$,*

$$\log P_{0,m}(r,N) = -N^{m+1}\int_{\Sigma_m} E_r(x)\, d_m x + o(N^{m+1}),$$

where

$$\int_{\Sigma_m} E_r(x)\, d_m x = \frac{2m\log r}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}.$$

Theorem 0.2. *For $r > 0$,*

$$\log P_{0,m}(r,N) \geq -N^{m+1}\int_{x\in\Sigma_m : E_r(x)\geq 0} E_r(x)\, d_m x + o(N^{m+1}),$$

$$\log P_{0,m}(r,N) \leq -N^{m+1}\int_{x\in\mathbb{R}^{m+} : \sum_{i=1}^{m} x_i \leq \alpha_0} E_r(x)\, d_m x + o(N^{m+1}),$$

where $\alpha_0 = \alpha_0(r,m) > 0$ is defined by

$$\alpha_0 = \alpha_0(r,m) = \begin{cases} 1 & \text{if } 2\log r + \sum_{k=2}^{m} 1/k \geq 0, \\ \text{the nonzero root of } \alpha = \dfrac{\alpha\log\alpha + (1-\alpha)\log(1-\alpha)}{2\log r + \sum_{k=2}^{m} 1/k} & \text{if } 2\log r + \sum_{k=2}^{m} 1/k < 0. \end{cases}$$

Here, when $m = 1$, we take $\sum_{k=2}^{m} 1/k = 0$.

Remark 0.3. Theorem 0.1 can be derived from Theorem 0.2 as, when $r \geq 1$, $\{x \in \Sigma_m : E_r(x) \geq 0\} = \Sigma_m$ and $\alpha_0(r,m) = 1$. In fact, we could have proved this general case directly, but the idea of the proof would turn out to be extremely difficult to follow.

Corollary 0.4. *In the case of $m = 1$, the following asymptotic formula for the logarithm of the hole probability over a disc exists for all $r > 0$:*

$$\log P_{0,1}(r, N) = -N^2 \int_0^{\alpha_0} E_r(x)\,dx + o(N^2);$$

here

$$\int_0^{\alpha_0} E_r(x)\,dx = \tfrac{1}{2}\alpha_0(2\log r + 1 - \log\alpha_0)$$

and $\alpha_0 = \alpha_0(r, 1) \in (0, 1]$ is given in Theorem 0.2.

Because of the simplicity of the one-dimensional case, we can obtain more about the hole probability of SU(2) Gaussian random polynomials:

Theorem 0.5. *If $U \subset \mathbb{C}$ is a bounded simply connected domain containing 0 and ∂U is a Jordan curve, let $\phi : D(0, 1) \to U$ be a biholomorphism given by the Riemann mapping theorem such that $\phi(0) = 0$ (thus ϕ is unique up to the composition of a unitary transformation of $\mathbb{C}$). Then the hole probability $P_{0,1}(U, N)$ of* SU(2) *Gaussian random polynomials of degree N over U satisfies*

$$\log P_{0,1}(U, N) \le -\big(\log|\phi'(0)| + \tfrac{1}{2}\big)N^2 + o(N^2).$$

Also, in dimension one, it makes sense to study the number of zeros in some set. So let the generalized hole probability $P_{k,1}(r, N)$ be the probability that an SU(2) Gaussian random polynomial of degree N has no more than k zeros in $D(0, r)$; then, the following theorem shows that the asymptotic formula of $\log P_{k,1}(r, N)$ exists:

Theorem 0.6. *For all $k \ge 0$ and $r > 0$,*

$$\log P_{k,1}(r, N) = -\tfrac{1}{2}\alpha_0(2\log r + 1 - \log\alpha_0)N^2 + o(N^2),$$

where $\alpha_0 = \alpha_0(r, 1) \in (0, 1]$ is given in Theorem 0.2.

We should remark here that in all the cases we consider, the event that some Gaussian random polynomial has zeros on the boundary of some open set is a null set, i.e., of zero probability. Therefore we do not distinguish between the (generalized) hole probability over an open set and that over its closure.

1. Background

We review in this section some background on SU($m + 1$) Gaussian random polynomials and the definition of our probability measures. Before that, we define two lexicographically ordered sets that will be consistently used as index sets throughout this paper.

Definition 1.1.
$$\Gamma_{m,N} := \{J = (j_1, \dots, j_m) \in [0, N]^m \cap \mathbb{Z}^m : 0 \le j_1 \le \dots \le j_m \le N\},$$
$$\Lambda_{m,N} := \{K = (k_1, \dots, k_m) \in [0, N]^m \cap \mathbb{Z}^m : |K| = k_1 + \dots + k_m \le N\}.$$

It is not difficult to show that $|\Gamma_{m,N}| = |\Lambda_{m,N}| = \binom{N+m}{m}$.

The tautological line bundle $\mathcal{O}(-1)$ over the complex projective space $\mathbb{CP}^m$ is a holomorphic line bundle with fibers

$$\mathcal{O}(-1)_{[x]} = \mathbb{C}\cdot x \quad \text{for all } [x] = [x_0 : \cdots : x_m] \in \mathbb{CP}^m.$$

Its dual bundle, denoted by $\mathcal{O}(1)$, is called the hyperplane section bundle, since $\mathcal{O}(1) = \mathcal{O}(H)$, where the divisor

$$H = \{[x] \in \mathbb{CP}^m : x_0 = 0\}$$

is a hyperplane in $\mathbb{CP}^m$. $H^0(\mathbb{CP}^m, \mathcal{O}(N))$, the space of holomorphic sections of the tensor bundle $\mathcal{O}(N) = \mathcal{O}(1)^{\otimes N}$, is isomorphic to ${}^h\mathcal{P}^N_{m+1}$, the space of $(m+1)$-variable homogeneous polynomials of degree N. The Fubini–Study metric h_{FS} on $\mathcal{O}(1)$ can be described in the following way: Over the open subset

$$U_0 = \{[x] = [x_0 : \cdots : x_m] \in \mathbb{CP}^m : x_0 \neq 0\} \subset \mathbb{CP}^m,$$

we have a local frame of $\mathcal{O}(1)$,

$$e([x]) = x_0.$$

Set

$$\|e([x])\|^2_{h_{\mathrm{FS}}} = \frac{|x_0|^2}{\sum_{i=0}^m |x_i|^2} = \frac{|x_0|^2}{\|x\|^2},$$

which is independent of the choice of representative x of $[x]$. In terms of the affine coordinates

$$z = (z_1, \dots, z_m) = \left(\frac{x_1}{x_0}, \dots, \frac{x_m}{x_0}\right)$$

over U_0,

$$\|e(z)\|^2_{h_{\mathrm{FS}}} = (1 + \|z\|^2)^{-1} = \left(1 + \sum_{i=1}^m |z_i|^2\right)^{-1},$$

which defines a metric with positive Chern curvature form

$$\omega_{\mathrm{FS}} = -\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log \|e(z)\|^2_{h_{\mathrm{FS}}} = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log (1 + |z_1|^2 + \cdots + |z_m|^2).$$

This induces a metric h^N_{FS} on the line bundle $\mathcal{O}(N)$ so that

$$\|e^{\otimes N}(z)\|^2_{h^N_{\mathrm{FS}}} = (1 + \|z\|^2)^{-N}.$$

With the frame $e^{\otimes N}$ over U_0, for any $s \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$, represented as $p(x_0, \dots, x_m) \in {}^h\mathcal{P}^N_{m+1}$, we have

$$p(x_0, \dots, x_m) = \frac{p(x_0, \dots, x_m)}{x_0^N} e^{\otimes N}([x]) = p(1, z_1, \dots, z_m) e^{\otimes N}([x]),$$

which implies that all the elements in $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ can be viewed over U_0 as polynomials in $(z_1, \dots, z_m)$ of degree at most N.

Since ω_{FS} is positive over $\mathbb{CP}^m$, we may take it as a polarized metric form on $\mathbb{CP}^m$, and the associated volume form is $dV = \omega_{\text{FS}}^m/m!$. Thus, the metric h_{FS}^N together with the volume form dV induce a Hermitian inner product on the space of holomorphic sections $H^0(\mathbb{CP}^m, \mathcal{O}(N))$: for all $s_1, s_2 \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$,

$$\langle\langle s_1, s_2\rangle\rangle := \int_{\mathbb{CP}^m} \langle s_1, s_2\rangle_{h_{\text{FS}}^N}\, dV.$$

With this inner product, there is an orthonormal basis $\{S_K^N\}_{K=(k_1,\dots,k_m)\in\Lambda_{m,N}}$ given in local affine coordinates $(z_1,\dots,z_m)$ over U_0 by

$$S_K^N(z) = \sqrt{(N+1)\cdots(N+m)}\sqrt{\binom{N}{K}}z^K,$$

where we adopt the notations

$$\binom{N}{K} = \frac{N!}{(N-|K|)!k_1!\cdots k_m!}, \quad z^K := z_1^{k_1}\cdots z_m^{k_m}.$$

Thus, $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ is equal to $\big\{s_N = \sum_{K\in\Lambda_{m,N}} c_K S_K^N : c = (c_K)_{K\in\Lambda_{m,N}} \in \mathbb{C}^{\binom{N+m}{m}}\big\}$. Endow $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ with the Gaussian probability measure γ_N defined by

$$d\gamma_N(s_N) := \pi^{-\binom{N+m}{m}} e^{-\|c\|^2}\, d_{2\binom{N+m}{m}}c,$$

where $\|c\|^2 = \sum_{K\in\Lambda_{m,N}} |c_K|^2$ and $d_{2\binom{N+m}{m}}c$ denotes the $2\binom{N+m}{m}$-dimensional Lebesgue measure. Then γ_N is characterized by the property that $\{c_K\}_{K\in\Lambda_{m,N}}$ consists of independent and identically distributed (i.i.d.) standard complex Gaussian random variables. Then $\big(H^0(\mathbb{CP}^m, \mathcal{O}(N)), \gamma_N\big)$ is called the ensemble of SU($m+1$) Gaussian random polynomials of degree N, since the random element s_N is distributionally invariant under SU($m+1$) transformations of $\mathbb{CP}^m$. Its hole probability over the polydisc $(D(0,r))^m \subset \mathbb{C}^m$ is

$$\begin{aligned} P_{0,m}(r,N) &= \gamma_N\big\{s_N \in H^0(\mathbb{CP}^m, \mathcal{O}(N)) : 0 \notin s_N\big((\bar{D}(0,r))^m\big)\big\} \\ &= \pi^{-\binom{N+m}{m}} \int_{c\in\mathbb{C}^{\binom{N+m}{m}}:0\notin s_N((\bar{D}(0,r))^m)} e^{-\|c\|^2}\, d_{2\binom{N+m}{m}}c \\ &= \pi^{-\binom{N+m}{m}} \int_{c\in\mathbb{C}^{\binom{N+m}{m}}:0\notin \tilde{s}_N((\bar{D}(0,r))^m)} e^{-\|c\|^2}\, d_{2\binom{N+m}{m}}c, \end{aligned}$$

where $\tilde{s}_N(z) = \sum_{K\in\Lambda_{m,N}} c_K \sqrt{\binom{N}{K}}z^K$. Hereafter, when considering hole probability, we work on $\tilde{s}_N$ instead of s_N for simplicity.

2. Preliminaries

Definition 2.1.

$$Q_{r,m}(N) := \sum_{K\in\Lambda_{m,N}} \log\Big[\binom{N}{K}r^{2|K|}\Big].$$

Lemma 2.2. $Q_{r,m}(N) = N^{m+1}\int_{\Sigma_m} E_r(x)\,d_m x + o(N^{m+1}) = \Big[\frac{2m\log r}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}\Big]N^{m+1} + o(N^{m+1}).$

Proof. We can prove inductively that, for $k \geq 1$,

$$\left(\frac{k}{e}\right)^k \leq k! \leq \frac{k^{k+1}}{e^{k-1}}$$

or, equivalently,

$$k \log k - k \leq \log k! \leq (k+1) \log k - (k-1). \tag{2-1}$$

Hence we have

$$-(k+1) \log N + (k-1) \leq k \log \frac{k}{N} - \log k! \leq -k \log N + k \quad \text{for } 0 \leq k \leq N. \tag{2-2}$$

For all $K = (k_1, \dots, k_m) \in \Lambda_{m,N}$,

$$\log\left[\binom{N}{K} r^{2|K|}\right] - NE_r\left(\frac{K}{N}\right) = \log N! + \sum_{i=1}^{m}\left(k_i \log \frac{k_i}{N} - \log k_i!\right) + \left[(N-|K|) \log \frac{N-|K|}{N} - \log (N-|K|)!\right],$$

Applying (2-1) and (2-2), we then get

$$\log\left[\binom{N}{K} r^{2|K|}\right] - NE_r\left(\frac{K}{N}\right) \geq (N \log N - N) - (N+m+1) \log N + (N-m-1) = -(m+1)(\log N + 1),$$

$$\log\left[\binom{N}{K} r^{2|K|}\right] - NE_r\left(\frac{K}{N}\right) \leq [(N+1) \log N - (N-1)] - N \log N + N = \log N + 1.$$

Hence, for all $K \in \Lambda_{m,N}$,

$$\left|\log\left[\binom{N}{K} r^{2|K|}\right] - NE_r\left(\frac{K}{N}\right)\right| \leq (m+1)(\log N + 1),$$

so

$$\left|Q_{r,m}(N) - N \sum_{K \in \Lambda_{m,N}} E_r\left(\frac{K}{N}\right)\right| \leq \sum_{K \in \Lambda_{m,N}} \left|\log\left[\binom{N}{K} r^{2|K|}\right] - NE_r\left(\frac{K}{N}\right)\right| \leq (m+1)(\log N + 1)\binom{N+m}{m} = o(N^{m+1}). \tag{2-3}$$

Take

$$\mathring{\Lambda}_{m,N} := \{K \in \Lambda_{m,N} : k_i \geq 1 \text{ for } 1 \leq i \leq m \text{ and } |K| \leq N - m - 1\} \subset \Lambda_{m,N}$$

and

$$\mathring{\Sigma}_m(N) := \bigcup_{K \in \mathring{\Lambda}_{m,N}} \left[\frac{k_1}{N}, \frac{k_1+1}{N}\right] \times \cdots \times \left[\frac{k_m}{N}, \frac{k_m+1}{N}\right] \subset \Sigma_m.$$

Then

$$|\mathring{\Lambda}_{m,N}| = \binom{N-m-1}{m},$$

$$|\Lambda_{m,N} \setminus \mathring{\Lambda}_{m,N}| = \binom{N+m}{m} - \binom{N-m-1}{m} = O(N^{m-1}),$$

$$\mathrm{Vol}_{\mathbb{R}^m}(\Sigma_m \setminus \mathring{\Sigma}_m(N)) = \frac{1}{m!} - N^{-m}\binom{N-m-1}{m} = O(N^{-1}).$$

Over Σ_m we have

$$|E_r| \le 2|\log r| + \frac{m+1}{e} = O(1);$$

hence

$$\left| N \sum_{K\in\Lambda_{m,N}} E_r\Big(\frac{K}{N}\Big) - N \sum_{K\in\mathring{\Lambda}_{m,N}} E_r\Big(\frac{K}{N}\Big) \right| \le N|\Lambda_{m,N}\setminus\mathring{\Lambda}_{m,N}| \sup_{\Sigma_m}|E_r| = O(N^m). \tag{2-4}$$

As

$$\sup_{\mathring{\Sigma}_m(N)} \|\nabla E_r\| \le O(\log N),$$

we have

$$\begin{aligned}
&\left| N \sum_{K\in\mathring{\Lambda}_{m,N}} E_r\Big(\frac{K}{N}\Big) - N^{m+1}\int_{\mathring{\Sigma}_m(N)} E_r(x)\,d_m x \right| \\
&\qquad \le N^{m+1} \sum_{K\in\mathring{\Lambda}_{m,N}} \int_{\left[\frac{k_1}{N},\frac{k_1+1}{N}\right]\times\cdots\times\left[\frac{k_m}{N},\frac{k_m+1}{N}\right]} \left| E_r\Big(\frac{K}{N}\Big) - E_r(x)\right| d_m x \\
&\qquad \le N^{m+1}\binom{N-m-1}{m} N^{-m} O(\log N) O(N^{-1}) \\
&\qquad = O(N^m \log N).
\end{aligned} \tag{2-5}$$

Moreover,

$$\left| N^{m+1}\int_{\mathring{\Sigma}_m(N)} E_r(x)\,d_m x - N^{m+1}\int_{\Sigma_m} E_r(x)\,d_m x \right| \le N^{m+1}\sup_{\Sigma_m}|E_r|\,\mathrm{Vol}_{\mathbb{R}^m}(\Sigma_m\setminus\mathring{\Sigma}_m(N)) = O(N^m). \tag{2-6}$$

Combining (2-3)–(2-6), we thus obtain

$$\begin{aligned}
Q_{r,m}(N) &= N^{m+1}\int_{\Sigma_m} E_r(x)\,d_m x + o(N^{m+1}) \\
&= N^{m+1}\int_{\Sigma_m} 2\sum_{i=1}^m x_i \log r - \left[\sum_{i=1}^m x_i\log x_i + \Big(1-\sum_{i=1}^m x_i\Big)\log\Big(1-\sum_{i=1}^m x_i\Big)\right] d_m x + o(N^{m+1}) \\
&= N^{m+1}\left[2m\log r\int_{\Sigma_m} x_1\,d_m x - (m+1)\int_{\Sigma_m} x_1\log x_1\,d_m x\right] + o(N^{m+1}) \\
&= \left[\frac{2m\log r}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}\right] N^{m+1} + o(N^{m+1}). \qquad \square
\end{aligned}$$

Remark 2.3. The scaled lattice $(1/N)\Lambda_{m,N} \subset \mathbb{R}^m$ tends to Σ_m. Hence Lemma 2.2 is in fact converting a Riemann sum into a Riemann integral and estimating the error. Such procedures will appear several times in this paper.

Remark 2.4. The function $E_r(x)$ in the above lemma can also be written as

$$E_r(x) = -b_{\{x\}}(z_r) + \log(1 + \|z_r\|^2),$$

where $z_r = (r, \dots, r) \in \mathbb{R}^m$ and $b_{\{x\}}$ is the exponential decay rate of the expected mass density of random L^2-normalized polynomials with some prescribed Newton polytope (see Theorem 1.2 and (78) in [Shiffman and Zelditch 2004]).

