ANALYSIS & PDE

Volume 7 No. 8 2014

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ANALYSIS AND PDE
Vol. 7, No. 8, 2014
dx.doi.org/10.2140/apde.2014.7.2009

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GLOBAL REGULARITY FOR A SLIGHTLY SUPERCRITICAL HYPERDISSIPATIVE NAVIER–STOKES SYSTEM

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We prove global existence of smooth solutions for a slightly supercritical hyperdissipative Navier–Stokes under the optimal condition on the correction to the dissipation. This proves a conjecture formulated by Tao.

1. Introduction

Let $d \geq 3$ and consider the generalized Navier–Stokes system

$$\begin{cases} \partial u/\partial t + (u \cdot \nabla)u + \nabla p + D_0^2 u = 0, \\ \nabla \cdot u = 0, \\ \int_{[0,2\pi]^d} u(t,x)\,dx = 0, \end{cases} \tag{1-1}$$

on $[0, 2\pi]^d$ with periodic boundary conditions, where D_0 is a Fourier multiplier with nonnegative symbol m. The Navier–Stokes system is recovered when $m(k) = |k|$. If

$$m(k) \geq c \frac{|k|^{(d+2)/4}}{G(|k|)}, \tag{1-2}$$

where $G : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that

$$\int_1^\infty \frac{ds}{sG(s)^4} = \infty, \tag{1-3}$$

and

$$\frac{G(x)}{|x|^{(d+2)/4}} \text{ is eventually nonincreasing,} \tag{1-4}$$

then in [Tao 2009] it is proved[1] that (1-1) has a global smooth solution for every smooth initial condition. The result has been extended to the two-dimensional case in [Katz and Tapay 2012].

A heuristic argument developed in [Tao 2009] and based on the comparison between the speed of propagation of a (possible) blow-up and the rate of dissipation suggests that regularity should still hold under the weaker condition

$$\int_1^\infty \frac{ds}{sG(s)^2} = \infty. \tag{1-5}$$

D. Barbato acknowledges the financial support of the research project "Stochastic Processes and Applications to Complex Systems" (CPDA123182) of the University of Padua.
MSC2010: primary 76D03, 76D05; secondary 35Q30, 35Q35.
Keywords: Navier–Stokes, dyadic model, global existence, slightly supercritical Navier–Stokes equations.

[1]The proof of that result is given in $\mathbb{R}^d$, but it can be easily extended to the periodic setting; see [Tao 2009, Remark 2.1].

The main result of this paper, contained in the following theorem, is a complete proof of this conjecture.

Theorem 1.1. *Let $d \geq 2$ and assume conditions (1-2), (1-4) and (1-5) hold for a nondecreasing function $G : [0, \infty) \to [0, \infty)$. Then (1-1) has a global smooth solution for every smooth initial condition.*

A simple version of this conjecture, when reformulated on a toy model, has been proved for the dyadic model in [Barbato et al. 2014]. Actually, for that model one could prove regularity in the full supercritical regime, with $m(k) = |k|$, as was done in [Barbato et al. 2011], but it was natural to develop there some of the main ideas on which also this paper is based. In fact, here we prove that the equations for the velocity can be reduced to a suitable dyadic-like model, but with infinitely many interactions. A more sophisticated version of the arguments of [Barbato et al. 2014] ensures regularity of this dyadic model and, in turn, of the solution of problem (1-1).

Our technique for proving Theorem 1.1 is flexible enough to include an additional critical parameter. Consider the generalized Leray α-model,

$$\begin{cases} \partial v/\partial t + (u \cdot \nabla) v + \nabla p + D_1 v = 0, \\ v = D_2 u, \\ \nabla \cdot v = 0, \\ \int_{[0,2\pi]^d} v(t,x)\, dx = \int_{[0,2\pi]^d} u(t,x)\, dx = 0, \end{cases} \tag{1-6}$$

where D_1 and D_2 are Fourier multipliers with nonnegative symbols m_1 and m_2.

Theorem 1.2. *Let $d \geq 2$ and $\alpha, \beta \geq 0$, and assume*

$$m_1(k) \geq c\frac{|k|^\alpha}{g(|k|)}, \qquad m_2(k) \geq c|k|^\beta, \qquad \alpha + \beta \geq \frac{d+2}{2},$$

where $g : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $x^{-\alpha} g(x)$ is eventually nonincreasing and

$$\int_1^\infty \frac{ds}{s g(s)} = \infty. \tag{1-7}$$

Then (1-6) has a global smooth solution for every smooth initial condition.

Under the assumptions of Theorem 1.1, if $\beta = 0$, $\alpha = (d+2)/2$, $g(x) = G(x)^2$, $m_2(k) = 1$, and $m_1(k) = m(k)^2$, then the assumptions of Theorem 1.2 are met. Therefore Theorem 1.1 follows immediately from Theorem 1.2, and it is sufficient to prove only the second result.

Our results hold as well when the problems are considered in $\mathbb{R}^d$, since in our method large scales play no significant role (see Remark 2.9).

The model (1-6) with $g \equiv 1$ was introduced by Olson and Titi [2007]. They proposed the idea that a weaker nonlinearity and a stronger viscous dissipation could work together to yield regularity. Their statement uses the stronger hypothesis $\alpha + \beta/2 \geq (d+2)/2$ though, and this result was later logarithmically improved in [Yamazaki 2012] with condition (1-3).

Our results are also relevant in view of the analysis in [Tao 2014, Remark 5.2], since they confirm that the condition (1-7) is optimal when general nonlinear terms with the same scaling are considered.

The proof of the above theorem is based on two crucial ideas. The first idea is that smoothness of (1-6) can be reduced to the smoothness of a suitable shell model, obtained by averaging the energy of a solution of (1-6) over dyadic shells in Fourier space. We believe that this reduction may be interesting beyond the scope of this paper. The second idea is that the overall contribution of energy and dissipation over large shells satisfies a recursive inequality. Under condition (1-7), dissipation significantly dumps the flow of energy towards small scales and ensures smoothness. This is a more sophisticated version of the result obtained in [Barbato et al. 2014], due to the larger number of interactions between shells.

The paper is organized as follows. In Section 2 we derive the *shell approximation* of a solution of (1-6). The recursive formula is obtained in Section 3. In Section 4 we deduce exponential decays of shell modes by the recursive formula. The Appendix contains a standard existence and uniqueness result for the sake of completeness.

2. From the generalized Fourier Navier–Stokes to the dyadic equation

This section contains one of the crucial steps in our approach. We show that the proof of Theorem 1.2 can be reduced to a proof of the decay of solutions of a suitable shell model. For simplicity and without loss of generality, from now on we assume that

$$m_1(k) = \frac{|k|^\alpha}{g(|k|)}, \qquad m_2(k) \geq |k|^\beta.$$

The shell approximation. The dynamics of our generalized version of the Navier–Stokes equation in Fourier decomposition are

$$\begin{cases} v_k' = -\dfrac{|k|^\alpha}{g(|k|)} v_k - i \displaystyle\sum_{h \in \mathbb{Z}^d \setminus \{0\}} \frac{\langle v_h, k\rangle}{|h|^\beta} P_k(v_{k-h}), \\ \langle v_k, k \rangle = 0, \\ v_{-k} = \overline{v_k}, \end{cases} \tag{2-1}$$

for $k \in \mathbb{Z}^d \setminus \{0\}$, where $P_k(w) := w - (\langle w, k\rangle / |k|^2) k$ and $v_0 = 0$. A solution is a family $(v_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ where each $v_k = v_k(t)$ is a differentiable map from $[0, \infty)$ to $\mathbb{C}^d$ satisfying (2-1) for all times.

As is common in Littlewood–Paley theory, let $\Phi : [0, \infty) \to [0, 1]$ be a smooth function such that $\Phi \equiv 1$ on $[0, 1]$, $\Phi \equiv 0$ on $[2, \infty)$, and Φ is strictly decreasing on $[1, 2]$. For $x \geq 0$, let $\psi(x) := \Phi(x) - \Phi(2x)$, so that ψ is a smooth bump function supported on $\left(\frac{1}{2}, 2\right)$ satisfying

$$\sum_{n=0}^{\infty} \psi\left(\frac{x}{2^n}\right) = 1 - \Phi(2x) \equiv 1, \qquad x \geq 1.$$

Notice that it is elementary to show that $\sqrt{\psi}$ is Lipschitz continuous.

