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LOW TEMPERATURE ASYMPTOTICS FOR QUASISTATIONARY DISTRIBUTIONS IN A BOUNDED DOMAIN

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We analyze the low temperature asymptotics of the quasistationary distribution associated with the overdamped Langevin dynamics (also known as the Einstein–Smoluchowski diffusion equation) in a bounded domain. This analysis is useful to rigorously prove the consistency of an algorithm used in molecular dynamics (the hyperdynamics) in the small temperature regime. More precisely, we show that the algorithm is exact in terms of state-to-state dynamics up to exponentially small factors in the limit of small temperature. The proof is based on the asymptotic spectral analysis of associated Dirichlet and Neumann realizations of Witten Laplacians. In order to widen the range of applicability, the usual assumption that the energy landscape is a Morse function has been relaxed as much as possible.

1. Introduction

The motivation of this work comes from the mathematical analysis of an algorithm used in molecular dynamics, called the hyperdynamics [Voter 1997]. The aim of this algorithm is to generate very efficiently the discrete state-to-state dynamics associated with a continuous state space, metastable, Markovian dynamics, by modifying the potential function. In Section 1A, we explain the principle of the algorithm and state the mathematical problem. In Section 1B, the main result of this article is given in a simple setting.

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1A. ***Molecular dynamics, hyperdynamics and the quasistationary distribution.*** Molecular dynamics calculations consist in simulating very long trajectories of a particle model of matter, in order to infer macroscopic properties from an atomic description. Examples include the study of the change of conformation of large molecules (such as proteins), with applications in biology, or the description of the motion of defects in materials.

In a constant-temperature environment, the dynamics used in practice contains stochastic terms which model thermostatting. The prototypical example, which is the focus of this work, is the overdamped Langevin dynamics,

$$dX_t = -\nabla f(X_t)\,dt + \sqrt{2\beta^{-1}}\,dB_t, \tag{1-1}$$

where $X_t \in \mathbb{R}^{3N}$ is the position vector of N particles, $f : \mathbb{R}^{3N} \to \mathbb{R}$ is the potential function (assumed to be smooth here), and $\beta^{-1} = k_B T$ with k_B the Boltzmann constant and T the temperature. The stochastic process B_t is a standard $3N$-dimensional Brownian motion. The dynamics (1-1) admits the canonical ensemble $\mu(dx) = Z^{-1}\exp(-\beta f(x))\,dx$ as an invariant probability measure.

To relate the macroscopic properties of matter to the microscopic phenomenon, one simulates the process $(X_t)_{t\geq 0}$ (or processes following related dynamics, like the Langevin dynamics) over very long times. The difficulty associated with such simulations is *metastability*, namely the fact that the stochastic process remains trapped for very long times in some regions of the configurational space, called the metastable states. The time step used to obtain stable discretization is typically 10^{-15} s, while the macroscopic timescales of interest range from a few microseconds to a few seconds. At the macroscopic level, the details of the dynamics $(X_t)_{t\geq 0}$ do not matter. The important information is the history of the visited metastable states, the so-called *state-to-state dynamics*.

The principle of the hyperdynamics algorithm [Voter 1997] is to modify the potential f in order to accelerate the exit from metastable states, while keeping a correct state-to-state dynamics. Here, we focus on one elementary brick of this dynamics, namely the exit event from a given metastable state.

In mathematical terms, the problem is as follows (we refer to [Le Bris et al. 2012] for the mathematical proofs of the statements below). Assuming that the process remains trapped for a very long time in a domain $\Omega_+ \subset \mathbb{R}^{3N}$ (Ω_+ is a metastable state,[1] as mentioned above), it is known that the process reaches a local equilibrium called the quasistationary distribution (QSD) ν attached to the domain Ω_+, before leaving it. We assume that Ω_+ is a smooth bounded domain in $\mathbb{R}^{3N}$. The probability distribution ν has support Ω_+ and is such that, for all smooth test function $\varphi : \mathbb{R}^{3N} \to \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{E}(\varphi(X_t)|\tau > t) = \int_{\Omega_+} \varphi\,d\nu, \tag{1-2}$$

where

$$\tau = \inf\{t > 0 : X_t \notin \Omega_+\}$$

is the first exit time from Ω_+ for X_t. The metastability of the well Ω_+ can be quantified through the rate of convergence of the limit in (1-2); in the following, it is assumed that this convergence is infinitely fast. From a PDEs viewpoint, ν has a density v with respect to the Boltzmann–Gibbs measure

[1] We use the notation Ω_+ since, in the following, we will need a subdomain Ω_- such that $\overline{\Omega}_- \subset \Omega_+$.

$\mu(dx) = e^{-\beta f(x)}\,dx$, v being the first eigenvector of the infinitesimal generator of the dynamics (1-1) with Dirichlet boundary conditions on $\partial\Omega_+$:

$$\begin{cases} -\nabla f \cdot \nabla v + \beta^{-1}\Delta v = -\lambda v & \text{in } \Omega_+, \\ v = 0 & \text{on } \partial\Omega_+, \end{cases} \tag{1-3}$$

where $-\lambda < 0$ is the first eigenvalue. In other words,

$$d\nu = \frac{1_{\Omega_+}(x)v(x)\exp(-\beta f(x))\,dx}{\int_{\Omega_+} v(x)\exp(-\beta f(x))\,dx}.$$

Starting from the QSD ν (namely if $X_0 \sim \nu$), the way the stochastic process X_t, solution to (1-1), leaves the well Ω_+ is known: the law of the pair of random variables (τ, X_τ) (exit time, exit point) is characterized by the following three properties, the first two of which are the building blocks of a Markovian transition starting from Ω_+:

(i) τ and X_τ are independent.

(ii) τ is exponentially distributed with parameter λ:

$$\tau \sim \mathcal{E}(\lambda), \tag{1-4}$$

where the notation $\sim$ is used to indicate the law of a random variable.

(iii) The exit point distribution has an analytic expression in terms of v: for all smooth test functions $\varphi : \partial\Omega_+ \to \mathbb{R}$,

$$\mathbb{E}^\nu(\varphi(X_\tau)) = -\frac{\int_{\partial\Omega_+} \varphi\partial_n(v\exp(-\beta f))\,d\sigma}{\beta\lambda\int_{\Omega_+} v(x)\exp(-\beta f(x))\,dx}, \tag{1-5}$$

where, for any smooth function $w : \Omega_+ \to \mathbb{R}$, $\partial_n w = \nabla w \cdot n$ denotes the outward normal derivative, σ is the Lebesgue measure on $\partial\Omega_+$ and $\mathbb{E}^\nu$ indicates the expectation for the stochastic process X_t following (1-1) and starting under the QSD, $X_0 \sim \nu$.

In practical cases of interest, the typical exit time is very large ($\mathbb{E}(\tau) = 1/\lambda$ is very large). The principle of the hyperdynamics is to modify the potential f in the state Ω_+ to lead to smaller exit times, while keeping a correct statistics on the exit points. Let us make this more precise, and let us consider the process $X_t^{\delta f}$ which evolves on a new potential $f + \delta f$:

$$dX_t^{\delta f} = -\nabla(f + \delta f)(X_t^{\delta f})\,dt + \sqrt{2\beta^{-1}}\,dB_t. \tag{1-6}$$

Instead of simulating $(X_t)_{t\geq 0}$ following the dynamics (1-1) and considering the associated random variables (τ, X_τ), the hyperdynamics algorithm consists in simulating $(X_t^{\delta f})_{t\geq 0}$ and considering the associated random variables $(\tau^{\delta f}, X^{\delta f}_{\tau^{\delta f}})$, where $\tau^{\delta f}$ is the first exit time from Ω_+ for $X_t^{\delta f}$.

The assertion underlying the hyperdynamics algorithm is the following: under appropriate assumptions on the perturbation δf, (i) the exit point distribution of $X_t^{\delta f}$ from Ω_+ is (almost) the same as the exit point distribution of X_t from Ω_+, and (ii) the exit time distribution for X_t can be inferred from the exit time distribution for $X_t^{\delta f}$ by a simple multiplicative factor (see (1-7)–(1-8) below).

More precisely, the assumptions on δf in [Voter 1997] can be stated as follows: (i) δf is sufficiently small that Ω_+ is still a metastable state for $X_t^{\delta f}$, and (ii) δf is zero on the boundary of Ω_+. The first hypothesis implies that we can assume that $X_0^{\delta f}$ is distributed according to the QSD $\nu^{\delta f}$ associated with (1-6) and Ω_+. The aim of this paper is to prove that, in the small temperature regime (namely $\beta \to \infty$) and under appropriate assumptions on δf, we indeed have the equality in law

$$(\tau, X_\tau) \overset{\mathcal{L}}{\simeq} (B\tau^{\delta f}, X^{\delta f}_{\tau^{\delta f}}), \tag{1-7}$$

where, in the left-hand side, $X_0 \sim \nu$ and, in the right-hand side, $X_0^{\delta f} \sim \nu^{\delta f}$. The so-called boost factor B has the expression

$$B = \frac{\int_{\Omega_+} \exp(-\beta f)}{\int_{\Omega_+} \exp(-\beta(f+\delta f))} = \int_{\Omega_+} \exp(\beta \delta f) \frac{\exp(-\beta(f+\delta f))}{\int_{\Omega_+} \exp(-\beta(f+\delta f))}. \tag{1-8}$$

The second formula is interesting because it shows that B can be approximated through ergodic averages on the process $(X_t^{\delta f})_{t\geq 0}$ (and this is actually exactly what is done in practice).

In view of the formulas (1-4)–(1-5) for the laws of the distributions of the two random variables exit time and exit point, a crucial point for the mathematical analysis of the hyperdynamics algorithm is to study how the first eigenvalue λ and the normal derivative $\partial_n v$ (v being the first eigenvector; see (1-3)) are modified when changing the potential f to $f + \delta f$. More precisely, we would like to check that, in the limit $\beta \to \infty$, $\lambda^{\delta f} = B\lambda$ and, up to a multiplicative constant, $\partial_n v^{\delta f} \propto \partial_n v$, where, with obvious notation, $(-\lambda^{\delta f}, v^{\delta f})$ denotes the first eigenvalue–eigenfunction pair solution to (1-3) when f is replaced by $f + \delta f$.

1B. *The main results in a simple setting.* Let us state the main results obtained in this paper in a simple and restricted setting. For the potential f, we assume that there exists a subdomain Ω_- such that $\overline{\Omega}_- \subset \Omega_+$ and:

(i) f and $f\big|_{\partial\Omega_+}$ are Morse functions, namely $\mathcal{C}^\infty$ functions with nondegenerate critical points;

(ii) $|\nabla f| \neq 0$ in $\overline{\Omega}_+ \setminus \Omega_-$, $\partial_n f > 0$ on $\partial\Omega_-$ and $\min_{\partial\Omega_+} f \geq \min_{\partial\Omega_-} f$;

(iii) the critical values of f in Ω_- are all distinct and the differences $f(U^{(1)}) - f(U^{(0)})$, where $U^{(0)}$ ranges over the local minima of $f\big|_{\Omega_-}$ and $U^{(1)}$ ranges over the critical points of $f\big|_{\Omega_-}$ with index 1, are all distinct;

(iv) the maximal value of f at critical points, denoted by $\mathrm{cvmax} = \max\{f(x) : x \in \Omega_+, |\nabla f(x)| = 0\} = \max\{f(x) : x \in \Omega_-, |\nabla f(x)| = 0\}$, satisfies

$$\min_{\partial\Omega_-} f - \mathrm{cvmax} > \mathrm{cvmax} - \min_{\Omega_-} f. \tag{1-9}$$

Concerning the perturbation δf, let us assume that $f + \delta f$ satisfies the same four above hypotheses as f, and that, in addition,

$$\delta f = 0 \quad \text{on } \Omega_+ \setminus \Omega_-.$$

Under these assumptions on f and δf, it can be shown that the first eigenvalue–eigenfunction pairs $(-\lambda, v)$ and $(-\lambda^{\delta f}, v^{\delta f})$, the respective solutions to (1-3) with the potential f and $f+\delta f$, satisfy the following estimate: for some positive constant c, in the limit $\beta \to \infty$,

$$\frac{\lambda^{\delta f}}{\lambda} = B(1 + \mathcal{O}(e^{-\beta c})),$$

where, we recall, B is defined by (1-8) and

$$\frac{\partial_n v\big|_{\partial\Omega_+}}{\|\partial_n v\|_{L^1(\partial\Omega_+)}} = \frac{\partial_n v^{\delta f}\big|_{\partial\Omega_+}}{\|\partial_n v^{\delta f}\|_{L^1(\partial\Omega_+)}} + \mathcal{O}(e^{-\beta c}) \quad \text{in } L^1(\partial\Omega_+).$$

These results are simple consequences of the general Theorem 2.4 below (see Corollary 2.9) together with Proposition 7.1 and Remark 7.2.

For readers who are familiar with the Agmon distance, let us note that condition (1-9) can actually be replaced by Hypothesis 2 (stated in Section 2) and condition (7-1). Condition (7-1) explicitly states that the potential function f on $\partial\Omega_-$ should be larger than the largest barrier (difference of potential between index-one critical points and local minima) within Ω_-.

1C. ***Outline of the article.*** The main result of this article, Theorem 2.4, gives general asymptotic formulas for the first eigenvalue λ and the normal derivative $\partial_n v$ in the limit of small temperature. This theorem will be proven under assumptions involving the low-lying spectra of Witten Laplacians on Ω_- and on $\Omega_+ \setminus \overline{\Omega}_-$. These assumptions hold for potentials satisfying the four conditions (i)–(iv) stated above, but they are also valid in much more general cases. In particular, we have in mind assumptions stated only in terms of Ω_+ (see Remark 7.4), or potentials not fulfilling the Morse assumption (see Section 7B).

The outline of the article is as follows: In Section 2, we specify our general assumptions and state the two main theorems, Theorem 2.4 and Theorem 2.10. In Section 3, exponential decay estimates for the eigenvectors in terms of Agmon distances are reviewed. In Section 4, approximate eigenvectors for the Dirichlet Witten Laplacians on Ω_+ are constructed in terms of eigenvectors for the Neumann Witten Laplacians on Ω_- and eigenvectors for the Dirichlet Witten Laplacians on the shell $\Omega_+ \setminus \overline{\Omega}_-$. Following the strategy of [Helffer et al. 2004; Helffer and Nier 2006; Le Peutrec 2009; 2010b; 2011; Le Peutrec et al. 2013], accurate approximations of singular values of the Witten differential $d_{f,h}$ are computed using matrix arguments in Section 5. Theorem 2.4 and Theorem 2.10 are finally proved in Section 6. The general assumptions used to prove the theorems are then thoroughly discussed and illustrated with various examples in Section 7. Our approach relies on the introduction of boundary Witten Laplacians (namely Witten Laplacians with Dirichlet or Neumann boundary conditions) and requires notions and notation of Riemannian differential geometry. A short presentation of these notions is given in the Appendix.

2. Assumptions and statements of the main results

In order to prove the main result, we first need to restate the eigenvalue problem (1-3) with the standard notation used in the framework of Witten Laplacians, which will be our central tool. It is easy to check that (λ, v) satisfies (1-3) if and only if (λ_1, u_1) satisfies

$$\Delta_{f,h}^{D,(0)}(\Omega_+)u_1 = \lambda_1 u_1$$

with

$$h = \frac{2}{\beta}, \quad \lambda_1 = \frac{4}{\beta}\lambda = 2h\lambda, \quad u_1 = \exp\big(-\tfrac{1}{2}\beta f\big)v = \exp\Big(-\frac{f}{h}\Big)v$$

and where $\Delta_{f,h}^{D,(0)}(\Omega_+)$ is the Witten Laplacian on zero-forms on $\Omega_+ \subset \mathbb{R}^d$, $d = 3N$, with homogeneous Dirichlet boundary conditions on $\partial\Omega_+$ (see (2-3) below for more general formulas on p-forms),

$$\Delta_{f,h}^{D,(0)}(\Omega_+)u_1 = (-h\nabla + \nabla f)\cdot((h\nabla + \nabla f)u_1) = -h^2\Delta u_1 + (|\nabla f|^2 - h\Delta f)u_1. \tag{2-1}$$

Notice that the operator $\Delta_{f,h}^{D,(0)}(\Omega_+)$ is a *positive* symmetric operator. We recall that Ω_+ is the metastable domain of interest, and Ω_- is a subdomain of Ω_+, where the potential f is modified in the hyperdynamics algorithm. We will thus study how the first eigenvalue λ_1 and eigenfunction u_1 of the Witten Laplacian $\Delta_{f,h}^{D,(0)}(\Omega_+)$ depend on $f\big|_{\Omega_-}$. We will state the results in a very general setting, namely for open, regular, bounded, connected subsets Ω_- and Ω_+ of a d-dimensional Riemannian manifold (M, g) such that $\overline{\Omega}_- \subset \Omega_+$.

The first assumption we make on f is the following:

Hypothesis 1. *The function $f : M \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function satisfying*

$$|\nabla f| > 0 \text{ on } \overline{\Omega}_+ \setminus \Omega_-, \quad \partial_n f > 0 \text{ on } \partial\Omega_- \quad \text{and} \quad \min_{\partial\Omega_+} f \geq \min_{\partial\Omega_-} f. \tag{2-2}$$

In (2-2), n denotes the unit normal vector on $\partial\Omega_-$ that points outward from Ω_-. This first assumption has simple consequences that will be used repeatedly.

Lemma 2.1. *Under Hypothesis 1, for all $x \in \overline{\Omega}_+ \setminus \Omega_-$,*

$$f(x) \geq \min_{\partial\Omega_-} f > \min_{\Omega_-} f = \min_{\Omega_+} f.$$

Proof. The last equality is a simple consequence of the fact that the critical points are in Ω_- and of the inequality $\min_{\partial\Omega_+} f \geq \min_{\partial\Omega_-} f$. Let us now consider the first inequality. Let us denote by $\gamma_x(t)$ the gradient trajectory $\dot{\gamma}_x = -\nabla f(\gamma_x)$ starting from $x \in \Omega_+$ $(\gamma_x(0) = x)$. Let us consider $x \in \overline{\Omega}_+ \setminus \Omega_-$ such that $f(x) < \min_{\partial\Omega_+} f$. Since $t \mapsto f(\gamma_x(t))$ is nonincreasing, $(\gamma_x(t))_{t\geq 0}$ remains in the bounded domain Ω_+ and is thus well defined for all positive times. Moreover, necessarily, the distance of $\gamma_x(t)$ to the set of critical points of f tends to 0 as $t \to \infty$. This implies that there exists $t_0 > 0$ such that $\gamma_x(t_0) \in \Omega_-$ and, thus, $f(x) = f(\gamma_x(0)) \geq f(\gamma_x(t_0)) \geq \min_{\partial\Omega_-} f$. This concludes the proof of the first inequality. The second inequality is a consequence of the assumption $\partial_n f > 0$ on $\partial\Omega_-$, and is proven by considering the trajectory $(\gamma_x(t))_{t\geq 0}$ with $x \in \arg\min_{\partial\Omega_-} f$. □

Remark 2.2. One can easily check, using the same arguments, that the condition $\partial_n f > 0$ on $\partial\Omega_+$, together with the two first conditions of Hypothesis 1, implies $\min_{\partial\Omega_+} f > \min_{\partial\Omega_-} f$.

The second assumption on f is:

Hypothesis 2. *There exists $c_0 > 0$ such that the set of critical points of f in Ω_+ is included in $\{f < \min_{\partial\Omega_+} f - c_0\}$:*

$$\{x \in \Omega_+ : \nabla f(x) = 0\} \subset \Big\{x \in \Omega_+ : f(x) < \min_{\partial\Omega_+} f - c_0\Big\}.$$

In addition to Hypotheses 1 and 2, our main results are stated under assumptions on the spectrum of the Witten Laplacians associated with f on Ω_- and $\Omega_+ \setminus \overline{\Omega}_-$ (see Hypotheses 3 and 4 below). We will discuss more explicit assumptions on f for which those additional hypotheses are satisfied in Section 7. Let us first define the Witten Laplacians. We refer the reader to [Witten 1982; Helffer and Sjöstrand 1985b; Cycon et al. 1987; Burghelea 1997; Zhang 2001] for introductory texts on the semiclassical analysis of Witten Laplacians and its famous application to Morse inequalities, and related results.

The Witten Laplacians are defined on $\bigwedge \mathcal{C}^\infty(M) = \bigoplus_{p=0}^{d} \bigwedge^p \mathcal{C}^\infty(M)$ as

$$\Delta_{f,h} = (d^*_{f,h} + d_{f,h})^2 = d^*_{f,h} d_{f,h} + d_{f,h} d^*_{f,h},$$
$$\text{where } d_{f,h} = e^{-f/h}(hd)e^{f/h} \text{ and } d^*_{f,h} = e^{f/h}(hd^*)e^{-f/h}. \tag{2-3}$$

On a domain $\Omega \subset M$ and for $m \in \mathbb{N}$, the Sobolev space $\bigwedge W^{m,2}(\Omega)$ is defined as the set of $u \in \Lambda L^2(\Omega)$ such that, locally, $\partial_x^\alpha u \in \bigwedge L^2(\Omega)$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \le m$ (this property does not depend on the local coordinate system $(x^1, \dots, x^d)$). When Ω is a regular bounded domain, $\bigwedge W^{m,2}(\Omega)$ coincides with the set of $u \in \bigwedge L^2$ such that there exists $\tilde{u} \in \bigwedge W^{m,2}(M)$ such that $\tilde{u}\big|_\Omega = u$. The spaces $\bigwedge W^{s,2}(\Omega)$ for $s \in \mathbb{R}$ are then defined by duality and interpolation. For $m = 1$, the quantity $\sqrt{\|u\|^2_{L^2(\Omega)} + \|du\|^2_{L^2(\Omega)} + \|d^*u\|^2_{L^2(\Omega)}}$ is equivalent to the $W^{1,2}(\Omega)$-norm. This is a well-known result when $\Omega = \mathbb{R}^d$. The extension to a regular, bounded domain is proved by using local charts and the reflexion principle; see [Taylor 1997; Chazarain and Piriou 1982].

In a regular, bounded domain Ω of M, various self-adjoint realizations of $\Delta_{f,h}$ can be considered:

- The Dirichlet realization $\Delta^D_{f,h}(\Omega)$ with domain

$$D(\Delta^D_{f,h}(\Omega)) = \Big\{\omega \in \bigwedge W^{2,2}(\Omega) : \boldsymbol{t}\omega\big|_{\partial\Omega} = 0,\ \boldsymbol{t}d^*_{f,h}\omega\big|_{\partial\Omega} = 0\Big\}.$$

This is the Friedrichs extension of the closed quadratic form

$$\mathcal{D}(\omega, \omega') = \langle d_{f,h}\omega, d_{f,h}\omega'\rangle_{L^2} + \langle d^*_{f,h}\omega, d^*_{f,h}\omega'\rangle_{L^2} \tag{2-4}$$

defined on the domain

$$\bigwedge W^{1,2}_D(\Omega) = \Big\{\omega \in \bigwedge W^{1,2}(\Omega) : \boldsymbol{t}\omega\big|_{\partial\Omega} = 0\Big\}.$$

Its restriction to zero-forms (functions) is simply the operator (2-1) on Ω with homogeneous Dirichlet boundary conditions. It is associated with the stochastic process (1-1) killed at the boundary.

- The Neumann realization $\Delta^N_{f,h}(\Omega)$ with domain

$$D(\Delta^N_{f,h})(\Omega) = \Big\{\omega \in \bigwedge W^{2,2}(\Omega) : \boldsymbol{n}\omega\big|_{\partial\Omega} = 0,\ \boldsymbol{n}d_{f,h}\omega\big|_{\partial\Omega} = 0\Big\}.$$

This is the Friedrichs extension of the closed quadratic form (2-4) defined on the domain

$$\bigwedge W^{1,2}_N(\Omega) = \Big\{\omega \in \bigwedge W^{1,2}(\Omega) : \boldsymbol{n}\omega\big|_{\partial\Omega} = 0\Big\}.$$

Its restriction to zero-forms (functions) is simply the operator (2-1) on Ω with homogeneous Neumann boundary conditions. It is associated with the stochastic process (1-1) reflected at the boundary.

We will handle exponentially small quantities and we shall use the following notation, which is convenient when comparing them.

Definition 2.3. Let $(E, \|\ \|)$ be a normed space. For two functions $a : \mathbb{R}_+ \to E$ and $b : \mathbb{R}_+ \to \mathbb{R}_+$, we write:

- $a(h) = \mathcal{O}(b(h))$ if there exist $h_0 > 0$ and $C > 0$ such that $\|a(h)\| \le Cb(h)$ for all $h \in (0, h_0)$;
- $a(h) = \tilde{\mathcal{O}}(b(h))$ if, for every $\varepsilon > 0$, $a(h) = \mathcal{O}(b(h)e^{\varepsilon/h})$, or, equivalently,

$$\forall \varepsilon > 0 \ \exists h_0 > 0 \ \exists C > 0 \ \forall h \in (0, h_0) \quad \|a(h)\| \le Cb(h)e^{\varepsilon/h}.$$

Notice that $a(h) = \tilde{\mathcal{O}}(b(h))$ is equivalent to $\limsup_{h\to 0} h \log(\|a(h)\|/b(h)) \le 0$. Note in particular the identity $\mathcal{O}(e^{-c_1/h})\tilde{\mathcal{O}}(e^{-c_2/h}) = \tilde{\mathcal{O}}(e^{-(c_1+c_2)/h}) = \mathcal{O}(e^{-c'/h})$ for any fixed $c' < c_1 + c_2$, independently of $h \in (0, h_0)$.

We are now in position to state the two additional hypotheses on f, which are stated as assumptions on the eigenvalues of Witten Laplacians on Ω_- and $\Omega_+ \setminus \overline{\Omega}_-$. We assume that there exist a constant $c_0 > 0$ and a function $\nu : (0, h_0) \to (0, +\infty)$ with

$$\forall \varepsilon > 0 \ \exists C_\varepsilon > 1 \quad \frac{1}{C_\varepsilon} e^{-\varepsilon/h} \le \nu(h) \le h, \tag{2-5}$$

or, equivalently,

$$\log\left(\frac{\nu(h)}{h}\right) \le 0 \quad \text{and} \quad \lim_{h\to 0} h\log(\nu(h)) = 0,$$

and such that the following hypotheses are fulfilled:

Hypothesis 3. *The Neumann Witten Laplacian defined on Ω_- and restricted to forms of degree* 0 *and* 1, $\Delta^{N,(p)}_{f,h}(\Omega_-)$, $p = 0, 1$, *satisfies*

$$\#\big[\sigma(\Delta^{N,(p)}_{f,h}(\Omega_-)) \cap [0, \nu(h)]\big] =: m^N_p(\Omega_-)\,, \tag{2-6}$$

$$\sigma(\Delta^{N,(p)}_{f,h}(\Omega_-)) \cap [0, \nu(h)] \subset [0, e^{-c_0/h}] \tag{2-7}$$

with $m^N_p(\Omega_-)$ independent of $h \in (0, h_0)$. Throughout, eigenvalues are counted with multiplicity, and the symbol # denotes the cardinal of a finite ensemble.

In addition, there exists in $\overline{\Omega}_-$ an open neighborhood $\mathcal{V}_-$ of $\partial\Omega_-$ such that any eigenfunction $\psi(h)$ of $\Delta^{N,(0)}_{f,h}(\Omega_-)$ associated with a small nonzero eigenvalue $\mu(h)$ (namely $0 < \mu(h) \le \nu(h)$) satisfies

$$\|\psi(h)\|_{L^2(\mathcal{V}_-)} = \tilde{\mathcal{O}}(\sqrt{\mu(h)}). \tag{2-8}$$

Hypothesis 4. *The Dirichlet Witten Laplacian on $\Omega_+ \setminus \overline{\Omega}_-$ restricted to one-forms satisfies*

$$\#\big[\sigma(\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \overline{\Omega}_-)) \cap [0, \nu(h)]\big] =: m_1^D(\Omega_+ \setminus \overline{\Omega}_-)\,, \tag{2-9}$$

$$\sigma(\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \overline{\Omega}_-)) \cap [0, \nu(h)] \subset [0, e^{-c_0/h}] \tag{2-10}$$

with $m_1^D(\Omega_+ \setminus \overline{\Omega}_-)$ independent of $h \in (0, h_0)$.

Our main results concern the smallest eigenvalue as well as properties of the associated eigenfunction of $\Delta_{f,h}^{D,(0)}(\Omega_+)$.

Theorem 2.4. *Assume Hypotheses 1, 2, 3, 4 and that $h \in (0, h_0)$ with $h_0 > 0$ small enough. The eigenvalues contained in $[0, \nu(h)]$ of the Dirichlet Witten Laplacians $\Delta_{f,h}^{D,(p)}(\Omega_+)$ for $p = 0, 1$, satisfy*:

$$\begin{aligned} m_0^D(\Omega_+) &:= \#\big[\sigma(\Delta_{f,h}^{D,(0)}(\Omega_+)) \cap [0, \nu(h)]\big] = m_0^N(\Omega_-), \\ m_1^D(\Omega_+) &:= \#\big[\sigma(\Delta_{f,h}^{D,(1)}(\Omega_+)) \cap [0, \nu(h)]\big] = m_1^N(\Omega_-) + m_1^D(\Omega_+ \setminus \overline{\Omega}_-), \\ &\sigma(\Delta_{f,h}^{D,(p)}(\Omega_+)) \cap [0, \nu(h)] \subset [0, e^{-c/h}]. \end{aligned}$$

Let $(u_k^{(1)})_{1 \le k \le m_1^D(\Omega_+ \setminus \overline{\Omega}_-)}$ be an orthonormal basis of the spectral subspace $\operatorname{Ran} 1_{[0,\nu(h)]}(\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \overline{\Omega}_-))$ *and set*

$$\kappa_f = \min_{\partial\Omega_+} f - \min_{\Omega_+} f.$$

The smallest eigenvalue of $\Delta_{f,h}^{D,(0)}(\Omega_+)$ satisfies, in the limit $h \to 0$,

$$\lim_{h \to 0} h \log \lambda_1^{(0)}(\Omega_+) = -2\kappa_f, \tag{2-11}$$

$$\lambda_1^{(0)}(\Omega_+) = \frac{h^2 \sum_{k=1}^{m_1^D(\Omega_+ \setminus \overline{\Omega}_-)} \big|\int_{\partial\Omega_+} e^{-f/h} u_k^{(1)}(n)(\sigma)\, d\sigma\big|^2}{\int_{\Omega_+} e^{-2f(x)/h}\, dx}(1 + \mathcal{O}(e^{-c/h})) \tag{2-12}$$

for some constant $c > 0$ and $u_k^{(1)}(n)(\sigma) = \boldsymbol{i}_n u_k^{(1)}(\sigma)$ with the interior product notation (A-1). Moreover, the nonnegative $L^2(\Omega_+)$-normalized eigenfunction $u_1^{(0)}$ satisfies

$$\bigg\| u_1^{(0)} - \frac{e^{-f/h}}{\big(\int_{\Omega_+} e^{-2f(x)/h}\, dx\big)^{1/2}} \bigg\|_{W^{2,2}(\Omega_+)} = \mathcal{O}(e^{-c/h}), \tag{2-13}$$

$$\bigg\| d_{f,h} u_1^{(0)} + \sum_{k=1}^{m_1^D(\Omega_+ \setminus \overline{\Omega}_-)} \frac{h \int_{\partial\Omega_+} e^{-f(\sigma)/h} u_k^{(1)}(n)(\sigma)\, d\sigma}{\big(\int_{\Omega_+} e^{-2f(x)/h}\, dx\big)^{1/2}} u_k^{(1)} \bigg\|_{W^{p,2}(\mathcal{V})} = \mathcal{O}(e^{-(\kappa_f + c_{\mathcal{V}})/h}) \tag{2-14}$$

for all $p \in \mathbb{N}$, where $\mathcal{V}$ is any neighborhood of $\partial\Omega_+$ lying in $\Omega_+ \setminus \overline{\Omega}_-$ and $c_{\mathcal{V}} > 0$ is a constant independent of p and h. The symbols $d\sigma$ and $n(\sigma)$, respectively, denote the infinitesimal volume on $\partial\Omega_+$ and the outward normal vector at $\sigma \in \partial\Omega_+$.

We would like to stress again that Theorem 2.4 does not require f to be a Morse function on Ω_+, nor on $\partial\Omega_+$.

Remark 2.5. It would be interesting for practical applications to relax the assumption $|\nabla f| > 0$ on $\overline{\Omega}_+ \setminus \Omega_-$ in Hypothesis 1 in order to be able to consider saddle points on $\partial\Omega_+$.

Remark 2.6. While proving these results, we will actually show that, necessarily, $m_1^D(\Omega_+ \setminus \overline{\Omega}_-) \neq 0$; see Remark 5.6 below.

Remark 2.7. All the terms in the sum in (2-14) are exponentially small, but at least one is larger than the remainder $\mathcal{O}(e^{-(\kappa_f + c_{\mathcal{V}})/h})$ (see (5-10) and Proposition 6.4). The number of terms which are indeed larger than the remainder depends on the precise value of $c_{\mathcal{V}}$, which depends on the geometry, the global topology of the domain and the function f (the possibility of several terms is discussed in Remarks 7.7 and 7.9 after Proposition 7.5). In particular, if f is a Morse function, the heights of the generalized critical points of index 2 along $\partial\Omega_+$ play a role.

Remark 2.8. In spectral theory, it is natural to work with complex-valued functions or complex-valued forms. In view of the probabilistic interpretation of our results, the above result is stated — and, actually, most of the analysis of this text is carried out — with real-valued functions or forms. One exception is Section 4A, which requires functional calculus and resolvents for complex spectral parameters. Notice that it is straightforward to write a complex-valued version of the previous results, by replacing the real scalar product by the hermitian scalar product. For example, in (2-14), this simply consists in changing $\int_{\partial\Omega_+} e^{-f(\sigma)/h} u_k^{(1)}(n)(\sigma)\, d\sigma$ to $\int_{\partial\Omega_+} e^{-f(\sigma)/h} \overline{u_k^{(1)}}(n)(\sigma)\, d\sigma$.

Note that the numerators in the estimates (2-12) and (2-14) of the eigenvalue $\lambda_1^{(0)}(\Omega_+)$ and of $d_{f,h} u_1^{(0)}$ depend only on the values of f and the geometry of Ω_+ around $\partial\Omega_+$. More precisely, they do not change when f is modified inside Ω_-. This allows us to understand the variations of $\lambda_1^{(0)}(\Omega_+)$ and $\partial_n u_1^{(0)}\big|_{\partial\Omega_+}$ with respect to f, which is needed in the hyperdynamics algorithm (see Section 1A).

Corollary 2.9. *Let f_1 and f_2 be two functions which fulfill Hypotheses 1, 2, 3 and 4. Let $\lambda_1^{(0)}(f_1)$ be the first eigenvalue of $\Delta_{f_1,h}^{D,(0)}(\Omega_+)$ associated with the nonnegative normalized eigenvector $u_1^{(0)}(f_1)$, and $\lambda_1^{(0)}(f_2)$ the first eigenvalue of $\Delta_{f_2,h}^{D,(0)}(\Omega_+)$ associated with the eigenvector $u_1^{(0)}(f_2)$. Assume additionally $f_1 = f_2$ in $\Omega_+ \setminus \overline{\Omega}_-$. The quantities $\lambda_1^{(0)}(f_{1,2})$ and $\partial_n[e^{-f_{1,2}/h} u_1^{(0)}(f_{1,2})]\big|_{\partial\Omega_+} = e^{-f_{1,2}/h}[\partial_n u_1^{(0)}(f_{1,2})]\big|_{\partial\Omega_+}$ satisfy*

$$\frac{\lambda_1^{(0)}(f_2)}{\lambda_1^{(0)}(f_1)} = \frac{\int_{\Omega_+} e^{-2f_1(x)/h}\, dx}{\int_{\Omega_+} e^{-2f_2(x)/h}\, dx}(1 + \mathcal{O}(e^{-c/h})), \tag{2-15}$$

$$\frac{\partial_n[e^{-f_2/h} u_1^{(0)}(f_2)]\big|_{\partial\Omega_+}}{\|\partial_n[e^{-f_2/h} u_1^{(0)}(f_2)]\|_{L^1(\partial\Omega_+)}} = \frac{\partial_n[e^{-f_1/h} u_1^{(0)}(f_1)]\big|_{\partial\Omega_+}}{\|\partial_n[e^{-f_1/h} u_1^{(0)}(f_1)]\|_{L^1(\partial\Omega_+)}} + \mathcal{O}(e^{-c/h}) \quad \text{in } L^1(\partial\Omega_+). \tag{2-16}$$

Other corollaries and variations of Theorem 2.4 are given in Section 6. Among the consequences, one can prove the following result when, additionally, $f\big|_{\partial\Omega_+}$ is a Morse function and $\partial_n f > 0$ on $\partial\Omega_+$.

Theorem 2.10. *Assume Hypotheses 1, 2, 3 and 4 and $h \in (0, h_0)$ with $h_0 > 0$ small enough. Assume moreover that $f\big|_{\partial\Omega_+}$ is a Morse function and $\partial_n f > 0$ on $\partial\Omega_+$. Then the first eigenvalue $\lambda_1^{(0)}(\Omega_+)$ of*

$\Delta_{f,h}^{D,(0)}(\Omega_+)$ *and the corresponding* $L^2(\Omega_+)$*-normalized nonnegative eigenfunction* $u_1^{(0)}$ *satisfy*

$$\lambda_1^{(0)}(\Omega_+) = \frac{\int_{\partial\Omega_+} 2\partial_n f(\sigma) e^{-2f(\sigma)/h}\, d\sigma}{\int_{\Omega_+} e^{-2f(x)/h}\, dx}(1+\mathcal{O}(h)), \tag{2-17}$$

$$-\frac{\partial_n[e^{-f/h}u_1^{(0)}]\big|_{\partial\Omega_+}}{\|\partial_n[e^{-f/h}u_1^{(0)}]\|_{L^1(\partial\Omega_+)}} = \frac{(2\partial_n f)e^{-2f/h}\big|_{\partial\Omega_+}}{\|(2\partial_n f)e^{-2f/h}\|_{L^1(\partial\Omega_+)}} + \mathcal{O}(h) \quad \text{in } L^1(\partial\Omega_+). \tag{2-18}$$

The proof of Theorem 2.4 is given in Proposition 3.12, Lemma 5.9, Proposition 6.1 and Proposition 6.8. The proof of Corollary 2.9 is given in Section 6D. The proof of Theorem 2.10 is given in Section 7A2.

3. A priori exponential decay and first consequences

By applying Agmon's type estimate (see, for example, [Helffer 1988; Dimassi and Sjöstrand 1999] for a general introduction) for boundary Witten Laplacians, we give here exponential decay estimates for the eigenvectors of $\Delta^N_{f,h}(\Omega_-)$, $\Delta^D_{f,h}(\Omega_+ \setminus \overline{\Omega}_-)$ and $\Delta^D_{f,h}(\Omega_+)$.

