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INVERSE SCATTERING WITH PARTIAL DATA ON ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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We prove a local support theorem for the radiation fields on asymptotically hyperbolic manifolds and use it to show that the scattering operator restricted to an open subset of the boundary of the manifold determines the manifold and the metric modulo isometries that are equal to the identity on the open subset where the scattering operator is known.

1. Introduction

We recall that the ball model of the hyperbolic space $\mathbb{H}^{n+1}$ is given by

$$\mathbb{B}^{n+1} = \{z \in \mathbb{R}^{n+1} : |z| < 1\} \quad \text{equipped with the metric } g = \frac{4\,dz^2}{(1-|z|^2)^2}.$$

It is well known that $(\mathbb{B}^{n+1}, g)$ is a complete manifold with constant curvature -1. On the other hand, $(\mathbb{B}^{n+1}, (1-|z|^2)^2 g)$ is the interior of a compact Riemannian manifold with boundary. This structure can be generalized by replacing $\mathbb{B}^{n+1}$ with the interior of a C^∞ compact manifold X, with boundary ∂X, of dimension $n+1$ and replacing $1-|z|^2$ with a function $\rho \in C^\infty(X)$ which defines ∂X; that is, $\rho > 0$ in the interior of X, $\{\rho = 0\} = \partial X$, and $d\rho \neq 0$ at ∂X. Such a function ρ will be called a boundary-defining function. We will denote the interior of X by $\mathring{X}$. If g is a Riemannian metric on $\mathring{X}$ such that

$$\rho^2 g = H \tag{1-1}$$

is C^∞ and nondegenerate up to ∂X then, according to [Mazzeo and Melrose 1987], g is complete and its sectional curvatures approach $-|d\rho|_H^2$ as $\rho \downarrow 0$. In particular, when

$$|d\rho|_{H^2} = 1 \quad \text{at } \partial X, \tag{1-2}$$

the sectional curvatures converge to -1 at the boundary. A Riemannian manifold $(\mathring{X}, g)$, where X is a compact C^∞ manifold with boundary and where (1-1) and (1-2) hold, is said to be an asymptotically hyperbolic manifold (AHM). Any compact C^∞ Riemannian manifold with boundary X can be equipped with such a metric.

We will study certain properties of the asymptotic behavior of solutions to the Cauchy problem for the wave equation on $(\mathring{X}, g)$. In particular, we will study the Friedlander radiation fields on AHM, and show that the support of the radiation fields restricted to an open subset of ∂X controls the support of the initial

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data of the Cauchy problem for the wave equation. Such theorems are usually called support theorems; see, for example, [Helgason 1999]. When $\mathring{X} = \mathbb{H}^{n+1}$, the radiation fields are given by the Lax–Phillips transform which involves the horocyclic Radon transform, and our support theorem generalizes the results of [Lax and Phillips 1982] to this setting.

We will use this result and adapt the boundary control theory of [Belishev 1987; Belishev and Kurylev 1992; Tataru 1995; 1999], and a refinement of the results of [Belishev and Kurylev 1992] due to Kurylev and Lassas [2002] and Katchalov, Kurylev and Lassas [Katchalov et al. 2001], to prove that the scattering operator restricted to a nonempty open set $\Gamma \subset \partial X$ determines (X, g) modulo isometries that are equal to the identity on Γ. There is a very large body of work on scattering and inverse scattering for Schrödinger operators, obstacle problems, etc., however much less is known about inverse scattering on manifolds. It was proved in [Sá Barreto 2005] that the scattering operator on the entire boundary of an AHM (X, g) determines the manifold and the metric modulo isometries that are the identity at ∂X. Guillarmou and Sá Barreto [2008] extended the result of [Sá Barreto 2005] to asymptotically complex hyperbolic manifolds. Isozaki, Kurylev and Lassas [Isozaki et al. 2010; 2013] studied the case of manifolds of cylindrical ends and asymptotically hyperbolic orbifolds; see also their survey paper [Isozaki et al. 2014]. One should also mention the book by Isozaki and Kurylev [2014], where they discuss spectral theory and inverse problems on AHM. If an AHM manifold is also Einstein, Guillarmou and Sá Barreto [2009] showed that the scattering matrix at one energy determines the manifold.

2. Preliminaries and statements of the results

We begin by recalling the definition of the radiation fields and the scattering operator. Let $u(t, z)$ satisfy the wave equation

$$\begin{gathered} \big(D_t^2 - \Delta_g - \tfrac{1}{4}n^2\big)u = 0 \quad \text{on } \mathbb{R}_\pm \times \mathring{X}, \\ u(0, z) = f_1, \quad D_t u(0, z) = f_2, \quad f_1,\ f_2 \in C_0^\infty(\mathring{X}). \end{gathered} \tag{2-1}$$

The spectrum of the Laplacian Δ_g, denoted by $\sigma(\Delta_g)$, was studied by [Mazzeo 1988; 1991; Mazzeo and Melrose 1987] and more recently by Bouclet [2013]. They showed that $\sigma(\Delta_g) = \sigma_{\mathrm{pp}}(\Delta_g) \cup \sigma_{\mathrm{ac}}(\Delta_g)$, where $\sigma_{\mathrm{pp}}(\Delta_g)$ is the finite point spectrum, $\sigma_{\mathrm{ac}}(\Delta_g)$ is the absolutely continuous spectrum and

$$\sigma_{\mathrm{ac}}(\Delta_g) = \big[\tfrac{1}{4}n^2, \infty\big), \quad \sigma_{\mathrm{pp}}(\Delta_g) \subset \big(0, \tfrac{1}{4}n^2\big). \tag{2-2}$$

The role of the factor $n^2/4$ in (2-1) is to shift the continuous spectrum of Δ_g to $[0, \infty)$.

Equation (2-1) has a conserved energy given by

$$\begin{gathered} E(u, \partial_t u)(t) = \int_X \big(|du(t)|^2 - \tfrac{1}{4}n^2|u(t)|^2 + |\partial_t u(t)|^2\big)\, d\,\mathrm{vol}_g, \\ E(u, \partial_t u)(0) = E(f_1, f_2) = \int_X \big(|df_1|^2 - \tfrac{1}{4}n^2[t]\,|f_1|^2 + |f_2|^2\big)\, d\,\mathrm{vol}_g\,. \end{gathered} \tag{2-3}$$

However, $E(f_1, f_2)$ is a nonnegative quadratic form only when projected onto $L^2_{\mathrm{ac}}(X)$. As in [Sá Barreto 2005], we define the energy space

$$H_E(X) = \{(f_1, f_2) : f_1,\ f_2 \in L^2(X),\ df_1 \in L^2(X) \text{ and } E(f_1, f_2) < \infty\}$$

and, if $\{\phi_j : 1 \le j \le N\}$ are the eigenfunctions of Δ_g, we define the projector

$$\mathcal{P}_{\mathrm{ac}} : L^2(X) \to L^2_{\mathrm{ac}}(X), \quad f \mapsto f - \sum_{j=1}^{N} \langle f, \phi_j \rangle \phi_j,$$

and the space $E_{\mathrm{ac}}(X) = \mathcal{P}_{\mathrm{ac}}(H_E(X))$.

The wave group induces a strongly continuous group of unitary operators,

$$U(t) : E_{\mathrm{ac}}(X) \to E_{\mathrm{ac}}(X), \quad (f_1, f_2) \mapsto (u(t), \partial_t u(t)).$$

Next we recall the definition of the forward and backward radiation fields from [Sá Barreto 2005]. We will work with a specific boundary defining function and, since our definition will depend on this choice, we will recall the construction from [Graham 2000]. Since any two defining functions of ∂X, ρ and $\tilde{\rho}$, satisfy $\rho = e^{\omega}\tilde{\rho}$ with $\omega \in C^\infty(X)$, if $H = \rho^2 g$ and $\tilde{H} = \tilde{\rho}^2 g$ then $H|_{\partial X} = e^{2\omega(0,y)}\tilde{H}|_{\partial X}$. Hence, $\rho^2 g|_{\partial X}$ determines a conformal class of metrics on ∂X. We have $H = \rho^2 g = e^{2\omega}\tilde{\rho}^2 g$, and so $H = e^{2\omega}\tilde{H}$. Since $d\rho = e^{\omega}(\tilde{\rho}\, d\omega + d\tilde{\rho})$, we have

$$|d\rho|^2_H = |d\tilde{\rho} + \tilde{\rho}\, d\omega|^2_{\tilde{H}} = |d\tilde{\rho}|^2_{\tilde{H}} + \tilde{\rho}^2 |d\omega|^2_{\tilde{H}} + 2\tilde{\rho}(\nabla_{\tilde{H}}\tilde{\rho})\omega.$$

Hence,

$$|dx|_H = 1 \qquad \text{if and only if} \qquad 2(\nabla_{\tilde{H}}\tilde{\rho})\omega + \tilde{\rho}|d\omega|^2_{\tilde{H}} = \frac{1}{\tilde{\rho}}(1 - |d\tilde{\rho}|^2_{\tilde{H}}), \quad \omega|_{\partial X} = 0.$$

Since, by assumption, $|d\tilde{\rho}|_{\tilde{H}} = 1$ at ∂X, this is a noncharacteristic ODE, and hence it has a solution in a neighborhood of ∂X. Notice that the function ρ is in principle defined only on a collar neighborhood of ∂X, but it can be extended to the whole manifold as a boundary-defining function.

The boundary-defining function ρ gives an identification between $[0, \varepsilon) \times \partial X$ and a collar neighborhood U of ∂X,

$$\Psi : [0, \varepsilon) \times \partial X \to U \subset X, \quad (x, y) \mapsto \exp(x\nabla_H \rho)(y),$$

where $\exp(x\nabla_H\rho)(y)$ just means that one follows the integral curve of $\nabla_H\rho$ starting at y for x units of time. In this case,

$$\begin{gathered} \Psi^* g = \frac{dx^2}{x^2} + \frac{h(x)}{x^2} \quad \text{on } (0, \varepsilon) \times \partial X, \quad h(0) = H|_{\partial X}, \\ \Psi = \mathrm{Id} \quad \text{on } \partial X, \end{gathered} \tag{2-4}$$

where $h(x)$ is a C^∞ family of metrics ∂X for $x \in [0, \varepsilon)$. From now on we will use this identification $U \sim [0, \varepsilon)_x \times \partial X$.

In the coordinates (2-4), for fixed $y \in \partial X$ the curve $\gamma(s) = (s, y)$ is a geodesic for the metric g, the distance between (x, y) and (x', y), $x < x'$, is $\log(x'/x)$, and if time t is the arc-length parameter then $t = \log x' - \log x$. So, to analyze global properties of $u(t, z)$ in space and time, it is convenient to work with an exponential compactification of $\mathbb{R} \ni t$, and we choose a function T such that $\{T = 0\} = \{t = 0\}$, $T = 1 - e^{-t}$ if $t > 1$, and $T = -1 + e^t$ if $t < -1$. Let $Y = [-1, 1] \times X$ be the compactified space; see Figure 1. The light cones will converge to the corners of the manifold Y and to separate them one blows up the intersection of ∂X with $\{T = -1\}$ and $\{T = 1\}$. This gives a manifold with corners $\tilde{Y}$, pictured in

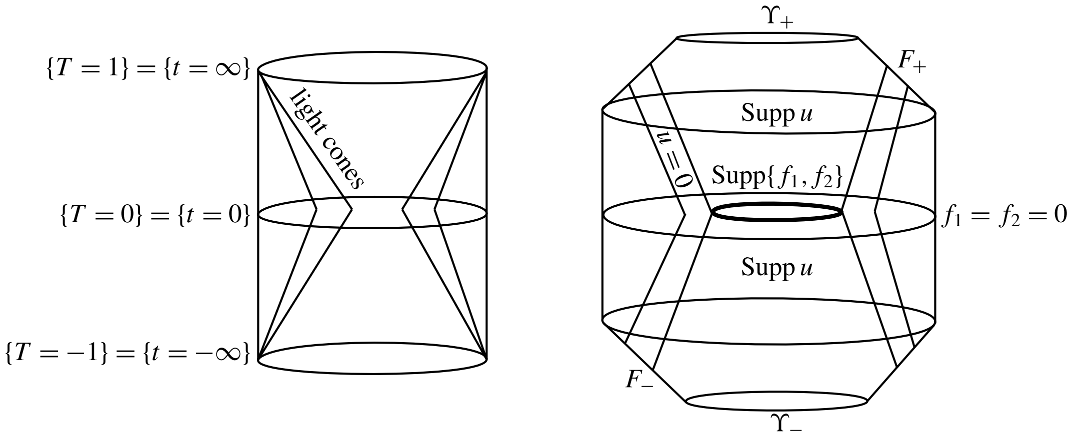

Figure 1. The manifold $Y=[-1,1]\times X$, and $\tilde{Y}$, its blow-up along $\partial X\times\{T=\pm 1\}$. All the light cones intersect at $\{x=0, T=\pm 1\}$ in Y, but in $\tilde{Y}$ they are separated at the faces F_+ and F_-.

Figure 1. In local coordinates, the blow-up is the equivalent of introducing polar coordinates $x=r\cos\theta$, $T\pm 1=r\sin\theta$.

It was proved in [Sá Barreto 2005] that, if $(f_1, f_2)\in C_0^\infty(X)$, the solution u to the wave equation (2-1) is in $C^\infty(\tilde{Y}\setminus(\overline{\Upsilon_+\cup\Upsilon_-}))$ (see Figure 1 for the definition of $\Upsilon_\pm$). The analysis of the behavior of $u(t,z)$ on the faces $\Upsilon_\pm$ give, among other things, information about the local energy decay, and will not be studied here. A similar discussion about the asymptotic solutions of the wave equation on de Sitter–Schwarzschild space, including the pictures, can be found in [Melrose et al. 2014a; 2014b]; see also [Vasy 2013].

Following Friedlander [1980; 2001], one defines the forward and backward radiation fields of u as

$$\begin{aligned}\mathcal{R}_+(f_1,f_2)&=x^{-n/2}\partial_t u|_{F_+\setminus\overline{\Upsilon}_+},\\ \mathcal{R}_-(f_1,f_2)&=x^{-n/2}\partial_t u|_{F_-\setminus\overline{\Upsilon}_-}.\end{aligned}$$

If we use projective coordinates x and $\tau_+=x/(1-T)$, valid near $F_+\setminus\overline{\Upsilon}_+$, and $\tau_-=x/(1+T)$, valid near $F_-\setminus\overline{\Upsilon}_-$, and set $s_+=\log\tau_+$ and $s_-=-\log\tau_-$, then, for $(f_1,f_2)\in C_0^\infty(\mathring{X})\times C_0^\infty(\mathring{X})$, the solution $u(t,z)$ to (2-1), with $z=(x,y)$, satisfies

$$\begin{aligned}V_+(x,s_+,y)&=x^{-n/2}u(s_+-\log x,x,y)\in C^\infty([0,\varepsilon)_x\times\mathbb{R}_{s_+}\times\partial X)\\ V_-(x,s,y)&=x^{-n/2}u(s_-+\log x,x,y)\in C^\infty([0,\varepsilon)_x\times\mathbb{R}_{s_-}\times\partial X).\end{aligned}\tag{2-5}$$

In these coordinates, the forward and backward radiation fields can be expressed as

$$\begin{aligned}&\mathcal{R}_+:C_0^\infty(\mathring{X})\times C_0^\infty(\mathring{X})\to C^\infty(\mathbb{R}\times\partial X),\quad \mathcal{R}_+(f_1,f_2)(s_+,y)=D_{s_+}V_+(0,s_+,y),\\ &\mathcal{R}_-:C_0^\infty(\mathring{X})\times C_0^\infty(\mathring{X})\to C^\infty(\mathbb{R}\times\partial X),\quad \mathcal{R}_-(f_1,f_2)(s_-,y)=D_{s_-}V_-(0,s_-,y).\end{aligned}\tag{2-6}$$

It was shown in [Sá Barreto 2005] that $\mathcal{R}_\pm$ extend to unitary operators

$$\mathcal{R}_\pm : E_{\mathrm{ac}}(X) \to L^2(\mathbb{R} \times \partial X), \quad (f_1, f_2) \mapsto \mathcal{R}_\pm(f_1, f_2), \tag{2-7}$$

where the measure on ∂X is the one induced by the metric h_0 defined in (2-4).

It follows from the definitions that $\mathcal{R}_\pm$ are translation representations of the wave group as in the Lax–Phillips theory [1989], i.e.,

$$\mathcal{R}_\pm(U(T)(f_1, f_2))(s, y) = \mathcal{R}_\pm(f_1, f_2)(s + T, y). \tag{2-8}$$

One can define the scattering operator

$$\mathcal{S} : L^2(\mathbb{R} \times \partial X) \to L^2(\mathbb{R} \times \partial X), \quad \mathcal{S} = \mathcal{R}_+ \circ \mathcal{R}_-^{-1}, \tag{2-9}$$

which is unitary in $L^2(\partial X \times \mathbb{R})$ and, in view of (2-8), commutes with translations in the s variable.

The scattering matrix $\mathcal{A}(\lambda)$ is defined by conjugating $\mathcal{S}$ with the Fourier transform in the s variable:

$$\mathcal{A}(\lambda) = \mathcal{F} \circ \mathcal{S} \circ \mathcal{F}^{-1}, \quad \mathcal{F} f(\lambda) = \int_{\mathbb{R}} e^{-i\lambda s} f(s)\, ds. \tag{2-10}$$

In particular, $\mathcal{S}$ determines $\mathcal{A}(\lambda)$, $\lambda \in \mathbb{R}$ and vice versa. It was proved in [Joshi and Sá Barreto 2000] that $\mathcal{A}(\lambda)$ continues meromorphically to $\mathbb{C} \setminus D$, where D is a discrete subset of $\mathbb{C}$.

As discussed above, the distance between (x, y) and (x', y), $x < x' < \varepsilon$, is $\log(x'/x)$. The finite speed of propagation for the wave equation implies that the solution $u(t, z)$ of (2-1) satisfies $u(t, z) = 0$ if $t < d_g(z, \operatorname{Supp}(f_1, f_2))$. In particular, if $f_1(x', y) = f_2(x', y) = 0$ for all $x' < \rho$, then $u(t, x) = 0$ for $x < x' < \rho$ and $t < \log(x'/x)$. This implies that $V_+(s, x, y) = x^{-n/2}\partial_t u(s - \log x, x, y) = 0$ provided $x < x' < \rho$ and $s = t + \log x < \log x' < \log \rho$. This shows that, if $f_1(x', y) = f_2(x', y) = 0$ in $x' \le \rho$, then $\mathcal{R}_+(f_1, f_2)(s, y) = 0$ for $s \le \log \rho$. The converse of this statement for initial data of the type $(0, f)$ was proved in [Sá Barreto 2005]: if $f \in L^2_{\mathrm{ac}}(X)$ and $\mathcal{R}_+(0, f)(s, y) = 0$ for $s \le \log \rho \ll 0$ and $y \in \partial X$, then $f(x, y) = 0$ in $x \le \rho$. Due to possible cancelations, one cannot expect the converse to be true for an arbitrary pair (f_1, f_2). In this paper we prove the following refinement of this result:

Theorem 2.1. *Let $\Gamma \subset \partial X$ be a nonempty open subset, let $f \in L^2_{\mathrm{ac}}(X)$ and let $s_0 \in \mathbb{R}$. Let $\varepsilon > 0$ be such that* (2-4) *holds in $(0, \varepsilon) \times \partial X$, and let $\bar\varepsilon = \min\{\varepsilon, e^{s_0}\}$. Then $\mathcal{R}_+(0, f)(s, y) = 0$ in $\{s < s_0,\ y \in \Gamma\}$ if and only if, for every $z = (x, y) \in (0, \bar\varepsilon) = U_{\bar\varepsilon}$,*

$$d_g(z, \operatorname{Supp} f) > \log \frac{e^{s_0}}{x}, \tag{2-11}$$

where d_g denotes the distance function with respect to the metric g and Supp f *denotes the support of f. Another way of stating* (2-11) *is to say that $f = 0$ on the set*

$$\mathcal{D}_{s_0}(\Gamma) = \left\{ z \in X : \exists q = (x, y) \in U_{\bar\varepsilon},\ d_g(z, q) < \log \frac{e^{s_0}}{x} \right\} = \bigcup_{(x,y)\in U_{\bar\varepsilon}} B\Big((x, y), \log \frac{e^{s_0}}{x}\Big), \tag{2-12}$$

where $B(p, r)$ denotes the open ball of radius r centered at p with respect to the metric g.

If $\Gamma = \partial X$ and $\bar{\varepsilon} = e^{s_0}$ then, for any $z = (\alpha, y)$ with $\alpha < e^{s_0}$, pick $q = (x, y)$ with $x < \alpha < e^{s_0}$. Then $d_g((\alpha, y), (x, y)) = \log(\alpha/x) < \log(e^{s_0}/x)$. Therefore, $\{(\alpha, y) : \alpha < e^{s_0},\ y \in \partial X\} \subset \mathcal{D}_{s_0}(\partial X)$, and hence Theorem 2.1 shows that, if $f \in L^2_{\mathrm{ac}}(X)$ and $\mathcal{R}_+(0, f)(s, y) = 0$ for $s \le s_0$ and $y \in \partial X$, then $f(x, y) = 0$ for $x < e^{s_0}$. This particular case of Theorem 2.1, when $\Gamma = \partial X$ and $\bar{\varepsilon} = e^{s_0}$ was proved in [Sá Barreto 2005].

Lax and Phillips [1982] proved Theorem 2.1 for the case when (X, g) is the hyperbolic space. In that case the radiation field is given in terms of the horocyclic Radon transform, and their result says that, if the integral of f over all horospheres tangent to points $(0, y)$ with $y \in \Gamma$ and radii less than or equal to $\frac{1}{2}e^{s_0}$ is equal to zero, then $f = 0$ in the region given by the union of these horocycles. It is useful to explain what the set $\mathcal{D}_{s_0}(\Gamma)$ is when (X, g) is the hyperbolic space, and verify that Theorem 2.1 implies the result of Lax and Phillips. It is easier to do the computations for the half-space model of hyperbolic space, which is given by

$$\mathbb{H}^{n+1} = \{(x, y) : x > 0,\ y \in \mathbb{R}^n\} \quad \text{with the metric } g = \frac{dx^2 + dy^2}{x^2}.$$

The distance function between $z = (x, y)$ and $w = (\alpha, y')$ satisfies

$$\cosh d_g(z, w) = \frac{x^2 + \alpha^2 + |y - y'|^2}{2x\alpha}.$$

Since $d_g(z, z') \le \log(e^{s_0}/\alpha)$, we obtain

$$\left(x - \tfrac{1}{2}e^{s_0}(1 + \alpha^2 e^{-2s_0})\right)^2 + |y - y'|^2 \le \tfrac{1}{4}e^{2s_0}(1 + \alpha^2 e^{-2s_0})^2 - \alpha^2 = \tfrac{1}{4}e^{2s_0}(1 - \alpha^2 e^{-2s_0})^2,$$

which corresponds to a ball $D(\alpha)$ centered at $\left(\frac{1}{2}e^{s_0}(1 + \alpha^2 e^{-2s_0}), y'\right)$ and radius $\frac{1}{2}e^{s_0}(1 - \alpha^2 e^{-2s_0})$. Since $\alpha < e^{s_0}$, we have $D(\alpha) \subset D(0)$, as shown in Figure 2. This ball is tangent to the plane $x = e^{s_0}$ at the point (e^{s_0}, y'). When $\alpha = 0$, the ball $D(0)$ has center $\left(\frac{1}{2}e^{s_0}, y'\right)$ and radius $\frac{1}{2}e^{s_0}$ and is also tangent to the plane $\{x = 0\}$. The boundary of $D(0)$ is called a horosphere since it is orthogonal to the geodesics emanating from the point $(0, y')$. When $\alpha = e^{s_0}$, $D(e^{s_0}) = (e^{s_0}, y')$. The set $\mathcal{D}_{s_0}(\Gamma)$ consists of the union

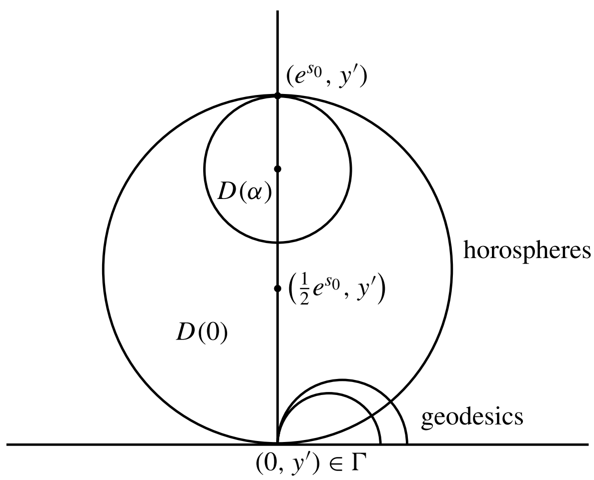

Figure 2. The horospheres tangent at $(0, y')$ and the balls $D(\alpha)$.

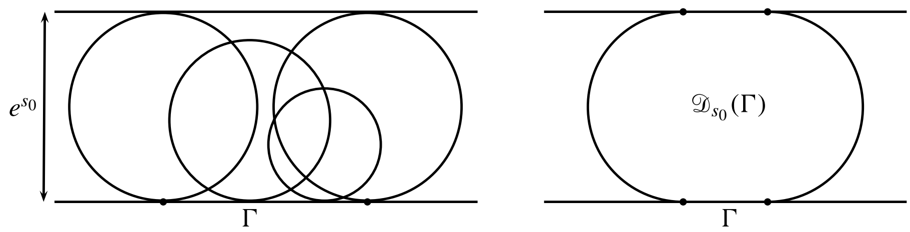

Figure 3. The set $\mathcal{D}_{s_0}(\Gamma)$ when (X, g) is the hyperbolic space is given by the union of horospheres tangent to points on Γ and radii less than or equal to $\frac{1}{2}e^{s_0}$.

of horospheres with radii less than or equal to $\frac{1}{2}e^{s_0}$ tangent to points $(0, y')$ with $y' \in \Gamma$; see Figure 3.

