ANALYSIS & PDE

Volume 8 No. 2 2015

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ANALYSIS AND PDE
Vol. 8, No. 2, 2015
dx.doi.org/10.2140/apde.2015.8.257

SMOOTH PARAMETRIC DEPENDENCE OF ASYMPTOTICS OF THE SEMICLASSICAL FOCUSING NLS

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We consider the one-dimensional focusing (cubic) nonlinear Schrödinger equation (NLS) in the semiclassical limit with exponentially decaying complex-valued initial data, whose phase is multiplied by a real parameter. We prove smooth dependence of the asymptotic solution on the parameter. Numerical results supporting our estimates of important quantities are presented.

1. Introduction

We consider the semiclassical focusing nonlinear Schrödinger (NLS) equation

$$i\varepsilon\,\partial_t q + \varepsilon^2\,\partial_x^2 q + 2|q|^2 q = 0 \tag{1}$$

with the initial data

$$q(x,0) = A(x)e^{\frac{i\mu}{\varepsilon}S(x)}, \quad A(x), S(x) \in \mathbb{R}, \quad \mu \geq 0, \tag{2}$$

in the asymptotic limit $\varepsilon \to 0$. Equation (1) is a well-known integrable system [Zakharov and Shabat 1972], and a lot of work has been done on this initial value problem (see below). The focus of the present study is on the parameter μ in the exponent of the initial data. For the specific data

$$A(x) = -\operatorname{sech} x, \quad S'(x) = -\tanh x, \quad \mu \geq 0, \tag{3}$$

studied in [Kamvissis et al. 2003; Tovbis et al. 2004], the solution undergoes a transition at $\mu = 2$. When $\mu < 2$, the Lax spectrum contains discrete eigenvalues numbering $O\left(\frac{1}{\varepsilon}\right)$, each eigenvalue giving rise to a soliton in the solution, which thus consists of both a radiative and a solitonic part. When $\mu \geq 2$, the spectrum is purely continuous and the solution is purely radiative (absence of solitons). We prove that the local wave parameters (branch points of the Riemann surface that represents the asymptotic solution locally in space-time) vary smoothly with μ, even at the critical value $\mu = 2$. Indeed, numerical experiments have shown absence of any noticeable transition in the behavior of the branch points at the critical value [Miller and Kamvissis 1998]. Theorem 4.5 establishes this fact rigorously.

The reason μ deserves special attention as a perturbation parameter is twofold. First, at the value of $\mu = 2$ there is a phase transition in the nature of the solution (there is a solitonic part when $\mu < 2$; see below). Perturbing μ across this value allows continuation of the validity of rigorously derived asymptotics [Tovbis et al. 2006] from the region $\mu \geq 2$ to the region $\mu < 2$. Ab initio derivation of such

Venakides thanks the NSF for supporting this work under grants NSF DMS-0707488 and NSF DMS-1211638.
MSC2010: primary 37K15, 37K40; secondary 35P25.
Keywords: NLS, semiclassical limit, Riemann–Hilbert problems.

asymptotics in the region $\mu < 2$ would be technically more demanding. Also, μ is a singularity of the RH contour in a way that cannot be remedied by contour deformations. Such a difficulty is absent when the perturbation parameters are space and time variables x and t. Indeed, the methods of [Tovbis and Venakides 2009; 2010] are applied in this work, and amongst the surprises which allow the methods to apply is a collection of explicit formulae for dependence on μ summarized in Lemmas 4.1–4.3.

Essential mathematical difficulties are encountered in the solution of the initial value problem (1) and (2) in general, and (1) and (3) in particular.

(1) The calculation of the scattering data at $t = 0$ is extremely delicate, as seen from the work of Klaus and Shaw [2002].

(2) The linearizing Zakharov–Shabat eigenvalue problem [1972] is *not self-adjoint*. This is in contrast to the self-adjointness of the initial value problem for the small-dispersion Korteweg–de Vries (KdV) equation, in which a systematic steepest descent procedure was developed by Deift, Venakides and Zhou [Deift et al. 1997] for calculating the asymptotic solution (see also [Deift et al. 1994]). The approach in [Deift et al. 1997] extended the original steepest descent analysis of Deift and Zhou [1993] for oscillatory Riemann–Hilbert problems by adding to it the g-function mechanism. A systematic procedure then obtained the KdV solution, which consists of waves that are fully nonlinear. These waves are typically modulated. In other words, the oscillations are rapid, exhibiting wavenumbers and frequencies in the small spatiotemporal scale that vary in the large scale in accordance with *modulation equations*.

(3) The system of modulation equations in the form of a set of PDEs for the NLS equation exhibits complex characteristics [Forest and Lee 1986]. Posed naturally as an initial value problem, the system is thus *ill-posed* and modulated NLS waves are unstable. The instability to the large-scale spatio-temporal variation of the wave parameters (modulational instability) is the primary source of problems in nonlinear fiber optical transmission, which is governed by the NLS equation.

In spite of the modulational instability, there exist initial data with a particular combination of A and S that evolve into a profile of modulated waves. The ordered structure of modulated nonlinear waves was first observed numerically by Miller and Kamvissis [1998], for the initial data of (3) with $\mu = 0$ and values of ε that allowed them to implement the multisoliton NLS formulae. Miller and Kamvissis observed the phenomenon of wave breaking (see below) and the formation of more complex wave structures past the break in this work. Later numerical findings by Ceniseros and Tian [2002], as well as by Cai, D. W. McLaughlin, and K. D. T.-R. McLaughlin [Cai et al. 2002], also detected the ordered structures.

These studies were followed by analytic work of Kamvissis, K. D. T.-R. McLaughlin and Miller [Kamvissis et al. 2003] for the same initial data ($\mu = 0$), the corresponding initial scattering data having been earlier calculated explicitly by Satsuma and Yajima [1974]. This work set forth a procedure aiming at the analytic determination of the observed phenomena that would practically extend the steepest descent procedure cited above to the non-self-adjoint case. Following a similar approach, Tovbis, Venakides and Zhou derived these phenomena rigorously from the initial data (3) with $\mu > 0$ [Tovbis et al. 2004]; the initial scattering data were previously obtained by Tovbis and Venakides [2000]. The asymptotic calculation of the wave solution was with error of order ε for points in space-time that are off the break point and off the caustic curves (see below). In further work [Tovbis et al. 2006; 2007], the same authors also derived the long-time behavior of the asymptotic solution for $\mu \geq 2$ and generalized the prebreak analysis to a wide class of initial data. The detailed asymptotic behavior in a neighborhood of the first break point was derived recently by Bertola and Tovbis [2013].

The rigorous derivation of the mechanism of the second break remains an open problem. Using a combination of theoretical and numerical arguments, Lyng and Miller [2007] obtained significant insights for initial data (3) with $\mu = 0$ when the solution is an N-soliton, where $N = O\left(\frac{1}{\varepsilon}\right)$. In particular, they identified a mathematical mechanism for the second break, which depends essentially on the discrete nature of the spectrum of the N-soliton and turns out to differ from the mechanism of the first break.

The asymptotic solution for *shock initial data*,

$$A = \text{constant}, \quad S'(x) = \operatorname{sign} x, \quad \mu > 0, \tag{4}$$

was derived globally in time by Buckingham and Venakides [2007].

The work of the present paper relies on the determinant form of the modulation equations of the NLS obtained by Tovbis and Venakides [2009]. The modulation equations are transcendental equations, not differential equations, and thus the modulational instability does not hinder the analysis. Tovbis and Venakides utilized the determinant form to study the variation of the asymptotic procedure as parameters of the Riemann–Hilbert problem, in particular the spatial and the time variables that are parameters in the Riemann–Hilbert problem analysis, change. They proved [2010] that, in the case of a regular break, the nonlinear steepest descent asymptotics can be "automatically" continued through the breaking curve (however, the expressions for the asymptotic solution will be different on different sides of the curve). Although the results are stated and proven for the focusing NLS equation, they can be reformulated for AKNS systems, as well as for the nonlinear steepest descent method in a more general setting. The present paper examines the variation of the procedure with respect to the parameter μ and proves that the variation is smooth even as μ crosses the critical value $\mu = 2$.

1A. ***Background: n-phase waves, inverse scattering, and the Riemann–Hilbert Problem.*** In order to make the study accessible beyond the group of experts in the subject, we give an overview of our understanding of the phenomenology of the time evolution of the semiclassical NLS equation and the mathematics that represents this phenomenology.

In the ideal (and necessarily unstable) scenario, in which modulated wave profiles persist in space-time, so does the separation in two space-time scales. In the large scale, a set of boundaries (breaking or caustic curves) divides the space-time half-plane (x, t), $t > 0$, into regions. Inside each region, and in the leading order as $\varepsilon \to 0$, the solution is an n-phase wave ($n = 0, 1, 2, 3, \ldots$), with wave periods and wavelengths in the small scale. The wave parameters vary in the large scale. The increase in n occurs typically as a new phase is generated at a point in space-time, due, for example, to wave-breaking (more precisely, to avert wave-breaking) or to two existing phases coming together. The newly generated oscillatory phase spreads in space with finite speed and the trace of its fronts in space-time constitutes the set of breaking curves.

An n-phase NLS wave is a solution of (1) which exhibits a "carrier" plane wave and n nonlinearly interacting wave-phases that control its oscillating amplitude. The wave is characterized by a set of $2n+2$ real wave parameters: $n+1$ frequencies and $n+1$ wavenumbers. In the scenario discussed above, waves with periods and wavelengths of order $O(\varepsilon)$ constitute the small space-time scale. The boundaries separating phases in space-time exist in the large scale, which is of order $O(1)$. These boundaries play the role of nonlinear caustic curves. The analytic wave profile of an n-phase wave is given explicitly in terms of an elliptic ($n=1$) or hyperelliptic ($n>1$) Riemann theta function, derived from a compact Riemann surface of genus n. This is true not only for the NLS but for most of the integrable wave equations studied. The $2n+2$ branch points of the Riemann surface are the wave parameters of choice that determine the $n+1$ frequencies and $n+1$ wavenumbers. In the case of the NLS, the 0-phase wave is simply a plane wave.

The initial data (2) have the structure of a modulated 0-phase wave. As t, increasing from zero, reaches a value $t = t_{\text{break}}$, the $n=0$ initial phase breaks at a caustic point in space-time. As described above, a wave-phase of higher n emerges then and spreads in space. As time increases, the endpoints of the spatial interval of existence of the new phase define the two caustic curves in space-time that emanate from the break point. The eventual breaking of this new phase is called the *second break*. The mechanism of the second break is fundamentally different from that of the first.

The *analytic description of n-phase waves* [Belokolos et al. 1994] is in terms of an n-phase Riemann theta function. This is an n-fold Fourier series obtained by summing $\exp\{2\pi i\, \boldsymbol{z}\cdot\boldsymbol{m} + \pi i\,(B\boldsymbol{m},\boldsymbol{m})\}$ over the multi-integer $\boldsymbol{m}$. B is an $n\times n$ matrix with positive definite imaginary part that gives the series exponential quadratic convergence. In the case of NLS waves, the matrix B arises from periods of the Riemann surface of the radical

$$R(z) = \left(\prod_{i=0}^{2n+1}(z-\alpha_i)\right)^{\frac{1}{2}}, \quad \text{where } \alpha_{2j+1} = \overline{\alpha}_{2j}; \tag{5}$$

the elements of B are linear combinations of the hyperelliptic integrals

$$\oint \frac{z^k}{R(z)}, \quad k = 0, 1, \dots, n-1,$$

along appropriate closed contours on the Riemann surface of R [Belokolos et al. 1994]. The series has a natural quasiperiodic structure in the n complex arguments $\boldsymbol{z} = (z_1, \dots, z_n)$. Each z_j is linear in x and t and represents a nonlinear phase of the wave; the wavenumbers and frequencies are expressed in terms of hyperelliptic integrals of the radical R and are thus functions of the α_{2j}, whose status as preferred parameters is obvious from (5).

The semiclassical limit procedure gives the emergent wave structure described above *without* any a priori ansatz of such structure. The radical $R(z)$ and the wave parameters arise naturally in the procedure. As mentioned above, these wave structures are modulated in space-time. In the large space-time scale, the branch points α_i vary and their number experiences a jump across the breaking curves. The branch points are calculated from the modulation equations in determinant form (they are transcendental equations, not partial differential equations, and thus there is no ill-posedness at hand). The number of branch points is obtained with the additional help of sign conditions that are obtained in the procedure.

Overview of scattering, inverse scattering, and the Riemann–Hilbert problem (RHP). The NLS was solved by Zakharov and Shabat [1972], who discovered a Lax pair that linearizes it. The Lax pair consists of two ordinary differential operators, one in the spatial variable x and the second in the time variable t.

The first operator of the Lax pair is a Dirac-type operator that is not self-adjoint. The corresponding eigenvalue problem (Zakharov–Shabat) is a 2×2 first-order *linear* ODE, with independent variable x. The NLS solution $q(x,t,\varepsilon)$ plays the role of a scatterer, entering in the off-diagonal entries of the ODE matrix. *Scattering data* are defined for those values of the spectral parameter z that produce bounded (Zakharov–Shabat) solutions. This happens when z is real (these solutions are called scattering states) and at the discrete set of proper eigenvalues z_j (the eigenvalues come in complex-conjugate pairs; the normalized L^2 solutions are called bound states). The *reflection coefficient* $r(z)$, $z\in\mathbb{R}$ provides a connection between the asymptotic behaviors of the scattering states as $x\to\pm\infty$. The *norming constants* c_j, corresponding to the proper eigenvalues z_j, provide the asymptotic behavior of the bound states as $x\to+\infty$.

