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NODAL SETS AND GROWTH EXPONENTS OF LAPLACE EIGENFUNCTIONS ON SURFACES

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We prove a result, announced by F. Nazarov, L. Polterovich and M. Sodin, that exhibits a relation between the average local growth of a Laplace eigenfunction on a closed surface and the global size of its nodal set. More precisely, we provide a lower and an upper bound to the Hausdorff measure of the nodal set in terms of the expected value of the growth exponent of an eigenfunction on disks of wavelength-like radius. Combined with Yau's conjecture, the result implies that the average local growth of an eigenfunction on such disks is bounded by constants in the semiclassical limit. We also obtain results that link the size of the nodal set to the growth of solutions of planar Schrödinger equations with small potential.

1. Introduction and main results

1.1. ***Nodal sets of Laplace eigenfunctions.*** Let (M, g) be a smooth, closed two-dimensional Riemannian manifold endowed with a C^∞ metric g. Let $\{\phi_\lambda\}$, $\lambda \nearrow \infty$, be any sequence of eigenfunctions of the negative-definite Laplace–Beltrami operator Δ_g:

$$\Delta_g \phi_\lambda + \lambda \phi_\lambda = 0. \tag{1.1.1}$$

In local coordinates, we write the Laplace–Beltrami operator as

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{g}\frac{\partial}{\partial x_j}\right).$$

The nodal set of ϕ_λ is the set

$$Z_\lambda := \{p \in M : \phi_\lambda(p) = 0\}.$$

It is known [Cheng 1976] that Z_λ is a smooth curve away from its finite singular set

$$S_\lambda := \{p \in M : \phi_\lambda(p) = \nabla\phi_\lambda(p) = 0\}.$$

Nodal sets of Laplace eigenfunctions have been of interest since the discovery of the Chladni patterns and their asymptotic properties as $\lambda \nearrow \infty$ have been intensively studied, notably in the context of quantum mechanics. In that setting, the square of a normalized eigenfunction ϕ_λ represents the probability density of a free particle in the pure state corresponding to ϕ_λ and Z_λ can be thought of as the set where such a particle is least likely to be found. Estimating the one-dimensional Hausdorff measure $\mathcal{H}^1(Z_\lambda)$ of

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the nodal set has thus been the subject of intense studies over the last three decades, sparked by the well-known conjecture of S.-T. Yau [1982; 1993]:

Conjecture 1.1.2. *Let (M, g) be a compact, C^∞ Riemannian manifold of dimension n. There exist positive constants c and C such that*

$$c\lambda^{1/2} \leq \mathcal{H}^{n-1}(Z_\lambda) \leq C\lambda^{1/2}.$$

Remark that this paper is concerned with the case $n = 2$ but that the conjecture has been stated for smooth manifolds of any dimension. A common intuition in spectral geometry is that a λ-eigenfunction behaves in many ways similarly to a trigonometric polynomial of degree $\lambda^{1/2}$. As such, one can understand Yau's conjecture as a broad generalization of the fundamental theorem of algebra: counting multiplicities, a polynomial of degree $\lambda^{1/2}$ will vanish $\lambda^{1/2}$ times. The conjecture has been proved by Donnelly and Fefferman [1988] for real analytic pairs (M, g) of any dimension. When M is a surface with a C^∞ metric, the lower bound was proved by Brüning [1978]. The current best upper bound of $\lambda^{3/4}$ obtained by [Donnelly and Fefferman 1990; Dong 1992] is still weaker than the conjectured one. Note that the current best exponent $\frac{3}{4}$ in dimension 2 gets much worse in higher dimensions. Indeed, for $n \geq 3$, the current best upper bound is $\lambda^{\sqrt{\lambda}}$ and has been obtained by Hardt and Simon [1989]. This hints that the methods used on surfaces are specific and cannot, in general, be easily extended to higher-dimensional manifolds, which is indeed the case for the results of this paper. For more details and a thorough survey of the most recent results on nodal sets of Laplace eigenfunctions, we refer to [Zelditch 2013].

1.2. ***An averaged measure of the local growth.*** Here and elsewhere in this article, given a ball $B(r)$ of radius r, αB will denote the concentric ball of radius αr. In any metric space, it is possible to measure the growth of a continuous function f by defining its *doubling exponent* $\beta(f, B)$ on a metric ball B by

$$\beta(f, B) := \log \frac{\sup_B |f|}{\sup_{\frac{1}{2}B} |f|}.$$

The simplest example is that of the polynomial x^n on the real interval $D = [-1, 1]$, for which the doubling exponent is the degree n, modulo a constant. Indeed, $\beta(x^n, [-1, 1]) = n \log 2$. Given two concentric balls B and αB, where $0 < \alpha < 1$, one can define the more general *α-growth exponent* $\beta(f, B; \alpha)$ by

$$\beta(f, B; \alpha) := \log \frac{\sup_B |f|}{\sup_{\alpha B} |f|}.$$

Albeit more general, the growth exponent can still be seen as the analog of the degree of a polynomial as showcased once again by the monomial x^n:

$$\beta(x^n, [-1, 1]; \alpha) = \log \frac{\sup_{[-1,1]} |x|^n}{\sup_{[-\alpha,\alpha]} |x|^n} = n \log(\alpha^{-1}).$$

It is worth mentioning that the growth exponent is itself a special case of the more general Bernstein index, which measures in a similar fashion the growth of a continuous function from one compact set to a strictly larger one. For more background on the Bernstein index, we refer to [Khovanskii and Yakovenko 1996; Roytwarf and Yomdin 1997].

The metric g turns M into a metric space, and it is natural to define similar exponents to measure the growth of eigenfunctions on metric disks on the surface. We write $B_p(r)$ for a metric disk centered at $p \in M$ and of radius r. Donnelly and Fefferman [1988] show that on a smooth manifold (M, g) of any dimension, the following holds for every ball B:

$$\beta(\phi_\lambda, B) \leq c\lambda^{1/2},$$

where $c = c(g, r, \alpha)$ is a positive constant depending only on the geometry of M, the radius r and the scaling factor α. From now on, we will restrict our attention to disks $B_p(r)$ of radius comparable to the wavelength: $r = k_0\lambda^{-1/2}$, where k_0 is a suitably small, positive constant. It turns out that, at this scale, the local study of an eigenfunction can be reduced to that of a solution of a planar Schrödinger equation (see Section 2.3), which is a central idea throughout this article. For simplicity, we write

$$\beta_p(\lambda) := \beta(\phi_\lambda, B_p(r); \alpha_0)$$

for the α_0-growth exponent of ϕ_λ and where α_0 is a geometric constant whose explicit value is given by (2.2.3). The quantity $\beta_p(\lambda)$ is by definition local, and motivated by Section 7.3 in [Nazarov et al. 2005], we make it global by defining the *average local growth* of a λ-eigenfunction, which is essentially the averaged L^1 norm of $\beta_p(\lambda)$:

$$A(\lambda) := \frac{1}{\mathrm{Vol}(M)} \int_M \beta_p(\lambda)\, \mathrm{d}V_g(p).$$

Thus, $A(\lambda)$ can be interpreted as the expected value of the α_0-growth exponent of an eigenfunction ϕ_λ on disks of wavelength radius.

1.3. ***Results.*** We recall the basic intuition of interpreting an eigenfunction ϕ_λ as a polynomial of degree λ. In the case of a polynomial, the degree controls both the growth and the number of zeros and it is thus natural to expect a similar link for eigenfunctions. Our main result proves Conjecture 7.1 of [Nazarov et al. 2005] and provides such a link by showing that the average local growth is comparable to the size of the nodal set Z_λ times the wavelength $\lambda^{-1/2}$.

Theorem 1. *Let (M, g) be a smooth, closed Riemannian manifold of dimension 2. There exist positive constants c_1 and c_2 such that*

$$c_1\lambda^{1/2}A(\lambda) \leq \mathcal{H}^1(Z_\lambda) \leq c_2\lambda^{1/2}(A(\lambda) + 1). \tag{1.3.1}$$

The theorem provides an interesting reformulation of Yau's conjecture for surfaces with smooth metric. Recall that, in this setting, the lower bound of Conjecture 1.1.2 is proven so that, in view of Theorem 1, the conjecture holds if and only if

$$A(\lambda) = O(1).$$

Also, since the conjecture is true in the analytic case, we immediately have that $A(\lambda) = O(1)$ in such a setting. In other words, on a surface with a real analytic metric, the average local growth of an eigenfunction on balls of small radius is bounded by a constant independent of the eigenvalue.

Finally, two other main results are of interest, namely Theorems 2.1.1 and 3.1.1, each providing a link between growth exponents and the size of nodal sets of solutions to a planar Schrödinger equation. The explicit statement of these results is respectively given at the beginning of Sections 2 and 3.

1.4. ***Outline of proof and organization of the paper.*** Nazarov et al. [2005, §7.3] suggested a heuristic for the proof of Theorem 1 that essentially consisted of the following four steps:

(i) Reduce an eigenfunction ϕ_λ to a solution F of a planar Schrödinger equation. This is done locally on a conformal coordinate patch by restricting ϕ_λ to a small disk of radius $\sim \lambda^{-1/2}$, which transforms the eigenvalue equation (1.1.1) into

$$\Delta F + qF = 0,$$

where Δ is the flat Laplacian and q is a smooth potential with small uniform norm.

(ii) Use Lemma 3.4 from [Nazarov et al. 2005] to express F as the composition $u \circ h$ of a harmonic function u with a K-quasiconformal homeomorphism h whose dilation factor K is controlled.

(iii) Extend to F and then to ϕ_λ some appropriate estimates linking the size of the nodal set of u with its growth exponent β. Such estimates are in the spirit of Lemma 2.13 in [Nazarov et al. 2005] (see also [Gelfond 1934; Robertson 1939; Khovanskii and Yakovenko 1996]) and relate the growth exponents of a harmonic function u on some disk with the number of change of signs of u on the boundary of either a larger or a smaller disk.

(iv) The last step is an integral-geometric argument based on a generalized Crofton formula that allows one to recover the global statement of Theorem 1 from the local estimates obtained in the previous steps.

This approach has been successful in obtaining the lower bound for the size of the nodal set in terms of the average local growth, that is, the left inequality of Theorem 1. The details are presented in Section 3. However, as first noticed by J. Bourgain, the same approach cannot be used for the other inequality. The problem roughly resides in step (iii), where we aim to extend to $F = u \circ h$ a result of the type

$$N_u(\partial D_-) \leq \beta_u(D^+),$$

where $N_u(\partial D_-)$ is the number of zeros of u on a circle ∂D_- that is strictly contained in a bigger disk D^+ on which the doubling exponent is computed. It is impossible to do so since we have no way to ensure that the K-quasiconformal map h will map the circle ∂D_- to another circle in the domain of F. It might in fact map a circle to a nonrectifiable curve, which prevents one from properly counting the zeros of F.

Based on a private communication with Nazarov, Polterovich and Sodin, we take a different route to prove the upper bound in Theorem 1, which is inspired by [Donnelly and Fefferman 1988]. More precisely, we keep steps (i) and (iv) but replace the intermediate steps by Theorem 2.1.1, which provides a convenient estimate linking the size of the nodal set of F on a small disk to its growth exponent on a bigger disk. This approach is presented in Section 2 and allows us to recover the remaining inequality of our main theorem. Theorem 2.1.1 thus plays a crucial role, and its proof is presented in Section 4. The general idea is to tile the domain of F into squares of rapid and slow growth and to then notice that: (a) the nodal set in

a square of slow growth is small and (b) there cannot be too many squares of rapid growth. The interested reader will also find further explanations detailing the structure of that proof in Section 4.2. Involved in the proof are notably the technical Proposition 4.2.1, which roughly proves statement (b) above, as well as the specialized Carleman estimate of Lemma 5.2.1, whose rather long derivations we respectively present in Sections 5 and 6. We conclude the article with a discussion and a few questions in Section 7.

Notation. Throughout the paper, we will denote positive numerical constants in the following fashion. Constants $c_1, c_2, \dots$ will be used in the statements of any result and may depend on the geometry of the manifold M but nothing else. In particular, they are independent of λ. Within proofs, we will use $a_1, a_2, \dots$ for numerical constants without any dependency and $b_1, b_2, \dots$ for constants that may depend on the geometry of the surface. Often, we merge many numerical constants together to simplify the sometimes heavy notation; for example, $a_5 = a_3^{-1} a_4 (4\pi)/\operatorname{Vol}(M)$. Finally, we reset the numeration for the constants a_i at each section.

We will use D to denote Euclidean disks and B for metric balls on the surface. Given the context, we either write $D(p, r)$ for a disk centered at p of radius r or just D_p if the radius is known. Finally, we will keep the convention that, given a positive constant a and a disk $D = D(p, r)$, aD denotes the concentric disk of radius ar. We write $\mathbb{D}$ for the open unit disk in $\mathbb{R}^2$.

2. Upper bound for the length of the nodal set

In this section, we prove the right inequality of Theorem 1, which provides an upper bound to the length of the nodal set in terms of the average local growth of an eigenfunction ϕ_λ. The main tool in the proof is the following, which links the size of the nodal set of a Schrödinger eigenfunction to its growth exponent:

Theorem 2.1.1. *Let $F : 3\mathbb{D} \to \mathbb{R}$ be a solution of*

$$\Delta F + qF = 0 \tag{2.1.2}$$

with the potential $q \in C^\infty(3\mathbb{D})$ satisfying $\|q\|_\infty = \sup_{3\mathbb{D}} |q| < \epsilon_0$. Let also

$$\beta := \beta\big(F, \tfrac{5}{2}\mathbb{D}; 10\big) = \log \frac{\sup_{\frac{5}{2}\mathbb{D}} |F|}{\sup_{\frac{1}{4}\mathbb{D}} |F|}.$$

Finally, denote by Z_F the nodal set $\{p \in 3\mathbb{D} : F(p) = 0\}$ of F. Then

$$\mathcal{H}^1\big(Z_F \cap \tfrac{1}{60}\mathbb{D}\big) \le c_3 \beta^*,$$

where $\beta^* := \max\{\beta, 1\}$ and c_3 is a positive constant.

We remark that we do not assume here that q has a constant sign. The proof of this theorem is presented in Section 5, and some information about the value of ϵ_0 is given at the end of Lemma 5.4.6.