Let $\xi = (\xi_1, \dots, \xi_m)$, where $\xi_i = (\xi_{i,0}, \dots, \xi_{i,N}) \in \mathbb{C}^{N+1}$ for $1 \le i \le m$.

Definition 2.5. $W_{m,N}(\xi)$ is the $\binom{N+m}{m} \times \binom{N+m}{m}$ matrix with rows indexed by $\Gamma_{m,N}$ and columns indexed by $\Lambda_{m,N}$ such that, for all $J = (j_1, \dots, j_m) \in \Gamma_{m,N}$, $K = (k_1, \dots, k_m) \in \Lambda_{m,N}$, the (J, K)-entry of $W_{m,N}(\xi)$ is $\xi_J^K = \xi_{1,j_1}^{k_1} \cdots \xi_{m,j_m}^{k_m}$.

The next lemma gives the formula for a "Vandermonde-type" determinant.

Lemma 2.6.
$$|\det W_{m,N}(\xi)| = \prod_{i=1}^{m} \prod_{0 \le j < k \le N} |\xi_{i,j} - \xi_{i,k}|^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}}.$$

Proof. For all $1 \le i \le m$ and $0 \le j < k \le N$, the rows of $W_{m,N}(\xi)$ involving $\xi_{i,j}$ correspond to the set

$$\Gamma_{m,N}^{i,j} = \{(j_1, \dots, j_m) \in \Gamma_{m,N} : j_i = j\},$$

while those rows involving $\xi_{i,k}$ correspond to the set

$$\Gamma_{m,N}^{i,k} = \{(j_1, \dots, j_m) \in \Gamma_{m,N} : j_i = k\}. \tag{2-7}$$

Let

$$\tilde{\Gamma}_{m,N}^{i,j} = \{(j_1, \dots, \hat{j}_i, \dots, j_m) \in [0, N]^{m-1} \cap \mathbb{Z}^{m-1} : 0 \le j_1 \le \cdots \le j_{i-1} \le j \le j_{i+1} \le \cdots \le j_m \le N\},$$
$$\tilde{\Gamma}_{m,N}^{i,k} = \{(j_1, \dots, \hat{j}_i, \dots, j_m) \in [0, N]^{m-1} \cap \mathbb{Z}^{m-1} : 0 \le j_1 \le \cdots \le j_{i-1} \le k \le j_{i+1} \le \cdots \le j_m \le N\};$$

then

$$|\Gamma_{m,N}^{i,j}| = |\tilde{\Gamma}_{m,N}^{i,j}| = \binom{j+i-1}{i-1}\binom{N-j+m-i}{m-i},$$
$$|\Gamma_{m,N}^{i,k}| = |\tilde{\Gamma}_{m,N}^{i,k}| = \binom{k+i-1}{i-1}\binom{N-k+m-i}{m-i}.$$

Since, for any $1 \le i \le m$,

$$\Gamma_{m,N} = \bigsqcup_{k=0}^{N} \Gamma_{m,N}^{i,k},$$

we have the equality

$$\sum_{k=0}^{N} \binom{k+i-1}{i-1}\binom{N-k+m-i}{m-i} = \binom{N+m}{m}. \tag{2-8}$$

Note that

$$\tilde{\Gamma}_{m,N}^{i,j} \cap \tilde{\Gamma}_{m,N}^{i,k} = \{(j_1, \dots, \hat{j}_i, \dots, j_m) \in [0, N]^{m-1} \cap \mathbb{Z}^{m-1} : 0 \le j_1 \le \cdots \le j_{i-1} \le j < k \le j_{i+1} \le \cdots \le j_m \le N\}$$

and

$$|\tilde{\Gamma}_{m,N}^{i,j} \cap \tilde{\Gamma}_{m,N}^{i,k}| = \binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i},$$

which means that there are $\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}$ pairs of rows; within each pair the only difference between two rows is $\xi_{i,j}$ instead of $\xi_{i,k}$. Therefore, for all $1 \le i \le m$ and $0 \le j < k \le N$,

$$(\xi_{i,j} - \xi_{i,k})^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}} \;\Big|\; \det W_{m,N}(\xi),$$

and thus

$$G_{m,N}(\xi) \mid \det W_{m,N}(\xi), \tag{2-9}$$

where

$$G_{m,N}(\xi) := \prod_{i=1}^{m} \prod_{0 \le j < k \le N} (\xi_{i,j} - \xi_{i,k})^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}}.$$

Furthermore, for all $1 \le i \le m$,

$$\begin{aligned}
\deg_{\xi_i} G_{m,N}(\xi) &= \sum_{0 \le j < k \le N} \binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i} \\
&= \sum_{k=1}^{N} \left[\sum_{j=0}^{k-1} \binom{j+i-1}{i-1} \right] \binom{N-k+m-i}{m-i} \\
&= \sum_{k=1}^{N} \binom{k-1+i}{i} \binom{N-k+m-i}{m-i} \\
&= \sum_{k-1=0}^{N-1} \binom{(k-1)+(i+1)-1}{(i+1)-1} \binom{(N-1)-(k-1)+(m+1)-(i+1)}{(m+1)-(i+1)} \\
&= \binom{(N-1)+(m+1)}{m+1} = \binom{N+m}{m+1},
\end{aligned} \tag{2-10}$$

where the second-to-last equality is due to (2-8). On the other hand, for all $1 \le i \le m$ and $1 \le k \le N$, the number of K in $\Lambda_{m,N}$ with $k_i = k$ is $\binom{N-k+m-1}{m-1}$; hence,

$$\deg_{\xi_i} \det W_{m,N}(\xi) = \sum_{k=1}^{N} k \binom{N-k+m-1}{m-1} = \binom{N+m}{m+1},$$

where the second equality is the special case $i = 1$ in (2-10). Therefore, for all $1 \le i \le m$,

$$\deg_{\xi_i} \det W_{m,N}(\xi) = \deg_{\xi_i} G_{m,N}(\xi). \tag{2-11}$$

By (2-9) and (2-11),

$$\det W_{m,N}(\xi) = C_{m,N} G_{m,N} = C_{m,N} \prod_{i=1}^{m} \prod_{0 \le j < k \le N} (\xi_{i,j} - \xi_{i,k})^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}},$$

where $C_{m,N}$ is a constant depending only on m and N. Consider the monomial

$$g_{m,N}(\xi) := \prod_{i=1}^{m} \prod_{k=1}^{N} \xi_{i,k}^{\sum_{j=0}^{k-1} \binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}} = \prod_{i=1}^{m} \prod_{k=1}^{N} \xi_{i,k}^{\binom{k+i-1}{i}\binom{N-k+m-i}{m-i}};$$

then

$$G_{m,N}(\xi) = \pm g_{m,N}(\xi) + \cdots .$$

In the Appendix, we show that the coefficient of $g_{m,N}$ in the expansion of $\det W_{m,N}(\xi)$ equals 1, and therefore $C_{m,N} = \pm 1$. □

3. Proof of Theorem 0.1

To prove Theorem 0.1, it suffices to prove separately the lower and upper bounds

$$-N^{m+1} \int_{\Sigma_m} E_r(x)\, d_m x + o(N^{m+1}) \le \log P_{0,m}(r, N) \le -N^{m+1} \int_{\Sigma_m} E_r(x)\, d_m x + o(N^{m+1})$$

Lower bound.

Proof of the lower bound in Theorem 0.1. Recall that $\tilde{s}_N(z) = \sum_{K \in \Lambda_{m,N}} c_K \sqrt{\binom{N}{K}} z^K$. Hence,

$$|\tilde{s}_N(z)| \ge |c_{(0,\dots,0)}| - \sum_{K \in \Lambda_{m,N} \setminus \{(0,\dots,0)\}} |c_K| \sqrt{\binom{N}{K}} r^{|K|} \quad \text{for all } z = (z_1, \dots, z_m) \in (\bar{D}(0, r))^m. \tag{3-1}$$

Consider the event $\Omega_{r,m,N}$:

(i) $|c_{(0,\dots,0)}| \ge \sqrt{N}$,

(ii) $|c_K| \le \dfrac{1}{2\sqrt{N} \sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}}, \quad K \in \Lambda_{m,N} \setminus \{(0, \dots, 0)\}.$

Then, if $\Omega_{r,m,N}$ occurs, by (3-1) we have that for all $z = (z_1, \dots, z_m) \in (\bar{D}(0, r))^m$,

$$\begin{aligned} |\tilde{s}_N(z)| &\ge \sqrt{N} - \sum_{K \in \Lambda_{m,N} \setminus \{(0,\dots,0)\}} \frac{\sqrt{\binom{N}{K}} r^{|K|}}{2\sqrt{N} \sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}} \\ &= \sqrt{N} - \sum_{K \in \Lambda_{m,N} \setminus \{(0,\dots,0)\}} \frac{1}{2\sqrt{N} \binom{|K|+m-1}{m-1}} \\ &= \sqrt{N} - \sum_{k=1}^{N} \frac{1}{2\sqrt{N}} = \frac{1}{2}\sqrt{N} > 0; \end{aligned}$$

hence

$$\begin{aligned} P_{0,m}(r, N) &\ge \gamma_N(\Omega_{r,m,N}) \\ &= \gamma_N(|c_{(0,\dots,0)}| \ge \sqrt{N}) \prod_{K \in \Lambda_{m,N} \setminus \{(0,\dots,0)\}} \gamma_N\left(|c_K| \le \frac{1}{2\sqrt{N} \sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}} \right), \end{aligned}$$

where $\gamma_N(|c_{(0,\dots,0)}| \geq \sqrt{N}) = e^{-N}$. Recall that for $K \in \Lambda_{m,N}\backslash\{(0,\dots,0)\}$ the standard complex Gaussian random variables c_K satisfy $\gamma_N(|c_K| \leq a) \geq \frac{1}{2}a^2$ whenever $a \leq 1$. Since

$$\frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}r^{|K|}\binom{|K|+m-1}{m-1}}} \leq 1$$

if $r \geq 1$, we thus have

$$\gamma_N\left(|c_K| \leq \frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}r^{|K|}\binom{|K|+m-1}{m-1}}}\right) \geq \frac{1}{2}\left[\frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}r^{|K|}\binom{|K|+m-1}{m-1}}}\right]^2 = \frac{1}{8N\binom{N}{K}r^{2|K|}\binom{|K|+m-1}{m-1}^2},$$

and

$$\log P_{0,m}(r,N) \geq -N - \sum_{K\in\Lambda_{m,N}\backslash\{(0,\dots,0)\}} \left\{\log 8 + \log N + 2\log\binom{|K|+m-1}{m-1} + \log\left[\binom{N}{K}r^{2|K|}\right]\right\}.$$

Since

$$\log\binom{|K|+m-1}{m-1} \leq \log\binom{N+m-1}{m-1} = O(\log N),$$

it follows that

$$\sum_{K\in\Lambda_{m,N}\backslash\{(0,\dots,0)\}} \left[\log 8 + \log N + 2\log\binom{|K|+m-1}{m-1}\right] = \binom{N+m}{m}O(\log N) = o(N^{m+1}).$$

Therefore,

$$\begin{aligned}\log P_{0,m}(r,N) &\geq -\sum_{K\in\Lambda_{m,N}\backslash\{(0,\dots,0)\}} \log\left[\binom{N}{K}r^{2|K|}\right] + o(N^{m+1})\\ &= -Q_{r,m}(N) + o(N^{m+1})\\ &= -N^{m+1}\int_{\Sigma_m} E_r(x)\,d_m x + o(N^{m+1}).\end{aligned}$$

□

Upper bound. Let $\delta > 0$ be small and $\kappa = 1 - \sqrt{\delta}$. We shall first treat δ as a small positive constant and at the end we will let $\delta \to 0+$. For the sake of clarity, all the constants C, the big O and little o terms listed throughout this paper will not depend on δ unless otherwise stated.

Definition 3.1. $$z_j(N) := \kappa r e^{2\pi\sqrt{-1}j/N+1} \quad \text{for } 0 \leq j \leq N.$$

For all $p \in \mathbb{Z}^+$, by division with remainder, $N+1 = q(N)p + l(N)$, where $q(N) \in \mathbb{Z}$, $q(N) \geq 0$ and $0 \leq l(N) < p$. For convenience, we drop the dependence on N when there is no chance of confusion. Since $N+1 = l(q+1) + (p-l)q$, for all $1 \leq i \leq m$, define $\xi_i = (\xi_{i,0},\dots,\xi_{i,N})$ by

$$\xi_{i,sp+t} = \begin{cases} z_{t(q+1)+s} & \text{if } 0 \leq t \leq l-1,\ 0 \leq s \leq q,\\ z_{l(q+1)+(t-l)q+s} & \text{if } l \leq t \leq p-1,\ 0 \leq s \leq q-1.\end{cases} \tag{3-2}$$

Intuitively, (3-2) gives a way to choose points $\xi_{i,j}$ $(j = 0, 1, \dots)$ one after another on the circle of radius κr such that the arguments of each two consecutive points differ approximately by $2\pi/p$. Denote the permutation of $N+1$ indices $\{0, \dots, N\}$ given by (3-2) by τ, i.e., $z_j = \xi_{i,\tau(j)}$ for $0 \le j \le N$ and $1 \le i \le m$.

For $t \in \{0, \dots, p-1\}$, denote

$$I_t = \begin{cases} \{t(q+1), \dots, t(q+1)+q\} & \text{if } 0 \le t \le l-1, \\ \{l(q+1)+(t-l)q, \dots, l(q+1)+(t-l)q+(q-1)\} & \text{if } l \le t \le p-1, \end{cases}$$

$$a_t = tq + \min\{t, l\} = \begin{cases} t(q+1) & \text{if } 0 \le t \le l-1, \\ l(q+1)+(t-l)q & \text{if } l \le t \le p-1. \end{cases}$$

$I_0, \dots, I_{p-1}$ give a partition of $\{0, \dots, N\}$. Again there is an implicit dependence on N for each term defined above, and we will indicate this dependence explicitly when necessary. Then

$$\tau(j) = (j - a_t)p + t = \begin{cases} [j - t(q+1)]p + t & \text{if } j \in I_t,\ 0 \le t \le l-1, \\ [j - l(q+1) - (t-l)q]p + t & \text{if } j \in I_t,\ l \le t \le p-1, \end{cases}$$

and, if $\{j(N)\}_{N=1}^{\infty}$ is a sequence satisfying $j(N) \in I_t(N)$ for all $N \ge 1$, then

$$|\tau_N(j(N)) - pj(N) + t(N+1)| \le 2p^2,$$

and therefore

$$\frac{\tau_N(j(N))}{N+1} - \left(p\frac{j(N)}{N+1} - t\right) = O(N^{-1}). \tag{3-3}$$

Lemma 3.2. *With the values of ξ_i given by* (3-2),

$$\log|\det W_{m,N}(\xi)| = m\binom{N+m}{m+1}\log(\kappa r) + \frac{\beta_m}{p}N^{m+1} + o(N^{m+1}),$$

where $\beta_m = (1/(m-1)!)\int_0^1 x^m \log[2\sin(\pi x)]\,dx$, which is finite for each $m \ge 1$ by the comparison test for improper integrals.

Proof. By Lemma 2.6,

$$\begin{aligned}
&\log|\det W_{m,N}(\xi)| \\
&\quad= \log\Bigg[\prod_{i=1}^{m}\prod_{0\le j<k\le N}|\xi_{i,j}-\xi_{i,k}|^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}}\Bigg] \\
&\quad= \sum_{i=1}^{m}\sum_{0\le j<k\le N}\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}\left\{\log\left|\frac{\xi_{i,j}}{\kappa r}-\frac{\xi_{i,k}}{\kappa r}\right| + \log(\kappa r)\right\} \\
&\quad= \sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\binom{\tau(j)+i-1}{i-1}\binom{N-\tau(k)+m-i}{m-i}\log\left|\frac{\xi_{i,\tau(j)}}{\kappa r}-\frac{\xi_{i,\tau(k)}}{\kappa r}\right| + m\binom{N+m}{m+1}\log(\kappa r) \\
&\quad= \sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\binom{\tau(j)+i-1}{i-1}\binom{N-\tau(k)+m-i}{m-i}\log\left|e^{2\pi\sqrt{-1}\frac{j}{N+1}} - e^{2\pi\sqrt{-1}\frac{k}{N+1}}\right| \\
&\qquad\qquad + m\binom{N+m}{m+1}\log(\kappa r),
\end{aligned}$$

where the second part of the third equality is due to (2-10). Now we are going to show that the first term after the last equality can be approximated by a double integral.

$$\begin{aligned}\sum_{i=1}^{m} \sum_{0\le\tau(j)<\tau(k)\le N} &\binom{\tau(j)+i-1}{i-1}\binom{N-\tau(k)+m-i}{m-i} \log\left|e^{2\pi\sqrt{-1}\frac{j}{N+1}} - e^{2\pi\sqrt{-1}\frac{k}{N+1}}\right| \\ &= \sum_{i=1}^{m} \sum_{0\le\tau(j)<\tau(k)\le N} \left[\frac{(\tau(j))^{i-1}}{(i-1)!} + o\big((\tau(j))^{i-1}\big)\right]\left[\frac{(N-\tau(k))^{m-i}}{(m-i)!} + o\big((N-\tau(k))^{m-i}\big)\right] \\ &\qquad\qquad \times \log\left|1 - e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}\right|. \end{aligned} \tag{3-4}$$

For all $0 \le j, k \le N$, $1 \le i \le m$, $0 \le u, v \le p-1$, denote

$$\begin{aligned} \mathcal{I}_{j,k,N} &= \left[\frac{j}{N+1}, \frac{j+1}{N+1}\right] \times \left[\frac{k}{N+1}, \frac{k+1}{N+1}\right], \\ L_{u,v,N} &= \{(j,k) \in I_u \times I_v : \tau(j) < \tau(k)\}, \\ T_{u,v}(N) &= \bigcup_{(j,k)\in L_{u,v,N}} \mathcal{I}_{j,k,N}, \\ \mathring{L}_{u,v,N} &= \{(j,k) \in L_{u,v,N} : j-k \ne \pm N \text{ and } j-k \ne \pm 1\} \subset L_{u,v,N}, \\ \mathring{T}_{u,v}(N) &= \bigcup_{(j,k)\in \mathring{L}_{u,v,N}} \mathcal{I}_{j,k,N} \subset T_{u,v}(N), \end{aligned}$$

and define a function over $\{(x, y) \in (0,1) \times (0,1) : x \ne y\}$ by

$$g^i_{u,v}(x, y) = (px-u)^{i-1}[1-(py-v)]^{m-i} \log\left|1 - e^{2\pi\sqrt{-1}(x-y)}\right|.$$

