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This paper is concerned with the well-posedness of the Cauchy problem for first order symmetric hyperbolic systems in the sense of Friedrichs. The classical theory says that if the coefficients of the system and if the coefficients of the symmetrizer are Lipschitz continuous, then the Cauchy problem is well posed in L^2. When the symmetrizer is log-Lipschitz or when the coefficients are analytic or quasianalytic, the Cauchy problem is well posed in C^∞. We give counterexamples which show that these results are sharp. We discuss both the smoothness of the symmetrizer and of the coefficients.

1. Introduction

We consider the well-posedness of the Cauchy problem for first order symmetric hyperbolic systems in the sense of Friedrichs [1954], who proved that if the coefficients of the system *and if* the coefficients of the symmetrizer are Lipschitz continuous, then the Cauchy problem is well posed in L^2. This has been extended to hyperbolic systems which admit Lipschitzian microlocal symmetrizers (see [Métivier 2014]).

The main objective of this paper is to discuss the necessity of these smoothness assumptions and to provide new counterexamples to the well-posedness. In the spirit of [Colombini and Spagnolo 1989; Colombini and Nishitani 1999], we make a detailed analysis of systems in space dimension one with coefficients which depend only on time. Even more, we concentrate our analysis on the 2×2 system

$$Lu := \partial_t u + \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \partial_x u = \partial_t u + A(t)u. \tag{1-1}$$

The symbol is assumed to be strongly hyperbolic or uniformly diagonalizable, which means that there is a bounded symmetrizer $S(t)$, with S^{-1} bounded, which is positive definite and such that $S(t)A(t)$ is symmetric. This is equivalent to the condition that there is $\delta > 0$ such that

$$\delta((a-d)^2 + b^2 + c^2) \le \tfrac{1}{4}(a-d)^2 + bc. \tag{1-2}$$

If the symmetrizer S and the coefficients are Lipschitz continuous then the Cauchy problem is well posed in L^2. Indeed, in this case, solutions on $[0, T] \times \mathbb{R}$ of $Lu = f$ satisfy

$$\|u(t)\|_{L^2} \le C(\|u(0)\|_{L^2} + \|Lu\|_{L^2}) \quad \text{with} \quad C = C_0 \exp\left(\int_0^T |\partial_t S(s)| \, ds\right). \tag{1-3}$$

Lipschitz smoothness of the symmetrizer is almost necessary for well-posedness in L^2, even for very smooth coefficients:

Theorem 1.1. *For each modulus of continuity ω such that $t^{-1}\omega(t) \to +\infty$ as $t \to 0$, there is a system* (1-1) *with coefficients in $\bigcap_{s>1} G^s([0, T])$, with a symmetrizer satisfying*

$$|S(t) - S(t')| \le C\omega(|t - t'|), \tag{1-4}$$

such that the Cauchy problem is ill posed in L^2 in the sense that there is no constant C such that the estimate (1-3) *is satisfied.*

Here and below, we denote by $G^s([0, T])$ the Gevrey class of functions of order s. They are C^∞ functions f such that, for some constant C which depends on f,

$$\|\partial_t^j f\|_{L^\infty} \le C^{j+1}(j!)^s \quad \text{for all } j \in \mathbb{N}.$$

This theorem extends to systems a similar result obtained in [Cicognani and Colombini 2006] for the strictly hyperbolic wave equation

$$\partial_t^2 u - a(t)\partial_x^2 u = f. \tag{1-5}$$

Indeed, there is a close parallel between the energy $|\partial_t u|^2 + a(t)|\partial_x u|^2$ for the wave equation and $(S(t)u, u)$ for the system, and, in this case, the smoothness of $S(t)$ plays a role analogous to the smoothness of a. For the wave equation, when a is log-Lipschitz, i.e., admits the modulus of continuity $\omega(t) = t|\ln t|$, the Cauchy problem is well posed in C^∞ with a loss of derivatives proportional to time [Colombini et al. 1979]. In intermediate cases between Lipschitz and log-Lipschitz, that is when $(t|\ln t|)^{-1}\omega(t) \to 0$ and $t^{-1}\omega(t) \to +\infty$, the loss of derivative is effective but is arbitrarily small on any interval [Cicognani and Colombini 2006]. The proof of these results extends immediately to systems (1-1) where the smoothness of the symmetrizer plays the role of the smoothness of the coefficient a.

The next result extends to systems the result in [Colombini et al. 1979; Colombini and Spagnolo 1989] showing that the log-Lipschitz smoothness of the symmetrizer is a sharp condition for the well-posedness in C^∞, even for C^∞ coefficients.

Theorem 1.2. *For each modulus of continuity ω satisfying $(t|\ln t|)^{-1}\omega(t) \to +\infty$ as $t \to 0$, there are systems* (1-1) *with C^∞ coefficients, with symmetrizers which satisfy the estimate* (1-4), *such that the Cauchy problem is ill posed in C^∞, meaning that, for all n and all $T > 0$, there is no constant C such that the estimate*

$$\|u\|_{L^2} \le C\|Lu\|_{H^n} \tag{1-6}$$

is satisfied for all $u \in C_0^\infty([0, T] \times \mathbb{R})$.

