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PARTIAL COLLAPSING AND THE SPECTRUM OF THE HODGE–DE RHAM OPERATOR

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Our goal is to calculate the limit spectrum of the Hodge–de Rham operator under the perturbation of collapsing one part of a manifold obtained by gluing together two manifolds with the same boundary. It appears to take place in the general problem of blowing up conical singularities as introduced by Mazzeo and Rowlett.

1. Introduction

This work takes place in the context of the spectral studies of singular perturbations of the metrics, as a means to know what are the topological or metrical meanings carried by the spectrum of geometric operators. We can mention in this direction, without exhaustivity, studies on the adiabatic limits [Mazzeo and Melrose 1990; Rumin 2000], on collapsing [Fukaya 1987; Lott 2002a; 2002b; 2004], on resolution blowups of conical singularities [Mazzeo 2006; Rowlett 2006; 2008] and on shrinking handles [Anné and Colbois 1995; Anné et al. 2009].

The present study can be considered as a generalization of the results of [Anné and Takahashi 2012], where we studied the limit of the spectrum of the Hodge–de Rham (or the Hodge–Laplace) operator under collapsing of one part of a connected sum.

In our previous work, we restricted the submanifold Σ used to glue the two parts to be a sphere. In fact, this problem is quite related to resolution blowups of conical singularities: the point is to measure the influence of the topology of the part which disappears and of the conical singularity created at the limit of the "big part". If we look at the situation from the "small part", we understand the importance of the *quasiasymptotically conical space* obtained from rescaling the small part and gluing an infinite cone; see the definition below in (1).

When Σ is the sphere $\mathbb{S}^n$, the conical singularity is quite simple. There are no *half-bound states* — called extended solutions in the sequel — on the quasiasymptotically conical space. Our result presented here takes care of these new possibilities and gives a general answer to the problem studied by Mazzeo and Rowlett. Indeed, in [Mazzeo 2006; Rowlett 2006; 2008], it is supposed that the spectrum of the operator on the quasiasymptotically conical space does not meet 0. Our study relaxes this hypothesis. It is done only with the Hodge–de Rham operator, but can easily be generalized.

Let us fix some notations.

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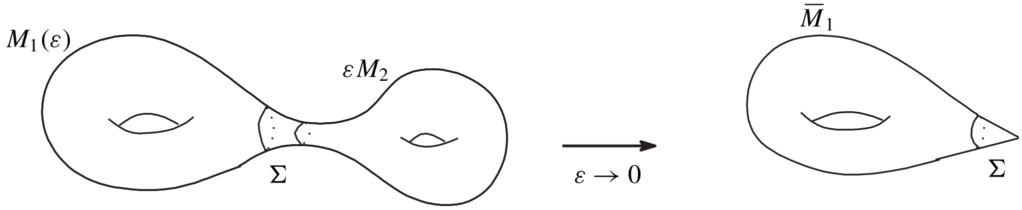

Figure 1. Partial collapsing of M_ε.

1.1. ***Set-up.*** Let M_1 and M_2 be two connected, oriented, compact manifolds with the same boundary Σ, a compact manifold of dimension $n \geq 2$. We denote by $m = n + 1$ the dimension of M_1 and M_2. We endow Σ with a fixed metric h.

Let $\overline{M}_1$ be the manifold with conical singularity obtained from M_1 by gluing M_1 to a cone $\mathscr{C} = [0, 1) \times \Sigma$; we write (r, y) for points on $\mathscr{C}$, and there exists on $\overline{M}_1 = M_1 \cup \mathscr{C}$ a metric $\bar{g}_1$ which equals $dr^2 + r^2 h$ on the smooth part $r > 0$ of the cone.

We choose on M_2 a metric g_2 which is "trumpet-like", i.e., M_2 is isometric near the boundary to $\left[0, \frac{1}{2}\right) \times \Sigma$ with the conical metric which equals $ds^2 + (1-s)^2 h$ if s is the coordinate defining the boundary by $s = 0$.

For any ε with $0 \leq \varepsilon < 1$, we define

$$\mathscr{C}_{\varepsilon,1} = \{(r, y) \in \mathscr{C} \mid r > \varepsilon\} \quad \text{and} \quad M_1(\varepsilon) = M_1 \cup \mathscr{C}_{\varepsilon,1}.$$

The goal of the following calculus is to determine the limit spectrum of the Hodge–de Rham operator acting on the differential forms of the Riemannian manifold

$$M_\varepsilon = M_1(\varepsilon) \cup_{\varepsilon.\Sigma} \varepsilon.M_2,$$

which is obtained by gluing together $(M_1(\varepsilon), g_1)$ and $(M_2, \varepsilon^2 g_2)$. By construction, these two manifolds have isometric boundary and the metric g_ε obtained on M_ε is smooth.

Remark 1. The common boundary Σ of dimension n has some topological obstructions. In fact, since Σ is the boundary of the oriented, compact manifold M_1, Σ is oriented cobordant to zero. So, by Thom's cobordism theory, all the Stiefel–Whitney and all the Pontrjagin numbers vanish (see C. T. C. Wall [1960] or [Milnor and Stasheff 1974, §18, p. 217]). Furthermore, this condition is also sufficient; that is, the inverse does hold.

In particular, it is impossible to take Σ^{4k} as the complex projective spaces $\mathbb{CP}^{2k}$ $(k \geq 1)$ because the Pontrjagin number $p_k(\mathbb{CP}^{2k})$ is nonzero.

1.2. ***Results.*** We can describe the limit spectrum as follows; it has two parts. One part comes from the big part, namely $\overline{M}_1$, and is expressed by the spectrum of a good extension of the Hodge–de Rham operator on this manifold with the conical singularity. This extension is self-adjoint and comes from an extension of the Gauss–Bonnet operators. All these extensions are classified by subspaces W of the total eigenspaces corresponding to the eigenvalues within $\left(-\frac{1}{2}, \frac{1}{2}\right)$ of an operator A acting on the boundary Σ. This point is developed in Section 2.2. The other part comes from the collapsing part, namely M_2, where the limit Gauss–Bonnet operator is taken with boundary conditions of Atiyah–Patodi–Singer-type. This point is developed in Section 2.3. This operator, denoted $\mathfrak{D}_2$ in the sequel, can also be seen on the quasiasymptotically conical space $\widetilde{M}_2$ already mentioned, namely

$$\widetilde{M}_2 = M_2 \cup ([1, \infty) \times \Sigma) \tag{1}$$

with the metric $dr^2 + r^2 h$ on the conical part. Only the zero eigenvalue is concerned with this part. In fact, the manifold M_ε has small eigenvalues, in contrast to [Anné and Takahashi 2012], and the multiplicity of 0 at the limit corresponds to the total eigenspaces of these small and null eigenvalues. Thus, our main theorem, which asserts the convergence of the spectrum, has two components.

Theorem A. *The set of all positive limit values is just equal to that of all positive spectrum of the Hodge–de Rham operator* $\Delta_{1,W}$ *on* $\overline{M}_1$, *where*

$$W \subset \bigoplus_{|\gamma|<\frac{1}{2}} \operatorname{Ker}(A - \gamma)$$

is the space of the elements that generate extended solutions on $\widetilde{M}_2$. *A precise definition is given in* (7).

Theorem B. *The multiplicity of* 0 *in the limit spectrum is given by the sum*

$$\dim \operatorname{Ker}(\Delta_{1,W}) + \dim \operatorname{Ker}(\mathfrak{D}_2) + i_{1/2},$$

where $i_{1/2}$ *denotes the dimension of the vector space* $\mathcal{I}_{1/2}$ *— see* (8) *— of extended solutions* ω *on* $\widetilde{M}_2$ *introduced by Carron* [2001b], *admitting on restriction to* $r = 1$ *a nontrivial component in* $\operatorname{Ker}\bigl(A - \frac{1}{2}\bigr)$.

1.3. *Comments.*

1.3.1. This result is also valid in dimension 2. In order to understand it, look at the following example. Let $I = [0, 1]$ and $M_1 = M_2 = \mathbb{S}^1 \times I$. We can shrink half of a torus: $\mathbb{S}^1 \times \mathbb{S}^1 = M_1 \cup_\Sigma M_1$ for $\Sigma = \mathbb{S}^1 \sqcup \mathbb{S}^1$. Then $\overline{M}_1$ is a 2-sphere with no harmonic 1-forms and $\widetilde{M}_2$ has no L^2-harmonic 1-forms. But $i_{1/2} = 2$. Indeed $\widetilde{M}_2$ is a cylinder with flat ends. With obvious coordinates (r, θ), $d\theta$ and $*(d\theta) \sim dr/r$ near ∞ give a base for extended solutions.

1.3.2. We choose, in our study, a simple metric to make explicit computations. This fact is not a restriction, as already explained in [Anné and Takahashi 2012], because of the result of Dodziuk [1982] which assures uniform control of the eigenvalues of geometric operators with regard to variations of the metric.

1.3.3. More examples are given in the last section of the paper.

2. Gauss–Bonnet operator

On a Riemannian manifold, the Gauss–Bonnet operator is defined as the operator $D = d + d^*$ acting on differential forms. It is symmetric and can have some closed extensions on manifolds with boundary or with conical singularities. We review these extensions in the cases involved in our study.

2.1. ***Gauss–Bonnet operator on M_ε.*** We recall that, on M_ε, a Gauss–Bonnet operator D_ε, Sobolev spaces and also a Hodge–de Rham operator Δ_ε can be defined as a general construction on any manifold $X = X_1 \cup X_2$, which is the union of two Riemannian manifolds with isometric boundaries (the details are given in [Anné and Colbois 1995]): if D_1 and D_2 are the Gauss–Bonnet operators "$d + d^*$" acting on the differential forms of each part, the quadratic form

$$q(\phi) = \int_{X_1} |D_1(\phi\restriction_{X_1})|^2 \, d\mu_{X_1} + \int_{X_2} |D_2(\phi\restriction_{X_2})|^2 \, d\mu_{X_2} \tag{2}$$

is well-defined and closed on the domain

$$\mathrm{Dom}(q) = \{\phi = (\phi_1, \phi_2) \in H^1(\Lambda T^* X_1) \times H^1(\Lambda T^* X_2) \mid \phi_1\restriction_{\partial X_1} =_{L^2} \phi_2\restriction_{\partial X_2}\}.$$

On this space, the total Gauss–Bonnet operator $D(\phi) = (D_1(\phi_1), D_2(\phi_2))$ is defined and self-adjoint. For this definition, we have to, in particular, identify $(\Lambda T^* X_1)\restriction_{\partial X_1}$ and $(\Lambda T^* X_2)\restriction_{\partial X_2}$. This can be done by decomposing the forms into tangential and normal parts (with inner normal); the equality above means then that the tangential parts are equal and the normal parts opposite. This definition generalizes the definition in the smooth case.

The Hodge–de Rham operator $(d+d^*)^2$ of X is then defined as the operator obtained by the polarization of the quadratic form q. This gives compatibility conditions between ϕ_1 and ϕ_2 on the common boundary. We do not give details on these facts, because our manifold is smooth. But we shall use this presentation for the quadratic form.

2.2. ***Gauss–Bonnet operator on $\overline{M}_1$.*** Let $D_{1,\min}$ be the closure of the Gauss–Bonnet operator defined on the smooth forms with compact support in the smooth part $M_1(0)$. For any such form ϕ_1, following [Brüning and Seeley 1988; Anné et al. 2009], on the cone $\mathscr{C}$ we write

$$\phi_1 = dr \wedge r^{-(n/2-p+1)} \beta_{1,\varepsilon} + r^{-(n/2-p)} \alpha_{1,\varepsilon}$$

and define $\sigma_1 = (\beta_1, \alpha_1) = U(\phi_1)$. The operator has, on the cone $\mathscr{C}$, the expression

$$U D_1 U^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\partial_r + \frac{1}{r} A\right) \quad \text{with} \quad A = \begin{pmatrix} \frac{1}{2}n - P & -D_0 \\ -D_0 & P - \frac{1}{2}n \end{pmatrix},$$

where P is the operator of degree, that is, $P\omega = p \cdot \omega$ for a p-form ω, and $D_0 = d_0 + d_0^*$ is the Gauss–Bonnet operator on the manifold (Σ, h), while the Hodge–de Rham operator has, in these coordinates, the expression

$$U \Delta_1 U^* = -\partial_r^2 + \frac{1}{r^2} A(A+1). \tag{3}$$

The closed extensions of the operator $D_1 = d + d^*$ on the manifold with conical singularity $\overline{M}_1$ have been studied in [Brüning and Seeley 1988; Lesch 1997]. They are classified by the spectrum of its *Mellin symbol*, which is here the operator with parameter $A + z$.

Spectrum of A. The spectrum of A was calculated in [Brüning and Seeley 1988, p. 703]. By their result, the spectrum of A is given by the values

$$\begin{cases} \pm\bigl(p-\tfrac{1}{2}n\bigr) & \text{with multiplicity } \dim H^p(\Sigma), \\ (-1)^{p+1}\tfrac{1}{2} \pm \sqrt{\mu^2+\bigl(\tfrac{1}{2}(n-1)-p\bigr)^2}, & \end{cases} \tag{4}$$

where p is any integer, $0 \le p \le n$, and μ^2 runs over the spectrum of the Hodge–de Rham operator on (Σ, h) acting on the coexact p-forms.