Theorem 2.1 can be explained in terms of the sojourn time along a geodesic. In this setting, the sojourn time plays the role of the distance function to the boundary of X and is closely related to the Busemann function used in differential geometry. Let $\gamma(t)$ be a geodesic, parametrized by the arc length, passing through $z = \gamma(0)$ and such that $\gamma(t) \to y \in \partial X$ as $t \to \infty$. We define

$$s(z, \gamma) = \lim_{t\to\infty}\bigl(t + \log x(\gamma(t))\bigr).$$

The relationship between the sojourn times and the radiation fields for nontrapping asymptotically hyperbolic manifolds was studied in [Sá Barreto and Wunsch 2005]. We have the following consequence of Theorem 2.1:

Corollary 2.2. *Let f and $\Gamma \subset \partial X$ satisfy the hypotheses of Theorem 2.1; then $f = 0$ on the set of points $z \in \mathring{X}$ such that there exists a geodesic $\gamma(t)$, parametrized by the arc length, with $\gamma(0) = z$, $\gamma(t) \to y \in \Gamma$ as $t \to \infty$, and $s(z, \gamma) < s_0$.*

Proof. Suppose there exists a geodesic $\gamma(t)$, parametrized by the arc length t, such that $\gamma(0) = z$, $\lim_{t\to\infty}\gamma(t) = y$ and

$$\lim_{t\to\infty}\bigl(t + \log x(\gamma(t))\bigr) = s < s_0.$$

Since t is the arc-length parameter, $d\bigl(z, \bigl(x(\gamma(t)), y\bigr)\bigr) \le t$ and $s < s_0$, there exists $T > 0$ such that, for $t > T$, $\gamma(t) \in U \sim [0, \varepsilon) \times \partial X$ where the coordinates (2-4) are valid and $t + \log x(\gamma(t)) < s_0$. Therefore, if $t > T$,

$$d\bigl(z, (x(t), y)\bigr) \le t < s_0 - \log x(\gamma(t)) = \log \frac{e^{s_0}}{x(\gamma(t))}.$$

Hence $z \in \mathcal{D}_{s_0}(\Gamma)$. □

Theorem 2.1 says that the support of the radiation field $\mathcal{R}_+(0, f)$ controls the support of the initial data $(0, f)$. We will use this result to adapt the boundary control method of [Belishev 1987; Belishev and Kurylev 1992; Kurylev and Lassas 2002; Katchalov et al. 2001] to study the inverse scattering problem with partial data.

Let $\Gamma \subset \partial X$ be an open subset and let $\mathcal{S}$ denote the scattering operator as in (2-9). We define the restriction of $\mathcal{S}$ to $\mathbb{R} \times \Gamma$ as

$$\mathcal{S}_\Gamma : L^2(\mathbb{R} \times \Gamma) \to L^2(\mathbb{R} \times \Gamma), \quad F \mapsto (\mathcal{S}F)|_\Gamma. \tag{2-13}$$

In other words, one starts with an $F \in L^2(\mathbb{R} \times \Gamma)$, finds the solution of the wave equation that has backward radiation field equal to F, then finds the corresponding forward radiation field, and restricts it to the subset $\mathbb{R} \times \Gamma$. We study the problem of determining (X, g) from $\mathcal{S}_\Gamma$. Recall that our definition of $\mathcal{S}$ depends on the choice of the product structure (2-4). In fact, the method used in [Graham 2000] and discussed above to construct the diffeomorphism (2-4) can also be used to show that, given two AHM (X_j, g_j), $j = 1, 2$, there exists $\varepsilon > 0$ such that (2-14) holds for both metrics. Recall that x is just the time through which one flows along the integral curves of $\nabla_H \rho$. One can take ε to the smallest one that works for both metrics, and one finds that there exist collar neighborhoods $U_j \subset X_j$ of ∂X_j and C^∞ diffeomorphisms

$$\Psi_j : (0, \varepsilon) \times \partial X_j \to U_j$$

such that

$$\Psi_j^* g_j = \frac{dx^2}{x^2} + \frac{h_j(x)}{x^2} \quad \text{in } (0, \varepsilon) \times \partial X_j, \quad h_j(0) = h_{j0}, \quad j = 1, 2, \tag{2-14}$$

where $h_j(x)$ is a C^∞ family of metrics on ∂X_j for $x \in [0, \varepsilon)$, and $\Psi_j = \mathrm{Id}$ on ∂X_j. In particular, if there exists an open set $\Gamma \subset \partial X_1 \cap \partial X_2$, as manifolds, then (2-14) holds on $(0, \varepsilon) \times \Gamma$, and $h_j(x)$ are C^∞ families of metrics on Γ. We prove the following:

Theorem 2.3. *Let (X_1, g_1) and (X_2, g_2) be connected, asymptotically hyperbolic manifolds and suppose there exists an nonempty open set $\Gamma \subset \partial X_1 \cap \partial X_2$ (as manifolds). Let x be such that (2-14) holds on a collar neighborhood of ∂X_j for $j = 1, 2$. Suppose that $h_1(0) = h_2(0)$ on Γ. Let $\mathcal{S}_{j,\Gamma}$, $j = 1, 2$, be the corresponding scattering operators restricted to Γ, and suppose that $\mathcal{S}_{1,\Gamma} = \mathcal{S}_{2,\Gamma}$. Then there exists a C^∞ diffeomorphism*

$$\Psi : X_1 \to X_2 \quad \textit{such that} \quad \Psi = \mathrm{Id} \ \textit{on} \ \Gamma \quad \textit{and} \quad \Psi^* g_2 = g_1. \tag{2-15}$$

Since we only know $\mathcal{S}$ on part of the boundary, we can only expect to recover information on the connected components of (X, g) that contain Γ, so we assume that X is connected. This result guarantees that the scattering operator restricted to Γ determines (X, g), including its topology and C^∞ structure, modulo isometries that are equal to the identity on Γ.

Theorem 2.3, and the method we use to prove it, are related to the question of reconstructing a compact Riemannian manifold with boundary from the Dirichlet-to-Neumann map (DTNM) for the wave equation. One may think of the scattering operator as the DTNM on the boundary at infinity. Belishev and Kurylev [1992] showed that the DTNM for the wave equation determines a compact manifold and its Riemannian metric using the boundary control method and a unique continuation result later proved by Tataru [1995; 1999]. Different proofs, which also rely on the result of Tataru, were given in [Katchalov et al. 2001]. This result of Tataru will be important in the proof of Theorem 2.1. The reconstruction of a compact manifold in the case where the Dirichlet-to-Neumann map is only known on part of the boundary was carried out by Kurylev and Lassas [2000] using a modification of the boundary control method; see also Section 4.4 of [Katchalov et al. 2001]. We will adapt the boundary control methods to this setting by using the radiation fields.

3. The proof of Theorem 2.1

The sufficiency of condition (2-11) in Theorem 2.1 is just a consequence of the finite speed of propagation for the wave equation.

Lemma 3.1. *Let $f \in L^2_{\mathrm{ac}}(X)$ be such that $d_g(z, \operatorname{Supp} f) > \log(e^{s_0}/x)$ for all $z = (x, y) \in (0, \bar{\varepsilon}) \times \Gamma$. Then $\mathcal{R}_+(0, f)(s, y) = 0$ if $s \le s_0$ and $y \in \Gamma$.*

Proof. Let $u(t, z)$ satisfy the wave equation (2-1) with initial data $(0, f)$. The finite speed of propagation for solutions of the wave equation guarantees that $u(t, z) = 0$ if $0 \le t < d_g(z, \operatorname{Supp} f)$. In particular, if $z = (x, y)$ with $x < \bar{\varepsilon}$, $y \in \Gamma$, then $u(t, x, y) = 0$ if $0 \le t \le s_0 - \log x < d_g(z, \operatorname{Supp} f)$. Since $s = t + \log x$, we have that $V_+(x, s, y) = x^{-n/2} u(s - \log x, x, y) = 0$ provided $\log x \le s \le s_0$, $x < \bar{\varepsilon}$, $y \in \Gamma$. This implies that $\mathcal{R}_+(0, f)(s, y) = 0$ if $s \le s_0$ and $y \in \Gamma$. □

We will first outline the proof of the converse, which is based on unique continuation arguments. We state three propositions, and indicate how to use them to prove the converse of Theorem 2.1. We will finish the proof of Theorem 2.1 at the end of the section, after we have proved the three propositions.

In the region where (2-4) holds, the Cauchy problem (2-1), with initial data $(0, f)$ translates into the following initial value problem for $V_+(x, s, y) = x^{-n/2} u(s + \log x, x, y)$:

$$
\begin{gathered}
P V_+(x, s, y) = 0 \quad \text{in } \log x < s,\ x < \varepsilon,\ y \in \partial X, \\
V_+(x, \log x, y) = 0, \quad D_s V_+(x, \log x, y) = x^{-n/2} f(x, y), \quad x < \varepsilon,\ y \in \partial X,
\end{gathered}
\tag{3-1}
$$

where

$$
P = -x^{-n/2-1}\big(D_t^2 - \Delta - \tfrac{1}{4}n^2\big)x^{n/2} = \partial_x(2\partial_s + x\partial_x) - x\Delta_h + A\partial_s + Ax\partial_x + \tfrac{1}{2}nA. \tag{3-2}
$$

Here, Δ_h is the (positive) Laplace operator on ∂X corresponding to the metric $h(x)$, in local y coordinates,

$$
\begin{gathered}
\Delta_h = -\frac{1}{\sqrt{\theta}} \partial_{y_i}(\sqrt{\theta}\, h^{ij} \partial_{y_j}), \\
\text{where} \quad h = (h_{ij}(x, y)), \quad h^{-1} = (h^{ij}(x, y)), \quad \theta = \det(h_{ij}) \quad \text{and} \quad A = \frac{1}{\sqrt{\theta}} \partial_x \sqrt{\theta}.
\end{gathered}
\tag{3-3}
$$

In the first proposition, we are interested in the behavior of $V_+(x, s, y)$ for x near $\{x = 0\}$ and $\{s = -\infty\}$. As in [Sá Barreto 2005], we work in the compactified space $\tilde{Y}$ — see Figure 1 — and set

$$
\mu = e^{-s_-/2} \quad \text{and} \quad \nu = e^{s_+/2}. \tag{3-4}
$$

This implies that $s = 2 \log \nu$ and $x = \mu\nu$. Notice that $\mu = \sqrt{\tau_+}$ and $\nu = \sqrt{\tau_-}$ and that, in these coordinates, the lateral face Σ of $\tilde{Y}$ is given by $\Sigma = \{\tau_+ = \tau_- = 0\} = \{\mu = \nu = 0\}$, and one may think of this as collapsing the lateral face Σ, as shown in Figure 4.

In coordinates (μ, ν, y), the operator P defined in (3-2) has the form

$$
\tilde{P} = \partial_\mu \partial_\nu - \mu\nu\Delta_h + \tfrac{1}{2}A(\mu\partial_\mu + \nu\partial_\nu) + \tfrac{1}{2}nA, \tag{3-5}
$$

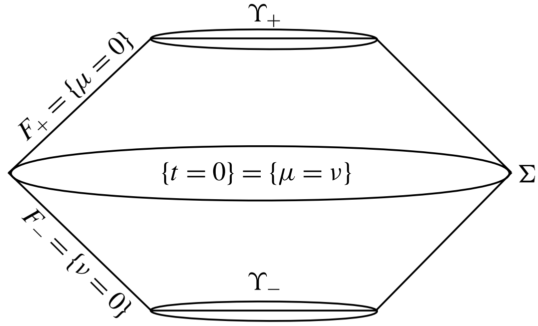

Figure 4. A compactification of $\mathbb{R}_t \times X$ with the face Σ collapsed.

where $h = h(\mu\nu)$, $A = A(\mu\nu, y)$. If

$$W(\mu, \nu, y) = V_+(\mu\nu, 2\log\nu, y) = (\mu\nu)^{-n/2} u\Big(\log\frac{\nu}{\mu}, \mu\nu, y\Big), \tag{3-6}$$

the Cauchy problem (3-1) becomes

$$\begin{gathered} \tilde{P}W = 0, \quad \mu,\ \nu \in (0, \varepsilon),\ y \in \partial X, \\ W(\mu, \mu, y) = 0, \quad \partial_\mu W(\mu, \mu, y) = -\mu^{-1-n} f(\mu^2, y). \end{gathered} \tag{3-7}$$

The fact that the initial data is of the form $(0, f)$ implies that the solution $u(t, z)$ to (2-1) satisfies $u(t, z) = -u(-t, z)$, and this implies that $W(\mu, \nu, y) = -W(\nu, \mu, y)$.

Proposition 3.2. *Let $f \in L^2_{\mathrm{ac}}(X)$ be such that $\mathcal{R}_+(0, f)(s, y) = 0$ in $\{s < s_0\} \times \Gamma$. Let u satisfy the initial value problem for the wave equation* (2-1) *with initial data* $(0, f)$, *and let $W(\mu, \nu, y)$ be defined as in* (3-6). *Then, in the sense of distributions $\partial_\mu^k W(\mu, \nu, y)|_{\{\mu=0\}} = 0$ in $[0, e^{s_0/2}) \times \Gamma$ and $\partial_\nu^k W(\mu, \nu, y)|_{\{\nu=0\}} = 0$ in $[0, e^{s_0/2}) \times \Gamma$ for $k = 0, 1, \ldots$. Moreover, for every $p \in \Gamma$ there exists $\delta > 0$ such that $W(\mu, \nu, y) = 0$ if $0 < \mu < \delta$, $0 < \nu < \delta$ and $|y - p| < \delta$.* (See Figure 5.)

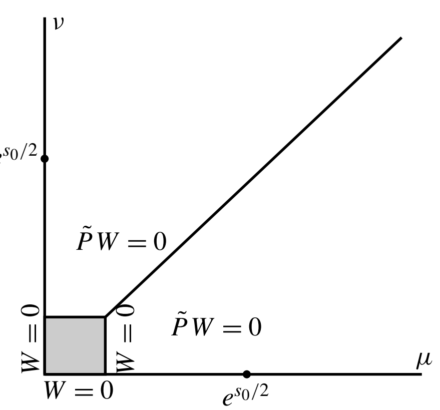

Figure 5. Unique continuation from infinity: if $\mathcal{R}_+(0, f)(s, y) = 0$ for $s \leq s_0$ and a.e. $y \in \Gamma$ then, for every $p \in \Gamma$, there exists $\delta > 0$ such that $W(\mu, \nu, y) = 0$ in the region shown provided that $|y - p| < \delta$.

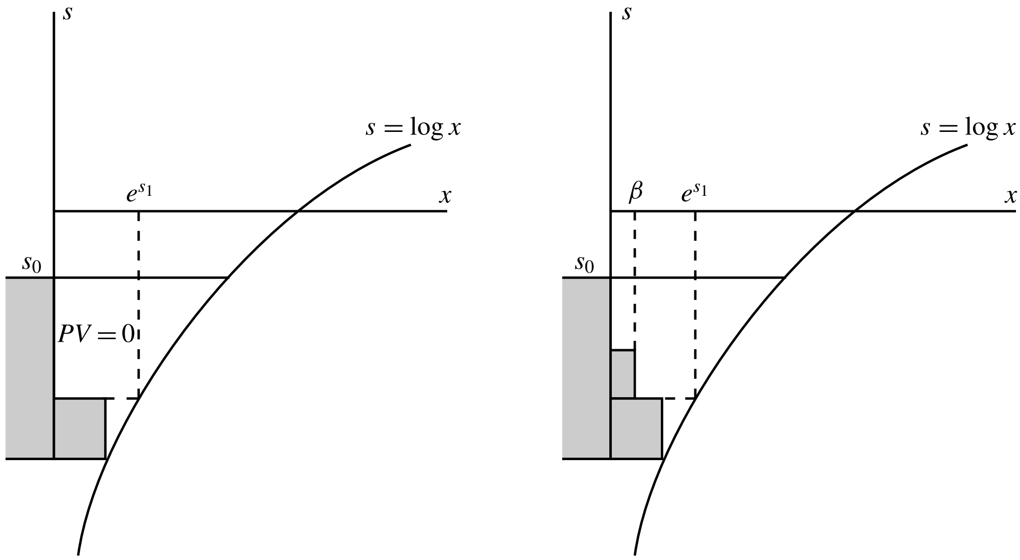

Figure 6. If $PV = 0$, and $V = 0$ in the dark region on the left, then $V = 0$ in the dark region on the right. This establishes unique continuation across the wedge $\{\log x < s < s_1,\ x < \delta,\ |y - p| < \delta\} \cup \{x \le 0,\ s < s_0,\ |y - p| < \delta\}$.

Next we need to show that we can increase the size of the neighborhood where $V_+ = 0$, and to do this we will use an iteration scheme involving the next two propositions. We will again use variables (x, s, y), and this time we will apply Hörmander's unique continuation theorem [1994b, Theorem 28.2.3], to prove:

Proposition 3.3. *Let $V(x, s, y) \in H^1_{\text{loc}}$ in the region $|x| < \varepsilon$, $y \in \Gamma$ and $s \in \mathbb{R}$, satisfy $PV = 0$, where P is given by* (3-2). *Let $s_1 < s_0$, $\delta > 0$ and $p \in \Gamma$ and suppose that*

$$V(x, s, y) = 0 \quad \textit{on } \{x \in (-\varepsilon, 0],\ s < s_0,\ y \in \Gamma\} \cup \{\log x < s < s_1,\ x < \delta,\ |y - p| < \delta\}.$$

Then there exists $\beta \in (0, \delta)$ such that $V(x, s, y) = 0$ if $x < \beta$, $|y - p| < \beta$ and $\log x < s < s_1 + \frac{1}{4}(s_0 - s_1)$. (Figure 6 illustrates the result.)

We know from Proposition 3.2 that $V_+(x, s, y) = 0$ for $x < \delta$, $|y - y_0| < \delta$ and $\log x \le s \le \log \delta$. We set $s_1 = \log \delta$. Proposition 3.3 shows that $V_+(s, x, y) = 0$ in $x < \beta < \delta$, $|y - y_0| < \beta < \delta$ and $s < s_1 + \frac{1}{4}(s_0 - s_1)$. In other words, $V_+(x, s, y) = 0$ in a larger interval in the s variable at the expense of shrinking the neighborhood of $\{x = 0,\ y = p\}$.

The second piece of the scheme is a consequence of a result of Tataru [1995; 1999], and it shows that, while the neighborhood of p might shrink, the neighborhood of $x = 0$ in fact does not. Figure 7 illustrates the result.

Proposition 3.4. *Let $u(t, z)$ satisfy* (2-1) *with initial data $f_1 = 0$, $f_2 = f \in L^2(X)$. Let $V_+(x, s, y) = x^{-n/2} u(s - \log x, x, y)$. Let $p \in \Gamma$, and suppose that there exist $s_2 \in \mathbb{R}$, $\gamma > 0$ and $\delta > 0$ such that $V_+(x, s, y) = 0$ if $0 < x < \gamma$, $\log x < s < s_2$ and $|y - p| < \delta$. Then $u(t, z) = 0$ if there is (x, y) with $x < \gamma$ and $|y - p| < \delta$ such that $|t| + d_g(z, (x, y)) < \log(e^{s_2}/x)$, where d_g is the distance with respect to the*

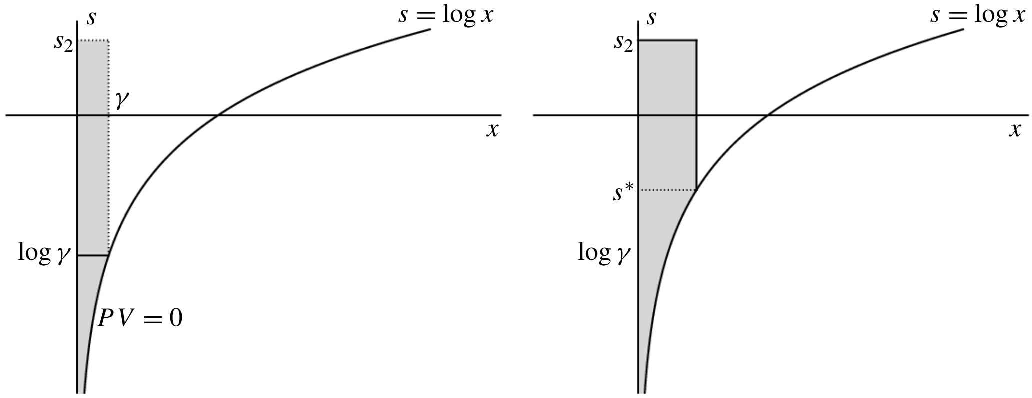

Figure 7. If $PV = 0$ and $V = 0$ in the dark region on the left, then $V = 0$ in the dark region on the right.

metric g. In particular, if $s^* < s_2$ is such that coordinates (2-4) *holds for $x < e^{s^*}$, then*

$$V_+(x, s, y) = 0 \quad \textit{if } |y - p| < \delta,\ 0 < x < e^{s^*} \textit{ and } \log x < s < s_2. \tag{3-8}$$

The idea is to iterate Propositions 3.3 and 3.4 to prove Theorem 2.1. We know from Proposition 3.2 that for any $p \in \Gamma$ there exists $\delta > 0$ such that

$$V_+(x, s, y) = 0 \quad \text{if } x < \delta,\ \log x < s < \log \delta,\ |y - p| < \delta.$$

Moreover, $V_+(x, s, y) = 0$ if $x < 0$, $s < s_0$ and $y \in \Gamma$. Applying Proposition 3.3 with $s_1 = \log \delta$, we find that there exists $\beta_1 < \delta$ such that

$$V_+(x, s, y) = 0 \quad \text{provided } x < \beta_1,\ |y - p| < \beta_1 \text{ and } \log x < s < \log \delta + \tfrac{1}{4}(s_0 - \log \delta).$$

Then Proposition 3.4 guarantees that there exists $s^* \ll 0$ independent of p such that

$$V_+(x, s, y) = 0 \quad \text{if } x < e^{s^*},\ |y - p| < \beta_1,\ s < s_2 = \log \delta + \tfrac{1}{4}(s_0 - \log \delta).$$

The main point is that, while the neighborhood of p shrinks from one step to the next, the neighborhood of $x = 0$ stays the same. Since $p \in \Gamma$ is arbitrary, it follows that in fact

$$V_+(x, s, y) = 0 \quad \text{if } x < e^{s^*},\ y \in \Gamma,\ s < s_2 = \log \delta + \tfrac{1}{4}(s_0 - \log \delta). \tag{3-9}$$

After using this argument n times, we find that

$$V_+(x, s, y) = 0 \quad \text{if } x < e^{s^*},\ y \in \Gamma,\ s < s_n = s_{n-1} + \tfrac{1}{4}(s_0 - s_{n-1}).$$

The sequence $\{s_n = s_{n-1} + \frac{1}{4}(s_0 - s_{n-1})\}$ is monotone and bounded by s_0. So it has a limit which is obviously equal to s_0. This implies that

$$V_+(x, s, y) = 0 \quad \text{if } x < e^{s^*},\ y \in \Gamma,\ s < s_0. \tag{3-10}$$

This does not quite yet prove Theorem 2.1, and the proof will be completed after the proof of Proposition 3.4. Now we will prove the three propositions above.