2.2. ***From the surface to the plane: the passage to Schrödinger eigenfunctions with small potential.*** Cover the surface M with a finite number N of conformal charts (U_i, ψ_i), $\psi_i : U_i \subset M \to V_i \subset \mathbb{R}^2$, $i \in I = \{1, \dots, N\}$. On each of these charts, the metric is conformally flat and there exist smooth positive

functions q_i such that $g = q_i(x, y)(dx^2 + dy^2)$. By compactness, we can find positive constants q_- and q^+ such that we have $0 < q_- < q_i < q^+$ for all $i = 1, \ldots, N$. The metric is thus pinched between scalings of the flat metric, and we have a local equivalence of various metric notions on M and in $\mathbb{R}^2$. In particular, given any subset $E \subset U_i$, the one-dimensional Hausdorff measures are equivalent:

$$b_1 \mathscr{H}^1(\psi_i(E)) \leq \mathscr{H}^1(E) \leq b_2 \mathscr{H}^1(\psi_i(E)). \tag{2.2.1}$$

In the same spirit, the Riemannian volume form on M and the Lebesgue measure $\mathrm{d}A$ in $\mathbb{R}^2$ are equivalent in the following sense: given any integrable function f on U_i, we have

$$b_3 \int_{V_i} f \, \mathrm{d}A \leq \int_{U_i} f \, \mathrm{d}V_g \leq b_4 \int_{V_i} f \, \mathrm{d}A. \tag{2.2.2}$$

Note that the explicit values of the constants $b_1, \ldots, b_4$ involve only the geometric constants q_- and q^+. We now let $B_p := B_p(k_0 \lambda^{-1/2}) \subset M$ be a metric disk and set

$$\alpha_0 := \frac{q_-}{5q^+}. \tag{2.2.3}$$

The value of the small positive constant k_0 will be fixed later. Recall that, at a point $p \in M$, the growth exponent $\beta_p(\lambda)$ of an eigenfunction ϕ_λ is defined by

$$\beta_p(\lambda) := \log \frac{\sup_{B_p} |\phi_\lambda|}{\sup_{\alpha_0 B_p} |\phi_\lambda|}.$$

2.3. ***Metric and Euclidean disks.*** In order to estimate $\beta_p(\lambda)$ from below, we define the Euclidean disks

$$D_p^+ := D_p(q_- k_0 \lambda^{-1/2}), \qquad D_p^- := \alpha_0 D_p(q^+ k_0 \lambda^{-1/2})$$

so that D_p^- is a proper subset of D_p^+. Note that, by a Euclidean disk $D_p(r)$ centered at $p \in M$, we mean the set $\{(x, y) : x^2 + y^2 \leq r^2\}$, where (x, y) are local conformal coordinates around p. The inclusions $B_p \supset D_p^+$ and $\alpha_0 B_p \subset D_p^-$ imply

$$\log \frac{\sup_{D_p^+} |\phi_\lambda|}{\sup_{D_p^-} |\phi_\lambda|} \leq \beta_p(\lambda).$$

In a conformal chart (U_i, ψ_i), the eigenvalue equation $\Delta_g \phi_\lambda + \lambda \phi_\lambda = 0$ becomes

$$\Delta \phi_\lambda + \lambda q_i \phi_\lambda = 0. \tag{2.3.1}$$

With the aim of using Theorem 2.1.1, we endow the disk $3\mathbb{D}$ with the complex coordinate $z = x + iy$, fix a scaling constant $\tau = 2q^+ \alpha_0$ and define a function $F = F_{\lambda, p} : 3\mathbb{D} \to \mathbb{R}$ by $F(z) = \phi_\lambda(\tau k_0 \lambda^{-1/2} z + p)$. The scaling allows us to absorb the spectral parameter λ in the potential. Indeed, we have

$$\Delta F = \tau^2 k_0^2 \lambda^{-1} \Delta \phi = (k_0 \tau)^2 \lambda^{-1} (-\lambda q_i) \phi_\lambda = -(k_0 \tau)^2 q_i F$$

so that F satisfies (2.1.2), where $q = (k_0 \tau)^2 q_i$ is a smooth potential whose supremum norm satisfies $\|q\|_\infty < \epsilon_0$ without loss of generality. Indeed, since the family of q_i is bounded, we can choose k_0 as

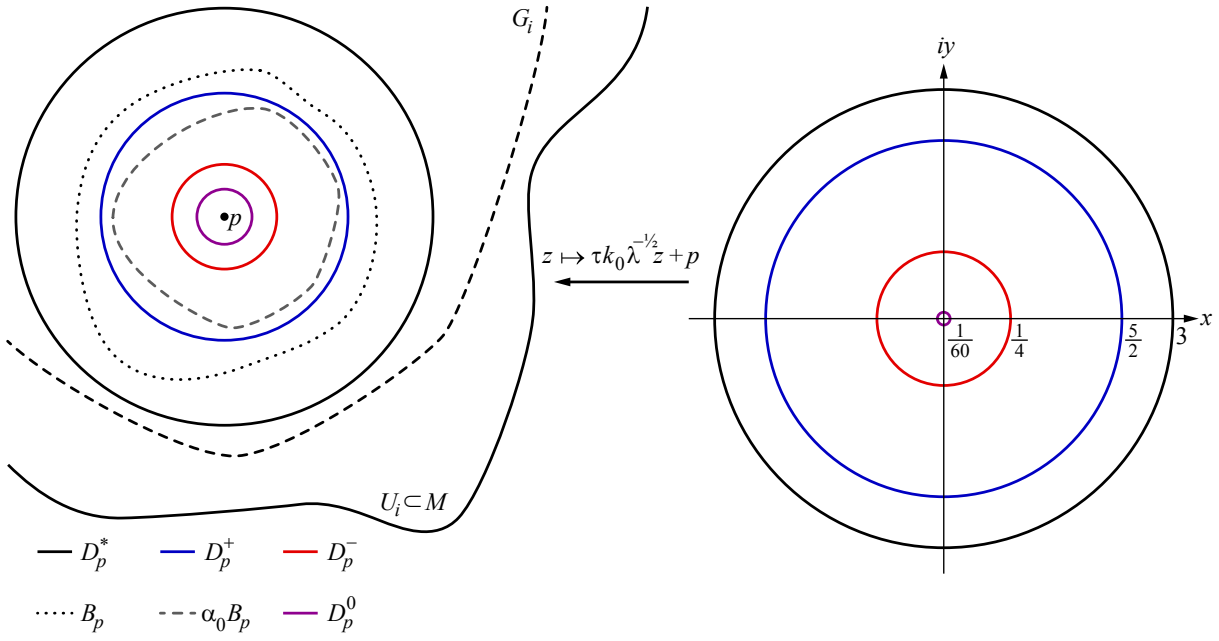

Figure 1. Mapping of Euclidean disks and metric balls within a conformal patch.

small as needed. The transformation $z \mapsto \tau k_0 \lambda^{-1/2} z + p$ induces the following correspondences between disks in $3\mathbb{D}$ and Euclidean disks centered at p:

$$\left\{|z| \le \tfrac{1}{4}\right\} \leftrightarrow D_p^-, \qquad \left\{|z| \le \tfrac{5}{2}\right\} \leftrightarrow D_p^+, \qquad \left\{|z| \le \tfrac{1}{60}\right\} \leftrightarrow D_p^0,$$

where $D_p^0 = D_p\left(\frac{1}{60}\tau k_0 \lambda^{-1/2}\right)$. As a consequence, we have

$$\beta^* < \beta + 1 = \log \frac{\sup_{D_p^+} |\phi_\lambda|}{\sup_{D_p^-} |\phi_\lambda|} + 1 \le \beta_p(\lambda) + 1. \tag{2.3.2}$$

It is important at this stage to remark that the construction of F is dependent on a fixed choice of conformal chart U_i both for the well-posedness of (2.3.1) as well as the very definition of the Euclidean disks. Thus, in order to allow the construction of $F = F_{\lambda,p}$ everywhere on the surface M, one has to choose k_0 small enough so that the disks $D_p^* := D_p(3k_0\tau\lambda^{-1/2})$, which are mapped onto $3\mathbb{D}$, are contained in at least one chart U_i for every $p \in M$. This allows the definition of the mapping $\sigma : M \to I = \{1, \dots, N\}$, which assigns to a point p a unique index $\sigma(p)$ such that $D_p^* \subset U_{\sigma(p)}$. The disjoint sets $G_i := \sigma^{-1}(i)$ form a partition of M. Figure 1 summarizes the setting we are in by presenting a sketch of the various correspondences between Euclidean disks in G_i and those in $3\mathbb{D}$.

We now turn to the study of the nodal set Z_λ. Recall that S_λ is the singular set of the eigenfunction ϕ_λ, and consider the sets $Z_\lambda(i) := \psi_i((Z_\lambda \setminus S_\lambda) \cap G_i) \subset \mathbb{R}^2$. Since S_λ is discrete, we have

$$\mathcal{H}^1(Z_\lambda) = \mathcal{H}^1(Z_\lambda \setminus S_\lambda) \le b_2 \sum_{i \in I} \mathcal{H}^1(Z_\lambda(i)). \tag{2.3.3}$$

Denote by Z_F the nodal set of F. By construction, we have

$$\mathscr{H}^1(Z_\lambda(i) \cap D_p^0) = (k_0\tau)\lambda^{-1/2}\mathscr{H}^1\big(Z_F \cap \tfrac{1}{60}\mathbb{D}\big).$$

Applying Theorem 2.1.1 and (2.3.2) now yields

$$\mathscr{H}^1(Z_\lambda(i) \cap D_p^0) \leq a_2\lambda^{-1/2}(\beta_p(\lambda) + 1). \tag{2.3.4}$$

We integrate the left-hand side of the last equation over the set G_i and use a generalized Crofton formula (see (6) in [Hug and Schneider 2002]) to get

$$\int_{G_i} \mathscr{H}^1(Z_\lambda(i) \cap D_p^0)\,\mathrm{d}A(p) = a_3\mathscr{H}^2(D_p^0)\mathscr{H}^1(Z_\lambda(i)) = a_4\lambda^{-1}\mathscr{H}^1(Z_\lambda(i)). \tag{2.3.5}$$

Recalling the equivalence (2.2.2) and combining (2.3.4) and (2.3.5) then gives

$$a_4\lambda^{-1}\mathscr{H}^1(Z_\lambda(i)) \leq a_2\lambda^{-1/2}\int_{G_i}(\beta_p(\lambda)+1)\,\mathrm{d}A(p) \leq (a_2b_3^{-1})\lambda^{-1/2}\int_{G_i}(\beta_p(\lambda)+1)\,\mathrm{d}V.$$

Simplifying readily gives

$$\mathscr{H}^1(Z_\lambda(i)) \leq a_5\lambda^{1/2}\int_{G_i}(\beta_p(\lambda)+1)\,\mathrm{d}V$$

so that

$$\begin{aligned}\mathscr{H}^1(Z_\lambda) &\leq b_2\sum_{i\in I}\mathscr{H}^1(Z_\lambda(i)) \leq a_6\lambda^{1/2}\sum_{i\in I}\int_{G_i}(\beta_p(\lambda)+1)\,\mathrm{d}V = a_6\lambda^{1/2}\int_M(\beta_p(\lambda)+1)\,\mathrm{d}V\\ &\leq c_2\lambda^{1/2}(A(\lambda)+1).\end{aligned}$$

3. Lower bound for the length of the nodal set

In this section, we prove the left inequality of Theorem 1. As was the case in the previous section, the central idea is once again the use of conformal coordinates on M and restriction to wavelength scales to reduce the local behavior of an eigenfunction ϕ_λ to that of F, a solution of a planar Schrödinger equation with small, smooth potential. The main result of this section is the following theorem, which suitably links the growth exponent of F with its nodal set:

Theorem 3.1.1. *Let $F : \overline{\mathbb{D}} \to \mathbb{R}$ be a solution of*

$$\Delta F + qF = 0 \tag{3.1.2}$$

in $\mathbb{D}$ and with the potential $q \in C^\infty(\mathbb{D})$ satisfying $\|q\|_\infty = \sup_{\overline{\mathbb{D}}}|q| < \epsilon_1$. Denote by $|Z_F(\mathbb{S}^1)|$ the number of zeros of F on the unit circle $\mathbb{S}^1$. Then

$$\log\frac{\sup_{\rho^+\mathbb{D}}|F|}{\sup_{\rho^-\mathbb{D}}|F|} \leq c_4(1 + |Z_F(\mathbb{S}^1)|),$$

where $0 < \rho^- < \rho^+ < \frac{1}{2}$ are fixed, small radii.

The value of ϵ_1 can be obtained in the proof of Lemma 3.3 in [Nazarov et al. 2005] while those of ρ^- and ρ^+ are given in the proof. The constant ρ^- depends on the geometry of the manifold. It is possible to get rid of this dependency if one wants Theorem 3.1.1 to be a stand-alone result. However, our aim is to prove the left inequality of Theorem 1, and as such, our choice of p^- makes the rest of the argument much simpler. Also, remark that, in contrast to Theorem 3.1.1 where F was defined on $\mathbb{D}$, the setting is now in $3\mathbb{D}$. This is an arbitrary choice made only in order to ease the writing of the respective proofs: confining Theorem 2.1.1 to the unit disk would have added even more complexity in the expression of the many constants needed to carry out the long proof.

3.2. ***Proof of Theorem 3.1.1.*** The general strategy is as follows: we first prove a similar kind of result for harmonic functions, and inspired by [Nazarov et al. 2005], we then express F as the composition of a harmonic function and a K-quasiconformal homeomorphism. Controlling the properties of the quasiconformal homeomorphism allows one to recover the desired result. We begin with a lemma that relates the growth of harmonic functions within a disk and its nodal set on the boundary.

Lemma 3.2.1. *Let $v \in C^\infty(\mathbb{D}) \cap C^0(\overline{\mathbb{D}})$ be harmonic in the open unit disk, and denote by N_v the number of changes of sign of v on the circle $|z| = 1$. Choose r_0 in $0 < r_0 < \frac{1}{2}$. Then*

$$\frac{\sup_{\frac{1}{2}\mathbb{D}}|v|}{\sup_{r_0\mathbb{D}}|v|} \le \left(\frac{c_5}{r_0}\right)^{N_v}, \tag{3.2.2}$$

where c_5 is a positive numerical constant.