The second operator of the Lax pair evolves the scattering states and the bound states in time and is again a 2×2 linear ODE system. The holding of the NLS equation guarantees that this evolution involves the action of a time-dependent unitary operator. As a result, the spectrum of the first Lax operator remains constant in time and the scattering data evolve in time through multiplication by simple explicit exponential propagators. The continuous spectrum contributes *radiation* to the solution of the NLS. The bound states contribute *solitons*.

Zakharov and Shabat [1972] developed the inverse scattering procedure for deriving $q(x,t)$ at any (x,t) given the scattering data at $t=0$. In the modern approach initiated by Shabat [1976], the procedure is recast into a *matrix Riemann–Hilbert problem* for a 2×2 matrix on the complex plane of the spectral variable z. One needs to determine the matrix $m(z)$ that is analytic on the closed complex plane, off an oriented contour Σ, that consists of the real axis and of small circles surrounding the eigenvalues. Modulo multiplication of its columns by normalizing factors $e^{\pm ixz/\varepsilon}$, the matrix $m(z)$ (the unknown of the problem at $t>0$) is a judiciously specified fundamental matrix solution of the eigenvalue problem of the first operator of the Lax pair (Zakharov–Shabat eigenvalue problem). In order to determine the matrix $m(z)$, one is given a jump condition on the contour Σ, and a normalization condition at $z\to\infty$,

$$m(z)=\begin{pmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\to \text{identity, as } z\to\infty;\quad m_+(z)=m_-(z)V,\ \text{ when } z\in\Sigma. \tag{6}$$

The subscripts $\pm$ indicate limits taken from the left/right of the contour. The 2×2 matrix $V=V(z,x,t,\varepsilon)$, defined on the jump contour and referred to as the *jump matrix*, is nonsingular and encodes the scattering data (see below). The space-time variables x,t (and the semiclassical parameter ε) are parameters in the problem.

The solution to (1) is given by the simple formula

$$q(x,t,\varepsilon)=\lim_{z\to\infty} zm_{12}(z,x,t,\varepsilon). \tag{7}$$

The results and the calculations of this study are in the asymptotic limit of the semiclassical parameter $\varepsilon\to 0$.

1B. ***Background: the semiclassical limit $\varepsilon \to 0$.*** The Riemann–Hilbert approach is a major tool in the asymptotic analysis of integrable systems, as established with the discovery of the steepest descent method [Deift and Zhou 1993; 1995] and its extension through the g-function mechanism [Deift et al. 1997; Tovbis et al. 2004]. The asymptotic methods via the RHP approach also apply to orthogonal polynomial asymptotics [Deift 1999; Deift et al. 1999a] and to random matrices [Baik et al. 1999; Deift et al. 1999b; Duits and Kuijlaars 2009; Ercolani and McLaughlin 2003].

The semiclassical asymptotic analysis of the highly oscillatory RHP is similar in spirit to the steepest descent method for integrals in the complex plane. Throughout the analysis, the quantities x, t, and μ enter as parameters. The semiclassical analysis, performed with the aid of the g-function mechanism [Deift et al. 1997; Tovbis et al. 2004], is constructive. An undetermined function $g(z)$ is introduced in the RHP through a simple transformation of the independent matrix variable of the RHP. The contour of the RHP, itself an unknown, is partitioned (in a way to be determined) into two types of interlacing subarcs. The jump matrix is manipulated differently in the two subarc types. In one of them (main arcs), the jump matrix is factored in a certain way. In the other type (complementary arcs), it is factored differently or is left as is. The g-function mechanism then imposes on appropriate entries of the jump matrix factors the condition of constancy in the complex spectral variable combined with boundedness as $\varepsilon \to 0$, while imposing decay as $\varepsilon \to 0$ on other entries. The constancy conditions are equalities and the decay conditions are sign conditions that act on exponents, forcing the decay of the corresponding exponential entries. Put together, these conditions constitute a scalar RHP for the function $g(z)$ (or, as below, on its sister function $h(z)$). The contour of the RHP, its partitioning, and finally the functions $g(z)$ and $h(z)$ follow from the analysis of this scalar RHP. This procedure allows the peeling-off of the leading order solution of the original matrix RHP and leaves behind a matrix RHP for the error. This RHP is solvable with the aid of a Neumann series.

The formulae for the conditions obtained through the g-function mechanism have an intuitive interpretation that arises from (2D) potential theory in the complex plane of the spectral parameter z. The main question is to determine the equilibrium measure for an energy functional [Deift 1999; Kamvissis and Rakhmanov 2005; Simon 2011] that depends parametrically on the variables x and t. The support of the measure depends on x and t. For problems with self-adjoint Lax operators (e.g., the Korteweg–de Vries equation in the small dispersion limit), the support is on the real line (typically a set of intervals as in [Lax and Levermore 1983a; 1983b; 1983c; Venakides 1985]). In a general non-self-adjoint case the supports are in the complex plane. This is the case with the (focusing) NLS under study, whose spatial Lax operator, the Zakharov–Shabat system, is of Dirac type and is non-self-adjoint.

The conditions obtained through the g-function mechanism are exactly the variational conditions for the equilibrium measure. Deriving these conditions rigorously as such is highly taxing, especially in the non-self-adjoint case. This is not needed though. The conditions are used essentially as an ansatz in the RHP. As long as the calculation confirms the ansatz, the whole procedure is rigorous.

In the cases of the (focusing) NLS in the semiclassical limit that have been worked out so far [Boutet de Monvel et al. 2011; Buckingham and Venakides 2007; Lyng and Miller 2007; Tovbis et al. 2004; 2007], the support of the equilibrium measure is a finite union of arcs in the complex plane with complex-conjugate symmetry. Denote the endpoints of the "main arcs" (see below) by $\{\alpha_j\}_{j=0}^{N'}$, with some finite $N' \in \mathbb{N}$. The analysis leads naturally to the representation of these points as the roots of a monic polynomial, the square root of which is exactly the radical in (5). This radical, a finite-genus Riemann surface, constitutes the passage to the periodic structure of the local waveform. We refer to these endpoints henceforth as "branch points". It is necessary to establish the existence and the number of the branch points for each pair (x,t) as well as the existence of the arcs, which provide the leading contribution to the expression of the local waveform.

The approach to obtaining the asymptotic solution from the initial data is to analyze the RHP for fixed x and t, thus treating the space and time variables as parameters. The smooth dependence of the branch points α_j on the parameters x and t for the semiclassical NLS with $q(x,0) = \operatorname{sech}(x)$, $\mu = 0$ was studied in [Kamvissis et al. 2003, pp. 148–162] by considering moment conditions. A different approach, to start with local behavior near the α_j, was applied in [Tovbis and Venakides 2009], leading to formulae of the form

$$\frac{\partial \alpha_j}{\partial x}(x,t) = -\frac{2\pi i \, \dfrac{\partial K}{\partial x}(\alpha_j, \vec{\alpha}, x, t)}{D(\vec{\alpha}) \displaystyle\oint_{\hat{\gamma}} \frac{f'(\zeta, x, t)}{(\zeta - \alpha_j(x,t)) R(\zeta, \vec{\alpha})} d\zeta}, \tag{8}$$

$$\frac{\partial \alpha_j}{\partial t}(x,t) = -\frac{2\pi i \, \dfrac{\partial K}{\partial t}(\alpha_j, \vec{\alpha}, x, t)}{D(\vec{\alpha}) \displaystyle\oint_{\hat{\gamma}} \frac{f'(\zeta, x, t)}{(\zeta - \alpha_j(x,t)) R(\zeta, \vec{\alpha})} d\zeta}, \tag{9}$$

where $\vec{\alpha} = \vec{\alpha}(x,t)$.

In addition to [Tovbis and Venakides 2009; 2010], this work is similar in spirit to [Kuijlaars and McLaughlin 2000], which pertains to random matrix theory. Those authors put a strong topology on the set of allowable potentials (a potential is analogous to the function f in the present work) and demonstrate that the so-called "regular case" is generic — i.e., if you find a potential with fixed genus and all other side-conditions are satisfied in a strict sense, then the same genus holds for all potentials in a neighborhood.

In this study, we obtain the following main results.

(1) We extend formulae (8) and (9) to include the dependence of the branch points and of the contour $\hat{\gamma} = \hat{\gamma}(\mu)$ on the external parameter μ,

$$\frac{\partial \alpha_j}{\partial \mu}(x,t,\mu) = -\frac{2\pi i \, \dfrac{\partial K}{\partial \mu}(\alpha_j, \vec{\alpha}, x, t, \mu)}{D(\vec{\alpha}) \displaystyle\oint_{\hat{\gamma}(\mu)} \frac{f'(\zeta, x, t, \mu)}{(\zeta - \alpha_j(x,t,\mu)) R(\zeta, \vec{\alpha})} d\zeta}, \tag{10}$$

where $\vec{\alpha} = \vec{\alpha}(x,t,\mu)$. We show that the dependence is smooth, meaning that the contour, the jump matrix, and the solution of the scalar RHP evolve smoothly (they are continuously differentiable) in μ.

(2) We simplify the expression for $\dfrac{\partial K}{\partial \mu}$ in (43).

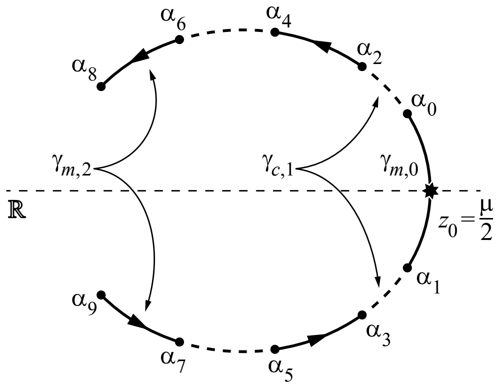

Figure 1. The RHP jump contour in the case of genus 4 with complex-conjugate symmetry in the notation of [Tovbis et al. 2004].

(3) We show good agreement of formula (10) with the dependence of the branch points on the parameter μ, obtained by the direct numerical solution of the system (19) (see Figure 3).

(4) We prove the preservation of genus of the asymptotic solution in an open interval of parameter μ. In particular, the genus is preserved (0 or 2) for all x and $t > 0$ for some open interval (which depends on x and t) for $\mu < 2$.

This paper is organized as follows. Section 2 contains definitions and prior results. Section 3 discusses the analyticity of f in μ and the differentiability of the branch points $\alpha_j = \alpha_j(\mu)$. Section 4 studies the μ-dependence of the quantities that appear in Theorem 4.5. Section 5 is devoted to sign conditions and the preservation of genus (Theorem 5.8). Section 6 provides numerical support for the main result of the paper. The Appendix supplies explicit formulations of all relevant expressions and the main result in the genus 0 and genus 2 cases.

2. Preliminaries

We consider a model scalar Riemann–Hilbert problem which arises in the process of the asymptotic solution of the semiclassical focusing NLS (1) with the initial condition (3). The input to the problem is a given function $f(z)$ that derives originally from the asymptotic limit of the scattering data for this initial value problem. The function $f(z)$ (see (21)) depends parametrically on the space and time variables, x and t. It also depends on the real parameter μ in the initial data of the NLS (3). The following properties of the function $f(z)$ are crucial for our calculations.

(1) $f(z)$ is analytic at all points of $\mathbb{C} \setminus \mathbb{R}$ except for branch cuts.

(2) $f(z)$ has a $z \ln z$ singularity at the point $z = \mu/2$.

(3) $f(z)$ is Schwarz-symmetric.

Other functions that satisfy these conditions are a priori admissible as inputs to our model problem; whether they too lead to solutions is a matter of calculation.

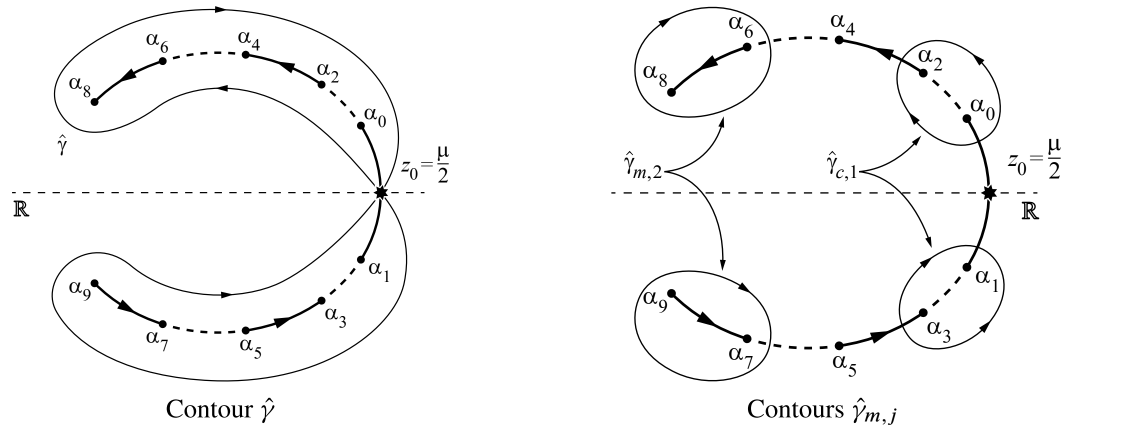

Figure 2. Contours of integration for the function $h(z)$ of (14). The point $z_0 = \mu/2$ is a point of nonanalyticity of $f(z)$ on γ.