Indeed, looking at the Gauss–Bonnet operator acting on even forms, they identify even forms on the cone with the sections $(\phi_0, \dots, \phi_n)$ of the total bundle $\Lambda T^*(\Sigma)$ by $\phi_0 + \phi_1 \wedge dr + \phi_2 + \phi_3 \wedge dr + \cdots$. These sections can also represent odd forms on the cone by $\phi_0 \wedge dr + \phi_1 + \phi_2 \wedge dr + \phi_3 + \cdots$. With these identifications, they have to study the spectrum of the following operator acting on sections of $\Lambda T^*(\Sigma)$:

$$S_0 = \begin{pmatrix} c_0 & d_0^* & 0 & \cdots & 0 \\ d_0 & c_1 & d_0^* & \ddots & \vdots \\ 0 & d_0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & c_{n-1} & d_0^* \\ 0 & \cdots & 0 & d_0 & c_n \end{pmatrix}$$

if $c_p = (-1)^{p+1}\bigl(p-\tfrac{1}{2}n\bigr)$. With the same identification, if we introduce the operator $\widetilde{S}_0$ having the same formula but on the diagonal the terms $\tilde{c}_p = (-1)^p\bigl(p-\tfrac{1}{2}n\bigr) = -c_p$, then the operator A can be written as

$$A = -(S_0 \oplus \widetilde{S}_0).$$

The expression of the spectrum of A is then a direct consequence of the computations of [Brüning and Seeley 1988].

Closed extensions of D_1. Let $D_{1,\max}$ be the maximal closed extension of D_1, with the domain

$$\operatorname{Dom}(D_{1,\max}) = \{\phi \in L^2(\overline{M}_1) \mid D_1\phi \in L^2(\overline{M}_1)\}.$$

If $\operatorname{Spec}(A) \cap \bigl(-\tfrac{1}{2}, \tfrac{1}{2}\bigr) = \varnothing$, then $D_{1,\max} = D_{1,\min}$. In particular, D_1 is essentially self-adjoint on the space of smooth forms with compact support away from the conical singularity.

Otherwise, the quotient $\operatorname{Dom}(D_{1,\max}) / \operatorname{Dom}(D_{1,\min})$ is isomorphic to

$$B := \bigoplus_{|\gamma|<\frac{1}{2}} \operatorname{Ker}(A-\gamma).$$

More precisely, by Lemma 3.2 of [Brüning and Seeley 1988], there exists a surjective linear map

$$\mathscr{L} : \operatorname{Dom}(D_{1,\max}) \to B$$

with $\operatorname{Ker}(\mathscr{L}) = \operatorname{Dom}(D_{1,\min})$. Furthermore, we have the estimate

$$\|u(r) - r^{-A}\mathscr{L}(\phi)\|^2_{L^2(\Sigma)} \le C(\phi)|r \log r|$$

for $\phi \in \operatorname{Dom}(D_{1,\max})$ and $u = U(\phi)$.

Now, for any subspace $W \subset B$, we can associate the operator $D_{1,W}$ with $\mathrm{Dom}(D_{1,W}) := \mathscr{L}^{-1}(W)$. As a result of [Brüning and Seeley 1988], all closed extensions of $D_{1,\min}$ are obtained by this way. Note that each $D_{1,W}$ defines a self-adjoint extension $\Delta_{1,W} = (D_{1,W})^* \circ D_{1,W}$ of the Hodge–de Rham operator, and, as a result, we have $(D_{1,W})^* = D_{1,\mathbb{I}(W^\perp)}$, where

$$\mathbb{I} = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}, \quad \text{so} \quad \mathbb{I}\begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}.$$

This extension is associated with the quadratic form $\phi \mapsto \|D\phi\|^2_{L^2}$ on the domain $\mathrm{Dom}(D_{1,W})$.

Finally, we recall the results of [Lesch 1997]. The operators $D_{1,W}$, and in particular $D_{1,\min}$ and $D_{1,\max}$, are elliptic and satisfy the singular estimate (SE) — see [Lesch 1997, p. 54] — so by Proposition 1.4.6 of [Lesch 1997] and the compactness of $\overline{M}_1$, they satisfy the *Rellich property*: the inclusion of $\mathrm{Dom}(D_{1,W})$ into $L^2(\overline{M}_1)$ is compact.

2.3. ***Gauss–Bonnet operator on M_2.*** We know, by the works of Carron [2001a; 2001b], following Atiyah, Patodi and Singer [Atiyah et al. 1975], that the operator D_2 admits a closed extension $\mathscr{D}_2$ with the domain defined by the global boundary condition

$$\Pi_{\le 1/2} \circ U = 0$$

if Π_I is the spectral projection of A relative to the interval I, and $\le \frac{1}{2}$ denotes the interval $\left(-\infty, \frac{1}{2}\right]$. Moreover, this extension is elliptic in the sense that the H^1-norm of elements of the domain is controlled by the norm of the graph. Indeed, this boundary condition is related to a problem on a complete unbounded manifold as follows:

Let $\widetilde{M}_2$ denote the large manifold obtained from M_2 by gluing a conical cylinder $\mathscr{C}_{1,\infty} = [1, \infty) \times \Sigma$ with metric $dr^2 + r^2 h$ and $\widetilde{D}_2$ its Gauss–Bonnet operator. A differential form on M_2 admits an L^2-harmonic extension on $\widetilde{M}_2$ precisely when the restriction on the boundary satisfies $\Pi_{\le 1/2} \circ U = 0$.

Indeed, from the harmonicity, these L^2-forms must satisfy $(\partial_r + (1/r)A)\sigma = 0$, or, if we decompose the form associated with the eigenspaces of A as $\sigma = \sum_{\gamma \in \mathrm{Spec}(A)} \sigma_\gamma$, then the equation imposes that for all $\gamma \in \mathrm{Spec}(A)$ there exists $\sigma^0_\gamma \in \mathrm{Ker}(A - \gamma)$ such that $\sigma_\gamma = r^{-\gamma}\sigma^0_\gamma$. This expression is in $L^2(\mathscr{C}_{1,\infty})$ if and only if $\gamma > \frac{1}{2}$ or $\sigma^0_\gamma = 0$.

It will be convenient to introduce the L^2-harmonic extension operator

$$\begin{aligned} P_2 : \Pi_{>1/2}(H^{1/2}(\Sigma)) &\to L^2(\Lambda T^* \mathscr{C}_{1,\infty}) \\ \sigma = \sum_{\substack{\gamma \in \mathrm{Spec}(A) \\ \gamma > \frac{1}{2}}} \sigma_\gamma &\mapsto P_2(\sigma) = U^* \Bigg(\sum_{\substack{\gamma \in \mathrm{Spec}(A) \\ \gamma > \frac{1}{2}}} r^{-\gamma} \sigma_\gamma \Bigg). \end{aligned}$$

This limit problem is of the category *nonparabolic at infinity* in the terminology of Carron — see particularly Theorem 2.2 of [Carron 2001b] and Proposition 5.1 of [Carron 2001a] — then, as a consequence of Theorem 0.4 of [Carron 2001b], we know that the kernel of $\mathscr{D}_2$ is of finite dimension and that the graph norm of the operator controls the H^1-norm (Theorem 2.1 of [Carron 2001b]).

Proposition 2. *There exists a constant $C > 0$ such that, for each differential form $\phi \in H^1(\Lambda T^*M_2)$ satisfying the boundary condition $\Pi_{\leq 1/2} \circ U(\phi) = 0$,*

$$\|\phi\|^2_{H^1(M_2)} \leq C\{\|\phi\|^2_{L^2(M_2)} + \|D_2\phi\|^2_{L^2(M_2)}\}.$$

As a consequence, the kernel of $\mathfrak{D}_2$, which is isomorphic to $\operatorname{Ker}(\widetilde{D}_2)$, is of finite dimension and can be mapped into the total space $\sum_p H^p(M_2)$ of the absolute cohomology.

A proof of this proposition can be obtained by the same way as Proposition 5 in [Anné and Takahashi 2012].

Extended solutions. Recall that for this type of operator, behind the L^2-solutions of $\widetilde{D}_2(\phi) = 0$ which correspond to the solutions of the elliptic operator of Proposition 2, Carron defined *extended* solutions which are included in the bigger space $\mathcal{W}$, defined as the closure of the space of smooth p-forms with compact support in $\widetilde{M}_2$ for the norm

$$\|\phi\|^2_{\mathcal{W}} := \|\phi\|^2_{L^2(M_2)} + \|D_2\phi\|^2_{L^2(\widetilde{M}_2)}.$$

A Hardy-type inequality describes the growth at infinity of an extended solution:

Lemma 3. *For a function $v \in C_0^\infty(e, \infty)$ and a real number λ, we have*

$$\left(\lambda + \tfrac{1}{2}\right)^2 \int_e^\infty \frac{v^2}{r^2}\,dr \leq \int_e^\infty \frac{1}{r^{2\lambda}} |\partial_r(r^\lambda v)|^2\,dr \quad \text{if } \lambda \neq -\tfrac{1}{2},$$
$$\frac{1}{4}\int_e^\infty \frac{v^2}{r^2|\log r|^2}\,dr \leq \int_e^\infty r|\partial_r(r^{-1/2}v)|^2\,dr \quad \text{if } \lambda = -\tfrac{1}{2}.$$

We remark now that, for a p-form ϕ with support in the infinite cone $\mathcal{C}_{e,\infty}$, we can write

$$\begin{aligned}\|D_2\phi\|^2_{L^2(\widetilde{M}_2)} &= \sum_{\lambda \in \operatorname{Spec}(A)} \int_e^\infty \left\| \left(\partial_r + \frac{\lambda}{r}\right)\sigma_\lambda \right\|^2_{L^2(\Sigma)} dr \\ &= \sum_{\lambda \in \operatorname{Spec}(A)} \int_e^\infty \frac{1}{r^{2\lambda}} \|\partial_r(r^\lambda \sigma_\lambda)\|^2_{L^2(\Sigma)}\,dr.\end{aligned}$$

Thus, as an application of Lemma 3, we see that a kernel of $\widetilde{D}_2$, which must be $\sigma_\lambda(r) = r^{-\lambda}\sigma_\lambda(1)$ on the infinite cone, satisfies the condition of growth at infinity of Lemma 3. For $\lambda > -\frac{1}{2}$ there is no restriction, since $r^{-2\lambda-2}$ is integrable near ∞ as well as for $\lambda = -\frac{1}{2}$: if $v = r^{1/2}v_0$ for large r then the integral $\int v^2/|r \log r|^2\,dr$ is convergent, so, if we require that $(1/r)\phi$ is in L^2 then, for any $\lambda < -\frac{1}{2}$,

$$\sigma_\lambda(1) = 0.$$

While the L^2-solutions correspond to the condition $\sigma_\lambda(1) = 0$ for any $\lambda \leq \frac{1}{2}$. As a consequence, the extended solutions which are not in L^2 correspond to boundary terms with components in the total eigenspaces related to the eigenvalues of A in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. In the case studied in [Anné and Takahashi 2012], there do not exist such eigenvalues and we had not to take care of extended solutions.

More precisely, we must introduce the Dirac–Neumann operator (see [Carron 2001a, paragraphe 2.a])

$$\begin{aligned} T : H^{k+1/2}(\Sigma) &\to H^{k-1/2}(\Sigma) \\ \sigma &\mapsto U \circ D_2(\mathscr{E}(\sigma))\restriction_{\Sigma}, \end{aligned} \tag{5}$$

where $\mathscr{E}(\sigma)$ is the solution of the Poisson problem

$$(D_2)^2(\mathscr{E}(\sigma)) = 0 \quad \text{on } M_2 \qquad \text{and} \qquad U \circ \mathscr{E}(\sigma)\restriction_{\Sigma} = \sigma \quad \text{on } \Sigma.$$

In the same way, one can define

$$\begin{aligned} T_{\mathscr{C}} : H^{k+1/2}(\Sigma) &\to H^{k-1/2}(\Sigma) \\ \sigma &\mapsto U \circ D_2(\widetilde{\mathscr{E}}(\sigma))\restriction_{\Sigma}, \end{aligned} \tag{6}$$

where $\widetilde{\mathscr{E}}(\sigma)$ is the solution of the Poisson problem

$$(D_2)^2(\widetilde{\mathscr{E}}(\sigma)) = 0 \quad \text{on } \mathscr{C}_{1,\infty} \qquad \text{and} \qquad U \circ \widetilde{\mathscr{E}}(\sigma)\restriction_{\Sigma} = \sigma \quad \text{on } \Sigma.$$

Then $\mathrm{Im}(T_{\mathscr{C}}) = \mathrm{Im}(\Pi_{>1/2})$ is a subspace of $\mathrm{Ker}(T_{\mathscr{C}}) = \mathrm{Im}(\Pi_{\geq -1/2})$. Carron [2001a] proved that this operator is continuous for $k \geq 0$. The L^2-solutions correspond to the boundary values in $\mathrm{Im}(T) \cap \mathrm{Im}(\Pi_{>1/2})$, while extended solutions correspond to the space $\mathrm{Ker}(T) \cap \mathrm{Im}(\Pi_{\geq -1/2})$. Carron also proved that, in the compact case, $\mathrm{Ker}(T) = \mathrm{Im}(T)$. We can now define the space W that appears in Theorem A:

$$W = \bigoplus_{|\gamma| < \frac{1}{2}} W_\gamma, \quad \text{where} \quad W_\gamma = \{\phi \in \mathrm{Ker}(A - \gamma) \mid \exists \eta \in \mathrm{Im}(\Pi_{>\gamma}) \;\; T(\phi + \eta) = 0\}. \tag{7}$$

Let us denote by

$$\mathscr{I}_{1/2} := (\mathrm{Ker}(T) \cap \mathrm{Im}(\Pi_{\geq 1/2}))/(\mathrm{Ker}(T) \cap \mathrm{Im}(\Pi_{>1/2})) \tag{8}$$

the space of extended solutions with nontrivial component on $\mathrm{Ker}\big(A - \frac{1}{2}\big)$.

