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CRITERIA FOR HANKEL OPERATORS TO BE SIGN-DEFINITE

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We show that the total multiplicities of negative and positive spectra of a self-adjoint Hankel operator H in $L^2(\mathbb{R}_+)$ with integral kernel $h(t)$ and of the operator of multiplication by the inverse Laplace transform of $h(t)$, the distribution $\sigma(\lambda)$, coincide. In particular, $\pm H \geq 0$ if and only if $\pm\sigma(\lambda) \geq 0$. To construct $\sigma(\lambda)$, we suggest a new method of inversion of the Laplace transform in appropriate classes of distributions. Our approach directly applies to various classes of Hankel operators. For example, for Hankel operators of finite rank, we find an explicit formula for the total numbers of their negative and positive eigenvalues.

1. Introduction

1.1. Hankel operators can be defined as integral operators

$$(Hf)(t) = \int_0^\infty h(t+s) f(s)\, ds \tag{1-1}$$

in the space $L^2(\mathbb{R}_+)$ with kernels h that depend on the sum of variables only. Of course H is symmetric if $h(t) = \overline{h(t)}$. In the fundamental paper [Megretskiĭ et al. 1995], A. V. Megretskiĭ, V. V. Peller, and S. R. Treil characterized the spectra of all bounded self-adjoint Hankel operators by a certain balance between the positive and negative parts of their spectra. The result of [Megretskiĭ et al. 1995] applies to *all* Hankel operators, and so it does not allow one to distinguish spectral properties of particular operators.

The cases where Hankel operators can be explicitly diagonalized are very scarce. We mention here the kernels $h(t) = t^{-1}$ [Carleman 1923], $h(t) = (t+1)^{-1}$ [Mehler 1881], and $h(t) = t^{-1}e^{-t}$ [Magnus 1950; Rosenblum 1958a; 1958b]. These kernels are treated in a unified way in [Yafaev 2010], where some new examples are also considered.

Our goal here is to find explicit expressions for the total numbers $N_+(H)$ and $N_-(H)$ of (strictly) positive and negative eigenvalues of self-adjoint Hankel operators H. Actually, we show that $N_\pm(H) = N_\pm(\Sigma)$, where Σ is the operator[1] of multiplication by the function (distribution) $\sigma(\lambda)$ obtained through the inversion of the Laplace transform

$$h(t) = \int_0^\infty e^{-t\lambda} \sigma(\lambda)\, d\lambda. \tag{1-2}$$

We call $\sigma(\lambda)$ the *sigma function* of a Hankel operator H or of its kernel $h(t)$.

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[1] To be more precise, we consider the quadratic forms (Hf, f) and $(\Sigma\varphi, \varphi)$ instead of the operators H and Σ.

In particular, we obtain necessary and sufficient conditions for the sign-definiteness of Hankel operators. Indeed, it formally follows from (1-2) that

$$(Hf, f) = \int_0^\infty d\lambda\, \sigma(\lambda) \left| \int_0^\infty f(t) e^{-t\lambda}\, dt \right|^2,$$

and hence $\pm H \geq 0$ if and only if $\pm\sigma \geq 0$. We usually discuss conditions for $H \geq 0$, but of course replacing H by $-H$ we obtain conditions for $H \leq 0$. Note that positive[2] distributions are always given by some measures, so that for positive Hankel operators H, representation (1-2) reduces to

$$h(t) = \int_0^\infty e^{-t\lambda}\, dm(\lambda), \tag{1-3}$$

where $dm(\lambda)$ is a (positive) measure on $[0, \infty)$.

1.2. If a function $\sigma(\lambda)$ is sufficiently regular (for example, bounded), then its Laplace transform (1-2) is analytic in the right half-plane and satisfies certain decay conditions for $|t| \to \infty$. For example, such simple functions as the characteristic functions of intervals or $h(t) = e^{-t^2}$ do not satisfy these conditions. A regular function $\sigma(\lambda)$ can be recovered from its Laplace transform $h(t)$ by the integral of $h(a + i\tau)$, $a > 0$, over $\tau \in \mathbb{R}$; alternatively, it can also be recovered (see, for example, [Paley and Wiener 1934, Section 13]) from the values of $h(t)$ for $t > 0$. These methods are not sufficient for our purposes since, for example, for $h(t) = t^k e^{-\alpha t}$, $k \in \mathbb{Z}_+$, $\operatorname{Re}\alpha > 0$ (such Hankel operators have rank k), the corresponding function

$$\sigma(\lambda) = \delta^{(k)}(\lambda - \alpha) \tag{1-4}$$

($\delta(\,\cdot\,)$ is the Dirac function) is a highly singular distribution, especially if $\operatorname{Im}\alpha \neq 0$.

Thus we are led to a solution of (1-2) for $\sigma(\lambda)$ in a class of distributions. Put

$$b(\xi) = \frac{1}{2\pi} \frac{\int_0^\infty h(t) t^{-i\xi}\, dt}{\int_0^\infty e^{-t} t^{-i\xi}\, dt}, \tag{1-5}$$

and let $s(x) = \sqrt{2\pi}\,(\Phi^* b)(x)$, where Φ is the Fourier transform. We show that the function

$$\sigma(\lambda) := s(-\ln\lambda) \tag{1-6}$$

satisfies (1-2). We call $b(\xi)$ the *b-function* and $s(x)$ the *sign function* (or *s-function*) of the Hankel operator H (or of its integral kernel $h(t)$). The sigma function $\sigma(\lambda)$ differs from $s(x)$ by a change of variables only. In specific examples we consider, functions $b(\xi)$ and $s(x)$ may be of a quite different nature. For instance, $s(x)$ may be a polynomial or, on the contrary, it may be a highly singular distribution such as a combination of delta functions and their derivatives. We emphasize that all our formulas are understood in the sense of distributions and of course no analyticity of $h(t)$ is required. From a purely formal point of view, our method of inversion of the Laplace transform is not too far from one of the methods described in [Paley and Wiener 1934], but the classes of functions (distributions) are quite different.

[2] We use the term “positive” instead of the more precise but lengthy term “nonnegative”.

The precise meaning of formula (1-5) requires some discussion. Observe that the denominator in (1-5) coincides with the numerator for the special case $h(t)=e^{-t}$. It equals $\Gamma(1-i\xi)$, and hence exponentially tends to zero as $|\xi|\to\infty$. Therefore $b(\xi)$ is a "nice" function of ξ only under very restrictive assumptions on the kernel $h(t)$. Thus, to cover natural examples, we are obliged to work with distributions $b(\xi)$ and $s(x)$. The choice of appropriate spaces of distributions is also very important. The Schwartz space $\mathcal{S}(\mathbb{R})'$ is too restrictive for our purposes, which is seen already in the example of finite-rank Hankel operators. In order to be able to divide in (1-5) by an exponentially decaying function, we assume that the numerator belongs to the class of distributions $C_0^\infty(\mathbb{R})'$. This means that the Fourier transform of the function $\theta(x)=e^x h(e^x)$ belongs to $C_0^\infty(\mathbb{R})'$, that is, θ belongs to the space $\mathcal{Z}'$ dual to the space $\mathcal{Z}=\mathcal{Z}(\mathbb{R})$ of analytic test functions. The class of distributions $h(t)$ such that the corresponding function θ is in $\mathcal{Z}'$ will be denoted by $\mathcal{Z}'_+$. It follows from (1-5) that $b\in C_0^\infty(\mathbb{R})'$ and $s\in\mathcal{Z}'$ if $h\in\mathcal{Z}'_+$.

A remarkable circumstance is that, in these classes, there is a one-to-one correspondence between kernels of Hankel operators and their sigma functions. To be precise, let us put $h^\natural(\lambda)=\lambda^{-1}\sigma(\lambda)$. We show that $h\in\mathcal{Z}'_+$ if and only if $h^\natural\in\mathcal{Z}'_+$, and the correspondence $h\mapsto h^\natural$ is a continuous one-to-one mapping of $\mathcal{Z}'_+$ onto itself. As an example, note that although the functions $h(t)=t^k e^{-\alpha t}$ and $h^\natural(\lambda)=\lambda^{-1}\delta^{(k)}(\lambda-\alpha)$ are of a completely different nature, both of them belong to the class $\mathcal{Z}'_+$.

In the case $h\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, the condition $h\in\mathcal{Z}'_+$ means that

$$\int_0^\infty |h(t)|(1+|\ln t|)^{-\kappa}\,dt<\infty \tag{1-7}$$

for some κ. Condition (1-7) is also quite general and does not require that the corresponding Hankel operator be bounded. For example, it admits kernels

$$h(t)=P(\ln t)t^{-1}, \tag{1-8}$$

where $P(x)$ is an arbitrary polynomial. Note that Hankel operators with such kernels are bounded for $P(x)=\mathrm{const}$ only.

1.3. Our study of spectral properties of Hankel operators H relies on their reduction to the operators S of multiplication by the corresponding sign functions. This reduction is given by a transformation which is, in a suitable sense, invertible but not unitary. Let B,

$$(Bg)(\xi)=\int_{-\infty}^{\infty} b(\xi-\eta)g(\eta)\,d\eta, \tag{1-9}$$

be the operator of convolution with the function (1-5), and let S be the operator of multiplication by $s(x)$ so that $S=\Phi^* B\Phi$. If $h(t)=\overline{h(t)}$, then $b(-\xi)=\overline{b(\xi)}$ and $s(x)=\overline{s(x)}$ so that the operators B and S are formally symmetric.

We establish the identity

$$(Hf,f)=(Bg,g)=(Su,u), \tag{1-10}$$

where

$$g(\xi)=\Gamma\big(\tfrac12+i\xi\big)\tilde f(\xi)=:(\Xi f)(\xi),\quad u(x)=(\Phi^* g)(x), \tag{1-11}$$

$\tilde{f}(\xi)$ is the Mellin transform of $f(t)$, and $\Gamma(\cdot)$ is the gamma function. We often write the identity (1-10) in short form as

$$H = \Xi^* B \Xi = \widehat{\Xi}^* S \widehat{\Xi}, \tag{1-12}$$

where $\widehat{\Xi} = \Phi^* \Xi$.

It follows from (1-12) that the total multiplicities of the strictly positive (negative) spectra of the operators H and B, or S, coincide:

$$N_\pm(H) = N_\pm(B) = N_\pm(S). \tag{1-13}$$

This result can be compared with Sylvester's inertia theorem, which states the same for Hermitian matrices H and B, or S, related by (1-12) provided the matrix Ξ, or $\widehat{\Xi}$, is invertible. In contrast to linear algebra, in our case the operators H and B, or S, are of a completely different nature and B and S (but not H) admit explicit spectral analysis.

Thus our calculation of the numbers $N_\pm(H)$ consists of two parts. The first is the construction of the sign function (distribution) $s(x)$. The second is the study of the operator S of multiplication by $s(x)$. Observe that since $s(x)$ is a distribution, the numbers $N_\pm(S)$ are not necessarily zero or infinity. We also note that $N_\pm(S) = N_\pm(\Sigma)$ because the functions $s(x)$ and $\sigma(\lambda)$ differ by the change of variables (1-6) only.

In particular, we see that the Hankel operator H is positive if and only if $B \geq 0$ or, equivalently, $S \geq 0$. This means that a Hankel operator H is positive if and only if its sign function $s(x)$ is positive. In some cases the calculation of the sign function is not necessary. Actually, we show that if $|b(\xi)| \to \infty$ as $|\xi| \to \infty$, then H is not sign-definite.

Under the assumption $h \in \mathscr{Z}'_+$, we prove the identity (1-10) for test functions $f(t)$ whose Mellin transforms $\tilde{f}$ are in $C_0^\infty(\mathbb{R})$. Then functions (1-11) belong to $C_0^\infty(\mathbb{R})$ and both sides of (1-10) are well defined. The condition $h \in \mathscr{Z}'_+$ is very general. It is satisfied for *all* bounded, but also for a wide class of unbounded, Hankel operators H. More than that, it is not even required that H be defined by formula (1-1) on some dense set. Therefore we work with quadratic forms (Hf, f), which is more convenient and yields more general results. This context allows us to accommodate distributions $h(t)$ as kernels of Hankel operators and makes the theory self-consistent. Note that for bounded operators H, the identity (1-10) extends to all elements $f \in L^2(\mathbb{R}_+)$.

1.4. Representation (1-2) does not require the positivity of H. If, however, $H \geq 0$, then combining our results with the Bochner–Schwartz theorem, we obtain that $\sigma(\lambda)\, d\lambda = dm(\lambda)$, where $dm(\lambda)$ is a positive measure on $\mathbb{R}_+$ ($m(\{0\}) = 0$). In this case, representation (1-2) reduces to (1-3), with the measure $dm(\lambda)$ satisfying for some $\varkappa$ the condition

$$\int_0^\infty (1 + |\ln \lambda|)^{-\varkappa} \lambda^{-1}\, dm(\lambda) < \infty, \tag{1-14}$$

which follows from the assumption $h \in \mathscr{Z}'_+$.

Recall that according to the Bernstein theorem (see the original paper [1929] or [Akhiezer 1965; Widder 1941]), the representation (1-3) is true if and only if the function $h(t)$ is completely monotonic. In contrast to this classical result, we link the representation (1-3) to the positivity of the Hankel operator H with kernel $h(t)$. This fact is not very surprising in view of the analogy with the discrete case when Hankel operators are given in the space $\ell^2(\mathbb{Z}_+)$ by infinite matrices with elements h_{n+m} where $n, m \in \mathbb{Z}_+$. Indeed, according to the classical Hamburger theorem (see, e.g., [Akhiezer 1965]), the positivity of a discrete Hankel operator is equivalent to the existence of a solution of the moment problem with moments h_n. In the continuous case, the role of the moment problem is played by the exponential representation (1-3).

We mention that Hankel operators H with kernels $h(t)$ admitting representation (1-3) were considered by H. Widom [1966] and J. S. Howland [1971]. Such kernels $h(t)$ and operators H are necessarily positive. Widom proved that H is bounded if and only if $m([0,\lambda)) = O(\lambda)$ as $\lambda \to 0$ and as $\lambda \to \infty$. In this case, $h(t) \le Ct^{-1}$ for some $C > 0$. Howland showed that H belongs to the trace class if and only if condition (1-14) is satisfied for $\varkappa = 0$.

1.5. A large part of this paper is devoted to applying the general theory to various classes of Hankel operators H, although we do not try to cover all possible cases. In some examples, the sign-definiteness of H can also be verified or refuted with the help of Bernstein's theorems. Note, however, that our approach yields additionally an explicit formula for the total numbers $N_\pm(H)$ of positive and negative eigenvalues of H.

In Section 5, we calculate $N_\pm(H)$ for Hankel operators H of finite rank. Then we consider two specific examples. The first one is given by the formula

$$h(t) = t^k e^{-\alpha t}, \quad \alpha > 0,\ k \ge -1. \tag{1-15}$$

Note that the Hankel operator H with such kernel has finite rank for $k \in \mathbb{Z}_+$ only. We show that H is positive if and only if $k \le 0$. The second class of kernels is defined by the formula

$$h(t) = e^{-t^r}, \quad r > 0. \tag{1-16}$$

It turns out that the corresponding Hankel operator is positive if and only if $r \le 1$.

Section 6 is devoted to a study of Hankel operators H with kernels $h(t)$ having a singularity at a single point $t_0 > 0$. In this case the operators H are compact, but both numbers $N_\pm(H)$ are infinite. We find the asymptotics of positive ($\lambda_n^{(+)}$) and negative ($\lambda_n^{(-)}$) eigenvalues of H as $n \to \infty$ for singularities of different strengths.

Finally, in Section 7, we consider perturbations of the Carleman operator $\boldsymbol{C}$, that is, of the Hankel operator with kernel t^{-1}, by various classes of compact Hankel operators V. The operator $\boldsymbol{C}$ can be explicitly diagonalized by the Mellin transform. We recall that it has the absolutely continuous spectrum $[0, \pi]$ of multiplicity 2. The Carleman operator plays a distinguished role in the theory of Hankel operators. In particular, it is important for us that its sign function $s(x)$ equals 1. As was pointed out by Howland [1992], Hankel operators are to a certain extent similar to differential operators. In this analogy, the Carleman operator $\boldsymbol{C}$ plays the role of the "free" Schrödinger operator D^2, $D = -id/dx$, in the space $L^2(\mathbb{R})$. Furthermore, Hankel operators $H = \boldsymbol{C} + V$ with "perturbed" kernels $h(t) = t^{-1} + v(t)$ can be compared to Schrödinger operators $D^2 + \mathsf{V}(x)$. The assumption that $v(t)$ decays sufficiently rapidly as

$t \to \infty$ and is not too singular as $t \to 0$ corresponds to a sufficiently rapid decay of the potential $\mathsf{V}(x)$ as $|x| \to \infty$.