In [Colombini and Nishitani 1999] the question of the well-posedness of the Cauchy problem is considered under the angle of the smoothness of the coefficients alone. In this aspect, the analysis is related to the analysis of the weakly hyperbolic wave equation (1-5) (see [Colombini et al. 1983]). If the coefficients are C^∞, the problem is well posed in all Gevrey classes G^s, but the well-posedness in C^∞ is obtained only when the coefficients are analytic or belong to a quasianalytic class. Indeed, the next theorem shows that this is sharp.

Theorem 1.3. *There are example of systems* (1-1) *on* $[0, T] \times \mathbb{R}$ *with uniformly hyperbolic symbols and coefficients in the intersection of the Gevrey classes $\bigcap G^s$ for $s > 1$, admitting continuous symmetrizers, such that the Cauchy problem is ill posed in C^∞.*

This theorem improves the similar result obtained in [Colombini and Nishitani 1999], where the counterexample had coefficients in $\bigcap G^s$ for $s > 2$. The same construction can be used to provide a similar improvement to the known result in [Colombini and Spagnolo 1982] for the wave equation:

Theorem 1.4. *There are nonnegative functions $a \in \bigcap_{s>1} G^s([0, T])$ such that the Cauchy problem for the weakly hyperbolic wave equation* (1-5) *is ill posed in C^∞.*

The theorems above show that the smoothness of *both* the coefficients *and* the symmetrizer play a role in the well-posedness in C^∞. The next theorem is a kind of interpolation between the two extreme cases of Theorem 1.2 and Theorem 1.3.

Theorem 1.5. *For all $s > 1$ and $\mu < 1 - 1/s$, there are examples of systems* (1-1) *on $[0, T] \times \mathbb{R}$, with uniformly hyperbolic symbols, coefficients in the Gevrey classes G^s and symmetrizers in the Hölder space C^μ, such that the Cauchy problem is ill posed in C^∞.*

This leaves open the question of the well-posedness in C^∞ when the coefficients belong to G^s and the symmetrizer to C^μ with $\mu + 1/s \geq 1$.

We end this introduction with several remarks about symmetrizers for the 2×2 system (1-1). For simplicity, we assume that the coefficients are real. Write

$$A(t) = \tfrac{1}{2}\mathrm{tr}A(t)\,\mathrm{Id} + A_1(t).$$

Then $A_1^2 = h\,\mathrm{Id}$ with $h = \tfrac{1}{4}(a-d)^2 + bc$ satisfying (1-2). In particular,

$$\Sigma(t) = A_1^*(t)A_1(t) + h(t)\,\mathrm{Id}$$

is a symmetrizer for A in the sense that Σ and $\Sigma A = \tfrac{1}{2}(\mathrm{tr}A)\Sigma + hA_1^* + hA_1$ are symmetric. In general, Σ is *not* a symmetrizer in the sense of Friedrichs, since it is not uniformly positive definite, unless $h > 0$, which means that the system is strictly hyperbolic. More precisely, $\Sigma \approx h\,\mathrm{Id}$. But Σ has the same smoothness as the coefficients of A.

On the other hand, since the system is uniformly diagonalizable, there are bounded symmetrizers $\Sigma_1(t)$ which are uniformly positive definite. For instance, $h^{-1}\Sigma$ is a bounded symmetrizer. More generally, writing

$$\tfrac{1}{2}(a-d) = h^{1/2}a_1, \quad b = b_1 h^{1/2}, \quad c = c_1 h^{1/2}, \tag{1-7}$$

one has $a_1^2 + b_1 c_1 \geq \delta(a_1^2 + b_1^2 + c_1^2) \geq \delta > 0$ and the symmetrizer is of the form

$$\Sigma_1 = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \quad \text{with} \quad 2a_1\beta = b_1\alpha - c_1\gamma. \tag{1-8}$$

Therefore, there is a cone of positive symmetrizers of dimension 2. Their smoothness depends on the smoothness of a_1, b_1, c_1, that is, of $h^{-1/2}A_1$. There might be better choices than others. For instance, if the system is symmetric, $\Sigma_1 = \mathrm{Id}$ is a very smooth symmetrizer. Our discussion below concerns the smoothness of both Σ and Σ_1 and their possible interplay.

