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ON THE BOUNDARY VALUE PROBLEM FOR THE SCHRÖDINGER EQUATION COMPATIBILITY CONDITIONS AND GLOBAL EXISTENCE

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We consider linear and nonlinear Schrödinger equations on a domain Ω with nonzero Dirichlet boundary conditions and initial data. We first study the linear boundary value problem with boundary data of optimal regularity (in anisotropic Sobolev spaces) with respect to the initial data. We prove well-posedness under natural compatibility conditions. This is essential for the second part, where we prove the existence and uniqueness of maximal solutions for nonlinear Schrödinger equations. Despite the nonconservation of energy, we also obtain global existence in several (defocusing) cases.

On étudie des équations de Schrödinger linéaires et non linéaires sur un domaine Ω avec donnée initiale et condition au bord de Dirichlet non nulles. Dans une première partie on étudie le problème linéaire pour des données au bord dans des espaces de Sobolev anisotropes de régularité optimale par rapport aux données de Cauchy. On démontre la nature bien posée du problème avec les conditions de compatibilité naturelles à tout ordre de régularité. Ces résultats sont essentiels pour établir des résultats de type Cauchy–Lipschitz pour le problème non linéaire, ceux ci font l'objet de la deuxième partie. Malgré la non conservation de l'énergie, on obtient des solutions globales en dimension 2.

Introduction

This article is a continuation of [Audiard 2013] on the initial boundary value problem for the (linear and nonlinear) Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = f, & (x,t) \in \Omega \times [0,T[, \\ u|_{t=0} = u_0, & x \in \Omega, \\ u|_{\partial\Omega\times[0,T]} = g, & (x,t) \in \partial\Omega \times [0,T[, \end{cases} \tag{IBVP}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a smooth open set. Our main purpose is to deal with boundary data of arguably optimal regularity, and in particular too rough to be dealt with by lifting arguments. When f depends on u we generically refer to the nonlinear Schrödinger equation as NLS. We will study nonlinearities that are essentially similar to $\lambda|u|^\alpha u$.

A classical tool to deal with the well-posedness of NLS is Strichartz estimates. It is well known that if $\Omega = \mathbb{R}^d$, the semigroup $e^{it\Delta}$ satisfies

$$\|e^{it\Delta}u_0\|_{L^p(\mathbb{R},\,L^q(\mathbb{R}^d))} \lesssim \|u_0\|_{L^2} \quad \text{when } \frac{2}{p} + \frac{d}{q} = \frac{d}{2},$$

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for $p, q \geq 2$, $q < \infty$ for $d = 2$ (see [Cazenave 2003] and [Keel and Tao 1998] for the endpoint), and more generally the *scale-invariant* estimates

$$\|e^{it\Delta}u_0\|_{L^p(\mathbb{R},\, L^q(\mathbb{R}^d))} \lesssim \|u_0\|_{H^s} \quad \text{when } \frac{2}{p}+\frac{d}{q}=\frac{d}{2}-s.$$

Similar estimates with $\frac{2}{p}+\frac{d}{q} \geq \frac{d}{2}-s$ are true on *bounded* time intervals and simple scaling considerations show that the condition $\frac{2}{p}+\frac{d}{q} \geq \frac{d}{2}-s$ is necessary. When $\frac{2}{p}+\frac{d}{q}-\frac{d}{2}+s=r>0$, they are often called Strichartz estimates with loss of r derivatives. The derivation of such estimates (and the associated well-posedness results) for NLS on a domain with the Dirichlet (or Neumann) laplacian has been intensively studied over the last decade in various geometric settings. We will only cite results in the case where Ω is the exterior of a nontrapping obstacle, since it is the one studied here. Roughly speaking, a nontrapping obstacle is an obstacle such that any ray propagating according to the laws of geometric optic leaves any compact set in finite time (for a mathematical definition of the rays, see [Melrose and Sjöstrand 1978]). In seminal work, Burq, Gérard and Tzvetkov [Burq et al. 2004] proved a local smoothing property similar to the one on $\mathbb{R}^d$ (see [Constantin and Saut 1988]) and deduced Strichartz estimates with loss of $\frac{1}{p}$ derivative. Since then numerous improvements were obtained [Anton 2008a; 2008b; Blair et al. 2008] and eventually led to scale-invariant Strichartz estimates: see Blair, Smith and Sogge [Blair et al. 2012] in the general nontrapping case ($s > 0$ and limited range of exponents), [Ivanovici 2010] for the exterior of a convex obstacle ($s = 0$, all exponents except endpoints). The methods used relied heavily on spectral localization and construction of parametrices. As such they are not very convenient for the study of nonhomogeneous boundary value problems when the boundary data are not smooth enough to reduce the problem to a homogeneous one.

On the other hand, Morawetz and virial identities have proved to be very robust tools to study linear and nonlinear Schrödinger equations. One of their first applications goes back to [Glassey 1977], and it has since been massively refined (as a tool of a much larger machinery) to the point where exhaustive attribution is now impossible (we may cite, at least, [Kenig and Merle 2006; Planchon and Vega 2009; Colliander et al. 2008]). Such tools only rely on differentiation and integration by parts; this makes them flexible enough to be used even with nonzero boundary data and part of our results rely on this approach.

As already mentioned, our aim is to treat Schrödinger equations on a domain with nonzero Dirichlet conditions. The case of dimension one is by now relatively well understood: the local Cauchy theory on intervals is essentially on par with the theory on $\mathbb{R}$ (see [Holmer 2005] for local existence in H^s, $0 \leq s \leq 1$, subcritical and critical nonlinearities). For $d \geq 2$, there are many fewer results. We might mention the classical linear results of [Lions and Magenes 1968b], which were based on lifting arguments and thus prevented boundary data of very low regularity. Indeed, if one takes a lifting Lg of the boundary data, then $u - Lg$ satisfies

$$\begin{cases} i\partial_t(u-Lg)+\Delta(u-Lg)=f-(i\partial_t+\Delta)Lg, \\ (u-Lg)|_{\partial\Omega}=0, \\ (u-Lg)|_{t=0}=u_0-Lg|_{t=0}, \end{cases}$$

so that $u \in C_T L^2$ would require $(i\partial_t + \Delta)Lg \in L^1_T L^2$. For a general L this would require $g \in L^1 H^{3/2}$, which is a loss of one derivative in space compared to our result (see below).

Bu and Strauss [2001] obtained the existence of global weak H^1 solutions for defocusing nonlinear Schrödinger equations with smooth (C^3) boundary data. In the important field of control theory, linear well-posedness and controllability in H^{-1} was obtained for Dirichlet data in L^2 when Ω is a smooth *bounded* domain. While optimal on bounded domains, this "loss" of one derivative on the boundary data is not natural in general. On the half line, it is generally believed that, for initial data $u_0 \in H^s(R^+)$, optimally $g \in H^{s/2+1/4}(\mathbb{R}^+)$ (see [Holmer 2005] for a discussion on this). This pair of spaces is considered to be optimal for at least two reasons: if one rescales solutions as $u(\lambda x, \lambda^2 t)$ both spaces scale as $\lambda^{s-1/2}$, and the space also appears in the famous Kato smoothing property for the Cauchy problem, $\|e^{it\partial_x^2} u_0\|_{L^\infty_x H_t^{s/2+1/4}} \lesssim \|u_0\|_{H^s}$ (see [Kenig et al. 1991]), which can be read as a trace estimate.

The natural generalization of $H^{s/2+1/4}(\mathbb{R}^+)$ in larger dimension is the *anisotropic* Sobolev space $H^{s+1/2,2}(\partial\Omega \times [0,T]) = L^2_T H^{s+1/2} \cap H_T^{s/2+1/4} L^2$ of functions that, roughly speaking, have twice more regularity in space than in time. We obtained in [Audiard 2012] well-posedness for the linear Schrödinger equation on the half space with boundary conditions having this regularity (and satisfying some Kreiss–Lopatinskii condition). However, the method relied quite heavily on the simple geometry of Ω. When Ω is the exterior of a nontrapping obstacle, a simple duality argument was used to obtain the following linear result:

Theorem 0.1 [Audiard 2013]. *For $f \in L^2_T H^{s-1/2}$ compactly supported, $g \in H_0^{s+1/2,2}(\partial\Omega \times [0,T])$, $u_0 \in H_D^s$, $-\frac{1}{2} < s \leq \frac{3}{2}$, the initial boundary value problem (IBVP) has a unique transposition solution. It satisfies*

$$\|u\|_{C_T H^s} \lesssim \|f\|_{L^2_T H^{s-1/2}} + \|g\|_{H_0^{s+1/2}} + \|u_0\|_{H_0^s}.$$

In the case $s = -\frac{1}{2}$, the result is true if $H^{-1/2}$ is replaced by $(H_D^{1/2})'$.