As shown in [Yafaev 2013], the results on the discrete spectrum of the operator H lying *above* its essential spectrum $[0, \pi]$ are close in spirit to the results on the discrete spectrum of the Schrödinger operator $D^2 + \mathsf{V}(x)$. On the other hand, the results on the negative spectrum of the Hankel operator H are drastically different. In particular, contrary to the case of differential operators with decaying coefficients, the finiteness of the negative spectrum of the Hankel operator H is not determined by the behavior of $v(t)$ at singular points $t = 0$ and $t = \infty$. As an example, consider the Hankel operator with kernel

$$h(t) = t^{-1} - \gamma e^{-t^r}, \quad r \in (0, 1).$$

Now the kernel of the perturbation is a function that decays faster than any power of t^{-1} as $t \to \infty$, and it has a finite limit as $t \to 0$. Nevertheless, we show that the negative spectrum of H is infinite if $\gamma > \gamma_0$ (here $\gamma_0 = \gamma_0(r)$ is an explicit constant), while H is positive if $\gamma \le \gamma_0$. Such a phenomenon has no analogy for Schrödinger operators with decaying potentials, although a somewhat similar effect (known as the Efimov effect) occurs for three-particle Schrödinger operators. Note, however, that for $\gamma > \gamma_0$, a new band of the continuous spectrum appears for three-particle systems, while in our case, the continuous spectrum of H is $[0, \pi]$ for all values of γ.

We also study perturbations of the Carleman operator $\boldsymbol{C}$ by Hankel operators V of finite rank. Here we obtain a striking result: the total numbers of negative eigenvalues of the operators $H = \boldsymbol{C} + V$ and V coincide.

As examples, we consider only bounded Hankel operators in this paper. However, our general results directly apply to a wide class of unbounded operators, such as Hankel operators with kernels (1-8); see [Yafaev 2014a].

1.6. Let us briefly describe the structure of the paper. In Section 2, we define the basic objects, establish the inversion formula (1-2), and obtain the main identity (1-10). Necessary information on bounded Hankel operators (including a continuous version of the Nehari theorem) is collected in Section 3. In Sections 2 and 3, we do not assume that the function h is real, i.e., the corresponding Hankel operator H is not necessarily symmetric. Spectral consequences of the formula (1-10) and, in particular, criteria for the sign-definiteness of Hankel operators are stated in Section 4. In Sections 5, 6, and 7, we apply the general theory to particular classes of Hankel operators.

Let us introduce some standard notation. We denote by Φ,

$$(\Phi u)(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} u(x) e^{-ix\xi}\, dx,$$

the Fourier transform. The space $\mathcal{Z} = \mathcal{Z}(\mathbb{R})$ of test functions is defined as the subset of the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R})$ which consists of functions φ admitting the analytic continuation to entire functions in the complex plane $\mathbb{C}$ and satisfying bounds

$$|\varphi(z)| \le C_n (1 + |z|)^{-n} e^{r|\operatorname{Im} z|}, \quad \text{for all } z \in \mathbb{C},$$

for some $r = r(\varphi) > 0$ and all n. Note that $\mathcal{Z}$ is invariant with respect to the complex conjugation $\varphi(z) \mapsto \varphi^*(z) = \overline{\varphi(\bar{z})}$. We recall that the Fourier transform Φ maps $\mathcal{Z} \to C_0^\infty(\mathbb{R})$ and that $\Phi^* : C_0^\infty(\mathbb{R}) \to \mathcal{Z}$. The dual classes of distributions (continuous antilinear functionals) are denoted by $\mathcal{S}'$, $C_0^\infty(\mathbb{R})'$, and $\mathcal{Z}'$, respectively. In general, for a linear topological space $\mathcal{L}$, we use the notation $\mathcal{L}'$ for its dual space.

We use the notation $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and $\langle \cdot, \cdot \rangle$ for the duality symbols in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R})$, respectively. They are always linear in the first argument and antilinear in the second argument. The letter C (sometimes with indices) denotes various positive constants whose precise values are inessential.

2. The main identity

2.1. Let us consider the Hankel operator H defined by equality (1-1) in the space $L^2(\mathbb{R}_+)$. Actually, it is more convenient to work with sesquilinear forms (Hf_1, f_2) instead of operators.

Before giving precise definitions, let us explain our construction at a formal level. It follows from (1-1) that

$$(Hf_1, f_2) = \int_0^\infty \int_0^\infty h(t+s) f_1(s) \overline{f_2(t)}\, dt\, ds = \int_0^\infty h(t) \overline{F(t)}\, dt =: \langle\!\langle h, F \rangle\!\rangle, \tag{2-1}$$

where

$$F(t) = \int_0^t \overline{f_1(s)} f_2(t-s)\, ds =: (\bar{f}_1 \star f_2)(t) \tag{2-2}$$

is the Laplace convolution of the functions $\bar{f}_1$ and f_2. Formula (2-1) allows us to consider h as a distribution with the test function F defined by (2-2). Thus the Hankel quadratic form will be defined by the relation

$$h[f_1, f_2] := \langle\!\langle h, \bar{f}_1 \star f_2 \rangle\!\rangle. \tag{2-3}$$

Let us introduce the test function

$$\Omega(x) = F(e^x) =: (\mathcal{R}F)(x) \tag{2-4}$$

and the distribution

$$\theta(x) = e^x h(e^x) \tag{2-5}$$

defined for $x \in \mathbb{R}$. Setting $t = e^x$ in (2-1), we see that

$$\langle\!\langle h, F \rangle\!\rangle = \int_{-\infty}^\infty \theta(x) \overline{\Omega(x)}\, dx =: \langle \theta, \Omega \rangle. \tag{2-6}$$

We are going to consider the form (2-6) on pairs F, h such that the corresponding test function Ω defined by (2-4) is an element of the space $\mathcal{Z}$ of analytic functions and the corresponding distribution θ defined by (2-5) is an element of the dual space $\mathcal{Z}'$. The set of all such F and h will be denoted by $\mathcal{Z}_+$ and $\mathcal{Z}'_+$, respectively; that is,

$$F \in \mathcal{Z}_+ \iff \Omega \in \mathcal{Z} \quad \text{and} \quad h \in \mathcal{Z}'_+ \iff \theta \in \mathcal{Z}'. \tag{2-7}$$

Of course, the topology in $\mathcal{Z}_+$ is induced by that in $\mathcal{Z}$ and $\mathcal{Z}'_+$ is dual to $\mathcal{Z}_+$. Note that $h \in \mathcal{Z}'_+$ if $h \in L^1_{\text{loc}}(\mathbb{R}_+)$ and integral (1-7) is convergent for some κ. In this case, the corresponding function (2-5) satisfies the condition

$$\int_{-\infty}^{\infty} |\theta(x)|(1+|x|)^{-\kappa}\,dx < \infty,$$

and hence $\theta \in \mathcal{S}' \subset \mathcal{Z}'$.

Define the unitary operator $U : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ by the equality

$$(Uf)(x) = e^{x/2} f(e^x). \tag{2-8}$$

Let the set $\mathcal{D}$ consist of functions $f(t)$ such that $Uf \in \mathcal{Z}$. Since

$$f(t) = t^{-1/2}(Uf)(\ln t)$$

and $\mathcal{Z} \subset \mathcal{S}$, we see that functions $f \in \mathcal{D}$ and their derivatives satisfy the estimates

$$|f^{(m)}(t)| = C_{n,m} t^{-1/2-m}(1+|\ln t|)^{-n}$$

for all n and m. Obviously, $f \in \mathcal{D}$ if and only if $\varphi(t) = t^{1/2} f(t)$ belongs to the class $\mathcal{Z}_+$.

Let us show that form (2-3) is correctly defined on functions $f_1, f_2 \in \mathcal{D}$. To that end, we have to verify that function (2-2) belongs to the space $\mathcal{Z}_+$ or, equivalently, that function (2-4) belongs to the space $\mathcal{Z}$. This requires some preliminary study, which will also allow us to derive a convenient representation for form (2-3).

Recall that the Mellin transform $M : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ is defined by the formula

$$(Mf)(\xi) = (2\pi)^{-1/2} \int_0^{\infty} f(t) t^{-1/2-i\xi}\,dt. \tag{2-9}$$

Of course, $M = \Phi U$, where Φ is the Fourier transform and U is operator (2-8). Since both Φ and U are unitary, the operator M is also unitary. The inversion of the formula (2-9) is given by the relation

$$f(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \tilde{f}(\xi) t^{-1/2+i\xi}\,d\xi, \quad \tilde{f} = Mf. \tag{2-10}$$

Let $\Gamma(z)$ be the gamma function. Recall that $\Gamma(z)$ is a holomorphic function in the right half-plane and $\Gamma(z) \neq 0$ for all $z \in \mathbb{C}$. According to the Stirling formula, the function $\Gamma(z)$ tends to zero exponentially as $|z| \to \infty$ parallel with the imaginary axis. To be more precise, we have

$$\Gamma(\alpha + i\lambda) = e^{\pi i(2\alpha-1)/4} \left(\frac{2\pi}{e}\right)^{1/2} \lambda^{\alpha-1/2} e^{i\lambda(\ln\lambda - 1)} e^{-\pi\lambda/2}(1 + O(\lambda^{-1})) \tag{2-11}$$

for a fixed $\alpha > 0$ and $\lambda \to +\infty$. Since $\Gamma(\alpha - i\lambda) = \overline{\Gamma(\alpha + i\lambda)}$, this yields also the asymptotics of $\Gamma(\alpha + i\lambda)$ as $\lambda \to -\infty$.

If $f_j \in \mathcal{D}$, $j = 1, 2$, then $\tilde{f}_j = Mf_j = \Phi U f_j \in C_0^{\infty}(\mathbb{R})$, and hence the functions $g_j(\xi)$ defined by formula (1-11) also belong to the class $C_0^{\infty}(\mathbb{R})$. Let us introduce the convolution of the functions g_1 and g_2,

$$(g_1 * g_2)(\xi) = \int_{-\infty}^{\infty} g_1(\xi - \eta) g_2(\eta)\,d\eta,$$

and set

$$(\mathcal{J}g)(\xi)=g(-\xi).$$

We have the following result.

Lemma 2.1. *Suppose that* $f_j\in\mathcal{D}$, $j=1,2$, *and define functions* $g_j(\xi)$ *by equality* (1-11). *Let the function* $\Omega(x)$ *be defined by formulas* (2-2) *and* (2-4). *Then*

$$(\Phi\Omega)(\xi)=(2\pi)^{-1/2}\Gamma(1+i\xi)^{-1}((\mathcal{J}\bar{g}_1)*g_2)(\xi). \tag{2-12}$$

Proof. Substituting (2-10) into (2-2), we see that

$$F(t)=(2\pi)^{-1}\int_0^t ds\int_{-\infty}^{\infty}\overline{\tilde{f}_1(\tau)}(t-s)^{-1/2-i\tau}\,d\tau\int_{-\infty}^{\infty}\tilde{f}_2(\sigma)s^{-1/2+i\sigma}\,d\sigma.$$

Observe that

$$\int_0^t (t-s)^{-1/2-i\tau}s^{-1/2+i\sigma}\,ds=t^{i(\sigma-\tau)}\frac{\Gamma\big(\frac12-i\tau\big)\Gamma\big(\frac12+i\sigma\big)}{\Gamma(1+i(\sigma-\tau))}.$$

Then using (1-11), we obtain the representation

$$\begin{aligned}F(t)&=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}t^{i(\sigma-\tau)}\Gamma(1+i(\sigma-\tau))^{-1}\overline{g_1(\tau)}g_2(\sigma)\,d\tau\,d\sigma\\&=(2\pi)^{-1}\int_{-\infty}^{\infty}t^{i\xi}\Gamma(1+i\xi)^{-1}((\mathcal{J}\bar{g}_1)*g_2)(\xi)\,d\xi,\end{aligned}$$

whence

$$\Omega(x)=(2\pi)^{-1}\int_{-\infty}^{\infty}e^{i\xi x}\Gamma(1+i\xi)^{-1}((\mathcal{J}\bar{g}_1)*g_2)(\xi)\,d\xi.$$

This is equivalent to formula (2-12). □

Observe that the function $\Gamma(1+i\xi)^{-1}$ on the right-hand side of (2-12) tends to infinity exponentially as $|\xi|\to\infty$. Nevertheless, $\Phi\Omega\in C_0^\infty(\mathbb{R})$ because $(\mathcal{J}\bar{g}_1)*g_2\in C_0^\infty(\mathbb{R})$ for $g_1, g_2\in C_0^\infty(\mathbb{R})$. Thus we have:

Corollary 2.2. *Let* $f_j\in\mathcal{D}$, $j=1,2$, *and let the function* $\Omega(x)$ *be defined by formulas* (2-2) *and* (2-4). *Then* $\Omega\in\mathcal{Z}$ *or, equivalently,* $F\in\mathcal{Z}_+$.

Now we are in a position to give the precise definition.

Definition 2.3. Let $h\in\mathcal{Z}'_+$ and $f_j\in\mathcal{D}$, $j=1,2$. Then $\bar{f}_1\star f_2\in\mathcal{Z}_+$ and the Hankel sesquilinear form is defined by the relation (2-3).

We shall see in Section 2.4 that $h\in\mathcal{Z}'_+$ is determined uniquely by the values $\langle h,\bar{f}_1\star f_2\rangle$ on $f_1, f_2\in\mathcal{D}$, that is, $h=0$ if $\langle h,\bar{f}_1\star f_2\rangle=0$ for all $f_1, f_2\in\mathcal{D}$.

Of course (2-3) can be rewritten as

$$h[f_1,f_2]=\langle\theta,\Omega\rangle, \tag{2-13}$$

where θ is distribution (2-5) and

$$\Omega(x)=(\bar{f}_1\star f_2)(e^x).$$

We sometimes write $h[f_1, f_2]$ as integral (2-1), keeping in mind that its precise meaning is given by Definition 2.3.

2.2. Our next goal is to show that (2-13) is the sesquilinear form of the convolution operator B, that is, it equals the right-hand side of (1-10). Here the representation of Lemma 2.1 for the function

$$G(\xi) := \sqrt{2\pi}\,\Gamma(1+i\xi)(\Phi\Omega)(\xi) \tag{2-14}$$

plays a crucial role.

Since θ is in $\mathcal{Z}'$, its Fourier transform $a = \Phi\theta$ is correctly defined as an element of $C_0^\infty(\mathbb{R})'$. Formally,

$$a(\xi) = (\Phi\theta)(\xi) = (2\pi)^{-1/2} \int_0^\infty h(t) t^{-i\xi}\, dt, \tag{2-15}$$

that is, $a(\xi)$ is the Mellin transform of the function $h(t)t^{1/2}$. Let $\Omega \in \mathcal{Z}$. Passing to the Fourier transforms and using notation (2-14), we see that

$$\langle \theta, \Omega \rangle = \langle a, \Phi\Omega \rangle = \langle b, G \rangle, \tag{2-16}$$

where $G \in C_0^\infty(\mathbb{R})$ and the distribution $b \in C_0^\infty(\mathbb{R})'$ is given by the relation

$$b(\xi) = (2\pi)^{-1/2} a(\xi) \Gamma(1-i\xi)^{-1}, \tag{2-17}$$

which is of course the same as (1-5). Thus we are led to the following.

Definition 2.4. Let $h \in \mathcal{Z}'_+$. The distribution $b \in C_0^\infty(\mathbb{R})'$ defined by formulas (2-5), (2-15), and (2-17) is called the b-function of the kernel $h(t)$ (or of the Hankel operator H). Its Fourier transform $s = \sqrt{2\pi}\,\Phi^* b \in \mathcal{Z}'$ is called the s-function or the sign function.

Recall that the distribution σ was defined by relation (1-6). It is convenient to also introduce

$$h^\natural(\lambda) = \lambda^{-1}\sigma(\lambda) = \lambda^{-1} s(-\ln\lambda). \tag{2-18}$$

The following assertion is an immediate consequence of formulas (2-5), (2-15), and (2-17).

Proposition 2.5. *The mappings*

$$h \mapsto \theta \mapsto a \mapsto b \mapsto s \mapsto h^\natural$$

yield one-to-one correspondences (*bijections*)

$$\mathcal{Z}'_+ \to \mathcal{Z}' \to C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R}) \to \mathcal{Z}' \to \mathcal{Z}'_+.$$

All of them, *as well as their inverse mappings*, *are continuous.*

Putting together equalities (2-6) and (2-16), we see that

$$\langle h, F \rangle = \langle b, G \rangle. \tag{2-19}$$

Combining this relation with Lemma 2.1 and Definitions 2.3, 2.4 and using notation (1-11), we obtain the main identity (1-10). To be more precise, we have the following result.

Theorem 2.6. *Suppose that $h \in \mathscr{Z}'_+$, and let $b \in C_0^\infty(\mathbb{R})'$ be the corresponding b-function. Let $f_j \in \mathscr{D}$, $j = 1, 2$, and let the functions g_j be defined by formula* (1-11). *Then $g_j \in C_0^\infty(\mathbb{R})$, and the representation*

$$\langle h, \bar{f}_1 \star f_2 \rangle = \langle b, (\mathscr{J}\bar{g}_1) * g_2 \rangle =: b[g_1, g_2] \tag{2-20}$$

holds.