3A. *Agmon identity.* We shall use an identity for boundary Witten Laplacians, proved in [Helffer and Nier 2006] in the Dirichlet case and in [Le Peutrec 2010b] in the Neumann case.

Lemma 3.1. *Let* Ω *be a regular bounded domain of* (M, g) *and let* $\Delta^D_{f,h}(\Omega)$ (*resp.* $\Delta^N_{f,h}(\Omega)$) *be the Dirichlet* (*resp. Neumann*) *realization of* $\Delta_{f,h}(\Omega)$*. Let* φ *be a real-valued Lipschitz function on* $\overline{\Omega}$*. Then, for any real-valued* $\omega \in D(\Delta^D_{f,h}(\Omega))$ (*resp.* $\omega \in D(\Delta^N_{f,h}(\Omega))$),

$$\begin{aligned}\langle \omega, e^{2\varphi/h}\Delta^D_{f,h}(\Omega)\omega\rangle_{L^2(\Omega)} &= h^2\|de^{\varphi/h}\omega\|^2_{L^2(\Omega)} + h^2\|d^*e^{\varphi/h}\omega\|^2_{L^2(\Omega)} + \langle(|\nabla f|^2 - |\nabla\varphi|^2 + h\mathcal{L}_{\nabla f} + h\mathcal{L}^*_{\nabla f})e^{\varphi/h}\omega, e^{\varphi/h}\omega\rangle_{L^2(\Omega)}\\ &\quad - h\int_{\partial\Omega}\langle\omega,\omega\rangle_{T^*_\sigma\Omega}e^{2\varphi(\sigma)/h}\frac{\partial f}{\partial n}(\sigma)\, d\sigma.\end{aligned}$$

$$\begin{aligned}\langle \omega, e^{2\varphi/h}\Delta^N_{f,h}(\Omega)\omega\rangle_{L^2(\Omega)} &= h^2\|de^{\varphi/h}\omega\|^2_{L^2(\Omega)} + h^2\|d^*e^{\varphi/h}\omega\|^2_{L^2(\Omega)} + \langle(|\nabla f|^2 - |\nabla\varphi|^2 + h\mathcal{L}_{\nabla f} + h\mathcal{L}^*_{\nabla f})e^{\varphi/h}\omega, e^{\varphi/h}\omega\rangle_{L^2(\Omega)}\\ &\quad + h\int_{\partial\Omega}\langle\omega,\omega\rangle_{T^*_\sigma\Omega}e^{2\varphi(\sigma)/h}\frac{\partial f}{\partial n}(\sigma)\, d\sigma.\end{aligned}$$

In the previous formulas, the notation $\mathcal{L}_X$ refers to the Lie derivative; see (A-2). We shall use this lemma with specific functions φ associated with the metric $|\nabla f|^2 g$.

Lemma 3.2. *Let* Ω *be an open subset of* M, $f \in \mathcal{C}^\infty(\overline{\Omega})$*, and let* d_{Ag} *be the geodesic pseudodistance on* $\overline{\Omega}$ *associated with the possibly degenerate metric* $|\nabla f|^2 g$*. The function* $(x, y) \mapsto d_{Ag}(x, y)$ *is Lipschitz* (*and thus almost everywhere differentiable*) *and satisfies*

$$\begin{aligned}&|\nabla_x d_{Ag}(x, y_0)| \le |\nabla f(x)| \quad \text{for all } y_0 \in \overline{\Omega} \text{ and for a.e. } x \in \Omega,\\ &|f(x) - f(y)| \le d_{Ag}(x, y) \quad \text{for all } x, y \in \overline{\Omega}.\end{aligned} \tag{3-1}$$

The equality $d_{Ag}(x, y) = |f(x) - f(y)|$ *occurs if there is an integral curve of* ∇f *joining* x *to* y*. Moreover, for any* $A \subset \overline{\Omega}$*, the function* $x \mapsto d_{Ag}(x, A)$ *(where* $d_{Ag}(x, A) = \inf_{a \in A} d_{Ag}(x, a)$*) is Lipschitz and satisfies*

$$|\nabla_x d_{Ag}(x, A)| \leq |\nabla f(x)| \quad \text{for a.e. } x \in \Omega.$$

Proof. The Lipschitz property comes from the triangular inequality for $d_{\mathrm{Ag}}(x, y)$. It carries over to $d_{\mathrm{Ag}}(x, A)$. The comparison between $|f(x) - f(y)|$ and $d_{\mathrm{Ag}}(x, y)$ comes from

$$|f(x) - f(y)| = \left| \int_0^1 \nabla f(\gamma(t)) \cdot \dot{\gamma}(t)\, dt \right| \leq \int_0^1 |\nabla f(\gamma(t))|\, |\dot{\gamma}(t)|\, dt = |\gamma|_{\mathrm{Ag}}$$

for any $\mathcal{C}^1$-path γ joining x to y and denoting by $|\gamma|_{\mathrm{Ag}}$ its length according to d_{Ag}. □

Remark 3.3. A detailed discussion about the equality $d_{\mathrm{Ag}}(x, y) = |f(x) - f(y)|$ when f is a Morse function, which involves the notion of generalized integral curves of ∇f, can be found in [Helffer and Sjöstrand 1985b].

3B. ***Exponential decay for the eigenvectors of*** $\boldsymbol{\Delta^{N,(p)}_{f,h}(\Omega_-)}$ ***(***$\boldsymbol{p = 0, 1}$***).*** Notice that, from Hypothesis 1, there exists an open set U such that

$$\overline{U} \subset \Omega_- \qquad \text{and} \qquad |\nabla f| \neq 0 \quad \text{in } \overline{\Omega}_- \setminus U. \tag{3-2}$$

The following proposition will be useful to prove that all the eigenvectors of $\Delta^{N,(p)}_{f,h}$ are exponentially small in the neighborhood of $\partial\Omega_-$ (see Proposition 3.5). It actually holds for any open set $U \subset \Omega_-$ which contains all the critical points, without the additional requirement $\overline{U} \subset \Omega_-$.

Proposition 3.4. *Let* U *be an open subset of* Ω_- *such that* $|\nabla f| \neq 0$ *in* $\overline{\Omega}_- \setminus U$ *and let* $d_{Ag}(x, U)$ *be the Agmon distance to* U *defined for* $x \in \Omega_-$*. There exists a constant* $C > 0$ *independent of* $h \in [0, h_0]$ *such that every normalized eigenvector* ω_{λ_h} *of* $\Delta^N_{f,h}(\Omega_-)$ *associated with an eigenvalue* $\lambda_h \in [0, \nu(h)]$ *satisfies*

$$\|e^{d_{Ag}(\cdot, U)/h}\omega_{\lambda_h}\|_{L^2(\Omega_- \setminus U)} \leq \|e^{d_{Ag}(\cdot, U)/h}\omega_{\lambda_h}\|_{L^2(\Omega_-)} \leq C,$$
$$\|e^{d_{Ag}(\cdot, U)/h}\omega_{\lambda_h}\|_{W^{1,2}(\Omega_- \setminus U)} \leq \|e^{d_{Ag}(\cdot, U)/h}\omega_{\lambda_h}\|_{W^{1,2}(\Omega_-)} \leq \frac{C}{h^{1/2}}.$$

Proof. The function $d_{\mathrm{Ag}}(\cdot, U)$ vanishes in $\overline{U}$ and satisfies the properties of Lemma 3.2 with $(\Omega, A) = (\Omega_-, \overline{U})$. Let us now apply Lemma 3.1 on $\Delta^N_{f,h}(\Omega_-)$ with the function $\varphi = (1 - \alpha h) d_{\mathrm{Ag}}(\cdot, U)$ (where α is a positive constant to be fixed later on) and a normalized eigenvector ω: $\Delta^N_{f,h}(\Omega_-)\omega = \lambda\omega$, where $\lambda \in [0, \nu(h)]$. With $\partial f/\partial n > 0$ on $\partial\Omega_-$, $\nu(h) \leq h$ and $|\nabla\varphi|^2 \leq (1 - \alpha h)|\nabla f|^2$ (for $h < 1/\alpha$), we obtain

$$0 \geq h^2 \|d e^{\varphi/h}\omega\|^2_{L^2(\Omega_-)} + h^2 \|d^* e^{\varphi/h}\omega\|^2_{L^2(\Omega_-)} + h[\alpha \langle e^{\varphi/h}\omega, |\nabla f|^2 e^{\varphi/h}\omega\rangle_{L^2(\Omega_-)} - C_f \|e^{\varphi/h}\omega\|^2_{L^2(\Omega_-)}]. \tag{3-3}$$

Here, we have used the fact that, for any vector field X, $\mathcal{L}_X + \mathcal{L}^*_X$ is a differential operator of order 0 involving derivatives of X and g that are uniformly bounded in $\overline{\Omega}_-$.

Using (3-2), choose α such that $\alpha \min_{x \in \overline{\Omega}_- \setminus U} |\nabla f(x)|^2 \geq 2C_f$ and add $2C_f h \|e^{\varphi/h}\omega\|^2_{L^2(U)}$ on both sides of the inequality (3-3). Using the fact that

$$2C_f h \geq 2C_f h \|\omega\|^2_{L^2(U)} = 2C_f h \|e^{\varphi/h}\omega\|^2_{L^2(U)},$$

one obtains

$$2C_f h \geq h^2\|de^{\varphi/h}\omega\|^2_{L^2(\Omega_-)} + h^2\|d^*e^{\varphi/h}\omega\|^2_{L^2(\Omega_-)} + C_f h\|e^{\varphi/h}\omega\|^2_{L^2(\Omega_-)}.$$

This implies $\|e^{(1-\alpha h)d_{\mathrm{Ag}}(\cdot\,,U)/h}\omega\|^2_{L^2(\Omega_-)} \leq 2$ and

$$\|(hd)e^{(1-\alpha h)d_{\mathrm{Ag}}(\cdot\,,U)/h}\omega\|^2_{L^2(\Omega_-)} + \|(hd)^*e^{(1-\alpha h)d_{\mathrm{Ag}}(\cdot\,,U)/h}\omega\|^2_{L^2(\Omega_-)} \leq 2C_f h.$$

Since $d_{\mathrm{Ag}}(\cdot\,, U)$ is a Lipschitz (and thus also bounded) function on $\overline{\Omega}_-$, this ends the proof. □

Here is a useful consequence of Proposition 3.4:

Proposition 3.5. *Let $(\psi_j^{(0)})_{1\leq j\leq m_0^N(\Omega_-)}$ (resp. $(\psi_k^{(1)})_{1\leq k\leq m_1^N(\Omega_-)}$) be an orthonormal basis of eigenvectors of $\Delta_{f,h}^{N,(0)}(\Omega_-)$ (resp. $\Delta_{f,h}^{N,(1)}(\Omega_-)$) associated with the eigenvalues lying in $[0,\nu(h)]$ (or, owing to Hypothesis 3, in $[0,e^{-c_0/h}]$). Let $U\subset\Omega_-$ be an open set satisfying (3-2). Let $\chi_-\in\mathcal{C}_0^\infty(\Omega_-)$ be a cut-off function such that $0\leq\chi_-\leq 1$ and $\chi_-\equiv 1$ on a neighborhood of U. The functions $v_j^{(0)}=\chi_-\psi_j^{(0)}$, $1\leq j\leq m_0^N(\Omega_-)$ (resp. one-forms $v_k^{(1)}=\chi_-\psi_k^{(1)}$, $1\leq k\leq m_1^N(\Omega_-)$) belong to the domain $D(\Delta_{f,h}^{D,(0)}(\Omega_+))$ (resp. $D(\Delta_{f,h}^{D,(1)}(\Omega_+))$) of the Dirichlet realization of $\Delta_{f,h}$ in Ω_+ and they satisfy: for $h\in[0,h_0]$,*

$$\sum_{j=1}^{m_0^N(\Omega_-)}\|\psi_j^{(0)}-v_j^{(0)}\|_{W^{1,2}(\Omega_-)} + \sum_{k=1}^{m_1^N(\Omega_-)}\|\psi_k^{(1)}-v_k^{(1)}\|_{W^{1,2}(\Omega_-)} = \mathcal{O}(e^{-c_{\chi_-}/h}),$$

$$(\langle v_j^{(0)}, v_{j'}^{(0)}\rangle_{L^2(\Omega_+)})_{j,j'} = \mathrm{Id}_{m_0^N(\Omega_-)} + \mathcal{O}(e^{-c_{\chi_-}/h}),\qquad (\langle v_k^{(1)}, v_{k'}^{(1)}\rangle_{L^2(\Omega_+)})_{k,k'} = \mathrm{Id}_{m_1^N(\Omega_-)} + \mathcal{O}(e^{-c_{\chi_-}/h}),$$

$$\langle v_j^{(0)}, \Delta_{f,h}^{D,(0)}(\Omega_+)v_j^{(0)}\rangle_{L^2(\Omega_+)} = \mathcal{O}(e^{-c_{\chi_-}/h}),\qquad \langle v_k^{(1)}, \Delta_{f,h}^{D,(1)}(\Omega_+)v_k^{(1)}\rangle_{L^2(\Omega_+)} = \mathcal{O}(e^{-c_{\chi_-}/h}),$$

where the $\mathcal{O}(e^{-c_{\chi_-}/h})$ remainders can be bounded from above by $C_{\chi_-}e^{-c_{\chi_-}/h}$ for some constants C_{χ_-}, $c_{\chi_-}>0$ independent of $h\in[0,h_0]$. Throughout, Id_m denotes the identity matrix of size $m\times m$.

Proof. Let ψ be a $L^2(\Omega_-)$-normalized eigenvector of $\Delta_{f,h}^{N,(p)}(\Omega_-)$, $p=0,1$, associated with the eigenvalue $\lambda=\mathcal{O}(e^{-c_0/h})$, and set $v=\chi_-\psi$. Since χ_- belongs to $\mathcal{C}_0^\infty(\Omega_-)$ the form $v=\chi_-\psi$ belongs to $D(\Delta_{f,h}^{D,(p)}(\Omega_+))$.

The $W^{1,2}(\Omega_-)$ estimates as well as the result on the Gram matrices are consequences of

$$\|\psi-v\|_{W^{1,2}(\Omega_-)} = \|(1-\chi_-)\psi\|_{W^{1,2}(\Omega_-)} \leq \|\psi\|_{W^{1,2}(\Omega_-\setminus\{\chi_-=1\})} \leq C'_{\chi_-}e^{-c'_{\chi_-}/h} \tag{3-4}$$

for some constants $c'_{\chi_-}>0$ and $C'_{\chi_-}>0$. The estimate (3-4) is derived from Proposition 3.4 by using the fact that there exists $c>0$ such that $d_{\mathrm{Ag}}(x,U)\geq c$ for all $x\in\Omega_-\setminus\{\chi_-=1\}$ (this is a consequence of (3-2)).

For the last estimate of Proposition 3.5, we use Lemma 3.1 with $\varphi=0$. Considering first the estimate on $\Delta_{f,h}^D$ with $\Omega=\Omega_+$, $\omega=v=\chi_-\psi$ and then the estimate on $\Delta_{f,h}^N$ with $\Omega=\Omega_-$, $\omega=\psi$, one obtains

$$\langle\chi_-\psi, \Delta_{f,h}^D(\Omega_+)\chi_-\psi\rangle_{L^2(\Omega_+)}$$
$$= h^2\|d\chi_-\psi\|^2_{L^2(\Omega_+)} + h^2\|d^*\chi_-\psi\|^2_{L^2(\Omega_+)} + \langle(|\nabla f|^2 + h\mathcal{L}_{\nabla f} + h\mathcal{L}^*_{\nabla f})\chi_-\psi, \chi_-\psi\rangle_{L^2(\Omega_+)} + 0 \tag{3-5}$$

and (since $\partial f/\partial n > 0$ on $\partial\Omega_-$)

$$e^{-c_0/h} \geq \lambda \geq h^2\|d\psi\|^2_{L^2(\Omega_-)} + h^2\|d^*\psi\|^2_{L^2(\Omega_-)} + \langle(|\nabla f|^2 + h\mathcal{L}_{\nabla f} + h\mathcal{L}^*_{\nabla f})\psi, \psi\rangle_{L^2(\Omega_-)}. \tag{3-6}$$

By considering the difference between (3-5) and (3-6), we thus have

$$\begin{aligned}
&\langle \chi_-\psi, \Delta^D_{f,h}(\Omega_+)\chi_-\psi\rangle_{L^2(\Omega_+)} \\
&\quad \leq e^{-c_0/h} + h^2(\|d\chi_-\psi\|^2_{L^2(\Omega_+)} - \|d\psi\|^2_{L^2(\Omega_-)}) + h^2(\|d^*\chi_-\psi\|^2_{L^2(\Omega_+)} - \|d^*\psi\|^2_{L^2(\Omega_-)}) \\
&\qquad + \big(\langle(|\nabla f|^2 + h\mathcal{L}_{\nabla f} + h\mathcal{L}^*_{\nabla f})\chi_-\psi, \chi_-\psi\rangle_{L^2(\Omega_+)} - \langle(|\nabla f|^2 + h\mathcal{L}_{\nabla f} + h\mathcal{L}^*_{\nabla f})\psi, \psi\rangle_{L^2(\Omega_-)}\big).
\end{aligned}$$

The last three terms in the right-hand side are all of order $\mathcal{O}(e^{-c_{\chi_-}/h})$. Indeed, for the first term (the two other terms are estimated in the same way),

$$\begin{aligned}
\big|\|d\chi_-\psi\|^2_{L^2(\Omega_+)} - \|d\psi\|^2_{L^2(\Omega_-)}\big| &= |\langle d(1-\chi_-)\psi, d(1+\chi_-)\psi\rangle_{L^2(\Omega_-)}| \\
&\leq C''_{\chi_-}\|\psi\|^2_{W^{1,2}(\Omega_-\setminus\{\chi_-=1\})} \leq C^{(3)}_{\chi_-}e^{-2c'_{\chi_-}/h},
\end{aligned}$$

using again (3-4). This proves the last estimate. □

According to the terminology of [Le Peutrec 2009], the property on the Gram matrices in Proposition 3.5 is equivalent to the almost orthonormality of the family $(v_j^{(p)})_{1\leq j\leq m_p^N(\Omega_-)}$, $p = 0, 1$, in $L^2(\Omega_+)$.

Definition 3.6. A finite family of h-dependent vectors $(u_k^h)_{1\leq k\leq N}$ in a Hilbert space $\mathcal{H}$ is almost orthonormal if the Gram matrix satisfies

$$(\langle u_j^h, u_k^h\rangle)_{1\leq j,k\leq N} = \mathrm{Id}_N + \mathcal{O}(e^{-c/h})$$

for some $c > 0$ independent of h.

We end this subsection with some remarks on the spectrum of $\Delta^{N,(0)}_{f,h}(\Omega_-)$, which we denote (as usual, in increasing order and with multiplicity) by $(\mu_k^{(0)}(\Omega_-))_{k\geq 1}$. The first eigenvalue of $\Delta^{N,(0)}_{f,h}(\Omega_-)$ is $\mu_1^{(0)}(\Omega_-) = 0$ associated with the eigenvector

$$\psi_1^{(0)} = \frac{e^{-f/h}}{\big(\int_{\Omega_-} e^{-2f(x)/h}\,dx\big)^{1/2}}.$$

One can prove that the second eigenvalue $\mu_2^{(0)}(\Omega_-)$ of $\Delta^{N,(0)}_{f,h}(\Omega_-)$ is exponentially large compared to $e^{-2\kappa_f/h}$, where we recall $\kappa_f = \min_{\partial\Omega_+} f - \min_{\Omega_+} f = \min_{\partial\Omega_+} f - \min_{\Omega_-} f$.

Proposition 3.7. *Let* cvmax *be the maximum critical value of f in Ω_-:*

$$\mathrm{cvmax} = \max\{f(x) : x \in \Omega_-, \nabla f(x) = 0\}.$$

Then the second eigenvalue $\mu_2^{(0)}(\Omega_-)$ of $\Delta^{N,(0)}_{f,h}(\Omega_-)$ satisfies

$$\liminf_{h\to 0} h\log(\mu_2^{(0)}(\Omega_-)) \geq -2(\mathrm{cvmax} - \min_{\Omega_-} f) \geq -2\kappa_f + 2c_0,$$

where c_0 denotes the positive constant used in Hypothesis 2.

Proof. The second inequality $-2(\mathrm{cvmax} - \min_{\Omega_-} f) \geq -2\kappa_f + 2c_0$ is of course a consequence of Hypothesis 2. To prove the first inequality, let us reason by contradiction and assume that there exists $\varepsilon_0 > 0$ and a sequence h_n such that $\lim_{n\to\infty} h_n = 0$ and

$$\min\{\sigma(\Delta_{f,h_n}^{N,(0)}(\Omega_-)) \setminus \{0\}\} \leq Ce^{-2(\mathrm{cvmax} - \min_{\Omega_-} f+\varepsilon_0)/h_n}.$$

To simplify the notation, let us drop the subscript n in h_n. Let $\psi_2^{(0)}$ be a normalized eigenfunction of $\Delta_{f,h}^{N,(0)}(\Omega_-)$ associated with $\mu_2^{(0)}(\Omega_-) > 0$. It is orthogonal to $\psi_1^{(0)}$ in $L^2(\Omega_-)$ and it satisfies: for any $\Omega \subset \Omega_-$,

$$\|d_{f,h}\psi_2^{(0)}\|^2_{L^2(\Omega)} \leq \|d_{f,h}\psi_2^{(0)}\|^2_{L^2(\Omega_-)} = \langle\psi_2^{(0)}, \Delta_{f,h}^{N,(0)}(\Omega_-)\psi_2^{(0)}\rangle_{L^2(\Omega_-)} = \mu_2^{(0)} \leq Ce^{-2(\mathrm{cvmax} - \min_{\Omega_-} f+\varepsilon_0)/h}.$$

In particular, for $\Omega = \{x \in \Omega_- : f(x) < \mathrm{cvmax} + \frac{1}{2}\varepsilon_0\}$, this gives

$$\begin{aligned}\|d(e^{(f-\min_{\Omega_-} f)/h}\psi_2^{(0)})\|^2_{L^2(\Omega)} &\leq h^{-2}\max_{x\in\Omega}|e^{(f(x)-\min_{\Omega_-} f)/h}|^2\|d_{f,h}\psi_2^{(0)}\|^2_{L^2(\Omega)}\\ &\leq Ch^{-2}e^{-2(\mathrm{cvmax} - \min_{\Omega_-} f+\varepsilon_0)/h}\max_{x\in\Omega}|e^{(f(x)-\min_{\Omega_-} f)/h}|^2 \leq C'e^{-\varepsilon_0/h}.\end{aligned}$$

Using the spectral gap estimate for the Neumann Laplacian in Ω (or equivalently the Poincaré–Wirtinger inequality on Ω), there is a constant C_h (depending on $\psi_2^{(0)}$) such that

$$\|\psi_2^{(0)} - C_he^{-(f-\min_{\Omega_-} f)/h}\|_{L^2(\Omega)} = \mathcal{O}(e^{-\varepsilon_0/(2h)}).$$

Equivalently, there is a constant C_h such that

$$\|\psi_2^{(0)} - C_h\psi_1^{(0)}\|_{L^2(\Omega)} = \mathcal{O}(e^{-\varepsilon_0/(2h)}). \tag{3-7}$$

Further, using Proposition 3.4 with $U = \{x \in \Omega_- : f(x) < \mathrm{cvmax} + \frac{1}{4}\varepsilon_0\} \subset \Omega$, and a lower bound on $d_{\mathrm{Ag}}(x, U)$ (see (3-4) for a similar argument), one obtains

$$\|\psi_1^{(0)}\|_{L^2(\Omega_-\setminus\Omega)} + \|\psi_2^{(0)}\|_{L^2(\Omega_-\setminus\Omega)} \leq C_{\varepsilon_0}e^{-c_{\varepsilon_0}/h}. \tag{3-8}$$

The two estimates (3-7) and (3-8) contradict the orthogonality of $\psi_2^{(0)}$ and $\psi_1^{(0)}$ in $L^2(\Omega_-)$ in the limit $h \to 0$ (actually $n \to \infty$). □

3C. *Exponential decay for the eigenvectors of $\Delta_{f,h}^{D,(p)}(\Omega_+ \setminus \bar{\Omega}_-)$.* In this section, we will check that $\sigma(\Delta_{f,h}^{D,(0)}(\Omega_+ \setminus \bar{\Omega}_-)) \cap [0, \nu(h)] = \varnothing$ and provide the same results as in the previous section for the eigenvectors of $\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \bar{\Omega}_-)$. Let us start with an equivalent of Proposition 3.4.

Proposition 3.8. *Let $\mathcal{V}$ be a subset of $\bar{\Omega}_+ \setminus \Omega_-$ such that $\partial\Omega_+ \subset \mathcal{V}$ and let $d_{Ag}(x, \mathcal{V})$ be the Agmon distance to $\mathcal{V}$ defined for $x \in \Omega_+ \setminus \bar{\Omega}_-$. There exists a constant $C > 0$ independent of $h \in [0, h_0]$ such that every normalized eigenvector ψ of $\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \bar{\Omega}_-)$ associated with an eigenvalue $\lambda \in [0, \nu(h)]$ satisfies*

$$\|e^{d_{Ag}(\cdot,\mathcal{V})/h}\psi\|_{W^{1,2}(\Omega_+\setminus\bar{\Omega}_-)} \leq \frac{C}{h}.$$

Proof. The proof follows ideas from [Dimassi and Sjöstrand 1999]. Using Lemma 3.1, the fact that $\lambda \le h$ and the assumption on the sign of the normal derivative of f on $\partial\Omega_-$ stated in Hypothesis 1, we have

$$0 \ge h^2\|de^{\varphi/h}\psi\|^2_{L^2(\Omega_+\setminus\bar\Omega_-)} + h^2\|d^*e^{\varphi/h}\psi\|^2_{L^2(\Omega_+\setminus\bar\Omega_-)} + \langle(|\nabla f|^2-|\nabla\varphi|^2)e^{\varphi/h}\psi, e^{\varphi/h}\psi\rangle_{L^2(\Omega_+\setminus\bar\Omega_-)} \\ - hC_f\|e^{\varphi/h}\psi\|^2_{L^2(\Omega_+\setminus\bar\Omega_-)} - h\int_{\partial\Omega_+}\langle\psi,\psi\rangle_{\bigwedge T^*_\sigma\Omega_+} e^{2\varphi(\sigma)/h}\frac{\partial f}{\partial n}(\sigma)\,d\sigma. \quad (3\text{-}9)$$

Using the trace theorem, there exists a constant $C_{\mathcal V}$ such that, for any $\omega \in \bigwedge W^{1,2}(\mathcal V)$,

$$\int_{\partial\Omega_+}\langle\omega,\omega\rangle_{\bigwedge T^*_\sigma\Omega_+}\,d\sigma \le C_{\mathcal V}[\|\omega\|^2_{L^2(\mathcal V)} + \|\omega\|_{W^{1,2}(\mathcal V)}\|\omega\|_{L^2(\mathcal V)}].$$

By applying this inequality to $\omega = e^{\varphi/h}\psi$ and using

$$\|\omega\|^2_{W^{1,2}(\mathcal V)} \le C_{\mathcal V}[\|\omega\|^2_{L^2(\mathcal V)} + \|d\omega\|^2_{L^2(\mathcal V)} + \|d^*\omega\|^2_{L^2(\mathcal V)}],$$

the last term of (3-9) is estimated by

$$\left|h\int_{\partial\Omega_+}\langle\psi,\psi\rangle_{\bigwedge T^*_\sigma\Omega_+} e^{2\varphi(\sigma)/h}\frac{\partial f}{\partial n}(\sigma)\,d\sigma\right| \le \tfrac12 h^2[\|de^{\varphi/h}\psi\|^2_{L^2(\mathcal V)} + \|d^*e^{\varphi/h}\psi\|^2_{L^2(\mathcal V)}] + C_{f,\mathcal V}\|e^{\varphi/h}\psi\|^2_{L^2(\mathcal V)} \\ \le \tfrac12 h^2[\|de^{\varphi/h}\psi\|^2_{L^2(\Omega_+\setminus\bar\Omega_-)} + \|d^*e^{\varphi/h}\psi\|^2_{L^2(\Omega_+\setminus\bar\Omega_-)}] + C_{f,\mathcal V}$$

since $\varphi \equiv 0$ on $\mathcal V$. Taking $\varphi = (1-\alpha h)d_{\mathrm{Ag}}(x,\mathcal V)$ in (3-9) gives (using $|\nabla\varphi|^2 \le (1-\alpha h)|\nabla f|^2$ and the inequality $\|e^{\varphi/h}\psi\|^2_{L^2(\Omega_+\setminus\bar\Omega_-)} = \|e^{\varphi/h}\psi\|^2_{L^2(\mathcal V)} + \|e^{\varphi/h}\psi\|^2_{L^2(\Omega_+\setminus\overline{\Omega_-\cup\mathcal V})} \le C'_{\mathcal V} + \|e^{\varphi/h}\psi\|^2_{L^2(\Omega_+\setminus\overline{\Omega_-\cup\mathcal V})}$)

$$C'_{f,\mathcal V} \ge \tfrac12 h^2[\|de^{\varphi/h}\psi\|^2_{L^2(\Omega_+\setminus\bar\Omega_-)} + \|d^*e^{\varphi/h}\psi\|^2_{L^2(\Omega_+\setminus\bar\Omega_-)}] \\ + h\big(\alpha\min_{x\in\Omega_+\setminus\overline{\Omega_-\cup\mathcal V}}|\nabla f(x)|^2 - C_f\big)\|e^{\varphi/h}\psi\|^2_{L^2(\Omega_+\setminus\overline{\Omega_-\cup\mathcal V})}.$$

By taking α large enough, this yields the exponential decay estimate

$$\|e^{d_{\mathrm{Ag}}(\cdot,\mathcal V)/h}\psi\|_{W^{1,2}(\Omega_+\setminus\bar\Omega_-)} \le \frac{C''_{f,\mathcal V}}{h}. \qquad \square$$

We are now in position to state the main result of this section, which can be seen as an equivalent of Proposition 3.5 for $\Delta^{D,(p)}_{f,h}(\Omega_+\setminus\bar\Omega_-)$.

Proposition 3.9. (1) *There is a constant $c > 0$ such that*

$$\sigma(\Delta^{D,(0)}_{f,h}(\Omega_+\setminus\bar\Omega_-)) \cap [0,c] = \varnothing \quad \textit{for all } h \in (0,h_0). \quad (3\text{-}10)$$

(2) *Let $(\psi^{(1)}_k)_{m^N_1(\Omega_-)+1\le k\le m^N_1(\Omega_-)+m^D_1(\Omega_+\setminus\bar\Omega_-)}$ be an orthonormal basis of eigenvectors of $\Delta^{D,(1)}_{f,h}(\Omega_+\setminus\bar\Omega_-)$ associated with the eigenvalues in $[0,\nu(h)]$, and let $\chi_+ \in \mathcal C^\infty(\bar\Omega_+)$ be such that $\chi_+ \equiv 1$ in a neighborhood of $\partial\Omega_+$ and $\chi_+ \equiv 0$ in a neighborhood of $\bar\Omega_-$. For all $k \in \{m^N_1(\Omega_-)+1,\dots,m^N_1(\Omega_-)+m^D_1(\Omega_+\setminus\bar\Omega_-)\}$, set $v^{(1)}_k = \chi_+\psi^{(1)}_k$. Then*

$$\sum_{k=m^N_1(\Omega_-)+1}^{m^N_1(\Omega_-)+m^D_1(\Omega_+\setminus\bar\Omega_-)} \|\psi^{(1)}_k - v^{(1)}_k\|_{W^{1,2}(\Omega_+\setminus\bar\Omega_-)} = \mathcal O(e^{-c_{\chi_+}/h}), \quad (3\text{-}11)$$

that is, the one-forms $v_k^{(1)}$ *are close to* $\psi_k^{(1)}$ *for* $k \in \{m_1^N(\Omega_-)+1, \dots, m_1^N(\Omega_-)+m_1^D(\Omega_+ \setminus \bar{\Omega}_-)\}$*. They are almost orthonormal in* $L^2(\Omega_+)$*:*

$$(\langle v_k^{(1)}, v_{k'}^{(1)} \rangle_{L^2(\Omega_+)})_{k,k'} = \mathrm{Id}_{m_1^D(\Omega_+ \setminus \bar{\Omega}_-)} + \mathcal{O}(e^{-c_{\chi_+}/h}).$$

Moreover, they belong to $D(\Delta_{f,h}^{D,(1)}(\Omega_+))$ *and they satisfy*

$$\langle v_k^{(1)}, \Delta_{f,h}^{D,(1)}(\Omega_+) v_k^{(1)} \rangle_{L^2(\Omega_+)} = \mathcal{O}(e^{-c_{\chi_+}/h}) \quad \text{and} \quad d_{f,h}^* v_k^{(1)} \equiv 0 \quad \text{in } \{\chi_+ = 1\}.$$

All the $\mathcal{O}(e^{-c_{\chi_+}/h})$ *remainders can be bounded from above by* $C_{\chi_+} e^{-c_{\chi_+}/h}$ *for some constants* $C_{\chi_+}, c_{\chi_+} > 0$ *independent of* $h \in [0, h_0]$*.*

Proof. (1) The lower bound on the spectrum of $\Delta_{f,h}^{D,(0)}(\Omega_+ \setminus \bar{\Omega}_-)$ comes from Lemma 3.1, used with $\varphi = 0$, and Hypothesis 1: for any function $\omega \in D(\Delta_{f,h}^{D,(0)}(\Omega_+ \setminus \bar{\Omega}_-))$,

$$\begin{aligned}&\langle \omega, \Delta_{f,h}^{D,(0)}(\Omega_+ \setminus \bar{\Omega}_-)\omega \rangle_{L^2(\Omega_+ \setminus \bar{\Omega}_-)} \\ &= h^2 \|d\omega\|^2_{L^2(\Omega_+ \setminus \bar{\Omega}_-)} + h^2 \|d^*\omega\|^2_{L^2(\Omega_+ \setminus \bar{\Omega}_-)} + \langle (|\nabla f|^2 + h\mathcal{L}_{\nabla f} + h\mathcal{L}^*_{\nabla f})\omega, \omega \rangle_{L^2(\Omega_+ \setminus \bar{\Omega}_-)} \geq C_f \|\omega\|^2_{L^2(\Omega_+ \setminus \bar{\Omega}_-)}.\end{aligned}$$

(2) Let us start by proving that $d_{f,h}^* v_k^{(1)} \equiv 0$ in $\{\chi_+ = 1\}$. Let ψ be an eigenvector of $\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \bar{\Omega}_-)$ associated with an eigenvalue $\lambda \in [0, \nu(h)]$. Then, $d_{f,h}^*\psi$ belongs to $D(\Delta_{f,h}^{D,(0)}(\Omega_+ \setminus \bar{\Omega}_-))$ and

$$\Delta_{f,h}^{D,(0)}(d_{f,h}^*\psi) = \lambda d_{f,h}^*\psi,$$

according to [Helffer and Nier 2006] (see also (4-3) below). Using now (3-10) and $\lambda \leq \nu(h) \leq h$, this implies

$$d_{f,h}^*\psi \equiv 0, \tag{3-12}$$

and thus $d_{f,h}^* v \equiv 0$ in $\{\chi_+ \equiv 1\}$.

All the other estimates are proved like in Proposition 3.5 as consequences of the exponential decay estimate for the eigenvector ψ, stated in Proposition 3.9, using a neighborhood $\mathcal{V} \subset \bar{\Omega}_+ \setminus \Omega_-$ of $\partial\Omega_+$ such that $\chi_+ \equiv 1$ in a neighborhood of $\bar{\mathcal{V}}$.

For example, for (3-11), using $d_{\mathrm{Ag}}(x, \mathcal{V}) \geq 2c'_{\chi_+} > 0$ for $x \in \operatorname{supp}(1-\chi_+)$, Proposition 3.9 provides

$$\|(1-\chi_+)\psi\|_{W^{1,2}(\Omega_+ \setminus \bar{\Omega}_-)} \leq C'_{\chi_+} e^{-c'_{\chi_+}/h}. \tag{3-13}$$

The proofs of the two other estimates on $\langle v_k^{(1)}, v_{k'}^{(1)} \rangle_{L^2(\Omega_+)}$ and $\langle v_k^{(1)}, \Delta_{f,h}^{D,(1)}(\Omega_+) v_k^{(1)} \rangle_{L^2(\Omega_+)}$ follow the same lines as in the proof of Proposition 3.5. ☐

3D. *Exponential decay for the eigenvectors of $\Delta_{f,h}^{D,(p)}(\Omega_+)$, ($p = 0, 1$).* We will use the two operators $\Delta_{f,h}^N(\Omega_-)$ and $\Delta_{f,h}^D(\Omega_+ \setminus \bar{\Omega}_-)$ to analyze the spectrum of $\Delta_{f,h}^D(\Omega_+)$.

Definition 3.10. On $\bigwedge L^2(\Omega_+) = \bigwedge L^2(\Omega_-) \oplus \bigwedge L^2(\Omega_+ \setminus \bar{\Omega}_-)$, let $\Delta_{f,h}^{\oplus}(\Omega_+)$ be the self-adjoint operator $\Delta_{f,h}^N(\Omega_-) \oplus \Delta_{f,h}^D(\Omega_+ \setminus \bar{\Omega}_-)$.

In other words, for any form u such that $u1_{\Omega_-} \in D(\Delta_{f,h}^N(\Omega_-))$ and $u1_{\Omega_+ \setminus \bar{\Omega}_-} \in D(\Delta_{f,h}^D(\Omega_+ \setminus \bar{\Omega}_-))$ (namely if $u \in D(\Delta_{f,h}^{\oplus}(\Omega_+))$),

$$\Delta_{f,h}^{\oplus}(\Omega_+)u = \Delta_{f,h}^N(\Omega_-)(u1_{\Omega_-}) + \Delta_{f,h}^D(\Omega_+ \setminus \bar{\Omega}_-)(u1_{\Omega_+ \setminus \bar{\Omega}_-}).$$

It is easy to check that the spectrum of $\Delta^{\oplus,(p)}_{f,h}(\Omega_+)$ is the union of the two spectra $\sigma(\Delta^{N,(p)}_{f,h}(\Omega_-))$ and $\sigma(\Delta^{D,(p)}_{f,h}(\Omega_+\backslash\overline{\Omega}_-))$. Bases of eigenvectors are given by the direct sum structure. In particular, we have

$$m^{\oplus}_p(\Omega_+) = m^N_p(\Omega_-) + m^D_p(\Omega_+ \setminus \overline{\Omega}_-),$$

where $m^{\oplus}_p(\Omega_+) = \#\big[\sigma(\Delta^{\oplus}_{f,h}(\Omega_+)) \cap [0, \nu(h)]\big]$ denotes the number of small eigenvalues of $\Delta^{\oplus}_{f,h}(\Omega_+)$.