Thanks to a virial identity, we also obtained a local smoothing property similar to the one in [Burq et al. 2004], which allowed us to derive Strichartz estimates with a loss of $\frac{1}{p}$ derivative. Well-posedness in $H^{1/2}$ for the expected range of nonlinearities followed by the usual fixed-point argument.

This work contained, however, a number of important limitations:

- The virial estimate was derived when Ω is the exterior of a strictly convex obstacle.
- Since the natural space for our virial estimate is $H^{1/2}$, the local well-posedness theorem was stated for $u_0 \in H_D^{1/2}$ rather than the energy space H^1.
- The linear well-posedness theorem was obtained for trivial compatibility conditions, $u_0 \in H_D^{1/2}(\Omega)$ and $g \in H_0^{1,2}(\partial\Omega \times [0,T])$.
- Since such conditions are certainly not preserved by the flow, continuation arguments were not available, so the existence of a maximal solution (let alone global solution) was out of reach.

The main purpose of this article is to lift most of the previous restrictions to provide a good local and global Cauchy theory in the energy space. Rather than the exterior of a convex compact obstacle, we will only assume that Ω is the exterior of a compact star-shaped obstacle. On the other hand, we do not improve the loss in the Strichartz estimates, so that we obtain local well-posedness for a range of nonlinearities essentially similar to $|u|^\alpha u$ with the limitation $\alpha < 2/(d-2)$ (the whole subcritical range is $\alpha < 4/(d-2)$). In the case where Ω^c is strictly convex, however, we improve it to $\alpha < 3/(d-2)$. These results are true for boundary data in the almost optimal space $H^{3/2+\varepsilon,2}$ and a discussion is included on the possibility to replace it by the optimal space. If one takes slightly smoother boundary data in $H^{2+\varepsilon,2}(\partial\Omega\times[0,T])$, we obtain global well-posedness for $\alpha < 2/(d-2)$ if Ω^c is star-shaped, and for the whole subcritical range $\alpha < 4/(d-2)$ if Ω^c is strictly convex. The existence of global solutions for $g \in H^{3/2+\varepsilon,2}$ is much more intricate, and is only obtained in dimension 2 with a quite technical limitation on α.

The presence of ε in the trace spaces can most likely be avoided up to lengthier computations that we chose to avoid for simplicity of the proofs (see Remarks 3.5, 3.8, 4.3).

Structure of the article.

- The functional spaces that we use are defined in Section 1, which also provide some useful trace and interpolation results.
- In Section 2 we define the natural compatibility conditions and we prove well-posedness for the linear IBVP when such conditions are met.
- In Section 3 we provide the basic modifications to the proof in [Audiard 2013] that give local smoothing through a virial estimate when Ω is star-shaped. The boundary data is assumed to be in the almost optimal space $H^{3/2+\varepsilon,2}$. We deduce Strichartz estimates at the H^1 level thanks to an interpolation argument; this section also includes a smoothing property on $\partial_n u$ that is essential for global existence issues.
- In Section 4 we prove the nonlinear well-posedness results stated above.
- The Appendix contains two elementary interpolation results.

1. Functional spaces and Strichartz estimates

Functional spaces. For $p \geq 1$ we denote by $L^p(\Omega)$ the usual Lebesgue spaces. If there is no ambiguity, when X is a Banach space we write

$$L^p([0,T],X) = L^p_T X, \quad L^p(\mathbb{R}^+,X) = L^p_t X.$$

For integer m we denote by $W^{m,p}(\Omega)$ the usual Sobolev spaces; $W_0^{m,p}$ is the closure of $C_c^\infty(\Omega)$ for the $W^{m,p}$ topology.

For $s \geq 0$, the space $W^{s,p}(\Omega)$ is defined by real interpolation; see [Tartar 2007, Sections 32 and 34]. When $p = 2$, the Sobolev spaces are denoted by H^s, H^s_0. For $s > 0$, we set $H^{-s}(\Omega) = (H^s_0(\Omega))'$.

For $s \geq 0$ and Δ_D the Dirichlet laplacian on Ω, the space H^s_D is the domain of $(1-\Delta_D)^{s/2}$. When $\frac{1}{2} < s \leq 1$, $H^s_D = H^s_0$, and when $0 \leq s < \frac{1}{2}$, $H^s_D = H^s$. The space $H^{1/2}_D$ does not coincide with $H^{1/2}_0 = H^{1/2}$ (it is the Lions–Magenes space $H^{1/2}_{00}$ but we will use the notation $H^{1/2}_D$).

The Besov spaces $B^s_{p,q}(\Omega)$ are the restrictions to Ω of functions in $B^s_{p,q}(\mathbb{R}^d)$ [Tartar 2007, Sections 32 and 34]. For $s \geq 0$, $s \notin \mathbb{N}$, we have $B^s_{p,p} = W^{s,p}$ (see [Bergh and Löfström 1976; Tartar 2007]). The spaces $B^s_{p,q,0}$ are defined as the closure of $C^\infty_c(\Omega)$ in $B^s_{p,q}$.

The anisotropic Sobolev spaces on $[0,T]\times\Omega$ are defined as

$$H^{s,2} = L^2([0,T], H^s(\Omega)) \cap H^{s/2}([0,T], L^2(\Omega)).$$

Anisotropic Besov spaces can be defined in a similar way (see [Amann 2009]):

$$B^{s,2}_{p,q,0} = L^p_T B^s_{p,q,0} \cap B^{s/2}_{p,q}([0,T], L^p(\Omega)).$$

Finally, we use the same definitions for functions defined on $\partial\Omega$ or $\partial\Omega\times[0,T]$ using local maps.

We recall in the following proposition the classical rules on embeddings and traces of functional spaces:

Proposition 1.1 (Sobolev embeddings and traces [Lions and Magenes 1968b; Triebel 1983]).

- *If $0 \leq sp < d$, $t \geq 0$, we have $B^{t+s}_{p,q}(\Omega) \hookrightarrow B^s_{p_1,q}$ when $\frac{1}{p_1} = \frac{1}{p} - \frac{s}{d}$.*
- *If $sp > d$, $W^{s,p} \hookrightarrow C^0(\overline{\Omega})$ and, if $sp < d$, then $W^{s,p} \hookrightarrow L^q(\Omega)$ when $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$.*
- *If $sp > 1$, the trace operator $C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$ extends continuously to*

$$W^{s,p}(\Omega) \to W^{s-1/p,p}(\partial\Omega).$$

- *For $0 \leq s' \leq \frac{s}{2}$, the anisotropic spaces $H^{s,2}(\Omega\times[0,T])$ are embedded in $H^{s'}_T H^{s-2s'}$.*
- *For $s > \frac{1}{2}$, the trace operator $H^{s,2}(\Omega\times[0,T]) \to H^{s-1/2,2}(\partial\Omega\times[0,T])$ is continuous.*
- *For $s > 1$, $\mathbb{O} = \Omega$ or $\partial\Omega$, there is a time-trace operator from the embedding*

$$H^{s,2}([0,T]\times\mathbb{O}) \hookrightarrow C([0,T], H^{s-1}(\mathbb{O})).$$

For $s_0, s_1 \geq 0$, we have the interpolation identity (see [Triebel 1983])

$$[B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1}]_{\theta,q} = B^{\theta s_0 + (1-\theta)s_1}_{p,q}.$$

Similar interpolation results are true for anisotropic Sobolev spaces. In [Lions and Magenes 1968b] it is proved that for $s > 0$, $\mathbb{O} = \Omega$ or $\partial\Omega$, $0 \leq \theta \leq 1$ and $t = \theta s$, $H^{t,2}([0,T]\times\mathbb{O}) = [L^2, H^{s,2}]_\theta$.

In addition to their nice interpolation properties, composition rules in Besov spaces are relatively simple: if $F(0) = 0$ and $|\nabla F(z)| \lesssim |z|^\alpha$, then for $0 < s < 1$, $1 \leq q \leq \infty$, $1 \leq p \leq r \leq \infty$, $\frac{1}{\sigma} + \frac{1}{r} = \frac{1}{p}$, we have

$$\|F(u)\|_{B^s_{p,q}} \lesssim \|u\|^\alpha_{L^{\alpha\sigma}} \|u\|_{B^s_{r,q}}; \tag{1-1}$$

this is Proposition 4.9.4 in [Cazenave 2003] when $\Omega = \mathbb{R}^d$, and it follows from the existence of a (universal) extension operator when Ω is an exterior domain; see [Amann 2009, Sections 4.1, 4.4].

Since anisotropic Besov spaces are more intricate and scarcely used in the article, we will cite their properties we need when relevant, pointing to the reference [Amann 2009].

Finally, we recall some Strichartz estimates known for the boundary value problem with homogeneous boundary condition.