Proof of Proposition 3.2. First we claim that, without loss of generality, we may assume that $f \in L^2_{\mathrm{ac}}(X) \cap C^\infty(\mathring{X})$. To do this we need to characterize the range $\mathcal{R}_+(0, f)$, $f \in L^2_{\mathrm{ac}}(X)$. Notice that the solution $u(t, z)$ of (2-1) with data $(0, f)$ satisfies $u(-t, z) = -u(t, z)$, and hence $V_+(s, x, y) = x^{-n/2} u(s - \log x, x, y)$ and $V_-(s, x, y) = x^{-n/2} u(s + \log x, x, y)$ satisfy

$$V_+(x, -s, y) = x^{-n/2} u(-s - \log x, x, y) = -V_-(x, s, y). \tag{3-11}$$

In particular, we have

$$\mathcal{R}_+(0, f)(-s, y) = -(\partial_s V_+)(0, -s, y) = \partial_s V_-(0, s, y) = \mathcal{R}_-(0, f)(s, y).$$

Similarly,

$$\mathcal{R}_+(h, 0)(-s, y) = -\mathcal{R}_-(h, 0)(s, y).$$

So, if $F = \mathcal{R}_+(h, f)$ satisfies $F^*(s, y) = F(-s, y)$, then

$$F^*(s, y) = -\mathcal{R}_-(h, 0)(s, y) + \mathcal{R}_-(0, f)(s, y).$$

We apply $\mathcal{S} = \mathcal{R}_+ \mathcal{R}_-^{-1}$ to this identity and obtain

$$\mathcal{S} F^* = -\mathcal{R}_+(h, 0) + \mathcal{R}_+(0, f),$$

and we conclude that

$$\begin{aligned} \tfrac{1}{2}(\mathcal{S} F^* + F) &= \mathcal{R}_+(0, f), \\ \tfrac{1}{2}(\mathcal{S} F^* - F) &= \mathcal{R}_+(h, 0). \end{aligned} \tag{3-12}$$

Hence, $\mathcal{S} F^* = F^*$ if and only if $\mathcal{R}_+(h, 0) = 0$, and thus $h = 0$. Similarly, $\mathcal{S} F^* = -F$ if and only if $\mathcal{R}_+(0, f) = 0$ and hence $f = 0$. Therefore, we conclude that

$$\begin{aligned} \{F \in L^2(\mathbb{R} \times \partial X) : \mathcal{S} F^* = F\} &= \{\mathcal{R}_+(0, f) : f \in L^2_{\mathrm{ac}}(X)\}, \\ \{F \in L^2(\mathbb{R} \times \partial X) : \mathcal{S} F^* = -F\} &= \{\mathcal{R}_+(h, 0) : (h, 0) \in E_{\mathrm{ac}}(X)\}. \end{aligned} \tag{3-13}$$

The same argument applied to the backward radiation field shows that

$$\begin{aligned} \{F \in L^2(\mathbb{R} \times \partial X) : F^* = \mathcal{S} F\} &= \{\mathcal{R}_-(0, f) : f \in L^2_{\mathrm{ac}}(X)\}, \\ \{F \in L^2(\mathbb{R} \times \partial X) : F^* = -\mathcal{S} F\} &= \{\mathcal{R}_-(h, 0) : (h, 0) \in E_{\mathrm{ac}}(X)\}. \end{aligned} \tag{3-14}$$

Since $\mathcal{R}_+(0, f)(s, y) = 0$ in $\{s < s_0\} \times \Gamma$, we may take the convolution of $\mathcal{R}_+(0, f)$ with $\psi_\delta(s) \in C_0^\infty(\mathbb{R})$ even and supported in $(-\delta, \delta)$, with $\int \psi_\delta(s)\, ds = 1$. If $F(s, y) = \mathcal{R}_+(0, f)(s, y)$ and $F(s, y) = 0$ for $s \le s_0$, and

$$H_\delta(s, y) = \psi_\delta * F(s, y) = \int_{\mathbb{R}} \psi_\delta(s - s') F(s', y)\, ds',$$

then $H_\delta(s,y)=0$ if $s\le s_0-\delta$ and, since ψ_δ is even,

$$H^*_\delta(s,y)=H_\delta(-s,y)=\int_{\mathbb{R}}\psi_\delta(-s-s')F(s',y)\,ds'=\int_{\mathbb{R}}\psi_\delta(s+s')F(s',y)\,ds'$$
$$=\int_{\mathbb{R}}\psi_\delta(s-s')F(-s',y)\,ds'=\psi_\delta*F^*.$$

But the scattering operator commutes with translations in s, and hence it commutes with convolutions in the variable s. Therefore, in view of (3-13),

$$\mathcal{S}H^*_\delta=\psi_\delta*\mathcal{S}F^*=\psi_\delta*F=H_\delta.$$

We then use (3-13) to show that there exists $f_\delta\in L^2_{\mathrm{ac}}(X)$ such that $H_\delta=\mathcal{R}_+(0,f_\delta)$. Since $\mathcal{R}_+$ is unitary, $\|F-H_\delta\|_{L^2(\mathbb{R}\times\partial X)}=\|f-f_\delta\|_{L^2(X)}$, and hence $\|f_\delta-f\|_{L^2(X)}\to 0$ as $\delta\to 0$. Moreover, since $\partial_s^2\mathcal{R}_+(0,f)=\mathcal{R}_+(0,(\Delta-n^2/4)f)$, it follows that, for every $k\ge 0$,

$$\partial_s^{2k}H_\delta(s,y)=\mathcal{R}_+\big(0,\big(\Delta-\tfrac14 n^2\big)^k f_\delta\big)\in L^2(\mathbb{R}\times\partial X),$$

and thus $(\Delta-n^2/4)^k f_\delta\in L^2(X)$ for all $k\ge 0$, using that $\mathcal{R}_+$ is unitary. Therefore, by elliptic regularity, $f_\delta\in C^\infty(\mathring{X})$. If one proves Theorem 2.1 for $f\in C^\infty(\mathring{X})\cap L^2_{\mathrm{ac}}(X)$, then we conclude that $f_\delta(z)=0$ for $z\in\mathcal{D}_{s_0-\delta}(\Gamma)$. But, since $f_\delta\to f$ as $\delta\to 0$, it follows that $f(z)=0$ in $\mathcal{D}_{s_0}(\Gamma)$.

Next we will show that, if $\mathcal{R}(0,f)(s,y)=0$ in $\{s<s_0\}\times\Gamma$, then in the sense of distributions W vanishes to infinite order at $\{\mu=0,\ \nu<e^{s_0/2}\}\times\Gamma\cup\{\nu=0,\ \mu<e^{s_0/2}\}\times\Gamma$. Recall that we are assuming that $f\in C^\infty(\mathring{X})$, so the solution W to (3-7) is C^∞ in the region $\{\mu>0,\ \nu>0\}$. The issue here is the behavior of W at $\{\mu=0\}\cup\{\nu=0\}$.

Notice that, if $F(\mu,y)=\mu^{-1-n}f(\mu^2,y)$, then

$$\int_0^\varepsilon\int_{\partial X}\mu|F(\mu,y)|^2\theta^{\frac12}(\mu^2,y)\,dy\,d\mu=\frac12\int_0^{\varepsilon^2}\int_{\partial X}|f(x,y)|^2x^{-n-1}\theta^{\frac12}(x,y)\,dy\,dx\le\tfrac12\|f\|^2_{L^2(X)}.\tag{3-15}$$

We know from Theorem 2.1 of [Sá Barreto 2005] that, if $f\in C_0^\infty(\mathring{X})\cap L^2_{\mathrm{ac}}(X)$, then W has a C^∞ extension up to $\{\mu=0\}\cup\{\nu=0\}$ and, since $\partial_s=\frac12(\nu\partial_\nu-\mu\partial_\mu)$, then, provided $f\in C_0^\infty(\mathring{X})\cap L^2_{\mathrm{ac}}(X)$,

$$\mathcal{R}_+(0,f)(2\log\nu,y)=\tfrac12[(\nu\partial_\nu-\mu\partial_\mu)W(\mu,\nu,y)]\big|_{\mu=0}=\tfrac12\nu\partial_\nu W(0,\nu,y),\tag{3-16}$$

and we want to show that this restriction makes sense for $f\in L^2_{\mathrm{ac}}(X)$. We will work in the region $\{\nu\ge\mu\}$, but since the solution to (3-7) is odd under the change $(\mu,\nu)\mapsto(\nu,\mu)$, the same holds for the backward radiation field in the region $\{\nu\le\mu\}$.

Again, we assume that $f\in C_0^\infty(\mathring{X})\cap L^2_{\mathrm{ac}}(X)$, and W satisfies (3-7). If one multiplies the equation $\tilde{P}W=0$ by $\nu\partial_\nu W-\mu\partial_\mu W$, one obtains the identity

$$\frac{1}{2\sqrt{h(\mu\nu,y)}}\partial_\mu\big[\big(\nu|\partial_\nu W|^2+\mu^2\nu|d_h(\mu\nu)W|^2\big)\sqrt{h}\big]-\frac{1}{2\sqrt{h(\mu\nu,y)}}\partial_\nu\big[\big(\mu|\partial_\mu W|^2+\nu^2\mu|d_h(\mu\nu)W|^2\big)\sqrt{h}\big]$$
$$+\mu\nu\delta_{h(\mu\nu)}((\nu\partial_\nu W-\mu\partial_\mu W)d_{h(\mu\nu)}V)+Q(W,\mu\partial_\mu W,\nu\partial_\nu W,\mu\nu\partial_{y_j}W)=0,$$

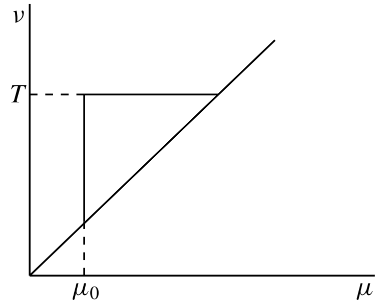

Figure 8. The region of integration in (3-17).

where $\delta_{h(\mu\nu)}$ is the divergence operator on the section ∂X dual to $d_{h(\mu\nu)}$ with respect to the metric $h(\mu\nu)$, and Q is a quadratic form. One then integrates this identity in the region $\Omega_{\mu_0,T} \times \partial X$, where $\Omega_{\mu_0,T} = \{\mu_0 \le \mu \le \nu,\ \mu \le \nu \le T\}$ is pictured in Figure 8, uses the divergence theorem and then the analogue of Gronwall's inequality, to arrive at the following inequality: for $0 \le \mu_0 \le T$, $T \in (0, e^{s_0/2})$, with T small enough that coordinates (2-4) hold for $x = \mu\nu$, there exists $C > 0$ which does not depend on f or W such that

$$\int_{\mu_0}^{T}\int_{\partial X}\big[(|W|^2+\mu|\partial_\mu W|^2+\mu\nu^2|d_{h(\mu\nu)}W|^2)\sqrt{\theta}(\mu\nu)\big]\big|_{\nu=T}\,dy\,d\mu \\ +\int_{\mu_0}^{T}\int_{\partial X}\big[(|W|^2+\nu|\partial_\nu W|^2+\mu^2\nu|d_{h(\mu\nu)}W|^2)\,\sqrt{\theta}(\mu\nu)\big]\big|_{\mu=\mu_0}\,dy\,d\nu \le C\|f\|^2_{L^2(X)}. \tag{3-17}$$

We refer the reader to the proof of Lemma 4.1 of [Sá Barreto 2005] for the details. In fact, this follows from equations (4.11), (4.14) and (4.15) of [Sá Barreto 2005], and (3-15) above.

We let

$$I(W,\mu_0,T)=\int_{\mu_0}^{T}\int_{\partial X}\big[(|W|^2+\nu|\partial_\nu W|^2+\mu^2\nu|d_{h(\mu\nu)}W|^2)\,\sqrt{\theta}(\mu\nu)\big]\big|_{\mu=\mu_0}\,dy\,d\nu.$$

If $f \in L^2_{\mathrm{ac}}(X)$ and if we take a sequence $f_j \in C_0^\infty(\mathring{X}) \cap L^2_{\mathrm{ac}}(X)$ with $\|f - f_j\|_{L^2(X)} \to 0$, (3-17) shows that, for fixed $\mu_0 \in [0, T]$,

$$I(W_j - W_k, \mu_0, T) \le C\|f_j - f_k\|^2_{L^2(X)},$$

and in particular, if $\mu_0 \in [0, T]$ and W is a solution of (3-7) with $f \in L^2_{\mathrm{ac}}(X)$, then, for $\mu_0 \in [0, T]$, the integral

$$\int_{\mu_0}^{T}\int_{\partial X}\nu|\partial_\nu W(\mu_0,\nu,y)|^2\sqrt{\theta}(\mu_0\nu,y)\,d\nu\,dy \le C\|f\|^2_{L^2(X)} \tag{3-18}$$

is well defined uniformly up to $\mu_0 = 0$. Since the radiation field is unitary, then in the sense of (3-18) the restriction $\nu\partial_\nu W(\mu_0, \nu, y)|_{\{\mu_0=0\}}$ is well defined, and hence (3-16) holds for $f \in L^2_{\mathrm{ac}}(X)$.

As was done in [Sá Barreto 2005], it is convenient to get rid of the term $A(\mu\partial_\mu + \nu\partial_\nu)$ in (3-5), by conjugating the operator by $\theta^{-1/4}$. Since Δ_h is the positive Laplacian, we find that, in local coordinates near a point $p \in \Gamma$,

$$\tilde{Q} = \theta^{1/4} \tilde{P} \theta^{-1/4} = \partial_\mu \partial_\nu + \mu\nu \sum_{i,j} h^{ij}(\mu\nu, y) \partial_{y_i} \partial_{y_j} + \mu\nu \sum_j \mathcal{B}_j(\mu\nu, y) \partial_{y_j} + C(\mu\nu, y), \tag{3-19}$$

where $C(\mu\nu, y)$ and $\mathcal{B}_j(\mu\nu, y)$ are C^∞, and $h^{-1} = (h^{ij})$ is the matrix associated with the metric h. Let $\tilde{W} = \theta^{1/4} W$; then $\tilde{Q}\tilde{W} = 0$. For $\phi(y) \in C_0^\infty(U)$, where $U \Subset \Gamma$ is such that (3-19) holds in $[0, \varepsilon] \times [0, \varepsilon] \times U$, let

$$G(\mu, \nu) = \int_{\partial X} \tilde{W}(\mu, \nu, y) \overline{\phi(y)} \, dy. \tag{3-20}$$

Notice that this is consistent with the conjugation of $\tilde{P}$ by $\theta^{1/4}$, and the factor $\theta^{1/2}$ is no longer present in the L^2 product. Let

$$Z(\mu\nu, y, D_y) = \tilde{Q} - \partial_\mu \partial_\nu = \mu\nu \sum_{i,j} h^{ij}(\mu\nu, y) \partial_{y_i} \partial_{y_j} + \mu\nu \sum_j \mathcal{B}_j(\mu\nu, y) \partial_{y_j} + C(\mu\nu, y),$$

and let $Z^*(\mu\nu, y, D_y)$ denote its adjoint with respect to the $L^2(\partial X)$ product defined by (3-20); then

$$\partial_\mu \partial_\nu G(\mu, \nu) = \int_{\partial X} \tilde{W}(\mu, \nu, y) \overline{Z^*(\mu\nu, y, D_y)\phi(y)} \, dy \tag{3-21}$$

It follows from (3-17) that there exists $C > 0$ such that

$$\begin{aligned} \int_0^T |\partial_\mu \partial_\nu G(\mu, T)|^2 \, d\mu &\le C \Big(\sum_{|\alpha| \le 2} \sup |\partial_y^\alpha \phi| \Big)^2 \|f\|^2_{L^2(X)}, \\ \int_{\mu_0}^T |\partial_\mu \partial_\nu G(\mu_0, \nu)|^2 \, d\nu &\le C \Big(\sum_{|\alpha| \le 2} \sup |\partial_y^\alpha \phi| \Big)^2 \|f\|^2_{L^2(X)} \quad \text{for } \mu_0 \in (0, T]. \end{aligned} \tag{3-22}$$

Let us write $K = \big(\sum_{|\alpha| \le 2} \sup |\partial_y^\alpha \phi| \big) \|f\|_{L^2(X)}$. Therefore, if $\delta < \mu < \varepsilon$,

$$|\partial_\nu G(\mu, \nu) - \partial_\nu G(\delta, \nu)| = \left| \int_\delta^\mu \partial_s \partial_\nu G(s, \nu) \, ds \right| \le CK(\mu - \delta)^{1/2}.$$

Hence, for $\nu > 0$,

$$\limsup_{\delta \to 0} |\partial_\nu G(\delta, \nu)| \le \liminf_{\mu \to 0} |\partial_\nu G(\mu, \nu)|,$$

so $\lim_{\mu \to 0} |\partial_\nu G(\mu, \nu)|$ exists. On the other hand, $\mathcal{R}_+(0, f)(s, y) = 0$ for $y \in \Gamma$ and $s \le s_0$, so according to (3-16) it follows that

$$\partial_\nu G(0, \nu) = 0, \quad \nu \in (0, T).$$

Now we use (3-22) to show that, if $0 \le \mu \le \nu \le T$, then there exists $C > 0$ such that

$$\begin{aligned} |\partial_\nu G(\mu, \nu)| = \left| \int_0^\mu \partial_s \partial_\nu G(s, \nu) \, ds \right| &\le \mu^{1/2} \Big(\int_0^\mu |\partial_s \partial_\nu G(s, \nu)|^2 \, ds \Big)^{\frac{1}{2}} \\ &\le \mu^{1/2} \Big(\int_0^\nu |\partial_s \partial_\nu G(s, \nu)|^2 \, ds \Big)^{\frac{1}{2}} \le CK\mu^{1/2}. \end{aligned} \tag{3-23}$$

Since $W(\mu, \mu, y) = 0$, we have, for $\mu \le \nu \le T$,

$$|G(\mu, \nu)| = \left| \int_\mu^\nu \partial_s G(\mu, s)\, ds \right| \le CK\mu^{1/2}(\nu - \mu). \tag{3-24}$$

This shows that, for every $\phi \in C_0^\infty(U)$,

$$\left| \int_{\partial X} \tilde{W}(\mu, \nu, y)\overline{\phi(y)}\, dy \right| \le CK\mu^{1/2},$$
$$\left| \int_{\partial X} \partial_\nu \tilde{W}(\mu, \nu, y)\overline{\phi(y)}\, dy \right| \le CK\mu^{1/2}.$$

Since $C_0^\infty(\mathbb{R}^2) \times C_0^\infty(U)$ spans $C_0^\infty(\mathbb{R}^2 \times U)$, it follows that for any $\psi(\mu, \nu, y)$, with $\mu, \nu \in [0, T]$,

$$\begin{aligned} \left| \int_{\partial X} \tilde{W}(\mu, \nu, y)\overline{\psi(\mu, \nu, y)}\, dy \right| &\le C\Big(\sum_{|\alpha| \le 2} \sup |\partial_y^\alpha \psi| \Big) \|f\|_{L^2(X)} \mu^{1/2}, \\ \left| \int_{\partial X} \partial_\nu \tilde{W}(\mu, \nu, y)\overline{\psi(\mu, \nu, y)}\, dy \right| &\le C\Big(\sum_{|\alpha| \le 2} \sup |\partial_y^\alpha \psi| \Big) \|f\|_{L^2(X)} \mu^{1/2}. \end{aligned} \tag{3-25}$$

Now we differentiate (3-21) with respect to ∂_ν. We have, for $\mu, \nu \in [0, T]$,

$$\partial_\nu \partial_\mu \partial_\nu G(\mu, \nu) = \int_{\partial X} \big[\partial_\nu \tilde{W}(\mu, \nu, y)\overline{Z^*(\mu\nu, y, D_y)\phi(y)} + \tilde{W}(\mu, \nu, y)\partial_\nu \overline{Z^*(\mu\nu, y, D_y))\phi(y)} \big]\, dy,$$

we apply (3-25) to $\psi(\mu, \nu, y) = Z^*(\mu\nu, y, D_y)\phi(y)$ and $\psi(\mu, \nu, y) = \partial_\nu Z^*(\mu\nu, y, D_y)\phi(y)$, and we conclude that

$$|\partial_\mu \partial_\nu^2 G(\mu, \nu, y)| \le C\Big(\sum_{|\alpha| \le 4} |\sup \partial_y^\alpha \phi| \Big) \|f\|_{L^2(X)} \mu^{1/2}$$

Let us denote $K_N(\phi) = \big(\sum_{|\alpha| \le N} |\sup \partial_y^\alpha \phi| \big) \|f\|_{L^2(X)}$. Since $\tilde{W}(\mu, \mu, y) = 0$, we have $\partial_\mu \partial_\nu G(\mu, \mu) = 0$, and so

$$|\partial_\mu \partial_\nu G(\mu, \nu)| = \left| \int_\mu^\nu \partial_\mu \partial_s^2 G(\mu, s)\, ds \right| \le K_4(\phi)\mu^{1/2}. \tag{3-26}$$

On the other hand, since $W(\mu, \mu, y) = 0$, it follows that $(\partial_\mu W)(\mu, \mu, y) = -(\partial_\nu W)(\mu, \mu, y)$. In particular, when $\nu = \mu$, we have

$$|\partial_\mu G(\mu, \mu)| \le CK_2(\phi)\mu^{1/2}$$

and, since

$$\partial_\mu G(\mu, \nu) = (\partial_\mu G)(\mu, \mu) + \int_\mu^\nu \partial_s \partial_\mu G(\mu, s)\, ds,$$

we have

$$|\partial_\mu G(\mu, \nu)| \le C(K_2(\phi) + K_4(\phi))\mu^{1/2}. \tag{3-27}$$

Proceeding as above, since $\partial_\nu G(0, \nu) = 0$, it follows from (3-26) that $|\partial_\nu G(\mu, \nu)| \le CK_4(\phi)\mu^{3/2}$ and, since $G(\mu, \mu) = 0$, we have $|G(\mu, \nu)| \le CK_4(\phi)\mu^{3/2}$ and $|\partial_\mu \partial_\nu^2 G(\mu, \nu)| \le CK_6(\phi)\mu^{3/2}$. Iterating this argument, and using the symmetry of W, we get that, for $k \geq 0$,

$$\partial_\mu^k G(0,\nu)=0, \quad \partial_\nu^k G(\mu,0)=0, \quad |(\partial_\mu G)(\mu,\mu)|=|(\partial_\nu G)(\mu,\mu)| \leq C\mu^k. \tag{3-28}$$

This shows that, in the sense of distributions, $\tilde{W}(\mu,\nu,y)$ vanishes to infinite order at

$$\{\mu=0,\ \nu<T\}\times\Gamma \cup \{\nu=0,\ \mu<T\}\times\Gamma,$$

where T has been chosen to be small enough that (2-4) holds for $x=\mu\nu<\varepsilon$. But this argument can be used finitely many times to show this holds for any $T\in(0,e^{s_0/2})$. In particular this shows that in the sense of distributions $\tilde{W}$ can be extended across the wedge $\{\mu=0\}\cup\{\nu=0\}$ so that

$$\begin{aligned} \tilde{Q}\tilde{W}&=0 \quad \text{in } (-e^{s_0/2},e^{s_0/2})\times(-e^{s_0/2},e^{s_0/2})\times\Gamma=\mathbb{O},\\ \tilde{W}&=0 \quad \text{in } \{\mu<0,\ 0\leq\nu<e^{s_0/2}\}\times\Gamma\cup\{\nu<0,\ 0\leq\mu<e^{s_0/2}\}\times\Gamma. \end{aligned} \tag{3-29}$$

From (3-17) we know more about the regularity of $\tilde{W}$. We also know that

$$\tilde{W}\in C^\infty\big(\mathbb{O}\setminus(\{\mu=0,\ \nu\geq 0\}\cup\{\nu=0,\ \mu\geq 0\})\big),$$

and in fact Hörmander's propagation of singularities theorem implies that

$$WF(\tilde{W})\subset\{\mu=0,\ \nu\geq 0,\ \xi_1=\xi_2=0\}\cup\{\nu=0,\ \mu\geq 0,\ \xi_1=\xi_2=0\}, \tag{3-30}$$

where ξ_1 and ξ_2 are dual to μ and ν respectively. If this were not true, singularities would propagate into the region where we know $\tilde{W}$ is C^∞. Indeed, the principal symbol of $\tilde{Q}$ is

$$q=-\xi_1\xi_2-\mu\nu h(\mu\nu,y,\eta),$$

and hence its bicharacteristics satisfy

$$\begin{aligned} &\dot{\mu}=-\xi_2, && \mu(0)=\mu_0, && \dot{\nu}=-\xi_1, && \nu(0)=\nu_0,\\ &\dot{\xi}_1=\nu(h+\mu\nu(\partial_x h)), && \xi_1(0)=\xi_{10}, && \dot{\xi}_2=\mu(h+\mu\nu(\partial_x h)), && \xi_2(0)=\xi_{20},\\ &\dot{y}_j=-\mu\nu\partial_{\eta_j}h, && y_j(0)=y_{j0}, && \dot{\eta}_j=\mu\nu\partial_{y_j}h, && \eta_j(0)=\eta_{j0}. \end{aligned}$$

Therefore, the bicharacteristics over $\mu=0$ satisfy $\mu=0$, $\xi_2=0$, $y=y_0$ and $\eta=\eta_0$ and

$$\dot{\nu}=-\xi_1, \quad \nu(0)=\nu_0, \quad \nu_0\geq 0, \qquad \dot{\xi}_1=\nu h(0,y_0,\eta_0), \quad \xi_1(0)=\xi_{10},$$

and hence, if we denote $h_0=h(0,y_0,\eta_0)$,

$$\nu(t)=\nu_0\cos(t\sqrt{h_0})-\frac{\xi_{10}}{\sqrt{h_0}}\sin(t\sqrt{h_0}), \quad \xi_1(t)=\xi_{10}\cos(t\sqrt{h_0})+\nu_0\sqrt{h_0}\sin(t\sqrt{h_0}).$$

If $(0,\nu_0,y_0,\xi_{10},0,\eta_0)\in WF(\tilde{W})$ with $\nu_0\geq 0$ and $\xi_{10}>0$, then $\nu(T)=-(\nu_0+\xi_1)/\sqrt{2}<0$ for $T=3\pi/(4\sqrt{h_0})$, and so the point

$$\Big(0,-\frac{1}{\sqrt{2}}(\nu_0+\xi_{10}),y_0,\frac{1}{\sqrt{2}}(-\xi_{10}+h_0\nu_0),0,\eta_0\Big)$$

lies in $WF(\tilde{W})$. On the other hand, if $\xi_{10}<0$, take $T=5\pi/(4\sqrt{h_0})$ and so

$$\Big(0, \frac{1}{\sqrt{2}}(-\nu_0+\xi_{10}), y_0, -\frac{1}{\sqrt{2}}(\xi_{10}+h_0\nu_0), 0, \eta_0\Big) \in WF(\tilde{W}).$$

But this is not possible, since $\tilde{W} \in C^\infty$ in $\{\nu < 0\}$. The same analysis applies to $\{\nu = 0,\ \mu \geq 0\}$.

The next step is to prove the following unique continuation result:

Lemma 3.5. *Let $\Gamma \subset \partial X$ be open and not empty. Let $W(\mu, \nu, y)$ satisfy* (3-17), *and let $\tilde{W} = \theta^{1/4} W$ satisfy* (3-29). *Then for any $p \in \Gamma$ there exists $\delta > 0$ such that $\tilde{W}(\mu, \nu, y) = 0$ provided $|\mu| < \delta$, $|\nu| < \delta$ and $|y - p| < \delta$.*

Proof. It is not clear that this result is a consequence of Theorem 1.1.2 of [Alinhac 1984], but (3-31) below is similar to the estimates in Section 4.1 of [Alinhac 1984]. As usual, the proof of this result is based on a Carleman estimate. However, we need to be quite careful when applying the Carleman estimate, which is proved for C_0^∞ functions, to $\tilde{W}$. In general, one would have to cut off and mollify $\tilde{W}$ and then apply Friedrich's lemma; see for example the proof of [Hörmander 1994b, Theorem 28.3.4]. This usually requires the solution to be in H^1_{loc}. However, here the regularity for $\tilde{W}$ is given by (3-17), which is not quite H^1_{loc} near $\{\mu = 0\}$ or $\{\nu = 0\}$. We will avoid cutting $\tilde{W}$ in the variables (μ, ν), as the commutator of $\tilde{Q}$ with the cut-off function would produce terms in $\partial_\mu W$ and $\partial_\nu W$, which we cannot yet control. However cut offs in the y do not offer any problem, since the commutator of $\tilde{Q}$ with a cut-off function in y only would produce terms like $\mu\nu\partial_{y_j}\tilde{W}$, which can be controlled by (3-17). We will prove the following Carleman inequality, which will be used to prove the stated unique continuation from infinity, and will also be used to improve the regularity of $\tilde{W}$.