Proposition 3.11. *Let U be an open set satisfying (3-2). Let $(\psi^{(p)}_k)_{1\le k\le m^D_p(\Omega_+)}$, $p = 0$ or 1, be an orthonormal basis of eigenvectors of $\Delta^{D,(p)}_{f,h}(\Omega_+)$ associated with the eigenvalues in $[0, \nu(h)]$, and let $\chi \in \mathcal{C}^\infty(\overline{\Omega}_+)$ be such that $\chi \equiv 1$ in a neighborhood of $\partial\Omega_+ \cup U$ and $\chi \equiv 0$ in a neighborhood of $\partial\Omega_-$. For all $k \in \{1, \dots, m^D_p(\Omega_+)\}$, set $v^{(p)}_k = \chi\psi^{(p)}_k$. The forms $v^{(p)}_k$ are close to $\psi^{(p)}_k$ for $k \in \{1, \dots, m^D_p(\Omega_+)\}$:*

$$\sum_{k=1}^{m^D_p(\Omega_+)} \|\psi^{(p)}_k - v^{(p)}_k\|_{W^{1,2}(\Omega_+)} = \mathcal{O}(e^{-c_\chi/h}).$$

They are almost orthonormal in $L^2(\Omega_+)$:

$$(\langle v^{(p)}_k, v^{(p)}_{k'}\rangle_{L^2(\Omega_+)})_{k,k'} = \mathrm{Id}_{m^D_p(\Omega_+)} + \mathcal{O}(e^{-c_\chi/h}).$$

Moreover, they belong to the domain $D(\Delta^{\oplus,(p)}_{f,h}(\Omega_+))$ and they satisfy

$$\langle v^{(p)}_k, \Delta^{\oplus,(p)}_{f,h}(\Omega_+) v^{(p)}_k\rangle_{L^2(\Omega_+)} = \mathcal{O}(e^{-c_\chi/h}).$$

All the $\mathcal{O}(e^{-c_\chi/h})$ remainders can be bounded from above by $C_\chi e^{-c_\chi/h}$ for some constants $C_\chi, c_\chi > 0$ independent of $h \in [0, h_0]$.

Proof. The proof for $p = 0$ follows the same lines as the proofs of Proposition 3.4 and Proposition 3.5, because the boundary term in Lemma 3.1 disappears for functions vanishing along $\partial\Omega_+$.

For $p = 1$, the boundary term has to be taken into account as we did in the proofs of Proposition 3.8 and Proposition 3.9. A neighborhood $\mathcal{V}$ of $\partial\Omega_+$ has to be introduced and the function φ used in Lemma 3.1 is $\varphi(x) = (1-\alpha h)d_{\mathrm{Ag}}(x, U \cup \mathcal{V})$ with $\alpha > 0$ large enough. □

Notice that the number $m^D_p(\Omega_+)$ of small eigenvalues for $\Delta^{D,(p)}_{f,h}(\Omega_+)$ is a priori dependent on h. We did not explicitly indicate this dependency since the result of the next section is that $m^D_p(\Omega_+)$ is actually independent of h.

3E. *On the number of small eigenvalues of $\Delta^{D,(p)}_{f,h}(\Omega_+)$.* Using the results of the three previous sections, one can show that the number $m^D_p(\Omega_+)$ of eigenvalues of $\Delta^{D,(p)}_{f,h}(\Omega_+)$ in $[0, \nu(h)]$, is actually independent of $h \in (0, h_0)$.

Proposition 3.12. *For $p \in \{0, 1\}$, the number of eigenvalues of $\Delta^{D,(p)}_{f,h}(\Omega_+)$ lying in $[0, \nu(h)]$ is given by*

$$m^D_p(\Omega_+) = m^N_p(\Omega_-) + m^D_p(\Omega_+ \setminus \overline{\Omega}_-),$$

where we recall (see (3-10)) that $m^D_0(\Omega_+ \setminus \overline{\Omega}_-) = 0$. Moreover all these eigenvalues are exponentially small, i.e., there exists $c'_0 > 0$ such that

$$\sigma(\Delta^{D,(p)}_{f,h}(\Omega_+)) \cap [0, \nu(h)] \subset [0, e^{-c'_0/h}] \quad \textit{for all } h \in (0, h_0),\ p = 0, 1.$$

Proof. This is obtained as an application of the min–max principle. Indeed, we know that the spectrum of $\Delta^{D,(p)}_{f,h}(\Omega_+)$ is given by the formula

$$\lambda_k^{(p)}(\Omega_+) = \sup_{\{\omega_1,\dots,\omega_{k-1}\}} Q(\omega_1,\dots,\omega_{k-1}) \quad \text{for } k \geq 1,$$

where

$$Q(\omega_1,\dots,\omega_{k-1}) = \inf_v \left\{ \frac{\langle v, \Delta^{D,(p)}_{f,h}(\Omega_+)v\rangle_{L^2(\Omega_+)}}{\|v\|^2_{L^2(\Omega_+)}} : v \in D(\Delta^{D,(p)}_{f,h}(\Omega_+)),\, v \in \operatorname{Span}(\omega_1,\dots,\omega_{k-1})^{\perp} \right\}.$$

By convention, for $k=1$, the supremum is taken over an empty set (and can thus be neglected). Using Proposition 3.5 and Proposition 3.9, one can build $m_p := m_p^N(\Omega_-) + m_p^D(\Omega_+ \setminus \overline{\Omega}_-)$, almost orthonormal vectors for which the Rayleigh quotients associated with $\Delta^{D,(p)}_{f,h}(\Omega_+)$ are exponentially small. Let us fix $\varepsilon > 0$ and consider $\{\omega_1,\dots,\omega_{m_p-1}\}$ such that $\lambda^{(p)}_{m_p}(\Omega_+) \leq Q(\omega_1,\dots,\omega_{m_p-1}) + \varepsilon$. Since, in the limit $h \to 0$, the m_p vectors built in Proposition 3.5 and Proposition 3.9 are linearly independent, there exists a linear combination $v \in D(\Delta^{D,(p)}_{f,h}(\Omega_+))$ of these vectors which is in $\operatorname{Span}(\omega_1,\dots,\omega_{m_p-1})^{\perp}$. Using the estimates on the Rayleigh quotients and the almost orthonormality of these vectors, one obtains that $\langle v, \Delta^{D,(p)}_{f,h}(\Omega_+)v\rangle_{L^2(\Omega_+)}/\|v\|^2_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h})$ for some positive constant c. This implies that $Q(\omega_1,\dots,\omega_{k-1}) = \mathcal{O}(e^{-c/h})$ and thus $\lambda^{(p)}_{m_p}(\Omega_+) = \mathcal{O}(e^{-c/h})$. Therefore, one gets $m_p^D(\Omega_+) \geq m_p = m_p^N(\Omega_-) + m_p^D(\Omega_+ \setminus \overline{\Omega}_-)$.

Similar reasoning on $\Delta^{\oplus,(p)}_{f,h}(\Omega_+)$ using Proposition 3.11 gives the opposite inequality $m_p^{\oplus}(\Omega_+) = m_p^N(\Omega_-) + m_p^D(\Omega_+ \setminus \overline{\Omega}_-) \geq m_p^D(\Omega_+)$. This ends the proof. □

4. Quasimodes for $\Delta^{D,(0)}_{f,h}(\Omega_+)$ and $\Delta^{D,(1)}_{f,h}(\Omega_+)$

In this section, we specify the quasimodes which will be useful for the analysis of the spectrum of $\Delta^{D,(0)}_{f,h}(\Omega_+)$ lying in $[0,\nu(h)]$. In our context, for $p = 0, 1$, a quasimode for $\Delta^{D,(p)}_{f,h}(\Omega_+)$ is simply a function v in the domain $D(\Delta^{D,(p)}_{f,h}(\Omega_+))$ such that $\langle v, \Delta^{D,(p)}_{f,h}(\Omega_+)v\rangle_{L^2(\Omega_+)}/\|v\|^2_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h})$. Quasimodes for $\Delta^{D,(0)}_{f,h}(\Omega_+)$ (resp. $\Delta^{D,(1)}_{f,h}(\Omega_+)$) will be built from the eigenvectors of $\Delta^{N,(0)}_{f,h}(\Omega_-)$ (resp. of $\Delta^{N,(1)}_{f,h}(\Omega_-)$ and $\Delta^{D,(1)}_{f,h}(\Omega_+ \setminus \overline{\Omega}_-)$).

4A. ***The restricted differential β.*** We recall here basic properties of boundary Witten Laplacians.

Proposition 4.1. *Let Ω be a regular bounded domain of (M,g) and consider the Dirichlet (resp. Neumann) realization $A = \Delta^D_{f,h}(\Omega)$ (resp. $A = \Delta^N_{f,h}(\Omega)$) of the Witten Laplacian with form domain $Q(A) = W^{1,2}_D(\Omega)$ (resp. $Q(A) = W^{1,2}_N(\Omega)$). The differential $d_{f,h}$ and codifferential $d^*_{f,h}$ satisfy the commutation property: for all $z \in \mathbb{C} \setminus \sigma(A)$ and $u \in Q(A)$,*

$$d_{f,h}(z-A)^{-1}u = (z-A)^{-1}d_{f,h}u \quad \textit{and} \quad d^*_{f,h}(z-A)^{-1}u = (z-A)^{-1}d^*_{f,h}u.$$

Consequently, for any $\ell \in \mathbb{R}_+$,

$$d_{f,h} \circ 1_{[0,\ell]}(A^{(p)}) = 1_{[0,\ell]}(A^{(p+1)}) \circ d_{f,h}, \quad \textit{and} \quad d^*_{f,h} \circ 1_{[0,\ell]}(A^{(p)}) = 1_{[0,\ell]}(A^{(p-1)}) \circ d^*_{f,h}, \tag{4-1}$$

where $A^{(p)}$ denotes the restriction of A to p-forms. Moreover, if $F_\ell^{(p)}$ denotes the spectral subspace $\operatorname{Ran} 1_{[0,\ell]}(A^{(p)})$, *the chain complex*

$$0 \longrightarrow F_\ell^{(0)} \longrightarrow \cdots \longrightarrow F_\ell^{(p-1)} \xrightarrow{-d_{f,h}} F_\ell^{(p)} \xrightarrow{-d_{f,h}} F_\ell^{(p+1)} \longrightarrow \cdots \longrightarrow F_\ell^{(d)} \longrightarrow 0 \tag{4-2}$$

is quasi-isomorphic to the relative (resp. absolute) Hodge–de Rham chain complex. The Witten codifferential $d^*_{f,h}$ implements the dual chain complex.

Relative and absolute homologies are standard notions in algebraic topology and Morse theory (see, for example, [Hatcher 2002; Milnor 1963]). Their translations to cohomology and boundary value Hodge theory is presented, for example, in [Taylor 1997; Schwarz 1995]. A quasi-isomorphism is a morphism of complexes which induces an isomorphism of homology groups.

We refer to [Chang and Liu 1995; Helffer and Nier 2006; Le Peutrec 2010b] for the adaptation to boundary cases of these well-known properties of Witten Laplacians [Cycon et al. 1987, Chapter 11].

Let us give two consequences of that result that are useful in our context. First, the following property, which was already used in the proof of Proposition 3.9, holds (using the notation of Proposition 4.1):

$$A^{(p)}\psi = \lambda\psi \implies \begin{cases} A^{(p+1)} d_{f,h}\psi = \lambda d_{f,h}\psi \\ A^{(p-1)} d^*_{f,h}\psi = \lambda d^*_{f,h}\psi \end{cases} \tag{4-3}$$

with the convention $A^{(-1)} = A^{(d+1)} = 0$. Secondly, we have the orthogonal decompositions

$$F_\ell = \operatorname{Ker}\big[A\big|_{F_\ell}\big] \overset{\perp}{\oplus} \operatorname{Ran}\big[d_{f,h}\big|_{F_\ell}\big] \overset{\perp}{\oplus} \operatorname{Ran}\big[d^*_{f,h}\big|_{F_\ell}\big], \tag{4-4}$$

$$\operatorname{Ran}\big[d^*_{f,h}\big|_{F_\ell}\big]^\perp = \operatorname{Ker}\big[d_{f,h}\big|_{F_\ell}\big] = \operatorname{Ker}\big[A\big|_{F_\ell}\big] \overset{\perp}{\oplus} \operatorname{Ran}\big[d_{f,h}\big|_{F_\ell}\big],$$

$$\operatorname{Ran}\big[d_{f,h}\big|_{F_\ell}\big]^\perp = \operatorname{Ker}\big[d^*_{f,h}\big|_{F_\ell}\big] = \operatorname{Ker}\big[A\big|_{F_\ell}\big] \overset{\perp}{\oplus} \operatorname{Ran}\big[d^*_{f,h}\big|_{F_\ell}\big],$$

where $F_\ell = \bigoplus_{p=0}^d F_\ell^{(p)}$. In our problem, we shall use the following notation:

Definition 4.2. Consider the Dirichlet realization $\Delta^D_{f,h}(\Omega_+)$ of $\Delta_{f,h}$ on Ω_+. For $p = 0, 1$, the operators $\Pi^{(p)}$ are the spectral projections

$$\Pi^{(p)} = 1_{[0,\nu(h)]}(\Delta^{D,(p)}_{f,h}(\Omega_+)), \quad p = 0, 1,$$

and their range is denoted by $F^{(p)}$. Moreover, the Witten differential $d_{f,h}$ restricted to $F^{(0)}$ is written as $\beta = d_{f,h}\big|_{F^{(0)}} : F^{(0)} \to F^{(1)}$, so that $\Delta^{D,(0)}_{f,h}(\Omega_+)\big|_{F^{(0)}} = \beta^*\beta$, where $\beta^* = d^*_{f,h}\big|_{F^{(1)}} : F^{(1)} \to F^{(0)}$.

A consequence of the commutation properties (4-1) is the identity

$$\beta = \Pi^{(1)} d_{f,h} = d_{f,h}\Pi^{(0)} = \Pi^{(1)}\beta\Pi^{(0)}. \tag{4-5}$$

Moreover, (4-4) becomes

$$F^{(0)} = \operatorname{Ran}[\beta^*], \quad \text{since } \operatorname{Ker}(\beta) = \{0\}$$

because $\beta u = d_{f,h}u = 0$ and $u = 0$ on $\partial\Omega$ imply $u = 0$, and

$$F^{(1)} = \operatorname{Ker}[\beta^*] \overset{\perp}{\oplus} \operatorname{Ran}[\beta] = \operatorname{Ker}[\Delta^{D,(1)}_{f,h}(\Omega_+)] \overset{\perp}{\oplus} \operatorname{Ran}(\beta) \overset{\perp}{\oplus} \operatorname{Ran}\big[d^*_{f,h}\big|_{F^{(2)}}\big]. \tag{4-6}$$

4B. ***Truncated eigenvectors.*** Let us recall the eigenvectors that have been introduced in Propositions 3.5 and 3.9:

- $(\psi_j^{(0)})_{1\le j\le m_0^N(\Omega_-)}$ are eigenvectors for the operator $\Delta_{f,h}^{N,(0)}(\Omega_-)$ associated with the eigenvalues $0=\mu_1^{(0)}(\Omega_-)\le C_0e^{-2(\kappa_f-c_0)/h}\le\mu_2^{(0)}(\Omega_-)\le\cdots\le\mu_{m_0^N(\Omega_-)}^{(0)}(\Omega_-)\le e^{-c_0/h}\le\nu(h)$. The first eigenvector $\psi_1^{(0)}$ associated with the eigenvalue $\mu_1^{(0)}(\Omega_-)=0$ is $\psi_1^{(0)}=e^{-f/h}1_{\Omega_-}/\big(\int_{\Omega_-}e^{-2f(x)/h}\,dx\big)^{1/2}$. The lower bound on $\mu_2^{(0)}(\Omega_-)$ stated above is valid for sufficiently small h and was proven in Proposition 3.7.
- $(\psi_k^{(1)})_{1\le k\le m_1^N(\Omega_-)}$ are eigenvectors for the operator $\Delta_{f,h}^{N,(1)}(\Omega_-)$ associated with the $m_1^N(\Omega_-)$ eigenvalues smaller than $\nu(h)$. Using (4-3), those eigenvectors can be labeled so that
$$\psi_k^{(1)}=(\mu_{k+1}^{(0)}(\Omega_-))^{-1/2}d_{f,h}\psi_{k+1}^{(0)}=(\mu_{k+1}^{(0)}(\Omega_-))^{-1/2}\beta\psi_{k+1}^{(0)}\quad\text{for } k\in\{1,\dots,m_0^N(\Omega_-)-1\}.$$
Notice that we may have $m_1^N(\Omega_-)=m_0^N(\Omega_-)-1$. If not, using (4-6), $\beta^*\psi_k^{(1)}=d_{f,h}^*\psi_k^{(1)}=0$ for $k\ge m_0^N(\Omega_-)$.
- $(\psi_k^{(1)})_{m_1^N(\Omega_-)+1\le k\le m_1^N(\Omega_-)+m_1^D(\Omega_+\setminus\bar\Omega_-)}$ are eigenvectors for the operator $\Delta_{f,h}^{D,(1)}(\Omega_+\setminus\bar\Omega_-)$ associated with the $m_1^D(\Omega_+\setminus\bar\Omega_-)$ eigenvalues smaller than $\nu(h)$. From (3-12) in the proof of Proposition 3.9, we know that $d_{f,h}^*\psi_k^{(1)}=\beta^*\psi_k^{(1)}=0$.

In Proposition 3.12 we proved that $m_0^D(\Omega_+)=m_0^N(\Omega_-)$ and $m_1^D(\Omega_+)=m_1^N(\Omega_-)+m_1^D(\Omega_+\setminus\bar\Omega_-)$. The families $(\psi_j^{(0)})_{1\le j\le m_0^D(\Omega_+)}$ and $(\psi_k^{(1)})_{1\le k\le m_1^D(\Omega_+)}$ are orthonormal bases of eigenvectors for $\Delta_{f,h}^{\oplus,(0)}(\Omega_+)$ and $\Delta_{f,h}^{\oplus,(1)}(\Omega_+)$, respectively, restricted to the spectral range $[0,\nu(h)]$. These two families will be used to construct quasimodes for the operator $\Delta_{f,h}^{D,(p)}(\Omega_+)$ restricted to the spectral range $[0,\nu(h)]$. This will require some appropriate truncations or extrapolations, detailed below.

Let us start with $\psi_1^{(0)}$ and let us introduce
$$\tilde\psi_1^{(0)}=\frac{e^{-f/h}1_{\Omega_+}(x)}{\big(\int_{\Omega_+}e^{-2f(x)/h}\,dx\big)^{1/2}}. \tag{4-7}$$
These two functions are exponentially close in $L^2(\Omega_+)$, that is,
$$\|\psi_1^{(0)}-\tilde\psi_1^{(0)}\|_{L^2(\Omega_+)}\le Ce^{-c/h},$$
owing to $f(x)\ge\min_{\partial\Omega_-}f>\min_{\Omega_+}f$ for all $x\in\bar\Omega_+\setminus\Omega_-$ and the following upper and lower bounds of the integral factor:

Lemma 4.3. *Let Ω be a regular bounded domain of (M,g) and let f belong to $\mathcal{C}^\infty(\bar\Omega)$ such that $\min_{\bar\Omega}f$ is achieved in Ω. Then there exists a constant $C_f>0$ such that*
$$\frac{1}{C_f}h^{d/2}e^{-2(\min_\Omega f)/h}\le\int_\Omega e^{-2f(x)/h}\,dx\le \mathrm{Vol}_g(\Omega)e^{-2(\min_\Omega f)/h},$$
where $\mathrm{Vol}_g(\Omega)$ denotes the volume of Ω for the metric g.

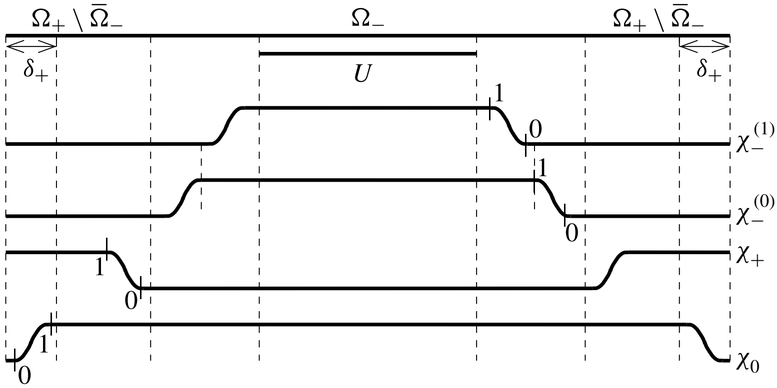

Figure 1. Positions of the domains Ω_+, Ω_- and U, and of the supports of the cut-off functions $\chi_-^{(0)}$, $\chi_-^{(1)}$, χ_+, χ_0.

Proof. The upper bound is obvious since $e^{-2f(x)/h} \le e^{-2(\min_\Omega f)/h}$ for all $x \in \Omega$. For the lower bound, write

$$\int_\Omega e^{-2f(x)/h}\,dx = \int_\Omega \int_{2f(x)}^{+\infty} e^{-t/h}\,\frac{dt}{h}\,dx = \int_{2\min_\Omega f}^{+\infty} \mathrm{Vol}_g(2f < t)e^{-t/h}\,\frac{dt}{h}$$
$$= e^{-2(\min_\Omega f)/h} \int_0^{+\infty} \mathrm{Vol}_g\big(2f < 2\min_\Omega f + hs\big)e^{-s}\,ds.$$

We assumed the existence of $x_0 \in \Omega$ such that $f(x_0) = \min_\Omega f$. Using the Taylor expansion of f around x_0, there exist $r > 0$, $h_0 > 0$ and $s_0 > 0$ such that the ball $B(x_0, (hs)^{1/2}/r)$ is included in $\{f < \min_\Omega f + \frac{1}{2}hs\}$ for all $s < s_0$ and $h < h_0$. Since $\mathrm{Vol}_g[B(x_0, (hs)^{1/2}/r)] \ge (hs)^{d/2}/C_r$, we get

$$\int_\Omega e^{-2f(x)/h}\,dx \ge \frac{1}{C_r} e^{-2(\min_\Omega f)/h} \int_0^{s_0} e^{-s}(hs)^{d/2}\,ds \ge \frac{h^{d/2}e^{-2(\min_\Omega f)/h}}{C_f}. \qquad \square$$

Compared to the standard Laplace estimate, the interest of Lemma 4.3 is that it holds even if the minimum of f is degenerate.

In all of what follows, U denotes a fixed subset of Ω_- satisfying (3-2). Let us introduce various cut-off functions, which all satisfy $0 \le \chi \le 1$. We refer to Figure 1 for an illustration of these cut-off functions with respect to the three sets $U \subset \Omega_- \subset \Omega_+$.

- $\chi_-^{(0)}$ and $\chi_-^{(1)}$ are two cut-off functions like χ_- in Proposition 3.5, that is, $\chi_-^{(p)} \in \mathcal{C}_0^\infty(\Omega_-)$ and $\chi_-^{(p)} \equiv 1$ in a neighborhood of U with the additional condition that $\chi_-^{(0)} \equiv 1$ in a neighborhood of $\operatorname{supp} \chi_-^{(1)}$.
- χ_+ is chosen as in Proposition 3.9, that is, $\chi_+ \in \mathcal{C}^\infty(\overline{\Omega}_+)$, $\chi_+ \equiv 1$ in a neighborhood of $\partial\Omega_+$ and $\chi_+ \equiv 0$ in a neighborhood of $\overline{\Omega}_-$. Let us introduce $c_+ > 0$ such that $\chi_+ \equiv 1$ on $\{x \in \overline{\Omega}_+ : d(x, \partial\Omega_+) \le c_+\}$.
- χ_0 belongs to $\mathcal{C}_0^\infty(\Omega_+)$, $\chi_0 \equiv 1$ in a neighborhood of $\overline{\Omega}_-$ and is chosen in such a way that its gradient is supported in $\{x \in \Omega_+ : d(x, \partial\Omega_+) \le \delta_+\}$, where $\delta_+ \in (0, c_+)$ will be fixed later.

We are now in position to introduce a family of quasimodes for the operator $\Delta_{f,h}^{D,(p)}(\Omega_+)$.

Definition 4.4. Let $\chi_-^{(0)}$, $\chi_-^{(1)}$, χ_+ and χ_0 be the cut-off functions defined above. Let $(\psi_j^{(0)})_{1\le j\le m_0^D(\Omega_+)}$ and $(\psi_k^{(1)})_{1\le k\le m_1^D(\Omega_+)}$ be the previously gathered families of eigenvectors of $\Delta_{f,h}^{N,(p)}(\Omega_-)$ and $\Delta_{f,h}^{D,(1)}(\Omega_+\setminus\overline{\Omega}_-)$, and finally let $\tilde{\psi}_1^{(0)}$ be given by (4-7). The families of vectors $(v_j^{(0)})_{1\le j\le m_0^D(\Omega_+)}$ and $(v_k^{(1)})_{1\le k\le m_1^D(\Omega_+)}$ are defined by:

- $v_1^{(0)} = \chi_0 \tilde{\psi}_1^{(0)}$;
- $v_j^{(0)} = \chi_-^{(0)} \psi_j^{(0)}$ for $j \in \{2, \dots, m_0^D(\Omega_+)\}$;
- $v_k^{(1)} = \chi_-^{(1)} \psi_k^{(1)}$ for $k \in \{1, \dots, m_1^N(\Omega_-)\}$;
- $v_k^{(1)} = \chi_+ \psi_k^{(1)}$ for $k \in \{m_1^N(\Omega_-)+1, \dots, m_1^D(\Omega_+)\}$.

Proposition 4.5. *The families $(v_j^{(0)})_{1\le j\le m_0^D(\Omega_+)}$ and $(v_k^{(1)})_{1\le k\le m_1^D(\Omega_+)}$ of Definition 4.4 satisfy:*

(1) *They are almost orthonormal in $L^2(\Omega_+)$:*

$$(\langle v_j^{(0)}, v_{j'}^{(0)}\rangle_{L^2(\Omega_+)})_{1\le j,j'\le m_0^D(\Omega_+)} = \mathrm{Id}_{m_0^D(\Omega_+)} + \mathcal{O}(e^{-c/h}),$$
$$(\langle v_k^{(1)}, v_{k'}^{(1)}\rangle_{L^2(\Omega_+)})_{1\le k,k'\le m_1^D(\Omega_+)} = \mathrm{Id}_{m_1^D(\Omega_+)} + \mathcal{O}(e^{-c/h})$$

for some constant $c > 0$ independent of δ_+.

(2) *The elements $v_j^{(0)}$, $1 \le j \le m_0^D(\Omega_+)$ (resp. $v_k^{(1)}$, $1 \le k \le m_1^D(\Omega_+)$) belong to $D(\Delta_{f,h}^{D,(0)}(\Omega_+))$ (resp. $\Delta_{f,h}^{D,(1)}(\Omega_+)$) and satisfy*

$$\langle v_j^{(0)}, \Delta_{f,h}^{D,(0)}(\Omega_+) v_j^{(0)}\rangle_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h}) \quad \textit{and} \quad \langle v_k^{(1)}, \Delta_{f,h}^{D,(1)}(\Omega_+) v_k^{(1)}\rangle_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h}),$$

respectively, for some constant $c > 0$ independent of δ_+.

(3) *Let us consider the spectral projections $\Pi^{(0)}$ and $\Pi^{(1)}$ associated with $\Delta_{f,h}^D(\Omega_+)$ introduced in Definition 4.2. The elements $v_j^{(0)}$, $1 \le j \le m_0^D(\Omega_+)$ (resp. $v_k^{(1)}$, $1 \le k \le m_1^D(\Omega_+)$) satisfy:*

$$\|v_j^{(0)} - \Pi^{(0)} v_j^{(0)}\|_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h}) \quad \textit{and} \quad \|v_k^{(1)} - \Pi^{(1)} v_k^{(1)}\|_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h}),$$

respectively, for some constant $c > 0$ independent of δ_+.

Proof. (1) The families $(\psi_j^{(0)})_{1\le j\le m_0^D(\Omega_+)}$ and $(\psi_k^{(1)})_{1\le k\le m_1^D(\Omega_+)}$ are orthonormal bases of eigenvectors of $\Delta_{f,h}^{\oplus,(0)}$ and $\Delta_{f,h}^{\oplus,(1)}$, respectively. Proposition 3.5 implies that the family $(\chi_-^{(0)}\psi_j^{(0)})_{1\le j\le m_0^D(\Omega_+)}$ is almost orthonormal. The estimate $\|\chi_0\tilde{\psi}_1^{(0)} - \chi_-^{(0)}\psi_1^{(0)}\|_{L^2(\Omega_+)} \le Ce^{-c/h}$ (which is a consequence of Lemma 4.3 and $f(x) \ge \min_{\partial\Omega_-} f > \min_{\Omega_+} f$ for all $x \in \overline{\Omega}_+ \setminus \Omega_-$) ends the proof of the almost orthonormality of $(v_j^{(0)})_{1\le j\le m_0^D(\Omega_+)}$. For $p = 1$, the two families $(v_k^{(1)} = \chi_-^{(1)}\psi_k^{(1)})_{1\le k\le m_1(\Omega_-)}$ and $(v_k^{(1)} = \chi_+\psi_k^{(1)})_{m_1^N(\Omega_-)+1\le k\le m_1^D(\Omega_+)}$ have disjoint supports and therefore lie in orthogonal subspaces of $L^2(\Omega_+)$. Also, the almost orthonormality of both families is again a consequence of the exponential decay of the $\psi_k^{(1)}$; see Proposition 3.5 and Proposition 3.11.

(2) With the chosen truncations, all the vectors $v_j^{(0)}$ (resp. $v_k^{(1)}$) belong to the domain $D(\Delta_{f,h}^{D,(0)}(\Omega_+))$ (resp. $D(\Delta_{f,h}^{D,(1)}(\Omega_+))$). In all cases except $p = 0$ and $k = 1$, we obtain, for $v = \chi\psi$ (we omit the index k and the superscript (p)) and $A\psi = \lambda\psi$, where $A = \Delta_{f,h}^N(\Omega_-)$ or $A = \Delta_{f,h}^D(\Omega_+ \setminus \overline{\Omega}_-)$,

$$\langle v, \Delta_{f,h}^D(\Omega_+)v\rangle_{L^2(\Omega_+)} = \|d_{f,h}v\|_{L^2}^2 + \|d_{f,h}^* v\|_{L^2(\Omega_+)}^2 \le \langle \psi, A\psi\rangle + C\|\psi\|_{W^{1,2}(\{\chi\ne 1\})}^2 \le Ce^{-c/h},$$

owing to $\langle\psi, A\psi\rangle = \lambda = \mathcal{O}(e^{-c_0/h})$ and to the estimates on $\psi - v$ given in Proposition 3.5 and Proposition 3.11. For $p = 0$ and $k = 1$, it is even simpler because $d_{f,h}\tilde{\psi}_1^{(0)} = 0$ implies

$$\langle v_1^{(0)}, \Delta_{f,h}^{D,(0)}(\Omega_+)v_1^{(0)}\rangle_{L^2(\Omega_+)} = \|d_{f,h}(\chi_0\tilde{\psi}_1^{(0)})\|^2_{L^2(\Omega_+)} = \|(hd\chi_0)\tilde{\psi}_1^{(0)}\|^2_{L^2(\Omega_+)} \leq Ce^{-c/h}$$

as a consequence of Lemma 4.3 (see (5-8) below for a more precise estimate).

(3) All the $v_j^{(0)}$ and $v_k^{(1)}$ satisfy $\langle v, Av\rangle_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h})$ with $A = \Delta_{f,h}^{D,(0)}(\Omega_+)$ or $A = \Delta_{f,h}^{D,(1)}(\Omega_+)$, and recall that $\Pi^{(0)}$ and $\Pi^{(1)}$ are the spectral projectors $1_{[0,\nu(h)]}(A)$. The last estimates are consequences of

$$\nu(h)\|1_{(\nu(h),+\infty)}(A)v\|^2_{L^2(\Omega_+)} \leq \langle v, Av\rangle_{L^2(\Omega_+)} \leq Ce^{-c/h}$$

together with the fact that $\lim_{h\to 0} h\log\nu(h) = 0$; see (2-5). □

In the next section, we will need these calculations:

Proposition 4.6. *The coefficients* $\langle v_k^{(1)}, d_{f,h}v_j^{(0)}\rangle_{L^2(\Omega_+)}$, $j \in \{1, \dots, m_0^D(\Omega_+)\}$, $k \in \{1, \dots, m_1^D(\Omega_+)\}$, *satisfy*:

(1) *For* $j = 1$ *and* $k \in \{1, \dots, m_1^N(\Omega_-)\}$, $\langle v_k^{(1)}, d_{f,h}v_1^{(0)}\rangle_{L^2(\Omega_+)} = 0$.

(2) *For* $j = 1$ *and* $k \in \{m_1^N(\Omega_-) + 1, \dots, m_1^D(\Omega_+)\}$,

$$\langle v_k^{(1)}, d_{f,h}v_1^{(0)}\rangle_{L^2(\Omega_+)} = -\frac{h\int_{\partial\Omega_+} e^{-f(\sigma)/h}\mathbf{i}_n\psi_k^{(1)}(\sigma)\,d\sigma}{\left(\int_{\Omega_+} e^{-2f(x)/h}\,dx\right)^{1/2}},$$

where $d\sigma$ *is the infinitesimal volume on* $\partial\Omega_+$ *and* $n(\sigma)$ *the outward normal vector at* $\sigma \in \partial\Omega_+$.

(3) *For* $j \in \{2, \dots, m_0^D(\Omega_+)\}$ *and* $k \in \{1, \dots, m_1^N(\Omega_-)\}$,

$$\langle v_k^{(1)}, d_{f,h}v_j^{(0)}\rangle_{L^2(\Omega_+)} = \sqrt{\mu_j^{(0)}(\Omega_-)}(\delta_{k,j-1} + \mathcal{O}(e^{-c/h})).$$

(4) *For* $j \in \{2, \dots, m_0^D(\Omega_+)\}$ *and* $k \in \{m_1^N(\Omega_-) + 1, \dots, m_1^D(\Omega_+)\}$, $\langle v_k^{(1)}, d_{f,h}v_j^{(0)}\rangle_{L^2(\Omega_+)} = 0$.

Proof. Cases (1) and (4) are due to the disjoint supports of $d_{f,h}v_j^{(0)}$ and $v_k^{(1)}$ (see Figure 1).

Case (3) comes from the computation

$$\begin{aligned} d_{f,h}v_j^{(0)} &= d_{f,h}(\chi_-^{(0)}\psi_j^{(0)}) = \chi_-^{(0)}d_{f,h}\psi_j^{(0)} + (hd\chi_-^{(0)}) \wedge \psi_j^{(0)} \\ &= \sqrt{\mu_j^{(0)}(\Omega_-)}\chi_-^{(0)}\psi_{j-1}^{(1)} + \psi_j^{(0)}hd\chi_-^{(0)}. \end{aligned}$$

The condition $\chi_-^{(0)} \equiv 1$ in a neighborhood of supp $\chi_-^{(1)}$ then leads to

$$\begin{aligned} \langle v_k^{(1)}, d_{f,h}v_j^{(0)}\rangle_{L^2(\Omega_+)} &= \langle \chi_-^{(1)}\psi_k^{(1)}, \sqrt{\mu_j^{(0)}(\Omega_-)}\psi_{j-1}^{(1)}\rangle_{L^2(\Omega_-)} \\ &= \sqrt{\mu_j^{(0)}(\Omega_-)}\delta_{k,j-1} + \sqrt{\mu_j^{(0)}(\Omega_-)}\|(1-\chi_-^{(1)})\psi_k^{(1)}\|_{L^2(\Omega_-)}, \end{aligned}$$

and we conclude with the exponential decay of $\psi_k^{(1)}$ given by (3-4) in the proof Proposition 3.5.

For case (2), we first use

$$d_{f,h} v_1^{(0)} = d_{f,h}(\chi_0 \tilde{\psi}_1^{(0)}) = \frac{e^{-f/h}}{\left(\int_{\Omega_+} e^{-2f(x)/h}\, dx\right)^{1/2}} h\, d\chi_0.$$

The assumption on the supports of χ_0 and χ_+ (see Figure 1) implies that $d\chi_0$ is supported in the interior of $\{x \in \overline{\Omega}_+ : \chi_+(x) = 1\}$, so that

$$\left(\int_{\Omega_+} e^{-2f(x)/h}\, dx\right)^{\frac{1}{2}} \langle v_k^{(1)}, d_{f,h} v_j^{(0)}\rangle = \langle \chi_+ \psi_k^{(1)}, e^{-f/h} h\, d\chi_0\rangle = \langle \psi_k^{(1)}, e^{-f/h} h\, d\chi_0\rangle.$$

The definition of the Hodge $\star$ operation gives

$$\langle \psi_k^{(1)}, e^{-f/h} h\, d\chi_0\rangle = h \int_{\Omega_+} d\chi_0 \wedge [\star(e^{-f/h}\psi_k^{(1)})] = -h \int_{\Omega_+ \setminus \overline{\Omega}_-} d(1-\chi_0) \wedge [\star(e^{-f/h}\psi_k^{(1)})].$$

We recall (see (3-12) in the proof of Proposition 3.9) that $d^*_{f,h}\psi_k^{(1)} = 0$ in $\Omega_+ \setminus \overline{\Omega}_-$, which means

$$d[\star(e^{-f/h}\psi_k^{(1)})] = (-1)^{1+1} \star \left[\frac{e^{-f/h}}{h} d^*_{f,h}\psi_k^{(1)}\right] = 0 \quad \text{in } \Omega_+ \setminus \overline{\Omega}_- .$$

Hence, we get

$$d(1-\chi_0) \wedge [\star(e^{-f/h}\psi_k^{(1)})] = d\big[(1-\chi_0) \wedge [\star(e^{-f/h}\psi_k^{(1)})]\big],$$

and Stokes' formula yields

$$\langle \psi_k^{(1)}, e^{-f/h} h\, d\chi_0\rangle = -h \int_{\partial\Omega_+} e^{-f/h} \star \psi_k^{(1)} = -h \int_{\partial\Omega_+} e^{-f/h} \boldsymbol{t}(\star\psi_k^{(1)}).$$

Using the relations (A-8), $\boldsymbol{t}\star = \star\boldsymbol{n}$, and (A-10) $\omega_1 \wedge (\star\boldsymbol{n}\omega_2) = \langle \omega_1, \boldsymbol{i}_n \omega_2\rangle_{\bigwedge^{p-1} T^*_\sigma \Omega_+} d\sigma$ along $\partial\Omega_+$ (where $d\sigma$ is the infinitesimal volume on $\partial\Omega_+$ and $n(\sigma)$ the outward normal vector at $\sigma \in \partial\Omega_+$) with $p = 1$, $\omega_1 = 1$ and $\omega_2 = \psi_k^{(1)}$, we get

$$\langle \psi_k^{(1)}, e^{-f/h} h\, d\chi_0\rangle = -h \int_{\partial\Omega_+} e^{-f(\sigma)/h} \boldsymbol{i}_n \psi_k^{(1)}(\sigma)\, d\sigma.$$

This concludes the proof of case (2), and of Proposition 4.6. □

5. Analysis of the restricted differential β

It is in this section that the assumption (2-8) is used. We assume that the open subset U of Ω_- that has been used to build the cut-off functions in the previous section satisfies (in addition to (3-2))

$$U \cup \mathcal{V}_- = \overline{\Omega}_-, \tag{5-1}$$

where $\mathcal{V}_-$ is the neighborhood of $\partial\Omega_-$ introduced in the assumption (2-8).

