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GLOBAL WELL-POSEDNESS ON THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

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As a continuation of our previous work, we consider the global well-posedness for the derivative nonlinear Schrödinger equation. We prove that it is globally well posed in the energy space, provided that the initial data $u_0 \in H^1(\mathbb{R})$ with $\|u_0\|_{L^2} < 2\sqrt{\pi}$.

1. Introduction

We study the following Cauchy problem of the nonlinear Schrödinger equation with derivative (DNLS):

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x(|u|^2 u), & t \in \mathbb{R},\ x \in \mathbb{R}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}). \end{cases} \tag{1-1}$$

It arises from studying the propagation of circularly polarized Alfvén waves in magnetized plasma with a constant magnetic field; see [Mio et al. 1976; Mjolhus 1976; Sulem and Sulem 1999] and the references therein. The equation in (1-1) is L^2-critical and completely integrable. The H^1-solution of (1-1) obeys the following mass, energy, and momentum conservation laws:

$$M(u(t)) := \int_{\mathbb{R}} |u(t,x)|^2\,dx = M(u_0), \tag{1-2}$$

$$E_D(u(t)) := \int_{\mathbb{R}} \left(|u_x(t,x)|^2 + \tfrac{3}{2}\operatorname{Im}|u(t,x)|^2 u(t,x)\overline{u_x(t,x)} + \tfrac{1}{2}|u(t,x)|^6 \right) dx = E_D(u_0), \tag{1-3}$$

$$P_D(u(t)) := \operatorname{Im}\int_{\mathbb{R}} \overline{u(t,x)}u_x(t,x)\,dx - \frac{1}{2}\int_{\mathbb{R}} |u(t,x)|^4\,dx = P_D(u_0). \tag{1-4}$$

Local well-posedness for the Cauchy problem (1-1) is well understood. It was proved in the energy space $H^1(\mathbb{R})$ in [Hayashi 1993; Hayashi and Ozawa 1992; 1994], and earlier by Guo and Tan [1991] and Tsutsumi and Fukuda [1980; 1981] in smooth spaces. See [Biagioni and Linares 2001; Takaoka 1999; 2001] for local well-posedness and ill-posedness results for rough data below the energy space.

The global well-posedness for (1-1) has also been widely studied. By using mass and energy conservation laws, and the gauge transformations, Hayashi and Ozawa [1994; Ozawa 1996] proved that (1-1) is globally well-posed in the energy space $H^1(\mathbb{R})$ under the condition

$$\|u_0\|_{L^2} < \sqrt{2\pi}. \tag{1-5}$$

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Here 2π is the mass of the ground state Q, which is the unique (up to some symmetries) positive solution of the elliptic equation

$$-Q_{xx}+Q-\tfrac{3}{16}Q^5=0. \tag{1-6}$$

As shown in [Weinstein 1983], $Q=2[\cosh(2x)]^{-1/2}$. Since Q is an optimizer for the Gagliardo–Nirenberg inequality (1-12), any function with mass strictly less than the mass of Q has positive energy.

Condition (1-5) was improved recently in [Wu 2013]. We proved that there exists a small constant $\varepsilon_*>0$ such that (1-1) is still globally well-posed in the energy space when the initial data satisfies $\|u_0\|_{L^2}<\sqrt{2\pi}+\varepsilon_*$. The result implies that, for (1-1), the ground state mass 2π is not the threshold of the global well-posedness and blow-up. This is different from the L^2-critical power-type Schrödinger equation (the nonlinearity $i\partial_x(|u|^2u)$ in (1-1) is replaced by $-\frac{3}{16}|u|^4u$); see [Wu 2013] for further discussion.

For related results on the well-posedness and stability theory for the derivative nonlinear Schrödinger equation (1-1), see [Colin and Ohta 2006; Colliander et al. 2001; 2002; Grünrock and Herr 2008; Guo and Wu 1995; Herr 2006; Miao et al. 2011; Nahmod et al. 2012; Takaoka 2001; Thomann and Tzvetkov 2010; Win 2010].

