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TUNNEL EFFECT FOR SEMICLASSICAL RANDOM WALKS

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We study a semiclassical random walk with respect to a probability measure with a finite number n_0 of wells. We show that the associated operator has exactly n_0 eigenvalues exponentially close to 1 (in the semiclassical sense), and that the others are $\mathcal{O}(h)$ away from 1. We also give an asymptotic of these small eigenvalues. The key ingredient in our approach is a general factorization result of pseudodifferential operators, which allows us to use recent results on the Witten Laplacian.

1. Introduction

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a smooth function and let $h \in]0, 1]$ denote a small parameter throughout. Under suitable assumptions specified later, the density $e^{-\phi(x)/h}$ is integrable and there exists $Z_h > 0$ such that $d\mu_h(x) = Z_h e^{-\phi(x)/h} dx$ defines a probability measure on $\mathbb{R}^d$. We can associate to μ_h the Markov kernel $t_h(x, dy)$ given by

$$t_h(x, dy) = \frac{1}{\mu_h(B(x, h))} \mathbb{1}_{|x-y|<h} \, d\mu_h(y). \tag{1-1}$$

From the point of view of random walks, this kernel can be understood as follows: Assume that at step n the walk is in x_n; then the point x_{n+1} is chosen in the small ball $B(x_n, h)$ uniformly at random with respect to $d\mu_h$. The probability distribution at time $n \in \mathbb{N}$ of a walk starting from x is given by the kernel $t_h^n(x, dy)$. The long-time behavior ($n \to \infty$) of the kernel $t_h^n(x, dy)$ carries information on the ergodicity of the random walk, and has many practical applications (we refer to [Lelièvre et al. 2010] for an overview of computational aspects). Observe that, if ϕ is a Morse function, then the density $e^{-\phi/h}$ concentrates at scale $\sqrt{h}$ around minima of ϕ, whereas the moves of the random walk are at scale h.

Another point of view comes from statistical physics and can be described as follows: One can associate to the kernel $t_h(x, dy)$ an operator $\boldsymbol{T}_h$ acting on the space C_0 of continuous functions going to zero at infinity by the formula

$$\boldsymbol{T}_h f(x) = \int_{\mathbb{R}^d} f(y) t_h(x, dy) = \frac{1}{\mu_h(B(x, h))} \int_{|x-y|<h} f(y) \, d\mu_h(y).$$

This defines a bounded operator on C_0, enjoying the Markov property ($\boldsymbol{T}_h(1) = 1$).

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The transpose $\boldsymbol{T}_h^\star$ of $\boldsymbol{T}_h$ is defined by duality on the set of bounded positive measures $\boldsymbol{M}_b^+$ (resp. bounded measures $\boldsymbol{M}_b$). If $d\nu$ is a bounded measure, we have

$$\boldsymbol{T}_h^\star(d\nu) = \left(\int_{\mathbb{R}^d} \mathbb{1}_{|x-y|<h} \mu_h(B(y,h))^{-1}\, d\nu(y) \right) d\mu_h. \tag{1-2}$$

Assume that a particle in $\mathbb{R}^d$ is distributed according to a probability measure $d\nu$; then $\boldsymbol{T}_h^\star(d\nu)$ represents its distribution after a move according to $t_h(x,dy)$, and the distribution after n steps is then given by $(\boldsymbol{T}_h^\star)^n(d\nu)$. The existence of a limit distribution is strongly related to the existence of an invariant measure. In the present context, one can easily see that $\boldsymbol{T}_h^\star$ admits the invariant measure

$$d\nu_{h,\infty}(x) = \widetilde{Z}_h \mu_h(B(x,h))\, d\mu_h(x),$$

where $\widetilde{Z}_h$ is chosen so that $d\nu_{h,\infty}$ is a probability. The aim of the present paper will be to prove the convergence of $(\boldsymbol{T}_h^\star)^n(d\nu)$ towards $d\nu_{h,\infty}$ when n goes to infinity for any probability measure $d\nu$, and to get precise information on the speed of convergence. Taking $d\nu(y) = \delta_x(y)$, it turns out that it is equivalent to study the convergence of $t_h^n(x,dy)$ towards $d\nu_{h,\infty}$. Note that, in the present setting, proving pointwise convergence (h being fixed) of $t_h^n(x,dy)$ towards the invariant measure is an easy consequence of a general theorem (see [Feller 1971, Theorem 2, p. 272]). The purpose of our approach is to get convergence in a stronger topology and to obtain precise information on the behavior with respect to the semiclassical parameter h.

Before going further, let us recall some elementary properties of $\boldsymbol{T}_h$ that will be useful in the sequel. First, we can see easily from its definition that the operator $\boldsymbol{T}_h$ can be extended as a bounded operator both on $L^\infty(d\nu_{h,\infty})$ and $L^1(d\nu_{h,\infty})$. From the Markov property and the fact that $d\nu_{h,\infty}$ is stationary, it is clear that

$$\|\boldsymbol{T}_h\|_{L^\infty(d\nu_{h,\infty})\to L^\infty(d\nu_{h,\infty})} = \|\boldsymbol{T}_h\|_{L^1(d\nu_{h,\infty})\to L^1(d\nu_{h,\infty})} = 1.$$

Hence, by interpolation, $\boldsymbol{T}_h$ defines also a bounded operator of norm 1 on $L^2(\mathbb{R}^d, d\nu_{h,\infty})$. Finally, observe that $\boldsymbol{T}_h$ is selfadjoint on $L^2(d\nu_{h,\infty})$ (thanks again to the Markov property).

Let us go back to the study of the sequence $(\boldsymbol{T}_h^\star)^n$ and explain the topology we use to study the convergence of this sequence. Instead of looking at this evolution on the full set of bounded measures, we restrict the analysis by introducing the stable Hilbert space

$$\mathscr{H}_h = L^2(d\nu_{h,\infty}) = \left\{ f \text{ measurable on } \mathbb{R}^d \text{ such that } \textstyle\int |f(x)|^2\, d\nu_{h,\infty} < \infty \right\}, \tag{1-3}$$

for which we have a natural injection with norm 1, $\mathscr{J} : \mathscr{H}_h \hookrightarrow \boldsymbol{M}_b$, when identifying an absolutely continuous measure $d\nu_h = f(x) d\nu_{h,\infty}$ with its density f. Using (1-2), we can see easily that $\boldsymbol{T}_h^\star \circ \mathscr{J} = \mathscr{J} \circ \boldsymbol{T}_h$. From this identification, $\boldsymbol{T}_h^\star$ (acting on $\mathscr{H}_h$) inherits the properties of $\boldsymbol{T}_h$:

$$\boldsymbol{T}_h^\star : \mathscr{H}_h \to \mathscr{H}_h \text{ is selfadjoint and continuous with operator norm 1.} \tag{1-4}$$

Hence, its spectrum is contained in the interval $[-1, 1]$. Moreover, we will see later that -1 is sufficiently far from the spectrum. Since we are interested in the convergence of $(\boldsymbol{T}_h^\star)^n$ in the L^2 topology, it is then sufficient for our purpose to give a precise description of the spectrum of $\boldsymbol{T}_h$ near 1.

Convergence of Markov chains to stationary distributions is a wide area of research with many applications. Knowing that a computable Markov kernel converges to a given distribution may be very useful in practice. In particular, it is often used to sample a given probability in order to implement Monte Carlo methods (see [Lelièvre et al. 2010] for numerous algorithms and computational aspects). However, most results giving a priori bounds on the speed of convergence for such algorithms hold for discrete state space (we refer to [Diaconis 2009] for a state of the art on Monte Carlo Markov chain methods).

This point of view is also used to track extremal points of any function by simulated annealing procedure. For example, this was used in [Holley and Stroock 1988] on finite state space and in [Holley et al. 1989; Miclo 1992] on continuous state space.

Relatedly, let us recall that the study of time-continuous processes is of current interest in statistical physics (see for instance the work of Bovier, Eckhoff, Gayrard and Klein [Bovier et al. 2004; 2005] on metastable states).

More recently, Diaconis and Lebeau [2009] obtained first results on discrete time processes on continuous state space. This approach was then further developed in [Diaconis et al. 2011] to get convergence results on the Metropolis algorithm on bounded domains of Euclidean space. Similar results were also obtained in [Lebeau and Michel 2010; Guillarmou and Michel 2011] in various geometric situations. In all these papers, the probability $d\mu_h$ is independent of h, which leads ultimately to a spectral gap of order h^2. Here, the situation is quite different and somehow "more semiclassical". This permits us to exhibit situations with very small spectral gap of order $e^{-c/h}$. The precise asymptotic of this gap (and more generally of the eigenvalues close to 1) is driven by the tunnel effect between wells (see [Helffer and Sjöstrand 1984] for results in the case of Schrödinger operators). In this paper, we shall compute accurately this quantity under the following assumptions on ϕ:

Hypothesis 1. *We suppose that ϕ is a Morse function with nondegenerate critical points and that there exist c, $R > 0$ and some constants $C_\alpha > 0$, $\alpha \in \mathbb{N}^d$ such that, for all $|x| \geq R$, we have*

$$|\partial_x^\alpha \phi(x)| \leq C_\alpha, \quad |\nabla\phi(x)| \geq c \quad \text{and} \quad \phi(x) \geq c|x| \qquad \text{for all } \alpha \in \mathbb{N}^d \setminus \{0\}.$$

In particular, there is a finite number of critical points.

Observe that functions ϕ satisfying this assumption are at most linear at infinity. It may be possible to relax this assumption to quadratic growth at infinity, and we guess our results hold true also in this context. However, it doesn't seem possible to get a complete proof with the class of symbols used in this paper.

Under the above assumption, it is clear that $d\mu_h(x) = Z_h e^{-\phi(x)/h}\,dx$ is a probability measure. For the following, we call $\mathcal{U}$ the set of critical points $\boldsymbol{u}$. We denote by $\mathcal{U}^{(0)}$ the set of minima of ϕ and by $\mathcal{U}^{(1)}$ the set of saddle points, i.e., the critical points with index 1 (note that this set may be empty). We also introduce $n_j = \sharp\,\mathcal{U}^{(j)}$, $j = 0, 1$, the number of elements of $\mathcal{U}^{(j)}$.

We shall first prove the following result:

Theorem 1.1. *There exist δ, $h_0 > 0$ such that the following assertions hold true for $h \in]0, h_0]$: First, $\sigma(\boldsymbol{T}_h^\star) \subset [-1+\delta, 1]$ and $\sigma_{\text{ess}}(\boldsymbol{T}_h^\star) \subset [-1+\delta, 1-\delta]$. Moreover, $\boldsymbol{T}_h^\star$ has exactly n_0 eigenvalues in $[1-\delta h, 1]$, which are in fact in $[1-e^{-\delta/h}, 1]$. Lastly, 1 is a simple eigenvalue for the eigenstate $\nu_{h,\infty} \in \mathcal{H}_h$.*

This theorem will be proved in the next section. The goal of this paper is to describe accurately the eigenvalues close to 1. We will see later that describing the eigenvalues of $\boldsymbol{T}_h^\star$ close to 1 has many common points with the spectral study of the so-called semiclassical Witten Laplacian (see Section 4). We introduce the following generic assumptions on the critical points of ϕ:

Hypothesis 2. *We suppose that the values $\phi(\boldsymbol{s})-\phi(\boldsymbol{m})$ are distinct for any $\boldsymbol{s}\in\mathcal{U}^{(1)}$ and $\boldsymbol{m}\in\mathcal{U}^{(0)}$.*

Note that this generic assumption could easily be relaxed at the cost of messy notation and less precise statements, following, e.g., [Hérau et al. 2011], and that we chose to focus in this article on other particularities of the problem.

Let us recall that, under the above assumptions, there exists a labeling of minima and saddle points, $\mathcal{U}^{(0)}=\{\boldsymbol{m}_k : k=1,\dots,n_0\}$ and $\mathcal{U}^{(1)}=\{\boldsymbol{s}_j : j=2,\dots,n_1+1\}$, which permits us to describe the low-lying eigenvalues of the Witten Laplacian (see [Helffer et al. 2004; Hérau et al. 2011], for instance). Observe that the enumeration of $\mathcal{U}^{(1)}$ starts with $j=2$, since we will need a fictional saddle point $\boldsymbol{s}_1=+\infty$. We shall recall this labeling procedure in the Appendix.

Let us denote by $1=\lambda_1^\star(h)>\lambda_2^\star(h)\geq\cdots\geq\lambda_{n_0}^\star(h)$ the n_0 largest eigenvalues of $\boldsymbol{T}_h^\star$. The main result of this paper is the following:

Theorem 1.2. *Under Hypotheses 1 and 2, there exists a labeling of minima and saddle points and constants $\alpha, h_0>0$ such that, for all $k=2,\dots,n_0$ and for any $h\in\,]0,h_0]$,*

$$1-\lambda_k^\star(h)=\frac{h}{(2d+4)\pi}\mu_k\sqrt{\left|\frac{\det\phi''(\boldsymbol{m}_k)}{\det\phi''(\boldsymbol{s}_k)}\right|}e^{-2S_k/h}(1+\mathcal{O}(h)),$$

where $S_k:=\phi(\boldsymbol{s}_k)-\phi(\boldsymbol{m}_k)$ (the Arrhenius number) and $-\mu_k$ denotes the unique negative eigenvalue of ϕ'' at $\boldsymbol{s}_k$.

Remark 1.3. The leading term in the asymptotic of $1-\lambda_k^\star(h)$ above is exactly (up to the factor $(2d+4)$) the one of the k-th eigenvalue of the Witten Laplacian on the 0-forms obtained in [Helffer et al. 2004]. This relationship will be transparent from the proof below.

As an immediate consequence of these results and of the spectral theorem, we get that the convergence to equilibrium holds slowly and that the system has a metastable regime. More precisely, we have the following result, whose proof can be found at the end of Section 5.

Corollary 1.4. *Let $d\nu_h$ be a probability measure in $\mathcal{H}_h$ and assume first that ϕ has a unique minimum. Then, using that $\sigma(\boldsymbol{T}_h^\star)\subset[-1+\delta,1-\delta h]$, it yields*

$$\|(\boldsymbol{T}_h^\star)^n(d\nu_h)-d\nu_{h,\infty}\|_{\mathcal{H}_h}=\mathcal{O}(h)\|d\nu_h\|_{\mathcal{H}_h} \tag{1-5}$$

for all $n\gtrsim|\ln h|h^{-1}$, which corresponds to the Ehrenfest time. But, if ϕ has several minima, we can write

$$(\boldsymbol{T}_h^\star)^n(d\nu_h)=\Pi d\nu_h+\mathcal{O}(h)\|d\nu_h\|_{\mathcal{H}_h} \tag{1-6}$$

for all $h^{-1}|\ln h|\lesssim n\lesssim e^{2S_{n_0}/h}$. Here, Π can be taken as the orthogonal projector on the n_0 functions $\chi_k(x)e^{-(\phi(x)-\phi(\boldsymbol{m}_k))/h}$, where χ_k is any cutoff function near $\boldsymbol{m}_k$.

On the other hand, we have, for any $n \in \mathbb{N}$,

$$\|(\boldsymbol{T}_h^\star)^n(d\nu_h) - d\nu_{h,\infty}\|_{\mathscr{H}_h} \leq (\lambda_2^\star(h))^n \|d\nu_h\|_{\mathscr{H}_h}, \tag{1-7}$$

where $\lambda_2^\star(h)$ is described in Theorem 1.2. Note that this inequality is optimal. In particular, for $n \gtrsim |\ln h| h^{-1} e^{2S_2/h}$, the right-hand side of (1-7) is of order $\mathcal{O}(h)\|d\nu_h\|_{\mathscr{H}_h}$.

Thus, for a reasonable number of iterations (which guarantees (1-5)), 1 seems to be an eigenvalue of multiplicity n_0; whereas, for a very large number of iterations, the system returns to equilibrium. Then, (1-6) is a metastable regime.

Since $t_h(x, dy)$ is absolutely continuous with respect to $d\nu_{h,\infty}$, then $(\boldsymbol{T}_h^\star)^n(\delta_{y=x}) = t_h^n(x, dy)$ belongs to $\mathscr{H}_h$ for any $n \geq 1$. Hence, the above estimate and the fact that $d\nu_{h,\infty}$ is invariant show that

$$\|t_h^n(x, dy) - d\nu_{h,\infty}\|_{\mathscr{H}_h} \leq (\lambda_2^\star(h))^{n-1} \|t_h(x, dy)\|_{\mathscr{H}_h}.$$

Moreover, the prefactor $\|t_h(x, dy)\|_{\mathscr{H}_h}$ could be easily computed but depends on x and h.

Throughout this paper, we use semiclassical analysis (see [Dimassi and Sjöstrand 1999; Martinez 2002; Zworski 2012] for expository books on this theory). Let us recall that a function $m : \mathbb{R}^d \to \mathbb{R}^+$ is an order function if there exists $N_0 \in \mathbb{N}$ and a constant $C > 0$ such that, for all $x, y \in \mathbb{R}^d$, $m(x) \leq C\langle x - y\rangle^{N_0} m(y)$. Here and throughout we use the notation $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$. This definition can be extended to functions $m : \mathbb{R}^d \times \mathbb{C}^{d'} \to \mathbb{R}^+$ by identifying $\mathbb{R}^d \times \mathbb{C}^{d'}$ with $\mathbb{R}^{d+2d'}$. Given an order function m on $T^*\mathbb{R}^d \simeq \mathbb{R}^{2d}$, we will denote by $S^0(m)$ the space of semiclassical functions on $T^*\mathbb{R}^d$ whose derivatives are all bounded by m, and by $\Psi^0(m)$ the set of corresponding pseudodifferential operators. For any $\tau \in]0, \infty]$ and any order function m on $\mathbb{R}^d \times \mathbb{C}^d$, we will denote by $S_\tau^0(m)$ the set of symbols which are analytic with respect to ξ in the strip $|\operatorname{Im}\xi| < \tau$ and bounded by some constant times $m(x, \xi)$ in this strip. We will denote by $S_\infty^0(m)$ the union over $\tau > 0$ of $S_\tau^0(m)$. We denote by $\Psi_\tau^0(m)$ the set of corresponding operators. Lastly, we say that a symbol p is classical if it admits an asymptotic expansion $p(x, \xi; h) \sim \sum_{j\geq 0} h^j p_j(x, \xi)$. We will denote by $S_{\tau,\mathrm{cl}}^0(m)$ and $S_{\mathrm{cl}}^0(m)$ the corresponding classes of symbols.

We will also need some matrix-valued pseudodifferential operators. Let $\mathcal{M}_{p,q}$ denote the set of real-valued matrices with p rows and q columns, and $\mathcal{M}_p = \mathcal{M}_{p,p}$. Let $\mathscr{A} : T^*\mathbb{R}^d \to \mathcal{M}_{p,q}$ be a smooth function. We will say that $\mathscr{A}$ is a (p, q)-matrix weight if $\mathscr{A}(x, \xi) = (a_{i,j}(x, \xi))_{i,j}$ and $a_{i,j}$ is an order function for every $i = 1, \dots, p$ and $j = 1, \dots, q$. If $p = q$, we will simply say that $\mathscr{A}$ is a q-matrix weight.

Given a (p, q)-matrix weight $\mathscr{A}$, we will denote by $S^0(\mathscr{A})$ the set of symbols $p(x, \xi) = (p_{i,j}(x, \xi))_{i,j}$ defined on $T^*\mathbb{R}^d$ with values in $\mathcal{M}_{p,q}$ such that, for all i, j, $p_{i,j} \in S^0(a_{i,j})$, and by $\Psi^0(\mathcal{M}_{p,q})$ the set of corresponding pseudodifferential operators. Obvious extensions of these definitions leads to the definition of matrix-valued symbols analytic w.r.t. to ξ and the corresponding operators, $S_\tau^0(\mathscr{A})$ and $\Psi_\tau^0(\mathscr{A})$. In the following, we shall mainly use the Weyl semiclassical quantization of symbols, defined by

$$\operatorname{Op}(p)u(x) = (2\pi h)^{-d} \int_{T^*\mathbb{R}^d} e^{ih^{-1}(x-y)\xi} p\big(\tfrac{1}{2}(x + y), \xi\big) u(y)\, dy\, d\xi \tag{1-8}$$

for $p \in S^0(\mathscr{A})$. We shall also use the following notations. Given two pseudodifferential operators A and B, we shall write $A = B + \Psi^k(m)$ if the difference $A - B$ belongs to $\Psi^k(m)$. At the level of symbols, we shall write $a = b + S^k(m)$ instead of $a - b \in S^k(m)$.

The preceding theorem is close — in the spirit and in the proof — to the ones given for the Witten Laplacian in [Helffer et al. 2004] and for the Kramers–Fokker–Planck operators in [Hérau et al. 2011]. In those works, the results are deeply linked with some properties inherited from a so-called supersymmetric structure, which allow the operators to be written as twisted Hodge Laplacians of the form

$$P = d_{\phi,h}^* A d_{\phi,h},$$

where $\boldsymbol{d}$ is the usual differential, $d_{\phi,h} = h\boldsymbol{d} + d\phi(x)\wedge = e^{-\phi/h} h\boldsymbol{d}\, e^{\phi/h}$ is the differential twisted by ϕ, and A is a constant matrix in $\mathscr{M}_d$. Here we are able to recover a supersymmetric-type structure, and the main ingredients for the study of the exponentially small eigenvalues are therefore available. This is contained in the following theorem, that we give in rather general context since it may be useful in other situations.

Let us introduce the d-matrix weights Ξ, $\mathscr{A} : T^*\mathbb{R}^d \to \mathscr{M}_d$ given by $\mathscr{A}_{i,j}(x,\xi) = (\langle \xi_i \rangle \langle \xi_j \rangle)^{-1}$, $\Xi_{i,j} = \delta_{i,j} \langle \xi_i \rangle$, and observe that $(\Xi \mathscr{A})_{i,j} = \langle \xi_j \rangle^{-1}$. In the following theorem, we state an exact factorization result, which will be the key point in our approach.