The main result of this section is the following:

Proposition 5.1. *The singular values of $\beta = d_{f,h}\big|_{F^{(0)}} : F^{(0)} \to F^{(1)}$, labeled in decreasing order, are given by*

$$s_j(\beta) = \sqrt{\mu^{(0)}_{m_0^D(\Omega_+)+1-j}(\Omega_-)}(1+\mathcal{O}(e^{-c/h})) \quad \textit{for } j \in \{1, \dots, m_0^D(\Omega_+)-1\},$$

$$s_{m_0^D(\Omega_+)}(\beta) = \frac{h\sqrt{\sum_{k=m_1^N(\Omega_-)+1}^{m_1^D(\Omega_+)} \left|\int_{\partial\Omega_+} e^{-f(\sigma)/h} \boldsymbol{i}_n \psi_k^{(1)}(\sigma)\, d\sigma\right|^2}}{\sqrt{\int_{\Omega_+} e^{-2f(x)/h}\, dx}}(1+\mathcal{O}(e^{-c/h}))$$

for some $c > 0$.

According to the notation of Section 4B, $(\mu_j^{(0)}(\Omega_-))_{1\le j\le m_0^D(\Omega_+)}$ are the eigenvalues of $\Delta_{f,h}^{N,(0)}(\Omega_-)$ and $(\psi_k^{(1)})_{m_1^N(\Omega_-)+1\le k\le m_1^D(\Omega_+)}$ are the eigenvectors of $\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \overline{\Omega}_-)$. Notice that, contrary to the eigenvalues of the operators considered in the previous sections which were labeled in increasing order, the singular values are naturally labeled in decreasing order. Of course, the singular values of β are related to the small eigenvalues of $\Delta_{f,h}^{D,(0)}(\Omega_+)$ through the relation

$$\sigma(\Delta_{f,h}^{D,(0)}(\Omega_+)) \cap [0, \nu(h)] = \{s_k(\beta)^2 : 1 \le k \le m_0^D(\Omega_+)\}, \tag{5-2}$$

since $\Delta_{f,h}^{D,(0)}\big|_{F^{(0)}} = \beta^*\beta$. Proposition 5.1 will thus be instrumental in proving Theorem 2.4.

The idea of the proof of Proposition 5.1 follows the linear algebra argument used in [Helffer et al. 2004; Helffer and Nier 2006; Le Peutrec 2010b; Le Peutrec et al. 2013] and well summarized in [Le Peutrec 2009]. Notice that $\beta = d_{f,h}\big|_{F^{(0)}}$ is a *finite-dimensional* linear operator. The proof then relies on the following fundamental property for singular values of matrices. Let us denote by $s_k(B)$, $k \in \{1, \dots, \max(n_0, n_1)\}$, the singular values of a matrix $B \in \mathcal{M}_{n_1,n_0}(\mathbb{C})$. Then, for any matrices $C_0 \in \mathcal{M}_{n_0}(\mathbb{C})$ and $C_1 \in \mathcal{M}_{n_1}(\mathbb{C})$,

$$s_k(BC_0) \le s_k(B)\|C_0\|, \quad s_k(C_1B) \le \|C_1\| s_k(B), \tag{5-3}$$

and, for any matrices $C_0 \in GL_{n_0}(\mathbb{C})$ and $C_1 \in GL_{n_1}(\mathbb{C})$,

$$\frac{1}{\|C_0^{-1}\|\,\|C_1^{-1}\|} s_k(B) \le s_k(C_1BC_0) \le \|C_0\|\,\|C_1\| s_k(B), \tag{5-4}$$

where $\|A\| = (\max\sigma(AA^T))^{1/2}$ denotes the spectral radius of a matrix A. The inequalities (5-3) are specific and simple cases of the Ky Fan inequalities (see, for example, [Simon 1979] for a generalization). In particular, when $C_p^*C_p = \mathrm{Id}_{n_p} + \mathcal{O}(\varepsilon)$ $(p = 0, 1)$, the k-th singular value of B is close to the k-th singular value of C_1BC_0, that is, $s_k(C_1BC_0) = s_k(B)(1+\mathcal{O}(\varepsilon))$. In particular, computing the singular values of β in almost orthonormal bases (according to Definition 3.6) changes every $s_k(\beta)$ into $s_k(\beta)(1+\mathcal{O}(e^{-c/h}))$. To analyze the singular values of β, we will use the almost orthonormal bases built in the previous section.

Remark 5.2. Our approach, which emphasizes the differential $d_{f,h}$ and allows almost orthonormal changes of bases, is very close to [Bismut and Zhang 1994] (see in particular their Section 6), where an isomorphism between the Thom–Smale complex and the Witten complex is constructed.[2] The interest of our technique, following [Helffer and Nier 2006; Le Peutrec 2010b; Le Peutrec et al. 2013], is that the

[2]F. Nier thanks J. M. Bismut for mentioning this point.

hierarchy of long range tunnel effects can be analyzed accurately using a Gauss elimination algorithm (see [Le Peutrec 2009]). This makes more explicit the inductive process which was used by Bovier, Eckhoff, Gayrard and Klein [Bovier et al. 2004; 2005]. Actually, the present analysis shows that the Thom–Smale transversality condition and the Morse condition are not necessary: introducing the suitable block structure associated with the assumed geometry of the tunnel effect (see in particular Hypothesis 2) suffices.

5A. *Structure of β.* The estimates $\|v_j^{(0)} - \Pi^{(0)} v_j^{(0)}\|_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h})$ and $\|v_k^{(1)} - \Pi^{(1)} v_k^{(1)}\|_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h})$ of Proposition 4.5 together with the results stated in Proposition 4.5(1) ensure that

$$\mathcal{B}^{(0)} = (\Pi^{(0)} v_j^{(0)})_{1\le j\le m_0^D(\Omega_+)} \quad \text{and} \quad \mathcal{B}^{(1)} = (\Pi^{(1)} v_k^{(1)})_{1\le k\le m_1^D(\Omega_+)}$$

are almost orthonormal bases of $F^{(0)}$ and $F^{(1)}$. The same holds for their dual bases (in $L^2(\Omega_+)$), denoted by $\mathcal{B}^{(0),*}$ and $\mathcal{B}^{(1),*}$. The matrix of $\beta = d_{f,h}\big|_{F^{(0)}} : F^{(0)} \to F^{(1)}$ in the bases $\mathcal{B}^{(0)}$, $\mathcal{B}^{(1),*}$ is given by

$$M(\beta, \mathcal{B}^{(0)}, \mathcal{B}^{(1),*}) = B = (b_{k,j})_{1\le k\le m_1^D(\Omega_+),\, 1\le j\le m_0^D(\Omega_+)} \quad \text{with} \quad b_{k,j} = \langle \Pi^{(1)} v_k^{(1)}, \beta \Pi^{(0)} v_j^{(0)} \rangle_{L^2(\Omega_+)}.$$

Remember that the coefficients are equivalently written, by using (4-5), as

$$b_{k,j} = \langle \Pi^{(1)} v_k^{(1)}, \beta \Pi^{(0)} v_j^{(0)} \rangle_{L^2(\Omega_+)} = \langle \Pi^{(1)} v_k^{(1)}, d_{f,h} v_j^{(0)} \rangle_{L^2(\Omega_+)} = \langle v_k^{(1)}, d_{f,h} \Pi^{(0)} v_j^{(0)} \rangle_{L^2(\Omega_+)}. \tag{5-5}$$

Following the various cases discussed in Proposition 4.6, where the scalar products $\langle v_k^{(1)}, d_{f,h} v_j^{(0)} \rangle_{L^2(\Omega_+)}$ were studied, we shall write the matrix B in block form:

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}, \quad \text{where} \quad \begin{cases} B_{1,1} = (\langle \Pi^{(1)} v_k^{(1)}, d_{f,h} v_1^{(0)} \rangle_{L^2(\Omega_+)})_{1\le k\le m_1^N(\Omega_-)}, \\ B_{1,2} = (\langle \Pi^{(1)} v_k^{(1)}, d_{f,h} v_j^{(0)} \rangle_{L^2(\Omega_+)})_{2\le j\le m_0^D(\Omega_+),\, 1\le k\le m_1^N(\Omega_-)}, \\ B_{2,1} = (\langle \Pi^{(1)} v_k^{(1)}, d_{f,h} v_1^{(0)} \rangle_{L^2(\Omega_+)})_{m_1^N(\Omega_-)+1\le k\le m_1^D(\Omega_+)}, \\ B_{2,2} = (\langle \Pi^{(1)} v_k^{(1)}, d_{f,h} v_j^{(0)} \rangle_{L^2(\Omega_+)})_{2\le j\le m_0^D(\Omega_+),\, m_1^N(\Omega_-)+1\le k\le m_1^D(\Omega_+)} \end{cases}$$

In the following, we will give some estimates of each of these blocks in the asymptotic regime $h \to 0$. We let

$$C_0 = 2\|\nabla f\|_{L^\infty(\operatorname{supp}(\nabla \chi_0))}. \tag{5-6}$$

Notice that $C_0 > 0$. We assume that $\delta_+ > 0$ is chosen so that

$$\delta_+ < \frac{\kappa_f}{C_0}. \tag{5-7}$$

The assumption (2-8) will be useful to study the blocks $B_{1,2}$ and $B_{2,2}$ and the parameter $\delta_+ > 0$ (see Figure 1) will be further adjusted when considering the blocks $B_{1,1}$ and $B_{2,1}$.

5B. *The blocks $B_{1,2}$ and $B_{2,2}$.* Estimates for both blocks rely on assumption (2-8). Let us start with $B_{1,2}$.

Lemma 5.3. *The coefficients of $B_{1,2}$ satisfy*

$$b_{k,j} = \langle \Pi^{(1)} v_k^{(1)}, d_{f,h} v_j^{(0)} \rangle_{L^2(\Omega_+)} = \sqrt{\mu_j^{(0)}(\Omega_-)}\,(\delta_{k,j-1} + \mathcal{O}(e^{-c/h}))$$

for $j \in \{2, \dots, m_0^D(\Omega_+)\}$ and $k \in \{1, \dots, m_1^N(\Omega_-)\}$.

Proof. Let us first estimate $\|d_{f,h} v_j^{(0)}\|_{L^2(\Omega_+)}$ by writing

$$d_{f,h} v_j^{(0)} = d_{f,h}(\chi_-^{(0)} \psi_j^{(0)}) = \chi_-^{(0)} d_{f,h} \psi_j^{(0)} + h\psi_j^{(0)} d\chi_-^{(0)} = \chi_-^{(0)} \sqrt{\mu_j^{(0)}(\Omega_-)}\, \psi_{j-1}^{(1)} + h\psi_j^{(0)} d\chi_-^{(0)}.$$

Since $\operatorname{supp} d\chi_-^{(0)} \subset \Omega_- \setminus U \subset \mathcal{V}_-$ (see (5-1)), (2-8) implies $\|d_{f,h} v_j^{(0)}\|_{L^2(\Omega_+)} = \tilde{\mathcal{O}}\big(\sqrt{\mu_j^{(0)}(\Omega_-)}\big)$. The difference

$$|\langle \Pi^{(1)} v_k^{(1)}, d_{f,h} v_j^{(0)} \rangle_{L^2(\Omega_+)} - \langle v_k^{(1)}, d_{f,h} v_j^{(0)} \rangle_{L^2(\Omega_+)}|$$

is thus bounded from above by

$$\|\Pi^{(1)} v_k^{(1)} - v_k^{(1)}\|_{L^2(\Omega_+)} \tilde{\mathcal{O}}\big(\sqrt{\mu_j^{(0)}(\Omega_-)}\big) \le C e^{-c'/(2h)} \sqrt{\mu_j^{(0)}(\Omega_-)},$$

owing to the estimate $\|\Pi^{(1)} v_k^{(1)} - v_k^{(1)}\| = \mathcal{O}(e^{-c'/h})$ obtained in Proposition 4.5(3). The result then comes from the expression of $\langle v_k^{(1)}, d_{f,h} v_j^{(0)} \rangle_{L^2(\Omega_+)}$ given in Proposition 4.6(3). □

The estimate of the block $B_{2,2}$ follows the same lines:

Lemma 5.4. *The coefficients of $B_{2,2}$ satisfy*

$$b_{k,j} = \langle \Pi^{(1)} v_k^{(1)}, d_{f,h} v_j^{(0)} \rangle_{L^2(\Omega_+)} = \mathcal{O}\big(\sqrt{\mu_j^{(0)}(\Omega_-)}\, e^{-c/h}\big)$$

for $j \in \{2, \dots, m_0^D(\Omega_+)\}$ and $k \in \{m_1^N(\Omega_-)+1, \dots, m_1^D(\Omega_+)\}$.

Proof. Using $\|d_{f,h} v_j^{(0)}\| = \tilde{\mathcal{O}}\big(\sqrt{\mu_j^{(0)}(\Omega_-)}\big)$ again, $\|\Pi^{(1)} v_k^{(1)} - v_k^{(1)}\| = \mathcal{O}(e^{-c'/h})$ and, according to Proposition 4.6(4), $\langle v_k^{(1)}, d_{f,h} v_j^{(0)} \rangle = 0$ we get $|b_{k,j}| \le C e^{-c'/(2h)} \sqrt{\mu_j^{(0)}(\Omega_-)}$. □

5C. *The block $B_{1,1}$*. In this section, the value of the parameter δ_+ is adjusted. This value will possibly be changed twice more: for the estimate of the block $B_{2,1}$ and in the final proof of Theorem 2.4; see Sections 6A and 6B. Remember that the constant c occurring in the remainders $\mathcal{O}(e^{-c/h})$ introduced in Proposition 4.5 does not depend on $\delta_+ > 0$.

Lemma 5.5. *For any $k \in \{1, \dots, m_1^N(\Omega_-)\}$, the matrix element $b_{k,1}$ satisfies*

$$b_{k,1} = \langle \Pi^{(1)} v_k^1, d_{f,h} v_1^{(0)} \rangle_{L^2(\Omega_+)} = \mathcal{O}(e^{-(\kappa_f + c - C_0\delta_+)/h}),$$

where $\kappa_f = \min_{\partial\Omega_+} f - \min_{\Omega_+} f$, and the constants $c > 0$ and $C_0 > 0$ (defined by (5-6)) are independent of $\delta_+ > 0$. In particular, when $\delta_+ > 0$ is chosen smaller than c/C_0, one gets

$$b_{k,1} = \mathcal{O}(e^{-(\kappa_f + c)/h})$$

for a positive constant c, which depends on δ_+.

Proof. Remember that $v_1^{(0)} = \chi_0 \tilde{\psi}_1^{(0)} = \chi_0 e^{-f/h} / \big(\int_{\Omega_+} e^{-2f(x)/h}\, dx\big)^{1/2}$, where $\nabla\chi_0$ is supported in $\{x \in \Omega_+ : d(x, \partial\Omega_+) < \delta_+\}$ (see Figure 1). The Witten differential of $v_1^{(0)}$ satisfies

$$d_{f,h} v_1^{(0)} = \frac{e^{-f/h}}{\big(\int_{\Omega_+} e^{-2f(x)/h}\, dx\big)^{1/2}} (h\, d\chi_0)$$

and its L^2-norm can be estimated by

$$\|d_{f,h}v_1^{(0)}\|^2_{L^2(\Omega_+)} \le C_{\chi_0}\frac{\int_{\mathrm{supp}(\nabla\chi_0)} e^{-2f(x)/h}\,dx}{\int_{\Omega_+} e^{-2f(x)/h}\,dx}.$$

With $f(x) \ge \min_{\partial\Omega_+} f - \frac{1}{2}C_0\delta_+$ for $x \in \mathrm{supp}(\nabla\chi_0)$ (where C_0 is defined by (5-6) and does not depend on δ_+) and the lower bound $\int_{\Omega_+} e^{-2f(x)/h}\,dx \ge h^{d/2}e^{-2(\min_{\Omega_+} f)/h}/C_1$ of Lemma 4.3, we get

$$\|d_{f,h}v_1^{(0)}\|^2_{L^2(\Omega_+)} \le C_1 h^{-d/2} e^{-2(\kappa_f - C_0\delta_+/2)/h} \le C_2 e^{-2(\kappa_f - C_0\delta_+)/h} \tag{5-8}$$

provided that h is small enough. Then, like in Lemma 5.3, using

$$|b_{k,1} - \langle v_k^{(1)}, d_{f,h}v_1^{(0)}\rangle_{L^2(\Omega_+)}| \le \|\Pi^{(1)}v_k^{(1)} - v_k^{(1)}\|_{L^2(\Omega_+)}\|d_{f,h}v_1^{(0)}\|_{L^2(\Omega_+)} \le C_3 e^{-c'/h}e^{-(\kappa_f - C_0\delta_+)/h},$$

the equality $\langle v_k^{(1)}, d_{f,h}v_1^{(0)}\rangle = 0$ (see Proposition 4.6(1)) yields the result. □

Remark 5.6. If $m_1^D(\Omega_+ \setminus \Omega_-) = 0$ (and thus $m_1^N(\Omega_-) = m_1^D(\Omega_+)$), the previous lemma shows that

$$\langle \Pi^{(0)}v_1^{(0)}, \beta^*\beta\Pi^{(0)}v_1^{(0)}\rangle_{F^{(0)}} = \|\beta\Pi^{(0)}v_1^{(0)}\|^2_{F^{(0)}} = \sum_{k=1}^{m_1^N(\Omega_-)} |b_{k,1}|^2(1 + \mathcal{O}(e^{-c/h})) = \mathcal{O}(e^{-(\kappa_f + c)/h}).$$

This implies that $\beta^*\beta$ (and therefore $\Delta_{f,h}^{D,(0)}(\Omega_+)$) has an eigenvalue of the order $\mathcal{O}(e^{-(\kappa_f+c)/h})$, which contradicts Lemma 5.9 below. Therefore, $m_1^D(\Omega_+ \setminus \Omega_-)$ is not zero.

5D. *The block $B_{2,1}$*. We shall first give an approximate expression for the coefficients of the column $B_{2,1}$.

Proposition 5.7. *For any $k \in \{m_1^N(\Omega_-)+1, \dots, m_1^D(\Omega_+)\}$, the matrix element*

$$b_{k,1} = \langle \Pi^{(1)}v_k^1, d_{f,h}v_1^{(0)}\rangle_{L^2(\Omega_+)}$$

satisfies

$$b_{k,1} = -\frac{h\int_{\partial\Omega_+} e^{-f(\sigma)/h}\boldsymbol{i}_n\psi_k^{(1)}(\sigma)\,d\sigma}{\left(\int_{\Omega_+} e^{-2f(x)/h}\,dx\right)^{1/2}} + \mathcal{O}(e^{-(\kappa_f+c)/h}), \tag{5-9}$$

where c is a positive constant which depends on $\delta_+ > 0$ chosen to be sufficiently small, and $\kappa_f = \min_{\partial\Omega_+} f - \min_{\Omega_+} f$. Moreover, these coefficients $b_{k,1}$ satisfy

$$\lim_{h\to 0} h\log\left[\sum_{k=m_1^N(\Omega_-)+1}^{m_1^D(\Omega_+)} |b_{k,1}|^2\right] = -2\kappa_f. \tag{5-10}$$

The estimate (5-10) shows that the approximation (5-9) is meaningful, in the sense that some of the coefficients $b_{k,1}$ are indeed larger than the error term $\mathcal{O}(e^{-(\kappa_f+c)/h})$. In particular, we have

$$\sum_{k=m_1^N(\Omega_-)+1}^{m_1^D(\Omega_+)} |b_{k,1}|^2 = \frac{h^2\sum_{k=m_1^N(\Omega_-)+1}^{m_1^D(\Omega_+)}\left(\int_{\partial\Omega_+} e^{-f(\sigma)/h}\boldsymbol{i}_n\psi_k^{(1)}(\sigma)\,d\sigma\right)^2}{\int_{\Omega_+} e^{-2f(x)/h}\,dx}(1 + \mathcal{O}(e^{-c/h})). \tag{5-11}$$

Proof. The first statement is proved like in Lemma 5.5, after recalling

$$\langle v_k^{(1)}, d_{f,h} v_1^{(0)}\rangle = -\frac{h \int_{\partial\Omega_+} e^{-f(\sigma)/h} \boldsymbol{i}_n \psi_k^{(1)}(\sigma)\, d\sigma}{\left(\int_{\Omega_+} e^{-2f(x)/h}\, dx\right)^{1/2}},$$

according to Proposition 4.6(2).

For the equality (5-10), the upper bound

$$\limsup_{h\to 0} h \log\Bigg[\sum_{k=m_1^N(\Omega_-)+1}^{m_1^D(\Omega_+)} |b_{k,1}|^2\Bigg] \le -2\kappa_f$$

is a consequence of

$$\left|\frac{\int_{\partial\Omega_+} e^{-f(\sigma)/h} \boldsymbol{i}_n \psi_k^{(1)}(\sigma)\, d\sigma}{\left(\int_{\Omega_+} e^{-2f(x)/h}\, dx\right)^{1/2}}\right| \le C \frac{\left(\int_{\partial\Omega_+} |\boldsymbol{i}_n \psi_k^{(1)}(\sigma)|^2\, d\sigma\right)^{1/2}}{\left(\int_{\Omega_+} e^{-2(f(x)-\min_{\Omega_+} f)/h}\, dx\right)^{1/2}} e^{-\kappa_f/h},$$

where the denominator is bounded from below by Lemma 4.3. The numerator is estimated by

$$\|\psi_k^{(1)}|_{\partial\Omega_+}\|_{L^2(\partial\Omega_+)} \le C\|\psi_k^{(1)}\|_{W^{1,2}(\mathcal{V})} = \mathcal{O}(h^{-1}) = \tilde{\mathcal{O}}(1)$$

owing to Proposition 3.8, since $d_{\mathrm{Ag}}(x, \mathcal{V}) = 0$ for $x \in \mathcal{V}$. Using Lemma 5.5, the lower bound for (5-10) is equivalent to

$$\liminf_{h\to 0} h \log\Bigg[\sum_{k=1}^{m_1^D(\Omega_+)} |b_{k,1}|^2\Bigg] \ge -2\kappa_f. \tag{5-12}$$

Since $b_{k,1} = \langle \Pi^{(1)} v_k^{(1)}, d_{f,h} \Pi^{(0)} v_1^{(0)}\rangle_{L^2(\Omega_+)}$ is the k-th component of $d_{f,h}\Pi^{(0)} v_1^{(0)} \in F^{(1)}$ in the almost orthonormal basis $\mathcal{B}^{(1),*}$ of $F^{(1)}$, the inequality (5-12) is equivalent to

$$\liminf_{h\to 0} h \log(\|d_{f,h}\Pi^{(0)} v_1^{(0)}\|^2_{L^2(\Omega_+)}) = \liminf_{h\to 0} h \log\big(\langle \Pi^{(0)} v_1^{(0)}, \Delta_{f,h}^{D,(0)}(\Omega_+)\Pi^{(0)} v_1^{(0)}\rangle_{L^2(\Omega_+)}\big) \ge -2\kappa_f.$$

With $\|\Pi^{(0)} v_1^{(0)}\|_{L^2(\Omega_+)} = 1 + \mathcal{O}(e^{-c/h})$, the last inequality is a consequence of

$$\liminf_{h\to 0} h \log\big[\min \sigma(\Delta_{f,h}^{D,(0)}(\Omega_+))\big] \ge -2\kappa_f,$$

which is proved in the next lemma. □

Remark 5.8. Using Lemma 5.5, the asymptotic result (5-10) is actually equivalent to

$$\lim_{h\to 0} h \log\Bigg[\sum_{k=1}^{m_1^D(\Omega_+)} |b_{k,1}|^2\Bigg] = -2\kappa_f.$$

We end this section with an estimate on the bottom of the spectrum of $\Delta_{f,h}^{D,(0)}(\Omega_+)$, which was used to conclude the proof of Proposition 5.7 above.

Lemma 5.9. *The bottom of the spectrum of* $\Delta^{D,(0)}_{f,h}(\Omega_+)$ *satisfies*

$$\lim_{h\to 0} h\log\bigl[\min\sigma(\Delta^{D,(0)}_{f,h}(\Omega_+))\bigr] = -2\kappa_f.$$

In particular, we have

$$\forall\varepsilon>0\ \exists C_\varepsilon>1\ \exists h_\varepsilon>0\ \forall h\in(0,h_\varepsilon]\quad \min\sigma(\Delta^{D,(0)}_{f,h}(\Omega_+))\ge \frac{1}{C_\varepsilon}e^{-2(\kappa_f+\varepsilon)/h}.$$

Proof. Let us introduce a function $w_1^{(0)}$ defined similarly to $v_1^{(0)}$ by $w_1^{(0)}=\tilde\chi_0\tilde\psi_1^{(0)}$, where $\tilde\chi_0$ is a $\mathcal{C}_0^\infty(\Omega_+)$ function, equal to 1 in a neighborhood of Ω_- and such that $d\tilde\chi_0$ is supported in $\{x\in\Omega_+ : d(x,\partial\Omega_+)\le\delta\}$. The estimate $\limsup_{h\to 0} h\log\bigl[\min\sigma(\Delta^{D,(0)}_{f,h}(\Omega_+))\bigr]\le -2\kappa_f$ is then a consequence of the computation

$$\langle w_1^{(0)},\Delta_{f,h}w_1^{(0)}\rangle_{L^2(\Omega_+)} = \|d_{f,h}w_1^{(0)}\|^2_{L^2(\Omega_+)} = \tilde{\mathcal{O}}(e^{-2(\kappa_f-C_0\delta)/h}) \tag{5-13}$$

by considering δ arbitrarily small. The last equality is proved like (5-8) above.

It remains to prove that $\liminf_{h\to 0} h\log\bigl[\min\sigma(\Delta^{D,(0)}_{f,h}(\Omega_+))\bigr]\ge -2\kappa_f$. The proof is very similar to that of Proposition 3.7. Assume on the contrary that there exists $\varepsilon_0>0$ and a sequence h_n such that $\lim_{n\to\infty}h_n=0$ and

$$\min\sigma(\Delta^{D,(0)}_{f,h_n}(\Omega_+))\le Ce^{-2(\kappa_f+\varepsilon_0)/h_n}.$$

To simplify the notation, let us drop the subscript n in h_n. The previous inequality means that there exists $v_h\in L^2(\Omega_+)$ and $\lambda_h\ge 0$ such that

$$\Delta^{D,(0)}_{f,h}v_h=\lambda_h v_h\quad\text{in }\ \Omega_+,\qquad v_h\big|_{\partial\Omega_+}=0,\qquad \|v_h\|_{L^2(\Omega_+)}=1, \tag{5-14}$$

$$\lambda_h=\langle v_h,\Delta^{D,(0)}_{f,h}(\Omega_+)v_h\rangle_{L^2(\Omega_+)}=\|d_{f,h}v_h\|^2_{L^2(\Omega_+)}\le Ce^{-2(\kappa_f+\varepsilon_0)/h}. \tag{5-15}$$

For a small $t>0$, let us consider the domain

$$\Omega_t=\Bigl\{x\in\Omega_+ : f(x)<\min_{\partial\Omega_+}f+t\Bigr\}.$$

With $d_{f,h}=e^{-(f-\min_{\Omega_+}f)/h}(hd)e^{(f-\min_{\Omega_+}f)/h}$, the estimate (5-15) implies

$$\begin{aligned}\|d(e^{(f-\min_{\Omega_+}f)/h}v_h)\|_{L^2(\Omega_t)} &\le h^{-1}\max_{x\in\Omega_t}e^{(f(x)-\min_{\Omega_+}f)/h}\|d_{f,h}v_h\|_{L^2(\Omega_t)}\\ &\le Ch^{-1}e^{-(\varepsilon_0-t)/h}=\mathcal{O}(e^{-\varepsilon_0/(2h)})\end{aligned} \tag{5-16}$$

as soon as $t<\frac12\varepsilon_0$.

For a given $t\in\bigl(0,\frac12\varepsilon_0\bigr)$, let us now prove that $\|v_h\|_{L^2(\Omega_t)}$ is close to 1, using the same reasoning as in the proof of Proposition 3.4. There exists is an open neighborhood $\mathcal{V}$ of $\{x\in\Omega_- : \nabla f(x)=0\}$ such that $\mathcal{V}\subset\Omega_t$ and

$$d_{\mathrm{Ag}}(\Omega_+\setminus\Omega_t,\mathcal{V})\ge c>0, \tag{5-17}$$

where c can be chosen independently of t, and ε_0 and is positive according to Hypothesis 2. Applying Lemma 3.1 with $\Omega=\Omega_+$ and $\varphi=(1-\alpha h)d_{\mathrm{Ag}}(\,\cdot\,,\mathcal{V})$, one gets, for $h<1/\alpha$ (similarly to (3-3)),

$$0\ge h^2\|d(e^{\varphi/h}v_h)\|^2_{L^2(\Omega_+)}+h\bigl[\alpha\langle e^{\varphi/h}v_h,|\nabla f|^2e^{\varphi/h}v_h\rangle_{L^2(\Omega_+)}-C_f\|e^{\varphi/h}v_h\|^2_{L^2(\Omega_+)}\bigr].$$

By choosing α sufficiently large that $\alpha \min_{\Omega_+\setminus\mathcal{V}} |\nabla f|^2 \geq 2C_f$, we get

$$0 \geq h^2\|d(e^{\varphi/h}v_h)\|^2_{L^2(\Omega_+)} + h\big[C_f\|e^{\varphi/h}v_h\|^2_{L^2(\Omega_+\setminus\mathcal{V})} - C_f\|e^{\varphi/h}v_h\|^2_{L^2(\mathcal{V})}\big].$$

Using the fact that $\|e^{\varphi/h}v_h\|^2_{L^2(\mathcal{V})} = \|v_h\|^2_{L^2(\mathcal{V})} \leq 1$, we obtain, by adding $2C_f h\|v_h\|^2_{L^2(\mathcal{V})}$ on both sides of the previous inequality,

$$2C_f h \geq 2C_f h\|v_h\|^2_{L^2(\mathcal{V})} \geq h^2\|d(e^{\varphi/h}v_h)\|^2_{L^2(\Omega_+)} + hC_f\|e^{\varphi/h}v_h\|^2_{L^2(\Omega_+)}.$$

This implies, in particular,

$$\|e^{d_{\mathrm{Ag}}(\,\cdot\,,\mathcal{V})/h}v_h\|^2_{L^2(\Omega_+)} \leq 2,$$

and thus, using (5-17),

$$\|v_h\|^2_{L^2(\Omega_+\setminus\Omega_t)} \leq Ce^{-c/h}.$$

This implies

$$\|e^{(f-\min_{\Omega_+} f)/h}v_h\|_{L^2(\Omega_t)} \geq \|v_h\|_{L^2(\Omega_t)} \geq 1 - Ce^{-c/h}, \tag{5-18}$$

where, we recall, c is independent of t and ε_0, supposed to be small enough.

The two estimates (5-16) and (5-18) lead to a contradiction. Indeed, let us now set $t = \frac{1}{4}\varepsilon_0$. The Poincaré–Wirtinger inequality or, equivalently, the spectral gap estimate for the Neumann Laplacian in $\Omega_{\varepsilon_0/4}$, implies that there exists a constant C_h such that

$$\|(e^{(f-\min_{\Omega_+} f)/h}v_h) - C_h\|_{L^2(\Omega_{\varepsilon_0/4})} = \mathcal{O}(e^{-\varepsilon_0/(2h)}),$$

and therefore

$$\|(e^{(f-\min_{\Omega_+} f)/h}v_h) - C_h\|_{W^{1,2}(\Omega_{\varepsilon_0/4})} = \mathcal{O}(e^{-\varepsilon_0/(2h)}).$$

Since $\Omega_{\varepsilon_0/4} \cap \partial\Omega_+$ has a nonempty interior U_{ε_0}, the trace theorem implies

$$\|(e^{(f-\min_{\Omega_+} f)/h}v_h) - C_h\|_{L^2(U_{\varepsilon_0})} = \mathcal{O}(e^{-\varepsilon_0/(2h)}).$$

Since $v_h\big|_{\partial\Omega_+} \equiv 0$ and since U_{ε_0} is fixed by ε_0 and independent of h, this implies $C_h = \mathcal{O}(e^{-\varepsilon_0/(2h)})$. We are led to

$$1 - Ce^{-c/h} \leq \|v_h\|_{L^2(\Omega_{\varepsilon_0/4})} \leq \|e^{(f-\min_{\Omega_+} f)/h}v_h\|_{L^2(\Omega_{\varepsilon_0/4})} \leq \|C_h\|_{L^2(\Omega_{\varepsilon_0/4})} + Ce^{-\varepsilon_0/(2h)} \leq C'e^{-\varepsilon_0/(2h)},$$

which is impossible when h is small enough. □

This lemma shows the equality (2-11) stated in Theorem 2.4.

5E. *Singular values of β.* We are now in position to complete the proof of Proposition 5.1.

Proof of Proposition 5.1. Let $e^{(0)} = (e_1^{(0)}, \dots, e^{(0)}_{m_0^D(\Omega_+)})$ (resp. $e^{(1)} = (e_1^{(1)}, \dots, e^{(1)}_{m_1^D(\Omega_+)})$) denote an orthonormal basis of $F^{(0)}$ (resp. of $F^{(1)}$) and let C_0 (resp. C_1) be the matrix of the change of basis from $e^{(0)}$ (resp. from $\mathcal{B}^{(1),*}$) to $\mathcal{B}^{(0)}$ (resp. to $e^{(1)}$). Let $A = M(\beta, e^{(0)}, e^{(1)})$ denote the matrix of β in the bases $e^{(0)}$ and $e^{(1)}$, so that

$$A = C_1 B C_0,$$

where, we recall, $B = M(\beta, \mathcal{B}^{(0)}, \mathcal{B}^{(1),*})$. Using the fact that $\mathcal{B}^{(0)}$ and $\mathcal{B}^{(1)}$ are almost orthonormal bases, the matrices C_0 and C_1 satisfy $C_p^* C_p = \mathrm{Id} + \mathcal{O}(\varepsilon)$, so that (according to (5-4))

$$s_j(\beta) = s_j(A) = s_j(C_1 B C_0) = s_j(B)(1 + \mathcal{O}(e^{-c/h})).$$

The singular values of β can be understood from those of B, up to exponentially small relative errors.