In this paper, we continue to consider the L^2-assumption on initial data and obtain the global well-posedness as follows:

Theorem 1.1. *For any $u_0\in H^1(\mathbb{R})$ with*

$$\int_{\mathbb{R}}|u_0(x)|^2\,dx<4\pi, \tag{1-7}$$

the Cauchy problem (1-1) *is globally well-posed in $H^1(\mathbb{R})$ and the solution u satisfies*

$$\|u\|_{L_t^\infty H_x^1}\le C(\|u_0\|_{H^1}).$$

As $2\pi=\|Q\|_{L^2}^2$, we notice that there is also a solitary wave solution whose mass is 4π, given by

$$u(t,x)=e^{3i/4\int_{-\infty}^{x+t}|W(y)|^2\,dy}e^{-it/4-ix/2}W(x+t), \tag{1-8}$$

where W is the ground state of the elliptic equation

$$-W_{xx}+\tfrac{1}{2}W^3-\tfrac{3}{16}W^5=0. \tag{1-9}$$

Up to some symmetries,

$$W(x)=2(x^2+1)^{-1/2}. \tag{1-10}$$

Therefore, Theorem 1.1 indicates that the Cauchy problem (1-1) is globally well-posed in $H^1(\mathbb{R})$ when $\|u_0\|_{L^2}<\|W\|_{L^2}$.

Compared to Q, W is polynomial decaying at infinity. Furthermore, W is an optimal function of the sharp Gagliardo–Nirenberg inequality (see [Agueh 2006])

$$\|f\|_{L^6}\le C_{\mathrm{GN}}\|f\|_{L^4}^{8/9}\|f_x\|_{L^2}^{1/9}, \tag{1-11}$$

where we wrote C_{GN} for the sharp constant $C_{\mathrm{GN}} = 3^{1/6}(2\pi)^{-1/9}$. This inequality plays an important role in the proof of our main theorem. There is also a comparison with another sharp Gagliardo–Nirenberg inequality (see [Weinstein 1983]),

$$\|f\|_{L^6}^6 \le \frac{4}{\pi^2}\|f\|_{L^2}^4\|f_x\|_{L^2}^2, \tag{1-12}$$

in which the equality is attained by Q, which was applied previously to prove the global well-posedness when $\|u_0\|_{L^2} < \sqrt{2\pi}$.

So there is an interesting problem of whether $\|W\|_{L^2}^2 = 4\pi$ is the mass threshold of the global well-posedness and blowup for (1-1). See Section 3 below for further discussion.

Now let us have a look at the strategy of the proof of Theorem 1.1. Developed by Hayashi and Ozawa, the gauge transformation is an important tool to study the derivative nonlinear Schrödinger equation. Let

$$v(t,x) := e^{-3i/4\int_{-\infty}^{x}|u(t,y)|^2\,dy}u(t,x); \tag{1-13}$$

then, from (1-1), v is the solution of

$$i\partial_t v + \partial_x^2 v = \tfrac{1}{2}i|v|^2 v_x - \tfrac{1}{2}iv^2\bar{v}_x - \tfrac{3}{16}|v|^4 v \tag{1-14}$$

with the initial data $v_0 = \exp\bigl(-\frac{3}{4}i\int_{-\infty}^{x}|u_0(y)|^2\,dy\bigr)u_0$. Moreover, v obeys the same mass conservation law as (1-2), the energy conservation law (1-3) becomes

$$E(v(t)) := \|v_x(t)\|_{L_x^2}^2 - \tfrac{1}{16}\|v(t)\|_{L_x^6}^6 = E(v_0), \tag{1-15}$$

and the momentum conservation law (1-4) becomes

$$P(v(t)) := \operatorname{Im}\int_{\mathbb{R}}\overline{v(t,x)}v_x(t,x)\,dx + \frac{1}{4}\int_{\mathbb{R}}|v(t,x)|^4\,dx = P(v_0). \tag{1-16}$$

From the argument used in [Wu 2013] to prove the global well-posedness for the DNLS, an important consideration is the usage of the momentum conservation law. We observe that the key point is to give a small control of the following term from (1-16):

$$\operatorname{Im}\int_{\mathbb{R}}\overline{v(t,x)}v_x(t,x)\,dx. \tag{1-17}$$

To be more precise, one may prove that

$$-\operatorname{Im}\int_{\mathbb{R}}\overline{v(t,x)}v_x(t,x)\,dx \le c\|v_x(t)\|_{L^2}\|v(t)\|_{L^2}, \tag{1-18}$$

where c is a positive constant. This is trivial for $c = 1$ by Hölder's inequality. Suppose that one can obtain the inequality with a suitable small constant c. Then the global well-posedness will follow. In [Wu 2013], by using the rigidity of the ground state Q, we proved that, if the mass is larger but close to 2π and there is a time sequence $\{t_n\}$ such that $\|v(t_n)\|_{H^1}$ tends to infinity, then $v(t_n)$ is close to Q up to some symmetries. Since Q is real-valued, (1-18) can be given for small $c > 0$.