Let $\mathbb{N}_0$ denote the set of nonnegative integers. For all $n \in \mathbb{N}_0$, we introduce the radial maps $\psi_n : \mathbb{R}^d \to [0, 1]$ defined by $\psi_n(x) = \psi(2^{-n}|x|)$. Notice that

$$\sum_{n \in \mathbb{N}_0} \psi_n(x) \equiv 1, \qquad x \in \mathbb{Z}^d \setminus \{0\}.$$

In Littlewood–Paley theory, one typically defines ψ_n for all $n \in \mathbb{Z}$, introduces objects like

$$P_n(x) := \sum_{k \in \mathbb{Z}^d} \psi_n(k) v_k e^{i\langle k,x\rangle},$$

and then proves that $u = \sum_n P_n$. Since these P_n are not orthogonal[2] this does not give a nice decomposition of energy, as

$$\sum_{n \in \mathbb{Z}} \|P_n\|_{L^2}^2 \neq \sum_{k \in \mathbb{Z}^d} |v_k|^2 = \|u\|_{L^2}^2.$$

Thus, instead of $P_n(x)$, we introduce a sort of *square-averaged* Littlewood–Paley decomposition. Let

$$X_n(t) := \Big(\sum_{k \in \mathbb{Z}^d} \psi_n(k) |v_k(t)|^2 \Big)^{\frac{1}{2}}, \qquad n \in \mathbb{N}_0,\ t \geq 0. \tag{2-2}$$

Then clearly

$$\sum_{n \in \mathbb{N}_0} X_n^2 = \sum_{k \in \mathbb{Z}^d} |v_k|^2 = \|u\|_{L^2}^2.$$

Remark 2.1. One major difference with respect to the usual Littlewood–Paley theory is that it is impossible to recover v from these X_n (as it was with the components $P_n(x)$), since they are averaged both in the physical space and over one shell of the frequency space.

We will denote by H^γ the Hilbert–Sobolev space of periodic functions with differentiation index γ, namely

$$H^\gamma = \Big\{ v = (v_k)_{k \in \mathbb{Z}^d} \ :\ \sum (1 + |k|^2)^\gamma |v_k|^2 < \infty \Big\}. \tag{2-3}$$

Definition 2.2. If (2-2) holds, we say that $X = (X_n(t))_{n \in \mathbb{N}_0, t \geq 0}$ is the *shell approximation* of v.

If $v \in H^\gamma$ and X is its shell approximation, then

$$\sum_n 2^{2\gamma n} X_n^2 = \sum_k \Big(\sum_n 2^{2\gamma n} \psi_n(k) \Big) |v_k|^2 \approx \sum_k |k|^{2\gamma} |v_k|^2 = \|v\|_{H^\gamma}^2. \tag{2-4}$$

Hence, $v(t) \in C^\infty$ if and only if $\sup_n 2^{\gamma n} X_n < \infty$ for every $\gamma > 0$. In view of Theorem A.1, Theorem 1.2 follows if we can prove:

Theorem 2.3. *Under the assumptions of Theorem 1.2, let $v(0)$ be smooth and periodic and let $m \geq 2 + d/2$. If v is a solution of (1-6) in H^m on its maximal interval of existence $[0, T_\star)$, X is its shell approximation and*

$$\sup_{[0,T_\star)} \sum 2^{2mn} X_n^2 < \infty,$$

then $T_\star = \infty$.

---

[2] They are in fact *almost orthogonal*, in the sense that $\langle P_n, P_m \rangle_{L^2} = 0$ whenever $|m - n| \geq 2$.

The shell solution. We want to write a system of equations for the shell approximation of a solution of (1-6). We give a more formal connection between (1-6) and its shell equation because we believe the notion will turn out to be useful beyond the scopes of the present work.

Define the set I to be those $(l, m, n) \in \mathbb{N}_0^3$ for which the difference between the two largest integers among l, m and n is at most 2.

We are now ready to introduce the shell model ODE for the energy of each shell (Equation (2-5)).

Definition 2.4 (shell solution). Let $X = (X_n)_{n\in\mathbb{N}_0}$ be a sequence of real-valued maps $X_n : [0, \infty) \to \mathbb{R}$. We say that X is a *shell solution* if there are two families of real-valued maps $\chi = (\chi_n)_{n\in\mathbb{N}_0}$ and $\phi = (\phi_{(l,m,n)})_{(l,m,n)\in I}$ such that

$$\frac{d}{dt}X_n^2(t) = -\chi_n(t)X_n^2(t) + \sum_{\substack{l,m\in\mathbb{N}_0\\(l,m,n)\in I}} \phi_{(l,m,n)}(t)X_l(t)X_m(t)X_n(t) \tag{2-5}$$

for all $n \in \mathbb{N}_0$ and $t > 0$, where the sum above is understood as absolutely convergent, and χ, ϕ satisfy the following:

(1) The family ϕ is *antisymmetric*, in the sense that

$$\phi_{(l,m,n)}(t) = -\phi_{(l,n,m)}(t), \quad (l, m, n) \in I,\ t \geq 0.$$

(2) There exist two positive constants c_1 and c_2 for which

$$\chi_n(t) \geq c_1 \frac{2^{\alpha n}}{g(2^{n+1})} \quad \text{and} \quad |\phi_{(l,m,n)}(t)| \leq c_2 2^{(d/2+1-\beta)\min\{l,m,n\}} \tag{2-6}$$

for all $(l, m, n) \in I$ and $t \geq 0$.

Remark 2.5. We will prove below that the shell approximation of a solution of (1-6) is a shell solution. It is easy to check that the dissipation term is local, as expected, due to the way the shell components of a solution interact in the model's dynamics. As for the nonlinear term, it turns out that the set I of the triples of indices (l, m, n) for which there may be interaction between the shell components l, m and n is quite small. This is basically because, in the Fourier space, three components may interact only if they are the sides of a triangle, and by the triangle inequality their lengths cannot be in three shells far away from each other.

Remark 2.6. To ensure that the sum in (2-5) is absolutely convergent, it is sufficient to assume that the sequence $(X_n(t))_{n\in\mathbb{N}_0}$ is square-summable (this will be a consequence of the energy inequality; see Definition 3.1). Indeed, if n is not the smallest index, then the sum is extended to a finite number of indices. Otherwise, $\phi_{(l,m,n)}$ is constant with respect to l, m.

Remark 2.7. The antisymmetric property is what makes the nonlinearity of (2-5) *formally* conservative. In fact, using antisymmetry, a change of variable ($m' = n$ and $n' = m$) and the fact that $(l, m', n') \in I$ if and only if $(l, n', m') \in I$, one could formally write

$$-\sum_{\substack{l,m,n\in\mathbb{N}_0\\(l,m,n)\in I}} \phi_{(l,m,n)} X_l X_m X_n = \sum_{\substack{l,m,n\in\mathbb{N}_0\\(l,m,n)\in I}} \phi_{(l,n,m)} X_l X_m X_n = \sum_{\substack{l,m',n'\in\mathbb{N}_0\\(l,n',m')\in I}} \phi_{(l,m',n')} X_l X_{m'} X_{n'}$$
$$= \sum_{\substack{l,m',n'\in\mathbb{N}_0\\(l,m',n')\in I}} \phi_{(l,m',n')} X_l X_{m'} X_{n'}.$$

If these sums are absolutely convergent, this would prove indeed that the expression itself is equal to zero.

Since these are infinite sums, these computations are not rigorous unless we know, for instance, that $\sum_n 2^{2\gamma n} X_n^2 < \infty$ with $\gamma \geq \frac{1}{3}\bigl(\frac{1}{2}d + 1 - \beta\bigr)$, as can be verified by an elementary computation.

The shell model as a shell approximation. The bounds on the coefficients given in Definition 2.4 are in the correct direction to prove regularity results (and hence Theorem 2.3). The following theorem, which is the main result of this section, shows that they capture the natural scaling of the shell interactions for the *physical* solutions.

Theorem 2.8. *If v is a solution of (1-6) on $[0, T]$ and X is its shell approximation, then X is a shell solution.*

Remark 2.9. At this stage it is easy to realize that our main results hold also in $\mathbb{R}^d$ with minimal changes. Indeed when passing to the shell approximation, all large frequencies are considered together in the first element of the shell model.