Then

$$|L_{u,v,N} \setminus \mathring{L}_{u,v,N}| \le 2N+2, \tag{3-5}$$

$$\mathrm{Vol}_{\mathbb{R}^2}(T_{u,v}(N) \setminus \mathring{T}_{u,v}(N)) \le O(N^{-1}), \tag{3-6}$$

$$\frac{1}{N+1} \le \left|\frac{j-k}{N+1}\right| \le \frac{N}{N+1} \qquad \text{for } (j,k) \in L_{u,v,N}, \tag{3-7}$$

$$\frac{1}{N+1} \le |x-y| \le \frac{N}{N+1} \qquad \text{for } (x,y) \in \mathring{T}_{u,v}(N), \tag{3-8}$$

$$|g^i_{u,v}(x,y)| \le O(\log N) \qquad \text{if } \frac{1}{N+1} \le |x-y| \le \frac{N}{N+1}, \tag{3-9}$$

$$\|\nabla g^i_{u,v}(x,y)\| \le O(N^{\frac{1}{2}}) \qquad \text{if } \frac{1}{\sqrt{N+1}} \le |x-y| \le 1 - \frac{1}{\sqrt{N+1}}. \tag{3-10}$$

From (3-3), we have

$$\sum_{0\le\tau(j)<\tau(k)\le N}(\tau(j))^{i-1}(N-\tau(k))^{m-i}\log\left|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}\right|$$
$$=(N+1)^{m-1}\sum_{0\le u,v\le p-1}\sum_{(j,k)\in L_{u,v,N}}\left[p\frac{j}{N+1}-u+O(N^{-1})\right]^{i-1}\left[1-\left(p\frac{k}{N+1}-v\right)+O(N^{-1})\right]^{m-i}$$
$$\times\log\left|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}\right|. \quad (3\text{-}11)$$

For all $0\le u,v\le p-1$, by (3-5), (3-7) and (3-9), we get

$$\sum_{(j,k)\in L_{u,v,N}}\left(p\frac{j}{N+1}-u\right)^{i-1}\left[1-\left(p\frac{k}{N+1}-v\right)\right]^{m-i}\log\left|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}\right|$$
$$=\sum_{(j,k)\in L_{u,v,N}}g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)$$
$$=\sum_{(j,k)\in \mathring{L}_{u,v,N}}g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)+O(N\log N). \quad (3\text{-}12)$$

Moreover,

$$\left|(N+1)^{-2}\sum_{(j,k)\in \mathring{L}_{u,v,N}}g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)-\iint_{\mathring{T}_{u,v}(N)}g^i_{u,v}(x,y)\,dx\,dy\right|$$
$$\le\sum_{(j,k)\in \mathring{L}_{u,v,N}}\iint_{\mathscr{I}_{j,k,N}}\left|g^i_{u,v}(x,y)-g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)\right|dx\,dy$$
$$=\sum_{\substack{(j,k)\in \mathring{L}_{u,v,N}\\ \frac{1}{\sqrt{N+1}}\le|\frac{j-k}{N+1}|\le1-\frac{1}{\sqrt{N+1}}}}\iint_{\mathscr{I}_{j,k,N}}\left|g^i_{u,v}(x,y)-g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)\right|dx\,dy$$
$$+\sum_{\substack{(j,k)\in \mathring{L}_{u,v,N}\\ |\frac{j-k}{N+1}|<\frac{1}{\sqrt{N+1}}\\ \text{or } |\frac{j-k}{N+1}|>1-\frac{1}{\sqrt{N+1}}}}\iint_{\mathscr{I}_{j,k,N}}\left|g^i_{u,v}(x,y)-g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)\right|dx\,dy. \quad (3\text{-}13)$$

Since

$$\#\left\{(j,k)\in \mathring{L}_{u,v,N}:\frac{1}{\sqrt{N+1}}\le\left|\frac{j-k}{N+1}\right|\le1-\frac{1}{\sqrt{N+1}}\right\}\le|\mathring{L}_{u,v,N}|=O(N^2),$$
$$\#\left\{(j,k)\in \mathring{L}_{u,v,N}:\left|\frac{j-k}{N+1}\right|<\frac{1}{\sqrt{N+1}}\text{ or }\left|\frac{j-k}{N+1}\right|>1-\frac{1}{\sqrt{N+1}}\right\}\le O(N^{\frac{3}{2}}),$$

(3-10) implies that

$$\sum_{\substack{(j,k)\in\mathring{L}_{u,v,N}\\ \frac{1}{\sqrt{N+1}}\le|\frac{j-k}{N+1}|\le 1-\frac{1}{\sqrt{N+1}}}} \iint_{\mathcal{I}_{j,k,N}} \left| g^i_{u,v}(x,y) - g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)\right| dx\,dy$$

$$\le O(N^2)\times(N+1)^{-2}\times\frac{\sqrt{2}}{N+1}\times \sup_{\frac{1}{\sqrt{N+1}}\le|x-y|\le 1-\frac{1}{\sqrt{N+1}}} \|\nabla g^i_{u,v}(x,y)\| \le O(N^{-\frac{1}{2}}), \quad (3\text{-}14)$$

and, by (3-8) and (3-9),

$$\sum_{\substack{(j,k)\in\mathring{L}_{u,v,N}\\ |\frac{j-k}{N+1}|<\frac{1}{\sqrt{N+1}}\\ \text{or } |\frac{j-k}{N+1}|>1-\frac{1}{\sqrt{N+1}}}} \iint_{\mathcal{I}_{j,k,N}} \left| g^i_{u,v}(x,y) - g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)\right| dx\,dy$$

$$\le O(N^{\frac{3}{2}})\times(N+1)^{-2}\times O(\log N) = O(N^{-\frac{1}{2}}\log N). \quad (3\text{-}15)$$

Let $T_{u,v}=\{(x,y)\in\mathbb{R}^2 : 0\le x-u/p\le y-v/p\le 1/p\}$. Since $g^i_{u,v}$ is L^1_{loc}, the measure $g^i_{u,v}(x,y)\,dx\,dy$ is absolutely continuous with respect to the Lebesgue measure. Therefore, by Lemma 3.3 below, we have that

$$\iint_{\mathring{T}_{u,v}(N)} g^i_{u,v}(x,y)\,dx\,dy - \iint_{T_{u,v}} g^i_{u,v}(x,y)\,dx\,dy = o(1) \quad \text{as } N\to\infty. \quad (3\text{-}16)$$

By (3-12)–(3-16),

$$\sum_{(j,k)\in L_{u,v,N}} \left(p\frac{j}{N+1}-u\right)^{i-1}\left[1-\left(p\frac{k}{N+1}-v\right)\right]^{m-i} \log\left|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}\right|$$

$$= (N+1)^2 \iint_{T_{u,v}} g^i_{u,v}(x,y)\,dx\,dy + o(N^2). \quad (3\text{-}17)$$

(3-17) and (3-11) imply that

$$\sum_{0\le\tau(j)<\tau(k)\le N} (\tau(j))^{i-1}(N-\tau(k))^{m-i}\log\left|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}\right|$$

$$= (N+1)^{m+1}\sum_{0\le u,v\le p-1}\iint_{T_{u,v}} g^i_{u,v}(x,y)\,dx\,dy + o(N^{m+1}), \quad (3\text{-}18)$$

(3-18) and (3-4) imply

$$\sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\binom{\tau(j)+i-1}{i-1}\binom{N-\tau(k)+m-i}{m-i}\log\left|e^{2\pi\sqrt{-1}\frac{j}{N+1}}-e^{2\pi\sqrt{-1}\frac{k}{N+1}}\right|$$

$$=\sum_{i=1}^{m}\sum_{0\le u,v\le p-1}\iint_{T_{u,v}}\frac{g^i_{u,v}(x,y)}{(i-1)!(m-i)!}\,dx\,dy + o(N^{m+1})$$

$$= \sum_{i=1}^{m} \sum_{0 \le u,v \le p-1} \iint_{T_{u,v}} \frac{[p(x-\frac{u}{p})]^{i-1}}{(i-1)!} \frac{[1-p(y-\frac{v}{p})]^{m-i}}{(m-i)!} \log\left|1-e^{2\pi\sqrt{-1}(x-y)}\right| dx\, dy + o(N^{m+1})$$

$$= \sum_{i=1}^{m} \sum_{0 \le u,v \le p-1} \iint_{T_{0,0}} \frac{(px)^{i-1}}{(i-1)!} \frac{(1-py)^{m-i}}{(m-i)!} \log\left|1-e^{2\pi\sqrt{-1}(x-y+\frac{u}{p}-\frac{v}{p})}\right| dx\, dy + o(N^{m+1})$$

$$= \sum_{i=1}^{m} \sum_{0 \le u \le p-1} \iint_{T_{0,0}} \frac{(px)^{i-1}}{(i-1)!} \frac{(1-py)^{m-i}}{(m-i)!} \log\left[\prod_{v=0}^{p-1} \left|e^{2\pi\sqrt{-1}\frac{v}{p}} - e^{2\pi\sqrt{-1}(x-y+\frac{u}{p})}\right|\right] dx\, dy + o(N^{m+1})$$

$$= p \sum_{i=1}^{m} \iint_{T_{0,0}} \frac{(px)^{i-1}}{(i-1)!} \frac{(1-py)^{m-i}}{(m-i)!} \log\left|1-e^{2\pi\sqrt{-1}(px-py)}\right| dx\, dy + o(N^{m+1})$$

$$= \frac{1}{p} \iint_{T} \sum_{i=1}^{m} \frac{x^{i-1}}{(i-1)!} \frac{(1-y)^{m-i}}{(m-i)!} \log\left|1-e^{2\pi\sqrt{-1}(x-y)}\right| dx\, dy + o(N^{m+1})$$

$$= \frac{1}{p(m-1)!} \iint_{T} (1+x-y)^{m-1} \log\left|1-e^{2\pi\sqrt{-1}(x-y)}\right| dx\, dy + o(N^{m+1}),$$

where $T = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le y \le 1\}$. After the change of variables $\tilde{x} = x - y$, $\tilde{y} = y$, T is mapped to $\widetilde{T} = \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 : -1 \le \tilde{x} \le 0,\ -\tilde{x} \le \tilde{y} \le 1\}$. Then

$$\begin{aligned}
\frac{1}{(m-1)!} \iint_{T} (1+x-y)^{m-1} &\log\left|1-e^{2\pi\sqrt{-1}(x-y)}\right| dx\, dy \\
&= \frac{1}{(m-1)!} \iint_{\widetilde{T}} (1+\tilde{x})^{m-1} \log\left|1-e^{2\pi\sqrt{-1}\tilde{x}}\right| d\tilde{x}\, d\tilde{y} \\
&= \frac{1}{(m-1)!} \int_{-1}^{0} (1+\tilde{x})^{m} \log\left|1-e^{2\pi\sqrt{-1}\tilde{x}}\right| d\tilde{x} \\
&= \frac{1}{(m-1)!} \int_{0}^{1} x^{m} \log\left|1-e^{2\pi\sqrt{-1}x}\right| dx \\
&= \frac{1}{(m-1)!} \int_{0}^{1} x^{m} \log[2\sin(\pi x)]\, dx \\
&= \beta_m;
\end{aligned}$$

hence,

$$\sum_{i=1}^{m} \sum_{0 \le \tau(j) < \tau(k) \le N} \binom{\tau(j)+i-1}{i-1} \binom{N-\tau(k)+m-i}{m-i} \log\left|e^{2\pi\sqrt{-1}\frac{j}{N+1}} - e^{2\pi\sqrt{-1}\frac{k}{N+1}}\right| = \frac{\beta_m}{p} N^{m+1} + o(N^{m+1}).$$

Thus,

$$\log|\det W_{m,N}(\xi)| = m\binom{N+m}{m+1} \log(\kappa r) + \frac{\beta_m}{p} N^{m+1} + o(N^{m+1}). \qquad \square$$

Lemma 3.3. $\lim_{N\to\infty} \mathrm{Vol}_{\mathbb{R}^2}(T_{u,v} \triangle \mathring{T}_{u,v}(N)) = 0$ *for any* $0 \le u, v \le p-1$, *where* $T_{u,v} \triangle \mathring{T}_{u,v}(N)$ *denotes the difference set of* $T_{u,v}$ *and* $\mathring{T}_{u,v}(N)$.

Proof. By (3-6), the statement in the lemma is equivalent to $\lim_{N\to\infty} \mathrm{Vol}_{\mathbb{R}^2}(T_{u,v} \triangle T_{u,v}(N)) = 0$, which follows from $\lim_{N\to\infty}(T_{u,v}(N)\setminus\partial T_{u,v}) = \mathring{T}_{u,v}$, as $T_{u,v}\triangle T_{u,v}(N) = (T_{u,v}\setminus T_{u,v}(N))\cup(T_{u,v}(N)\setminus T_{u,v})$. Since $T_{u,v} \setminus T_{u,v}(N) \subset [\mathring{T}_{u,v} \setminus (T_{u,v}(N) \setminus \partial T_{u,v})] \cup \partial T_{u,v}$,

$$\begin{aligned}\mathrm{Vol}_{\mathbb{R}^2}(T_{u,v} \setminus T_{u,v}(N)) &\le \mathrm{Vol}_{\mathbb{R}^2}(\mathring{T}_{u,v} \setminus (T_{u,v}(N) \setminus \partial T_{u,v})) + \mathrm{Vol}_{\mathbb{R}^2}(\partial T_{u,v})\\ &= \iint_{\mathbb{R}^2} \mathbb{1}_{\mathring{T}_{u,v}\setminus(T_{u,v}(N)\setminus\partial T_{u,v})}\, dx\, dy\\ &\le \iint_{\mathbb{R}^2} \bigl|\mathbb{1}_{\mathring{T}_{u,v}} - \mathbb{1}_{T_{u,v}(N)\setminus\partial T_{u,v}}\bigr|\, dx\, dy;\end{aligned}$$

the last line tends to 0 by Fatou's lemma. A similar proof works for $T_{u,v}(N) \setminus T_{u,v}$. Therefore, it remains to prove $\lim_{N\to\infty}(T_{u,v}(N) \setminus \partial T_{u,v}) = \mathring{T}_{u,v}$.

First we'll show that $\limsup_{N\to\infty} T_{u,v}(N) \subset T_{u,v}$. For all $(x, y) \in \limsup_{N\to\infty} T_{u,v}(N)$, there exists a sequence $\{N_n\}_{n=1}^{\infty} \to \infty$ such that, for any $n \ge 1$, there exists $(j(N_n), k(N_n)) \in I_u(N_n) \times I_v(N_n)$ with $\tau_{N_n}(j(N_n)) < \tau_{N_n}(k(N_n))$ and with $(x, y) \in \mathcal{I}_{j(N_n),k(N_n),N_n}$. Then $\lim_{n\to\infty} j(N_n)/(N_n+1) = x$ and $\lim_{n\to\infty} k(N_n)/(N_n+1) = y$. Since $0 \le \tau_{N_n}(j(N_n))/(N_n+1) < \tau_{N_n}(k(N_n))/(N_n+1) \le N_n/(N_n+1)$ and $(j(N_n), k(N_n)) \in I_u(N_n) \times I_v(N_n)$, (3-3) implies that

$$0 \le p \lim_{n\to\infty} j(N_n)/(N_n+1) - u \le p \lim_{n\to\infty} k(N_n)/(N_n+1) - v \le 1.$$

Hence $0 \le px - u \le py - v \le 1$ and $(x, y) \in T_{u,v}$.

Next we will prove $\mathring{T}_{u,v} \subset \liminf_{N\to\infty} T_{u,v}(N)$. For all $(x, y) \in \mathring{T}_{u,v}$, $0 < x-u/p < y-v/p < 1/p$. Then there exist $0 < \epsilon_1, \epsilon_2, \eta_1, \eta_2 < 1/p$ such that $x = u/p+\epsilon_1 = (u+1)/p-\eta_1$ and $y = v/p+\epsilon_2 = (v+1)/p-\eta_2$. For each $N > 0$, define $j(N) = \lfloor (N+1)x \rfloor$ and $k(N) = \lfloor (N+1)y \rfloor$. When N is large enough, $j(N) = \lfloor (N+1)(u/p+\epsilon_1) \rfloor = uq(N) + \lfloor ul(N)/p + \epsilon_1(N+1) \rfloor \ge uq(N) + \min\{u, l(N)\} = a_u$, while

$$\begin{aligned} j(N) = \Bigl\lfloor (N+1)\Bigl(\frac{u+1}{p} - \eta_1\Bigr) \Bigr\rfloor &= (u+1)q(N) + \Bigl\lfloor (u+1)\frac{l(N)}{p} - \eta_1(N+1) \Bigr\rfloor\\ &\le (u+1)q(N) + \min\{u+1, l(N)\} - 1 = a_{u+1} - 1\end{aligned}$$

for $0 \le u < p-1$, which indicates that $j(N) \in I_u(N)$. Similarly, $k(N) \in I_v(N)$ for N large. Moreover, $\lim_{N\to\infty} \tau(j(N))/(N+1) = p \lim_{N\to\infty} j(N)/(N+1) - u = p \lim_{N\to\infty} \lfloor (N+1)x \rfloor/(N+1) - u = px - u$; similarly, $\lim_{N\to\infty} \tau(k(N))/(N+1) = py - v$. And, since $0 < px-u < py-v < 1$, for N large enough we have $0 < \tau(j(N))/(N+1) < \tau(k(N))/(N+1) < 1$, so $0 < \tau(j(N)) < \tau(k(N)) \le N$. Thus, by the definition of $j(N)$ and $k(N)$, we have, for N large, $(x, y) \in \mathcal{I}_{j(N),k(N),N} \subset \bigcup_{(j,k)\in L_{u,v,N}} \mathcal{I}_{j,k,N} = T_{u,v}(N)$, which implies that $(x, y) \in \liminf_{N\to\infty} T_{u,v}(N)$.