2. The counterexamples

We consider systems of the form

$$LU := \partial_t U + \begin{pmatrix} 0 & a(t) \\ b(t) & 0 \end{pmatrix} \partial_x U \tag{2-1}$$

with a and b real. We always assume that it is uniformly strongly hyperbolic, that is, that $\sigma = a/b > 0$ and $1/\sigma$ are bounded. Our goal is to contradict the estimates (1-3) and (1-6). We contradict the analogous estimates which are obtained by Fourier transform in x, and, more precisely, we construct sequences of functions u_k, v_k and f_k in $C^\infty([0, T])$, vanishing near $t = 0$, satisfying

$$\partial_t u_k + i h_k a(t) v_k = f_k, \quad \partial_t v_k + i h_k b(t) u_k = 0 \tag{2-2}$$

with $h_k \to +\infty$ and such that

$$\frac{\|f_k\|_{L^2}}{\|(u_k, v_k)\|_{L^2}} \to 0 \quad \text{as } k \to \infty \tag{2-3}$$

in the first case, or, for all j and l,

$$\frac{\|h_k^j \partial_t^l f_k\|_{L^2}}{\|(u_k, v_k)\|_{L^2}} \to 0 \quad \text{as } k \to \infty \tag{2-4}$$

in the second case. Moreover, the support of these function is contained in an interval $I_k = [t_k, t_k']$ with $0 < t_k < t_k'$ and $t_k' \to 0$, showing that the problem is ill posed on any interval $[0, T]$ with $T > 0$.

Exponentially amplified solutions of the wave equation. In this section we review and adapt the construction of [Colombini and Spagnolo 1989]. The key remark is that the function $\underline{w}_\varepsilon(t) = e^{-\varepsilon\phi(t)} \cos t$ satisfies

$$\partial_t^2 \underline{w}_\varepsilon + \underline{\alpha}_\varepsilon \underline{w}_\varepsilon = 0 \tag{2-5}$$

if

$$\phi(t) = \int_0^t (\cos s)^2 \, ds, \quad \underline{\alpha}_\varepsilon(t) = 1 + 2\varepsilon \sin 2t - \varepsilon^2 (\cos t)^4. \tag{2-6}$$

The important property of the $\underline{w}_\varepsilon$ is their exponential decay at $+\infty$. More precisely,

$$e^{\varepsilon t/2} \underline{w}_\varepsilon(t) = e^{\varepsilon \sin(2t)/4} \cos t \quad \text{is } 2\pi\text{-periodic}$$

and

$$\underline{w}_\varepsilon(t + 2\pi) = e^{-\varepsilon\pi}\,\underline{w}_\varepsilon(t). \tag{2-7}$$

Next, one symmetrizes and localizes this solution. More precisely, consider $\chi \in C^\infty(\mathbb{R})$, supported in $]{-7\pi}, 7\pi[$, odd, equal to 1 on $[-6\pi, -2\pi]$ and thus equal to -1 on $[2\pi, 6\pi]$, and such that, for all t, $0 \le \chi(t) \le 1$ and $|\partial_t \chi(t)| \le 1$. For $\nu \in \mathbb{N}$, let

$$\Phi_\nu(t) = \int_0^t \chi_\nu(s)(\cos s)^2\,ds, \quad \chi_\nu(t) = \chi\left(\frac{t}{\nu}\right). \tag{2-8}$$

For $\varepsilon > 0$, $w_{\varepsilon,\nu}(t) = e^{\varepsilon\Phi_\nu(t)}\cos t$ satisfies

$$\partial_t^2 w_{\varepsilon,\nu} + \alpha_{\varepsilon,\nu} w_{\varepsilon,\nu} = 0, \tag{2-9}$$

where

$$\alpha_{\varepsilon,\nu}(t) = 1 + \varepsilon\chi_\nu \sin 2t - \varepsilon\Phi_\nu'' - (\varepsilon\Phi_\nu')^2$$
$$= 1 + 2\varepsilon\chi_\nu \sin 2t - \varepsilon\chi_\nu'(\cos t)^2 - \varepsilon^2\chi_\nu^2(\cos t)^4. \tag{2-10}$$

For $\varepsilon \le \varepsilon_0 = \frac{1}{10}$ and for all ν,

$$|\alpha_{\varepsilon,\nu} - 1| \le \tfrac{1}{2}, \tag{2-11}$$

and we always assume below that the condition $\varepsilon \le \varepsilon_0$ is satisfied. We note also that $\alpha_{\varepsilon,\nu} = 1$ for $|t| \ge 7\nu\pi$, since χ_ν vanishes there.

The final step is to localize the solution in $[-6\nu\pi, 6\nu\pi]$. Introduce an odd cut-off function $\zeta(t)$ supported in $]{-6\pi}, 6\pi[$ and equal to 1 for $|t| \le 4\pi$, and let

$$\tilde{w}_{\varepsilon,\nu}(t) = \zeta\left(\frac{t}{\nu}\right)w_{\varepsilon,\nu}(t). \tag{2-12}$$

This function is supported in $[-6\nu\pi, 6\nu\pi]$ and equal to $w_{\varepsilon,\nu}$ on $[-4\nu\pi, 4\nu\pi]$. Then

$$f_{\varepsilon,\nu} = \partial_t^2 \tilde{w}_{\varepsilon,\nu} + \alpha_{\varepsilon,\nu} \tilde{w}_{\varepsilon,\nu} = 2\nu^{-1}\zeta'\left(\frac{t}{\nu}\right)\partial_t w_{\varepsilon,\nu} + \nu^{-2}\zeta''\left(\frac{t}{\nu}\right)w_{\varepsilon,\nu} \tag{2-13}$$

is supported in $[-6\nu\pi, -4\nu\pi] \cup [4\nu\pi, 6\nu\pi]$.