Theorem 1.5. *Let $p(x,\xi;h) \in S^0_\infty(1)$ and let $P_h = \operatorname{Op}(p)$. Suppose that $p(x,\xi;h) = p_0(x,\xi) + S^0_\infty(h)$ and that, for all $(x,\xi) \in \mathbb{R}^{2d}$, $p(x,\xi;h)$ is real. Let ϕ satisfy Hypotheses 1 and 2 and assume that the following assumptions hold true*:

(i) $P_h(e^{-\phi/h}) = 0$;

(ii) *for all $x \in \mathbb{R}^d$, the function $\xi \in \mathbb{R}^d \mapsto p(x,\xi;h)$ is even*;

(iii) *for all $\delta > 0$, there exists $\alpha > 0$ such that, for all $(x,\xi) \in T^*\mathbb{R}^d$, $d(x,\mathscr{U})^2 + |\xi|^2 \geq \delta$ implies $p_0(x,\xi) \geq \alpha$*;

(iv) *for any critical point $\boldsymbol{u} \in \mathscr{U}$, we have*

$$p_0(x,\xi) = |\xi|^2 + |\nabla\phi(x)|^2 + r(x,\xi)$$

with $r(x,\xi) = \mathcal{O}(|(x-\boldsymbol{u},\xi)|^3)$ near $(\boldsymbol{u},0)$.

Then, for $h > 0$ small enough, there exists a symbol $q \in S^0(\Xi\mathscr{A})$ satisfying the following properties:

First, $P_h = d_{\phi,h}^* Q^* Q d_{\phi,h}$ with $Q = \operatorname{Op}(q)$.

Next, $q(x,\xi;h) = q_0(x,\xi) + S^0(h\Xi\mathscr{A})$ and, for any critical point $\boldsymbol{u} \in \mathscr{U}$, we have

$$q_0(x,\xi) = \operatorname{Id} + \mathcal{O}(|(x-\boldsymbol{u},\xi)|).$$

If we assume additionally that $r(x,\xi) = \mathcal{O}(|(x-\boldsymbol{u},\xi)|^4)$, then $q_0(x,\xi) = \operatorname{Id} + \mathcal{O}(|(x-\boldsymbol{u},\xi)|^2)$ near $(\boldsymbol{u},0)$ for any critical point $\boldsymbol{u} \in \mathscr{U}$.

Lastly, if $p \in S^0_{\mathrm{cl}}(1)$ then $q \in S^0_{\mathrm{cl}}(\Xi\mathscr{A})$.

As already mentioned, we decided in this paper not to give results in the most general case so that technical aspects do not hide the main ideas. Nevertheless, we would like to mention here some possible generalizations of the preceding result.

First, it should certainly be possible to use more general order functions and to prove factorization results for symbols in other classes (for instance $S^0(\langle(x,\xi)\rangle^2)$). This should allow us to see the supersymmetric structure of the Witten Laplacian as a special case of our result. In other words, the symbol $p(x,\xi;h) = |\xi|^2 + |\nabla\phi(x)|^2 - h\Delta\phi(x)$ would satisfy assumptions (i) to (iv) above.

The analyticity of the symbol p with respect to the variable ξ is certainly not necessary in order to get a factorization result (it suffices to take a nonanalytic q in the conclusion to see it). Nevertheless, since our approach consists in conjugating the operator by $e^{-\phi/h}$, it seems difficult to deal with nonanalytic symbols. Moreover, using a regularization procedure in the proof the above theorem, it is certainly possible to prove that the symbol q above can be chosen in a class $S^0_\tau(\Xi\mathscr{A})$ for some $\tau > 0$. Using this additional property, it may be possible to prove some Agmon estimates, construct more accurate quasimodes (on the 1-forms), and then to prove a full asymptotic expansion in Theorem 1.2.

A more delicate question should be to get rid of the parity assumption (ii). It is clear that this assumption is not necessary (take $q(x,\xi) = \langle\xi\rangle^{-2}\big(\mathrm{Id} + \mathrm{diag}(\xi_i/\langle\xi\rangle)\big)$ in the conclusion), but it seems difficult to prove a factorization result without it. For instance, if we consider the case $\phi = 0$ in dimension 1 (which doesn't fit exactly in our framework but enlightens the situation) then $P_h = hD_x$ cannot be smoothly factorized simultaneously on the left and on the right.

As will be seen in the proof below, the operator Q (as well as Q^*Q) above is not unique. Trying to characterize the set of all possible Q should be also a question of interest.

The optimality of assumption (iv) should be questioned. Expanding q_0 near $(\boldsymbol{u},0)$, we can see that we necessarily have

$$p_0(x,\xi) = |q_0(\boldsymbol{u},0)(\xi - i\nabla\phi)|^2 + \mathscr{O}(|(x-\boldsymbol{u},\xi)|^3)$$

near any critical point. In assumption (iv) we consider the case $q_0(\boldsymbol{u},0) = \mathrm{Id}$, but it could easily be relaxed to any invertible matrix $q_0(\boldsymbol{u},0)$.

Lastly, we shall mention that, for semiclassical differential operators of order 2, a supersymmetric structure (in the class of differential operators) was established by Hérau, Hitrik and Sjöstrand [Hérau et al. 2013]. This result requires fewer assumptions, but doesn't hold true in any good class of symbols.

The plan of the article is the following. In the next section we analyze the structure of the operator $\boldsymbol{T}_h^\star$ and prove the first results on the spectrum stated in (1-1). In Section 3 we prove Theorem 1.5 and apply it to the case of the random walk operator. In Section 4, we prove some preliminary spectral results, and in Section 5 we prove Theorem 1.2.

2. Structure of the operator and first spectral results

In this section, we analyze the structure of the spectrum of the operator $\boldsymbol{T}_h^\star$ on the space $\mathscr{H}_h = L^2(d\nu_{h,\infty})$ (see (1-3)). But it is more convenient to work with the standard Lebesgue measure than with the measure $d\nu_{h,\infty}$. We then introduce the Maxwellian $\mathcal{M}_h$, defined by

$$d\nu_{h,\infty} = \mathcal{M}_h(x)\,dx, \quad \text{so that} \quad \mathcal{M}_h = \widetilde{Z}_h \mu_h(B_h(x)) Z_h e^{-\phi(x)/h}, \tag{2-1}$$

and we make the change of function

$$\mathcal{U}_h u(x) := \mathcal{M}_h^{-1/2}(x) u(x),$$

where $\mathcal{U}_h$ is unitary from $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d, dx)$ to $\mathcal{H}_h$. Letting

$$T_h := \mathcal{U}_h^* \boldsymbol{T}_h^\star \mathcal{U}_h, \tag{2-2}$$

the conjugated operator acting in $L^2(\mathbb{R}^d)$, we have

$$\begin{aligned} T_h u(x) &= Z_h \mathcal{M}_h^{-1/2}(x) e^{-\phi(x)/h} \int_{\mathbb{R}^d} \mathbb{1}_{|x-y|<h} \mathcal{M}_h^{1/2}(y) \mu_h(B(y,h))^{-1} u(y)\,dy \\ &= \left(\frac{Z_h e^{-\phi(x)/h}}{\mu_h(B(x,h))} \right)^{\frac12} \int_{|x-y|<h} u(y) \left(\frac{Z_h e^{-\phi(y)/h}}{\mu_h(B(y,h))} \right)^{\frac12} dy. \end{aligned}$$

We let

$$a_h(x) = (\alpha_d h^d)^{1/2} \left(\frac{Z_h e^{-\phi(x)/h}}{\mu_h(B(x,h))} \right)^{\frac12},$$

and define the operator $\mathbb{G}$ by

$$\mathbb{G} u(x) = \frac{1}{\alpha_d h^d} \int_{|x-y|<h} u(y)\,dy \tag{2-3}$$

where $\alpha_d = \operatorname{vol}(B(0,1))$ denotes the Euclidean volume of the unit ball, so that, with these notations, the operator T_h is

$$T_h = a_h \mathbb{G} a_h, \tag{2-4}$$

i.e.,

$$T_h u(x) = a_h(x) \mathbb{G}(a_h u)(x).$$

We note that

$$a_h^{-2}(x) = \frac{\mu_h(B(x,h)) e^{\phi(x)/h}}{\alpha_d h^d Z_h} = \frac{1}{\alpha_d h^d} \int_{|x-y|<h} e^{(\phi(x)-\phi(y))/h}\,dy = e^{\phi(x)/h} \mathbb{G}(e^{-\phi/h})(x). \tag{2-5}$$

We now collect some properties of $\mathbb{G}$ and a_h.

One simple but fundamental observation is that $\mathbb{G}$ is a semiclassical Fourier multiplier, $\mathbb{G} = G(hD) = \operatorname{Op}(G)$, where

$$G(\xi) = \frac{1}{\alpha_d} \int_{|z|<1} e^{iz\cdot\xi}\,dz \quad \text{for all } \xi \in \mathbb{R}^d. \tag{2-6}$$

Lemma 2.1. *The function G is analytic on $\mathbb{C}^d$ and enjoys the following properties*:

(i) $G : \mathbb{R}^d \to \mathbb{R}$.

(ii) *There exists* $\delta > 0$ *such that* $G(\mathbb{R}^d) \subset [-1+\delta, 1]$. *Near* $\xi = 0$, *we have*

$$G(\xi) = 1 - \beta_d |\xi|^2 + \mathcal{O}(|\xi|^4),$$

where $\beta_d = (2d+4)^{-1}$. *For any* $r > 0$, $\sup_{|\xi| \geq r} |G(\xi)| < 1$ *and* $\lim_{|\xi| \to \infty} G(\xi) = 0$.

(iii) *For all* $\tau \in \mathbb{R}^d$, *we have* $G(i\tau) \in \mathbb{R}$, $G(i\tau) \geq 1$ *and, for any* $r > 0$, $\inf_{|\tau| \geq r} G(i\tau) > 1$.

(iv) *For all* $\xi, \tau \in \mathbb{R}^d$ *we have* $|G(\xi + i\tau)| \leq G(i\tau)$.

Proof. The function G is analytic on $\mathbb{C}^d$ since it is the Fourier transform of a compactly supported distribution. The fact that $G(\mathbb{R}^d) \subset \mathbb{R}$ is clear using the change of variable $z \mapsto -z$. The second item was shown in [Lebeau and Michel 2010].

We now prove (iii). The fact that $G(i\tau)$ is real for any $\tau \in \mathbb{R}^d$ is clear. Moreover, one can see easily that $\tau \mapsto G(i\tau)$ is radial, so that there exists a function $\Gamma : \mathbb{R} \to \mathbb{R}$ such that, for all $\tau \in \mathbb{R}^d$, $G(i\tau) = \Gamma(|\tau|)$. Simple computations show that Γ enjoys the following properties:

- Γ is even;
- Γ is strictly increasing on $\mathbb{R}_+$;
- $\Gamma(0) = 1$.

This leads directly to the claimed properties for $G(i\tau)$.

Finally, the fact that $|G(\xi + i\tau)| \leq G(i\tau)$ for all $\xi, \tau \in \mathbb{R}^d$ is trivial, since $|e^{iz\cdot(\xi+i\tau)}| = e^{-z\cdot\tau}$ for all $z \in \mathbb{R}^d$. $\square$

Lemma 2.2. *There exist* $c_1, c_2 > 0$ *such that* $c_1 < a_h(x) < c_2$ *for all* $x \in \mathbb{R}^d$ *and* $h \in]0, 1]$. *Moreover, the functions* a_h *and* a_h^{-2} *belong to* $S^0(1)$ *and have classical expansions* $a_h = a_0 + ha_1 + \cdots$ *and* $a_h^{-2} = a_0^{-2} + \cdots$. *In addition,*

$$a_0(x) = G(i\nabla\phi(x))^{-1/2},$$
$$a_1(x) = G(i\nabla\phi(x))^{-3/2} \frac{1}{4\alpha_d} \int_{|z|<1} e^{-\nabla\phi(x)\cdot z} \langle \phi''(x)z, z\rangle \, dz.$$

Lastly, there exist $c_0, R > 0$ *such that, for all* $|x| \geq R$, $a_h^{-2}(x) \geq 1 + c_0$ *for* $h > 0$ *small enough.*

Proof. By a simple change of variable, we have

$$a_h^{-2}(x) = \frac{1}{\alpha_d} \int_{|z|<1} e^{(\phi(x)-\phi(x+hz))/h} \, dz.$$

Since there exists $C > 0$ such that $|\nabla\phi(x)| \leq C$ for all $x \in \mathbb{R}^d$, we can find some constants $c_1, c_2 > 0$ such that $c_1 < a_h(x)^{-2} < c_2$ for all $x \in \mathbb{R}^d$ and $h \in]0, 1]$. Moreover, thanks to the bounds on the derivatives of ϕ, we get easily that derivatives of a_h^{-2} are also bounded. This shows that a_h^{-2} belongs to $S^0(1)$ and, since it is bounded from below by $c_1 > 0$, we get immediately that $a_h \in S^0(1)$.

On the other hand, by simple Taylor expansion, we get that a_h and a_h^{-2} have classical expansions and the required expressions for a_0 and a_1. Since $|\nabla\phi(x)| \geq c > 0$ for x large enough, it follows from

Lemma 2.1(iii) that there exist c_0, $R > 0$ such that, for all $|x| \geq R$, $G(i\nabla\phi(x)) \geq 1 + 2c_0$, and hence $a_h^{-2}(x) \geq 1 + c_0$ for $h > 0$ sufficiently small. □

Since we want to study the spectrum near 1, it will be convenient to introduce

$$P_h := 1 - T_h. \tag{2-7}$$

Using (2-4) and (2-5), we get

$$P_h = a_h(V_h(x) - G(hD_x))a_h \tag{2-8}$$

with $V_h(x) = a_h^{-2}(x) = e^{\phi/h}G(hD_x)(e^{-\phi/h})$. As a consequence of the previous lemmas, we get the following proposition for P_h:

Proposition 2.3. *The operator P_h is a semiclassical pseudodifferential operator whose symbol $p(x, \xi; h)$ in $S^0_\infty(1)$ admits a classical expansion that reads $p = p_0 + hp_1 + \cdots$ with*

$$p_0(x, \xi) = 1 - G(i\nabla\phi(x))^{-1}G(\xi) \geq 0 \quad \text{and} \quad p_1(x, \xi) = G_1(x)G(\xi),$$

where

$$G_1(x) = -G(i\nabla\phi(x))^{-2}\frac{1}{2\alpha_d}\int_{|z|<1} e^{-\nabla\phi(x)\cdot z}\langle\phi''(x)z, z\rangle\, dz = -\beta_d\Delta\phi(\boldsymbol{u}) + \mathcal{O}(|x - \boldsymbol{u}|),$$

near any $\boldsymbol{u} \in \mathcal{U}$.

Proof. The fact that p belongs to $S^0_\infty(1)$ and admits a classical expansion is clear thanks to Lemma 2.1 and Lemma 2.2. From the standard pseudodifferential calculus in Weyl quantization, the symbol p satisfies

$$\begin{aligned} p(x, \xi; h) &= 1 - a_0^2 G - 2a_0a_1Gh - \frac{h}{2i}a_0\{G, a_0\} - \frac{h}{2i}\{a_0, a_0G\} + S^0(h^2) \\ &= 1 - a_0^2G - 2a_0a_1Gh + S^0(h^2). \end{aligned}$$

Combined with Lemma 2.2, this leads to the required expressions for p_0 and p_1.

Finally, the nonnegativity of p_0 comes from the formula

$$p_0 = G(i\nabla\phi(x))^{-1}\big((1 - G(\xi)) + \big(G(i\nabla\phi(x)) - 1\big)\big),$$

and Lemma 2.1, which implies that $1 - G(\xi) \geq 0$ and $G(i\nabla\phi(x)) - 1 \geq 0$. □

We finish this section with the following proposition, which is a part of Theorem 1.1.

Proposition 2.4. *There exist δ, $h_0 > 0$ such that the following assertions hold true for $h \in]0, h_0]$: First, $\sigma(T_h) \subset [-1 + \delta, 1]$ and $\sigma_{\mathrm{ess}}(T_h) \subset [-1 + \delta, 1 - \delta]$. Second, 1 is a simple eigenvalue for the eigenfunction $\mathcal{M}_h^{1/2}$.*

Proof. We start by proving $\sigma(T_h) \subset [-1 + \delta, 1]$. From (1-4), we already know that $\sigma(T_h) \subset [-1, 1]$. Moreover, Lemma 2.1(ii)–(iii) implies $0 \leq a_0(x) \leq 1$ and $G(\mathbb{R}^d) \subset [-1 + \nu, 1]$ for some $\nu > 0$. Thus, we deduce that the symbol $\tau_h(x, \xi)$ of the pseudodifferential operator $T_h \in \Psi^0(1)$ satisfies

$$\tau_h(x, \xi) \geq -1 + \nu + \mathcal{O}(h).$$

Then, Gårding's inequality yields

$$T_h \geq -1 + \tfrac{1}{2}\nu$$

for h small enough. Summing up, we obtain $\sigma(T_h) \subset [-1+\delta, 1]$.

Let us prove the assertion about the essential spectrum. Let $\chi \in C_0^\infty(\mathbb{R}^d; [0, 1])$ be equal to 1 on $B(0, R)$, where $R > 0$ is as in Lemma 2.2. Since $\mathbb{G} = G(hD) \in \Psi^0(1)$ and $\lim_{|\xi|\to\infty} G(\xi) = 0$, the operator

$$T_h - (1-\chi)T_h(1-\chi) = \chi T_h + T_h\chi - \chi T_h \chi$$

is compact. Hence, $\sigma_{\mathrm{ess}}(T_h) = \sigma_{\mathrm{ess}}((1-\chi)T_h(1-\chi))$. Now, for all $u \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}\langle (1-\chi)T_h(1-\chi)u, u\rangle &= \langle \mathbb{G}a_h(1-\chi)u, a_h(1-\chi)u\rangle \\ &\leq \|a_h(1-\chi)u\|^2 \leq (1+c_0)^{-1}\|u\|^2,\end{aligned}$$

since $\|\mathbb{G}\|_{L^2\to L^2} \leq 1$ and $|a_h(1-\chi)| \leq (1+c_0)^{-1/2}$, thanks to Lemma 2.1(iii) and Lemma 2.2. As a consequence, there exists $\delta > 0$ such that $\sigma_{\mathrm{ess}}(T_h) \subset [-1+\delta, 1-\delta]$.

To finish the proof, it remains to show that 1 is a simple eigenvalue. Let $k_h(x, y)$ denotes the distribution kernel of T_h. From (2-3), (2-4) and Lemma 2.2, there exists $\varepsilon > 0$ such that, for all $x, y \in \mathbb{R}^d$,

$$k_h(x, y) \geq \varepsilon h^{-d} \mathbb{1}_{|x-y|<h}. \tag{2-9}$$

We now consider $\widetilde{T}_h = T_h + 1$. Since $\|T_h\| = 1$, the operator $\widetilde{T}_h$ is bounded and nonnegative. Moreover, $\mathcal{M}_h^{1/2}$ is clearly an eigenvector associated to the eigenvalue $\|\widetilde{T}_h\| = 2$. On the other hand, (2-9) implies that $\widetilde{T}_h$ is positivity-preserving (this means that $u(x) \geq 0$ almost everywhere and $u \neq 0$ implies $\widetilde{T}_h u(x) \geq 0$ almost everywhere and $\widetilde{T}_h u \neq 0$). Furthermore, $\widetilde{T}_h$ is ergodic (in the sense that, for any $u, v \in L^2(\mathbb{R}^d)$ nonnegative almost everywhere and not the zero function, there exists $n \geq 1$ such that $\langle u, \widetilde{T}_h^n v\rangle > 0$). Indeed, let u, v be two such functions. We have $\langle u, \widetilde{T}_h^n v\rangle \geq \langle u, T_h^n v\rangle$, where, by (2-9), the distribution kernel of T_h^n satisfies

$$k_h^{(n)}(x, y) \geq \varepsilon_n h^{-d} \mathbb{1}_{|x-y|<(n-1)h}$$

with $\varepsilon_n > 0$. Thus, if $n \geq 1$ is chosen such that $\mathrm{dist}(\text{ess-supp}(u), \text{ess-supp}(v)) < nh$, we have $\langle u, \widetilde{T}_h^n v\rangle > 0$. Lastly, the above properties of $\widetilde{T}_h$ and the Perron–Frobenius theorem (see Theorem XIII.43 of [Reed and Simon 1978]) imply that 1 is a simple eigenvalue of T_h. □

3. Supersymmetric structure

In this section, we prove that the operator $\mathrm{Id} - \boldsymbol{T}_h^\star$ admits a supersymmetric structure and prove Theorem 1.5. We showed in the preceding section that

$$\mathrm{Id} - \boldsymbol{T}_h^\star = \mathcal{U} P_h \mathcal{U}^*$$

and, before proving Theorem 1.5, we state and prove as a corollary the main result on the operator P_h. Recall here that $\beta_d = (2d+4)^{-1}$ and $\Xi\mathcal{A}$ is the matrix symbol defined by $\Xi\mathcal{A}_{i,j} = \langle \xi_j\rangle^{-1}$ for all $i, j = 1, \dots, d$.

Corollary 3.1. *There exists a classical symbol $q \in S^0_{\mathrm{cl}}(\Xi\mathscr{A})$ such that the following holds true: First, $P_h = L_\phi^* L_\phi$ with $L_\phi = Q d_{\phi,h} a_h$ and $Q = \mathrm{Op}(q)$. Second, $q = q_0 + \Psi^0(h\Xi\mathscr{A})$ with $q_0(x,\xi) = \beta_d^{1/2}\,\mathrm{Id} + \mathscr{O}(|(x-\boldsymbol{u},\xi)|^2)$ for any critical point $\boldsymbol{u} \in \mathscr{U}$.*

Proof. Since we know that $P_h = a_h(V_h(x) - \mathbb{G})a_h$, we only have to prove that $\beta_d^{-1}\widetilde{P}_h$ satisfies the assumptions of Theorem 1.5, where

$$\widetilde{P}_h = V_h(x) - G(hD). \tag{3-1}$$

Assumption (i) is satisfied by construction.

Observe that, thanks to Proposition 2.3, $\widetilde{P}_h$ is a pseudodifferential operator and, since the variables x and ξ are separated, its symbol in any quantization is given by $\tilde{p}_h(x,\xi) = V_h(x) - G(\xi)$. Moreover, Lemma 2.2 and Proposition 2.3 show that $\tilde{p}_h$ admits a classical expansion $\tilde{p} = \sum_{j=0}^{\infty} h^j \tilde{p}_j$ with $\tilde{p}_j$, $j \geq 1$, depending only on x, and $\tilde{p}_0(x,\xi) = G(i\nabla\phi(x)) - G(\xi)$. Hence, it follows from Lemma 2.1 that $\tilde{p}$ satisfies assumptions (ii) and (iii).