Proof of Lemma 3. Let $v \in C_0^\infty(e, \infty)$; by integration by parts and the Cauchy–Schwarz inequality, we obtain, for $\lambda \neq -\frac{1}{2}$,

$$\begin{aligned} \int_e^\infty \frac{v^2}{r^2}\,dr &= \int_e^\infty \frac{1}{r^{2\lambda+2}} |r^\lambda v|^2\,dr = \int_e^\infty \partial_r \left\{ \frac{-1}{(2\lambda+1) r^{2\lambda+1}} \right\} |r^\lambda v|^2\,dr \\ &= \int_e^\infty \left\{ \frac{1}{(2\lambda+1) r^{2\lambda+1}} \right\} 2(r^\lambda v)\partial_r(r^\lambda v)\,dr = \int_e^\infty \frac{2}{(2\lambda+1)} \frac{v}{r} \cdot r^{-\lambda} \partial_r(r^\lambda v)\,dr \\ &\leq \frac{2}{|2\lambda+1|} \sqrt{\int_e^\infty \frac{v^2}{r^2}\,dr} \cdot \sqrt{\int_e^\infty |r^{-\lambda}\partial_r(r^\lambda v)|^2\,dr}, \end{aligned}$$

which gives directly the first result of Lemma 3.

The second one is obtained in the same way:

$$\begin{aligned}\int_e^\infty \frac{v^2}{r^2|\log r|^2}\,dr &= \int_e^\infty \left(\frac{v}{\sqrt{r}}\right)^2 \frac{1}{r|\log r|^2}\,dr = \int_e^\infty \left(\frac{v}{\sqrt{r}}\right)^2 \partial_r\left(\frac{-1}{\log r}\right)dr \\ &= \int_e^\infty \frac{2v}{\sqrt{r}}\partial_r\left(\frac{v}{\sqrt{r}}\right)\cdot\frac{1}{\log r}\,dr = \int_e^\infty \frac{2v}{r\log r}\cdot\sqrt{r}\,\partial_r\left(\frac{v}{\sqrt{r}}\right)dr \\ &\le 2\sqrt{\int_e^\infty \frac{v^2}{r^2|\log r|^2}\,dr}\cdot\sqrt{\int_e^\infty \left|\sqrt{r}\,\partial_r\left(\frac{v}{\sqrt{r}}\right)\right|^2 dr}. \qquad \square\end{aligned}$$

3. Notations and tools

Let q_ε be the quadratic form defined on M_ε by the formula (2); to write a form ϕ_ε in $\mathrm{Dom}(q_\varepsilon)$, we use, as in [Anné et al. 2009], the following change of scales:

$$\phi_{1,\varepsilon} := \phi_\varepsilon\restriction_{M_1(\varepsilon)} \quad \text{and} \quad \phi_{2,\varepsilon} := \varepsilon^{m/2-p}\phi_\varepsilon\restriction_{M_2}.$$

We write, on the cone $\mathscr{C}_{\varepsilon,1}$,

$$\phi_{1,\varepsilon} = dr\wedge r^{-(n/2-p+1)}\beta_{1,\varepsilon} + r^{-(n/2-p)}\alpha_{1,\varepsilon}$$

and define $\sigma_{1,\varepsilon} = (\beta_{1,\varepsilon}, \alpha_{1,\varepsilon}) = U(\phi_{1,\varepsilon})$.

On the other part, it is more convenient to define $r = 1 - s$ for $s \in \left[0, \frac{1}{2}\right]$ and write $\phi_{2,\varepsilon} = dr\wedge r^{-(n/2-p+1)}\beta_{2,\varepsilon} + r^{-(n/2-p)}\alpha_{2,\varepsilon}$ near the boundary. Then we can define, for $r \in \left[\frac{1}{2}, 1\right]$ (the boundary of M_2 corresponds to $r = 1$),

$$\sigma_{2,\varepsilon}(r) = (\beta_{2,\varepsilon}(r), \alpha_{2,\varepsilon}(r)) = U(\phi_{2,\varepsilon})(r).$$

The L^2-norm, for a p-form on M_1 supported in the cone $\mathscr{C}_{\varepsilon,1}$, has the expression

$$\|\phi_\varepsilon\|^2_{L^2(M_\varepsilon)} = \int_{M_1(\varepsilon)} |\sigma_{1,\varepsilon}|^2\,d\mu_{g_1} + \int_{M_2} |\phi_{2,\varepsilon}|^2\,d\mu_{g_2}$$

and the quadratic form in our study is

$$q_\varepsilon(\phi_\varepsilon) = \int_{M_\varepsilon} |(d+d^*)\phi_\varepsilon|^2_{g_\varepsilon}\,d\mu_{g_\varepsilon} = \int_{M_1(\varepsilon)} |UD_1U^*(\sigma_{1,\varepsilon})|^2\,d\mu_{g_1} + \frac{1}{\varepsilon^2}\int_{M_2} |D_2(\phi_{2,\varepsilon})|^2\,d\mu_{g_2}. \tag{9}$$

The compatibility condition for the quadratic form is $\varepsilon^{1/2}\alpha_{1,\varepsilon}(\varepsilon) = \alpha_{2,\varepsilon}(1)$ and $\varepsilon^{1/2}\beta_{1,\varepsilon}(\varepsilon) = \beta_{2,\varepsilon}(1)$, or

$$\sigma_{2,\varepsilon}(1) = \varepsilon^{1/2}\sigma_{1,\varepsilon}(\varepsilon). \tag{10}$$

The compatibility condition for the Hodge–de Rham operator, of the first order, is obtained by expressing that $D\phi_\varepsilon \sim (UD_1U^*\sigma_{1,\varepsilon}, \varepsilon^{-1}UD_2U^*\sigma_{2,\varepsilon})$ belongs to the domain of D. In terms of σ, it gives

$$\sigma'_{2,\varepsilon}(1) = \varepsilon^{3/2}\sigma'_{1,\varepsilon}(\varepsilon). \tag{11}$$

To understand the limit problem, we proceed to several estimates.

3.1. ***Expression of the quadratic form.*** For any ϕ such that the component ϕ_1 is supported in the cone $\mathscr{C}_{\varepsilon,1}$, one has, with $\sigma_1 = U(\phi_1)$ and by the same calculus as in [Anné et al. 2009]:

$$\int_{\mathscr{C}_{\varepsilon,1}} |D_1\phi|^2\,d\mu_{g_\varepsilon} = \int_\varepsilon^1 \left\| \left(\partial_r + \frac{1}{r}A\right)\sigma_1 \right\|^2_{L^2(\Sigma)} dr = \int_\varepsilon^1 \left[\|\sigma_1'\|^2_{L^2(\Sigma)} + \frac{2}{r}(\sigma_1', A\sigma_1)_{L^2(\Sigma)} + \frac{1}{r^2}\|A\sigma_1\|^2_{L^2(\Sigma)} \right] dr.$$

3.2. ***Limit problem.*** As a Hilbert space, we introduce

$$\mathscr{H}_\infty := L^2(\overline{M}_1) \oplus \operatorname{Ker}(\widetilde{D}_2) \oplus \mathscr{I}_{1/2} \tag{12}$$

with the space $\mathscr{I}_{1/2}$ as defined in (8), and the limit operator

$$\Delta_{1,W} \oplus 0 \oplus 0$$

with W as defined in (7).

Finally, let us define:

- *A cut-off function* ξ_1 on M_1 around the conical singularity,

$$\xi_1(r) = \begin{cases} 1 & \text{if } 0 \le r \le \frac{1}{2}, \\ 0 & \text{if } 1 \le r. \end{cases} \tag{13}$$

- *The prolongation operator*

$$\begin{aligned} P_\varepsilon : H^{1/2}(\Sigma) &\to H^1(\mathscr{C}_{\varepsilon,1}) \\ \sigma = \sum_{\gamma\in\operatorname{Spec}(A)} \sigma_\gamma &\mapsto P_\varepsilon(\sigma) = U^*\Bigg(\sum_{\gamma\in\operatorname{Spec}(A)} \varepsilon^{\gamma-1/2} r^{-\gamma}\sigma_\gamma \Bigg). \end{aligned} \tag{14}$$

We remark that, restricted to $\operatorname{Im}(\Pi_{>1/2})$, $P_\varepsilon(\sigma)$ is the transplant on $M_1(\varepsilon)$ of $P_2(\sigma)$ (see Section 2.3); then there exists a constant $C > 0$ such that, for all $\sigma \in \operatorname{Im}(\Pi_{>1/2})$,

$$\|P_2(\sigma)\|^2_{L^2(\mathscr{C}_{1,1/\varepsilon})} = \|P_\varepsilon(\sigma)\|^2_{L^2(\mathscr{C}_{\varepsilon,1})} \le C \sum_{\gamma>\frac{1}{2}} \|\sigma_\gamma\|^2_{L^2(\Sigma)} = C\|\sigma\|^2_{L^2(\Sigma)}, \tag{15}$$

and also that, if $\psi_2 \in \operatorname{Dom}(\mathscr{D}_2)$, then $\big(\xi_1 P_\varepsilon(U(\psi_2\restriction_\Sigma)),\, \psi_2\big)$ defines an element of $H^1(M_\varepsilon)$.

4. Proof of the spectral convergence

We denote by $\lambda_N(\varepsilon)$, $N \ge 1$, the spectrum of the total Hodge–de Rham operator of M_ε and by λ_N, $N \ge 1$, the spectrum of the limit operator defined in Section 3.2.

4.1. ***Upper bound:*** $\limsup_{\varepsilon\to 0} \lambda_N(\varepsilon) \le \lambda_N$**.** With the min–max formula, which says that

$$\lambda_N(\varepsilon) = \inf_{\substack{E\subset\operatorname{Dom}(D_\varepsilon)\\ \dim E=N}} \inf\Bigg\{ \sup_{\substack{\phi\in E\\ \|\phi\|=1}} \int_{M_\varepsilon} |D_\varepsilon\phi|^2_{g_\varepsilon}\,d\mu_{g_\varepsilon} \Bigg\},$$

we have to describe how to transplant eigenforms of the limit problem on M_ε.

We describe this transplantation term by term. For the first term, we use the same ideas as in [Anné et al. 2009].

For an eigenform ϕ of $\Delta_{1,W}$ corresponding to the eigenvalue λ, $U(\phi)$ can be decomposed on an orthonormal base $\{\sigma_\gamma\}_\gamma$ of eigenforms of A and each component can be expressed by the Bessel functions. For $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, it has the form

$$\{c_\gamma r^{\gamma+1} F_\gamma(\lambda r^2) + d_\gamma r^{-\gamma} G_\gamma(\lambda r^2)\}\sigma_\gamma,$$

where F_γ, G_γ are entire functions satisfying $F_\gamma(0) = G_\gamma(0) = 1$ and c_γ, d_γ are constants.

We remark that $c_\gamma r^{\gamma+1} F_\gamma(\lambda r^2)\sigma_\gamma \in \mathrm{Dom}(D_{1,\min})$ and $d_\gamma r^{-\gamma}(G_\gamma(\lambda r^2) - G_\gamma(0))\sigma_\gamma \in \mathrm{Dom}(D_{1,\min})$. So we can write $\phi = \phi_0 + \bar{\phi}$ with

$$\phi_0 \in \mathrm{Dom}(D_{1,\min}) \quad \text{and} \quad U(\bar{\phi})(r) = \xi_1(r) \sum_{\substack{\gamma \in \mathrm{Spec}(A) \\ |\gamma| < \frac{1}{2}}} d_\gamma r^{-\gamma} \sigma_\gamma.$$

By the definition of $D_{1,\min}$, ϕ_0 can be approached, with the operator norm, by a sequence of smooth forms $\phi_{0,\varepsilon}$ with compact support in $M_1(\varepsilon)$.

By the definition of W, we know that $\sum_{|\gamma|<1/2} d_\gamma \sigma_\gamma \in W$. So there exists $\phi_{2,\gamma} \in \mathrm{Ker}(D_2)$ such that $U(\phi_{2,\gamma}(1)) - d_\gamma \sigma_\gamma \in \mathrm{Im}(\Pi_{>\gamma})$. We remark finally that, by the definition (14), we can write $U(\bar{\phi})(r) = \xi_1(r) \sum_{|\gamma|<1/2} \varepsilon^{1/2-\gamma} P_\varepsilon(d_\gamma \sigma_\gamma)$.