Passing to the Fourier transforms on the right-hand side of (2-20) and using

$$\Phi^*((\mathscr{J}\bar{g}_1) * g_2) = (2\pi)^{1/2}\overline{\Phi^* g_1}\Phi^* g_2,$$

we obtain:

Corollary 2.7. *Let $s \in \mathscr{Z}'$ be the sign function of h, and let $u_j = \Phi^* g_j = \Phi^* \Xi f_j \in \mathscr{Z}$. Then*

$$\langle h, \bar{f}_1 \star f_2 \rangle = \langle s, u_1^* u_2 \rangle =: s[u_1, u_2]. \tag{2-21}$$

Loosely speaking, equalities (2-20) and (2-21) mean that

$$\langle h, \bar{f}_1 \star f_2 \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b(\xi - \eta) g_1(\eta)\overline{g_2(\xi)}\, d\xi\, d\eta = \int_{-\infty}^{\infty} s(x) u_1(x)\overline{u_2(x)}\, dx. \tag{2-22}$$

In the particular case $h(t) = t^{-1}$ when $H = \boldsymbol{C}$ is the Carleman operator, we have

$$\theta(x) = 1, \quad a(\xi) = (2\pi)^{1/2}\delta(\xi), \quad b(\xi) = \delta(\xi), \quad s(x) = 1, \tag{2-23}$$

and hence (2-22) yields

$$\langle h, \bar{f}_1 \star f_2 \rangle = \int_{-\infty}^{\infty} g_1(\xi)\overline{g_2(\xi)}\, d\xi = \int_{-\infty}^{\infty} \left|\Gamma\left(\tfrac{1}{2} + i\xi\right)\right|^2 \tilde{f}_1(\xi)\overline{\tilde{f}_2(\xi)}\, d\xi,$$

where

$$\left|\Gamma\left(\tfrac{1}{2} + i\xi\right)\right|^2 = \frac{\pi}{\cosh(\pi\xi)}. \tag{2-24}$$

This leads to the familiar diagonalization of the Carleman operator.

2.3. According to Proposition 2.5, the distribution $h^\natural$ determines uniquely the distribution h. Let us now obtain an explicit formula for the mapping $h^\natural \mapsto h$. This requires some auxiliary information.

Let $\boldsymbol{\Gamma}_\alpha : C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R})$, $\alpha > 0$, be the operator of multiplication by the function $\Gamma(\alpha + i\xi)$. Making the change of variables $t = e^{-x}$ in the definition of the gamma function, we see that

$$\Gamma(\alpha + i\lambda) = \int_0^\infty e^{-t} t^{\alpha + i\lambda - 1}\, dt = \int_{-\infty}^{\infty} e^{-e^{-x}} e^{-\alpha x} e^{-ix\lambda}\, dx, \quad \alpha > 0,$$

and hence

$$(2\pi)^{-1}\int_{-\infty}^{\infty} e^{ix\lambda}\Gamma(\alpha + i\lambda)\, d\lambda = e^{-e^{-x}} e^{-\alpha x}. \tag{2-25}$$

It follows that

$$(\Phi^* \boldsymbol{\Gamma}_\alpha \Phi \Omega)(x) = \int_{-\infty}^{\infty} e^{\alpha(y - x)} e^{-e^{y - x}}\Omega(y)\, dy. \tag{2-26}$$

Let us also introduce the operator L_α:

$$(\mathsf{L}_\alpha F)(\lambda) = \lambda^\alpha \int_0^\infty e^{-t\lambda} t^{\alpha-1} F(t)\, dt, \quad \lambda > 0, \quad \alpha > 0. \tag{2-27}$$

Obviously, $\mathsf{L}_\alpha F \in C^\infty(\mathbb{R}_+)$ for all bounded functions $F(t)$ and, in particular, for $F \in \mathcal{X}_+$. Note that L_α is the Laplace operator L,

$$(\mathsf{L} F)(\lambda) = \int_0^\infty e^{-t\lambda} F(t)\, dt, \tag{2-28}$$

sandwiched by the weights λ^α and $t^{\alpha-1}$.

The following result yields the whole scale of spaces where the Laplace operator L acts as an isomorphism. Recall that the operator $\mathcal{R}$ defined by (2-4) is a one-to-one mapping of $\mathcal{X}_+$ onto $\mathcal{X}$.

Lemma 2.8. *For all $\alpha > 0$, the identity*

$$\mathsf{L}_\alpha = \mathcal{R}^{-1} \mathcal{J} \Phi^* \mathbf{\Gamma}_\alpha \Phi \mathcal{R} \tag{2-29}$$

holds. In particular, L_α and its inverse are the one-to-one continuous mappings of $\mathcal{X}_+$ onto itself.

Proof. Putting $\Omega(y) = (\mathcal{R}F)(y) = F(e^y)$ in (2-26) and making the change of variables $t = e^y$, we find that

$$(\Phi^* \mathbf{\Gamma}_\alpha \Phi \mathcal{R} F)(x) = e^{-\alpha x} \int_0^\infty e^{-e^{-x} t} t^{\alpha-1} F(t)\, dt.$$

Now making the change of variables $\lambda = e^{-x}$, we arrive at the identity (2-29).

Consider the right-hand side of (2-29). All mappings $\mathcal{R} : \mathcal{X}_+ \to \mathcal{X}$, $\Phi : \mathcal{X} \to C_0^\infty(\mathbb{R})$, $\mathbf{\Gamma}_\alpha : C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R})$, $\Phi^* : C_0^\infty(\mathbb{R}) \to \mathcal{X}$, $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ are bijections. All of them as well as their inverses are continuous. Therefore the identity (2-29) ensures the same result for the operator $\mathsf{L}_\alpha : \mathcal{X}_+ \to \mathcal{X}_+$. □

The adjoint operators L_α^* are defined by the relation $\langle \mathsf{L}_\alpha F, \psi \rangle = \langle F, \mathsf{L}_\alpha^* \psi \rangle$, where $F \in \mathcal{X}_+$ and $\psi \in \mathcal{X}_+'$ are arbitrary. According to (2-27), they are formally given by the relation

$$(\mathsf{L}_\alpha^* \psi)(t) = t^{\alpha-1} \int_0^\infty e^{-t\lambda} \lambda^\alpha \psi(\lambda)\, d\lambda, \quad t > 0. \tag{2-30}$$

By duality, the next assertion follows from Lemma 2.8.

Theorem 2.9. *For all $\alpha > 0$, the operators L_α^* as well as their inverses are the one-to-one continuous mappings of $\mathcal{X}_+'$ onto itself.*

To recover $h(t)$, we proceed from formula (2-19). Passing to the Fourier transforms, we can write it as

$$\langle h, F \rangle = (2\pi)^{-1/2} \langle s, \Phi^* G \rangle,$$

where G is defined by formulas (2-4), (2-14), that is, $G = (2\pi)^{1/2} \mathbf{\Gamma}_1 \Phi \mathcal{R} F$. Therefore, using the identity (2-29) for $\alpha = 1$, we see that

$$\langle h, F \rangle = \langle s, \mathcal{J} \mathcal{R} \mathsf{L}_1 F \rangle = \int_{-\infty}^\infty s(x) \overline{(\mathsf{L}_1 F)(e^{-x})}\, dx.$$

Making the change of variables $\lambda = e^{-x}$ and using notation (2-18), we obtain the identity

$$\langle h, F\rangle = \langle h^{\natural}, \mathsf{L}_1 F\rangle.$$

Passing here to adjoint operators and taking into account that $F \in \mathcal{X}_+$ is arbitrary, we find that

$$h = \mathsf{L}_1^* h^{\natural} \quad \text{or} \quad h = \mathsf{L}^* \sigma, \tag{2-31}$$

where $\sigma(\lambda) = \lambda h^{\natural}(\lambda)$. In view of (2-30), this gives the precise sense to formula (1-2).

Let us state the result obtained.

Theorem 2.10. *Let $h \in \mathcal{X}'_+$, and let $s \in \mathcal{X}'$ be the corresponding sign function* (*see Definition 2.4*). *Define the distribution $h^{\natural}$ by formula* (2-18). *Then $h^{\natural} \in \mathcal{X}'_+$ and h can be recovered from $h^{\natural}$ or σ by formulas* (2-31).

We emphasize that in the roundabout $h \mapsto h^{\natural} \mapsto h$, the mapping $h \mapsto h^{\natural}$ and its inverse $h^{\natural} \mapsto h$ are the one-to-one continuous mappings of the set $\mathcal{X}'_+$ onto itself.

Let us also give a direct expression of $u(x) = (\Phi^* g)(x)$ in terms of $f(t)$.

Lemma 2.11. *Suppose that $f \in \mathcal{D}$ and put $\varphi(t) = t^{1/2} f(t)$. Let $g(\xi)$ be defined by formula* (1-11) *and $u(x) = (\Phi^* g)(x)$. Then*

$$u(x) = (\mathsf{L}_{1/2}\varphi)(e^{-x}). \tag{2-32}$$

Proof. Since $(\mathcal{R}\varphi)(x) = (Uf)(x)$, it follows from formula (2-29) for $\alpha = \frac{1}{2}$ that

$$(\mathcal{R}^{-1}\mathcal{J}\Phi^* \mathbf{\Gamma}_{1/2}\Phi U f)(\lambda) = (\mathsf{L}_{1/2}\varphi)(\lambda).$$

The left-hand side here equals $(\mathcal{R}^{-1}\mathcal{J}u)(\lambda)$, which after the change of variables $\lambda = e^{-x}$ yields (2-32). □

Now we can rewrite identity (2-21) in a slightly different way.

Corollary 2.12. *Let $h \in \mathcal{X}'_+$, and let the distribution $h^{\natural} \in \mathcal{X}'_+$ be defined by formula* (2-18). *Then for arbitrary $f_j \in \mathcal{D}$, $j = 1, 2$, we have*

$$\langle h, \bar{f}_1 \star f_2\rangle = \langle h^{\natural}, \overline{\mathsf{L}_{1/2}\varphi_1}\mathsf{L}_{1/2}\varphi_2\rangle, \quad \textit{where } \varphi_j(t) = t^{1/2} f_j(t). \tag{2-33}$$

Proof. It suffices to make the change of variables $x = -\ln\lambda$ in the right-hand side of (2-22) and to take equality (2-32) into account. □

We emphasize that according to Lemma 2.8, $\mathsf{L}_{1/2}\varphi_j \in \mathcal{X}_+$, and hence $\overline{\mathsf{L}_{1/2}\varphi_1}\mathsf{L}_{1/2}\varphi_2 \in \mathcal{X}_+$. Thus the right-hand side of (2-33) is correctly defined.

2.4. Finally, we check that a distribution $h \in \mathcal{X}'_+$ is determined uniquely by the values $\langle h, \bar{f}_1 \star f_2\rangle$ on $f_1, f_2 \in \mathcal{D}$. First we consider convolution operators. Let us introduce the shift in the space $L^2(\mathbb{R})$:

$$(T(\tau)g)(\xi) = g(\xi - \tau), \quad \tau \in \mathbb{R}. \tag{2-34}$$

Since

$$(g_1 * g_2)(\xi) = \int_{-\infty}^{\infty} (T(\tau)g_1)(\xi) g_2(\tau)\, d\tau \quad \text{for all } g_1, g_2 \in C_0^\infty(\mathbb{R}),$$

we have the formula

$$\langle b, (\mathcal{J}\bar{g}_1) * g_2\rangle = \int_{-\infty}^{\infty} \langle b, T(\tau)\mathcal{J}\bar{g}_1\rangle \overline{g_2(\tau)}\, d\tau, \tag{2-35}$$

where for $b \in C_0^\infty(\mathbb{R})'$ the function $\langle b, T(\tau)\mathcal{J}\bar{g}_1\rangle$ is infinitely differentiable in $\tau \in \mathbb{R}$.

The following assertion is quite standard.

Lemma 2.13. *Let $b \in C_0^\infty(\mathbb{R})'$. Suppose that $\langle b, (\mathcal{J}\bar{g}_1) * g_2\rangle = 0$ for all $g_1, g_2 \in C_0^\infty(\mathbb{R})$. Then $b = 0$.*

Proof. If $\langle b, (\mathcal{J}\bar{g}_1) * g_2\rangle = 0$ for all $g_2 \in C_0^\infty(\mathbb{R})$, then $\langle b, T(\tau)\mathcal{J}\bar{g}_1\rangle = 0$ for all $\tau \in \mathbb{R}$ according to formula (2-35). In particular, for $\tau = 0$ we have $\langle b, \mathcal{J}\bar{g}_1\rangle = 0$, whence $b = 0$ because $g_1 \in C_0^\infty(\mathbb{R})$ is arbitrary. ☐

Next we pass to Hankel operators.

Proposition 2.14. *Let $h \in \mathcal{Z}_+'$. Suppose that $\langle\!\langle h, \bar{f}_1 \star f_2\rangle\!\rangle = 0$ for all $f_1, f_2 \in \mathcal{D}$. Then $h = 0$.*

Proof. Let $b \in C_0^\infty(\mathbb{R})'$ be the b-function of h (see Definition 2.4). For arbitrary $g_1, g_2 \in C_0^\infty(\mathbb{R})$, we can construct $f_1, f_2 \in \mathcal{D}$ by formula (1-11). Since $\langle\!\langle h, \bar{f}_1 \star f_2\rangle\!\rangle = 0$, it follows from the identity (2-20) that $\langle b, (\mathcal{J}\bar{g}_1) * g_2\rangle = 0$. Therefore $b = 0$ according to Lemma 2.13. Now Proposition 2.5 implies that $h = 0$. ☐

3. Bounded Hankel operators

Our main goal here is to show that the condition $h \in \mathcal{Z}_+'$ is satisfied for all bounded Hankel operators H.

3.1. In this section we a priori only assume that $h \in C_0^\infty(\mathbb{R}_+)'$ and consider the Hankel form (2-3) on functions $f_1, f_2 \in C_0^\infty(\mathbb{R}_+)$. Let $T_+(\tau)$, where $\tau \geq 0$, be the restriction of the shift (2-34) on its invariant subspace $L^2(\mathbb{R}_+)$. Since

$$(\bar{f}_1 \star f_2)(t) = \int_0^\infty (T_+(\tau)\bar{f}_1)(t) f_2(\tau)\, d\tau \quad \text{for all } f_1, f_2 \in C_0^\infty(\mathbb{R}_+),$$

for all $h \in C_0^\infty(\mathbb{R}_+)'$ we have the formula

$$\langle\!\langle h, \bar{f}_1 \star f_2\rangle\!\rangle = \int_0^\infty \langle\!\langle h, T_+(\tau)\bar{f}_1\rangle\!\rangle \overline{f_2(\tau)}\, d\tau. \tag{3-1}$$

Here the function $\langle\!\langle h, T_+(\tau)\bar{f}_1\rangle\!\rangle$ is infinitely differentiable in $\tau \in \mathbb{R}_+$, and this function, as well as all its derivatives, has finite limits as $\tau \to 0$. In the theory of Hankel operators, formula (3-1) plays the role of formula (2-35) for convolution operators.

The proof of the following assertion is almost the same as that of Lemma 2.13.

Proposition 3.1. *Let $h \in C_0^\infty(\mathbb{R}_+)'$. Suppose that $\langle\!\langle h, \bar{f}_1 \star f_2\rangle\!\rangle = 0$ for all $f_1, f_2 \in C_0^\infty(\mathbb{R}_+)$. Then $h = 0$.*

Proof. If $\langle\!\langle h, \bar{f}_1 \star f_2\rangle\!\rangle = 0$ for all $f_2 \in C_0^\infty(\mathbb{R}_+)$, then $\langle\!\langle h, T_+(\tau)\bar{f}_1\rangle\!\rangle = 0$ for all $\tau \in [0, \infty)$ according to formula (3-1). In particular, for $\tau = 0$ we have $\langle\!\langle h, \bar{f}_1\rangle\!\rangle = 0$, which implies that $h = 0$ because $f_1 \in C_0^\infty(\mathbb{R}_+)$ is arbitrary. ☐

Of course Propositions 2.14 and 3.1 differ only by the set of functions on which the Hankel form is considered.

Assume now that

$$|\langle h, \bar{f} \star f\rangle| \le C\|f\|^2 \quad \text{for all } f \in C_0^\infty(\mathbb{R}_+). \tag{3-2}$$

Then there exists a bounded operator H such that

$$(Hf_1, f_2) = \langle h, \bar{f}_1 \star f_2\rangle \quad \text{for all } f_1, f_2 \in C_0^\infty(\mathbb{R}_+). \tag{3-3}$$

We call H the Hankel operator associated to the Hankel form $\langle h, \bar{f}_1 \star f_2\rangle$.

3.2. It is possible to characterize Hankel operators by some commutation relations. A presentation of such results for discrete Hankel operators acting in the space of sequences $l^2(\mathbb{Z}_+)$ can be found in [Power 1982, §1.1].