Lemma 3.6. *Let $p \in \Gamma$, and let $\tilde{Q}$ be the operator defined in* (3-19). *For $0 < \nu_0 \leq e^{s_0/2}$, let*

$$\Omega_\varepsilon = \{(\mu, \nu, y) : |\mu| < \varepsilon,\ |\nu| \leq \nu_0,\ |y - p| < 2\varepsilon\}, \quad \Sigma_{1,\varepsilon} = \big\{\nu = \nu_0,\ 0 \leq \mu \leq \tfrac{1}{2}\varepsilon,\ |y - p| < 2\varepsilon\big\},$$
$$\Omega_\varepsilon^+ = \{(\mu, \nu, y) \in \Omega_\varepsilon : \mu \geq 0,\ \nu \geq 0\}, \quad \Sigma_{2,\varepsilon} = \big\{\mu = \tfrac{1}{2}\varepsilon,\ 0 \leq \nu \leq \nu_0,\ |y - p| < 2\varepsilon\big\}.$$

Let $C_0 = \sup_{\Omega_\varepsilon} |C|$, where C is the zeroth order term of $\tilde{Q}$. Let $\gamma > 0$ be such that $\gamma C_0^2 \nu_0^3$ is small enough, and let $\varphi_a(\mu, \nu, y) = \mu + \gamma\nu + \frac{1}{2}a\gamma|y - p|^2$, where $a = 0$ or $a = 1$. Then there exist $\varepsilon_0 > 0$, $M > 0$ such that if $0 < \varepsilon < \varepsilon_0$ and $k \geq \frac{1}{4}$, then the following estimate holds for all $v(\mu, \nu, y) \in C^\infty(\Omega_\varepsilon)$ supported in $\{(\mu, \nu, y) :\ \mu \geq 0,\ \nu \geq 0,\ |y - p| \leq \varepsilon\}$:

$$\begin{aligned} M\|\varphi^{-k}\tilde{Q}v\| &+ Mk\int_{\Sigma_{1,\varepsilon}} [\mu\nu\varphi^{-1}|\nabla_y\varphi^{-k}v|^2 + k^2\varphi^{-3-2k}|v|^2]\, d\mu\, dy \\ &\qquad + Mk\int_{\Sigma_{2,\varepsilon}} [\mu\nu\varphi^{-1}|\nabla_y\varphi^{-k}v|^2 + k^2\varphi^{-3-2k}|v|^2]\, d\nu\, dy \\ &\geq k^3\|\varphi^{-k-2}v\|^2 + k^2\|\varphi^{-1}\partial_\mu\varphi^{-k}v\|^2 + k^2\|\varphi^{-1}\partial_\nu\varphi^{-k}v\|^2 + k\|(\mu+\gamma\nu)^{1/2}\varphi^{-1/2}\nabla_y\varphi^{-k}v\|^2, \end{aligned} \tag{3-31}$$

where $\|v\|^2 = \int_{\Omega_\varepsilon^+} |v|^2\, d\mu\, d\nu\, dy$.

Proof. The estimate with $a = 0$ was proved in [Sá Barreto 2005]. We are doing it again here for the convenience of the reader, and we will use it to improve the regularity of $\tilde{W}$. But this estimate with $a = 0$ is not strong enough to prove the unique continuation result, for which we need the estimate with $a = 1$. We will use $\varphi = \varphi_a$ in the proof to simplify the already heavy notation.

Without loss of generality, we assume that $p=0$ and that v is real-valued. We know from (3-19) that

$$\tilde{Q}(\mu,\nu,y,\partial_\mu,\partial_\nu,\partial_y)=\partial_\mu\partial_\nu+\mu\nu\sum_{i,j=1}^{n}h^{ij}(\mu\nu,y)\partial_{y_i}\partial_{y_j}+\mu\nu\sum_{j=1}^{n}\mathcal{B}_j(\mu\nu,y)\partial_{y_j}+C(\mu\nu,y).$$

As usual, we define $\tilde{Q}_k=\varphi^{-k}\tilde{Q}\varphi^k$ and, since $\partial_\mu\varphi=1$, $\partial_\nu\varphi=\gamma$ and $\partial_{y_j}\varphi=a\gamma y_j$, we have

$$\tilde{Q}_k=\varphi^{-k}\tilde{Q}\varphi^k=\tilde{Q}(\mu,\nu,y,\partial_\mu+k\varphi^{-1},\partial_\nu+k\gamma\varphi^{-1},\partial_y+ka\gamma y\varphi^{-1}),$$

and we write

$$\tilde{Q}_k=\mathcal{Q}_k+k\mathcal{L},$$

with

$$\mathcal{L}=\varphi^{-1}(\partial_\nu+\gamma\partial_\mu),$$

$$\begin{aligned}\mathcal{Q}_k=\partial_\mu\partial_\nu+\gamma(k^2-k)\varphi^{-2}+\mu\nu h^{ij}(\mu\nu,y)(\partial_{y_i}+ka\gamma y_i\varphi^{-1})(\partial_{y_j}+ka\gamma y_j\varphi^{-1})&\\ +\mu\nu\mathcal{B}_j(\partial_{y_j}+ka\gamma y_j\varphi^{-1})+C,&\end{aligned}$$

where we used the notation $\sum_{ij=1}^{n}A_{ij}B_{ij}=A_{ij}B_{ij}$ and $D_jE_j=\sum_{j=1}^{n}D_jE_j$ to indicate sums over repeated indices. Therefore,

$$\|\tilde{Q}_kv\|^2=\|\mathcal{Q}_kv\|^2+k^2\|\mathcal{L}v\|^2+2k\langle\mathcal{Q}_kv,\mathcal{L}v\rangle, \tag{3-32}$$

where

$$\langle u,v\rangle=\int_{\Omega_\varepsilon^+}uv\,dy\,d\mu\,d\nu\quad\text{and}\quad\|v\|^2=\langle v,v\rangle.$$

The first term of (3-32) is positive and we will compute $k^2\|\mathcal{L}v\|^2+2k\langle\mathcal{Q}_kv,\mathcal{L}v\rangle$. Since v is supported in $\{\mu\geq 0,\ \nu\geq 0\}$, we will assume that $\mu\geq 0$ and $\nu\geq 0$ in the computations below. We will also use M for a generic constant. The first term of $\langle\mathcal{Q}_kv,\mathcal{L}v\rangle$ is

$$\begin{aligned}&\langle\partial_\mu\partial_\nu v,\varphi^{-1}(\partial_\nu+\gamma\partial_\mu)v\rangle\\ &\quad=\frac{1}{2}\int_{\Omega_\varepsilon^+}\varphi^{-1}(\partial_\mu(\partial_\nu v)^2+\gamma\partial_\nu(\partial_\mu v)^2)\,dy\,d\mu\,d\nu\\ &\quad=\frac{1}{2}\int_{\Omega_\varepsilon^+}(\partial_\mu(\varphi^{-1}(\partial_\nu v)^2)+\partial_\nu(\gamma\varphi^{-1}\partial_\mu v)^2)\,dy\,d\mu\,d\nu+\frac{1}{2}\int_{\Omega_\varepsilon^+}\varphi^{-2}(\gamma^2(\partial_\mu v)^2+(\partial_\nu v)^2)\,dy\,d\mu\,d\nu\\ &\quad\geq\tfrac{1}{2}(\gamma^2\|\varphi^{-1}\partial_\mu v\|^2+\|\varphi^{-1}\partial_\nu v\|^2).\end{aligned} \tag{3-33}$$

Here we used that v and all its derivatives vanish at $\{\mu=0\}\cup\{\nu=0\}$, and the boundary terms in $\Sigma_{j,\varepsilon}$, $j=1,2$ are nonnegative. The next term is

$$\begin{aligned}&\gamma(k^2-k)\langle\varphi^{-2}v,\varphi^{-1}(\gamma\partial_\mu+\partial_\nu)v\rangle\\ &\quad=\tfrac{1}{2}\gamma(k^2-k)\int_{\Omega_\varepsilon^+}\varphi^{-3}(\gamma\partial_\mu+\partial_\nu)v^2\,dy\,d\mu\,d\nu\\ &\quad=\tfrac{1}{2}\gamma(k^2-k)\int_{\Omega_\varepsilon^+}(\gamma\partial_\mu+\partial_\nu)(\varphi^{-3}v^2)\,dy\,d\mu\,d\nu+3\gamma^2(k^2-k)\int_{\Omega_\varepsilon^+}\varphi^{-4}|v|^2\,d\mu\,dy\\ &\quad=\tfrac{1}{2}\gamma(k^2-k)\int_{\Sigma_{1,\varepsilon}}\varphi^{-3}v^2\,d\mu\,dy+\tfrac{1}{2}\gamma^2(k^2-k)\int_{\Sigma_{2,\varepsilon}}\varphi^{-3}v^2\,d\nu\,dy+3\gamma^2(k^2-k)\|\varphi^{-2}v\|^2.\end{aligned} \tag{3-34}$$

Since we want to prove (3-31) for all $k \geq \frac{1}{4}$, we need to get rid of the negative term $-3k\gamma^2\|\varphi^{-2}v\|^2$ in (3-34). To do this we use the term $\|\varphi^{-1}\partial_\nu v\|^2$ from (3-33). Notice that $\varphi^{-1}\partial_\nu v = \partial_\nu(\varphi^{-1}v) + \gamma\varphi^{-2}v$, and hence

$$(\varphi^{-1}\partial_\nu v)^2 \geq \gamma^2\varphi^{-4}v^2 + 2\gamma\varphi^{-2}v\partial_\nu(\varphi^{-1}v) = \gamma^2\varphi^{-4}v^2 + \gamma\varphi^{-1}\partial_\nu(\varphi^{-1}v)^2.$$

Therefore,

$$\|\varphi^{-1}\partial_\nu v\|^2 \geq 2\gamma^2\|\varphi^{-2}v^2\|^2,$$

and so

$$3\gamma^2(k^2-k)\|\varphi^{-2}v\|^2 + \tfrac{7}{16}\|\varphi^{-1}\partial_\nu v\|^2 \geq 3\gamma^2\big(k^2-k+\tfrac{7}{24}\big)\|\varphi^{-2}v\|^2 \geq \tfrac{3}{8}k^2\gamma^2\|\varphi^{-2}v\|^2.$$

Hence, the first two terms satisfy

$$\begin{aligned}\langle\partial_\mu\partial_\nu v, \varphi^{-1}(\partial_\nu+\gamma\partial_\nu)v\rangle &+ (k^2-k)\langle\varphi^{-2}v, \varphi^{-1}(\gamma\partial_\mu+\partial_\nu)v\rangle \\ &\geq \tfrac{1}{2}\gamma^2\|\varphi^{-1}\partial_\mu v\|^2 + \tfrac{1}{16}\|\varphi^{-1}\partial_\nu v\|^2 + \tfrac{3}{8}k^2\gamma^2\|\varphi^{-2}v\|^2 \\ &\qquad + \tfrac{1}{2}(k^2-k)\int_{\Sigma_{1,\varepsilon}}\varphi^{-3}v^2\,d\mu\,dy + \tfrac{1}{2}\gamma^2(k^2-k)\int_{\Sigma_{2,\varepsilon}}\varphi^{-3}v^2\,d\nu\,dy.\end{aligned}\tag{3-35}$$

To estimate the third term, we integrate by parts in y_j, recalling that v is compactly supported in the y variable in the interior of Ω_ε^+. We use that h^{ij} is symmetric to write it as

$$\begin{aligned}\langle\mu\nu h^{ij}&(\partial_{y_i}+ka\gamma y_i\varphi^{-1})(\partial_{y_j}+ka\gamma y_j\varphi^{-1})v, \mathscr{L}v\rangle \\ &= \frac{1}{2}\int_{\Omega_\varepsilon^+}\mu\nu h^{ij}\,[(\partial_{y_i}+ka\gamma y_i\varphi^{-1})(\partial_{y_j}+ka\gamma y_j\varphi^{-1})v]\mathscr{L}v\,dy\,d\mu\,d\nu \\ &\qquad + \frac{1}{2}\int_{\Omega_\varepsilon^+}\mu\nu h^{ij}\,[(\partial_{y_j}+kay_j\varphi^{-1})(\partial_{y_i}+ka\gamma y_i\varphi^{-1})v]\mathscr{L}v\,dy\,d\mu\,d\nu = I + II,\end{aligned}$$

where

$$\begin{aligned}I = -\frac{1}{2}\int_{\Omega_\varepsilon^+}&\mu\nu h^{ij}(\partial_{y_j}v+ka\gamma y_j\varphi^{-1}v)[(\partial_{y_i}-ka\gamma y_i\varphi^{-1})\mathscr{L}v]\,dy\,d\mu\,d\nu \\ &- \frac{1}{2}\int_{\Omega_\varepsilon^+}\mu\nu h^{ij}(\partial_{y_i}v+ka\gamma y_i\varphi^{-1}v)[(\partial_{y_j}-ka\gamma y_j\varphi^{-1})\mathscr{L}v]\,dy\,d\mu\,d\nu,\end{aligned}$$

$$II = -\int_{\Omega_\varepsilon^+}[\partial_{y_i}(\mu\nu h^{ij})](\partial_{y_j}v+ka\gamma y_j\varphi^{-1}v)\mathscr{L}v\,dy\,d\mu\,d\nu.$$

We can bound II from below by using that

$$\begin{aligned}\partial_{y_i}(\mu\nu h^{ij})(\partial_{y_j}v+ka\gamma y_j\varphi^{-1}v)\mathscr{L}v &\geq -M(\mu\nu)^{3/4}|\partial_{y_j}v+ka\gamma y_j\varphi^{-1}v|\,(\mu\nu)^{1/4}|\mathscr{L}v| \\ &\geq -M\big((\mu\nu)^{3/2}|\nabla_y v|^2 + k^2a^2\gamma^2(\mu\nu)^{3/2}|y|^2\varphi^{-2}v^2 + (\mu\nu)^{1/2}|\mathscr{L}v|^2\big).\end{aligned}$$

Hence,

$$II \geq -M\big(\|(\mu\nu)^{3/4}\nabla_y v\|^2 + \gamma^2k^2a^2\|(\mu\nu)^{3/4}|y|\varphi^{-1}v\|^2 + \|(\mu\nu)^{1/4}\mathscr{L}v\|^2\big).\tag{3-36}$$

Using that

$$(\partial_{y_i}-ka\gamma y_i\varphi^{-1})\mathscr{L}v = \mathscr{L}(\partial_{y_i}-ka\gamma y_i\varphi^{-1})v - a\gamma y_j\varphi^{-1}\mathscr{L}v - 2ka\gamma^2 y_i\varphi^{-3}v,$$

we write $I = I_1 + I_2$, where

$$\begin{aligned} I_1 &= -\frac{1}{2}\int_{\Omega_\varepsilon^+} \mu\nu h^{ij}(\partial_{y_j} v + ka\gamma y_j\varphi^{-1}v)[\mathscr{L}(\partial_{y_i} v - ka\gamma y_i\varphi^{-1}v)]\, dy\, d\mu\, d\nu \\ &\quad -\frac{1}{2}\int_{\Omega_\varepsilon^+} \mu\nu h^{ij}(\partial_{y_i} v + ka\gamma y_i\varphi^{-1}v)[\mathscr{L}(\partial_{y_j} v - ka\gamma y_j\varphi^{-1}v)]\, dy\, d\mu\, d\nu \\ I_2 &= a\int_{\Omega_\varepsilon^+} \mu\nu h^{ij}(\partial_{y_j} v + ka\gamma y_j\varphi^{-1}v)(\gamma y_i\varphi^{-1}\mathscr{L}v + 2k\gamma^2 y_i\varphi^{-3}v)\, dy\, d\mu\, d\nu. \end{aligned} \tag{3-37}$$

To bound the term I_2 from below, we write

$$\begin{aligned} &\mu\nu h^{ij}(\partial_{y_j} v + ka\gamma y_j\varphi^{-1}v)(\gamma y_i\varphi^{-1}\mathscr{L}v + 2k\gamma^2 y_i\varphi^{-3}v) \\ &\qquad \geq -M|y|^{1/2}\mu\nu\varphi^{-1}(|\partial_{y_i} v| + ka\gamma|y|\varphi^{-1}|v|)|y|^{1/2}(\gamma|\mathscr{L}v| + ka\gamma^2\varphi^{-2}|v|) \\ &\qquad \geq -M\big(|y|(\mu\nu)^2\varphi^{-2}|\nabla_y v|^2 + \gamma^2|y|(\mathscr{L}v)^2 + k^2a^2\gamma^2|y|^3(\mu\nu)^2\varphi^{-4}|v|^2 + k^2a^2\gamma^4|y|\varphi^{-4}|v|^2\big). \end{aligned}$$

Therefore,

$$I_2 \geq -Ma\big(\||y|^{1/2}\mu\nu\varphi^{-1}\nabla_y v\|^2 + \gamma^2\||y|^{1/2}\mathscr{L}v\|^2 + k^2a^2\gamma^2\||y|^{3/2}\mu\nu\varphi^{-2}v\|^2 + k^2a^2\gamma^4\||y|^{1/2}\varphi^{-2}v\|^2\big) \tag{3-38}$$

Next we consider the term I_1. Since $\mathscr{L} = \varphi^{-1}(\partial_\mu + \partial_\nu)$, integrating by parts in μ and ν we conclude that the term I_1 satisfies

$$\begin{aligned} I_1 &= -\frac{1}{2}\int_{\Omega_\varepsilon^+} \mu\nu h^{ij}\mathscr{L}\big[(\partial_{y_j} v + ka\gamma y_j\varphi^{-1}v)(\partial_{y_i} v - ka\gamma y_i\varphi^{-1}v)\big]\, dy\, d\mu\, d\nu \\ &= -\frac{1}{2}\int_{\Omega_\varepsilon^+} (\gamma\partial_\mu + \partial_\nu)\big[(\mu\nu\varphi^{-1}h^{ij})(\partial_{y_j} v + ka\gamma y_j\varphi^{-1}v)(\partial_{y_i} v - ka\gamma y_i\varphi^{-1}v)\big]\, dy\, d\mu\, d\nu \\ &\qquad + \frac{1}{2}\int_{\Omega_\varepsilon^+} \big[(\gamma\partial_\mu + \partial_\nu)(\mu\nu\varphi^{-1}h^{ij})\big](\partial_{y_j} v + ka\gamma y_j\varphi^{-1}v)(\partial_{y_i} v - ka\gamma y_i\varphi^{-1}v)\, dy\, d\mu\, d\nu \\ &= -\frac{1}{2}\int_{\Sigma_{1,\varepsilon}} \mu\nu\varphi^{-1}h^{ij}((\partial_{y_i} + ka\gamma y_j\varphi^{-1})v)((\partial_{y_j} - ka\gamma y_j\varphi^{-1})v)\, d\mu\, dy \\ &\qquad - \frac{\gamma}{2}\int_{\Sigma_{2,\varepsilon}} \mu\nu\varphi^{-1}h^{ij}((\partial_{y_i} + ka\gamma y_j\varphi^{-1})v)((\partial_{y_j} - ka\gamma y_j\varphi^{-1})v)\, d\nu\, dy \\ &\qquad\quad + \frac{1}{2}\int_{\Omega_\varepsilon^+} \big[(\gamma\partial_\mu + \partial_\nu)(\mu\nu\varphi^{-1}h^{ij})\big](\partial_{y_j} v + ka\gamma y_j\varphi^{-1}v)(\partial_{y_i} v - ka\gamma y_i\varphi^{-1}v)\, dy\, d\mu\, d\nu. \end{aligned}$$

Notice that

$$\begin{aligned} (\gamma\partial_\mu + \partial_\nu)(\mu\nu h^{ij}(\mu\nu, y)\varphi^{-1}) &= \big[(\gamma\nu + \mu)\varphi^{-1} - 2\gamma\mu\nu\varphi^{-2}\big]h^{ij} + (\mu + \gamma\nu)\mu\nu\varphi^{-1}(\partial_x h^{ij}) \\ &= \varphi^{-2}\Big[\big((\mu + \gamma\nu)\big(\mu + \gamma\nu + \tfrac{1}{2}a\gamma|y|^2\big) - 2\gamma\mu\nu\big)h^{ij}(\mu\nu, y) \\ &\qquad + \mu\nu(\mu + \gamma\nu)\big(\mu + \gamma\nu + \tfrac{1}{2}a\gamma|y|^2\big)(\partial_x h^{ij})(\mu\nu, y)\Big] \\ &= \varphi^{-2}\Big[\big(\mu^2 + \gamma^2\nu^2 + \tfrac{1}{2}a\gamma(\mu + \gamma\nu)|y|^2\big)h^{ij}(\mu\nu, y) \\ &\qquad + \mu\nu(\mu + \gamma\nu)\big(\mu + \gamma\nu + \tfrac{1}{2}a\gamma|y|^2\big)(\partial_x h^{ij})(\mu\nu, y)\Big]. \end{aligned} \tag{3-39}$$

Hence,

$$|(\gamma\partial_\mu+\partial_\nu)(\mu\nu h^{ij}(\mu\nu,y)\varphi^{-1})|\le M\varphi^{-1}(\mu+\gamma\nu). \tag{3-40}$$

On the other hand, since h^{ij} is positive definite, we know that there exists $M>0$ such that

$$h^{ij}W_iW_j\ge M|W|^2,\quad W\in\mathbb{R}^n, \tag{3-41}$$

We conclude from (3-39), (3-40), (3-41) and the symmetry of h^{ij} that, for ε small enough, there exists M such that

$$\begin{aligned}[(\partial_\mu+\partial_\nu)&(\mu\nu h^{ij}\varphi^{-1})](\partial_{y_j}v+ka\gamma y_j\varphi^{-1}v)(\partial_{y_j}v-ka\gamma y_j\varphi^{-1}v)\\ &=[(\partial_\mu+\partial_\nu)(\mu\nu h^{ij}\varphi^{-1})](\partial_{y_i}v\partial_{y_j}v-k^2a^2\gamma^2y_iy_j\varphi^{-2}v^2)\\ &\ge M(\mu+\gamma\nu)\varphi^{-1}|\nabla_yv|^2-Mk^2a^2(\mu+\gamma\nu)\gamma^2|y|^2\varphi^{-3}|v|^2. \end{aligned}\tag{3-42}$$

Hence, for ε small enough,

$$\begin{aligned}I_1\ge M\|\varphi^{-1/2}(\mu+\gamma\nu)^{1/2}\nabla_yv\|^2&-Mk^2a^2\gamma^2\||y|(\mu+\gamma\nu)^{1/2}\varphi^{-3/2}v\|^2\\ &-M\int_{\Sigma_{1,\varepsilon}}\mu\nu(\varphi^{-1}|\nabla_yv|^2+k^2a^2\varphi^{-3}|y|^2v^2)\,d\mu\,dy\\ &-M\int_{\Sigma_{2,\varepsilon}}\mu\nu(\varphi^{-1}|\nabla_yv|^2+k^2a^2\varphi^{-3}|y|^2v^2)\,d\nu\,dy.\end{aligned}\tag{3-43}$$

We write the last term of $\langle\mathcal{Q}_kv,\mathcal{L}v\rangle$ as

$$\begin{aligned}\langle\mu\nu&\mathcal{B}_j(\partial_{y_j}+ka\gamma y_j\varphi^{-1})v+Cv,\mathcal{L}v\rangle\\ &=\langle\mu\nu\varphi^{-1/2}\mathcal{B}_j(\partial_{y_j}+ka\gamma y_j\varphi^{-1})v+\varphi^{-1/2}Cv,\varphi^{1/2}\mathcal{L}v\rangle\\ &\ge-\|\varphi^{1/2}\mathcal{L}v\|^2-\|C\varphi^{-1/2}v\|^2-Mk^2a^2\gamma^2\||y|\mu\nu\varphi^{-3/2}v\|^2-M\|(\mu\nu)\varphi^{-1/2}\nabla_yv\|^2,\end{aligned}\tag{3-44}$$