The proof of Theorem 2.8 can be found at the end of this section. It is based on Propositions 2.10–2.11 below, which give the actual definitions of χ and ϕ and prove their properties.

Proposition 2.10. *Let X be the shell approximation of a solution v. Define $\chi_n(t)$ for $n \in \mathbb{N}_0$ and $t \geq 0$ by*

$$\chi_n(t) := \begin{cases} \dfrac{2}{X_n^2(t)} \displaystyle\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \psi_n(k) \frac{|k|^\alpha}{g(|k|)} |v_k(t)|^2 & \text{if } X_n(t) \neq 0, \\[2ex] \dfrac{2^{\alpha n - \alpha + 1}}{g(2^{n+1})} & \text{if } X_n(t) = 0. \end{cases} \tag{2-7}$$

Then

$$\chi_n(t) \geq \frac{2^{\alpha n - \alpha + 1}}{g(2^{n+1})}, \qquad n \in \mathbb{N}_0,\ t \geq 0.$$

Proof. Fix $n \in \mathbb{N}_0$ and $t \geq 0$. The map ψ_n is supported on $\{x \in \mathbb{Z}^d : 2^{n-1} < |x| < 2^{n+1}\}$ and g is nondecreasing, so

$$\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \psi_n(k) \frac{|k|^\alpha}{g(|k|)} |v_k(t)|^2 \geq \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \psi_n(k) \frac{2^{(n-1)\alpha}}{g(2^{n+1})} |v_k(t)|^2 = \frac{2^{(n-1)\alpha}}{g(2^{n+1})} X_n^2(t),$$

where we used (2-2). By (2-7) we get the result. □

We finally turn our attention to the antisymmetry property and an upper bound for $\phi_{(l,m,n)}(t)$:

Proposition 2.11. *Let X be the shell approximation of a solution v. Define $\phi_{(l,m,n)}(t)$ for all $l, m, n \in \mathbb{N}_0$ and $t \geq 0$ as*

$$\phi_{(l,m,n)}(t) := \frac{2}{X_l(t)X_m(t)X_n(t)} \sum_{\substack{h,k\in\mathbb{Z}^d \\ h\neq 0}} \psi_l(h)\psi_m(k-h)\psi_n(k) \frac{\operatorname{Im}\{\langle v_h(t), k\rangle\langle v_{k-h}(t), v_k(t)\rangle\}}{|h|^\beta} \tag{2-8}$$

(*unless $X_l(t)X_m(t)X_n(t) = 0$, in which case $\phi_{(l,m,n)}(t) := 0$*). *Then*:

(1) $\phi_{(l,m,n)}(t) = 0$ *for all $(l, m, n) \notin I$ and all $t \geq 0$.*

(2) $\phi_{(l,m,n)}(t) = -\phi_{(l,n,m)}(t)$ *for all $l, m, n \in \mathbb{N}_0$ and all $t \geq 0$.*

(3) *For any $\beta \geq 0$ there exists a constant $c_3 > 0$ depending only on d, β and ψ such that*

$$|\phi_{(l,m,n)}(t)| \leq c_3 2^{(d/2+1-\beta)\min\{l,m,n\}}, \quad (l, m, n) \in I,\ t \geq 0. \tag{2-9}$$

For the proof we need a couple of lemmas:

Lemma 2.12. *Suppose $v = (v_k)_{k\in\mathbb{Z}^d}$ is a complex field over $\mathbb{Z}^d$ such that, for all $k \in \mathbb{Z}^d$, $\langle k, v_k\rangle = 0$ and $\overline{v_k} = v_{-k}$. Then, for all $h \in \mathbb{Z}^d$,*

$$\sum_{k\in\mathbb{Z}^d} \psi_m(k-h)\psi_n(k) \operatorname{Im}\{\langle v_h, k\rangle\langle v_{k-h}, v_k\rangle\} = -\sum_{k\in\mathbb{Z}^d} \psi_m(k)\psi_n(k-h) \operatorname{Im}\{\langle v_h, k\rangle\langle v_{k-h}, v_k\rangle\}.$$

Proof. Consider the left-hand side. By performing the change of variable $k' = h - k$, we obtain

$$\begin{aligned}
\psi_m(k-h) &= \psi_m(-k') = \psi_m(k'),\\
\psi_n(k) &= \psi_n(h-k') = \psi_n(k'-h),\\
\langle v_h, k\rangle &= \langle v_h, h-k'\rangle = -\langle v_h, k'\rangle,\\
\langle v_{k-h}, v_k\rangle &= \langle v_{-k'}, v_{h-k'}\rangle = \langle \overline{v_{k'}}, \overline{v_{k'-h}}\rangle = \langle v_{k'-h}, v_{k'}\rangle.
\end{aligned}$$

The sum for $k \in \mathbb{Z}^d$ is equivalent to the sum for $k' \in \mathbb{Z}^d$, and this concludes the proof. □

Lemma 2.13. *Let v be a solution and X its shell approximation. Then, for all $a, b, c \in \mathbb{N}_0$ and all $t \geq 0$,*

$$\sum_{h\in\mathbb{Z}^d} \psi_a(h)|v_h(t)| \sum_{k\in\mathbb{Z}^d} \sqrt{\psi_b(k)\psi_c(k-h)}\,|v_k(t)|\,|v_{k-h}(t)| \leq 2^{d(a+3)/2} X_a(t)X_b(t)X_c(t).$$

Proof. By the Cauchy–Schwarz inequality and formula (2-2), we have that, for all $h \in \mathbb{Z}^d$,

$$\sum_{k\in\mathbb{Z}^d} \sqrt{\psi_b(k)\psi_c(k-h)}\,|v_k(t)|\,|v_{k-h}(t)| \leq X_b(t)X_c(t).$$

Then, let S_a denote the intersection of $\mathbb{Z}^d$ and the support of ψ_a. By inscribing S_a in a cube, we can bound its cardinality by $|S_a| \leq (2^{a+2}+1)^d \leq 2^{(a+3)d}$, so

$$\sum_{k\in\mathbb{Z}^d} \psi_a(k)|v_k(t)| \leq \left(|S_a| \sum_{k\in S_a} \psi_a^2(k)v_k^2(t)\right)^{\frac{1}{2}} \leq (2^{(a+3)d})^{1/2} X_a(t),$$

where we used the fact that $\psi_a(k) \le 1$. $\square$

Proof of Proposition 2.11. Consider Equation (2-8), the definition of $\phi_{(l,m,n)}$. By applying Lemma 2.12, for fixed t we immediately conclude that

$$\phi_{(l,n,m)} = -\phi_{(l,m,n)}, \quad l, m, n \in \mathbb{N}_0,$$

and in particular that $\phi_{(l,m,m)} = 0$.

Moreover, for all choices of h and k, the arguments of ψ_l, ψ_m and ψ_n are the sides of a triangle in $\mathbb{R}^d$, so by the triangle inequality the size of the largest (without loss of generality k) is at most twice the size of the second largest (without loss of generality h). On the other hand, for all $j \in \mathbb{N}_0$ the support of ψ_j is $\{x \in \mathbb{R}^d : 2^{j-1} < |x| < 2^{j+1}\}$. Thus, whenever $\psi_l(h)\psi_n(k) \neq 0$, necessarily $n \le l+2$, since

$$2^{n-1} < |k| \le 2|h| < 2^{l+2}.$$

This proves that $\phi_{(l,m,n)} = 0$ outside the set I defined before Definition 2.4.

Finally, we prove inequality (2-9) for $(l, m, n) \in I$ with $m < n$. We will consider separately the two cases $n - m > 2$ and $n - m \in \{1, 2\}$, starting with the former.