Let us define a bounded operator Q in the space $L^2(\mathbb{R}_+)$ by the equality

$$(Qf)(t) = -2e^{-t}\int_0^t e^s f(s)\,ds.$$

Note that

$$Q = -2\int_0^\infty T_+(\tau)e^{-\tau}\,d\tau. \tag{3-4}$$

Lemma 3.2. *Let* (3-2) *hold. Then the operator H defined by formula* (3-3) *satisfies the commutation relations*

$$HT_+(\tau) = T_+(\tau)^* H \quad \textit{for all } \tau \ge 0 \tag{3-5}$$

and

$$HQ = Q^* H. \tag{3-6}$$

Proof. Since

$$(T_+(\tau)\bar{f}_1) \star f_2 = \bar{f}_1 \star (T_+(\tau) f_2) \quad \text{for all } \tau \ge 0,$$

relation (3-5) directly follows from (3-3). By virtue of formula (3-4), relation (3-6) is a consequence of (3-5). □

Below we need the Nehari theorem; see the original paper [1957] or [Peller 2003, Chapter 1, §1; Power 1982, Chapter 1, §2]. We formulate it in the Hardy space $\mathbb{H}_+^2(\mathbb{R})$ of functions analytic in the upper half-plane. We denote by $\widehat{Q}$ the operator of multiplication by the function $(\mu - i)/(\mu + i)$ in this space. Clearly, $\widehat{Q} = \Phi^* Q \Phi$.

Theorem 3.3 [Nehari 1957]. *Let $\omega \in L^\infty(\mathbb{R})$, and let an operator $\widehat{H}$ in the space $\mathbb{H}_+^2(\mathbb{R})$ be defined by the relation*

$$(\widehat{H}\hat{f}_1, \hat{f}_2) = \int_{-\infty}^\infty \omega(\mu)\hat{f}_1(-\mu)\overline{\hat{f}_2(\mu)}\,d\mu \quad \textit{for all } \hat{f}_1, \hat{f}_2 \in \mathbb{H}_+^2(\mathbb{R}). \tag{3-7}$$

Then $\widehat{H}$ is bounded and $\widehat{H}\widehat{Q} = \widehat{Q}^*\widehat{H}$. Conversely, if $\widehat{H}$ is a bounded operator in $\mathbb{H}_+^2(\mathbb{R})$ and $\widehat{H}\widehat{Q} = \widehat{Q}^*\widehat{H}$, then there exists a function $\omega \in L^\infty(\mathbb{R})$ such that representation (3-7) *holds.*

The following assertion can be regarded as a translation of this theorem into the space $L^2(\mathbb{R}_+)$. Recall that, by the Paley–Wiener theorem, the Fourier transform $\Phi : \mathbb{H}^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+)$ is the unitary operator. Since

$$\int_{-\infty}^{\infty} (\mu + i)^{-1} e^{-i\mu t}\, d\mu = -2\pi i e^{-t}$$

for $t > 0$ and this integral is zero for $t < 0$, we have the relation

$$I + Q = \Phi \widehat{Q} \Phi^*. \tag{3-8}$$

Theorem 3.4. *If $h = (2\pi)^{-1/2}\Phi\omega$, where $\omega \in L^\infty(\mathbb{R})$ (in this case $h \in \mathcal{S}' \subset C_0^\infty(\mathbb{R}_+)'$), then estimate* (3-2) *is true and the operator H in the space $L^2(\mathbb{R}_+)$ defined by formula* (3-3) *satisfies the commutation relation* (3-5). *Conversely, if a bounded operator H satisfies* (3-5), *then representation* (3-3) *holds with $h = (2\pi)^{-1/2}\Phi\omega$ for some $\omega \in L^\infty(\mathbb{R})$.*

Proof. Since

$$(\Phi^*(\bar{f}_1 \star f_2))(\mu) = \sqrt{2\pi}\, \overline{(\mathcal{J}\hat{f}_1)(\mu)}\, \hat{f}_2(\mu) \quad \text{for all } f_1, f_2 \in C_0^\infty(\mathbb{R}_+),$$

where $\hat{f}_1 = \Phi^* f_1$, $\hat{f}_2 = \Phi^* f_2$, and $(\mathcal{J}\hat{f}_1)(\mu) = \hat{f}_1(-\mu)$, we have

$$\langle h, \bar{f}_1 \star f_2 \rangle = \sqrt{2\pi}\, \langle \Phi^* h, \overline{(\mathcal{J}\hat{f}_1)}\, \hat{f}_2 \rangle \quad \text{for all } h \in \mathcal{S}'. \tag{3-9}$$

Therefore, estimate (3-2) is satisfied if $\Phi^* h \in L^\infty(\mathbb{R})$. Relation (3-5) for the corresponding Hankel operator H follows from Lemma 3.2.

Conversely, if a bounded operator H satisfies relation (3-5), then by virtue of (3-4), it also satisfies relation (3-6). Hence it follows from (3-8) that $\widehat{H}\widehat{Q} = \widehat{Q}^*\widehat{H}$, where $\widehat{H} = \Phi^* H \Phi$ is a bounded operator in the space $\mathbb{H}^2_+(\mathbb{R})$. Thus, by Theorem 3.3, there exists a function $\omega \in L^\infty(\mathbb{R})$ such that representation (3-7) holds. This means that

$$(Hf_1, f_2) = \int_{-\infty}^{\infty} \omega(\mu) \hat{f}_1(-\mu) \overline{\hat{f}_2(\mu)}\, d\mu \quad \text{for all } f_1, f_2 \in L^2(\mathbb{R}_+). \tag{3-10}$$

If $h = (2\pi)^{-1/2}\Phi\omega$, then the right-hand sides in (3-9) and (3-10) coincide. This yields representation (3-3). □

Corollary 3.5. *For a bounded operator H in the space $L^2(\mathbb{R}_+)$, commutation relations* (3-5) *and* (3-6) *are equivalent.*

Proof. As was already noted, (3-6) follows from (3-5) according to formula (3-4). Conversely, if H satisfies (3-5), then representation (3-3) holds according to Theorem 3.4. Thus it remains to use Lemma 3.2. □

Recall that *a function $\omega \in L^\infty(\mathbb{R})$ such that $\Phi\omega = \sqrt{2\pi}\, h$ is called the symbol of a bounded Hankel operator H with kernel $h(t)$*. In view of formula (3-7), if $\omega \in \mathbb{H}^\infty_-(\mathbb{R})$, that is, ω admits an analytic continuation to a bounded function in the lower half-plane, then the Hankel operator $\widehat{H}$ equals 0, and hence $H = \Phi\widehat{H}\Phi^* = 0$. Therefore the symbol is defined up to a function in the class $\mathbb{H}^\infty_-(\mathbb{R})$.

3.3. Now we are in a position to check that the condition $h \in \mathfrak{X}'_+$ is satisfied for all bounded Hankel operators. By (2-7), it means that distribution (2-5) belongs to the class $\mathfrak{X}'$. We shall verify the stronger inclusion $\theta \in \mathcal{S}'$.

To that end, it suffices to check that, for some $N \in \mathbb{Z}_+$ and some $\kappa \in \mathbb{R}$,

$$|\langle\theta, \Omega\rangle| \le C \sum_{n=0}^{N} \max_{x\in\mathbb{R}} \big((1+|x|)^\kappa |\Omega^{(n)}(x)|\big) \quad \text{for all } \Omega \in C_0^\infty(\mathbb{R}). \tag{3-11}$$

Putting $F(t) = \Omega(\ln t)$, we see that (3-11) is equivalent to the estimate

$$|\langle h, F\rangle| \le C \sum_{n=0}^{N} \max_{t\in\mathbb{R}_+} \big((1+|\ln t|)^\kappa t^n |F^{(n)}(t)|\big), \qquad F \in C_0^\infty(\mathbb{R}_+). \tag{3-12}$$

Let us make some comments on this condition. If $h \in L^1_{\mathrm{loc}}(\mathbb{R})$, then estimate (3-12) for $N = 0$ is equivalent to the convergence of integral (1-7) for the same values of κ. If H is Hilbert–Schmidt, that is,

$$\int_0^\infty |h(t)|^2 t\, dt < \infty,$$

then integral (1-7) converges for any $\kappa > \frac{1}{2}$. Similarly, if $|h(t)| \le Ct^{-1}$, then integral (1-7) converges for any $\kappa > 1$.

For the proof of (3-12) in the general case, we use the following elementary result. Its proof is given in Appendix A.

Lemma 3.6. *If $F \in C_0^\infty(\mathbb{R}_+)$, then for an arbitrary $\kappa > 2$, the estimate*

$$\|\Phi^* F\|_{L^1(\mathbb{R})} \le C(\kappa) \sum_{n=0}^{2} \max_{t\in\mathbb{R}_+} \big((1+|\ln t|)^\kappa t^n |F^{(n)}(t)|\big) \tag{3-13}$$

holds.

Corollary 3.7. *If $h = \Phi\omega$, where $\omega \in L^\infty(\mathbb{R})$, then estimate* (3-12) *holds for $N = 2$ and an arbitrary $\kappa > 2$.*

Proof. It suffices to combine the estimates

$$|\langle h, F\rangle| = |\langle\omega, \Phi^* F\rangle| \le \|\omega\|_{L^\infty(\mathbb{R})} \|\Phi^* F\|_{L^1(\mathbb{R})}$$

and (3-13). □

Since, by Theorem 3.4, for a bounded Hankel operator H, its kernel h equals $\Phi\omega$ for some $\omega \in L^\infty(\mathbb{R})$, we arrive at the following result.

Theorem 3.8. *Suppose that $h \in C_0^\infty(\mathbb{R}_+)'$ and that condition* (3-2) *is satisfied. Then estimate* (3-12) *holds for $N = 2$ and an arbitrary $\kappa > 2$; in particular, $h \in \mathfrak{X}'_+$.*

The following simple example shows that for $N = 0$, estimate (3-12) is in general violated (for all κ).

Example 3.9. Let $h(t) = e^{-it^2}$. Then the corresponding Hankel operator H is bounded because according to the formula $e^{-i(t+s)^2} = e^{-it^2} e^{-i2ts} e^{-is^2}$, it is a product of three bounded operators. Since $h \in L^\infty(\mathbb{R}_+)$, estimate (3-12) for $N = 0$ is equivalent to the convergence of integral (1-7) for the same value of κ. However, this integral diverges at infinity for all κ.

Let us show that for $h(t) = e^{-it^2}$, condition (3-12) is satisfied for $N = 1$ and $\kappa = 0$. Integrating by parts, we see that

$$\int_0^\infty h(t)\overline{F(t)}\,dt = -\int_0^\infty h_1(t)\overline{F'(t)}\,dt, \tag{3-14}$$

where the function $h_1(t) = \int_0^t e^{-is^2} ds$ is bounded. Therefore, the integral on the right-hand side of (3-14) is bounded by $\max_{t \in \mathbb{R}_+}\big((1+|\ln t|)^\kappa t|F'(t)|\big)$ for any $\kappa > 1$.

Note that for $h(t) = e^{-it^2}$, the symbol of H equals $\omega(\mu) = \sqrt{\pi} e^{-\pi i/4} e^{i\mu^2/4}$. More generally, one can consider the class of symbols $\omega(\mu)$ such that $\omega \in C^\infty(\mathbb{R})$, $\omega(\mu) = e^{i\omega_0 \mu^\alpha}$, $\omega_0 > 0$ for large positive μ and $\omega(\mu) = 0$ for large negative μ. Of course, Hankel operators with such symbols are bounded. Using the stationary phase method, we find that for $\alpha > 1$, the corresponding kernel $h(t)$ has the asymptotics

$$h(t) \sim h_0 t^\beta e^{i\sigma t^\gamma}, \quad t \to \infty, \tag{3-15}$$

where $\beta = (1-\alpha/2)(\alpha-1)^{-1}$, $\gamma = \alpha(\alpha-1)^{-1}$, and h_0, $\sigma = \bar{\sigma}$ are some constants. Moreover, $h(t)$ is a bounded function on all finite intervals. Similarly to Example 3.9, it can be checked that for such kernels, condition (3-12) is satisfied for $N = 1$ but not for $N = 0$. The same conclusion is true for $\alpha \in (0, 1)$, because in this case the asymptotic relation (3-15) holds for $t \to 0$.

3.4. Here we shall show that, for bounded Hankel operators H, the representations (2-20) and (2-21) extend to all $f_1, f_2 \in L^2(\mathbb{R}_+)$. By Theorem 3.8, we have $h \in \mathfrak{X}'_+$. Let b and s be the corresponding b- and s-functions (see Definition 2.4). Recall that the operator Ξ is defined by formula (1-11). We denote by K the operator of multiplication by the function $\sqrt{\cosh(\pi\xi)/\pi}$ in the space $L^2(\mathbb{R})$. It follows from identity (2-24) and the unitarity of the Mellin transform (2-9) that

$$\|K\Xi f\| = \|f\|,$$

and hence the operator $K\Xi : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ is unitary. Therefore, in view of the identities (2-20) and (2-21), we have the following result.

Lemma 3.10. *The inequalities* (3-2),

$$|\langle b, (\mathcal{J}\bar{g}) * g\rangle| \le C\|Kg\|^2 \quad \text{for all } g \in C_0^\infty(\mathbb{R}), \tag{3-16}$$

and

$$|\langle s, u^* u\rangle| \le C\|K\Phi u\|^2 \quad \text{for all } u \in \mathfrak{X} \tag{3-17}$$

are equivalent. The Hankel operator corresponding to form (2-3) *is bounded if and only if one of equivalent estimates* (3-2), (3-16), *or* (3-17) *is satisfied.*

These estimates can be formulated in a slightly different way. Let us introduce the space $\mathcal{E} \subset L^2(\mathbb{R})$ of exponentially decaying functions with the norm $\|g\|_{\mathcal{E}} = \|Kg\|$. Then the space $\mathcal{W} = \Phi^*\mathcal{E}$ consists of functions $u(x)$ admitting the analytic continuation $u(z)$ in the strip $\operatorname{Im} z \in (-\pi/2, \pi/2)$; moreover, functions $u(x+iy)$ have limits in $L^2(\mathbb{R})$ as $y \to \pm\pi/2$. The identity

$$\|\Phi u\|_{\mathcal{E}}^2 = (2\pi)^{-1} \int_{-\infty}^{\infty} \left(\left| u\left(x + i\frac{\pi}{2}\right)\right|^2 + \left| u\left(x - i\frac{\pi}{2}\right)\right|^2 \right) dx =: \|u\|_{\mathcal{W}}^2$$

defines the Hilbert norm on $\mathcal{W}$. We call $\mathcal{W}$ the exponential Sobolev space because it is contained in the standard Sobolev spaces $\mathsf{H}^l(\mathbb{R})$ for all l. The operators $\Xi : L^2(\mathbb{R}_+) \to \mathcal{E}$ and $\widehat{\Xi} := \Phi^*\Xi : L^2(\mathbb{R}_+) \to \mathcal{W}$ are of course unitary. Obviously, $\|Kg\|$ and $\|K\Phi u\|$ on the right-hand sides of (3-16) and (3-17) can be replaced by $\|g\|_{\mathcal{E}}$ and $\|u\|_{\mathcal{W}}$, respectively. Note that the inclusions $f \in L^2(\mathbb{R}_+)$, $g = \Xi f \in \mathcal{E}$, and $u = \widehat{\Xi} f \in \mathcal{W}$ are equivalent.

Recall that the operator B is defined by formula (1-9) and $(Su)(x) = s(x)u(x)$. If one of the equivalent estimates (3-2), (3-16), or (3-17) is satisfied, then all operators $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $B : \mathcal{E} \to \mathcal{E}'$, and $S : \mathcal{W} \to \mathcal{W}'$ are bounded. Using that relations $f_n \to f$ in $L^2(\mathbb{R}_+)$, $g_n = \Xi f_n \to g = \Xi f$ in $\mathcal{E}$, and $u_n = \Phi^* g_n \to u = \Phi^* g$ in $\mathcal{W}$ are equivalent, we extend (2-20) and (2-21) to all $f \in L^2(\mathbb{R}_+)$. Thus we have obtained the following result.

Proposition 3.11. *If one of equivalent estimates* (3-2), (3-16), *or* (3-17) *is satisfied*, *then the identities*

$$(Hf_1, f_2) = (Bg_1, g_2) = (Su_1, u_2), \quad g_j = \Xi f_j, \quad u_j = \Phi^* g_j$$

are true for all $f_1, f_2 \in L^2(\mathbb{R}_+)$.

Let K_l be the operator of multiplication by the function $(1+\xi^2)^{l/2}$. Then estimates (3-16) or (3-17) are satisfied provided

$$|\langle b, (\mathcal{J}\bar{g}) * g\rangle| \le C_l \|K_l g\|^2 \quad \text{or} \quad |\langle s, |u|^2\rangle| \le C_l \|u\|^2_{\mathsf{H}^l(\mathbb{R})}, \tag{3-18}$$

for some l; in this case

$$C = C_l \pi \max_{\xi \in \mathbb{R}} \bigl((1+\xi^2)^l (\cosh(\pi\xi))^{-1}\bigr).$$

3.5. In terms of the sign function, it is possible to give simple sufficient conditions for the boundedness and compactness of Hankel operators.

Proposition 3.12. *A Hankel operator H is bounded if its sign function satisfies the condition*

$$s \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}). \tag{3-19}$$

If $s \in L^\infty(\mathbb{R})$ and $s(x) \to 0$ as $|x| \to \infty$, then H is compact.