In conclusion, we have

$$\mathring{T}_{u,v} \subset \liminf_{N\to\infty} T_{u,v}(N) \subset \limsup_{N\to\infty} T_{u,v}(N) \subset T_{u,v},$$

from which

$$\lim_{N\to\infty}(T_{u,v}(N)\setminus\partial T_{u,v})=\mathring{T}_{u,v}.\qquad\square$$

Let $\zeta=(\zeta_J)^t_{J\in\Gamma_{m,N}}=(\tilde{s}_N(\xi_J))^t_{J\in\Gamma_{m,N}}=(\tilde{s}_N(\xi_{1,j_1},\dots,\xi_{m,j_m}))^t_{J\in\Gamma_{m,N}}$ be an $\binom{N+m}{m}$-dimensional mean zero complex Gaussian random vector. Let its covariance matrix be Σ; then, for all $J=(j_1,\dots,j_m)$, $J'=(j'_1,\dots,j'_m)\in\Gamma_{m,N}$ and

$$\begin{aligned}\Sigma_{J,J'}&=\mathbb{E}_N(\zeta_J\bar{\zeta}_{J'})=\mathbb{E}_N(\tilde{s}_N(\xi_J)\overline{\tilde{s}_N(\xi_{J'})})\\&=\sum_{K\in\Lambda_{m,N}}\Bigl[\sqrt{\tbinom{N}{K}}\xi_J^K\Bigr]\Bigl[\sqrt{\tbinom{N}{K}}\bar{\xi}_{J'}^K\Bigr]\\&=\sum_{K\in\Lambda_{m,N}}\binom{N}{K}(\xi_J\bar{\xi}_{J'})^K\\&=(1+\xi_J\bar{\xi}_{J'})^N\\&=(1+\xi_{1,j_1}\bar{\xi}_{1,j'_1}+\cdots+\xi_{m,j_m}\bar{\xi}_{m,j'_m})^N,\end{aligned}$$

where $\mathbb{E}_N$ denotes the expectation with respect to the probability measure γ_N.

Lemma 3.4. *With the assignment of ξ as in* (3-2),

$$\log(\det\Sigma)=Q_{\kappa r,m}(N)+\frac{2\beta_m}{p}N^{m+1}+o(N^{m+1}).$$

Proof. $\Sigma=V_{m,N}(\xi)V^*_{m,N}(\xi)$, where $V_{m,N}(\xi)=\Bigl(\sqrt{\tbinom{N}{K}}\xi_J^K\Bigr)_{J\in\Gamma_{m,N},K\in\Lambda_{m,N}}$ is an $\binom{N+m}{m}\times\binom{N+m}{m}$ matrix. Thus

$$\det\Sigma=|\det V_{m,N}(\xi)|^2=\prod_{K\in\Lambda_{m,N}}\binom{N}{K}|\det W_{m,N}(\xi)|^2.$$

By Lemma 3.2,

$$\begin{aligned}\log(\det\Sigma)&=\sum_{K\in\Lambda_{m,N}}\log\binom{N}{K}+2\log|\det W_{m,N}(\xi)|\\&=\sum_{K\in\Lambda_{m,N}}\log\binom{N}{K}+2m\binom{N+m}{m+1}\log(\kappa r)+\frac{2\beta_m}{p}N^{m+1}+o(N^{m+1})\\&=\sum_{K\in\Lambda_{m,N}}\log\binom{N}{K}+2\sum_{K\in\Lambda_{m,N}}|K|\log(\kappa r)+\frac{2\beta_m}{p}N^{m+1}+o(N^{m+1})\\&=Q_{\kappa r,m}(N)+\frac{2\beta_m}{p}N^{m+1}+o(N^{m+1}).\qquad\square\end{aligned}$$

As $\log|\tilde{s}_N(z)|$ is plurisubharmonic in a neighborhood of $(\bar{D}(0,r))^m$, we have

$$\begin{aligned}
\log \prod_{J\in\Gamma_{m,N}} |\zeta_J| &= \sum_{J\in\Gamma_{m,N}} \log|\tilde{s}_N(\xi_J)| \\
&\le \sum_{J\in\Gamma_{m,N}} \int\!\cdots\!\int_{(\partial D(0,r))^m} \log|\tilde{s}_N(u)| \prod_{i=1}^m P_r(\xi_{i,j_i},u_i)\, d\sigma_r(u_1)\cdots d\sigma_r(u_m) \\
&= (N+1)^m \int\!\cdots\!\int_{(\partial D(0,r))^m} \log|\tilde{s}_N(u)| \Big[\sum_{J\in\Gamma_{m,N}} \prod_{i=1}^m \frac{P_r(\xi_{i,j_i},u_i)}{N+1} \\
&\qquad\qquad - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\, d_m x \Big] d\sigma_r(u_1)\cdots d\sigma_r(u_m) \\
&\quad + (N+1)^m \int\!\cdots\!\int_{(\partial D(0,r))^m} \log|\tilde{s}_N(u)| \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\, d_m x\, d\sigma_r(u_1)\cdots d\sigma_r(u_m), \qquad (3\text{-}19)
\end{aligned}$$

where $P_r(\xi,u) = (r^2-|\xi|^2)/(|u-\xi|^2)$ is the Poisson kernel of $D(0,r)$, $d\sigma_r$ is the Haar measure on $\partial D(0,r)$, $d_m x$ is the Lebesgue measure on $\mathbb{R}^m$, and

$$H = \bigcup_{0\le t_1,\ldots,t_m\le p-1} H_{t_1,\ldots,t_m} := \bigcup_{0\le t_1,\ldots,t_m\le p-1} \Big\{ x=(x_1,\ldots,x_m)\in\mathbb{R}^m : 0\le x_1-\frac{t_1}{p}\le\cdots\le x_m-\frac{t_m}{p}\le\frac{1}{p}\Big\}.$$

Let I and II be the two summands on the right-hand side of (3-19). Then

$$\begin{aligned}
I \le (N+1)^m \max_{u\in(\partial D(0,r))^m} \Big| \sum_{J\in\Gamma_{m,N}} \prod_{i=1}^m \frac{P_r(\xi_{i,j_i},u_i)}{N+1} - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\, d_m x \Big| \\
\times \int\!\cdots\!\int_{(\partial D(0,r))^m} \big|\log|\tilde{s}_N(u)|\big|\, d\sigma_r(u_1)\cdots d\sigma_r(u_m). \qquad (3\text{-}20)
\end{aligned}$$

First we estimate $\int\cdots\int_{(\partial D(0,r))^m} \big|\log|\tilde{s}_N(u)|\big|\, d\sigma_r(u_1)\cdots d\sigma_r(u_m)$.

Lemma 3.5. $$\gamma_N\Big(\sup_{u\in(\partial D(0,r))^m} |\tilde{s}_N(u)| < 1\Big) \le e^{-Q_{r,m}(N)}.$$

Proof. $$\tilde{s}_N(u) = \sum_{K\in\Lambda_{m,N}} c_K \sqrt{\binom{N}{K}} u^K \quad\Rightarrow\quad \frac{\partial^K}{\partial u^K}\tilde{s}_N(0) = K!\sqrt{\binom{N}{K}} c_K,$$

where $\partial^K/\partial u^K$ refers to $(\partial^{k_1}/\partial u_1^{k_1})\cdots(\partial^{k_m}/\partial u_1^{k_m})$ and $K! = k_1!\cdots k_m!$.

By Cauchy's integral formula,

$$\frac{\partial^K}{\partial u^K}\tilde{s}_N(0) = \frac{K!}{(2\pi\sqrt{-1})^m} \int\!\cdots\!\int_{(\partial D(0,r))^m} \frac{\tilde{s}_N(u)}{\prod_{i=1}^m u_i^{k_i+1}}\, du_1\cdots du_m,$$

so

$$c_K = \binom{N}{K}^{-\frac{1}{2}} \frac{1}{(2\pi\sqrt{-1})^m} \int\!\cdots\!\int_{(\partial D(0,r))^m} \frac{\tilde{s}_N(u)}{\prod_{i=1}^m u_i^{k_i+1}}\, du_1 \cdots du_m,$$

and thus

$$|c_K| \le \frac{\sup_{u\in(\partial D(0,r))^m} |\tilde{s}_N(u)|}{\sqrt{\binom{N}{K}} r^{|K|}} \quad \text{for all } K \in \Lambda_{m,N}.$$

Therefore, $\sup_{u\in(\partial D(0,r))^m} |\tilde{s}_N(u)| < 1$ would imply that, for all $K \in \Lambda_{m,N}$,

$$|c_K| \le \left[\binom{N}{K} r^{2|K|}\right]^{-\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}\gamma_N\Big(\sup_{u\in(\partial D(0,r))^m} |\tilde{s}_N(u)| < 1\Big) &\le \prod_{K\in\Lambda_{m,N}} \gamma_N\left(|c_K| \le \left[\binom{N}{K} r^{2|K|}\right]^{-\frac{1}{2}}\right)\\ &\le \prod_{K\in\Lambda_{m,N}} \left[\binom{N}{K} r^{2|K|}\right]^{-1}\\ &= e^{-Q_{r,m}(N)}. \qquad \square\end{aligned}$$

The next lemma follows directly from the first part of [Shiffman et al. 2008, Theorem 3.1], but here we provide a self-contained proof without using the language of sections and metrics.

Lemma 3.6. *Given $U \subset \mathbb{C}^m$ open and bounded with $\sup_{z\in\overline{U}} \|z\| = R > 0$, for all $\eta > 0$,*

$$\gamma_N\Big\{\sup_{z\in\overline{U}} |\tilde{s}_N(z)| > (1+R^2)^{\frac{N}{2}} e^{\eta N}\Big\} \le e^{-e^{\eta N}} \quad \text{for } N \gg 1.$$

Proof. By the Cauchy–Schwartz inequality,

$$\begin{aligned}\sup_{z\in\overline{U}} |\tilde{s}_N(z)| = \sup_{z\in\overline{U}} \Big| \sum_{K\in\Lambda_{m,N}} c_K \sqrt{\binom{N}{K}} z^K \Big| &\le \|c\| \sup_{z\in\overline{U}} \Big[\sum_{K\in\Lambda_{m,N}} \binom{N}{K} |z|^{2K}\Big]^{\frac{1}{2}}\\ &= \|c\| \sup_{z\in\overline{U}} (1+\|z\|^2)^{\frac{N}{2}}\\ &= \|c\|(1+R^2)^{\frac{N}{2}},\end{aligned}$$

so

$$\gamma_N\Big\{\sup_{z\in\overline{U}} |\tilde{s}_N(z)| > (1+R^2)^{\frac{N}{2}} e^{\eta N}\Big\} \le \gamma_N\{\|c\| > e^{\eta N}\} = e^{-e^{2\eta N}} \sum_{k=0}^{\binom{N+m}{m}-1} \frac{e^{(2\eta N)k}}{k!};$$

hence,

$$\begin{aligned}\log \gamma_N\Big\{\sup_{z\in\overline{U}} |\tilde{s}_N(z)| > (1+R^2)^{\frac{N}{2}} e^{\eta N}\Big\} &\le -e^{2\eta N} + \log\binom{N+m}{m} + (2\eta N)\Big[\binom{N+m}{m} - 1\Big]\\ &\le -e^{\eta N} \quad \text{for } N \gg 1. \qquad \square\end{aligned}$$

Lemma 3.7. $$\int\cdots\int_{(\partial D(0,r))^m} \big|\log|\tilde{s}_N(u)|\big|\, d\sigma_r(u_1)\cdots d\sigma_r(u_m) \le CN/\delta^m$$

for some constant C outside an event of probability at most $e^{-e^N}+e^{-Q_{\kappa r,m}(N)}$.

Proof. Applying Lemma 3.6 to $U=(D(0,r))^m$, we have

$$\gamma_N\Big\{\sup_{u\in(\partial D(0,r))^m}|\tilde{s}_N(u)|>(1+mr^2)^{\frac{N}{2}}e^{\eta N}\Big\}\le\gamma_N\Big\{\sup_{u\in(\bar{D}(0,r))^m}|\tilde{s}_N(u)|>(1+mr^2)^{\frac{N}{2}}e^{\eta N}\Big\}\le e^{-e^{\eta N}}. \tag{3-21}$$

Therefore, taking $\eta=1$, outside an event of probability at most e^{-e^N} we have

$$\log^+|\tilde{s}_N(u)|\le\tfrac{1}{2}N\log(1+mr^2)+N\quad\text{on }(\partial D(0,r))^m,$$

so

$$\int\cdots\int_{(\partial D(0,r))^m}\log^+|\tilde{s}_N(u)|\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)\le\tfrac{1}{2}N\log(1+mr^2)+N. \tag{3-22}$$

Applying Lemma 3.5 to the distinguished boundary $(\partial D(0,\kappa r))^m$, we have, outside an event of probability at most $e^{-Q_{\kappa r,m}(N)}$, $\sup_{u\in(\partial D(0,\kappa r))^m}|\tilde{s}_N(u)|\ge 1$, i.e., there exists some $\eta\in(\partial D(0,\kappa r))^m$ such that $|\tilde{s}_N(\eta)|\ge 1$ and

$$\begin{aligned}0\le\log|\tilde{s}_N(\eta)|&\le\int\cdots\int_{(\partial D(0,r))^m}\log|\tilde{s}_N(u)|\prod_{i=1}^m P_r(\eta_i,u_i)\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\&=\int\cdots\int_{(\partial D(0,r))^m}\log^+|\tilde{s}_N(u)|\prod_{i=1}^m P_r(\eta_i,u_i)\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\&\qquad-\int\cdots\int_{(\partial D(0,r))^m}\log^-|\tilde{s}_N(u)|\prod_{i=1}^m P_r(\eta_i,u_i)\,d\sigma_r(u_1)\cdots d\sigma_r(u_m).\end{aligned} \tag{3-23}$$

Since for all $1\le i\le m$, $|\eta_i|=\kappa r=(1-\sqrt{\delta})r$ and $|u_i|=r$ we have $\sqrt{\delta}/2\le P_r(\eta_i,u_i)\le 2/\sqrt{\delta}$, (3-23) implies that, outside an event of probability at most $e^{-Q_{\kappa r,m}(N)}$,

$$\begin{aligned}\Big(\frac{\sqrt{\delta}}{2}\Big)^m&\int\cdots\int_{(\partial D(0,r))^m}\log^-|\tilde{s}_N(u)|\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\&\le\Big(\frac{2}{\sqrt{\delta}}\Big)^m\int\cdots\int_{(\partial D(0,r))^m}\log^+|\tilde{s}_N(u)|\,d\sigma_r(u_1)\cdots d\sigma_r(u_m).\end{aligned} \tag{3-24}$$

Combining (3-22) and (3-24), we get that, outside an event of probability at most $e^{-e^N}+e^{-Q_{\kappa r,m}(N)}$,

$$\begin{aligned}&\int\cdots\int_{(\partial D(0,r))^m}\big|\log|\tilde{s}_N(u)|\big|\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\&\quad=\int\cdots\int_{(\partial D(0,r))^m}\log^+|\tilde{s}_N(u)|\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)+\int\cdots\int_{(\partial D(0,r))^m}\log^-|\tilde{s}_N(u)|\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)\end{aligned}$$

$$\leq \Big[1+\Big(\frac{4}{\delta}\Big)^m\Big] \underset{(\partial D(0,r))^m}{\int\cdots\int} \log^+|\tilde{s}_N(u)|\, d\sigma_r(u_1)\cdots d\sigma_r(u_m)$$

$$\leq \Big[1+\Big(\frac{4}{\delta}\Big)^m\Big]\Big[\frac{1}{2}N\log(1+mr^2)+N\Big] = \frac{CN}{\delta^m}. \qquad \square$$

Lemma 3.8. $\displaystyle \max_{u\in(\partial D(0,r))^m}\bigg|\sum_{J\in\Gamma_{m,N}}\prod_{i=1}^m \frac{P_r(\xi_{i,j_i},u_i)}{N+1} - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d_m x\bigg| \leq \frac{o(1)}{\delta^{\frac{1}{2}(m+1)}}.$

Proof. For all $u\in(\partial D(0,r))^m$,

$$\begin{aligned}
&\bigg|\sum_{J\in\Gamma_{m,N}}\prod_{i=1}^m \frac{P_r(\xi_{i,j_i},u_i)}{N+1} - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d_m x\bigg| \\
&\quad = \bigg|\sum_{\tau(J)\in\Gamma_{m,N}}\prod_{i=1}^m \frac{P_r(\xi_{i,\tau(j_i)},u_i)}{N+1} - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d_m x\bigg| \\
&\quad \leq \sum_{0\leq t_1,\dots,t_m\leq p-1}\bigg|\sum_{\substack{J\in I_{t_1}\times\cdots\times I_{t_m}\\ \tau(J)\in\Gamma_{m,N}}}\prod_{i=1}^m \frac{P_r(z_{j_i},u_i)}{N+1} - \int_{H_{t_1,\dots,t_m}} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d_m x\bigg| \\
&\quad \leq \sum_{0\leq t_1,\dots,t_m\leq p-1}\bigg|\sum_{\substack{J\in I_{t_1}\times\cdots\times I_{t_m}\\ \tau(J)\in\Gamma_{m,N}}}\prod_{i=1}^m \frac{P_r(z_{j_i},u_i)}{N+1} - \int_{H_{t_1,\dots,t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d_m x\bigg| \\
&\qquad\qquad + \sum_{0\leq t_1,\dots,t_m\leq p-1}\bigg|\int_{H_{t_1,\dots,t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d_m x \\
&\qquad\qquad\qquad - \int_{H_{t_1,\dots,t_m}} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d_m x\bigg|,
\end{aligned} \tag{3-25}$$

where $H_{t_1,\dots,t_m}(N)=\bigcup_{J\in I_{t_1}\times\cdots\times I_{t_m}:\,\tau(J)\in\Gamma_{m,N}}\Big[\frac{j_1}{N+1},\frac{j_1+1}{N+1}\Big]\times\cdots\times\Big[\frac{j_m}{N+1},\frac{j_m+1}{N+1}\Big]$.