Theorem 1.2 [Burq et al. 2004; Ivanovici 2010]. *If Ω is the exterior of a nontrapping obstacle, then for any $T>0$,*

$$\|e^{it\Delta_D}u_0\|_{L^p_tL^q}\lesssim\|u_0\|_{L^2}\quad\textit{when}\ \frac{1}{p}+\frac{d}{q}=\frac{d}{2},\ p\geq 2. \tag{1-2}$$

If Ω is the exterior of a strictly convex obstacle then

$$\|e^{it\Delta_D}u_0\|_{L^p_TL^q}\lesssim\|u_0\|_{L^2}\quad\textit{when}\ \frac{2}{p}+\frac{d}{q}=\frac{d}{2},\ p>2. \tag{1-3}$$

2. Linear well-posedness

In this section, we assume that Ω is the exterior of a compact nontrapping obstacle. We recall what we meant by "transposition solution" in Theorem 0.1:

Definition 2.1. Let $\chi\in C_c^\infty(\mathbb{R}^d)$, $f\in L^2_TH^{-1}(\Omega)$. We say that u is a *transposition solution* of the problem

$$\begin{cases} i\partial_tu+\Delta u=\chi f\in L^2_TH^{-1},\\ u|_{t=0}=u_0\in (H_D^{1/2}(\Omega))',\\ u|_{\partial\Omega\times[0,T]}=g\in L^2([0,T]\times\partial\Omega)\end{cases} \tag{2-1}$$

when $u\in C_T(H_D^{1/2})'$ and, for any $f_1\in L^1_TH_D^{1/2}$, if v is the solution of

$$\begin{cases} i\partial_tv+\Delta v=f_1,\\ v|_{t=T}=0,\\ v|_{\partial\Omega\times[0,T]}=0,\end{cases} \tag{2-2}$$

then we have the identity

$$\int_0^T\langle u,f_1\rangle_{(H_D^{1/2})',H_D^{1/2}}\,dt=\int_0^T\langle f,\chi v\rangle_{H^{-1},H_0^1}\,dt+\int_0^T(g,\partial_nv)_{L^2(\partial\Omega)}\,dt+i\langle u_0,v(0)\rangle_{(H_D^{1/2})',H_D^{1/2}}, \tag{2-3}$$

where $\langle\cdot,\cdot\rangle_{X,X'}$ is the duality bracket.

In [Audiard 2013] we obtained by derivation/interpolation arguments well-posedness for (u_0,g) in $H^s_D\times H_0^{s+1/2,2}$; the aim of this section is to extend it to $(u_0,f,g)\in H^s\times H^{s-1/2,2}\times H^{s+1/2,2}$ for any $s\geq-\frac{1}{2}$, under natural compatibility conditions that we derive now.

Compatibility conditions. We consider the linear initial boundary value problem (IBVP)

$$\begin{cases} i\partial_tu+\Delta u=f, & (x,t)\in\Omega\times[0,T[,\\ u|_{t=0}=u_0, & x\in\Omega,\\ u|_{\partial\Omega\times[0,T]}=g, & (x,t)\in\partial\Omega\times[0,T[.\end{cases} \tag{2-4}$$

Local compatibility. If $u_0\in H^s$, $g\in H^{s+1/2,2}$, $s>\frac{1}{2}$, then u_0 has a trace on $\partial\Omega$ and g has a trace at $t=0$; the identity $u|_{t=0}|_{\partial\Omega}=u|_{\partial\Omega}|_{t=0}$ imposes the zeroth-order compatibility condition

$$u_0|_{\partial\Omega}=g|_{t=0}. \tag{CC0}$$

The next compatibility conditions are defined inductively: set $\varphi_0 = u_0$, $\varphi_{n+1} = \frac{1}{i}(\partial_t^n f|_{t=0} - \Delta\varphi_n)$; the k-th order compatibility condition is

$$\partial_t^k g|_{t=0} = \varphi_k|_{\partial\Omega}, \tag{CCk}$$

which must be satisfied if $u_0 \in H^s(\Omega)$, $g \in H^{s+1/2,2}(\partial\Omega\times[0,T])$, $f \in H^{s-1/2,2}(\Omega\times[0,T])$, $s > 2k+\frac{1}{2}$.

Global compatibility. If $s = \frac{1}{2}$, there is a more subtle compatibility condition, the so-called "global compatibility condition": thanks to local maps, we can assume that u_0, g are defined by a collection of $(u_0^j, f^j g^j)_{1\le j\le J}$ defined on $\mathbb{R}^{d-1}\times\mathbb{R}^+$ ($\mathbb{R}^+$ corresponds to the t-variable for g^j and normal space variable for u_0^j, f^j); we say that (u_0, g) satisfy the zeroth-order global compatibility condition when

$$\forall 1 \le j \le J \quad \int_0^\infty \int_{\mathbb{R}^{d-1}} |u_0^j(x', h) - g^j(x', h^2)|^2 \, dx' \, \frac{dh}{h} < \infty; \tag{CCG0}$$

similarly, we define the global compatibility conditions of order k for $s = \frac{1}{2} + 2k$ as

$$\forall 1 \le j \le J \quad \int_0^\infty \int_{\mathbb{R}^{d-1}} |\varphi_k^j(x', h) - \partial_t^k g^j(x', h^2)|^2 \, dx' \, \frac{dh}{h} < \infty, \tag{CCGk}$$

It is standard [Lions and Magenes 1968a] that (CCk) is stronger than (CCGk).

In what follows, we say that $(u_0, f, g) \in H^s \times H^{s-1/2,2} \times H^{s+1/2,2}$ "satisfy the compatibility conditions" when all conditions that make sense are satisfied, namely (CCk) holds for $k < \frac{s}{2} - \frac{1}{4}$, and also (CCG$k$) if $s = \frac{1}{2} + 2k$.

Theorem 2.2. *For $-\frac{1}{2} < s \le \frac{3}{2}$, let $(u_0, f, g) \in H^s \times L_T^2 H^{s-1/2} \times H^{s+1/2,2}$ be such that f is compactly supported and (u_0, f, g) satisfy the compatibility conditions; then the solution of (IBVP) is in $C_T H^s$.*

For $s > \frac{3}{2}$ and $(u_0, f, g) \in H^s \times H^{s-1/2,2} \times H^{s+1/2,2}$ satisfying the compatibility conditions, $u \in C_T H^s$.

The spirit of the proof is relatively similar to the classical argument of [Rauch and Massey 1974] for hyperbolic boundary value problems. Let us describe it and where the difficulty lies: the natural idea is to consider Δu, which is formally a solution of a similar boundary value problem; the low regularity theorem implies $\Delta u \in C_T(H_D^{1/2})'$, and we conclude, by an elliptic regularity argument, that $u \in C_T H^{3/2}$. However, due to the weak setting it is not clear that Δu is actually a solution of the expected boundary value problem. For "trivial" compatibility conditions it is sufficient to approximate the initial data by $(u_{0,n}, g_n, f_n) \in C_c^\infty(\Omega)\times C_c^\infty(\partial\Omega\times]0,T])\times C_c^\infty(\overline{\Omega}\times]0,T])$ that automatically satisfy the compatibility conditions at any order. In general, the existence of smooth data that satisfy the compatibility conditions at a sufficient order will be done in Lemma 2.4.

Lemma 2.3. *If $(u_0, f, g) \in H^{3/2} \times L_T^2 H^1 \times H^{2,2}$ with f compactly supported and (CC0) satisfied, the unique transposition solution of (IBVP) belongs to $C_T H^{3/2}$.*

For $k \ge 2$, if $(u_0, f, g) \in H^{2k-1/2} \times H^{2k-1,2} \times H^{2k,2}$, f compactly supported and (CCj), $0 \le j \le k-1$ satisfied, the unique transposition solution of (IBVP) belongs to $C_T H^{2k-1/2}$.