Proof. Let u be the harmonic conjugate of v such that $u(0) = 0$. Then the function

$$f(z) = \sum_{n=0}^{\infty} \xi_n z^n = u(z) + i\,v(z)$$

is holomorphic in the closed unit disk $\{|z| \le 1\}$. Suppose that

$$\sup_{r_0\mathbb{D}}|v| = \max_{|z|=r_0}|v| = 1.$$

The harmonic function v changes sign $2p = N_v$ times on the circle $|z| = 1$, where p is a nonnegative integer. Also, let $\mu_p := \max\{|\xi_0|, |\xi_1|, \ldots, |\xi_p|\}$. By [Robertson 1939, Theorem 1, (iii)], we have

$$|f(re^{i\theta})| < c(p)\mu_p(1-r)^{-2p-1}, \quad r < 1, \tag{3.2.3}$$

where $c(p) > 0$ is a constant depending on p that will be given explicitly later. (Robertson actually proves (3.2.3) in our current setting and then uses a limiting argument to obtain a slightly different statement.)

The classical Schwarz formula says that, for a function g holomorphic on the open disk $r_0\mathbb{D}$ and continuous on the boundary $\{|z| = r_0\}$, we have

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}(g(r_0e^{i\theta}))\frac{r_0e^{i\theta}+z}{r_0e^{i\theta}-z}\,\mathrm{d}\theta + i\ \mathrm{Im}(g(0)), \quad |z| < r_0.$$

Since $f = u + iv$ is holomorphic, so is $g = v - iu$ and we obviously have $|f| = |g|$ so that the following inequality holds for all $|z| \leq \frac{1}{2}r_0$:

$$|f(z)| = |g(z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}(g(r_0 e^{i\theta})) \frac{r_0 e^{i\theta} + z}{r_0 e^{i\theta} - z} \, \mathrm{d}\theta + iu(0) \right|$$
$$\leq \frac{1}{2\pi} \max_{|z|=r_0} |v| \int_0^{2\pi} \frac{r_0 + |z|}{|r_0 - z|} \, \mathrm{d}\theta \leq 3 =: a_1.$$

Applying Cauchy's inequality for holomorphic functions to $f = \sum_{n=0}^{\infty} \xi_n z^n$ on the open disk of radius $\frac{1}{2}r_0$,

$$|\xi_n| \leq \left(\frac{r_0}{2}\right)^{-n} \sup_{|z|=r_0/2} |f(z)| = a_1 \left(\frac{2}{r_0}\right)^n.$$

Hence, we have $\mu_p \leq a_1 (2/r_0)^p$. Setting $r = \frac{1}{2}$ in (3.2.3) now yields

$$\left| f\left(\tfrac{1}{2} e^{i\theta}\right) \right| \leq c(p) \mu_p 2^{2p+1} \leq 2a_1 c(p) \left(\frac{2}{r_0}\right)^p 2^{2p} \leq 2a_1 c(p) \left(\frac{4}{r_0}\right)^{2p},$$

which in turn means

$$\sup_{\frac{1}{2}\mathbb{D}} |v| = \max_{|z|=1/2} |v| \leq \left| f\left(\tfrac{1}{2} e^{i\theta}\right) \right| \leq 2a_1 c(p) \left(\frac{4}{r_0}\right)^{2p}.$$

Going back to [Robertson 1939], we use the explicit value of the constant $c(p)$ to get the bound

$$c(p) = 2^{2p} + \frac{(2p)!}{(p!)^2} = 2^{2p} + \binom{2p}{p} \leq 2^{2p} + \left(\frac{2pe}{p}\right)^p = 2^{2p} + (2e)^p \leq 2(2e)^{2p}.$$

Since we assumed that $\sup_{r_0\mathbb{D}} |v| = 1$, we have

$$\frac{\sup_{\frac{1}{2}\mathbb{D}} |v|}{\sup_{r_0\mathbb{D}} |v|} \leq 4a_1 \left(\frac{8e}{r_0}\right)^{2p}.$$

Suppose now that $\sup_{r_0\mathbb{D}} |v| = \tau \neq 1$, and let as before $f = u + iv$ be the holomorphic function built from v and its harmonic conjugate u. Define $\tilde{f} = \tilde{u} + i\tilde{v}$ by $\tilde{f} = \tau^{-1} f$. Then $\sup_{r_0\mathbb{D}} |\tilde{v}| = 1$ and

$$\frac{\sup_{\frac{1}{2}\mathbb{D}} |v|}{\sup_{r_0\mathbb{D}} |v|} = \frac{\tau \sup_{\frac{1}{2}\mathbb{D}} |\tilde{v}|}{\tau \sup_{r_0\mathbb{D}} |\tilde{v}|} \leq 4a_1 \left(\frac{8e}{r_0}\right)^{2p} \leq \left(\frac{c_5}{r_0}\right)^{2p}. \qquad \square$$

We now prove Theorem 3.1.1. By Lemmas 3.3 and 3.4 in [Nazarov et al. 2005], there exist a K-quasiconformal homeomorphism $h : \mathbb{D} \to \mathbb{D}$ with $h(0) = 0$, a harmonic function $v : \mathbb{D} \to \mathbb{R}$ and a solution φ to (3.1.2) such that $F = \varphi \cdot (v \circ h)$. Moreover, the function φ is positive and satisfies

$$1 - a_2 \epsilon_1 \leq \varphi \leq 1.$$

Finally, the dilation factor of the quasiconformal map h satisfies

$$1 \leq K \leq 1 + a_3 \|q\|_\infty \leq a_4.$$

We refer the reader to [Nazarov et al. 2005] for the precise values of the various constants stated above. We recall Mori's theorem (see Section III.C in [Ahlfors 1966] or [Nazarov et al. 2005]) for K-quasiconformal homeomorphisms:

$$\tfrac{1}{16}|z_1 - z_2|^K \le |h(z_1) - h(z_2)| \le 16|z_1 - z_2|^{1/K}.$$

Since the origin is a fixed point of h, we have

$$\tfrac{1}{16}|z|^K \le |h(z)| \le 16|z|^{1/K}, \quad z \in \mathbb{D}.$$

Fix a small radius $\rho^+ = \left(\frac{1}{32}\right)^{a_4}$, and consider the circle $\{|z| = \rho^+\}$. For such z, Mori's theorem gives $|h(z)| \le 16(\rho^+)^{1/K} \le \frac{1}{2}$ so that

$$h(\rho^+\mathbb{D}) \subset \tfrac{1}{2}\mathbb{D}.$$

Now, set $\rho^- := \frac{1}{5}\rho^+(q_-/q^+)^2$. The image by h of the circle $\{|z| = \rho^-\}$ contains the circle of radius $\frac{1}{16}(\rho^-)^K \ge \frac{1}{16}(\rho^-)^{a_4} =: r_0$. As a consequence, we have

$$r_0\mathbb{D} \subset h((\rho^-)\mathbb{D}).$$

Since $F = \varphi \cdot (v \circ h)$, the bounds on φ and the above inclusions imply

$$\frac{\sup_{\rho^+\mathbb{D}}|F|}{\sup_{\rho^-\mathbb{D}}|F|} \le a_5 \frac{\sup_{\rho^+\mathbb{D}}|v \circ h|}{\sup_{\rho^-\mathbb{D}}|v \circ h|} \le a_5 \frac{\sup_{\frac{1}{2}\mathbb{D}}|v|}{\sup_{r_0\mathbb{D}}|v|},$$

where $a_5 = (1 - a_2\epsilon_1)^{-1}$. Since φ is positive and h is a homeomorphism, the number N_F of sign changes of F on the unit circle is the same as that of v. Applying Lemma 3.2.1 now yields

$$\frac{\sup_{\rho^+\mathbb{D}}|F|}{\sup_{\rho^-\mathbb{D}}|F|} \le a_5\left(\frac{c_5}{r_0}\right)^{N_F}.$$

Since the number $|Z_F(\mathbb{S}^1)|$ of zeros of F on the unit circle is bounded below by N_F, taking the logarithm on both sides yields

$$\log \frac{\sup_{\rho^+\mathbb{D}}|F|}{\sup_{\rho^-\mathbb{D}}|F|} \le c_4(1 + |Z_F(\mathbb{S}^1)|),$$

where $c_4 = \max\{a_5, c_5/r_0\}$.

3.3. ***A lower bound for the nodal set in terms of the average local growth.*** In order to recover the right inequality of Theorem 1, we propose an argument that is very similar to the one developed in Section 2. It thus helps to refer to that section when reading the remainder of this one. The aim is to apply Theorem 3.1.1 to a function F that has been built from an eigenfunction ϕ_λ and to then apply an integral-geometric argument to recover the desired result. We begin with the same setting as that of Section 2.2 and then define the Euclidean disks

$$D_p^+ := D_p(q^+ k_0 \lambda^{-1/2}), \qquad D_p^- := \alpha_0 D_p(q^- k_0 \lambda^{-1/2}).$$

The last two definitions employ the same notation as in the previous section, but the radii of the disks are different. The inclusions $B_p \subset D_p^+$ and $\alpha B_p \supset D_p^-$ imply

$$\beta_p(\lambda) \leq \log \frac{\sup_{D_p^+}|\phi_\lambda|}{\sup_{D_p^-}|\phi_\lambda|}. \tag{3.3.1}$$

Let $\tau := q^+/\rho^+$ be a scaling constant, endow the unit disk with the complex coordinate $z = x + iy$ and define $F_{\lambda,p} = F : \mathbb{D} \to \mathbb{R}$ by $F(z) = F(\tau k_0 \lambda^{-1/2} z + p)$. The function F solves (3.1.2), and the potential q satisfies $\|q\|_\infty < \min\{\epsilon_0, \epsilon_1\}$ without loss of generality, choosing k_0 small enough. Recalling that $\rho^- = \frac{1}{5}\rho^+(q^-/q^+)^2$, we remark that the mapping $z \mapsto \tau k_0 \lambda^{-1/2} z + p$ induces the bijections

$$\{|z| \leq \rho^+\} \leftrightarrow D_p^+, \qquad \{|z| \leq \rho^-\} \leftrightarrow D_p^-.$$

An immediate consequence is

$$\log \frac{\sup_{\rho^+\mathbb{D}}|F|}{\sup_{\rho^-\mathbb{D}}|F|} = \log \frac{\sup_{D_p^+}|\phi_\lambda|}{\sup_{D_p^-}|\phi_\lambda|} \geq \beta_p(\lambda). \tag{3.3.2}$$

Notice that, for F to be properly defined on $\mathbb{D}$, the Euclidean disk $D_p^0 := D_p(\tau k_0 \lambda^{-1/2})$ must lie completely within some conformal chart U_i. Hence, to ensure that the above construction can be carried through for any $p \in M$, we choose k_0 small enough that $D_p(\tau k_0 \lambda^{-1/2})$ is a proper subset of at least one conformal chart U_i for every $p \in M$. This allows one to define the map $\sigma : M \to I = \{1, \ldots, N\}$ that assigns to $p \in M$ a unique index $\sigma(p)$ such that $D_p(\tau k_0 \lambda^{-1/2}) \subset U_{\sigma(p)}$. Once again, the sets $G_i := \sigma^{-1}(i) \subset U_i$ form a partition of M. Now consider the sets $Z_\lambda(i) := \psi_i((Z_\lambda \setminus S_\lambda) \cap G_i)$, $i = 1, \ldots, N$. Then

$$\mathcal{H}^1(Z_\lambda) = \mathcal{H}^1(Z_\lambda \setminus S_\lambda) \geq b_1 \sum_{i \in I} \mathcal{H}^1(Z_\lambda(i)). \tag{3.3.3}$$

Denote by $|Z_{p,\lambda}(i)|$ the number of intersection points of the circle ∂D_p^0 with $Z_\lambda(i)$. By construction, the following equality holds outside from the singular set, that is, almost everywhere:

$$|Z_{p,\lambda}(i)| = |Z_F(\mathbb{S}^1)|. \tag{3.3.4}$$

Applying Theorem 3.1.1 and (3.3.2) now yields

$$\beta_p(\lambda) \leq c_4(1 + |Z_{p,\lambda}(i)|) \tag{3.3.5}$$

outside of S_λ. We integrate the left-hand side of the last equation over the set G_i and use a generalized Crofton formula [Hug and Schneider 2002, (6)] to get

$$\int_{G_i \setminus S_\lambda} |Z_\lambda(\partial D_p^0)| \, \mathrm{d}A(p) = a_2 \mathcal{H}^1(\partial D_p^0)\mathcal{H}^1(Z_\lambda(i)) = a_3 \lambda^{-1/2} \mathcal{H}^1(Z_\lambda(i)). \tag{3.3.6}$$

Notice that, in contrast to the previous use of an analogous Crofton formula in Section 2, we have now integrated, over all planar rigid motions, the cardinality of the intersection of a one-dimensional rotation-invariant submanifold — namely the circle ∂D_p^0 — with the one-dimensional nodal set.

It is now straightforward to conclude

$$\begin{aligned} A(\lambda) &= \frac{1}{\mathrm{Vol}(M)} \sum_{i\in I} \int_{G_i\setminus S_\lambda} \beta_p(\lambda)\, \mathrm{d}V_g \\ &\le (b_4c_4)\left(1 + \frac{1}{\mathrm{Vol}(M)} \sum_{i\in I} \int_{G_i\setminus S_\lambda} |Z_{p,\lambda}(i)|\, \mathrm{d}A(p)\right) \\ &= a_5\left(1 + \frac{a_3}{\mathrm{Vol}(M)}\lambda^{-1/2} \sum_{i\in I} \mathscr{H}^1(Z_\lambda(i))\right) \\ &\le a_6(1 + \lambda^{-1/2}\mathscr{H}^1(Z_\lambda)) \\ &\le c_1\mathscr{H}^1(Z_\lambda)\lambda^{-1/2}, \end{aligned}$$

where the last inequality uses the fact that the lower bound in Yau's conjecture holds for surfaces, preventing $\lambda^{-1/2}\mathscr{H}^1(Z_\lambda)$ to be too small.

4. Nodal set and growth of planar Schrödinger eigenfunctions with small potential

This section is dedicated to the proof of Theorem 2.1.1. We start with a function $F : 3\mathbb{D} \to \mathbb{R}$ that satisfies the equation $\Delta F + qF = 0$ on $3\mathbb{D}$. The potential q is smooth and has a small uniform norm: $\|q\|_\infty < \epsilon_0$. Recall that

$$\beta = \log \frac{\sup_{\frac{5}{2}\mathbb{D}} |F|}{\sup_{\frac{1}{4}\mathbb{D}} |F|}$$

and that $\beta^* = \max\{\beta, 1\}$.