Therefore, provided ε_0 is small enough, we deduce from equations (3-35), (3-36), (3-38), (3-43) and (3-44) that

$$\begin{aligned}k^2\|\mathcal{L}v\|^2+2k\langle\mathcal{Q}_kv,\mathcal{L}v\rangle&+Mk\int_{\Sigma_{1,\varepsilon}}(\mu\nu\varphi^{-1}|\nabla_yv|^2+k^2\varphi^{-3}v^2)\,d\mu\,dy\\ &+Mk\int_{\Sigma_{2,\varepsilon}}(\mu\nu\varphi^{-1}|\nabla_yv|^2+k^2\varphi^{-3}v^2)\,d\nu\,dy\\ \ge\tfrac12k\gamma^2\|\varphi^{-1}\partial_\mu v\|^2&+\tfrac1{16}k\|\varphi^{-1}\partial_\nu v\|^2+\int_{\Omega_\varepsilon^+}(k^2-kMF_1(\mu,\nu,y))|\mathcal{L}v|^2\,d\mu\,d\nu\,dy\\ &+k\int_{\Omega_\varepsilon^+}|\nabla_yv|^2(M_1(\mu+\gamma\nu)\varphi^{-1}-MF_2(\mu,\nu,y))\,d\mu\,d\nu\,dy\\ &+k\int_{\Omega_\varepsilon^+}k^2\gamma^4\varphi^{-4}v^2\big(\tfrac38-MF_3(\mu,\nu,y)\big)\,d\mu\,d\nu\,dy-k\int_{\Omega_\varepsilon^+}|C|\varphi^{-1}v^2\,d\mu\,d\nu\,dy,\end{aligned}\tag{3-45}$$

where

$$
\begin{aligned}
F_1(\mu,\nu,y)&=(\mu\nu)^{1/2}+\gamma^2|y|+\varphi,\\
F_2(\mu,\nu,y)&=(\mu\nu)^{3/2}+|y|(\mu\nu)^2\varphi^{-2}+(\mu\nu)^2\varphi^{-1},\\
F_3(\mu,\nu,y)&=(\mu\nu)^{3/2}|y|^2\varphi^2+|y|^3(\mu\nu)^2+\gamma^2|y|+|y|^2(\mu+\gamma\nu)\varphi+|y|^2(\mu\nu)^2\varphi.
\end{aligned}
$$

The term involving C is the most problematic. Recall that $\varphi=\mu+\gamma\nu+\frac{1}{2}a\gamma|y|^2$ and, since $|\mu|\le\varepsilon$, $|y|\le\varepsilon$ and $\nu\le\nu_0$, it follows that $\varphi\le\varepsilon+\gamma\nu_0+\frac{1}{2}a\gamma\varepsilon^2$. Therefore, if $C_0=\sup_{\Omega_\varepsilon}|C|$,

$$\tfrac{3}{8}k^2\gamma^2\varphi^{-4}-|C|^2\varphi^{-1}\ge\varphi^{-4}\big(\tfrac{3}{8}k^2\gamma^2-C_0^2\varphi^3\big)\ge\varphi^{-4}\big(\tfrac{3}{8}k^2\gamma^2-9C_0^2\big(\varepsilon^3+\gamma^3\nu_0^3+\tfrac{1}{8}a\gamma^3\varepsilon^6\big)\big).$$

If one picks γ such that $9\gamma C_0^2\nu_0^3<\frac{3}{256}$, then

$$\tfrac{3}{8}k^2-9\gamma C_0^2\nu_0^3\ge\tfrac{3}{16}k^2\quad\text{for all }\ k\ge\tfrac{1}{4},$$

and therefore

$$\tfrac{3}{8}k^2\gamma^2\varphi^{-4}-|C|^2\varphi^{-1}\ge\varphi^{-4}\big(\tfrac{3}{16}k^2\gamma^2-9C_0^2\big(\varepsilon^3+\tfrac{1}{8}a\gamma^3\varepsilon^6\big)\big)\quad\text{for all }\ k\ge\tfrac{1}{4}.$$

Notice also that $\mu\le\varphi$, and hence the coefficient of $|\nabla v|^2$ in (3-45) satisfies

$$
\begin{aligned}
&M_1(\mu+\gamma\nu)\varphi^{-1}-M((\mu\nu)^{3/2}+|y|(\mu\nu)^2\varphi^{-2}+(\mu\nu)^2\varphi^{-1})\\
&\qquad\ge\varphi^{-1}\big(M_1(\mu+\gamma\nu)-M|((\mu\nu)^{3/2}\varphi+|y|\mu\nu^2+(\mu\nu)^2)\big)\\
&\qquad\ge\tfrac{1}{2}M_1(\mu+\gamma\nu)\varphi^{-1}\quad\text{for }\ \varepsilon_0\ \text{ small enough.}
\end{aligned}
$$

One can then pick ε_0, such that for every $\varepsilon\in(0,\varepsilon_0)$,

$$
\begin{aligned}
k^2\|\mathscr{L}v\|^2+2k\langle\mathscr{Q}_kv,\mathscr{L}v\rangle&+Mk\int_{\Sigma_{1,\varepsilon}}(\mu\nu\varphi^{-1}|\nabla_yv|^2+k^2\varphi^{-3}v^2)\,d\mu\,dy\\
&+Mk\int_{\Sigma_{2,\varepsilon}}(\mu\nu\varphi^{-1}|\nabla_yv|^2+k^2\varphi^{-3}v^2)\,d\nu\,dy\\
\ge M\big(k\|(\mu+\gamma\nu)^{1/2}\varphi^{-1/2}\nabla_yv\|^2&+k^2\|\mathscr{L}v\|^2+k\|\varphi^{-1}\partial_\mu v\|^2+k\|\varphi^{-1}\partial_\nu v\|^2+k^3\gamma^2\|\varphi^{-2}v\|^2\big),
\end{aligned}
$$

This ends the proof of Lemma 3.6. □

Next we want to use (3-31) to prove Lemma 3.5. Let $\chi\in C_0^\infty\big(\{|y|<\varepsilon/4\}\big)$, $\chi=1$ on $\{|y|\le\varepsilon/8\}$. Let $V(\mu,\nu,y)=\chi(y)\tilde{W}(\mu,\nu,y)$. We choose $\psi(y)$ to be a C_0^∞ function supported in $\{|y|<\varepsilon/4\}$ with $\int\psi(y)\,dy=1$, and define $\psi_\delta(y)=(\delta)^{-n}\psi(y/\delta)$, $\delta>0$. Then, for δ small enough,

$$V_\delta=\psi_\delta*'V\in C_0^\infty(\Omega_{2\varepsilon})\quad\text{is supported in}\quad\big\{\mu\ge0,\ \nu\ge0,\ |y|\le\tfrac{1}{2}\varepsilon\big\}.$$

where $*'$ denotes convolution in the y variable. To see that, let $\zeta(\mu,\nu)\in C_0^\infty$; then the Fourier transform $\widehat{\zeta V}_\delta$ satisfies

$$\widehat{\zeta V}_\delta(\xi_1,\xi_2,\eta)=\hat{\psi}(\delta\eta)(\widehat{\zeta V})(\xi_1,\xi_2,\eta),$$

which in view of (3-30) is rapidly decaying in any conic neighborhood of a point $(\xi_{10},\xi_{20},\eta_0)\neq0$. Hence $V_\delta\in C^\infty$, and (3-31) holds for V_δ. Now we would like to take the limit of (3-31) for V_δ as $\delta\to0$.

Notice that $\varphi \geq \varepsilon$ on $\Sigma_{2,\varepsilon}$ and $\varphi \geq \gamma\nu_0$ on $\Sigma_{1\varepsilon}$ and, in view of (3-17),

$$\begin{aligned}
&\int_{\Sigma_{1,\varepsilon}} [\mu\nu|\nabla_y\varphi^{-k}\tilde{W}|^2 + k^2|\varphi^{-2-k}\tilde{W}|^2]\, d\mu\, dy < M(\gamma\nu_0)^{-k},\\
&\int_{\Sigma_{2,\varepsilon}} [\mu\nu|\nabla_y\varphi^{-k}\tilde{W}|^2 + k^2|\varphi^{-2-k}\tilde{W}|^2]\, d\nu\, dy < M\varepsilon^{-k},
\end{aligned} \tag{3-46}$$

and these terms in (3-31) do not offer any problem when passing to the limit.

One cannot use (3-31) with $a = 0$ to prove Lemma 3.5, however we will use it here to show that

$$\begin{gathered}
(\mu+\gamma\nu)^{-k}\nabla V, \quad (\mu+\gamma\nu)^{-k-1}\partial_\nu V, \quad (\mu+\gamma\nu)^{-k-1}\partial_\mu V,\\
(\mu+\gamma\nu)^{-k-2}V \in L^2(\Omega_\varepsilon) \quad \text{with } k \geq \tfrac{1}{4}.
\end{gathered} \tag{3-47}$$

For now, we take $a = 0$ and $\varphi = \mu+\gamma\nu$. We know from (3-17) that $\tilde{W}$, $[\mu\nu(\mu+\gamma\nu)]^{1/2}\nabla_y\tilde{W} \in L^2(\Omega_\varepsilon)$. Since $\mu\gamma\nu \leq \frac{1}{2}(\mu+\gamma\nu)^2$, it follows that $\gamma(\mu+\gamma\nu)^{-1}(\mu\nu)^2 \leq (\mu+\gamma\nu)\mu\nu$, and hence one can apply Friedrich's lemma — see, for example, Lemma 17.1.5 of [Hörmander 1994a] — to show that

$$\lim_{\delta\to 0}\big\|(\mu+\gamma\nu)^{-1/4}\mu\nu[(h^{ij}\partial_{y_i}\partial_{y_j} + \mathcal{B}_j\partial_{y_j})\psi_\delta *' V - \psi_\delta *' (h^{ij}\partial_{y_i}\partial_{y_j} + \mathcal{B}_j\partial_{y_j} V)]\big\| = 0 \tag{3-48}$$

We also know from (3-17) that, for fixed $T > 0$, $\mu^{1/2}\partial_\mu\tilde{W}(\mu, T, y) \in L^2([0, T]\times\partial X)$. Hence the same holds for V and for V_δ for all $\delta > 0$. One can easily show that

$$\mu(\partial_\mu V_\delta)^2 \geq \tfrac{1}{4}\mu^{-1}(\log\mu)^{-2}V_\delta^2 - \partial_\mu((-\log\mu)^{-1}V_\delta^2).$$

Since V_δ vanishes to infinite order at $\mu = 0$, if we integrate the above on $\big[0, \frac{1}{2}\varepsilon\big]\times\partial X$ we obtain

$$\int_{\partial X}\Big(\log\frac{2}{\varepsilon}\Big)^{-1} V_\delta\big(\tfrac{1}{2}\varepsilon, T, y\big)\, dy + \int_0^T\int_{\partial X}\mu(\partial_\mu V_\delta)^2\, dy\, d\mu \geq \int_0^T\int_{\partial X}\mu^{-1}(\log\mu)^{-2}V_\delta^2\, dy\, d\mu. \tag{3-49}$$

Since, in view of (3-17), the left-hand side is finite for V, if one applies (3-49) to $V_\delta - V_{\delta'}$ it follows that V_δ is a Cauchy sequence in the norm given by the right-hand side of (3-49). So it converges and, since V_δ converges weakly to V, we conclude that $\mu^{-1/2}|\log\mu|^{-1}V \in L^2(\Omega_\varepsilon)$, and in particular

$$(\mu+\nu)^{-1/4}V \in L^2(\Omega_\varepsilon). \tag{3-50}$$

Since $\tilde{Q}$ is given by (3-19), it follows from (3-48) and (3-50) that

$$\lim_{\delta\to 0}\big\|(\mu+\nu)^{-1/4}(\tilde{Q}(\psi_\delta *' V) - \psi_\delta *' (\tilde{Q}V))\big\| = 0. \tag{3-51}$$

Since $\tilde{Q}\tilde{W} = 0$, it follows that

$$\tilde{Q}V = \tilde{Q}(\chi(y)\tilde{W}) = \mu\nu h^{ij}(\tilde{W}\partial_{y_i}\partial_{y_j}\chi + 2\partial_{y_i}\chi\partial_{y_j}\tilde{W}) + \mu\nu(\mathcal{B}_j\partial_{y_j}\chi)\tilde{W}.$$

So we conclude that, in view of (3-17), $(\mu+\nu)^{-1/4}\tilde{Q}V \in L^2(\Omega_\varepsilon)$ and hence

$$\lim_{\delta\to 0}\|(\mu+\gamma\nu)^{-k}\tilde{Q}V_\delta\|_{L^2(\Omega_\varepsilon)} = \|(\mu+\gamma\nu)^{-k}\tilde{Q}V\| < \infty, \quad k = \tfrac{1}{4}. \tag{3-52}$$

Therefore (3-31), with $a=0$ and $k=\frac{1}{4}$, holds for V in place of V_δ, and in particular we conclude that (3-47) holds for $k=\frac{1}{4}$ (notice that in this case $(\mu+\gamma\nu)\varphi^{-1}=1$). We then apply the argument used above to show that (3-31) holds for $k=\frac{1}{4}+1$, and hence (3-47) holds for $k=\frac{1}{4}+1$, and by induction and interpolation, this shows that (3-47) holds for all $k\geq\frac{1}{4}$.

Now we use the same argument with $\varphi=\varphi_1=\mu+\gamma\nu+\frac{1}{2}\gamma|y|^2$. Notice that in this case $\varphi\geq\mu+\gamma\nu$ and we have from (3-47) that

$$\varphi^{-k}\nabla_y V,\quad \varphi^{-k-1}\partial_\nu V,\quad \varphi^{-k-1}\partial_\mu V,\quad \varphi^{-k-2}V\in L^2(\Omega_\varepsilon),\quad k\geq\tfrac{1}{4}. \tag{3-53}$$

Since φ depends on y, it is not clear how to apply Friedrich's lemma in the bootstrapping argument above to prove (3-53), as one would have to analyze the commutator of the convolution and the weight, which is of course singular. But, given (3-53), Friedrich's lemma can be easily applied and we conclude that (3-31) holds for V and $\varphi=\varphi_1$. In particular we conclude from (3-46) that, for ε small enough,

$$Mk^3\varepsilon^{-k}+C\|\varphi^{-k}\tilde{Q}\chi(y)\tilde{W}\|^2\geq k^3\|\varphi^{-2-k}\chi(y)\tilde{W}\|^2. \tag{3-54}$$

Now we really use the power of (3-31) with $a=1$: since $\tilde{Q}\tilde{W}=0$, and $\chi=1$ for $|y|\leq\frac{1}{8}\varepsilon$, $\tilde{Q}(\chi(y)\tilde{W})=[\tilde{Q},\chi(y)]\tilde{W}$ is supported in $|y|\geq\frac{1}{8}\varepsilon$, and hence $\varphi\geq\lambda\varepsilon^2$ on the support of $\tilde{Q}V$, where $\lambda=\frac{1}{128}\gamma$. We deduce from (3-54) that, for ε small enough, there exists $C=C(\tilde{W})>0$ such that

$$C(\lambda\varepsilon^2)^{-2k}\geq\|\varphi^{-2-k}\chi(y)\tilde{W}\|^2.$$

Hence,

$$\left\|\left(\frac{\varphi}{\lambda\varepsilon^2}\right)^{-k}\chi(y)\tilde{W}\right\|\leq C,\quad k>1,$$

and therefore $\tilde{W}(\mu,\nu,y)=0$ if $\varphi\leq\lambda\varepsilon^2$, and in particular $\tilde{W}=0$ if $0\leq\mu\leq\frac{1}{3}\lambda\varepsilon^2$, $0\leq\gamma\nu\leq\frac{1}{3}\lambda\varepsilon^2$ and $\gamma|y|^2\leq\frac{1}{3}\lambda\varepsilon^2$. This ends the proof of Lemma 3.5, and consequently the proof of Proposition 3.2. □

Notice that since $\nu_0\in(0,e^{s_0/2})$ is arbitrary, this result also establishes regularity for $\tilde{W}$, and in particular it shows that $\tilde{W}\in H^1_{\mathrm{loc}}$.

Proof of Proposition 3.3. We will use Hörmander's unique continuation theorem, and we will find a function whose level surfaces are strictly pseudoconvex. The key point here is that the coefficients of the operator P defined in (3-2) do not depend on s, and hence P is invariant under translations in the variable s. Let

$$\varphi(x,s,y)=-x-\kappa(s-s_1)-|y-p|^2,\quad \text{where } \kappa>0 \text{ small will be chosen later.}$$

Since, for $|y-p|<\delta$, $V=0$ if $x\in(-\varepsilon,0]$ and $s<s_0$, or if $x<\delta$ and $\log x<s<s_1$, we have — see Figure 6 —

$$V(x,s,y)=0\quad \text{if } \varphi>0,\ -\varepsilon<x<\delta,\ \text{and } |y-p|<\delta. \tag{3-55}$$

The principal symbol of the operator P is

$$p=-2\sigma\xi-x\xi^2-xh(x,y,\eta), \tag{3-56}$$

where (ξ, σ, η) are the dual variables to (x, s, y). Since $\nabla\varphi(x, s, y) = (-1, -\kappa, -2(y-p))$, we have

$$p(x, s, y, \nabla\varphi(x, s, y)) = -2\kappa - x\big(1 + h(x, y, 2(y-p))\big). \tag{3-57}$$

If $|y-p| < \beta$ is small enough and $x > -\kappa$, then $x\big(1 + h(x, y, 2(y-p))\big) > -\frac{3}{2}\kappa$, and hence $p(x, s, y, \nabla\varphi) < -\frac{1}{2}\kappa$. Therefore φ is not characteristic at (x, s, y) if $x > -\kappa$ and $|y-p| < \beta$, for small enough β.

The Hamilton vector field of p is

$$H_p = -2\xi\partial_s - 2(\sigma + x\xi)\partial_x - xH_h + (\xi^2 + h + x\partial_x h)\partial_\xi, \tag{3-58}$$

where H_h denotes the Hamilton vector field of $h(x, y, \eta)$ in the variables (y, η). Hence,

$$(H_p\varphi)(x, s, y, \xi, \sigma, \eta) = 2(\sigma + x\xi) + 2\kappa\xi + xH_h|y-p|^2 \quad \text{and}$$

$$\begin{aligned}(H_p^2\varphi)&(x, s, y, \xi, \sigma, \eta)\\ &= -2(\sigma + x\xi)(2\xi + H_h|y-p|^2 + x\partial_x H_h|y-p|^2) - (xH_h)^2|y-p|^2 + 2(\kappa + x)(\xi^2 + h + x\partial_x h).\end{aligned}$$

If $H_p\varphi = 0$, it follows that

$$\begin{aligned}H_p^2\varphi(x, s, y, \xi, \sigma, \eta) = 2(x+3\kappa)\xi^2 + 2\xi((x+\kappa)H_h|y-p|^2 + \kappa x\partial_x H_h|y-p|^2) + 2(\kappa + x)(h + x\partial_x h)\\ + x((H_h|y-p|^2)^2 + xH_h|y-p|^2\partial_x H_h|y-p|^2 - xH_h^2|y-p|^2).\end{aligned}$$

If $|y-p| < \beta$ is small enough, there exists $C > 0$ depending on h only such that

$$|H_p|y-p|^2| \le C\beta|\eta| \quad \text{and} \quad |\partial_x H_p|y-p|^2| \le C\beta|\eta|.$$

If we impose that $-\frac{1}{2}\kappa < x < \beta$, it follows that there exists $\varepsilon_0 > 0$ depending on h such that, if $\beta, \kappa \in (0, \varepsilon_0)$ are small, then there exists $C > 0$ such that

$$h + x\partial_x h \ge C|\eta|^2,$$

and hence

$$\begin{aligned}H_p^2\varphi(x, s, p, \xi, \sigma, \eta) &\ge \kappa C(\xi^2 - \beta|\xi||\eta| + |\eta|^2)\\ &\ge C\kappa(\xi^2 + |\eta|^2) \qquad \text{if } -\tfrac{1}{2}\kappa < x < \beta,\ |y-p| < \beta \text{ and } \kappa, \delta \in (0, \varepsilon_0).\end{aligned}$$

So we conclude that there exists $\varepsilon_0 > 0$ depending on h such that

$$\begin{aligned}p(x, s, y, \xi, \sigma, \eta) = H_p\varphi(x, s, y, \xi, \sigma, \eta) = 0 \quad \Longrightarrow \quad H_p^2\varphi(x, s, y, \xi, \sigma, \eta) > 0\\ \text{if } (\xi, \sigma, \eta) \ne 0,\ -\tfrac{1}{2}\kappa < x < \beta,\ |y-p| < \beta,\ \kappa, \beta \in (0, \varepsilon_0).\end{aligned} \tag{3-59}$$

Since P is of second order, we deduce from (3-57) and (3-59) that the level surfaces of φ are strictly pseudoconvex in the region

$$-\tfrac{1}{2}\kappa < x < \beta,\ |y-p| < \beta \quad \text{provided } \kappa,\ \beta \in (0, \varepsilon_0); \tag{3-60}$$

see for example the first paragraph of Section 28.4 of [Hörmander 1994b]. As mentioned above, the fact that the coefficients of P do not depend on s imply that the conditions in (3-60) do not depend on s. Now we appeal to Theorem 28.2.3 and Proposition 28.3.3 of [Hörmander 1994b] and conclude that, if

$$Y = \left\{-\tfrac{1}{4}\kappa < x < \tfrac{1}{2}\beta,\ |y-p| < \tfrac{1}{\sqrt{2}}\beta,\ |s-s_1| < s_0 - s_1\right\},$$

then there exist $C > 0$, $\tau_0 > 0$ and $\lambda > 0$ large such that, if $\psi = e^{\lambda\varphi}$,

$$C\|e^{\tau\psi}Pv\|^2 \geq \tau^2\|e^{\tau\psi}v\|^2 + \tau\|e^{\tau\psi}v\|^2_{H^1} \quad \text{for all } v \in C_0^\infty(Y) \text{ and } \tau \geq \tau_0 > 0. \tag{3-61}$$

Let $\theta \in C_0^\infty(Y)$ with $\theta = 1$ if $-\frac{1}{8}\kappa < x < \frac{1}{4}\beta$, $|y-p| < \frac{1}{2}\beta$ and $|s-s_1| < \frac{3}{4}(s_0-s_1)$. Since $PV = 0$, it follows that

$$P(\theta V) = [P, \theta]V.$$

But, for $(x, s, y) \in Y$, $V(x, s, y)$ is supported in the region $x > 0$, $s > s_1$, so we conclude that

$$P(\theta(x, s, y)V) \text{ is supported in } (x, s, y) \in Y,\ x \geq \tfrac{1}{4}\beta, \text{ or } s - s_1 \geq \tfrac{3}{4}(s_0 - s_1), \text{ or } |y-p| \geq \tfrac{1}{2}\beta.$$

Therefore, by the definition of φ we have

$$\varphi(x, s, y) \leq -\min\left\{\tfrac{1}{4}\beta,\ \tfrac{1}{4}3\kappa(s_0 - s_1),\ \tfrac{1}{4}\beta^2\right\} \quad \text{on the support of } P(\theta V). \tag{3-62}$$

Pick κ small so that $\min\left\{\frac{1}{4}\beta, \frac{3}{4}\kappa(s_0-s_1), \frac{1}{4}\beta^2\right\} = \frac{3}{4}\kappa(s_0-s_1) = \gamma$. We deduce from (3-61) and (3-62) that

$$\tau^2\|e^{\tau(e^{\lambda\varphi} - e^{-\lambda\gamma})}\theta V\|^2 \leq C, \quad \tau > \tau_0.$$

We remark that, due to Friedrich's lemma, one can apply (3-61) to θV even though V is not C^∞; see [Hörmander 1994b]. Therefore, $\theta V = 0$ if $e^{\lambda\varphi} - e^{-\lambda\gamma} > 0$, so $\theta V = 0$ if $\varphi > -\gamma$. So we deduce that

$$\theta V(x, s, y) = 0 \quad \text{provided } \kappa(s - s_1) < \tfrac{1}{3}\gamma,\ 0 < x < \tfrac{1}{3}\gamma\ |y-p|^2 < \tfrac{1}{3}\gamma.$$

In particular,

$$V(x, s, y) = 0 \quad \text{provided } s < s_1 + \tfrac{1}{4}(s_0 - s_1),\ 0 < x < \tfrac{1}{3}\gamma,\ |y-p|^2 < \tfrac{1}{3}\gamma. \tag{3-63}$$

This concludes the proof of Proposition 3.3. □

Proof of Proposition 3.4. The key point in the proof is the following consequence of Tataru's theorem [1995; 1999]; see also [Hörmander 1997; Robbiano and Zuily 1998]. Let Ω be a C^∞ manifold equipped with a C^∞ Riemannian metric g. Let L be a first-order C^∞ operator that does not depend on t. If $u(t, z)$ is a C^∞ function that satisfies

$$(D_t^2 - \Delta_g + L(z, D_z))u = 0 \quad \text{in } (-\tilde{T}, \tilde{T}) \times \Omega,$$
$$u(t, z) = 0 \text{ in a neighborhood of } \{z_0\} \times (-T, T),\ T < \tilde{T},$$

then

$$u(t, z) = 0 \quad \text{if } |t| + d_g(z, z_0) < T, \tag{3-64}$$

where d_g is the distance measured with respect to the metric g.

Since the initial data of (2-1) is $(0, f)$, $u(t, z) = -u(-t, z)$. If $0 < x < \gamma$, $\log x < s < s_1$, and $|y-p| < \delta$, it follows from the definition of V_+ that

$$u(t, x, y) = 0 \quad \text{if } 0 < x < \gamma,\ |y-p| < \delta \text{ and } |t| \le s_2 - \log x = \log \frac{e^{s_2}}{x}.$$

Applying (3-64) with $z_0 = (x, y)$, we obtain

$$u(t, z) = 0 \quad \text{provided } |t| + d_g(z; (x, y)) < \log \frac{e^{s_2}}{x} \text{ with } 0 < x < \delta,\ |y-p| < \delta.$$

If $z = (\alpha, y)$ with $e^{s^*} > \alpha > x$, $d_g((x, y); (\alpha, y)) = \log(\alpha/x)$, it follows from (3-64) that

$$u(t, (\alpha, y)) = 0 \quad \text{if } t + \log \frac{\alpha}{x} < \log \frac{e^{s_2}}{x}.$$

In particular this guarantees that $u(t, \alpha, y) = 0$ if $0 < t < \log(e^{s_2}/\alpha)$ and, since $s = t + \log \alpha$, we have $V_+(\alpha, s, y) = 0$ if $\alpha < e^{s^*}$, $\log x < s < s_2$ and $|y-p| < \delta$. This ends the proof of Proposition 3.4. □

Proof of Theorem 2.1. As promised at the beginning of the section, we shall now finish the proof of Theorem 2.1. We start with (3-10), which says that $V_+(x, s, y) = x^{-n/2} u(s - \log x, x, y)$ satisfies $V_+(x, s, y) = 0$ if $y \in \Gamma$, $x < e^{s^*}$ and $\log x < s < s < s_0$.