The proof is postponed until after the following approximation lemma:

Lemma 2.4. *For* $(u_0, f, g) \in H^{3/2}(\Omega) \times L^2([0,T], H^1(\Omega)) \times H^{2,2}([0,T] \times \partial\Omega)$ *satisfying* (CC0), *there exists a sequence* $(u_{0,k}, f_k, g_k) \in H^2 \times H^{2,2} \times H^{5/2,2}$ *satisfying* (CC0) *such that*

$$\|(u_0, f, g) - (u_{0,k}, f_k, g_k)\|_{H^{3/2} \times L^2_T H^1 \times H^{2,2}} \to_k 0.$$

Proof. By density of smooth functions in Sobolev spaces, there exists (v_k, f_k, g_k) smooth such that $(v_k, f_k, g_k) \to_k (u_0, f, g)$ $(H^{3/2} \times L^2_T H^1 \times H^{2,2})$; however, the sequence a priori does not satisfy (CC0). Let us modify $u_{0,k} = v_k + \varphi_k$; it is sufficient to construct $\varphi_k \in H^2(\Omega)$ such that $\|\varphi_k\|_{H^{3/2}} \to_k 0$ and

$$\varphi_k|_{\partial\Omega} = g_k|_{t=0} - v_k|_{\partial\Omega}. \tag{2-5}$$

This is an underdetermined system on $(\partial_n^j \varphi_k)_{0 \le j \le 1}$ that we close by imposing $\partial_k \varphi_k = 0$: we define $\varphi_k \in H^2$ as the lifting of $(g_k|_{t=0} - v_k|_{\partial\Omega}, 0)$. From standard trace theory, there exists a lifting operator

$$\begin{aligned} L:\ H^{3/2}(\partial\Omega) &\to H^2(\Omega) \\ b &\mapsto v \quad \text{such that } v|_{\partial\Omega} = b,\ \partial_n v = 0, \end{aligned}$$

that extends continuously as a lifting operator $H^1 \to H^{3/2}$ (on the half space in Fourier variables $\xi = (\xi', \xi_d)$ one may take $\widehat{Lb} = \hat{b}(\xi') h(\xi_d / \sqrt{1 + |\xi'|^2}) / \sqrt{1 + |\xi'|^2}$ with h smooth and compactly supported, $\int h\, d\xi_1 = 1$, $\int \xi_1 h\, d\xi_1 = 0$; see [Lions and Magenes 1968a] for more details). In particular, we have $\|g_k|_{t=0} - v_k|_{\partial\Omega}\|_{H^1} \to \|g|_{t=0} - u_0|_{\partial\Omega}\|_{H^1} = 0$, which implies $\|\varphi_k\|_{H^{3/2}} \to 0$. □

Proof of Lemma 2.3. We first detail the case $s = \frac{3}{2}$ and will deal with $s = -\frac{1}{2} + 2k$, $k \in \mathbb{N}$ by induction. Let u be the solution of (IBVP). If (CC0) is satisfied, then there exists $(u_{0,k}, g_k, f_k)$ as in Lemma 2.4, and we call the associated solutions u_k. Since $\|u_k - u\|_{C_T(H_D^{1/2})'} \to_k 0$, it is sufficient to prove the convergence of u_k in $C_T H^{3/2}$. We first check that $u_k \in C_T H^2$. Let $\tilde{g}_k \in H^{3,2}(\Omega \times [0,T])$ be a lifting (for its existence, see [Lions and Magenes 1968b, chapitre 4, section 2]) such that

$$\begin{cases} \tilde{g}_k|_{\partial\Omega \times [0,T]} = g_k, \\ \Delta \tilde{g}_k|_{\partial\Omega \times [0,T]} = f_k|_{\partial\Omega \times [0,T]} - i\partial_t g_k. \end{cases}$$

We define

$$w_k = e^{it\Delta_D}(u_{0,k} - \tilde{g}_k|_{t=0}) + \int_0^t e^{i(t-s)\Delta_D}(f_k - i\partial_t \tilde{g}_k - \Delta\tilde{g}_k)\, ds,$$

the solution of the homogeneous IBVP with initial data $u_{0,k} - \tilde{g}_k|_{t=0}$ and forcing term $f_k - i\partial_t \tilde{g}_k - \Delta\tilde{g}_k$, so that $u_k = w_k + \tilde{g}_k$. The embedding $H^{3,2} \hookrightarrow C_T H^2$ and (CC0) then imply $u_{0,k} - \tilde{g}_k|_{t=0} \in H_D^2$ and $f_k - i\partial_t \tilde{g}_k - \Delta\tilde{g}_k \in L_T^1 H_D^2$, thus $w_k \in C_T H_D^2$ and $u_k = w_k + \tilde{g}_k \in C_T H^2$. In particular, $\Delta u_k \in C_T L^2$ and we can now check that it is the transposition solution of the IBVP

$$\begin{cases} i\partial_t v_k + \Delta v_k = \Delta f_k, & (x,t) \in \Omega \times [0,T[, \\ v_k|_{t=0} = \Delta u_{0,k}, & x \in \Omega, \\ v_k|_{\partial\Omega \times [0,T]} = -i\partial_t g_k + f_k|_{\partial\Omega \times [0,T]}; \end{cases} \tag{2-6}$$

that is to say (2-3) is satisfied with data $(\Delta u_{0,k}, \Delta f_k, -i\partial_t g_k + f_k|_{\partial\Omega \times [0,T]})$.

Let $\varphi\in C^\infty([0,T],C_c^\infty(\Omega))$; we set $w=\int_T^t e^{i(t-s)\Delta_D}\Delta\varphi\,ds$ the solution of the dual boundary value problem with data $\Delta\varphi$. By definition of u_k,

$$\begin{aligned}\iint_{\Omega\times[0,T]}\Delta u_k\overline{\varphi}\,dx\,dt&=\iint_{\Omega\times[0,T]}u_k\overline{\Delta\varphi}\,dx\,dt\\&=\iint_{\Omega\times[0,T]}f_k\overline{w}\,dx\,dt+i\int_\Omega u_{0,k}\overline{w(0)}\,dx+\iint_{\partial\Omega\times[0,T]}g_k\overline{\partial_n w}\,dS\,dt.\end{aligned}$$

Now, since $w=\Delta\int_T^t e^{i(t-s)\Delta_D}\varphi\,ds:=\Delta v$, where $v\in C^1H_D^2$, we can write

$$\begin{aligned}&\iint_{\Omega\times[0,T]}\Delta u_k\overline{\varphi}\,dx\,dt\\&=\iint_{[0,T]\times\Omega}f_k\overline{\Delta v}\,dx\,dt+i\int_\Omega u_{0,n}\overline{\Delta v(0)}dx+\iint_{\partial\Omega\times[0,T]}g_k\overline{\partial_n\Delta v}\,dS\,dt\\&=\iint_{\Omega\times[0,T]}\Delta f_k\bar{v}\,dx\,dt+i\int_\Omega\Delta u_{0,k}\overline{v(0)}\,dx+i\int_{\partial\Omega}u_{0,k}\partial_n\overline{v(0)}\,dx\\&\qquad+\iint_{\partial\Omega\times[0,T]}g_k\partial_n(\overline{-i\partial_t v+\varphi})+f_k\overline{\partial_n v}\,dS\,dt\\&=\iint_{\Omega\times[0,T]}\Delta f_k\bar{v}\,dx\,dt+\iint_{\partial\Omega\times[0,T]}(f_k-i\partial_t g_k)\partial_n\bar{v}\,dS\,dt+i\int_\Omega\Delta u_{0,k}\overline{v(0)}\,dx\\&\qquad+i\int_{\partial\Omega}u_{0,k}\partial_n\overline{v(0)}\,dS+i\Big[\int_{\partial\Omega}g_k\overline{\partial_n v}\,dS\Big]_0^T\\&=\iint_{\Omega\times[0,T]}\Delta f_k\bar{v}\,dx\,dt+\iint_{\partial\Omega\times[0,T]}(f_k-i\partial_t g_k)\partial_n\bar{v}\,dS\,dt+i\int_\Omega\Delta u_{0,k}\overline{v(0)}\,dx,\end{aligned}$$

where in the last equality we used (CC0) and the cancellation of $v|_{t=T}$. Since the equality is true for arbitrary φ, by density of $C^\infty([0,T],C_c^\infty(\Omega))$ in $L_T^1H_D^{1/2}$ we obtain that Δu_k is the transposition solution of (2-6), and Δu_k converges in $C_T(H_D^{1/2})'$ since $\Delta u_{0,k},\Delta f_k, i\partial_t g_k-f_k|_{\partial\Omega\times[0,T]}$ converge in $(H^{1/2})'_D\times L_T^2H^{-1}\times L^2$. Arguing as in the end of proof of [Audiard 2013, Proposition 6], we obtain the convergence of u_k in $C_TH^{3/2}$ and its limit is u by uniqueness of the limit. This settles the case $s=\frac{3}{2}$.