In this paper, we present a different argument to prove the bound (1-18) under the suitable but explicit assumption of L^2-norm of the initial data. Our method here does not need to use the property of the ground state Q of (1-6). As was previously mentioned, it depends heavily on the sharp Gagliardo–Nirenberg inequality (1-11). This is to be expected, since the norms involved in the inequality (1-11) are strongly related to the energy and momentum conservation laws.

Let us expand our argument. If $\|v(t)\|_{H^1}$ tends to infinity, then, by the momentum and energy conservation laws, (1-18) is approximately

$$\tfrac{1}{4}\|v(t)\|_{L^4}^4 \approx -\operatorname{Im}\int_{\mathbb{R}} \overline{v(t,x)}\, v_x(t,x)\,dx \le c\|v_x(t)\|_{L^2}\|v(t)\|_{L^2} \approx c\|v_0\|_{L^2}\|v(t)\|_{L^6}^3.$$

So, to obtain the small bound c, we turn to consider the quantity

$$f(t) := \frac{\|v(t)\|_{L^4}^4}{\|v(t)\|_{L^6}^3}.$$

Indeed, we shall prove that f^2 obeys some cubic inequality. Thus, the condition for global well-posedness is transformed to finding the solution to an elementary cubic equation.

This paper is organized as follows. In Section 2, we present the proof of Theorem 1.1. In Section 3, we discuss some related problems.

2. The proof of Theorem 1.1

Let v be the function in (1-13), which is the solution of the equation (1-14). Note that

$$u_x = e^{3i/4\int_{-\infty}^{x}|v(t,y)|^2\,dy}\big(\tfrac{3}{4}i|v|^2 v + v_x\big).$$

Therefore, by the sharp Gagliardo–Nirenberg inequality (1-12) and mass conservation law, for any $t \in \mathbb{R}$,

$$\begin{aligned}\|u_x(t)\|_{L^2} &\le \|v_x(t)\|_{L^2} + \frac{3}{4}\|v(t)\|_{L^6}^3 \le \|v_x(t)\|_{L^2} + \frac{3}{2\pi}\|v(t)\|_{L^2}^2\|v_x(t)\|_{L^2}\\ &\le \Big(1+\frac{3}{2\pi}\|u_0\|_{L^2}^2\Big)\|v_x(t)\|_{L^2}.\end{aligned}$$

That is, the boundedness of v in H^1-norm implies the boundedness of u in H^1-norm. Therefore, to prove the theorem, we may consider the function v in (1-13) instead. To simplify the notations, we set

$$E_0 = E(v_0), \quad P_0 = P(v_0), \quad m_0 = M(v_0).$$

Furthermore, we assume $m_0 > 2\pi$. Otherwise, the global well-posedness has been proved in [Hayashi and Ozawa 1994; Wu 2013].

Let $(-T_-(v_0), T_+(v_0))$ be the maximal lifespan of the solution v of (1-14). To prove Theorem 1.1, it is sufficient to obtain the (indeed uniformly) a priori estimate of the solutions in H^1-norm. That is,

$$\sup_{t\in(-T_-(v_0),T_+(v_0))}\|v_x(t)\|_{L^2} < +\infty.$$

As in [Wu 2013], we argue by contradiction. Suppose that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ with limit $-T_-(v_0)$ or $T_+(v_0)$ such that

$$\|v_x(t_n)\|_{L^2} \to +\infty \quad \text{as } n\to\infty. \tag{2-1}$$

Then, from the energy conservation law, we also have

$$\|v(t_n)\|_{L^6} \to +\infty \quad \text{as } n \to \infty.$$

Let us define the sequence $\{f_n\}_{n=1}^{\infty}$ by

$$f_n = \frac{\|v(t_n)\|_{L^4}^4}{\|v(t_n)\|_{L^6}^3};$$

then we have both the lower and upper bounds of f_n as follows:

Lemma 2.1. *There exists a sequence ε_n, with $\varepsilon_n \to 0$ as $n \to \infty$, such that*

$$2C_{\mathrm{GN}}^{-9/2} + \varepsilon_n \le f_n \le \sqrt{m_0}. \tag{2-2}$$

Proof of Lemma 2.1. From Hölder's inequality, we have

$$\|v(t_n)\|_{L^4}^4 \le \|v(t_n)\|_{L^2}\|v(t_n)\|_{L^6}^3 = \sqrt{m_0}\|v(t_n)\|_{L^6}^3,$$

and thus

$$f_n \le \sqrt{m_0}.$$

On the other hand, from the sharp Gagliardo–Nirenberg inequality (1-11) and the energy conservation law (1-15), we have

$$\begin{aligned} f_n \ge \frac{\left(C_{\mathrm{GN}}^{-6}\|v(t_n)\|_{L^6}^6\|v_x(t_n)\|_{L^2}^{-2/3}\right)^{3/4}}{\|v(t_n)\|_{L^6}^3} &= C_{\mathrm{GN}}^{-9/2}\frac{\|v(t_n)\|_{L^6}^{3/2}}{\|v_x(t_n)\|_{L^2}^{1/2}} \\ &= 2C_{\mathrm{GN}}^{-9/2}\frac{\|v(t_n)\|_{L^6}^{3/2}}{\left(\|v(t_n)\|_{L^6}^6 + 16E_0\right)^{1/4}} \\ &= 2C_{\mathrm{GN}}^{-9/2} + \varepsilon_n, \end{aligned}$$

where

$$\varepsilon_n := 2C_{\mathrm{GN}}^{-9/2}\frac{\|v(t_n)\|_{L^6}^{3/2} - \left(\|v(t_n)\|_{L^6}^6 + 16E_0\right)^{1/4}}{\left(\|v(t_n)\|_{L^6}^6 + 16E_0\right)^{1/4}}.$$

By the mean value theorem, we have

$$\varepsilon_n = O\left(\|v(t_n)\|_{L^6}^{-6}\right) \to 0.$$

This proves the lemma. □

By Lemma 2.1, and $\|v(t_n)\|_{L^4}^4 = f_n\|v(t_n)\|_{L^6}^3$, we have

$$\|v(t_n)\|_{L^4} \to +\infty \quad \text{as } n \to \infty.$$

In the spirit of [Banica 2004], we define

$$\phi(t,x) = e^{i\alpha x}v(t,x),$$

where the parameter α depends on t and is given below. Then $\phi_x(t,x) = e^{i\alpha x}(i\alpha v(t,x) + v_x(t,x))$, and thus

$$\|\phi_x\|_{L^2}^2 = \|v_x\|_{L^2}^2 + 2\alpha \operatorname{Im} \int \bar{v} v_x \, dx + \alpha^2 \|v\|_{L^2}^2.$$

Subtracting $\frac{1}{16}\|\phi\|_{L^6}^6 = \frac{1}{16}\|v\|_{L^6}^6$ from both sides yields

$$E(\phi) = E(v) + 2\alpha \operatorname{Im} \int \bar{v} v_x \, dx + \alpha^2 \|v\|_{L^2}^2.$$

By the mass and energy conservation laws (1-2) and (1-15), this gives

$$-2\alpha \operatorname{Im} \int \overline{v(t,x)} v_x(t,x) \, dx = -E(\phi(t)) + \alpha^2 m_0 + E_0. \tag{2-3}$$

On the other hand, using (1-11), we have

$$\begin{aligned} E(\phi(t_n)) &= \|\phi_x(t_n)\|_{L^2}^2 - \tfrac{1}{16}\|\phi(t_n)\|_{L^6}^6 \\ &\geq C_{\mathrm{GN}}^{-18} \|\phi(t_n)\|_{L^6}^{18} \|\phi(t_n)\|_{L^4}^{-16} - \tfrac{1}{16}\|\phi(t_n)\|_{L^6}^6 \\ &= \big(C_{\mathrm{GN}}^{-18} \|v(t_n)\|_{L^6}^{12} \|v(t_n)\|_{L^4}^{-16} - \tfrac{1}{16}\big)\|\phi(t_n)\|_{L^6}^6 \\ &= \big(C_{\mathrm{GN}}^{-18} f_n^{-4} - \tfrac{1}{16}\big)\|v(t_n)\|_{L^6}^6. \end{aligned}$$