Now we recall that $V_+(x, s, y) = x^{-n/2} u(s - \log x, x, y)$ and so, if $w = (\alpha, p)$ with $0 < \alpha < e^{s^*}$ and $p \in \Gamma$, then the solution $u(t, z)$ vanishes in a neighborhood of $\{w\} \times (0, \log(e^{s_0}/\alpha))$. Again we use that the data is of the form $(0, f)$, and hence $u(-t, z) = -u(t, z)$. So in fact $u(t, z)$ vanishes in a neighborhood of $\{w\} \times (-\log(e^{s_0}/\alpha), \log(e^{s_0}/\alpha))$. Therefore, by (3-64),

$$u(t, z) = \partial_t u(t, z) = 0 \quad \text{if } |t| + d_g(z, w) < \log \frac{e^{s_0}}{\alpha}.$$

In particular, when $t = 0$ we find that $\partial_t u(0, z) = f(z) = 0$ provided $d_g(z, w) < \log(e^{s_0}/\alpha)$, and this concludes the proof of Theorem 2.1. □

Final remarks. The following result will be useful in the next section:

Corollary 3.7. *Let (X, g) be a connected AHM and let $\Gamma \subset \partial X$ be open, $\Gamma \neq \varnothing$. If $f \in L^2_{\mathrm{ac}}(X)$ and $\mathcal{R}_+(0, f)(s, y) = 0$ in $\mathbb{R} \times \Gamma$, then $f = 0$. Similarly, if $(h, 0) \in E_{\mathrm{ac}}(X)$ and $\mathcal{R}_+(h, 0)(s, y) = 0$ in $\mathbb{R} \times \Gamma$, then $h = 0$.*

Proof. If $\mathcal{R}_+(0, f)(s, y) = 0$ in $\mathbb{R} \times \Gamma$, then $f(z) = 0$ if $z \in \mathcal{D}_{s_0}(\Gamma)$ for every s_0. Since the distance between any two points in the interior of X is finite, it follows that $f = 0$.

Suppose $F = \mathcal{R}_+(h, 0)(s, y) = 0$ in $\mathbb{R} \times \Gamma$. As in the proof of Proposition 3.2, by taking the convolution of F with $\phi \in C_0^\infty(\mathbb{R})$, even, we may assume that $(\Delta_g - n^2/4)^k h \in L^2_{\mathrm{ac}}(X)$ for every $k \ge 0$. Let $u(t, z)$ satisfy (2-1) with initial data $(h, 0)$ and let $V = \partial_t u$. Then V satisfies (2-1) with initial data $(0, (\Delta_g - n^2/4)h)$ and $\mathcal{R}_+(0, (\Delta_g - n^2/4)h)(s, y) = 0$ in $\mathbb{R} \times \Gamma$. But, as we have shown, this implies that $(\Delta_g - n^2/4)h = 0$. Since $(h, 0) \in E_{\mathrm{ac}}(X)$, this implies that $h = 0$. □

One should remark that this result can be proved by applying a result of Mazzeo [1991]; see also [Vasy and Wunsch 2005]. The solution to (2-1) with initial data $(0, f)$ is odd and, since $\mathcal{R}_+(0, f)(s, y) = 0$ for $s \in \mathbb{R}$, $y \in \Gamma$, it follows that $\mathcal{R}_-(0, f)(s, y) = 0$ for $s \in \mathbb{R}$, $y \in \Gamma$. Taking the Fourier transform in t, we find that

$$\big(\Delta_g - \lambda^2 - \tfrac{1}{4}n^2\big)\hat{u}(\lambda, z) = 0$$

and, using that $\mathcal{R}_+(0, f)(s, y) = \mathcal{R}_-(0, f)(s, y) = 0$, one deduces that $\hat{u}(\lambda, z)$ vanishes to infinite order at Γ, using a formal power series argument as in the proof of Proposition 3.4 of [Graham and Zworski 2003]. Theorem 14 of [Mazzeo 1991] implies that $\hat{u} = 0$ and hence $u = 0$. In particular, $f = 0$.

4. The control space

As we saw in (3-13) and (3-14), the ranges of the forward and backward radiation fields

$$\mathcal{R}_\pm(0, L^2_{\mathrm{ac}}(X)) = \{\mathcal{R}_\pm(0, f) : f \in L^2_{\mathrm{ac}}(X)\}$$

are closed subspaces of $L^2(\mathbb{R} \times \partial X)$ and are characterized by the scattering operator. Moreover, since $\mathcal{R}_\pm$ are unitary, $\|\mathcal{R}_\pm(0, f)\|_{L^2(\mathbb{R}\times\partial X)} = \|f\|_{L^2(X)}$. The main goal of this section is to show that the ranges $\{\mathcal{R}_\pm(0, f)|_{\mathbb{R}\times\Gamma}\}$ are determined by the restriction of the scattering operator to Γ, as defined in (2-13). Throughout the remainder of the paper we shall write

$$L^2(\mathbb{R} \times \Gamma) = \{F|_{\mathbb{R}\times\Gamma} : F \in L^2(\mathbb{R} \times \partial X)\}.$$

The key observation is:

Lemma 4.1. *If $F = \mathcal{R}_+(h, f) \in L^2(\mathbb{R} \times \Gamma)$, then*

$$\|f\|_{L^2(X)} = \big\langle F, \tfrac{1}{2}(F + S_\Gamma F^*)\big\rangle,$$

and in particular $\|f\|_{L^2(X)}$ is determined by $\mathcal{S}_\Gamma F$.

Proof. If $F(s, y) = \mathcal{R}_+(h, f) \in L^2(\mathbb{R} \times \Gamma)$, so in particular F is supported in $\mathbb{R} \times \Gamma$ then, according to (3-12) and the fact that $\mathcal{R}_+$ is unitary,

$$\begin{aligned}\big\langle F, \tfrac{1}{2}(F + \mathcal{S}_\Gamma F^*)\big\rangle = \big\langle F, \tfrac{1}{2}(F + (\mathcal{S}F^*)|_{\mathbb{R}\times\Gamma})\big\rangle &= \big\langle F, \tfrac{1}{2}(F + \mathcal{S}F^*)\big\rangle \\ &= \langle \mathcal{R}_+(h, f), \mathcal{R}_+(0, f)\rangle = \|f\|^2_{L^2(X)}.\end{aligned}$$ □

This suggests that

$$\mathbb{C}^n_+\big(\mathcal{R}_+(0, f)|_{\mathbb{R}\times\Gamma}\big) = \|f\|_{L^2(X)}$$

defines a norm on the space $\{\mathcal{R}_+(0, f)|_{\mathbb{R}\times\Gamma} : f \in L^2_{\mathrm{ac}}(X)\}$. We shall prove that it does and, moreover, the norm is determined by $\mathcal{S}_\Gamma$.

Theorem 4.2. *Let $\Gamma \subset \partial X$ be a nonempty open subset such that $\partial X \setminus \Gamma$ does not have empty interior. The space*

$$\mathcal{M}(\Gamma)^\pm = \{\mathcal{R}_\pm(0, f)|_{\mathbb{R}\times\Gamma} : f \in L^2_{\mathrm{ac}}(X)\},$$

equipped with norm $\mathbb{C}^n_\pm$ defined by

$$\mathbb{C}^n_\pm(\mathcal{R}_\pm(0, f)|_{\mathbb{R}\times\Gamma}) = \|f\|_{L^2(X)}, \tag{4-1}$$

is a Hilbert space determined by $\mathcal{S}_\Gamma$.

Proof. We shall work with the forward radiation field. The proof of the result for $\mathcal{R}_-$ is identical. Since $\mathcal{R}_+$ is linear, the triangle inequality for the $L^2(X)$-norm implies that $\mathbb{C}^n_+$ is a norm, and that

$$\big\langle \mathcal{R}_+(0,f)|_{\mathbb{R}\times\Gamma}, \mathcal{R}_+(0,h)|_{\mathbb{R}\times\Gamma}\big\rangle_{\mathbb{C}^n_+} = \langle f,h\rangle_{L^2(X)}$$

is an inner product. Since $\mathcal{R}_+$ is continuous and $L^2_{\mathrm{ac}}(X)$ is complete, it follows that $(\mathcal{M}(\Gamma)^+, \mathbb{C}^n_+)$ is a Hilbert space. We need to show that it is determined by $\mathcal{S}_\Gamma$. We recall from (3-12) that if $F=\mathcal{R}_+(f,h)$ then

$$\tfrac{1}{2}(F+\mathcal{S}F^*)|_{\mathbb{R}\times\Gamma} = \mathcal{R}_+(0,h)|_{\mathbb{R}\times\Gamma}. \tag{4-2}$$

So, if $F\in L^2(\mathbb{R}\times\Gamma)$, then $F^*\in L^2(\mathbb{R}\times\Gamma)$ and hence $(F+\mathcal{S}F^*)|_{\mathbb{R}\times\Gamma} = F+\mathcal{S}_\Gamma F^*$. We shall let

$$\begin{aligned} \mathcal{L}: L^2(\mathbb{R}\times\Gamma) &\to L^2(\mathbb{R}\times\Gamma) \\ F &\mapsto \tfrac{1}{2}(F+\mathcal{S}_\Gamma F^*). \end{aligned} \tag{4-3}$$

Since $\mathcal{S}$ is unitary, it follows that $\|\mathcal{L}\|\le 1$. Since $\mathcal{R}_+$ is unitary, given $F\in L^2(\mathbb{R}\times\Gamma)$ there exists $(f,h)\in E_{\mathrm{ac}}(X)$ such that $\mathcal{R}_+(f,h)=F$. We can say the following about such initial data:

Lemma 4.3. *Let $\Gamma\subset\partial X$ be a nonempty open subset such that $\partial X\setminus\Gamma$ contains an open set $\mathbb{O}$, and let $h\in L^2_{\mathrm{ac}}(X)$. Then there exists at most one f such that $(f,0)\in E_{\mathrm{ac}}(X)$ and $\mathcal{R}_+(f,h)$ is supported in $\mathbb{R}\times\Gamma$. Moreover, the set*

$$\mathscr{C}(\Gamma)=\{h\in L^2_{\mathrm{ac}}(X) : \text{there exists } (f,0)\in E_{\mathrm{ac}}(X) \text{ such that } \mathcal{R}_+(f,h)(s,y)=0,\ y\in\partial X\setminus\Gamma\}$$

is dense in $L^2_{\mathrm{ac}}(X)$.

Proof. First, if $\mathcal{R}_+(f_1,h)$ and $\mathcal{R}_+(f_2,h)$ are supported in $\mathbb{R}\times\overline{\Gamma}$, then $\mathcal{R}_+(f_1-f_2,0)$ is supported in $\mathbb{R}\times\overline{\Gamma}$, but this implies that $\mathcal{R}_+(f_1-f_2,0)=0$ in $\mathbb{R}\times\mathbb{O}$, and so Corollary 3.7 implies that $f_1=f_2$.

If $v\in L^2_{\mathrm{ac}}(X)$ is such that $\langle v,h\rangle_{L^2(X)}=0$ for all $h\in\mathscr{C}(\Gamma)$ then, since $\mathcal{R}_+$ is unitary, for all $(f,0)\in E_{\mathrm{ac}}(X)$,

$$\langle v,h\rangle_{L^2(X)} = \langle \mathcal{R}_+(0,v), \mathcal{R}_+(f,h)\rangle_{L^2(\mathbb{R}\times\partial X)}$$

Since $h\in\mathscr{C}(\Gamma)$ is arbitrary, it follows that

$$\langle \mathcal{R}_+(0,v), F\rangle_{L^2(\mathbb{R}\times\partial X)}=0 \quad \text{for all } F\in L^2(\mathbb{R}\times\Gamma).$$

Hence $\mathcal{R}_+(0,v)=0$ on $\mathbb{R}\times\Gamma$ and, by Corollary 3.7, $v=0$. □

Lemma 4.4. *If $\Gamma\subset\partial X$ is open, nonempty and $\partial X\setminus\Gamma$ contains an open subset, then the map $\mathcal{L}$ is injective and has dense range.*

Proof. If $F=\mathcal{R}_+(f,h)\in L^2(\mathbb{R}\times\Gamma)$, then $\mathcal{L}F=\mathcal{R}_+(0,h)|_{\mathbb{R}\times\Gamma}$. If $\mathcal{L}F=0$ then $\mathcal{R}_+(0,h)=0$ on $\mathbb{R}\times\Gamma$. It follows from Corollary 3.7 that $h=0$, and hence $F=\mathcal{R}(f,0)$. Since there exists an open subset $\mathbb{O}\subset(\partial X\setminus\Gamma)$, and F is supported in $\mathbb{R}\times\overline{\Gamma}$, it follows that $F=\mathcal{R}_+(f,0)=0$ in $\mathbb{R}\times\mathbb{O}$, and again by Corollary 3.7, $f=0$ and so $F=0$.

Now we prove that its range is dense. Let $H \in L^2(\mathbb{R}\times\Gamma)$ be orthogonal to the range of $\mathscr{L}$. Suppose that $H = \mathcal{R}_+(h_1, h_2)$, with $(h_1, h_2) \in E_{\mathrm{ac}}(X)$. Then for every $F = \mathcal{R}_+(f, h) \in L^2(\mathbb{R}\times\Gamma)$, $h \in \mathscr{C}(\Gamma)$,

$$\begin{aligned} 0 = \langle H, (F + \mathscr{S}F^*)|_{\mathbb{R}\times\Gamma}\rangle_{L^2(\mathbb{R}\times\Gamma)} = \langle H, F + \mathscr{S}F^*\rangle_{L^2(\mathbb{R}\times\Gamma)} &= \langle H, \mathcal{R}_+(0, h)\rangle_{L^2(\mathbb{R}\times\partial X)} \\ &= \langle \mathcal{R}_+(h_1, h_2), \mathcal{R}_+(0, h)\rangle_{L^2(\mathbb{R}\times\partial X)} \\ &= \langle h_2, h\rangle_{L^2(X)}. \end{aligned}$$

Since $\mathscr{C}(\Gamma)$ is dense in $L^2_{\mathrm{ac}}(X)$, $h_2 = 0$. Hence $H = \mathcal{R}_+(h_1, 0) = 0$ on $\mathbb{R}\times\mathbb{O}$, and so $H = 0$. □

We shall let

$$\mathscr{F}^+(\Gamma) = \mathscr{L}(L^2(\mathbb{R}\times\Gamma)) = \{\mathcal{R}_+(0, f)|_{\mathbb{R}\times\Gamma} : f \in \mathscr{C}(\Gamma)\}, \tag{4-4}$$

and equip $\mathscr{F}^+(\Gamma)$ with the norm given by Lemma 4.1,

$$\mathbb{C}^n_+(\mathcal{R}_+(0, f)) = \|f\|_{L^2(X)}.$$

Thus $(\mathscr{F}^+(\Gamma), \mathbb{C}^n_+)$ is a normed vector space and, since $\mathscr{C}(\Gamma)$ is dense in $L^2(X)$, $\mathscr{F}^+(\Gamma)$ is dense in $(\mathscr{M}^+(\Gamma), \mathbb{C}^n_+)$. Hence $(\mathscr{M}^+(\Gamma), \mathbb{C}^n_+)$ is the completion of $(\mathscr{F}^+(\Gamma), \mathbb{C}^n_+)$ into a Hilbert space, and therefore it is determined by $\mathscr{S}_\Gamma$. Notice that the completion of $\mathscr{F}^+(\Gamma)$ with the $L^2(\mathbb{R}\times\Gamma)$-norm is $L^2(\mathbb{R}\times\Gamma)$. But

$$\|\mathcal{R}_+(0, h)|_{(\mathbb{R}\times\Gamma)}\|_{L^2(\mathbb{R}\times\Gamma)} \le \|h\|_{L^2(X)},$$

so $\mathbb{C}^n_+$ is a stronger norm and $(\mathscr{M}^+(\Gamma), \mathbb{C}^n_+)$ is a smaller space. This ends the proof of Theorem 4.2. □

5. Proof of Theorem 2.3

The operators $\mathscr{S}_{j,\Gamma}$, $j = 1, 2$ were defined in terms of boundary-defining functions for which (2-14) holds for both g_1 and g_2 in $U_j \sim [0, \varepsilon)\times\partial X_j$. In particular,

$$\Psi_j^* g_j = \frac{dx^2}{x^2} + \frac{h_j(x)}{x^2} \quad \text{on } (0, \varepsilon)\times\Gamma, \quad h_1(0) = h_2(0) = h_0 \text{ on } \Gamma. \tag{5-1}$$

Our first step will be to prove that there exists $\varepsilon > 0$ such that the tensors $h_1(x)$ and $h_2(x)$ are equal on $[0, \varepsilon)\times\Gamma$. Once this is done, if $\Psi_j : [0, \varepsilon)\times\partial X_j \to U_j$, $j = 1, 2$, are the maps that satisfy (2-14), and if $W_{1,\varepsilon} = \Psi_1([0, \varepsilon)\times\Gamma)$, $W_{2,\varepsilon} = \Psi_{2,\varepsilon}([0, \varepsilon)\times\Gamma)$, then

$$\Psi_1^*(g_1|_{W_{1,\varepsilon}}) = \Psi_2^*(g_2|_{W_{2,\varepsilon}}) \quad \text{on } [0, \varepsilon)\times\Gamma. \tag{5-2}$$

Since $\Psi_j = \mathrm{Id}$ on Γ, $j = 1, 2$, this implies that

$$\Psi_\varepsilon = \Psi_2\circ\Psi_1^{-1} : W_{1,\varepsilon} \mapsto W_{2,\varepsilon}, \quad (\Psi_2\circ\Psi_1)^{-1} g_2 = g_1, \quad \Psi_\varepsilon = \mathrm{Id} \text{ on } \Gamma \tag{5-3}$$

gives an isometry between neighborhoods of Γ.

The local diffeomorphism. We will prove that, if $h_j(x)$ are such that (5-1) holds, then $h_1(x) = h_2(x)$ on $[0, \varepsilon) \times \Gamma$, and hence this gives the map Ψ_ε defined in (5-3). Our first step in this construction will be:

Proposition 5.1. *Let (X_1, g_1), (X_2, g_2) and Γ satisfy the hypotheses of Theorem 2.3, and denote by $\mathcal{R}_{j,\pm}(s, y, x', y')$ the Schwartz kernels of $\mathcal{R}_{j,\pm}$ acting on $(0, f)$. Then there exists $\epsilon > 0$ such that* (2-14) *holds on $[0, \varepsilon) \times \partial X_j$, $j = 1, 2$, and*

$$\begin{aligned} h_1(x, y, dy) &= h_2(x, y, dy) && \text{if } x \in [0, \varepsilon),\ y \in \Gamma, \\ \mathcal{R}_{1,\pm}(s, y, x', y') &= \mathcal{R}_{2,\pm}(s, y, x', y') && \text{if } y, y' \in \Gamma,\ x' < \varepsilon. \end{aligned} \tag{5-4}$$

Proof. The proof of Proposition 5.1 is an adaptation of the boundary control method of [Belishev 1987; Belishev and Kurylev 1992] to this setting. By working on an open subset of Γ if necessary, we may assume that $\partial X \setminus \Gamma$ does not have empty interior. As in [Sá Barreto 2005], pick $x_1 < \varepsilon$ and consider the spaces

$$\begin{aligned} \mathcal{M}^+_{x_1}(\Gamma) &= \{F \in \mathcal{M}^+(\Gamma) : F(s, y) = 0,\ s \le \log x_1\}, \\ \mathcal{M}^-_{x_1}(\Gamma) &= \{F \in \mathcal{M}^-(\Gamma) : F(s, y) = 0,\ s \ge -\log x_1\}, \end{aligned}$$

and let

$$\mathcal{P}^+_{x_1} : \mathcal{M}^+(\Gamma) \to \mathcal{M}^+_{x_1}(\Gamma) \quad \text{and} \quad \mathcal{P}^-_{x_1} : \mathcal{M}^-(\Gamma) \to \mathcal{M}^-_{x_1}(\Gamma) \tag{5-5}$$

denote the respective orthogonal projections with respect to the norms $\mathbb{C}^n_\pm$ defined in (4-1). Since $\mathcal{M}^\pm(\Gamma)$ and $\mathcal{M}^\pm_{x_1}(\Gamma)$ are determined by $\mathcal{S}_\Gamma$, the projections $\mathcal{P}^\pm_{x_1}$ are also determined by $\mathcal{S}_\Gamma$. Notice that $(\mathcal{P}^+_{x_1} F)(s, y)$ is not necessarily equal to $H(s - \log x_1)F(s, y)$, where H is the Heaviside function, as $H(s - \log x_1)F(s, y)$ may not be in $\mathcal{M}^+(\Gamma)$.

In view of finite speed of propagation and Theorem 2.1,

$$\begin{aligned} \mathcal{M}^+_{x_1}(\Gamma) &= \{\mathcal{R}_+(0, h)|_{\mathbb{R}\times\Gamma} : h \in L^2_{\mathrm{ac}}(X),\ h(z) = 0,\ z \in \mathcal{D}_{\log x_1}(\Gamma)\}, \\ \mathcal{M}^-_{x_1}(\Gamma) &= \{\mathcal{R}_-(0, h)|_{\mathbb{R}\times\Gamma} : h \in L^2_{\mathrm{ac}}(X),\ h(z) = 0,\ z \in \mathcal{D}_{\log x_1}(\Gamma)\}. \end{aligned}$$

As in [Sá Barreto 2005], the key to proving Proposition 5.1 is to understand the effect of the projectors $\mathcal{P}^\pm_{x_1}$ on the initial data. First we deal with the case where Δ_{g_j}, $j = 1, 2$, have no eigenvalues. In this case, $L^2(X_j) = L^2_{\mathrm{ac}}(X_j)$.

Lemma 5.2. *Let (X, g) be an asymptotic hyperbolic manifold such that Δ_g has no eigenvalues. Let x be such that* (2-4) *holds in $(0, \epsilon) \times \partial X$. For $x_1 \in (0, \epsilon)$, let $\mathcal{P}^+_{x_1}$ denote the orthogonal projector defined in* (5-5)*. Let χ_{x_1} be the characteristic function of the set $X_{x_1} = X \setminus \mathcal{D}_{\log x_1}(\Gamma)$. Then, for every $f \in L^2_{\mathrm{ac}}(X) = L^2(X)$,*

$$\mathcal{P}^+_{x_1}(\mathcal{R}_+(0, f)|_{\mathbb{R}\times\Gamma}) = \mathcal{R}_+(0, \chi_{x_1} f)|_{\mathbb{R}\times\Gamma}.$$

Proof. Since $\mathcal{P}^+_{x_1}$ is a projector, there exists $f_{x_1} \in L^2(X)$ such that

$$\mathcal{P}^+_{x_1}(\mathcal{R}_+(0, f)|_{\mathbb{R}\times\Gamma}) = \mathcal{R}_+(0, f_{x_1})|_{\mathbb{R}\times\Gamma}$$

and, for every $h \in L^2(X)$ supported in X_{x_1},

$$\big\langle \mathcal{R}_+(0, f_{x_1})|_{\mathbb{R}\times\Gamma}, \mathcal{R}_+(0, h)|_{\mathbb{R}\times\Gamma} \big\rangle_{\mathbb{C}^n_+} = \langle f_{x_1}, h \rangle_{L^2(X)} = \langle f, h \rangle_{L^2(X)}.$$

Hence $f_{x_1} = \chi_{x_1} f$. □

Next we will analyze the singularities $\mathcal{R}_+(0, \chi_{x_1} f)$ at $\{s = \log x_1\}$ and, as in the proof of Proposition 3.2, we may assume that f is C^∞. In the case where $\Gamma = \partial X$, χ_{x_1} is the characteristic function of the set $\{x \geq x_1\}$ and the singularities of $\mathcal{R}_+(0, \chi_{x_1} f)$ can be computed using the plane wave expansion of the solution to the Cauchy problem

$$PV = 0, \quad V|_{s=\log x} = 0 \quad \text{and} \quad \partial_s V|_{s=\log x} = f(x, y)\chi_{x_1}, \tag{5-6}$$

where P is the operator defined in (3-2). In this case, one just writes

$$V(x, s, y) = V^+(x, s, y) + V^-(x, s, y), \quad \text{where}$$
$$V^+(x, s, y) \sim \sum_{j=1}^{\infty} v_j^+(x, y)(s - \log x_1)_+^j \quad \text{and} \quad V^-(x, s, y) \sim \sum_{j=1}^{\infty} v_j^-(x, y)(2\log x - x_1 - s)_+^j,$$

where $s = \log x_1$ and $s = 2\log x + \log x_1$ correspond to the forward and backward waves emanating from $\{x = x_1,\ s = \log x\}$. One then computes the coefficients of the expansion by using a series of transport equations. The wave $V^-(x, s, y)$ goes towards the interior and will hit $\{x = 0\}$ for $s > \log x_1$, but the wave $V^+(x, s, y)$ will intersect $\{x = 0\}$ at $s = \log x_1$. The first coefficient in the expansion of $V^+(x, s, y)$ is given by $v_1^+(x, y) = \frac{1}{2}(|h|^{1/4}(x_1, y)/|h|^{1/4}(x, y))x_1^{-n/2-1} f(x_1, y)$. Since (3-16) is well defined for L^2_{ac} initial data, $\mathcal{R}_+(0, \chi_{x_1} f) = \partial_s V(x, s, y)|_{\{x=0\}}$, and hence near $\{s = \log x_1\}$ one has an expansion

$$\mathcal{R}_+(0, \chi_{x_1} f) \sim \frac{1}{2}\frac{|h|^{1/4}(x_1, y)}{|h|^{1/4}(0, y)} x_1^{-n/2-1} f(x_1, y)(s - \log x_1)_+^0 + \sum_{j=1}^{\infty} v_j(0, y)^+(s - \log x_1)_+^j. \tag{5-7}$$

We refer the reader to the proof of Lemma 8.9 of [Sá Barreto 2005] for details.