Case 1. Since $m < n - 2$ and $(l, m, n) \in I$, we have $m = \min\{l, m, n\}$ and $|l - n| \le 2$. This means in particular that typically $|k - h| < |k|$ for all the nonzero terms of the sum in (2-8), so it is convenient to substitute $\langle v_h, k\rangle = \langle v_h, k - h\rangle$ in the equation to obtain the bound

$$|\phi_{(l,m,n)}| \le \frac{2}{X_l X_m X_n} \sum_{\substack{h,k\in\mathbb{Z}^d\\ h\neq 0}} \psi_l(h)\psi_m(k-h)\psi_n(k) \frac{|v_h|\,|k-h|\,|v_{k-h}|\,|v_k|}{|h|^\beta}.$$

By the definition of ψ_l, either $\psi_l(h) = 0$ or $|h| \ge 2^{l-1} \ge 2^m$. Applying this and the change of variable $k' = k - h$, one gets

$$|\phi_{(l,m,n)}| \le \frac{2^{1-\beta m}}{X_l X_m X_n} \sum_{k'\in\mathbb{Z}^d} \psi_m(k')|k'|\,|v_{k'}| \sum_{h\in\mathbb{Z}^d} \psi_l(h)\psi_n(k'+h)|v_h|\,|v_{k'+h}|.$$

In the same way, we can substitute $|k'| \le 2^{m+1}$ and apply Lemma 2.13 (recall that $\psi \le 1$, so $\psi \le \sqrt{\psi}$) to get

$$|\phi_{(l,m,n)}| \le 2^{1-\beta m+m+1+d(m+3)/2}.$$

Since in the present case $\min\{l, m, n\} = m$, this proves inequality (2-9) with $c_3 = 2^{2+3d/2}$.

Case 2. Suppose now that $n - m \in \{1, 2\}$ and $(l, m, n) \in I$; then $l \le n + 2$ and $\min\{l, m, n\} \ge l - 4$. In this case it is l that can be small with respect to m and n, so we take the terms in l and h outside the internal sum:

$$|\phi_{(l,m,n)}| \le \frac{2}{X_l X_m X_n} \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \frac{\psi_l(h)}{|h|^\beta} \left| \sum_{k\in\mathbb{Z}^d} \psi_m(k-h)\psi_n(k)\,\mathrm{Im}\{\langle v_h, k\rangle\langle v_{k-h}, v_k\rangle\} \right|.$$

The idea is to exploit the cancellations in the sum over k that happen when $k-h$ and k are switched. By Lemma 2.12 and the bound $|k| \le 2^{n+1}$ for k in the support of ψ_m or ψ_n,

$$\begin{aligned}|\phi_{(l,m,n)}| &\le \frac{2}{X_l X_m X_n} \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \frac{\psi_l(h)}{|h|^\beta}\frac{1}{2}\Big|\sum_{k\in\mathbb{Z}^d}\big(\psi_m(k-h)\psi_n(k)-\psi_m(k)\psi_n(k-h)\big)\operatorname{Im}\{\langle v_h,k\rangle\langle v_{k-h},v_k\rangle\}\Big| \\ &\le \frac{2^{n+1}}{X_l X_m X_n} \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \frac{\psi_l(h)|v_h|}{|h|^\beta}\sum_{k\in\mathbb{Z}^d}\big|\psi_m(k-h)\psi_n(k)-\psi_m(k)\psi_n(k-h)\big||v_{k-h}||v_k|.\end{aligned}$$

We turn our attention to the term $\psi_m(k-h)\psi_n(k)-\psi_m(k)\psi_n(k-h)$ and show that it is small. Let L denote the Lipschitz constant of the function $\psi^{1/2}$. Then, for all $h, k \in \mathbb{Z}^d$ and all $m, n \in \mathbb{N}_0$ such that $m \ge n-2$,

$$\begin{aligned}&\big|\sqrt{\psi_m(k-h)\psi_n(k)}-\sqrt{\psi_m(k)\psi_n(k-h)}\big| \\ &\qquad= \big|\sqrt{\psi_m(k-h)\psi_n(k)}-\sqrt{\psi_m(k)\psi_n(k)}+\sqrt{\psi_m(k)\psi_n(k)}-\sqrt{\psi_m(k)\psi_n(k-h)}\big| \\ &\qquad\le L\frac{|h|}{2^m}\sqrt{\psi_n(k)}+L\frac{|h|}{2^n}\sqrt{\psi_m(k)} \le L\frac{|h|}{2^{n-3}}.\end{aligned}$$

Moreover, by symmetry with respect to m and n,

$$\sum_{k\in\mathbb{Z}^d}\big(\sqrt{\psi_m(k-h)\psi_n(k)}+\sqrt{\psi_m(k)\psi_n(k-h)}\big)|v_{k-h}||v_k| = 2\sum_{k\in\mathbb{Z}^d}\sqrt{\psi_m(k-h)\psi_n(k)}\,|v_{k-h}||v_k|,$$

so that

$$|\phi_{(l,m,n)}| \le \frac{2^5 L}{X_l X_m X_n}\sum_{h\in\mathbb{Z}^d\setminus\{0\}}|h|^{1-\beta}\psi_l(h)|v_h|\sum_{k\in\mathbb{Z}^d}\sqrt{\psi_m(k-h)\psi_n(k)}\,|v_{k-h}||v_k|.$$

By the usual bound $2^{l-1} \le |h| \le 2^{l+1}$ and since $\beta \ge 0$, we see that $|h|^{1-\beta} \le 2^{l(1-\beta)+1+\beta}$ so, by Lemma 2.13,

$$|\phi_{(l,m,n)}| \le 2^5 2^{(1-\beta)l+1+\beta}2^{(l+3)d/2}L \le 2^{(d/2+1-\beta)(l-4)+9-3\beta+11d/2}L.$$

Since in the present case $\min\{l,m,n\} \ge l-4$, this proves inequality (2-9) with $c_3 = 2^{9+11d/2-3\beta}L$. $\square$

Finally we have all the ingredients to prove the main theorem of this section:

Proof of Theorem 2.8. A direct computation using (2-2) and (2-1) shows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}X_n^2 &= \operatorname{Re}\sum_{k\in\mathbb{Z}^d}\psi_n(k)\langle v_k', v_k\rangle \\ &= -\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\psi_n(k)\frac{|k|^\alpha}{g(|k|)}|v_k|^2 + \operatorname{Im}\sum_{k\in\mathbb{Z}^d}\sum_{h\in\mathbb{Z}^d\setminus\{0\}}\psi_n(k)\frac{\langle v_h,k\rangle}{|h|^\beta}\langle P_k(v_{k-h}),v_k\rangle \\ &= -\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\psi_n(k)\frac{|k|^\alpha}{g(|k|)}|v_k|^2 + \sum_{\substack{h,k\in\mathbb{Z}^d\\ h\neq 0}}\psi_n(k)\frac{\operatorname{Im}\{\langle v_h,k\rangle\langle v_{k-h},v_k\rangle\}}{|h|^\beta}.\end{aligned}$$

To deal with the first sum, define χ as in Proposition 2.10. By applying (2-7) for $X_n(t) \neq 0$ and (2-2) for $X_n(t) = 0$, we see that in both cases

$$2 \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \psi_n(k) \frac{|k|^\alpha}{g(|k|)} |v_k|^2 = \chi_n(t) X_n^2(t).$$

Now consider the second sum. Since the terms with $h = k$ give no contribution, we can apply

$$\sum_{l \in \mathbb{N}_0} \psi_l(h) = \sum_{m \in \mathbb{N}_0} \psi_m(k-h) = 1, \qquad h, k \in \mathbb{Z}^d,\ 0 \neq h \neq k,$$

to get

$$\begin{aligned}\sum_{\substack{h,k \in \mathbb{Z}^d \\ h \neq 0}} \psi_n(k) \frac{\operatorname{Im}\{\langle v_h, k\rangle \langle v_{k-h}, v_k\rangle\}}{|h|^\beta} &= \sum_{\substack{h,k \in \mathbb{Z}^d \\ h \neq 0}} \sum_{l,m \in \mathbb{N}_0} \psi_l(h)\psi_m(k-h)\psi_n(k) \frac{\operatorname{Im}\{\langle v_h, k\rangle \langle v_{k-h}, v_k\rangle\}}{|h|^\beta} \\ &= \sum_{l,m \in \mathbb{N}_0} \sum_{\substack{h,k \in \mathbb{Z}^d \\ h \neq 0}} \psi_l(h)\psi_m(k-h)\psi_n(k) \frac{\operatorname{Im}\{\langle v_h, k\rangle \langle v_{k-h}, v_k\rangle\}}{|h|^\beta},\end{aligned}$$

where it was possible to exchange the order of summation because the middle expression is clearly absolutely convergent.