For all $0\leq t_1,\dots,t_m\leq p-1$,

$$\begin{aligned}
&\bigg|\sum_{\substack{J\in I_{t_1}\times\cdots\times I_{t_m}\\ \tau(J)\in\Gamma_{m,N}}}\prod_{i=1}^m \frac{P_r(z_{j_i},u_i)}{N+1} - \int_{H_{t_1,\dots,t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d_m x\bigg| \\
&\leq \sum_{\substack{J\in I_{t_1}\times\cdots\times I_{t_m}\\ \tau(J)\in\Gamma_{m,N}}} \int_{\left[\frac{j_1}{N+1},\frac{j_1+1}{N+1}\right]\times\cdots\times\left[\frac{j_m}{N+1},\frac{j_m+1}{N+1}\right]}\bigg|\prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i) - \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}\frac{j_i}{N+1}},u_i)\bigg|\,d_m x \\
&\leq \frac{(q+1)^m}{(N+1)^m} m \sup_{|\omega|=\kappa r,\,|u|=r}[P_r(\omega,u)]^{m-1} \sup_{|\omega|\leq\kappa r,\,|u|=r}\bigg|\frac{\partial P_r(\omega,u)}{\partial\omega}\bigg|\frac{2\pi\kappa r}{N+1}
\end{aligned}$$

$$\le \frac{C}{p^m \delta^{\frac{1}{2}(m+1)}(N+1)},$$

so

$$\sum_{0\le t_1,\dots,t_m\le p-1} \left| \sum_{\substack{J\in I_{t_1}\times\cdots\times I_{t_m}\\ \tau(J)\in\Gamma_{m,N}}} \prod_{i=1}^m \frac{P_r(z_{j_i},u_i)}{N+1} - \int_{H_{t_1,\dots,t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d_m x \right| \\ \le \frac{C}{\delta^{\frac{1}{2}(m+1)}(N+1)} = \frac{o(1)}{\delta^{\frac{1}{2}(m+1)}}. \quad (3\text{-}26)$$

To bound the second term in (3-25), we need the following statement, which can be proved in a similar way as Lemma 3.3:

$$\lim_{N\to\infty} \mathrm{Vol}_{\mathbb{R}^m}(H_{t_1,\dots,t_m}(N) \triangle H_{t_1,\dots,t_m}) = 0 \quad \text{for any } 0\le t_1,\dots,t_m\le p-1.$$

Hence,

$$\sum_{0\le t_1,\dots,t_m\le p-1} \left| \int_{H_{t_1,\dots,t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d_m x - \int_{H_{t_1,\dots,t_m}} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d_m x \right| \\ \le \sum_{0\le t_1,\dots,t_m\le p-1} \mathrm{Vol}_{R^m}(H_{t_1,\dots,t_m}(N)\triangle H_{t_1,\dots,t_m}) \Big[\sup_{|\omega|=\kappa r,|u|=r} P_r(\omega,u)\Big]^m \\ \le \sum_{0\le t_1,\dots,t_m\le p-1} o(1)\Big(\frac{2}{\sqrt{\delta}}\Big)^m = \frac{o(1)}{\delta^{\frac{1}{2}m}}. \quad (3\text{-}27)$$

This $o(1)$ may depend on p.

By (3-25), (3-26) and (3-27) the lemma is proved. □

Combining (3-20), Lemma 3.7 and Lemma 3.8, we have, outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r,m}(N)}$,

$$I \le (N+1)^m \frac{o(1)}{\delta^{\frac{1}{2}(m+1)}} \frac{CN}{\delta^m} = \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}.$$

By changing the order of integration,

$$II = (N+1)^m \int_H \int\!\cdots\!\int_{(\partial D(0,r))^m} \log|\tilde{s}_N(u)| \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)\,d_m x.$$

If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$, $\log|\tilde{s}_N(u)|$ is harmonic in u_i in a neighborhood of $\bar{D}(0,r)$ for each fixed $(u_1,\dots,\hat{u}_i,\dots,u_m)$ in $(\bar{D}(0,r))^{m-1}$. Applying the mean value theorem for harmonic functions, we get

$$II = (N+1)^m \int_H \underset{(\partial D(0,r))^m}{\int\cdots\int} \log|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, u_2, \dots, u_m)| \times \prod_{i=2}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i)\, d\sigma_r(u_2)\cdots d\sigma_r(u_m)\, d_m x$$

$$= \cdots = (N+1)^m \int_H \log|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \dots, \kappa r e^{2\pi\sqrt{-1}x_m})|\, d_m x.$$

Define

$$\Xi = \int_H \log|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \dots, \kappa r e^{2\pi\sqrt{-1}x_m})|\, d_m x, \tag{3-28}$$

which is a complex random variable. Thus we have proved:

Lemma 3.9. *If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$ then*

$$\log \prod_{J\in\Gamma_{m,N}} |\zeta_J| \le \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + (N+1)^m \Xi$$

outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r,m}(N)}$.

Replacing $\Gamma_{m,N}$ by $\Gamma_{m,N}^{(\varrho)} = \{J = (j_1, \dots, j_m) \in [0,N]^m \cap \mathbb{Z}^m : 0 \le j_{\varrho(1)} \le \cdots \le j_{\varrho(m)} \le N\}$, where ϱ can be any element in S_m, the permutation group of m letters, similar results hold and we have counterparts for Lemma 3.4 and Lemma 3.9, which we state without proof.

Lemma 3.10. *Denote the covariance matrix of the random vector $(\zeta_J^{(\varrho)} = \tilde{s}_N(\xi_J))^t_{J\in\Gamma_{m,N}^{(\varrho)}}$ by $\Sigma^{(\varrho)}$. Then*

$$\log(\det \Sigma^{(\varrho)}) = Q_{\kappa r,m}(N) + \frac{2\beta_m}{p} N^{m+1} + o(N^{m+1}).$$

For all $\varrho \in S_m$, let

$$H^{(\varrho)} = \bigcup_{0\le t_1,\dots,t_m\le p-1} H^{(\varrho)}_{t_1,\dots,t_m} := \bigcup_{0\le t_1,\dots,t_m\le p-1} \left\{ x = (x_1,\dots,x_m) \in \mathbb{R}^m : 0 \le x_{\varrho(1)} - \frac{t_{\varrho(1)}}{p} \le \cdots \le x_{\varrho(m)} - \frac{t_{\varrho(m)}}{p} \le \frac{1}{p} \right\},$$

and define the random variable

$$\Xi^{(\varrho)} = \int_{H^{(\varrho)}} \log\left|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \dots, \kappa r e^{2\pi\sqrt{-1}x_m})\right| d_m x.$$

Then:

Lemma 3.11. *If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$ then*

$$\log \prod_{J\in\Gamma_{m,N}^{(\varrho)}} |\zeta_J^{(\varrho)}| \le \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + (N+1)^m \Xi^{(\varrho)}$$

outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r,m}(N)}$.

The last ingredient we need to prove the upper bound is the following lemma:

Lemma 3.12 [Nishry 2010, Lemma 4.6]. *Let s, $t>0$ and $N\in\mathbb{N}^+$ be such that $\log(t^N/s)\ge N$; then*

$$\mathrm{Vol}_{\mathbb{R}^N}\left\{(r_1,\dots,r_N)\in\mathbb{R}^N : 0\le r_j\le t \ \text{and}\ \prod_{j=1}^{N} r_j\le s\right\}\le \frac{s}{(N-1)!}\log^N\left(\frac{t^N}{s}\right).$$

Proof of the upper bound in Theorem 0.1. If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$ then, by the mean value property of pluriharmonic functions,

$$\begin{aligned}\sum_{\varrho\in S_m}\Xi^{(\varrho)} &= \sum_{\varrho\in S_m}\int_{H^{(\varrho)}}\log\bigl|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1},\dots,\kappa r e^{2\pi\sqrt{-1}x_m})\bigr|\,d_m x\\ &= \int_{\bigcup_{\varrho\in S_m}H^{(\varrho)}}\log\bigl|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1},\dots,\kappa r e^{2\pi\sqrt{-1}x_m})\bigr|\,d_m x\\ &= \int_0^1\!\!\cdots\!\int_0^1\log\bigl|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1},\dots,\kappa r e^{2\pi\sqrt{-1}x_m})\bigr|\,dx_1\cdots dx_m\\ &= \int\!\!\cdots\!\int_{(\partial D(0,\kappa r))^m}\log|\tilde{s}_N(\omega_1,\dots,\omega_m)|\,d\sigma_{\kappa r}(\omega_1)\cdots d\sigma_{\kappa r}(\omega_m)\\ &= \log|\tilde{s}_N(0,\dots,0)|\\ &= \log|c_{(0,\dots,0)}|;\end{aligned}$$

the second equality holds because, for distinct $\varrho_1,\varrho_2\in S_m$, $H^{(\varrho_1)}\cap H^{(\varrho_2)}$ is of m-dimensional Lebesgue measure zero. Then,

$$\begin{aligned}P_{0,m}(r,N) &= \gamma_N\{0\notin\tilde{s}_N((\bar{D}(0,r))^m)\}\\ &= \gamma_N\{(\log|c_{(0,\dots,0)}|>2m!\log N)\cap(0\notin\tilde{s}_N((\bar{D}(0,r))^m))\}\\ &\qquad\qquad + \gamma_N\{(\log|c_{(0,\dots,0)}|\le 2m!\log N)\cap(0\notin\tilde{s}_N((\bar{D}(0,r))^m))\}\\ &\le \gamma_N(|c_{(0,\dots,0)}|>N^{2m!})+\gamma_N\left\{\left(\sum_{\varrho\in S_m}\Xi^{(\varrho)}\le 2m!\log N\right)\cap(0\notin\tilde{s}_N((\bar{D}(0,r))^m))\right\}\\ &\le e^{-N^{4m!}}+\gamma_N\left\{\bigcup_{\varrho\in S_m}(\Xi^{(\varrho)}\le 2\log N)\cap(0\notin\tilde{s}_N((\bar{D}(0,r))^m))\right\}\\ &\le e^{-N^{4m!}}+\sum_{\varrho\in S_m}\gamma_N\{(\Xi^{(\varrho)}\le 2\log N)\cap(0\notin\tilde{s}_N((\bar{D}(0,r))^m))\}.\end{aligned}$$

Lemma 3.9 implies

$$\begin{aligned}\gamma_N\big\{(\Xi \le 2\log N)\cap\big(0\notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\big)\big\}\\ &\le e^{-e^N}+e^{-Q_{\kappa r,m}(N)}+\gamma_N\Big\{\log\prod_{J\in\Gamma_{m,N}}|\zeta_J|\le \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\Big\}\\ &= e^{-e^N}+e^{-Q_{\kappa r,m}(N)}+\gamma_N\Big\{\prod_{J\in\Gamma_{m,N}}|\zeta_J|\le \exp\Big\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\Big\}\Big\}.\end{aligned}$$

Define

$$\mathscr{E}_{m,N}=\Big\{\zeta=(\zeta_J)_{J\in\Gamma_{m,N}}\in\mathbb{C}^{\binom{N+m}{m}}:\prod_{J\in\Gamma_{m,N}}|\zeta_J|\le\exp\Big\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\Big\}\Big\},$$

and

$$\mathscr{F}_{m,N}=\big\{\zeta=(\zeta_J)_{J\in\Gamma_{m,N}}\in\mathscr{E}_{m,N}:|\zeta_J|\le(2+2mr^2)^{\frac{N}{2}}\ \ \forall J\in\Gamma_{m,N}\big\}\subset\mathscr{E}_{m,N},$$

which can both be treated as subsets in $\mathbb{C}^{\binom{N+m}{m}}$ and events in the probability space $\big(H^0(\mathbb{CP}^m,\mathcal{O}(N)),\gamma_N\big)$. Thus,

$$\begin{aligned}\gamma_N\big\{(\Xi \le 2\log N)\cap\big(0\notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\big)\big\}\\ &\le e^{-e^N}+e^{-Q_{\kappa r,m}(N)}+\gamma_N(\mathscr{E}_{m,N})\\ &\le e^{-e^N}+e^{-Q_{\kappa r,m}(N)}+\gamma_N(\mathscr{E}_{m,N}\setminus\mathscr{F}_{m,N})+\gamma_N(\mathscr{F}_{m,N})\end{aligned}\tag{3-29}$$

and

$$\begin{aligned}\gamma_N(\mathscr{E}_{m,N}\setminus\mathscr{F}_{m,N}) &\le \gamma_N\big\{|\zeta_J|>(2+2mr^2)^{\frac{N}{2}}\ \text{ for some }\ J\in\Gamma_{m,N}\big\}\\ &\le\gamma_N\big\{\sup_{\omega\in(\partial D(0,\kappa r))^m}|\tilde{s}_N(\omega)|>(2+2mr^2)^{\frac{N}{2}}\big\}\\ &\le\gamma_N\big\{\sup_{\omega\in(\bar{D}(0,r))^m}|\tilde{s}_N(\omega)|>(1+mr^2)^{\frac{N}{2}}2^{\frac{N}{2}}\big\}\\ &\le e^{-2^{\frac{N}{2}}},\end{aligned}\tag{3-30}$$

where the last inequality is due to Lemma 3.6. Then Lemma 3.4 gives

$$\begin{aligned}\gamma_N(\mathscr{F}_{m,N}) &= \frac{1}{\pi^{\binom{N+m}{m}}\det\Sigma}\int_{\mathscr{F}_{m,N}}e^{-\zeta^*\Sigma^{-1}\zeta}\,d_{2\binom{N+m}{m}}\zeta\\ &\le\exp\Big\{-\Big[Q_{\kappa r,m}(N)+\frac{2\beta_m}{p}N^{m+1}\Big]+o(N^{m+1})\Big\}\pi^{-\binom{N+m}{m}}\operatorname{Vol}_{\mathbb{C}^{\binom{N+m}{m}}}(\mathscr{F}_{m,N}).\end{aligned}$$

Change into polar coordinates and note that

$$\begin{aligned}&\mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathscr{F}_{m,N})\\&=\mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}\left\{(x_J)_{J\in\Gamma_{m,N}}\in\left[0,(2+2mr^2)^{\frac{N}{2}}\right]^{\binom{N+m}{m}}:\prod_{J\in\Gamma_{m,N}}x_J\le\exp\left\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\right\}\right\};\end{aligned}$$

then

$$\begin{aligned}\gamma_N(\mathscr{F}_{m,N})&\le 2^{\binom{N+m}{m}}\exp\left\{-\left[Q_{\kappa r,m}(N)+\frac{2\beta_m}{p}N^{m+1}\right]+o(N^{m+1})\right\}\\&\qquad\times\exp\left\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\right\}\mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathscr{F}_{m,N})\\&=2^{\binom{N+m}{m}}\exp\left\{-\left[Q_{\kappa r,m}(N)+\frac{2\beta_m}{p}N^{m+1}\right]+\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\right\}\mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathscr{F}_{m,N}).\end{aligned}$$

Since $\binom{N+m}{m}\frac{1}{2}N\log(2+2mr^2)-[o(N^{m+1})/\delta^{\frac{3}{2}m+\frac{1}{2}}+2(N+1)^m\log N]>\binom{N+m}{m}$ for N large (up to now p, δ are constants), we can apply Lemma 3.12 and get:

$$\begin{aligned}&\mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathscr{F}_{m,N})\\&\quad\le\frac{\exp\{o(N^{m+1})/\delta^{\frac{3}{2}m+\frac{1}{2}}+2(N+1)^m\log N\}}{\left[\binom{N+m}{m}-1\right]!}\left\{\binom{N+m}{m}\frac{N}{2}\log(2+2mr^2)\right.\\&\qquad\qquad\left.-\left[\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\right]\right\}^{\binom{N+m}{m}}\\&\quad\le\frac{\exp\{o(N^{m+1})/\delta^{\frac{3}{2}m+\frac{1}{2}}+2(N+1)^m\log N\}}{2^{\binom{N+m}{m}}\left[\binom{N+m}{m}-1\right]!}\left[N\binom{N+m}{m}\log(2+2mr^2)\right]^{\binom{N+m}{m}};\end{aligned}$$

then,

$$\begin{aligned}\gamma_N(\mathscr{F}_{m,N})\le\frac{\exp\{o(N^{m+1})/\delta^{\frac{3}{2}m+\frac{1}{2}}+2(N+1)^m\log N-[Q_{\kappa r,m}(N)+\frac{2\beta_m}{p}N^{m+1}]\}}{\left[\binom{N+m}{m}-1\right]!}\\\times\left[N\binom{N+m}{m}\log(2+2mr^2)\right]^{\binom{N+m}{m}},\end{aligned}$$

so

$$\begin{aligned}\log\gamma_N(\mathscr{F}_{m,N})&\le\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N-\left[Q_{\kappa r,m}(N)+\frac{2\beta_m}{p}N^{m+1}\right]\\&\qquad+\binom{N+m}{m}\log\left[N\binom{N+m}{m}\log(2+2mr^2)\right]-\log\left[\binom{N+m}{m}-1\right]!\\&=-Q_{\kappa r,m}(N)-\frac{2\beta_m}{p}N^{m+1}+\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}.\end{aligned}\tag{3-31}$$

By Lemma 2.2, (3-29), (3-30) and (3-31),

$$\begin{aligned}&\gamma_N\{(\Xi \le 2\log N)\cap\big(0\notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\big)\}\\&\quad\le e^{-e^N}+e^{-Q_{\kappa r,m}(N)}+e^{-2^{\frac{N}{2}}}+\exp\bigg\{-Q_{\kappa r,m}(N)-\frac{2\beta_m}{p}N^{m+1}+\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\bigg\}\\&\quad\le \exp\bigg\{-\min\bigg\{\frac{2m\log(\kappa r)}{(m+1)!}+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k},\ \frac{2m\log(\kappa r)}{(m+1)!}+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}+\frac{2\beta_m}{p}\bigg\}N^{m+1}+\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\bigg\}.\end{aligned}$$

Similarly, for all $\varrho\in S_m$,

$$\begin{aligned}&\gamma_N\{(\Xi^{(\varrho)} \le 2\log N)\cap\big(0\notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\big)\}\\&\quad\le \exp\bigg\{-\min\bigg\{\frac{2m\log(\kappa r)}{(m+1)!}+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k},\ \frac{2m\log(\kappa r)}{(m+1)!}+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}+\frac{2\beta_m}{p}\bigg\}N^{m+1}+\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\bigg\};\end{aligned}$$

thus,

$$\begin{aligned}&P_{0,m}(r,N)\\&\quad\le e^{-N^{4m!}}\\&\qquad+m!\exp\bigg\{-\min\bigg\{\frac{2m\log(\kappa r)}{(m+1)!}+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k},\ \frac{2m\log(\kappa r)}{(m+1)!}+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}+\frac{2\beta_m}{p}\bigg\}N^{m+1}+\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\bigg\}\\&\quad=\exp\bigg\{-\min\bigg\{\frac{2m\log(\kappa r)}{(m+1)!}+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k},\ \frac{2m\log(\kappa r)}{(m+1)!}+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}+\frac{2\beta_m}{p}\bigg\}N^{m+1}+\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\bigg\},\end{aligned}$$

so

$$\begin{aligned}&\log P_{0,m}(r,N)\\&\qquad\le -\min\bigg\{\frac{2m\log(\kappa r)}{(m+1)!}+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k},\ \frac{2m\log(\kappa r)}{(m+1)!}+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}+\frac{2\beta_m}{p}\bigg\}N^{m+1}+\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}},\end{aligned}$$

and thus

$$\limsup_{N\to\infty}\frac{\log P_{0,m}(r,N)}{N^{m+1}}\le -\min\bigg\{\frac{2m\log(\kappa r)}{(m+1)!}+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k},\ \frac{2m\log(\kappa r)}{(m+1)!}+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}+\frac{2\beta_m}{p}\bigg\}.$$

Let $p\to\infty$; then

$$\limsup_{N\to\infty}\frac{\log P_{0,m}(r,N)}{N^{m+1}}\le -\bigg[\frac{2m\log(\kappa r)}{(m+1)!}+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}\bigg].$$

Let $\delta\to 0+$; then $\kappa=1-\sqrt{\delta}\to 1$, so

$$\limsup_{N\to\infty}\frac{\log P_{0,m}(r,N)}{N^{m+1}}\le -\bigg[\frac{2m\log r}{(m+1)!}+\frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}\bigg].$$

Hence,

$$\log P_{0,m}(r,N) \le -\left[\frac{2m\log r}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}\right]N^{m+1} + o(N^{m+1}).$$

Thus, Theorem 0.1 is proved. □

4. Proof of Theorem 0.2

The proof of Theorem 0.2 is quite similar to that of Theorem 0.1. We only need to make some slight modifications in picking "determining exponents" and "sampling points".