Let $\phi_{2,\varepsilon} = \sum_{|\gamma|<1/2} \varepsilon^{1/2-\gamma} \phi_{2,\gamma}$ and

$$\phi_\varepsilon = \left(\phi_{0,\varepsilon} + \xi_1 P_\varepsilon\left(\sum_{\substack{\gamma \in \mathrm{Spec}(A) \\ |\gamma| < \frac{1}{2}}} \varepsilon^{1/2-\gamma} U(\phi_{2,\gamma}(1))\right), \phi_{2,\varepsilon}\right) \in H^1(M_\varepsilon).$$

It is a good transplantation: $\|\phi_{2,\varepsilon}\| \to 0$ as the term added on $M_1(\varepsilon)$ (indeed, a term of the sum $\xi_1 \varepsilon^{1/2-\gamma} P_\varepsilon(U\phi_{2,\gamma}(1) - d_\gamma \sigma_\gamma)$ corresponds to some $\gamma' > \gamma$; if $\gamma' > \frac{1}{2}$ it is $O(\varepsilon^{1/2-\gamma})$ by (15), if $\gamma' < \frac{1}{2}$ it is $O(\varepsilon^{\gamma'-\gamma})$, and if $\gamma' = \frac{1}{2}$ it is $O(\varepsilon^{1/2-\gamma}\sqrt{|\log \varepsilon|})$). Moreover, they are harmonic, up to ξ_1.

For the two last ones, we shrink the infinite cone on M_1 and cut with the function ξ_1, already defined in (13).

Finally, if $\mathrm{Ker}\left(A - \frac{1}{2}\right) \neq \{0\}$, then, for each nonzero element $[\bar{\sigma}^{1/2}] \in \mathcal{I}_{1/2}$, there exists ψ_2 with $D_2(\psi_2) = 0$ on M_2 that has the boundary value $\bar{\sigma}^{1/2}$ modulo $\mathrm{Im}(\Pi_{>1/2})$. Then, we can construct a *quasimode* as follows:

$$\psi_\varepsilon := |\log \varepsilon|^{-1/2}\left(\xi_1.\left\{r^{-1/2} U^*(\bar{\sigma}^{1/2}) + P_\varepsilon(U(\psi_2)\restriction_\Sigma - \bar{\sigma}^{1/2})\right\}, \psi_2\right). \tag{16}$$

The L^2-norm of this element is uniformly bounded from above and below, and

$$\lim_{\varepsilon \to 0} \|\psi_\varepsilon\|_{L^2(M_\varepsilon)} = \|\bar{\sigma}^{1/2}\|_{L^2(\Sigma)}.$$

Moreover, it satisfies $q(\psi_\varepsilon) = O(|\log \varepsilon|^{-1})$, giving then a "small eigenvalue", as well as the elements of $\mathrm{Ker}(\mathcal{D}_2)$ and of $\mathrm{Ker}(\Delta_{1,W})$.

Note, as an aside, that it is remarkable that the same construction, for an extended solution with corresponding boundary value in $\mathrm{Ker}(A - \gamma)$, $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, does not give a quasimode: indeed, if ψ_2 is such a solution, the transplanted element will be

$$\psi_\varepsilon = \big(\xi_1.\big\{r^{-\gamma} U^*(\bar\sigma^\gamma) + \varepsilon^{1/2-\gamma} P_\varepsilon(U(\psi_2)\restriction_\Sigma - \bar\sigma^\gamma)\big\}, \varepsilon^{1/2-\gamma}\psi_2\big),$$

for which $q(\psi_\varepsilon)$ does not converge to 0 as $\varepsilon \to 0$.

To conclude the estimate of the upper bounds, we have only to verify that these transplanted forms have a Rayleigh–Ritz quotient comparable to the initial one and that the orthogonality is almost conserved by transplantation.

4.2. ***Lower bound:*** $\liminf_{\varepsilon\to 0} \lambda_N(\varepsilon) \geq \lambda_N$. We first proceed for one index. We know, by Section 4.1, that for each N the family $\{\lambda_N(\varepsilon)\}_{\varepsilon>0}$ is bounded; set

$$\lambda := \liminf_{\varepsilon\to 0} \lambda_N(\varepsilon).$$

There exists a sequence $\{\varepsilon_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty}\lambda_N(\varepsilon_i) = \lambda$. For each i, let ϕ_i be a normalized eigenform relative to $\lambda_i = \lambda_N(\varepsilon_i)$.

4.2.1. *On the regular part of $\overline{M}_1$.*

Lemma 4. *For our given family ϕ_i, the family $\{(1-\xi_1).\phi_{1,i}\}_{i\in\mathbb{N}}$ is bounded in $H^1_0(M_1(0), g_1)$.*

Then it remains to study $\xi_1.\phi_{1,i}$, which can be expressed with the polar coordinates. We remark that the quadratic form of these forms is uniformly bounded.

4.2.2. *Estimates of the boundary term.* The expression above can be decomposed with respect to the eigenspaces of A; in the following calculus, we suppose that $\sigma_1(1) = 0$:

$$\begin{aligned}
&\int_\varepsilon^1 \left[\|\sigma_1'\|^2_{L^2(\Sigma)} + \frac{2}{r}(\sigma_1', A\sigma_1)_{L^2(\Sigma)} + \frac{1}{r^2}\|A\sigma_1\|^2_{L^2(\Sigma)}\right] dr \\
&\qquad = \int_\varepsilon^1 \left[\|\sigma_1'\|^2_{L^2(\Sigma)} + \partial_r\left(\frac{1}{r}(\sigma_1, A\sigma_1)_{L^2(\Sigma)}\right) + \frac{1}{r^2}\big\{(\sigma_1, A\sigma_1)_{L^2(\Sigma)} + \|A\sigma_1\|^2_{L^2(\Sigma)}\big\}\right] dr \\
&\qquad = \int_\varepsilon^1 \left[\|\sigma_1'\|^2_{L^2(\Sigma)} + \frac{1}{r^2}(\sigma_1, (A+A^2)\sigma_1)_{L^2(\Sigma)}\right] dr - \frac{1}{\varepsilon}(\sigma_1(\varepsilon), A\sigma_1(\varepsilon))_{L^2(\Sigma)}.
\end{aligned}$$

This shows that the quadratic form controls the boundary term if the operator A is negative but $(A + A^2)$ is nonnegative. The latter condition is satisfied exactly on the orthogonal complement of the spectral space corresponding to the interval $(-1, 0)$. By applying $\xi_1.\phi_{1,i}$ to this fact, we obtain the following lemma:

Lemma 5. *Let $\Pi_{\leq -1}$ be the spectral projection of the operator A relative to the interval $(-\infty, -1]$. There exists a constant $C > 0$ such that, for any $i \in \mathbb{N}$,*

$$\|\Pi_{\leq -1} \circ U(\phi_{1,i}(\varepsilon_i))\|_{H^{1/2}(\Sigma)} \leq C\sqrt{\varepsilon_i}.$$

In view of Proposition 2, we also want a control of the components of σ_1 associated with the eigenvalues of A in $\left(-1, \frac{1}{2}\right]$. The number of these components is finite and we can work term by term. So we write, on $\mathscr{C}_{\varepsilon,1}$,

$$\sigma_1(r) = \sum_{\gamma\in\mathrm{Spec}(A)} \sigma_1{}^{\gamma}(r) \quad \text{with} \quad A\sigma_1{}^{\gamma}(r) = \gamma\sigma_1{}^{\gamma}(r)$$

and we suppose again $\sigma_1(1) = 0$. From the equation $(\partial_r + A/r)\sigma_1^{\gamma} = r^{-\gamma}\partial_r(r^{\gamma}\sigma_1^{\gamma})$ and the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}
\|\varepsilon^{\gamma}\sigma_1^{\gamma}(\varepsilon)\|^2_{L^2(\Sigma)} &= \left\| \int_{\varepsilon}^{1} \partial_r(r^{\gamma}\sigma_1^{\gamma})\, dr \right\|^2_{L^2(\Sigma)} \\
&\le \left\{ \int_{\varepsilon}^{1} \left\| r^{\gamma}\cdot\left(\partial_r + \frac{1}{r}A\right)\sigma_1^{\gamma}(r) \right\|_{L^2(\Sigma)} dr \right\}^2 \\
&\le \int_{\varepsilon}^{1} r^{2\gamma}\, dr \cdot \int_{\varepsilon_i}^{1} \left\| \partial_r(\sigma_1^{\gamma}) + \frac{\gamma}{r}(\sigma_1^{\gamma}) \right\|^2_{L^2(\Sigma)} dr.
\end{aligned}$$

Thus, if the quadratic form is bounded, there exists a constant $C > 0$ such that

$$\|\sigma_1^{\gamma}(\varepsilon)\|^2_{L^2(\Sigma)} \le \begin{cases} C\varepsilon^{-2\gamma}(1-\varepsilon^{2\gamma+1})/(2\gamma+1) & \text{if } \gamma \ne -\frac{1}{2}, \\ C\varepsilon|\log\varepsilon| & \text{if } \gamma = -\frac{1}{2}. \end{cases} \tag{17}$$

This gives:

Lemma 6. *Let Π_I be the spectral projection of the operator A relative to the interval I. There exist constants α, $C > 0$ such that, for any $i \in \mathbb{N}$,*

$$\|\Pi_{(-1,0)} \circ U(\phi_{1,i}(\varepsilon_i))\|_{H^{1/2}(\Sigma)} \le C\varepsilon_i^{\alpha}.$$

Here, $0 < \alpha < \frac{1}{2}$ satisfies that $-\alpha$ is larger than any negative eigenvalue of A.

With the compatibility condition (10) and the ellipticity of A, the estimate above gives also:

Lemma 7. *With the same notation, there exist constants β, $C > 0$ such that, for any $i \in \mathbb{N}$*

$$\|\Pi_{[0,1/2)} \circ U(\phi_{2,i}(1))\|_{H^{1/2}(\Sigma)} \le C\varepsilon_i^{\beta}.$$

Here, $\frac{1}{2} - \beta$ is the largest nonnegative eigenvalue of A strictly smaller than $\frac{1}{2}$ (if there is no such eigenvalue, we put $\beta = \frac{1}{2}$).

Finally, we study $\sigma_1^{1/2}$ for our family of forms (the parameter i is omitted in the notation). It satisfies, for $\varepsilon_i < r < \frac{1}{2}$, the equation

$$\left(-\partial_r^2 + \frac{3}{4r^2}\right)\sigma_1^{1/2} = \lambda_i\sigma_1^{1/2}.$$

The solutions of this equation can be expressed in terms of the Bessel and the Neumann functions: there exist entire functions F, G with $F(0) = G(0) = 1$ and differential forms c_i, d_i in $\mathrm{Ker}\big(A - \frac{1}{2}\big)$ such that

$$\sigma_1^{1/2}(r) = c_i r^{3/2}F(\lambda_i r^2) + d_i\left\{ r^{-1/2}G(\lambda_i r^2) + \frac{2}{\pi}\log(r) r^{3/2}F(\lambda_i r^2) \right\} \tag{18}$$

(see [Anné et al. 2009, Lemma 4]). The fact that the L^2-norm is bounded gives that $\|c_i\|_{L^2}^2 + |\log \varepsilon_i|\,\|d_i\|_{L^2}^2$ is bounded. Finally, by substituting this estimate in the expression above, we have

$$\|\sigma_1^{1/2}(\varepsilon_i)\|^2_{L^2(\Sigma)} = O\left(\frac{1}{\varepsilon_i|\log \varepsilon_i|}\right).$$

With the compatibility condition (10), we obtain:

Lemma 8. *There exists a constant $C > 0$ such that, for any $i \in \mathbb{N}$,*

$$\|\Pi_{\{1/2\}} \circ U(\phi_{2,i})(1)\|_{H^{1/2}(\Sigma)} \le \frac{C}{\sqrt{|\log \varepsilon_i|}}.$$

4.2.3. *Convergence of $\phi_{2,i}$.* Let us now define, in general, $\tilde{\phi}_{2,\varepsilon}$ as the form obtained by the prolongation of $\phi_{2,\varepsilon}$ by $\sqrt{\varepsilon}\xi_1(\varepsilon r)\phi_{1,\varepsilon}(\varepsilon r)$ on the infinite cone $\mathscr{C}_{1,\infty}$. A change of variables gives that

$$\|\tilde{\phi}_{2,\varepsilon}\|_{L^2(\mathscr{C}_{1,\infty})} = \|\xi_1\phi_{1,\varepsilon}\|_{L^2(\mathscr{C}_{\varepsilon,1})},$$

while

$$\int_{\widetilde{M}_2} |\widetilde{D}_2(\tilde{\phi}_{2,\varepsilon})|^2\, d\mu = \varepsilon^2 \int_{\mathscr{C}_{\varepsilon,1}} |D_1(\xi_1\phi_{1,\varepsilon})|^2\, d\mu_{g_1} + \int_{M_2} |D_2(\phi_{2,\varepsilon})|^2\, d\mu_{g_2}.$$

Thus, by the definition of ϕ_i, the family $\{\tilde{\phi}_{2,i}\}_{i\in\mathbb{N}}$ is bounded in $\mathscr{W}$ and $\int_{\mathscr{C}_{1,\infty}} |\widetilde{D}_2(\tilde{\phi}_{2,i})|^2\, d\mu = O(\varepsilon_i^2)$. The work of Carron [2001b] gives us that $\|\tilde{\phi}_{2,i}(1)\|_{H^{1/2}(\Sigma)}$ is bounded and the following:

Proposition 9. *There exists a subfamily of the family $\{\tilde{\phi}_{2,i}\}_{i\in\mathbb{N}}$ which converges in $L^2(M_2, g_2)$. Its limit $\tilde{\phi}_2$ defines an extended solution on $\widetilde{M}_2$, i.e., $\widetilde{D}_2(\tilde{\phi}_2) = 0$ and $\tilde{\phi}_2\restriction_\Sigma \in \operatorname{Ker}(T) \cap \operatorname{Im}(\Pi_{\ge -1/2})$.*

We still denote by $\tilde{\phi}_{2,i}$ the subfamily obtained.