Lemma 2.1. *For all j, there is a constant C_j such that, for all $\varepsilon \le \varepsilon_0$ and all $\nu \ge 1$,*

$$\|\partial_t^j f_{\varepsilon,\nu}\|_{L^2} \le C_j \nu^{-1} e^{-\varepsilon\nu\pi} \|\tilde{w}_{\varepsilon,\nu}\|_{L^2}. \tag{2-14}$$

Proof. By symmetry, it is sufficient to estimate $f_{\varepsilon,\nu}$ for $t \ge 0$, that is, on $[4\nu\pi, 6\nu\pi]$. On $[2\nu\pi, 6\nu\pi]$, $\chi_\nu = -1$, hence $\Phi_\nu - \phi$ is constant and

$$w_{\varepsilon,\nu}(t) = c_{\nu,\varepsilon}\underline{w}_\varepsilon(t), \quad c_{\nu,\varepsilon} = e^{\varepsilon\Phi_\nu(2\nu\pi)}.$$

Moreover, on this interval $\alpha_{\varepsilon,\nu} = \underline{\alpha}_\varepsilon$ is bounded with derivatives bounded independently of ε, and hence

$$\|\partial_t^j f_{\varepsilon,\nu}\|_{L^2} \le C_j \nu^{-1} c_{\nu,\varepsilon} \|(\underline{w}_\varepsilon, \partial_t \underline{w}_\varepsilon)\|_{L^2([4\nu\pi, 6\nu\pi])}.$$

By (2-7), this implies

$$\|\partial_t^j f_{\varepsilon,\nu}\|_{L^2} \le C_j \nu^{-1} c_{\nu,\varepsilon} e^{-\varepsilon\nu\pi} \|(\underline{w}_\varepsilon, \partial_t \underline{w}_\varepsilon)\|_{L^2([2\nu\pi, 4\nu\pi])}.$$

On the other hand,

$$\|w_{\varepsilon,\nu}\|_{L^2} \geq c_{\nu,\varepsilon}\|\underline{w}_\varepsilon\|_{L^2([2\nu\pi,4\nu\pi])}.$$

Therefore, it is sufficient to prove that there is a constant C such that, for all ν and ε,

$$\|(\underline{w}_\varepsilon, \partial_t\underline{w}_\varepsilon)\|_{L^2([2\nu\pi,4\nu\pi])} \leq C\|\underline{w}_\varepsilon\|_{L^2([2\nu\pi,4\nu\pi])}.$$

Using (2-7) again, one has

$$\|(\underline{w}_\varepsilon, \partial_t\underline{w}_\varepsilon)\|^2_{L^2([2\nu\pi,4\nu\pi])} = \sum_{k=0}^{\nu-1} e^{-2(\varepsilon k+\nu)\pi}\|(\underline{w}_\varepsilon, \partial_t\underline{w}_\varepsilon)\|^2_{L^2([0,2\pi])}$$

and

$$\|\underline{w}_\varepsilon\|^2_{L^2([2\nu\pi,4\nu\pi])} = \sum_{k=0}^{\nu-1} e^{-2(\varepsilon k+\nu)\pi}\|\underline{w}_\varepsilon\|^2_{L^2([0,2\pi])}.$$

On $[0, 2\pi]$, the H^1 norms of the $\underline{w}_\varepsilon$ are uniformly bounded, while their L^2 norms remain larger than a positive constant. $\qquad\square$

Construction of the coefficients and of the solutions. For $k \geq 1$, let $\rho_k = k^{-2}$. We consider intervals $I_k = [t_k, t'_k]$ and $J_k = [t'_k, t_{k-1}]$ of the same length $\rho_k = t'_k - t_t = t_{k-1} - t'_k$, starting at $t_0 = 2\sum_{k=1}^\infty \rho_k$, and thus such that $t_k \to 0$.

The functions a and b are defined on $]0, t_0]$ as follows: we fix a function $\beta \in C^\infty(\mathbb{R})$ supported in $]0, 1[$ and with sequences ε_k, ν_k and δ_k to be chosen later on;

$$\text{on } I_k, \quad \begin{cases} a(t) = \delta_k\alpha_{\varepsilon_k,\nu_k}(-8\pi\nu_k + 16\pi(t-t_k)\nu_k/\rho_k), \\ b(t) = \delta_k, \end{cases} \tag{2-15}$$

$$\text{on } J_k, \quad a(t) = b(t) = \delta_k + (\delta_{k-1}-\delta_k)\beta((t-t'_k)/\rho_k).$$

Because $\alpha_{\varepsilon,\nu} = 1$ for $|t| \geq 7\nu\pi$, the coefficient a equals δ_k near the endpoints of I_k. The use of the function β on J_k makes a smooth transition between δ_k and δ_{k-1}. Therefore, a and b are C^∞ on $]0, t_0]$. The coefficients will be chosen so that they extend smoothly up to $t = 0$.