Finally, it follows from Lemma 2.1(ii) that, near $(\boldsymbol{u},0)$ (for any $\boldsymbol{u} \in \mathscr{U}$), we have

$$\tilde{p}(x,\xi) = \beta_d(|\xi|^2 + |\nabla\phi(x)|^2) + \mathscr{O}(|(x-\boldsymbol{u},\xi)|^4) + S^0(h),$$

so that we can apply Theorem 1.5 in the case where $r = \mathscr{O}(|(x-\boldsymbol{u},\xi)|^4)$. Taking into account the multiplication by a_h completes the proof for P_h. $\square$

Proof of Theorem 1.5. Given a symbol $p \in S^0(1)$ we recall first the well-known left and right quantizations

$$\mathrm{Op}^l(p)u(x) = (2\pi h)^{-d} \int_{T^*\mathbb{R}^d} e^{ih^{-1}(x-y)\xi} p(x,\xi)u(y)\,dy\,d\xi \tag{3-2}$$

and

$$\mathrm{Op}^r(p)u(x) = (2\pi h)^{-d} \int_{T^*\mathbb{R}^d} e^{ih^{-1}(x-y)\xi} p(y,\xi)u(y)\,dy\,d\xi. \tag{3-3}$$

If $p(x,y,\xi)$ belongs to $S^0(1)$, we define $\widetilde{\mathrm{Op}}(p)$, by

$$\widetilde{\mathrm{Op}}(p)(u)(x) = (2\pi h)^{-d} \int_{T^*\mathbb{R}^d} e^{ih^{-1}(x-y)\xi} p(x,y,\xi)u(y)\,dy\,d\xi, \tag{3-4}$$

We recall the formula allowing us to pass from one of these quantizations to the other. If $p(x,y,\xi)$ belongs to $S^0(1)$, then $\widetilde{\mathrm{Op}}(p) = \mathrm{Op}^l(p_l) = \mathrm{Op}^r(p_r)$ with

$$p_l(x,\xi) = (2\pi h)^{-d} \int_{T^*\mathbb{R}^d} e^{ih^{-1}z(\xi'-\xi)} p(x,x-z,\xi')\,d\xi'\,dz, \tag{3-5}$$

and

$$p_r(y,\xi) = (2\pi h)^{-d} \int_{T^*\mathbb{R}^d} e^{ih^{-1}z(\xi'-\xi)} p(y+z,y,\xi')\,d\xi'\,dz. \tag{3-6}$$

Recall that we introduced the d-matrix weight $\mathscr{A} : T^*\mathbb{R}^d \to \mathscr{M}_d$ given by $\mathscr{A}_{i,j}(x,\xi) = (\langle\xi_i\rangle\langle\xi_j\rangle)^{-1}$. Suppose that p satisfies the hypotheses of Theorem 1.5: $P = \mathrm{Op}(p)$ with $p \in S^0_\infty(1)$, $p(x,\xi;h) = p_0(x,\xi) + S^0(h)$ such that:

(i) $P(e^{-\phi/h}) = 0$;

(ii) for all $x \in \mathbb{R}^d$, the function $\xi \in \mathbb{R}^d \mapsto p(x, \xi; h)$ is even;

(iii) for all $\delta > 0$, there exists $\alpha > 0$ such that, for all $(x, \xi) \in T^*\mathbb{R}^d$, $d(x, \mathcal{U})^2 + |\xi|^2 \geq \delta$ implies $p_0(x, \xi) \geq \alpha$;

(iv) near any critical points $\boldsymbol{u} \in \mathcal{U}$ we have

$$p_0(x, \xi) = |\xi|^2 + |\nabla\phi(x)|^2 + r(x, \xi)$$

with either $r = \mathcal{O}(|(x - \boldsymbol{u}, \xi)|^3)$ (assumption (A2)), or $r = \mathcal{O}(|(x - \boldsymbol{u}, \xi)|^4)$ (assumption (A2′)).

The symbol p may depend on h, but we omit this dependence in order to lighten the notations.

The proof goes in several steps. First we prove that there exists a symbol $\hat{q} \in S^0_\infty(\mathcal{A})$ such that

$$P_h = d^*_{\phi,h} \widehat{Q} d_{\phi,h}, \quad \text{where } \widehat{Q} = \mathrm{Op}(\hat{q}).$$

In a moment we shall prove that the operator $\widehat{Q}$ can be chosen so that $\widehat{Q} = Q^* Q$ for some pseudodifferential operator Q satisfying some good properties.

Let us start with the first step. For this purpose we need the following lemma:

Lemma 3.2. *Let $p \in S^0_\infty(1)$ and $P_h = \mathrm{Op}(p)$. Assume that, for all $x \in \mathbb{R}$, the function $\xi \mapsto p(x, \xi; h)$ is even. Suppose also that $P_h(e^{-\phi/h}) = 0$. Then, there exists $\hat{q} \in S^0_\infty(\mathcal{A})$ such that $P_h = d^*_{\phi,h} \widehat{Q} d_{\phi,h}$ with $\widehat{Q} = \mathrm{Op}(\hat{q})$. Moreover, if p has a principal symbol then so does $\hat{q}$, and if $p \in S^0_{\infty,\mathrm{cl}}$ then $\hat{q} \in S^0_{\infty,\mathrm{cl}}$.*

Remark 3.3. Since $P_h(e^{-\phi/h}) = 0$, it is quite clear that P_h can be factorized by $d_{\phi,h}$ on the right. On the other hand, the fact that P_h can be factorized by $d^*_{\phi,h}$ on the left necessarily implies that $P^*_h(e^{-\phi/h}) = 0$. At first glance, there is no reason for this identity to hold true, since we don't suppose in the above lemma that P_h is selfadjoint. This is actually verified for the following reason. Start from $\mathrm{Op}(p)(e^{-\phi/h}) = 0$; then, taking the conjugate and using the fact that ϕ is real, we get

$$\mathrm{Op}(\bar{p}(x, -\xi))(e^{-\phi/h}) = 0.$$

Hence, the parity assumption on p implies that $\mathrm{Op}(p)^*(e^{-\phi/h}) = 0$.

Proof of Lemma 3.2. The fundamental, simple remark is that, if a is a symbol such that $a(x, \xi) = b(x, \xi) \cdot \xi$, then the operator $\mathrm{Op}^l(a)$ can be factorized by hD_x on the right: $\mathrm{Op}^l(a) = \mathrm{Op}^l(b) \cdot hD_x$, whereas the right quantization of a can be factorized on the left: $\mathrm{Op}^r(a) = hD_x \cdot \mathrm{Op}^r(b)$. We have to implement this simple idea, dealing with the fact that our operator is twisted by $e^{\phi/h}$.

Introduce the operator $P_{\phi,h} = e^{\phi/h} P_h e^{-\phi/h}$. Then, for any $u \in \mathcal{S}(\mathbb{R}^d)$,

$$P_{\phi,h}u(x) = (2\pi h)^{-d} \int e^{ih^{-1}(x-y)\xi} e^{h^{-1}(\phi(x)-\phi(y))} p\big(\tfrac{1}{2}(x + y), \xi\big) u(y)\, dy\, d\xi.$$

We now use the Kuranishi trick. Let $\theta(x, y) = \int_0^1 \nabla\phi(tx + (1 - t)y)\, dt$. Then $\phi(x) - \phi(y) = (x - y) \cdot \theta(x, y)$ and

$$P_{\phi,h}u(x) = (2\pi h)^{-d} \int e^{ih^{-1}(x-y)(\xi - i\theta(x,y))} p\big(\tfrac{1}{2}(x + y), \xi\big) u(y)\, dy\, d\xi.$$

Since $p \in S^0_\infty$, a simple change of integration path shows that $P_{\phi,h}$ is a bounded pseudodifferential operator $P_{\phi,h} = \widetilde{\mathrm{Op}}(\tilde{p}_\phi)$ with

$$\tilde{p}_\phi(x, y, \xi) = p\big(\tfrac{1}{2}(x + y), \xi + i\theta(x, y)\big).$$

To get the expression of $P_{\phi,h}$ in left quantization, it suffices then to apply (3-5) to get $P_{\phi,h} = \mathrm{Op}^l(p_\phi)$ with

$$\begin{aligned} p_\phi(x,\xi) &= (2\pi h)^{-d} \int_{\mathbb{R}^{2d}} e^{ih^{-1}(\xi'-\xi)(x-z)} p\big(\tfrac{1}{2}(x+z), \xi' + i\theta(x,z)\big)\, d\xi'\, dz \\ &= (2\pi h)^{-d} \int_{\mathbb{R}^{2d}} e^{ih^{-1}\xi'(x-z)} p\big(\tfrac{1}{2}(x+z), \xi' + \xi + i\theta(x,z)\big)\, d\xi'\, dz. \end{aligned}$$

Observe that for any smooth function $g : \mathbb{R}^d \to \mathbb{R}$ we have

$$g(\xi) - g(0) = \sum_{j=1}^{d} \int_0^1 \xi_j \partial_{\xi_j} g(\gamma_j^\pm(s,\xi))\, ds \tag{3-7}$$

with $\gamma_j^+(s,\xi) = (\xi_1, \dots, \xi_{j-1}, s\xi_j, 0, \dots, 0)$ and $\gamma_j^-(s,\xi) = (0, \dots, 0, s\xi_j, \xi_{j+1}, \dots, \xi_d)$. A very simple observation is that, for any $(x,\xi) \in T^*\mathbb{R}^d$ and any $s \in [0, 1]$, we have $x \cdot \gamma_j^\pm(s,\xi) = \gamma_j^\pm(s,x) \cdot \xi$. This will be used often in the sequel.

Let us go back to the study of p_ϕ. Since $P_h(e^{-\phi/h}) = 0$, we have $p_\phi(x, 0) = 0$ and, by (3-7), we get

$$p_\phi(x,\xi) = \sum_{j=1}^{d} \xi_j \check{q}^\pm_{\phi,j}(x,\xi) = \sum_{j=1}^{d} \xi_j \check{q}_{\phi,j}(x,\xi)$$

with $\check{q}_{\phi,j} = \frac{1}{2}(\check{q}^+_{\phi,j} + \check{q}^-_{\phi,j})$ and

$$\check{q}^\pm_{\phi,j}(x,\xi) = (2\pi h)^{-d} \int_{\mathbb{R}^{2d}} e^{ih^{-1}\xi'(x-z)} \int_0^1 \partial_{\xi_j} p\big(\tfrac{1}{2}(x+z), \xi' + \gamma_j^\pm(s,\xi) + i\theta(x,z)\big)\, ds\, dz\, d\xi',$$

where the above integral has to be understood as an oscillatory integral. Since $\partial^\alpha_\xi p$ is bounded for any α, integration by parts with respect to ξ' and z shows that $\check{q}^\pm_{\phi,j} \in S^0_\infty(1)$. Moreover, by definition of $\gamma_j^\pm$ we have

$$\xi_j \check{q}^\pm_{\phi,j} = (2\pi h)^{-d} \int_{\mathbb{R}^{2d}} e^{ih^{-1}\xi'(x-z)} c_j^\pm(x,z,\xi)\, dz\, d\xi$$

with $c_j^\pm(x,z,\xi) = p\big(\frac{1}{2}(x+z), \xi' + \gamma_j^\pm(1,\xi) + i\theta(x,z)\big) - p\big(\frac{1}{2}(x+z), \xi' + \gamma_j^\pm(0,\xi) + i\theta(x,z)\big)$. This symbol is clearly in $S^0_\infty(1)$, so that integration by parts as before shows that $\xi_j \check{q}^\pm_{\phi,j} \in S^0_\infty(1)$. Since ξ_j and $\check{q}^\pm_{\phi,j}$ are both scalar, this proves that $\check{q}^\pm_{\phi,j} \in S^0_\infty(\langle \xi_j \rangle^{-1})$.

Observe now that

$$P_h = e^{-\phi/h} P_{\phi,h} e^{\phi/h} = e^{-\phi/h} \mathrm{Op}^l\big(\tfrac{1}{2}(\check{q}^+_\phi + \check{q}^-_\phi)\big) \cdot \Big(\frac{h}{i}\nabla_x\Big) e^{\phi/h} = e^{-\phi/h} \widetilde{Q} e^{\phi/h} \cdot d_{\phi,h}$$

with $\widetilde{Q}=\frac{1}{2}(\widetilde{Q}^{+}+\widetilde{Q}^{-})$ and $\widetilde{Q}^{\pm}=\mathrm{Op}^{l}(\check{q}^{\pm})$. Let $\widetilde{Q}^{\pm}_{\phi}=e^{-2\phi/h}\,\mathrm{Op}^{l}(\check{q}^{\pm}_{\phi})e^{2\phi/h}$; then $\widetilde{Q}^{\pm}_{\phi}=\widetilde{\mathrm{Op}}(\tilde{q}^{\pm}_{\phi})$ with $\tilde{q}^{\pm}_{\phi}=(\tilde{q}^{\pm}_{\phi,1},\dots,\tilde{q}^{\pm}_{\phi,d})$ and

$$
\begin{aligned}
&\tilde{q}^{\pm}_{\phi,j}(x,y,\xi)=\check{q}^{\pm}_{\phi,j}(x,\xi-2i\theta(x,y))\\
&=(2\pi h)^{-d}\int_{\mathbb{R}^{2d}}e^{ih^{-1}\xi'(x-z)}\int_0^1\partial_{\xi_j}p\big(\tfrac{1}{2}(x+z),\xi'+\gamma_j^{\pm}(s,\xi)-2i\gamma_j^{\pm}(s,\theta(x,y))+i\theta(x,z)\big)\,ds\,dz\,d\xi',
\end{aligned}
$$

and it follows from (3-6) that $\widetilde{Q}_{\phi}=\mathrm{Op}^{r}(\check{q}_{\phi})$ with $\check{q}_{\phi}=\check{q}^{+}_{\phi}+\check{q}^{-}_{\phi}$, $\check{q}^{\pm}_{\phi}=(\check{q}^{\pm}_{\phi,1},\dots,\check{q}^{\pm}_{\phi,d})$ and

$$
\begin{aligned}
&\check{q}^{\pm}_{\phi,j}(x,\xi)\\
&=(2\pi h)^{-d}\int_{\mathbb{R}^{2d}}e^{ih^{-1}(\xi'-\xi)u}\tilde{q}^{\pm}_{\phi,j}(x+u,x,\xi')\,du\,d\xi'\\
&=(2\pi h)^{-2d}\int_{\mathbb{R}^{4d}}\int_0^1 e^{ih^{-1}[(\xi'-\xi)u+(x+u-z)\eta]}\\
&\qquad\times\partial_{\xi_j}p\big(\tfrac{1}{2}(x+u+z),\eta+\gamma_j^{\pm}(s,\xi')-2i\gamma_j^{\pm}(s,\theta(x+u,x))+i\theta(x+u,z)\big)\,ds\,dz\,du\,d\xi'\,d\eta.
\end{aligned}
$$

Make the change of variables $z=x+v$ and $\nu=\gamma_j^{\pm}(s,\xi')+\eta$; the above equation yields

$$
\begin{aligned}
\check{q}^{\pm}_{\phi,j}(x,\xi)=(2\pi h)^{-2d}\int_{\mathbb{R}^{4d}}\int_0^1 &e^{ih^{-1}[(\xi'-\xi)u+(u-v)(\nu-\gamma_j^{\pm}(s,\xi'))]}\\
&\times\partial_{\xi_j}p\big(x+\tfrac{1}{2}(u+v),\nu+\psi_j^{\pm}(s,x,u,v)\big)\,ds\,du\,dv\,d\nu\,d\xi'
\end{aligned}
$$

with $\psi_j^{\pm}(s,x,u,v)=i\theta(x+u,x+v)-2i\gamma_j^{\pm}(s,\theta(x+u,x))$.

Define $\hat{p}^2(x,z)=\int e^{-iz\xi}p(x,\xi)\,d\xi$, the Fourier transform of p with respect to the second variable, and observe that, since $\xi\mapsto p(x,\xi)$ is even, so is $z\mapsto\hat{p}^2(x,z)$. Using the above notations, we have

$$
\partial_{\eta_j}p(x,\eta)=\frac{i}{(2\pi)^d}\int_{\mathbb{R}^d}e^{iz\eta}z_j\hat{p}^2(x,z)\,dz,
$$

and we get

$$
\begin{aligned}
\check{q}^{\pm}_{\phi,j}(x,\xi)=\frac{i}{(2\pi)^d(2\pi h)^{2d}}\int_{\mathbb{R}^{4d}\times[0,1]\times\mathbb{R}^d}&z_j e^{ih^{-1}[(u-v+hz)\nu+(\xi'-\xi)u-(u-v)\gamma_j^{\pm}(s,\xi')]}\\
&\times\hat{p}^2\big(x+\tfrac{1}{2}(u+v),z\big)e^{iz\psi_j^{\pm}(s,x,u,v)}\,du\,dv\,d\xi'\,d\nu\,ds\,dz.
\end{aligned}
$$

Let $\mathscr{F}_{h,\nu\mapsto v}$ denote the semiclassical Fourier transform with respect to ν, and $\overline{\mathscr{F}}_{h,u\mapsto\nu}$ its inverse. Writing

$$
f_{s,x,v,z}(u)=z_j\hat{p}^2\big(x+\tfrac{1}{2}(u+v),z\big)e^{ih^{-1}[(\xi'-\xi)u-(u-v)\gamma_j^{\pm}(s,\xi')]}e^{iz\psi_j^{\pm}(s,x,u,v)},
$$

we get

$$\begin{aligned}\breve{q}^{\pm}_{\phi,j}(x,\xi) &= \frac{i}{(2\pi)^d(2\pi h)^d}\int_{\mathbb{R}^{2d}\times[0,1]\times\mathbb{R}^d} \mathscr{F}_{h,v\mapsto v}\overline{\mathscr{F}}_{h,u\mapsto v}(f_{s,x,v,z})(v-hz)\,dv\,d\xi'\,ds\,dz\\ &= \frac{i}{(2\pi)^d(2\pi h)^d}\int_{\mathbb{R}^{2d}\times[0,1]\times\mathbb{R}^d} z_j e^{ih^{-1}[(\xi'-\xi)(v-hz)+h\gamma_j^{\pm}(s,z)\xi']}\\ &\qquad\qquad \times \hat{p}^2\big(x+v-\tfrac{1}{2}hz, z\big)e^{i\psi_j^{\pm}(s,x,v-hz,v)z}\,dv\,d\xi'\,ds\,dz,\end{aligned}$$

where we have used the fact that $\gamma_j^{\pm}(s,\xi')z=\gamma_j^{\pm}(s,z)\xi'$. Similarly, integrating with respect to ξ' and v, we obtain

$$\breve{q}^{\pm}_{\phi,j}(x,\xi)=\frac{i}{(2\pi)^d}\int_{[0,1]\times\mathbb{R}^d} z_j e^{i\gamma_j^{\pm}(s,z)\xi}\,\hat{p}^2\big(x+h\big(\tfrac{1}{2}z-\gamma_j^{\pm}(s,z)\big),z\big)e^{\varphi_j^{\pm}(s,z)}\,ds\,dz$$

with $\varphi_j^{\pm}(s,z)=iz\psi_j^{\pm}\big(s,x,-h\gamma_j^{\pm}(s,z),h(z-\gamma_j^{\pm}(s,z))\big)$. From the definition of $\psi_j^{\pm}$, we get

$$\begin{aligned}\varphi_j^{\pm}(s,z) &= 2z\gamma_j^{\pm}\big(s,\theta(x-h\gamma_j^{\pm}(s,z),x)\big)-z\theta\big(x-h\gamma_j^{\pm}(s,z),x+h(z-\gamma_j^{\pm}(s,z))\big)\\ &= 2\gamma_j^{\pm}(s,z)\theta(x-h\gamma_j^{\pm}(s,z),x)-z\theta\big(x-h\gamma_j^{\pm}(s,z),x+h(z-\gamma_j^{\pm}(s,z))\big),\end{aligned}$$

and, since θ is defined by $\phi(x)-\phi(y)=(x-y)\theta(x,y)$, it follows easily that

$$\varphi_j^{\pm}(s,z)=\frac{1}{h}\Big(2\phi(x)-\phi(x-h\gamma_j^{\pm}(s,z))-\phi\big(x+h(z-\gamma_j^{\pm}(s,z))\big)\Big).$$

Let us write $\rho_j^{\pm}(x,s,z)=\hat{p}^2\big(x+h\big(\tfrac{1}{2}z-\gamma_j^{\pm}(s,z)\big),z\big)$; then

$$\breve{q}^{\pm}_{\phi,j}(x,0)=\frac{i}{(2\pi)^d}\int_{[0,1]\times\mathbb{R}^d} z_j\,\rho_j^{\pm}(x,s,z)e^{\varphi_j^{\pm}(s,z)}\,ds\,dz. \tag{3-8}$$

Observe now that we have the identities

$$\begin{aligned}\gamma_j^{\pm}(1-s,-z)&=-(z-\gamma_j^{\mp}(s,z)),\\ \tfrac{1}{2}z-\gamma_j^{\pm}(s,z)&=-\tfrac{1}{2}z-\gamma_j^{\mp}(1-s,-z)\end{aligned} \tag{3-9}$$

for all $s\in[0,1]$, $z\in\mathbb{R}^d$. In particular, since $\hat{p}^2$ is even with respect to the second variable, we get

$$\rho_j^{\pm}(x,1-s,-z)e^{\varphi_j^{\pm}(1-s,-z)}=\rho_j^{\mp}(x,s,z)e^{\varphi_j^{\mp}(s,z)}.$$

As a consequence, by the change of variables $(s,z)\mapsto(1-s,-z)$ in (3-8), we get $\breve{q}^{+}_{\phi,j}(x,0)=-\breve{q}^{-}_{\phi,j}(x,0)$, and hence $\breve{q}_{\phi}(x,0)=0$. Since $\breve{q}_{\phi,j}$ belongs to $S^0_\infty(\langle\xi_j\rangle^{-1})$ for all j, we get by the same trick as for the right factorization that there exists some symbol $q=(\bar{q}_{j,k})\in S^0_\infty(\mathscr{A})$ such that $\breve{q}_{\phi,j}(x,\xi)=\sum_{k=1}^d\xi_k\bar{q}_{j,k}(x,\xi)$. Since we use right quantization, it follows that, for all $u\in\mathscr{S}(\mathbb{R}^d,\mathbb{C}^d)$,

$$\operatorname{Op}^r(\breve{q}_\phi)u=\frac{h}{i}\operatorname{div}\operatorname{Op}^r(\bar{q})u=hD_x^*\operatorname{Op}^r(\bar{q})u,$$

where we have used the matrix-valued symbol $q=(q_{j,k})$. Consequently, for all $u\in\mathscr{S}(\mathbb{R}^d)$,

$$P_hu=e^{\phi/h}\operatorname{Op}^r(\breve{q}_\phi)e^{-\phi/h}d_{\phi,h}u=d^*_{\phi,h}e^{\phi/h}\operatorname{Op}^r(\bar{q})e^{-\phi/h}d_{\phi,h}u.$$

Using again the analyticity of $\bar{q}$, there exists $\hat{q} \in S^0_\infty(A)$ such that

$$\widehat{Q} := e^{\phi/h}\operatorname{Op}^r(\bar{q})e^{-\phi/h} = \operatorname{Op}(\hat{q}),$$

and the factorization is proved. The fact that $\hat{q}$ admits an expansion in powers of h follows easily from the above computations, since it is the case for p. □

Let us apply Lemma 3.2 to $P_h = \operatorname{Op}(p)$. Then, there exists a symbol $\hat{q} \in S^0_\infty(\mathscr{A})$ such that

$$P_h = d^*_{\phi,h}\widehat{Q}d_{\phi,h},$$

with $\widehat{Q} = \operatorname{Op}(\hat{q})$ and $\hat{q} = \hat{q}_0 + S^0(h)$. Now the strategy is the following: we will modify the operator $\widehat{Q}$ so that the new $\widehat{Q}$ is selfadjoint, nonnegative and $\widehat{Q}$ can be written as the square of a pseudodifferential operator, $\widehat{Q} = Q^*Q$.