4.2.4. *Convergence near the singularity.* Now we use the fact that eigenforms satisfy an equation which imposes a local form. We concentrate on $\gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. If we write

$$\phi_{1,i}^{[-1/2,1/2]} = \sum_{\gamma\in[-1/2,1/2]} U^*\sigma_1^\gamma(r),$$

the terms σ_1^γ satisfy the equations

$$\left(-\partial_r^2 + \frac{\gamma(1+\gamma)}{r^2}\right)\sigma_1^\gamma = \lambda_i\sigma_1^\gamma.$$

The solutions of this equation can be expressed in term of the Bessel functions: there exist entire functions F, G with $F(0) = G(0) = 1$ and differential forms $c_{\gamma,i}$, $d_{\gamma,i}$ in $\operatorname{Ker}(A - \gamma)$ such that

$$\sigma_1^\gamma(r) = \begin{cases} c_{\gamma,i}r^{\gamma+1}F_\gamma(\lambda_i r^2) + d_{\gamma,i}(r^{-\gamma}G_\gamma(\lambda_i r^2)), & |\gamma| < \frac{1}{2}, \\ c_{1/2,i}r^{3/2}F_{1/2}(\lambda_i r^2) + d_{1/2,i}(r^{-1/2}G_{1/2}(\lambda_i r^2) + \dfrac{2}{\pi}\log(r)r^{3/2}F_{1/2}(\lambda_i r^2)), & \gamma = \frac{1}{2}, \\ c_{-1/2,i}r^{1/2}F_{-1/2}(\lambda_i r^2) + d_{-1/2,i}(r^{1/2}\log(r)G_{-1/2}(\lambda_i r^2)), & \gamma = -\frac{1}{2}. \end{cases} \tag{19}$$

The lemmas of the previous subsections give us the result that the families $c_{\gamma,i}$ and $d_{\gamma,i}$ are bounded and, by extraction, we can suppose that they converge. In the case of $\gamma = \frac{1}{2}$, we have more: $\|d_{1/2,i}\|_{L^2(\Sigma)} = O(|\log \varepsilon_i|^{-1/2})$.

But, turning back to the family of the last proposition, we also know that the family $\sqrt{\varepsilon_i}\xi_1(\varepsilon_i r)\phi_{1,i}(\varepsilon_i r)$ converges to 0 on any sector $1 \leq r \leq R$, according to the explicit form of $\sigma_1^\gamma(r)$. As a consequence, the form $\tilde{\phi}_2$ has no component for $\gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and is indeed an L^2-solution. We have proved:

Proposition 10. *The form $\tilde{\phi}_2$ in Proposition 9 has no component for $\gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. If we set $\phi_2 := \tilde{\phi}_2 \restriction_{M_2}$, there exists a subfamily of $\{\phi_{2,i}\}_i$ which converges to ϕ_2 as $i \to \infty$, and it satisfies*

$$\phi_2 \in \mathrm{Dom}(\mathscr{D}_2), \quad \|\phi_2\|_{L^2(M_2,g_2)} \leq 1 \quad \textit{and} \quad D_2(\phi_2) = 0.$$

Moreover, the harmonic prolongation of $\sqrt{\varepsilon_i}\xi_1(\varepsilon_i r)\phi_{1,i}(\varepsilon_i r)$,

$$\bar{\phi}_{2,i} = \mathscr{E}(\sqrt{\varepsilon_i}\xi_1(\varepsilon_i r)\phi_{1,i}(\varepsilon_i r)),$$

minimizes the norm of $D_2(\phi_2)$. As a consequence, $\|D_2(\bar{\phi}_{2,i})\|_{L^2(M_2)} = O(\varepsilon_i)$ implies

$$\|T(\sqrt{\varepsilon_i}\phi_{1,i}(\varepsilon_i))\|_{H^{-1/2}(\Sigma)} = O(\varepsilon_i)$$

with the Dirac–Neumann operator T defined in (5).

But, by Lemmas 5 and 6, we know that $\|\Pi_{<-1/2}(\phi_{1,i}(\varepsilon))\|_{H^{1/2}(\Sigma)} = O(\sqrt{\varepsilon})$. The continuity of T thus gives $\|T \circ \Pi_{\geq -1/2}(\phi_{1,i}(\varepsilon_i))\|_{H^{-1/2}(\Sigma)} = O(\sqrt{\varepsilon_i})$. To obtain consequences of this result for the term $\Pi_{[-1/2,1/2]}(\phi_{1,i}(\varepsilon_i))$, we must make sense of the possibility of working modulo $\mathrm{Im}(T)$. In the following, for simplicity of notation, we identify the spectral projection Π_I of A for the interval I with $U^*\Pi_I U$.

Proposition 11. *The space $T(\mathrm{Im}(\Pi_{>1/2}) \cap H^{1/2}(\Sigma))$ is closed in $H^{-1/2}(\Sigma)$, as a consequence of the work of Carron. Let us define $B(\phi)$ for $\phi \in \mathrm{Im}(\Pi_{[-1/2,1/2]})$ as the orthogonal projection of $T(\phi)$ onto the orthogonal complement of this space. Then B is linear and satisfies*:

- $\|B\phi\|_{H^{-1/2}(\Sigma)} \leq \|T\phi\|_{H^{-1/2}(\Sigma)}$.
- *If $B(\phi) = 0$, there exists an $\eta \in \mathrm{Im}(\Pi_{>1/2})$ such that $T(\phi + \eta) = 0$.*

Proof. To prove that $T(\mathrm{Im}(\Pi_{>1/2}) \cap H^{1/2}(\Sigma))$ is closed, we must recall some facts contained in [Carron 2001a]. Let us denote here $T_{\mathscr{C}}$ the operator constructed as T, but for the infinite part $\mathscr{C}_{1,\infty}$. Then $\mathrm{Im}(T_{\mathscr{C}}) = \mathrm{Im}(\Pi_{>1/2})$ is a subspace of $\mathrm{Ker}(T_{\mathscr{C}}) = \mathrm{Im}(\Pi_{\geq -1/2})$. We know that $T + T_{\mathscr{C}}$ is an elliptic operator of order 1 on Σ which is compact. As a consequence, $\mathrm{Ker}(T + T_{\mathscr{C}})$ is finite-dimensional, $(T + T_{\mathscr{C}})(H^{1/2}(\Sigma))$ is a closed subspace of $H^{-1/2}(\Sigma)$ and $T + T_{\mathscr{C}}$ admits a continuous parametrix $Q : H^{-1/2}(\Sigma) \to H^{1/2}(\Sigma)$ such that

$$Q \circ (T + T_{\mathscr{C}}) = \mathrm{Id} - \Pi_{\mathrm{Ker}(T+T_{\mathscr{C}})},$$

where $\Pi_{\mathrm{Ker}(T+T_{\mathscr{C}})}$ denotes the orthogonal projection onto $\mathrm{Ker}(T + T_{\mathscr{C}})$ for the inner product of $H^{1/2}(\Sigma)$. We can now prove that $T(\mathrm{Im}\,\Pi_{>1/2} \cap H^{1/2}(\Sigma))$ is closed.

Let $\{\sigma_i\}_i$ be a sequence of elements in $\mathrm{Im}(\Pi_{>1/2}) \cap H^{1/2}(\Sigma)$ such that $T(\sigma_i)$ converges, and let $\psi = \lim_{i\to\infty} T(\sigma_i)$. We can suppose that

$$\sigma_i \in (\mathrm{Ker}(T) \cap \mathrm{Im}(\Pi_{>1/2}) \cap H^{1/2}(\Sigma))^{\perp}.$$

We have $\operatorname{Im}(\Pi_{>1/2}) \cap H^{1/2}(\Sigma) \subset \operatorname{Ker}(T_{\mathscr{C}})$. Then $(T+T_{\mathscr{C}})\sigma_i = T(\sigma_i)$ converges and $\tau_i = Q \circ (T+T_{\mathscr{C}})\sigma_i$ converges; let $\tau = \lim_{i\to\infty} \tau_i$. Thus,

$$\sigma_i = \tau_i + e_i \quad \text{with} \quad \tau_i \in \operatorname{Ker}(T+T_{\mathscr{C}})^{\perp},\ e_i \in \operatorname{Ker}(T+T_{\mathscr{C}}).$$

The sequence $\{e_i\}_i$ must be bounded, unless we can extract a subsequence $\|e_i\| \to \infty$, so it is true also for $\|\sigma_i\|$ and, by extraction, we can suppose that the bounded sequence $e_i/\|\sigma_i\|$ converges, since it lives in a finite-dimensional space. Let e' be this limit; then $e' = \lim e_i/\|\sigma_i\|$ also and $e' \in \operatorname{Im}(\Pi_{>1/2}) \cap H^{1/2}(\Sigma)$.

Finally, e' satisfies $\|e'\| = 1$, and

$$e' \in \operatorname{Ker}(T+T_{\mathscr{C}}) \quad \text{and} \quad e' \in \operatorname{Ker}(T_{\mathscr{C}}),$$

as well as e_i and σ_i, which implies $T(e') = 0$. Thus, $e' = \lim \sigma_i/\|\sigma_i\| \in \operatorname{Im}(\Pi_{>1/2}) \cap H^{1/2}(\Sigma) \cap \operatorname{Ker}(T)$. But, by the assumption on σ_i, e' must be orthogonal to this space, which is a contradiction.

So, e_i is a bounded sequence in a finite-dimensional space; by extraction, we can suppose that it converges. Then σ_i admits a convergent subsequence, and let σ denote its limit; then

$$\sigma \in \operatorname{Im}(\Pi_{>1/2}) \cap H^{1/2}(\Sigma) \quad \text{and} \quad \psi = T(\sigma). \qquad \square$$

As an application of Proposition 11, we have

$$\|B \circ \Pi_{[-1/2,1/2]}(\phi_{1,i}(\varepsilon_i))\|_{H^{-1/2}(\Sigma)} = O(\sqrt{\varepsilon_i}).$$

This is the sum of few terms. We remark that the term with $c_{\gamma,i}$ is in fact always $O(\sqrt{\varepsilon_i})$. For the same reason, we can freeze the function G at 0, where its value is 1. So we can say

$$\left\| \varepsilon_i^{1/2} \log(\varepsilon_i) B \circ U^*(d_{-1/2,i}) + \sum_{|\gamma|<\frac{1}{2}} \varepsilon_i^{-\gamma} B \circ U^*(d_{\gamma,i}) + \varepsilon_i^{-1/2} B \circ U^*(d_{1/2,i}) \right\|_{H^{-1/2}(\Sigma)} = O(\sqrt{\varepsilon_i}), \quad (20)$$

while all the other terms, which behave like r^{δ} with $\delta > \frac{1}{2}$, occur in an expression belonging to $\operatorname{Dom}(D_{1,\min})$.

In fact, we have the following result:

Proposition 12. *One can write $\Pi_{(-1/2,1/2]} \circ U(\xi_1\phi_{1,i}) = \bar{\sigma}_{1,i} + \sigma_{0,i}$ with the bounded sequence $U^*(\sigma_{0,i})$ in $\operatorname{Dom}(D_{1,\min})$ and $\bar{\sigma}_{1,i} = \bar{\sigma}_{1,i}^{<1/2} + \bar{\sigma}_{1,i}^{1/2}$ satisfies that there exists a subfamily of $\bar{\sigma}_{1,i}^{<1/2}$ which converges to $\sum_{\gamma\in(-1/2,1/2)} r^{-\gamma}\sigma_\gamma$ as $i \to \infty$ with $\sum_{\gamma\in(-1/2,1/2)} \sigma_\gamma \in W$, while*

$$\bar{\sigma}_{1,i}^{1/2} \sim \frac{1}{\sqrt{|\log \varepsilon_i|}} r^{-1/2} \bar{\sigma}_{1/2} \quad \text{for some } \bar{\sigma}_{1/2} \in \operatorname{Ker}\bigl(A - \tfrac{1}{2}\bigr).$$

Thus, $\bar{\sigma}_{1,i}^{1/2}$ concentrates on the singularity.