4.1. ***A configuration of disks and annuli.*** We start with some notation for disks and annuli within our main setting, which takes place in the disk $3\mathbb{D}$. We denote a finite set of *small disks* by

$$D_\nu = D(z_\nu, \delta) \subset \tfrac{1}{60}\mathbb{D}, \quad 1 \le \nu \le N,$$

where the radius $\delta > 0$ is suitably small. We will say that such a set of small disks is γ-*separated* if it satisfies $|z_\mu - z_\nu| \ge 2\gamma\delta$ for all $\mu \ne \nu$, where γ is some positive constant. One has to understand the γ-separation condition as disjointness after a scaling of factor γ. For example, in Figure 2, the disks D_1 and D_2 are γ-separated while the pair D_ν and D_N is not.

For a small $0 < a \ll 1$, we now let $D_\nu(a) := (1-2a)D_\nu$ and define the following annuli:

- $A_\nu = \{(1-2a)\delta < |z - z_\nu| < (1-a)\delta\}$,
- $A_{\nu'} = \{(1-3a)\delta < |z - z_\nu| < (1-4/(3a))\delta\}$,
- $A_{\nu''} = \left\{\left(1 - \frac{3}{2}a\right)\delta < |z - z_\nu| < (1-a)\delta\right\}$.

We regroup the collection of annuli A_ν under $A = \bigcup_\nu A_\nu$. Figure 3 provides a close-up of the various annuli defined above.

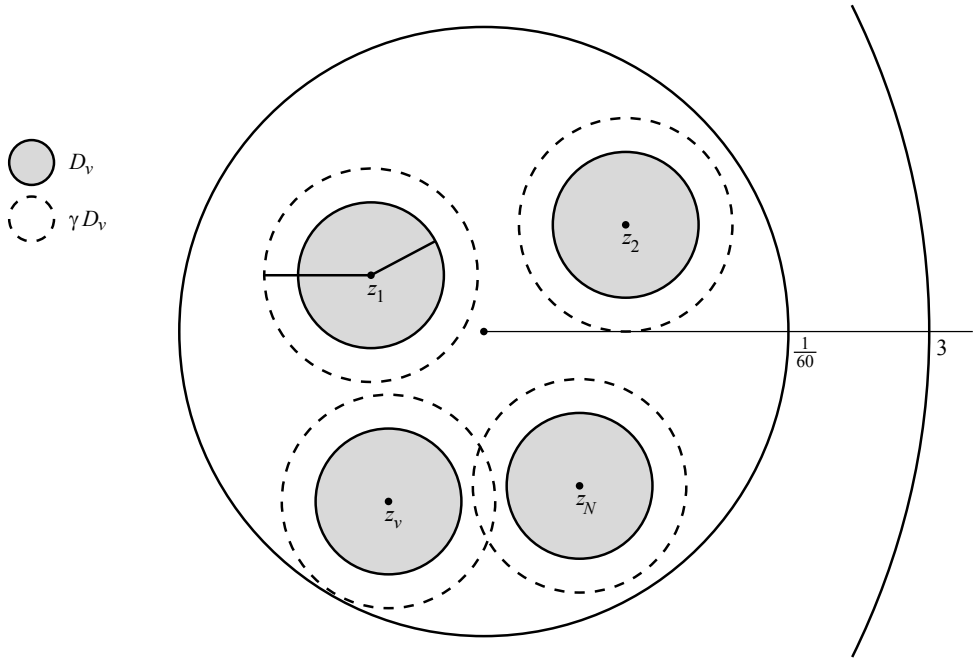

Figure 2. A finite set of disks D_v and scaled disks within $\frac{1}{60}\mathbb{D}$.

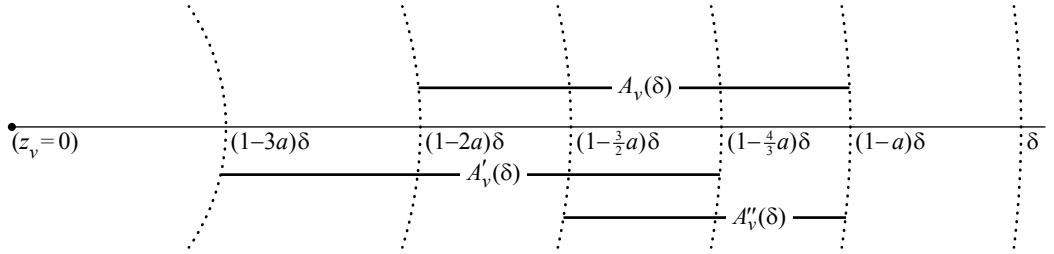

Figure 3. Various annuli within a disk D_v of radius δ centered in z_v.

Given $M > 0$, we say that a disk D_v is a *disk of M-rapid growth* or simply a *rapid disk* if

$$M \int_{A_{v'}} F^2 \leq \int_{A_{v''}} F^2. \tag{4.1.1}$$

We say the radius δ is β^*-related if it satisfies

$$\delta < \tfrac{1}{60}, \qquad \delta\beta^* < \tfrac{1}{2}. \tag{4.1.2}$$

Finally, we fix the separation constant to $\gamma := \delta^{-1/2}$.

4.2. ***Intermediate results.*** We first state a result that shows that, if the potential is small enough and if we fix the growth threshold M sufficiently high, there cannot be too many disks of rapid growth. In fact, it turns out that the number of such disks is bounded above by a constant times the growth exponent β^*.

Proposition 4.2.1. *Suppose that the radius of a collection of γ-separated small disks in $\frac{1}{60}\mathbb{D}$ satisfies the constraints* (4.1.2), *and let $\mathcal{N} = \mathcal{N}(M)$ denote the number of such disks that are of M-rapid growth. Then*

$$\mathcal{N} \leq c_5 \beta^*,$$

provided that $\|q\|_\infty < \epsilon_0$ and $M > M_0$, where c_5, ϵ_0 and M_0 are positive constants.

The rather long proof, inspired by that of Proposition 4.7 in [Donnelly and Fefferman 1990], is presented in Section 5. The next result is Proposition 5.14 in the same reference and links the growth condition and the local length of the nodal set.

Proposition 4.2.2. *Suppose that the disk of radius ϵ centered at z_μ is not M_0-rapid, that is,*

$$\int_{(1-\frac{3}{2}a)\epsilon<|z-z_\mu|<(1-a)\epsilon} F^2 \leq M^{-1} \int_{(1-3a)\epsilon<|z-z_\mu|<(1-\frac{4}{3}a)\epsilon} F^2$$

holds. Then

$$\mathcal{H}^1(Z_F \cap D(z_\mu, c_6\epsilon)) \leq c_7\epsilon,$$

where $c_6, c_7 > 0$ are positive constants.

The last two propositions allow us to lay out a general strategy to prove Theorem 2.1.1. Indeed, we now know that there cannot be too many disks of rapid growth and the nodal set of a slow disk cannot be too big. Conjugating those two ideas in the right way will allow us to bound the global length of the nodal set by the growth exponent of F.

The proof is based on an iterative process that will be indexed by $k = 0, 1, 2, \ldots$. We begin the first step $k = 0$ by fixing some $\delta(0)$ satisfying the constraints (4.1.2) and then divide the square $P = \left\{(x, y) : |x|, |y| < \frac{1}{60}\right\}$ into a grid of squares whose sides have length $\delta(0)$. We distribute those smaller squares into two categories. The *rapid squares* $R_i(0)$, $i = 1, 2, \ldots, r(0)$, are those that contain at least one point $z_i(0) \in R_i(0)$ such that $D_i = D(z_i, \delta)$ is a disk of M-rapid growth of the function F. Here we have fixed $M = M_0$ to allow the use of Proposition 4.2.1. If that condition is not satisfied, we consider the square to be a *slow square* and label it $S_j(0)$, $j = 1, 2, \ldots, s(0)$.

We now proceed to the next step $k = 1$ and set $\delta(1) = \frac{1}{2}\delta(0)$. We bisect the rapid squares $R_i(0)$ of the previous step into four smaller squares and split those newly obtained squares into rapid squares $R_i(1)$, $i = 1, 2, \ldots, r(1)$, and slow squares $S_j(1)$, $j = 1, 2, \ldots, s(1)$, depending on whether they include a point that is the center of an M-rapid disk of radius $\delta(1)$. Note that the slow squares of the previous step are left untouched. Figure 4 gives a representation of the tiling process.

We repeat the process so that, at step k, we have $\delta(k) = 2^{-k}\delta(0)$ as well as some rapid squares $R_i(k)$ and slow squares $S_j(k)$. Let $I(k) = \{1, 2, \ldots, r(k)\}$ be the indexing set of the rapid squares obtained at step k. To simplify notation, we will sometimes write δ instead of $\delta(0)$ in what follows and until the end of the section.

Lemma 4.2.3. *Denote by $|I(k)|$ the cardinality of the finite set $I(k)$, i.e., the number of rapid squares at step k. There exists a constant $c_8 > 0$ such that, for each step $k = 0, 1, 2, \ldots$, we have*

$$|I(k)| \leq c_8 \delta^{-1} \beta^*.$$

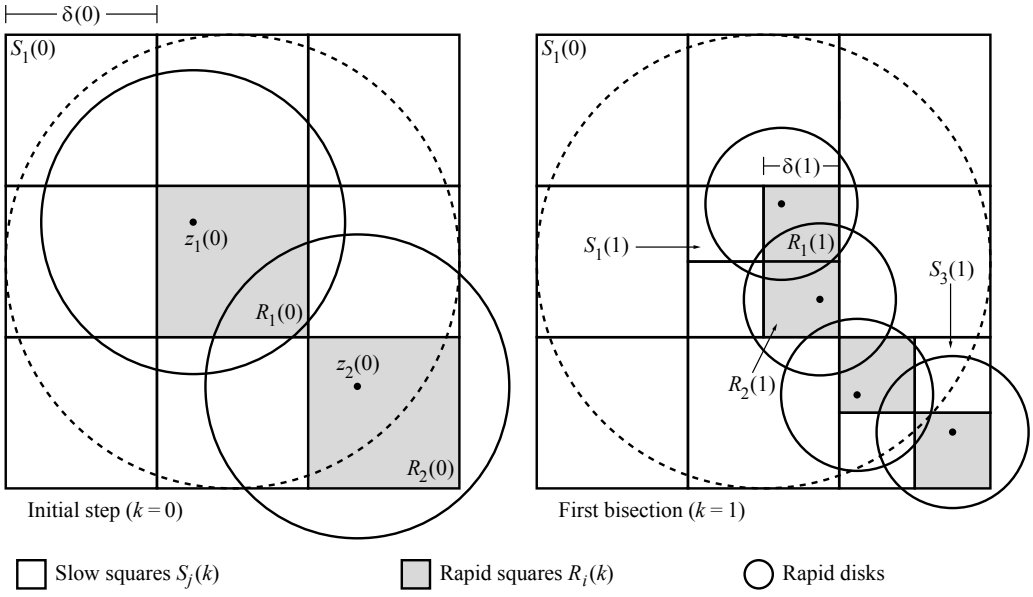

Figure 4. Iterative tiling of P in rapid and slow squares.

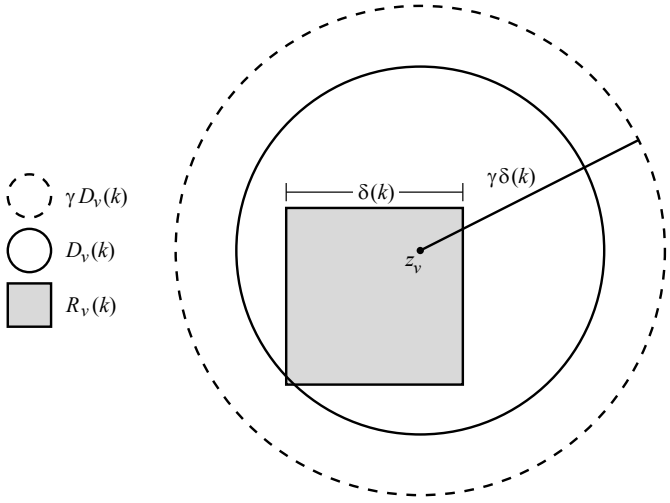

Figure 5. A close-up of a rapid square.

Proof. Recall that $\delta(k) := 2^{-k}\delta(0)$. Since $\delta(0)$ satisfies the constraints (4.1.2), it follows that $\delta(k)$ is β^*-related for all $k \in \mathbb{N} \cup \{0\}$.

We choose some $\nu \in I(k)$ and recall that there is one rapid-growth disk $D_\nu(k)$ whose center z_ν lies in $R_\nu(k)$. Notice that, since $\gamma\delta(k) > \sqrt{2}\delta(k)$, we have $R_\nu(k) \subset \gamma D_\nu(k)$ as shown in Figure 5.