First observe that, since P_h is selfadjoint,

$$P_h = \tfrac{1}{2}(P_h + P_h^*) = d^*_{\phi,h}\tfrac{1}{2}(\widehat{Q} + \widehat{Q}^*)d_{\phi,h},$$

so that we can assume in the following that $\widehat{Q}$ is selfadjoint. This means that the partial operators $\widehat{Q}_{j,k} = \operatorname{Op}(\hat{q}_{j,k})$ satisfy $\widehat{Q}^*_{j,k} = \widehat{Q}_{k,j}$ (or, at the level of symbols, $\hat{q}_{k,j} = \overline{\hat{q}_{j,k}}$). For $k = 1,\dots,d$, let us write $d^k_{\phi,h} = h\partial_k + \partial_k\phi(x)$. Then

$$P_h = \sum_{j,k=1}^{d}(d^j_{\phi,h})^*\widehat{Q}_{j,k}d^k_{\phi,h}. \tag{3-10}$$

We would like to take the square root of $\widehat{Q}$ and show that it is still a pseudodifferential operator. The problem is that we don't even know if $\widehat{Q}$ is nonnegative. Nevertheless, we can use the nonuniqueness of operators $\widehat{Q}$ such that (3-10) holds to go to a situation where $\widehat{Q}$ is close to a diagonal operator with nonnegative partial operators on the diagonal. The starting point of this strategy is the commutation relation

$$[d^j_{\phi,h}, d^k_{\phi,h}] = 0 \quad \text{for all } j,\ k \in \{1,\dots,d\}, \tag{3-11}$$

which holds thanks to $d^j_{\phi,h} = e^{-\phi/h}h\partial_j e^{\phi/h}$ and Schwarz' theorem. Hence, for any bounded operator B, we have

$$P_h = d^*_{\phi,h}\widehat{Q}^{\mathrm{mod},\bullet}d_{\phi,h} = \sum_{j,k=1}^{d}(d^j_{\phi,h})^*\widehat{Q}^{\mathrm{mod},\bullet}_{j,k}d^k_{\phi,h}, \tag{3-12}$$

with $\widehat{Q}^{\mathrm{mod},\bullet} = \widehat{Q} + \mathscr{B}^\bullet$, $\bullet \in \{0,\infty\}$ for some $\mathscr{B}^\bullet$ having one of the two following forms:

- (*exchange between three coefficients*) For any $j_0, k_0, n \in \{1,\dots,d\}$, the operator $\mathscr{B}^\infty(j_0,k_0,n;B) = (\mathscr{B}^\infty_{j,k})_{j,k=1,\dots,d}$ is defined by

$$\begin{aligned}
\mathscr{B}^\infty_{j,k} &= 0 \quad \text{if } (j,k) \notin \{(n,n),(j_0,k_0),(k_0,j_0)\},\\
\mathscr{B}^\infty_{j_0,k_0} &= -(d^n_{\phi,h})^*Bd^n_{\phi,h} \quad \text{and} \quad \mathscr{B}^\infty_{k_0,j_0} = (\mathscr{B}^\infty_{j_0,k_0})^*,\\
\mathscr{B}^\infty_{n,n} &= (d^{j_0}_{\phi,h})^*Bd^{k_0}_{\phi,h} + (d^{k_0}_{\phi,h})^*B^*d^{j_0}_{\phi,h}.
\end{aligned} \tag{3-13}$$

When $j_0 = k_0$, we use the convention that $\mathcal{B}^{\infty}_{j_0,j_0} = -(d^n_{\phi,h})^*(B + B^*)d^n_{\phi,h}$. Such modifications will be used away from the critical points.

- (*exchange between four coefficients*) For any $j_0, k_0, k_1 \in \{1, \dots, d\}$, the operator $\mathcal{B}^0(j_0, k_0, k_1; B) = (\mathcal{B}^0_{j,k})_{j,k=1,\dots,d}$ is defined by

$$\begin{aligned} \mathcal{B}^0_{j,k} &= 0 \quad \text{if } (j,k) \notin \{(j_0,k_0),(k_0,j_0),(j_0,k_1),(k_1,j_0)\}, \\ \mathcal{B}^0_{j_0,k_0} &= -Bd^{k_1}_{\phi,h} \quad \text{and} \quad \mathcal{B}^0_{k_0,j_0} = (\mathcal{B}^0_{j_0,k_0})^*, \\ \mathcal{B}^0_{j_0,k_1} &= Bd^{k_0}_{\phi,h} \quad \text{and} \quad \mathcal{B}^0_{k_1,j_0} = (\mathcal{B}^0_{j_0,k_1})^*. \end{aligned} \tag{3-14}$$

Such modifications will be used near the critical points.

Recall that the d-matrix weights $\mathcal{A}$ and $\Xi\mathcal{A}$ are given by $\mathcal{A}_{j,k} = \langle \xi_j \rangle^{-1} \langle \xi_k \rangle^{-1}$ and $(\Xi\mathcal{A})_{j,k} = \langle \xi_k \rangle^{-1}$. Using the preceding remark, we can prove the following:

Lemma 3.4. *Let $\widehat{Q} = \operatorname{Op}(\hat{q})$, where $\hat{q} \in S^0(\mathcal{A})$ is a Hermitian symbol with $\hat{q}(x,\xi;h) = \hat{q}_0(x,\xi) + S^0(h\mathcal{A})$. We let $P = d^*_{\phi,h} \widehat{Q} d_{\phi,h}$ and let $p(x,\xi;h) = p_0(x,\xi) + S^0(h) \in S^0(1)$ be its symbol. Assume that the following assumptions hold*:

(A1) *For all $\delta > 0$, there exists $\alpha > 0$ such that, for all $(x,\xi) \in T^*\mathbb{R}^d$, $|\xi|^2 + d(x,\mathcal{U})^2 \geq \delta$ implies $p_0(x,\xi) \geq \alpha$.*

(A2) *Near $(\boldsymbol{u}, 0)$, for any critical point $\boldsymbol{u} \in \mathcal{U}$, we have*

$$p_0(x,\xi) = |\xi|^2 + |\nabla\phi(x)|^2 + r(x,\xi) \tag{3-15}$$

with $r(x,\xi) = \mathcal{O}(|(x - \boldsymbol{u}, \xi)|^3)$.

Then, for h small enough, there exists a symbol $q \in S^0(\Xi\mathcal{A})$ such that

$$P_h = d^*_{\phi,h} Q^* Q d_{\phi,h},$$

with $Q = \operatorname{Op}(q)$ and

$$q(x,\xi;h) = \operatorname{Id} + \mathcal{O}(|(x - \boldsymbol{u}, \xi)|) + S^0(h) \tag{3-16}$$

near $(\boldsymbol{u}, 0)$ for any $\boldsymbol{u} \in \mathcal{U}$. Moreover, $Q = F \operatorname{Op}(\Xi^{-1})$ for some $F \in \Psi^0(1)$ that is invertible and selfadjoint with $F^{-1} \in \Psi^0(1)$.

If, additionally to the previous assumptions, we suppose:

(A2′) *the remainder term in (3-15) satisfies $r(x,\xi) = \mathcal{O}(|(x - \boldsymbol{u}, \xi)|^4)$;*

then

$$q(x,\xi;h) = \operatorname{Id} + \mathcal{O}(|(x - \boldsymbol{u}, \xi)|^2) + S^0(h) \tag{3-17}$$

near $(\boldsymbol{u}, 0)$.

Finally, if $\hat{q} \in S^0_{\mathrm{cl}}(\mathcal{A})$ then $q \in S^0_{\mathrm{cl}}(\Xi\mathcal{A})$.

Proof. In the following, we assume that ϕ has a unique critical point $\boldsymbol{u}$ and that $\boldsymbol{u}=0$. Using some cutoff in space, we can always make this assumption without loss of generality. Given $\varepsilon>0$, let $w_0, w_1, \dots, w_d \in S^0(1)$ be nonnegative functions such that

$$w_0+w_1+\cdots+w_d=1 \tag{3-18}$$

whose support satisfies

$$\operatorname{supp}(w_0)\subset\{|\xi|^2+|\nabla\phi(x)|^2\le 2\varepsilon\},$$

and, for all $\ell\ge 1$,

$$\operatorname{supp}(w_\ell)\subset\Big\{|\xi|^2+|\nabla\phi(x)|^2\ge\varepsilon \text{ and } |\xi_\ell|^2+|\partial_\ell\phi(x)|^2\ge\frac{1}{2d}(|\xi|^2+|\nabla\phi(x)|^2)\Big\}.$$

Let us decompose $\widehat{Q}$ according to these truncations:

$$\widehat{Q}=\sum_{\ell=0}^{d}\widehat{Q}^\ell \tag{3-19}$$

with $\widehat{Q}^\ell := \operatorname{Op}(w_\ell\hat{q})$ for all $\ell\ge 0$. We will modify each of the operators $\widehat{Q}^\ell$ separately, using the following modifiers. For $j_0, k_0, n\in\{1,\dots,d\}$ and $\beta\in S^0(\langle\xi_{j_0}\rangle^{-1}\langle\xi_{k_0}\rangle^{-1}\langle\xi_n\rangle^{-2})$ we write for short

$$\mathscr{B}^\infty(j_0,k_0,n;\beta):=\mathscr{B}^\infty(j_0,k_0,n;\operatorname{Op}(\beta)),$$

where the right-hand side is defined by (3-13). In the same way, given $j_0,k_0,k_1\in\{1,\dots,d\}$ and $\beta\in S^0(\langle\xi_{j_0}\rangle^{-1}\langle\xi_{k_0}\rangle^{-1}\langle\xi_{k_1}\rangle^{-1})$ we write for short

$$\mathscr{B}^0(j_0,k_0,k_1;\beta):=\mathscr{B}^0(j_0,k_0,k_1;\operatorname{Op}(\beta)),$$

where the right-hand side is defined by (3-14). Observe that any operator of one of these two forms belongs to $\Psi^0(\mathscr{A})$. Let $\mathscr{M}(\mathscr{A})\subset\Psi^0(\mathscr{A})$ be the vector space of bounded operators on $L^2(\mathbb{R}^d)^d$ generated by these operators. Then, (3-12) says exactly that

$$P_h=d_{\phi,h}^*(\widehat{Q}+\mathscr{M})d_{\phi,h}\quad\text{for any } \mathscr{M}\in\mathscr{M}(\mathscr{A}). \tag{3-20}$$

Step 1. We first remove the terms of order 1 near the origin. More precisely, we show that there exists $\mathscr{M}^0\in\mathscr{M}(\mathscr{A})$ such that

$$\check{Q}^0:=\widehat{Q}^0+\mathscr{M}^0=\operatorname{Op}(\check{q}^0)+\Psi^0(h\mathscr{A}), \tag{3-21}$$

where $\check{q}^0\in S^0(\mathscr{A})$ satisfies, near $(0,0)\in T^*\mathbb{R}^d$,

$$\check{q}^0(x,\xi)=w_0(x,\xi)(\operatorname{Id}+\rho(x,\xi)) \tag{3-22}$$

with $\rho\in S(\mathscr{A})$ such that:

- $\rho(x,\xi)=\mathscr{O}(|(x,\xi)|)$ under the assumption (A2);
- $\rho(x,\xi)=\mathscr{O}(|(x,\xi)|^2)$ under the assumption (A2′).

From (3-10), we have

$$p_0(x,\xi)=\sum_{j,k=1}^{d}\hat{q}_{0;j,k}(x,\xi)(\xi_j+i\partial_j\phi(x))(\xi_k-i\partial_k\phi(x)),$$

where $\hat{q}_0=(\hat{q}_{0;j,k})_{j,k}$ denotes the principal symbol of $\hat{q}$. Expanding $\hat{q}_0$ near the origin, we get

$$\hat{q}_0(x,\xi)=\hat{q}_0(0,0)+\nu(x,\xi)$$

with $\nu(x,\xi)=\mathcal{O}(|(x,\xi)|)$. Then, we deduce

$$p_0(x,\xi)=\sum_{j,k=1}^{d}(\hat{q}_{0;j,k}(0,0)+\nu_{j,k}(x,\xi))(\xi_j+i\partial_j\phi(x))(\xi_k-i\partial_k\phi(x)). \tag{3-23}$$

Identifying (3-15) and (3-23), we obtain $\hat{q}_{0;j,k}(0,0)=\delta_{j,k}$, which establishes (3-21)–(3-22) under the assumption (A2).

Suppose now that (A2′) is satisfied. Identifying (3-15) and (3-23) as before, we obtain

$$\sum_{j,k=1}^{d}\nu_{j,k}(x,\xi)(\xi_j+i\partial_j\phi(x))(\xi_k-i\partial_k\phi(x))=\mathcal{O}(|(x,\xi)|^4). \tag{3-24}$$

Defining $A:=\operatorname{Hess}(\phi)(0)$, we have $\partial_j\phi(x)=(Ax)_j+\mathcal{O}(x^2)$. Then, (3-24) becomes

$$\sum_{j,k=1}^{d}\nu_{j,k}(x,\xi)(\xi_j+i(Ax)_j)(\xi_k-i(Ax)_k)=\mathcal{O}(|(x,\xi)|^4). \tag{3-25}$$

Let us introduce the new variables $\eta=\xi+iAx$ and $\bar{\eta}=\xi-iAx$. Then, (3-25) reads

$$\sum_{j,k=1}^{d}\nu_{j,k}(x,\xi)\eta_j\bar{\eta}_k=\mathcal{O}(|(x,\xi)|^4)=\mathcal{O}(|(\eta,\bar{\eta})|^4). \tag{3-26}$$

On the other hand, since A is invertible, there exist some complex numbers $\alpha^n_{j,k},\tilde{\alpha}^n_{j,k}$ for $j,k,n=1,\dots,d$ such that

$$\nu_{j,k}(x,\xi)=\sum_{n=1}^{d}(\alpha^n_{j,k}\bar{\eta}_n+\tilde{\alpha}^n_{j,k}\eta_n)+\mathcal{O}(|(\eta,\bar{\eta})|^2). \tag{3-27}$$

Combined with (3-26), this yields $\sum_{j,k,n=1}^{d}(\alpha^n_{j,k}\bar{\eta}_n+\tilde{\alpha}^n_{j,k}\eta_n)\eta_j\bar{\eta}_k=\mathcal{O}(|(\eta,\bar{\eta})|^4)$ and, since the left-hand side is a polynomial of degree 3 in $(\eta,\bar{\eta})$, it follows that

$$\sum_{j,k,n=1}^{d}(\alpha^n_{j,k}\bar{\eta}_n+\tilde{\alpha}^n_{j,k}\eta_n)\eta_j\bar{\eta}_k=0 \tag{3-28}$$

for any $\eta\in\mathbb{C}^d$. Hence, uniqueness of coefficients of polynomials of $(\eta,\bar{\eta})$ implies

$$\alpha^n_{j,k}+\alpha^k_{j,n}=0\quad\text{for all }\ j,\ k,\ n\in\{1,\dots,d\}. \tag{3-29}$$

In particular, $\alpha_{j,k}^{k}=0$. On the other hand, $\tilde{\alpha}_{j,k}^{n}=\overline{\alpha_{k,j}^{n}}$ for all j,k,n since $\widehat{Q}$ is selfadjoint.

Now, we define

$$\check{Q}^{0}:=\widehat{Q}^{0}+\mathcal{M}^{0}\quad\text{with}\quad \mathcal{M}^{0}:=\sum_{j_0,k_0=1}^{d}\sum_{n=k_0+1}^{d}\alpha_{j_0,k_0}^{n}\mathcal{B}^{0}(j_0,k_0,n;w_0).$$

It follows from symbolic calculus that $\check{Q}^{0}=\operatorname{Op}(\check{q}^{0})$, with $\check{q}^{0}\in S^{0}(\mathscr{A})$ given by

$$\begin{aligned}\check{q}_{j,k}^{0}=w_0\Big(\hat{q}_{0;j,k}-\sum_{n>k}\alpha_{j,k}^{n}(\xi_n-i\partial_n\phi(x))+\sum_{n<k}\alpha_{j,n}^{k}(\xi_n-i\partial_n\phi(x))\\ -\sum_{n>j}\overline{\alpha_{k,j}^{n}}(\xi_n+i\partial_n\phi(x))+\sum_{n<j}\overline{\alpha_{k,n}^{j}}(\xi_n+i\partial_n\phi(x))\Big)+S^{0}(h\mathscr{A})\end{aligned}$$

for any j,k. Moreover, from (3-29) and $\xi_n+i\partial_n\phi(x)=\eta_n+\mathcal{O}(|x|^2)$ near $(0,0)$, we get

$$\check{q}_{j,k}^{0}=w_0\Big(\hat{q}_{0;j,k}-\sum_{n=1}^{d}\alpha_{j,k}^{n}\bar{\eta}_n-\sum_{n=1}^{d}\overline{\alpha_{k,j}^{n}}\eta_n+\tilde{\rho}_{j,k}\Big)+S^{0}(h\mathscr{A})$$

with $\tilde{\rho}\in S^{0}(\mathscr{A})$ such that $\tilde{\rho}=\mathcal{O}(|(x,\xi)|^2)$ near the origin. Using the identity $\hat{q}_{0;j,k}=\delta_{j,k}+\nu_{j,k}$ together with (3-27), we get

$$\check{q}_{j,k}^{0}=w_0(\delta_{j,k}+\rho_{j,k})+S^{0}(h\mathscr{A})$$

with $\rho\in S^{0}(\mathscr{A})$ such that $\rho=\mathcal{O}(|(x,\xi)|^2)$ near the origin. This implies (3-21)–(3-22) under the assumption (A2′), and achieves the proof of Step 1.

Step 2. We now remove the antidiagonal terms away from the origin. More precisely, we show that there exist some $\mathcal{M}^{\ell}\in\mathcal{M}(\mathscr{A})$ and some diagonal symbols $\tilde{q}^{\ell}\in S^{0}(\mathscr{A})$ such that

$$\widetilde{Q}^{\ell}:=\widehat{Q}^{\ell}+\mathcal{M}^{\ell}=\operatorname{Op}(w_\ell\tilde{q}^{\ell})+\Psi^{0}(h\mathscr{A})\tag{3-30}$$

for any $\ell\in\{1,\dots,d\}$.

For $j_0,k_0,\ell\in\{1,\dots,d\}$ with $j_0\neq k_0$, let $\beta_{j_0,k_0,\ell}$ be defined by

$$\beta_{j_0,k_0,\ell}(x,\xi):=\frac{w_\ell(x,\xi)\hat{q}_{j_0,k_0}(x,\xi)}{|\xi_\ell|^2+|\partial_\ell\phi(x)|^2}.$$

By the support properties of w_ℓ, we have $\beta_{j_0,k_0,\ell}\in S^{0}(\langle\xi_{j_0}\rangle^{-1}\langle\xi_{k_0}\rangle^{-1}\langle\xi_\ell\rangle^{-2})$, so $\mathcal{B}^{\infty}(j_0,k_0,\ell;\beta_{j_0,k_0,\ell})$ belongs to $\mathcal{M}(\mathscr{A})$. Defining

$$\mathcal{M}^{\ell}:=\sum_{j_0\neq k_0}\mathcal{B}^{\infty}(j_0,k_0,\ell;\beta_{j_0,k_0,\ell}),$$

the pseudodifferential calculus gives

$$(d_{\phi,h}^{\ell})^{*}\operatorname{Op}(\beta_{j_0,k_0,\ell})d_{\phi,h}^{\ell}=\operatorname{Op}(w_\ell\hat{q}_{j_0,k_0})+\Psi^{0}(h\langle\xi_{j_0}\rangle^{-1}\langle\xi_{k_0}\rangle^{-1}),$$

which implies

$$\widehat{Q}^{\ell}+\mathcal{M}^{\ell}=\operatorname{Op}(w_\ell\hat{q})+\mathcal{M}^{\ell}=\operatorname{Op}(w_\ell\tilde{q}^{\ell})+\Psi^{0}(h\mathscr{A})$$

with $\tilde{q}^\ell \in S^0(\mathscr{A})$ diagonal. This proves (3-30).

Step 3. Let us now prove that we can modify each $\widetilde{Q}^\ell$ so that its diagonal coefficients are suitably bounded from below. More precisely, we claim that there exist $c>0$ and $\widetilde{\mathcal{M}}^\ell \in \mathcal{M}(\mathscr{A})$ such that

$$\breve{Q}^\ell := \widetilde{Q}^\ell + \widetilde{\mathcal{M}}^\ell = \mathrm{Op}(\breve{q}^\ell) + \Psi^0(h\mathscr{A}) \tag{3-31}$$

with $\breve{q}^\ell$ diagonal and $\breve{q}^\ell_{i_0,i_0}(x,\xi) \geq c w_\ell(x,\xi)\langle \xi_{i_0}\rangle^{-2}$ for all $i_0 \in \{1,\dots,d\}$.