This is quite similar to the choice in [Colombini and Nishitani 1999], except that we introduce a new sequence ε_k, which is crucial to control the Hölder continuity of $\sigma = a/b$.

We use the family (2-12) to construct solutions of the system supported in I_k for k large. On I_k, b is constant and (2-2) reads

$$\partial_t u_k + ih_k\delta_k\alpha_k v_k = f_k, \quad \partial_t v_k + ih_k\delta_k u_k = 0 \tag{2-16}$$

with

$$\alpha_k(t) = \alpha_{\varepsilon_k,\nu_k}\left(-8\pi\nu_k + \frac{16\pi(t-t_k)\nu_k}{\rho_k}\right).$$

Therefore, a C^∞ solution of (2-2) supported in I_k is

$$u_k(t) = i\,\partial_t\tilde{w}_{\varepsilon_k,\nu_k}\left(-8\pi\nu_k + \frac{16\pi(t-t_k)\nu_k}{\rho_k}\right), \quad v_k(t) = \tilde{w}_{\varepsilon_k,\nu_k}\left(-8\pi\nu_k + \frac{16\pi(t-t_k)\nu_k}{\rho_k}\right) \tag{2-17}$$

with

$$f_k(t) = 16i\pi\left(\frac{v_k}{\rho_k}\right)f_{\varepsilon_k,v_k}\left(-8\pi v_k + \frac{16\pi(t-t_k)v_k}{\rho_k}\right) \tag{2-18}$$

provided that

$$h_k = \frac{16\pi v_k}{\rho_k\delta_k}. \tag{2-19}$$

3. Properties of the coefficients

We always assume that

$$\varepsilon_k \leq \varepsilon_0, \quad \varepsilon_k v_k \to +\infty, \quad \delta_k \to 0. \tag{3-1}$$

Conditions for blow-up.

Lemma 3.1. *If*

$$(\rho_k)^{-1}e^{-\varepsilon_k v_k\pi} \to 0, \tag{3-2}$$

then the blow-up property in L^2, (2-3), is satisfied.

Proof. By Lemma 2.1,

$$\|f_k\|_{L^2} \leq C\rho_k^{-1}e^{-\varepsilon_k v_k\pi}\|v_k\|_{L^2}. \qquad \square$$

Lemma 3.2. *If*

$$\frac{1}{\varepsilon_k v_k}\ln\left(\frac{h_k v_k}{\rho_k}\right) \to 0, \tag{3-3}$$

then the blow-up property in C^∞, (2-4), is satisfied.

Proof. By Lemma 2.1, one has

$$\frac{\|\partial_t^l h_k^j f_k\|_{L^2}}{\|(u_k,v_k)\|_{L^2}} \leq C_l v_k^{-1} h_k^j\left(\frac{16\pi v_k}{\rho_k}\right)^{l+1}e^{-\varepsilon_k v_k\pi}.$$

This tends to 0 if

$$\varepsilon_k v_k\pi - j\ln h_k - (l+1)\ln\left(\frac{v_k}{\rho_k}\right) \to +\infty.$$

If (3-3) is satisfied, this is true for all j and l. $\qquad \square$

Smoothness of the coefficients.

Lemma 3.3. *If*

$$\frac{\ln(v_k/\rho_k)}{|\ln(\delta_k\varepsilon_k)|} \to 0, \tag{3-4}$$

then the functions a and b are C^∞ up to $t = 0$.

Proof. Both a and b are $O(\delta_k)$ and thus converge to 0 when $t \to 0$. Moreover, for $j \geq 1$,

$$|\partial_t^j a| \leq C_j \begin{cases} \delta_k \varepsilon_k (\nu_k/\rho_k)^j & \text{on } I_k, \\ \delta_k \rho_k^{-j} & \text{on } J_k. \end{cases}$$

The worst situation occurs on I_k and the right-hand side is bounded if

$$j \ln\left(\frac{\nu_k}{\rho_k}\right) - |\ln(\delta_k \varepsilon_k)|$$

is bounded from above. This is true for all j under the assumption (3-4), implying that a is C^∞ on $[0, t_0]$. The proof for b is similar and easier. $\qquad\square$

Next, we investigate the possible Gevrey regularity of the coefficients. For that we need to make a special choice of the cut-off functions χ and β which occur in the construction of a and b. We can choose them in a class contained in $\bigcap_{s>1} G^s$ and containing compactly supported functions (see, e.g., [Mandelbrojt 1952]). We choose them so that there is a constant C such that, for all j,

$$\sup_t (|\partial_t^j \chi(t)| + |\partial_t^j \beta(t)|) \leq C^{j+1} j! (\ln j)^{2j}. \tag{3-5}$$

Lemma 3.4. *If* (3-5) *is satisfied then, for* $j \geq 1$,

$$\sup_{t \in I_k \cup J_k} (|\partial_t^j a(t)| + |\partial_t^j b(t)|) \leq K^{j+1} \delta_k \varepsilon_k \left(\left(\frac{\nu_k}{\rho_k}\right)^j + \left(\frac{1}{\rho_k}\right)^j j! (\ln j)^{2j}\right). \tag{3-6}$$