Now define ϕ as in Proposition 2.11. By applying (2-8) or (2-2), depending on $X_l(t)X_m(t)X_n(t)$ being positive or zero, we see that, for all $l, m, n \in \mathbb{N}_0$ and $t \geq 0$,

$$2 \sum_{\substack{h,k \in \mathbb{Z}^d \\ h \neq 0}} \psi_l(h)\psi_m(k-h)\psi_n(k) \frac{\operatorname{Im}\{\langle v_h, k\rangle \langle v_{k-h}, v_k\rangle\}}{|h|^\beta} = \phi_{(l,m,n)}(t) X_l(t) X_m(t) X_n(t).$$

Putting it all together we get

$$\frac{d}{dt} X_n^2(t) = -\chi_n(t) X_n^2(t) + \sum_{l,m \in \mathbb{N}_0} \phi_{(l,m,n)}(t) X_l(t) X_m(t) X_n(t), \qquad n \in \mathbb{N}_0,\ t \geq 0.$$

Finally, recalling by Proposition 2.11 that $\phi \equiv 0$ outside I, we may restrict the scope of the sum and obtain (2-5). The required properties of the coefficients χ and ψ follow again from Propositions 2.10–2.11. □

3. From the dyadic equation to the recursive inequality

In view of the results of the previous section, we can now concentrate on shell solutions and forget (1-6). In this section we proceed as in [Barbato et al. 2014] and deduce a recursive inequality between the tails of energy and dissipation. Clearly here, due to the more complex nonlinear interaction, the relation is less trivial than in [Barbato et al. 2014].

Definition 3.1. A shell solution X satisfies the *energy inequality* on $[0, T]$ if $\sum_n X_n^2(0)$ is finite and

$$\sum_{n \in \mathbb{N}_0} X_n^2(t) + \int_0^t \sum_{n \in \mathbb{N}_0} \chi_n(s) X_n^2(s)\, ds \leq \sum_{n \in \mathbb{N}_0} X_n^2(0), \qquad t \in [0, T]. \tag{3-1}$$

Definition 3.2. Let X be a shell solution and define the sequences of real-valued maps $(F_n)_{n\in\mathbb{N}_0}$ and $(d_n)_{n\in\mathbb{N}_0}$ for $t \geq 0$ by

$$F_n(t) := \sum_{k\geq n} X_k^2(t), \quad d_n(t) := \left(F_n(t) + \sum_{h\geq n} \int_0^t \chi_h(s) X_h^2(s)\, ds\right)^{\frac{1}{2}}.$$

We will call $(F_n)_{n\in\mathbb{N}_0}$ the *tail* of X and $(d_n)_{n\in\mathbb{N}_0}$ the *energy bound* of X.

The recursive inequality between the tails and the energy bound is given in the next result.

Proposition 3.3. *Let X be a shell solution that satisfies the energy inequality on a time interval $[0, t]$, let $(d_n)_{n\in\mathbb{N}_0}$ be its sequence of energy bounds, and set $\lambda = 2^\alpha$.*

Then there is a positive constant $c_4 > 0$, not depending on t, such that, for all $n \in \mathbb{N}_0$,

$$d_n^2(t) \leq F_n(0) + c_4 \sum_{l=0}^{n-1} \frac{\bar{d}_l}{\lambda^{n-l}} \sum_{m\geq n-2} \frac{g(2^{m+1})}{\lambda^{m-n}} (d_m^2(t) - d_{m+1}^2(t)), \tag{3-2}$$

where $\bar{d}_l := \max_{s\in[0,t]} d_l(s)$.

Proof. Fix $n \in \mathbb{N}_0$. Differentiate $\sum_{h=0}^{n-1} X_h^2$ using (2-5):

$$\frac{d}{dt} \sum_{h=0}^{n-1} X_h^2 = -\sum_{h=0}^{n-1} \chi_h X_h^2 + \sum_{\substack{l,m,h\in\mathbb{N}_0 \\ (l,m,h)\in I \\ h\leq n-1}} \phi_{(l,m,h)} X_l X_m X_h.$$

Apply Lemma 3.4 below to the second sum and integrate on $[0, t]$ to obtain

$$\sum_{h=0}^{n-1} X_h^2(t) - \sum_{h=0}^{n-1} X_h^2(0) = -\int_0^t \sum_{h=0}^{n-1} \chi_h X_h^2\, ds - \int_0^t \sum_{\substack{(l,m,h)\in I \\ m<n\leq h}} \phi_{(l,m,h)} X_l X_m X_h\, ds$$

so that, by the energy inequality (3-1),

$$F_n(t) + \int_0^t \sum_{h\geq n} \chi_h(s) X_h^2(s)\, ds \leq F_n(0) + \int_0^t \sum_{\substack{(l,m,h)\in I \\ m<n\leq h}} \phi_{(l,m,h)} X_l(s) X_m(s) X_h(s)\, ds,$$

where the F_n are the tails of X and $F_n(0) < \infty$ by hypothesis. Thus, by the definition of d_n (Definition 3.2),

$$d_n^2(t) \leq F_n(0) + \int_0^t \sum_{\substack{(l,m,h)\in I \\ m<n\leq h}} \phi_{(l,m,h)} X_l(s) X_m(s) X_h(s)\, ds.$$

Recall that $\alpha + \beta \geq \frac{1}{2}d + 1$, hence the bound (2-6) for ϕ yields $\phi_{(l,m,h)} \leq c_2 \lambda^{\min\{l,m,h\}}$. Therefore

$$d_n^2(t) \leq F_n(0) + \int_0^t \sum_{\substack{(l,m,h)\in I \\ m<n\leq h}} c_2 \lambda^{\min\{l,m\}} |X_l(s) X_m(s) X_h(s)|\, ds.$$

It is convenient to split the set over which the sum is taken into the sets $\{l < m\}$ and $\{m \le l\}$:

$$\begin{aligned}
\sum_{\substack{(l,m,h)\in I\\ m<n\le h}} \lambda^{\min\{l,m\}}|X_l X_m X_h| &\le \sum_{\substack{(l,m,h)\in I\\ l<m<n\le h}} \lambda^l |X_l X_m X_h| + \sum_{\substack{(l,m,h)\in I\\ m<n\le h\\ m\le l}} \lambda^m |X_l X_m X_h| \\
&\le \sum_{\substack{(l,m,h)\in I\\ l<m<n\le h}} \lambda^l |X_l X_m X_h| + \sum_{\substack{(l,m,h)\in I\\ l<n\le h\\ l\le m}} \lambda^l |X_l X_m X_h| \\
&\le 2 \sum_{\substack{(l,m,h)\in I\\ l<n\le h\\ l\le m}} \lambda^l |X_l X_m X_h| \le 2\sum_{l=0}^{n-1} \lambda^l \bar{d}_l \sum_{h\ge n} \sum_{m=h-2}^{h+2} |X_m X_h|.
\end{aligned}$$

Apply the Cauchy–Schwarz inequality to get

$$2\sum_{h\ge n} \sum_{m=h-2}^{h+2} |X_h X_m| \le \sum_{h\ge n} \sum_{m=h-2}^{h+2} (X_h^2 + X_m^2) \le 10 \sum_{m\ge n-2} X_m^2.$$

Then by the bound on χ in (2-6), on all $[0, t]$,

$$\sum_{m\ge n-2} X_m^2 \le c_1^{-1} \sum_{m\ge n-2} \frac{g(2^{m+1})}{\lambda^m} \chi_m X_m^2.$$

Finally the integral of $\chi_m X_m^2$ can be bounded as follows, since $F_m(t)$ is nonincreasing with respect to m:

$$d_m^2(t) - d_{m+1}^2(t) = F_m(t) - F_{m+1}(t) + \int_0^t \chi_m(s) X_m^2(s)\, ds \ge \int_0^t \chi_m(s) X_m^2(s)\, ds.$$

Putting it all together we obtain

$$d_n^2(t) \le F_n(0) + 10\frac{c_2}{c_1} \sum_{l=0}^{n-1} \frac{\bar{d}_l}{\lambda^{-l}} \sum_{m\ge n-2} \frac{g(2^{m+1})}{\lambda^m} (d_m^2(t) - d_{m+1}^2(t)),$$

thus proving (3-2) with $c_4 = 10c_2/c_1$. □

Lemma 3.4. *Let X be a shell solution; then, for all $n \in \mathbb{N}_0 \setminus \{0\}$ and $s \in [0, t]$,*

$$\sum_{\substack{(l,m,h)\in I\\ h\le n-1}} \phi_{(l,m,h)} X_l X_m X_h = - \sum_{\substack{(l,m,h)\in I\\ m\le n-1<h}} \phi_{(l,m,h)} X_l X_m X_h. \tag{3-3}$$