Thus,

$$\bigcup_{\nu \in I(k)} R_\nu(k) \subset \bigcup_{\nu \in I(k)} \gamma D_\nu(k).$$

We now choose a *maximal* subcollection of disjoint disks γD_ν and denote by $I^*(k) \subset I(k)$ the corresponding set of indices. Notice that disjointness of two scaled disks γD_ν and γD_μ is equivalent to γ-separation of D_ν and D_μ. By maximality, for $\mu \in I(k) \setminus I^*(k)$, there exists $\nu \in I^*(k)$ such that $|z_\mu - z_\nu| \le 2\gamma\delta(k)$. In this case and for all $z \in \gamma D_\mu(k)$, we thus have

$$|z - z_\nu| \le |z - z_\mu| + |z_\mu - z_\nu| \le \gamma\delta(k) + 2\gamma\delta(k) < 4\gamma\delta(k).$$

As a consequence, we get the inclusion $\gamma D_\mu(k) \subset 4\gamma D_\nu(k)$, where μ represents a disk excluded from the maximal subset. This in turn means

$$\bigcup_{\nu\in I(k)} \gamma D_\nu(k) \subset \bigcup_{\nu\in I^*(k)} 4\gamma D_\nu(k).$$

Hence,

$$\bigcup_{\nu\in I(k)} R_\nu(k) \subset \bigcup_{\nu\in I^*(k)} 4\gamma D_\nu(k).$$

We compare the respective areas of the regions covered by the last inclusion and get $|I(k)|\delta^2(k) \le 16\pi\gamma^2\delta^2(k)|I^*(k)|$. By Proposition 4.2.1, $|I^*(k)| \le c_5\beta^*$ and we finally get

$$|I(k)| \le 16\pi\gamma^2|I^*(k)| \le 16\pi c_5\gamma^2\beta^* = c_8\delta^{-1}(0)\beta^*,$$

which concludes the proof since I is precisely the set indexing the rapid squares. □

Lemma 4.2.4. *Denote by $|J(k)|$ the number of slow squares $S_j(k)$ obtained at step k. Then, for any $k = 0, 1, 2, \ldots$, we have*

$$|J(k)| \le 4c_8\delta^{-1}\beta^*.$$

Proof. By construction, we have $|J(k)| \le 4|I(k-1)| \le 4c_8\delta^{-1}\beta^*$. □

Lemma 4.2.5. *There exists a constant c_9 such that, for each slow square $S_j(k)$ and each $k = 0, 1, 2, \ldots$,*

$$\mathcal{H}^1(Z_F \cap S_j(k)) \le c_9 2^{-k}\delta.$$

Proof. If z_μ lies in some slow square $S_i(k)$, then the disk $D(z_\mu, \delta(k))$ is slow, which means it satisfies

$$\int_{(1-3a)\delta<|z-z_\mu|<(1-\frac{4}{3}a)\delta} F^2 > M^{-1} \int_{(1-\frac{3}{2}a)\delta<|z-z_\mu|<(1-a)\delta} F^2.$$

By Proposition 4.2.2, we thus have

$$\mathcal{H}^1(Z_F \cap \mathcal{D}(z_\mu, c_6 2^{-k}\delta)) \le c_7 2^{-k}\delta,$$

which holds for all $z_\mu \in S_j(k)$. We can now pick a finite collection of $N_0 = N_0(c_6)$ points $z_l \in S_j(k)$ such that the reunion of the associated disks $D(z_l, c_6 2^{-k}\delta)$ cover $S_j(k)$. The collection being finite, we have

$$\mathcal{H}^1(Z_F \cap S_j(k)) \le \sum_{l=1}^{N_0} \mathcal{H}^1(Z_F \cap D(z_l, c_6 2^{-k}\delta)) \le (N_0 c_7)2^{-k}\delta = c_9 2^{-k}\delta.$$ □

The next result is exactly Lemma 6.3 in [Donnelly and Fefferman 1990].

Lemma 4.2.6. *The union* $\bigcup_{j\in J(k),\ k\in\mathbb{N}\cup\{0\}} S_j(k)$ *covers the whole square*

$$P = \left\{|x|, |y| \leq \tfrac{1}{60}\right\}$$

except for the singular set $\mathcal{S}_F := \{z \in P : F(z) = \nabla F(z) = 0\}$.

The last lemma allows us to discard the singular set when studying the length of the nodal set of F.

Lemma 4.2.7. *Let* $\mathcal{S}_F$ *be the singular set of* F *in* P. *Then*

$$\mathcal{H}^1(\mathcal{S}) = 0.$$

Proof. It is well-known (see for instance [Bers 1955; Han and Lin 2007]) that the singular set $\mathcal{S}$ of a F is a submanifold of codimension 2, which means here that it is a finite set of points, whence $\mathcal{H}^1(\mathcal{S}) = 0$. □

We are now ready to complete the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. Using all of the above lemmas, we have

$$\begin{aligned}\mathcal{H}^1\left(Z_F \cap \tfrac{1}{60}\mathbb{D}\right) &\leq \mathcal{H}^1(Z_F \cap P) \underset{6,7}{=} \sum_{k=0}^{\infty} \sum_{j\in J(k)} \mathcal{H}^1(Z_F \cap S_j(k)) \\ &\underset{5}{\leq} c_9\delta \sum_{k=0}^{\infty} \sum_{j\in J(k)} 2^{-k} \underset{4}{\leq} (c_9\delta)4c_8\delta^{-1}\beta^* \sum_{k=0}^{\infty} 2^{-k} \\ &= 4c_8c_9\beta^* \sum_{k=0}^{\infty} 2^{-k} \leq c_3\beta^*.\end{aligned}$$

□

5. Proof of Proposition 4.2.1

We divide the rather long proof into six subsections. The treatment is based on the proof of Proposition 4.7 in [Donnelly and Fefferman 1990].

5.1. ***Setting.*** Using the same hypotheses, we will actually prove a slightly different statement. We let $t := \beta + 1$. It follows from the fact that $\delta\beta^* < \frac{1}{2}$ that

$$\delta t < 1. \tag{5.1.1}$$

We normalize F by the condition $\sup_{3\mathbb{D}}|F| = 1$, which has no effect whatsoever on the growth exponent. Finally, we can choose the uniform norm of the potential to be conveniently small: $\|q\|_\infty < \epsilon_0 < 1$. We will show that there exists a constant $c_5 > 0$ such that, for a large enough $M = M_0$, the number $\mathcal{N} = \mathcal{N}(M)$ of γ-separated, M-rapid disks satisfies

$$\mathcal{N} < c_5 t,$$

which implies the result since $t \leq 2\beta^* = 2\max\{\beta, 1\}$. We recall that we are still in the setting of disks and annuli described in Section 4.1, that is, we have an arbitrary, finite collection of open disks $D_\nu \subset \frac{1}{60}\mathbb{D}$,

$1 \le \nu \le N$, each of radius δ. Moreover, the collection of disks is γ-separated: the disks are mutually disjoint after a scaling of factor γ,

$$|z_\mu - z_\nu| \ge 2\gamma\delta \quad \text{for all } \mu \ne \nu,$$

where $\gamma = \delta^{-1/2}$.

5.2. ***A Carleman-type estimate.*** The starting point of the proof is (2.4) of [Donnelly and Fefferman 1990], which is an estimate in the spirit of Carleman, relating the weighted L^2 norm of a function to that of some of its derivatives.

Lemma 5.2.1. *Let $t > 0$, and define*

$$P(z) := \prod_\nu (z - z_\nu).$$

There exists a constant $c_{10} > 0$ such that, for any $f \in C_0^\infty\big(3\mathbb{D} \setminus \bigcup_\nu D_\nu(a)\big)$, we have

$$\int_{3\mathbb{D}} |\Delta f|^2 |P|^{-2} e^{t|z|^2} \ge c_{10}\left(t^2 \int_{3\mathbb{D}} |f|^2 |P|^{-2} e^{t|z|^2} + \delta^{-2} \int_A |\nabla f|^2 |P|^{-2} e^{t|z|^2}\right). \tag{C1}$$

The rather long development of that inequality is postponed to Section 6. Our first goal is to replace $|\nabla f|^2$ by $|f|^2$ in the right-hand side of the Carleman estimate. To do so, we will need the next two lemmas.

Lemma 5.2.2. *There exist positive constants c_i, $i = 11, \dots, 14$, such that, for any $w_1, w_2 \in A_\nu$,*

$$\text{(i)} \quad c_{11} \le \frac{e^{t|w_1|^2}}{e^{t|w_2|^2}} \le c_{12}, \qquad \text{(ii)} \quad c_{13} \le \frac{|P(w_1)|}{|P(w_2)|} \le c_{14}.$$

Proof. Since $w_1, w_2 \in \frac{1}{60}\mathbb{D}$, we have

$$\big|t|w_1|^2 - t|w_2|^2\big| = t\big|(|w_1| - |w_2|)(|w_1 + w_2|)\big| \le t\big||w_1| - |w_2|\big| \le t|w_1 - w_2| \le 2t\delta.$$

Since $t\delta \le 1$, the result (i) now follows from exponentiation.

We now prove (ii). We have

$$\begin{aligned}\big|\log|P(w_1)| - \log|P(w_2)|\big| &= \left|\sum_\mu \log|w_1 - z_\mu| - \sum_\mu \log|w_2 - z_\mu|\right| \\ &\le \big|\log|w_1 - z_\nu| - \log|w_2 - z_\nu|\big| + \sum_{\mu\ne\nu} \big|\log|w_1 - z_\mu| - \log|w_2 - z_\mu|\big|.\end{aligned}$$

We first consider the first term of the right-hand side of the above inequality. Suppose without loss of generality that w_1 is farther from z_ν than w_2, that is, $|w_1 - z_\nu| = \max\{|w_1 - z_\nu|, |w_2 - z_\nu|\}$. Then since both w_1 and w_2 belong to the annulus A_ν, we have

$$\begin{aligned}\big|\log|w_1 - z_\nu| - \log|w_2 - z_\nu|\big| &= \log|w_1 - z_\nu| - \log|w_2 - z_\nu| \\ &\le \log(1-a)\delta - \log(1-2a)\delta \\ &= \log\frac{1-a}{1-2a} = a_2,\end{aligned}$$

where $a_2 > 0$. It now remains to estimate $\sum_{\mu\neq\nu}\big|\log|w_1 - z_\mu| - \log|w_2 - z_\mu|\big|$. By the mean value theorem applied to $w \mapsto |w - z_\mu|$, there exists some point $w \in \{(1-\tau)w_1 + \tau w_2 : 0 \le \tau \le 1\}$ such that

$$\big|\log|w_1 - z_\mu| - \log|w_2 - z_\mu|\big| = |w - z_\mu|^{-1}|w_1 - w_2|.$$

The triangle inequality also implies $|z_\mu - z_\nu| \le |w - z_\mu| + |w - z_\nu| \le 2|w - z_\mu|$, whence $|w - z_\mu|^{-1} \le 2|z_\mu - z_\nu|$ and

$$\big|\log|w_1 - z_\mu| - \log|w_2 - z_\mu|\big| \le 2\frac{|w_1 - w_2|}{|z_\mu - z_\nu|} \le \frac{4\delta}{|z_\mu - z_\nu|}.$$

We now have

$$\sum_{\mu\neq\nu}\big|\log|w_1 - z_\mu| - \log|w_2 - z_\mu|\big| \le 4\delta\sum_{\mu\neq\nu}|z_\mu - z_\nu|^{-1}. \tag{5.2.3}$$

For $z \in \gamma D_\mu$, $\mu \neq \nu$, we have $|z - z_\nu| + |z_\mu - z_\nu| \le 2|z_\mu - z_\nu|$, from which we easily get

$$\int_{\gamma D_\mu}|z - z_\nu|^{-1} \ge \frac{1}{2}\int_{\gamma D_\mu}\frac{1}{|z_\mu - z_\nu|} = \frac{\pi(\gamma\delta)^2}{2|z_\mu - z_\nu|}.$$

We define $E_\nu := \bigcup_{\mu\neq\nu}\gamma D_\mu$, and we now have

$$4\delta\sum_{\mu\neq\nu}|z_\mu - z_\nu|^{-1} \le \frac{8\delta}{\pi(\gamma\delta)^2}\sum_{\mu\neq\nu}\int_{\gamma D_\mu}|z - z_\nu|^{-1} = \frac{8}{\pi\gamma^2\delta}\int_{E_\nu}|z - z_\nu|^{-1}. \tag{5.2.4}$$

Let B_ν be the disk centered at z_ν whose total area is the same as E_ν; that is, $\mathrm{Area}(B_\nu) = \mathrm{Area}(E_\nu) = (N-1)\pi(\gamma\delta)^2$. Remark that the maximum number of γ-separated disks of radius δ in $3\mathbb{D}$ is of the order $(\gamma\delta)^{-2}$; that is, there exists a positive constant c, independent of γ and δ, such that the cardinality N of our collection of disks satisfies $N < c(\gamma\delta)^{-2}$. We consequently have

$$\int_{E_\nu}|z - z_\nu|^{-1} \le \int_{B_\nu}|z - z_\nu|^{-1} \le 4\sqrt{\mathrm{Area}(E_\nu)} \le 4\sqrt{\pi N}\gamma\delta \le 4\sqrt{c\pi}. \tag{5.2.5}$$

Combining (5.2.3), (5.2.4) and (5.2.5) now gives

$$\sum_{\mu\neq\nu}\big|\log|w_1 - z_\mu| - \log|w_2 - z_\mu|\big| \le \frac{32\sqrt{c\pi}}{\pi\gamma^2\delta} = \frac{a_3}{\gamma^2\delta} = a_3$$

since $\gamma = \delta^{-1/2}$. Finally,

$$\big|\log|P(w_1)| - \log|P(w_2)|\big| \le a_2 + a_3,$$

from which the result follows via exponentiation. □

The second lemma is a Poincaré-like inequality.

Lemma 5.2.6. *Suppose $f \in C^\infty(A_\nu)$ and vanishes on the inner boundary $|z| = (1-2a)\delta$ of A_ν. Then*

$$\int_{A_\nu}|\nabla f|^2 \ge \frac{c_{15}}{\delta^2}\int_{A_\nu}|f|^2, \tag{5.2.7}$$

where c_{15} is a positive constant.

Proof. We introduce polar coordinates (r, θ) on A_ν. Since $f((1-2a)\delta, \theta) \equiv 0$, the fundamental theorem of calculus yields

$$f(r,\theta) = \int_{(1-2a)\delta}^{r} \frac{\partial f}{\partial s}(s,\theta)\, ds.$$

Hence,

$$\int_{A_\nu} |f|^2 \, \mathrm{d}A = \int_0^{2\pi} \int_{(1-2a)\delta}^{(1-a)\delta} \left(\int_{(1-2a)\delta}^{r} \frac{\partial f}{\partial s}(s,\theta)\, ds \right)^2 r\, dr\, d\theta.$$

By Cauchy–Schwarz, we have

$$\left(\int_{(1-2a)\delta}^{r} \frac{\partial f}{\partial s}(s,\theta)\, ds \right)^2 \le \int_{(1-2a)\delta}^{r} \left(\frac{\partial f}{\partial s} \right)^2 ds \int_{(1-2a)\delta}^{r} 1^2\, ds \le a\delta \int_{(1-2a)\delta}^{r} \left(\frac{\partial f}{\partial s} \right)^2 ds.$$

Consequently,

$$\begin{aligned} \int_{A_\nu} |f|^2 &\le a\delta \int_0^{2\pi} \int_{(1-2a)\delta}^{(1-a)\delta} \int_{(1-2a)\delta}^{(1-a)\delta} \left(\frac{\partial f}{\partial s} \right)^2 ds\, r\, dr\, d\theta \\ &\le a\delta \int_0^{2\pi} \frac{r^2}{2}\Big|_{(1-2a)\delta}^{(1-a)\delta} \int_{(1-2a)\delta}^{(1-a)\delta} \left(\frac{\partial f}{\partial s} \right)^2 \frac{s}{(1-2a)\delta}\, ds\, d\theta \\ &\le c_{15}\delta^2 \int_0^{2\pi} \int_{(1-2a)\delta}^{(1-a)\delta} |\nabla f|^2 s\, ds\, d\theta = c_{15}\delta^2 \int_{A_\nu} |\nabla f|^2. \end{aligned}$$

□

Fix one $w_\nu \in A_\nu$ for all $1 \le \nu \le N$. Then, for each ν, we have

$$\begin{aligned} \int_{A_\nu} |\nabla f|^2 |P|^{-2} e^{t|z|^2} &\ge (c_{11}c_{13}^2) \frac{e^{t|w_\nu|^2}}{|P(w_\nu)|^2} \int_{A_\nu} |\nabla f|^2 \ge (c_{11}c_{13}^2 c_{15}) \frac{e^{t|w_\nu|^2}}{\delta^2 |P(w_\nu)|^2} \int_{A_\nu} f^2 \\ &\ge (c_{11}^2 c_{13}^4 c_{15}) \delta^{-2} \int_{A_\nu} f^2 |P|^{-2} e^{t|z|^2}, \end{aligned}$$

where we have used, respectively, Lemmas 5.2.2, 5.2.6 and then 5.2.2 again. The Carleman estimate (C1) thus becomes

$$\int_{3\mathbb{D}} |\Delta f|^2 |P|^{-2} e^{t|z|^2} \ge a_4 \left(t^2 \int_{3\mathbb{D}} f^2 |P|^{-2} e^{t|z|^2} + \delta^{-4} \int_A f^2 |P|^{-2} e^{t|z|^2} \right), \tag{C2}$$

where $a_4 := \min\{c_{11}^2 c_{13}^4 c_{15}, c_{10}\}$.