Proof. On I_k we take advantage of the explicit form (2-10) of $\alpha_{\varepsilon,\nu}$: it is a finite sum of sine and cosines with coefficients of the form $\chi(t/\nu)$. Scaled on I_k, each derivative of the trigonometric functions yields a factor ν_k/ρ_k, while the derivatives of χ_{ν_k} have only a factor $1/\rho_k$. Since χ' and χ^2 satisfy estimates similar to (3-5), we conclude that a satisfies

$$|\partial_t^j a(t)| \leq \varepsilon_k \delta_k K^j \sum_{l \leq j} \left(\frac{\nu_k}{\rho_k}\right)^{j-l} C^{l+1} l! (\ln l)^{2l},$$

implying the estimate (3-6) on I_k. On I_k, b is constant. On J_k things are clear by scaling, since the coefficients are functions of $\beta((t - t_k')/\rho_k)$. $\qquad\square$

To estimate quantities such as $\delta_k (\nu_k/\rho_k)^j$, we use the following inequalities for $a > 0$ and $x \geq 1$:

$$e^{-x} x^a \leq a^a \tag{3-7}$$

and

$$e^{-e^x} x^a \leq \begin{cases} |\ln a|^a & \text{when } a \geq e, \\ 1 & \text{when } a \leq e. \end{cases} \tag{3-8}$$

Corollary 3.5. *Suppose that* $\delta_k = e^{-\eta_k}$ *and that, for* $s > s' > 1$,

$$\left(\frac{\nu_k}{\rho_k}\right) \leq C \eta_k^s \quad \text{and} \quad \left(\frac{1}{\rho_k}\right)^j \leq C \eta_k^{s'-1}. \tag{3-9}$$

Then the coefficients belong to the Gevrey class G^s.

If, for some $p > 0$ and $q > 0$,

$$\eta_k \geq e^{k^q} \quad and \quad \left(\frac{v_k}{\rho_k}\right) \leq Ck^p \eta_k, \tag{3-10}$$

then the coefficients belong to $\bigcap_{s>1} G^s$.

Proof. We neglect ε_k and only use the bound $\varepsilon_k \leq \varepsilon_0$. In the first case, we obtain from (3-7) that

$$\delta_k \left(\frac{v_k}{\rho_k}\right)^j \leq e^{-\eta_k} (C\eta_k)^{sj} \leq (C'j)^{js}, \quad \delta_k \left(\frac{1}{\rho_k}\right)^j \leq (C''j)^{j(s'-1)},$$

implying that

$$|\partial_t^j (a, b)| \leq K^{j+1} j^{sj}.$$

In the second case, combining (3-7) and (3-8)

$$e^{-\eta_k} \left(\frac{v_k}{\rho_k}\right)^j \leq C'^j j^j k^{pj} e^{-\eta_k/2} \leq C''^j j^j (1 + \ln j)^{pj/q}.$$

Using (3-8) again for the second term, we obtain that

$$|\partial_t^j (a, b)| \leq K^{j+1} j^j (\ln j)^{rj}$$

with $r = \max\{p, 4\}/q$. In particular, the right-hand side is estimated by $K_s^{k+1} j^{js}$ for all $s > 1$, proving that the functions a and b belong to $\bigcap_{s>1} G^s$. $\qquad\square$

Smoothness of the symmetrizer.

Lemma 3.6. *Suppose that ω is a continuous and increasing function on $[0, 1]$ such that $t^{-1}\omega(t)$ is decreasing. If*

$$\varepsilon_k \leq \omega\left(\frac{\rho_k}{v_k}\right) \tag{3-11}$$

then $\sigma = a/b$ satisfies

$$|\sigma(t) - \sigma(t')| \leq C\omega(|t - t'|). \tag{3-12}$$

In particular, if $\mu \leq 1$ and

$$\limsup_k \varepsilon_k \left(\frac{v_k}{\rho_k}\right)^\mu < +\infty, \tag{3-13}$$

then σ is Hölder continuous of exponent μ. If

$$\varepsilon_k \left(\frac{v_k}{\rho_k}\right) \leq C \ln\left(\frac{v_k}{\rho_k}\right)^\theta, \tag{3-14}$$

then $\omega(t) = t|\ln t|^\theta$ is a modulus of continuity for σ.