Now Lemmas 5.3, 5.4, 5.5 and Proposition 5.7 can be gathered (using the block structure of B introduced in Section 5A), in the asymptotic regime $h \to 0$, as

$$B = \left(\begin{array}{c|cccc}
\mathcal{O}(b_{k_0,1}e^{-c/h}) & b_{1,2} & \mathcal{O}(b_{2,3}e^{-c/h}) & \dots & \mathcal{O}(b_{m_0^D-1,m_0^D}e^{-c/h}) \\
\vdots & \mathcal{O}(b_{1,2}e^{-c/h}) & b_{2,3} & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \mathcal{O}(b_{m_0^D-1,m_0^D}e^{-c/h}) \\
\mathcal{O}(b_{k_0,1}e^{-c/h}) & \mathcal{O}(b_{1,2}e^{-c/h}) & \mathcal{O}(b_{2,3}e^{-c/h}) & \dots & b_{m_0^D-1,m_0^D} \\
\mathcal{O}(b_{k_0,1}e^{-c/h}) & \mathcal{O}(b_{1,2}e^{-c/h}) & \mathcal{O}(b_{2,3}e^{-c/h}) & \dots & \mathcal{O}(b_{m_0^D-1,m_0^D}e^{-c/h}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathcal{O}(b_{k_0,1}e^{-c/h}) & \mathcal{O}(b_{1,2}e^{-c/h}) & \mathcal{O}(b_{2,3}e^{-c/h}) & \dots & \mathcal{O}(b_{m_0^D-1,m_0^D}e^{-c/h}) \\
\hline
b_{m_1^N+1,1} & \mathcal{O}(b_{1,2}e^{-c/h}) & \mathcal{O}(b_{2,3}e^{-c/h}) & \dots & \mathcal{O}(b_{m_0^D-1,m_0^D}e^{-c/h}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_{m_1^D,1} & \mathcal{O}(b_{1,2}e^{-c/h}) & \mathcal{O}(b_{2,3}e^{-c/h}) & \dots & \mathcal{O}(b_{m_0^D-1,m_0^D}e^{-c/h})
\end{array}\right),$$

where we used m_0^D (resp. m_1^N, m_1^D) instead of $m_0^D(\Omega_+)$ (resp. $m_1^N(\Omega_-)$, $m_1^D(\Omega_+)$) and where k_0 is a (possibly h-dependent) index such that $|b_{k_0,1}| = \max_{m_1^N+1 \le k \le m_1^D} |b_{k,1}|$. By Gaussian elimination (see [Le Peutrec 2009] for more details), one can find a matrix $R \in \mathcal{M}_{m_1^D}(\mathbb{R})$ with $\|R\| = \mathcal{O}(e^{-c/h})$ such that

$$(\mathrm{Id}_{m_1^D} + R)B = \tilde{B} = \left(\begin{array}{c|c}
\begin{matrix} 0(m_0^D - 1, 1) \\ 0(m_1^N - m_0^D + 1, 1) \end{matrix} & \begin{matrix} \tilde{B}_{1,2} \\ 0(m_1^N - m_0^D + 1, m_0^D - 1) \end{matrix} \\
\hline
\tilde{B}_{3,1} & 0(m_1^D - m_1^N, m_0^D - 1)
\end{array}\right)$$

with

$$\tilde{B}_{3,1} = \begin{pmatrix} b_{m_1^N+1,1} \\ \vdots \\ b_{m_1^D,1} \end{pmatrix} \quad \text{and} \quad \tilde{B}_{1,2} = \begin{pmatrix}
b_{1,2}(1+\mathcal{O}(e^{-c/h})) & 0 & \cdots & 0 \\
0 & \ddots & & \vdots \\
\vdots & & \ddots & 0 \\
0 & \cdots & 0 & b_{m_0^D-1,m_0^D}(1+\mathcal{O}(e^{-c/h}))
\end{pmatrix},$$

where $0(i, j)$ is the null matrix in $\mathcal{M}_{i,j}(\mathbb{R})$. We deduce that the singular values of B are approximated (up to exponentially small relative error terms) by the ones of $\tilde{B}$, which are given by its block structure. We find (recall that the singular values are labeled in decreasing order):

$$s_j(B) = |b_{m_0^D-j,m_0^D-j+1}|(1 + \mathcal{O}(e^{-c/h})) \quad \text{for } j \in \{1, \dots, m_0^D - 1\}$$

and

$$s_{m_0^D}(B)^2 = \Bigg[\sum_{k=m_1^N+1}^{m_1^D} |b_{k,1}|^2\Bigg](1+\mathcal{O}(e^{-c/h})).$$

We conclude the proof of Proposition 5.1 using the approximate values of $b_{k,k+1}$ ($k \in \{1,\dots,m_0^D - 1\}$) and $b_{k,1}$ ($k \in \{m_1^N + 1,\dots,m_1^D\}$) given in Lemma 5.3 and Proposition 5.7:

$$|b_{m_0^D-j,m_0^D-j+1}| = \sqrt{\mu^{(0)}_{m_0^D-j+1}(\Omega_-)}(1+\mathcal{O}(e^{-c/h})) \quad \text{for } j \in \{1,\dots,m_0^D-1\},$$

$$\sum_{k=m_1^N+1}^{m_1^D} |b_{k,1}|^2 = \frac{h^2 \sum_{k=m_1^N+1}^{m_1^D} \big(\int_{\partial\Omega_+} e^{-f(\sigma)/h} \boldsymbol{i}_n \psi_k^{(1)}(\sigma)\, d\sigma\big)^2}{\int_{\Omega_+} e^{-2f(x)/h}\, dx} + \mathcal{O}(e^{-(2\kappa_f+c)/h}).$$

In particular, for h small enough, we indeed have

$$|b_{m_0^D-1,m_0^D}|^2 \ge \cdots \ge |b_{1,2}|^2 \ge \sum_{k=m_1^N+1}^{m_1^D} |b_{k,1}|^2,$$

the last inequality being a consequence of (5-10) and $|b_{1,2}|^2 = \mu_2^{(0)}(\Omega_-)(1+\mathcal{O}(e^{-c/h})) \ge C_\varepsilon e^{-2(\kappa_f-c_0)/h}$ using Proposition 3.7. □

6. Proof of Theorem 2.4 and two corollaries

Proposition 5.1 already provides a precise asymptotic result on the exponentially small eigenvalues of $\Delta^{D,(0)}_{f,h}(\Omega_+)$, using (5-2):

$$\lambda_j^{(0)}(\Omega_+) = s_{m_0^D(\Omega_+)+1-j}(\beta)^2 = \mu_j^{(0)}(\Omega_-)(1+\mathcal{O}(e^{-c/h})) \quad \text{for } j \in \{2,\dots,m_0^D(\Omega_+)\}, \tag{6-1}$$

$$\lambda_1^{(0)}(\Omega_+) = s_{m_0^D(\Omega_+)}(\beta)^2 = \frac{h^2 \sum_{k=m_1^N(\Omega_-)+1}^{m_1^D(\Omega_+)} \big(\int_{\partial\Omega_+} e^{-f(\sigma)/h} \boldsymbol{i}_n \psi_k^{(1)}(\sigma)\, d\sigma\big)^2}{\int_{\Omega_+} e^{-2f(x)/h}\, dx}(1+\mathcal{O}(e^{-c/h})), \tag{6-2}$$

the second estimate being a consequence of Proposition 5.7 (see (5-11)). This is essentially the result of Theorem 2.4 about $\lambda_1^{(0)}(\Omega_+)$ (see (2-12)); it remains to show that the basis $(\psi_k^{(1)})_{m_1^N(\Omega_-)+1\le k\le m_1^D(\Omega_+)}$ in (6-2) (which was introduced in Section 4B) can be replaced by *any* orthonormal basis $(u_k^{(1)})_{1\le k\le m_1^D(\Omega_+\setminus\Omega_-)}$ of $\operatorname{Ran} 1_{[0,\nu(h)]}(\Delta^{D,(1)}_{f,h}(\Omega_+ \setminus \overline{\Omega}_-))$. This will be done in Section 6C.

In addition, it also remains to prove the estimates (2-13) and (2-14) on the eigenvector $u_1^{(0)}$ associated with the smallest eigenvalue $\lambda_1^{(0)}(\Omega_+)$. This will be the subject of Sections 6A and 6B. We recall that the spectral subspace associated with $\lambda_1^{(0)}(\Omega_+)$ is one-dimensional (since $\lambda_2^{(0)}(\Omega_+) \ge \lambda_1^{(0)}(\Omega_+)e^{c/h}$). We thus have

$$u_1^{(0)} = \frac{\Pi_0 v_1^{(0)}}{\|\Pi_0 v_1^{(0)}\|_{L^2(\Omega_+)}}, \tag{6-3}$$

where Π_0 denotes the spectral projection associated with $\lambda_1^{(0)}(\Omega_+)$:

$$\Pi_0 = 1_{\{\lambda_1^{(0)}(\Omega_+)\}}(\Delta_{f,h}^{D,(0)}(\Omega_+)). \tag{6-4}$$

The fact that $\Pi_0 v_1^{(0)} \neq 0$ follows from the fact that $\Pi_0 \Pi^{(0)} = \Pi_0$ and the estimate, for small h,

$$\begin{aligned}\frac{\langle \Pi^{(0)} v_1^{(0)}, \Delta_{f,h}^{D,(0)} \Pi^{(0)} v_1^{(0)}\rangle_{L^2(\Omega_+)}}{\|\Pi^{(0)} v_1^{(0)}\|^2_{L^2(\Omega_+)}} &= \frac{\|d_{f,h}\Pi^{(0)} v_1^{(0)}\|^2_{L^2(\Omega_+)}}{\|\Pi^{(0)} v_1^{(0)}\|^2_{L^2(\Omega_+)}} = \|\beta \Pi^{(0)} v_1^{(0)}\|^2_{L^2(\Omega_+)}(1+\mathcal{O}(e^{-c/h})) \\ &= \sum_{k=m_1^N(\Omega_-)+1}^{m_1^D(\Omega_+)} |b_{k,1}|^2 (1+\mathcal{O}(e^{-c/h})) \\ &= \lambda_1^{(0)}(\Omega_+)(1+\mathcal{O}(e^{-c/h})) \le \lambda_2^{(0)}(\Omega_+) e^{-c/h}\end{aligned} \tag{6-5}$$

for some positive constant c. The second and third equalities are consequences of the almost orthonormality of the bases $\mathcal{B}^{(0)}$ and $\mathcal{B}^{(1),*}$ (see Proposition 4.5). The third one comes from (6-2) and (5-11). The last inequality is a consequence of (6-1) and Proposition 3.7.

Finally, Section 6D is devoted to two corollaries of Theorem 2.4.

6A. ***Approximation of*** $\boldsymbol{u_1^{(0)}}$**.** Let us first prove the estimate (2-13) on $u_1^{(0)}$.

Proposition 6.1. *There exists* $c > 0$ *such that*

$$\left\| u_1^{(0)} - \frac{e^{-f/h}}{\left(\int_{\Omega_+} e^{-2f(x)/h}\,dx\right)^{1/2}} \right\|_{W^{2,2}(\Omega_+)} = \mathcal{O}(e^{-c/h}).$$

Proof. Since $\|v_1^{(0)} - e^{-f/h}/\left(\int_{\Omega_+} e^{-2f(x)/h}\,dx\right)^{1/2}\|_{W^{2,2}(\Omega_+)} = \mathcal{O}(e^{-c/h})$ (which is a simple consequence of Lemma 4.3), it suffices to prove $\|u_1^{(0)} - v_1^{(0)}\|_{W^{2,2}(\Omega_+)} = \mathcal{O}(e^{-c/h})$.

Let us first prove the result in the $L^2(\Omega_+)$-norm. From (6-5), we have $\|d_{f,h}\Pi^{(0)} v_1^{(0)}\|^2_{L^2(\Omega_+)} \le \lambda_2^{(0)}(\Omega_+)e^{-c/h}$, and thus

$$\begin{aligned}\lambda_2^{(0)}(\Omega_+)\left\| 1_{[\lambda_2^{(0)}(\Omega_+),+\infty)}(\Delta_{f,h}^{D,(0)}(\Omega_+))\Pi^{(0)} v_1^{(0)}\right\|^2_{L^2(\Omega_+)} &\le \langle \Pi^{(0)} v_1^{(0)}, \Delta_{f,h}^{D,(0)}(\Omega_+)\Pi^{(0)} v_1^{(0)}\rangle_{L^2(\Omega_+)} \\ &\le \lambda_2^{(0)}(\Omega_+)e^{-c/h}.\end{aligned}$$

Since $\Pi_0 = \Pi_0\Pi^{(0)}$, we deduce

$$\left\|\Pi_0 v_1^{(0)} - \Pi^{(0)} v_1^{(0)}\right\|_{L^2(\Omega_+)} = \left\| 1_{[\lambda_2^{(0)}(\Omega_+),+\infty)}(\Delta_{f,h}^{D,(0)}(\Omega_+))\Pi^{(0)} v_1^{(0)}\right\|_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h}).$$

Using in addition the facts that $\|\Pi^{(0)} v_1^{(0)} - v_1^{(0)}\|_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h})$ and $\|v_1^{(0)}\|_{L^2(\Omega_+)} = 1+\mathcal{O}(e^{-c/h})$ (see Proposition 4.5), this proves

$$\|u_1^{(0)} - v_1^{(0)}\|_{L^2(\Omega_+)} = \mathcal{O}(e^{-c'/h}). \tag{6-6}$$

The estimate in the $W^{2,2}(\Omega_+)$-norm is then obtained by a bootstrap argument that will be used many times again below. The following equations hold:

$$\begin{cases}\Delta^{(0)}_{f,h}u^{(0)}_1=\lambda^{(0)}_1(\Omega_+)u^{(0)}_1,\\ u^{(0)}_1\big|_{\partial\Omega_+}=0,\end{cases}\qquad\text{and}\qquad\begin{cases}\Delta^{(0)}_{f,h}v^{(0)}_1=g_h,\\ v^{(0)}_1\big|_{\partial\Omega_+}=0,\end{cases}$$

where g_h is defined by the equation $g_h=\Delta^{(0)}_{f,h}v^{(0)}_1$ and, using the same arguments as in the proof of (5-8), $\|g_h\|_{L^2(\Omega_+)}=\mathcal{O}(e^{-(\kappa_f-C_0\delta_+)/h})$. Recall that, by the assumption (5-7), δ_+ is small enough that $C_0\delta_+<\kappa_f$, and thus $\|g_h\|_{L^2(\Omega_+)}=\mathcal{O}(e^{-c/h})$. We then deduce that, with Δ_{H} denoting the Hodge Laplacian (A-3), $u^{(0)}_1-v^{(0)}_1$ solves

$$\begin{cases}\Delta^{(0)}_{\mathrm{H}}(u^{(0)}_1-v^{(0)}_1)=\tilde g_h,\\ (u^{(0)}_1-v^{(0)}_1)\big|_{\partial\Omega_+}=0.\end{cases}$$

Again, $\tilde g_h$ is defined by the first equation. Using the formula (A-6), which relates the Hodge and the Witten Laplacians and the estimate (6-6), $\|\tilde g_h\|_{L^2(\Omega_+)}=\mathcal{O}(e^{-c'/h})$. The elliptic regularity of the Dirichlet Hodge Laplacian then implies $\|u^{(0)}_1-v^{(0)}_1\|_{W^{2,2}(\Omega_+)}=\mathcal{O}(e^{-c'/h})$. □

6B. *Approximation of $d_{f,h}u^{(0)}_1$*. We now consider $d_{f,h}u^{(0)}_1$. In this section, we will first prove (2-14) using for the $u^{(1)}_k$ the special basis considered in Section 5. This will be generalized to any orthonormal basis of $\operatorname{Ran} 1_{[0,\nu(h)]}(\Delta^{D,(1)}_{f,h}(\Omega_+\setminus\overline{\Omega}_-))$ in the next section.

Let us start with an estimate in the $L^2(\Omega_+)$-norm.

Proposition 6.2. *Let $\mathcal{B}^*_1=(w_k)_{1\le k\le m^D_1(\Omega_+)}$ be the basis of $F^{(1)}=\operatorname{Ran} 1_{[0,\nu(h)]}(\Delta^{D,(1)}_{f,h}(\Omega_+))$ dual (in $L^2(\Omega_+)$) to $\mathcal{B}_1=(\Pi^{(1)}v^{(1)}_k)_{1\le k\le m^D_1(\Omega_+)}$. Then the eigenvector $u^{(0)}_1$ of $\Delta^{D,(0)}_{f,h}(\Omega_+)$ given by (6-3) satisfies*

$$\Bigg\|d_{f,h}u^{(0)}_1-\sum_{k=m^N_1(\Omega_-)+1}^{m^D_1(\Omega_+)}b_{k,1}w_k\Bigg\|_{L^2(\Omega_+)}=\mathcal{O}(e^{-(\kappa_f+c)/h})\tag{6-7}$$

for some $c>0$ and where the coefficients $b_{k,1}$ are defined by (5-5).

Proof. By definition of the matrix $B=M(\beta,\mathcal{B}^{(0)},\mathcal{B}^{(1)*})$,

$$d_{f,h}(\Pi^{(0)}v^{(0)}_1)=\beta(\Pi^{(0)}v^{(0)}_1)=\sum_{k=1}^{m^D_1(\Omega_+)}b_{k,1}w_k=\sum_{k=m^N_1(\Omega_-)+1}^{m^D_1(\Omega_+)}b_{k,1}w_k+r_h$$

with $\|r_h\|_{L^2(\Omega_+)}=\mathcal{O}(e^{-(\kappa_f+c)/h})$, this estimate being a consequence of the almost orthonormality of the one-forms w_k, and of Lemma 5.5. Equation (6-7) is thus equivalent to:

$$\big\|d_{f,h}(u^{(0)}_1-\Pi^{(0)}v^{(0)}_1)\big\|_{L^2(\Omega_+)}=\mathcal{O}(e^{-(\kappa_f+c)/h}).$$

Notice that

$$u^{(0)}_1-\Pi^{(0)}v^{(0)}_1=\|\Pi_0v^{(0)}_1\|^{-1}_{L^2(\Omega_+)}(\Pi_0-\Pi^{(0)})v^{(0)}_1+(\|\Pi_0v^{(0)}_1\|^{-1}_{L^2(\Omega_+)}-1)\Pi^{(0)}v^{(0)}_1.$$

We recall that $\|\Pi_0 v_1^{(0)}\|_{L^2(\Omega_+)} = 1+\mathcal{O}(e^{-c/h})$ and $\|d_{f,h}\Pi^{(0)} v_1^{(0)}\|_{L^2(\Omega_+)} = \|\beta\Pi^{(0)} v_1^{(0)}\|_{L^2(\Omega_+)} = \tilde{\mathcal{O}}(e^{-\kappa_f/h})$ (see (6-5)). This implies that

$$\left\|d_{f,h}(u_1^{(0)} - \Pi^{(0)} v_1^{(0)})\right\|_{L^2(\Omega_+)} = \left\|d_{f,h}((\Pi_0 - \Pi^{(0)})v_1^{(0)})\right\|_{L^2(\Omega_+)}(1+\mathcal{O}(e^{-c/h})) + \mathcal{O}(e^{-(\kappa_f+c)/h}).$$

Moreover, using the fact that $\Pi_0\Pi^{(0)} = \Pi_0$ and $\Pi^{(0)} - \Pi_0 = 1_{[\lambda_2^{(0)}(\Omega_+),+\infty)}(\Delta_{f,h}^{D,(0)}(\Omega_+))$ commutes with $\Delta_{f,h}^{D,(0)}(\Omega_+)$,

$$\begin{aligned}
\|d_{f,h}((\Pi_0 - \Pi^{(0)})v_1^{(0)})\|_{L^2(\Omega_+)}^2 &= \langle(\Pi^{(0)} - \Pi_0)v_1^{(0)}, \Delta_{f,h}^{D,(0)}(\Omega_+)(\Pi^{(0)} - \Pi_0)v_1^{(0)}\rangle_{L^2(\Omega_+)} \\
&= \|\beta\Pi^{(0)} v_1^{(0)}\|_{L^2(\Omega_+)}^2 - \lambda_1^{(0)}(\Omega_+)\|\Pi_0 v_1^{(0)}\|_{L^2(\Omega_+)}^2 \\
&= \lambda_1^{(0)}(\Omega_+)(1+\mathcal{O}(e^{-c/h})) - \lambda_1^{(0)}(\Omega_+)(1+\mathcal{O}(e^{-c/h})) \\
&= \mathcal{O}(e^{-2(\kappa_f+c')/h}).
\end{aligned}$$

The third equality is obtained from (6-5) and the last one from the estimate on the bottom of the spectrum in Lemma 5.9. This concludes the proof of (6-7). □

To perform a bootstrap argument to extend the previous result to stronger norms, we need an intermediate lemma:

Lemma 6.3. *For any $p \in \mathbb{N}$, there exists $C_p > 0$ and $N_p \in \mathbb{N}$ such that*

$$\|u\|_{W^{p,2}(\Omega_+)} \le C_p h^{-N_p}\|u\|_{L^2(\Omega_+)} \quad \textit{for all } u \in F^{(1)} = \operatorname{Ran} 1_{[0,\nu(h)]}(\Delta_{f,h}^{D,(1)}(\Omega_+)).$$

Proof. Let us introduce an orthonormal basis $(e_k)_{1\le k\le m_1^D(\Omega_+)}$ of eigenvectors of $\Delta_{f,h}^{D,(1)}(\Omega_+)$ associated with the small eigenvalues $\lambda_k^{(1)}(\Omega_+) \le \nu(h)$, so $\Delta_{f,h}^{D,(1)} e_k = \lambda_k^{(1)} e_k$. We have

$$\|d_{f,h}e_k\|_{L^2(\Omega_+)}^2 + \|d_{f,h}^* e_k\|_{L^2(\Omega_+)}^2 = \lambda_k^{(1)} \le \nu(h).$$

For any $u \in F^{(1)}$, there exist some reals $(u_k)_{1\le k\le m_1^D(\Omega_+)}$ such that

$$u = \sum_{k=1}^{m_1^D(\Omega_+)} u_k e_k \quad \text{with} \quad \sum_{k=1}^{m_1^D(\Omega_+)} |u_k|^2 = \|u\|_{L^2(\Omega_+)}^2.$$

Lemma 6.3 will be proven if one can show that, for all $p \in \mathbb{N}$, there exist $C_p > 0$ and $N_p \in \mathbb{N}$ such that $\|e_k\|_{W^{p,2}(\Omega_+)} \le C_p h^{-N_p}$ for all $k \in \{1, \dots, m_1^D(\Omega_+)\}$. From

$$4\||\nabla f|e_k\|_{L^2(\Omega_+)}^2 + 2\|d_{f,h}e_k\|_{L^2(\Omega_+)}^2 + 2\|d_{f,h}^* e_k\|_{L^2(\Omega_+)}^2 \ge h^2[\|de_k\|_{L^2(\Omega_+)}^2 + \|d^* e_k\|_{L^2(\Omega_+)}^2]$$

(which is obtained from the formulas (A-4) and (A-5) that relate $d_{f,h}$ to d and $d_{f,h}^*$ to d^*), we deduce $\|e_k\|_{W^{1,2}(\Omega_+)} \le Ch^{-1}$. Then the equation $\Delta_{f,h}^{D,(1)}(\Omega_+)e_k = \lambda_k^{(1)} e_k$ can be written

$$\begin{cases} \Delta_H^{(1)} e_k = r_k(h) \\ \mathbf{t}e_k\big|_{\partial\Omega_+} = 0, \quad \mathbf{t}d^* e_k\big|_{\partial\Omega_+} = \rho_k(h) \end{cases}$$

with $\|r_k(h)\|_{L^2(\Omega_+)} + \|\rho_k(h)\|_{W^{1/2,2}(\partial\Omega_+)} = \mathcal{O}(h^{-2})$. The estimate on $\rho_k(h)$ follows from $0 = \mathbf{t}d_{f,h}^* e_k = h\mathbf{t}d^* e_k + \mathbf{i}_{\nabla f} e_k$, so that $\|\rho_k(h)\|_{W^{1/2,2}(\partial\Omega_+)} = h^{-1}\|\mathbf{i}_{\nabla f} e_k\|_{W^{1/2,2}(\partial\Omega_+)} \le Ch^{-1}\|e_k\|_{W^{1,2}(\Omega_+)} \le C'h^{-2}$. The estimate on $r_k(h)$ comes from the relation (A-6) between the Hodge and the Witten Laplacians. The elliptic regularity of the above system (see, for example, [Schwarz 1995, Theorem 2.2.6]) implies $\|e_k\|_{W^{2,2}(\Omega_+)} = \mathcal{O}(h^{-2})$. Finally, the result for a general $p \in \mathbb{N}$ is obtained by a bootstrap argument. □

We are now in position to restate the result of Proposition 6.2 in terms of the $W^{p,2}(\mathcal{V})$-norm.

Proposition 6.4. *Let $(\psi_k^{(1)})_{m_1^N(\Omega_-)+1\le k\le m_1^D(\Omega_+)}$ be the orthonormal basis of eigenvectors chosen in Section 4B and let χ_+ be the cut-off function of Definition 4.4. For any $p \in \mathbb{N}$, there exists a constant $C_p > 0$ such that*

$$\Big\| d_{f,h} u_1^{(0)} - \sum_{k=m_1^N(\Omega_-)+1}^{m_1^D(\Omega_+)} b_{k,1} \psi_k^{(1)} \Big\|_{W^{p,2}(\mathcal{V})} \le C_p e^{-(\kappa_f + c)/h},$$

where $\mathcal{V}$ is any neighborhood of $\partial\Omega_+$ contained in $\{\chi_+ = 1\}$, c is a positive constant and, we recall (see Proposition 5.7), the coefficients $b_{k,1}$ defined by (5-5) satisfy

$$b_{k,1} = -\frac{h \int_{\partial\Omega_+} e^{-f(\sigma)/h} \mathbf{i}_n \psi_k^{(1)}(\sigma)\, d\sigma}{\big(\int_{\Omega_+} e^{-2f(x)/h}\, dx\big)^{1/2}} + \mathcal{O}(e^{-(\kappa_f + c)/h}).$$

Proof. From Proposition 6.2 and Lemma 6.3, we deduce

$$\Big\| d_{f,h} u_1^{(0)} - \sum_{k=m_1^N(\Omega_-)+1}^{m_1^D(\Omega_+)} b_{k,1} w_k \Big\|_{W^{p,2}(\Omega_+)} \le C_p h^{-N_p} e^{-(\kappa_f + c)/h} \le C_p' e^{-(\kappa_f + c/2)/h}.$$

Since, by the almost orthonormality of the family $(\Pi^{(1)} v_k^{(1)})_{1\le k\le m_1^D(\Omega_+)}$, $\|w_k - \Pi^{(1)} v_k^{(1)}\|_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h})$ and $\max\{|b_{k,1}|, m_1^N(\Omega_-)+1 \le k \le m_1^D(\Omega_+)\} = \tilde{\mathcal{O}}(e^{-\kappa_f/h})$ (see Proposition 5.7), Lemma 6.3 also leads to

$$\Big\| d_{f,h} u_1^{(0)} - \sum_{k=m_1^N(\Omega_-)+1}^{m_1^D(\Omega_+)} b_{k,1} \Pi^{(1)} v_k^{(1)} \Big\|_{W^{p,2}(\Omega_+)} \le C_p'' e^{-(\kappa_f + c/2)/h}.$$

By recalling the definition of $v_k^{(1)} = \chi_+ \psi_k^{(1)}$, it suffices now to check that $\|v_k^{(1)} - \Pi^{(1)} v_k^{(1)}\|_{W^{p,2}(\Omega_+)}$ is of order $\mathcal{O}(e^{-c'/h})$ for some $c' > 0$. We already know

$$\|v_k^{(1)} - \Pi^{(1)} v_k^{(1)}\|_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h})$$

from Proposition 4.5.

For the $W^{1,2}(\Omega_+)$ estimates, notice that

$$\|d_{f,h} v_k^{(1)}\|_{L^2(\Omega_+)}^2 + \|d_{f,h}^* v_k^{(1)}\|_{L^2(\Omega_+)}^2 = \langle v_k^1, \Delta_{f,h}^{D,(1)}(\Omega_+) v_k^{(1)} \rangle_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h})$$

(again from Proposition 4.5), while $\Pi^{(1)} v_k^{(1)} \in F^{(1)} = \operatorname{Ran} 1_{[0,\nu(h)]}(\Delta_{f,h}^{D,(1)}(\Omega_+))$ implies

$$\|d_{f,h} \Pi^{(1)} v_k^{(1)}\|_{L^2(\Omega_+)}^2 + \|d_{f,h}^* \Pi^{(1)} v_k^{(1)}\|_{L^2(\Omega_+)}^2 = \langle \Pi^{(1)} v_k^1, \Delta_{f,h}^{D,(1)}(\Omega_+) \Pi^{(1)} v_k^{(1)} \rangle_{L^2(\Omega_+)} = \mathcal{O}(e^{-c/h}).$$

We deduce

$$\begin{aligned}
&\|d(v_k^{(1)}-\Pi^{(1)}v_k^{(1)})\|_{L^2(\Omega_+)}^2+\|d^*(v_k^{(1)}-\Pi^{(1)}v_k^{(1)})\|_{L^2(\Omega_+)}^2\\
&\quad\le \frac{2}{h^2}\big[\|d_{f,h}(v_k^{(1)}-\Pi^{(1)}v_k^{(1)})\|_{L^2(\Omega_+)}^2+\|d_{f,h}^*(v_k^{(1)}-\Pi^{(1)}v_k^{(1)})\|_{L^2(\Omega_+)}^2+2\||\nabla f|(v_k^{(1)}-\Pi^{(1)}v_k^{(1)})\|_{L^2(\Omega_+)}^2\big]\\
&\quad\le \frac{Ce^{-2c/h}}{h^2}.
\end{aligned}$$

This gives the $W^{1,2}$ estimate $\|v_k^{(1)}-\Pi^{(1)}v_k^{(1)}\|_{W^{1,2}(\Omega_+)}=\tilde{\mathcal{O}}(e^{-c/h})$.

The $W^{p,2}$ estimates ($p\ge 2$) are then obtained by an argument based on the elliptic regularity of the (nonhomogeneous) Dirichlet Hodge Laplacian. On the one hand, $\|\Pi^{(1)}v_k^{(1)}\|_{L^2(\Omega_+)}=1+\mathcal{O}(e^{-c/h})$, $\Pi^{(1)}v_k^{(1)}\in F^{(1)}$ and $\|\Delta_{f,h}^{D,(1)}|_{F^{(1)}}\|=\mathcal{O}(e^{-c/h})$ (see Proposition 3.12) imply that $\|\Delta_{f,h}^{D,(1)}\Pi^{(1)}v_k^{(1)}\|_{L^2(\Omega_+)}=\mathcal{O}(e^{-c/h})$. Lemma 6.3 can then be used to obtain $\|\Delta_{f,h}^{D,(1)}\Pi^{(1)}v_k^{(1)}\|_{W^{p,2}(\Omega_+)}=\tilde{\mathcal{O}}(e^{-c/h})$ for any integer p. Here, $\|\Delta_{f,h}^{D,(1)}|_{F^{(1)}}\|=\sup_{u\in F^{(1)}}(\|\Delta_{f,h}^{D,(1)}u\|_{L^2(\Omega_+)}/\|u\|_{L^2(\Omega_+)})$ is simply the spectral radius of the finite-dimensional operator $\Delta_{f,h}^{D,(1)}:F^{(1)}\to F^{(1)}$. On the other hand, Lemma 6.5 below implies $\|\Delta_{f,h}^{D,(1)}v_k^{(1)}\|_{W^{p,2}(\Omega_+)}=\|\Delta_{f,h}^{D,(1)}(\chi_+\psi_k^{(1)})\|_{W^{p,2}(\Omega_+)}=\mathcal{O}(e^{-c/h})$ for any integer p, using the arguments of the proofs of Proposition 3.5 or 3.9 to get the estimate on the truncated eigenvector from the exponential decay of the eigenvector. Thus, for $p\ge 1$, if $\|(v_k^{(1)}-\Pi^{(1)}v_k^{(1)})\|_{W^{p,2}(\Omega_+)}=\tilde{\mathcal{O}}(e^{-c/h})$ then the difference $v_k^{(1)}-\Pi^{(1)}v_k^{(1)}$ satisfies

$$\begin{cases}\Delta_H^{(1)}(v_k^{(1)}-\Pi^{(1)}v_k^{(1)})=r_k(h),\\ \mathbf{t}(v_k^{(1)}-\Pi^{(1)}v_k^{(1)})=0,\quad \mathbf{t}d^*(v_k^{(1)}-\Pi_k^{(1)}v_k^{(1)})=\varrho_k(h)\end{cases}$$

with $\|r_k(h)\|_{W^{p,2}(\Omega_+)}=\tilde{\mathcal{O}}(e^{-c/h})$ and $\|\varrho_k(h)\|_{W^{p-1/2,2}(\Omega_+)}=\tilde{\mathcal{O}}(e^{-c/h})$.

This implies $\|(v_k^{(1)}-\Pi^{(1)}v_k^{(1)})\|_{W^{p+2,2}(\Omega_+)}=\tilde{\mathcal{O}}(e^{-c/h})$. A bootstrap argument (induction on p) thus shows that, for any p, $\|v_k^{(1)}-\Pi^{(1)}v_k^{(1)}\|_{W^{p,2}(\Omega_+)}=\tilde{\mathcal{O}}(e^{-c/h})\le\mathcal{O}(e^{-c'/h})$ for any $c'<c$. □

We end this section with an estimate on the exponential decay (in a neighborhood of supp χ_+) of the eigenvectors of $\Delta_{f,h}^{D,(1)}(\Omega_+\setminus\overline{\Omega}_-)$ in $\mathcal{C}^\infty$ norm. This is a refinement of Proposition 3.8, which was needed in the previous proof.

Lemma 6.5. *For every $\varepsilon\in(0,1)$, there exists a function $\varphi_\varepsilon\in\mathcal{C}_0^\infty(\Omega_+\setminus\overline{\Omega}_-)$ such that, for all $x\in\overline{\Omega}_+\setminus\Omega_-$,*

$$\begin{gathered}|\nabla\varphi_\varepsilon(x)|\le(1-\varepsilon)|\nabla f(x)|,\\ d(x,\partial\Omega_+\cup\partial\Omega_-)\le\tfrac{1}{2}\varepsilon\implies\varphi_\varepsilon(x)=0,\\ \varphi_\varepsilon(x)\ge 0\quad\text{and}\quad d_{Ag}(x,\partial\Omega_+\cup\partial\Omega_-)-C\varepsilon\le\varphi_\varepsilon(x),\end{gathered}$$

where $C>0$ is a constant independent of ε. For every $p\in\mathbb{N}$, and once φ_ε is fixed, there exist $C_{\varepsilon,p}>0$ and $N_p>0$ independent of $h\in[0,h_0]$ such that every normalized eigenvector ψ of $\Delta_{f,h}^{D,(1)}(\Omega_+\setminus\overline{\Omega}_-)$ associated with an eigenvalue $\lambda\in[0,\nu(h)]$ satisfies

$$\|e^{\varphi_\varepsilon/h}\psi\|_{W^{p,2}(\Omega_+\setminus\overline{\Omega}_-)}\le C_{\varepsilon,p}h^{-N_p}.$$

As explained in the proof, we cannot state this result with φ_ε equal to the Agmon distance to a neighborhood of $\partial\Omega_+$ as in Proposition 3.8 because the Agmon distance is not a sufficiently regular function.

Proof. The function $\varphi_\varepsilon \in \mathcal{C}_0^\infty(\Omega_+ \setminus \overline{\Omega}_-)$ is built as an accurate enough mollified version of $\theta_\varepsilon(x) = (1-2\varepsilon)d_{\mathrm{Ag}}(x, \mathcal{V}_+^\varepsilon \cup \mathcal{V}_-^\varepsilon)$, where

$$\mathcal{V}_\pm^\varepsilon = \{x \in \overline{\Omega}_+ \setminus \Omega_- : d(x, \partial\Omega_\pm) \le \varepsilon\}.$$

Indeed, the function θ_ε is a Lipschitz function such that

$$|\nabla\theta_\varepsilon(x)| \le (1-2\varepsilon)|\nabla f(x)| \quad \text{a.e.},$$
$$d(x, \partial\Omega_+ \cup \partial\Omega_-) \le \varepsilon \implies \theta_\varepsilon(x) = 0,$$
$$d(x, \partial\Omega_+ \cup \partial\Omega_-) - C_1\varepsilon \le \theta_\varepsilon(x) \le d(x, \partial\Omega_+ \cup \partial\Omega_-)$$

hold in $\overline{\Omega}_+ \setminus \Omega_-$, with $C_1 \ge 0$ independent of ε. Since θ_ε fulfills uniform Lipschitz estimates and $|\nabla f(x)| \ge c > 0$ on $\overline{\Omega}_+ \setminus \Omega_-$, all the properties of φ_ε are obtained by considering the convolution of θ_ε with a mollifier with a sufficiently small compact support. We cannot simply take $\varphi_\varepsilon = d_{\mathrm{Ag}}(x, \partial\Omega_+ \cup \partial\Omega_-)$, or even $\varphi_\varepsilon = d_{\mathrm{Ag}}(\,\cdot\,, \mathcal{V}_+^\varepsilon \cup \mathcal{V}_-^\varepsilon)$, because the argument requires us to consider high-order derivatives of φ_ε.

Let ψ be a normalized eigenvector of $\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \overline{\Omega}_-)$ associated with an eigenvalue $\lambda \in [0, \nu(h)]$. We already know from Proposition 3.8 that

$$\|e^{\varphi_\varepsilon/h}\psi\|_{W^{1,2}(\Omega_+\setminus\overline{\Omega}_-)} \le C_\varepsilon h^{-1}. \tag{6-8}$$

The argument to obtain the estimates in $W^{p,2}(\Omega_+ \setminus \overline{\Omega}_-)$-norms is based on a bootstrap argument, using the elliptic regularity of nonhomogeneous Dirichlet boundary problems for the Hodge Laplacian.

Indeed, we have

$$e^{-\varphi_\varepsilon/h}\Delta_{f,h}e^{\varphi_\varepsilon/h} = \Delta_{f,h} - h\mathcal{L}_{\nabla\varphi_\varepsilon} + h\mathcal{L}_{\nabla\varphi_\varepsilon}^* - |\nabla\varphi_\varepsilon|^2,$$

and thus

$$\Delta_{f,h}(e^{\varphi_\varepsilon/h}\psi) = \lambda e^{\varphi_\varepsilon/h}\psi - he^{\varphi_\varepsilon/h}\mathcal{L}_{\nabla\varphi_\varepsilon}\psi + he^{\varphi_\varepsilon/h}\mathcal{L}_{\nabla\varphi_\varepsilon}^*\psi - |\nabla\varphi_\varepsilon|^2 e^{\varphi_\varepsilon/h}\psi.$$

Using the fact that $\Delta_{f,h} = h^2(dd^* + d^*d) + h(\mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*) + |\nabla f|^2$, we obtain

$$\Delta_H v = h^{-2}(\lambda v - he^{\varphi_\varepsilon/h}\mathcal{L}_{\nabla\varphi_\varepsilon}e^{-\varphi_\varepsilon/h}v + he^{\varphi_\varepsilon/h}\mathcal{L}_{\nabla\varphi_\varepsilon}^*e^{-\varphi_\varepsilon/h}v - |\nabla\varphi_\varepsilon|^2 v - h\mathcal{L}_{\nabla f}v - h\mathcal{L}_{\nabla f}^*v - |\nabla f|^2 v), \tag{6-9}$$

where

$$v = e^{\varphi_\varepsilon/h}\psi.$$

For the boundary conditions, we have, of course,

$$\boldsymbol{t}v = 0, \tag{6-10}$$

and

$$0 = \boldsymbol{t}d_{f,h}^*\psi = e^{\varphi_\varepsilon/h}\boldsymbol{t}d_{f,h}^*\psi = \boldsymbol{t}d_{f,h}^*e^{\varphi_\varepsilon/h}\psi + e^{\varphi_\varepsilon/h}\boldsymbol{t}\boldsymbol{i}_{\nabla\varphi_\varepsilon}\psi.$$

The condition $\varphi_\varepsilon = 0$ in a neighborhood of $\partial\Omega_+ \cup \partial\Omega_-$ implies $\nabla\varphi_\varepsilon = 0$ on $\partial\Omega_+ \cup \partial\Omega_-$, and thus $\boldsymbol{t}\boldsymbol{i}_{\nabla\varphi_\varepsilon}\psi = 0$. Since $d_{f,h}^* = hd^* + \boldsymbol{i}_{\nabla f}$, we thus obtain

$$\boldsymbol{t}d^*v = -\frac{1}{h}\boldsymbol{i}_{\nabla f}v. \tag{6-11}$$

By considering the boundary value problem (6-9)–(6-11) and using the $W^{1,2}(\Omega_+ \setminus \bar{\Omega}_-)$ estimate (6-8), we thus obtain, by the elliptic regularity of the Dirichlet Hodge Laplacian,

$$\|e^{\varphi_\varepsilon/h}\psi\|_{W^{2,2}(\Omega_+\setminus\bar{\Omega}_-)} \le C_{2,\varepsilon}h^{-3}.$$

This is due to the fact that the right-hand side in (6-9) (resp. (6-11)) is a differential operator of order 1 (resp. 0). The $W^{p,2}(\Omega_+ \setminus \bar{\Omega}_-)$ estimates for $p \ge 3$ are then obtained by induction on p. □

6C. *Change of basis in $F^{(1)}$*. In the previous sections, the estimates (2-12) and (2-14) of the eigenvalue $\lambda_1^{(0)}$ and $d_{f,h}u_1^{(0)}$ in a neighborhood of $\partial\Omega_+$ have been proven with the basis $(\psi_k^{(1)})_{m_1^N(\Omega_-)+1\le k\le m_1^D(\Omega_+)}$ of $\operatorname{Ran} 1_{[0,\nu(h)]}(\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \bar{\Omega}_-))$. The aim of this section is to show that the estimates (2-12) and (2-14) are valid for any almost orthonormal basis (according to Definition 3.6)

$$(u_k^{(1)})_{1\le k\le m_1^D(\Omega_+\setminus\bar{\Omega}_-)} \quad \text{of} \quad \operatorname{Ran} 1_{[0,\nu(h)]}(\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \bar{\Omega}_-)).$$

The next proposition thus concludes the proof of Theorem 2.4.