Lower bound.

Definition 4.1.

$$\Lambda_{m,N}(r) := \left\{K \in \Lambda_{m,N} : \binom{N}{K} r^{2|K|} \ge 1\right\} \subset \Lambda_{m,N},$$
$$R_{r,m}(N) := \sum_{K\in\Lambda_{m,N}(r)} \log\left[\binom{N}{K} r^{2|K|}\right].$$

Lemma 4.2.

$$\log P_{0,m}(r,N) \ge -R_{r,m}(N) + o(N^{m+1}).$$

Proof. Consider the following event $\Omega_{r,m,N}$:

(i) $|c_{(0,\dots,0)}| \ge \sqrt{N}$,

(ii) $|c_K| \le \dfrac{1}{2\sqrt{N}\sqrt{\binom{N}{K}}r^{|K|}\binom{|K|+m-1}{m-1}}, \quad K \in \Lambda_{m,N}(r)\backslash\{(0,\dots,0)\}$,

(iii) $|c_K| \le \dfrac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}}, \quad K \in \Lambda_{m,N}\backslash\Lambda_{m,N}(r)$.

Then, when $\Omega_{r,m,N}$ occurs, for all $z \in (\bar{D}(0,r))^m$,

$$\begin{aligned}
|\tilde{s}_N(z)| &\ge \sqrt{N} - \sum_{K\in\Lambda_{m,N}(r)\backslash\{(0,\dots,0)\}} \frac{\sqrt{\binom{N}{K}}r^{|K|}}{2\sqrt{N}\sqrt{\binom{N}{K}}r^{|K|}\binom{|K|+m-1}{m-1}} - \sum_{K\in\Lambda_{m,N}\backslash\Lambda_{m,N}(r)} \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}}\\
&= \sqrt{N} - \sum_{K\in\Lambda_{m,N}\backslash\{(0,\dots,0)\}} \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}}\\
&= \sqrt{N} - \sum_{k=1}^{N}\frac{1}{2\sqrt{N}}\\
&= \tfrac{1}{2}\sqrt{N} > 0.
\end{aligned}$$

Thus,

$$
\begin{aligned}
P_{0,m}(r,N) &\geq \gamma_N(\Omega_{r,m,N}) \\
&= \gamma_N(|c_{(0,\dots,0)}| \geq \sqrt{N}) \prod_{K\in\Lambda_{m,N}(r)\setminus\{(0,\dots,0)\}} \gamma_N\left(|c_K| \leq \frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}} r^{|K|}\binom{|K|+m-1}{m-1}}\right) \\
&\qquad \times \prod_{K\in\Lambda_{m,N}\setminus\Lambda_{m,N}(r)} \gamma_N\left(|c_K| \leq \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}}\right) \\
&\geq e^{-N} \prod_{K\in\Lambda_{m,N}(r)\setminus\{(0,\dots,0)\}} \frac{1}{8N\binom{N}{K} r^{2|K|}\binom{|K|+m-1}{m-1}^2} \prod_{K\in\Lambda_{m,N}\setminus\Lambda_{m,N}(r)} \frac{1}{8N\binom{|K|+m-1}{m-1}^2},
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\log P_{0,m}(r,N) \\
&\quad \geq -N - \sum_{K\in\Lambda_{m,N}(r)\setminus\{(0,\dots,0)\}} \log\left[\binom{N}{K} r^{2|K|}\right] - \sum_{K\in\Lambda_{m,N}(r)\setminus\{(0,\dots,0)\}} \log\left[8N\binom{|K|+m-1}{m-1}^2\right] \\
&\quad = -\sum_{K\in\Lambda_{m,N}(r)\setminus\{(0,\dots,0)\}} \log\left[\binom{N}{K} r^{2|K|}\right] + o(N^{m+1}) \\
&\quad = -R_{r,m}(N) + o(N^{m+1}). \qquad \square
\end{aligned}
$$

Upper bound. For some $\alpha \in (0,1]$, we can define the index sets $\Lambda_{m,\lfloor\alpha N\rfloor}$ and $\Gamma_{m,\lfloor\alpha N\rfloor}$, and the $\binom{\lfloor\alpha N\rfloor+m}{m} \times \binom{\lfloor\alpha N\rfloor+m}{m}$ matrix

$$W_{m,\lfloor\alpha N\rfloor}(\xi) = (\xi_J^K)_{J\in\Gamma_{m,\lfloor\alpha N\rfloor},\ K\in\Lambda_{m,\lfloor\alpha N\rfloor}}.$$

We also assign the values of the variables $(\xi_{i,j})_{0\leq i\leq m,\ 0\leq j\leq\lfloor\alpha N\rfloor}$ by the points on $\partial D(0,\kappa r)$ in a way similar to in Section 3, except that we replace N by $\lfloor\alpha N\rfloor$. Then we have the following lemma:

Lemma 4.3. $$\log|\det W_{m,\lfloor\alpha N\rfloor}(\xi)| = m\binom{\lfloor\alpha N\rfloor+m}{m+1}\log(\kappa r) + \frac{\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1} + o(N^{m+1}).$$

The word $\zeta = (\zeta_J)^t_{J\in\Gamma_{m,\lfloor\alpha N\rfloor}} = (\tilde{s}_N(\xi_J))^t_{J\in\Gamma_{m,\lfloor\alpha N\rfloor}}$ is a dimension-$\binom{\lfloor\alpha N\rfloor+m}{m}$ mean zero complex Gaussian random vector with covariance matrix

$$\Sigma = V_{m,N,\alpha}(\xi) V^*_{m,N,\alpha}(\xi),$$

where $V_{m,N,\alpha}(\xi) = \left(\sqrt{\binom{N}{K}}\xi_J^K\right)_{J\in\Gamma_{m,\lfloor\alpha N\rfloor},\ K\in\Lambda_{m,N}}$ is a $\binom{\lfloor\alpha N\rfloor+m}{m} \times \binom{N+m}{m}$ matrix.

Definition 4.4. $$Q_{r,m,\alpha}(N) := \sum_{K\in\Lambda_{m,\lfloor\alpha N\rfloor}} \log\left[\binom{N}{K} r^{2|K|}\right].$$

Lemma 4.5. $$\log\det\Sigma \geq Q_{\kappa r,m,\alpha}(N) + \frac{2\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1} + o(N^{m+1}).$$

Proof. By the Cauchy–Binet identity, summing over the $\binom{\lfloor\alpha N\rfloor+m}{m}\times\binom{\lfloor\alpha N\rfloor+m}{m}$ minors of $V_{m,N,\alpha}(\xi)$,

$$\det\Sigma=\sum_M|\det M|^2\geq\left|\det\left(\sqrt{\binom{N}{K}}\xi_J^K\right)_{J\in\Gamma_{m,\lfloor\alpha N\rfloor},\ K\in\Lambda_{m,\lfloor\alpha N\rfloor}}\right|^2=\prod_{K\in\Lambda_{m,\lfloor\alpha N\rfloor}}\binom{N}{K}|\det W_{m,\lfloor\alpha N\rfloor}(\xi)|^2,$$

so

$$\begin{aligned}\log\det\Sigma&\geq\sum_{K\in\Lambda_{m,\lfloor\alpha N\rfloor}}\log\binom{N}{K}+2m\binom{\lfloor\alpha N\rfloor+m}{m+1}\log(\kappa r)+\frac{2\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1}+o(N^{m+1})\\&=\sum_{K\in\Lambda_{m,\lfloor\alpha N\rfloor}}\log\left[\binom{N}{K}(\kappa r)^{2|K|}\right]+\frac{2\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1}+o(N^{m+1})\\&=Q_{\kappa r,m,\alpha}(N)+\frac{2\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1}+o(N^{m+1}).\end{aligned}$$ □

The following lemma is a counterpart of Lemma 3.9. The proof is similar.

Lemma 4.6. *If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$ then*

$$\log\prod_{J\in\Gamma_{m,\lfloor\alpha N\rfloor}}|\zeta_J|\leq\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+(\lfloor\alpha N\rfloor+1)^m\,\Xi$$

outside an event of probability at most $e^{-e^N}+e^{-R_{\kappa r,m}(N)}$, where the complex random variable Ξ is defined in (3-28).

By the same trick of permutation as in Section 3, we can get an upper bound estimate for $P_{0,m}(r,N)$:

$$\begin{aligned}P_{0,m}(r,N)\leq e^{-N^{4m!}}+m!\Big\{&e^{-e^N}+e^{-R_{\kappa r,m}(N)}+e^{-2^{\frac{N}{2}}}\\&+\exp\left[-Q_{\kappa r,m,\alpha}(N)-\frac{2\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1}+\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\right]\Big\}.\end{aligned}\tag{4-1}$$

Punch line of the proof. In order to prove Theorem 0.2, it suffices to compute $R_{r,m}(N)$ and $Q_{r,m,\alpha}(N)$ asymptotically. We follow the same idea as in Lemma 2.2.

The scaled lattice $(1/N)\Lambda_{m,N}(r)$ corresponds to the set

$$\{x=(x_1,\dots,x_m)\in\Sigma_m:E_r(x)\geq 0\}$$

and $(1/N)\Lambda_{r,m,\alpha}(N)$ corresponds to the set

$$\left\{x=(x_1,\dots,x_m)\in\mathbb{R}^{m+}:\sum_{i=1}^m x_i\leq\alpha\leq 1\right\}.$$

So we have

$$R_{r,m}(N)=\sum_{K\in\Lambda_{m,N}(r)}\log\left[\binom{N}{K}r^{2|K|}\right]=N^{m+1}\int_{\substack{x\in\Sigma_m\\E_r(x)\geq 0}}E_r(x)\,d_mx+o(N^{m+1}),\tag{4-2}$$

$$Q_{r,m,\alpha}(N)=\sum_{K\in\Lambda_{m,\lfloor\alpha N\rfloor}}\log\left[\binom{N}{K}r^{2|K|}\right]=N^{m+1}\int_{\substack{x\in\mathbb{R}^{m+}\\\sum_{i=1}^m x_i\leq\alpha}}E_r(x)\,d_mx+o(N^{m+1}).\tag{4-3}$$

Moreover, if we go through the proof of Lemma 2.2, we find that the $o(N^{m+1})$ terms in (4-2) and (4-3) are uniform if $r\leq c$ for some constant $c>0$, which implies that, when r is replaced by $\kappa r=(1-\sqrt{\delta})r$, the remainder won't depend on δ.

Proof of Theorem 0.2. The lower bound proof is already implied by Lemma 4.2 and (4-2). To prove the upper bound, by (4-1) and (4-3),

$$\log P_{0,m}(r,N)$$
$$\leq -N^{m+1}\min\left\{\int_{\substack{x\in\Sigma_m\\E_{\kappa r}(x)\geq 0}}E_{\kappa r}(x)\,d_mx,\ \int_{\substack{x\in\mathbb{R}^{m+}\\\sum_{i=1}^m x_i\leq\alpha}}E_{\kappa r}(x)\,d_mx+\frac{2\beta_m\alpha^{m+1}}{p}\right\}+\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}.$$

Similarly as in Section 3, we obtain

$$\begin{aligned}\log P_{0,m}(r,N)&\leq -N^{m+1}\min\left\{\int_{\substack{x\in\Sigma_m\\E_r(x)\geq 0}}E_r(x)\,d_mx,\ \int_{\substack{x\in\mathbb{R}^{m+}\\\sum_{i=1}^m x_i\leq\alpha}}E_r(x)\,d_mx\right\}+o(N^{m+1})\\&=-N^{m+1}\int_{\substack{x\in\mathbb{R}^{m+}\\\sum_{i=1}^m x_i\leq\alpha}}E_r(x)\,d_mx+o(N^{m+1}).\end{aligned}$$

Now we must find a proper $\alpha_0=\alpha_0(r,m)\in(0,1]$ which maximizes $\int_{x\in\mathbb{R}^{m+},\sum_{i=1}^m x_i\leq\alpha}E_r(x)\,d_mx$. For this purpose, we consider the function defined on $(0,1]$ by

$$\Upsilon(\alpha):=\int_{x\in\mathbb{R}^{m+}:\sum_{i=1}^m x_i\leq\alpha}E_r(x)\,d_mx.$$

Then

$$\begin{aligned}\Upsilon(\alpha)&=2m\log r\int_{\substack{x\in\mathbb{R}^{m+}\\\sum_{i=1}^m x_i\leq\alpha}}x_1\,d_mx-m\int_{\substack{x\in\mathbb{R}^{m+}\\\sum_{i=1}^m x_i\leq\alpha}}x_1\log x_1\,d_mx\\&\qquad-\int_{\substack{x\in\mathbb{R}^{m+}\\\sum_{i=1}^m x_i\leq\alpha}}\left(1-\sum_{i=1}^m x_i\right)\log\left(1-\sum_{i=1}^m x_i\right)d_mx\\&=2m\log r\frac{\alpha^{m+1}}{(m+1)!}-m\frac{\alpha^{m+1}}{(m+1)!}\left[\log\alpha-\sum_{k=2}^{m+1}\frac{1}{k}\right]-\frac{1}{(m-1)!}\int_0^\alpha(1-x)x^{m-1}\log(1-x)\,dx,\end{aligned}$$

$$\Upsilon'(\alpha)=\frac{\alpha^{m-1}}{(m-1)!}\left\{\left(2\log r+\sum_{k=2}^m\frac{1}{k}\right)\alpha-[\alpha\log\alpha+(1-\alpha)\log(1-\alpha)]\right\},$$

where we take $\sum_{k=2}^{m} 1/k = 0$ when $m = 1$. So, if $2\log r + \sum_{k=2}^{m} 1/k \geq 0$, $\Upsilon'(\alpha) \geq 0$ over $(0, 1]$, thus $\max_{(0,1]} \Upsilon = \Upsilon(1)$ and therefore $\alpha_0 = 1$.

If $2\log r + \sum_{k=2}^{m} 1/k < 0$, $\big(2\log r + \sum_{k=2}^{m} 1/k\big)\alpha = \alpha \log \alpha + (1-\alpha)\log(1-\alpha)$ has a unique nonzero root $\alpha_0 \in (0, 1)$, and

$$\max_{(0,1]} \Upsilon = \Upsilon(\alpha_0) = \int_{\substack{x\in\mathbb{R}^{m+} \\ \sum_{i=1}^{m} x_i \leq \alpha_0}} E_r(x)\, d_m x = \frac{1}{(m+1)!}\left[(1-\alpha_0^m)\log(1-\alpha_0) + \sum_{k=1}^{m} \frac{\alpha_0^k}{k}\right]. \tag{4-4}$$

This concludes the proof. □

Remark 4.7. The proofs of Theorems 0.1 and 0.2 also work for a general polydisc $\prod_{i=1}^{m} D(0, r_i)$. For example, if $r = (r_1, \ldots, r_m) \in [1, \infty)^m$, the function E_r in Theorem 0.1 would be

$$E_r(x) = 2\sum_{i=1}^{m} x_i \log r_i - \left[\sum_{i=1}^{m} x_i \log x_i + \left(1 - \sum_{i=1}^{m} x_i\right)\log\left(1 - \sum_{i=1}^{m} x_i\right)\right]$$

and $\int_{\Sigma_m} E_r(x)\, d_m x$ would equal $(2/(m+1)!)\sum_{i=1}^{m} \log r_i + (1/m!)\sum_{k=2}^{m+1} 1/k$.

5. Hole probability of SU(2) polynomials

Proof of Corollary 0.4. When $r \geq 1$, $\alpha_0 = 1$. The result follows from Theorem 0.1.

When $0 < r < 1$, for $x \in \mathbb{R}^+$,

$$E_r(x) = 2x\log r - [x\log x + (1-x)\log(1-x)] \geq 0 \quad \Longleftrightarrow \quad 0 \leq x \leq \alpha_0.$$

By Theorem 0.2,

$$\log P_{0,1}(r, N) = -N^2 \int_0^{\alpha_0} E_r(x)\, dx + o(N^2),$$

where the value of the integral in the corollary is due to (4-4) and the fact that

$$2\alpha_0 \log r = \alpha_0 \log \alpha_0 + (1-\alpha_0)\log(1-\alpha_0). \qquad \square$$

Proof of Theorem 0.5. As ∂U is a Jordan curve, by Carathéodory's theorem ϕ can be extended to a homeomorphism $\bar{D}(0, 1) \to \bar{U}$. We still use ϕ to denote the extended map. Thus, $\tilde{s}_N(z) = \sum_{k=0}^{N} c_k \binom{N}{k}^{1/2} z^k$ is nonvanishing over $\bar{U}$ if and only if $t_N(\omega) := \sum_{k=0}^{N} c_k \binom{N}{k}^{1/2} (\phi(\omega))^k$ is nonvanishing over $\bar{D}(0, 1)$, where $t_N \in \mathcal{O}(D(0, 1)) \cap \mathscr{C}(\bar{D}(0, 1))$.