For $\ell, i_0 \in \{1,\dots,d\}$, let $\beta_{i_0,\ell}$ be defined by

$$\beta_{i_0,\ell}(x,\xi) := \frac{w_\ell(x,\xi)}{2(|\xi_\ell|^2 + |\partial_\ell \phi(x)|^2)}\Big(\tilde{q}^\ell_{i_0,i_0}(x,\xi) - \frac{\gamma}{1+|\xi_{i_0}|^2 + |\partial_{i_0}\phi(x)|^2}\Big),$$

where $\gamma>0$ will be specified later. The symbol $\beta_{i_0,\ell}$ belongs to $S^0(\langle \xi_{i_0}\rangle^{-2}\langle \xi_\ell\rangle^{-2})$, so $\mathscr{B}^\infty(i_0,i_0,\ell;\beta_{i_0,\ell})$ is in $\mathcal{M}(\mathscr{A})$. Defining

$$\widetilde{\mathcal{M}}^\ell := \sum_{i_0\neq\ell} \mathscr{B}^\infty(i_0,i_0,\ell;\beta_{i_0,\ell}),$$

the symbolic calculus shows that $\widetilde{Q}^\ell + \widetilde{\mathcal{M}}^\ell = \mathrm{Op}(\breve{q}^\ell) + \Psi^0(h\mathscr{A})$ with $\breve{q}^\ell$ diagonal and

$$\breve{q}^\ell_{i_0,i_0}(x,\xi) = \frac{\gamma w_\ell(x,\xi)}{1+|\xi_{i_0}|^2+|\partial_{i_0}\phi(x)|^2} \quad \text{for all } i_0 \neq \ell. \tag{3-32}$$

It remains to prove that we can choose $\gamma>0$ above, so that $\breve{q}^\ell_{\ell,\ell}(x,\xi) \geq c w_\ell(x,\xi)\langle \xi_\ell\rangle^{-2}$. Thanks to assumption (A1), there exists $\alpha>0$ such that

$$p_0(x,\xi) \geq \alpha \quad \text{for all } (x,\xi) \in \operatorname{supp}(w_\ell). \tag{3-33}$$

On the other hand, a simple commutator computation shows that $\mathrm{Op}(w_\ell)P_h = d^*_{\phi,h}\widehat{Q}^\ell d_{\phi,h} + \Psi^0(h)$. Combined with (3-20), (3-30) and the definition of $\breve{q}^\ell$, this yields

$$\mathrm{Op}(w_\ell)P_h = d^*_{\phi,h}\widetilde{Q}^\ell d_{\phi,h} + \Psi^0(h) = d^*_{\phi,h}\,\mathrm{Op}(\breve{q}^\ell) d_{\phi,h} + \Psi^0(h),$$

and then

$$(w_\ell p_0)(x,\xi) = \sum_{i_0=1}^{d} \breve{q}^\ell_{i_0,i_0}(x,\xi)(|\xi_{i_0}|^2 + |\partial_{i_0}\phi(x)|^2) + S^0(h).$$

Now, using (3-32), we get

$$(w_\ell p_0)(x,\xi) = \breve{q}^\ell_{\ell,\ell}(x,\xi)(|\xi_\ell|^2 + |\partial_\ell\phi(x)|^2) + \gamma(d-1)w_\ell(x,\xi) + S^0(h).$$

Combining this relation with (3-33) and choosing $\gamma = \alpha/(2d)$, we obtain

$$\breve{q}^\ell_{\ell,\ell}(x,\xi) \geq \frac{\alpha w_\ell(x,\xi)}{2(|\xi_\ell|^2+|\partial_\ell\phi(x)|^2)} + S^0(h\langle \xi_\ell\rangle^{-2}). \tag{3-34}$$

Thus, $\breve{q}^\ell_{\ell,\ell}$ satisfies the required lower bound and (3-31) follows.

Step 4. Lastly, we take the square root of the modified operator. Let us define

$$\check{Q} := \sum_{\ell=0}^{d} \check{Q}^{\ell} \in \Psi^0(\mathscr{A}), \tag{3-35}$$

with $\check{Q}^{\ell}$ defined above. Thanks to (3-20), we have $P_h = (d_{\phi,h})^* \widehat{Q} d_{\phi,h} = (d_{\phi,h})^* \check{Q} d_{\phi,h}$ and it follows from the preceding constructions that the principal symbol $\check{q}$ of $\check{Q}$ satisfies

$$\check{q}(x,\xi) \geq w_0(x,\xi)(\mathrm{Id} + \mathscr{O}(|(x,\xi)|)) + c \sum_{\ell \geq 1} w_{\ell}(x,\xi)\, \mathrm{diag}(\langle \xi_j \rangle^{-2}).$$

Shrinking $c > 0$ and the support of w_0 if necessary, it follows that

$$\check{q}(x,\xi) \geq c\, \mathrm{diag}(\langle \xi_j \rangle^{-2}).$$

Letting $E = \mathrm{Op}(\Xi)\check{Q}\,\mathrm{Op}(\Xi)$, and $e \in S^0(1)$ be the symbol of E, the pseudodifferential calculus gives $e(x,\xi;h) = e_0(x,\xi) + S^0(h)$ with

$$e_0(x,\xi) \geq c\, \mathrm{diag}(\langle \xi_j \rangle \langle \xi_j \rangle^{-2} \langle \xi_j \rangle) = c\, \mathrm{Id}, \tag{3-36}$$

so that, for $h > 0$ small enough, $e(x,\xi) \geq \frac{1}{2} c\, \mathrm{Id}$. Hence, we can adapt the proof of Theorem 4.8 of [Helffer and Nier 2005] to our semiclassical setting to get that $F := E^{1/2}$ belongs to $\Psi^0(1)$ and that $F^{-1} \in \Psi^0(1)$. Then, $\check{Q} = Q^* Q$ with $Q := F\,\mathrm{Op}(\Xi^{-1})$ and, by construction, $Q \in \Psi^0(\Xi\mathscr{A})$.

In addition, as in Theorem 4.8 of [Helffer and Nier 2005], we can show that $F = \mathrm{Op}(e_0^{1/2}) + \Psi^0(h)$, so that $Q = \mathrm{Op}(q_0) + \Psi^0(h\Xi\mathscr{A})$ with $q_0 = e_0^{1/2}\Xi^{-1}$. If, moreover, $\hat{q}$ admits a classical expansion, then $\check{q} \in S^0_{\mathrm{cl}}(\mathscr{A})$, and the same argument shows that both e and q admit classical expansions.

Let us now study q_0 near $(\boldsymbol{u}, 0)$. For (x,ξ) close to $(\boldsymbol{u}, 0)$ we have $\Xi = \mathrm{Id} + \mathscr{O}(|\xi|^2)$ and $\check{q}_0 = \mathrm{Id} + \rho(x,\xi)$, so

$$e_0(x,\xi) = \Xi \check{q}_0 \Xi = \mathrm{Id} + \rho(x,\xi) + \mathscr{O}(|\xi|^2),$$

and we get easily $q_0 = e_0^{1/2}\Xi^{-1} = \mathrm{Id} + \mathscr{O}(|\xi|^2 + \rho(x,\xi))$, which proves (3-16) and (3-17). ☐

This completes the proof of Theorem 1.5. ☐

4. Quasimodes on k-forms and first exponential-type eigenvalue estimates

Pseudodifferential Hodge–Witten Laplacian on the 0-forms. This part is devoted to the rough asymptotic of the small eigenvalues of P_h and to the construction of associated quasimodes. From Theorem 1.5, this operator has the expression

$$P_h = a_h d^*_{\phi,h} G d_{\phi,h} a_h, \tag{4-1}$$

where G is the matrix of pseudodifferential operators

$$G = (\mathrm{Op}(g_{j,k}))_{j,k} := Q^* Q = \mathrm{Op}(\Xi)^{-1} F^* F\, \mathrm{Op}(\Xi)^{-1}.$$

Using Corollary 3.1 and that G is selfadjoint, we remark that $g_{j,k} \in S^0(\langle \xi_j \rangle^{-1} \langle \xi_k \rangle^{-1})$ and $g_{j,k} = \overline{g_{k,j}}$. Thus, P_h can be viewed as a Hodge–Witten Laplacian on 0-forms (or a Laplace–Beltrami operator) with the pseudodifferential metric G^{-1}. In the following, we will then use the notation $P^{(0)} := P_h$.

Since $a_h(\boldsymbol{u}) = 1 + \mathcal{O}(h)$ and $g(\boldsymbol{u}, 0) = \beta_d \operatorname{Id} + \mathcal{O}(h)$ for all the critical points $\boldsymbol{u} \in \mathcal{U}$, it is natural to consider the operator with the coefficients a_h and G frozen at 1 and $\beta_d \operatorname{Id}$, respectively. For that, let $\Omega^p(\mathbb{R}^d)$, $p = 1, \dots, d$, be the space of C^∞ p-forms on $\mathbb{R}^d$. We then define

$$P^W = d^*_{\phi,h} d_{\phi,h} + d_{\phi,h} d^*_{\phi,h}, \tag{4-2}$$

the semiclassical Witten Laplacian on the de Rham complex, and $P^{W,(p)}$, its restriction to the p-forms. This operator has been intensively studied (see, e.g., [Helffer and Sjöstrand 1985; Cycon et al. 2008; Bovier et al. 2004; 2005; Helffer et al. 2004]), and a lot is known concerning its spectral properties. In particular, from Lemma 1.6 and Proposition 1.7 of [Helffer and Sjöstrand 1985], we know that there are n_0 exponentially small (real nonnegative) eigenvalues, and that the others are above h/C.

From [Helffer et al. 2004; Hérau et al. 2011], we have good normalized quasimodes for $P^{W,(0)}$ associated to all minima of ϕ. For $k \in \{1, \dots, n_0\}$, they are given by

$$f_k^{W,(0)}(x) = \chi_{k,\varepsilon}(x) b_k^{(0)}(h) e^{-(\phi(x)-\phi(\boldsymbol{m}_k))/h},$$

where $b_k^{(0)}(h) = (\pi h)^{-d/4} \det(\operatorname{Hess}\phi(\boldsymbol{m}_k))^{1/4}(1 + \mathcal{O}(h))$, and where the $\chi_{k,\varepsilon}$ are cutoff functions localized in sufficiently large areas containing $\boldsymbol{m}_k \in \mathcal{U}^{(0)}$. In fact, we need large support (associated to level sets of ϕ) and properties for the cutoff functions $\chi_{k,\varepsilon}$, so that the refined analysis of the next section can be done. We postpone to the Appendix the construction of the cutoff functions, the definition of $\varepsilon > 0$, refined estimates on this family $(f_k^{W,(0)})_k$, and in particular the fact that it is a quasiorthonormal free family of functions, following closely [Helffer et al. 2004; Hérau et al. 2011].

We now define the quasimodes associated to $P^{(0)}$ in the following way:

$$f_k^{(0)}(x) := a_h(x)^{-1} f_k^{W,(0)}(x) = a_h(x)^{-1} b_k^{(0)}(h) \chi_{k,\varepsilon}(x) e^{-(\phi(x)-\phi(\boldsymbol{m}_k))/h} \tag{4-3}$$

for $1 \le k \le n_0$. We then have:

Lemma 4.1. *The system $(f_k^{(0)})_k$ is free, and there exists $\alpha > 0$ independent of ε such that*

$$\langle f_k^{(0)}, f_{k'}^{(0)} \rangle = \delta_{k,k'} + \mathcal{O}(h) \quad \textit{and} \quad P^{(0)} f_k^{(0)} = \mathcal{O}(e^{-\alpha/h}).$$

Remark 4.2. For this result to be true, it would have been sufficient to take truncation functions with smaller support (say in a small neighborhood of each minimum $\boldsymbol{m}_k$). We emphasize again that the more complicated construction for the quasimodes is justified by their later use.

Proof. First, observe that

$$\langle f_k^{(0)}, f_{k'}^{(0)} \rangle = \langle a_h^{-2} f_k^{W,(0)}, f_{k'}^{W,(0)} \rangle = \delta_{kk'} + \langle (a_h^{-2} - 1) f_k^{W,(0)}, f_{k'}^{W,(0)} \rangle.$$

Moreover, near any minimum $\boldsymbol{m}_k$, $a_h^{-2} - 1 = \mathcal{O}(h + |x - \boldsymbol{m}_k|^2)$ and $\phi(x) - \phi(\boldsymbol{m}_k)$ is quadratic, so

$$\|(a_h^{-2} - 1) f_k^{W,(0)}\| = \mathcal{O}(h), \tag{4-4}$$

which proves the first statement. For the last statement, it is enough to notice that

$$P^{(0)} f_k^{(0)} = a_h^* d_{\phi,h}^* Q^* Q d_{\phi,h} f_k^{W,(0)}$$

and apply Lemma A.3. □

We now prove a first rough spectral result on $P^{(0)}$, using the preceding lemma.

Proposition 4.3. *The operator $P^{(0)}$ has exactly n_0 exponentially small (real nonnegative) eigenvalues, and the remaining part of its spectrum is in $[\varepsilon_0 h, +\infty[$ for some $\varepsilon_0 > 0$.*

Usually, this type of result is a consequence of an IMS formula. It is possible to do that here (with effort) but we prefer to give a simpler proof using what we know about $P^{W,(0)}$. The following proof is based on the spectral theorem and the maxi-min principle.

Proof. Thanks to Proposition 2.4, the spectrum of $P^{(0)}$ is discrete in $[0, \delta]$ and its j-th eigenvalue is given by

$$\sup_{\dim E = j-1} \inf_{u \in E^\perp, \|u\|=1} \langle P^{(0)} u, u \rangle. \tag{4-5}$$

Lemma 4.1 directly implies

$$\langle P^{(0)} f_k^{(0)}, f_{k'}^{(0)} \rangle \leq \| P^{(0)} f_k^{(0)} \| \| f_{k'}^{(0)} \| = \mathcal{O}(e^{-\alpha/h})$$

for some $\alpha > 0$. Using the almost orthogonality of the $f_k^{(0)}$, (4-5) and $P^{(0)} \geq 0$, we deduce that $P^{(0)}$ has at least n_0 eigenvalues that are exponentially small.

We now want to prove that the remaining part of the spectrum of $P^{(0)}$ is above $\varepsilon_0 h$ for some $\varepsilon_0 > 0$ small enough. For this, we set

$$\mathscr{E} := \operatorname{Vect}\{ f_k^{W,(0)} : k = 1, \dots, n_0 \},$$

and we consider $u \in a_h^{-1} \mathscr{E}^\perp$ with $\|u\| = 1$. We have, again,

$$\langle P^{(0)} u, u \rangle = \| F \operatorname{Op}(\Xi^{-1}) d_{\phi,h} a_h u \|^2 \geq \varepsilon_0 \| \operatorname{Op}(\Xi^{-1}) d_{\phi,h} a_h u \|^2 \tag{4-6}$$

for some $\varepsilon_0 > 0$ independent of h, which may change from line to line. For the last inequality, we have used that $\|F^{-1}\|$ is uniformly bounded since $F^{-1} \in \Psi^0(1)$. On the other hand, using $0 \leq P^{W,(1)} = -h^2 \Delta \otimes \operatorname{Id} + \mathcal{O}(1)$, we notice that

$$\operatorname{Op}(\Xi^{-1})^2 \geq (-h^2 \Delta + 1)^{-1} \otimes \operatorname{Id} \geq \varepsilon_0 (P^{W,(1)} + 1)^{-1}$$

for some (other) $\varepsilon_0 > 0$. Therefore, using the classical intertwining relations

$$(P^{W,(1)} + 1)^{-1/2} d_{\phi,h} = d_{\phi,h} (P^{W,(0)} + 1)^{-1/2},$$

and the fact that $P^{W,(0)} = d_{\phi,h}^* d_{\phi,h}$ on 0-forms, we get

$$\begin{aligned}\langle P^{(0)} u, u \rangle \geq \varepsilon_0 \| (P^{W,(1)} + 1)^{-1/2} d_{\phi,h} a_h u \|^2 &= \varepsilon_0 \| d_{\phi,h} (P^{W,(0)} + 1)^{-1/2} a_h u \|^2 \\ &= \varepsilon_0 \langle P^{W,(0)} (P^{W,(0)} + 1)^{-1} a_h u, a_h u \rangle. \end{aligned} \tag{4-7}$$

Now, let $\mathcal{F}$ be the eigenspace of $P^{W,(0)}$ associated to the n_0 exponentially small eigenvalues, and let $\Pi_{\mathcal{E}}$ (resp. $\Pi_{\mathcal{F}}$) be the orthogonal projectors onto $\mathcal{E}$ (resp. $\mathcal{F}$). Then, from Proposition 1.7 of [Helffer and Sjöstrand 1985] (see also Theorem 2.4 of [Helffer and Sjöstrand 1984]), we have $\|\Pi_{\mathcal{E}}-\Pi_{\mathcal{F}}\| = \mathcal{O}(e^{-\alpha/h})$. Moreover, since the (n_0+1)-st eigenvalue of $P^{W,(0)}$ is of order h, the spectral theorem gives

$$P^{W,(0)}(P^{W,(0)}+1)^{-1} \geq \varepsilon_0 h(1-\Pi_{\mathcal{F}}) + \mathcal{O}(e^{-\alpha/h}) \geq \varepsilon_0 h(1-\Pi_{\mathcal{E}}) + \mathcal{O}(e^{-\alpha/h}).$$

Then, using $a_h u \in \mathcal{E}^{\perp}$, $\|u\| = 1$ and Lemma 2.2, (4-7) becomes

$$\langle P^{(0)}u, u\rangle \geq \varepsilon_0 h\|a_h u\|^2 + \mathcal{O}(e^{-\alpha/h}) \geq \tfrac{1}{2}c_1\varepsilon_0 h.$$

Finally, this estimate and (4-5) imply that $P^{(0)}$ has at most n_0 eigenvalues below $\frac{1}{2}c_1\varepsilon_0 h$. Taking $\frac{1}{2}c_1\varepsilon_0$ as the new value of ε_0 gives the result. □

Pseudodifferential Hodge–Witten Laplacian on the* 1*-forms. Since we want to follow a supersymmetric approach to prove the main theorem of this paper, we have to build an extension $P^{(1)}$ of $P^{(0)}$ defined on 1-forms which satisfies properties similar to those of $P^{W,(1)}$. To do this, we use the following coordinates for $\omega \in \Omega^1(\mathbb{R}^d)$ and $\sigma \in \Omega^2(\mathbb{R}^d)$:

$$\omega = \sum_{j=1}^{d} \omega_j(x)dx_j, \quad \sigma = \sum_{j<k} \sigma_{j,k}(x)dx_j \wedge dx_k,$$

and we extend the matrix $\sigma_{j,k}$ as a function with values in the space of antisymmetric matrices. Recall that the exterior derivative satisfies

$$(d^{(1)}\omega)_{j,k} = \partial_{x_j}\omega_k - \partial_{x_k}\omega_j \quad \text{and} \quad (d^{*(1)}\sigma)_j = -\sum_k \partial_{x_k}\sigma_{k,j}. \tag{4-8}$$

In the previous section, we saw that $P^{(0)}$ can be viewed as the Hodge–Witten Laplacian on 0-forms with a pseudodifferential metric G^{-1}. It is then natural to consider the corresponding Hodge–Witten Laplacian on 1-forms. Thus, mimicking the construction in the standard case, we define

$$P^{(1)} := Qd_{\phi,h}a_h^2 d_{\phi,h}^* Q^* + (Q^{-1})^* d_{\phi,h}^* M d_{\phi,h} Q^{-1}, \tag{4-9}$$

where M is the linear operator acting on $\Omega^2(\mathbb{R}^d)$ with coefficients

$$M_{(j,k),(a,b)} := \tfrac{1}{2}\operatorname{Op}(a_h^2(g_{j,a}g_{k,b} - g_{k,a}g_{j,b})). \tag{4-10}$$

Note that M is well-defined on $\Omega^2(\mathbb{R}^d)$ (i.e., $M\sigma$ is antisymmetric if σ is antisymmetric) since $M_{(k,j),(a,b)} = M_{(j,k),(b,a)} = -M_{(j,k),(a,b)}$. Furthermore, we deduce from the properties of $g_{j,k}$ that

$$M_{(j,k),(a,b)} \in \Psi^0(\langle\xi_j\rangle^{-1}\langle\xi_k\rangle^{-1}\langle\xi_a\rangle^{-1}\langle\xi_b\rangle^{-1}). \tag{4-11}$$

Remark 4.4. When G^{-1} is a true metric (and not a matrix of pseudodifferential operators), the operator $P^{(1)}$ defined in (4-9) is the usual Hodge–Witten Laplacian on 1-forms. Our construction is then an extension to the pseudodifferential case. Generalizing these structures to p-forms, it should be possible to define a Hodge–Witten Laplacian on the total de Rham complex. It could also be possible to define such an operator using only abstract geometric quantities (and not explicit formulas like (4-10)).

On the other hand, a precise choice for the operator M is not relevant in the present paper. Indeed, for the study of the small eigenvalues of $P^{(0)}$, only the first part (in (4-9)) of $P^{(1)}$ is important (see Lemma 4.7 below). The second part is only used to make the operator $P^{(1)}$ elliptic. Thus, any M satisfying (4-11) and $M_{(j,k),(a,b)} \geq \varepsilon \operatorname{Op}(\langle \xi_j \rangle^{-2} \langle \xi_k \rangle^{-2}) \otimes \operatorname{Id}$ should probably work.

We first show that $P^{(1)}$ acts diagonally (at the first order), as is the case for $P^{W,(1)}$.