The unknown of the problem is a function $h(z)$ that has the following properties.

(1) The function $h(z)$ is Schwarz-reflexive.

(2) The function $h(z)$ compensates for the points of nonanalyticity of $f(z)$ in the sense that $h + f$ is analytic at these points with only one exception, the point $z = \mu/2$.

(3) The function $h + f$ is analytic in $\overline{\mathbb{C}} \setminus \gamma$, where γ is a contour to be determined that passes through point $z = \mu/2$ and is symmetric with respect to the real axis.

(4) The function $h(z)$ exhibits constant (independent of z) jumps across subarcs of the contour γ. This is made more specific below.

Remark. A cleaner formulation of the RHP could be achieved with the unknown $f + h$ instead of h. Indeed, the function

$$g(z) = \tfrac{1}{2}(h(z) + f(z))$$

jumps only across the contour γ, providing a cleaner RHP formulation. Yet the results are in terms of the function $h(z)$; hence our choice in its favor.

Remark. Based on the previous remark, we still refer to the contour γ as the contour of the RHP. Since the contour itself is one of the unknowns, we refer to the problem as a "free contour RHP", in analogy to the well-known "free boundary value problems".

We now set the precise conditions for the contour γ and the jumps across it.

(1) The contour γ is a finite-length, non-self-intersecting arc that is symmetric with respect to the real axis. It intersects the real axis at a point $z = \mu/2$ (we refer to this point as z_0) to be discussed below.

(2) The contour γ is oriented from its endpoint in the lower complex half-plane to its endpoint in the upper half-plane.

(3) For some integer N, we consider $2N+1$ distinct points of the contour in the upper half-plane, including the contour endpoint; we also consider their complex conjugates in the lower half-plane. We label the points in the upper half-plane with even indices $\{\alpha_{2i}\}_{i=0}^{2N}$ that increase in the direction of orientation of the contour and we label the points in the lower half-plane with odd indices $\{\alpha_{2i+1}\}_{i=0}^{2N}$ that decrease in the direction of orientation. Clearly, the sequence of points in the direction of orientation are

$$\underbrace{\alpha_{4N+1}, \alpha_{4N-1}, \alpha_{4N-3}, \dots, \alpha_3, \alpha_1}_{\text{lower half-plane}}, \underbrace{\alpha_0, \alpha_2, \dots, \alpha_{4N-4}, \alpha_{4N-2}, \alpha_{4N}}_{\text{upper half-plane}},$$

α_{4N+1} and α_{4N} are the endpoints of the contour γ, and

$$\alpha_{2i+1} = \overline{\alpha_{2i}}, \quad i = 0, 1, 2, \dots, 2N. \tag{11}$$

The jumps of the RHP are defined on the arcs into which the contour is partitioned by these points. Two alternative types of RHP jumps are imposed; each arc is labeled as a main arc or a complementary arc, respectively. The two arc types interlace along the contour and the contour has main arcs at both ends. All arcs inherit their orientation from the contour γ.

It is trivial to check that arcs which are complex-conjugate to each other are either both main or both complementary. It is convenient to lump such an arc pair into one entity; in the following definitions, we retain the terms *main arc* and *complementary arc* for such arc pairs, by abuse of vocabulary.

(1) We define as main arcs $\gamma_{m,j}$, where $j = 0, 1, \dots, N$:

$$\gamma_{m,0} = [\alpha_1, \alpha_0], \quad \gamma_{m,j} = [\alpha_{4j-2}, \alpha_{4j}] \cup [\alpha_{4j+1}, \alpha_{4j-1}], \quad j = 1, \dots, N.$$

Thus, a main arc consists of a single arc when $j = 0$ and a union of two arcs when $j > 0$.

(2) We define as complementary arcs $\gamma_{c,j}$, where $j = 1, \dots, N$:

$$\gamma_{c,j} = [\alpha_{4j-4}, \alpha_{4j-2}] \cup [\alpha_{4j-1}, \alpha_{4j-3}], \quad j = 1, \dots, N.$$

The jump conditions of the RHP are given by

$$\begin{cases} h_+(z) + h_-(z) = 2W_j & \text{on } \gamma_{m,j}, \ j = 0, 1, \dots, N, \\ h_+(z) - h_-(z) = 2\Omega_j & \text{on } \gamma_{c,j}, \ j = 1, \dots, N, \\ h(z) + f(z) \text{ is analytic in } \overline{\mathbb{C}} \setminus \gamma, \end{cases} \tag{12}$$

where W_j and Ω_j are real constants with a normalization $W_0 = 0$.

We end our formulation of the free contour RHP for the function $h(z)$ by reiterating what the knowns and what the unknowns of the problem are. The contour γ is unknown, except for the requirement of passing through the singular point $z_0 = \mu/2$. The positions of the $4N+2$ partitioning points are unknown. As we have formulated the problem so far, the value of the integer N is free. This freedom is lifted if important additional conditions ("sign conditions") are imposed on the RHP, as occurs in the case of NLS [Tovbis et al. 2004]. The sign conditions guarantee the decay of certain jump matrix entries. In the presence of these conditions, the number of points turns into an important unknown; the n-phase wave represented has $n = 2N$. Finally, the real constants in the jump conditions are unknown. To summarize, the only known data is the function $f(z; \mu, \beta)$.

Our main concern is the dependence of the solution of the problem on the parameter μ. Any other parameters (space and time if the RHP arises from the focusing NLS) are collectively labeled β. A multiparameter family of such functions f will be discussed below. The specific function f that corresponds to the focusing NLS equation with initial data (3) is given in the beginning of Section 3.

Definition 2.1. Let

$$\vec{\alpha} = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{4N+1}\}. \tag{13}$$

We say

$$\gamma \in \Gamma(\vec{\alpha}, \mu)$$

if a contour γ satisfies all the conditions set above and shown in Figure 1.

Note that for fixed $\vec{\alpha}$ and μ, the contour γ, aside from passing through $z = \alpha_j$ and $z = \mu/2$, is free to deform continuously within a domain of analyticity of f. Thus for a fixed f, the notation $\gamma = \gamma(\vec{\alpha}, \mu)$ may indicate the general element of the set $\Gamma(\vec{\alpha}, \mu)$. The following lemma is an immediate consequence of our definitions.

Lemma 2.2. *Consider $\gamma_0 = \gamma_0(\vec{\alpha}_0, \mu_0) \in \Gamma(\vec{\alpha}_0, \mu_0)$.*

There exist open neighborhoods of $\vec{\alpha}_0$ and μ_0 such that for all $\vec{\alpha}$ in the neighborhood of $\vec{\alpha}_0$, for all μ in the neighborhood of μ_0 there is a contour $\gamma(\vec{\alpha}, \mu) \in \Gamma(\vec{\alpha}, \mu)$.

Definition 2.3. We say that

$$\hat{\gamma} \in \hat{\Gamma}(\gamma, \vec{\alpha}, \mu)$$

if $\hat{\gamma}$ is a non-self-intersecting closed contour around $\gamma \in \Gamma(\vec{\alpha}, \mu)$ within the domain of analyticity of f except at $z_0 = \mu/2$, with complex-conjugate symmetry $\overline{\hat{\gamma}} = \hat{\gamma}$.

We define $\hat{\gamma}_{m,j}$ and $\hat{\gamma}_{c,j}$ similarly.

Remark 2.4. By considering the loop contours $\hat{\gamma}$, $\hat{\gamma}_{m,j}$, $\hat{\gamma}_{c,j}$, the explicit dependence of the contours on the end points $\vec{\alpha}$ is removed (for example in (32)–(35)). So even though $\gamma = \gamma(\vec{\alpha}, \mu)$, in all our evaluations below $\hat{\gamma} = \hat{\gamma}(\mu)$.

Remark 2.5. Lemma 2.2 implies that if $\hat{\gamma}_0 \in \hat{\Gamma}(\gamma_0, \vec{\alpha}_0, \mu_0)$, then there is a contour $\gamma \in \Gamma(\vec{\alpha}, \mu)$ such that $\hat{\gamma} \in \hat{\Gamma}(\gamma, \vec{\alpha}, \mu)$ for all $\vec{\alpha}$ and μ in some open neighborhoods of $\vec{\alpha}_0$ and μ_0.

Definition 2.6. We denote the RHP (12) as

$$\mathrm{RHP}(\gamma, \vec{\alpha}, \mu, f),$$

where $\gamma \in \Gamma(\vec{\alpha}, \mu)$.

The solution of the RHP (12), $h(z)$, can be found explicitly [Tovbis et al. 2004]:

$$h(z)=\frac{R(z)}{2\pi i}\left[\oint_{\hat\gamma}\frac{f(\zeta)}{(\zeta-z)R(\zeta)}\,d\zeta+\sum_{j=1}^{N}\oint_{\hat\gamma_{m,j}}\frac{W_j}{(\zeta-z)R(\zeta)}\,d\zeta+\sum_{j=1}^{N}\oint_{\hat\gamma_{c,j}}\frac{\Omega_j}{(\zeta-z)R(\zeta)}\,d\zeta\right], \tag{14}$$

or, in determinant form [Tovbis and Venakides 2009],

$$h(z)=\frac{R(z)}{D}K(z), \tag{15}$$

where z lies inside of $\hat\gamma$ and outside all $\hat\gamma_{c,j}$ and $\hat\gamma_{m,j}$, and where

$$K(z)=\frac{1}{2\pi i}\begin{vmatrix}\oint_{\hat\gamma_{m,1}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{m,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{m,1}}\frac{d\zeta}{(\zeta-z)R(\zeta)}\\ \vdots & \ddots & \vdots & \vdots\\ \oint_{\hat\gamma_{m,N}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{m,N}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{m,N}}\frac{d\zeta}{(\zeta-z)R(\zeta)}\\ \oint_{\hat\gamma_{c,1}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{c,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{c,1}}\frac{d\zeta}{(\zeta-z)R(\zeta)}\\ \vdots & \ddots & \vdots & \vdots\\ \oint_{\hat\gamma_{c,N}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{c,N}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{c,N}}\frac{d\zeta}{(\zeta-z)R(\zeta)}\\ \oint_{\hat\gamma}\frac{f(\zeta)\,d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma}\frac{\zeta^{N-1}f(\zeta)\,d\zeta}{R(\zeta)} & \oint_{\hat\gamma}\frac{f(\zeta)\,d\zeta}{(\zeta-z)R(\zeta)}\end{vmatrix} \tag{16}$$

and

$$D=\det(A), \tag{17}$$

with

$$A=\begin{pmatrix}\oint_{\hat\gamma_{m,1}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{m,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)}\\ \vdots & \ddots & \vdots\\ \oint_{\hat\gamma_{m,N}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{m,N}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)}\\ \oint_{\hat\gamma_{c,1}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{c,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)}\\ \vdots & \ddots & \vdots\\ \oint_{\hat\gamma_{c,N}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{c,N}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)}\end{pmatrix}. \tag{18}$$

The arc end points $\{\alpha_j\}$ satisfy the system

$$K(\alpha_j)=0,\quad j=0,1,\dots,4N+1. \tag{19}$$

The dependence on x and t was considered in [Tovbis and Venakides 2009]. This is a simpler situation when the jump contour γ in the RHP (12) is independent of the parameters.

The main related results in [Tovbis and Venakides 2009] are the determinant form (15) and:

Theorem 2.7. *Let $f(z,\vec{\beta})$, where $\vec{\beta}\in B\subset\mathbb{R}^m$. For all $\vec{\beta}\in B$ assume $f(z,\vec{\beta})$ is analytic on $S\in\mathbb{C}$. Moreover, $\gamma\setminus S$ consists of no more than finitely many points and f is continuous on γ. The modulation equations* (19) *imply the system of* $4N+2$ *differential equations*

$$(\alpha_j)_{\beta_k} = -\frac{2\pi i\frac{\partial}{\partial\beta_k}K(\alpha_j)}{D\oint_{\hat{\gamma}}\frac{f'(\zeta)}{(\zeta-\alpha_j)R(\zeta)}d\zeta}. \tag{20}$$

In particular, one gets (8) and (9) for parameters x and t. Note that the contour γ is assumed independent of parameters x and t explicitly. The dependence on these parameters comes in through the branch points $\vec{\alpha}=\vec{\alpha}(x,t)$.