In the case studied here, when $\Gamma \neq \partial X$, this is not so clear since χ_{x_1} is the characteristic function of $X_{x_1} = X \setminus \mathcal{D}_{\log x_1}(\Gamma)$, which is a more complicated set. However, if x_1 is small enough, the boundary of X_{x_1} contains $\Gamma_{x_1} = \{(x_1, y) : y \in \Gamma\}$. We will show that the singularities of $\mathcal{R}_+(0, \chi_{x_1} f)$ at $\{s = \log x_1,\ y \in \Gamma\}$ can be computed as in the previous case. The singularities of $\chi_{x_1} f$ lie on the set

$$\partial\mathcal{D}_{\log x_1} = \{z \in \mathring{X} : \text{there exists } (\bar{x}, \bar{y}) \in U_{\bar{\varepsilon}} \text{ such that } d_g(z, (\bar{x}, \bar{y})) = \log x_1 - \log \bar{x}\}$$

Since $\mathring{X}$ is complete, there exists a geodesic γ joining $z \in \partial\mathcal{D}_{\log x_1}$ and $(\bar{x}, \bar{y})$ such that

$$\gamma(0) = z, \quad \gamma(\bar{t}) = (\bar{x}, \bar{y}) \quad \text{and} \quad \bar{t} = d_g(z, (\bar{x}, \bar{y})).$$

One can think of this in terms of the wave equation with γ being the projection of a null bicharacteristic of $p = \frac{1}{2}(\tau^2 - x^2\xi^2 - x^2 h(x, y, \eta))$ in $\{p = 0,\ \tau = 1\}$ starting at z and going to $(\bar{x}, \bar{y})$. If one then sets $s = t + \log x$ it follows that, along this bicharacteristic, $s = t + \log x(\gamma(t))$. Hence, at $\hat{t}$, $s(\bar{t}) = \log x_1$. In these coordinates (we are using ξ by abuse of notation but we should use $\tilde{\xi}$, where $\tilde{\xi} = \xi - \tau/x$),

$$\{p = 0,\ \tau = 1\} = \left\{p = \sigma\xi + \tfrac{1}{2}x\xi^2 + \tfrac{1}{2}xh(x, y, \eta) = 0,\ \sigma = 1\right\}$$

and we have that, for $1+x\xi \neq 0$,

$$\frac{ds}{dx}=\frac{\xi}{1+x\xi},\quad \frac{d\xi}{dx}=-\frac{\xi^2+h+x\partial_x h}{2(1+x\xi)},\quad \frac{d\eta}{dx}=-\frac{x\partial_y h}{2(1+x\xi)},\quad \frac{dy}{dx}=-\frac{x\partial_\eta h}{2(1+x\xi)}.$$

So, unless $\xi=\eta=0$, $ds/dx \neq 0$. But, if $\xi=\eta=0$ at a point then, by uniqueness, $\xi=\eta=0$ along the curve. In the latter case $s=\log x_1$, $y=\bar{y}\in\Gamma$ along the curve. If $\xi\neq 0$, the geodesic will reach $\{x=0\}$ for $s\neq \log x_1$. So we conclude that (5-7) holds for $y\in\Gamma_1$, where $\overline{\Gamma}_1$ is a compact subset of Γ. The precise propagation of singularities is given by:

Lemma 5.3. *Let x be a defining function of ∂X such that* (2-4) *holds. Let $\mathcal{M}^+(\Gamma)\ni F=\mathcal{R}_+(0,f)|_{\mathbb{R}\times\Gamma}$ with f smooth. Let $\Theta(x_1,s,y)=\frac{1}{2}x_1^{-n/2-1}f(x_1,y)(|h|^{1/4}(x_1,y)/|h|^{1/4}(0,y))(s-\log x_1)_+^0$. There exists $\varepsilon>0$ such that, for any $x_1\in(0,\varepsilon)$,*

$$\mathcal{P}^+_{x_1}F(s,y)-\Theta(x_1,s,y)\in H^1_{\mathrm{loc}}(\mathbb{R}\times\Gamma). \tag{5-8}$$

Since $\mathcal{P}^+_{x_1}$ and $\mathcal{M}^+(\Gamma)$ are determined by $\mathcal{S}_\Gamma$ in view of (5-8), $\Theta(x_1,s,y)$ is determined by $\mathcal{S}_\Gamma$ provided $x_1\in(0,\varepsilon)$ and $y\in\Gamma$. By assumption in Theorem 2.3, $h_{0,1}=h_{0,2}$ on Γ. Therefore, $|h_1|(0,y)=|h_2|(0,y)$, $y\in\Gamma$ and, since $F=\mathcal{R}_+(0,f)|_{\mathbb{R}\times\Gamma}$ in Lemma 5.3, we obtain the following result:

Corollary 5.4. *Let (X_1,g_1) and (X_2,g_2) be asymptotically hyperbolic manifolds satisfying the hypothesis of Theorem 2.3. Moreover, assume that Δ_{g_j}, $j=1,2$, have no eigenvalues. Let $\mathcal{R}_{j,\pm}$, $j=1,2$, denote the corresponding forward or backward radiation fields defined in coordinates in which* (2-4) *holds. Then there exists an $\epsilon>0$ such that, for $(x,y)\in[0,\epsilon)\times\Gamma$,*

$$\begin{aligned}|h_1|^{1/4}(x,y)\mathcal{R}^{-1}_{1,-}F(x,y)&=|h_2|^{1/4}(x,y)\mathcal{R}^{-1}_{2,-}F(x,y)\quad \textit{for all } F\in\mathcal{M}^-(\Gamma),\\ |h_1|^{1/4}(x,y)\mathcal{R}^{-1}_{1,+}F(x,y)&=|h_2|^{1/4}(x,y)\mathcal{R}^{-1}_{2,+}F(x,y)\quad \textit{for all } F\in\mathcal{M}^+(\Gamma).\end{aligned} \tag{5-9}$$

Proposition 5.1 easily follows from this result. Indeed, since

$$\mathcal{R}^{-1}_{j,-}\left(\frac{\partial^2}{\partial s^2}F\right)=\left(\Delta_{g_j}-\tfrac{1}{4}n^2\right)\mathcal{R}^{-1}_{j,-}F, \tag{5-10}$$

if we apply Corollary 5.4 to $\partial_s^2 F$ we obtain

$$|h_1|^{1/4}(x,y)\left(\Delta_{g_1}-\tfrac{1}{4}n^2\right)\mathcal{R}^{-1}_{1,-}F(x,y)=|h_2|^{1/4}(x,y)\left(\Delta_{g_2}-\tfrac{1}{4}n^2\right)\mathcal{R}^{-1}_{2,-}F(x,y). \tag{5-11}$$

If $\mathcal{R}^{-1}_{1,-}F=(0,f)$, where $F\in\mathcal{M}(\Gamma)^-$ is arbitrary and the metrics have no eigenvalues, equations (5-9) and (5-11) give

$$|h_1|^{1/4}(x,y)\left(\Delta_{g_1}-\tfrac{1}{4}n^2\right)f(x,y)=|h_2|^{1/4}(x,y)\left(\Delta_{g_2}-\tfrac{1}{4}n^2\right)\frac{|h_1|^{1/4}(x,y)}{|h_2|^{1/4}(x,y)}f(x,y) \tag{5-12}$$

for all $f\in C_0^\infty((0,\epsilon)\times\Gamma)\cap L^2_{\mathrm{ac}}(X)$. Therefore the operators on both sides of (5-12) are equal. In particular, the coefficients of the principal parts of Δ_{g_1} are equal to those of Δ_{g_2}, and hence the tensors h_1 and h_2 from (2-4) are equal. This proves that

$$\begin{aligned}\mathcal{R}^{-1}_{1,-}(s,y,x',y')&=\mathcal{R}^{-1}_{2,-}(s,y,x',y'),\quad && y,y'\in\Gamma,\ x'\in[0,\varepsilon),\\ h_1(x,y,dy)&=h_2(x,y,dy), && y\in\Gamma,\ x\in[0,\varepsilon),\end{aligned} \tag{5-13}$$

and of course the same holds for the forward radiation field. Since $\mathcal{R}_\pm$ are unitary, $\mathcal{R}_\pm^{-1} = \mathcal{R}_\pm^*$, and hence this determines the kernel of $\mathcal{R}_\pm$. This proves Proposition 5.1 in the case of no eigenvalues.

Now we remove the assumption that there are no eigenvalues. We need to show that, if $\mathcal{S}_{1,\Gamma} = \mathcal{S}_{2,\Gamma}$, then the eigenvalues of Δ_{g_1} and Δ_{g_2} are equal, and the eigenfunctions can be reordered in such a way that their traces are equal on Γ. In fact they agree to infinite order at Γ. To show that, we need to appeal to the stationary version of scattering theory, and we have to recall the relationship between the scattering operator, the scattering matrix and the resolvent from [Sá Barreto 2005]. It was shown in [Joshi and Sá Barreto 2000] that $\mathcal{A}(\lambda)$, defined in (2-10), continues meromorphically to $\mathbb{C} \setminus D$, where D is a discrete set. The eigenvalues of Δ_g correspond to poles of $\mathcal{A}(\lambda)$ on the negative imaginary axis. Proposition 3.6 of [Graham and Zworski 2003] states that, if $\lambda_0 \in i\mathbb{R}_-$ is such that $\frac{1}{4}n^2 + \lambda_0^2$ is an eigenvalue of Δ_g, then the scattering matrix $\mathcal{A}(\lambda)$ has a pole at λ_0 and its residue is given by

$$\operatorname{Res}_{\lambda_0} A(\lambda) = \begin{cases} \Pi_{\lambda_0} & \text{if } \; -i\lambda_0 \notin \frac{1}{2}\mathbb{N}, \\ \Pi_{\lambda_0} - P_l & \text{if } \; -i\lambda_0 = \frac{1}{2}l, \; l \in \mathbb{N}, \end{cases} \tag{5-14}$$

where P_l is a differential operator whose coefficients depend on derivatives of the tensor h at ∂X, and the Schwartz kernel of Π_{λ_0} is

$$K(\Pi_{\lambda_0})(y, y') = -2i\lambda_0 \sum_{j=1}^{N_0} \phi_j^0 \otimes \phi_j^0(y, y'), \tag{5-15}$$

where N_0 is the multiplicity of the eigenvalue $\frac{1}{4}n^2 + \lambda_0^2$, the ϕ_j, $1 \le j \le N_0$, are the corresponding orthonormalized eigenfunctions and $\phi_j^0(y)$ is defined by

$$\phi_j^0(y) = x^{-n/2-\lambda_0}\phi_j(x, y)|_{x=0}. \tag{5-16}$$

Since $A_{1,\Gamma} = A_{2,\Gamma}$, $\lambda \in \mathbb{R} \setminus 0$, it follows from Theorem 1.2 of [Joshi and Sá Barreto 2000] that, in coordinates where (2-14) is satisfied, all derivatives of h_1 and h_2 agree at $x = 0$ on Γ. Therefore the operators $P_{l,j}$ in (5-14) corresponding to (X_j, g_j) are the same in Γ. Then (5-14), (5-15), and the meromorphic continuation of the scattering matrix show that Δ_{g_1} and Δ_{g_2} have the same eigenvalues with the same multiplicity. Moreover, (5-15) implies that if ϕ_j and ψ_j, $1 \le j \le N_0$, are orthonormal sets of eigenfunctions of Δ_{g_1} and Δ_{g_2}, respectively, corresponding to the eigenvalue $\frac{1}{4}n^2 + \lambda_0^2$, then there exists a constant orthogonal $(N_0 \times N_0)$-matrix A such that $\Phi^0\big|_\Gamma = A\Psi^0\big|_\Gamma$, where $(\Phi^0)^T = (\phi_1^0, \phi_2^0, \dots, \phi_{N_0}^0)$ and $(\Psi^0)^T = (\psi_1^0, \psi_2^0, \dots, \psi_{N_0}^0)$. So, by redefining one set of eigenfunctions from, let us say, Ψ to $A\Psi$, where $\Psi^T = (\psi_1, \psi_2, \dots, \psi_{N_0})$, we may assume that

$$\phi_j^0(y) = \psi_j^0(y), \quad y \in \Gamma, \quad j = 1, 2, \dots, N_0. \tag{5-17}$$

Note that this does not change the orthonormality of the eigenfunctions in X_2 because A is orthogonal. Denote the eigenvalues of Δ_{g_1} and Δ_{g_2}, which we know are equal, by

$$\mu_j = \tfrac{1}{4}n^2 + \lambda_j^2, \quad \lambda_j \in i\mathbb{R}_-, \quad 1 \le j \le N. \tag{5-18}$$

They are also ordered so that $\mu_1 \le \mu_2 \le \dots \le \mu_N$.

Again, we use that the singularities of $\chi_{x_1} f$ at Γ_{x_1} produce the singularities of $\mathcal{R}_+(0, \chi_{x_1} f)$ at $\{s = \log x_1, y \in \Gamma\}$ and expand the solution to (2-1) with initial data, $(0, \chi_{x_1} f)$. However, in this case $L^2(X) \neq L^2_{\mathrm{ac}}(X)$ and hence Lemma 5.2 is not valid, and we have to replace it by the following:

Lemma 5.5. *Let (X, g) be an asymptotic hyperbolic manifold and let ϕ_j, $1 \le j \le N$, denote the orthonormal set of eigenfunctions of Δ_g. Let x be such that* (2-4) *holds in $(0, \epsilon) \times \partial X$. For $x_1 \in (0, \epsilon)$, let $\mathcal{P}^+_{x_1}$ denote the orthogonal projector defined in* (5-5). *Let χ_{x_1} be the characteristic function of the set $X_{x_1} = X \setminus \mathcal{D}_{\log x_1}(\Gamma)$. There exists ε_0 such that, if $\varepsilon < \varepsilon_0$, then for every $f \in L^2_{\mathrm{ac}}(X)$ there exists $\alpha(x_1, f)$, which is a linear function of f, such that*

$$\mathcal{P}^+_{x_1}(\mathcal{R}_+(0, f)|_{\mathbb{R}\times\Gamma}) = \mathcal{R}_+\Bigl(0, \chi_{x_1}\Bigl(f - \sum_{j=1}^{N} \alpha_j(x_1, f)\phi_j\Bigr)\Bigr)\Big|_{\mathbb{R}\times\Gamma}.$$

Proof. Let $h \in L^2_{\mathrm{ac}}(X)$ be supported in X_{x_1}. This means that $\langle h, \chi_{x_1}\phi_j\rangle = 0$ for $1 \le j \le N$. Then, since $\mathcal{P}^+_{x_1}$ is a projector, there exists $f_{x_1} \in L^2_{\mathrm{ac}}(X)$, supported in X_{x_1}, such that $\mathcal{P}^+_{x_1}(\mathcal{R}_+(0, f)|_{\mathbb{R}\times\Gamma}) = \mathcal{R}_+(0, f_{x_1})|_{\mathbb{R}\times\Gamma}$ and, for every $h \in L^2_{\mathrm{ac}}(X)$ supported in X_{x_1},

$$\bigl\langle \mathcal{R}_+(0, f_{x_1})|_{\mathbb{R}\times\Gamma}, \mathcal{R}_+(0, h)|_{\mathbb{R}\times\Gamma}\bigr\rangle_{\mathbb{C}^n_+} = \langle f_{x_1}, h\rangle_{L^2(X)} = \langle f, h\rangle_{L^2(X)}.$$

Hence $\langle (f_{x_1} - f), h\rangle = 0$ for all $h \in C_0^\infty(X) \cap L^2_{\mathrm{ac}}(X)$ supported in X_{x_1}. We claim that there exist $\alpha_j = \alpha_j(x_1, f) \in \mathbb{C}$ such that

$$f_{x_1} - \chi_{x_1} f - \chi_{x_1} \sum_{j=1}^{N} \alpha_j \phi_j = 0 \quad \text{for } x_1 \text{ small enough.}$$

If such a formula were to hold, since $\langle f_{x_1}, \chi_{x_1}\phi_j\rangle = 0$ one would have to have

$$\langle f, \chi_{x_1}\phi_k\rangle_{L^2(X)} = \sum_{j=1}^{N} \alpha_j \langle \chi_{x_1}\phi_j, \chi_{x_1}\phi_k\rangle_{L^2(X)}.$$

This gives a linear system of equations

$$M\alpha = F, \quad \alpha^T = (\alpha_1, \dots, \alpha_N), \quad F^T = (F_1(x_1), \dots, F_N(x_1)),$$
$$M_{jk}(x_1) = \langle \chi_{x_1}\phi_j, \chi_{x_1}\phi_k\rangle_{L^2(X)}, \quad F_k(x_1) = \langle f, \chi_{x_1}\phi_k\rangle_{L^2(X)}.$$

Since the eigenfunctions are orthonormal, for $x_1 = 0$ we have $M_{jk}(0) = \delta_{jk}$. Therefore, there exists $\varepsilon_0 > 0$, which depends on the matrix M, and hence only on the eigenfunctions and not on f, such that the system has a solution if $x_1 < \varepsilon_0$. Notice that, since $f \in L^2_{\mathrm{ac}}(X)$, for $x_1 = 0$ we have $F_k(0) = 0$, and hence $\alpha(0, f) = 0$.

With this choice of α_j, the function

$$G = f_{x_1} - \chi_{x_1} f - \chi_{x_1} \sum_{j=1}^{N} \alpha_j \phi_j$$

is supported in X_{x_1} and $\langle G, \phi_j\rangle_{L^2(X)} = 0$, so $G \in L^2_{\mathrm{ac}}(X)$. But at the same time $\langle F, h\rangle_{L^2(X)} = 0$ for all $h \in L^2_{\mathrm{ac}}(X)$ supported in X_{x_1}. Therefore $\langle G, G\rangle_{L^2(X)} = 0$, and so $G = 0$. ☐

As in [Sá Barreto 2005], we shall denote

$$T(x_1)f = \sum_j \alpha_j(x_1, f)\phi_j.$$

Since $\alpha(0, f) = 0$, $T(0) = 0$. Therefore one can pick ε small so that

$$\|T(x_1)\| < \tfrac{1}{2} \quad \text{for } x_1 < \varepsilon. \tag{5-19}$$

In this case, Lemma 5.3 and Corollary 5.4 have to be substituted by:

Lemma 5.6. *Let (X, g) be an asymptotically hyperbolic manifold, and let x be a defining function of ∂X such that* (2-4) *holds. Let ϕ_j, $1 \le j \le N$, denote the eigenfunctions of Δ_g and let $T(x_1)$ be defined as above. Let $F \in \mathcal{M}^+(\Gamma)$, $F = \mathcal{R}_+(0, f)|_{\mathbb{R}\times\Gamma}$ with f smooth and let*

$$\Xi(x_1, s, y) = \tfrac{1}{2} x_1^{-n/2-1} \frac{|h|^{1/4}(x_1, y)}{|h|^{1/4}(0, y)} \big[(\mathrm{Id} - T(x_1))f\big](x_1, y)(s - \log x_1)_+^0.$$

There exists $\varepsilon > 0$ such that, for any $x_1 \in (0, \varepsilon)$,

$$\mathcal{P}^+_{x_1} F(s, y) - \Xi(x_1, s, y) \in H^1_{\mathrm{loc}}(\mathbb{R}\times\Gamma). \tag{5-20}$$

Corollary 5.7. *Let (X_1, g_1) and (X_2, g_2) be asymptotically hyperbolic manifolds satisfying the hypothesis of Theorem 2.3. Let $\mathcal{R}_{j,\pm}$, $j = 1, 2$, denote the corresponding forward or backward radiation fields defined in coordinates in which* (2-4) *holds. Then there exists an $\varepsilon > 0$ such that, for $(x, y) \in (0, \varepsilon)\times\Gamma$,*

$$\begin{aligned} |h_1|^{1/4}(x, y)(\mathrm{Id} - T_1(x))\mathcal{R}^{-1}_{1,-}F(x, y) &= |h_2|^{1/4}(x, y)(\mathrm{Id} - T_2(x))\mathcal{R}^{-1}_{2,-}F(x, y) \quad \textit{for all } F \in \mathcal{M}^-(\Gamma),\\ |h_1|^{1/4}(x, y)(\mathrm{Id} - T_1(x))\mathcal{R}^{-1}_{1,+}F(x, y) &= |h_2|^{1/4}(x, y)(\mathrm{Id} - T_2(x))\mathcal{R}^{-1}_{2,+}F(x, y) \quad \textit{for all } F \in \mathcal{M}^+(\Gamma). \end{aligned} \tag{5-21}$$

We write $\mathcal{R}^{-1}_{j,-}F(x, y) = f_j(x, y)$, and pick ε small so that (5-19) holds. We apply (5-21) to f_1 and f_2 and to $\big(\Delta_{g_1} - \frac{1}{4}n^2\big)f_1$ and $\big(\Delta_{g_2} - \frac{1}{4}n^2\big)f_2$ for $(x, y) \in [0, \varepsilon)\times\Gamma$ and find that

$$\begin{gathered} |h_1(x)|^{1/4}(\mathrm{Id} - T_1(x))f_1 = |h_2(x)|^{1/4}(\mathrm{Id} - T_2(x))f_2,\\ |h_1(x)|^{1/4}(\mathrm{Id} - T_1(x))\big(\Delta_{g_1} - \tfrac{1}{4}n^2\big)f_1(x, y) = |h_2(x)|^{1/4}(\mathrm{Id} - T_2(x))\big(\Delta_{g_2} - \tfrac{1}{4}n^2\big)f_2(x, y). \end{gathered} \tag{5-22}$$

Therefore,

$$f_2(x, y) = (\mathrm{Id} - T_2(x))^{-1}\frac{|h_1|^{1/4}}{|h_2|^{1/4}}(\mathrm{Id} - T_1(x))f_1(x, y) = \frac{|h_1|^{1/4}}{|h_2|^{1/4}}f_1(x, y) + K(x)f_1(x, y),$$

where K is a compact operator. If one substitutes this into the second equation in (5-22), one obtains

$$|h_1|^{1/4}(\mathrm{Id} - T_1)\big(\Delta_{g_1} - \tfrac{1}{4}n^2\big)f_1 = |h_2|^{1/4}(\mathrm{Id} - T_2)\big(\Delta_{g_2} - \tfrac{1}{4}n^2\big)\bigg(\frac{|h_1|^{1/4}}{|h_2|^{1/4}}f_1 + Kf_1\bigg)$$

Hence,

$$\big(\Delta_{g_1} - \tfrac{1}{4}n^2\big)f_1(x, y) - \frac{|h_2|^{1/4}}{|h_1|^{1/4}}\big(\Delta_{g_2} - \tfrac{1}{4}n^2\big)\bigg(\frac{|h_1|^{1/4}}{|h_2|^{1/4}}f_1\bigg)(x, y) = (\mathcal{K}f_1)(x, y),$$

where $\mathcal{K}$ is a compact operator. Since the operator on the left-hand side is a differential operator, and the operator on the right-hand side is compact, they both must be equal to zero. As above, we conclude that in coordinates (x, y), the coefficients of the operators Δ_{g_1} are equal to those of Δ_{g_2}. Hence, we must have $h_1(x, y, dy) = h_2(x, y, dy)$.

We still have to show that (5-4) holds in the case where eigenvalues exist. Let $F \in \mathcal{M}^+(\Gamma)$, and let $f_j = \mathcal{R}_{j,+}^{-1} F$. Let v_j satisfy (2-1) with initial data $(0, f_j)$. Let $V_j(x, s, y) = x^{-n/2} v_j(s - \log x, x, y)$. Since $\mathcal{R}_+(0, f_j) = F$, we have $\partial_s V_j(0, s, y) = F$. Since $\Delta_{g_1} = \Delta_{g_2}$ in $(0, \varepsilon) \times \Gamma$, for P as defined in (3-2),

$$\begin{gathered} P(V_1 - V_2) = 0 \quad \text{in } \log x < s,\ x < \varepsilon,\ y \in \Gamma \\ (V_1 - V_2)(x, \log x, y) = 0, \quad \partial_s(V_1 - V_2)(x, \log x, y) = f_1(x, y) - f_2(x, y) \text{ on } x < \varepsilon,\ y \in \Gamma, \\ \partial_s(V_1 - V_2)(0, s, y) = 0, \quad y \in \Gamma,\ s \in \mathbb{R}. \end{gathered} \tag{5-23}$$

Now we apply Propositions 3.2, 3.3 and 3.4 as in the proof of Theorem 2.1, to conclude that there exists s^* such that

$$V_1(x, s, y) = V_2(x, s, y) \quad \text{provided } x < e^{s^*},\ y \in \Gamma,\ s \in \mathbb{R}.$$

We then apply Tataru's theorem, as in the argument used in the final step of the proof of Theorem 2.1, to conclude that $f_1(z) - f_2(z) = 0$ for every $z \in (0, \varepsilon) \times \Gamma$ such that there exists $(x, y) \in (0, e^{s^*}) \times \Gamma$ with $d(z, (x, y)) < e^s/x$. In particular this shows that $f_1 = f_2$ in $(0, \varepsilon) \times \Gamma$. One cannot say that $f_1 = f_2$ on X since (5-23) only holds on $(0, \varepsilon) \times \Gamma$. Since F is arbitrary, (5-4) follows. □

Since $h_1(x) = h_2(x)$ on $[0, \varepsilon) \times \Gamma$, this finishes the construction of the map Ψ_ε defined in (5-3). We will use both equalities in (5-4) to extend Ψ_ε to a global diffeomorphism $\Psi : X_1 \to X_2$ satisfying (2-15).