Lemma 4.5. *The operator $P^{(1)} \in \Psi^0(1)$ is selfadjoint on $\Omega^1(\mathbb{R}^d)$. Moreover,*

$$P^{(1)} = P^{(0)} \otimes \operatorname{Id} + \Psi^0(h). \tag{4-12}$$

Proof. We begin by estimating the first part of $P^{(1)}$,

$$P_1^{(1)} := Q d_{\phi,h} a_h^2 d_{\phi,h}^* Q^*.$$

Let $q_{j,k} \in S^0(\langle \xi_k \rangle^{-1})$ denote the symbol of the coefficients of Q and let $d_{\phi,h}^j = h\partial_j + (\partial_j \phi)$. Using the composition rules of matrices, a direct computation gives

$$(P_1^{(1)})_{j,k} = \sum_a \operatorname{Op}(q_{j,a}) d_{\phi,h}^a a_h^2 (d_{\phi,h}^* Q^*)_k = \sum_{a,b} \operatorname{Op}(q_{j,a}) d_{\phi,h}^a a_h^2 (d_{\phi,h}^b)^* \operatorname{Op}(\overline{q_{k,b}}). \tag{4-13}$$

We then deduce that $P_1^{(1)}$ is a selfadjoint operator on $\Omega^1(\mathbb{R}^d)$ with coefficients of class $\Psi^0(1)$. Moreover, this formula implies

$$(P_1^{(1)})_{j,k} = \sum_{a,b} \operatorname{Op}(a_h^2 q_{j,a} \overline{q_{k,b}}) d_{\phi,h}^a (d_{\phi,h}^b)^* + \Psi^0(h). \tag{4-14}$$

It remains to study

$$P_2^{(1)} := (Q^{-1})^* d_{\phi,h}^* M d_{\phi,h} Q^{-1}.$$

Let $q_{j,k}^{-1} \in S^0(\langle \xi_j \rangle)$ denote the symbol of the coefficients of Q^{-1}. The formulas of (4-8), the definition (4-10) and the composition rules of matrices imply

$$\begin{aligned}
(P_2^{(1)})_{j,k} &= \sum_\alpha \operatorname{Op}(\overline{q_{\alpha,j}^{-1}}) (d_{\phi,h}^* M d_{\phi,h} Q^{-1})_{\alpha,k} \\
&= -\sum_{a,\alpha} \operatorname{Op}(\overline{q_{\alpha,j}^{-1}}) (d_{\phi,h}^a)^* (M d_{\phi,h} Q^{-1})_{(a,\alpha),k} \\
&= -\sum_{a,b,\alpha,\beta} \operatorname{Op}(\overline{q_{\alpha,j}^{-1}}) (d_{\phi,h}^a)^* M_{(a,\alpha),(b,\beta)} (d_{\phi,h} Q^{-1})_{(b,\beta),k} \\
&= -\sum_{a,b,\alpha,\beta} \operatorname{Op}(\overline{q_{\alpha,j}^{-1}}) (d_{\phi,h}^a)^* M_{(a,\alpha),(b,\beta)} (d_{\phi,h}^b \operatorname{Op}(q_{\beta,k}^{-1}) - d_{\phi,h}^\beta \operatorname{Op}(q_{b,k}^{-1})) \\
&= -2 \sum_{a,b,\alpha,\beta} \operatorname{Op}(\overline{q_{\alpha,j}^{-1}}) (d_{\phi,h}^a)^* M_{(a,\alpha),(b,\beta)} d_{\phi,h}^b \operatorname{Op}(q_{\beta,k}^{-1}),
\end{aligned} \tag{4-15}$$

where we have used that $M_{(a,\alpha),(b,\beta)} = -M_{(a,\alpha),(\beta,b)}$. By (4-11), a typical term of these sums satisfies

$$\operatorname{Op}(\overline{q^{-1}_{\alpha,j}})(d^a_{\phi,h})^* M_{(a,\alpha),(b,\beta)} d^b_{\phi,h} \operatorname{Op}(q^{-1}_{\beta,k}) \in \Psi^0(\langle\xi_\alpha\rangle\langle\xi_a\rangle\langle\xi_a\rangle^{-1}\langle\xi_\alpha\rangle^{-1}\langle\xi_b\rangle^{-1}\langle\xi_\beta\rangle^{-1}\langle\xi_b\rangle\langle\xi_\beta\rangle),$$

and then $P^{(1)}_2 \in \Psi^0(1)$. On the other hand, using $g_{j,k} = \overline{g_{k,j}}$ and (4-10), we get

$$\begin{aligned}
(P^{(1)}_2)^*_{j,k} &= -\sum_{a,b,\alpha,\beta} \operatorname{Op}(\overline{q^{-1}_{\beta,k}})(d^b_{\phi,h})^* \operatorname{Op}(\overline{a^2_h(g_{a,b}g_{\alpha,\beta} - g_{\alpha,b}g_{a,\beta})}) d^a_{\phi,h} \operatorname{Op}(q^{-1}_{\alpha,j}) \\
&= -\sum_{a,b,\alpha,\beta} \operatorname{Op}(\overline{q^{-1}_{\beta,k}})(d^b_{\phi,h})^* \operatorname{Op}(a^2_h(g_{b,a}g_{\beta,\alpha} - g_{b,\alpha}g_{\beta,a})) d^a_{\phi,h} \operatorname{Op}(q^{-1}_{\alpha,j}) \\
&= -\sum_{a,b,\alpha,\beta} \operatorname{Op}(\overline{q^{-1}_{\alpha,k}})(d^a_{\phi,h})^* \operatorname{Op}(a^2_h(g_{a,b}g_{\alpha,\beta} - g_{a,\beta}g_{\alpha,b})) d^b_{\phi,h} \operatorname{Op}(q^{-1}_{\beta,j}) = (P^{(1)}_2)_{k,j},
\end{aligned}$$

so that $P^{(1)}_2$ is selfadjoint on $\Omega^1(\mathbb{R}^d)$. Finally, (4-11) and (4-15) yield

$$\begin{aligned}
(P^{(1)}_2)_{j,k} &= \sum_{a,b} \operatorname{Op}\biggl(a^2_h \sum_{\alpha,\beta} \overline{q^{-1}_{\alpha,j}} q^{-1}_{\beta,k}(g_{a,b}g_{\alpha,\beta} - g_{a,\beta}g_{\alpha,b})\biggr)(d^a_{\phi,h})^* d^b_{\phi,h} + \Psi^0(h) \\
&= \sum_{a,b} \operatorname{Op}(a^2_h g_{a,b}\delta_{j,k} - a^2_h q_{j,b}\overline{q_{k,a}})(d^a_{\phi,h})^* d^b_{\phi,h} + \Psi^0(h),
\end{aligned} \tag{4-16}$$

since

$$\sum_j g_{a,j} q^{-1}_{j,b} = \overline{q_{b,a}} + S^0(h\langle\xi_a\rangle^{-1}) \quad \text{and} \quad \sum_j q_{a,j} q^{-1}_{j,b} = \delta_{a,b} + S^0(h),$$

which follow from $GQ^{-1} = Q^*$ and $QQ^{-1} = \mathrm{Id}$.

Summing up the previous properties of $P^{(1)}_\bullet$, the operator $P^{(1)} = P^{(1)}_1 + P^{(1)}_2 \in \Psi^0(1)$ is selfadjoint on $\Omega^1(\mathbb{R}^d)$. Lastly, combining (4-14) and (4-16), we obtain

$$\begin{aligned}
P^{(1)} &= \sum_{a,b} (d^a_{\phi,h})^* \operatorname{Op}(a^2_h g_{a,b}) d^b_{\phi,h} \otimes \mathrm{Id} + \Psi^0(h) = a_h d^*_{\phi,h} G d_{\phi,h} a_h \otimes \mathrm{Id} + \Psi^0(h) \\
&= P^{(0)} \otimes \mathrm{Id} + \Psi^0(h),
\end{aligned} \tag{4-17}$$

and the lemma follows. □

The next result compares $P^{(1)}$ and $P^{W,(1)}$.

Lemma 4.6. *There exist some pseudodifferential operators $(R_k)_{k=0,1,2}$ such that*

$$P^{(1)} = \beta_d P^{W,(1)} + R_0 + R_1 + R_2,$$

where the remainder terms enjoy the following properties:

(i) *R_0 is a $d \times d$ matrix whose coefficients are finite sums of terms of the form*

$$(d^a_{\phi,h})^*(\operatorname{Op}(r_0) + \Psi^0(h)) d^b_{\phi,h}$$

with $a, b \in \{1, \dots, d\}$ and $r_0 \in S^0(1)$ satisfying $r_0(x,\xi) = \mathcal{O}(|(x - \boldsymbol{u}, \xi)|^2)$ near $(\boldsymbol{u}, 0)$, $\boldsymbol{u} \in \mathcal{U}$;

(ii) *R_1 is a matrix whose coefficients are finite sums of terms of the form $h\operatorname{Op}(r_1)d^a_{\phi,h}$ or $h(d^a_{\phi,h})^*\operatorname{Op}(r_1)$ with $a\in\{1,\dots,d\}$ and $r_1\in S^0(1)$ satisfying $r_1(x,\xi)=\mathcal{O}(|(x-\boldsymbol{u},\xi)|)$ near $(\boldsymbol{u},0)$, $\boldsymbol{u}\in\mathcal{U}$;*

(iii) *$R_2\in\Psi^0(h^2)$.*

Proof. As in the proof of Lemma 4.5, we use the decomposition $P^{(1)}=P^{(1)}_1+P^{(1)}_2$. From Corollary 3.1 and Lemma 2.2, the coefficients appearing in these operators satisfy

$$\begin{aligned}
a_h&=\tilde a+S^0(h)\in S^0(1),\\
q_{a,b}&=\tilde q_{a,b}+S^0(h\langle\xi_b\rangle^{-1})\in S^0(\langle\xi_b\rangle^{-1}),\\
q^{-1}_{a,b}&=\tilde q^{-1}_{a,b}+S^0(h\langle\xi_a\rangle)\in S^0(\langle\xi_a\rangle),\\
M_{(j,k),(a,b)}&=\operatorname{Op}(\tilde m_{(j,k),(a,b)})+\Psi^0(h\langle\xi_j\rangle^{-1}\langle\xi_k\rangle^{-1}\langle\xi_a\rangle^{-1}\langle\xi_b\rangle^{-1})
\end{aligned}$$

with $\tilde m_{(j,k),(a,b)}\in S^0(\langle\xi_j\rangle^{-1}\langle\xi_k\rangle^{-1}\langle\xi_a\rangle^{-1}\langle\xi_b\rangle^{-1})$ and

$$\begin{aligned}
\tilde a&=1+\mathcal{O}(|(x-\boldsymbol{u},\xi)|^2), &\qquad \tilde m_{(j,k),(a,b)}&=\tfrac12\beta_d^2(\delta_{j,a}\delta_{k,b}-\delta_{k,a}\delta_{j,b})+\mathcal{O}(|(x-\boldsymbol{u},\xi)|^2),\\
\tilde q_{a,b}&=\beta_d^{1/2}\delta_{a,b}+\mathcal{O}(|(x-\boldsymbol{u},\xi)|^2), &\qquad \tilde q^{-1}_{a,b}&=\beta_d^{-1/2}\delta_{a,b}+\mathcal{O}(|(x-\boldsymbol{u},\xi)|^2)
\end{aligned}$$

near $(\boldsymbol{u},0)$, $\boldsymbol{u}\in\mathcal{U}$. Then, making commutations in (4-13) and (4-15), we obtain the desired result. ☐

We now make the link between the eigenvalues of $P^{(0)}$ and $P^{(1)}$. For that, we will use the so-called intertwining relations, which are a fundamental tool in the supersymmetric approach. Recall that, thanks to Theorem 1.5, $P^{(0)}$ can be written as

$$P^{(0)}=L^*_\phi L_\phi\quad\text{with}\quad L_\phi=Qd_{\phi,h}a_h. \tag{4-18}$$

We obtain the following result:

Lemma 4.7. *On 0-forms, we have*

$$L_\phi P^{(0)}=P^{(1)}L_\phi=L_\phi L^*_\phi L_\phi. \tag{4-19}$$

Moreover, for all $\lambda\in\mathbb{R}\setminus\{0\}$, the operator $L_\phi:\ker(P^{(0)}-\lambda)\to\ker(P^{(1)}-\lambda)$ is injective. Finally, $L_\phi(\ker(P^{(0)}))=\{0\}$.

Proof. Let us first prove (4-19). Using (4-9), (4-18) and the usual cohomology rule (i.e., $d^2_{\phi,h}=0$), we have

$$\begin{aligned}
P^{(1)}L_\phi&=L_\phi L^*_\phi L_\phi+(Q^{-1})^*d^*_{\phi,h}Md_{\phi,h}Q^{-1}Qd_{\phi,h}a_h\\
&=L_\phi L^*_\phi L_\phi+(Q^{-1})^*d^*_{\phi,h}Md_{\phi,h}d_{\phi,h}a_h\\
&=L_\phi L^*_\phi L_\phi=L_\phi P^{(0)}.
\end{aligned}\tag{4-20}$$

Now, let $u\neq0$ be an eigenfunction of $P^{(0)}$ associated to $\lambda\in\mathbb{R}$. In particular, $\|L_\phi u\|^2=\lambda\|u\|^2$ vanishes if and only if $\lambda=0$. Moreover, (4-19) yields

$$P^{(1)}L_\phi u=L_\phi P^{(0)}u=\lambda L_\phi u.$$

This implies the second part of the lemma. ☐

We shall now study more precisely the small eigenvalues of $P^{(1)}$. Recall that s_j, $j = 2, \dots, n_1+1$, denote the saddle points (of index 1) of ϕ. Again, we will stick to the analysis already made for the Witten Laplacian on 1-forms $P^{W,(1)}$, for which we recall the following properties. From Lemma 1.6 and Proposition 1.7 of [Helffer and Sjöstrand 1985], the operator $P^{W,(1)}$ is selfadjoint, positive and has exactly n_1 exponentially small (nonzero) eigenvalues (counted with multiplicities). We next recall the construction of associated quasimodes made in Definition 4.3 of [Helffer et al. 2004]. Let u_j denote a normalized fundamental state of $P^{W,(1)}$ restricted to an appropriate neighborhood of s_j with Dirichlet boundary conditions. The quasimodes $f_j^{W,(1)}$ are then defined by

$$f_j^{W,(1)} := \|\theta_j u_j\|^{-1}\theta_j(x)u_j(x), \tag{4-21}$$

where θ is a well-chosen C_0^∞ localization function around s_j. Since the $f_j^{W,(1)}$ have disjoint support, we immediately deduce

$$\langle f_j^{W,(1)}, f_{j'}^{W,(1)}\rangle = \delta_{j,j'}. \tag{4-22}$$

In particular, the family $\{f_j^{W,(1)} : j = 2, \dots, n_1+1\}$ is a free family of 1-forms. Furthermore, Theorem 1.4 of [Helffer and Sjöstrand 1985] implies that these quasimodes have a WKB expression,

$$f_j^{W,(1)}(x) = \theta_j(x) b_j^{(1)}(x,h) e^{-\phi_{+,j}(x)/h}, \tag{4-23}$$

where $b_j^{(1)}(x,h)$ is a normalization 1-form having a semiclassical asymptotic, and $\phi_{+,j}$ is the phase associated to the outgoing manifold of $\xi^2 + |\nabla_x\phi(x)|^2$ at $(s_j, 0)$. Moreover, the phase function $\phi_{+,j}$ satisfies the eikonal equation $|\nabla_x\phi_{+,j}|^2 = |\nabla_x\phi|^2$ and $\phi_{+,j}(x) \sim |x - s_j|^2$ near s_j. For other properties of $\phi_{+,j}$, we refer to [Helffer and Sjöstrand 1985]. On the other hand, Lemma 1.6 and Proposition 1.7 of [Helffer and Sjöstrand 1985] imply that there exists $\alpha > 0$ independent of ε such that

$$P^{W,(1)} f_j^{W,(1)} = \mathcal{O}(e^{-\alpha/h}). \tag{4-24}$$

Lastly, we deduce from Proposition 1.7 of [Helffer and Sjöstrand 1985] that there exists $\nu > 0$ such that

$$\langle P^{W,(1)}u, u\rangle \geq \nu h\|u\|^2 \tag{4-25}$$

for all $u \perp \operatorname{Vect}\{f_j^{W,(1)} : j = 2, \dots, n_1+1\}$.

Now, let us define the quasimodes associated to $P^{(1)}$ by

$$f_j^{(1)}(x) := \beta_d^{1/2}(Q^*)^{-1} f_j^{W,(1)} \tag{4-26}$$

for $2 \leq j \leq n_1+1$. Note that this is possible since $(Q^*)^{-1} \in \Psi^0(\langle\xi\rangle)$. Using that $(Q^*)^{-1}$ is close to $\beta_d^{-1/2}\operatorname{Id}$ microlocally near $(s_j, 0)$, we will prove that they form a good, approximately normalized and orthogonal family of quasimodes for $P^{(1)}$.

Lemma 4.8. *The system $(f_j^{(1)})_j$ is free and, for all $j, j' = 2, \dots, n_1+1$, we have*

$$\|f_j^{(1)} - f_j^{W,(1)}\| = \mathcal{O}(h), \quad \langle f_j^{(1)}, f_{j'}^{(1)}\rangle = \delta_{j,j'} + \mathcal{O}(h) \quad \text{and} \quad P^{(1)} f_j^{(1)} = \mathcal{O}(h^2).$$

Proof. From (4-26) and Corollary 3.1, we have

$$f_j^{(1)} - f_j^{W,(1)} = (\beta_d^{1/2}(Q^*)^{-1} - \mathrm{Id}) f_j^{W,(1)} = \mathrm{Op}(r) f_j^{W,(1)}$$

with $r \in S^0(\langle \xi \rangle^2)$ such that, modulo $S^0(h\langle \xi \rangle^2)$, $r(x,\xi) = \mathcal{O}(|(x-\boldsymbol{u},\xi)|^2)$ near $(\boldsymbol{u},0)$, $\boldsymbol{u} \in \mathcal{U}$. Moreover, using Taylor expansion and symbolic calculus, we can write

$$r(x,\xi) = \sum_{|\alpha+\beta| \in \{0,2\}} h^{1-|\alpha+\beta|/2} r_{\alpha,\beta}(x,\xi)(x-\boldsymbol{s}_j)^\alpha \xi^\beta$$

with $r_{\alpha,\beta} \in S^0(\langle \xi \rangle^2)$. Combined with the WKB form of the $f_j^{W,(1)}$ given in (4-23) (and, in particular, with $\phi_{+,j}(x) \sim |x-\boldsymbol{s}_j|^2$ near $\boldsymbol{s}_j$), it shows that

$$\mathrm{Op}(r) f_j^{W,(1)} = \mathcal{O}(h), \tag{4-27}$$

which proves the first statement.

The second statement is a direct consequence of the above estimate and (4-22).

For the last estimate, we follow the same strategy. Thanks to Lemma 4.6, we have

$$P^{(1)} f_j^{(1)} = \beta_d P^{W,(1)} f_j^{W,(1)} + \beta_d P^{W,(1)}(f_j^{(1)} - f_j^{W,(1)}) + R_0 f_j^{(1)} + R_1 f_j^{(1)} + R_2 f_j^{(1)}. \tag{4-28}$$

Proceeding as above, we write

$$P^{W,(1)}(f_j^{(1)} - f_j^{W,(1)}) = P^{W,(1)}(\beta_d^{1/2}(Q^*)^{-1} - \mathrm{Id}) f_j^{W,(1)},$$

where, using (4-2), Corollary 3.1 and the pseudodifferential calculus, the corresponding operator can be decomposed as

$$P^{W,(1)}(\beta_d^{1/2}(Q^*)^{-1} - \mathrm{Id}) = \mathrm{Op}\Big(\sum_{|\alpha+\beta| \in \{0,2,4\}} h^{2-|\alpha+\beta|/2} \tilde{r}_{\alpha,\beta}(x,\xi;h)(x-\boldsymbol{s}_j)^\alpha \xi^\beta \Big)$$

for some $\tilde{r}_{\alpha,\beta} \in S^0(\langle \xi \rangle^3)$. Thus, as in (4-27), we deduce

$$\beta_d P^{W,(1)}(f_j^{(1)} - f_j^{W,(1)}) = \mathcal{O}(h^2). \tag{4-29}$$

In the same way, we deduce from Lemma 4.6 that, for any $p = 0, 1, 2$,

$$R_p \beta_d^{1/2}(Q^*)^{-1} = \mathrm{Op}\Big(\sum_{|\alpha+\beta| \in \{0,2,4\}} h^{2-|\alpha+\beta|/2} r^p_{\alpha,\beta}(x,\xi;h)(x-\boldsymbol{s}_j)^\alpha \xi^\beta \Big)$$

with $r^p_{\alpha,\beta} \in S^0(\langle \xi \rangle^3)$. Thus,

$$R_p f_j^{(1)} = R_p \beta_d^{1/2}(Q^*)^{-1} f_j^{W,(1)} = \mathcal{O}(h^2). \tag{4-30}$$

Combining (4-28) with the estimates (4-24), (4-29) and (4-30), we obtain $P^{(1)} f_j^{(1)} = \mathcal{O}(h^2)$ and this concludes the proof of the lemma. □

The following proposition is the analogue of Proposition 4.3.

Proposition 4.9. *The operator $P^{(1)}$ has exactly n_1 $\mathcal{O}(h^2)$ (real) eigenvalues, and the remaining part of the spectrum is in $[\varepsilon_1 h, +\infty[$ for some $\varepsilon_1 > 0$.*

The idea of the proof is to consider separately the regions of the phase space close to the critical points $\mathcal{U}$ and away from this set. In the first one, we approximate $P^{(1)}$ by $P^{W,(1)}$ using that $Q \simeq \beta_d^{1/2}\,\mathrm{Id}$ microlocally near $(\boldsymbol{u}, 0)$, $\boldsymbol{u} \in \mathcal{U}$. In the second one, we use that (the symbol of) $P^{(1)}$ is elliptic by (4-12).

We start this strategy with a pseudodifferential IMS formula. For $\eta > 0$ fixed, let $\chi_0 \in C_0^\infty(\mathbb{R}^{2d}; [0,1])$ be supported in a neighborhood of size η of $\mathcal{U}$ and such that $\chi_0 = 1$ near $\mathcal{U}$ and $\chi_\infty := (1-\chi_0^2)^{1/2} \in C^\infty(\mathbb{R}^{2d})$. In particular,

$$\chi_0^2(x,\xi) + \chi_\infty^2(x,\xi) = 1 \quad \text{for all } (x,\xi) \in \mathbb{R}^{2d}. \tag{4-31}$$

In the sequel, the remainder terms may depend on η, but C will denote a positive constant independent of η, which may change from line to line. Using Lemma 4.5 and the shorthand $\mathrm{Op}(a) = \mathrm{Op}(a) \otimes \mathrm{Id}$, the pseudodifferential calculus gives

$$\begin{aligned}
P^{(1)} &= \tfrac{1}{2}(\mathrm{Op}(\chi_0^2 + \chi_\infty^2) P^{(1)} + P^{(1)} \mathrm{Op}(\chi_0^2 + \chi_\infty^2)) \\
&= \tfrac{1}{2}(\mathrm{Op}(\chi_0)^2 P^{(1)} + P^{(1)} \mathrm{Op}(\chi_0)^2) + \tfrac{1}{2}(\mathrm{Op}(\chi_\infty)^2 P^{(1)} + P^{(1)} \mathrm{Op}(\chi_\infty)^2) + \Psi^0(h^2) \\
&= \mathrm{Op}(\chi_0) P^{(1)} \mathrm{Op}(\chi_0) + \mathrm{Op}(\chi_\infty) P^{(1)} \mathrm{Op}(\chi_\infty) \\
&\qquad\qquad + \tfrac{1}{2}[\mathrm{Op}(\chi_0), [\mathrm{Op}(\chi_0), P^{(1)}]] + \tfrac{1}{2}[\mathrm{Op}(\chi_\infty), [\mathrm{Op}(\chi_\infty), P^{(1)}]] + \mathcal{O}(h^2) \\
&= \mathrm{Op}(\chi_0) P^{(1)} \mathrm{Op}(\chi_0) + \mathrm{Op}(\chi_\infty) P^{(1)} \mathrm{Op}(\chi_\infty) + \mathcal{O}(h^2).
\end{aligned} \tag{4-32}$$

In the previous estimate, we have crucially used that $\mathrm{Op}(\chi_\bullet) \otimes \mathrm{Id}$ are matrices of pseudodifferential operators collinear to the identity.