The main related result in [Tovbis and Venakides 2010] is:

Theorem 2.8. *Let the nonlinear steepest descent asymptotics for solution $q(x,t,\varepsilon)$ of the NLS* (1) *be valid at some point (x_0,t_0). If (x_*,t_*) is an arbitrary point, connected with (x_0,t_0) by a piecewise-smooth path Σ, if the contour $\gamma(x,t)$ of the RHP* (12) *does not interact with singularities of $f(z)$ as (x,t) varies from (x_0,t_0) to (x_*,t_*) along Σ, and if all the branch points are bounded and stay away from the real axis, then the nonlinear steepest descent asymptotics* (*with the proper choice of genus*) *are also valid at (x_*,t_*).*

We extend Theorem 2.7 and make partial progress in the direction of Theorem 2.8 in the case when the jump contour explicitly depends on the parameter μ. We require that the point of logarithmic singularity of $f=f(z,\mu)$, $z_0=\mu/2$ be always on γ. Additionally, we prove preservation of genus for all $x>0$, $t>0$, $\mu>0$, under certain conditions which guarantee that the parameters are away from asymptotic solution breaks (see Theorem 5.8). In particular, the genus is preserved in a neighborhood of the special value of the parameter $\mu=2$. Thus we obtain that for all $x>0$, $t>0$ (except on the first breaking curve), there is a small neighborhood such that for all $\mu<2$ in the neighborhood, the genus is the same as for $\mu=2$, where it is known to be 0 or 2.

3. μ-dependence in the semiclassical focusing NLS

To apply the methods from [Tovbis and Venakides 2009] we need analyticity of $f(z,\mu)$ in the parameter μ.

The function $f(z)$, obtained in [Tovbis et al. 2004] from a semiclassical approximation of the exactly derived scattering data for the NLS with initial condition (3) [Tovbis and Venakides 2000], is given by

$$f(z,\mu,x,t)=\left(\frac{\mu}{2}-z\right)\left[\frac{\pi i}{2}+\ln\left(\frac{\mu}{2}-z\right)\right]+\frac{z+T}{2}\ln(z+T)+\frac{z-T}{2}\ln(z-T)$$
$$-T\tanh^{-1}\frac{T}{\mu/2}-xz-2tz^2+\frac{\mu}{2}\ln 2\quad\text{when }\Im z\geq 0, \tag{21}$$

and

$$f(z)=\overline{f(\overline{z})}\quad\text{when }\Im z<0, \tag{22}$$

where the branch cuts are chosen as follows: for $0<\mu<2$, the logarithmic branch cuts are from $z=\mu/2$ along the real axis to $+\infty$, from $z=T$ to 0 and along the real axis to $+\infty$, and from $z=-T$ to 0 and along the real axis to $-\infty$; for $\mu\geq 2$, the branch cuts are from $z=T$ to $+\infty$ and from $z=-T$ to $-\infty$ along the real axis, where

$$T=T(\mu)=\sqrt{\frac{\mu^2}{4}-1},\quad \Im T\geq 0. \tag{23}$$

For $\mu\geq 2$, $T\geq 0$ is real and for $0<\mu<2$, T is purely imaginary with $\Im T>0$.

Lemma 3.1. *$f(z,\mu)$ and $f'(z,\mu)$ are analytic in μ for $\mu>0$, $x>0$, $t>0$, for all z, $\Im z\neq 0$, $z\notin[-T,T]$.*

Proof. Consider

$$f'(z,\mu)=-\frac{\pi i}{2}-\ln\left(\frac{\mu}{2}-z\right)+\frac{1}{2}\ln\left(z^2-\frac{\mu^2}{4}+1\right)-x-4tz, \tag{24}$$

which is analytic in $\mu>0$, for $\Im z\neq 0$, $z\notin[-T,T]$.

For $\mu>0$, $\mu\neq 2$, $f(z,\mu)$ is clearly analytic in μ for $\Im z\neq 0$. At $\mu=2$ ($T=0$) we find the power series of $f(z,\mu)$ in T and show that it contains only even powers. Since

$$T^{2k}=\left(\frac{\mu^2}{4}-1\right)^k=\frac{(\mu+2)^k}{4^k}(\mu-2)^k, \tag{25}$$

it will show analyticity of $f(z,\mu)$ in μ.

Start with expanding basic terms in series at $T=0$:

$$\frac{1}{\mu/2}=\sqrt{1+T^2}^{\,-1}=\sum_{k=0}^{\infty}c_kT^{2k},\quad \ln(z\pm T)=z\ln z\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\pm\frac{T}{z}\right)^n. \tag{26}$$

Then the logarithmic terms in (21) become

$$\begin{aligned}\frac{z+T}{2}\ln(z+T)+\frac{z-T}{2}\ln(z-T)&=z\ln z-z\sum_{n\text{ is even}}\frac{1}{n}\left(\frac{T}{z}\right)^n+T\sum_{n\text{ is odd}}\frac{1}{n}\left(\frac{T}{z}\right)^n\\&=z\ln z+\sum_{k=1}^{\infty}\frac{1}{2k(2k-1)z^{2k-1}}T^{2k},\end{aligned} \tag{27}$$

which has only even powers of T and is analytic in μ for $\Im z\neq 0$. Next we consider the inverse hyperbolic tangent term in (21), and taking into account that $\tanh^{-1}z$ is an odd function,

$$\begin{aligned}T\tanh^{-1}\frac{T}{\mu/2}&=T\tanh^{-1}\frac{T}{\sqrt{1+T^2}}=T\tanh^{-1}\sum_{k=0}^{\infty}c_kT^{2k+1}\\&=T\sum_{k=0}^{\infty}\tilde{c}_kT^{2k+1}=\sum_{k=0}^{\infty}\tilde{c}_kT^{2k+2},\end{aligned} \tag{28}$$

which also has only even powers of T.

So $f(z,x,t,\mu)$ is analytic in μ for $\mu>0$, $x>0$, $t>0$, $\Im z\neq 0$, $z\notin[-T(\mu),T(\mu)]$. ☐

Then, for $\Im z > 0$,

$$\frac{\partial f}{\partial \mu}(z,\mu) = \frac{\pi i}{4} + \frac{1}{2}\ln\left(\frac{\mu}{2} - z\right) + \ln 2 + \frac{\mu}{8T}\left[\ln(z+T) - \ln(z-T) - 2\tanh^{-1}\frac{2T}{\mu}\right], \tag{29}$$

where $\tanh^{-1} x = x + O(x^3)$, as $x \to 0$; then

$$\frac{\partial f}{\partial \mu}(z,\mu) = \frac{\pi i}{4} + \frac{1}{2}\ln\left(\frac{\mu}{2} - z\right) + \ln 2 + \frac{\mu}{4z} - \frac{1}{2} + O(T),\ T \to 0. \tag{30}$$

So $\mu = 2$ is a removable singularity for $f_\mu(z,\mu)$ and

$$\lim_{\substack{\mu\to 2\\ T\to 0}} \frac{\partial f}{\partial \mu}(z,\mu) = \frac{\pi i}{4} + \frac{1}{2}\ln(1-z) + \ln 2 + \frac{1}{2z} - \frac{1}{2}, \tag{31}$$

which is analytic in z for $\Im z \neq 0$.

Remark 3.2. The jump of $f(z)$ is caused by the Schwarz reflection (22) on the real axis and is linear in z since $\Im f$ is a linear function on the real axis (as a limit) near $\mu/2$ with $\Im f(\mu/2) = 0$ [Tovbis et al. 2004].

4. Parametric dependence of the scalar RHP

The main difficulty is the dependence of $f(z)$ (thus the RHP (12)) and the modulation equations (19) on parameter μ, which also controls the logarithmic branch point $z = \mu/2$ on the contour $\hat\gamma$. We show that the dependence on μ is smooth.

To solve $\vec K(\vec\alpha,\mu) = \vec 0$, we need nondegeneracy of the system and apply the implicit function theorem. The following technical lemma simplifies expressions for partial derivatives in μ of (14) and (16).

Lemma 4.1. *Let the function f be given by* (21), *and consider a contour $\gamma_0 = \gamma(\vec\alpha,\mu_0) \in \Gamma(\vec\alpha,\mu_0)$ having fixed arc end points $\vec\alpha$. There is an open neighborhood of μ_0 such that for all μ in the neighborhood of μ_0, there is $\hat\gamma(\mu) \in \hat\Gamma(\gamma,\vec\alpha,\mu)$, and for all $j = 0,1,\dots,4N+1$, $n \in \mathbb{N}$,*

$$\frac{\partial}{\partial\mu}\oint_{\hat\gamma(\mu)} \frac{\zeta^n f(\zeta,\mu)\,d\zeta}{R(\zeta,\vec\alpha)} = \oint_{\hat\gamma(\mu)} \frac{\zeta^n \frac{\partial f(\zeta,\mu)}{\partial\mu}\,d\zeta}{R(\zeta,\vec\alpha)}, \tag{32}$$

$$\frac{\partial}{\partial\mu}\oint_{\hat\gamma(\mu)} \frac{f(\zeta,\mu)\,d\zeta}{(\zeta-\alpha_j)R(\zeta,\vec\alpha)} = \oint_{\hat\gamma(\mu)} \frac{\frac{\partial f(\zeta,\mu)}{\partial\mu}\,d\zeta}{(\zeta-\alpha_j)R(\zeta,\vec\alpha)}, \tag{33}$$

$$\frac{\partial}{\partial\mu}\oint_{\hat\gamma_{m,k}} \frac{\zeta^n d\zeta}{R(\zeta,\vec\alpha)} = 0,\quad \frac{\partial}{\partial\mu}\oint_{\hat\gamma_{m,k}} \frac{d\zeta}{(\zeta-\alpha_j)R(\zeta,\vec\alpha)} = 0,\ k = 1,2,\dots,N, \tag{34}$$

$$\frac{\partial}{\partial\mu}\oint_{\hat\gamma_{c,k}} \frac{\zeta^n d\zeta}{R(\zeta,\vec\alpha)} = 0,\quad \frac{\partial}{\partial\mu}\oint_{\hat\gamma_{c,k}} \frac{d\zeta}{(\zeta-\alpha_j)R(\zeta,\vec\alpha)} = 0,\ k = 1,2,\dots,N. \tag{35}$$

Proof. The idea of the proof is to consider finite differences and take the limit as $\Delta\mu \to 0$. The complication is that both the integrands and the contours of integration depend on μ.

Denote the integral on the left in (32) by I_1:

$$I_1(\mu) = \oint_{\hat{\gamma}(\mu)} \frac{\zeta^n f(\zeta,\mu)}{R(\zeta,\vec{\alpha})}\, d\zeta, \tag{36}$$

where $\hat{\gamma}(\mu) \in \hat{\Gamma}(\gamma,\vec{\alpha},\mu)$. Consider

$$\frac{I_1(\mu+\Delta\mu) - I_1(\mu)}{\Delta\mu}, \tag{37}$$

with small real $\Delta\mu \neq 0$. There are two logarithmic branch cuts near the contours of integration, in $f(z,\mu)$ and in $f(z,\mu+\Delta\mu)$, with both branch cuts chosen from $z_0(\mu) = \mu/2$ and $z_0(\mu+\Delta\mu)$ horizontally to the right along the real axis. Additionally, these functions have a jump on the real axis for $z < \mu/2$ from Schwarz symmetry.

We choose some fixed points δ_1 and δ_2 to be real, satisfying

$$\delta_1 < \frac{\mu}{2} - \frac{|\Delta\mu|}{2} < \frac{\mu}{2} + \frac{|\Delta\mu|}{2} < \delta_2. \tag{38}$$

Both contours of integration $\hat{\gamma}(\mu)$, $\hat{\gamma}(\mu+\Delta\mu)$ are pushed to the real axis near z_0 and split into

$$[\delta_1,\delta_2] := [\delta_1 + i0, \delta_2 + i0] \cup [\delta_2 - i0, \delta_1 - i0] \tag{39}$$

and its complement. On the complement, we can also deform both contours to coincide. So $\hat{\gamma}(\mu+\Delta\mu) = \hat{\gamma}(\mu)$.

Across $[\delta_1,\delta_2]$, $f(z,\mu)$ has a jump $\pi i|z_0(\mu) - z|$ and $f(z,\mu+\Delta\mu)$ has a jump $\pi i|z_0(\mu+\Delta\mu) - z|$. So contributions near z_0 in both cases are small.

Then

$$\frac{I_1(\mu+\Delta\mu) - I_1(\mu)}{\Delta\mu} = \frac{1}{\Delta\mu}\left(\oint_{\hat{\gamma}(\mu+\Delta\mu)} \frac{\zeta^n f(\zeta,\mu+\Delta\mu)}{R(\zeta,\vec{\alpha})}\, d\zeta - \oint_{\hat{\gamma}(\mu)} \frac{\zeta^n f(\zeta,\mu)}{R(\zeta,\vec{\alpha})}\, d\zeta\right); \tag{40}$$

we add and subtract $\displaystyle\oint_{\hat{\gamma}(\mu)} \frac{\zeta^n f(\zeta,\mu+\Delta\mu)}{R(\zeta,\vec{\alpha})}\, d\zeta$, giving

$$(40) = \frac{1}{\Delta\mu}\left(\oint_{\hat{\gamma}(\mu+\Delta\mu)-\hat{\gamma}(\mu)} \frac{\zeta^n f(\zeta,\mu+\Delta\mu)}{R(\zeta,\vec{\alpha})}\, d\zeta + \oint_{\hat{\gamma}(\mu)} \frac{\zeta^n (f(\zeta,\mu+\Delta\mu) - f(\zeta,\mu))}{R(\zeta,\vec{\alpha})}\, d\zeta\right). \tag{41}$$

The first integral is 0, because $\hat{\gamma}(\mu+\Delta\mu) = \hat{\gamma}(\mu)$.