For $s=-\frac{1}{2}+2k$, $k\geq 2$, we argue by induction. Let us introduce the boundary value problems

$$\begin{cases}i\partial_t v+\Delta v=\Delta^m f, & (x,t)\in\Omega\times[0,T[,\\ v|_{t=0}=\Delta^m u_0, & x\in\Omega,\\ v|_{\partial\Omega\times[0,T]}=\psi_m|_{\partial\Omega\times[0,T]},\end{cases}\tag{IBVPm}$$

where ψ_m is defined inductively by $\psi_0=g$, $\psi_{j+1}=\Delta^j f|_{\partial\Omega\times[0,T]}-i\partial_t\psi_j$. We assume that (u_0,f,g) in $H^{-1/2+2k}\times H^{-1+2k,2}\times H^{2k,2}$ satisfy (CCj), $0\leq j\leq k-1$, and $\Delta^j u$ is a solution of (IBVPj) for $0\leq j\leq k-1$. In particular, $\Delta^{k-1}u$ is a solution of (IBVP$k-1$) and the previous argument implies that $\Delta^{k-1}u\in C_TH^{3/2}$ if $(\Delta^{k-1}u_0,\Delta^{k-1}f,\psi_{k-1})$ belong to $H^{3/2}\times L_T^2H^1\times H^{2,2}$ and satisfy the compatibility condition $\psi_{k-1}|_{t=0}=\Delta^{k-1}u_0|_{\partial\Omega}$. The first condition is clear, since[1] $\psi_j\in H^{2k-j}(\partial\Omega\times[0,T])$,

[1] Actually, the careful reader may note that the regularity of the boundary data only requires $f\in H^{2m-3/2+\varepsilon,2}$, $\varepsilon>0$, rather than $H^{2m-1,2}$. This is not important as the dispersive estimates in next section require the full regularity $f\in H^{2m-1,2}$.

and for the compatibility condition we may note that

$$\forall j \geq 1 \quad \begin{aligned} \psi_j &= (-i\partial_t)^j g + \sum_{p=0}^{j-1} (-i\partial_t)^p \Delta^{j-1-p} \frac{f}{i}\Big|_{\partial\Omega\times[0,T]}, \\ \varphi_j &= (i\Delta)^j u_0 + \sum_{p=0}^{j-1} \partial_t^{j-1-p} (i\Delta)^p f|_{t=0}, \end{aligned}$$

so that $\psi_{k-1}|_{t=0} = \Delta^{k-1} u_0$ is equivalent to (CC$k-1$). Thus

$$\Delta^{k-1} u \in C_T H^{3/2} \qquad \text{and} \qquad \Delta^j u|_{\partial\Omega} = \psi_j \in H^{2(k-j)} \hookrightarrow C_T H^{2(k-j)-1}, \quad 0 \leq j \leq k-2,$$

so that, by elliptic regularity, $u \in C_T H^{2k-1/2}$. □

We can now conclude this section:

Proof of Theorem 2.2. We have obtained well-posedness for $s = -\frac{1}{2}, \frac{3}{2}$. The case $-\frac{1}{2} \leq s \leq \frac{3}{2}$ follows by interpolation if we check that $H^s \times H^{s+1/2,2} \times L^2_T H^{s-1/2}$ with compatibility condition is the interpolated space between $(H_D^{1/2})' \times L^2 \times L^2_T H^{-1/2}$ and $H^{3/2} \times H^{2,2} \times L^2_T H^1$ with compatibility condition; this is proved in Lemma A.2 in the Appendix.

For $s \geq \frac{3}{2}$, let $m \in \mathbb{N}$ be such that $-\frac{1}{2} + 2m \leq s < -\frac{1}{2} + 2(m+1)$. The case of equality is Lemma 2.3; in the case of strict inequality we recall that $\Delta^m u$ is a solution of (IBVPm), where it is easily seen that if $(f, g) \in H^{s-1/2,2}(\Omega\times[0,T]) \times H^{s+1/2}(\partial\Omega\times[0,T])$ then $\psi_m \in H^{s+1/2-2m}$. Since $-\frac{1}{2} \leq s-2m \leq \frac{3}{2}$, we have from the previous case that $\Delta^m u \in C_T H^{s-2m}$; the regularity of u follows by elliptic regularity. □

3. Dispersive estimates

From now on we assume that Ω^c is star-shaped; up to translation we can also assume that it is star-shaped with respect to 0.

Local smoothing. Let us first recall the key virial identity:

Proposition 3.1 [Audiard 2013]. *If u is a smooth solution of (IBVP), $h \in C^k(\Omega)$, $\nabla^k h$ bounded for $1 \leq k \leq 4$, and $I(u) = 2\,\mathrm{Im} \int_\Omega \nabla h \cdot \nabla u \bar{u}\, dx$, then, setting $\nabla_\tau = \nabla - n\partial_n$,*

$$\frac{d}{dt} I(u(t)) = 4\,\mathrm{Re} \int_\Omega \mathrm{Hess}(h)(\nabla u, \overline{\nabla u}) - \tfrac{1}{4}|u|^2 \Delta^2 h + \nabla h \cdot \nabla u \bar{f} + \tfrac{1}{2}\bar{u}\Delta h f\, dx$$
$$+ \mathrm{Re} \int_{\partial\Omega} 2\partial_n h |\nabla_\tau u|^2 - 2\partial_n h |\partial_n u|^2 - 2i\partial_n h \partial_t u \bar{u}\, dS + \mathrm{Re} \int_{\partial\Omega} -2\bar{u}\Delta h \partial_n u + |u|^2 \partial_n \Delta h\, dS.$$

For the choice $h(x) = \sqrt{1+|x|^2}$, we have $\mathrm{Hess}(h) \geq 1/(1+|x|^2)^{3/2}$, $\partial_n h \leq 0$ (because Ω is star-shaped); this leads to the following result:

Proposition 3.2. *For any $\varepsilon > 0$, $(u_0, f, g) \in H^{1/2}(\Omega) \times L^2(\Omega\times[0,T]) \times H^{1+\varepsilon,(1+\varepsilon)/2}(\partial\Omega\times[0,T])$ that satisfy (CCG0), f compactly supported, we have*

$$\left\| \frac{\nabla u}{(1+|x|^2)^{3/4}} \right\|_{L^2([0,T],L^2(\Omega))} + \|\partial_n u\|_{L^2(\partial\Omega\times[0,T])} \lesssim (\|u_0\|_{H^{1/2}} + \|f\|_{L^2} + \|g\|_{H^{1+\varepsilon,2}}).$$

Remark 3.3. The constant in $\lesssim$ depends on ε, T and the size of $\mathrm{supp}(f)$, and blows up if $\varepsilon \to 0$, $T \to \infty$ or $\mathrm{supp}(f) \to \Omega$. We chose not to emphasize this as it will not matter in the rest of the article.

Proof. The proof was essentially done in [Audiard 2013] for a strictly convex obstacle; we write it out since it must be slightly modified for the case of a star-shaped obstacle. We use that f is compactly supported to absorb the term $\int \nabla h \nabla u \overline{f}\, dx$ in $\int \mathrm{Hess}(h)(\nabla u \overline{\nabla u})\, dx$, and Ω^c is star-shaped thus $\partial_n h \le 0$ (n is the outer normal of Ω), so integration in time gives

$$\left\| \frac{\nabla u}{(1+|x|^2)^{3/4}} \right\|^2_{L^2(\Omega\times[0,T])} \lesssim \|u\|^2_{L^2(\Omega\times[0,T])} + \|f\|^2_{L^2(\Omega\times[0,T])} + \|g\|^2_{H^{1+\varepsilon,2}(\partial\Omega\times[0,T])} + |I(u(T))| + |Iu_0|.$$

To estimate $|I(u(T))|+|I(u(0))|$ the main issue is that $\nabla u \in (H_D^{1/2})'$, which is slightly larger than $H^{-1/2}$. Following the notations of Lemma A.2, we first remark that the assumptions of the lemma imply $(u_0, g) \in X^{1/2}$ and we use the lifting operator $H^{s,s/2} \to H^{s+1/2,s/2+1/4}(\Omega\times[0,T])$, $g \mapsto R_1 g$. If $(u_0, g) \in X^{3/4}$, then $(u_0 - R_1 g|_{t=0}, u(T) - R_1 g|_{t=T}) \in (H_0^1(\Omega))^2$, while, if $(u_0, g) \in X^{1/3}$, then $(u_0 - R_1 g|_{t=0}, u(T) - R_1 g|_{t=T}) \in (H^{1/6}(\Omega))^2$, thus by interpolation

$$(u_0, g) \in X^{1/2} \implies (u_0 - \tilde{g}|_{t=0}, u(T) - \tilde{g}|_{t=T}) \in (H_D^{1/2}(\Omega))^2.$$

This implies for $t \in [0, T]$

$$\left| \int_\Omega \overline{u(t) - R_1 g(t)} \nabla u \cdot \nabla h \, dx \right| \lesssim \|u\|_{C([0,T],H^{1/2})} \|g\|_{H^{1,2}}$$