Since

$$\begin{bmatrix} t_N(0) \\ t_N'(0) \\ \vdots \\ t_N^{(N)}(0) \end{bmatrix} = A \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix},$$

where A is an $(N+1)\times(N+1)$ lower triangular matrix with diagonal entries $\Big\{k!\sqrt{\binom{N}{k}}(\phi'(0))^k\Big\}_{0\le k\le N}$, $(t_N(0)\dots\ t_N^{(N)}(0))^t$ is Gaussian with covariance matrix AA^*. Then

$$\det(AA^*)=|\det A|^2=\prod_{k=0}^{N}\Big[k!^2\binom{N}{k}|\phi'(0)|^{2k}\Big]\neq 0 \tag{5-1}$$

because ϕ is a biholomorphism.

We again define $\kappa=1-\sqrt{\delta}$. Then, if $\sup_{\partial D(0,\kappa)}|t_N|<1$, for $0\le k\le N$,

$$|t_N^{(k)}(0)|=\left|\frac{k!}{2\pi\sqrt{-1}}\int_{\partial D(0,\kappa)}\frac{t_N(u)}{u^{k+1}}\,du\right|\le\frac{k!}{\kappa^k}.$$

Therefore,

$$\begin{aligned}\gamma_N(\sup_{\partial D(0,\kappa)}|t_N|<1)&\le\gamma_N\Big\{(t_N(0),\dots,t_N^{(N)}(0))\in\prod_{k=0}^{N}\bar D\Big(0,\frac{k!}{\kappa^k}\Big)\Big\}\\&=\frac{1}{\pi^{N+1}\det(AA^*)}\int_{\prod_{k=0}^{N}\bar D(0,k!/\kappa^k)}\exp\{-\eta^*(AA^*)^{-1}\eta\}\,d_{2(N+1)}\eta\\&\le\frac{\pi^{N+1}\prod_{k=0}^{N}(k!/\kappa^k)^2}{\pi^{N+1}\det(AA^*)}.\end{aligned}$$

By (5-1),

$$\begin{aligned}\gamma_N\Big(\sup_{\partial D(0,\kappa)}|t_N|<1\Big)&\le\frac{\prod_{k=0}^{N}(k!/\kappa^k)^2}{\prod_{k=0}^{N}\big[k!^2\binom{N}{k}|\phi'(0)|^{2k}\big]}\\&=\Big\{\prod_{k=0}^{N}\Big[\binom{N}{k}(\kappa|\phi'(0)|)^{2k}\Big]\Big\}^{-1}\\&=\exp\{-Q_{\kappa|\phi'(0)|,1}(N)\}\\&=\exp\big\{-\big(\log|\phi'(0)|+\log\kappa+\tfrac12\big)N^2+o(N^2)\big\},\end{aligned}$$

where the last equality is due to Lemma 2.2.

Similarly as in Lemma 3.9, we can show that if $t_N|_{\bar D(0,1)}\neq 0$ then, outside an event of probability at most $e^{-e^N}+\exp\{-Q_{\kappa|\phi'(0)|,1}(N)\}=\exp\big\{-\big(\log|\phi'(0)|+\log\kappa+\frac12\big)N^2+o(N^2)\big\}$,

$$\log\prod_{j=0}^{N}|t_N(z_j)|\le\frac{o(N^2)}{\delta^2}+(N+1)\log|c_0|,$$

where $z_j=\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}}$, $0\le j\le N$.

Now, $(t_N(z_0)\ \cdots\ t_N(z_N))^t$ is complex Gaussian with covariance matrix

$$\Sigma = \big(\mathbb{E}_N(t_N(z_j)\overline{t_N(z_j)})\big)_{0\le i,j\le N} = \bigg(\sum_{k=0}^{N}\binom{N}{k}(\phi(z_i))^k(\overline{\phi(z_j)})^k\bigg)_{0\le i,j\le N}$$

$$= \begin{bmatrix} \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_0) & \cdots & \sqrt{\binom{N}{N}}(\phi(z_0))^N \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_N) & \cdots & \sqrt{\binom{N}{N}}(\phi(z_N))^N \end{bmatrix} \begin{bmatrix} \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_0) & \cdots & \sqrt{\binom{N}{N}}(\phi(z_0))^N \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_N) & \cdots & \sqrt{\binom{N}{N}}(\phi(z_N))^N \end{bmatrix}^*$$

and

$$\det\Sigma = \prod_{k=0}^{N}\binom{N}{k}\prod_{0\le i<j\le N}|\phi(z_i)-\phi(z_j)|^2,$$

so

$$\log\det\Sigma = \sum_{k=0}^{N}\log\binom{N}{k} + 2\sum_{0\le i<j\le N}\log|\phi(z_i)-\phi(z_j)|. \tag{5-2}$$

Next we will show that

$$2\sum_{0\le i<j\le N}\log|\phi(z_i)-\phi(z_j)| = N^2\iint_{(\partial D(0,\kappa))^2}\log|\phi(u_1)-\phi(u_2)|\,d\sigma_\kappa(u_1)\,d\sigma_\kappa(u_2)+o_\delta(N^2), \tag{5-3}$$

where $o_\delta(N^2)$ denotes a lower-order term depending on δ.

Since

$$2\sum_{0\le i<j\le N}\log|\phi(z_i)-\phi(z_j)| = 2(N+1)^2\sum_{0\le i<j\le N}\frac{1}{(N+1)^2}\log\Big|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}})-\phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})\Big|$$

and

$$\iint_{(\partial D(0,\kappa))^2}\log|\phi(u_1)-\phi(u_2)|\,d\sigma_\kappa(u_1)\,d\sigma_\kappa(u_2) = \int_0^1\int_0^1\log\Big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\Big|\,dx\,dy$$
$$= 2\iint_{0\le x\le y\le 1}\log\Big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\Big|\,dx\,dy,$$

it suffices to show that

$$\bigg|\sum_{0\le i<j\le N}\frac{1}{(N+1)^2}\log\Big|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}})-\phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})\Big| - \iint_{0\le x\le y\le 1}\log\Big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\Big|\,dx\,dy\bigg| = o_\delta(1).$$

Since ϕ is a biholomorphism, we set

$$\inf_{\bar{D}(0,\kappa)} |\phi'| = a(\delta) > 0.$$

And, by Cauchy's inequality, we have

$$\sup_{\bar{D}(0,\kappa)} |\phi'| \leq O(\delta^{-1}).$$

For each N, define

$$\Delta(N) = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i < j \leq N\},$$

the "far from diagonal" indices $FD(N)$ to be the set of those $(i, j) \in \Delta(N)$ such that

$$\begin{cases} \lfloor\sqrt{N+1}\rfloor + i \leq j \leq N - \lfloor\sqrt{N+1}\rfloor + i & \text{if } 0 \leq i \leq \lfloor\sqrt{N+1}\rfloor, \\ \lfloor\sqrt{N+1}\rfloor + i \leq j \leq N & \text{if } \lfloor\sqrt{N+1}\rfloor < i \leq N - \lfloor\sqrt{N+1}\rfloor, \\ j \in \varnothing & \text{if } i > N - \lfloor\sqrt{N+1}\rfloor, \end{cases}$$

with

$$\mathcal{FD}(N) = \bigcup_{(i,j)\in FD(N)} \mathcal{I}_{i,j,N}$$

(recall the definition of $\mathcal{I}_{i,j,N}$ on page 1935), and the "near diagonal" indices to be

$$D(N) = \Delta(N) \setminus FD(N).$$

Then

$$|D(N)| = O(N^{\frac{3}{2}})$$

and, for $(i, j) \in FD(N)$,

$$\frac{i}{N+1} - \frac{j}{N+1} \geq (N+1)^{-\frac{1}{2}} \bmod 1.$$

So,

$$\Bigg| \sum_{0\leq i<j\leq N} \frac{1}{(N+1)^2} \log \left|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})\right| - \iint_{0\leq x\leq y\leq 1} \log \left|\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})\right| dx\, dy \Bigg|$$

$$\le \sum_{(i,j)\in D(N)} \frac{1}{(N+1)^2}\Big|\log\big|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}})-\phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})\big|\Big|$$
$$+\sum_{(i,j)\in FD(N)}\int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}}\int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}}\Big|\log\big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\big|$$
$$-\log\big|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}})-\phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})\big|\Big|\,dx\,dy$$
$$+\Big|\iint_{\mathscr{F}\mathscr{D}(N)}\log\big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\big|\,dx\,dy$$
$$-\iint_{0\le x\le y\le 1}\log\big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\big|\,dx\,dy\Big|.$$

Let I, II and III be the summands of the last expression.

For all $(i,j)\in D(N)$,

$$\frac{a(\delta)}{N+1}\le\big|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}})-\phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})\big|\le O(1),$$

so

$$\Big|\log\big|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}})-\phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})\big|\Big|\le|\log a(\delta)|+\log(N+1),$$

and thus

$$I\le\frac{O(N^{\frac{3}{2}})}{N^2}[|\log a(\delta)|+\log(N+1)]=o_\delta(1).$$

Since

$$\sup_{x-y\ge(N+1)^{-\frac{1}{2}}\bmod 1}\big\|\nabla\log\big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\big|\big\|\le\frac{O(\delta^{-1})}{a(\delta)(N+1)^{-\frac{1}{2}}}=\frac{O(N^{\frac{1}{2}})}{\delta a(\delta)},$$

we have

$$II\le\frac{N^2}{(N+1)^2}\sup_{x-y\ge(N+1)^{-\frac{1}{2}}\bmod 1}\big\|\nabla\log\big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\big|\big\|\,O(N^{-1})$$
$$\le\frac{O(N^{-\frac{1}{2}})}{\delta a(\delta)}=o_\delta(1).$$

By a similar argument as in Lemma 3.3, we have

$$\lim_{N\to\infty}\mathrm{Vol}_{\mathbb{R}^2}\big(\mathscr{F}\mathscr{D}(N)\,\triangle\,\{(x,y)\in\mathbb{R}^2:0\le x\le y\le 1\}\big)=0.$$

Furthermore, (5-4) and (5-5) below indicate that the function $\log\big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\big|$ is L^1 over $[0,1]^2$, so

$$III\le o_\delta(1).$$

Thus, we have proved (5-3).

For $u_1, u_2 \in D(0,1)$, define

$$\psi(u_1,u_2)=\begin{cases}\dfrac{\phi(u_1)-\phi(u_2)}{u_1-u_2} & \text{if } u_1\neq u_2,\\ \phi'(u_1) & \text{if } u_1=u_2.\end{cases}$$

Then ψ is continuous and nonzero in $D(0,1)\times D(0,1)$. Moreover, by the removable singularity theorem, ψ is holomorphic in u_1 as well as u_2. Therefore, $\log|\psi|$ is pluriharmonic in $D(0,1)\times D(0,1)$. By the mean value equality,

$$\begin{aligned}&\int_{\partial D(0,\kappa)}\int_{\partial D(0,\kappa)}\log|\phi(u_1)-\phi(u_2)|\,d\sigma_\kappa(u_1)\,d\sigma_\kappa(u_2)\\ &\quad=\int_{\partial D(0,\kappa)}\int_{\partial D(0,\kappa)}\log|\psi(u_1,u_2)|\,d\sigma_\kappa(u_1)\,d\sigma_\kappa(u_2)+\int_{\partial D(0,\kappa)}\int_{\partial D(0,\kappa)}\log|u_1-u_2|\,d\sigma_\kappa(u_1)d\sigma_\kappa(u_2)\\ &\quad=\log|\psi(0,0)|+\log\kappa+\int_{\partial D(0,1)}\int_{\partial D(0,1)}\log|u_1-u_2|\,d\sigma_1(u_1)\,d\sigma_1(u_2)\\ &\quad=\log|\phi'(0)|+\log\kappa+\int_{\partial D(0,1)}\int_{\partial D(0,1)}\log|u_1-u_2|\,d\sigma_1(u_1)\,d\sigma_1(u_2),\end{aligned}\tag{5-4}$$

and

$$\begin{aligned}\int_{\partial D(0,1)}\int_{\partial D(0,1)}\log|u_1-u_2|\,d\sigma_1(u_1)\,d\sigma_1(u_2)&=\int_0^1\int_0^1\log\left|e^{2\pi\sqrt{-1}x}-e^{2\pi\sqrt{-1}y}\right|dx\,dy\\ &=\int_0^1\log\left|1-e^{2\pi\sqrt{-1}x}\right|dx\\ &=\int_{\partial D(0,1)}\log|1-z|\,d\sigma_1(z)\\ &=0,\end{aligned}\tag{5-5}$$

where the last equality is due to Lebesgue's dominated convergence theorem.

Equations (5-2)–(5-5) show that

$$\begin{aligned}\log\det\Sigma&=\sum_{k=0}^{N}\log\binom{N}{k}+(\log|\phi'(0)|+\log\kappa)N^2+o_\delta(N^2)\\ &=\big(\log|\phi'(0)|+\log\kappa+\tfrac{1}{2}\big)N^2+o_\delta(N^2).\end{aligned}$$

The remaining part is similar to Section 3. □

Remark 5.1. For $U=D(0,r)$, ϕ is a rotation composed with a scaling by r, so $|\phi'(0)|=r$. Thus, the upper bound in Theorem 0.5 is $-\big(\log r+\frac{1}{2}\big)N^2+o(N^2)$, which agrees with Corollary 0.4 in the case $r\geq 1$.

6. Generalized hole probabilities of SU(2) polynomials

If $n(r,N)$ denotes the number of zeros of $\tilde{s}_N(z)$ in $\bar{D}(0,r)$, counting multiplicity, then the hole probability $P_{0,1}(r,N)$ is just the first term of a sequence of probabilities

$$P_{k,1}(r,N) = \gamma_N\{n(r,N) \le k\}, \quad k \ge 0.$$

We call $P_{k,1}(r,N)$ a generalized hole probability because, compared with the large degree or total number of zeros in $\mathbb{C}$ of the polynomial $\tilde{s}_N$, any finite number k is negligible. It has the status of almost having no zeros in $D(0,r)$. And, by Theorem 0.6, it turns out that the generalized hole probabilities are numerically almost equal to the regular one.

Proof of Theorem 0.6. Equation (3-21) implies that, for all $\eta > 0$,

$$\gamma_N\left\{\int_{\partial D(0,r)} \log|\tilde{s}_N(u)|\,d\sigma_r(u) > \frac{N}{2}\log(1+r^2) + \eta N\right\} \le e^{-e^{\eta N}} \quad \text{for } N \gg 1. \tag{6-1}$$

We follow the notations in Section 4, except this time $m = 1$ and we take the number of partitions to be $p = 1$. The corresponding restatement of Lemma 4.6 is

$$\gamma_N\left\{\log\prod_{j=0}^{\lfloor\alpha_0 N\rfloor}|\zeta_j| > \frac{o(N^2)}{\delta^2} + (\lfloor\alpha_0 N\rfloor + 1)\int_{\partial D(0,r)} \log|\tilde{s}_N(u)|\,d\sigma_r(u)\right\} \le e^{-e^N} + e^{-R_{\kappa r,1}(N)},$$

where $\zeta_j = \tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}j/(\lfloor\alpha_0 N\rfloor+1)})$, $0 \le j \le \lfloor\alpha_0 N\rfloor$. Here we do not need to assume $0 \notin \tilde{s}_N(\bar{D}(0,r))$ as we do in Lemma 4.6; the counterpart of *II* in (3-19) is

$$II = (\lfloor\alpha_0 N\rfloor + 1)\int_{\partial D(0,r)} \log|\tilde{s}_N(u)| \int_H P_r(\kappa r e^{2\pi\sqrt{-1}x}, u)\,dx\,d\sigma_r(u).$$

Since $m = 1$ and $p = 1$, $H = [0,1] \subset \mathbb{R}$, so

$$\begin{aligned} II &= (\lfloor\alpha_0 N\rfloor + 1)\int_{\partial D(0,r)} \log|\tilde{s}_N(u)| \int_0^1 P_r(\kappa r e^{2\pi\sqrt{-1}x}, u)\,dx\,d\sigma_r(u) \\ &= (\lfloor\alpha_0 N\rfloor + 1)\int_{\partial D(0,r)} \log|\tilde{s}_N(u)|\,d\sigma_r(u). \end{aligned}$$

Therefore, for all $\eta > 0$ small enough,

$$\begin{aligned} &\gamma_N\left\{\int_{\partial D(0,r)} \log|\tilde{s}_N(u)|\,d\sigma_r(u) \le \frac{N}{2}\log(1+r^2) - \eta N\right\} \\ &\le e^{-e^N} + e^{-R_{\kappa r,1}(N)} + \gamma_N\left\{\prod_{j=0}^{\lfloor\alpha_0 N\rfloor}|\zeta_j| \le \exp\left\{\frac{o(N^2)}{\delta^2} + (\lfloor\alpha_0 N\rfloor + 1)\left[\frac{N}{2}\log(1+r^2) - \eta N\right]\right\}\right\}. \end{aligned} \tag{6-2}$$

Following the steps (3-29)–(3-31), we can show that

$$\log\gamma_N\left\{\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\,d\sigma_r(u)\le\frac{N}{2}\log(1+r^2)-\eta N\right\}$$
$$\le N(\lfloor\alpha_0N\rfloor+1)[\log(1+r^2)-2\eta]-Q_{\kappa r,1,\alpha_0}(N)-2\beta_1\alpha_0^2N^2+\frac{o(N^2)}{\delta^2},$$
$$Q_{\kappa r,1,\alpha_0}(N)\sim N^2\int_0^{\alpha_0}E_r(x)\,dx=\tfrac{1}{2}\alpha_0[2\log\kappa r+1-\log\alpha_0]N^2,$$
$$\begin{aligned}\beta_1&=\int_0^1 x\log[2\sin(\pi x)]\,dx\\&=\int_0^1(x-\tfrac{1}{2})\log[2\sin(\pi x)]\,dx+\frac{1}{2}\int_0^1\log[2\sin(\pi x)]\,dx\\&=\int_{-\frac{1}{2}}^{\frac{1}{2}}x\log\big[2\sin\pi\big(x+\tfrac{1}{2}\big)\big]\,dx+\frac{1}{2}\int_0^1\log[2\sin(\pi x)]\,dx\\&=\int_{-\frac{1}{2}}^{\frac{1}{2}}x\log[2\cos(\pi x)]\,dx+\frac{1}{2}\int_0^1\log[2\sin(\pi x)]\,dx.\end{aligned}$$