Thus

$$\frac{I_1(\mu+\Delta\mu) - I_1(\mu)}{\Delta\mu} = \oint_{\hat{\gamma}(\mu)} \frac{\zeta^n \frac{f(\zeta,\mu+\Delta\mu) - f(\zeta,\mu)}{\Delta\mu}}{R(\zeta,\vec{\alpha})}\, d\zeta. \tag{42}$$

The last step is to take the limit as $\Delta\mu \to 0$ and to interchange it with the integral. The contour of integration is split into two: a small neighborhood near z_0 and its complement. For the integral near z_0, by a direct computation it can be shown that the limit can be passed under the integral. The integral over the second part of the contour has the integrand uniformly bounded in μ, since $\log(\zeta-\mu/2)$ in $\partial f/\partial\mu$ is uniformly bounded away from $\mu/2$, so the limit and the integral can be interchanged. This completes the proof for the first integral (32).

The second integral (33) is done similarly. The rest of the integrals (34), (35) are independent of μ since the only dependence on μ sits in $z_0(\mu)\in\gamma_{m,0}$. $\square$

Using Lemma 4.1,

$$\frac{\partial K}{\partial\mu}(\alpha_j,\vec{\alpha},\mu)=\frac{1}{2\pi i}\begin{vmatrix} \oint_{\hat{\gamma}_{m,1}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,1}}\frac{d\zeta}{(\zeta-\alpha_j)R(\zeta)} \\ \vdots & \ddots & \vdots & \vdots \\ \oint_{\hat{\gamma}_{m,N}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,N}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,N}}\frac{d\zeta}{(\zeta-\alpha_j)R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,1}}\frac{d\zeta}{(\zeta-\alpha_j)R(\zeta)} \\ \vdots & \ddots & \vdots & \vdots \\ \oint_{\hat{\gamma}_{c,N}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,N}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,N}}\frac{d\zeta}{(\zeta-\alpha_j)R(\zeta)} \\ \oint_{\hat{\gamma}}\frac{f_\mu(\zeta)\,d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}}\frac{\zeta^{N-1}f_\mu(\zeta)\,d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}}\frac{f_\mu(\zeta)\,d\zeta}{(\zeta-\alpha_j)R(\zeta)} \end{vmatrix}, \tag{43}$$

where f_μ is given by (29).

Lemma 4.2. *Let f be given by* (21) *and consider a contour $\gamma_0=\gamma(\vec{\alpha}_0,\mu_0)\in\Gamma(\vec{\alpha}_0,\mu_0)$ having arc end points $\vec{\alpha}_0$. Then there are open neighborhoods of $\vec{\alpha}_0$ and μ_0 such that for all $\vec{\alpha}$ and μ in the neighborhoods of $\vec{\alpha}_0$ and μ_0, respectively, there is a contour $\gamma=\gamma(\vec{\alpha},\mu)\in\Gamma(\vec{\alpha},\mu)$ and*

$$K_j(\vec{\alpha},\mu):=K(\alpha_j,\vec{\alpha},\mu),\quad j=0,1,\ldots,4N+1, \tag{44}$$

is continuously differentiable in $\vec{\alpha}$ and in μ.

Proof. Since $\gamma_0\in\Gamma(\vec{\alpha}_0,\mu_0)$ by Lemma 2.2, there are neighborhoods of $\vec{\alpha}_0$ and μ_0 such that for all $\vec{\alpha}$ and μ in the neighborhoods of $\vec{\alpha}_0$ and μ_0, respectively, there is a contour $\gamma=\gamma(\vec{\alpha},\mu)\in\Gamma(\vec{\alpha},\mu)$.

$K_j(\vec{\alpha},\mu)$ is analytic in $\vec{\alpha}$ by the determinant structure and the integral entries (16), where explicit dependence on $\vec{\alpha}$ is only in the $R(z,\vec{\alpha})$ term, which is analytic away from $z=\alpha_j$, $j=0,\ldots,4N+1$.

The integrals in the last row of the matrix in (43) involve the function f_μ given by (29), which is integrable near $z=\mu/2$, and hence the whole determinant is continuous in μ. Thus $K_j(\alpha_j,\vec{\alpha},\mu)$ is continuously differentiable in $\vec{\alpha}$ and in μ. $\square$

By Lemma 4.2, the modulation equations (19)

$$K_j(\vec{\alpha},\mu)=K(\alpha_j,\vec{\alpha},\mu)=0 \tag{45}$$

are smooth in $\vec{\alpha}$ and in the parameter μ. Next we want to solve this system for $\vec{\alpha}=\vec{\alpha}(\mu)$ and derive smoothness in μ by the implicit function theorem.

For the next lemma we need $K'(z,\vec{\alpha},\mu)=\dfrac{dK}{dz}(z,\vec{\alpha},\mu)$:

$$K'(z,\vec{\alpha},\mu)=\frac{1}{2\pi i}\begin{vmatrix} \oint_{\hat{\gamma}_{m,1}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,1}}\frac{d\zeta}{(\zeta-z)^2R(\zeta)} \\ \vdots & \ddots & \vdots & \vdots \\ \oint_{\hat{\gamma}_{m,N}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,N}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,N}}\frac{d\zeta}{(\zeta-z)^2R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,1}}\frac{d\zeta}{(\zeta-z)^2R(\zeta)} \\ \vdots & \ddots & \vdots & \vdots \\ \oint_{\hat{\gamma}_{c,N}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,N}}\frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,N}}\frac{d\zeta}{(\zeta-z)^2R(\zeta)} \\ \oint_{\hat{\gamma}}\frac{f(\zeta)\,d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}}\frac{\zeta^{N-1}f(\zeta)\,d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}}\frac{f(\zeta)\,d\zeta}{(\zeta-z)^2R(\zeta)} \end{vmatrix}, \tag{46}$$

where z is inside of $\hat{\gamma}(\mu)$ and inside of $\hat{\gamma}_{m,j}$ and $\hat{\gamma}_{c,j}$ or $\hat{\gamma}_{c,j+1}$.

Lemma 4.3. *Let f be given by* (21) *and consider a contour $\gamma_0\in\Gamma(\vec{\alpha}_0,\mu_0)$, where $\vec{\alpha}_0$ and μ_0 satisfy*

$$\vec{K}(\vec{\alpha}_0,\mu_0)=\vec{0}.$$

Assume that for $\vec{\alpha}_0=\{\alpha_j^0\}_{j=0}^{4N+1}$,

$$\lim_{z\to\alpha_j^0}K'(z,\vec{\alpha}_0,\mu_0)\neq 0,\qquad j=0,1,\dots,4N+1.$$

Then the modulation equations

$$\vec{K}(\vec{\alpha},\mu)=\vec{0}$$

can be uniquely solved for $\vec{\alpha}=\vec{\alpha}(\mu)$, which is continuously differentiable for all μ in some open neighborhood of μ_0 and $\vec{\alpha}(\mu_0)=\vec{\alpha}_0$.

Proof. $\vec{K}$ is continuously differentiable in $\vec{\alpha}$ and in μ by Lemma 4.2.

As shown in [Tovbis and Venakides 2009], the matrix

$$\left\{\frac{\partial\vec{K}}{\partial\vec{\alpha}}\right\}_{j,l}=\left\{\frac{\partial K(\alpha_j)}{\partial\alpha_l}\right\}_{j,l}$$

is diagonal and

$$\frac{\partial K(\alpha_j)}{\partial\alpha_j}=\frac{3}{2}D\lim_{z\to\alpha_j^0}\left(\frac{h(z)}{R(z)}\right)'=\frac{3}{2}\lim_{z\to\alpha_j^0}K'(z,\vec{\alpha},\mu)\neq 0. \tag{47}$$

So

$$\det\left|\frac{\partial\vec{K}}{\partial\vec{\alpha}}(\vec{\alpha}_0)\right|=\prod_j\frac{\partial K(\alpha_j)}{\partial\alpha_j}\neq 0, \tag{48}$$

under the assumptions. By the implicit function theorem, $\vec{\alpha}(\mu)$ are uniquely defined in some neighborhood of μ_0 and smooth in μ. Note that $\vec{\alpha}(\mu_0)=\vec{\alpha}_0$ by assumption. □

Remark 4.4. The condition $\lim_{z\to\alpha_j^0} K'(z,\vec{\alpha}_0,\mu_0)\neq 0$, $j=0,1,\dots,4N+1$, in Lemma 4.3 is equivalent to

$$\lim_{z\to\alpha_j^0}\frac{h'(z,\vec{\alpha}_0,\mu_0)}{R(z,\vec{\alpha}_0)}\neq 0,\qquad j=0,1,\dots,4N+1.$$

All quantities below depend on parameters x and t. We assume that for the rest of the paper x and t are fixed.

Theorem 4.5 (μ-perturbation in genus N). *Consider a finite-length non-self-intersecting contour γ_0 in the complex plane consisting of a finite union of oriented arcs*

$$\gamma_0=\left(\bigcup\gamma_{m,j}\right)\cup\left(\bigcup\gamma_{c,j}\right)\in\Gamma(\vec{\alpha}_0,\mu_0)$$

with the distinct arc end points $\vec{\alpha}_0$ and depending on parameter μ (see Figure 1). Assume $\vec{\alpha}_0$ and μ_0 satisfy a system of equations

$$\vec{K}(\vec{\alpha}_0,\mu_0)=\vec{0},$$

and f is given by (21). *Let $\gamma=\gamma(\vec{\alpha},\mu)$ be a contour of an RHP which seeks a function $h(z)$ which satisfies the conditions*

$$\begin{cases} h_+(z)+h_-(z)=2W_j & \text{on } \gamma_{m,j},\ j=0,1,\dots,N,\\ h_+(z)-h_-(z)=2\Omega_j & \text{on } \gamma_{c,j},\ j=1,\dots,N,\\ h(z)+f(z) \text{ is analytic in } \overline{\mathbb{C}}\setminus\gamma, \end{cases}\tag{49}$$

where $\Omega_j=\Omega_j(\vec{\alpha},\mu)$ and $W_j=W_j(\vec{\alpha},\mu)$ are real constants (with normalization $W_0=0$) whose numerical values will be determined from the RH conditions. Assume that there is a function $h(z,\vec{\alpha}_0,\mu_0)$ which satisfies (49) *and suppose $h'(z,\vec{\alpha}_0,\mu_0)/R(z,\vec{\alpha}_0)\neq 0$ for all z on γ_0.*

Then there is a contour $\gamma(\vec{\alpha},\mu)\in\Gamma(\vec{\alpha},\mu)$ such that the solution $\vec{\alpha}=\vec{\alpha}(\mu)$ of the system

$$\vec{K}(\vec{\alpha},\mu)=\vec{0}\tag{50}$$

and $h(z,\vec{\alpha}(\mu),\mu)$ which solves (49) *are uniquely defined and continuously differentiable in μ in some open neighborhood of μ_0.*

Moreover,

$$\frac{\partial\alpha_j}{\partial\mu}(\mu)=-\frac{2\pi i\,\dfrac{\partial K}{\partial\mu}(\alpha_j(\mu),\vec{\alpha}(\mu),\mu)}{D(\vec{\alpha}(\mu),\mu)\displaystyle\oint_{\hat{\gamma}(\mu)}\frac{f'(\zeta,\mu)}{(\zeta-\alpha_j(\mu))R(\zeta,\vec{\alpha}(\mu))}\,d\zeta},\tag{51}$$

$$\frac{\partial h}{\partial\mu}(z,\mu)=\frac{R(z,\vec{\alpha}(\mu))}{2\pi i}\oint_{\hat{\gamma}(\mu)}\frac{\dfrac{\partial f}{\partial\mu}(\zeta,\mu)}{(\zeta-z)R(\zeta,\vec{\alpha}(\mu))}\,d\zeta,\tag{52}$$

where z is inside of $\hat{\gamma}$.

Furthermore, $\Omega_j(\mu) = \Omega_j(\vec{\alpha}(\mu), \mu)$ *and* $W_j(\mu) = W_j(\vec{\alpha}(\mu), \mu)$ *are defined and continuously differentiable in* μ *in some open neighborhood of* μ_0*, and*

$$\frac{\partial \Omega_j}{\partial \mu}(\mu) = -\frac{1}{D}\begin{vmatrix} \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \\ \vdots & \ddots & \vdots \\ \oint_{\hat{\gamma}_{m,N}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,N}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \\ \vdots & \ddots & \vdots \\ \oint_{\hat{\gamma}_{c,j-1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,j-1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}} \frac{f_\mu(\zeta)}{R(\zeta,\vec{\alpha})}d\zeta & \cdots & \oint_{\hat{\gamma}} \frac{\zeta^{N-1}f_\mu(\zeta)}{R(\zeta,\vec{\alpha})}d\zeta \\ \oint_{\hat{\gamma}_{c,j+1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,j+1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \\ \vdots & \ddots & \vdots \\ \oint_{\hat{\gamma}_{c,N}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,N}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \end{vmatrix},$$

$$\frac{\partial W_j}{\partial \mu}(\mu) = -\frac{1}{D}\begin{vmatrix} \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \\ \vdots & \ddots & \vdots \\ \oint_{\hat{\gamma}_{m,j-1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,j-1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}} \frac{f_\mu(\zeta)}{R(\zeta,\vec{\alpha})}d\zeta & \cdots & \oint_{\hat{\gamma}} \frac{\zeta^{N-1}f_\mu(\zeta)}{R(\zeta,\vec{\alpha})}d\zeta \\ \oint_{\hat{\gamma}_{m,j+1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,j+1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \\ \vdots & \ddots & \vdots \\ \oint_{\hat{\gamma}_{m,N}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,N}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \\ \vdots & \ddots & \vdots \\ \oint_{\hat{\gamma}_{c,N}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,N}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \end{vmatrix}, \tag{53}$$

where $R(\zeta) = R(\zeta, \vec{\alpha})$, $f(\zeta) = f(\zeta, \mu)$, $f_\mu(\zeta) = \dfrac{\partial f}{\partial \mu}(\zeta, \mu)$, *and* $D = D(\vec{\alpha}(\mu))$.