Proof. The term $\bar{\sigma}_{1,i}$ comes from the expression obtained in (20), while $\sigma_{0,i}$ is the sum of all the other terms.

We then concentrate on (20). First, we gather the terms concerning the same eigenvalue and still denote by $d_{\gamma,i}$ the sum of all the terms with the same eigenvalue. Let $-\frac{1}{2} \le \gamma_p < \cdots < \gamma_0 \le \frac{1}{2}$ be the eigenvalues of A in $\bigl[-\frac{1}{2}, \frac{1}{2}\bigr]$.

We then define the limit d_γ as

$$d_\gamma := \begin{cases} \lim_{i\to\infty} d_{\gamma,i}, & \gamma \neq \frac{1}{2}, \\ \lim_{i\to\infty} \sqrt{|\log \varepsilon_i|}\, d_{1/2,i}, & \gamma = \frac{1}{2}, \end{cases}$$

and put $E_\gamma = \mathrm{Ker}(A-\gamma)$.

Indeed, we can, step by step, decompose $d_{\gamma,i}$ into a part in $\mathrm{Ker}(B\circ U^*)$ and a part which exhibits a smaller behavior in ε_i.

• *First step: in $E_{1/2}$.* Multiplying (20) by $\sqrt{\varepsilon_i}$, we obtain that $\|B\circ U^*(d_{1/2,i})\|_{H^{-1/2}(\Sigma)} = O(\varepsilon_i^{1/2-\gamma_1})$. We decompose $d_{1/2,i} = (1/\sqrt{|\log \varepsilon_i|}\,)d^{(0)}_{1/2,i} + d^{\perp}_{1/2,i}$ along $\mathrm{Ker}(B\circ U^*\restriction_{E_{1/2}})$ and its orthogonal complement in $E_{1/2}$. Then, $\|B\circ U^*(d_{1/2,i})\|_{H^{-1/2}(\Sigma)} = O(\varepsilon_i^{1/2-\gamma_1})$ implies $\|d^{\perp}_{1/2,i}\|_{H^{1/2}(\Sigma)} = O(\varepsilon_i^{1/2-\gamma_1})$. So,

$$d_{1/2} = \lim_{i\to\infty} \sqrt{|\log \varepsilon_i|}\, d_{1/2,i} = \lim_{i\to\infty} d^{(0)}_{1/2,i} \in \mathrm{Ker}(B\circ U^*)$$

and, if we write $d^{\perp}_{1/2,i} = \varepsilon_i^{1/2-\gamma_1} d_i^{(1)}$ and reintroduce this in (20), then it has the new expression

$$\left\| \varepsilon_i^{1/2} \log(\varepsilon_i) B\circ U^*(d_{-1/2,i}) + \sum_{j=2}^{p} \varepsilon_i^{-\gamma_j} B\circ U^*(d_{\gamma_j,i}) + \varepsilon_i^{-\gamma_1} B\circ U^*(d_i^{(1)} + d_{\gamma_1,i}) \right\|_{H^{-1/2}(\Sigma)} = O(\sqrt{\varepsilon_i}).$$

• *Second step: in $E_{1/2}\oplus E_{\gamma_1}$.* Multiplying by $\varepsilon_i^{\gamma_1}$ in the above, we obtain that

$$\|B\circ U^*(d_i^{(1)} + d_{\gamma_1,i})\|_{H^{-1/2}(\Sigma)} = O(\varepsilon_i^{\gamma_1-\gamma_2}). \tag{21}$$

We decompose $d_i^{(1)} + d_{\gamma_1,i} = d^{(0)}_{\gamma_1,i} + d^{\perp}_{\gamma_1,i}$ along $\mathrm{Ker}(B\circ U^*\restriction_{E_{1/2}\oplus E_{\gamma_1}})$ and its orthogonal complement in $E_{1/2}\oplus E_{\gamma_1}$.

Now, (21) says that $\|d^{\perp}_{\gamma_1,i}\|_{H^{1/2}(\Sigma)} = O(\varepsilon_i^{\gamma_1-\gamma_2})$, so $d_{\gamma_1} = \lim_{i\to\infty} d_{\gamma_1,i} = \lim_{i\to\infty} \Pi_{\{\gamma_1\}}(d^{(0)}_{\gamma_1,i})$ and, as $d^{(0)}_{\gamma_1,i} \in \mathrm{Ker}(B\circ U^*\restriction_{E_{1/2}\oplus E_{\gamma_1}})$, extracting from $\Pi_{\{1/2\}}(d^{(0)}_{\gamma_1,i})$ a convergent subsequence, we can say that there exists an $e_{1/2}\in E_{1/2}$ such that

$$d_{\gamma_1} + e_{1/2} \in \mathrm{Ker}(B\circ U^*).$$

On the other hand, if we can write

$$d^{\perp}_{\gamma_1,i} = \varepsilon_i^{\gamma_1-\gamma_2} d_i^{(2)},$$

then the new expression of (20) is

$$\left\| \varepsilon_i^{1/2} \log(\varepsilon_i) B\circ U^*(d_{-1/2,i}) + \sum_{j=3}^{p} \varepsilon_i^{-\gamma_j} B\circ U^*(d_{\gamma_j,i}) + \varepsilon_i^{-\gamma_2} B\circ U^*(d_i^{(2)} + d_{\gamma_2,i}) \right\|_{H^{-1/2}(\Sigma)} = O(\sqrt{\varepsilon_i}).$$

We can continue in this way until the term concerning γ_p. It constructs terms

$$\begin{aligned} d^{(0)}_{\gamma_k,i} &\in (E_{1/2}\oplus\cdots\oplus E_{\gamma_k}) \cap \mathrm{Ker}(B\circ U^*), \\ d_i^{(k+1)} &\in E_{1/2}\oplus\cdots\oplus E_{\gamma_k} \end{aligned}$$

with $0 \le k \le p$. If we decompose $d^{(0)}_{\gamma_k,i} = \sum_{j=0}^{k} d^{\gamma_k(0)}_{\gamma_j,i}$ and $d^{(k+1)}_i = \sum_{j=0}^{k} d^{(k+1)}_{\gamma_j,i}$, then

$$d_{1/2,i} = \frac{1}{\sqrt{|\log \varepsilon_i|}} d^{(0)}_{1/2,i} + \varepsilon_i^{1/2-\gamma_1} d^{\gamma_1(0)}_{1/2,i} + \varepsilon_i^{1/2-\gamma_2} d^{\gamma_2(0)}_{1/2,i} + \cdots + \varepsilon_i \log(\varepsilon_i) d^{(p+1)}_{1/2,i},$$
$$d_{\gamma_1,i} = \Pi_{\{\gamma_1\}}(d^{(0)}_{\gamma_1,i}) + \varepsilon_i^{\gamma_1-\gamma_2} d^{\gamma_2(0)}_{\gamma_1,i} + \varepsilon_i^{\gamma_1-\gamma_3} d^{\gamma_3(0)}_{\gamma_1,i} + \cdots.$$

Now, because all the families involved here (finite in number) are bounded in a finite-dimensional space, we can suppose, by successive extractions, that they converge. We have

$$d_\gamma = \lim_{\varepsilon_i \to 0} \Pi_{\{\gamma\}}\big(d^{(0)}_{\gamma,i}\big).$$

This means that there exist elements $\bar{\sigma}_\gamma = d_\gamma \in \operatorname{Ker}(A-\gamma)$, $|\gamma| \le \frac{1}{2}$, such that there exists an $\eta_\gamma \in \operatorname{Im}(\Pi_{>\gamma})$ with

$$(T \circ U^*)(\bar{\sigma}_\gamma + \eta_\gamma) = 0,$$

and, if we denote

$$\Pi_{(\gamma,1/2]}(\eta_\gamma) = \sum_{\mu>\gamma} \eta^\mu_\gamma,$$

then we obtain

$$\Pi_{(-1/2,1/2]} \circ U(\phi_{1,i}(r)) \sim \sum_{-\frac{1}{2} \le \mu < \gamma < \frac{1}{2}} r^{-\gamma}(\bar{\sigma}_\gamma + \varepsilon_i^{\gamma-\mu} \eta^\gamma_\mu) + r^{-1/2} \Big\{ |\log \varepsilon_i|^{-1/2} \bar{\sigma}_{1/2} + \sum_{-\frac{1}{2} \le \mu < \frac{1}{2}} \varepsilon_i^{1/2-\mu} \eta^{1/2}_\mu \Big\}.$$

Here, the term $\varepsilon_i^{-\mu}$ has to be replaced by $\varepsilon_i^{1/2} \log \varepsilon_i$ in the case of $\mu = -\frac{1}{2}$. □

4.2.5. *Conclusions on the side of M_1.* We now decompose $\phi_{1,i} = \phi_{1,\varepsilon_i}$ near the singularity as follows: Let

$$\xi_1 \phi_{1,\varepsilon_i} = \xi_1 \big\{ \phi^{\le -1/2}_{1,i} + \phi^{(-1/2,1/2]}_{1,i} + \phi^{>1/2}_{1,i} \big\}$$

according to the decomposition, on the cone, of σ_1 along the eigenvalues of A respectively less than $-\frac{1}{2}$, in $\big(-\frac{1}{2}, \frac{1}{2}\big]$ and greater than $\frac{1}{2}$.

We first remark that the expression and the convergence of $\phi^{(-1/2,1/2]}_{1,i}$ are given by the preceding Proposition 12.

Now $\phi^{>1/2}_{1,i}$ and $\widetilde{\psi}_{1,i} = \xi_1 P_{\varepsilon_i}\big(\Pi_{>1/2} \circ U(\phi_{2,i}(1))\big)$ have the same boundary value. But, by Propositions 9 and 10, we have

$$\lim_{i\to\infty} U(\phi_{2,i}(1)) = U(\phi_2(1)) \in \operatorname{Im}(\Pi_{>1/2}) \quad \text{for the norm of } H^{1/2}(\Sigma).$$

So, $\xi_1 \phi^{>1/2}_{1,i} - \widetilde{\psi}_{1,i}$ can be considered in $H^1(M_1(0))$ by a prolongation by 0 and:

Proposition 13. *By uniform continuity of P_{ε_i}, and the convergence property just recalled,*

$$\lim_{i\to\infty} \|\widetilde{\psi}_{1,i} - \xi_1 P_{\varepsilon_i}(U(\phi_2 \restriction_\Sigma))\|_{L^2(M_1(\varepsilon_i))} = 0.$$

On the other hand, $\xi_1 P_{\varepsilon_i}(U(\phi_2\restriction_\Sigma))$ *converges weakly to* 0 *on the open manifold* $M_1(0)$; *more precisely, for any fixed* η *with* $0<\eta<1$,

$$\lim_{i\to\infty}\|\xi_1 P_{\varepsilon_i}(U(\phi_2\restriction_\Sigma))\|_{L^2(M_1(\eta))}=0.$$

We remark finally that the boundary value of $\phi_{1,i}^{\le -1/2}$ is small. For this term we introduce the cut-off function taken in [Anné et al. 2009],

$$\xi_{\varepsilon_i}(r)=\begin{cases}1 & \text{if } 2\sqrt{\varepsilon_i}\le r,\\ (1/\log\sqrt{\varepsilon_i})\log(2\varepsilon_i/r) & \text{if } 2\varepsilon_i\le r\le 2\sqrt{\varepsilon_i},\\ 0 & \text{if } r\le 2\varepsilon_i.\end{cases}$$

Proposition 14. $$\lim_{i\to\infty}\|(1-\xi_{\varepsilon_i})\xi_1\phi_{1,i}^{\le -1/2}\|_{L^2(M_1(\varepsilon_i))}=0.$$

This is a consequence of the estimates of Lemmas 5 and 6; we remark that, by the same argument, we obtain also $\|\xi_1\phi_{1,i}^{\le -1/2}(r)\|_{L^2(\Sigma)}\le C\sqrt{r}$, so

$$\|(1-\xi_{\varepsilon_i})\xi_1\phi_{1,i}^{\le -1/2}\|_{L^2(M_1(\varepsilon_i))}=O(\varepsilon_i^{1/4}).$$

Proposition 15. *The forms*

$$\psi_{1,i}=(1-\xi_1)\phi_{1,i}+(\xi_1\phi_{1,i}^{>1/2}-\widetilde{\psi}_{1,i})+\xi_{\varepsilon_i}\xi_1\phi_{1,i}^{\le -1/2}+\xi_1 U^*(\overline{\sigma}_{0,i}^{1/2})$$

belong to $\mathrm{Dom}(D_{1,\min})$ *and define a bounded family.*

Proof. We will show that each term is bounded. For the last one, it is a consequence of Proposition 12. For the first one, it is already done in Lemma 4. For the second one, we note that

$$\begin{aligned} f_i &:= \left(\partial_r+\frac{A}{r}\right)U(\xi_1\phi_{1,i}^{>1/2}-\widetilde{\psi}_{1,i})\\ &=\xi_1\left(\partial_r+\frac{A}{r}\right)(U\phi_{1,i}^{>1/2})+\partial_r(\xi_1)U\big(\phi_{1,i}^{>1/2}-P_{\varepsilon_i}(\Pi_{>1/2}\phi_{2,i}(1))\big) \end{aligned}\tag{22}$$

is uniformly bounded in $L^2(\overline{M}_1)$, because of (15). This estimate (15) shows also that the L^2-norm of $\xi_1\phi_{1,i}^{>1/2}-\tilde{\psi}_{1,i}$ is bounded.