The construction of the global diffeomorphism. First we need to show that if the eigenfunctions are reordered such that (5-17) holds, then in fact $\phi_{j,1}(x, y) = \phi_{j,2}(x, y)$ on $(0, \varepsilon) \times \Gamma$. To prove this we have to appeal again to the stationary scattering theory. We know from [Joshi and Sá Barreto 2000] that the operator

$$E_+(\lambda)\psi(\lambda, y) = \widehat{\mathcal{R}_+(0, \psi)}(\lambda, y) = \int_{\mathbb{R}} e^{-i\lambda s} \mathcal{R}_+(0, f)(s, y)\, ds,$$

continues meromorphically to $\mathbb{C} \setminus D$, where D is a discrete subset. Since their Schwartz kernels satisfy $E_1(\lambda, y', x, y) = E_2(\lambda, y', x, y)$ for $x \in [0, \varepsilon)$ and $y, y' \in \Gamma$, $\lambda \in \mathbb{R}$, this equality must remain for $\mathbb{C} \setminus D$.

We also know from equation (3.15) of [Graham and Zworski 2003] that $\frac{1}{4}n^2 + \lambda_0^2$ is an eigenvalue of Δ_g if and only if $\lambda_0 \in i\mathbb{R}_-$ is a pole of $E(\lambda, y, z)$, with the same multiplicity, and its residue is given by

$$\frac{1}{2i\lambda_0} \sum_{k=1}^{K} \phi_k^0(y) \phi_k(z), \quad y \in \partial X,\ z \in X, \tag{5-24}$$

where $\phi_k^0(y)$ is defined in (5-16) and K is the multiplicity of the eigenvalue. We know from (5-17) and (5-18) that the eigenvalues and the traces of the eigenfunctions are equal. So if $\phi_k^{(j)}(x', y')$ $j = 1, 2$,

$1 \leq k \leq K$, denote the eigenfunctions, we must have

$$\sum_{k=1}^{K} (\phi_k^{(1)}(x', y') - \phi_k^{(2)}(x', y'))\phi_k^0(y) = 0, \quad x' \in [0, \varepsilon),\ y, y' \in \Gamma.$$

Since the points (x', y'), $x' \in [0, \varepsilon)$ and $y, y' \in \Gamma$ are arbitrary and can be independently chosen, we must have

$$\phi_k^{(1)}(x', y') = \phi_k^{(2)}(x', y') \quad \text{for all } x' \in [0, \varepsilon),\ y' \in \Gamma. \tag{5-25}$$

We know that the Schwartz kernels of the radiation fields $\mathcal{R}_{j,\pm}$, $j = 1, 2$, acting on data $(0, f)$, and the metric tensors $h_j(x, y, dy)$, $j = 1, 2$, satisfy (5-4). However, if $\phi \in C_0^\infty((0, \varepsilon) \times \Gamma)$ and $(\phi, 0) \in E_{\mathrm{ac}}(X_j)$, then

$$\partial_s \mathcal{R}_{j,\pm}(\phi, 0)(s, y) = \mathcal{R}_{j,\pm}\big(0, \big(\Delta_{g_j} - \tfrac{1}{4}n^2\big)\phi\big)(s, y).$$

Since ϕ is compactly supported, $\mathcal{R}_{+}\big(0, \big(\Delta_{g_j} - \frac{1}{4}n^2\big)\phi\big)(s, y) = 0$ for $s \ll 0$. So,

$$\begin{aligned}
\mathcal{R}_{j,+}(\phi, \psi) &= \mathcal{R}_{j,+}(0, \psi) + \int_{-\infty}^{s} \mathcal{R}_{j,+}\big(0, \big(\Delta_{g_j} - \tfrac{1}{4}n^2\big)\phi\big)(\tau, y)\, d\tau, \\
\mathcal{R}_{j,-}(\phi, \psi) &= \mathcal{R}_{j,-}(0, \psi) + \int_{s}^{\infty} \mathcal{R}_{j,-}\big(0, \big(\Delta_{g_j} - \tfrac{1}{4}n^2\big)\phi\big)(\tau, y)\, d\tau,
\end{aligned}$$

provided $(\phi, \psi) \in (C_0^\infty((0, \varepsilon) \times \Gamma) \times C_0^\infty((0, \varepsilon) \times \Gamma)) \cap E_{\mathrm{ac}}(X_j)$. Since we know from (5-13) that $\Delta_{g_1} = \Delta_{g_2}$ on $[0, \varepsilon) \times \Gamma$, and we also know from (5-25) that

$$\begin{aligned}
\mathcal{A}((0, \varepsilon) \times \Gamma) &\doteq \big(C_0^\infty((0, \varepsilon) \times \Gamma) \times C_0^\infty((0, \varepsilon) \times \Gamma)\big) \cap E_{\mathrm{ac}}(X_1) \\
&= \big(C_0^\infty((0, \varepsilon) \times \Gamma) \times C_0^\infty((0, \varepsilon) \times \Gamma)\big) \cap E_{\mathrm{ac}}(X_2),
\end{aligned}$$

we deduce that

$$\mathcal{R}_{1,\pm}(\phi, \psi)(s, y) = \mathcal{R}_{2,\pm}(\phi, \psi)(s, y), \quad (s, y) \in \mathbb{R} \times \Gamma,\ (\phi, \psi) \in \mathcal{A}((0, \varepsilon) \times \Gamma). \tag{5-26}$$

But $\mathcal{R}_\pm$ are unitary operators, and so their inverses are equal to their adjoints, and we deduce from (5-26) that the Schwartz kernels of the full operators $\mathcal{R}_{j,\pm}$ acting on $\mathcal{A}((0, \varepsilon) \times \Gamma)$ are determined by the scattering operator $\mathcal{S}_\Gamma$. We conclude that if $F \in L^2(\mathbb{R} \times \Gamma)$, and if $\mathcal{R}_{j,\pm}^{-1}\big|_\Gamma : L^2(\mathbb{R} \times \Gamma) \to E_{\mathrm{ac}}(X_1)|_{(0,\varepsilon)\times\Gamma}$, $j = 1, 2$, is given by

$$F(s, y) \mapsto (\phi_j, \psi_j) = (u_j(0), \partial_t u_j(0))\big|_{(0,\varepsilon)\times\Gamma},$$

then $(\phi_1, \psi_1) = (\phi_2, \psi_2)$. Here $u_j(t, z)$ denotes the solution to the Cauchy problems for the wave equation (2-1) for the metric g_j. But, on the other hand, $\mathcal{R}_{j,\pm}$ are translation representations of the wave group, and therefore

$$\mathcal{R}_{j,+}^{-1}\big|_\Gamma F(s + t) = (u_j(t), \partial_t u(t)),$$

where $u_j(t)$ satisfies (2-1) with initial data $(\phi, \psi) = \mathcal{R}_{j,\pm}^{-1}\big|_\Gamma \in \mathcal{A}((0, \varepsilon) \times \Gamma)$. We conclude that, if $u_j(t, z)$ solves (2-1) for the metric g_j, with initial data supported in $(0, \varepsilon) \times \Gamma$, then $u_1(t, z) = u_2(t, z)$, provided $z \in (0, \varepsilon) \times \Gamma$. This implies that, if $U_j(t, z, z')$ is the forward fundamental solution of the Cauchy problem for the wave equation in (X_j, g_j), then

$$U_1(t, z, z') = U_2(t, z, z'), \quad z,\ z' \in (0, \varepsilon) \times \Gamma,\ t > 0. \tag{5-27}$$

By Duhamel's principle, if

$$\begin{gathered}\big(D_t^2 - \Delta_{g_j} - \tfrac{1}{4}n^2\big)\tilde{U}_j(t, t', z, z') = \delta(x, y)\delta(t - t') \quad \text{in } X_j \times \mathbb{R}, \\ \tilde{U}_j(0) = \partial_t \tilde{U}_j(0) = 0,\end{gathered} \tag{5-28}$$

then

$$\tilde{U}_1(t, t'z, z') = \tilde{U}_2(t, t', z, z'), \quad t,\ t' \in \mathbb{R}_+,\ z,\ z' \in (0, \varepsilon) \times \Gamma. \tag{5-29}$$

So we have reduced the extension of the diffeomorphism to the following:

Proposition 5.8. *Let (X_1, g_1) and (X_2, g_2) be AHM such that*:

(A) *There exists a nonempty open subset $\Gamma \subset \partial X_1 \cap \partial X_2$ as manifolds and an open subset $\mathbb{O} \sim \Gamma \times (0, \varepsilon)$ such that $\mathbb{O} \subset \mathring{X}_1 \cap \mathring{X}_2$ as manifolds.*

(B) *The metric tensors g_j, $j = 1, 2$, satisfy $g_1 = g_2$ on $\mathbb{O}$.*

(C) *If $\tilde{U}_j(t, t', z, z')$, $j = 1, 2$ is the forward fundamental solution of the wave equation in (X_j, g_j), $j = 1, 2$, defined in* (5-28), *then $U_1(t, t', z, z') = U_2(t, t', z, z')$ for $t, t' \in \mathbb{R}_+$ and $z, z' \in \mathbb{O}$.*

Then there exists

$$\Psi : X_1 \to X_2 \quad \textit{such that} \quad \Psi^* g_2 = g_1 \ \textit{and} \ \Psi = \mathrm{Id} \ \textit{ in } \mathbb{O}. \tag{5-30}$$

This is similar to the inverse boundary value problem with data on part of the boundary, studied for example in [Katchalov et al. 2001; Kurylev and Lassas 2000], except that we are not dealing with boundary control but control from an open set in the interior. A somewhat similar problem for closed manifolds was studied in [Krupchyk et al. 2008]. Lassas and Oksanen [2014] also dealt with a problem of this nature. This is also related to the problem studied by Lassas, Taylor and Uhlmann on complete real analytic manifolds without boundary M_j, $j = 1, 2$, where the Green functions for the Laplace operator agree on $U \times U$, with $U \subset M_1 \cap M_2$; see Theorem 4.1 of [Lassas et al. 2003]. The difference here is that we do not have real analyticity of the manifolds, but we are dealing with the wave equation instead of the Laplace equation.

Proof. We adapt the proof of Theorem 4.33 in [Katchalov et al. 2001]. Instead of working with X_1 and X_2, we will fix $X = X_1$ and reconstruct $(X, g) = (X_1, g_1)$ from (A), (B) and (C). Of course, we are reconstructing (X_2, g_2) as well. First of all, we observe that an AHM has a uniform radius of injectivity for the geodesic flow. In other words, there exists a $\rho_0 > 0$ such that, if $S_pX = \{v \in T_pX : \|v\|_g = 1\}$, the map

$$\exp_p : [0, \rho_0) \times S_pX \to X, \quad (t, v) \mapsto \exp_p(tv),$$

is well defined for all $p \in X$. We pick a point $p \in \mathbb{O}$ and let $\rho \in (0, \rho_0)$ be such that the geodesic ball $B(p, \rho) \subset \mathbb{O}$. Let $f(t, z) \in C_0^\infty(\mathbb{R} \times B(p, \rho))$, $f(t, z) = 0$ for $t < 0$, and let $u^f(t, z)$ be the solution to

$$\begin{gathered} \big(D_t^2 - \Delta_g - \tfrac{1}{4}n^2\big)u^f(t, z) = f(t, z) \quad \text{in } \mathbb{R} \times X, \\ u^f(0) = \partial_t u^f(0) = 0. \end{gathered} \tag{5-31}$$

From the hypothesis (C) above, we know $u^f(t, z)$ for $z \in B(p, \rho)$, $t > 0$. We then define the map

$$\mathscr{B}(T) : C_0^\infty((0, T) \times B(p, \rho)) \to C^\infty((0, T) \times B(p, \rho)), \quad f \mapsto u^f\big|_{(0,T)\times B(p,\rho)}. \tag{5-32}$$

For $T > 0$ we will work with the space of functions

$$\mathscr{C}_0 = \mathscr{C}_0(p, \rho, T) \doteq \{\phi \in C_0^\infty((0, T] \times B(p, \rho)) : \phi(T) = 0\},$$

and the quotient space

$$\mathscr{C} = \mathscr{C}(p, \rho, T) \doteq \mathscr{C}_0 \big/ \big(D_t^2 - \Delta_g - \tfrac{1}{4}n^2\big)\mathscr{C}_0.$$

In other words,

$$\mathscr{C} = \{[\psi] : \psi \in \mathscr{C}_0\}, \quad \text{where } [\psi] = \Big\{\phi \in \mathscr{C}_0 : \text{there is } \zeta \in \mathscr{C}_0 \text{ such that } \phi = \psi + \big(D_t^2 - \Delta_g - \tfrac{1}{4}n^2\big)\zeta\Big\}.$$

Since we know g in $\mathbb{O}$, the space $\mathscr{C}$ is determined by hypotheses (A), (B) and (C).

For $\phi \in \mathscr{C}$, let u^ϕ be the solution to (5-31) in $\mathbb{R} \times X$. We define the map

$$C_T : \mathscr{C} \to C_0^\infty(X), \quad \phi \mapsto u^\phi(T, z).$$

The formal adjoint of this map is given by

$$C_T^* : \{w \in C_0^\infty(\{z \in X : d_g(z, B(p, \rho)) < T\})\} \to \mathscr{C}, \quad w \mapsto v|_{(0,T)\times B(p,\rho)},$$

where v is the solution to the Cauchy problem

$$\begin{gathered} \big(D_t^2 - \Delta_g - \tfrac{1}{4}n^2\big)v(t, z) = 0 \quad \text{in } \{t < T\} \times X, \\ v(T, z) = 0, \quad \partial_t v(T, z) = w. \end{gathered} \tag{5-33}$$

As in the boundary control method, we define

$$S_T = C_T^* C_T : \mathscr{C} \to \mathscr{C}.$$

The next step is to prove a Blagovestchenskii-type identity to show that S_T is determined by the map $\mathscr{B}(2T)$, which the map defined in (5-32) but in the time interval $(0, 2T)$, and hence is determined from (A), (B) and (C). Let $\phi(t, z)$, $\psi(t, z) \in \mathscr{C}$ and let $u^\phi(t, z)$, $u^\psi(t, z)$ be the solutions to (5-31), with left-hand side ϕ and ψ respectively. Let

$$W(s, t) = \int_X u^\phi(t, z)u^\psi(s, z)\, d\,\mathrm{vol}_g(z).$$

Notice that this integration is defined over the entire manifold. But, after integrating by parts, we obtain

$$\begin{aligned}(\partial_t^2-\partial_s^2)W(s,t)&=\int_X(\phi(t,z)u^{\psi}(s,z)-u^{\phi}(t,z)\psi(s,z))\,d\operatorname{vol}_g(z)\\&=\int_X[\phi(t,z)\mathcal{B}(T)\psi(s,z)-\psi(s,z)\mathcal{B}(T)\phi(t,z)]\,d\operatorname{vol}_g(z),\\ W(0,t)=\partial_sW(0,t)&=0,\qquad W(s,0)=\partial_tW(s,0)=0,\end{aligned}$$

and, since ϕ and ψ are supported in $(0,T)\times B(p,\rho)$, the last integration is restricted to $B(p,\rho)$. We can find $W(T,T)$ explicitly in terms of d'Alembert's formula, but we need to extend ϕ and ψ to the interval $(0,2T)$. As in [Belishev and Kurylev 1992], we define $\tilde{\phi}$ and $\tilde{\psi}$ to be the odd extensions of ϕ and ψ across $t=T$, in other words

$$\tilde{\phi}(t)=\begin{cases}\phi(t) & \text{if } t\in(0,T),\\ -\phi(2T-t) & \text{if } t\in(T,2T),\end{cases}$$

and similarly for $\tilde{\psi}$. This gives

$$W(T,T)=\int_0^T\int_t^{2T-t}\left(\int_X\big(\tilde{\phi}(t,z)\mathcal{B}(2T)(\tilde{\psi})(s,z)-\mathcal{B}(2T)(\tilde{\phi})(t,z)\tilde{\psi}(s,z)\big)\,d\operatorname{vol}_g(z)\right)ds\,dt$$

Since $\tilde{\psi}(s,z)$ is odd with respect to $s=T$, it follows that

$$\begin{aligned}W_j(T,T)&=\int_0^T\int_t^{2T-t}\int\tilde{\phi}(t,z)\mathcal{B}(2T)(\tilde{\psi})(s,z)\,d\operatorname{vol}_g(z)\,ds\,dt\\&=\int_0^T\int_X\phi(t,z)\left(\int_t^{2T-t}\mathcal{B}(2T)\tilde{\psi}(s,z)\,ds\right)d\operatorname{vol}_g(z)\,dt.\end{aligned}$$

On the other hand, since

$$W(T,T)=\langle C_T\phi,C_T\psi\rangle=\langle\phi,C_T^*C_T\psi\rangle,$$

it follows that

$$C_T^*C_T\psi(t,z)=\int_t^{2T-t}\mathcal{B}(2T)\tilde{\psi}(s,z)\,ds.$$

Now we define the following inner product in the space $\mathscr{C}$:

$$\langle\phi,\psi\rangle_{\mathscr{C}}=\langle u^{\phi}(T,z),u^{\psi}(T,z)\rangle_{L^2(X)}.$$

As shown above, this is determined by the map $\mathcal{B}$. We need to show that this is a nondegenerate inner product. First we show that the range $\{u^{\phi}(T):\phi\in\mathscr{C}\}$ is dense in the space

$$L^2\big(\{z\in X_j:d(z,B(p,\rho))\le T\}\big)=\big\{u\in L^2(X_j):\operatorname{Supp}(u)\subset\{z:d(z,B(p,\rho))\le T\}\big\}.$$

Suppose that $w\in L^2(\{z\in X_j:d(z,B(p,\rho))\le T\})$ is such that

$$\langle w,u^{\phi}(T)\rangle=0\quad\text{for all }\phi\in\mathscr{C}.$$

Let v satisfy (5-33) and let u^ϕ satisfy (5-31) with right-hand side equal to ϕ. Integrating the identity

$$v\big(D_t^2-\Delta_g-\tfrac{1}{4}n^2\big)u^\phi-u^\phi\big(D_t^2-\Delta_g-\tfrac{1}{4}n^2\big)v=v(t,z)\phi(t,z)$$

in the domain of influence of ϕ and w, we find that

$$\int_{B(p,\rho)\times(0,T)} v(t,z)\phi(t,z)\,dt\,d\operatorname{vol}_g(z)=0 \quad \text{for all } \phi\in\mathscr{C}. \tag{5-34}$$

But, again using the fact that v satisfies (5-33), we see that

$$\int_{B(p,\rho)\times(0,T)} v(t,z)\big(D_t^2-\Delta_g-\tfrac{1}{4}n^2\big)\phi(t,z)\,dt\,d\operatorname{vol}_g(z)=0 \quad \text{for all } \phi\in\mathscr{C}_0.$$

This means that (5-34) is satisfied for every $\phi\in\mathscr{C}_0$, and hence $v(t,z)=0$ in $(0,T)\times B(p,\rho)$. Now the odd extension $\tilde{v}(t,z)$ of $v(t,z)$ across $t=T$ satisfies (5-33) in $(0,2T)\times\{z:d(z,B(p,\rho))<T+\rho\}$ and $\tilde{v}(t,z)=0$ in $(0,2T)\times B(p,\rho)$. An application of Tataru's theorem implies that $\tilde{v}(t,z)=0$ if $|t|+d(z,B(p,\rho))\le T$ for any $q\in B(p,\rho)$. In particular, this implies that $w(z)=\partial_t v(T,z)=0$ provided $d(z,B(p,\rho))\le T$, and hence $w=0$.

Now suppose that $\phi\in\mathscr{C}$ is such that $\langle\phi,\psi\rangle_{\mathscr{C}}=0$ for every $\psi\in\mathscr{C}$. From the previous discussion, it follows that $u^\phi(T)=0$. Then

$$\tilde{u}(t,z)=\begin{cases} u^\phi(t,z) & \text{if } t<T,\\ -u^\phi(2T-t,z) & \text{if } t>T\end{cases}$$

satisfies

$$\begin{aligned}\big(D_t^2-\Delta_g-\tfrac{1}{4}n^2\big)\tilde{u}&=\tilde{\phi} \quad \text{in } \mathbb{R}\times X_j\\ \tilde{u}&=0 \quad \text{in } \mathbb{R}\times\{z:d(z,B(p,\rho))>T\}.\end{aligned}$$

Again, Tataru's theorem and finite speed of propagation implies that $u^\phi\in C_0^\infty((0,T]\times B(p,\rho))$ and $u^\phi(T)=0$. This of course means that $u^\phi\in\mathscr{C}_0$, and hence $[\phi]=0$.

Next we define $\overline{\mathscr{C}}$ as the Hilbert space given by the closure of $\mathscr{C}$ with the norm given by the inner product $\langle\phi,\psi\rangle_{\mathscr{C}}$, and set up a scheme which is very similar to the one used in the proof of Lemma 5.3, which is of course similar to the arguments used in [Belishev and Kurylev 1992; Katchalov et al. 2001]. For $\tau\in(0,T)$ define

$$\overline{\mathscr{C}}_\tau=\{\phi\in\overline{\mathscr{C}}:\phi(t,z)=0,\ t<\tau\},$$

and let

$$\mathscr{P}_\tau:\overline{\mathscr{C}}\to\overline{\mathscr{C}}_\tau$$

be the orthogonal projection to $\overline{\mathscr{C}}_\tau$. Then, using propagation of singularities (and here we do not have to project onto the continuous spectrum), and that the choices for $t=0$ and $t=T$ are arbitrary, we recover the metric tensor g and the fundamental solution of wave equation in $B(p,r)$, where $r=r(p)$ is the radius of injectivity of $\exp_p$. In other words, we recover

$$g(z),\quad z\in B(p,r) \quad \text{and} \quad \tilde{U}(t,t',z,z'),\quad t,\,t'\in\mathbb{R},\ z,\,z'\in B(p,r),\ r=r(p).$$

We repeat the process for every $p \in \mathbb{O}$, and we would like to define $\mathcal{M} = \bigcup_{p\in\mathbb{O}} B(p, r(p))$. However, we have to make sure the inclusion map $\imath : \mathcal{M} \hookrightarrow X$ is injective, which would guarantee that $\imath(\mathcal{M})$ is an open embedded submanifold of X. Therefore we need to identify the points that are in $B(p, r(p))$ and $B(q, r(q))$. In Section 4.4.9 of [Katchalov et al. 2001], since they are working on a compact manifold, they use the family of eigenfunctions to do that. Here the precise analogue is to use $\tilde{U}(t, t', z, z')$, and we shall say that $z \in B(p, r(p))$, and $w \in B(q, r(q))$ are equivalent, and we denote $z \equiv w$ if $\tilde{U}(t, t', z, z') = \tilde{U}(t, t', w, z')$ for all $t, t' > 0$ and $z' \in \mathbb{O}$. In this case, the points z and w correspond to the same point in X. This is the equivalent of saying that $u^\phi(t, z) = u^\phi(t, w)$ for all $t \in \mathbb{R}$ and for all $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{O})$. We also use the same identification for points in $\mathbb{O}$ and $B(p, r(p))$, $p \in \mathbb{O}$. With this identification, we set $\mathbb{O}_1 = \big(\bigcup_{p\in\mathbb{O}} B(p, r(p))\big) \cup \mathbb{O}$.

We have constructed an open C^∞ submanifold $\mathbb{O}_1 \subset X$ such that $\mathbb{O} = \mathbb{O}_0 \subset \mathbb{O}_1$ and such that hypotheses (A), (B) and (C) are satisfied for $\mathbb{O}_1$. Now we repeat the process for $\mathbb{O}_1$. Thus we obtain a sequence of C^∞ open submanifolds $\mathbb{O}_j \subset X$ satisfying $\mathbb{O}_j \subset \mathbb{O}_{j+1} \subset X$, $j = 0, 1, \ldots,$ and satisfying the hypotheses (A), (B) and (C) above. As in Section 4.4.9 of [Katchalov et al. 2001], we claim that for any compact subset $K \subset X$ there exists $J \in \mathbb{N}$ such that $K \subset \mathbb{O}_J$. To see that, we observe that, since (X, g) is complete, there exists $M > 0$ such that, for any $p \in K$, $\delta < \varepsilon$ and $\Gamma' \Subset \Gamma$, $d_g(p, \Gamma' \times \{\delta\}) \leq M$. We also assume that $\delta < \delta_0$, where δ_0 is the radius of injectivity of X. Since X is complete, given a point $p \in K$ there is a geodesic $\mu(s)$, parametrized by the arc length $0 \leq s \leq L \leq M$, joining p to a point $z \in \Gamma' \times \delta$. Let $x_0 = z$ and $x_k = \mu(k\delta)$, with $k = 0, 1, \ldots, [L/\delta] = J$. By definition $x_0 = z \in \Gamma \times \{\delta\} \subset \mathbb{O} = \widetilde{\mathbb{O}}_0$. Suppose that $x_k \in \widetilde{\mathbb{O}}_k$; then there exists $\rho > 0$ such that $B(x_k, \rho) \subset \mathbb{O}_k$ but, since δ is less than the radius of injectivity, $\overline{B(x_k, \delta)} \subset \mathbb{O}_{k+1}$ and, since s is the arc length, in particular $x_{k+1} \in \mathbb{O}_{k+1}$. By induction it follows that $p \in \mathbb{O}_{J+1} \subset \mathbb{O}_{[M/\delta]}$.

This shows that we can reconstruct $(\mathring{X}, g)$ from (A), (B) and (C). But we know a priori that (X, g) is an AHM, and so $\mathring{X}$ can be compactified into a C^∞ with boundary, and there exists a defining function x of ∂X for which (2-4) holds. The construction of the function x shows that the compactification is uniquely defined modulo diffeomorphisms that are equal to the identity in $\mathbb{O}$. $\square$

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