Proof. By Lemma 4.3, there is a contour $\gamma(\vec{\alpha},\mu)\in\Gamma(\vec{\alpha},\mu)$ for all μ in some neighborhood of μ_0 and the $\alpha_j(\mu)$ are continuously differentiable in μ. The formula for $\partial\alpha_j/\partial\mu$ is derived similarly as in [Tovbis and Venakides 2009]. We differentiate the modulation equations $K(\alpha_j)=K(\alpha_j,\vec{\alpha},\mu)=0$ which define $\vec{\alpha}=\vec{\alpha}(\mu)$ with respect to μ,

$$\sum_{l=0}^{4N+1}\frac{\partial K(\alpha_j)}{\partial\alpha_l}\frac{\partial\alpha_l}{\partial\mu}+\frac{\partial K}{\partial\mu}(\alpha_j)=0, \tag{54}$$

where the matrix $\left\{\dfrac{\partial K(\alpha_j)}{\partial\alpha_l}\right\}_{j,l}$ is diagonal [Tovbis and Venakides 2009], so

$$\frac{\partial K(\alpha_j)}{\partial\alpha_j}\frac{\partial\alpha_j}{\partial\mu}=-\frac{\partial K(\alpha_j)}{\partial\mu}. \tag{55}$$

Since

$$\frac{\partial K(\alpha_j)}{\partial\alpha_j}=\frac{D(\vec{\alpha},\mu)}{2\pi i}\oint_{\hat{\gamma}(\mu)}\frac{f'(\zeta,\mu)}{(\zeta-\alpha_j)R(\zeta,\vec{\alpha})}\,d\zeta, \tag{56}$$

we arrive at the evolution equations for the α_j:

$$\frac{\partial\alpha_j}{\partial\mu}=-\frac{2\pi i\,\dfrac{\partial K}{\partial\mu}(\alpha_j)}{D(\vec{\alpha},\mu)\displaystyle\oint_{\hat{\gamma}(\mu)}\frac{f'(\zeta,\mu)}{(\zeta-\alpha_j)R(\zeta,\vec{\alpha})}\,d\zeta},\qquad j=0,\dots,4N+1. \tag{57}$$

Next we compute $\dfrac{\partial h}{\partial\mu}$, which satisfies the scalar RHP

$$\begin{cases} h_{\mu,+}(z)+h_{\mu,-}(z)=0, & z\in\gamma_{m,j},\ \ j=0,1,\dots,N,\\ h_\mu(z)+f_\mu(z) \text{ is analytic in } \overline{\mathbb{C}}\setminus\gamma. \end{cases} \tag{58}$$

Then

$$\frac{\partial h}{\partial\mu}(z,\mu)=\frac{R(z,\vec{\alpha}(\mu))}{2\pi i}\oint_{\hat{\gamma}(\mu)}\frac{\dfrac{\partial f}{\partial\mu}(\zeta,\mu)}{(\zeta-z)R(\zeta,\vec{\alpha}(\mu))}\,d\zeta, \tag{59}$$

where z is inside of $\hat{\gamma}$. The integrand $\dfrac{\partial f}{\partial\mu}(\zeta,\mu)$ behaves like $\log(\zeta-z_0)$ near $\zeta=z_0$, and therefore is integrable.

Constants W_j and Ω_j are found from the linear system [Tovbis et al. 2004]

$$\oint_{\hat{\gamma}(\mu)}\frac{\zeta^n f(\zeta,\mu)}{R(\zeta,\vec{\alpha})}\,d\zeta+\sum_{j=1}^{N}\oint_{\hat{\gamma}_{c,j}}\frac{\zeta^n\Omega_j}{R(\zeta,\vec{\alpha})}\,d\zeta+\sum_{j=1}^{N}\oint_{\hat{\gamma}_{m,j}}\frac{\zeta^n W_j}{R(\zeta,\vec{\alpha})}\,d\zeta=0,\quad n=0,\dots,N-1. \tag{60}$$

Differentiating in μ and using Lemma 4.1 leads to

$$\oint_{\hat{\gamma}(\mu)}\frac{\zeta^n f_\mu(\zeta,\mu)}{R(\zeta,\vec{\alpha})}\,d\zeta+\sum_{j=1}^{N}\oint_{\hat{\gamma}_{c,j}}\frac{\zeta^n(\Omega_j)_\mu}{R(\zeta,\vec{\alpha})}\,d\zeta+\sum_{j=1}^{N}\oint_{\hat{\gamma}_{m,j}}\frac{\zeta^n(W_j)_\mu}{R(\zeta,\vec{\alpha})}\,d\zeta=0,\quad n=0,\dots,N-1, \tag{61}$$

or in matrix form,

$$\begin{pmatrix} \oint_{\hat\gamma_{m,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{m,1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \\ \vdots & \ddots & \vdots \\ \oint_{\hat\gamma_{m,N}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{m,N}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \\ \oint_{\hat\gamma_{c,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{c,1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \\ \vdots & \ddots & \vdots \\ \oint_{\hat\gamma_{c,N}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{c,N}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} \end{pmatrix}^T \begin{pmatrix} \frac{\partial \vec W}{\partial\mu} \\ \frac{\partial\vec\Omega}{\partial\mu} \end{pmatrix} = - \begin{pmatrix} \oint_{\hat\gamma(\mu)} \frac{f_\mu(\zeta,\mu)}{R(\zeta,\vec\alpha)}\,d\zeta \\ \vdots \\ \oint_{\hat\gamma(\mu)} \frac{\zeta^{N-1} f_\mu(\zeta,\mu)}{R(\zeta,\vec\alpha)}\,d\zeta \end{pmatrix}. \tag{62}$$

So $\frac{\partial\Omega_j}{\partial\mu}$ and $\frac{\partial W_j}{\partial\mu}$ satisfy (53). Note that $D \neq 0$ for distinct α_j's [Tovbis and Venakides 2009]. □

Remark 4.6. In [Tovbis and Venakides 2009], the case was considered where the contour γ is independent of external parameters x and t and the dependence of f on these parameters is linear. Here we apply the methods of that paper to the case of a dependence on the parameter μ when the jump contour explicitly passes through $z = \mu/2$, a point of singularity of f. Despite this more complicated dependence on μ, the resulting formulae are the same. The main reason is Lemma 4.1, which allows us to find partial derivatives with respect to μ of contour integrals involving dependence on μ in both integrands and contours of integration.

Remark 4.7. Theorem 4.5 guarantees that the solution of the RHP (49) is uniquely continued with respect to external parameters. Additional sign conditions on $\Im h$ need to be satisfied for h to correspond to an asymptotic solution of the NLS as in [Tovbis et al. 2004]. The sign conditions have to be satisfied near γ and additionally on semi-infinite complementary arcs connecting the arc end points of γ to ∞.

5. Sign conditions and preservation of genus

If the scalar RHP (12) is implemented in the asymptotic solution of the semiclassical NLS, certain sign conditions must be satisfied. Specifically, $\Im h(z) = 0$ on $\gamma_{m,j}$, $\Im h(z) < 0$ on both sides of $\gamma_{m,j}$, and $\Im h(z) \geq 0$ on $\gamma_{c,j}$ (see Definition 5.3 below). In this section we investigate the preservation of the sign structure of $\Im h$ under perturbations of μ.

Definition 5.1. Define $\gamma^\infty = \gamma^\infty(\vec\alpha,\mu)$ as an extension of a contour $\gamma(\vec\alpha,\mu) \in \Gamma(\vec\alpha,\mu)$ as $\gamma^\infty(\vec\alpha,\mu) = (\infty, \alpha_{4N+1}] \cup \gamma(\vec\alpha,\mu) \cup [\alpha_{4N}, \infty)$. Both additional arcs are considered as a complementary arc $\gamma_{c,N+1} = (\infty, \alpha_{4N+1}] \cup [\alpha_{4N}, \infty)$, and assume $\gamma_{c,N+1} = \overline{\gamma_{c,N+1}}$, so $\gamma^\infty = \overline{\gamma^\infty}$. With a slight abuse of notation we write $\gamma^\infty(\vec\alpha,\mu) \in \Gamma(\vec\alpha,\mu)$.

Lemma 5.2. *If the conditions of Theorem 4.5 hold on $\gamma^\infty(\vec\alpha_0,\mu_0) \in \Gamma(\vec\alpha_0,\mu_0)$ for $\vec K(\vec\alpha_0,\mu_0) = \vec 0$, the statement of the theorem holds on $\gamma^\infty(\vec\alpha,\mu)$, where $\vec K(\vec\alpha,\mu) = \vec 0$.*

Proof. The proof is unchanged since f is analytic near the additional semi-infinite arcs in $\gamma_{c,N+1}$ and the jump condition on the additional complementary arc $\gamma_{c,N+1}$ is taken to be zero ($\Omega_{N+1} = 0$) [Tovbis et al. 2004]. □

Note that the conditions in Lemma 5.2 are more restrictive since $\gamma \subset \gamma^\infty$.

Definition 5.3. A function h satisfies sign conditions on γ^∞ if $\Im h(z) = 0$ if $z \in \gamma_{m,j}$, $\Im h(z) < 0$ on both sides of $\gamma_{m,j}$ for all $j = 0, \dots, N$, and $\Im h(z) \geq 0$ if $z \in \gamma_{c,j}$ for all $j = 1, \dots, N+1$. We then write $h \in \mathrm{SC}(\gamma^\infty)$.

Note that the zero sign conditions ($\Im h(z) = 0$) on $\gamma_{m,j}$ are satisfied automatically through the construction of $h(z)$ by (14) in the case of h solving an RHP (49). We only need to check preservation of negative signs of $\Im h$ on both sides of the main arcs $\gamma_{m,j}$ and the nonnegativity of $\Im h$ on the complementary arcs $\gamma_{c,j}$, especially on the semi-infinite arcs $(\infty, \alpha_{4N+1}]$ and $[\alpha_{4N}, \infty)$.

Remark 5.4. Introducing the sign conditions in Definition 5.3 requires us to revisit Lemma 2.2, since the main arcs $\gamma_{m,j}$ are now rigid (nondeformable like the complementary arcs) due to the requirement for $\Im h$ to be negative on both sides of $\gamma(\vec{\alpha}(\mu), \mu)$. It has been established [Kamvissis et al. 2003, Lemma 5.2.1; Tovbis et al. 2007, Theorem 3.2] that all the contours persist under deformations of parameters x and t provided all sign inequalities are satisfied. This can be adapted to the deformations of μ. The danger of a main arc splitting into several disconnected branches as we perturb μ is averted by the fact that in the limit as $\mu \to \mu_0$, nonlinear local behavior would be produced near the main arcs, while the condition $h'/R \neq 0$ on γ implies linear local behavior. Thus Lemma 2.2 is valid even with the added new sign conditions. Thus we only need to show that the sign conditions are satisfied.

Theorem 5.5. *Let f be defined by* (21). *Let $\vec{K}(\vec{\alpha}_0, \mu_0) = \vec{0}$, $\gamma_0^\infty \in \Gamma(\vec{\alpha}_0, \mu_0)$ and assume h solves* $\mathrm{RHP}(\gamma_0^\infty, \vec{\alpha}_0, \mu_0, f)$ *with $h'(z, \mu_0)/R(z, \mu_0) \neq 0$ for all $z \in \gamma_0^\infty$, and $h \in \mathrm{SC}(\gamma_0^\infty)$.*

Then there is an open neighborhood of μ_0 where for all μ, there is an h which solves $\mathrm{RHP}(\gamma^\infty, \vec{\alpha}, \mu, f)$ *with $\gamma^\infty = \gamma^\infty(\vec{\alpha}, \mu)$, $\vec{K}(\vec{\alpha}, \mu) = \vec{0}$, $h'(z, \mu)/R(z, \mu) \neq 0$ for all $z \in \gamma^\infty$, and $h \in \mathrm{SC}(\gamma^\infty)$.*

Proof. Take any μ in a small enough open neighborhood of μ_0. There are two things we need to prove in addition to Lemma 5.2: $h'(z, \mu)/R(z, \vec{\alpha}(\mu)) \neq 0$ on γ^∞ and the sign conditions of $\Im h$ on γ^∞.