Combining this with (2-3) gives

$$-2\alpha \operatorname{Im} \int \overline{v(t_n,x)} v_x(t_n,x) \, dx \leq \big(\tfrac{1}{16} - C_{\mathrm{GN}}^{-18} f_n^{-4}\big)\|v(t_n)\|_{L^6}^6 + \alpha^2 m_0 + E_0,$$

which implies, for $\alpha > 0$,

$$-\operatorname{Im} \int \overline{v(t_n,x)} v_x(t_n,x) \, dx \leq \frac{1}{2\alpha}\big(\tfrac{1}{16} - C_{\mathrm{GN}}^{-18} f_n^{-4}\big)\|v(t_n)\|_{L^6}^6 + \frac{1}{2}\alpha m_0 + \frac{1}{2\alpha} E_0. \tag{2-4}$$

For convenience, we define β_n as

$$\beta_n := m_0^{-1}\big(\tfrac{1}{16} - C_{\mathrm{GN}}^{-18} f_n^{-4}\big)\|v(t_n)\|_{L^6}^6.$$

We split this into two cases:

Case 1: $\beta_n < 1$ for infinitely many n. This implies that, for such n,

$$\big(\tfrac{1}{16} - C_{\mathrm{GN}}^{-18} f_n^{-4}\big)\|v(t_n)\|_{L^6}^6 < m_0.$$

Therefore, from (2-4), we have

$$-\operatorname{Im} \int \overline{v(t_n,x)} v_x(t_n,x) \, dx \leq \frac{1}{2\alpha} m_0 + \frac{1}{2}\alpha m_0 + \frac{1}{2\alpha} E_0. \tag{2-5}$$

In particular, choosing $\alpha = 1$, we obtain

$$-\operatorname{Im} \int \overline{v(t_n,x)} v_x(t_n,x) \, dx \leq m_0 + \tfrac{1}{2} E_0. \tag{2-6}$$

By the momentum conservation law (1-16), we have

$$\tfrac{1}{4}\|v(t_n)\|_{L^4}^4 = -\operatorname{Im}\int \overline{v(t_n,x)}\,v_x(t_n,x)\,dx + P_0. \tag{2-7}$$

Hence, combining this with (2-6) and (2-7), we obtain

$$\|v(t_n)\|_{L^4}^4 \le 2(2m_0 + E_0 + 2P_0).$$

This contradicts $\|v(t_n)\|_{L^4} \to +\infty$, and thus we can rule out this case.

Case 2: $\beta_n \ge 1$ for all sufficiently large n. In this case, we set $\alpha = \alpha(t_n) = \sqrt{\beta_n}$. Then (2-4) becomes

$$-\operatorname{Im}\int \overline{v(t_n,x)}\,v_x(t_n,x)\,dx \le \tfrac{1}{4}\sqrt{m_0(1-16C_{\mathrm{GN}}^{-18} f_n^{-4})}\,\|v(t_n)\|_{L^6}^3 + \tfrac{1}{2}\beta_n^{-1/2}E_0. \tag{2-8}$$

By (2-7) and (2-8),

$$\|v(t_n)\|_{L^4}^4 \le \sqrt{m_0(1-16C_{\mathrm{GN}}^{-18} f_n^{-4})}\,\|v(t_n)\|_{L^6}^3 + 2\beta_n^{-1/2}E_0 + 4P_0,$$

which implies that

$$f_n \le \sqrt{m_0(1-16C_{\mathrm{GN}}^{-18} f_n^{-4})} + (2\beta_n^{-1/2}E_0 + 4P_0)\|v(t_n)\|_{L^6}^{-3}.$$

This provides the inequality

$$f_n^6 \le m_0 f_n^4 - 16 m_0 C_{\mathrm{GN}}^{-18} + f_n^4 \mathcal{R}_n, \tag{2-9}$$

where

$$\mathcal{R}_n = 2\sqrt{m_0(1-16C_{\mathrm{GN}}^{-18} f_n^{-4})}\,(2\beta_n^{-1/2}E_0+4P_0)\|v(t_n)\|_{L^6}^{-3} + (2\beta_n^{-1/2}E_0+4P_0)^2\|v(t_n)\|_{L^6}^{-6}.$$

Since $\beta_n \ge 1$ and $0 \le 1 - 16C_{\mathrm{GN}}^{-18} f_n^{-4} \le 1$, we have

$$\mathcal{R}_n \le 2\sqrt{m_0}(2E_0+4P_0)\|v(t_n)\|_{L^6}^{-3} + (2E_0+4P_0)^2\|v(t_n)\|_{L^6}^{-6} = O\big(\|v(t_n)\|_{L^6}^{-3}\big).$$