5.3. ***A suitable cut-off for F.*** We now apply the previous estimate to $f = \theta F$, where θ is a suitable cut-off. More precisely, the cut-off θ satisfies the following properties:

(i) $0 \le \theta \le 1$, $\theta \in C_0^\infty\big(2\mathbb{D} \setminus \bigcup_\nu D_\nu\big)$,

(ii) $\theta(z) \equiv 1$ on $\big\{z : |z| < 1,\ |z - z_\nu| > \big(1 - \frac{3}{2}a\big)\delta\big\}$,

(iii) $|\nabla\theta| + |\Delta\theta| \le a_5$ on $\{|z| > 1\}$,

(iv) $|\nabla\theta| \le a_6\delta^{-1}$ and $|\Delta\theta| \le a_7\delta^{-2}$ for $|z - z_\nu| \le \big(1 - \frac{3}{2}a\big)\delta$.

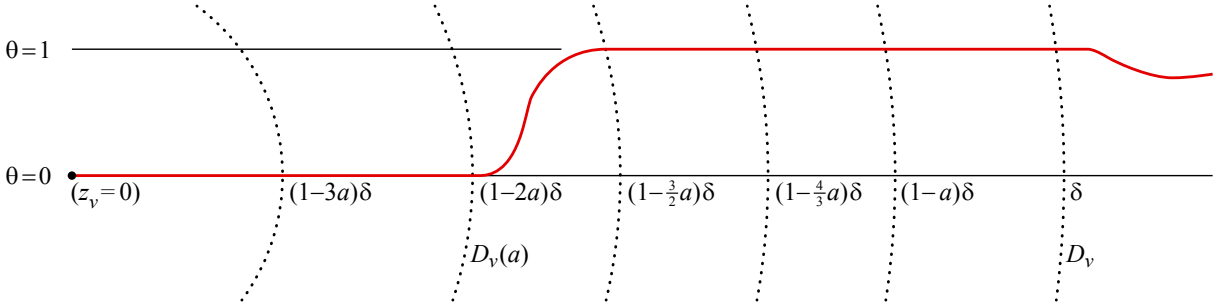

Figure 6. A smooth cut-off θ defined on $2\mathbb{D}$.

The property (iv) allows us to control the growth properties of the cut-off in terms of the radius δ of the disks. Figure 6 summarizes the property of the cut-off.

Using the properties of θ, we have the following:

Lemma 5.3.1. *Let F and θ be as defined in our current setting. Then*

$$|\Delta(\theta F)| \leq 5(q^2F^2 + |\nabla\theta|^2|\nabla F|^2 + F^2|\Delta\theta|^2).$$

Proof. The proof is a simple computation:

$$\begin{aligned}
|\Delta(\theta F)|^2 &= |\theta\Delta F + 2(\nabla\theta\cdot\nabla F) + F\Delta\theta|^2 \\
&\leq (|\theta\Delta F| + 2|\nabla\theta||\nabla F| + |F\Delta\theta|)^2 \\
&\leq 5(\theta^2|-qF|^2 + |\nabla\theta|^2|\nabla F|^2 + F^2|\Delta\theta|^2) \\
&\leq 5(q^2F^2 + |\nabla\theta|^2|\nabla F|^2 + F^2|\Delta\theta|^2).
\end{aligned}$$

□

Applying (C2) to θF now yields

$$\int_{2\mathbb{D}} |\Delta\theta F|^2|P|^{-2}e^{t|z|^2} \geq a_4\left(t^2\int_{2\mathbb{D}} |\theta F|^2|P|^{-2}e^{t|z|^2} + \delta^{-4}\int_A |\theta F|^2|P|^{-2}e^{t|z|^2}\right).$$

Using Lemma 5.3.1 to estimate the left-hand side of the above equation, we now get

$$\int_{2\mathbb{D}} \left(q^2F^2+|\nabla\theta|^2|\nabla F|^2+F^2|\Delta\theta|^2\right)|P|^{-2}e^{t|z|^2} \geq \frac{a_4}{5}\left(t^2\int_{2\mathbb{D}} |\theta F|^2|P|^{-2}e^{t|z|^2}+\delta^{-4}\int_A |\theta F|^2|P|^{-2}e^{t|z|^2}\right).$$

Now, since our potential is small, $\|q\|_\infty < \epsilon_0$, the first term of the left-hand side can without loss of generality (by picking a smaller constant if needed) be absorbed by the right-hand side, yielding

$$\int_{2\mathbb{D}} \left(|\nabla\theta|^2|\nabla F|^2 + F^2|\Delta\theta|^2\right)|P|^{-2}e^{t|z|^2} \geq a_8\left(t^2\int_{2\mathbb{D}} |\theta F|^2|P|^{-2}e^{t|z|^2} + \delta^{-4}\int_A |\theta F|^2|P|^{-2}e^{t|z|^2}\right). \quad \text{(C3)}$$

The remainder of the proof consists mostly of improvements of the left- and right-hand sides of this last estimate.

5.4. ***Using elliptic theory to improve the left-hand side of*** **(C3).** We now work on the left-hand side of the last Carleman estimate. By the definition of the cut-off θ, we have $|\nabla\theta| = |\Delta\theta| \equiv 0$ on $2\mathbb{D} \setminus \big(A = \bigcup_\nu A_\nu \cup \{1 \le |z| \le 2\}\big)$ so that it makes sense to write (LHS) $= I + \sum_\nu I_\nu$, where

$$I = \int_{1<|z|<2} \zeta(z), \qquad I_\nu = \int_{A_\nu} \zeta(z)$$

and

$$\zeta(z) = \big(|\nabla\theta|^2|\nabla F|^2 + F^2|\Delta\theta|^2\big)|P|^{-2}e^{t|z|^2}.$$

The following lemma uses elliptic theory to improve estimates of both I and I_ν:

Lemma 5.4.1. *There exist positive constants c_{11} and c_{12} such that*

$$\text{(i)} \quad I \le c_{16}e^{4t}\max_{|z|\ge1}|P|^{-2}\int_{3/4<|z|<9/4} F^2, \qquad \text{(ii)} \quad I_\nu \le c_{17}\delta^{-4}\max_{A_\nu}(|P|^{-2}e^{t|z|^2})\int_{A'_\nu} F^2.$$

Proof. Recalling the various assumptions on the cutoff θ, we immediately have

$$\begin{aligned} I &\le a_5e^{4t}\max_{1\le|z|\le2}|P|^{-2}\int_{1<|z|<2}(F^2+|\nabla F|^2) \\ &= a_5e^{4t}\max_{1\le|z|\le2}|P|^{-2}\|F\|^2_{H^1(\Omega')}, \end{aligned} \tag{5.4.2}$$

where $H^1 = W^{1,2}$ is the habitual Sobolev space and $\Omega' = \{1 < |z| < 2\}$. We now apply Theorem 8.8 in [Gilbarg and Trudinger 1998] with $L = \Delta$, $u = F$ and $f = -qF$ to get

$$\begin{aligned} \|F\|_{W^{2,2}(\Omega')} &\le a_9\big(\|F\|_{L^2(\Omega)} + \|qF\|_{L^2(\Omega)}\big) \\ &\le a_9\max\{1, \operatorname{Area}(\Omega)\epsilon_0\}\|F\|_{L^2(\Omega)} \\ &= a_{10}\|F\|_{L^2(\Omega)}, \end{aligned}$$

which holds for any subdomain Ω such that $\Omega' \Subset \Omega$, that is,

$$\sup_{x\in\partial\Omega,\ y\in\Omega'}|x-y| > 0.$$

We set $\Omega := \big\{\frac{3}{4} < |z| < \frac{9}{4}\big\}$ so that the above condition is satisfied. Since $\|\cdot\|_{W^{1,2}} \le \|\cdot\|_{W^{2,2}}$, we have

$$\|F\|^2_{H^1(\Omega')} \le a_{10}^2\|F\|^2_{L^2(\Omega)}$$

so that estimate (5.4.2) becomes

$$I \le (a_5a_{10}^2)e^{4t}\max_{1\le|z|\le2}|P|^{-2}\|F\|^2_{L^2(\Omega)} = c_{11}e^{4t}\max_{|z|\ge1}|P|^{-2}\int_{3/4<|z|<9/3}F^2.$$

We now prove the second part of the lemma. We define $\bar{A}_\nu := (1-2a)\delta < |z-z_\nu| < (1-3a/2)\delta \subset A_\nu$. Since $\theta(z) \equiv 1$ for $(1-3a/2)\delta < |z| < (1-a)\delta$, we have

$$\begin{aligned} I_\nu &\le \max_{A_\nu}\big(|P|^{-2}e^{t|z|^2}\big)\int_{\bar{A}_\nu}(|\nabla\theta|^2|\nabla F|^2 + F^2|\Delta\theta|^2) \\ &\le \max\{a_6^2, a_7^2\}\max_{A_\nu}\big(|P|^{-2}e^{t|z|^2}\big)\bigg[\int_{\bar{A}_\nu}\delta^{-2}|\nabla F|^2 + \int_{\bar{A}_\nu}\delta^{-4}F^2\bigg]. \end{aligned} \tag{5.4.3}$$

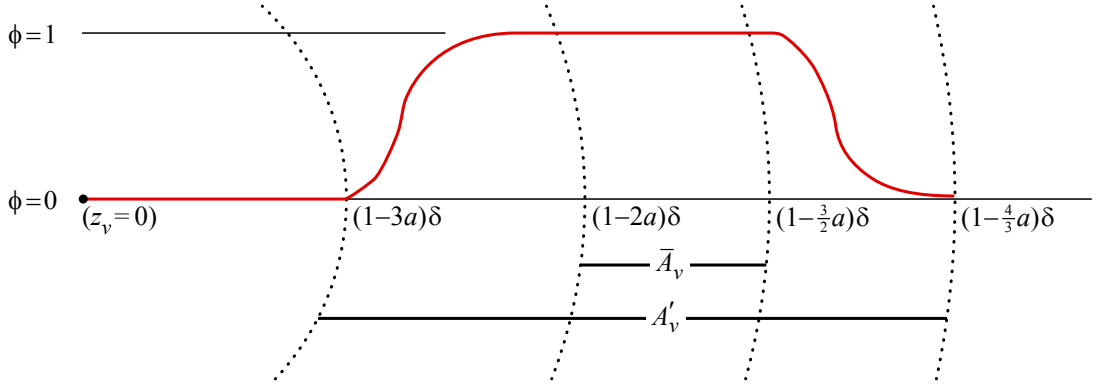

Figure 7. A second cutoff ϕ on the annuli.

Our goal is now to get rid of the gradient in the first integral of the last equation above. To do so, we set $\bar{I}_\nu := \int_{\bar{A}_\nu} |\nabla F|^2$ and introduce another cutoff $\phi \in C_0^\infty(A'_\nu)$ that satisfies

(i) $0 \le \phi \le 1$,

(ii) $\phi(z) \equiv 1$ on $\bar{A}_\nu$,

(iii) $|\nabla \phi| \le a_{11}(\phi\delta^{-1})$.

Figure 7 summarizes the properties of this cutoff function.