Assume $h'(z, \mu_0)/R(z, \vec{\alpha}(\mu_0)) \neq 0$ on γ_0^∞. Then there is a constant $C > 0$ such that

$$\left| \frac{h'(z, \mu_0)}{R(z, \vec{\alpha}(\mu_0))} \right| > C$$

for all $z \in \gamma_0$. Consider the solution $h(z, \mu)$ of $\mathrm{RHP}(\gamma^\infty, \vec{\alpha}, \mu)$, where $\vec{K}(\vec{\alpha}, \mu) = \vec{0}$. By Theorem 4.5 and Lemma 5.2, such a function exists and is continuously differentiable in μ. Moreover, $h'(z, \mu)$ is continuous in μ. Since γ is a compact set in $\mathbb{C}$ and $h'(z, \mu)/R(z, \vec{\alpha}(\mu))$ is continuous in z and μ, we have $h'(z, \mu)/R(z, \vec{\alpha}(\mu)) \neq 0$ for all $z \in \gamma$.

To show that $h'(z, \mu)/R(z, \vec{\alpha}(\mu)) \neq 0$ holds on γ^∞, we will now make use of the following properties of $f(z)$. On the real axis (in the nontangential limit from the upper half-plane),

$$\Im f(z + i0) = \lim_{\delta \to 0^+} f(z + i\delta), \quad z \in \mathbb{R},$$

is a piecewise linear function [Tovbis et al. 2004]

$$\Im f(z+i0) = \begin{cases} \frac{\pi}{2}\left(\frac{\mu}{2} - |z|\right) & \text{if } z < \frac{\mu}{2}, \\ \frac{\pi}{2}\left(z - \frac{\mu}{2}\right) & \text{if } z \geq \frac{\mu}{2}, \end{cases} \tag{63}$$

and since $g(z)$ is real on the real axis, $\Im h(z+i0) = -\Im f(z+i0)$. It is important for us that $|\Im h(z)|$ can be bounded away from zero as $z \to \infty$.

Similarly,

$$\Im f'(z+i0) = \begin{cases} \frac{\pi}{2} & \text{if } z \leq 0, \\ -\frac{\pi}{2} & \text{if } 0 < z \leq \frac{\mu}{2}, \\ \frac{\pi}{2} & \text{if } z > \frac{\mu}{2}, \end{cases}$$

and since $g'(z)$ is real on the real axis, $\Im h'(z+i0) = -\Im f'(z+i0)$.

Recall that $\gamma^\infty = (\infty, \alpha_{4N+1}] \cup \gamma \cup [\alpha_{4N}, \infty)$. The semi-infinite arcs $(\infty, \alpha_{4N+1}]$ and $[\alpha_{4N}, \infty)$ can be pushed to the real axis as $(-\infty - i0, -\mu/2 - i0) \cup [-\mu/2 - i0, \alpha_{4N+1}]$ and $[\alpha_{4N}, -\mu/2 + i0] \cup (-\mu/2 + i0, -\infty + i0)$, respectively.

On $[-\mu/2 - i0, \alpha_{4N+1}]$ and $[\alpha_{4N}, -\mu/2 + i0]$, we have $h'(z,\mu)/R(z, \vec{\alpha}(\mu)) \neq 0$ by continuity on a compact set. Finally, $\Im h'(z,\mu) = -\pi/2$ and $R(z,\vec{\alpha}) \in \mathbb{R}$ for all $z \in (-\mu/2 + i0, -\infty + i0)$. So $h'(z,\mu)/R(z,\vec{\alpha}(\mu)) \neq 0$ for all $z \in (-\mu/2 + i0, -\infty + i0)$. The interval $(-\infty - i0, -\mu/2 - i0)$ is done similarly. So $h'(z,\mu)/R(z,\vec{\alpha}(\mu)) \neq 0$ for all $z \in \gamma^\infty$, for any μ in the neighborhood of μ_0.

Let $h \in \mathrm{SC}(\gamma_0^\infty)$. Then $h \in \mathrm{SC}(\gamma(\mu))$ by continuity of h in z and μ, compactness of γ, and harmonicity of $\Im h$ combined with $h'(z,\mu)/R(z,\vec{\alpha}(\mu)) \neq 0$ for all $z \in \gamma^\infty$, which guarantees that the (negative) signs near the main arcs $\hat{\gamma}_{m,j}$ are preserved. On the semi-infinite arcs $(-\infty - i0, -\mu/2 - i0)$ and $(-\mu/2 + i0, -\infty + i0)$, $\Im h(z) = (\pi/2)(|z| - \mu/2)$ is positive and $[\alpha_{4N}, -\mu/2]$ and $[-\mu/2, \alpha_{4N+1}]$ are compact. So $\Im h \geq 0$ on $\gamma^\infty(\mu)$, that is, $h \in \mathrm{SC}(\gamma^\infty(\mu))$. □

Definition 5.6. We define the (finite) genus $G = G(\mu)$ of the asymptotic solution of the semiclassical one-dimensional focusing NLS with initial condition defined through $f(z,\mu)$ as (finite) $N \in \mathbb{N}$ if there exists an asymptotic solution of the NLS through the solution $h(z,\mu)$ of $\mathrm{RHP}(\gamma^\infty, \vec{\alpha}, \mu, f)$ with $\vec{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{4N+1})$, such that $h'(z,\mu)/R(z) \neq 0$ for all $z \in \gamma^\infty$ and the sign conditions of h on γ^∞ are satisfied: $h \in \mathrm{SC}(\gamma^\infty)$.

Remark 5.7. This definition of the genus of the asymptotic solution coincides with the genus of the (limiting) hyperelliptic Riemann surface of $R(z)$.

Theorem 5.8 (preservation of genus). *Suppose that for μ_0, the genus of the asymptotic solution of the NLS with initial condition defined through $f(z,\mu_0)$ in* (21) *is $G(\mu_0)$.*

Then there is an open neighborhood of μ_0 such that, for all μ in the neighborhood of μ_0, the genus of the asymptotic solution of the NLS with initial condition defined through $f(z,\mu)$ is preserved: $G(\mu) = G(\mu_0)$.

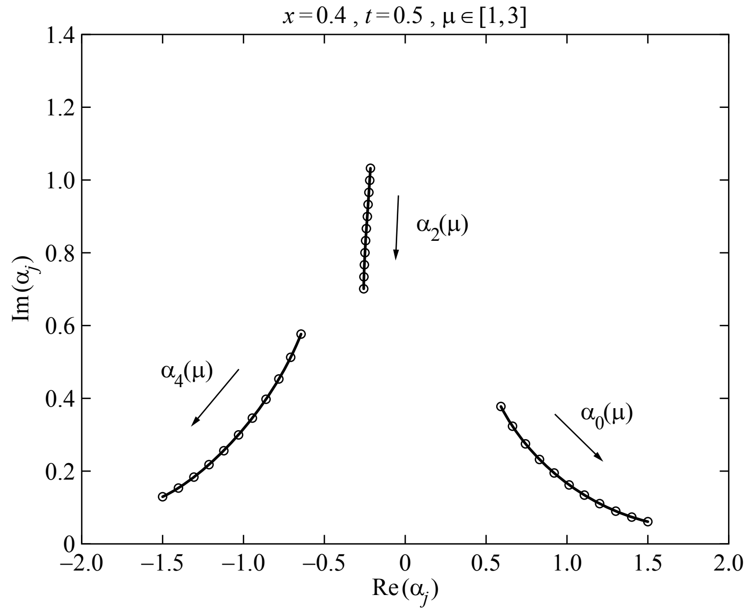

Figure 3. Comparison of μ evolution of $\vec{\alpha} = (\alpha_0, \alpha_2, \alpha_4)$ using (82) (solid lines) and (83) (circles).

Proof. Follows from Theorem 5.5 and Definition 5.6. □

Corollary 5.9. *Fix x and $t > t_0$, where $t_0(x)$ is the time of the first break in the asymptotic solution. Then in some open neighborhood of $\mu = 2$, the genus of the solution is* 2*.*

Proof. For $\mu = 2$ and $t > t_0(x)$ the genus is 2 for all x [Tovbis et al. 2004]. By Theorem 5.8, the genus is preserved in some open neighborhood of $\mu = 2$, including some open interval for $\mu < 2$. □

6. Numerics

Figure 3 compares solutions of (50) and (51) in genus 2 (see also (82) and (83) in the Appendix for more explicit expressions). The solutions are practically indistinguishable in the figure, with absolute difference less than 10^{-3} for $\mu \in [1, 3]$. This interval includes the critical value $\mu = 2$, which is the transition between the (solitonless) pure radiation case ($\mu \geq 2$) and the region with solitons ($0 < \mu < 2$). Computations are based on the code we developed for long-time studies of an obstruction in the g-function mechanism [Belov and Venakides 2015].

Appendix

A1. *Genus* 0 *region.* It was shown in [Tovbis et al. 2004] that for all $\mu > 0$ and for all x, there is a breaking curve $t = t_0(x)$ in the (x, t) plane. The region $0 \leq t < t_0(x)$ has genus 0 in the sense of genus of the underlying Riemann surface for the square root

$$R(z, \alpha_0) = \sqrt{(z - \alpha_0)(z - \alpha_1)}, \quad \alpha_1 = \overline{\alpha}_0,$$

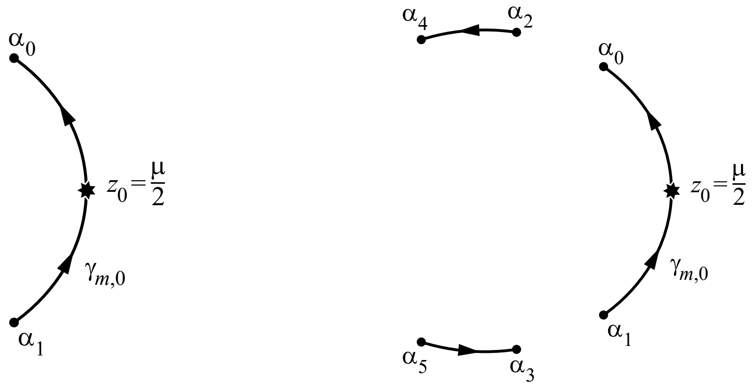

Figure 4. The jump contour in the case of genus 0 (left diagram) and genus 2 (right) with complex-conjugate symmetry in the notation of [Tovbis et al. 2004].

where the branch cut is chosen along the main arc connecting α_0 and $\alpha_1 = \overline{\alpha}_0$ through $z = \mu/2$, and the branch is fixed by $R(z) \to -z$ as $z \to +\infty$. The asymptotic solution of the NLS is expressed in terms of $\alpha_0 = \alpha_0(x, t, \mu)$.

All expressions in the genus 0 region ($N = 0$) have a simpler form. In particular,

$$h(z, \alpha_0, \mu) = \frac{R(z, \alpha_0)}{2\pi i} \oint_{\hat{\gamma}(\mu)} \frac{f(\zeta, \mu)\, d\zeta}{(\zeta - z) R(\zeta, \alpha_0)}, \tag{64}$$

$$K(z, \alpha_0, \mu) = \frac{1}{2\pi i} \oint_{\hat{\gamma}(\mu)} \frac{f(\zeta, \mu)\, d\zeta}{(\zeta - z) R(\zeta, \alpha_0)}, \tag{65}$$

and with a slight abuse of notation,

$$K(\alpha_0, \mu) := K(\alpha_0, \alpha_0, \mu) = \frac{1}{2\pi i} \oint_{\hat{\gamma}(\mu)} \frac{f(\zeta, \mu) d\zeta}{(\zeta - \alpha_0) R(\zeta, \alpha_0)}, \tag{66}$$

$$\frac{\partial K}{\partial \mu}(\alpha_0, \mu) := \frac{\partial K}{\partial \mu}(\alpha_0, \alpha_0, \mu) = \frac{1}{2\pi i} \oint_{\hat{\gamma}(\mu)} \frac{f_\mu(\zeta, \mu) d\zeta}{(\zeta - \alpha_0) R(\zeta, \alpha_0)}. \tag{67}$$

Theorem A.1 (μ-perturbation in genus 0). *Consider a finite-length non-self-intersecting oriented arc $\gamma_0 = [\alpha_0(\mu_0), \overline{\alpha}_0(\mu_0)] \in \Gamma(\vec{\alpha}, \mu_0)$ in the complex plane with the distinct end points ($\alpha_0 \neq \overline{\alpha}_0$) and depending on a parameter μ* (see Figure 4). *Assume α_0 and μ_0 satisfy the equation*

$$K(\alpha_0, \mu_0) = 0,$$

and f is given by (21). *Let $\gamma = \gamma(\vec{\alpha}, \mu)$ be the contour of an RHP which seeks a function $h(z)$ which satisfies the conditions*

$$\begin{cases} h_+(z) + h_-(z) = 0 \text{ on } \gamma, \\ h(z) + f(z) \text{ is analytic in } \overline{\mathbb{C}} \setminus \gamma. \end{cases} \tag{68}$$

Assume that there is a function $h(z, \alpha_0, \mu_0)$ which satisfies (68) *and suppose $\dfrac{h'(z, \alpha_0, \mu_0)}{R(z, \alpha_0)} \neq 0$ for all z on γ_0.*

Then there is a contour $\gamma(\vec{\alpha},\mu) \in \Gamma(\vec{\alpha},\mu)$ such that the solution $\alpha_0(\mu)$ of the equation

$$K(\alpha_0,\mu) = 0 \tag{69}$$

and $h(z,\alpha(\mu),\mu)$ which solves (68) *are uniquely defined and continuously differentiable in μ in some open neighborhood of μ_0.*

Moreover,

$$\frac{\partial\alpha_0}{\partial\mu}(\mu) = -\frac{2\pi i\,\dfrac{\partial K}{\partial\mu}(\alpha_0(\mu),\mu)}{\displaystyle\oint_{\hat{\gamma}(\mu)} \frac{f'(\zeta,\mu)}{(\zeta-\alpha_0(\mu))R(\zeta,\alpha_0(\mu))}\,d\zeta} \tag{70}$$

and

$$\frac{\partial h}{\partial\mu}(z,\mu) = \frac{R(z,\alpha_0(\mu))}{2\pi i}\oint_{\hat{\gamma}(\mu)} \frac{\dfrac{\partial f}{\partial\mu}(\zeta,\mu)}{(\zeta-z)R(\zeta,\alpha_0(\mu))}\,d\zeta, \tag{71}$$

where z is inside of $\hat{\gamma}$.