Proof. By using (2-6) and noticing that $\min(l, m, h) \le n - 1$, we see that by the definition of shell solutions (Definition 2.4) the left-hand side of (3-3) is an absolutely convergent sum. Therefore we can exploit the cancellations due to the antisymmetry of ϕ, as in Remark 2.7. Indeed

$$\sum_{\substack{(l,m,h)\in I\\ h\le n-1}} \phi_{(l,m,h)} X_l X_m X_h = \sum_{\substack{(l,m,h)\in I\\ m<h\le n-1}} \phi_{(l,m,h)} X_l X_m X_h + \sum_{\substack{(l,m,h)\in I\\ h\le n-1\\ m>h}} \phi_{(l,m,h)} X_l X_m X_h \tag{3-4}$$

and

$$\sum_{\substack{(l,m,h)\in I\\ h\leq n-1\\ m>h}} \phi_{(l,m,h)} X_l X_m X_h = -\sum_{\substack{(l,m,h)\in I\\ h\leq n-1\\ m>h}} \phi_{(l,h,m)} X_l X_m X_h = -\sum_{\substack{(l,h',m')\in I\\ m'\leq n-1\\ h'>m'}} \phi_{(l,m',h')} X_l X_{m'} X_{h'}$$
$$= -\sum_{\substack{(l,m',h')\in I\\ m'\leq n-1\\ m'<h'}} \phi_{(l,m',h')} X_l X_{m'} X_{h'}. \tag{3-5}$$

By substituting (3-5) into (3-4) the conclusion follows. □

4. Solving the recursion

In this section we complete the proof of our main result. In the previous section we have shown a recursive inequality involving the energy bounds of a shell solution. The following theorem shows that shell solutions are smooth. By Theorem 2.8, the shell approximation of a solution of (1-6) is a shell solution; hence Theorem 2.3 holds, and in turn Theorem 1.2 holds as well.

Theorem 4.1. *Let X be a shell solution satisfying the energy inequality on $[0, t)$. If $\sup_n 2^{mn}|X_n(0)| < \infty$ for every $m \geq 1$, then*

$$\sup_{s\in[0,t]} \sup_n 2^{mn}|X_n(s)| < \infty \quad \textit{for every } m \geq 1.$$

Let $b_n = g(2^{n+1})^{-1}$, $n \geq 0$; then the assumptions of Theorem 1.2 for g, in terms of the sequence b, are

- $(b_n)_{n\in\mathbb{N}_0}$ is nonincreasing,
- $(\lambda^n b_n)_{n\in\mathbb{N}_0}$ is nondecreasing, and
- $\sum_n b_n = \infty$.

Let X be a shell solution as in the statement of Theorem 4.1, denote by $(d_n)_{n\in\mathbb{N}_0}$ and $(F_n)_{n\in\mathbb{N}_0}$ the energy bound and the tail of X (see Definition 3.2), and set $\bar{d}_n = \sup_{[0,t]} d_n(t)$ for every n. Set

$$Q_n = \sum_{j=0}^{n-1} \frac{\bar{d}_j}{\lambda^{n-j}} \quad \text{and} \quad R_n(t) = \sum_{j\geq n} \frac{d_j(t)^2 - d_{j+1}(t)^2}{\lambda^{j-n} b_j},$$

where $\lambda = 2^\alpha$ as in the previous section. We recall that, by Proposition 3.3,

$$d_n(t)^2 \leq F_n(0) + c_4 Q_n R_{n-2}(t). \tag{4-1}$$

We now collect some properties of the quantities R_n, Q_n, $\bar{d}_n$ that will be crucial in the proof of Theorem 4.1.

Lemma 4.2. (1) *For every $1 \leq m_1 \leq m_2$ and $t > 0$,*

$$\min\{R_{m_1}(t), R_{m_1+1}(t), \dots, R_{m_2}(t)\} \leq \frac{\lambda}{\lambda-1} \frac{d_{m_1}(t)^2}{\sum_{n=m_1}^{m_2} b_n}. \tag{4-2}$$

(2) *For every $t > 0$, $\liminf_n R_n(t) = 0$.*

(3) $\bar d_n \downarrow 0$ *as* $n \to \infty$.

(4) $Q_n \to 0$ *as* $n \to \infty$.

(5) $(Q_n)_{n\ge 1}$ *is eventually nonincreasing.*

Proof. Since $\lambda^n b_n$ is nondecreasing, we know that $b_n - \lambda^{-1} b_{n-1} \ge 0$. Hence, by exchanging the sums,

$$\sum_{n=m_1}^{\infty} (b_n - \lambda^{-1} b_{n-1}) R_n(t) = \sum_{k=m_1}^{\infty} \frac{d_k(t)^2 - d_{k+1}(t)^2}{\lambda^k b_k} \sum_{n=m_1}^{k} \big(\lambda^n b_n - \lambda^{n-1} b_{n-1}\big) \le \sum_{k=m_1}^{\infty} (d_k(t)^2 - d_{k+1}(t)^2) \le d_{m_1}(t)^2.$$

If $m_2 \ge m_1$, since $(b_n)_{n\ge 1}$ is nonincreasing,

$$\sum_{n=m_1}^{m_2} (b_n - \lambda^{-1} b_{n-1}) R_n(t) \ge \min\{R_{m_1}(t), \dots, R_{m_2}(t)\} \sum_{n=m_1}^{m_2} (b_n - \lambda^{-1} b_{n-1}) \ge \frac{\lambda-1}{\lambda} \left(\sum_{n=m_1}^{m_2} b_n \right) \min\{R_{m_1}(t), \dots, R_{m_2}(t)\}.$$

The claim $\liminf_n R_n(t) = 0$ follows from (4-2), since $d_n(t) \le d_1(t)$ for every n, and since, by the assumptions on $(b_n)_{n\ge 1}$, we can find a sequence $(m_k)_{k\ge 1}$ such that $\sum_{n=m_k}^{m_{k+1}-1} b_n \uparrow \infty$.

To prove that $\bar d_n \downarrow 0$, we notice that the sequence $(m_k)_{k\ge 1}$ mentioned above does not depend on t; hence, using the monotonicity of $(d_n(t))_{n\ge 1}$ and formula (4-2), we can prove that $\liminf_n \bar d_n = 0$, and hence $\bar d_n \downarrow 0$ by monotonicity. Once we know that $\bar d_n \downarrow 0$, an easy and standard argument proves that $Q_n \to 0$.

To prove that $(Q_n)_{n\ge 1}$ is eventually nonincreasing, we notice that, since $(\bar d_n)_{n\ge 1}$ is nonincreasing,

$$(Q_{n+1} - Q_n) = \frac{1}{\lambda}(Q_n - Q_{n-1}) + \frac{1}{\lambda}(\bar d_n - \bar d_{n-1}) \le \frac{1}{\lambda}(Q_n - Q_{n-1}).$$

In view of the above inequality, it is sufficient to show that for some m the difference $Q_m - Q_{m-1}$ is nonpositive. This is true because otherwise the sequence $(Q_n)_{n\ge 1}$ would be nondecreasing, in contradiction with $Q_n \to 0$ and $Q_n \ge 0$. □

Given $\theta > 0$ and $n_0 \ge 1$, define by recursion the sequence

$$n_{k+1} = 2 + \min\left\{ n \ge n_k - 1 : \sum_{j=n_k-1}^{n} b_j \ge \theta \lambda^{-k/4} \right\}. \tag{4-3}$$