Proof. The first statement is obvious because under (3-19), the second estimate (3-18) is satisfied with $l > \frac{1}{2}$. To prove the second statement, we observe that the operator $S\Phi^* K^{-1}$ is compact because both S

and K^{-1} are operators of multiplication by bounded functions which tend to zero at infinity. Since the operator $K\,\Xi : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ is bounded, it follows from the identity (1-12) that the operator

$$H = (\Xi^*\Phi)S(\Phi^*\Xi) = \Xi^*\Phi(S\Phi^*K^{-1})(K\,\Xi)$$

is also compact. □

Condition (3-19) is of course *not* necessary for the boundedness of H. For example, in view of formula (1-4) for Hankel operators H of finite rank, the sign function is a singular distribution.

4. Criteria for sign-definiteness

In this section we suppose that $h(t) = \overline{h(t)}$ so that the operator H is formally symmetric. The results of Section 2 allow us to give simple necessary and sufficient conditions for a Hankel operator H to be positive or negative. Moreover, they also provide convenient tools for the calculation of the total multiplicity of the negative and positive spectra of H. We often state our results only for the negative spectrum. The corresponding results for the positive spectrum are obtained if H is replaced by $-H$.

4.1. Actually, we consider the problem in terms of Hankel quadratic forms rather than Hankel operators. This is both more general and more convenient. As usual, we take a distribution $h \in \mathcal{Z}'_+$ and introduce the b-function $b \in C_0^\infty(\mathbb{R})'$ and the s-function $s \in \mathcal{Z}'$ as in Definition 2.4.

Below we use the following natural notation. Let $\mathsf{h}[\varphi, \varphi]$ be a real quadratic form defined on a linear set D. We denote by $N_\pm(\mathsf{h}; \mathsf{D})$ the maximal dimension of linear sets $\mathcal{M}_\pm \subset \mathsf{D}$ such that $\pm\mathsf{h}[\varphi, \varphi] > 0$ for all $\varphi \in \mathcal{M}_\pm$, $\varphi \neq 0$. This means that there exists a linear set $\mathcal{M}_\pm$, $\dim \mathcal{M}_\pm = N_\pm(\mathsf{h})$, such that $\pm\mathsf{h}[\varphi, \varphi] > 0$ for all $\varphi \in \mathcal{M}_\pm$, $\varphi \neq 0$; and for every linear set $\mathcal{M}'_\pm$ with $\dim \mathcal{M}'_\pm > N_\pm(\mathsf{h})$ there exists $\varphi \in \mathcal{M}'_\pm$, $\varphi \neq 0$, such that $\pm\mathsf{h}[\varphi, \varphi] \le 0$. We apply this definition to the forms $h[f, f] = \langle h, \bar{f} \star f\rangle$ defined on $\mathcal{D}$, to $b[g, g] = \langle b, (\mathcal{J}\bar{g}) * g\rangle$ defined on $C_0^\infty(\mathbb{R})$, and to $s[u, u] = \langle s, |u|^2\rangle$ defined on $\mathcal{Z}$. Of course, if D is dense in a Hilbert space $\mathcal{H}$ and $\mathsf{h}[\varphi, \varphi]$ is semibounded and closed on D, then for the self-adjoint operator H corresponding to h, we have $N_\pm(\mathsf{H}) = N_\pm(\mathsf{h}; \mathsf{D})$.

Observe that formula (1-11) establishes a one-to-one correspondence between the sets $\mathcal{D}$ and $C_0^\infty(\mathbb{R})$. Moreover, the Fourier transform establishes a one-to-one correspondence between the sets $C_0^\infty(\mathbb{R})$ and $\mathcal{Z}$. Therefore the following assertion is a direct consequence of Theorem 2.6.

Theorem 4.1. *Let $h \in \mathcal{Z}'_+$. Then*

$$N_\pm(h; \mathcal{D}) = N_\pm(b; C_0^\infty(\mathbb{R})) = N_\pm(s; \mathcal{Z}).$$

In particular, we have:

Theorem 4.2. *Let $h \in \mathcal{Z}'_+$. Then $\pm\langle h, \bar{f} \star f\rangle \ge 0$ for all $f \in \mathcal{D}$ if and only if $\pm\langle b, (\mathcal{J}\bar{g}) * g\rangle \ge 0$ for all $g \in C_0^\infty(\mathbb{R})$, or $\pm\langle s, u^*u\rangle \ge 0$ for all $u \in \mathcal{Z}$.*

4.2. A calculation of the form $s[u, u]$ on analytic functions $u \in \mathcal{Z}$ is not always convenient. Therefore it is desirable to replace the class $\mathcal{Z}$, for example, by the class $C_0^\infty(\mathbb{R})$. Such a replacement is not obvious because for $u \in C_0^\infty(\mathbb{R})$ we only have $g = \Phi u \in \mathcal{Z}$. In this case $(Mf)(\xi) = \Gamma\big(\tfrac{1}{2} + i\xi\big)^{-1} g(\xi)$ need not even belong to $L^2(\mathbb{R})$, so that $f \notin L^2(\mathbb{R}_+)$. Nevertheless, under the additional assumption $s \in \mathcal{S}'$, we have the following assertion.

Lemma 4.3. *If $s \in \mathcal{S}'$, then $N_\pm(s;\mathcal{Z}) = N_\pm(s; C_0^\infty(\mathbb{R})) = N_\pm(s;\mathcal{S})$.*

Proof. Since $\Phi : \mathcal{Z} \to C_0^\infty(\mathbb{R})$, $\Phi^* : C_0^\infty(\mathbb{R}) \to \mathcal{Z}$ and $\mathcal{S}$ is invariant with respect to the Fourier transform Φ, it suffices, for example, to show that $N_\pm(s;\mathcal{Z}) = N_\pm(s;\mathcal{S})$. The inequality $N_\pm := N_\pm(s;\mathcal{S}) \ge N_\pm(s;\mathcal{Z})$ is obvious because $\mathcal{Z} \subset \mathcal{S}$.

Let us prove the opposite inequality. Consider for definiteness the sign "+". Let $\mathcal{L}_+ \subset \mathcal{S}$, and let $s[u,u] > 0$ for all $u \in \mathcal{L}_+$, $u \ne 0$. Suppose first that $N := \dim \mathcal{L}_+ < \infty$ and choose elements $u_1, \dots, u_N \in \mathcal{L}_+$ such that $s[u_j, u_k] = \delta_{j,k}$ for all $j, k = 1, \dots, N$. Let us construct elements $u_j^{(\epsilon)} \in \mathcal{Z}$ such that $u_j^{(\epsilon)} \to u_j$ and hence $u_j^{(\epsilon)} \bar{u}_k^{(\epsilon)} \to u_j \bar{u}_k$ in $\mathcal{S}$ as $\epsilon \to 0$ for $j, k = 1, \dots, N$. Since $s \in \mathcal{S}'$, we see that $s[u_j^{(\epsilon)}, u_k^{(\epsilon)}] \to \delta_{j,k}$ as $\epsilon \to 0$. For an arbitrary $\sigma > 0$, we can choose ϵ such that $\bigl|s[u_j^{(\epsilon)}, u_k^{(\epsilon)}] - \delta_{j,k}\bigr| \le \sigma$. Then for arbitrary $\lambda_1, \dots, \lambda_N \in \mathbb{C}$, we have

$$s\Bigl[\sum_{j=1}^N \lambda_j u_j^{(\epsilon)}, \sum_{j=1}^N \lambda_j u_j^{(\epsilon)}\Bigr] = \sum_{j=1}^N |\lambda_j|^2 s[u_j^{(\epsilon)}, u_j^{(\epsilon)}] + 2\,\mathrm{Re} \sum_{\substack{j,k=1\\ j\ne k}}^N \lambda_j \bar{\lambda}_k s[u_j^{(\epsilon)}, u_k^{(\epsilon)}]$$

$$\ge (1-\sigma)\sum_{j=1}^N |\lambda_j|^2 - 2\sigma \sum_{\substack{j,k=1\\ j\ne k}}^N \lambda_j \bar{\lambda}_k \ge \bigl(1-(2N-1)\sigma\bigr)\sum_{j=1}^N |\lambda_j|^2.$$

Thus elements $u_1^{(\epsilon)}, \dots, u_N^{(\epsilon)}$ are linearly independent if $(2N-1)\sigma < 1$. The same inequality shows that $s[u,u] > 0$ on all vectors $u \ne 0$ in the space $\mathcal{L}_+^{(\epsilon)}$ spanned by $u_1^{(\epsilon)}, \dots, u_N^{(\epsilon)}$.

If $N_+ = \infty$, then the same construction works on every finite-dimensional subspace of $\mathcal{L}_+$ where $s[u,u] > 0$. This yields a space $\mathcal{L}_+^{(\epsilon)} \subset \mathcal{Z}$ of arbitrarily large dimension where $s[u,u] > 0$. □

Putting together this lemma with Theorem 4.1, we obtain the following result.

Theorem 4.4. *Let $h \in \mathcal{Z}_+'$. Suppose that $b \in \mathcal{S}'$ or, equivalently, that $s \in \mathcal{S}'$. Then $N_\pm(h;\mathcal{D}) = N_\pm(s; C_0^\infty(\mathbb{R}))$.*

In many cases the following consequence of Theorem 4.4 is convenient. According to Proposition 3.12, under the assumptions of Theorem 4.5, H is defined as the bounded self-adjoint operator corresponding to the form $\langle h, \bar{f} \star f\rangle$. Therefore $N_\pm(h;\mathcal{D}) = N_\pm(H)$ is the total multiplicity of the (strictly) positive spectrum for the sign "+" and of the (strictly) negative spectrum for the sign "−" of the operator H. For definiteness, we consider the negative spectrum.

Theorem 4.5. *Let $h \in \mathcal{Z}_+'$, and let the corresponding sign function satisfy condition* (3-19). *If $s(x) \ge 0$, then the operator H is positive. If $s(x) \le -s_0 < 0$ for almost all x in some interval $\Delta \subset \mathbb{R}$, then $N_-(H) = \infty$.*

Proof. If $s(x) \ge 0$, then $H \ge 0$ according to the second relation in (2-22).

Let $s(x) \le -s_0 < 0$ for $x \in \Delta$, and let N be arbitrary. Choose a function $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi(x) = 1$ for $x \in [-\delta, \delta]$ and $\varphi(x) = 0$ for $x \notin [-2\delta, 2\delta]$, where $\delta = \delta_N$ is a sufficiently small number. Let points

$\alpha_j \in \Delta$, $j = 1, \dots, N$, be such that $\alpha_{j+1} - \alpha_j = \alpha_j - \alpha_{j-1}$ for $j = 2, \dots, N-1$. Set $\Delta_j = (\alpha_j - \delta, \alpha_j + \delta)$, $\widetilde{\Delta}_j = (\alpha_j - 2\delta, \alpha_j + 2\delta)$. For a sufficiently small δ, we may suppose that $\widetilde{\Delta}_j \subset \Delta$ for all $j = 1, \dots, N$ and that $\widetilde{\Delta}_{j+1} \cap \widetilde{\Delta}_j = \varnothing$ for $j = 1, \dots, N-1$. We set $\varphi_j(x) = \varphi(x - \alpha_j)$. Since $s(x) \le -s_0 < 0$ for $x \in \Delta$, we have

$$\langle s, |\varphi_j|^2 \rangle = \int_{-\infty}^{\infty} s(x) |\varphi_j(x)|^2 \, dx \le -2\delta s_0 < 0. \tag{4-1}$$

The functions $\varphi_1, \dots, \varphi_N$ have disjoint supports, and hence $\langle s, |u|^2 \rangle < 0$ for an arbitrary nontrivial linear combination u of the functions φ_j. Therefore, combining Theorem 4.1 and Theorem 4.4, we obtain the second statement of the theorem. □

Theorem 4.5 can be reformulated, although in a weaker form, in terms of the functions $b(\xi)$ and even $h(t)$. Suppose, for example, that

$$b \in L^1(\mathbb{R}). \tag{4-2}$$

Then $b(\xi)$'s Fourier transform $s(x)$ is a continuous function which tends to 0 as $|x| \to \infty$. The convolution operator B defined by formula (1-9) is bounded in $L^2(\mathbb{R})$ and self-adjoint, and

$$\operatorname{spec}(B) = [\min_{x \in \mathbb{R}} s(x), \max_{x \in \mathbb{R}} s(x)].$$

The result below follows directly from Theorem 4.5. Note that by Proposition 3.12, under (4-2) the operator H is compact.

Proposition 4.6. *Under* (4-2), *the Hankel operator H is positive if and only if $s(x) \ge 0$. If $\min_{x \in \mathbb{R}} s(x) < 0$, then H necessarily has an infinite number of negative eigenvalues.*

In particular, condition (4-2) is satisfied if

$$h(t) = \frac{\theta(\ln t)}{t}, \quad \text{where } \theta \in \mathcal{Z}.$$

In this case $a = \Phi\theta \in C_0^\infty(\mathbb{R})$, and hence $b \in C_0^\infty(\mathbb{R})$.

4.3. For the proof that a Hankel operator is not sign-definite, it is sometimes not even necessary to calculate the sign function $s(x)$ (the Fourier transform of $b(\xi)$). It turns out that if $b(\xi)$ grows as $|\xi| \to \infty$, then the form $b[g, g] = \langle b, \mathcal{J}\bar{g} * g \rangle$ cannot be sign-definite. More precisely, we have the following statement about convolutions with growing kernels $b(-\xi) = \overline{b(\xi)}$.

Theorem 4.7. *Let $b = b_0 + b_\infty$, where $b_0 \in C^p(\mathbb{R})'$ for some $p \in \mathbb{Z}_+$ and $b_\infty \in L^\infty_{\mathrm{loc}}(\mathbb{R})$. Suppose that there exists a sequence of intervals $\Delta_n = (r_n - \sigma_n, r_n + \sigma_n)$, where $r_n \to \infty$ (or equivalently $r_n \to -\infty$) and the sequence σ_n is bounded such that*

$$\lim_{n \to \infty} \sigma_n^l \min_{\xi \in \Delta_n} \operatorname{Re} b_\infty(\xi) = \infty \quad \text{or} \quad \lim_{n \to \infty} \sigma_n^l \max_{\xi \in \Delta_n} \operatorname{Re} b_\infty(\xi) = -\infty, \tag{4-3}$$

where $l = 2$ if $p = 0$ or $p = 1$ and $l = p + 1$ if $p \ge 2$. Then for both signs, $N_\pm(b; C_0^\infty(\mathbb{R})) \ge 1$.

Proof. Since b can be replaced by $-b$, we can assume that, for example, the first condition (4-3) is satisfied. Pick a real even function $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi(\xi) \ge 0$, $\varphi(\xi) = 1$ for $|\xi| \le \frac{1}{4}$, and $\varphi(\xi) = 0$ for $|\xi| \ge \frac{1}{2}$, and set

$$g_n(\xi) = \varphi\left(\frac{\xi - \frac{r_n}{2}}{\sigma_n}\right) \pm \varphi\left(\frac{\xi + \frac{r_n}{2}}{\sigma_n}\right). \tag{4-4}$$

An easy calculation shows that

$$((\mathcal{J} g_n) * g_n)(\xi) = 2\sigma_n \psi\left(\frac{\xi}{\sigma_n}\right) \pm \sigma_n \psi\left(\frac{\xi - r_n}{\sigma_n}\right) \pm \sigma_n \psi\left(\frac{\xi + r_n}{\sigma_n}\right), \tag{4-5}$$

where $\psi = (\mathcal{J}\varphi) * \varphi \in C_0^\infty(\mathbb{R})$. The function $\psi(\xi)$ is also even, with $\psi(\xi) \ge 0$, $\psi(\xi) \ge \frac{1}{8}$ for $|\xi| \le \frac{1}{8}$, and $\psi(\xi) = 0$ for $|\xi| \ge 1$.