From Lemma 2.1, we have

$$f_n^4\mathcal{R}_n = O\big(\|v(t_n)\|_{L^6}^{-3}\big) \to 0 \quad \text{as } n \to \infty.$$

Thus, for any small fixed $\epsilon > 0$, by choosing n large enough we have $f_n^4\mathcal{R}_n \le \epsilon$. Hence (2-9) becomes

$$f_n^6 \le m_0 f_n^4 - 16 m_0 C_{\mathrm{GN}}^{-18} + \epsilon. \tag{2-10}$$

Let $X = f_n^2$; then (2-10) becomes the inequality

$$X^3 - m_0 X^2 + b \le 0, \tag{2-11}$$

where $b = 16 m_0 C_{\mathrm{GN}}^{-18} - \epsilon > 0$. Let

$$F(X) = X^3 - m_0 X^2 + b;$$

then $F(X)$ attains its minimum value at $\frac{2}{3}m_0$ in the region $[0, \infty)$. Therefore, there are two positive solutions X_1 and X_2 of the equation

$$X^3 - m_0X^2 + b = 0 \tag{2-12}$$

if and only if $F\left(\frac{2}{3}m_0\right) < 0$. In other words, the inequality (2-11) has no solution in the region $[0, +\infty)$ if and only if

$$F\left(\tfrac{2}{3}m_0\right) > 0. \tag{2-13}$$

Hence, this leads to a contradiction under the condition (2-13).

Condition (2-13) is equivalent to

$$\tfrac{8}{27}m_0^3 - \tfrac{4}{9}m_0^3 + b > 0.$$

Since ϵ is arbitrarily small, this reduces to

$$\tfrac{8}{27}m_0^3 - \tfrac{4}{9}m_0^3 + 16m_0C_{\mathrm{GN}}^{-18} > 0,$$

which yields

$$m_0 < 6\sqrt{3}C_{\mathrm{GN}}^{-9} = 6\sqrt{3}\frac{1}{3^{3/2}(2\pi)^{-1}} = 4\pi.$$

Therefore, (1-14) is globally well-posed when $m_0 < 4\pi$. This proves the theorem.

One may expect to get some profit from the restriction $X \in (4C_{\mathrm{GN}}^{-9}, m_0)$ (rather than $[0, +\infty)$) given by Lemma 2.1. However, we cannot get any more from it. To see this, we note that, in the case $m_0 \geq 4\pi$, (2-11) is solved in the region $[0, +\infty)$ by

$$X_1 < X < X_2,$$

and we claim that

$$4C_{\mathrm{GN}}^{-9} < X_1 < X_2 < m_0. \tag{2-14}$$

Indeed, when $m_0 \geq 4\pi$,

$$\tfrac{2}{3}m_0 \geq \tfrac{8}{3}\pi > 4C_{\mathrm{GN}}^{-9} = \frac{8}{3\sqrt{3}}\pi,$$

and, for small ϵ, we have

$$F(4C_{\mathrm{GN}}^{-9}) = 64C_{\mathrm{GN}}^{-27} - \epsilon > 0,$$

which together imply that $4C_{\mathrm{GN}}^{-9} < X_1$. Similarly, since

$$\tfrac{2}{3}m_0 < m_0 \quad \text{and} \quad F(m_0) = b > 0,$$

we have $X_2 < m_0$. In conclusion, we have (2-14). Therefore, the inequality (2-11) is always solvable in the region of $(4C_{\mathrm{GN}}^{-9}, m_0)$ when $m_0 \geq 4\pi$, and so we can not obtain the contradiction from the restriction of $(4C_{\mathrm{GN}}^{-9}, m_0)$. We show this case graphically in Figure 1.