For the third one, we use the estimate due to the expression of the quadratic form. The estimate that $\int_{\mathscr{C}_{r,1}}|D_1(\xi_1\phi^{\le -1/2})|^2\,d\mu\le\Lambda$ gives that

$$\|\sigma_1^{\le -1/2}(r)\|_{L^2(\Sigma)}^2\le\Lambda r|\log r|\tag{23}$$

by the same argument as in Lemmas 5 and 6. Now

$$\begin{aligned}\|D_1(\xi_{\varepsilon_i}\xi_1\phi_{1,i}^{\le -1/2})\|_{L^2(\overline{M}_1)} &\le \|\xi_{\varepsilon_i}D_1(\xi_1\phi_{1,i}^{\le -1/2})\|_{L^2(\overline{M}_1)}+\||d\xi_{\varepsilon_i}|\cdot\xi_1\phi_{1,i}^{\le -1/2}\|_{L^2(\overline{M}_1)}\\ &\le \|D_1(\xi_1\phi_{1,i}^{\le -1/2})\|_{L^2(\mathscr{C}_{\varepsilon_i,1})}+\||d\xi_{\varepsilon_i}|\cdot\xi_1\phi_{1,i}^{\le -1/2}\|_{L^2(\mathscr{C}_{\varepsilon_i,\sqrt{\varepsilon_i}})}.\end{aligned}$$

The first term is bounded and, with $|A| \geq \frac{1}{2}$ for this term, and the estimate (23), we have

$$\| |d\xi_{\varepsilon_i}|\xi_1 \phi_{1,i}^{\leq -1/2}\|^2_{L^2(\mathscr{C}_{\varepsilon_i,\sqrt{\varepsilon_i}})} \leq \frac{4\Lambda}{|\log \varepsilon_i|^2} \int_{\varepsilon_i}^{\sqrt{\varepsilon_i}} \frac{\log r}{r}\, dr \leq \tfrac{3}{2}\Lambda.$$

This completes the proof. □

In fact, the decomposition used here is almost orthogonal:

Lemma 16. *There exists $\beta > 0$ such that*

$$(\phi_{1,i}^{>1/2} - \widetilde{\psi}_{1,i}, \widetilde{\psi}_{1,i})_{L^2(M_1(\varepsilon_i))} = O(\varepsilon_i^\beta).$$

Proof. If we decompose the terms into the eigenspaces of A, we see that only the eigenvalues in $\left(\frac{1}{2}, \infty\right)$ are involved. With $f_i = \sum_{\gamma > \frac{1}{2}} f^\gamma$ and $U(\phi_{1,i}^{>1/2} - \widetilde{\psi}_{1,i}) = \sum_{\gamma > \frac{1}{2}} \phi_0^\gamma$, equation (22) and the fact that $(\phi_{1,i}^{>1/2} - \widetilde{\psi}_{1,i})(\varepsilon_i) = 0$ imply

$$\phi_0^\gamma(r) = r^{-\gamma} \int_{\varepsilon_i}^r \rho^\gamma f^\gamma(\rho)\, d\rho.$$

Then, for each eigenvalue $\gamma > \frac{1}{2}$ of A,

$$\begin{aligned}
(\phi_0^\gamma, \tilde{\psi}_{1,i}^\gamma)_{L^2(\mathscr{C}_{\varepsilon_i,1})} &= \varepsilon_i^{\gamma-1/2} \int_{\varepsilon_i}^1 r^{-2\gamma} \int_{\varepsilon_i}^r \rho^\gamma (\sigma_\gamma, f^\gamma(\rho))_{L^2(\Sigma)}\, d\rho\, dr \\
&= \varepsilon_i^{\gamma-1/2} \int_{\varepsilon_i}^1 \frac{r^{-2\gamma+1}}{2\gamma-1} \cdot r^\gamma \cdot (\sigma_\gamma, f^\gamma(r))_{L^2(\Sigma)}\, dr + \frac{\varepsilon_i^{\gamma-1/2}}{2\gamma-1} \int_{\varepsilon_i}^1 \rho^\gamma (\sigma_\gamma, f^\gamma(\rho))_{L^2(\Sigma)}\, d\rho.
\end{aligned}$$

Thus, if $\gamma > \frac{3}{2}$, we have the upper bound

$$\begin{aligned}
&|(\phi_0^\gamma, \widetilde{\psi}_{1,i}^\gamma)_{L^2(\mathscr{C}_{\varepsilon_i,1})}| \\
&\quad \leq \varepsilon_i^{\gamma-1/2} \int_{\varepsilon_i}^1 \frac{r^{-\gamma+1}}{2\gamma-1} |(\sigma_\gamma, f^\gamma(r))_{L^2(\Sigma)}|\, dr + \frac{\varepsilon_i^{\gamma-1/2}}{(2\gamma-1)\sqrt{2\gamma+1}} \|\sigma_\gamma\|_{L^2(\Sigma)} \cdot \|f^\gamma\|_{L^2(\mathscr{C}_{\varepsilon_i,1})} \\
&\quad \leq C\varepsilon_i^{\gamma-\frac{1}{2}} \|\sigma_\gamma\|_{L^2(\Sigma)} \frac{\varepsilon_i^{(-2\gamma+3)/2}}{(2\gamma-1)\sqrt{2\gamma-3}} \|f^\gamma\|_{L^2(\mathscr{C}_{\varepsilon_i,1})} + \frac{\varepsilon_i^{\gamma-1/2}}{(2\gamma-1)\sqrt{2\gamma+1}} \|\sigma_\gamma\|_{L^2(\Sigma)} \cdot \|f^\gamma\|_{L^2(\mathscr{C}_{\varepsilon_i,1})},
\end{aligned}$$

while, for $\gamma = \frac{3}{2}$, the first term is $O(\varepsilon_i\sqrt{|\log \varepsilon_i|})$ and, for $\frac{1}{2} < \gamma < \frac{3}{2}$, it is $O(\varepsilon_i^{\gamma-1/2})$. In short, we have

$$|(\phi_0^\gamma, \widetilde{\psi}_{1,i}^\gamma)_{L^2(\mathscr{C}_{\varepsilon_i,1})}| \leq C\varepsilon_i^\beta \|\sigma_\gamma\|_{L^2(\Sigma)} \cdot \|f^\gamma\|_{L^2(\mathscr{C}_{\varepsilon_i,1})}$$

if $\beta > 0$ satisfies $\gamma \geq \beta + \frac{1}{2}$ for all eigenvalues γ of A in $\left(\frac{1}{2}, \infty\right)$. This estimate gives Lemma 16. □

Corollary 17. *There exists in $\{\psi_{1,i} + \phi_{1,i}^{(-1/2,1/2)}\}_i$ a subfamily which converges in L^2 to a form ϕ_1 in* $\mathrm{Dom}(D_{1,W})$ *that satisfies on the open manifold $M_1(0)$ the equation $\Delta\phi_1 = \lambda\phi_1$. Moreover,*

$$\|\phi_1\|^2_{L^2(M_1(0))} + \|\tilde{\phi}_2\|^2_{L^2(\widetilde{M}_2)} + \|\overline{\sigma}_{1/2}\|^2_{L^2(\Sigma)} = 1, \tag{24}$$

where $\tilde{\phi}_2$ is the prolongation of ϕ_2 by $P_2(\phi_2\restriction_\Sigma)$ on $\widetilde{M}_2$, and $\overline{\sigma}_{1/2}$ is given by Proposition 12.

Proof. Indeed, the family $\{\psi_{1,i}+\phi_{1,i}^{(-1/2,1/2)}\}_i$ is bounded in $\mathrm{Dom}(D_{1,\max})$; one can then extract a subfamily which converges in $L^2(\overline{M}_1, \bar{g}_1)$. But we know that $\widetilde{\psi}_{1,i}$ converges to 0 in any $M_1(\eta)$; the conclusion follows. We obtain also, with the help of Lemma 16, that

$$1-\{\|\phi_1\|^2_{L^2(M_1(0))}+\|\phi_2\|^2_{L^2(M_2)}\}=\lim_{i\to\infty}\left\{\|\widetilde{\psi}_{1,i}\|^2_{L^2(M_1(\varepsilon_i))}+\left\|\xi_1 U^*\left(\frac{1}{\sqrt{|\log\varepsilon_i|}}r^{-1/2}\overline{\sigma}_{1/2}\right)\right\|^2_{L^2(M_1(\varepsilon_i))}\right\}.$$

We remark that, by Proposition 13, $\phi_2=0$ implies $\lim_{i\to\infty}\|\widetilde{\psi}_{1,i}\|_{L^2(M_1(\varepsilon_i))}=0$. In fact, one has, by (15),

$$\lim_{i\to\infty}\|\widetilde{\psi}_{1,i}\|_{L^2(M_1(\varepsilon_i))}=\|P_2(U\phi_2\restriction_\Sigma)\|_{L^2(\widetilde{M}_2)}. \tag{25}$$

Finally, one has

$$\lim_{i\to\infty}\left\|\xi_1 U^*\left(\frac{1}{\sqrt{|\log\varepsilon_i|}}r^{-1/2}\overline{\sigma}_{1/2}\right)\right\|_{L^2(M_1(\varepsilon_i))}=\|\overline{\sigma}_{1/2}\|_{L^2(\Sigma)}.$$

□

4.3. *Lower bound, the end.* Now let $\{\phi_1(\varepsilon),\dots,\phi_N(\varepsilon)\}$ be an orthonormal family of eigenforms of the Hodge–de Rham operator associated with the eigenvalues $\lambda_1(\varepsilon),\dots,\lambda_N(\varepsilon)$. We can use the same procedure of extraction for all the families. This gives, in the limit domain, a family $\{(\phi_1^j,\phi_2^j,\overline{\sigma}^j_{1/2})\}_{1\le j\le N}$. We already know, by Corollary 17, that each element has norm 1. If we show that they are orthogonal, then we are done, by applying the min–max formula to the limit problem (12).

Lemma 18. *The limit family is orthonormal in $\mathcal{H}_\infty$.*

Proof. If we follow the procedure for one index, up to terms converging to zero, we have decomposed the eigenforms $\phi_j(\varepsilon)$ on M_ε into three terms:

$$\Phi^j_\varepsilon=\psi_{1,i}+\phi_{1,i}^{(-1/2,1/2)},\quad \widetilde{\Phi}^j_\varepsilon=\widetilde{\psi}_{1,i},\quad\text{and}\quad \overline{\Phi}^j_\varepsilon=U^*\left(\frac{1}{\sqrt{|\log\varepsilon|}}r^{-1/2}\overline{\sigma}^j_{1/2}\right). \tag{26}$$

Let $a\neq b$ be two indices. If we apply Lemma 16 to any linear combination of $\phi_a(\varepsilon)$ and $\phi_b(\varepsilon)$, we obtain that

$$\lim_{i\to\infty}\{(\Phi^a_{\varepsilon_i},\widetilde{\Phi}^b_{\varepsilon_i})_{L^2(M_1(\varepsilon_i))}+(\Phi^b_{\varepsilon_i},\widetilde{\Phi}^a_{\varepsilon_i})_{L^2(M_1(\varepsilon_i))}\}=0.$$

If we apply (25), we obtain

$$\lim_{i\to\infty}\{(\widetilde{\Phi}^a_{\varepsilon_i},\widetilde{\Phi}^b_{\varepsilon_i})_{L^2(M_1(\varepsilon_i))}+(\phi^a_{2,\varepsilon},\phi^b_{2,\varepsilon})_{L^2(M_2)}\}=(\tilde{\phi}^a_2,\tilde{\phi}^b_2)_{L^2(\widetilde{M}_2)}.$$

Then finally, from $(\phi_a(\varepsilon),\phi_b(\varepsilon))_{L^2(M_\varepsilon)}=0$, we conclude that

$$(\phi^a_1,\phi^b_1)_{L^2(\overline{M}_1)}+(\phi^a_2,\phi^b_2)_{L^2(\widetilde{M}_2)}+(\overline{\sigma}^a_{1/2},\overline{\sigma}^b_{1/2})_{L^2(\Sigma)}=0.$$

□

Proposition 19. *The multiplicity of* 0 *in the limit spectrum is given by the sum*

$$\dim\mathrm{Ker}(\Delta_{1,W})+\dim\mathrm{Ker}(\mathfrak{D}_2)+i_{1/2},$$

where $i_{1/2}$ denotes the dimension of the vector space $\mathcal{I}_{1/2}$ — see (8) *— of extended solutions ω on $\widetilde{M}_2$ introduced by Carron* [2001b], *corresponding to a boundary term on restriction to $r=1$ with nontrivial component in* $\mathrm{Ker}\big(A-\frac{1}{2}\big)$.

If the limit value λ is nonzero, then it belongs to the positive spectrum of the Hodge–de Rham operator $\Delta_{1,W}$ on $\overline{M}_1$, with the space W as defined in (7).