Since $|\langle b_0, g\rangle| \le C\|g\|_{C^p}$, it follows from (4-5) that

$$|\langle b_0, (\mathcal{J} g_n) * g_n\rangle| \le C\sigma_n^{1-p}. \tag{4-6}$$

Moreover, again according to (4-5), we have

$$\langle b_\infty, (\mathcal{J} g_n) * g_n\rangle = 2\sigma_n^2 \int_{-\infty}^{\infty} b_\infty(\sigma_n \eta)\psi(\eta)\, d\eta \pm 2\sigma_n^2 \int_{-\infty}^{\infty} \operatorname{Re} b_\infty(\sigma_n \eta + r_n)\psi(\eta)\, d\eta. \tag{4-7}$$

The first term on the right-hand side is $O(\sigma_n^2)$. For the second one, we use the estimate

$$32 \int_{-\infty}^{\infty} \operatorname{Re} b_\infty(\sigma_n \eta + r_n)\psi(\eta)\, d\eta \ge \min_{|\xi - r_n| \le \sigma_n} \operatorname{Re} b_\infty(\xi). \tag{4-8}$$

Let us first choose the sign "+" in (4-4). Then using representation (4-7) and putting together estimates (4-6) and (4-8), we obtain the lower bound

$$\langle b, (\mathcal{J} g_n) * g_n\rangle \ge -c(\sigma_n^{1-p} + \sigma_n^2) + \frac{\sigma_n^2}{16} \min_{|\xi - r_n| \le \sigma_n} \operatorname{Re} b_\infty(\xi).$$

If $p = 0$ or $p = 1$, then under the first condition in (4-3), the right-hand side here tends to $+\infty$ as $n \to \infty$. If $p \ge 2$, it is bounded from below by

$$\sigma_n^{1-p}\left(-c + \frac{\sigma_n^l}{16} \min_{|\xi - r_n| \le \sigma_n} \operatorname{Re} b_\infty(\xi)\right),$$

where the expression in the brackets tends again to $+\infty$. Therefore $\langle b, (\mathcal{J} g_n) * g_n\rangle > 0$ for sufficiently large n. Similarly choosing the sign "−" in (4-4), we see that $\langle b, (\mathcal{J} g_n) * g_n\rangle < 0$ for sufficiently large n. □

Corollary 4.8. *Instead of condition* (4-3), *assume that*

$$\lim_{|\xi| \to \infty} \operatorname{Re} b_\infty(\xi) = \infty \quad \text{or} \quad \lim_{|\xi| \to \infty} \operatorname{Re} b_\infty(\xi) = -\infty.$$

Then for both signs, $N_\pm(b; C_0^\infty(\mathbb{R})) \ge 1$.

In contrast to Theorem 4.7, there are no restrictions in Corollary 4.8 on the parameter p in the assumption $b_0 \in C^p(\mathbb{R})'$. On the other hand, condition (4-3) permits $\operatorname{Re} b(\xi)$ to tend to $\pm\infty$ only on some system of intervals. Moreover, the lengths of these intervals may tend to zero. In this case, however, the growth of $\operatorname{Re} b(\xi)$ and the decay of these lengths should be correlated and there are restrictions on admissible values of the parameters p and l.

Unlike Theorem 4.5, Theorem 4.7 does not guarantee that $N = \infty$; see Section 5.4 for a discussion of various possible cases.

4.4. Theorem 4.2 can be combined with the Bochner–Schwartz theorem (see, e.g., Theorem 3 in [Gel'fand and Vilenkin 1964, Chapter II, §3]). It states that a distribution $b \in C_0^\infty(\mathbb{R})'$ satisfying the condition $\langle b, \mathcal{J}\bar{g} * g\rangle \geq 0$ for all $g \in C_0^\infty(\mathbb{R})$ (such b are sometimes called distributions of positive type) is the Fourier transform

$$b(\xi) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ix\xi}\, d\mathsf{M}(x)$$

of a positive measure $d\mathsf{M}(x)$ such that

$$\int_{-\infty}^{\infty} (1 + |x|)^{-\varkappa}\, d\mathsf{M}(x) < \infty \tag{4-9}$$

for some $\varkappa$ (that is, of at most polynomial growth at infinity). In particular, this ensures that $b \in \mathcal{S}'$.

Theorem 4.2 implies that if $\langle h, \bar{f} \star f\rangle \geq 0$ for all $f \in \mathcal{D}$, then the distribution b related to h by Definition 2.4 is of positive type. This means that the sign function $s(x)$ of $h(t)$ is determined by the measure $d\mathsf{M}(x)$:

$$\langle s, \varphi\rangle = \int_{-\infty}^{\infty} \overline{\varphi(x)}\, d\mathsf{M}(x), \quad \varphi \in \mathcal{S},$$

that is, $s(x)\, dx = d\mathsf{M}(x)$. Let us define the measure

$$dm(\lambda) = \lambda\, d\mathsf{M}(-\ln \lambda), \quad \lambda \in \mathbb{R}_+. \tag{4-10}$$

It is easy to see that condition (4-9) is equivalent to condition (1-14) on measure (4-10). In terms of distribution (1-6), we have $\sigma(\lambda)\, d\lambda = dm(\lambda)$. Therefore, Theorem 2.10 leads to the following result.

Theorem 4.9. *Let $h \in \mathcal{Z}_+'$ and $\langle h, \bar{f} \star f\rangle \geq 0$ for all $f \in \mathcal{D}$. Then $h(t)$ admits the representation* (1-3) *with a positive measure $dm(\lambda)$ on $\mathbb{R}_+$ satisfying for some $\varkappa$ condition* (1-14).

The representation (1-3) is of course a particular case of (1-2). It is much more precise than (1-2), but requires the positivity of $\langle h, \bar{f} \star f\rangle$. Theorem 4.9 shows that the positivity of $\langle h, \bar{f} \star f\rangle$ imposes very strong conditions on $h(t)$. Actually, we have:

Corollary 4.10. *Let $h \in \mathcal{Z}_+'$ and $\langle h, \bar{f} \star f\rangle \geq 0$ for all $f \in \mathcal{D}$. Then $h \in C^\infty(\mathbb{R}_+)$ and*

$$(-1)^n h^{(n)}(t) \geq 0 \tag{4-11}$$

for all $t > 0$ and all $n = 0, 1, 2, \dots$ (such functions are called completely monotonic). The function $h(t)$ admits an analytic continuation in the right half-plane $\operatorname{Re} t > 0$, *and it is uniformly bounded in every strip*

$\operatorname{Re} t \in (t_1, t_2)$, *where* $0 < t_1 < t_2 < \infty$. *Moreover, for some* $\varkappa \in \mathbb{R}$ *and* $C > 0$, *we have the estimate*

$$h(t) \le C t^{-1}(1 + |\ln t|)^{\varkappa}, \quad t > 0. \tag{4-12}$$

All these assertions are direct consequences of the representation (1-3). In particular, under condition (1-14), we have

$$h(t) \le C \max_{\lambda \ge 0}\big(e^{-t\lambda}\lambda(1 + |\ln \lambda|)^{\varkappa}\big),$$

which yields (4-12).

Recall that according to the Bernstein theorem (see, e.g., Theorems 5.5.1 and 5.5.2 in [Akhiezer 1965]), condition (4-11) implies that the function $h(t)$ admits the representation (1-3) with some positive measure $dm(\lambda)$ on $[0, \infty)$. Note that condition (4-11) does not impose any restrictions on the measure $dm(\lambda)$ (except that the integral (1-3) is convergent for all $t > 0$).

Under the positivity assumption, the identity (2-21) takes a more precise form.

Proposition 4.11. *Let* $h \in \mathcal{Z}'_+$ *and* $\langle h, \bar{f} \star f\rangle \ge 0$ *for all* $f \in \mathcal{D}$. *Then there exists a positive measure* $d\mathsf{M}(x)$ *satisfying condition* (4-9) *for some* $\varkappa$ *such that*

$$\langle h, \bar{f}_1 \star f_2\rangle = \int_{-\infty}^{\infty} u_1(x)\overline{u_2(x)}\, d\mathsf{M}(x)$$

for all $f_j \in \mathcal{D}$, $j = 1, 2$, *and* $u_j = \Phi^* \Xi f_j \in \mathcal{Z}$, *where the mapping* Ξ *is defined by* (1-11).

5. Applications and examples

5.1. Consider first self-adjoint Hankel operators H of finite rank. Recall that integral kernels of Hankel operators of finite rank are given (this is the Kronecker theorem — see, e.g., Sections 1.3 and 1.8 of [Peller 2003]) by the formula

$$h(t) = \sum_{m=1}^{M} P_m(t) e^{-\alpha_m t}, \tag{5-1}$$

where $\operatorname{Re}\alpha_m > 0$ and $P_m(t)$ are polynomials of degree K_m. If H is self-adjoint, that is, $h(t) = \overline{h(t)}$, then the set $\{\alpha_1, \dots, \alpha_M\}$ consists of points lying on the real axis and pairs of points symmetric with respect to it. Let $\operatorname{Im}\alpha_m = 0$ for $m = 1, \dots, M_0$ and $\operatorname{Im}\alpha_m > 0$, $\alpha_{M_1+m} = \bar{\alpha}_m$ for $m = M_0 + 1, \dots, M_0 + M_1$. Thus $M = M_0 + 2M_1$; of course the cases $M_0 = 0$ or $M_1 = 0$ are not excluded. The condition $h(t) = \overline{h(t)}$ also requires that $P_m(t) = \overline{P_m(t)}$ for $m = 1, \dots, M_0$ and $P_{M_1+m}(t) = \overline{P_m(t)}$ for $m = M_0 + 1, \dots, M_0 + M_1$. As is well known and as we shall see below,

$$\operatorname{rank} H = \sum_{m=1}^{M} K_m + M =: r.$$

For $m = 1, \dots, M_0$, we denote by $\mathsf{p}_m = \bar{\mathsf{p}}_m$ the coefficient at t^{K_m} in the polynomial $P_m(t)$.

The following assertion yields an explicit formula for the numbers $N_\pm(H)$. Its proof relies on formula (1-4) for the sigma function of the kernel $h(t) = t^k e^{\alpha t}$ and on the identity $N_\pm(H) = N_\pm(S)$. The detailed proof is given in [Yafaev 2015].

Theorem 5.1. *For $m = 1, \dots, M_0$, set*

$$\begin{cases} \mathcal{N}_+^{(m)} = \mathcal{N}_-^{(m)} = \dfrac{K_m + 1}{2} & \text{if } K_m \text{ is odd,} \\ \mathcal{N}_+^{(m)} - 1 = \mathcal{N}_-^{(m)} = \dfrac{K_m}{2} & \text{if } K_m \text{ is even and } \mathsf{p}_m > 0, \\ \mathcal{N}_+^{(m)} = \mathcal{N}_-^{(m)} - 1 = \dfrac{K_m}{2} & \text{if } K_m \text{ is even and } \mathsf{p}_m < 0. \end{cases} \tag{5-2}$$

Then the total numbers $N_\pm(H)$ of (*strictly*) *positive and negative eigenvalues of the operator H are given by the formula*

$$N_\pm(H) = \sum_{m=1}^{M_0} \mathcal{N}_\pm^{(m)} + \sum_{m=M_0+1}^{M_0+M_1} K_m + M_1. \tag{5-3}$$

Formula (5-2) shows that every pair

$$P_m(t)e^{-\alpha_m t} + P_{m+M_1}(t)e^{-\alpha_{m+M_1} t}, \quad m = M_0 + 1, \dots, M_0 + M_1, \tag{5-4}$$

of complex conjugate terms in (5-1) yields $K_m + 1$ positive and $K_m + 1$ negative eigenvalues. The contribution of every real term $P_m(t)e^{-\alpha_m t}$, where $m = 1, \dots, M_0$, also consists of equal numbers $(K_m + 1)/2$ of positive and negative eigenvalues if the degree K_m of the polynomial $P_m(t)$ is odd. If K_m is even, then there is one more positive (negative) eigenvalue if $\mathsf{p}_m > 0$ ($\mathsf{p}_m < 0$). In particular, in the question considered, there is no "interference" between different terms $P_m(t)e^{-\alpha_m t}$, $m = 1, \dots, M_0$, and pairs (5-4) in representation (5-1) of the kernel $h(t)$.

According to (5-3), the operator H cannot be sign-definite if $M_1 > 0$. Moreover, according to (5-2), $\mathcal{N}_\pm^{(m)} = 0$ for $m = 1, \dots, M_0$ if and only if $K_m = 0$ and $\mp\mathsf{p}_m > 0$. Therefore we have the following result.

Corollary 5.2. *A Hankel operator H of finite rank is positive* (*negative*) *if and only if its kernel is given by the formula*

$$h(t) = \sum_{m=1}^{M_0} \mathsf{p}_m e^{-\alpha_m t},$$

where $\alpha_m > 0$ and $\mathsf{p}_m > 0$ ($\mathsf{p}_m < 0$) for all $m = 1, \dots, M_0$.

Corollary 5.2 admits different proofs which avoid formula (5-3). For example, one can use that although the functions $P_m(t)e^{-\alpha_m t}$ are analytic in the right half-plane $\operatorname{Re} t > 0$, they are bounded for $t = \tau + i\sigma$ as $\sigma \to \infty$ for a constant $P_m(t)$ only. Therefore, according to Corollary 4.10, such Hankel operators cannot be positive. Alternatively, using formula (5-15) below for the b-function of the kernel $t^k e^{-\alpha t}$, one can deduce Corollary 5.2 from Theorem 4.7.

Let us compare formula (5-3) with the result of [Megretskiĭ et al. 1995]. In application to finite-rank operators H, this general result implies that the spectra of Hankel operators are characterized by the following condition: the multiplicities of eigenvalues $\lambda \neq 0$ and $-\lambda$ of H do not differ by more than 1. This condition and formula (5-3) mean that there is a certain balance between positive and negative spectra of finite-rank Hankel operators. Nevertheless, neither of these results ensures another one.

5.2. Consider now Hankel operators H with kernels (1-15). Since the case $k = 0, 1, \dots$ (finite-rank Hankel operators) has been discussed in the previous subsection, here we suppose that $k \neq 0, 1, \dots$. If $k > -1$, condition (1-7) is satisfied for all κ, and the operators H are compact (actually, they belong to much better classes of operators). If $k = -1$, then condition (1-7) is satisfied for $\kappa > 1$, and the operators H are bounded but not compact.

Let us calculate the b- and s-functions of kernels (1-15). If $k > -1$, then function (2-15) equals

$$a(\xi) = (2\pi)^{-1/2} \int_0^\infty t^k e^{-\alpha t} t^{-i\xi}\, dt = (2\pi)^{-1/2} \alpha^{-1-k+i\xi} \Gamma(1+k-i\xi), \tag{5-5}$$

and hence function (2-17) equals

$$b(\xi) = \alpha^{-1-k+i\xi} \frac{\Gamma(1+k-i\xi)}{2\pi \Gamma(1-i\xi)}. \tag{5-6}$$

If $k = -1$, then in accordance with formulas (5-5) and (5-6), we have

$$a(\xi) = (2\pi)^{-1/2} \alpha^{i\xi} \lim_{\varepsilon \to +0} \Gamma(\varepsilon - i\xi), \quad b(\xi) = (2\pi)^{-1} \alpha^{i\xi} i(\xi + i0)^{-1}.$$

This yields the expression

$$\begin{cases} s(x) = 0 & \text{if } x > \beta, \\ s(x) = 1 & \text{if } x < \beta, \end{cases} \quad \text{where } \beta = -\ln \alpha, \tag{5-7}$$

for the function $s = \sqrt{2\pi}\, \Phi^* b$. Formula (5-7) remains true for the Carleman operator $\boldsymbol{C}$ (the Hankel operator with kernel $h(t) = t^{-1}$) when $\alpha = 0$. Indeed, in this case, according to (2-23), the sign function $s(x)$ equals 1.

Next, we calculate the Fourier transform of function (5-6). Assume first that $k \in (-1, 0)$. Then (see, e.g., formula (1.5.12) in [Erdélyi et al. 1953])

$$\int_0^\infty t^{-k-1} (t+1)^{-1+i\xi}\, dt = \frac{\Gamma(-k)\Gamma(1+k-i\xi)}{\Gamma(1-i\xi)}.$$

Making here the change of variables $t + 1 = \alpha^{-1} e^{-x}$, we find that

$$\frac{1}{\Gamma(-k)} \int_{-\infty}^\infty (e^{-x} - \alpha)_+^{-k-1} e^{-ix\xi}\, dx = \alpha^{-1-k-i\xi} \frac{\Gamma(1+k-i\xi)}{\Gamma(1-i\xi)}.$$

Passing now to the inverse Fourier transform, we see that for $k \in (-1, 0)$ the sign function $s(x) = s_k(x)$ of kernel (1-15) equals

$$s(x) = \frac{1}{\Gamma(-k)} (e^{-x} - \alpha)_+^{-k-1}. \tag{5-8}$$

Let us verify that this formula remains true for all noninteger k. To that end, we assume that (5-8) holds for some noninteger $k > -1$ and check it for $k_1 = k + 1$. Since

$$\Gamma(1+k_1-i\xi) = (k_1 - i\xi)\Gamma(1+k-i\xi),$$

we have

$$s_{k_1}(x) = \alpha^{-1}(k_1 - \partial)s_k(x).$$

Substituting here formula (5-8) for $s_k(x)$ and differentiating this expression, we obtain formula (5-8) for $s_{k_1}(x)$. This concludes the proof of relation (5-8) for all $k \geq -1$.

Lemma 5.3. *Let $h(t)$ be given by formula* (1-15), *where $k \notin \mathbb{Z}_+$. Then the sign function is determined by relation* (5-8).

Actually, relation (5-8) remains true for $k \in \mathbb{Z}_+$ if one takes into account that the distribution $(e^{-x} - \alpha)_+^{-k-1}$ has poles at integer points. For example, for $k = 0$ we have $s(x) = \alpha^{-1}\delta(x - \beta)$.

Obviously, $s(x) = 0$ for $x > \beta = -\ln\alpha$. If $k = -1$, then $s(x) = 1$ for $x < \beta$. If $k \in (-1, 0)$, then $s(x) \geq 0$ and $s \in L^1(\mathbb{R})$. Therefore it follows from Theorem 4.5 that $H \geq 0$.