Remark 6.6. We thus prove a slightly more general result than the one stated in Theorem 2.4, since it is only required that $(u_k^{(1)})_{1\le k\le m_1^D(\Omega_+\setminus\bar{\Omega}_-)}$ is an *almost* orthonormal basis of $\operatorname{Ran} 1_{[0,\nu(h)]}(\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \bar{\Omega}_-))$.

Remark 6.7. All the results below extend to complex-valued eigenbases, by simply replacing the real scalar product by the hermitian scalar product.

Proposition 6.8. *Let $\lambda_1^{(0)}$ be the first eigenvalue of $\Delta_{f,h}^{D,(0)}(\Omega_+)$ and $u_1^{(0)}$ the associated $L^2(\Omega_+)$-normalized nonnegative eigenfunction. For any almost orthonormal basis*

$$(u_k^{(1)})_{1\le k\le m_1^D(\Omega_+\setminus\bar{\Omega}_-)} \quad \textit{of} \quad \operatorname{Ran} 1_{[0,\nu(h)]}(\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \bar{\Omega}_-)),$$

the approximate expressions (2-12) and (2-14) for $\lambda_1^{(0)}$ and $d_{f,h}u_1^{(0)}$ hold true.

Proof. Let $(u_k^{(1)})_{1\le k\le m_1^D(\Omega_+\setminus\bar{\Omega}_-)}$ be an almost orthonormal basis of $\operatorname{Ran} 1_{[0,\nu(h)]}(\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \bar{\Omega}_-))$. Then there exists a matrix $C(h) = (c_{k,k'})_{1\le k,k'\le m_1^D(\Omega_+\setminus\bar{\Omega}_-)}$ such that

$$\begin{aligned} &C(h)C(h)^* = \mathrm{Id}_{m_1^D(\Omega_+\setminus\bar{\Omega}_-)} + \mathcal{O}(e^{-c/h}), \quad C(h)^*C(h) = \mathrm{Id}_{m_1^D(\Omega_+\setminus\bar{\Omega}_-)} + \mathcal{O}(e^{-c/h}), \\ &\text{and} \quad \psi^{(1)}_{k+m_1^N(\Omega_-)} = \sum_{k'=1}^{m_1^D(\Omega_+\setminus\bar{\Omega}_-)} c_{k,k'}u_{k'}^{(1)} \quad \text{for all } k \in \{1, \dots, m_1^D(\Omega_+ \setminus \bar{\Omega}_-)\}. \end{aligned} \tag{6-12}$$

Here, $C(h)^*$ denotes the transpose of the matrix $C(h)$.

Let L_1 (resp. L_2) be a continuous linear mapping from $\operatorname{Ran} 1_{[0,\nu(h)]}(\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \bar{\Omega}_-))$, the finite-dimensional space endowed with the scalar product of $L^2(\Omega_+ \setminus \bar{\Omega}_-)$, to $\mathbb{R}$ (resp. to some vector space E).

Then, using (6-12),

$$\sum_{k=m_1^N(\Omega_-)+1}^{m_1^D(\Omega_+)} L_1(\psi_k^{(1)})L_2(\psi_k^{(1)}) = \sum_{k,k_1,k_2=1}^{m_1^D(\Omega_+\setminus\bar\Omega_-)} c_{k,k_1}c_{k,k_2}L_1(u_{k_1}^{(1)})L_2(u_{k_2}^{(1)})$$
$$= \sum_{k'=1}^{m_1^D(\Omega_+\setminus\bar\Omega_-)} L_1(u_{k'}^{(1)})L_2(u_{k'}^{(1)}) + \mathcal{O}(\|L_1\|\|L_2\|e^{-c/h}), \tag{6-13}$$

where $\|L_1\|$ and $\|L_2\|$ denote the operator norms of the linear mappings L_1 and L_2.

The estimate (2-12) is then a consequence of (6-2) and (6-13) with

$$L_1 = L_2 : \operatorname{Ran} 1_{[0,\nu(h)]}(\Delta_{f,h}^{D,(1)}(\Omega_+\setminus\bar\Omega_-)) \to \mathbb{R}, \quad u \mapsto -\frac{\int_{\partial\Omega_+} e^{-f(\sigma)/h}\boldsymbol{i}_n u(\sigma)\,d\sigma}{\left(\int_{\Omega_+} e^{-2f(x)/h}\,dx\right)^{1/2}}$$

with $\|L_1\| = \|L_2\| = \tilde{\mathcal{O}}(e^{-\kappa_f/h})$ due to $\lambda_1^{(0)}(\Omega_+) = \tilde{\mathcal{O}}(e^{-2\kappa_f/h})$ (see (6-2)) and the orthonormality of the basis $(\psi^{(1)}_{k+m_1^N(\Omega_-)})_{1\le k\le m_1^D(\Omega_+\setminus\Omega_-)}$. The estimate (2-14) is a consequence of Proposition 6.4 and of (6-13) with L_1 like before and

$$L_2 : \operatorname{Ran} 1_{[0,\nu(h)]}(\Delta_{f,h}^{D,(1)}(\Omega_+\setminus\bar\Omega_-)) \to \bigwedge^1 W^{p,2}(\mathcal{V}), \quad u \mapsto u\big|_{\mathcal{V}}$$

with $\|L_2\| = \tilde{\mathcal{O}}(1)$ according to Lemma 6.3 applied with $\Delta_{f,h}^{D,(1)}(\Omega_+\setminus\bar\Omega_-)$ instead of $\Delta_{f,h}^{D,(1)}(\Omega_+)$. ☐

6D. *Corollaries.* The estimate (2-14) contains accurate information about the trace $\partial_n u_1^{(0)}\big|_{\partial\Omega_+}$:

Corollary 6.9. *Let $n : \sigma \mapsto n(\sigma)$ be the outward normal vector field on $\partial\Omega_+$ and let $\partial_n = \boldsymbol{i}_n d$ be the outward normal derivative for functions. For any almost orthonormal basis $(u_k^{(1)})_{1\le k\le m_1^D(\Omega_+\setminus\bar\Omega_-)}$ of* $\operatorname{Ran} 1_{[0,\nu(h)]}(\Delta_{f,h}^{D,(1)}(\Omega_+\setminus\bar\Omega_-))$, *the normal derivative of the nonnegative and normalized first eigenfunction $u_1^{(0)}$ of $\Delta_{f,h}^{D,(0)}(\Omega_+)$ satisfies*

$$\partial_n u_1^{(0)}(\sigma) \le 0 \quad \text{for all } \sigma \in \partial\Omega_+$$

and

$$\left\| \partial_n u_1^{(0)}\big|_{\partial\Omega_+} + \sum_{k=1}^{m_1^D(\Omega_+\setminus\bar\Omega_-)} \frac{\int_{\partial\Omega_+} e^{-f(\sigma)/h}\boldsymbol{i}_n u_k^{(1)}(\sigma)\,d\sigma}{\left(\int_{\Omega_+} e^{-2f(x)/h}\,dx\right)^{1/2}} \boldsymbol{i}_n u_k^{(1)} \right\|_{W^{p,2}(\partial\Omega_+)} = \mathcal{O}(e^{-(\kappa_f+c)/h}) \quad \text{for all } p\in\mathbb{N}$$

for some $c > 0$ independent of p.

Proof. The sign condition for $\partial_n u_1^{(0)}(\sigma)$ is a consequence of $u_1^{(0)} \ge 0$ in Ω_+ and $u_1^{(0)}\big|_{\partial\Omega_+} = 0$.

The trace theorem with (2-14) implies

$$d_{f,h}u_1^{(0)}\big|_{\partial\Omega_+} = -h \sum_{k=1}^{m_1^D(\Omega_+\setminus\bar\Omega_-)} \frac{\int_{\partial\Omega_+} e^{-f(\sigma)/h}\boldsymbol{i}_n u_k^{(1)}(\sigma)\,d\sigma}{\left(\int_{\Omega_+} e^{-2f(x)/h}\,dx\right)^{1/2}} u_k^{(1)} + \mathcal{O}(e^{-(\kappa_f+c)/h})$$

in any Sobolev space $W^{p,2}(\partial\Omega_+)$. Recalling

$$d_{f,h}u_1^{(0)} = hdu_1^{(0)} + u_1^{(0)}df \quad \text{and} \quad u_1^{(0)}\big|_{\partial\Omega_+} = 0$$

yields the result. □

Proof of Corollary 2.9. First, note that the equality

$$\partial_n[e^{-f_{1,2}/h}u_1^{(0)}(f_{1,2})]\big|_{\partial\Omega_+} = e^{-f_{1,2}/h}[\partial_n u_1^{(0)}(f_{1,2})]\big|_{\partial\Omega_+}$$

is simply due to the Dirichlet boundary condition $u_1^{(0)}\big|_{\partial\Omega_+} = 0$. The identity (2-15) is then a direct consequence of (2-12), since the same basis $(u_k^{(1)})_{1\le k\le m_1^D(\Omega_+\setminus\overline{\Omega}_-)}$ can be picked for f_1 and f_2 because these two functions coincide on $\Omega_+\setminus\overline{\Omega}_-$.

Second, for (2-16), it is more convenient to write (2-14) with f_j, $j=1,2$, in the form

$$\left(\int_{\Omega_+} e^{-2f_j(x)/h}\,dx\right)^{\frac{1}{2}} d_{f_j,h}u_1^{(0)}(f_j) = -h\sum_{k=1}^{m_1^D(\Omega_+\setminus\overline{\Omega}_-)}\left(\int_{\partial\Omega_+} e^{-f_j(\sigma)/h}\mathbf{i}_n u_k^{(1)}(\sigma)\,d\sigma\right)u_k^{(1)} + \mathcal{O}(e^{-(\min_{\partial\Omega_+} f_j+c)/h}),$$

the estimate being true in any Sobolev space $\bigwedge^1 W^{p,2}(\mathcal{V})$. Using the fact that $f_1\equiv f_2\equiv f$ in $\Omega_+\setminus\overline{\Omega}_-$, taking the trace along $\partial\Omega_+$ and multiplying by $e^{-(f-\min_{\partial\Omega_+} f)/h}$, which is less than 1 on $\partial\Omega_+$, and then by $e^{(\min_{\partial\Omega_+} f)/h}$, lead to

$$\left(\int_{\Omega_+} e^{-2f_j(x)/h}\,dx\right)^{\frac{1}{2}} e^{-(f-2\min_{\partial\Omega_+} f)/h}\partial_n u_1^{(0)}(f_j)\big|_{\partial\Omega_+} = -\sum_{k=1}^{m_1^D(\Omega_+\setminus\overline{\Omega}_-)}\left(\int_{\partial\Omega_+} e^{-(f(\sigma)-\min_{\partial\Omega_+}f)/h}\mathbf{i}_n u_k^{(1)}(\sigma)\,d\sigma\right)e^{-(f-\min_{\partial\Omega_+} f)/h}\mathbf{i}_n u_k^{(1)} + \mathcal{O}(e^{-c/h}),$$

the estimate being true in $L^1(\partial\Omega_+)$. The left-hand side is negative and its L^1-norm is thus given by the absolute value of its integral. Let us estimate this norm, using Lemma 4.3 and Lemma 5.9: for any positive ε,

$$\begin{aligned}
-\left(\int_{\Omega_+} e^{-2f_j(x)/h}\,dx\right)^{\frac{1}{2}}&\int_{\partial\Omega_+} e^{-(f-2\min_{\partial\Omega_+} f)/h}\partial_n u_1^{(0)}(f_j)(\sigma)\,d\sigma\\
&= \sum_{k=1}^{m_1^D(\Omega_+\setminus\overline{\Omega}_-)}\left(\int_{\partial\Omega_+} e^{-(f(\sigma)-\min_{\partial\Omega_+}f)/h}\mathbf{i}_n u_k^{(1)}(\sigma)\,d\sigma\right)^2 + \mathcal{O}(e^{-c/h})\\
&= e^{(2\min_{\partial\Omega_+}f)/h}\lambda_1^{(0)}(f_1)h^{-2}\int_{\Omega_+} e^{-2f_1(x)/h}\,dx + \mathcal{O}(e^{-c/h})\\
&\ge C_\varepsilon e^{(2\min_{\partial\Omega_+}f)/h}e^{-2(\kappa_f+\varepsilon)/h}h^{-2}\frac{1}{C_{f_1}}h^{d/2}e^{-(2\min_{\Omega_+} f_1)/h} + \mathcal{O}(e^{-c/h})\\
&= C_\varepsilon e^{-2\varepsilon/h}\frac{h^{-2+d/2}}{C_{f_1}} + \mathcal{O}(e^{-c/h}) \ge Ce^{-c/(2h)}.
\end{aligned}$$

Thus,

$$-\frac{e^{-f_j/h}\partial_n u_1^{(0)}(f_j)\big|_{\partial\Omega_+}}{\big\|e^{-f_j/h}\partial_n u_1^{(0)}(f_j)\big|_{\partial\Omega_+}\big\|_{L^1(\partial\Omega_+)}} = \frac{\sum_{k=1}^{m_1^D(\Omega_+\setminus\bar\Omega_-)}\big(\int_{\partial\Omega_+} e^{-(f(\sigma)-\min_{\partial\Omega_+}f)/h}\boldsymbol{i}_n u_k^{(1)}(\sigma)\,d\sigma\big)e^{-(f-\min_{\partial\Omega_+}f)/h}\boldsymbol{i}_n u_k^{(1)}}{\sum_{k=1}^{m_1^D(\Omega_+\setminus\bar\Omega_-)}\big(\int_{\partial\Omega_+} e^{-(f(\sigma)-\min_{\partial\Omega_+}f)/h}\boldsymbol{i}_n u_k^{(1)}(\sigma)\,d\sigma\big)^2} + \mathcal{O}(e^{-c/(2h)}).$$

This concludes the proof, since the right-hand side does not depend on f_j. □

7. About Hypotheses 3 and 4

We have chosen to set the Hypotheses 3 and 4 in terms of some spectral properties of the Witten Laplacians $\Delta^N_{f,h}(\Omega_-)$ and $\Delta^D_{f,h}(\Omega_-\setminus\bar\Omega_-)$ in order to be general enough and to cover possible further advances about the low spectrum of Witten Laplacians. These hypotheses can actually be translated into very explicit and simple geometric conditions on the function f when f is a Morse function such that $f\big|_{\partial\Omega_+}$ is a Morse function. We recall that a Morse function is a $\mathcal{C}^\infty$ function whose critical points are all nondegenerate. Section 7A is devoted to a verification of Hypotheses 3 and 4 when f and $f\big|_{\partial\Omega_+}$ are Morse functions, using the results of [Helffer and Nier 2006; Le Peutrec 2010b]. Theorem 2.10 is then obtained as a consequence of the accurate results under the Morse conditions and the estimates stated in Corollary 2.9.

Finally, Section 7B is devoted to a discussion about potentials that are not Morse functions. In particular, examples of functions f which are not Morse functions and for which Hypotheses 3 and 4 hold are presented.

7A. *The case of a Morse function f.*

7A1. *Verifying Hypotheses 3 and 4.* Let us first specify the assumptions which allow us to use the results of [Helffer and Nier 2006; Le Peutrec 2010b], in addition to Hypotheses 1 and 2, which were already explicitly formulated in terms of the function f:

Hypothesis 5. *The functions f and $f\big|_{\partial\Omega_+}$ are Morse functions.*

Hypothesis 6. *The critical values of f are all distinct and the differences $f(U^{(1)})-f(U^{(0)})$, where $U^{(0)}$ ranges over the local minima of f and $U^{(1)}$ ranges over the critical points of f with index* 1, *are all distinct.*

Although $f\big|_{\partial\Omega_-}$ is not assumed to be a Morse function (see the discussion below), Hypotheses 1, 5 and 6 ensure that the results of [Helffer and Nier 2006; Le Peutrec 2010b] on small eigenvalues of $\Delta^D_{f,h}(\Omega_+)$, $\Delta^N_{f,h}(\Omega_-)$ and $\Delta^D_{f,h}(\Omega_+\setminus\bar\Omega_-)$ apply. Following [Le Peutrec 2010b], Hypothesis 6 is useful to get accurate scaling rates for the small eigenvalues of $\Delta^{N,(0)}_{f,h}(\Omega_-)$. In particular, the information on the size of the second eigenvalue $\mu_2^{(0)}(\Omega_-)>\mu_1^{(0)}(\Omega_-)=0$ of $\Delta^{N,(0)}_{f,h}(\Omega_-)$ is important to prove (2-8) in Hypothesis 3. Hypothesis 6 also implies that f has a unique global minimum. Hypothesis 6 could certainly be relaxed.

Let us recall the general results of [Helffer and Nier 2006; Le Peutrec 2010b] on the number and the scaling of small eigenvalues for boundary Witten Laplacians in a regular domain Ω (see also [Chang and Liu 1995; Laudenbach 2011] for related results). The potential f is assumed to be a Morse function f on Ω such that $|\nabla f| \neq 0$ on $\partial\Omega$ and $f\big|_{\partial\Omega}$ is also a Morse function. The notion of critical points with index p for f has to be extended as follows, in order to take into account points on the boundary $\partial\Omega$.

• *In the interior* Ω*:* A generalized critical point with index p is, as usual, a critical point at which the Hessian of f has p negative eigenvalues. It is a local minimum for $p=0$, a saddle point for $p=1$ and a local maximum for $p=\dim M=d$.

• *Along the boundary* $\partial\Omega$ *in the Dirichlet case:* A generalized critical point with index $p\geq 1$ is a critical point σ of $f\big|_{\partial\Omega}$ with index $p-1$ such that the outward normal derivative is positive ($\partial_n f(\sigma)>0$). Therefore, along the boundary, there is no generalized critical point with index 0, and critical points with index 1 coincide with the local minima σ of $f\big|_{\partial\Omega}$ such that $\partial_n f(\sigma)>0$. Intuitively, this definition can be understood by interpreting the homogeneous Dirichlet boundary conditions as an extension of the potential by $-\infty$ outside Ω.

• *Along the boundary* $\partial\Omega$ *in the Neumann case:* A generalized critical point with index p is a critical point σ of $f\big|_{\partial\Omega}$ with index p such that the outward normal derivative is negative ($\partial_n f(\sigma)<0$). Therefore, along the boundary, a generalized critical point with index 0 is a local minimum of $f\big|_{\overline{\Omega}}$ and a critical point with index 1 is a saddle point σ of $f\big|_{\partial\Omega}$ such that $\partial_n f(\sigma)<0$. Intuitively, this definition can be understood by interpreting the homogeneous Neumann boundary conditions as an extension of the potential by $+\infty$ outside Ω.

The number of generalized critical points in Ω with index p is denoted by $\tilde{m}_p^D(\Omega)$ or $\tilde{m}_p^N(\Omega)$, depending on whether the boundary Witten Laplacian on Ω with Dirichlet or Neumann boundary conditions is considered.

One result of [Helffer and Nier 2006; Le Peutrec 2010b] says that, for $\nu(h)=h^{6/5}$, one has, for the Dirichlet Witten Laplacian,

$$\#\big[\sigma(\Delta_{f,h}^{D,(p)}(\Omega))\cap[0,\nu(h)]\big]=\tilde{m}_p^D(\Omega),\qquad \sigma(\Delta_{f,h}^{D,(p)}(\Omega))\cap[0,\nu(h)]\subset[0,e^{-c_0/h}],$$

and, for the Neumann boundary Witten Laplacian,

$$\#\big[\sigma(\Delta_{f,h}^{N,(p)}(\Omega))\cap[0,\nu(h)]\big]=\tilde{m}_p^N(\Omega),\qquad \sigma(\Delta_{f,h}^{N,(p)}(\Omega))\cap[0,\nu(h)]\subset[0,e^{-c_0/h}]$$

for some positive constant c_0. These results rely, like in [Cycon et al. 1987] for the boundaryless case, on the introduction of an h-dependent partition of unity and a rough analysis of boundary local models.

Let us now apply these general results in our context. Under Hypotheses 1 and 5, we have:

- $\tilde{m}_p^N(\Omega_-)$ is the number of critical points with index p in the interior of Ω_-.
- $\tilde{m}_p^D(\Omega_+\setminus\overline{\Omega}_-)$ is the number of critical points σ with index $p-1$ of $f\big|_{\partial\Omega_+}$ such that $\partial_n f(\sigma)>0$. In particular, $\tilde{m}_0^D(\Omega_+\setminus\overline{\Omega}_-)=0$, and $\tilde{m}_1^D(\Omega_+\setminus\overline{\Omega}_-)$ is the number of local minima of $f\big|_{\partial\Omega_+}$ with positive normal derivatives.

- $\tilde{m}_p^D(\Omega_+)$ is the number of critical points with index p in the interior of Ω_- plus the number of critical points σ of $f\big|_{\partial\Omega_+}$ with index $p-1$ such that $\partial_n f(\sigma) > 0$. For $p = 0$, $\tilde{m}_0^D(\Omega_+)$ equals $\tilde{m}_0^N(\Omega_-)$ while $\tilde{m}_1^D(\Omega_+)$ is $m_1^N(\Omega_-)$ augmented by the number of local minima of $f\big|_{\partial\Omega_+}$ with positive normal derivatives.

As already mentioned above, we can use the results of [Helffer and Nier 2006; Le Peutrec 2010b] without assuming that $f\big|_{\partial\Omega_-}$ is a Morse function. The reason is that $\partial_n f > 0$ on $\partial\Omega_-$ and, thus, there is no generalized critical point on $\partial\Omega_-$ associated with $\Delta_{f,h}^{N,(p)}(\Omega_-)$ and $\Delta_{f,h}^{D,(p)}(\Omega_+ \setminus \overline{\Omega}_-)$.

In summary, using these results, conditions (2-6), (2-7), (2-9) and (2-10) are fulfilled with $\nu(h) = h^{6/5}$, some $c_0 > 0$ and $m_p^{N,D}(\Omega) = \tilde{m}_p^{N,D}(\Omega)$, $p \in \{0, 1\}$ and $\Omega = \Omega_-$ or $\Omega = \Omega_+ \setminus \overline{\Omega}_-$. Hence, all the conditions of Hypotheses 3 and 4 are satisfied except (2-8). Note in particular that two of the results in Theorem 2.4,

$$m_0^D(\Omega_+) = m_0^N(\Omega_-) \quad \text{and} \quad m_1^D(\Omega_+) = m_1^N(\Omega_-) + m_1^D(\Omega_+ \setminus \overline{\Omega}_-),$$

are consistent with the relations on the numbers of generalized critical points:

$$\tilde{m}_0^D(\Omega_+) = \tilde{m}_0^N(\Omega_-) \quad \text{and} \quad \tilde{m}_1^D(\Omega_+) = \tilde{m}_1^N(\Omega_-) + \tilde{m}_1^D(\Omega_+ \setminus \overline{\Omega}_-).$$

As explained in the proof below, Hypothesis 6 is particularly useful to verify condition (2-8) in Hypothesis 3.

The following proposition thus yields a simple set of assumptions on f such that Theorem 2.4 holds:

Proposition 7.1. *Assume Hypotheses 1, 5 and 6 and let $\mathcal{U}^{(0)}$ (resp. $\mathcal{U}^{(1)}$) denote the set of critical points with index 0 (resp. 1) of $f\big|_{\Omega_-}$. Let us consider the Agmon distance d_{Ag} introduced in Lemma 3.2. Then the inequality*

$$d_{Ag}(\partial\Omega_-, \mathcal{U}^{(0)}) > \max_{U^{(1)} \in \mathcal{U}^{(1)},\, U^{(0)} \in \mathcal{U}^{(0)}} f(U^{(1)}) - f(U^{(0)}) \tag{7-1}$$

implies (2-8). As a consequence, the inequality (7-1) together with Hypotheses 1, 2, 5 and 6 are sufficient conditions for the results of Theorem 2.4 and its corollaries to hold.

Figures 2 and 3 give examples of functions f for which the inequality (7-1) together with Hypotheses 1, 2, 5 and 6 are fulfilled. Figure 4 is an example of a function f which satisfies Hypotheses 1, 2, 5 and 6, but not the inequality (7-1).

Remark 7.2. Since $d_{\mathrm{Ag}}(x, y) \geq |f(x) - f(y)|$ (see (3-1)), the condition (1-9) given in the introduction is a sufficient condition for (7-1). Condition (1-9) also implies Hypothesis 2. Thus, a set of sufficient conditions for Theorem 2.4 to hold is Hypotheses 1, 5 and 6 together with (1-9). This is indeed the simple setting presented in the introduction (see the four assumptions stated in Section 1B).

Remark 7.3. It may happen that $\mathcal{U}^{(1)} = \varnothing$. In this case, the inequality (7-1) is automatically satisfied, and there are no exponentially small nonzero eigenvalue for $\Delta_{f,h}^{N,(0)}(\Omega_-)$. Consistently, (2-8) is a void condition in this case.

Proof of Proposition 7.1. By the previous discussion, it only remains to prove that Hypotheses 1, 5 and 6 together with (7-1) imply (2-8) for the proposition to hold. According to [Le Peutrec 2010b], the smallest

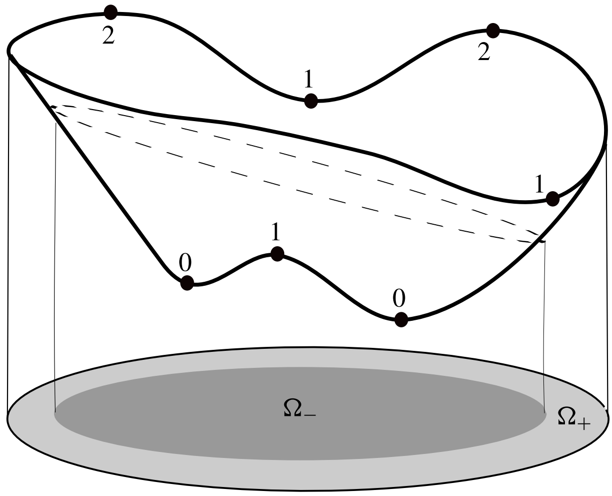

Figure 2. A two-dimensional example where the inequality (7-1) together with Hypotheses 1, 2, 5 and 6 are fulfilled. The generalized critical points are labeled by their indices.

nonzero eigenvalue of $\Delta^{N,(0)}_{f,h}(\Omega_-)$ (namely $\mu^{(0)}_2(\Omega_-)$), satisfies, under Hypotheses 5 and 6, the inequality

$$\lim_{h\to 0} h\log(\mu^{(0)}_2(\Omega_-)) = -2(f(U^{(1)}_{j_1}) - f(U^{(0)}_{j_0})) \geq -2 \max_{U^{(1)}\in\mathcal{U}^{(1)},\, U^{(0)}\in\mathcal{U}^{(0)}} f(U^{(1)}) - f(U^{(0)}),$$

where $U^{(0)}_{j_0}$ and $U^{(1)}_{j_1}$ are two critical points of index 0 and 1, respectively.

Let us now consider the exponential decay near $\partial\Omega_-$ of an eigenfunction of $\Delta^{N,(0)}_{f,h}(\Omega_-)$ associated with a nonzero, exponentially small eigenvalue. A stronger version of Proposition 3.4 can be given because under Hypotheses 1, 5 and 6 the critical points of $f\big|_{\Omega_-}$ which are not local minima are not associated with small eigenvalues of $\Delta^{N,(0)}_{f,h}(\Omega_-)$ (they are so-called *nonresonant* wells; see [Helffer and Sjöstrand 1985a]). Indeed, when U is a critical point of $f\big|_{\Omega_-}$ with $U \notin \mathcal{U}^{(0)}$, the local model of $\Delta^{D,(0)}_{f,h}(B(U,r))$ has his spectrum included in $[h/C(U,r), +\infty)$ for $r > 0$ small enough (see, for example, [Cycon et al. 1987]). Then, Corollary 2.2.7 of [Helffer and Sjöstrand 1985a] implies that any normalized eigenfunction $\psi(h)$ of $\Delta^{N,(0)}_{f,h}(\Omega_-)$ associated with an eigenvalue $\mu(h) \in [0, e^{-c_0/h}]$ satisfies

$$\forall\varepsilon > 0\ \exists C_\varepsilon > 0\ \forall x \in \Omega_- \quad |\psi_h(x)| \leq C_\varepsilon(e^{-(d_{\mathrm{Ag}}(x,\mathcal{U}_0)+\varepsilon)/h})$$

(compare with the result of Proposition 3.4). Hence, condition (7-1) implies that, in a small neighborhood $\mathcal{V}_-$ of $\partial\Omega_-$, the eigenfunction $\psi(h)$ is estimated by

$$\begin{aligned}\|\psi(h)\|_{L^2(\mathcal{V}_-)} &= \tilde{\mathcal{O}}(e^{-d_{\mathrm{Ag}}(\mathcal{V}_-,\mathcal{U}^{(0)})/h}) \leq C\exp\biggl(-\frac{\max_{U^{(1)}\in\mathcal{U}^{(1)},\,U^{(0)}\in\mathcal{U}^{(0)}} f(U^{(1)}) - f(U^{(0)}) + c}{h}\biggr)\\ &\leq \tilde{\mathcal{O}}\Bigl(\sqrt{\mu^{(0)}_2(\Omega_-)}\Bigr) \leq \tilde{\mathcal{O}}(\sqrt{\mu(h)})\end{aligned}$$

provided that $\mu(h) \neq \mu^{(0)}_1(\Omega_-) = 0$. This is exactly (2-8). □

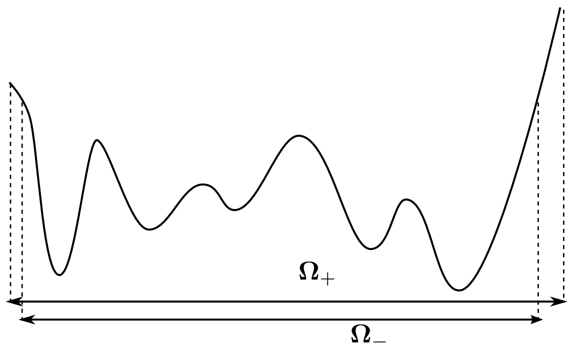

Figure 3. A one-dimensional example where the inequality (7-1) together with Hypotheses 1, 2, 5 and 6 are fulfilled.

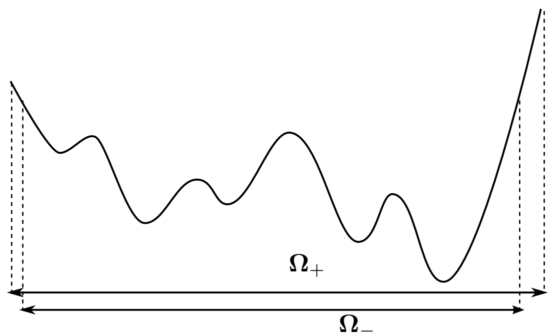

Figure 4. A one-dimensional example where Hypotheses 1, 2, 5 and 6 are fulfilled, but the inequality (7-1) is not satisfied. The condition (7-1) would be fulfilled with a lower local minimum on the left-hand side, for example (see Figure 3).

Remark 7.4 (assumptions in terms of Ω_+ only). Let us assume that Hypotheses 2, 5 and 6 hold. Then, it is easy to check that, if

$$\partial_n f\big|_{\partial\Omega_+} > 0 \tag{7-2}$$

and

$$d_{\mathrm{Ag}}(\partial\Omega_+, \mathcal{U}^{(0)}) > \max_{U^{(1)}\in\mathcal{U}^{(1)},\, U^{(0)}\in\mathcal{U}^{(0)}} f(U^{(1)}) - f(U^{(0)}), \tag{7-3}$$

then there exists a regular open domain Ω_- such that $\overline{\Omega}_- \subset \Omega_+$ and Hypothesis 1 and condition (7-1) hold. Indeed, conditions (7-2) and (7-3) are open and allow small deformation from Ω_+ to some subset Ω_-. Note that condition (7-2) implies that this small deformation can be chosen so that all the critical points of f are indeed in Ω_-; this is exactly Hypothesis 1. As a consequence, under Hypotheses 2, 5 and 6 and assumptions (7-2) and (7-3), the results of Theorem 2.4 hold for a well-chosen domain Ω_- such that $\overline{\Omega}_- \subset \Omega_+$.

In addition, following Remark 7.2 above, it is easy to check that the inequality

$$\min_{\partial\Omega_+} f - \text{cvmax} > \text{cvmax} - \min_{\Omega_+} f \tag{7-4}$$

is a sufficient condition for (7-3). It also implies Hypothesis 2. Thus, under Hypotheses 5 and 6 and the two assumptions (7-2) and (7-4), the results of Theorem 2.4 hold for a well-chosen domain Ω_- such that $\overline{\Omega}_- \subset \Omega_+$.

7A2. *Proof of Theorem 2.10.* In this section, more explicit formulas for $\lambda_1^{(0)}(\Omega_+)$ and $\partial_n(e^{-f/h}u_1^{(0)})$ are given under the Morse assumption on f and $f\big|_{\partial\Omega_+}$. We shall prove:

Proposition 7.5. *Assume Hypotheses 1, 2, 5, 6, the condition (7-1) and, moreover,*

$$\partial_n f > 0 \quad \text{on } \partial\Omega_+ . \tag{7-5}$$

Then the first eigenvalue $\lambda_1^{(0)}(\Omega_+)$ of $\Delta_{f,h}^{D,(0)}(\Omega_+)$ satisfies

$$\lambda_1^{(0)}(\Omega_+) = \sum_{k=1}^{m_1^D(\Omega_+\setminus\overline{\Omega}_-)} \left(\frac{h\det(\operatorname{Hess} f)(U_0)}{\pi\det\big(\operatorname{Hess} f\big|_{\partial\Omega_+}\big)(U_k^{(1)})}\right)^{\frac{1}{2}} 2\partial_n f(U_k^{(1)}) e^{-2(f(U_k^{(1)})-f(U_0))/h}(1+\mathcal{O}(h)) \tag{7-6}$$

$$= \frac{\int_{\partial\Omega_+} 2\partial_n f(\sigma) e^{-2f(\sigma)/h}\, d\sigma}{\int_{\Omega_+} e^{-2f(x)/h}\, dx}(1+\mathcal{O}(h)), \tag{7-7}$$

where U_0 is the (unique) global minimum of f in Ω_+ and the $U_k^{(1)}$ are the local minima of $f\big|_{\partial\Omega_+}$. Moreover, the normalized nonnegative eigenfunction $u_1^{(0)}$ of $\Delta_{f,h}^{D,(0)}(\Omega_+)$ associated with $\lambda_1^{(0)}(\Omega_+)$ satisfies

$$-\frac{\partial_n[e^{-f/h}u_1^{(0)}]\big|_{\partial\Omega_+}}{\|\partial_n[e^{-f/h}u_1^{(0)}]\|_{L^1(\partial\Omega_+)}} = \frac{(2\partial_n f)e^{-2f/h}\big|_{\partial\Omega_+}}{\|(2\partial_n f)e^{-2f/h}\|_{L^1(\partial\Omega_+)}} + \mathcal{O}(h) \quad \text{in } L^1(\partial\Omega_+). \tag{7-8}$$

Remark 7.6. The hypothesis $\partial_n f > 0$ on $\partial\Omega_+$ ensures that the set of all the local minima $U_k^{(1)}$ of $f\big|_{\partial\Omega_+}$ coincides with the set of generalized critical points with index 1 for $\Delta_{f,h}^{D}(\Omega_+\setminus\overline{\Omega}_-)$. The results of Proposition 7.5 also hold under the more general assumption that $\partial_n f(\sigma) > 0$ when $\sigma\in\partial\Omega_+$ is such that $f(\sigma) \le \min_{\partial\Omega_+} f + \varepsilon_0$ for some $\varepsilon_0 > 0$, by adapting the arguments below.

Remark 7.7. It is possible to write explicitly a first-order approximation for the probability density $-\partial_n(e^{-f/h}u_1^{(0)})\big|_{\partial\Omega_+}/\|\partial_n(e^{-f/h}u_1^{(0)})\|_{L^1(\partial\Omega_+)}$, in the spirit of the approximation (7-6) for $\lambda_1^{(0)}(\Omega_+)$. This approximation uses second-order Taylor expansions of f around the local minima $U_k^{(1)}$; see (7-20) below. More precisely, this approximation becomes

$$-\frac{\partial_n[e^{-f/h}u_1^{(0)}]\big|_{\partial\Omega_+}}{\|\partial_n[e^{-f/h}u_1^{(0)}]\|_{L^1(\partial\Omega_+)}} = \frac{\sum_{k=1}^{m_1^D(\Omega_+\setminus\overline{\Omega}_-)} t_k(h)G_k(h)}{\sum_{k=1}^{m_1^D(\Omega_+\setminus\overline{\Omega}_-)} t_k(h)} + \mathcal{O}(h), \tag{7-9}$$

where the $G_k(h)$ are Gaussian densities centered at the $U_k^{(1)}$ and the weights $t_k(h)$ are such that $\lim_{h\to 0} h\log t_k(h) = -f(U_k^{(1)})$. When $f\big|_{\partial\Omega_+}$ has a unique global minimum, the sums in (7-6) and (7-9) reduce to a single term.

Remark 7.8. As explained in Remark 7.4 above, it is again possible to write a set of assumptions in terms of Ω_+ only. In particular, the results of Proposition 7.5 hold under Hypotheses 2, 5 and 6 and assumptions (7-2) and (7-3).

Remark 7.9. It is possible to extend our analysis to the case of an h-dependent function $f = f_h$ such that our assumptions are verified with uniform constants. For example, the results hold if the values $f(U_k^{(1)})$ of f at the local minima $U_k^{(1)}$ are moved in an $\mathcal{O}(h)$ range without changing $f - f(U_k^{(1)})$ locally. This would change the coefficients $t_k(h)$ in (7-9) accordingly by $\mathcal{O}(1)$ factors.

Most of our effort will be devoted to the proof of Proposition 7.5. Let us first conclude the proof of Theorem 2.10 using the result of Proposition 7.5.

Proof of Theorem 2.10. Let f be a function such that Hypotheses 1, 2, 3 and 4 are satisfied. Let us assume moreover that $f\big|_{\partial\Omega_+}$ is a Morse function and $\partial_n f > 0$ on $\partial\Omega_+$. It is possible to build a $\mathcal{C}^\infty$ function $\tilde{f}$ such that $\tilde{f} = f$ on $\Omega_+ \setminus \Omega_-$ and Hypotheses 1, 2, 5 and 6 and condition (7-1) are satisfied by $\tilde{f}$. This relies in particular on the fact that Morse functions are dense in $\mathcal{C}^\infty$ functions. The condition (7-1) may require us to slightly change the local minimal values of the Morse function $\tilde{f}$.