Proof. On J_k, $\tilde{\sigma} = \sigma - 1$ vanishes and, on I_k,

$$\tilde{\sigma} = \varepsilon_k \alpha_{\varepsilon_k, \nu_k} \left(-8\pi \nu_k + \frac{16\pi(t - t_k)\nu_k}{\rho_k} \right),$$

and thus

$$|\tilde{\sigma}| \le C\varepsilon_k, \quad |\partial_t \tilde{\sigma}| \le \frac{C\varepsilon_k \nu_k}{\rho_k}. \tag{3-15}$$

Hence, for t and t' in I_k,

$$|\tilde{\sigma}(t) - \tilde{\sigma}(t')| \le C\varepsilon_k \min\left\{ 1, \frac{|t - t'|\nu_k}{\rho_k} \right\}.$$

If $\rho_k/\nu_k \le |t - t'|$, we use the first estimate and

$$|\tilde{\sigma}(t) - \tilde{\sigma}(t')| \le C\varepsilon_k \le C\omega\left(\frac{\rho_k}{\nu_k} \right) \le C\omega(|t - t'|).$$

If $|t - t'| \le \rho_k/\nu_k$, we use the second estimate and the monotonicity of $t^{-1}\omega(t)$:

$$|\tilde{\sigma}(t) - \tilde{\sigma}(t')| \le C\varepsilon_k \left(\frac{\nu_k}{\rho_k} \right) |t - t'| \le C\left(\frac{\nu_k}{\rho_k} \right) \omega\left(\frac{\rho_k}{\nu_k} \right) |t - t'| \le C\omega(|t - t'|).$$

This shows that (3-12) is satisfied when t and t' belong to the same interval I_k.

If t belongs to I_k and $t' \in J_k$, then $\tilde{\sigma}(t') = \tilde{\sigma}(t'_k) = 0$ and

$$|\tilde{\sigma}(t) - \tilde{\sigma}(t')| \le C\omega(|t - t'_k|) \le C\omega(|t - t'|).$$

Similarly, if $t < t'$ and t and t' do not belong to the same $I_k \cup J_k$, there are endpoints t_j and t_l such that $t_j \le t \le t_{j-1} \le t_l \le t' \le t_{l-1}$. Since $\tilde{\sigma}$ vanishes at the endpoints of I_k and on J_k,

$$|\tilde{\sigma}(t) - \tilde{\sigma}(t')| \le C|\tilde{\sigma}(t) - \tilde{\sigma}(t_j)| + |\tilde{\sigma}(t) - \tilde{\sigma}(t_j)|$$
$$\le C\omega(|t - t_{j-1}|) + C\omega(|t_l - t'|) \le C\omega(|t - t'|),$$

and the lemma is proved. $\qquad \square$

4. Proof of the theorems

We now adapt the choice of the parameters ε_k, ν_k and δ_k so that the coefficients and the symmetrizer satisfy the properties stated in the different theorems. We will choose two increasing functions, f and g, on $\{x \ge 1\}$ and define ε_k and δ_k in terms of ν_k through the relations

$$\frac{\varepsilon_k \nu_k}{\rho_k} = f\left(\frac{\nu_k}{\rho_k} \right), \quad \delta_k = e^{-\eta_k}, \quad \eta_k = g\left(\frac{\nu_k}{\rho_k} \right). \tag{4-1}$$

Recall that $\rho_k = k^{-2}$. The sequence of integers ν_k will be chosen to converge to $+\infty$, and thus $\nu_k/\rho_k \to +\infty$. The conditions (3-1) are satisfied if, at $+\infty$,

$$f(x) \ll x, \quad g(x) \to +\infty. \tag{4-2}$$

Here $\phi(x) \ll \psi(x)$ means that $\psi(x)/\phi(x) \to \infty$. In particular, the first condition implies that $\varepsilon_k \to 0$, so that the condition $\varepsilon \leq \varepsilon_0$ is certainly satisfied if k is large enough.

One has

$$|\ln(\delta_k \varepsilon_k)| = \eta_k + \ln\left(\frac{\nu_k}{\rho_k}\right) + \ln f\left(\frac{\nu_k}{\rho_k}\right).$$

Hence, by Lemma 3.3, the coefficients a and b are C^∞ when

$$\ln x \ll g(x) \ll x, \tag{4-3}$$

since with (4-2) this implies that $|\ln(\delta_k \varepsilon_k)| \sim \eta_k \gg \ln(\nu_k/\rho_k)$.

Proof of Theorem 1.1. Given the modulus of continuity ω, we choose $f(x) = x\omega(x^{-1})$. The assumption on ω is that f is increasing and $f(x) \to +\infty$ at infinity. The essence of the theorem is that f can grow to infinity as slowly as one wants. Lemma 3.6 implies that ω is a modulus of continuity for $\sigma = a/b$. By Lemma 3.1, the blow-up property (2-3) occurs when

$$k^2 e^{-k^{-2} f(k^2 \nu_k)\pi} \to 0.$$

This condition is satisfied if ν_k satisfies

$$f(k^2 \nu_k) \geq k^3. \tag{4-4}$$

Let $f_1(x) = \min\{f(x), \ln x\}$. We choose $g(x) = x/f_1(x)$ and ν_k such that

$$2k^3 \leq f_1(k^2 \nu_k) \leq 4k^3.$$

Note that this implies (4-4). We show that the conditions (3-10) are satisfied with $p = q = 3$ and $C = 4$ and a suitable choice of ν_k, so that, by Corollary 3.5, the coefficients belong to $\bigcap_{s>1} G^s$ and the theorem is proved.