On the other hand, an integration by parts formally gives

$$\begin{aligned}\left| \int_\Omega \overline{R_1 g(t)} \nabla u \cdot \nabla h \, dx \right| &\le \left| \int_\Omega u \,\mathrm{div}(\overline{R_1 g(t)} \nabla h)\, dx \right| + \left| \int_{\partial\Omega} g \overline{R_1 g(t)} \partial_n h \, dx \right| \\ &\le C_\varepsilon (\|u(t)\|_{H^{1/2-\varepsilon}} \|R_1 g(t)\|_{H^{1/2+\varepsilon}} + \|g(t)\|^2_{L^2}) \\ &\le C_\varepsilon (\|u\|_{C_T H^{1/2}} \|g\|_{H^{1+\varepsilon,2}} + \|g\|^2_{H^{1+\varepsilon,2}}),\end{aligned}$$

so that by a density argument we obtain

$$\begin{aligned}\left\| \frac{\nabla u}{(1+|x|^2)^{3/4}} \right\|_{L^2(\Omega\times[0,T])} &\le C_{\varepsilon,T} (\|u\|_{C_T\, H^{1/2}} + \|g\|_{H^{1+\varepsilon,2}} + \|f\|_{L^2}) \\ &\le C_{\varepsilon,T} (\|u_0\|_{H^{1/2}} + \|f\|_{L^2} + \|g\|_{H^{1+\varepsilon,2}}). \end{aligned} \tag{3-1}$$

The estimate on $\|\partial_n u\|_{L^2}$ cannot in general be obtained directly from the virial identity with $h = \sqrt{1+|x|^2}$ since we may have, for some $x \in \partial\Omega$, $\partial_n h = x \cdot n / \sqrt{1+|x|^2} = 0$. However, once local smoothing has been obtained it is quite simple to derive an estimate on $\partial_n u$. The argument that we give now is essentially the same as the one from [Planchon and Vega 2009] for the homogeneous case. Using the identity from Proposition 3.1 with some h smooth and compactly supported such that $\partial_n h < 0$, we obtain

$$\|\partial_n u\|^2_{L^2} \lesssim |I(u(T))| + |I(u_0)| + \|u\|^2_{L^2} + \|f\|^2_{L^2} + \|g\|_{H^{1+\varepsilon,2}} + \int_0^T \int_\Omega \mathrm{Hess}(h)(\nabla u, \overline{\nabla u})\, dx\, dt.$$

The integral of $\mathrm{Hess}(h)(\nabla u, \nabla u)\,dx$ is no longer positive; however, since h is compactly supported, it is controlled thanks to (3-1). $\square$

We can now state the local smoothing property for more general regularity:

Corollary 3.4. *Let $\varepsilon>0$, $\frac{1}{2}\le s<2$, $(u_0, f, g)\in H^s(\Omega)\times H^{s-1/2,2}(\Omega\times[0,T])\times H^{s+1/2+\varepsilon,2}(\partial\Omega\times[0,T])$ satisfying the compatibility conditions, f compactly supported, $\varepsilon>0$; then the solution $u\in C_TH^s$ of (IBVP) has the local smoothing property*

$$\left\|\frac{u}{(1+|x|^2)^{3/4}}\right\|_{L^2_TH^{s+1/2}}+\|\partial_n u\|_{H^{s-1/2,2}}\lesssim \|u_0\|_{H^s}+\|g\|_{H^{s+\varepsilon+1/2,2}}+\|f\|_{H^{s-1/2,2}}.$$

Proof. The case $s=\frac{1}{2}$ is Proposition 3.2. For $s=\frac{5}{2}$, we have already seen that Δu is a solution of the IBVP with forcing term Δf, initial conditions Δu_0 and boundary data $-i\partial_t g+f|_{\partial\Omega\times[0,T]}$, thus the local smoothing implies

$$\begin{aligned}\left\|\frac{\nabla\Delta u}{(1+|x|^2)^{3/4}}\right\|_{L^2(\Omega\times[0,T])}&\lesssim \|u_0\|_{H^{5/2}}+\|f\|_{L^2_TH^2}+\|g\|_{H^{3+\varepsilon,2}}+\|f\|_{H^{1+\varepsilon,2}(\partial\Omega\times[0,T])}\\&\lesssim \|u_0\|_{H^{5/2}}+\|f\|_{L^2_TH^2}+\|g\|_{H^{3+\varepsilon,2}}+\|f\|_{H^{3/2+\varepsilon,2}(\Omega\times[0,T])}\\&\lesssim \|u_0\|_{H^{5/2}}+\|f\|_{H^{2,2}(\Omega\times[0,T])}+\|g\|_{H^{3+\varepsilon,(3+\varepsilon)/2}}.\end{aligned}$$

Elliptic regularity then implies the estimate on $\|u/(1+|x|^2)^{3/4}\|_{H^3}$. The control of $\|\partial_n u\|_{H^{2,2}}$ requires a bit more care, since we cannot directly use the estimate on $\partial_n\Delta u$: for $x_0\in\partial\Omega$, we use local coordinates $(y_1,\dots,y_d)$ such that, on a neighbourhood U of x_0, $\partial\Omega\cap U=\{y_d=0\}$ and $\Omega\cap U\subset\{y_d>0\}$, and we define the differential operators $D_k=\varphi(y_1,\dots,y_{d-1})\psi(y_d)\partial_{y_k}$, $1\le k\le d-1$, with φ, ψ such that $\mathrm{supp}(\varphi\psi)\subset U$ and $\psi=1$ on a neighbourhood of 0. Setting $D_k=0$ outside U, the D_k define second-order differential operators on Ω and, by restriction, on $\partial\Omega$. For $1\le k,\ p\le d-1$, it can be checked as for Δu that $u_{kp}=D_kD_pu$ is the transposition solution of

$$\begin{cases}i\partial_t w+\Delta w=D_kD_pf+[\Delta, D_kD_p]u,\\ w|_{t=0}=D_kD_pu_0,\\ w|_{\partial\Omega}=D_kD_pg,\end{cases}$$

where the commutator $[\Delta,\ D_kD_p]$ is a third-order differential operator. The virial identity gives

$$\begin{aligned}\frac{dI(u_{kp})}{dt}=4\,\mathrm{Re}\int_\Omega &\mathrm{Hess}(h)(\nabla u_{kp},\nabla\bar u_{kp})-\tfrac{1}{4}|u_{kp}|^2\Delta^2h+\nabla h\cdot\nabla u_{kp}(\overline{D_kD_pf+[\Delta,D_kD_p]u})\,dx\\&+2\,\mathrm{Re}\int_\Omega \bar u_{kp}\Delta h(D_kD_pf+[\Delta,D_kD_p]u)\,dx\\&+\mathrm{Re}\int_{\partial\Omega}2\partial_nh|\nabla_\tau u_{kp}|^2-2\partial_nh|\partial_nu_{kp}|^2-2i\partial_nh\partial_tu_{kp}\bar u_{kp}\,dS\\&+\mathrm{Re}\int_{\partial\Omega}-2\bar u_{kp}\Delta h\partial_nu_{kp}+|u_{kp}|^2\partial_n\Delta h\,dS,\end{aligned}$$

Choosing h compactly supported such that $\partial_n h < 0$ on $\operatorname{supp} D_k$ as in the proof of Proposition 3.2 gives an estimate on $\|\partial_n u_{kp}\|_{L^2(\partial\Omega\times[0,T])}$, provided the new terms induced by $[\Delta, D_k D_p]u$ are controlled; this last point is a consequence of the local smoothing

$$\left|4\int_0^T\int_\Omega \nabla h\cdot\nabla u_{kp}\overline{[\Delta, D_k D_p]u}+\tfrac{1}{2}\bar{u}_{kp}\Delta h[\Delta, D_k D_p]u\,dx\right|dt \lesssim \|u_{kp}\|_{L^2_T H^1}\|u\|_{L^2_T H^3}$$
$$\lesssim \|u_0\|^2_{H^{5/2}}+\|f\|^2_{H^{2,2}}+\|g\|^2_{H^{3+\varepsilon,2}}.$$

This gives $\|\partial_n u_{kp}\|_{L^2} \lesssim \|u_0\|_{H^{5/2}}+\|f\|_{H^{2,2}}+\|g\|_{H^{3+\varepsilon,2}}$. Since $\psi = 1$ on a neighbourhood of 0 and $\partial_n = \partial_{y_d}$ on U, we have $\partial_n D_k D_p = D_k D_p \partial_n$, so that

$$\|D_k D_p \partial_n u\|_{L^2(\partial\Omega\times[0,T])} \lesssim \|u_0\|_{H^{5/2}}+\|f\|_{H^{2,2}}+\|g\|_{H^{3+\varepsilon,2}}.$$

Finally, since $\|\partial_n u(t)\|_{H^1} \lesssim \|u(t)\|_{H^{5/2}}$ and using a partition of unity, we get

$$\|\partial_n u\|_{L^2_T H^2} \lesssim \|u_0\|_{H^{5/2}}+\|f\|_{H^{2,2}}+\|g\|_{H^{3+\varepsilon,2}}.$$