The function $\tilde{f}$ now fulfills all the requirements of Proposition 7.5 and thus, with obvious notation,

$$\tilde{\lambda}_1^{(0)}(\Omega_+) = \frac{\int_{\partial\Omega_+} 2\partial_n f(\sigma) e^{-2f(\sigma)/h}\, d\sigma}{\int_{\Omega_+} e^{-2\tilde{f}(x)/h}\, dx}(1 + \mathcal{O}(h))$$

and

$$-\frac{\partial_n[e^{-\tilde{f}/h}\tilde{u}_1^{(0)}]\big|_{\partial\Omega_+}}{\|\partial_n[e^{-\tilde{f}/h}\tilde{u}_1^{(0)}]\|_{L^1(\partial\Omega_+)}} = \frac{(2\partial_n f)e^{-2f/h}\big|_{\partial\Omega_+}}{\|(2\partial_n f)e^{-2f/h}\|_{L^1(\partial\Omega_+)}} + \mathcal{O}(h) \quad \text{in } L^1(\partial\Omega_+).$$

Here, we have used the fact that $\tilde{f} = f$ on $\Omega_+ \setminus \Omega_-$. Notice that the function $\tilde{f}$ satisfies Hypotheses 1, 2, 3 and 4 by the results of the previous section. We thus conclude the proof by referring to Corollary 2.9. ☐

The proof of Proposition 7.5 is done in two steps: We first apply Theorem 2.4 using a very specific basis of $\operatorname{Ran} 1_{[0,\nu(h)]}(\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \overline{\Omega}_-))$ to get estimates of

$$\lambda_1^{(0)}(\Omega_+) \quad \text{and} \quad -\frac{\partial_n(e^{-f/h}u_1^{(0)})\big|_{\partial\Omega_+}}{\|\partial_n(e^{-f/h}u_1^{(0)})\|_{L^1(\partial\Omega_+)}}$$

in terms of second-order Taylor expansions of f around the local minima $U_k^{(1)}$ (see (7-6) and (7-20)). We then show that these expansions coincide with (7-7) and (7-8).

Before this, we explain how to build the almost orthonormal basis of $\operatorname{Ran} 1_{[0,\nu(h)]}(\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \overline{\Omega}_-))$ that is needed to prove our results. This construction relies heavily on the Morse assumption on f and $f\big|_{\partial\Omega_+}$ (see Hypothesis 5). We need the results of [Helffer and Nier 2006, Chapter 4] on approximate formulas for a basis of the eigenspace of $\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \overline{\Omega}_-)$ associated with $\mathcal{O}(e^{-c_0/h})$ eigenvalues (see also [Le Peutrec 2010a] for a more general analysis). In what follows, it is assumed that Hypotheses 1, 5 and 6 and condition (7-5) hold. The one-forms of that basis are constructed via a WKB expansion around each local minimum $U_k^{(1)}$ of $f\big|_{\partial\Omega_+}$ $(1 \le k \le m_1^D(\Omega_+ \setminus \overline{\Omega}_-))$. In a neighborhood $\mathcal{V}_k$ of $U_k^{(1)}$, consider the function φ_k defined in a neighborhood of $U_k^{(1)}$ as follows: We assume that all the $\mathcal{V}_k$ are disjoint subsets of $\bar{\Omega}_+ \setminus \bar{\Omega}_-$. The function φ_k satisfies the eikonal equation

$$|\nabla\varphi_k|^2 = |\nabla f|^2, \quad \varphi_k\big|_{\partial\Omega_+} = (f - f(U_k^{(1)}))\big|_{\partial\Omega_+}, \quad \partial_n\varphi_k\big|_{\partial\Omega_+} = -\partial_n f\big|_{\partial\Omega_+}.$$

In the neighborhood $\mathcal{V}_k$, one can build coordinates $(x', x^d) = (x^1, \dots, x^{d-1}, x^d)$ such that:

- The open set Ω_+ looks like a half-space:

$$\Omega_+ \cap \mathcal{V}_k = \{(x', x^d) : |x'| \le r,\ x^d < 0\},$$
$$\partial\Omega_+ \cap \mathcal{V}_k = \{(x', x^d) : |x'| \le r,\ x^d = 0\}.$$

- The metric has the form $g_{d,d}(x)(dx^d)^2 + \sum_{i,j=1}^{d-1} g_{i,j}(x)\, dx^i\, dx^j$ with $g_{i,j}(0) = \delta_{i,j}$ (notice that a different normalization of $g_{d,d}(0)$ was used in [Helffer and Nier 2006]).
- The coordinates (x', x^d) are Morse coordinates both for f and φ_k:

$$f(x) - f(U_k^{(1)}) = \partial_n f(U_k^{(1)})x^d + \frac{1}{2}\sum_{j=1}^{d-1}\lambda_j (x^j)^2, \quad \varphi_k(x) = -\partial_n f(U_k^{(1)})x^d + \frac{1}{2}\sum_{j=1}^{d-1}\lambda_j (x^j)^2, \tag{7-10}$$

  where the λ_j are the eigenvalues of $\mathrm{Hess}\big(f\big|_{\partial\Omega_+}\big)(U_k^{(1)})$.

In [Helffer and Nier 2006] a local self-adjoint realization of $\Delta_{f,h}^{(1)}$ around $U_k^{(1)}$ is introduced with the same boundary conditions along $\partial\Omega_+$ as for $\Delta_{f,h}^{D,(1)}(\Omega_+)$, with a unique exponentially small eigenvalue $\zeta_k(h) = \mathcal{O}(e^{-c_k/h})$. A corresponding approximate eigenvector is given by the WKB expansion (in the limit of small h)

$$z_k^{\mathrm{wkb},(1)}(x,h) = a_k(x,h)e^{-\varphi_k(x)/h}, \quad \text{where } a_k(x,h) \sim a_{k,0}(x)\, dx^d + \sum_{\ell=1}^{\infty} b_{k,\ell} h^\ell \tag{7-11}$$

with $b_{k,\ell} = \sum_{j=1}^{d} a_{k,\ell,j}(x)\, dx^j$ and $a_{k,0}(0) = 1$. The symbol $\sim$ stands for the equality of asymptotic expansions. Let $z_k^{(1)}$ be the eigenvector of the self-adjoint realization of $\Delta_{f,h}^{(1)}$ around $U_k^{(1)}$ introduced above, associated with $\zeta_k(h)$ and normalized by $\boldsymbol{i}_{\partial_{x^d}} z_k^{(1)}(0) = \boldsymbol{i}_{\partial_{x^d}} z_k^{\mathrm{wkb},(1)}(0)$. It is shown in [Helffer and Nier 2006, Proposition 4.3.2(b,d)] that the estimates

$$\forall \alpha \in \mathbb{N}^d \ \ \exists C_\alpha > 0 \ \ \exists N_\alpha \in \mathbb{N} \quad |\partial_x^\alpha z_k^{(1)}(x)| \le C_\alpha h^{-N_\alpha} e^{-\varphi_k(x)/h}, \tag{7-12}$$
$$\forall N \in \mathbb{N} \ \ \forall \alpha \in \mathbb{N}^d \ \ \exists C_{\alpha,N} > 0 \quad |\partial_x^\alpha (z_k^{\mathrm{wkb},(1)} - z_k^{(1)})(x)| \le C_{N,\alpha} h^N e^{-\varphi_k(x)/h} \tag{7-13}$$

hold for all x in a neighborhood $\mathcal{V}'_k \subset \mathcal{V}_k$ of $U_k^{(1)}$. Notice that the one-forms $z_k^{\mathrm{wkb},(1)}$ and $z_k^{(1)}$ are real-valued. By taking a cut-off function $\chi_k \in \mathcal{C}_0^\infty(\mathcal{V}'_k)$ with $\chi_k \equiv 1$ in a neighborhood of $U_k^{(1)}$, a normalized quasimode for $\Delta_{f,h}^{D,(1)}(\Omega_+ \setminus \Omega_-)$ is given by

$$w_k^{(1)} = \frac{\chi_k z_k^{(1)}}{\|\chi_k z_k^{(1)}\|_{L^2(\mathcal{V}'_k)}}.$$

The set of functions $(w_k^{(1)})_{k\in\{1,\dots,m_1^D(\Omega_+\setminus\bar{\Omega}_-)\}}$ is orthonormal, owing to the disjoint supports of the functions $(\chi_k)_{k\in\{1,\dots,m_1^D(\Omega_+\setminus\bar{\Omega}_-)\}}$. According to [Helffer and Nier 2006, Proposition 6.6], those quasimodes belong to the form domain of $\Delta^{D,(1)}_{f,h}(\Omega_+ \setminus \overline{\Omega}_-)$, and there exist two constants $C, c > 0$ such that

$$\|d_{f,h} w_k^{(1)}\|^2_{L^2(\Omega_+)} + \|d^*_{f,h} w_k^{(1)}\|^2_{L^2(\Omega_+)} \le C e^{-c/h} \tag{7-14}$$

holds for all $k \in \{1, \dots, m_1^D(\Omega_+ \setminus \overline{\Omega}_-)\}$. In addition, the estimates (7-12) and (7-13) with $\zeta_k(h) = \mathcal{O}(e^{-c_k/h})$ imply that the $w_k^{(1)}$ solve

$$\begin{cases} \Delta^{(1)}_{f,h} w_k^{(1)} = r_k \quad \text{on } \Omega_+ \setminus \overline{\Omega}_-, \\ t w_k^{(1)}\big|_{\partial\Omega_+ \cup \partial\Omega_-} = 0, \quad t d^*_{f,h} w_k^{(1)}\big|_{\partial\Omega_-} = 0, \quad t d^*_{f,h} w_k^{(1)}\big|_{\partial\Omega_+} = \rho_k, \end{cases} \tag{7-15}$$

where r_k and ρ_k satisfy

$$\forall p \in \mathbb{N} \;\; \exists C_p > 0 \;\; \forall k \in \{1, \dots, m_1^D(\Omega_+ \setminus \overline{\Omega}_-)\} \quad \|r_k\|_{W^{p,2}(\overline{\Omega}_+ \setminus \Omega_-)} + \|\rho_k\|_{W^{p+1/2,2}(\partial\Omega_+)} \le C_p e^{-c'/h} \tag{7-16}$$

for some $c' > 0$. The construction of the almost orthonormal basis of $\operatorname{Ran} 1_{[0,\nu(h)]}(\Delta^{D,(1)}_{f,h}(\Omega_+ \setminus \overline{\Omega}_-))$ is completed with the next lemma.

Lemma 7.10. *Assume Hypotheses 1, 5 and 6 and condition (7-5), and set*

$$u_k^{(1)} = 1_{[0,\nu(h)]}(\Delta^{D,(1)}_{f,h}(\Omega_+ \setminus \overline{\Omega}_-)) w_k^{(1)}$$

for any $k \in \{1, \dots, m_1^D(\Omega_+ \setminus \overline{\Omega}_-)\}$. Then $(u_k^{(1)})_{k \in \{1,\dots,m_1^D(\Omega_+ \setminus \overline{\Omega}_-)\}}$ is an almost orthonormal basis of $\operatorname{Ran} 1_{[0,\nu(h)]}(\Delta^{D,(1)}_{f,h}(\Omega_+ \setminus \overline{\Omega}_-))$.

Moreover,

$$\exists c > 0 \;\; \forall p \in \mathbb{N} \;\; \exists C_p > 0 \;\; \forall k \in \{1, \dots, m_1^D(\Omega_+ \setminus \overline{\Omega}_-)\} \quad \|u_k^{(1)} - w_k^{(1)}\|_{W^{p,2}(\Omega_+ \setminus \overline{\Omega}_-)} \le C_p e^{-c/h} \tag{7-17}$$

for all sufficiently small h.

Proof. Let us introduce $v_k^{(1)} = u_k^{(1)} - w_k^{(1)}$ for $k \in \{1, \dots, m_1^D(\Omega_+ \setminus \overline{\Omega}_-)\}$. The one-form $v_k^{(1)}$ belongs to the form domain of $\Delta^{D,(1)}_{f,h}(\Omega_+ \setminus \overline{\Omega}_-)$ and the spectral theorem leads to

$$\nu(h)\|v_k^{(1)}\|^2_{L^2(\Omega_+ \setminus \overline{\Omega}_-)} \le \|d_{f,h} w_k^{(1)}\|^2_{L^2(\Omega_+ \setminus \overline{\Omega}_-)} + \|d^*_{f,h} w_k^{(1)}\|^2_{L^2(\Omega_+ \setminus \overline{\Omega}_-)} \le C e^{-c/h} \le C e^{-c_1/h}$$

owing to (7-14) and $\sigma(\Delta^{D,(1)}_{f,h}(\Omega_+ \setminus \overline{\Omega}_-)) \cap [0, \nu(h)] \subset [0, e^{-c_0/h}]$. With (2-5), this implies that $\|v_k^{(1)}\|^2_{L^2(\Omega_+ \setminus \overline{\Omega}_-)} = \mathcal{O}(e^{-c_2/h})$. By using

$$h^2(\|d v_k^{(1)}\|^2_{L^2(\Omega_+ \setminus \overline{\Omega}_-)} + \|d^* v_k^{(1)}\|^2_{L^2(\Omega_+ \setminus \overline{\Omega}_-)}) \\ \le 2\|d_{f,h} v_k^{(1)}\|^2_{L^2(\Omega_+ \setminus \overline{\Omega}_-)} + 2\|d^*_{f,h} v_k^{(1)}\|^2_{L^2(\Omega_+ \setminus \overline{\Omega}_-)} + C\|v_k^{(1)}\|^{(2)}_{L^2(\Omega_+ \setminus \overline{\Omega}_-)},$$

we obtain

$$\|v_k^{(1)}\|^2_{W^{1,2}(\Omega_+ \setminus \overline{\Omega}_-)} = \mathcal{O}(h^{-2} e^{-c_2/h}) = \mathcal{O}(e^{-c_2/(2h)}).$$

Thus, the almost orthonormality property of $(u_k^{(1)})_{k \in \{1,\dots,m_1^D(\Omega_+ \setminus \overline{\Omega}_-)\}}$ is due to the orthonormality of $(w_k^{(1)})_{k \in \{1,\dots,m_1^D(\Omega_+ \setminus \overline{\Omega}_-)\}}$.

The $W^{p,2}$ estimates (7-17) are then obtained by a bootstrap argument (induction on p) using the elliptic regularity of the Hodge Laplacian. With $\Delta^{D,(1)}_{f,h}(\Omega_+\setminus\Omega_-)u^{(1)}_k=\tilde{\mathcal{O}}(e^{-c_0/h})$ in any $W^{p,2}$ (see Lemma 6.3), (7-15) leads to

$$\begin{cases}\Delta_H v^{(1)}_k = r'_k(h)-h^{-2}(\Delta_{f,h}-h^2\Delta_H)v^{(1)}_k,\\ \mathbf{t}v^{(1)}_k|_{\partial\Omega_+\cup\partial\Omega_-}=0,\quad \mathbf{t}d^*v^{(1)}_k|_{\partial\Omega_-}=0,\quad \mathbf{t}d^*v^{(1)}_k|_{\partial\Omega_+}=-h^{-1}\rho_k-h^{-1}\boldsymbol{i}_{\nabla f}v^{(1)}_k,\end{cases}$$

where $\|r'_k(h)\|_{W^{p,2}(\Omega_+\setminus\bar\Omega_-)}$ satisfies the same estimate (7-16) as $\|r_k(h)\|_{W^{p,2}(\Omega_+\setminus\bar\Omega_-)}$. Using the fact that the zeroth-order differential operator $\Delta_{f,h}-h^2\Delta_H=|\nabla f|^2+h(\mathcal{L}_{\nabla f}+\mathcal{L}^*_{\nabla f})$ is bounded in L^∞-norm, we thus obtain the $W^{p,2}$ estimates (7-17) by induction on p. □

Proof of Proposition 7.5. Let us apply Theorem 2.4 and Corollary 6.9 to the almost orthonormal basis $(u^{(1)}_k)_{1\le k\le m^D_1(\Omega_+\setminus\bar\Omega_-)}$ introduced in Lemma 7.10 (see Remark 6.6). From the estimate (7-17) and the fact that $\lim_{h\to0}h\log\lambda^{(0)}_1(\Omega_+)=-2\kappa_f$, we deduce

$$\lambda^{(0)}_1(\Omega_+)=\frac{h^2\sum_{k=1}^{m^D_1(\Omega_+\setminus\bar\Omega_-)}\left(\int_{\partial\Omega_+}e^{-f/h}\boldsymbol{i}_n w^{(1)}_k(\sigma)\,d\sigma\right)^2}{\int_{\Omega_+}e^{-2f(x)/h}\,dx}(1+\mathcal{O}(e^{-c/h})),$$

$$\partial_n u^{(0)}_1|_{\partial\Omega_+}=-\sum_{k=1}^{m^D_1(\Omega_+\setminus\bar\Omega_-)}\frac{\int_{\partial\Omega_+}e^{-f(\sigma)/h}\boldsymbol{i}_n w^{(1)}_k(\sigma)\,d\sigma}{\left(\int_{\Omega_+}e^{-2f(x)/h}\,dx\right)^{1/2}}\boldsymbol{i}_n w^{(1)}_k+\mathcal{O}(e^{-(\kappa_f+c)/h}),$$

where the last remainder term is measured in $W^{p,2}(\partial\Omega_+)$-norm for any $p\in\mathbb{N}$. In particular, we deduce

$$\frac{e^{-f/h}}{\left(\int_{\Omega_+}e^{-2f(x)/h}\,dx\right)^{1/2}}\partial_n u^{(0)}_1|_{\partial\Omega_+}=-\sum_{k=1}^{m^D_1(\Omega_+\setminus\bar\Omega_-)}\left(\int_{\partial\Omega_+}\theta_k(\sigma)\,d\sigma\right)\theta_k+\mathcal{O}(e^{-(2\kappa_f+c)/h})\quad\text{in } L^1(\partial\Omega_+)$$

and

$$\lambda^{(0)}_1(\Omega_+)=h^2\sum_{k=1}^{m^D_1(\Omega_+\setminus\bar\Omega_-)}\left(\int_{\partial\Omega_+}\theta_k\,d\sigma\right)^2(1+\mathcal{O}(e^{-c/h})),\tag{7-18}$$

where $\theta_k=\left(e^{-f/h}/\left(\int_{\Omega_+}e^{-2f(x)/h}\,dx\right)^{1/2}\right)\boldsymbol{i}_n w^{(1)}_k|_{\partial\Omega_+}$.

Using $\partial_n u^{(0)}_1|_{\partial\Omega_+}\le 0$ and the fact that the θ_k have disjoint supports, the following estimates hold:

$$\begin{aligned}\left(\int_{\Omega_+}e^{-2f(x)/h}\,dx\right)^{-\frac12}\left\|e^{-f/h}\partial_n u^{(0)}_1|_{\partial\Omega_+}\right\|_{L^1(\partial\Omega_+)}&=\sum_{k=1}^{m^D_1(\Omega_+\setminus\bar\Omega_-)}\left(\int_{\partial\Omega_+}\theta_k(\sigma)\,d\sigma\right)^2+\mathcal{O}(e^{-(2\kappa_f+c)/h})\\&=h^{-2}\lambda^{(0)}_1(\Omega_+)(1+\tilde{\mathcal{O}}(e^{-c/h})).\end{aligned}$$

In the last equality, we used (2-11) to get a lower bound on $\lambda^{(0)}_1(\Omega_+)$. By recalling that the Dirichlet boundary condition $u^{(0)}_1|_{\partial\Omega_+}=0$ implies

$$\partial_n[e^{-f/h}u^{(0)}_1]|_{\partial\Omega_+}=e^{-f/h}\partial_n u^{(0)}_1|_{\partial\Omega_+},$$

we thus get

$$-\frac{\partial_n[e^{-f/h}u_1^{(0)}]\big|_{\partial\Omega_+}}{\|\partial_n[e^{-f/h}u_1^{(0)}]\|_{L^1(\partial\Omega_+)}} = \frac{\sum_{k=1}^{m_1^D(\Omega_+\setminus\bar{\Omega}_-)}\big(\int_{\partial\Omega_+}\theta_k\,d\sigma\big)\theta_k}{\sum_{k=1}^{m_1^D(\Omega_+\setminus\bar{\Omega}_-)}\big(\int_{\partial\Omega_+}\theta_k\,d\sigma\big)^2} + \tilde{\mathcal{O}}(e^{-c/h}) \quad \text{in } L^1(\partial\Omega_+). \tag{7-19}$$

In order to get estimates from (7-18) and (7-19) in terms of f, it remains to approximate the quantities θ_k and $\int_{\partial\Omega_+}\theta_k\,d\sigma$ in the limit $h\to 0$. Recall that

$$\theta_k = \frac{e^{-f/h}}{\big(\int_{\Omega_+}e^{-2f(x)/h}\,dx\big)^{1/2}}\boldsymbol{i}_n w_k^{(1)}\big|_{\partial\Omega_+} \quad\text{and}\quad w_k^{(1)} = \frac{\chi_k z_k^{(1)}}{\|\chi_k z_k^{(1)}\|_{L^2(\mathcal{V}_k')}}.$$

The estimates are obtained using the Laplace method and the WKB expansion (7-11) together with (7-13) to approximate $z_k^{(1)}$.

- $\int_{\Omega_+}e^{-2f(x)/h}\,dx$: A direct application of the Laplace method gives

$$\int_{\Omega_+}e^{-2f(x)/h}\,dx = e^{-2f(U_0)/h}(\pi h)^{d/2}\big(\det(\operatorname{Hess} f)(U_0)\big)^{-1/2}(1+\mathcal{O}(h)),$$

  where U_0 is the unique global minimum of f.

- $\|\chi_k z_k^{(1)}\|_{L^2(\mathcal{V}_k')}$: Recall the coordinates around $U_k^{(1)}$ used in (7-10) and (7-11). Using these coordinates and (7-13), there is a $\mathcal{C}_0^\infty(\{x^d\le 0\})$ function $\alpha(x,h)\sim\sum_{k=0}^\infty \alpha_k(x)h^k$ with $\alpha_0(0)=1$ such that

$$\begin{aligned}\|\chi_k z_k^{(1)}\|^2_{L^2(\mathcal{V}_k')} &= \int_{\{x^d\le 0\}} e^{-2\varphi_k(x)/h}\alpha(x,h)\,dx^1\cdots dx^d\\ &= \int_{\{x^d\le 0\}} e^{2\partial_n f(U_k^{(1)})x^d/h}e^{-\sum_{j=1}^{d-1}\lambda_j(x^j)^2/h}\alpha(x,h)\,dx^1\cdots dx^d\\ &= \frac{h}{2\partial_n f(U_k^{(1)})}\frac{(\pi h)^{(d-1)/2}}{\sqrt{\lambda_1\cdots\lambda_{d-1}}}(1+\mathcal{O}(h))\\ &= \frac{(\pi h)^{(d+1)/2}}{2\pi\,\partial_n f(U_k^{(1)})\big(\det(\operatorname{Hess} f\big|_{\partial\Omega_+})(U_k^{(1)})\big)^{1/2}}(1+\mathcal{O}(h)).\end{aligned}$$

  We applied the Laplace method to get the estimate of the integral (using the fact that $\partial_n f(U_k^{(1)})>0$ by (7-5)).

- θ_k: On the one hand, using $f(x)=f(U_k^{(1)})+\partial_n f(U_k^{(1)})x^d+\frac{1}{2}\sum_{j=1}^{d-1}\lambda_j(x^j)^2$ in a neighborhood of $U_k^{(1)}$ (see (7-10)), we have, on $\partial\Omega_+$ (so that $x^d=0$),

$$\chi_k\frac{e^{-f/h}}{\big(\int_{\Omega_+}e^{-f(x)/h}\,dx\big)^{1/2}} = \chi_k e^{-(f(U_k^{(1)})-f(U_0))/h}(\pi h)^{-d/4}\big(\det(\operatorname{Hess} f)(U_0)\big)^{1/4}e^{-\sum_{j=1}^{d-1}\lambda_j(x^j)^2/(2h)}(1+\mathcal{O}(h)).$$

On the other hand, the function $\boldsymbol{i}_n w_k^{(1)}\big|_{\partial\Omega_+} = \chi_k \boldsymbol{i}_n z_k^{(1)}\big|_{\partial\Omega_+} / \|\chi_k z_k^{(1)}\|_{L^2(\mathcal{V}_k')}$ satisfies

$$\boldsymbol{i}_n w_k^{(1)}\big|_{\partial\Omega_+} = \chi_k \frac{\sqrt{2\pi\,\partial_n f(U_k^{(1)})}\big(\det\big(\operatorname{Hess} f\big|_{\partial\Omega_+}\big)(U_k^{(1)})\big)^{1/4}}{(\pi h)^{(d+1)/4}} e^{-\sum_{j=1}^{d-1}\lambda_j (x^j)^2/(2h)}(1+\mathcal{O}(h)).$$

From these two estimates, θ_k satisfies

$$\theta_k = A_k \chi_k e^{-\sum_{j=1}^{d-1}\lambda_j (x^j)^2/h}(1+\mathcal{O}(h)),$$

where

$$A_k = \frac{\sqrt{2\pi\,\partial_n f(U_k^{(1)})}}{(\pi h)^{(2d+1)/4}} \big(\det\big(\operatorname{Hess} f\big|_{\partial\Omega_+}\big)(U_k^{(1)})\big)^{1/4} \big(\det(\operatorname{Hess} f)(U_0)\big)^{1/4} e^{-(f(U_k^{(1)})-f(U_0))/h}.$$

- $\int_{\partial\Omega_+}\theta_k$: The Laplace method implies that

$$\int e^{-\sum_{j=1}^{d-1}\lambda_j (x^j)^2/h}\,dx^1\cdots dx^{d-1} = \frac{(\pi h)^{(d-1)/2}}{\sqrt{\lambda_1\cdots\lambda_{d-1}}}(1+\mathcal{O}(h)) = (\pi h)^{(d-1)/2}\big(\det\big(\operatorname{Hess} f\big|_{\partial\Omega_+}\big)(U_k^{(1)})\big)^{-1/2}(1+\mathcal{O}(h)).$$

  We thus obtain

$$\int_{\partial\Omega_+}\theta_k = \frac{\sqrt{2\pi\,\partial_n f(U_k^{(1)})}\big(\det(\operatorname{Hess} f)(U_0)\big)^{1/4}}{(\pi h)^{3/4}\big(\det\big(\operatorname{Hess} f\big|_{\partial\Omega_+}\big)(U_k^{(1)})\big)^{1/4}} e^{-(f(U_k^{(1)})-f(U_0))/h}(1+\mathcal{O}(h)).$$

Putting together the above information and using (7-18) and (7-19) finally implies

$$\lambda_1^{(0)}(\Omega_+) = \sqrt{\frac{h\det(\operatorname{Hess} f)(U_0)}{\pi}} \sum_{k=1}^{m_1^D(\Omega_+\setminus\bar\Omega_-)} \frac{2\partial_n f(U_k^{(1)})}{\sqrt{\det\big(\operatorname{Hess} f\big|_{\partial\Omega_+}\big)(U_k^{(1)})}} e^{-2(f(U_k^{(1)})-f(U_0))/h}(1+\mathcal{O}(h)),$$

which is exactly (7-6), and

$$-\frac{\partial_n[e^{-f/h}u_1^{(0)}]\big|_{\partial\Omega_+}}{\big\|\partial_n[e^{-f/h}u_1^{(0)}]\big\|_{L^1(\partial\Omega_+)}} = \frac{\sum_{k=1}^{m_1^D(\Omega_+\setminus\bar\Omega_-)} \partial_n f(U_k^{(1)}) e^{-2(f(U_k^{(1)})-f(U_0))/h}\chi_k e^{-\sum_{j=1}^{d-1}\lambda_j (x^j)^2/h}}{(\pi h)^{(d-1)/2}\sum_{k'=1}^{m_1^D(\Omega_+\setminus\bar\Omega_-)}\Big(\partial_n f(U_{k'}^{(1)})/\sqrt{\det\big(\operatorname{Hess} f\big|_{\partial\Omega_+}\big)(U_{k'}^{(1)})}\Big) e^{-2(f(U_{k'}^{(1)})-f(U_0))/h}}(1+\mathcal{O}(h)). \tag{7-20}$$

We thus obtain estimates of $\lambda_1^{(0)}(\Omega_+)$ and $-\partial_n(e^{-f/h}u_1^{(0)})\big|_{\partial\Omega_+}/\|\partial_n(e^{-f/h}u_1^{(0)})\|_{L^1(\partial\Omega_+)}$ in terms of second-order Taylor expansions of f around the local minima $U_k^{(1)}$. This ends the first step of the proof.

Actually, the two estimates (7-6) and (7-20) can be rewritten in a simpler form using the Laplace method again. By recalling the equality $f(x) = f(U_k^{(1)}) + \partial_n f(U_k^{(1)})x^d + \frac{1}{2}\sum_{j=1}^{d-1}\lambda_j (x^j)^2$ in a neighborhood of $U_k^{(1)}$, the Laplace method gives, by similar computations to those performed above,

$$\frac{\int_{\partial\Omega_+} 2\partial_n f(\sigma)e^{-2f(\sigma)/h}\,d\sigma}{\int_{\Omega_+} e^{-2f(x)/h}\,dx} = \sqrt{\frac{h\det(\operatorname{Hess} f)(U_0)}{\pi}} \sum_{k=1}^{m_1^D(\Omega_+\setminus\overline{\Omega}_-)} \frac{2\partial_n f(U_k^{(1)})}{\sqrt{\det\big(\operatorname{Hess} f\big|_{\partial\Omega_+}\big)(U_k^{(1)})}} e^{-2(f(U_k^{(1)})-f(U_0))/h}(1+\mathcal{O}(h)),$$

$$\frac{(2\partial_n f)e^{-2f/h}\big|_{\partial\Omega_+}}{\|(2\partial_n f)e^{-2f/h}\|_{L^1(\partial\Omega_+)}} = \frac{\sum_{k=1}^{m_1^D(\Omega_+\setminus\overline{\Omega}_-)} \partial_n f(U_k^{(1)})e^{-2(f(U_k^{(1)})-f(U_0))/h}\chi_k e^{-\sum_{j=1}^{d-1}\lambda_j(x^j)^2/h}}{(\pi h)^{(d-1)/2}\sum_{k'=1}^{m_1^D(\Omega_+\setminus\overline{\Omega}_-)}(\partial_n f(U_{k'}^{(1)})/\sqrt{\det\big(\operatorname{Hess} f\big|_{\partial\Omega_+}\big)(U_{k'}^{(1)})})e^{-2(f(U_{k'}^{(1)})-f(U_0))/h}} + \mathcal{O}(h),$$

where the last remainder term is measured in $L^1(\partial\Omega_+)$-norm. Comparing with the two estimates (7-6) and (7-20) above, we thus obtain (7-7) and (7-8). This concludes the proof. □

7B. *Beyond Morse assumptions.* In this section, we discuss Hypotheses 3 and 4 for functions f which do not fulfill the Morse assumptions of Hypothesis 5 above. In Sections 7B2 and 7B3, we present two examples (respectively in dimension 1 and 2) of functions f which do not fulfill Hypothesis 5 but for which Hypotheses 3 and 4 still hold true. Section 7B1 is first devoted to a few remarks that will be useful in the examples we will discuss below.

7B1. *General remarks.* First, we will use the duality between the chain complexes associated with $d_{f,h}$ and $d_{f,h}^*$. More precisely, conjugating with the Hodge $\star$-operator exchanges p- and $(\dim M-p)$-forms, d and d^*, f and $-f$, Neumann and Dirichlet boundary conditions. This was used extensively in [Le Peutrec 2011; Le Peutrec et al. 2013].

Second, the following lemma will also be useful. It is a variant of Proposition 3.7.

Lemma 7.11. *Let Ω be a regular bounded domain of the Riemannian manifold (M,g) and let $f\in\mathcal{C}^\infty(\overline{\Omega})$ be such that $(\nabla f)^{-1}(\{0\})$ has a unique nonempty connected component in Ω.*

- *If $\partial_n f\big|_{\partial\Omega}>0$ then the two first eigenvalues of $\Delta_{f,h}^{N,(0)}(\Omega)$ satisfy*

$$\mu_1^{(0)}(\Omega)=0 \quad \text{and} \quad \lim_{h\to 0} h\log\mu_2^{(0)}(\Omega)=0.$$

- *If $\partial_n f\big|_{\partial\Omega}<0$ and $|\nabla f|^2-h\Delta f\geq 0$ in Ω for all $h\in(0,h_0)$, then the first eigenvalue of $\Delta_{f,h}^{D,(0)}(\Omega)$ satisfies*

$$\lim_{h\to 0} h\log\lambda_1^{(0)}(\Omega)=0.$$

Proof. Up to the addition of a constant to the function f (which only affects the normalization of $e^{-f/h}$), one may assume without loss of generality that $f\equiv 0$ on $(\nabla f)^{-1}(\{0\})$ (using the connectedness assumption on $(\nabla f)^{-1}(\{0\})$). Then, $f\geq 0$ in Ω when $\partial_n f\big|_{\partial\Omega}>0$, and $f\leq 0$ when $\partial_n f\big|_{\partial\Omega}<0$.

The fact that $\mu_1^{(0)}(\Omega)=0$ is obvious, by considering the associated eigenvector $e^{-f/h}$. The Witten Laplacian acting on functions is the Schrödinger-type operator

$$\Delta_{f,h}^{(0)}=-h^2\Delta+|\nabla f|^2-h(\Delta f).$$

Since the function $|\nabla f|^2-h\Delta f$ is uniformly bounded in $\overline{\Omega}$, the two inequalities

$$\limsup_{h\to 0} h\log\mu_2^{(0)}(\Omega)\le 0 \quad\text{and}\quad \limsup_{h\to 0} h\log\lambda_1^{(0)}(\Omega)\le 0$$

are consequences of the min–max principle. For the Dirichlet case, any fixed nonzero function in $\mathcal{C}_0^\infty(\Omega)$ will provide an $\mathcal{O}(1)$ Rayleigh quotient. For the Neumann case, consider two regular functions $\chi_1,\ \chi_2\in\mathcal{C}_0^\infty(\Omega)$ such that $\operatorname{supp}\chi_1\cap\operatorname{supp}\chi_2=\varnothing$ and $\|\chi_1\|_{L^2(\Omega)}=\|\chi_2\|_{L^2(\Omega)}=1$, and take $\psi_h=\alpha_1(h)\chi_1+\alpha_2(h)\chi_2$ such that $\|\psi_h\|_{L^2}^2=|\alpha_1(h)|^2+|\alpha_2(h)|^2=1$ and $\langle\psi_h,e^{-f/h}\rangle_{L^2(\Omega)}=0$. We get $\langle\psi_h,\Delta_{f,h}^{N,(0)}\psi_h\rangle_{L^2(\Omega)}=\mathcal{O}(1)$ and the min–max principle applied to $\Delta_{f,h}^{N,(0)}(\Omega)$ on the orthogonal of $e^{-f/h}$ yields $\mu_2^{(0)}(\Omega)=\mathcal{O}(1)$ as $h\to 0$.

Let us first consider the case where $\partial_n f\big|_{\partial\Omega}<0$ and $|\nabla f|^2-h\Delta f\ge 0$. It remains to prove that $\liminf_{h\to 0} h\log\lambda_1^{(0)}(\Omega)\ge 0$. Let ω be a normalized eigenfunction associated with $\lambda_1^{(0)}(\Omega)$, so $\Delta_{f,h}^{D,(0)}(\Omega)\omega=\lambda_1^{(0)}(\Omega)\omega$ and $\|\omega\|_{L^2(\Omega)}=1$. Using Lemma 3.1 with $\varphi=0$ and the Poincaré inequality, we get

$$\lambda_1^{(0)}(\Omega)\ge h^2\|\nabla\omega\|_{L^2(\Omega)}^2\ge C_\Omega h^2.$$

This concludes the proof in the case $\partial_n f\big|_{\partial\Omega}<0$ and $|\nabla f|^2-h\Delta f\ge 0$.

Let us now consider the case $\partial_n f\big|_{\partial\Omega}>0$. It remains to prove that $\liminf_{h\to 0} h\log\mu_2^{(0)}(\Omega)\ge 0$. Let us reason by contradiction, by assuming that there exists $c>0$ and a sequence $(h_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} h_n=0 \quad\text{and}\quad \mu_2^{(0)}(\Omega)\le e^{-c/h_n} \quad\text{with } c>0.$$

Notice that $\mu_2^{(0)}(\Omega)$ depends on n. Let us introduce ω_n, a normalized eigenfunction associated with $\mu_2^{(0)}(\Omega)$, so $\Delta_{f,h_n}^{N,(0)}\omega_n=\mu_2^{(0)}(\Omega)\omega_n$ and $\|\omega_n\|_{L^2(\Omega)}=1$. Notice that $\int_\Omega\omega_n e^{-f/h_n}=0$. For $\varepsilon>0$, consider the open set

$$K_\varepsilon=\bigl\{x\in\overline{\Omega}: d\bigl(x,(\nabla f)^{-1}(\{0\})\bigr)<\varepsilon\bigr\},$$

so that $\overline{K}_\varepsilon$ is contained in Ω for $\varepsilon\in(0,\varepsilon_0)$ and ε_0 sufficiently small. Take a partition of unity $\chi_1^2+\chi_2^2\equiv 1$ in $\overline{\Omega}$ such that $\chi_i\in\mathcal{C}^\infty(\overline{\Omega})$, $\chi_1\equiv 1$ in a neighborhood of $K_{\varepsilon/2}$ and $\operatorname{supp}\chi_1\subset K_\varepsilon$. The IMS localization formula (see, for example, [Cycon et al. 1987]) gives

$$\begin{aligned} e^{-c/h_n} &\ge \langle\omega_n,\Delta_{f,h_n}^{N,(0)}(\Omega)\omega_n\rangle_{L^2(\Omega)}\\ &=\langle\chi_1\omega_n,\Delta_{f,h_n}^{N,(0)}(\Omega)\chi_1\omega_n\rangle_{L^2(\Omega)}+\langle\chi_2\omega_n,\Delta_{f,h_n}^{N,(0)}(\Omega)\chi_2\omega_n\rangle_{L^2(\Omega)}-h_n^2\sum_{j=1}^{2}\|\omega_n\nabla\chi_j\|_{L^2(\Omega)}^2. \end{aligned}\tag{7-21}$$

The lower bound (which is a consequence of $|\nabla f|^2>0$ on $\operatorname{supp}\chi_2$ and $\partial_n\chi_2=0$ on $\partial\Omega$)

$$\langle\chi_2\omega_n,\Delta_{f,h_n}^{N,(0)}(\Omega)\chi_2\omega_n\rangle_{L^2(\Omega)}\ge\langle\chi_2\omega_n,|\nabla f|^2\chi_2\omega_n\rangle_{L^2(\Omega)}-Ch_n\|\chi_2\omega_n\|_{L^2(\Omega)}^2\ge\frac{1}{C_\varepsilon}\|\chi_2\omega_n\|_{L^2(\Omega)}^2$$

for n sufficiently large together with (7-21) implies

$$\forall \delta > 0 \ \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \quad \|\omega_n\|^2_{L^2(K_\varepsilon)} \geq 1 - \delta.$$

Since $(\nabla f)^{-1}(\{0\})$ is assumed to be connected and, for every point of the open set K_ε, the gradient flow associated with f defines a path to $(\nabla f)^{-1}(\{0\})$, K_ε is a connected open set. The function $v_n = \omega_n\big|_{K_\varepsilon}$ belongs to $W^{1,2}(K_\varepsilon)$ with

$$h_n^2 e^{-2C\varepsilon^2/h_n} \|de^{f/h_n} v_n\|^2_{L^2(K_\varepsilon)} \leq \|d_{f,h} v_n\|^2_{L^2(K_\varepsilon)} \leq e^{-c/h_n},$$

thanks to the fact that

$$\exists C > 0 \ \forall x \in K_\varepsilon \quad 0 \leq f(x) \leq C\varepsilon^2.$$

By choosing $\varepsilon > 0$ so that $c - 2C\varepsilon^2 > 0$, the spectral gap estimate for the Neumann Laplacian in Ω (or equivalently the Poincaré–Wirtinger inequality in Ω) provides a constant C_n such that

$$\lim_{n\to\infty} \|e^{f/h_n} v_n - C_n\|_{L^2(K_\varepsilon)} = 0.$$

We thus deduce

$$\lim_{n\to\infty} \|\omega_n - C_n e^{-f/h_n}\|_{L^2(K_\varepsilon)} = 0 \quad \text{with} \quad \|\omega_n\|^2_{L^2(K_\varepsilon)} \geq 1 - \delta, \ \|\omega_n\|_{L^2(\Omega)} = 1.$$

For $\delta < 1$, this is in contradiction with $\int_\Omega \omega_n e^{-f/h_n} = 0$. □

7B2. *A one-dimensional example.* In this section, we exhibit a simple one-dimensional example of a function f satisfying Hypotheses 3 and 4 though not being a Morse function. An extension is then briefly discussed.