Lemma 4.10. *There exists $\delta_\eta > 0$, which may depend on η, such that*

$$\mathrm{Op}(\chi_\infty) P^{(1)} \mathrm{Op}(\chi_\infty) \geq \delta_\eta \mathrm{Op}(\chi_\infty)^2 + \mathcal{O}(h^\infty). \tag{4-33}$$

Moreover, there exists $C > 0$ such that, for all $\eta > 0$,

$$\mathrm{Op}(\chi_0) P^{(1)} \mathrm{Op}(\chi_0) \geq (\beta_d - C\eta) \mathrm{Op}(\chi_0) P^{W,(1)} \mathrm{Op}(\chi_0) - (C\eta h + \mathcal{O}(h^2)). \tag{4-34}$$

Proof. We first estimate $P^{(1)}$ outside of the critical points $\mathcal{U}$. Since χ_∞ vanishes near $\mathcal{U}$, Proposition 2.3 yields that there exist $\delta_\eta > 0$ and $\tilde{p}_\eta \in S^0(1)$ (which may depend on η) such that $p = \tilde{p}_\eta$ in a vicinity of the support of χ_∞ and $\tilde{p}_\eta(x,\xi) \geq 2\delta_\eta$ for all $(x,\xi) \in \mathbb{R}^{2d}$. Then, Lemma 4.5 and the pseudodifferential calculus (in particular, the Gårding inequality) imply

$$\begin{aligned}
\mathrm{Op}(\chi_\infty) P^{(1)} \mathrm{Op}(\chi_\infty) &= \mathrm{Op}(\chi_\infty) P^{(0)} \mathrm{Op}(\chi_\infty) + \mathrm{Op}(\chi_\infty) \mathcal{O}(h) \mathrm{Op}(\chi_\infty) \\
&= \mathrm{Op}(\chi_\infty) \mathrm{Op}(\tilde{p}_\eta) \mathrm{Op}(\chi_\infty) + \mathrm{Op}(\chi_\infty) \mathcal{O}(h) \mathrm{Op}(\chi_\infty) + \mathcal{O}(h^\infty) \\
&\geq \mathrm{Op}(\chi_\infty)(2\delta_\eta + \mathcal{O}(h)) \mathrm{Op}(\chi_\infty) + \mathcal{O}(h^\infty),
\end{aligned}$$

which implies (4-33) for h small enough. Here, we have identified as before A with $A \otimes \mathrm{Id}$ for scalar operators A.

We now consider $\mathrm{Op}(\chi_0)P^{(1)}\,\mathrm{Op}(\chi_0)$. Thanks to Lemma 4.6, we can write

$$\mathrm{Op}(\chi_0)P^{(1)}\,\mathrm{Op}(\chi_0)=\beta_d\,\mathrm{Op}(\chi_0)P^{W,(1)}\,\mathrm{Op}(\chi_0)+\sum_{k=0}^{2}\mathrm{Op}(\chi_0)R_k\,\mathrm{Op}(\chi_0).$$

Let $\tilde{\chi}_0\in C_0^\infty(\mathbb{R}^{2d};[0,1])$ be supported in a neighborhood of size η of $(\boldsymbol{u},0)$, $\boldsymbol{u}\in\mathcal{U}$, and such that $\tilde{\chi}_0=1$ near the support of χ_0. Then, for $\omega\in\Omega^1(\mathbb{R}^d)$, $\langle R_0\,\mathrm{Op}(\chi_0)\omega,\mathrm{Op}(\chi_0)\omega\rangle$ is a finite sum of terms of the form

$$\tilde{r}_0=\langle (d^a_{\phi,h})^*(\mathrm{Op}(r_0)+\Psi^0(h))d^b_{\phi,h}\,\mathrm{Op}(\chi_0)\omega_j,\mathrm{Op}(\chi_0)\omega_k\rangle. \tag{4-35}$$

Using functional analysis and pseudodifferential calculus, we get

$$\begin{aligned}|\tilde{r}_0|&=|\langle(\mathrm{Op}(r_0\tilde{\chi}_0)+\Psi^0(h))d^b_{\phi,h}\,\mathrm{Op}(\chi_0)\omega_j,d^a_{\phi,h}\,\mathrm{Op}(\chi_0)\omega_k\rangle|+\mathcal{O}(h^\infty)\|\omega\|^2\\&\le(\|\mathrm{Op}(r_0\tilde{\chi}_0)\|+\mathcal{O}(h))\|d^b_{\phi,h}\,\mathrm{Op}(\chi_0)\omega_j\|\,\|d^a_{\phi,h}\,\mathrm{Op}(\chi_0)\omega_k\|+\mathcal{O}(h^\infty)\|\omega\|^2\\&\le(\|\mathrm{Op}(r_0\tilde{\chi}_0)\|+\mathcal{O}(h))\langle P^{W,(0)}\,\mathrm{Op}(\chi_0)\omega,\mathrm{Op}(\chi_0)\omega\rangle+\mathcal{O}(h^\infty)\|\omega\|^2.\end{aligned} \tag{4-36}$$

Recall now that, for $a\in S^0(1)$,

$$\|\mathrm{Op}(a)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}=\|a\|_{L^\infty(\mathbb{R}^{2d})}+\mathcal{O}(h)$$

(see, e.g., [Zworski 2012, Theorem 13.13]). Thus, using that $\tilde{\chi}_0$ is supported in a neighborhood of size η of $(\boldsymbol{u},0)$ at which r_0 vanishes yields $\|\mathrm{Op}(r_0\tilde{\chi}_0)\|\le C\eta$, and (4-36) implies

$$|\langle R_0\,\mathrm{Op}(\chi_0)\omega,\mathrm{Op}(\chi_0)\omega\rangle|\le C\eta\langle P^{W,(0)}\,\mathrm{Op}(\chi_0)\omega,\mathrm{Op}(\chi_0)\omega\rangle+\mathcal{O}(h^\infty)\|\omega\|^2. \tag{4-37}$$

As before, $\langle R_1\,\mathrm{Op}(\chi_0)\omega,\mathrm{Op}(\chi_0)\omega\rangle$ is a finite sum of terms of the form

$$\tilde{r}_1=\langle\Psi^0(h)d^a_{\phi,h}\,\mathrm{Op}(\chi_0)\omega_j,\mathrm{Op}(\chi_0)\omega_k\rangle \tag{4-38}$$

or its complex conjugate. These terms can be estimated as

$$\begin{aligned}|\tilde{r}_1|&\le Ch\|d^a_{\phi,h}\,\mathrm{Op}(\chi_0)\omega_j\|\,\|\omega\|\\&\le\eta\|d^a_{\phi,h}\,\mathrm{Op}(\chi_0)\omega_j\|^2+\mathcal{O}(h^2)\|\omega\|^2\\&\le\eta\langle P^{W,(0)}\,\mathrm{Op}(\chi_0)\omega,\mathrm{Op}(\chi_0)\omega\rangle+\mathcal{O}(h^2)\|\omega\|^2,\end{aligned}$$

and then

$$|\langle R_1\,\mathrm{Op}(\chi_0)\omega,\mathrm{Op}(\chi_0)\omega\rangle|\le C\eta\langle P^{W,(0)}\,\mathrm{Op}(\chi_0)\omega,\mathrm{Op}(\chi_0)\omega\rangle+\mathcal{O}(h^2)\|\omega\|^2. \tag{4-39}$$

Combining Lemma 4.6 with the estimates (4-37), (4-39) and $R_2\in\Psi^0(h^2)$, we obtain

$$\mathrm{Op}(\chi_0)P^{(1)}\,\mathrm{Op}(\chi_0)\ge\beta_d\,\mathrm{Op}(\chi_0)P^{W,(1)}\,\mathrm{Op}(\chi_0)-C\eta\,\mathrm{Op}(\chi_0)P^{W,(0)}\,\mathrm{Op}(\chi_0)-\mathcal{O}(h^2).$$

Since $P^{W,(1)}=P^{W,(0)}\otimes\mathrm{Id}+\Psi^0(h)$ (see Equation (1.9) of [Helffer and Sjöstrand 1985], for example), this inequality gives (4-34). ☐

Let Π denote the orthogonal projection onto $\mathrm{Vect}\{f_j^{(1)}:j=2,\dots,n_1+1\}$. Using the previous lemma and its proof, we can describe the action of $P^{(1)}$ on Π:

Lemma 4.11. *The rank of* Π *is* n_1 *for* h *small enough. Moreover*,

$$P^{(1)}\Pi = \mathcal{O}(h^2) \quad \textit{and} \quad \Pi P^{(1)} = \mathcal{O}(h^2). \tag{4-40}$$

Finally, there exists $\varepsilon_1 > 0$ *such that*

$$(1-\Pi)P^{(1)}(1-\Pi) \geq \varepsilon_1 h(1-\Pi) \tag{4-41}$$

for h *small enough.*

Proof. Since the functions $f_j^{(1)}$ are almost orthogonal (i.e., $\langle f_j^{(1)}, f_{j'}^{(1)}\rangle = \delta_{j,j'} + \mathcal{O}(h)$), the rank of Π is n_1. Moreover, (4-40) is a direct consequence of Lemma 4.8.

We now give the lower bound for $P^{(1)}$ on the range of $1-\Pi$. Let $\mathscr{E}^{(1)}$ denote the space spanned by the $f_k^{W,(1)}$, $k = 2, \dots, n_1+1$ and $\mathscr{F}^{(1)}$ the eigenspace associated to the n_1 first eigenvalues of $P^{W,(1)}$. Let $\Pi_{\mathscr{E}^{(1)}}$, $\Pi_{\mathscr{F}^{(1)}}$ denote the corresponding orthogonal projectors. It follows from [Helffer and Sjöstrand 1985] that $\|\Pi_{\mathscr{E}^{(1)}} - \Pi_{\mathscr{F}^{(1)}}\| = \mathcal{O}(e^{-c/h})$ for some $c > 0$. On the other hand, it follows from the first estimate of Lemma 4.8 that $\|\Pi - \Pi_{\mathscr{E}^{(1)}}\| = \mathcal{O}(h)$. Combining these two estimates, we get

$$\|\Pi - \Pi_{\mathscr{F}^{(1)}}\| = \mathcal{O}(h).$$

Using this bound and the spectral properties of $P^{W,(1)}$, we get

$$P^{W,(1)} \geq \nu h - \nu h \Pi_{\mathscr{F}^{(1)}} \geq \nu h - \nu h \Pi + \mathcal{O}(h^2) \tag{4-42}$$

for some $\nu > 0$. From (4-23) and integration by parts, we also have $\mathrm{Op}(\chi_0)\Pi = \Pi + \mathcal{O}(h^\infty)$. Estimate (4-42) together with (4-31), (4-32), (4-33) and (4-34) give

$$\begin{aligned}
P^{(1)} &= \mathrm{Op}(\chi_0)P^{(1)}\,\mathrm{Op}(\chi_0) + \mathrm{Op}(\chi_\infty)P^{(1)}\,\mathrm{Op}(\chi_\infty) + \mathcal{O}(h^2) \\
&\geq (\beta_d - C\eta)\,\mathrm{Op}(\chi_0)P^{W,(1)}\,\mathrm{Op}(\chi_0) + \delta_\eta\,\mathrm{Op}(\chi_\infty)^2 - (C\eta h + \mathcal{O}(h^2)) \\
&\geq \nu h(\beta_d - C\eta)\,\mathrm{Op}(\chi_0)^2 - \nu h(\beta_d - C\eta)\Pi + \delta_\eta\,\mathrm{Op}(\chi_\infty)^2 - (C\eta h + \mathcal{O}(h^2)) \\
&\geq \nu h(\beta_d - C\eta) - \nu h(\beta_d - C\eta)\Pi - (C\eta h + \mathcal{O}(h^2)).
\end{aligned} \tag{4-43}$$

Thus, taking $\eta > 0$ small enough and applying $1-\Pi$, we finally obtain (4-41) for some $\varepsilon_1 > 0$. ☐

Proof of Proposition 4.9. From Proposition 2.4 and Lemma 4.5, the operator $P^{(1)}$ is bounded and its essential spectrum is above some positive constant independent of h. Next, the maxi-min principle together with (4-40) implies that $P^{(1)}$ has at least $\mathrm{rank}(\Pi) = n_1$ eigenvalues below Ch^2. In the same way, (4-41) yields that $P^{(1)}$ has at most n_1 eigenvalues below $\varepsilon_1 h$. Finally,

$$P^{(1)} = (1-\Pi)P^{(1)}(1-\Pi) + \Pi P^{(1)}(1-\Pi) + (1-\Pi)P^{(1)}\Pi + \Pi P^{(1)}\Pi \geq -Ch^2$$

proves that all the spectrum of $P^{(1)}$ is above $-Ch^2$. ☐

5. Eigenspace analysis and proof of the main theorem

Now we want to project the preceding quasimodes onto the generalized eigenspaces associated to exponentially small eigenvalues, and prove the main theorem. Recall that we have built in the preceding section quasimodes $f_k^{(0)}$, $k = 1, \dots, n_0$, for $P^{(0)}$ with good support properties. To each quasimode we will associate a function in $E^{(0)}$, the eigenspace associated to the $\mathcal{O}(h^2)$ eigenvalues. For this, we first define the spectral projector

$$\Pi^{(0)} = \frac{1}{2\pi i} \int_\gamma (z - P^{(0)})^{-1} \, dz, \tag{5-1}$$

where $\gamma = \partial B\big(0, \frac{1}{2}\varepsilon_0 h\big)$ and $\varepsilon_0 > 0$ is defined in Proposition 4.3. From the fact that $P^{(0)}$ is selfadjoint, we get that

$$\Pi^{(0)} = \mathcal{O}(1).$$

For the following, we denote the corresponding projection by

$$e_k^{(0)} = \Pi^{(0)}(f_k^{(0)}).$$

Lemma 5.1. *The system $(e_k^{(0)})_k$ is free and spans $E^{(0)}$. Further, there exists $\alpha > 0$ independent of ε such that*

$$e_k^{(0)} = f_k^{(0)} + \mathcal{O}(e^{-\alpha/h}) \quad \textit{and} \quad \langle e_k^{(0)}, e_{k'}^{(0)} \rangle = \delta_{k,k'} + \mathcal{O}(h).$$

Proof. The proof follows [Helffer and Sjöstrand 1985] (see also [Dimassi and Sjöstrand 1999]). We sketch it for the sake of completeness and to give the necessary modifications. Using (5-1) and the Cauchy formula, we get

$$\begin{aligned} e_k^{(0)} - f_k^{(0)} = \Pi^{(0)} f_k^{(0)} - f_k^{(0)} &= \frac{1}{2\pi i} \int_\gamma (z - P^{(0)})^{-1} f_k^{(0)} \, dz - \frac{1}{2\pi i} \int_\gamma z^{-1} f_k^{(0)} \, dz \\ &= \frac{1}{2\pi i} \int_\gamma (z - P^{(0)})^{-1} z^{-1} P^{(0)} f_k^{(0)} \, dz. \end{aligned}$$

Since $P^{(0)}$ is selfadjoint and according to Proposition 4.3, we have

$$\|(z - P^{(0)})^{-1}\| = \mathcal{O}(h^{-1})$$

uniformly for $z \in \gamma$. Using also the second estimate in Lemma 4.1, this yields

$$\|(z - P^{(0)})^{-1} z^{-1} P^{(0)} f_k^{(0)}\| = \mathcal{O}(h^{-2} e^{-\alpha/h}),$$

and, after integration,

$$\|e_k^{(0)} - f_k^{(0)}\| = \mathcal{O}(h^{-1} e^{-\alpha/h}).$$

Decreasing α, we obtain the first estimate of the lemma. In particular, this implies that the family $(e_k^{(0)})_k$ is free. Using that $E^{(0)}$ is of dimension n_0, the family $(e_k^{(0)})_k$ spans $E^{(0)}$.

For the last equality of the lemma, we just have to notice that

$$\langle e_k^{(0)}, e_{k'}^{(0)} \rangle = \langle f_k^{(0)}, f_{k'}^{(0)} \rangle + \mathcal{O}(e^{-\alpha/h}) = \delta_{k,k'} + \mathcal{O}(h) + \mathcal{O}(e^{-\alpha/h}) = \delta_{k,k'} + \mathcal{O}(h),$$

according to Lemma 4.1. The proof is complete. □

We can do a similar study for the analysis of $P^{(1)}$, for which we know that exactly n_1 (real) eigenvalues are $\mathcal{O}(h^2)$, and among them at least $n_0 - 1$ are exponentially small. Note that there is no particular reason for the remaining ones to also be exponentially small.

To the family of quasimodes $(f_j^{(1)})_j$, we now associate a family of functions in $E^{(1)}$, the eigenspace associated to the $\mathcal{O}(h^2)$ eigenvalues for $P^{(1)}$. By the spectral properties of the selfadjoint operator $P^{(1)}$, its spectral projector onto $E^{(1)}$ is given by

$$\Pi^{(1)} = \frac{1}{2\pi i} \int_\gamma (z - P^{(1)})^{-1} \, dz, \tag{5-2}$$

where $\gamma = \partial B\big(0, \frac{1}{2}\varepsilon_1 h\big)$, with ε_1 defined in Proposition 4.9. In the sequel, we write

$$e_j^{(1)} = \Pi^{(1)}(f_j^{(1)}).$$

Mimicking the proof of Lemma 5.1, one can show that the family $(e_j^{(1)})_j$ satisfies the following estimates:

Lemma 5.2. *The system $(e_j^{(1)})_j$ is free and spans $E^{(1)}$. Further, we have*

$$e_j^{(1)} = f_j^{(1)} + \mathcal{O}(h) \quad \textit{and} \quad \langle e_j^{(1)}, e_{j'}^{(1)} \rangle = \delta_{j,j'} + \mathcal{O}(h).$$

Thanks to the preceding lemmas, the families $(e_k^{(0)})_k$ and $(e_j^{(1)})_j$ are orthonormal, apart from an $\mathcal{O}(h)$ factor. To accurately compute the eigenvalues of $P^{(0)}$ and prove the main theorem, we need more precise estimates of exponential type. For this, we will use the intertwining relation $L_\phi P^{(0)} = P^{(1)} L_\phi$.

More precisely, we denote by L the $n_1 \times n_0$ matrix of this restriction of L_ϕ with respect to the bases $(e_j^{(1)})_j$ and $(e_k^{(0)})_k$:

$$L_{j,k} := \langle e_j^{(1)}, L_\phi e_k^{(0)} \rangle. \tag{5-3}$$

The classical way (e.g., [Helffer et al. 2004; Helffer and Sjöstrand 1985]) to compute the exponentially small eigenvalues of $P^{(0)}$ is to then accurately compute the singular values of L. For this, we first state a refined lemma about exponential estimates.

Lemma 5.3. *There exists $\alpha > 0$ independent of ε such that*

$$L_\phi L_\phi^* f_j^{(1)} = \mathcal{O}(e^{-\alpha/h}), \tag{5-4}$$

and also a smooth 1-form $r_j^{(1)}$ such that

$$L_\phi^*(e_j^{(1)} - f_j^{(1)}) = L_\phi^* r_j^{(1)} \quad \textit{and} \quad r_j^{(1)} = \mathcal{O}(e^{-\alpha/h}).$$

Proof. We first note that

$$\begin{aligned} L_\phi L_\phi^* f_j^{(1)} &= \beta_d^{1/2} L_\phi a_h d_{\phi,h}^* Q^* (Q^*)^{-1} f_j^{W,(1)} \\ &= \beta_d^{1/2} L_\phi a_h (d_{\phi,h}^* f_j^{W,(1)}). \end{aligned} \tag{5-5}$$

On the other hand, (4-2) and (4-24) give

$$\|d_{\phi,h}^{*} f_j^{W,(1)}\|^2 \leq \|d_{\phi,h}^{*} f_j^{W,(1)}\|^2 + \|d_{\phi,h} f_j^{W,(1)}\|^2 = \langle P^{W,(1)} f_j^{W,(1)}, f_j^{W,(1)}\rangle = \mathcal{O}(e^{-\alpha/h})$$

for some $\alpha > 0$ independent of ε. Since a_h and L_ϕ are uniformly bounded operators, (5-5) provides the required estimate.

Now we show the second and third equalities, following closely the proof of Lemma 5.1. Using (5-1), the intertwining relation (see Lemma 4.7) and the Cauchy formula, we have

$$\begin{aligned} L_\phi^*(e_j^{(1)} - f_j^{(1)}) &= L_\phi^* \Pi^{(1)} f_j^{(1)} - L_\phi^* f_j^{(1)} \\ &= \Pi^{(0)} L_\phi^* f_j^{(1)} - L_\phi^* f_j^{(1)} \\ &= \frac{1}{2\pi i}\int_\gamma (z - P^{(0)})^{-1} L_\phi^* f_j^{(1)}\,dz - \frac{1}{2\pi i}\int_\gamma z^{-1} L_\phi^* f_j^{(1)}\,dz \\ &= \frac{1}{2\pi i}\int_\gamma (z - P^{(0)})^{-1} z^{-1} P^{(0)} L_\phi^* f_j^{(1)}\,dz, \end{aligned} \tag{5-6}$$

where $\gamma = \partial B\big(0, \frac{1}{2}\min(\varepsilon_0, \varepsilon_1)h\big)$. Using again Lemma 4.7, this becomes

$$\begin{aligned} L_\phi^*(e_j^{(1)} - f_j^{(1)}) &= \frac{1}{2\pi i}\int_\gamma (z - P^{(0)})^{-1} z^{-1} L_\phi^* L_\phi L_\phi^* f_j^{(1)}\,dz \\ &= L_\phi^* \frac{1}{2\pi i}\int_\gamma (z - P^{(1)})^{-1} z^{-1} L_\phi L_\phi^* f_j^{(1)}\,dz. \end{aligned}$$

We then let

$$r_j^{(1)} = \left(\frac{1}{2\pi i}\int_\gamma (z - P^{(1)})^{-1} z^{-1}\,dz\right) L_\phi L_\phi^* f_j^{(1)}, \tag{5-7}$$

and the preceding equality reads

$$L_\phi^*(e_j^{(1)} - f_j^{(1)}) = L_\phi^* r_j^{(1)}. \tag{5-8}$$

Moreover, as in proof of Lemma 5.1, we have

$$\frac{1}{2\pi i}\int_\gamma (z - P^{(1)})^{-1} z^{-1}\,dz = \mathcal{O}(h^{-1}).$$

Combining with (5-4), this shows that $r_j^{(1)} = \mathcal{O}(e^{-\alpha/h})$ for some (new) $\alpha > 0$. □

We begin the study of the matrix L with the following lemma:

Lemma 5.4. *There exists $\alpha' > 0$ such that, if $\varepsilon > 0$ is sufficiently small and fixed, we have, for all $2 \leq j \leq n_1 + 1$ and $2 \leq k \leq n_0$,*

$$L_{j,k} = \langle f_j^{(1)}, L_\phi f_k^{(0)}\rangle + \mathcal{O}(e^{-(S_k + \alpha')/h}).$$

Moreover, $L_{j,1} = 0$ for all $2 \leq j \leq n_1 + 1$.