The time regularity of $\partial_n u$ can be obtained in a similar way by considering the IBVP satisfied by $\partial_t u$; the application of Proposition 3.2 requires $\partial_t f \in L^2(\Omega\times[0,T])$ and $\partial_t u|_{t=0} = i\Delta u_0 - if|_{t=0} \in H^{1/2}$, both of which are ensured by $f \in H^{2,2}$. Since $\partial_t\partial_n = \partial_n\partial_t$, the local smoothing property gives directly

$$\|\partial_t\partial_n u\|_{L^2(\partial\Omega\times[0,T])} \lesssim \|u_0\|_{H^{5/2}}+\|f\|_{H^{2,2}}+\|g\|_{H^{3,2}}.$$

The result for $\frac{1}{2}\le s<2$ then follows by a (nontrivial) interpolation argument similar to Lemma A.2 that we sketch now: Setting

$$Y^\alpha = \{(u_0, f, g) \in H^\alpha \times H^{\alpha-1/2,2}\times H^{\alpha+1/2,2} \text{ that satisfy the compatibility conditions}\},$$

it is sufficient to prove $[Y^{1/2}, Y^{5/2}]_\theta \supset Y^{2\theta+1/2}$ for $\theta < \frac{3}{4}$. To get rid of the link between u_0, f and g, let us define $H^{2,2}_{(0)}(\Omega\times[0,T]) = \{f \in H^{2,2} : f|_{\partial\Omega\times\{0\}} = 0\}$. Clearly

$$Y^{5/2} \supset \{(u_0, f, g) \in H^{5/2}\times H^{2,2}_{(0)}\times H^{3,2} \text{ with (CC0), (CCG1)}\} := Y^{5/2}_{(0)}.$$

The key point of $Y^{5/2}_{(0)}$ is that $f|_{t=0} \in H^1_0$, so that the $(f^j)_{1\le j\le J}$ introduced in the description of global compatibility conditions automatically satisfy $\int_0^\infty\int_{\mathbb{R}^{d-1}} |f^j(x', h)|^2\,dx'\,dh/h < \infty$. Therefore the conditions (CC0), (CCG1) only involve u_0 and g, and

$$Y^{5/2}_{(0)} = \{(u_0, g) \in H^{5/2}\times H^{3,2} \text{ with (CC0), (CCG1)}\}\times H^{2,2}_{(0)}.$$

For $\theta < \frac{3}{4}$, we have, from Proposition A.4, $[L^2, H^{2,2}_{(0)}\}]_\theta = H^{2\theta,2}(\Omega\times[0,T])$. As a consequence, setting $X^{3/2} = \{(u_0, g) \in H^{5/2}\times H^{3,2}$ with (CC0), (CCG1)$\}$ (as in Lemma A.2), we are reduced to checking that $[X^{1/2}, X^{3/2}]_\theta = X^{1/2+\theta}$, which can be done as in Lemma A.2. □

Remark 3.5. The loss of regularity on the boundary data can be avoided up to an arbitrary loss on the local smoothing. Indeed for $(u_0, f, g) \in H^{1/2+\varepsilon}\times H^{\varepsilon,2}\times H^{1+\varepsilon,2}$, the virial estimate implies $u \in L^2_T H^1$, and from an argument similar to Corollary 3.4 we find that, for $\frac{1}{2}+\varepsilon \le s < 2$, we have $(u_0, f, g) \in H^s\times H^{s-1/2,2}\times H^{s+1/2,2}$, then $u \in L^2_T H^{s+1/2-\varepsilon}$.

We choose to focus on the case where we lose some regularity on the boundary data because it avoids the use of peculiar numerology for the Strichartz estimates and well-posedness theorems in the rest of the article; however, we will continue to discuss this alternative approach in Remarks 3.8 and 4.3.

The estimate is restricted to functions f compactly supported near $\partial\Omega$. For the well-posedness results of next section we will also need smoothing of the normal derivative when f is supported "away from $\partial\Omega$":

Proposition 3.6. *Let w be the solution of the homogeneous boundary value problem*

$$\begin{cases} i\partial_t w + \Delta_D w = f, \\ w|_{t=0} = 0, \\ w|_{\partial\Omega} = 0; \end{cases}$$

then w satisfies the estimate

$$\|\partial_n w\|_{H^{1/2,2}(\partial\Omega\times[0,T])} \lesssim \|f\|_{B^{1,2}_{3/2,2,0}}.$$

Proof. From the Strichartz estimate in [Burq et al. 2004], we have

$$\|w\|_{C_T H^{1/2}_D \cap L^3 W_0^{1/2,3}} \lesssim \|f\|_{L^{3/2}_T W_0^{1/2,3/2}}.$$

The virial identity gives

$$\|\partial_n w\|^2_{L^2(\partial\Omega\times[0,T])} \lesssim \|u\|_{C_T H^{1/2}_D} + \|u\|_{L^3_T W_0^{1/2,3}} \|f\|_{L^{3/2} W_0^{1/2,3/2}} \lesssim \|f\|^2_{L^{3/2} W_0^{1/2,3/2}},$$

and similarly, using the same differentiation arguments as in Corollary 3.4, we get[2]

$$\|\partial_n w\|_{H^{2,2}(\partial\Omega\times[0,T])} \lesssim \|f\|_{L^{3/2}_T W_0^{5/2,3/2,2} \cap W_T^{5/4,3/2} L^{3/2}}.$$

Let us recall that, for $s \geq 0$, $s \notin \mathbb{N}$, $B^{s,2}_{3/2,3/2,0}(\Omega\times[0,T]) = W_T^{s/2,3/2} L^{3/2} \cap L^{3/2}_T W_0^{s,3/2}$. Using real interpolation with parameter $\theta = \frac{1}{4}$ and $q = 2$ gives the expected result, as a consequence of

$$[L^{3/2} W_0^{1/2,3/2}, L^{3/2}_T W_0^{5/2,3/2} \cap W_T^{5/4,3/2} L^{3/2}]_{1/4,2} \supset [B^{1/2,2}_{3/2,3/2,0}, B^{5/2,2}_{3/2,3/2,0}]_{1/4,2} = B^{1,2}_{3/2,2,0}.$$

The first inclusion is clear, and the equality follows from the interpolation of anisotropic Sobolev spaces; see the book of H. Amann [2009], Section 3.3 for the interpolation of anisotropic spaces on $\mathbb{R}^d$ and Section 4.4 for domains with corner. □

Strichartz estimates. We deduce in this section Strichartz estimates (with loss of derivatives) from the local smoothing. Following the terminology of admissible pair (those (p,q) such that $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$), we say that (p,q) is a weakly admissible pair if

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2}. \tag{3-2}$$

[2] When differentiating in time, we obtain $\partial_t u|_{t=0} = -if|_{t=0} \in W_0^{7/6,3/2} \hookrightarrow H_0^1 \hookrightarrow H_D^{1/2}$, thus the initial data for the problem satisfied by $\partial_t u$ is smooth enough to use the virial identity.

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, such that Ω^c is star-shaped with respect to* 0. *For $\varepsilon > 0$, $T < \infty$, $\frac{1}{2} \leq s < 2$, $(u_0, f, g) \in H^s \times H^{s-1/2,1/4} \times H^{s+1/2+\varepsilon,2}$ satisfying the compatibility conditions, f compactly supported, and any weakly admissible (p,q) with $p, q > 2$, the solution $u \in C_T H^1$ satisfies*

$$\|u\|_{L^p([0,T],W^{s,q}(\Omega))} \lesssim \|u_0\|_{H^s} + \|g\|_{H^{s+1/2+\varepsilon}} + \|f\|_{H^{s-1/2,1/4}}.$$

Proof. The argument from [Burq et al. 2004, Proposition 2.14] can be used with no meaningful modification (see also [Audiard 2013, Corollary 1]). Let us sketch it briefly: we decompose $u = \chi u + (1 - \chi u)$, χ compactly supported, $\chi = 1$ near $\partial\Omega \cup \mathrm{supp}(f)$. From the local smoothing property, $\chi u \in L^2_T H^{s+1/2} \cap L^\infty_T H^s$, we have by (complex) interpolation that $u \in L^p_T H^{s+1/p}$. The Sobolev embedding $H^{s+1/p} \hookrightarrow W^{s,q}$ with $\frac{1}{q} = \frac{1}{2} - \frac{1}{dp}$ and the local smoothing property from Corollary 3.4 imply $\chi u \in L^p_T W^{s,q}$.