A2. ***Genus 2 region.*** We now consider the genus 2 region ($N=2$), with underlying Riemann surface for the square root

$$R(z) = \sqrt{(z-\alpha_0)(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)(z-\alpha_4)(z-\alpha_5)},$$

where the branch cut is chosen along the main arcs connecting α_0 and α_1, α_2 and α_4, α_5 and α_3; and the branch is fixed by $R(z) \to -z^3$ as $z \to +\infty$.

Taking into account the complex-conjugate symmetry

$$\alpha_1 = \overline{\alpha}_0, \quad \alpha_3 = \overline{\alpha}_2, \quad \alpha_5 = \overline{\alpha}_4, \tag{72}$$

we have

$$h(z) = \frac{R(z)}{2\pi i}\left[\oint_{\hat{\gamma}} \frac{f(\zeta)}{(\zeta-z)R(\zeta)}\,d\zeta + \oint_{\hat{\gamma}_m} \frac{W}{(\zeta-z)R(\zeta)}\,d\zeta + \oint_{\hat{\gamma}_c} \frac{\Omega}{(\zeta-z)R(\zeta)}\,d\zeta\right], \tag{73}$$

where z is inside of $\hat{\gamma}$, $\hat{\gamma}_m$ is a loop around the main arc $\gamma_m = [\alpha_2,\alpha_4]\cup[\alpha_5,\alpha_3]$, and $\hat{\gamma}_c$ is a loop around the complementary arc $\gamma_c = [\alpha_0,\alpha_2]\cup[\alpha_3,\alpha_1]$ (see Figure 4). The real constants W and Ω solve the system

$$\begin{cases} \displaystyle\oint_{\hat{\gamma}} \frac{f(\zeta)}{R(\zeta)}\,d\zeta + \Omega\oint_{\hat{\gamma}_c} \frac{d\zeta}{R(\zeta)} + W\oint_{\hat{\gamma}_m} \frac{d\zeta}{R(\zeta)} = 0, \\ \displaystyle\oint_{\hat{\gamma}} \frac{\zeta f(\zeta)}{R(\zeta)}\,d\zeta + \Omega\oint_{\hat{\gamma}_c} \frac{\zeta}{R(\zeta)}\,d\zeta + W\oint_{\hat{\gamma}_m} \frac{\zeta}{R(\zeta)}\,d\zeta = 0. \end{cases} \tag{74}$$

Other useful expressions written explicitly in the genus 2 region are

$$K(z)=\frac{1}{2\pi i}\begin{vmatrix} \oint_{\hat{\gamma}_m}\frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_m}\frac{\zeta d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_m}\frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \oint_{\hat{\gamma}_c}\frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_c}\frac{\zeta d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_c}\frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \oint_{\hat{\gamma}}\frac{f(\zeta)\,d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}}\frac{\zeta f(\zeta)\,d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}}\frac{f(\zeta)\,d\zeta}{(\zeta-z)R(\zeta)} \end{vmatrix} \tag{75}$$

and

$$K(z)=\frac{1}{2\pi i}\left[\oint_{\hat{\gamma}}\frac{f(\zeta)}{(\zeta-z)R(\zeta)}\,d\zeta+\oint_{\hat{\gamma}_m}\frac{W}{(\zeta-z)R(\zeta)}\,d\zeta+\oint_{\hat{\gamma}_c}\frac{\Omega}{(\zeta-z)R(\zeta)}\,d\zeta\right], \tag{76}$$

where z is inside of $\hat{\gamma}$; and

$$\frac{\partial K}{\partial\mu}(\alpha_j,\vec{\alpha},\mu)=\frac{1}{2\pi i}\begin{vmatrix} \oint_{\hat{\gamma}_m}\frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_m}\frac{\zeta\, d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_m}\frac{d\zeta}{(\zeta-\alpha_j)R(\zeta)} \\ \oint_{\hat{\gamma}_c}\frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_c}\frac{\zeta\, d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_c}\frac{d\zeta}{(\zeta-\alpha_j)R(\zeta)} \\ \oint_{\hat{\gamma}}\frac{f_\mu(\zeta)\,d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}}\frac{\zeta f_\mu(\zeta)\,d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}}\frac{f_\mu(\zeta)\,d\zeta}{(\zeta-\alpha_j)R(\zeta)} \end{vmatrix} \tag{77}$$

or

$$\frac{\partial K}{\partial\mu}(\alpha_j,\vec{\alpha},\mu)=\frac{1}{2\pi i}\left[\oint_{\hat{\gamma}}\frac{f_\mu(\zeta)}{(\zeta-\alpha_j)R(\zeta)}\,d\zeta+\oint_{\hat{\gamma}_m}\frac{W_\mu}{(\zeta-\alpha_j)R(\zeta)}\,d\zeta+\oint_{\hat{\gamma}_c}\frac{\Omega_\mu}{(\zeta-\alpha_j)R(\zeta)}\,d\zeta\right], \tag{78}$$

where f_μ is given by (29). The real constants W_μ and Ω_μ solve the system

$$\begin{cases} \oint_{\hat{\gamma}}\frac{f_\mu(\zeta)}{R(\zeta)}d\zeta+\Omega_\mu\oint_{\hat{\gamma}_c}\frac{d\zeta}{R(\zeta)}+W_\mu\oint_{\hat{\gamma}_m}\frac{d\zeta}{R(\zeta)}=0, \\ \oint_{\hat{\gamma}}\frac{\zeta f_\mu(\zeta)}{R(\zeta)}\,d\zeta+\Omega_\mu\oint_{\hat{\gamma}_c}\frac{\zeta}{R(\zeta)}\,d\zeta+W_\mu\oint_{\hat{\gamma}_m}\frac{\zeta}{R(\zeta)}\,d\zeta=0. \end{cases} \tag{79}$$

Also,

$$D=\begin{vmatrix} \oint_{\hat{\gamma}_m}\frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_m}\frac{\zeta\, d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_c}\frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_c}\frac{\zeta\, d\zeta}{R(\zeta)} \end{vmatrix}. \tag{80}$$

Theorem A.2 (μ-perturbation in genus 2). *Consider a finite-length non-self-intersecting contour γ_0 in the complex plane consisting of a union of oriented arcs $\gamma_0=\gamma_m\cup\gamma_c\cup[\alpha_0,\overline{\alpha}_0]$ with the distinct arc end points $\vec{\alpha}_0=(\alpha_0,\alpha_2,\alpha_4)$ in the upper half-plane and depending on a parameter μ* (see Figure 4). *Assume $\vec{\alpha}_0$ and μ_0 satisfy a system of equations*

$$\begin{cases} K(\alpha_0, \vec{\alpha}_0, \mu_0) = 0, \\ K(\alpha_2, \vec{\alpha}_0, \mu_0) = 0, \\ K(\alpha_4, \vec{\alpha}_0, \mu_0) = 0, \end{cases}$$

and f is given by (21)*. Let $\gamma = \gamma(\vec{\alpha}, \mu)$ be the contour of an RHP which seeks a function $h(z)$ which satisfies the conditions*

$$\begin{cases} h_+(z) + h_-(z) = 0 & \text{on } \gamma_{m,0} = [\alpha_0, \overline{\alpha}_0], \\ h_+(z) + h_-(z) = 2W & \text{on } \gamma_m, \\ h_+(z) - h_-(z) = 2\Omega & \text{on } \gamma_c, \\ h(z) + f(z) \text{ is analytic in } \overline{\mathbb{C}} \setminus \gamma, \end{cases} \tag{81}$$

where $\Omega = \Omega(\vec{\alpha}, \mu)$ and $W = W(\vec{\alpha}, \mu)$ are real constants whose numerical values will be determined from the RH conditions. Assume that there is a function $h(z, \vec{\alpha}_0, \mu_0)$ which satisfies (81) *and suppose $h'(z, \vec{\alpha}_0, \mu_0)/R(z, \vec{\alpha}_0) \neq 0$ for all z on γ_0.*

Then there is a contour $\gamma(\vec{\alpha}, \mu) \in \Gamma(\vec{\alpha}, \mu)$ such that the solution $\vec{\alpha} = \vec{\alpha}(\mu)$ of the system

$$\begin{cases} K(\alpha_0, \vec{\alpha}, \mu) = 0, \\ K(\alpha_2, \vec{\alpha}, \mu) = 0, \\ K(\alpha_4, \vec{\alpha}, \mu) = 0 \end{cases} \tag{82}$$

and $h(z, \vec{\alpha}(\mu), \mu)$ which solves (81) *are uniquely defined and continuously differentiable in μ in some neighborhood of μ_0.*

Moreover,

$$\begin{aligned} \frac{\partial \alpha_0}{\partial \mu}(x, t, \mu) &= -\frac{2\pi i \dfrac{\partial K}{\partial \mu}(\alpha_0, \vec{\alpha}, \mu)}{D \displaystyle\oint_{\hat{\gamma}(\mu)} \frac{f'(\zeta)}{(\zeta - \alpha_0) R(\zeta)} \, d\zeta}, \\ \frac{\partial \alpha_2}{\partial \mu}(x, t, \mu) &= -\frac{2\pi i \dfrac{\partial K}{\partial \mu}(\alpha_2, \vec{\alpha}, \mu)}{D \displaystyle\oint_{\hat{\gamma}(\mu)} \frac{f'(\zeta)}{(\zeta - \alpha_2) R(\zeta)} \, d\zeta}, \\ \frac{\partial \alpha_4}{\partial \mu}(x, t, \mu) &= -\frac{2\pi i \dfrac{\partial K}{\partial \mu}(\alpha_4, \vec{\alpha}, \mu)}{D \displaystyle\oint_{\hat{\gamma}(\mu)} \frac{f'(\zeta)}{(\zeta - \alpha_4) R(\zeta)} \, d\zeta}, \end{aligned} \tag{83}$$

$$\frac{\partial h}{\partial \mu}(z, x, t, \mu) = \frac{R(z)}{2\pi i} \oint_{\hat{\gamma}(\mu)} \frac{\dfrac{\partial f}{\partial \mu}(\zeta)}{(\zeta - z) R(\zeta)} \, d\zeta, \tag{84}$$

where z is inside of $\hat{\gamma}$.

Furthermore, $\Omega(\mu)=\Omega(\vec{\alpha}(\mu),\mu)$ and $W(\mu)=W(\vec{\alpha}(\mu),\mu)$ are defined and continuously differentiable in μ in some open neighborhood of μ_0, and

$$\frac{\partial\Omega}{\partial\mu}(x,t,\mu)=-\frac{1}{D}\begin{vmatrix}\oint_{\hat{\gamma}_m}\frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_m}\frac{\zeta\,d\zeta}{R(\zeta)}\\ \oint_{\hat{\gamma}}\frac{f_\mu(\zeta)}{R(\zeta)}d\zeta & \oint_{\hat{\gamma}}\frac{\zeta f_\mu(\zeta)}{R(\zeta)}d\zeta\end{vmatrix}, \tag{85}$$

$$\frac{\partial W}{\partial\mu}(x,t,\mu)=-\frac{1}{D}\begin{vmatrix}\oint_{\hat{\gamma}}\frac{f_\mu(\zeta)}{R(\zeta)}d\zeta & \oint_{\hat{\gamma}}\frac{\zeta f_\mu(\zeta)}{R(\zeta)}d\zeta\\ \oint_{\hat{\gamma}_c}\frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_c}\frac{\zeta\,d\zeta}{R(\zeta)}\end{vmatrix}, \tag{86}$$

where $\alpha_j=\alpha_j(x,t,\mu)$, $R(\zeta)=R(\zeta,\vec{\alpha}(x,t,\mu))$, $f(\zeta)=f(\zeta,x,t,\mu)$, $f_\mu(\zeta)=\dfrac{\partial f}{\partial\mu}(\zeta,x,t,\mu)$, and $D=D(\vec{\alpha}(x,t,\mu))$.

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Received 29 Nov 2012. Revised 13 Jun 2014. Accepted 9 Jan 2015.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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ANALYSIS & PDE

Volume 8 No. 2 2015

2157-5045(2015)8:2;1-C