If $k > 0$, then distribution (5-8) does not have a definite sign. Therefore it can be deduced from Theorem 4.2 that the corresponding Hankel operator also is not sign-definite.

Alternatively, for the proof of this result we can use Corollary 4.8. Formula (2-11) implies that function (5-6) has the asymptotics

$$b(\xi) = (2\pi)^{-1}\alpha^{-1-k-i\xi}(-i\xi)^k(1 + O(|\xi|^{-1})), \quad |\xi| \to \infty. \tag{5-9}$$

Making the dilation transformation in (1-15), we can suppose that $\alpha = 1$. Then we have

$$\operatorname{Re} b(\xi) = (2\pi)^{-1}\cos\left(\frac{\pi k}{2}\right)\xi^k + O(\xi^{k-1}), \quad \xi \to +\infty. \tag{5-10}$$

Since $\cos(\pi k/2) \neq 0$ unless k is an integer odd number, this expression tends to $\pm\infty$ if $\pm\cos(\pi k/2) > 0$. Thus Corollary 4.8 for the case $b = b_\infty$ ensures that the Hankel operator H is not sign-definite.

Let us summarize the results obtained.

Proposition 5.4. *The Hankel operator with kernel* (1-15) *is positive for $k \in [-1, 0]$, and it is not sign-definite for $k > 0$.*

Actually, using relation (5-8), one can calculate explicitly the numbers $N_\pm(H)$ for all values of k (see [Yafaev 2014b]).

Explicit formulas for the sign functions can also be used to treat more complicated Hankel operators. For example, in view of (5-7), the following assertion directly follows from Theorem 4.5.

Example 5.5. The Hankel operator with kernel

$$h(t) = t^{-1}(e^{-\alpha_1 t} - \gamma e^{-\alpha_2 t}), \quad \gamma \geq 0,$$

is positive if and only if $\alpha_2 \geq \alpha_1 \geq 0$ and $\gamma \leq 1$.

5.3. In this subsection, we consider the Hankel operator H with kernel (1-16). Condition (1-7) is now fulfilled for all κ, and the operator H belongs of course to the Hilbert–Schmidt class (actually, to much better classes). Observe that

$$a(\xi) = (2\pi)^{-1/2} \int_0^\infty e^{-t^r} t^{-i\xi}\, dt = (2\pi)^{-1/2} r^{-1} \Gamma\Big(\frac{1-i\xi}{r}\Big)$$

and define, as usual, the function $b(\xi)$ by formula (2-17) so that

$$b(\xi) = (2\pi r)^{-1} \frac{\Gamma\Big(\dfrac{1-i\xi}{r}\Big)}{\Gamma(1-i\xi)}. \tag{5-11}$$

Consider first the case $r > 1$. It follows from the Stirling formula (2-11) that for all $r > 1$, the modulus of function (5-11) exponentially grows and the periods of its oscillations tend to zero only logarithmically as $|\xi| \to \infty$. Therefore, Theorem 4.7 implies that the Hankel operator with kernel (1-16) is not sign-definite.

The Hankel operator H with kernel $h(t) = e^{-t^2}$ can also be treated (see Appendix B) in a completely different way, which is perhaps also of some interest. This method shows that both positive and negative spectra of the operator H are infinite.

If $r = 1$, then $h(t) = e^{-t}$ yields a positive Hankel operator of rank 1.

Let us now consider the case $r < 1$. Again according to the Stirling formula (2-11), function (5-11) belongs to $L^1(\mathbb{R})$, so that its Fourier transform

$$s(x) = (2\pi r)^{-1} \int_{-\infty}^{\infty} \frac{\Gamma\Big(\dfrac{1-i\xi}{r}\Big)}{\Gamma(1-i\xi)} e^{ix\xi}\, d\xi =: I_r(x) \tag{5-12}$$

is a continuous function which tends to 0 as $|x| \to \infty$. Therefore, by Proposition 4.6, the corresponding Hankel operator H is nonnegative if and only if $I_r(x) \geq 0$ for all $x \in \mathbb{R}$.

It turns out that $I_r(x) \geq 0$. Surprisingly, we have not found a proof of this fact in the literature, but it follows from our results. Only for $r = \frac{1}{2}$, integral (5-12) can be explicitly calculated. Indeed, according to formula (1.2.15) of [Erdélyi et al. 1953],

$$\frac{\Gamma(2(1-i\xi))}{\Gamma(1-i\xi)} = 2^{1-2i\xi} \pi^{-1/2} \Gamma\big(\tfrac{3}{2} - i\xi\big).$$

Therefore it follows from formula (2-25) that

$$I_{1/2}(x) = 2^{-1} \pi^{-1/2} e^{3x/2} e^{-e^x/4}, \tag{5-13}$$

which is of course positive.

For an arbitrary $r \in (0, 1)$, one can proceed from the Bernstein theorem on completely monotonic functions (see Section 4.4). Observe that if

$$\psi(t) = t^{-p} e^{-t^r}, \quad p \geq 0, \tag{5-14}$$

then

$$\psi'(t) = -pt^{-p-1}e^{-t^r} - rt^{-p+r-1}e^{-t^r}.$$

Further differentiations of $\psi(t)$ change the sign and yield sums of terms having the form (5-14). Thus the function $h(t) = e^{-t^r}$ satisfies, for all n, condition (4-11), and hence admits the representation (1-3) with some positive measure $dm(\lambda)$. It follows from (1-3) that

$$(Hf, f) = \int_0^\infty |(\mathsf{L}f)(\lambda)|^2 \, dm(\lambda) \geq 0, \quad \text{for all } f \in C_0^\infty(\mathbb{R}_+),$$

where L is the Laplace transform (2-28). Since the operator H is bounded, this implies that $H \geq 0$.

Thus we have obtained the following result.

Proposition 5.6. *The Hankel operator with kernel* (1-16) *is positive for* $r \in (0, 1]$, *and it is not sign-definite for* $r > 1$.

Putting together this result with Theorem 4.5, we see that integral (5-12) is positive for all $r \in (0, 1)$. Our indirect proof of this fact looks curiously enough.

5.4. Let us now discuss convolution operators with growing kernels $b(\xi)$. We emphasize that condition (4-3) does not guarantee that the numbers $N_\pm(b; C_0^\infty(\mathbb{R}))$ are infinite. Indeed, consider the kernel $h(t) = t^k e^{-\alpha t}$, where k is a positive integer. Formula (5-6) shows that for $\operatorname{Im}\alpha = 0$, the corresponding b-function

$$b(\xi) = (2\pi)^{-1}\alpha^{-1-k+i\xi}(1 - i\xi)\cdots(k - i\xi) \tag{5-15}$$

has power asymptotics as $|\xi| \to \infty$. According to Theorem 5.1, the positive and negative spectra of the Hankel operator H with the kernel $h(t)$ are finite; for example, H has exactly $(k+1)/2$ positive and negative eigenvalues if k is odd. Moreover, if $\operatorname{Im}\alpha \neq 0$, then in view of (5-15), the function $b(\xi)$ exponentially grows as $\xi \to +\infty$ or $\xi \to -\infty$. Nevertheless, the Hankel operator H with kernel $h(t) = t^k(e^{-\alpha t} + e^{-\bar{\alpha} t})$ has exactly $k+1$ positive and negative eigenvalues.

On the other hand, for kernel (5-11), where $r = 2$, we have $N_\pm(b; C_0^\infty(\mathbb{R})) = \infty$. This follows from Theorem 4.1 because, by Proposition B.1, the Hankel operator with kernel $h(t) = e^{-t^2}$ has an infinite number of positive and negative eigenvalues.

A similar phenomenon occurs for Hankel operators with nonsmooth kernels. This is discussed in the next section. However, in general, the calculation of the numbers $N_\pm(b; C_0^\infty(\mathbb{R}))$ for convolutions with kernels $b(\xi)$ growing and oscillating at infinity looks like an open problem.

6. Hankel operators with nonsmooth kernels

According to Corollary 4.10, a Hankel operator H can be sign-definite only for kernels $h \in C^\infty(\mathbb{R}_+)$. Here we show that if $h(t)$ or one of its derivatives $h^{(l)}(t)$ has a jump discontinuity at some point $t_0 > 0$, then H has an infinite number of both positive and negative eigenvalues accumulating to zero. Moreover, we calculate their asymptotic behavior.

6.1. We start with a distributional kernel. Let the symbol (see the definition in Section 3.2) of the Hankel operator H be defined by the formula $\omega(\mu) = e^{it_0\mu}$. Then $h(t) = (2\pi)^{-1/2}(\Phi\omega)(t) = \delta(t - t_0)$. It follows from (1-1) that $H = 0$ for $t_0 \leq 0$ and

$$(Hf)(t) = f(t_0 - t)$$

for $t_0 > 0$, which we suppose from now on. For such $h(t)$, condition (3-12) is satisfied for $N = 0$ and $\kappa = 0$.

The operator H admits an explicit spectral analysis. Indeed, observe first that $(Hf)(t) = 0$ for $t > t_0$ and hence $L^2(t_0, \infty) \subset \operatorname{Ker} H$. Since $H^2 f = f$ for $f \in L^2(0, t_0)$, the restriction of H on its invariant subspace $L^2(0, t_0)$ may have only ± 1 as eigenvalues. Obviously, the eigenspace $\mathcal{H}_\pm$ of H corresponding to the eigenvalue ± 1 consists of all functions $f(t)$ such that $f(t) = \pm f(t_0 - t)$. Since

$$\mathcal{H}_+ \oplus \mathcal{H}_- \oplus L^2(t_0, \infty) = L^2(\mathbb{R}_+),$$

the spectrum of H consists of the eigenvalues $0, 1, -1$ of infinite multiplicity each.

6.2. For a compact operator H, let us denote by $\lambda_n^{(+)}$ $(-\lambda_n^{(-)})$ its positive (negative) eigenvalues. Positive (negative) eigenvalues are of course enumerated in decreasing (increasing) order with multiplicities taken into account.

Let us start with the explicit kernel

$$h(t) = (t_0 - t)^l \ \text{ for } \ t \leq t_0, \quad h(t) = 0 \ \text{ for } \ t > t_0, \tag{6-1}$$

where l is one of the numbers $l = 0, 1, \dots$. Then

$$(Hf)(t) = \int_0^{t_0 - t} (t_0 - t - s)^l f(s)\, ds, \quad t \in (0, t_0),$$

and $(Hf)(t) = 0$ for $t \geq t_0$. For such $h(t)$, the symbol equals

$$\omega(\mu) = \int_0^{t_0} e^{i\mu t}(t_0 - t)^l\, dt = l!\,(i\mu)^{-l-1}\Big(e^{i\mu t_0} - \sum_{k=0}^{l} \frac{1}{k!}(i\mu t_0)^k\Big).$$

It is a smooth function oscillating as $|\mu| \to \infty$.

It follows from (1-5) that the b-function of the operator H equals

$$b(\xi) = \frac{l!\, t_0^{l+1-i\xi}}{2\pi\Gamma(l + 2 - i\xi)} \tag{6-2}$$

(if $h(t) = \delta(t - t_0)$, then this formula is true with $l = -1$). So according to Theorem 4.7, we have $N_\pm(H) > 0$. Actually, the spectrum of H consists of an infinite number of positive and negative eigenvalues denoted by $\lambda_n^{(\pm)}$, and we will find their asymptotic behavior as $n \to \infty$.

Let us consider the spectral problem $Hf = \lambda f$, that is,

$$\int_0^{t_0 - t} (t_0 - t - s)^l f(s)\, ds = \lambda f(t), \quad t \in (0, t_0). \tag{6-3}$$

Differentiating this equation k times, we find that

$$(-1)^k l(l-1)\cdots(l-k+1)\int_0^{t_0-t}(t_0-t-s)^{l-k}f(s)\,ds=\lambda f^{(k)}(t) \tag{6-4}$$

for $k=1,\dots,l$. Differentiating (6-4), where $k=l$ once more, we see that

$$l!\,f(t)=\lambda(-1)^{l+1}f^{(l+1)}(t_0-t),\quad t\in(0,t_0). \tag{6-5}$$

Setting $t=t_0$ in (6-3) and (6-4), we obtain the boundary conditions

$$f(t_0)=f'(t_0)=\cdots=f^{(l)}(t_0)=0. \tag{6-6}$$

Conversely, if a function $f(t)$ satisfies (6-5) and boundary conditions (6-6), it satisfies also (6-3). This leads to the following intermediary result.

Lemma 6.1. *Let the operator A be defined on the Sobolev class $\mathsf{H}^{l+1}(0,t_0)$ by the equation*

$$(Af)(t)=(-1)^{l+1}f^{(l+1)}(t_0-t). \tag{6-7}$$

Considered with boundary conditions (6-6), *it is self-adjoint in the space $L^2(0,t_0)$, and its eigenvalues $\alpha_n^{(\pm)}$ are linked to eigenvalues $\lambda_n^{(\pm)}$ of the Hankel operator H with kernel* (6-1) *by the equation $\alpha_n^{(\pm)}=l!\,(\lambda_n^{(\pm)})^{-1}$.*

6.3. Clearly, A^2 is a differential operator and the asymptotic behavior of its eigenvalues is described by the Weyl formula. However, to find the eigenvalue asymptotics of the operator A, we have to distinguish between positive and negative eigenvalues. For this reason, it is convenient to introduce an auxiliary operator $\widetilde{A}$ with symmetric (with respect to the point 0) spectrum having the same eigenvalue asymptotics as A.

We define $\widetilde{A}$ by the same formula (6-7) as A but consider it on functions in $\mathsf{H}^{l+1}(0,t_0/2)\oplus\mathsf{H}^{l+1}(t_0/2,t_0)$ satisfying the boundary conditions

$$f^{(k)}(0)=f^{(k)}\Big(\frac{t_0}{2}-0\Big),\quad f^{(k)}\Big(\frac{t_0}{2}+0\Big)=f^{(k)}(t_0), \tag{6-8}$$

where $k=0,\dots,l$ for l even, and

$$f^{(k)}(0)=f^{(k)}\Big(\frac{t_0}{2}-0\Big)=0,\quad f^{(k)}\Big(\frac{t_0}{2}+0\Big)=f^{(k)}(t_0)=0, \tag{6-9}$$

where $k=0,\dots,(l-1)/2$ for l odd. The operator $\widetilde{A}$ is self-adjoint in the space $L^2(0,t_0/2)\oplus L^2(t_0/2,t_0)$, and it is determined by the matrix

$$\widetilde{A}=\begin{pmatrix}0 & A_{1,2}\\ A_{2,1} & 0\end{pmatrix},\quad A_{1,2}=A_{2,1}^*, \tag{6-10}$$

where $A_{2,1}:L^2(0,t_0/2)\to L^2(t_0/2,t_0)$. The operator $A_{2,1}$ is again given by relation (6-7) on functions in $\mathsf{H}^{l+1}(0,t_0/2)$ satisfying conditions (6-8) or (6-9) at the points 0 and $t_0/2-0$. It follows from formula

(6-10) that the spectrum of the operator $\widetilde{A}$ is symmetric with respect to the point 0 and consists of eigenvalues $\pm a_n$, where a_n^2 are eigenvalues of the operator $A_{2,1}^* A_{2,1} =: \boldsymbol{A}$.

An easy calculation shows that $\boldsymbol{A}$ is the differential operator $\boldsymbol{A} = (-1)^{l+1}\partial^{2l+2}$ in the space $L^2(0, t_0/2)$ defined on functions in the class $\mathsf{H}^{2l+2}(0, t_0/2)$ satisfying the boundary conditions $f^{(k)}(0) = f^{(k)}(t_0/2)$, where $k = 0, \dots, 2l+1$ for l even, and the boundary conditions

$$f^{(k)}(0) = f^{(k)}\left(\frac{t_0}{2}\right) = f^{(l+1+k)}(0) = f^{(l+1+k)}\left(\frac{t_0}{2}\right) = 0,$$

where $k = 0, \dots, (l-1)/2$ for l odd. The asymptotic formula for the eigenvalues a_n^2 of $\boldsymbol{A}$ is given by the Weyl formula, that is,

$$a_n = (2\pi t_0^{-1} n)^{l+1}(1 + O(n^{-1})).$$

Let us now observe that the operators A and $\widetilde{A}$ are self-adjoint extensions of a symmetric operator A_0 with finite deficiency indices $(2l+2, 2l+2)$. For example, A_0 can be defined by formula (6-7) on C^∞-functions vanishing in some neighborhoods of the points 0, $t_0/2$, and t_0. Therefore, the operators A and $\widetilde{A}$ have the same spectral asymptotics. Taking Lemma 6.1 into account, we obtain the following result.