Proposition 7.12. *Consider a function $f \in \mathcal{C}^\infty(\overline{\Omega}_+)$, $\Omega_+ = (a_+, b_+)$ with $a_+ < b_+$ two real numbers, such that*

$$f^{-1}(0) = (f')^{-1}(0) = [a_1, b_1], \quad -\infty < a_+ < a_1 \leq b_1 < b_+ < +\infty,$$
$$f'(a_+) < 0 \quad \text{and} \quad f'(b_+) > 0.$$

Then, for any $\Omega_- = (a_-, b_-)$ such that $a_+ < a_- < a_1 \leq b_1 < b_- < b_+$, Hypotheses 3 and 4 are valid with $m_0^N(\Omega_-) = 1$, $m_1^N(\Omega_-) = 0$ and $m_1^D(\Omega_+ \setminus \overline{\Omega}_-) = 2$.

Notice that, for this example, Hypotheses 1 and 2 are also satisfied, which means that the results of Theorem 2.4 are valid.

Proof. On an interval I with the Euclidean metric, the one-forms can be written as $u^{(1)} = u_1(x)\,dx$. The Witten Laplacians $\Delta^{(p)}_{f,h}(I)$ with $p = 0, 1$ are then given by

$$\Delta^{(0)}_{f,h}(I)u^{(0)} = (-h^2\partial_{x,x} + |\partial_x f|^2 - h(\partial_{x,x} f))u^{(0)},$$
$$\Delta^{(1)}_{f,h}(I)(u_1 dx) = \big[(-h^2\partial_{x,x} + |\partial_x f|^2 + h(\partial_{x,x} f))u_1\big]\,dx.$$

The Dirichlet boundary conditions are given by

$$u^{(0)} = 0 \quad \text{on } \partial I \qquad \text{and} \qquad -h\partial_x u_1 + (\partial_x f)u_1 = 0 \quad \text{on } \partial I,$$

while the Neumann boundary conditions are given by

$$h\partial_x u^{(0)} + (\partial_x f)u^{(0)} = 0 \quad \text{on } \partial I \qquad \text{and} \qquad u_1 = 0 \quad \text{on } \partial I.$$

This is a particular case of the general duality recalled at the beginning of Section 7B1. Let us now check Hypotheses 3 and 4.

First, $e^{-f/h}$ belongs to the kernel of $\Delta^{N,(0)}_{f,h}(\Omega_-)$. A direct application of Lemma 7.11 shows that (2-6) holds for $p=0$ with $m_0^N(\Omega_-)=1$. Second, by the duality argument, proving that (2-6) holds for $p=1$ with $m_1^N(\Omega_-)=0$ is equivalent to proving that there are no exponentially small eigenvalues for $\Delta^{D,(0)}_{-f,h}(\Omega_-)$ (notice that f has been changed to $-f$). But this is a consequence of the second part of Lemma 7.11, since f is convex. Finally, note that the condition (2-8) is empty, since the only exponentially small eigenvalue of $\Delta^{N,(0)}_{f,h}(\Omega_-)$ is 0. This shows that Hypothesis 3 holds.

The open set $\Omega_+ \setminus \overline{\Omega}_-$ is the disjoint union of the two open intervals (a_+, a_-) and (b_-, b_+). On each of them, $\partial_x f$ does not vanish and the Morse assumptions of Hypothesis 5 are satisfied. On (a_+, a_-) (resp. (b_-, b_+)), f has one generalized critical point of index 1 at a_+ (resp. at b_+). Therefore, using the results of [Helffer and Nier 2006] (see Section 7A1), (2-9) holds with $m_1^D(\Omega_+ \setminus \overline{\Omega}_-) = 2$. This shows that Hypothesis 4 holds. □

It is not difficult to treat the case when $f \in \mathcal{C}^\infty([a_+, b_+])$ has a finite number of critical intervals,

$$(f')^{-1}(\{0\}) = \bigcup_{n=1}^{2N+1} [a_n, b_n], \qquad a_+ < a_1 \le b_1 < \cdots < a_{2N+1} \le b_{2N+1} < b_+,$$

with $f'(a_+) < 0$ and $f'(b_+) > 0$. Again, $\Omega_- = (a_-, b_-)$, with $a_+ < a_- < a_1 < b_{2N+1} < b_- < b_+$. The local problems around every $[a_n, b_n]$ can be studied with the help of the duality argument and Lemma 7.11. Using an argument based on a partition of unity, one can check that (2-6) and (2-9) hold with $m_0^N(\Omega_-) = 2N+1$, $m_1^N(\Omega_-) = 2N$ and $m_1^D(\Omega_+ \setminus \overline{\Omega}_-) = 2N+2$. Hypothesis 1 is of course satisfied. Ensuring that Hypothesis 2 and condition (2-8) hold then requires us to correctly choose the heights of the critical values. They hold, for example, when $\max_{1\le n\le 2N+1} f(a_i) < \min\{f(a_+),\ f(b_+)\}$ and when $f(a_1)$ and $f(b_{2N+1})$ are the two smallest critical values.

7B3. *A two-dimensional example.* This example is inspired by the work of [Bismut 1986; Helffer and Sjöstrand 1987; 1988] on Bott inequalities. We consider the following $\mathcal{C}^\infty$ radial functions in $\mathbb{R}^2$:

$$\varphi_{\text{in}}(x) = e^{-1/(|x|^2-1)^2} 1_{[0,1]}(|x|),$$
$$\varphi_{\text{ext}} \equiv 0 \quad \text{for } |x| \le 1, \qquad \varphi_{\text{ext}} \text{ strictly convex in } \{|x| > 1\}.$$

The domain Ω_+ is the disc $D((-R,0), 2R)$ and Ω_- the disc $D((-R,0), 2R-1)$ with $R > 3$. The function f is defined by $f(x) = \varphi_{\text{in}}(x) + \varphi_{\text{ext}}\left(\frac{1}{2}x\right)$. The level sets of the function f are represented in Figure 5.

Proposition 7.13. *When $R > 3$ is chosen large enough, the above triple (Ω_+, Ω_-, f) fulfills Hypotheses 1, 2, 3 and 4 with $m_0^N(\Omega_-) = 1$, $m_1^N(\Omega_-) = 1$ and $m_1^D(\Omega_+ \setminus \overline{\Omega}_-) = 1$.*

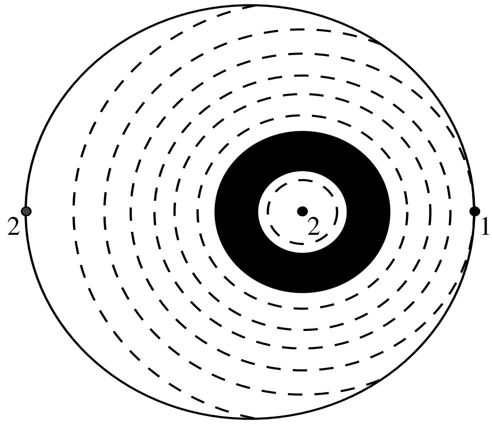

Figure 5. Only Ω_+ is represented. The level sets of f are represented by dashed lines. The black area is the 0 level set. The dots indicate the generalized critical points, together with their indices (for Dirichlet boundary conditions).

Proof. Thanks to the convexity assumption on $x \mapsto \varphi_{\text{ext}}\big(\frac{1}{2}x\big)$ and its local behavior around $\{|x| = 2\}$, Hypotheses 1 and 2 hold for $R > 3$ large enough.

The choice of non-0-centered disks for Ω_+ and Ω_- while f is a radial function implies that $f\big|_{\partial\Omega_+}$ has a unique local minimum and therefore, using the results recalled in Section 7A1, (2-9) is satisfied with $m_1^D(\Omega_+ \setminus \overline{\Omega}_-) = 1$. This shows that Hypothesis 4 holds.

The fact that (2-6) holds for $p = 0$ with $m_0^N(\Omega_-) = 1$ is a direct application of Lemma 7.11. This also implies that the condition (2-8) is void. It only remains to prove that (2-6) holds for $p = 1$ with $m_1^N(\Omega_-) = 1$. We will actually prove that (2-6) holds for $p = 2$ with $m_2^N(\Omega_-) = 1$. Then the quasi-isomorphism with the absolute cohomology of the disc (see Section 4A) gives $m_2^N(\Omega_-) - m_1^N(\Omega_-) + m_0^N(\Omega_-) = 1$, which indeed implies $m_1^N(\Omega_-) = 1$. Moreover, by the duality argument, (2-6) holds for $p = 2$ with $m_2^N(\Omega_-) = 1$ if (2-9) holds for $p = 0$ with $m_0^D(\Omega_-) = 1$, f being changed into $-f$. The proof of this claim will conclude the demonstration.

In the rest of this proof, $m_0^D(\Omega_-)$ denotes the number of small eigenvalues for $\Delta_{-f,h}^{D,(0)}(\Omega_-)$. The function $-f$ has a local minimum at $x = (0, 0)$. Applying the min–max principle with a quasimode $\chi(x)e^{f(x)/h}$, where χ is a smooth nonnegative function such that $\chi \equiv 1$ on $\big\{|x| \le \frac{1}{4}\big\}$ and $\chi \equiv 0$ on $\big\{|x| \ge \frac{1}{2}\big\}$, implies that $m_0^D(\Omega_-) \ge 1$.

Let us now consider $\omega \in D(\Delta_{-f,h}^{D,(0)}(\Omega_-))$, a normalized eigenvector associated with an exponentially small eigenvalue, so $\langle \omega, \Delta_{-f,h}^{D,(0)}(\Omega_-)\omega\rangle_{L^2(\Omega_-)} \le e^{-c/h}$ for some $c > 0$. Let $\chi_1^2 + \chi_2^2 = 1$ be a partition of unity on Ω_- with $\chi_1^2 \equiv 1$ on $\{|x| \le \varepsilon\}$ and $\chi_1^2 \equiv 0$ on $\{|x| \ge 2\varepsilon\}$ (for $\varepsilon < \frac{1}{4}$). The IMS localization formula gives

$$\langle \omega, \Delta_{-f,h}^{D,(0)}(\Omega_-)\omega\rangle_{L^2(\Omega_-)} = \langle \chi_1\omega, \Delta_{-f,h}^{D,(0)}(\Omega_-)\chi_1\omega\rangle_{L^2(\Omega_-)} + \langle \chi_2\omega, \Delta_{-f,h}^{D,(0)}(\Omega_-)\chi_2\omega\rangle_{L^2(\Omega_-)} - h^2 \sum_{j=1}^{2} \|\omega\nabla\chi_j\|_{L^2(\Omega_-)}^2. \tag{7-22}$$

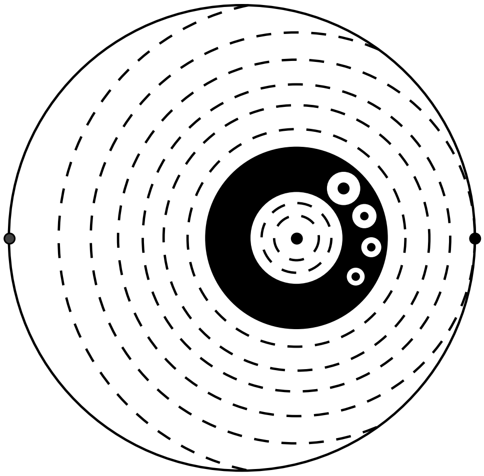

Figure 6. A variant of Figure 5 with $N = 4$. The supports of the additional terms in f (compared with Figure 5) are represented by the white disks.

The second term of the right-hand side equals $\langle \chi_2\omega, \Delta^{D,(0)}_{-f,h}(\Omega)\chi_2\omega\rangle_{L^2(\Omega_-\setminus\Omega_\varepsilon)}$ with $\Omega_\varepsilon = \{x \in \Omega_- : |x| \le \varepsilon\}$. Our choice of the function $f(x) = \varphi_{\text{in}}(x) + \varphi_{\text{ext}}\big(\frac{1}{2}x\big)$ ensures that, for $h \in (0, h_0)$ with h_0 small enough, $|\nabla f|^2 + h\Delta f$ is nonnegative on $\Omega_- \setminus \Omega_\varepsilon$. The second part of Lemma 7.11 thus implies that there exists a function ν of h such that

$$\langle \chi_2\omega, \Delta^{D,(0)}_{-f,h}(\Omega)\chi_2\omega\rangle_{L^2(\Omega_-\setminus\Omega_\varepsilon)} \ge \nu(h)\|\chi_2\omega\|^2_{L^2(\Omega_-\setminus\Omega_\varepsilon)}$$

with $\liminf_{h\to 0} h \log \nu(h) = 0$. In addition, exponential decay estimates based on the Agmon identity imply that $\sum_{j=1}^{2} \|\omega\nabla\chi_j\|^2_{L^2(\Omega_-)} = \mathcal{O}(e^{-c/h})$, since $|\nabla f| > 0$ on $\operatorname{supp}(\chi_1) \cup \operatorname{supp}(\chi_2)$ (this is obtained by adapting the arguments of Proposition 3.4, for example). By using the IMS localization formula (7-22), we thus obtain that $\|\chi_2\omega\|_{L^2(\Omega\setminus\Omega_\varepsilon)}$ goes to zero when h goes to zero, and thus that $\lim_{h\to 0}\|\chi_1\omega\|_{L^2(\Omega_-)} = \lim_{h\to 0}\|\omega\|_{L^2(\Omega_\varepsilon)} = 1$. Using then the same argument as in the end of the proof of the first part of Lemma 7.11, we obtain that, for sufficiently small ε, $\lim_{h\to 0}\|\omega - C_h e^{f/h}\|_{L^2(\Omega_\varepsilon)} = 0$ for some constant $C_h \in \mathbb{R}$. The two limits $\lim_{h\to 0}\|\omega\|_{L^2(\Omega_\varepsilon)} = 1$ and $\lim_{h\to 0}\|\omega - C_h e^{f/h}\|_{L^2(\Omega_\varepsilon)} = 0$ imply that, in the asymptotic $h \to 0$, ω cannot be orthogonal to $\chi e^{f/h}$ (recall that $\chi \equiv 1$ on Ω_ε), which is in the spectral subspace associated with exponentially small eigenvalues. This concludes the proof. □

It is not difficult to adapt the previous argument to the case when the function f has several local maxima. Set $(x_0, r_0) = (0, 1)$ and consider a finite number of points and radii $(x_k, r_k)_{1\le k\le N}$ such that the open discs $D(x_k, r_k)$, $k = 0, \dots, N$, are all disjoint and included in $D(0, 2)$. Let us consider the function $f(x) = \varphi_{\text{ext}}\big(\frac{1}{2}x\big) + \sum_{k=0}^{N} \varphi_{\text{in}}((x - x_k)/r_k)$ (see Figure 6). Then Hypotheses 1, 2, 3 and 4 hold with $m_0^N(\Omega_-) = 1$, $m_1^N(\Omega_-) = N + 1$ and $m_1^D(\Omega_+ \setminus \overline{\Omega}_-) = 1$.

Remark 7.14. Interestingly, one can extend the last example to build a function f for which Hypothesis 3 is *not* satisfied. Consider an infinite sequence $(x_k, r_k)_{k\in\mathbb{N}}$ with $x_0 = 0$ and $r_0 = 1$ such that the open discs $D(x_k, r_k)$, $k \ge 0$, are all disjoint and included in $D(0, 2)$. Take the function $f(x) =$

$\varphi_{\text{ext}}\big(\frac{1}{2}x\big) + \sum_{k=0}^{\infty}(r_k^k/(1+k^2))\varphi_{\text{in}}((x-x_k)/r_k)$ in the domain $\Omega_- = D((-R,0), 2R-1)$ with $R > 3$ large enough. By Lemma 7.11, we know $m_0^N(\Omega_-) = 1$, while quasimodes associated with every x_k show that the number of eigenvalues of $\Delta_{f,h}^{N,(2)}(\Omega_-)$ (or equivalently $\Delta_{-f,h}^{D,(0)}(\Omega_-)$) lying in $[0, e^{-\delta/h}]$ is larger than any fixed $n \in \mathbb{N}$ for h sufficiently small. Using, as in the proof of Proposition 7.13, the identity $m_2^N(\Omega_-) - m_1^N(\Omega_-) + m_0^N(\Omega_-) = 1$, the number of eigenvalues of $\Delta_{f,h}^{N,(1)}(\Omega_-)$ lying in $[0, e^{-\delta/h}]$ is thus also larger than any $n \in \mathbb{N}$ for h sufficiently small. Thus Hypothesis 3 is not satisfied.

Actually, there are up to now no satisfactory necessary and sufficient conditions which guarantee that Witten Laplacians with general $\mathcal{C}^\infty$ potentials have a finite number of exponentially small eigenvalues.

Appendix: Riemannian geometry formulas

For the sake of completeness and in order to help the reader not so familiar with those tools, here is a list of formulas of Riemannian geometry which were used in this text. We refer the reader, for example, to [Abraham and Marsden 1978; Cycon et al. 1987; Gallot et al. 2004; Sternberg 1964; Goldberg 1970] for introductory texts in differential and Riemannian geometry. We also consider here only real-valued differential forms (the extension to complex-valued differential forms is easy).

Let (M, g) be a d-dimensional Riemannian manifold. The tangent (resp. cotangent) bundle is denoted by TM (resp. T^*M) and its fiber over $x \in M$ by T_xM (resp. T_x^*M). The exterior algebra over T_x^*M is $\bigwedge T_x^*M = \bigoplus_{p=0}^d \bigwedge^p T_x^*M$ endowed with the exterior product $\wedge$, and the associated fiber bundle is denoted by $\bigwedge T^*M = \bigoplus \bigwedge^p T^*M$. The exterior product of p elements $(\varphi_i)_{1\le i\le p}$ of T_x^*M is defined by

$$\varphi_1 \wedge \cdots \wedge \varphi_p = \sum_{\sigma \in \mathfrak{S}_{\{1,\dots,p\}}} \epsilon_{\{1,\dots,p\}}(\sigma)\varphi_{\sigma(1)} \otimes \cdots \otimes \varphi_{\sigma(p)},$$

where $\epsilon_E(\pi)$ is the signature of the permutation $\pi \in \mathfrak{S}_E$. Differential forms are sections of this fiber bundle and their regularity is encoded by the notation: $\bigwedge \mathcal{C}^\infty(M)$ is the set of $\mathcal{C}^\infty$-differential forms, $\bigwedge L^2(M)$ is the set of L^2-differential forms, and so on. This notation was used in the present text for the sake of conciseness. A more standard and general notation would be $\mathcal{C}^\infty(M; \bigwedge T^*M)$, where $\mathcal{C}^\infty(M; E)$ more generally stands for the set of $\mathcal{C}^\infty$ sections of the differential fiber bundle (E, Π) on M with $\Pi : E \to M$ (a section $x \mapsto s(x)$ satisfies $\Pi(s(x)) = x$).

In a local coordinate system $(x^1, \dots, x^d)$, a basis of $\bigwedge^p T_x^*M$ is formed by the elements

$$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_p}, \qquad I = \{i_1, \dots, i_p\}, \qquad i_1 < \cdots < i_p.$$

Here and in the following, $I = \{i_1, \dots, i_p\}$ denotes a subset of $\{1, \dots, d\}$ with $\#I = p$ elements, which can be described equivalently as an ordered p-tuple $(i_1, \dots, i_p)$ with $i_1 < \cdots < i_p$.

A differential form $\omega \in \bigwedge^p T^*M$ is written

$$\omega = \sum_{\#I=p} \omega_I(x)\, dx^I,$$

and its differential is given by

$$d\omega = \sum_{\#I=p} \partial_{x^i}\omega_I(x)\, dx^i \wedge dx^I.$$

Remember that the exterior product is bilinear associative and antisymmetric:

$$\omega_1 \wedge \omega_2 = (-1)^{p_1 p_2} \omega_2 \wedge \omega_1, \quad \omega_i \in \bigwedge^{p_i} T_x^* M.$$

The differential and the $\wedge$ product satisfy $d \circ d = 0$ and

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{p_1} \omega_1 \wedge d\omega_2, \quad \omega_i \in \bigwedge^{p_i} \mathcal{C}^\infty(M).$$

A $\mathcal{C}^\infty$ vector field X on M is a $\mathcal{C}^\infty$ section of TM, that is, $X \in \mathcal{C}^\infty(M; TM)$. The interior product $\boldsymbol{i}_X$ is the local operation defined for $X_x \in T_x M$ and $\omega_x \in \bigwedge^p T_x^* M$ by

$$\boldsymbol{i}_{X_x} \omega_x(T_2, \dots, T_p) = \omega_x(X_x, T_2, \dots, T_p) \quad \text{for all } T_2, \dots, T_p \in T_x M. \tag{A-1}$$

For $X \in \mathcal{C}^\infty(M; TM)$ and $\omega_i \in \bigwedge^{p_i} \mathcal{C}^\infty(M)$, one has

$$\boldsymbol{i}_X(\omega_1 \wedge \omega_2) = (\boldsymbol{i}_X \omega_1) \wedge \omega_2 + (-1)^{p_1} \omega_1 \wedge (\boldsymbol{i}_X \omega_2).$$

When $\Phi : M \to N$ is a $\mathcal{C}^\infty$ map, Φ_* denotes the functorial push-forward and Φ^* the functorial pull-back. For a $\mathcal{C}^\infty$ map Φ and two forms ω_1, ω_2, one has

$$\Phi^*(d\omega_1) = d(\Phi^* \omega_1), \quad \Phi^*(\omega_1 \wedge \omega_2) = (\Phi^* \omega_1) \wedge (\Phi^* \omega_2).$$

When Φ is a diffeomorphism, ω a p-form and X a vector field,

$$\Phi^* \boldsymbol{i}_X \omega = \boldsymbol{i}_{\Phi^* X} \Phi^* \omega.$$

When Φ is a diffeomorphism given by the exponential map of a vector field X, we can define the Lie derivative

$$\mathcal{L}_X \omega = \frac{d}{dt} (e^{tX})^* \omega \Big|_{t=0} \quad \text{for } \omega \in \bigwedge \mathcal{C}^\infty(M). \tag{A-2}$$

The Lie derivative satisfies

$$\mathcal{L}_X(\omega_1 \wedge \omega_2) = (\mathcal{L}_X \omega_1) \wedge \omega_2 + \omega_1 \wedge (\mathcal{L}_X \omega_2),$$

and Cartan's magic formula says

$$\mathcal{L}_X = \boldsymbol{i}_X \circ d + d \circ \boldsymbol{i}_X.$$

Differential forms $d\omega$ with degree $p+1$ can be integrated along a $(p+1)$-chain, or more specifically a $(p+1)$-dimensional submanifold with boundary; let us write it as C with boundary ∂C. Stokes' formula is written

$$\int_C d\omega = \int_{\partial C} \omega,$$

and it is the ground for de Rham's cohomology.

The Riemannian structure adds the pointwise dependent scalar product $g(x)$ given by

$$\langle S, T \rangle_{T_x M} = \sum_{1 \le i, j \le d} g_{i,j}(x) S^i T^j$$

with a dual metric $(g^{i,j}(x))_{1\le i,j\le d} := g(x)^{-1}$ defined on T_x^*M. This is also written with Einstein's conventions as

$$g = g_{i,j}\, dx^i\, dx^j, \quad g_{i,j} g^{j,k} = \delta_i^k.$$

Both $g(x)$ and $g(x)^{-1}$ are extended by tensor product to $\bigwedge T_x M$ and $\bigwedge T_x^* M$: for $\omega, \omega' \in \bigwedge^p \mathcal{C}^\infty(M)$,

$$\langle \omega', \omega\rangle_{\bigwedge^p T_x^*M} = \sum_{\#I=p}\sum_{\#J=p}\left(\prod_{k=1}^{p} g^{i_k,j_k}\right)\omega'_I\omega_J,$$

where $I=\{i_1,\dots,i_p\}$ $(i_1<\cdots<i_p)$ and $J=\{j_1,\dots,j_p\}$ $(j_1<\cdots<j_p)$. The Riemannian infinitesimal volume (denoted simply by dx in the text) is in an oriented local coordinate system:

$$d\,\mathrm{Vol}_g(x) = (\det g)^{1/2}\, dx^1\wedge\cdots\wedge dx^d = (\det g)^{1/2}\, dx^1\cdots dx^d.$$

Those scalar products as nondegenerate bilinear forms allow identifications between forms and vectors. Here are examples: when $\omega = \omega_i(x)dx^i$ is a one-form, the vector $\omega^\#$ is given by $(\omega^\#)^i = g^{i,j}\omega_j$; when $X = X^i\partial_{x^i}$ is a vector field, $X^\flat$ is the one-form defined by $(X^\flat)_i = g_{i,j}X^j$. As an application, the gradient for a function is nothing but $\nabla f = (df)^\#$. Similarly, the Hessian of a function f at a point x, initially defined as a bilinear form, can be viewed a linear map of T_xM.

Another duality between forms of complementary degrees $p+p' = d = \dim M$ is provided by the Hodge $\star$ operator. When the Riemannian manifold (M,g) is orientable (locally this is always the case), the operator $\star : \bigwedge^p \mathcal{C}^\infty(M) \to \bigwedge^{d-p}\mathcal{C}^\infty(M)$ is defined by

$$\int \langle\omega',\omega\rangle_{\bigwedge^p T_x^*M}\, d\,\mathrm{Vol}_g(x) = \int \omega'\wedge(\star\omega), \quad \omega,\ \omega'\in\bigwedge^p\mathcal{C}^\infty(M).$$

In a coordinate system it is given by

$$(\star\omega)_J = \sum_I \delta_{I\cup J}^{\{1,\dots,d\}}\epsilon_{\{1,\dots,d\}}(I,J)(\det g)^{1/2}(\omega^\#)^I, \quad \begin{cases} I=\{i_1,\dots,i_p\},\ i_1<\cdots<i_p,\\ J=\{j_1,\dots,j_{d-p}\},\ j_1<\cdots<j_{d-p},\\ (I,J)=(i_1,\dots,i_p,\ j_1,\dots,j_{d-p}),\end{cases}$$

where $\delta_A^B = 1$ when $A=B$ and $\delta_A^B=0$ otherwise. We have the additional properties, for $\omega,\omega'\in\bigwedge^p\mathcal{C}^\infty(M)$,

$$\begin{aligned} \star(\lambda\omega+\omega') &= \lambda\star\omega + \star\omega', \quad \lambda\in\mathcal{C}^\infty(M),\\ \star\star\,\omega &= (-1)^{p(d+1)}\omega,\\ \omega\wedge(\star\omega') &= \omega'\wedge(\star\omega),\\ \star 1 &= d\,\mathrm{Vol}_g(x) \quad \text{(assuming } M \text{ is oriented).}\end{aligned}$$

The codifferential d^* is defined as the formal adjoint of the differential $d : \bigwedge\mathcal{C}^\infty(M)\to\bigwedge\mathcal{C}^\infty(M)$,

$$\langle d\omega,\omega'\rangle = \langle\omega, d^*\omega'\rangle.$$

With the Hodge $\star$ operator (do the identification on a compact oriented manifold without boundary with $\int_M d\eta = 0$),

$$\begin{cases} \star d^*\omega = (-1)^p d\star\omega, \\ \star d\omega = (-1)^{p+1} d^*\star\omega, \\ d^*\omega = (-1)^{pd+d+1}\star d\star\omega \end{cases} \qquad \text{for all } \omega \in \bigwedge^p \mathcal{C}^\infty(M).$$

The Hodge Laplacian is then given by

$$\Delta_{\mathrm{H}} = (d+d^*)^2 = dd^* + d^*d. \tag{A-3}$$

It is possible to write d^* and Δ_{H} in a coordinate system. For example,

$$(d^*\omega)_I = -g^{i,j}\delta^J_{i\cup I}\epsilon_J(i,I)\nabla_j\omega_J, \quad (i,I) = (i,i_1,\dots,i_{p-1}),$$
$$\nabla_j\omega_J = \partial_{x^j}\omega_J - \sum_{\ell=1}^p \omega_{I\cup\{k\}\setminus i_\ell}\epsilon_{I\cup\{k\}\setminus i_\ell}(i_1,\dots,i_{\ell-1},k,i_{\ell+1},\dots i_p)\Gamma^k_{i_\ell\, j},$$
$$\Gamma^k_{i_\ell,j} = \tfrac{1}{2}g^{k,m}(\partial_{x^{i_\ell}}g_{j,m} + \partial_{x^j}g_{m,i_\ell} - \partial_{x^m}g_{i_\ell,j}),$$

where one recognizes the covariant derivative ∇_j associated with the metric g (the Levi–Civita connection) and the Christoffel symbols $\Gamma^j_{k\,\ell}$. The writing of Δ_{H} involves the Riemann curvature tensor and is known as Weitzenbock's formula. We wrote the above example to convince the unfamiliar reader that the explicit writing in a coordinate system is not always more informative than the intrinsic formula.

Here is the example of the Witten Laplacian, $\Delta_{f,h} = (d_{f,h}+d^*_{f,h})^2 = d^*_{f,h}d_{f,h} + d_{f,h}d^*_{f,h}$:

$$d_{f,h} = e^{-f/h}(hd)e^{f/h} = hd + df\wedge, \tag{A-4}$$
$$d^*_{f,h} = e^{f/h}(hd^*)e^{-f/h} = hd^* + \boldsymbol{i}_{\nabla f}, \tag{A-5}$$
$$\begin{aligned}\Delta_{f,h} &= d_{f,h}d^*_{f,h} + d^*_{f,h}d_{f,h} = (hd+df\wedge)(hd^*+\boldsymbol{i}_{\nabla f}) + (hd^*+\boldsymbol{i}_{\nabla f})(hd+df\wedge)\\ &= h^2(dd^*+d^*d) + [(df\wedge)\circ\boldsymbol{i}_{\nabla f} + \boldsymbol{i}_{\nabla f}\circ(df\wedge)] + h[d\boldsymbol{i}_{\nabla f} + \boldsymbol{i}_{\nabla f}d] + h[(df\wedge)\circ d^* + d^*\circ(df\wedge)]\\ &= h^2\Delta_{\mathrm{H}} + |\nabla f|^2 + h(\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}),\end{aligned} \tag{A-6}$$

where we used $\boldsymbol{i}_X(df\wedge\omega) = df(X)\omega - df\wedge(\boldsymbol{i}_X\omega)$ with $X = \nabla f$, Cartan's magic formula and an easy identification of $\mathcal{L}^*_{\nabla f}$. No explicit computation of d^* or the Hodge Laplacian is necessary to understand the structure of the Witten Laplacian. In particular, $\mathcal{L}_X + \mathcal{L}^*_X$ is clearly a zeroth-order differential operator because in a coordinate system the formal adjoint of $a^j(x)\partial_{x^j}$ in $L^2(\mathbb{R}^d, \varrho(x)\,dx)$ equals $-a^j(x)\partial_{x^j} + b[a,\varrho](x)$, where $b[a,\varrho]$ is the multiplication by a function of x. The operator $\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}$ is not the local action of a tensor field on M because it does not follow the change of coordinates rule for tensors. Actually, one can give a meaning to the general expression

$$\Delta^{(p)}_{f,h} = h^2\Delta^{(p)}_{\mathrm{H}} + |\nabla f|^2 - h(\Delta f) + 2h(\mathrm{Hess}\, f)_p,$$

where $(\mathrm{Hess}\, f)_p$ is an element of the curvature tensor algebra (see [Jammes 2012] and references therein).

Let us conclude this appendix with integration by parts formulas in the case of a manifold with a boundary. All these formulas rely first on Stokes' formula $\int_\Omega d\omega = \int_{\partial\Omega}\omega$ when $\omega \in \bigwedge^{d-1}\mathcal{C}^\infty(\overline{\Omega})$.

Note that the right-hand side of Stokes' formula may equivalently (and more explicitly) be written $\int_{\partial\Omega}\omega=\int_{\partial\Omega}j^*\omega$, where $j:\partial\Omega\to\bar{\Omega}$ is the natural embedding map (a trace along $\partial\Omega$ is taken and, pointwise, $j^*\omega_x$ is evaluated only on $(d-1)$-vectors tangent to $\partial\Omega$). Another expression taken initially from [Schwarz 1995] is also convenient. For $\sigma\in\partial\Omega$ let $n(\sigma)$ be the outward normal vector and write, for any element $X\in T_\sigma M$, $X=X_T+X_n n$.

For $\omega\in\bigwedge^p\mathcal{C}^\infty(\bar{\Omega})$, define $\boldsymbol{t}\omega$ and $\boldsymbol{n}\omega=\omega-\boldsymbol{t}\omega$ by

$$\forall\sigma\in\partial\Omega\ \ \forall X_1,\dots,X_p\in T_\sigma\Omega\quad \boldsymbol{t}\omega(X_1,\dots,X_p)=\omega(X_{1,T},\dots,X_{p,T}).$$

If $(x^1,\dots,x^d)=(x',x^d)$ is a coordinate system in a neighborhood $\mathcal{V}$ of $\sigma_0\in\partial\Omega$ such that $\Omega\cap\mathcal{V}$ is given locally by $\{x^d<0\}$, $\partial\Omega\cap\mathcal{V}$ by $\{x^d=0\}$ and $n=\partial_{x^d}$, then a p-form can be written

$$\omega=\sum_{\substack{\#I=p\\ d\notin I}}\omega_I\,dx^I+\sum_{\substack{\#I'=p-1\\ d\notin I'}}\omega_{I'}\,dx^{I'}\wedge dx^d,$$

and the operators $\boldsymbol{t}$ and $\boldsymbol{n}$ act as

$$\boldsymbol{t}\omega=\sum_{\substack{\#I=p\\ d\notin I}}\omega_I\,dx^I,\qquad \boldsymbol{n}\omega=\sum_{\substack{\#I'=p-1\\ d\notin I'}}\omega_{I'}\,dx^{I'}\wedge dx^d.$$

Stokes' formula can be written now as $\int_\Omega d\omega=\int_{\partial\Omega}\boldsymbol{t}\omega$ for $\omega\in\bigwedge^{d-1}\mathcal{C}^\infty(\bar{\Omega})$, but contrary to the operator j^* the operator $\boldsymbol{t}$ makes sense in a collar neighborhood of $\partial\Omega$; locally $\boldsymbol{t}\omega_{(x',x^d)}=\boldsymbol{t}\omega_{(x',0)}$ by definition. In particular, the formula

$$\boldsymbol{t}\,d\omega=d\boldsymbol{t}\omega$$

makes sense for any $\omega\in\bigwedge\mathcal{C}^\infty(\bar{\Omega})$ and it is rather easy to check with the above coordinates description. One also gets, in the same way,

$$\boldsymbol{t}\omega=\boldsymbol{i}_n(n^\flat\wedge\omega)\quad\text{for }\omega\in\bigwedge\mathcal{C}^\infty(\bar{\Omega}),\tag{A-7}$$

$$\star\boldsymbol{n}=\boldsymbol{t}\star,\qquad \star\boldsymbol{t}=\boldsymbol{n}\star,\tag{A-8}$$

$$\boldsymbol{t}d=d\boldsymbol{t},\qquad \boldsymbol{n}d^*=d^*\boldsymbol{n},\tag{A-9}$$

$$\boldsymbol{t}\omega_1\wedge\star\boldsymbol{n}\omega_2=\langle\omega_1,\boldsymbol{i}_n\omega_2\rangle_{\bigwedge^p T^*_\sigma\Omega}\times d\,\mathrm{Vol}_{g,\partial\Omega}\quad\text{for }\omega_i\in\bigwedge^p\mathcal{C}^\infty(\bar{\Omega}),\tag{A-10}$$

where we recall that $d\,\mathrm{Vol}_{g,\partial\Omega}(X_1,\dots,X_{d-1})=d\,\mathrm{Vol}_g(n,X_1,\dots,X_{d-1})$.

The above formulas, for example lead to the following integration by parts for $\omega_1,\omega_2\in\bigwedge^p\mathcal{C}^\infty(\bar{\Omega})$:

$$\langle d_{f,h}\omega_1,d_{f,h}\omega_2\rangle_{L^2(\Omega)}+\langle d^*_{f,h}\omega_1,d^*_{f,h}\omega_2\rangle_{L^2(\Omega)}\\ =\langle\omega_1,\Delta_{f,h}\omega_2\rangle_{L^2(\Omega)}+h\int_{\partial\Omega}(\boldsymbol{t}\omega_2)\wedge\star\boldsymbol{n}d_{f,h}\omega_1-h\int_{\partial\Omega}(\boldsymbol{t}d^*_{f,h}\omega_1)\wedge(\star\boldsymbol{n}\omega_2).$$

This shows, for example, that $\Delta^D_{f,h}$ (resp. $\Delta^N_{f,h}$) with its form domain $W^{1,2}_D=\{\omega\in\bigwedge W^{1,2}:\boldsymbol{t}\omega=0\}$ (resp. $W^{1,2}_N=\{\omega\in\bigwedge W^{1,2}:\boldsymbol{n}\omega=0\}$) is associated with the Dirichlet form $\|d_{f,h}\omega\|^2+\|d^*_{f,h}\omega\|^2$. Interpreting the weak formulation of $\Delta_{f,h}\omega=f$ leads to the operator domains $D(\Delta^D_{f,h})$ and $D(\Delta^N_{f,h})$ (we refer the reader to [Helffer and Nier 2006] for details). The boundary terms of Lemma 3.1 are obtained in a very similar way.

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