Proof. We first treat the case $k = 1$. Since $f_1^{(0)}$ is collinear to $a_h^{-1}e^{-\phi/h}$, it belongs to $\ker(P^{(0)})$. Then, $e_1^{(0)} = \Pi^{(0)} f_1^{(0)} = f_1^{(0)}$ satisfies $L_\phi e_1^{(0)} = 0$ from Lemma 4.7. In particular, $L_{j,1} = 0$ for all $2 \le j \le n_1 + 1$.

We now assume $2 \le k \le n_0$. Using Lemma 4.7 and the definition of $e_\bullet^{(\star)}$, we can write

$$\begin{aligned}
L_{j,k} &= \langle e_j^{(1)}, L_\phi e_k^{(0)}\rangle = \langle e_j^{(1)}, L_\phi \Pi^{(0)} f_k^{(0)}\rangle = \langle e_j^{(1)}, \Pi^{(1)} L_\phi f_k^{(0)}\rangle \\
&= \langle \Pi^{(1)} e_j^{(1)}, L_\phi f_k^{(0)}\rangle = \langle e_j^{(1)}, L_\phi f_k^{(0)}\rangle = \langle f_j^{(1)}, L_\phi f_k^{(0)}\rangle + \langle e_j^{(1)} - f_j^{(1)}, L_\phi f_k^{(0)}\rangle \\
&= \langle f_j^{(1)}, L_\phi f_k^{(0)}\rangle + \langle L_\phi^*(e_j^{(1)} - f_j^{(1)}), f_k^{(0)}\rangle.
\end{aligned}$$

From Lemma 5.3, this becomes

$$\begin{aligned}
L_{j,k} &= \langle f_j^{(1)}, L_\phi f_k^{(0)}\rangle + \langle L_\phi^* r_j^{(1)}, f_k^{(0)}\rangle \\
&= \langle f_j^{(1)}, L_\phi f_k^{(0)}\rangle + \langle r_j^{(1)}, L_\phi f_k^{(0)}\rangle.
\end{aligned} \tag{5-9}$$

Now, since Q is bounded and according to Lemma A.3, we have

$$L_\phi f_k^{(0)} = Q d_{\phi,h} f^{W,(0)} = \mathcal{O}(e^{-(S_k - C\varepsilon)/h}).$$

Using Lemma 5.3 again, this yields

$$\langle r_j^{(1)}, L_\phi f_k^{(0)}\rangle = \mathcal{O}(e^{-(S_k + \alpha - C\varepsilon)/h}) \tag{5-10}$$

with $\alpha > 0$ independent of ε. Taking $\varepsilon > 0$ small enough, the lemma follows from (5-9) and (5-10). □

Now we recall the explicit computation of the matrix L. This is just a consequence of the study of the corresponding Witten Laplacian.

Lemma 5.5. *For all* $2 \le j \le n_1 + 1$ *and* $2 \le k \le n_0$, *we have*

$$L_{k,k} = \left(\frac{h}{(2d+4)\pi}\right)^{1/2} \mu_k^{1/2} \left|\frac{\det \phi''(\boldsymbol{m}_k)}{\det \phi''(\boldsymbol{s}_k)}\right|^{1/4} e^{-S_k/h}(1 + \mathcal{O}(h)) =: h^{1/2}\ell_k(h) e^{-S_k/h}$$

and

$$L_{j,k} = \mathcal{O}(e^{-(S_k + \alpha')/h}) \quad \textit{for all } j \ne k,$$

where $S_k := \phi(\boldsymbol{s}_k) - \phi(\boldsymbol{m}_k)$ *and* $-\mu_k$ *denotes the unique negative eigenvalue of* ϕ'' *at* $\boldsymbol{s}_k$.

Proof. First, we note that

$$\langle f_j^{(1)}, L_\phi f_k^{(0)}\rangle = \beta_d^{1/2}\langle f_j^{W,(1)}, d_{\phi,h} f_k^{W,(0)}\rangle,$$

by (4-3), (4-26) and $L_\phi = Q d_{\phi,h} a_h$. Thus, Lemma 5.4 implies

$$L_{j,k} = \beta_d^{1/2}\langle f_j^{W,(1)}, d_{\phi,h} f_k^{W,(0)}\rangle + \mathcal{O}(e^{-(S_k + \alpha')/h}).$$

The first term is exactly the approximate singular value of $d_{\phi,h}$ computed in [Helffer et al. 2004]. The result is then a direct consequence of Proposition 6.4 of [Helffer et al. 2004]. □

Now we are able to compute the singular values of L (i.e., the eigenvalues of $(L^*L)^{1/2}$).

Lemma 5.6. *There exists $\alpha' > 0$ such that the singular values $\nu_k(L)$ of L, enumerated in a suitable order, satisfy*

$$\nu_k(L) = |L_{k,k}|(1 + \mathcal{O}(e^{-\alpha'/h})) \quad \textit{for all } 1 \leq k \leq n_0.$$

Proof. Since the first column of L consists of zeros, we get $\nu_1 = 0$. Moreover, the other singular values of L are those of the reduced matrix L' with entries $L'_{j,k} = L_{j+1,k+1}$ for $1 \leq j \leq n_1$ and $1 \leq k \leq n_0 - 1$. We shall now use that the dominant term in each column of L' lies on the diagonal. Define the $(n_0 - 1) \times (n_0 - 1)$ diagonal matrix D by

$$D := \operatorname{diag}(L_{k+1,k+1} : k = 1, \dots, n_0 - 1).$$

Notice that D is invertible, thanks to the ellipticity of $\ell_{k+1}(h)$, and that $\nu_k(D) = |L_{k+1,k+1}|$. We also define the $n_1 \times (n_0 - 1)$ characteristic matrix of L'

$$U = (\delta_{j,k})_{j,k}.$$

From Lemma 5.5, there is a constant $\alpha' > 0$ such that

$$L' = (U + \mathcal{O}(e^{-\alpha'/h}))D. \tag{5-11}$$

The Fan inequalities (see, for example, Theorem 1.6 of [Simon 1979]) therefore give

$$\nu_k(L') \leq (1 + \mathcal{O}(e^{-\alpha'/h}))\nu_k(D). \tag{5-12}$$

To get the opposite estimate, we remark that $U^*U = \mathrm{Id}_{n_0-1}$. Then, (5-11) implies

$$D = (1 + \mathcal{O}(e^{-\alpha'/h}))U^*L',$$

and, as before,

$$\nu_k(D) \leq (1 + \mathcal{O}(e^{-\alpha'/h}))\nu_k(L'). \tag{5-13}$$

The lemma follows from $\nu_{k+1}(L) = \nu_k(L')$, (5-12), (5-13) and $\nu_k(D) = |L_{k+1,k+1}|$. $\square$

Now, Theorem 1.2 is a direct consequence of the explicit computations of Lemma 5.5 and of the following equivalent formulation:

Lemma 5.7. *The nonzero exponentially small eigenvalues of P_h are of the form*

$$h(\ell_k^2(h) + \mathcal{O}(h))e^{-2S_k/h} \quad \textit{for } 2 \leq k \leq n_0.$$

Proof. According to Lemma 5.1 and Lemma 5.2, the bases $(e_k^{(0)})_k$ and $(e_j^{(1)})_j$ of $E^{(0)}$ and $E^{(1)}$ respectively are orthonormal up to $\mathcal{O}(h)$ small errors. Let $(\tilde{e}_k^{(0)})_k$ and $(\tilde{e}_j^{(1)})_j$ be the corresponding orthonormalizations (obtained by taking square roots of the Gramians), which differ from the original bases by $\mathcal{O}(h)$ small recombinations. Then, with respect to the new bases, the matrix of L_ϕ takes the form $\widetilde{L} = (1 + \mathcal{O}(h))L(1 + \mathcal{O}(h))$. Using the Fan inequalities, we see that the conclusion of Lemma 5.6 is also valid for $\widetilde{L}$ (note that there is no need to have exponentially small errors here). Since the matrix of the restriction of $P^{(0)}$ to $E^{(0)}$ with respect to the basis $(\tilde{e}_k^{(0)})_k$ is given by $\widetilde{L}^*\widetilde{L}$, the lemma follows. $\square$

We end this section by showing that the main theorems stated in Section 1 imply the metastability of the system.

Proof of Corollary 1.4. We first prove (1-5) and (1-7). If ϕ has a unique minimum, Theorem 1.1 gives

$$\|(\boldsymbol{T}_h^\star)^n(d\nu_h) - d\nu_{h,\infty}\|_{\mathscr{H}_h} \le (1-\delta h)^n \|d\nu_h\|_{\mathscr{H}_h} = e^{n\ln(1-\delta h)+|\ln h|} h \|d\nu_h\|_{\mathscr{H}_h}.$$

Using that $n\ln(1-\delta h) \sim -\delta h n$, this estimate yields

$$\|(\boldsymbol{T}_h^\star)^n(d\nu_h) - d\nu_{h,\infty}\|_{\mathscr{H}_h} \le h\|d\nu_h\|_{\mathscr{H}_h}$$

for $n \gtrsim |\ln h| h^{-1}$. In the same way, if ϕ has several minima, Theorem 1.2 implies

$$\|(\boldsymbol{T}_h^\star)^n(d\nu_h) - d\nu_{h,\infty}\|_{\mathscr{H}_h} \le (\lambda_2^\star(h))^n \|d\nu_h\|_{\mathscr{H}_h} = e^{n\ln(\lambda_2^\star(h))+|\ln h|} h \|d\nu_h\|_{\mathscr{H}_h}.$$

Using now that $n\ln(\lambda_2^\star(h)) \sim n(\lambda_2^\star(h)-1) \sim -Cnhe^{-S_2/h}$ for some $C>0$, this estimate yields

$$\|(\boldsymbol{T}_h^\star)^n(d\nu_h) - d\nu_{h,\infty}\|_{\mathscr{H}_h} \le h\|d\nu_h\|_{\mathscr{H}_h}$$

for $n \gtrsim |\ln h| h^{-1} e^{S_2/h}$.

It remains to show (1-6). From Theorem 1.1, Theorem 1.2 and the proof of (1-5), we can write

$$(\boldsymbol{T}_h^\star)^n(d\nu_h) = \sum_{k=1}^{n_0} (\lambda_k^\star(h))^n \Pi_k d\nu_h + \mathcal{O}(h)\|d\nu_h\|_{\mathscr{H}_h},$$

for $n \gtrsim |\ln h| h^{-1}$. Here, Π_k is the spectral projector of $\boldsymbol{T}_h^\star$ associated to the eigenvalue $\lambda_k^\star(h)$. If we assume in addition that $n \lesssim e^{2S_{n_0}/h}$, then $(\lambda_k^\star(h))^n = 1+\mathcal{O}(h)$ for any $k=1,\dots,n_0$. Thus, the previous equation becomes

$$(\boldsymbol{T}_h^\star)^n(d\nu_h) = \Pi^{(0)} d\nu_h + \mathcal{O}(h)\|d\nu_h\|_{\mathscr{H}_h}, \tag{5-14}$$

since $\Pi^{(0)} = \Pi_1 + \cdots + \Pi_{n_0}$. Let

$$g_k(x) := \frac{\chi_k(x) e^{-(\phi(x)-\phi(\boldsymbol{m}_k))/h}}{\|\chi_k e^{-(\phi-\phi(\boldsymbol{m}_k))/h}\|}.$$

From (A-1), we immediately get $g_k = f_k^{W,(0)} + \mathcal{O}(h)$. Moreover, as in (4-4), we have

$$\|f_k^{(0)} - f_k^{W,(0)}\| = \|(a_h^{-1}-1) f_k^{W,(0)}\| = \mathcal{O}(h).$$

Combining with Lemma 5.1, we deduce

$$g_k = e_k^{(0)} + \mathcal{O}(h). \tag{5-15}$$

Using Lemma 5.1 one more time, the bases $(e_k^{(0)})_k$ and $(g_k)_k$ of $\operatorname{Im}\Pi^{(0)}$ and $\operatorname{Im}\Pi$, respectively, are almost orthogonal, in the sense that

$$\langle e_k^{(0)}, e_{k'}^{(0)}\rangle = \delta_{k,k'} + \mathcal{O}(h) \quad \text{and} \quad \langle g_k, g_{k'}\rangle = \delta_{k,k'} + \mathcal{O}(h).$$

This then yields

$$\Pi = \Pi^{(0)} + \mathcal{O}(h), \tag{5-16}$$

and (1-6) follows from (5-14). □

Appendix: Quasimodes, truncation procedure and labeling

In this appendix, we gather from [Helffer et al. 2004; Hérau et al. 2011] the refined construction of quasimodes on 0-forms for the Witten Laplacian, and the labeling procedure linking each minima with a saddle point of index 1. We recall briefly the construction proposed in [Hérau et al. 2011] (which was in the Fokker–Planck case there) but in a generic situation where all $\phi(\boldsymbol{s})-\phi(\boldsymbol{m})$ are distinct for $\boldsymbol{m}$ in the set of minima and $\boldsymbol{s}$ in the set of saddle points of ϕ.

In the following, we will denote by $\mathscr{L}(\sigma)=\{x\in\mathbb{R}^n:\phi(x)<\sigma\}$ the sublevel set associated to the value $\sigma\in\mathbb{R}$. Let $\boldsymbol{s}$ be a saddle point of ϕ and $B(\boldsymbol{s},r)=\{x\in\mathbb{R}^n:|x-\boldsymbol{s}|<r\}$. Then, for $r>0$ small enough, the set

$$B(\boldsymbol{s},r)\cap\mathscr{L}(\phi(\boldsymbol{s}))=\{x\in B(\boldsymbol{s},r):\phi(x)<\phi(\boldsymbol{s})\}$$

has precisely 2 connected components, $C_j(\boldsymbol{s},r)$ with $j=1,2$.

Definition A.1. We say that $\boldsymbol{s}\in\mathbb{R}^n$ is a separating saddle point (ssp) if it is either ∞ or it is a usual saddle point such that $C_1(\boldsymbol{s},r)$ and $C_2(\boldsymbol{s},r)$ are contained in different connected components of the set $\{x\in\mathbb{R}^n:\phi(x)<\phi(\boldsymbol{s})\}$. We denote by SSP the set of ssps.

We also introduce the set of separating saddle values (ssv), $\mathrm{SSV}=\{\phi(\boldsymbol{s}):\boldsymbol{s}\in\mathrm{SSP}\}$ with the convention that $\phi(\infty)=+\infty$.

A connected component E of the sublevel set $\mathscr{L}(\sigma)$ will be called a critical component if either $\partial E\cap\mathrm{SSP}\neq\varnothing$ or $E=\mathbb{R}^n$.

Let us now explain the way we label the critical points. We first order the saddle points in the following way. We recall from [Helffer et al. 2004] that $\sharp\mathrm{SSV}=n_0$ and then enumerate the ssvs in a decreasing way: $\infty=\sigma_1>\sigma_2>\cdots>\sigma_{n_0}$. To each ssv σ_j we can associate a unique ssp: we define $\boldsymbol{s}_1=\infty$ and, for any $j=2,\dots,n_0$, we let $\boldsymbol{s}_j$ be the unique ssp such that $\phi(\boldsymbol{s}_j)=\sigma_j$ (note that this $\boldsymbol{s}_j$ is unique thanks to Hypothesis 2).

Then we can proceed to the labeling of minima. We denote by $\boldsymbol{m}_1$ the global minimum of ϕ, $E_1=\mathbb{R}^d$ and by $S_1=\phi(\boldsymbol{s}_1)-\phi(\boldsymbol{m}_1)=+\infty$ the critical Arrhenius value.

Next we observe that the sublevel set $\mathscr{L}(\sigma_2)=\{x\in\mathbb{R}^n:\phi(x)<\sigma_2\}$ is the union of two critical components, with one containing $\boldsymbol{m}_1$. The remaining connected component of the sublevel set $\mathscr{L}(\sigma_2)$ will be denoted by E_2 and its minimum by $\boldsymbol{m}_2$. To the pair $(\boldsymbol{m}_2,\boldsymbol{s}_2)$ of critical points we associate the Arrhenius value $S_2=\phi(\boldsymbol{s}_2)-\phi(\boldsymbol{m}_2)$.

Continuing the labeling procedure, we decompose the sublevel set $\mathscr{L}(\sigma_3)$ into its connected components and perform the labeling as follows: we omit all those components that contain the already labeled minima $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$. Some of these components may be noncritical. There is only one critical one remaining, and we denote it by E_3. We then let $\boldsymbol{m}_3$ be the point of global minimum of the restriction of ϕ to E_3 and $S_3=\phi(\boldsymbol{s}_3)-\phi(\boldsymbol{m}_3)$.

We go on with this procedure, proceeding in the order dictated by the elements of the set SSV, arranged in the decreasing order, until all n_0 local minima $\boldsymbol{m}$ have been enumerated. In this way we have associated each local minima to one ssp: to each local minimum $\boldsymbol{m}_k$, there is one critical component E_k containing $\boldsymbol{m}_k$, and one ssp $\boldsymbol{s}_k$. We emphasize that in this procedure some of the saddle points (the noncritical ones) may not have been enumerated. For convenience, we enumerate these remaining saddle points from n_0+1 to n_1+1. Note that, with this labeling, $\mathcal{U}^{(1)} = \{\boldsymbol{s}_2, \dots, \boldsymbol{s}_{n_1+1}\}$. We then have

$$\text{minima} = \{\boldsymbol{m}_1, \dots, \boldsymbol{m}_{n_0}\}, \quad \text{SSP} = \{\boldsymbol{s}_1 = \infty, \boldsymbol{s}_2, \dots, \boldsymbol{s}_{n_0}\}.$$

We summarize the preceding discussion in the following proposition:

Proposition A.2. *The families of minima $\mathcal{U}^{(0)} = \{\boldsymbol{m}_k : k = 1, \dots, n_0\}$, separating saddle points $\{\boldsymbol{s}_k : k = 1, \dots, n_0\}$ and connected sets $\{E_k : k = 1, \dots, n_0\}$ satisfy*:

(i) $\boldsymbol{s}_1 = \infty$, $E_1 = \mathbb{R}^n$ *and $\boldsymbol{m}_1$ is the global minimum of ϕ.*

(ii) *For every $k \geq 2$, $\overline{E}_k$ is compact, E_k is the connected component containing $\boldsymbol{m}_k$ in*

$$\{x \in \mathbb{R}^n : \phi(x) < \phi(\boldsymbol{s}_k)\}$$

and $\phi(\boldsymbol{m}_k) = \min_{E_k} \phi$.

(iii) *If $\boldsymbol{s}_{k'} \in E_k$ for some $k, k' \in \{1, \dots, n_0\}$, then $k' > k$.*

To ensure that the eigenvalues $\lambda_k^\star$ are decreasing, if necessary we relabel the pairs of minima and critical saddle points so that the sequence S_k is decreasing.

Using [Helffer et al. 2004; Hérau et al. 2011], we shall now introduce suitable refined quasimodes, adapted to the local minima of ϕ and the simplified labeling, described in Proposition A.2. Let $\varepsilon_0 > 0$ be such that the distance between critical points is larger than $10\varepsilon_0$ and such that, for every critical point $\boldsymbol{u}$ and $k \in \{1, \dots, n_0\}$, we have either $\boldsymbol{u} \in \overline{E}_k$ or $\operatorname{dist}(\boldsymbol{u}, \overline{E}_k) \geq 10\varepsilon_0$. Also let $C_0 > 1$, to be defined later, and note that ε_0 may also be taken smaller later. For $0 < \varepsilon < \varepsilon_0$ we build a family of functions $\chi_{k,\varepsilon}$, $k \in \{1, \dots, n_0\}$ as follows: for $k = 1$, we let $\chi_{1,\varepsilon} = 1$ and, for $k \geq 2$, we consider the open set $E_{k,\varepsilon} = E_k \setminus \overline{B(\boldsymbol{s}_k, \varepsilon)}$, and let $\chi_{k,\varepsilon}$ be a C_0^∞-cutoff function supported in $E_{k,\varepsilon} + B(0, \varepsilon/C_0)$ and equal to 1 in $E_{k,\varepsilon} + B(0, \varepsilon/(2C_0))$. Then, we define the quasimodes for $1 \leq k \leq n_0$ by

$$f_k^{W,(0)} = b_k(h)\chi_{k,\varepsilon}(x)e^{-(\phi(x)-\phi(\boldsymbol{m}_k))/h}, \tag{A-1}$$

where b_k is a normalization constant, given thanks to the stationary phase theorem by

$$b_k(h) = (\pi h)^{-d/4} \det(\operatorname{Hess} \phi(\boldsymbol{m}_k))^{1/4}(b_{k,0} + h b_{k,1} + \cdots), \quad b_{k,0} = 1.$$

Then, for ε_0 small enough and C_0 large enough, there exists $C > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, we have the following lemma:

Lemma A.3. *The system $(f_k^{W,(0)})$ is free and there exists $\alpha > 0$ uniform in $\varepsilon < \varepsilon_0$ such that*

$$\langle f_k^{W,(0)}, f_{k'}^{W,(0)} \rangle = \delta_{k,k'} + \mathcal{O}(e^{-\alpha/h}), \quad d_{\phi,h} f_k^{W,(0)} = \mathcal{O}(e^{-(S_k - C\varepsilon)/h}),$$

and, in particular,

$$P^{W,(0)} f_k^{W,(0)} = \mathcal{O}(e^{-\alpha/h}).$$

Proof. This is a direct consequence of the statement and proof of Proposition 5.3 in [Hérau et al. 2011]. □

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