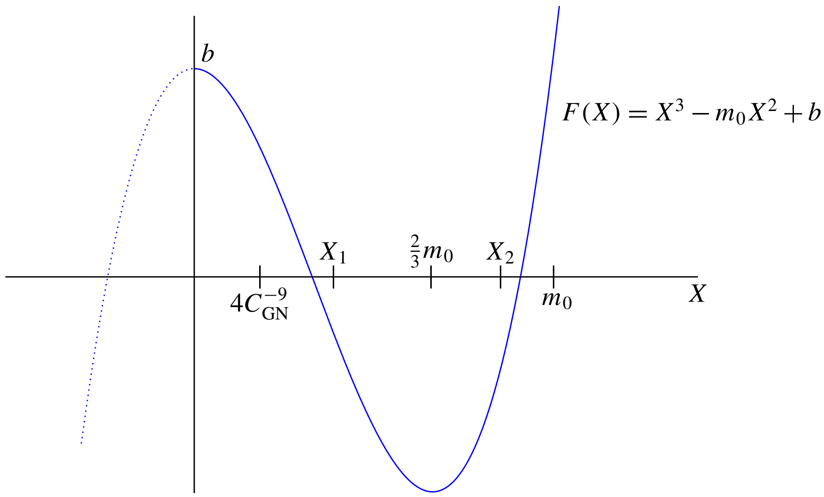

Figure 1. Graph of $F(X)$.

3. Further discussion

In this section, we would like to make a few remarks and indicate some related problems which remain open.[1]

First of all, whether or not the mass $M(W) = 4\pi$ is the mass threshold for global well-posedness of (1-1) is not resolved in this paper. To understand the problem, we make some remarks on W and the equation (1-9) in the following.

As shown in [Colin and Ohta 2006; Guo and Wu 1995], (1-14) has a two-parameter family of solitary wave solutions,

$$v_{\omega,c} = \phi_{\omega,c}(x+ct)e^{i\omega t-(ic/2)(x+ct)}, \tag{3-1}$$

where $(\omega, c) \in \mathbb{R}^2$ and $\phi_{\omega,c}$ is a positive solution of the elliptic equation

$$-\partial_{xx}\phi + \left(\omega - \tfrac{1}{4}c^2\right)\phi + \tfrac{1}{2}c\phi^3 - \tfrac{3}{16}\phi^5 = 0. \tag{3-2}$$

When $c^2 < 4\omega$, $\phi_{\omega,c}$ can be written as

$$\phi_{\omega,c}(x) = \left\{\frac{\sqrt{\omega}}{4\omega - c^2}\left[\cosh(\sqrt{4\omega - c^2}\,x) - \frac{c}{2\sqrt{\omega}}\right]\right\}^{-\frac{1}{2}}.$$

Guo and Wu [1995] proved that the solitary wave solutions (3-1) are orbitally stable when $c < 0$ and $c^2 < 4\omega$. This was extended by Colin and Ohta [2006], who proved the orbital stability for any $c^2 < 4\omega$.

Now we consider the other cases. From Pohožaev's identity, there is no solution for (3-2) when $4\omega \le c^2$ and $c \le 0$, and, from [Berestycki and Lions 1983] (see Section 6, Theorem 5), when $c^2 > 4\omega$ (3-2) has no positive solution which vanishes at infinity. Hence, the only remaining case is the "zero mass" case,

[1] Part of the contents in this section are from discussions with Soonsik Kwon.

$c^2 = 4\omega$ and $c > 0$. Thus, the "zero mass" case can be regarded as the endpoint case in the family of the solitary wave solutions (3-1).

For the endpoint case $c^2 = 4\omega$ and $c > 0$,

$$-\partial_{xx}\phi + \tfrac{1}{2}c\phi^3 - \tfrac{3}{16}\phi^5 = 0$$

is exactly solved by

$$W_c(x) = c^{1/2}W(cx),$$

where W is as defined in (1-10). Moreover,

$$\|W_c\|_{L^2}^2 = \|W\|_{L^2}^2 = 4\pi.$$

So it is an interesting problem whether the solitary wave solution (1-8) is orbitally stable or unstable, which is not covered in [Colin and Ohta 2006; Guo and Wu 1995]. See [Ohta 2014] for related studies.

The existence of the finite-time blow-up solution is also an open problem for (1-1). There are some related results on the generalized derivative nonlinear Schrödinger equation,

$$i\partial_t u + \partial_x^2 u = i|u|^{2\sigma}\partial_x u, \quad \sigma > 1. \tag{3-3}$$

This is a mass supercritical equation. See [Ambrose and Simpson 2014; Hao 2007; Liu et al. 2013b] for local and stability theories. Numerical simulations by Liu, Simpson and Sulem [Liu et al. 2013a] suggest the existence of finite-time blow-up solutions for (3-3). However, a rigorous proof remains to be found.

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