The definition of Q_n and the fact that the sequence $(\bar d_n)_{n\ge 1}$ is nonincreasing yield the following recursive formula for Q_{n_k}:

$$Q_{n_{k+1}} = \frac{1}{\lambda^{n_{k+1}-n_k}} Q_{n_k} + \sum_{j=n_k}^{n_{k+1}-1} \frac{\bar d_j}{\lambda^{n_{k+1}-j}} \le \frac{1}{\lambda} Q_{n_k} + c \bar d_{n_k}, \tag{4-4}$$

for a constant $c > 0$ depending only on λ. Moreover, if we choose n_0 large enough that $(Q_n)_{n\geq 0}$ is nonincreasing,

$$d_{n_{k+1}}(t)^2 \leq d_n(t)^2 \leq F_n(0) + c_4 Q_n R_{n-2}(t) \leq F_{n_k}(0) + c_4 Q_{n_k} R_{n-2}(t)$$

for each $n \in \{n_k + 1, \ldots, n_{k+1}\}$; hence, by formula (4-2) and the definition of the sequence $(n_k)_{k\geq 1}$,

$$\begin{aligned} d_{n_{k+1}}(t)^2 &\leq F_{n_k}(0) + c_4 Q_{n_k} \min\{R_{n_k-1}, \ldots, R_{n_{k+1}-2}\} \\ &\leq F_{n_k}(0) + c Q_{n_k} \frac{d_{n_k-1}(t)^2}{\sum_{n_k-1}^{n_{k+1}-2} b_j} \leq F_{n_k}(0) + c \frac{\lambda^{k/4}}{\theta} Q_{n_k} d_{n_k-1}(t)^2 \end{aligned}$$

and, in conclusion,

$$\bar{d}^2_{n_{k+1}} \leq F_{n_k}(0) + c \frac{\lambda^{k/4}}{\theta} Q_{n_k} \bar{d}^2_{n_k-1}. \tag{4-5}$$

Lemma 4.3 (initial step of the cascade). *Given $M > 0$, there are $n_0 \geq 1$ and $\theta > 0$ such that*

$$Q_{n_k} \leq \lambda^{-k/2} \quad \textit{and} \quad \bar{d}^2_{n_k} \leq \lambda^{-Mk},$$

for all $k \geq 0$.

Proof. Without loss of generality we can choose M large (depending only on the value of λ; see the end of the proof). Choose n_0 large enough that $(Q_n)_{n\geq n_0}$ is nonincreasing and

$$Q_{n_0-i} \leq \epsilon, \quad \bar{d}_{n_0-i} \leq \epsilon, \quad i = 0, 1, \qquad \text{and} \qquad \lambda^{Mn} F_n(0) \leq \epsilon, \quad n \geq n_0,$$

for a number $\epsilon \in (0, 1)$ suitably chosen below. We will prove by induction that

$$Q_{n_{k-i}} \leq \lambda^{-(k-i)/2}, \quad \bar{d}^2_{n_{k-i}} \leq \lambda^{-M(k-i)}, \quad i = 0, 1, \; k \geq 1. \tag{4-6}$$

For the initial step of the induction ($k = 1$), we notice that, by (4-4) and (4-5),

$$\begin{aligned} Q_{n_1} &\leq \frac{1}{\lambda} Q_{n_0} + c\bar{d}_{n_0} \leq \frac{\epsilon}{\lambda} + c\epsilon \leq \frac{1}{\lambda^{1/2}}, \\ \bar{d}^2_{n_1} &\leq F_{n_0}(0) + \frac{c}{\theta} Q_{n_0} \bar{d}^2_{n_0-1} \leq \epsilon + \frac{c}{\theta}\epsilon^3 \leq \lambda^{-M}, \end{aligned}$$

if we choose ϵ small enough, depending on the values of λ, M and θ.

Assume now that (4-6) holds for some $k \geq 1$, and let us prove that the same holds for $k+1$. To this end it is sufficient to give the estimate for $Q_{n_{k+1}}$ and $\bar{d}^2_{n_{k+1}}$. Again by (4-4), (4-5) and the induction hypothesis, and since $(n_k)_{k\geq 0}$ is increasing by definition,

$$\begin{aligned} Q_{n_{k+1}} &\leq \frac{1}{\lambda} Q_{n_k} + c\bar{d}_{n_k} \leq \lambda^{-k/2-1} + c\lambda^{-Mk/2} \leq \lambda^{-(k+1)/2}, \\ \bar{d}^2_{n_{k+1}} &\leq F_{n_k}(0) + c\frac{\lambda^{k/4}}{\theta} Q_{n_k} \bar{d}^2_{n_k-1} \leq \epsilon\lambda^{-Mk} + \frac{c}{\theta}\lambda^{-k/4}\lambda^{-M(k-1)} \leq \lambda^{-M(k+1)}, \end{aligned}$$

if M is large (depending on λ), and ϵ is small and θ is large (depending only on M, λ). □

Before giving the last step of the proof of Theorem 4.1, we show a property of the sequence $(n_k)_{k\geq 0}$. The proof is the same as [Barbato et al. 2014, Lemma 11]; we give the details for completeness.

Lemma 4.4. *Given $n_0 \geq 1$ and $\theta > 0$, consider the sequence defined in (4-3). For infinitely many k, $n_{k+1} = n_k + 1$. In particular, $b_{n_k-1} \geq \theta\lambda^{-k/4}$ for all such k.*

Proof. Assume by contradiction that there is r such that $n_{k+1} \geq n_k + 2$ for $k \geq r$. On the one hand

$$\sum_{j=n_k-1}^{n_{k+1}-3} b_j \leq \theta\lambda^{-k/4},$$

and summing up over $k \geq r$ yields

$$\sum_{k\geq r}\sum_{j=n_k-1}^{n_{k+1}-3} b_j < \infty \quad \Longrightarrow \quad \sum_k b_{n_k-2} = \infty.$$

On the other hand, $b_{n_k-2} \leq b_{n_k-3} \leq \theta\lambda^{-(k-1)/4}$ and the series $\sum_k b_{n_k-2}$ converges. □

Lemma 4.5 (cascade recursion). *For every $M > 0$ there is $c_M > 0$ such that*

$$\bar{d}_n^2 \leq c_M\lambda^{-Mn}, \qquad Q_n \leq c_M\lambda^{-n}.$$

Proof. There is no loss of generality if we assume M is large. Let n_0, θ be the values provided by Lemma 4.3. By Lemma 4.3 and Lemma 4.4 there are infinitely many $k \geq 1$ such that

$$b_{n_k-1} \geq \theta\lambda^{-k/4}, \qquad Q_{n_k} \leq \lambda^{-k/2}, \qquad \bar{d}_{n_k}^2 \leq \lambda^{-Mk}. \tag{4-7}$$

Let k_0 be one such index, taken sufficiently large (the size of k_0 will be chosen at the end of the proof). We will prove by induction that

$$\bar{d}_{n_{k_0}+m}^2 \leq c\lambda^{-Mm}, \qquad Q_{n_{k_0}+m} \leq c'\lambda^{-m}, \qquad b_{n_{k_0}-1+m} \geq \theta\lambda^{-k_0/4-m}, \tag{4-8}$$

for a suitable choice of the constants $c > 0$, $c' > 0$. We first notice that there is nothing to prove concerning $b_{n_{k_0}-1+m}$, since this is a straightforward consequence of the choice of k_0 and the monotonicity of $(\lambda^n b_n)_{n\geq 1}$.

The initial step $m = 0$ holds, since the inequalities in (4-7) hold for the index k_0. For $m = 1$,

$$\bar{d}_{n_{k_0}+1}^2 \leq \bar{d}_{n_{k_0}}^2 \leq c\lambda^{-M},$$

$$Q_{n_{k_0}+1} = \frac{1}{\lambda}Q_{n_{k_0}} + \frac{1}{\lambda}\bar{d}_{n_{k_0}} \leq \frac{1}{\lambda}(\lambda^{-k_0/2} + \lambda^{-Mk_0/2}) \leq \frac{c'}{\lambda},$$

if $c = \lambda^{-M(k_0-1)}$ and $c' \geq \lambda^{-k_0/2} + \lambda^{-Mk_0/2}$.

Assume that (4-8) holds for $1, \ldots, m$, for some $m \geq 1$. By definition,

$$\begin{aligned} Q_{n_{k_0}+m+1} = Q_{n_{k_0}}\lambda^{-(m+1)} + \sum_{j=n_{k_0}}^{n_{k_0}+m} \frac{\bar{d}_j}{\lambda^{n_{k_0}+m+1-j}} &\leq \lambda^{-k_0/2-(m+1)} + \sqrt{c}\lambda^{-(m+1)}\sum_{j=0}^{m}\lambda^{-(M/2-1)j} \\ &\leq \left(\lambda^{-k_0/2} + \frac{\lambda}{\lambda-1}\sqrt{c}\right)\lambda^{-(m+1)} \\ &\leq c'\lambda^{-(m+1)} \end{aligned}$$

if $c' = \lambda^{-k_0/2} + \lambda(\lambda-1)^{-1}\sqrt{c}$ (the previous constraint on c' is satisfied by this choice).