Using Green's identity and since ϕ vanishes on the boundary of A'_ν, we notice that

$$\int_{A'_\nu} q\phi F^2 = -\int_{A'_\nu} \phi F \Delta F = \int_{A'_\nu} \nabla(\phi F)\cdot \nabla F = \int_{A'_\nu} F(\nabla F \cdot \nabla \phi) + \int_{A'_\nu} \phi |\nabla F|^2.$$

Thus, since $\|q\|_\infty < 1$, we get

$$\int_{A'_\nu} \phi |\nabla F^2| \le \int_{A'_\nu} \phi F^2 + \frac{a_{11}}{\delta} \int_{A'_\nu} \phi |F| \|\nabla F\|. \tag{5.4.4}$$

Now, for any nonnegative numbers a, b and c and $k > 0$, we have the elementary inequality $abc \le \frac{1}{2}(ab^2/k - kac^2)$, which we apply to our setting to get

$$\phi\left(\frac{|F|}{\delta}\right)\|\nabla F\| \le \frac{1}{2}\left(\frac{\phi F^2}{k\delta^2} + k\phi|\nabla F|^2\right).$$

We integrate over A'_ν and then choose k small enough to absorb $\frac{1}{2}(k\phi|\nabla F|^2)$ in the left-hand side of (5.4.4) so that it becomes

$$\int_{A'_\nu} \phi |\nabla F|^2 \le \max\left\{1, \frac{a_{11}}{2k}\right\} \frac{1}{\delta^2} \int_{A'_\nu} \phi F^2.$$

Going back to the definition of $\bar{I}_\nu$, we now have

$$\bar{I}_\nu = \int_{\bar{A}_\nu} |\nabla F|^2 \le \int_{A'_\nu} \phi |\nabla F|^2 \le a_{12} \frac{1}{\delta^2} \int_{A'_\nu} \phi F^2 \le a_{12}\delta^{-2} \int_{A'_\nu} F^2.$$

Plugging this into (5.4.3) yields

$$\begin{aligned} I_\nu &\le \max\{a_6^2, a_7^2\} \max_{A_\nu}\big(|P|^{-2}e^{t|z|^2}\big)\bigg[\delta^{-2}\bar{I}_\nu + \int_{\bar{A}_\nu} \delta^{-4}F^2\bigg] \\ &\le \big(\max\{a_6^2, a_7^2\}\max\{1, a_{12}\}\big) \max_{A_\nu}\big(|P|^{-2}e^{t|z|^2}\big)\delta^{-4}\int_{\bar{A}_\nu} F^2 \\ &\le c_{17} \max_{A_\nu}\big(|P|^{-2}e^{t|z|^2}\big)\delta^{-4}\int_{A'_\nu} F^2. \end{aligned}$$

□

By Lemma 5.2.2, we have

$$\max_{A_\nu}\big(|P|^{-2}e^{t|z|^2}\big) \le a_{13}\min_{A_\nu}\big(|P|^{-2}e^{t|z|^2}\big).$$

Applying the estimates of Lemma 5.4.1 to the left-hand side of (C3) then gives

$$\text{(LHS)} = I + \sum_\nu I_\nu \le a_{14}\bigg(e^{4t}\max_{|z|\ge 1}|P|^{-2}\int_{3/4<|z|<9/4} F^2 + \delta^{-4}\sum_\nu \min_{A_\nu}\big(|P|^{-2}e^{t|z|^2}\big)\int_{A'_\nu} F^2\bigg), \quad (5.4.5)$$

where $a_{14} = \max\{c_{16}, c_{16}a_{13}\}$.

The next lemma introduces the growth exponent β of F in an expression that links the L^2 norms of F on two annuli of different sizes.

Lemma 5.4.6. *There exists a positive constant c_{18} such that*

$$\int_{3/4<|z|<9/4} F^2 \le c_{18}e^{2\beta}\int_{1/4<|z|<1/2} F^2.$$

Proof. First, recall that the potential q satisfies $\|q\|_\infty < \epsilon_0$. On the one hand,

$$\iint_{3/4<|z|<9/4} F^2\,\mathrm{d}A < \iint_{\frac{5}{2}\mathbb{D}} F^2\,\mathrm{d}A \le \left(\frac{25\pi}{4}\right)\sup_{\frac{5}{2}\mathbb{D}} F^2. \quad (5.4.7)$$

On the other hand, the definition of the growth exponent yields

$$\sup_{\frac{5}{2}\mathbb{D}} F^2 = e^{2\beta}\sup_{\frac{1}{4}\mathbb{D}} F^2. \quad (5.4.8)$$

Following an approach similar to Lemma 4.9 in [Nazarov et al. 2005], we now represent F as the sum of its Green potential and Poisson integral. More precisely, for $|z| < \frac{1}{4}$ and given any fixed radius $\rho \in \left(\frac{1}{4}, \frac{1}{2}\right]$,

$$F(z) = \iint_{\rho\mathbb{D}} p(\zeta)F(\zeta)G_\rho(z,\zeta)\,\mathrm{d}A(\zeta) + \int_{\rho\mathbb{S}^1} F(\zeta)P_\rho(z,\zeta)\,ds(\zeta), \quad (5.4.9)$$

where $G_\rho(z,\zeta) = \log|(\rho^2 - z\bar{\zeta})/\rho(z-\zeta)|$ and $P_\rho(z,\zeta) = (\rho^2 - |z|^2)/|\zeta - z|^2$. We write I_1 and I_2, respectively, for the double integral and the (line) integral above and notice that

$$F^2 = I_1^2 + 2I_1I_2 + I_2^2 \le 4(I_1^2 + I_2^2). \quad (5.4.10)$$

Using Cauchy–Schwartz, we get the upper bound

$$\begin{aligned} I_1^2 &\le \iint_{\rho\mathbb{D}} p^2(\zeta)F^2(\zeta)\,\mathrm{d}A(\zeta) \iint_{\rho\mathbb{D}} G_\rho^2(z,\zeta)\,\mathrm{d}A(\zeta) \\ &\le a_{15} \iint_{\rho\mathbb{D}} p^2(\zeta)F^2(\zeta)\,\mathrm{d}A(\zeta) \le a_{15}\|p\|_\infty^2 \iint_{\rho\mathbb{D}} F^2(\zeta)\,\mathrm{d}A(\zeta) \\ &\le a_{15}\epsilon_0^2 \iint_{\frac{1}{2}\mathbb{D}} F^2(\zeta)\,\mathrm{d}A(\zeta). \end{aligned} \tag{5.4.11}$$

In the above, we have $a_{15} = \sup_{\rho\in(1/4,1/2]} \sup_{z\in\frac{1}{4}\mathbb{D}} \iint_{\rho\mathbb{D}} G_\rho^2(z,\zeta)\,\mathrm{d}A(\zeta)$. Similarly,

$$I_2^2 \le \int_{\rho\,\mathbb{S}^1} F^2(\zeta)\,ds(\zeta) \int_{\rho\mathbb{S}^1} P_\rho^2(z,\zeta)\,ds(\zeta) \le a_{16} \int_{\rho\mathbb{S}^1} F^2(\zeta)\,ds(\zeta) \tag{5.4.12}$$

with $a_{16} = \sup_{\rho\in(1/4,1/2]} \sup_{z\in\frac{1}{4}\mathbb{D}} \int_{\rho\mathbb{S}^1} P_\rho^2(z,\zeta)\,ds(\zeta)$. Now, recalling that the representation of F in (5.4.9) holds for any $|z| \le \frac{1}{4}$ and substituting (5.4.11) and (5.4.12) into (5.4.10), we get

$$\sup_{z\in\frac{1}{4}\mathbb{D}} F^2 \le a_{17}\left(\epsilon_0^2 \iint_{\frac{1}{2}\mathbb{D}} F^2\,\mathrm{d}A + \int_{\rho\mathbb{S}^1} F^2\,ds\right) \quad \text{for all } \rho \in \left(\tfrac{1}{4}, \tfrac{1}{2}\right]$$

with $a_{17} = 4\max\{a_{15}, a_{16}\}$. Averaging over all ρ yields

$$\begin{aligned} \sup_{z\in\frac{1}{4}\mathbb{D}} F^2 &\le a_{17}\frac{16}{3\pi}\left(\epsilon_0^2 \iint_{\frac{1}{2}\mathbb{D}} F^2\,\mathrm{d}A + \iint_{1/4<|z|<1/2} F^2\,\mathrm{d}A\right) \\ &= a_{18}\epsilon_0^2 \iint_{\frac{1}{4}\mathbb{D}} F^2\,\mathrm{d}A + a_{18}(1+\epsilon_0^2) \iint_{1/4<|z|<1/2} F^2\,\mathrm{d}A \\ &\le \left(\frac{a_{18}\pi}{16}\right)\epsilon_0^2 \sup_{z\in\frac{1}{4}\mathbb{D}} F^2 + a_{18}(1+\epsilon_0^2) \iint_{1/4<|z|<1/2} F^2\,\mathrm{d}A \\ &= a_{19}\epsilon_0^2 \sup_{z\in\frac{1}{4}\mathbb{D}} F^2 + a_{18}(1+\epsilon_0^2) \iint_{1/4<|z|<1/2} F^2\,\mathrm{d}A. \end{aligned} \tag{5.4.13}$$

Hence,

$$(1-a_{19}\epsilon_0^2) \sup_{z\in\frac{1}{4}\mathbb{D}} F^2 \le a_{18}(1+\epsilon_0^2) \iint_{1/4<|z|<1/2} F^2\,\mathrm{d}A.$$

It suffices to choose ϵ_0 small enough so that $(1-a_{19}\epsilon_0^2)$ is positive to finally obtain

$$\sup_{z\in\frac{1}{4}\mathbb{D}} F^2 \le \frac{a_{18}(1+\epsilon_0^2)}{1-a_{19}\epsilon_0^2} \iint_{1/4<|z|<1/2} F^2\,\mathrm{d}A. \tag{5.4.14}$$

Linking (5.4.7), (5.4.8) and (5.4.14) together concludes the proof. □

To finalize our estimate of the left-hand side of (C3), we need one last lemma.

Lemma 5.4.15. *Let N be the number of disks D_v in our collection, that is, $N = \deg P$. Then there exists a positive constant c_{19} such that*

$$\max_{z\ge 1} |P|^{-2} \le e^{-c_{19}N} \min_{|z|\le\frac{1}{2}} |P|^{-2}.$$

Proof. For $|z| \geq 1$, we have

$$\frac{1}{|z-z_\nu|} \leq \frac{1}{|z|-|z_\nu|} \leq \frac{1}{1-1/60} = \frac{60}{59}$$

while, for $|z| \leq 1/2$, we have

$$\frac{1}{|z-z_\nu|} \geq \frac{1}{|z|+|z_\nu|} \geq \frac{1}{1/2+1/60} = \frac{60}{31}.$$

As a consequence,

$$\max_{|z|\geq 1} |P|^{-2} \leq \left(\tfrac{60}{59}\right)^{2\deg P} = \left(\tfrac{31}{59}\right)^{2\deg P} \left(\tfrac{60}{31}\right)^{2\deg P} \leq \left(\tfrac{31}{59}\right)^{2\deg P} \min_{|z|\leq 1/2} |P|^{-2}.$$

We set $c_{19} = -2\log\left(\frac{31}{59}\right)$ to conclude the proof. □

Applying the results of the last two lemmas to (5.4.5), we obtain a final estimate for the left-hand side of (C3):

$$\begin{aligned}(\text{LHS}) &\leq a_{14}\left(e^{4t}e^{-c_{19}N} \min_{|z|\leq 1/2} |P|^{-2} c_{18} e^{2\beta} \int_{1/4<|z|<1/2} F^2 + \delta^{-4} \sum_\nu \min_{A_\nu}\left(|P|^{-2}e^{t|z|^2}\right) \int_{A'_\nu} F^2\right) \\ &\leq a_{20}\left(e^{6t-c_{19}N} \min_{|z|\leq 1/2} |P|^{-2} \int_{1/4<|z|<1/2} F^2 + \delta^{-4} \sum_\nu \min_{A_\nu}\left(|P|^{-2}e^{t|z|^2}\right) \int_{A'_\nu} F^2\right), \qquad (5.4.16)\end{aligned}$$

where $a_{20} = a_{14}\max\{c_{18}, 1\}$, since $\beta < t$.

5.5. ***Improving the right-hand side of*** **(C3).** Recalling that $t > 1$ as well as the various properties of the cut-off, we estimate the right-hand side of (C3):

$$\begin{aligned}a_8^{-1}(\text{RHS}) &= t^2 \int_{2\mathbb{D}} |\theta F|^2 |P|^{-2} e^{t|z|^2} + \delta^{-4} \sum_\nu \int_{A_\nu} |\theta F|^2 |P|^{-2} e^{t|z|^2} \\ &\geq \left(\frac{\pi}{4}\right) \min_{|z|\leq 1/2} |P|^{-2} \int_{|z|<1/2} F^2 + \delta^{-4} \sum_\nu \min_{A_\nu}\left(|P|^{-2}e^{t|z|^2}\right) \int_{A_\nu} F^2 \\ &\geq a_{21}\left(\min_{|z|<1/2} |P|^{-2} \int_{1/4<|z|<1/2} F^2 + \delta^{-4} \sum_\nu \min_{A_\nu}\left(|P|^{-2}e^{t|z|^2}\right) \int_{A''_\nu} F^2\right), \qquad (5.5.1)\end{aligned}$$

where $a_{21} = \min\{\pi/4, 1\}$.

5.6. ***Conclusion.*** At last, putting together the estimates (C3), (5.4.16) and (5.5.1) yields

$$\begin{aligned}e^{6t-c_{19}N} \min_{|z|\leq 1/2} |P|^{-2} \int_{1/4<|z|<1/2} F^2 + \delta^{-4} \sum_\nu \min_{A_\nu}\left(|P|^{-2}e^{t|z|^2}\right) \int_{A'_\nu} F^2 \\ \geq a_{22}\left(\min_{|z|<1/2} |P|^{-2} \int_{1/4<|z|<1/2} F^2 + \delta^{-4} \sum_\nu \min_{A_\nu}\left(|P|^{-2}e^{t|z|^2}\right) \int_{A''_\nu} F^2\right),\end{aligned}$$

where $a_{22} = a_{21}a_8/a_{20}$. Recall that a disk D_ν is said to be M-rapid if

$$M \int_{A'_\nu} F^2 \leq \int_{A''_\nu} F^2.$$

Suppose now that all the disks of our collection are M-rapid, i.e., that $\mathcal{N} = N$, and assume without loss of generality that $a_{22} > 1$ (otherwise, the argument still works: it suffices to pick a larger M). We get

$$e^{6t-c_{19}N} \min_{|z|\le 1/2} |P|^{-2} \int_{1/4<|z|<1/2} F^2 + \delta^{-4} \sum_{\nu} \min_{A_\nu}\big(|P|^{-2} e^{t|z|^2}\big) \int_{A'_\nu} F^2$$
$$\ge \min_{|z|<1/2} |P|^{-2} \int_{1/4<|z|<1/2} F^2 + M\delta^{-4} \sum_{\nu} \min_{A_\nu}\big(|P|^{-2} e^{t|z|^2}\big) \int_{A'_\nu} F^2. \quad (5.6.1)$$

We get a contradiction if $N > \frac{6}{c_{19}} t \iff c_{19}N > c_6 t$, and the proof is completed.

6. An inequality in the spirit of Carleman

Carleman estimates are known to be useful in obtaining unique continuation results as well as growth estimates (see for instance [Koch and Tataru 2001]). It is thus not surprising that the estimate (C1) has played a crucial role in the proof of the growth estimate presented in the previous section. For completeness, we present here one way to obtain such an inequality, which follows very closely the approach taken in Section 2 of [Donnelly and Fefferman 1990].