Indeed, since $f_1(k^2 \nu_k) \leq 4k^3$, the condition $\nu_k/\rho_k \leq 4k^3 \eta_k$ is satisfied. Moreover, since $\ln(k^2 \nu_k) \geq 2k^3$,

$$\nu_k \geq k^{-2} e^{2k^3} \geq e^{k^3}$$

for k large. $\qquad\square$

Proof of Theorem 1.2. The proof is similar. Given the modulus of continuity ω, we choose $f(x) = x\omega(x^{-1})$. The assumption on ω is now that

$$\ln x \ll f(x). \tag{4-5}$$

The essence of the theorem is now that $f(x)/\ln x$ can grow to infinity as slowly as one wants. By Lemma 3.6, ω is a modulus of continuity for $\sigma = a/b$.

By Lemma 3.2, the blow-up property (2-4) is satisfied if

$$\ln h_k = \eta_k + \ln\left(\frac{\nu_k}{\rho_k}\right) + \ln(16\pi) \ll \varepsilon_k \nu_k;$$

that is, if

$$\rho_k f\left(\frac{\nu_k}{\rho_k}\right) \gg g\left(\frac{\nu_k}{\rho_k}\right) + \ln\left(\frac{\nu_k}{\rho_k}\right). \tag{4-6}$$

Let $\psi(x) = f(x)/\ln x$ and $g(x) = \sqrt{\psi(x)}\,\ln x$. Then

$$\psi(x) \gg 1, \quad \ln x \ll g(x) \ll f(x).$$

Therefore, the condition (4-6) is satisfied when $\rho_k\sqrt{\psi}(v_k/\rho_k) \to +\infty$, and for that it is sufficient to choose v_k such that

$$\psi(k^2 v_k) \geq k^5. \tag{4-7}$$

The condition $g(x) \gg \ln x$ implies that the coefficients are C^∞, and the theorem is proved. □

Proof of Theorem 1.5. With $s > 1$ and $0 < \mu < 1 - 1/s$, we choose

$$g(x) = x^{1/s} \ll f(x) = x^{1-\mu}. \tag{4-8}$$

The choice of f implies that $\sigma = a/b \in C^\mu$. The choice of g implies that

$$\frac{v_k}{\rho_k} \leq \left(g\left(\frac{v_k}{\rho_k}\right)\right)^s = \eta_k^s.$$

With $s' \in \,]1, s[$, the condition

$$\rho_k^{-1} \leq \eta_k^{s'-1}$$

is satisfied when $k^2 \leq (k^2 v_k)^{(s'-1)/s}$, that is, when

$$v_k \geq k^{2p}, \quad \text{where} \quad p = \frac{1+s-s'}{s'-1}. \tag{4-9}$$

In this case, Corollary 3.5 implies that the coefficients a and b belong to the Gevrey class G^s.

The blow-up property (2-4) is satisfied when (4-6) holds, that is, when

$$k^{-2}(k^2 v_k)^{1-\mu} \gg (k^2 v_k)^{1/s},$$

which is true if

$$v_k \geq k^{2q}, \quad \text{where} \quad q = \frac{\mu + 1/s}{1 - \mu - 1/s}.$$

Therefore, if $v_k \geq k^{2\max\{p,q\}}$, the system satisfies the conclusions of Theorem 1.5. □

Proof of Theorem 1.3. The analysis above shows that if one looks for coefficients in $\bigcap_{s>1} G^s$, one must choose g, and thus f, close to x. We choose here

$$g(x) = \frac{x}{(\ln x)^2} \ll f(x) = \frac{x}{\ln x} \ll x$$

Since $f(x)/x \to 0$ at infinity, the symmetrizer is continuous up to $t = 0$, but not in C^μ for any $\mu > 0$.

The ill-posedness in C^∞ is again guaranteed by the condition (4-6), that is, $\ln(k^2 v_k) \gg k^2$. In particular, it is satisfied when

$$v_k \geq e^{k^3}. \tag{4-10}$$

By Corollary 3.5, to finish the proof of Theorem 1.3 it is sufficient to show that one can choose ν_k satisfying (4-10) such that $\nu_k/\rho_k \leq 4k^6\eta_k$. This condition reads $\ln(k^2\nu_k) \leq 2k^3$, or

$$\nu_k \leq k^{-2}e^{2k^3}$$

which is compatible with (4-10) if k is large enough. $\qquad\square$

Proof of Theorem 1.4. Let $a \in \bigcap_{s>1} G^s$ denote the coefficient constructed for the proof of Theorem 1.3. The definition (2-15) shows that $a \geq 0$, and indeed $a > 0$, for $t > 0$. The functions ν_k defined at (2-17) are supported in I_k and are solutions of the wave equation (1-5) with source term f_k, and we have shown that

$$\frac{\|h_k^j \partial_t^l f_k\|_{L^2}}{\|\nu_k\|_{L^2}} \to 0 \quad \text{as } k \to \infty. \qquad\square$$

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