Since $\int_{-\frac{1}{2}}^{0}x\log[2\cos(\pi x)]\,dx$ and $\int_0^{\frac{1}{2}}x\log[2\cos(\pi x)]\,dx$ both converge and $x\log[2\cos(\pi x)]$ is odd,

$$\beta_1=\frac{1}{2}\int_0^1\log[2\sin(\pi x)]\,dx=\frac{1}{2}\int_{\partial D(0,1)}\log|1-z|\,d\sigma_1(z),$$

which equals 0 as in (5-5). Thus

$$\log\gamma_N\left\{\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\,d\sigma_r(u)\le\frac{N}{2}\log(1+r^2)-\eta N\right\}$$
$$\le-\frac{1}{2}\alpha_0[1+2\log(\kappa r)-\log\alpha_0-2\log(1+r^2)+4\eta]N^2+\frac{o(N^2)}{\delta^2}.\quad(6\text{-}3)$$

On the other hand,

$$R_{\kappa r,1}(N)\sim N^2\int_{E_{\kappa r}(x)\ge0}E_{\kappa r}(x)\,dx.\quad(6\text{-}4)$$

Combining (6-2)–(6-4) and letting $\delta\to0+$, we get

$$\begin{aligned}&\log\gamma_N\left\{\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\,d\sigma_r(u)\le\frac{N}{2}\log(1+r^2)-\eta N\right\}\\&\le-\min\left\{\tfrac{1}{2}\alpha_0[1+2\log r-\log\alpha_0-2\log(1+r^2)+4\eta],\tfrac{1}{2}\alpha_0[1+2\log r-\log\alpha_0]\right\}N^2+o(N^2)\\&=-\tfrac{1}{2}\alpha_0[1+2\log r-\log\alpha_0-2\log(1+r^2)+4\eta]N^2+o(N^2),\end{aligned}\quad(6\text{-}5)$$

for $0<\eta<\frac{1}{2}\log(1+r^2)$. Since

$$\int_{E_r(x)\ge0}E_r(x)\,dx=\tfrac{1}{2}\alpha_0[1+2\log r-\log\alpha_0]>0\quad\text{and thus}\quad 1+2\log r-\log\alpha_0>0,$$

we can choose $0<\eta<\frac{1}{2}\log(1+r^2)$ close to $\frac{1}{2}\log(1+r^2)$ such that

$$1+2\log r-\log\alpha_0-2\log(1+r^2)+4\eta>0.$$

Therefore, (6-5) makes sense. Denote

$$F_\eta(r)=\tfrac{1}{2}\alpha_0[1+2\log r-\log\alpha_0-2\log(1+r^2)+4\eta];$$

then we have, for $0<\eta<\frac{1}{2}\log(1+r^2)$,

$$\gamma_N\left\{\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\,d\sigma_r(u)\le\frac{N}{2}\log(1+r^2)-\eta N\right\}\le e^{-F_\eta(r)N^2+o(N^2)}. \tag{6-6}$$

Let $\rho>1$, to be determined. By discarding a null set, we may assume $\tilde{s}_N(0)\neq 0$, $0\notin\tilde{s}_N(\partial D(0,r))$ and $0\notin\tilde{s}_N(\partial D(0,\rho^{-1}r))$. So, by Jensen's formula (cf. [Hough et al. 2009, (7.2.11)]), almost surely,

$$\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\,d\sigma_r(u)=\log|c_0|+\int_0^r\frac{n(t,N)}{t}\,dt, \tag{6-7}$$

$$\int_{\partial D(0,\rho^{-1}r)}\log|\tilde{s}_N(u)|\,d\sigma_{\rho^{-1}r}(u)=\log|c_0|+\int_0^{\rho^{-1}r}\frac{n(t,N)}{t}\,dt. \tag{6-8}$$

Since $n(r,N)$ is increasing with respect to r, (6-7) and (6-8) imply

$$\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\,d\sigma_r(u)-\int_{\partial D(0,\rho^{-1}r)}\log|\tilde{s}_N(u)|\,d\sigma_{\rho^{-1}r}(u)=\int_{\rho^{-1}r}^r\frac{n(t,N)}{t}\,dt\le n(r,N)\log\rho,$$

and thus

$$n(r,N)\ge\frac{1}{\log\rho}\left[\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\,d\sigma_r(u)-\int_{\partial D(0,\rho^{-1}r)}\log|\tilde{s}_N(u)|\,d\sigma_{\rho^{-1}r}(u)\right]. \tag{6-9}$$

By (6-1), for $\eta_1>0$, outside an event of probability at most $e^{-e^{\eta_1 N}}$,

$$\int_{\partial D(0,\rho^{-1}r)}\log|\tilde{s}_N(u)|\,d\sigma_{\rho^{-1}r}(u)\le\frac{N}{2}\log(1+\rho^{-2}r^2)+\eta_1 N, \tag{6-10}$$

By (6-6), for $0<\eta_2<\frac{1}{2}\log(1+r^2)$, outside an event of probability at most $e^{-F_{\eta_2}(r)N^2+o(N^2)}$,

$$\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\,d\sigma_r(u)\ge\frac{N}{2}\log(1+r^2)-\eta_2 N. \tag{6-11}$$

By (6-9)–(6-11), outside an event of probability at most $e^{-e^{\eta_1 N}}+e^{-F_{\eta_2}(r)N^2+o(N^2)}$,

$$n(r,N)\ge\frac{N}{\log\rho}\left[\frac{1}{2}\log(1+r^2)-\frac{1}{2}\log(1+\rho^{-2}r^2)-(\eta_1+\eta_2)\right].$$

Therefore,

$$\gamma_N\left\{n(r,N)<\frac{N}{\log\rho}\left[\tfrac{1}{2}\log(1+r^2)-\tfrac{1}{2}\log(1+\rho^{-2}r^2)-(\eta_1+\eta_2)\right]\right\}\le e^{-e^{\eta_1 N}}+e^{-F_{\eta_2}(r)N^2+o(N^2)},$$

where the right-hand side is independent of ρ. We need to choose proper ρ, η_1 and η_2.

For all $\tau > 0$, we set

$$\frac{1}{\log \rho}\Big[\frac{1}{2}\log(1+r^2)-\frac{1}{2}\log(1+\rho^{-2}r^2)-(\eta_1+\eta_2)\Big]=\tau,$$
$$\eta_1+\eta_2=\eta_\tau(\rho):=\frac{1}{2}\log(1+r^2)-\frac{1}{2}\log(1+\rho^{-2}r^2)-\tau\log\rho.$$

If $\tau > 0$ is small enough, let $\rho_0(\tau) := \sqrt{(1-\tau)/\tau}\, r > 1$; then

$$\eta_\tau'(\rho)=\frac{\rho^{-3}r^2}{1+\rho^{-2}r^2}-\frac{\tau}{\rho}=\frac{(1-\tau)r^2-\tau\rho^2}{\rho(\rho^2+r^2)}\begin{cases}>0 & \text{when } 1<\rho<\rho_0,\\ =0 & \text{when } \rho=\rho_0,\\ <0 & \text{when } \rho>\rho_0,\end{cases}$$

and thus

$$\begin{aligned}(\eta_1+\eta_2)_{\max} &= \eta_\tau(\rho_0(\tau))\\ &= \frac{1}{2}\log(1+r^2)-\frac{1}{2}\log\Big(1+\frac{\tau}{1-\tau}\Big)-\tau\Big[\frac{1}{2}\log(1-\tau)-\frac{1}{2}\log\tau+\log r\Big]\\ &= \tfrac{1}{2}\log(1+r^2)+\tfrac{1}{2}\log(1-\tau)-\tfrac{1}{2}\tau\log(1-\tau)+\tfrac{1}{2}\tau\log\tau-\tau\log r\\ &= \tfrac{1}{2}\log(1+r^2)+\tfrac{1}{2}[\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r].\end{aligned}$$

For a fixed $r > 0$, we can choose smaller $\tau > 0$ if necessary so that

$$-\tfrac{1}{2}\log(1+r^2)<\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r<0.$$

This is possible since

$$\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r<0 \quad \text{if } 0<\tau<\alpha_0$$

and

$$\lim_{\tau\to 0+}[\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r]=0.$$

Thus, for such τ and the corresponding $\rho_0 = \rho_0(\tau)$,

$$\tfrac{1}{4}\log(1+r^2)<\eta_1+\eta_2=\eta_\tau(\rho_0)<\tfrac{1}{2}\log(1+r^2).$$

In this case, for all $0 < \eta_1 < \frac{1}{4}\log(1+r^2)$,

$$0<\eta_2=\tfrac{1}{2}\log(1+r^2)+\tfrac{1}{2}[\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r]-\eta_1<\tfrac{1}{2}\log(1+r^2),$$
$$\begin{aligned}\gamma_N\{n(r,N)<\tau N\}&=\gamma_N\Big\{n(r,N)<\frac{N}{\log\rho_0}\Big[\frac{1}{2}\log(1+r^2)-\frac{1}{2}\log(1+\rho_0^{-2}r^2)-(\eta_1+\eta_2)\Big]\Big\}\\ &\le e^{-e^{\eta_1 N}}+e^{-F_{\eta_2}(r)N^2+o(N^2)}.\end{aligned}$$

Fix any $k \ge 0$; when N is large enough, $k < \tau N$,

$$\begin{aligned}&\exp\big\{-\tfrac{1}{2}\alpha_0(1+2\log r-\log\alpha_0)N^2+o(N^2)\big\}=P_{0,1}(r,N)\le P_{k,1}(r,N)\le\gamma_N\{n(r,N)<\tau N\}\\ &\le e^{-e^{\eta_1 N}}+\exp\big\{-\tfrac{1}{2}\alpha_0\big\{(1+2\log r-\log\alpha_0)+2[\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r]-4\eta_1\big\}N^2+o(N^2)\big\}.\end{aligned}$$

Therefore,

$$-\tfrac{1}{2}\alpha_0(1+2\log r-\log\alpha_0)\le\liminf_{N\to\infty}\frac{\log P_{k,1}(r,N)}{N^2}\le\limsup_{N\to\infty}\frac{\log P_{k,1}(r,N)}{N^2}$$
$$\le-\tfrac{1}{2}\alpha_0\big\{(1+2\log r-\log\alpha_0)+2[\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r]-4\eta_1\big\}.$$

Let $\eta_1\to 0+$ and then $\tau\to 0+$; we have

$$\lim_{N\to\infty}\frac{\log P_{k,1}(r,N)}{N^2}=-\tfrac{1}{2}\alpha_0(1+2\log r-\log\alpha_0)$$

or, equivalently,

$$\log P_{k,1}(r,N)\sim-\tfrac{1}{2}\alpha_0(1+2\log r-\log\alpha_0)N^2. \qquad \square$$

Appendix

We now prove the following lemma:

Lemma A.1. *The coefficient of* $g_{m,N}(\xi)$ *in* $\det W_{m,N}(\xi)$ *equals* 1.

Proof. Let $\mathcal{S}_{m,N}$ be the set of bijections from $\Gamma_{m,N}$ to $\Lambda_{m,N}$ and, for all $\sigma\in\mathcal{S}_{m,N}$, $J\in\Gamma_{m,N}$, write $\sigma(J)=(\sigma_1(J),\dots,\sigma_m(J))$. Then

$$\det W_{m,N}(\xi)=\sum_{\sigma\in\mathcal{S}_{m,N}}\operatorname{sgn}(\sigma)\prod_{J\in\Gamma_{m,N}}\xi_J^{\sigma(J)}=\sum_{\sigma\in\mathcal{S}_{m,N}}\operatorname{sgn}(\sigma)\prod_{J\in\Gamma_{m,N}}\xi_{1,j_1}^{\sigma_1(J)}\cdots\xi_{m,j_m}^{\sigma_m(J)}.$$

To find those $\sigma\in\mathcal{S}_{m,N}$ ending up with $g_{m,N}(\xi)$, it is equivalent to find σ satisfying, for all $1\le i\le m$,

$$\sum_{J\in\Gamma_{m,N}^{i,k}}\sigma_i(J)=\begin{cases}\binom{k+i-1}{i}\binom{N-k+m-i}{m-i} & 1\le k\le N,\\ 0 & k=0,\end{cases}\tag{A-1}$$

where the set $\Gamma_{m,N}^{i,k}$ is defined in (2-7). We are going to prove by induction that

$$\sigma(J)=(j_1,j_2-j_1,\dots,j_m-j_{m-1})\quad\text{for all } J\in\Gamma_{m,N}.\tag{A-2}$$

First of all, similarly to $\Gamma_{m,N}^{i,k}$, we introduce

$$\Lambda_{m,N}^{i,k}=\{(k_1,\dots,k_m)\in\Lambda_{m,N}:k_1+\cdots+k_i=k\};$$

then

$$\Lambda_{m,N}=\bigsqcup_{k=0}^{N}\Lambda_{m,N}^{i,k}\quad\text{for all } 1\le i\le m,$$

and

$$|\Lambda_{m,N}^{i,k}|=\binom{k+i-1}{i-1}\binom{N-k+m-i}{m-i}=|\Gamma_{m,N}^{i,k}|.$$

When $i=1$, (A-1) shows that, for $0 \le k \le N$,

$$\sum_{J \in \Gamma_{m,N}^{1,k}} \sigma_1(J) = k\binom{N-k+m-1}{m-1}, \tag{A-3}$$

where the number of terms in the summation on the left is $|\Gamma_{m,N}^{1,k}| = \binom{N-k+m-1}{m-1} = |\Lambda_{m,N}^{1,k}|$ for all $0 \le k \le N$. Then

$$\begin{aligned}
k=0 \text{ in (A-3)} &\implies \sigma(\Gamma_{m,N}^{1,0}) = \Lambda_{m,N}^{1,0} \implies \sigma\Bigl(\bigsqcup_{k=1}^{N} \Gamma_{m,N}^{1,k}\Bigr) = \bigsqcup_{k=1}^{N} \Lambda_{m,N}^{1,k},\\
k=1 \text{ in (A-3)} &\implies \sigma(\Gamma_{m,N}^{1,1}) = \Lambda_{m,N}^{1,1} \implies \sigma\Bigl(\bigsqcup_{k=2}^{N} \Gamma_{m,N}^{1,k}\Bigr) = \bigsqcup_{k=2}^{N} \Lambda_{m,N}^{1,k},\\
&\ \ \vdots\\
k=N \text{ in (A-3)} &\implies \sigma(\Gamma_{m,N}^{1,N}) = \Lambda_{m,N}^{1,N},
\end{aligned}$$

so

$$\sigma_1(J) = j_1 \quad \text{for all } J \in \Gamma_{m,N}.$$

Now assume, for some $1 \le i \le m-1$, that $(\sigma_1+\cdots+\sigma_i)(J) = j_i$ for all $J \in \Gamma_{m,N}$. Then, for any $1 \le k \le N$,

$$\begin{aligned}
\sum_{J \in \Gamma_{m,N}^{i+1,k}} (\sigma_1+\cdots+\sigma_{i+1})(J) &= \sum_{J \in \Gamma_{m,N}^{i+1,k}} [j_i + \sigma_{i+1}(J)]\\
&= \sum_{j=0}^{k} j|\Gamma_{m,N}^{i,j} \cap \Gamma_{m,N}^{i+1,k}| + \binom{k+i}{i+1}\binom{N-k+m-i-1}{m-i-1}\\
&= \sum_{j=0}^{k} j\binom{j+i-1}{i-1}\binom{N-k+m-i-1}{m-i-1} + \binom{k+i}{i+1}\binom{N-k+m-i-1}{m-i-1}\\
&= k\binom{k+i}{i}\binom{N-k+m-i-1}{m-i-1},
\end{aligned}$$

where the second term on the second line of the calculation comes from (A-1). And, for $k=0$,

$$\sum_{J \in \Gamma_{m,N}^{i+1,0}} (\sigma_1+\cdots+\sigma_{i+1})(J) = \sum_{J \in \Gamma_{m,N}^{i+1,0}} [j_i + \sigma_{i+1}(J)] = 0.$$

So, for all $0 \le k \le N$,

$$\sum_{J \in \Gamma_{m,N}^{i+1,k}} (\sigma_1+\cdots+\sigma_{i+1})(J) = k\binom{k+i}{i}\binom{N-k+m-i-1}{m-i-1}, \tag{A-4}$$

where the number of terms in the summation on the left is $|\Gamma^{i+1,k}_{m,N}| = \binom{k+i}{i}\binom{N-k+m-i-1}{m-i-1} = |\Lambda^{i+1,k}_{m,N}|$ for all $0 \le k \le N$.

$$
\begin{aligned}
k = 0 \text{ in (A-4)} &\implies \sigma(\Gamma^{i+1,0}_{m,N}) = \Lambda^{i+1,0}_{m,N} \implies \sigma\Big(\bigsqcup_{k=1}^{N} \Gamma^{i+1,k}_{m,N}\Big) = \bigsqcup_{k=1}^{N} \Lambda^{i+1,k}_{m,N},\\
k = 1 \text{ in (A-4)} &\implies \sigma(\Gamma^{i+1,1}_{m,N}) = \Lambda^{i+1,1}_{m,N} \implies \sigma\Big(\bigsqcup_{k=2}^{N} \Gamma^{i+1,k}_{m,N}\Big) = \bigsqcup_{k=2}^{N} \Lambda^{i+1,k}_{m,N},\\
&\vdots\\
k = N \text{ in (A-4)} &\implies \sigma(\Gamma^{i+1,N}_{m,N}) = \Lambda^{i+1,N}_{m,N},
\end{aligned}
$$

so

$$(\sigma_1 + \cdots + \sigma_{i+1})(J) = j_{i+1} \quad \text{for all } J \in \Gamma_{m,N}.$$

Thus, (A-2) is proved. And it is trivial to check that the σ defined in (A-2) satisfies all the equations in (A-1). This means that there is only one $\sigma \in \mathcal{S}_{m,N}$ that ends up with $g_{m,N}(\xi)$, and it turns out to be order-preserving. Therefore,

$$\det W_{m,N}(\xi) = g_{m,N}(\xi) + \cdots . \qquad \square$$

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