Lemma 6.2. *The eigenvalues of the Hankel operator $H = H(t_0)$ with kernel* (6-1) *have the asymptotic behavior*

$$\lambda_n^{(\pm)} = l!\,(2\pi)^{-l-1} t_0^{l+1} n^{-l-1}(1 + O(n^{-1})). \tag{6-11}$$

Remark 6.3. It is interesting that the asymptotic coefficient in (6-11) is proportional to t_0^{l+1}, where t_0 is the jump point. However, this fact is not surprising, because the operators $H(t_0)$ are related by the equation $H(t_0) = t_0^{l+1}\mathsf{D}(t_0)^* H(1)\mathsf{D}(t_0)$, where $\mathsf{D}(t_0)$, $(\mathsf{D}(t_0)f)(t) = \sqrt{t_0}\, f(t_0 t)$, is the unitary operator of dilations.

Remark 6.4. In the case $l = 0$ we have the explicit formulas

$$\lambda_n^{(+)} = (2\pi)^{-1} t_0\bigl(n - \tfrac{3}{4}\bigr)^{-1}, \quad \lambda_n^{(-)} = (2\pi)^{-1} t_0\bigl(n - \tfrac{1}{4}\bigr)^{-1}, \quad n = 1, 2, \dots .$$

6.4. Now we extend the asymptotics (6-11) to general Hankel operators whose kernels (or their derivatives) have jumps of continuity at a single positive point. To that end, we combine Lemma 6.2 with Theorem 7.4 in Chapter 6 of [Peller 2003]. This theorem implies that singular values $s_n(V)$ of a Hankel operator V satisfy the bound

$$s_n(V) = o(n^{-l-1})$$

if V has a symbol belonging to the Besov class $\boldsymbol{B}_{p,p}^{l+1}(\mathbb{R})$, where $p = (l+1)^{-1}$. By the Weyl theorem on the stability of the power asymptotics of eigenvalues, adding such an operator V to the Hankel operator with kernel (6-1) cannot change the leading asymptotic term in formula (6-11). This yields the following result.

Theorem 6.5. *Let $l \in \mathbb{Z}_+$, and let $v(t)$ be the Fourier transform of a function in the Besov class $\boldsymbol{B}^{l+1}_{p,p}(\mathbb{R})$, where $p = (l+1)^{-1}$. Set*

$$h(t) = h_0(t_0 - t)^l + v(t)$$

for $t \le t_0$ and $h(t) = v(t)$ for $t > t_0$. Then eigenvalues of the Hankel operator H have the asymptotics

$$\lambda_n^{(\pm)} = |h_0| l! \, (2\pi)^{-l-1} t_0^{l+1} n^{-l-1} (1 + o(1))$$

as $n \to \infty$.

We emphasize that under the assumptions of this theorem, the leading terms in the asymptotics of the positive and negative eigenvalues are the same. Of course if $h(t)$ becomes smoother (l increases), then eigenvalues of the Hankel operator H decrease faster as $n \to \infty$. Observe that for $l = 0$ (when the kernel itself is discontinuous), the Hankel operator H does not belong to the trace class.

We finally note that, under assumptions close to those of Theorem 6.5, the asymptotic behavior of the singular values of the Hankel operator H was obtained long ago in [Glover et al. 1990] by a completely different method.

7. Perturbations of the Carleman operator

In this section we consider operators $H = H_0 + V$, where H_0 is the Carleman operator $\boldsymbol{C}$ (or a more general operator) and the perturbation V belongs to one of the classes introduced in Section 5. Various objects related to the operator H_0 will be endowed with the index "0", and objects related to the operator V will be endowed with the index "v".

7.1. For perturbations V of finite rank, we have the following result.

Theorem 7.1. *Let the sign function $s_0(x)$ of a Hankel operator H_0 be bounded and positive. Let the kernel $v(t)$ of V be given by the formula*

$$v(t) = \sum_{m=1}^{M} P_m(t) e^{-\alpha_m t},$$

where $P_m(t)$ is a polynomial of degree K_m. Put $H = H_0 + V$ and define the numbers $\mathcal{N}_-^{(m)}$ by formula (5-2). *Then $N_-(H)$ is given by formula* (5-3).

Corollary 7.2. *Under the assumptions of Theorem 7.1, we have $N_-(H) = N_-(V)$. In particular, $H \ge 0$ if and only if $V \ge 0$.*

Of course, in the case $H_0 = 0$, Theorem 7.1 reduces to Theorem 5.1. Since for the Carleman operator $\boldsymbol{C}$ the sign function equals 1, Theorem 7.1 applies to $H_0 = \boldsymbol{C}$.

The inequality $N_-(H) \le N_-(V)$ is obvious because $H_0 \ge 0$. On the other hand, the opposite inequality $N_-(H) \ge N_-(V)$ looks surprising because the operator H_0, which may have the continuous spectrum, is much "stronger" than the operator V of finite rank. At a heuristic level, the equality $N_-(H) = N_-(V)$ can be explained by the fact that the supports of the sign functions $s_0(x)$ and $s_v(x)$ are essentially disjoint.

Very loosely speaking, this means that the operators H_0 and V "live in orthogonal subspaces", and hence the positive operator H_0 does not affect the negative spectrum of V. The detailed proof of Theorem 7.1, as well as that of Theorem 5.1, is given in [Yafaev 2015].

7.2. Let $\boldsymbol{C}$ be the Carleman operator, and let V be the Hankel operator with kernel

$$v(t) = t^k e^{-\alpha t}, \quad \alpha > 0,\ k > -1. \tag{7-1}$$

The operator V is compact, and hence the essential spectrum $\mathrm{spec}_{\mathrm{ess}}(H_\gamma)$ of the operator

$$H_\gamma = \boldsymbol{C} - \gamma V, \quad \gamma \in \mathbb{R}, \tag{7-2}$$

coincides with the interval $[0, \pi]$. Since the sign function of the operator $\boldsymbol{C}$ equals 1, the sign function s_γ of the operator H_γ equals

$$s_\gamma(x) = 1 - \gamma s_v(x),$$

where the function $s_v(x)$ is given by formula (5-8).

Let first $k \in (-1, 0)$. Observe that $s_v(x)$ is continuous for $x < \beta = -\ln\alpha$ and $s_v(x) \to +\infty$ as $x \to \beta - 0$ but $s_v \in L^1(\mathbb{R})$. Thus the function $s_\gamma(x)$ goes to $-\infty$ as $x \to \beta - 0$ for all $\gamma > 0$, and hence it follows from Theorem 4.5 that the operator H_γ has infinite negative spectrum for all $\gamma > 0$.

In the case $k > 0$, we use the formula

$$b(\xi) = \delta(\xi) + b_v(\xi) \tag{7-3}$$

and apply Theorem 4.7 (more precisely, Corollary 4.8) with $b_0(\xi) = \delta(\xi)$ and $b_\infty(\xi) = b_v(\xi)$. Since $b_0 \in C(\mathbb{R})'$ and b_∞ has the asymptotic behavior (5-9), the operator H_γ has a negative spectrum for all $\gamma \neq 0$.

Let us summarize the results obtained.

Proposition 7.3. *Let $H_\gamma = \boldsymbol{C} - \gamma V$, where V is the Hankel operator with kernel* (7-1). *Then*:

(1) *If $k \in (-1, 0)$ and $\gamma > 0$, then the operator H_γ has an infinite number of negative eigenvalues.*

(2) *If $k > 0$, then the operator H_γ has at least one negative eigenvalue for all $\gamma \neq 0$.*

7.3. The result below directly follows from Theorem 4.5.

Proposition 7.4. *Suppose that the sign function $s_v(x)$ of a Hankel operator V is continuous and $s_v(x) \to 0$ as $|x| \to \infty$. Then the operator H_γ defined by formula* (7-2) *is positive if and only if*

$$\gamma s_v(x) \leq 1 \quad \textit{for all } x \in \mathbb{R}.$$

If this condition is not satisfied, then H_γ has infinite negative spectrum.

We note that by Proposition 3.12, under the assumption of Proposition 7.4 on the sign function s_v, the operator V is compact, and hence $\mathrm{spec}_{\mathrm{ess}}(H_\gamma) = [0, \infty)$. Of course, this assumption on s_v is satisfied if $b_v \in L^1(\mathbb{R})$.

Example 7.5. Let $v(t) = e^{-t^r}$, where $r < 1$. We have seen in Section 5.3 that its sign function $s_v(x)$ equals $I_r(x)$, where $I_r(x)$ is integral (5-12). Recall that $I_r(x)$ is a nonnegative continuous function of $x \in \mathbb{R}$ and $I_r(x) \to 0$ as $|x| \to \infty$. Set

$$\nu_r = \max_{x \in \mathbb{R}} I_r(x).$$

Then $H_\gamma \geq 0$ if $\gamma \leq \nu_r^{-1}$, and the operator H_γ has infinite negative spectrum for all $\gamma > \nu_r^{-1}$. Using the explicit formula (5-13), it is easy to calculate $\nu_{1/2} = 3\sqrt{6/\pi}\, e^{-3/2}$.

In the case $r > 1$ we use formula (7-3). As shown in Section 5.3, the modulus of the function $b_v(\xi)$ exponentially grows and the periods of its oscillations tend to zero only logarithmically as $|\xi| \to \infty$. Therefore Theorem 4.7 yields the following result.

Proposition 7.6. *Let* $v(t) = e^{-t^r}$, *where* $r > 1$. *Then the operator* (7-2) *has at least one negative eigenvalue for all* $\gamma \neq 0$.

Thus the results on the negative spectrum of the operator $H_\gamma = \boldsymbol{C} - \gamma V$, where $v(t) = e^{-t^r}$, are qualitatively different for $r < 1$, $r = 1$, and $r > 1$.

Appendix A: Proof of Lemma 3.6

Set

$$F_\kappa^{(n)} = \max_{t \in \mathbb{R}_+} \big(\langle \ln t \rangle^\kappa t^n |F^{(n)}(t)| \big),$$

where for shortness we use the notation $\langle x \rangle = (1 + |x|)$.

Let us first consider $(\Phi^* F)(\lambda)$ for $\lambda \in (-1, 1) =: I$. We have

$$\sqrt{2\pi}\,(\Phi^* F)(\lambda) = \int_0^a F(t) e^{i\lambda t}\, dt + \int_a^\infty F(t) e^{i\lambda t}\, dt, \quad a = |\lambda|^{-1/2}.$$

The first integral on the right-hand side is bounded by $F_0^{(0)} |\lambda|^{-1/2}$, which belongs to $L^1(I)$. In the second integral, we integrate by parts:

$$\int_a^\infty F(t) e^{i\lambda t}\, dt = i\lambda^{-1} F(a) e^{i\lambda a} + i\lambda^{-1} \int_a^\infty F'(t) e^{i\lambda t}\, dt. \tag{A-1}$$

The first term here is bounded by $C|\lambda|^{-1} \langle \ln \lambda \rangle^{-\kappa} F_\kappa^{(0)}$, which belongs to $L^1(I)$ if $\kappa > 1$. The second term is bounded by

$$|\lambda|^{-1} \int_a^\infty t^{-1} \langle \ln t \rangle^{-\kappa}\, dt\, F_\kappa^{(1)} \leq C |\lambda|^{-1} \langle \ln \lambda \rangle^{-\kappa+1} F_\kappa^{(1)}.$$

It belongs to $L^1(I)$ if $\kappa > 2$. Thus, for all $\kappa > 2$, we have

$$\|\Phi^* F\|_{L^1(I)} \leq C(\kappa) (F_0^{(0)} + F_\kappa^{(1)}). \tag{A-2}$$

Next, we consider $(\Phi^* F)(\lambda)$ for $|\lambda| \geq 1$. Integrating by parts, we see that

$$\sqrt{2\pi}\,(\Phi^* F)(\lambda) = i\lambda^{-1} \int_0^a F'(t) e^{i\lambda t}\, dt + i\lambda^{-1} \int_a^\infty F'(t) e^{i\lambda t}\, dt. \tag{A-3}$$

The first term here is bounded by

$$|\lambda|^{-1}\int_0^a t^{-1}\langle\ln t\rangle^{-\kappa}\,dt\,F_\kappa^{(1)}\le C|\lambda|^{-1}\langle\ln\lambda\rangle^{-\kappa+1}F_\kappa^{(1)}.$$

It belongs to $L^1(\mathbb{R}\setminus I)$ if $\kappa>2$. In the second integral in (A-3) we once more integrate by parts, that is, we use formula (A-1) with $F(t)$ replaced by $F'(t)$. The function $\lambda^{-2}F'(a)$ is bounded by $|\lambda|^{-3/2}F_0^{(1)}$. For the second term, we use the estimate

$$\left|\lambda^{-2}\int_a^\infty F''(t)e^{i\lambda t}\,dt\right|\le\lambda^{-2}\int_a^\infty t^{-2}\,dt\,F_0^{(2)}=|\lambda|^{-3/2}F_0^{(2)}.$$

Therefore the second term in (A-3) also belongs to $L^1(\mathbb{R}\setminus I)$. Thus, for all $\kappa>2$, we have

$$\|\Phi^*F\|_{L^1(\mathbb{R}\setminus I)}\le C(\kappa)(F_\kappa^{(1)}+F_0^{(2)}). \tag{A-4}$$

Putting together (A-2) and (A-4), we obtain estimate (3-13).

Appendix B: The Gaussian kernel

Here we return to the Hankel operator H with kernel $h(t)=e^{-t^2}$ considered in Section 5.3. Since $e^{-(t+s)^2}=e^{-t^2}e^{-2ts}e^{-s^2}$, we have the identity

$$(Hf,f)=(\mathsf{L}\psi,\psi), \tag{B-1}$$

where $\psi(t)=e^{-t^2/2}f(t/\sqrt{2})/\sqrt{2}$ and L is the Laplace transform defined in the space $L^2(\mathbb{R}_+)$ by formula (2-28). We shall use (B-1) essentially in the same way as the main identity (1-10). It follows from equality (2-29) for $\alpha=\frac{1}{2}$ that $\mathsf{L}=M^*\mathcal{J}\boldsymbol{\Gamma}_{1/2}M$, where M is the Mellin transform. Therefore the spectrum of L consists of the interval $[-\gamma,\gamma]$, where, according to (2-24),

$$\gamma=\max_{\xi\in\mathbb{R}}\bigl|\Gamma\bigl(\tfrac{1}{2}+i\xi\bigr)\bigr|=\sqrt{\pi}\max_{\xi\in\mathbb{R}}(\cosh(\pi\xi))^{-1}=\sqrt{\pi}.$$

This allows us to check the following assertion.

Proposition B.1. *The Hankel operator H with kernel $h(t)=e^{-t^2}$ has an infinite number of positive and negative eigenvalues.*

Proof. Fix some $\mu\in(0,\sqrt{\pi})$. For an arbitrary N, let $\Delta_1^{(+)},\dots,\Delta_N^{(+)}\subset(\mu,\sqrt{\pi})$ and $\Delta_1^{(-)},\dots,\Delta_N^{(-)}\subset(-\sqrt{\pi},-\mu)$ be closed mutually disjoint intervals. Choose functions $\varphi_j^{(\pm)}$ in the spectral intervals $\Delta_j^{(\pm)}$ of the operator L and such that $\|\varphi_j^{(\pm)}\|=1$, $j=1,\dots N$. Let $\varphi^{(\pm)}=\sum_{j=1}^N\alpha_j\varphi_j^{(\pm)}$ be a linear combination of the functions $\varphi_1^{(\pm)},\dots,\varphi_N^{(\pm)}$. Then

$$\pm(\mathsf{L}\varphi^{(\pm)},\varphi^{(\pm)})=\pm\sum_{j=1}^N|\alpha_j|^2(\mathsf{L}\varphi_j^{(\pm)},\varphi_j^{(\pm)})\ge\mu\sum_{j=1}^N|\alpha_j|^2\|\varphi_j^{(\pm)}\|^2=\mu\|\varphi^{(\pm)}\|^2. \tag{B-2}$$

For an arbitrary $\varepsilon>0$, we can choose $\psi_j^{(\pm)}\in C_0^\infty(\mathbb{R}_+)$ such that $\|\psi_j^{(\pm)}-\varphi_j^{(\pm)}\|<\varepsilon$ for all $j=1,\dots,N$. Since the functions $\varphi_j^{(\pm)}$ are orthogonal, the functions $\psi_j^{(\pm)}$ are linearly independent if ε is small enough.

Moreover, it follows from (B-2) that

$$\pm(\mathsf{L}\psi^{(\pm)}, \psi^{(\pm)}) \geq 2^{-1}\mu\|\psi^{(\pm)}\|^2 \tag{B-3}$$

if $\psi^{(\pm)} = \sum_{j=1}^{N} \alpha_j \psi_j^{(\pm)}$ and ε is small.

Set now $f^{(\pm)}(t) = \sqrt{2}\, e^{t^2} \psi^{(\pm)}(\sqrt{2}t)$. Then $f^{(\pm)} \in L^2(\mathbb{R}_+)$, and according to the identity (B-1), inequality (B-3) implies that $\pm(Hf^{(\pm)}, f^{(\pm)}) > 0$ on the linear subspace of such functions $f^{(\pm)}$ (except $f^{(\pm)} = 0$). This subspace has dimension N. Hence the operator H has at least N positive and N negative eigenvalues. Since N is arbitrary, this concludes the proof. □

We emphasize that the operator H is compact while the operator L has the continuous spectrum. Nevertheless the total multiplicities of their positive and negative spectra are the same (infinite).

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