By (4-1) and (4-2) we have that, for every $n \geq 2$,

$$d_{n+1}(t)^2 \leq F_{n+1}(0) + c_4 Q_{n+1} R_{n-1}(t) \leq F_{n+1}(0) + c_4 Q_{n+1} \frac{\bar{d}_{n-1}^2}{b_{n-1}};$$

hence, using the inequality for $Q_{n_{k_0}+m+1}$ already proved and the induction hypothesis,

$$\begin{aligned}
\bar{d}^2_{n_{k_0}+m+1} &\leq F_{n_{k_0}+m+1}(0) + c_4 Q_{n_{k_0+m+1}} \frac{\bar{d}^2_{n_{k_0}+m-1}}{b_{n_{k_0}+m-1}} \\
&\leq c\lambda^{-M(m+1)} \left(\lambda^{M(n_{k_0}+m+1)} F_{n_{k_0}+m+1}(0) + \frac{c_4}{\theta} c' \lambda^{2M+k_0/4} \right) \\
&\leq c 2^{-M(m+1)},
\end{aligned}$$

where the last inequality follows if k_0 is large enough since $\lambda^n F_n(0) \to 0$ by assumption, and by our choice of c, c' we have that $\lambda^{k_0/4} c' \to 0$ as $k_0 \to \infty$. □

Appendix A: Local existence and uniqueness

Consider the generalized system (1-6), under the same assumptions of Theorem 1.2. Assume[3] for simplicity that $m_1(k) = |k|^\alpha / g(|k|)$. Denote by V_m the subspace of H^m (see (2-3)) of divergence-free vector fields with mean zero. Our main theorem on local existence and uniqueness for (1-6) is as follows:

Theorem A.1. *Let $m \geq 2 + \frac{1}{2}d$ and $v_0 \in V_m$. Then there are $T > 0$ and a unique solution v of (1-6) on $[0, T]$ with initial condition v_0 such that*

$$v \in L^\infty([0, T]; V_m) \cap \mathrm{Lip}([0, T]; V_{m-\alpha}) \cap C([0, T]; V_m^{\mathrm{weak}}), \qquad \int_0^T \|D_1^{1/2} v\|_m^2 \, dt < \infty, \tag{A-1}$$

where V_m^{weak} is the space V_m with the weak topology. Moreover, v is right-continuous with values in V_m for the strong topology.

If $T_\star$ is the maximal time of existence of the solution starting from v_0, then either $T_\star = \infty$ or

$$\limsup_{t \uparrow T_\star} \|v(t)\|_m = \infty.$$

The proof of the theorem is based on a proof of existence of a local unique solution for the Euler equation taken from [Majda and Bertozzi 2002, Section 3.2]. The idea is that we cannot use the D_1 operator as a replacement for the Laplacian, since in general D_1 may not have smoothing properties (indeed, it is easy to adapt the counterexample in [Barbato et al. 2014, Remark 15] to D_1 on $\mathbb{R}^d$ or on the d-dimensional torus). Likewise we do not use any smoothing properties of D_2, so that our proof includes the case $\beta = 0$. The result is by no means optimal, but fits the needs of our paper.

[3]Existence and uniqueness can be proved also in the general case $m_1(k) \geq |k|^\alpha g(|k|)^{-1}$. A simple assumption that keeps our proof almost unchanged is a control from above, say $m(k) \leq |k|^\beta$ for some $\beta \geq \alpha$.

We work on the torus $[0, 2\pi]^d$, although the proof, essentially unchanged, works in $\mathbb{R}^d$. Denote by H the projection of $L^2([0, 2\pi]^d)$ onto divergence-free vector fields, and, for every $s > 0$, denote by V_s the projection of the Sobolev space $H^s([0, 2\pi]^d)$ onto divergence-free vector fields. We will denote by $\|\cdot\|_H$ and $\langle\cdot,\cdot\rangle_H$ the norm and the scalar product in H, and by $\|\cdot\|_s$ and $\langle\cdot,\cdot\rangle_s$ the norm and the scalar product in V_s.

We denote by $\widehat{B}(v_1, v_2)$ the (Leray) projection of the nonlinearity, namely

$$\widehat{B}(v_1, v_2) = \Pi_{\text{Leray}}[(D_2^{-1} v_1 \cdot \nabla) v_2].$$

Since $\beta \geq 0$, $\|D_2^{-1} v\|_s \leq \|v\|_s$ for every $s \in \mathbb{R}$. Hence (see for instance [Kato 1972] or [Constantin and Foiaş 1988]), for every $m \geq 1 + [d/2]$, there exists $c_m > 0$ such that

$$\|\widehat{B}(v_1, v_2)\|_m \leq c_m \|v_1\|_m \|v_2\|_{m+1},$$
$$\langle\widehat{B}(v_1, v_2), v_2\rangle_m \leq c_m \|v_1\|_m \|v_2\|_m^2.$$

In the rest of the section we briefly outline the proof of Theorem A.1, following [Majda and Bertozzi 2002, Section 3.2]. The proof of the following result is a slight modification of the arguments to prove [Majda and Bertozzi 2002, Theorem 3.4].

Proposition A.2. *Given an integer $m \geq 2 + d/2$, there exists a number $c_\star > 0$ such that for every $v_0 \in V_m$, if $T < c_\star/\|v_0\|_m$, there is a unique solution of (1-6) with initial condition v_0. Moreover, $v_\epsilon \to v$ in $C([0, T]; V_{m'})$ for $m' < m$ and in $C([0, T]; V_m^{\text{weak}})$, the inequalities in (A-1) hold for v, and for any $\epsilon > 0$,*

$$\sup_{[0,T]} \|v_\epsilon\|_m \leq \frac{\|v_0\|_m}{1 - c_\star T \|v_0\|_m}. \tag{A-2}$$

Unfortunately, at this stage, we cannot prove the analog of [Majda and Bertozzi 2002, Theorem 3.5] for our v, namely that v is continuous in time for the strong topology of V_m. The reason is that their proof uses either the reversibility of the Euler equation (which we do not have due to the presence of D_1), or the smoothing of the Laplace operator, which we do not have here either (as already mentioned). On the other hand, we can prove right-continuity:

Lemma A.3. *The solution v from Proposition A.2 is right-continuous with values in V_m for the strong topology, and dv/dt is right-continuous with values in $V_{m-\alpha}$.*

Proof. Given $t \in [0, T]$, the same computations leading to (A-2) yield

$$\sup_{[0,t]} \|v(s)\|_m \leq \|v_0\|_m + \frac{c_\star t \|v_0\|_m^2}{1 - c_\star t \|v_0\|_m};$$

therefore $\limsup_{t\downarrow 0} \|v(t)\|_m \leq \|v_0\|_m$. On the other hand, by weak continuity, $\|v_0\|_m \leq \liminf_{t\downarrow 0} \|v(t)\|_m$ and v is right-continuous at 0. Uniqueness for (1-6) and the same argument applied to $t \in (0, T]$ yield right-continuity in t. □

Nevertheless, we can still define a maximal solution and a maximal time of existence. Given $v_0 \in V_m$, let $T_\star$ be the maximal time of existence of the solution starting from v_0, that is the supremum over all $T > 0$ such that there exists a solution v of (1-6) on $[0, T]$ with $v(0) = u_0$, v right-continuous with values in V_m, continuous with values in V_m^{weak} and with dv/dt right-continuous with values in $V_{m-\alpha}$. Due to uniqueness, any two such solutions coincide on the common interval of definition.

Proposition A.4. *Given $v_0 \in V_m$, if $T_\star$ is the maximal time of existence of the solution starting from v_0, then either $T_\star = \infty$ or*

$$\limsup_{t \uparrow T_\star} \|v(t)\|_m = \infty.$$

Proof. Assume by contradiction that $T_\star < \infty$ and that $M := \sup_{t<T_\star} \|v(t)\|_m < \infty$. Let $T_0 = T_\star - c_\star/(4M)$, and start a solution with initial condition $v(T_0)$ at time T_0. By Proposition A.2 there is a solution of (1-6) on a time span of length at least $c_\star/(2\|v(T_0)\|_m) \geq c_\star/(2M)$, hence at least up to time $T_0 + c_\star/(2M) > T_\star$. By uniqueness, this solution is equal to v up to time $T_\star$. □

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Received 24 Jul 2014. Accepted 14 Dec 2014.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from MSP.

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ANALYSIS & PDE

Volume 7 No. 8 2014