6.1. ***An elementary inequality in a weighted Hilbert space.*** We let $\mathcal{D} \subset \mathbb{C}$ be open and bounded and $\varphi : \mathcal{D} \to \mathbb{R}$ be a smooth real-valued function. Let also $\mathcal{H} = L^2(\mathcal{D}, e^{-\varphi}\, dx\, dy)$ be the Hilbert space of complex-valued square-integrable functions on $\mathcal{D}$ with respect to the weight $e^{-\varphi}$. Finally, let $u \in C_0^\infty(\mathcal{D}) \subset H$. We introduce the differential operators

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \bar\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \qquad \bar\partial^* := e^{-\varphi}\partial(e^{-\varphi}\,\cdot\,).$$

Easy computations allow one to verify the following facts:

(i) For any real-valued function ψ, $\bar\partial\partial\psi = \frac{1}{4}\Delta\psi$.

(ii) By the Cauchy–Riemann equations, u is holomorphic if and only if $\bar\partial u = 0$.

(iii) $\bar\partial^*$ is the adjoint operator of $\bar\partial$.

(iv) $[\bar\partial, \bar\partial^*]u = \big(\frac{1}{4}\Delta\varphi\big)u$, where the interior of the parentheses acts on u by multiplication.

Lemma 6.1.1. *Let $\Phi : \mathcal{D} \to \mathbb{R}$ be a smooth, positive function. Then*

$$\int_{\mathcal{D}} |\bar\partial u|^2 \Phi \ge \int_{\mathcal{D}} \tfrac{1}{4}(\Delta \log \Phi)|u|^2 \Phi,$$

where the integrals are taken with respect to the usual Lebesgue measure, that is, not in the weighted Hilbert space $\mathcal{H}$.

Proof. Put $\varphi := -\log \Phi$, i.e., $e^{-\varphi} = \Phi$. In the following, the norms and inner products are taken in the Hilbert space $\mathcal{H}$:

$$\begin{aligned} 0 \le \|\bar\partial^* u\|^2 = (\bar\partial^* u, \bar\partial^* u) &= (\bar\partial\bar\partial^* u, u) \\ &= (\bar\partial^*\bar\partial u, u) + ([\bar\partial, \bar\partial^*]u, u) \\ &= (\bar\partial u, \bar\partial u) + ([\bar\partial, \bar\partial^*]u, u) \\ &= \|\bar\partial u\|^2 + \int_{\mathbb{D}} \left(\tfrac{1}{4}\Delta\varphi\right)|u|^2 e^{-\varphi}. \end{aligned}$$

Thus, $\|\bar\partial u\|^2 \ge -\frac{1}{4}\int_{\mathbb{D}} (\Delta\varphi)|u|^2 e^{-\varphi} = \frac{1}{4}\int_{\mathbb{D}} (\Delta \log \Phi)|u|^2\Phi$. □

6.2. ***A specialized choice of weight function.*** The remainder of the section aims to specialize the choice of Φ in order to obtain a more refined inequality. In particular, we will build a weight function that has singularities on a crucial set of points. In the following, a is a small, positive constant: $0 < a \ll 1$.

Lemma 6.2.1. *There exists a function* $\Psi_0(z)$, *defined for* $|z| > (1-2a)$, *such that*

(i) $a_1 \le \Psi_0(z) \le a_2$, *where* $a_1, a_2 > 0$,

(ii) $\Psi_0(z) \equiv 1$ *on* $\{|z| > 1\}$,

(iii) $\Delta \log \Psi_0 \ge 0$ *on* $\{|z| > (1-2a)\}$,

(iv) $\Delta \log \Psi_0 \ge a_3 > 0$ *on* $\{1-2a < |z| < 1-a\}$.

Proof. First, choose $\psi_0(z)$ to be a radial function, i.e., depending only on $r = |z|$. Let $h(r) \ge 0$ be smooth and such that $h(r) \ge a_3$ for $1-2a < r < 1-a$ and $h(r) = 0$ for $|z| > 1 - a/2$. Now consider the radial Laplacian

$$\Delta \log \psi_0(r) = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\log \psi_0(r),$$

which has smooth coefficients on $r > 1-2a$. By the fundamental theorem for ordinary differential equations, we let $\log \psi_0(r)$ be the solution of the second-order ODE

$$\begin{cases} \Delta \log \psi_0(r) = h(r), \\ \log \psi_0(1) = 0, \\ \log \psi_0{}'(1) = 0. \end{cases}$$

The function ψ_0 satisfies all the requirements. □

We now let $D_\nu := \{z : |z - z_\nu| < \delta\}$, $1 \le \nu \le N$, denote a finite collection of disks in the open unit disk $\mathbb{D}$ and let $D_\nu(a)$ be the closure of $(1-2a)D_\nu$. Define $\Phi_0 : \mathbb{C} \setminus \bigcup_\nu D_\nu(a)$ by

$$\Phi_0(z) = \begin{cases} 1 & \text{if } z \notin \bigcup_\nu D_\nu, \\ \psi_0\left(\dfrac{z - z_\nu}{\delta}\right) & \text{if } z \in D_\nu. \end{cases}$$

We have that $\log \Phi_0(z) = \log \Psi_0(w(z))$, where $w(z) = (z - z_\nu)/\delta$ and $w'(z) = 1/\delta$. Thus,

$$\Delta \log \Phi_0(z) = \frac{1}{\delta^2}\Delta \log \psi_0(w(z)) \ge a_3$$

for $z \in A_\nu = \{(1-2a)\delta < |z| < (1-a)\delta\}$. By Lemma 6.2.1, we have

(i) $a_1 \leq \Phi_0(z) \leq a_2$,

(ii) $\Delta \log \Phi_0 \geq 0$ for all $z \in \mathbb{C} \setminus \bigcup_\nu D_\nu(a)$,

(iii) $\Delta \log \Phi_0 \geq a_3/\delta^2$ for all $z \in A_\nu(\delta)$.

Let $t > 0$ be a constant, and denote by A the union $\bigcup_\nu A_\nu(\delta)$. We want to apply Lemma 6.1.1 to $\Phi(z) := \Phi_0(z)e^{t|z|^2}$. For $u \in C_0^\infty\big(\mathbb{C} \setminus \bigcup_\nu D_\nu(a)\big)$, we assume that $\mathscr{D}$ is a bounded domain such that $\operatorname{supp} u \subset \mathscr{D}$ and $A \subset \mathscr{D} \subset \mathbb{C} \setminus \bigcup_\nu D_\nu(a)$. Applying the lemma gives

$$\int_{\mathscr{D}} |\bar\partial u|^2 \Phi_0(z) e^{t|z|^2} \geq \int_{\mathscr{D}} \tfrac{1}{4}\big(\Delta \log \Phi_0 e^{t|z|^2}\big)|u|^2 \Phi_0 e^{t|z|^2}. \tag{6.2.2}$$

But $\log \Phi_0 e^{t|z|^2} = \log \Phi_0 + t|z|^2$, and the right-hand side of the above inequality satisfies

$$\begin{aligned}(\text{RHS}) &= \left[\int_A + \int_{\mathscr{D}\setminus A}\right] \tfrac{1}{4}(\Delta \log \Phi_0)|u|^2 \Phi_0(z) e^{t|z|^2} + t\int_{\mathscr{D}} |u|^2 \Phi_0(z) e^{t|z|^2} \\ &\geq \frac{a_3}{\delta^2}\int_A |u|^2 \Phi_0(z) e^{t|z|^2} + t\int_{\mathscr{D}} |u|^2 \Phi_0(z) e^{t|z|^2}.\end{aligned}$$

Since Φ_0 is bounded, we get

$$\int_{\mathscr{D}} |\bar\partial u|^2 e^{t|z|^2} \geq \frac{a_4}{\delta^2}\int_A |u|^2 e^{t|z|^2} + a_5 t \int_{\mathscr{D}} |u|^2 e^{t|z|^2}. \tag{6.2.3}$$

Define the holomorphic function $P(z) := \prod_\nu (z - z_\nu)$, and replace $u \mapsto u/P$. Then

$$\bar\partial\Big(\frac{u}{P}\Big) = \frac{\bar\partial u P - u\bar\partial P}{P^2} = \frac{\bar\partial u}{P}.$$

Since $u/P \in C_0^\infty(\mathscr{D})$, (6.2.3) becomes

$$\int_{\mathscr{D}} |\bar\partial u|^2 |P|^{-2} e^{t|z|^2} \geq \frac{a_4}{\delta^2}\int_A |u|^2 |P|^{-2} e^{t|z|^2} + a_5 t \int_{\mathscr{D}} |u|^2 |P|^{-2} e^{t|z|^2}. \tag{6.2.4}$$

All of the above discussion is valid for $u : \mathscr{D} \to \mathbb{C}$. We now choose $f : \mathscr{D} \to \mathbb{R}$. We have $|\bar\partial f| = |\partial f| = |\nabla f|$. We choose $u = \partial f$, whence $\bar\partial u = \bar\partial\partial f = \frac{1}{4}\Delta f$, which yields

$$\int_{\mathscr{D}} |\Delta f|^2 |P|^{-2} e^{t|z|^2} \geq \frac{a_6}{\delta^2}\int_A |\nabla f|^2 |P|^{-2} e^{t|z|^2} + a_7 t \int_{\mathscr{D}} |\bar\partial f|^2 |P|^{-2} e^{t|z|^2}.$$

We work on the last integral. Applying Lemma 6.1.1 to $\Phi = |P|^{-2} e^{t|z|^2}$, we get

$$\int_{\mathscr{D}} |\bar\partial f|^2 |P|^{-2} e^{t|z|^2} \geq \frac{1}{4}\int_{\mathscr{D}} \big(\Delta \log\big(|P|^{-2} e^{t|z|^2}\big)\big)|f|^2 |P|^{-2} e^{t|z|^2}.$$

Also, $\log |P|^{-2} = -\log \prod_\nu |z - z_\nu|^2 = -\sum_\nu \log |z - z_\nu|^2$, whence

$$\Delta \log\big(|P|^{-2} e^{t|z|^2}\big) = -\sum_\nu \delta(z - z_\nu) + 4t,$$

where δ is the Dirac delta, meaning that the sum above vanishes on $\mathcal{D}$. Thus,

$$\int_{\mathcal{D}} |\bar{\partial} f|^2 |P|^{-2} e^{t|z|^2} \geq t \int_{\mathcal{D}} |f|^2 |P|^{-2} e^{t|z|^2}.$$

Finally, (6.2.4) becomes the desired Carleman estimate

$$\int_{\mathcal{D}} |\Delta f|^2 |P|^{-2} e^{t|z|^2} \geq \frac{a_6}{\delta^2} \int_A |\nabla f|^2 |P|^{-2} e^{t|z|^2} + a_7 t^2 \int_{\mathcal{D}} |f|^2 |P|^{-2} e^{t|z|^2}, \tag{C1}$$

which holds for any $f \in C_0^\infty\big(\mathbb{R}^2 \setminus \bigcup_\nu D_\nu(a)\big)$, with $\mathcal{D}$ a bounded open set such that $A \subset \mathcal{D} \subset \bigcup_\nu D_\nu(a)$.

7. Discussion

7.1. ***Higher dimensions.*** In this paper, we have studied eigenfunctions of the Laplace–Beltrami operator on closed C^∞ surfaces and have underlined a natural interpretation of Yau's conjecture in light of Theorem 1. Since the conjecture is expected to hold in any dimension, it is natural to ask:

Question 7.1.1. Does Theorem 1 hold for a compact, smooth manifold of dimension $n \geq 3$?

It seems reasonable to expect that the result holds in higher dimensions. On the one hand, as previously stated, Yau's conjecture on the size of nodal sets is formulated for manifolds of any dimension. On the other hand, some fundamental results for the growth exponents of eigenfunctions are known to hold in any dimension, most notably the Donnelly–Fefferman growth bound

$$\beta(\phi_\lambda, B) = \log \frac{\sup_B |\phi_\lambda|}{\sup_{\frac{1}{2}B} |\phi_\lambda|} \leq c\sqrt{\lambda}, \tag{7.1.2}$$

where B is any metric ball (see for instance [Donnelly and Fefferman 1988; Mangoubi 2013; Nazarov et al. 2005]). However, the approach we have used relies crucially on the reduction of an eigenfunction ϕ_λ to a planar solution F to a Schrödinger equation, a transformation made possible by the existence of local conformal coordinates, a fact that does not generalize in dimension $n \geq 3$. One would therefore need to follow a fundamentally different approach to prove a result in the spirit of Theorem 1 in that setting. Nazarov et al. [2005] give a simpler proof of the growth bound (7.1.2) in the setting of closed surfaces. A generalization of that proof in higher dimensions has been given by Mangoubi [2013], notably using a clever extension of eigenfunctions on an n-dimensional manifold M to harmonic functions on the $(n+1)$-dimensional manifold $M \times \mathbb{R}$ (see also [Lin 1991; Jerison and Lebeau 1999; Nazarov et al. 2005]). We believe that a similar treatment could be useful in attempting to generalize Theorem 1.

7.2. ***How to measure the growth: generalization to L^q norms.*** Our measure of the growth of eigenfunctions has been made through growth exponents defined on small metric disks on which we have taken the L^∞ norm. Indeed, we recall that

$$\beta_p(\lambda) = \log \frac{\sup_B |\phi_\lambda|}{\sup_{\alpha_0} B |\phi_\lambda|},$$

where B is a metric ball of small radius centered at $p \in M$. For $1 \leq q \leq \infty$, define the more general q-growth exponent $\beta_p^q(\lambda)$ of an eigenfunction ϕ_λ:

$$\beta_p^q(\lambda) := \log \frac{\|\phi_\lambda\|_{L^q(B)}}{\|\phi_\lambda\|_{L^q(\alpha_0 B)}},$$

where B is once again a suitably small metric ball centered at p. Notice that $\beta_p(\lambda) = \beta_p^\infty(\lambda)$. Consider the average of such quantities on the surface; that is, define

$$B^q(\lambda) := \frac{1}{\mathrm{Vol}(M)} \int_M \beta_p^q(\lambda)\, \mathrm{d}V_g,$$

and then ask:

Question 7.2.1. For which $q \in [1, \infty)$, if any, do we have the following analogue of Theorem 1:

$$cB^q(\lambda)\lambda^{1/2} \leq \mathcal{H}^1(Z_\lambda) \leq C(B^q(\lambda)\lambda^{1/2} + 1)?$$

Keeping our setting of closed surfaces, it would suffice to prove analogues of Theorems 2.1.1 and 3.1.1 for q-growth exponents of planar Schrödinger eigenfunctions to answer positively the last question, but there does not seem to be an obvious way to tackle this problem.

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