Proof. The last process, with, in particular, (25) and (16), in fact constructs an element in the limit Hilbert space

$$\mathcal{H}_\infty := L^2(\overline{M}_1) \oplus \mathrm{Ker}(\widetilde{D}_2) \oplus \mathcal{I}_{1/2}.$$

This process is clearly *isometric* in the sense that, if we have an orthonormal family $\{\phi_j(\varepsilon_i)\}_j$ $(1 \le j \le N)$, we obtain at the limit an orthonormal family, where $\mathcal{H}_\infty$ is defined as an orthogonal sum of the Hilbert spaces. And, if we begin with eigenforms of Δ_{ε_i}, we obtain at the limit eigenforms of $\Delta_{1,W} \oplus \{0\} \oplus \{0\}$. The last calculus implies that $\liminf_{i\to\infty} \lambda_N(\varepsilon_i) \ge \lambda_N$. □

Remark 20. In order to understand this result, it is important to remember when the eigenvalue $\frac{1}{2}$ occurs in the spectrum of A. By the expression (4), we find that it occurs exactly:

- For n even, if $\frac{3}{4}$ is an eigenvalue of the Hodge–de Rham operator Δ_Σ acting on coexact forms of degree $\frac{1}{2}n$ or $\frac{1}{2}n-1$ of the submanifold Σ.
- For n odd, if 0 is an eigenvalue of Δ_Σ on forms of degree $\frac{1}{2}(n-1)$ or $\frac{1}{2}(n+1)$, but also if 1 is an eigenvalue on coexact forms of degree $\frac{1}{2}(n-1)$ on Σ.

A dilation of the metric on Σ allows us to avoid positive eigenvalues, but harmonic forms of degree $\frac{1}{2}(n-1)$ or $\frac{1}{2}(n+1)$ on Σ can not be avoided.

Moreover, Carron [2001a, Theorem 0.6] has proved that the extended index depends only on geometry at infinity: these harmonic forms on Σ will indeed create half-bound states, and then small eigenvalues will always appear.

5. Harmonic forms and small eigenvalues

It would be interesting to know how many small (but nonzero) eigenvalues appear. For this purpose, we can use the topological meaning of harmonic forms.

5.1. *Cohomology groups.* The topology of M_ε is independent of $\varepsilon \neq 0$ and can be understood by the Mayer–Vietoris exact sequence:

$$\cdots \longrightarrow H^p(M_\varepsilon) \xrightarrow{\text{res}} H^p(M_1(\varepsilon)) \oplus H^p(M_2) \xrightarrow{\text{dif}} H^p(\Sigma) \xrightarrow{\text{ext}} H^{p+1}(M_\varepsilon) \longrightarrow \cdots.$$

As already mentioned, the space $\mathrm{Ker}(\mathcal{D}_2) \oplus \mathcal{I}_{1/2}$ can be mapped into $H^*(M_2)$. More precisely, Hausel, Hunsicker and Mazzeo [Hausel et al. 2004, Theorem 1.A, p. 490] have proved that the space of the L^2-harmonic forms $\mathcal{H}^k_{L^2}(\widetilde{M}_2)$ on $\widetilde{M}_2$ is given by

$$\mathcal{H}^k_{L^2}(\widetilde{M}_2) \cong \begin{cases} H^k(M_2, \Sigma) & \text{if } k < \frac{1}{2}(n+1), \\ \mathrm{Im}(H^{(n+1)/2}(M_2, \Sigma) \to H^{(n+1)/2}(M_2)) & \text{if } k = \frac{1}{2}(n+1), \\ H^k(M_2) & \text{if } k > \frac{1}{2}(n+1). \end{cases} \tag{27}$$

We note that the space of L^2-harmonic forms is equal to that of L^2-harmonic fields, or the Hodge cohomology group, since $\widetilde{M}_2$ is complete.

For $\overline{M}_1$, we can use the results of Cheeger [1980; 1983]. Following his work, we know that the intersection cohomology groups $IH^*(\overline{M}_1)$ of $\overline{M}_1$ coincide with $\operatorname{Ker}(D_{1,\max} \circ D_{1,\min})$ if $H^{n/2}(\Sigma) = 0$. We also know that

$$IH^p(\overline{M}_1) \cong \begin{cases} H^p(M_1(\varepsilon)) & \text{if } p \leq \frac{1}{2}n, \\ H_c^p(M_1(\varepsilon)) & \text{if } p \geq \frac{1}{2}n + 1. \end{cases} \tag{28}$$

These results can be used for our study only if $D_{1,\max}$ and $D_{1,\min}$ coincide. This occurs if and only if A has no eigenvalues in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$. As a consequence of the expression of the eigenvalues of A, recalled in (4), this is the case if and only if:

- *for n odd*, the operator Δ_Σ has no eigenvalues in (0, 1) on coexact forms of degree $\frac{1}{2}(n-1)$;
- *for n even*, the operator Δ_Σ has no eigenvalues in $\left(0, \frac{3}{4}\right)$ on coexact forms of degree $\frac{1}{2}n$ or $\frac{1}{2}n - 1$, and $H^{n/2}(\Sigma) = 0$.

Thus, if $D_{1,\max} = D_{1,\min}$, which implies $H^{n/2}(\Sigma) = 0$ in the case where n is even, then the map

$$H^{n/2}(M_\varepsilon) \xrightarrow{\text{res}} H^{n/2}(M_1(\varepsilon)) \oplus H^{n/2}(M_2)$$

is surjective, and then any small eigenvalue in this degree must come from an element of $\operatorname{Ker}(\mathscr{D}_2) \oplus \mathscr{I}_{1/2}$ sent to 0 in $H^{n/2}(M_2)$. In this case also, the map

$$H^{n/2+1}(M_\varepsilon) \xrightarrow{\text{res}} H^{n/2+1}(M_1(\varepsilon)) \oplus H^{n/2+1}(M_2)$$

is injective, so there may exist small eigenvalues in this degree.

5.2. ***Some examples.*** We exhibit a general procedure to construct new examples as follows: Let W_i, $i = 1, 2$, be two compact Riemannian manifolds with boundary Σ_i and dimension $n_i + 1$ such that $n_1 + n_2 = n \geq 2$. We can apply our result to $M_1 := W_1 \times \Sigma_2$ and $M_2 := \Sigma_1 \times W_2$. The manifold M_ε is always diffeomorphic to $M = M_1 \cup M_2$.

For instance, let v_2 be the volume form of (Σ_2, h_2). It defines a harmonic form on M_1, and this form will appear in the limit spectrum if, transplanted onto $\overline{M}_1$, it defines an element in the domain of the operator $\Delta_{1,W}$.

In the notation introduced in Section 2.2, this element corresponds to $\beta = 0$ and $\alpha = r^{n/2-n_2} v_2$, and the expression of A gives that

$$A(\beta, \alpha) = \left(n_2 - \tfrac{1}{2}n\right)(\beta, \alpha).$$

If $\frac{1}{2}n - n_2 > 0$, then (β, α) is in the domain of $D_{1,\max} \circ D_{1,\min}$, and, if $n_2 = \frac{1}{2}n$, it is in the domain of $\Delta_{1,W}$ for the eigenvalue 0 of A.

So, if we know that $H^{n_2}(M) = 0$ or, more generally, $\dim H^{n_2}(M) < \dim H^{n_2}(\Sigma_2)$ in the case where Σ_2 is not connected, then this element will create a small eigenvalue on M_ε. If D^k denotes the unit ball in $\mathbb{R}^k$, this is the case for

$$W_1 = D^{n_1+1} \quad \text{and} \quad W_2 = D^{n_2+1} \quad \text{for } n_2 \leq n_1.$$

Then, $M = \mathbb{S}^{n_1+n_2+1}$ and we obtain:

Corollary 21. *For any degree k and any $\varepsilon > 0$, there exists a metric on $\mathbb{S}^m$ such that the Hodge–de Rham operator acting on k-forms admits an eigenvalue smaller than ε. We can see that, for $k < \frac{1}{2}m$, it is in the spectrum of coexact forms, and, by duality, for $k \geq \frac{1}{2}m$ it is in the spectrum of exact k-forms.*

Indeed, the case $k < \frac{1}{2}m$ is a direct application, as explained above. We see that our quasimode is coclosed. Thus, in the case where m is even, if ω is an eigenform of degree $\frac{1}{2}m - 1$ with small eigenvalue, then $d\omega$ is a closed eigenform with the same eigenvalue and degree $\frac{1}{2}m$. Finally, the case $k > \frac{1}{2}m$ is obtained by Hodge duality. We remark that in the case $k = 0$ we recover *Cheeger's dumbbell*, and also that this result has been proved by Guerini [2004] with another deformation, although he did not give the convergence of the spectrum.

By the surgery of the previous case, we obtain, for

$$W_1 := \mathbb{S}^{n_1} \times [0, 1] \quad \text{and} \quad W_2 := D^{n_2+1} \quad \text{for } 0 \leq n_2 < n_1 \text{ and } n = n_1 + n_2 \geq 2,$$

that $\Sigma_1 = \mathbb{S}^{n_1} \sqcup \mathbb{S}^{n_1}$, $\Sigma_2 = \mathbb{S}^{n_2}$ and $M = \mathbb{S}^{n_1} \times \mathbb{S}^{n_2+1}$. The volume form $v_2 \in H^{n_2}(\Sigma_2)$ again defines a harmonic form on $\overline{M}_1$ and, since $H^{n_2}(\mathbb{S}^{n_1} \times \mathbb{S}^{n_2+1}) = 0$, if $n_2 < n_1$, then v_2 defines a small eigenvalue on n_2-forms of M_ε.

Thus, by the duality, we obtain:

Corollary 22. *For any $k, l \geq 0$ with $0 \leq k-1 < l$ and any $\varepsilon > 0$, there exists a metric on $\mathbb{S}^l \times \mathbb{S}^k$ such that the Hodge–de Rham operator acting on $(k-1)$-forms and on $(l+1)$-forms admits an eigenvalue smaller than ε.*

This corollary is also a consequence of the previous one: we know that there exists a metric on $\mathbb{S}^k$ whose Hodge–de Rham operator admits a small eigenvalue on $(k-1)$-forms, and this property is maintained on $\mathbb{S}^l \times \mathbb{S}^{k+1}$.

With the same construction, we can exchange the roles of M_1 and M_2: the two volume forms of $\mathbb{S}^{n_1} \sqcup \mathbb{S}^{n_1}$ create one n_1-form with small but nonzero eigenvalue on $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2+1}$ if $n_1 \leq n_2 + 1$. By the duality, we obtain an (n_2+1)-form with small eigenvalue. So, with new notations, we have obtained:

Corollary 23. *For any $k < l$ with $k + l \geq 3$ and any $\varepsilon > 0$, there exists a metric on $\mathbb{S}^l \times \mathbb{S}^k$ such that the Hodge–de Rham operator acting on l-forms and on k-forms admits a positive eigenvalue smaller than ε.*

More generally, by repeating the $(k-1)$-dimensional surgery L times, we obtain the following:

Proposition 24 [Sha and Yang 1991]. *The connected sum of L copies of the product spheres, $\sharp_{i=1}^{L}(\mathbb{S}^k \times \mathbb{S}^l)$, can be decomposed as follows*:

$$\mathop{\sharp}_{i=1}^{L} (\mathbb{S}^k \times \mathbb{S}^l) \cong \left(\mathbb{S}^{k-1} \times \left(\mathbb{S}^{l+1} \setminus \coprod_{i=0}^{L} D_i^{l+1}\right)\right) \cup_\partial \left(D^k \times \coprod_{i=0}^{L} \mathbb{S}_i^l\right).$$

Remark 25. J.-P. Sha and D. Yang [1991] constructed a Riemannian metric of positive Ricci curvature on this manifold. More generally, see also [Wraith 2007].

In a similar way, using Proposition 24, we can obtain the small positive eigenvalues on the connected sum of L copies of the product spheres $\sharp_{i=1}^{L}(\mathbb{S}^k \times \mathbb{S}^l)$.

All these examples use the spectrum of $\overline{M}_1$. We can obtain also examples using the reduced L^2-cohomology group of $\widetilde{M}_2$, which is given by (27) [Hausel et al. 2004].

Suppose now that $n = \dim \Sigma$ is odd. Then, we have the long exact sequence

$$\cdots \to H^k(M_2, \Sigma) \to H^k(M_2) \to H^k(\Sigma) \to H^{k+1}(M_2, \Sigma) \to \cdots.$$

For $k = \frac{1}{2}(n-1)$, the space $H^k(M_2, \Sigma)$ is isomorphic to the reduced L^2-cohomology group of $\widetilde{M}_2$. If $H^{(n-1)/2}(\Sigma)$ is nontrivial, then any nontrivial harmonic k-form on Σ will create an extended solution, corresponding to an eigenvector of A with eigenvalue $\frac{1}{2}$.

For example, take $\Sigma = \mathbb{S}^k \times \mathbb{S}^{k+1}$ for $k = \frac{1}{2}(n-1)$; then $H^k(\Sigma)$ is nontrivial. Any nontrivial form $\omega \in H^k(\Sigma)$ sent to $0 \in H^{k+1}(M_2, \Sigma)$ comes from an element $\tilde{\omega} \in H^k(M_2)$ which is not in the reduced L^2-cohomology group of $\widetilde{M}_2$, by (27).

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