The function $(1-\chi)u$ extended by 0 outside $\mathrm{supp}(1-\chi)$ satisfies a Schrödinger equation on $\mathbb{R}^d$, and the usual Strichartz estimates on $\mathbb{R}^d$ imply (by a standard but nontrivial argument that originates in [Staffilani and Tataru 2002])

$$\|(1-\chi)u\|_{L^{2p}([0,T],W^{s,q})} \lesssim \|u_0\|_{H^s} + \|g\|_{H^{s+1/2+\varepsilon,2}} + \|f\|_{H^{s-1/2,1/4}}.$$

From $L^{2p}([0,T]) \subset L^p([0,T])$ we obtain the expected estimate. □

Remark 3.8. Following the observations of Remark 3.5, we could also prove an alternate Strichartz estimate with optimal boundary data in $H^{s+1/2,2}$ but $\frac{1}{p} + \frac{d}{q} = \frac{d}{2} + \frac{2\varepsilon}{p}$, simply by using the embedding $H^{s+1/2-\varepsilon} \hookrightarrow W^{s,q_1}$, $1/q_1 = \frac{1}{2} - \left(\frac{1}{2} - \varepsilon\right)/d$.

4. Nonlinear well-posedness

We consider here nonlinear IBVPs of the form

$$\begin{cases} i\partial_t u + \Delta u = F(u), & (x,t) \in \Omega \times [0,T[, \\ u|_{t=0} = u_0, & x \in \Omega, \\ u|_{\partial\Omega\times[0,T]} = g, & (x,t) \in \partial\Omega \times [0,T[, \end{cases} \tag{NLS}$$

with the following assumptions on $F \in C^1(\mathbb{C})$: there exists $\alpha > 0$ such that

$$|F(z)| \lesssim |z|(1 + |z|^\alpha), \tag{4-1}$$

$$|\nabla F(z)| \lesssim (1 + |z|)^\alpha. \tag{4-2}$$

For the smoothness of the flow we will assume $F \in C^2(\mathbb{C})$ and

$$|\nabla^2 F(z)| \lesssim (1 + |z|)^{\max(\alpha-1,0)} \tag{4-3}$$

Local well-posedness. Since our first result is local in time, we define

$$H^{3/2+\varepsilon,2}_{\mathrm{loc}}(\mathbb{R}^+ \times \partial\Omega) = \{g : \chi(t)g \in H^{3/2+\varepsilon,2}(\mathbb{R}^+_t \times \partial\Omega) \ \text{ for all } \ \chi \in C^\infty_c(\mathbb{R}^+)\}.$$

We say that $u \in C_T H^1$ is a local solution to (NLS) if it satisfies $i\partial_t u + \Delta u = F(u)$ in the sense of distributions (for $u \in C_T H^1$ all quantities in the equality make sense), $u|_{\partial\Omega\times[0,T]} = g$ in the usual sense of traces and $u|_{t=0} = u_0$.

Theorem 4.1. *If F satisfies (4-1)–(4-2), then for any $(u_0, g) \in H^1(\Omega) \times H^{3/2+\varepsilon,2}_{\mathrm{loc}}(\mathbb{R}^+ \times \Omega)$ satisfying (CC0) and $\alpha < 2/(d-2)$, there exists a unique maximal solution $u \in C_{T^*}H^1$ of (NLS).*

The solution is causal in the sense that $u(t)$ only depends of u_0 and $g|_{s\le t}$, and, if $T^* < \infty$, then $\lim_{t\to T^*} \|u(t)\|_{H^1} = +\infty$.

If F satisfies (4-3) and $d \le 3$, then for any $T < T^*$ the solution map is Lipschitz from bounded sets of $H^1(\Omega) \times H^{3/2+\varepsilon,2}(\mathbb{R}^+ \times \Omega)$ to $C([0,T], H^1)$.

It will be convenient to introduce $\tilde{u}$, the solution of

$$\begin{cases} i\partial_t \tilde{u} + \Delta\tilde{u} = F(\tilde{g}), & (x,t) \in \Omega \times [0,T[,\\ \tilde{u}|_{t=0} = u_0, & x \in \Omega,\\ \tilde{u}|_{\partial\Omega\times[0,T]} = g, \ (x,t) \in \partial\Omega \times [0,T[, \end{cases} \tag{4-4}$$

where $\tilde{g} \in H^{2,2}(\Omega \times [0,T])$ is a compactly supported lifting of g. Thus u must satisfy

$$u = \tilde{u} + \int_0^t e^{i(t-s)\Delta_D}(F(u) - F(\tilde{g}))(s)\,ds \quad \text{for all } t \in [0,T].$$

Choose q_0 such that $(2, q_0)$ is weakly admissible. According to Theorems 2.2 and 3.7, we have $\tilde{u} \in C_T H^1 \cap L^2_T W^{1,q_0}$ if $F(\tilde{g}) \in H^{1/2,2}$. Actually $F(\tilde{g})$ is smoother than needed:

Lemma 4.2. *For $\varphi \in H^{2,2}(\Omega \times [0,T])$ and F satisfying (4-1)–(4-2), $F(\varphi) \in H^{1,2}$.*

Proof. It is clear that $F(\varphi) \in L^2_T L^2$; indeed

$$\|F(\varphi)\|_{L^2_T L^2} \lesssim \|\varphi\|_{L^2_T L^2} + \|\varphi\|^{1+\alpha}_{L^{2(1+\alpha)}} \lesssim \|\varphi\|_{L^2_T H^1}(1 + \|\varphi\|^{\alpha}_{L^2_T H^1}).$$

Since $\alpha < 2/(d-2)$, there exist p, q satisfying

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad \min\Big(\frac{\alpha}{2}, \frac{1}{d}\Big) \ge \frac{1}{p} > \frac{\alpha(d-2)}{2d}, \quad \frac{1}{q} > \frac{d-2}{2d},$$

and Hölder's inequality gives, for any $t \in [0,T]$,

$$\begin{aligned} \|\nabla F(\varphi)(t)\|_{L^2(\Omega)} &\lesssim \|(1 + |\varphi|^{\alpha})\nabla\varphi\|_{L^2}\\ &\lesssim \|\varphi\|_{H^1} + \|\varphi\|^{\alpha}_{L^{\alpha p}}\|\nabla\varphi\|_{L^q}\\ &\lesssim \|\varphi\|_{H^1} + \|\varphi\|^{\alpha}_{H^1}\|\varphi\|_{H^2}, \end{aligned}$$

where we used the Sobolev embedding $H^1 \hookrightarrow L^q$, $2 \le q \le 2d/(d-2)$ (or $q < \infty$ if $d = 2$). From the embedding $H^{2,2} \hookrightarrow C_T H^1$ we deduce, by taking the L^2_T norm,

$$\|\nabla F(\varphi)\|_{L^2_T H^1} \lesssim \|\varphi\|_{L^2_T H^1} + \|\varphi\|^{\alpha}_{L^\infty_T H^1}\|\varphi\|_{L^2_T H^2} \lesssim \|\varphi\|_{H^{2,2}}(1 + \|\varphi\|^{\alpha}_{H^{2,2}}).$$

For the time regularity we have, using Hölder's inequalities again,

$$\begin{aligned} \|F(\varphi(t)) - F(\varphi(s))\|_{L^2(\Omega)} &\lesssim \|\varphi(t) - \varphi(s)\|_{L^2} + \big\||\varphi(t)| + |\varphi(s)|\big\|^{\alpha}_{L^{\alpha p}}\|\varphi(t) - \varphi(s)\|_{L^q}\\ &\lesssim \|\varphi(t) - \varphi(s)\|_{L^2} + \big\||\varphi(t)| + |\varphi(s)|\big\|^{\alpha}_{H^1}\|\varphi(t) - \varphi(s)\|_{H^1}, \end{aligned}$$

thus the embedding $H^{2,2} \hookrightarrow H^{1/2}([0,T], H^1(\Omega))$ gives

$$\begin{aligned}\|F(\varphi)\|^2_{\dot H^{1/2}_T L^2} &= \iint_{[0,T]^2} \frac{\|F(\varphi(t)) - F(\varphi(s))\|^2_{L^2}}{|t-s|^2}\, ds\, dt \\ &\lesssim \|\varphi\|^2_{H^{1/2}L^2} + \|\varphi\|^{2\alpha}_{L^\infty_T H^1} \|\varphi\|^2_{\dot H^{1/2}_T H^1} \\ &\lesssim \|\varphi\|^2_{H^{2,2}}(1 + \|\varphi\|^{2\alpha}_{H^{2,2}}). \qquad \square\end{aligned}$$

Proof of Theorem 4.1. Uniqueness: The uniqueness can be done as in the case of homogeneous Dirichlet boundary conditions from [Burq et al. 2004]. If u_1 and u_2 are two solutions in $C_{T^*}H^1$, then $w = u_1 - u_2$ is a solution of

$$\begin{cases} i\partial_t w + \Delta w = F(u_1) - F(u_2), & (x,t) \in \Omega \times [0,T[, \\ w|_{t=0} = 0, & x \in \Omega, \\ \tilde w|_{\partial\Omega\times[0,T]} = 0, & (x,t) \in \partial\Omega \times [0,T[. \end{cases}$$

This is a homogeneous boundary value problem for which the Strichartz estimates (1-2) give, for (p,q) weakly admissible as in (3-2), (r', s') weakly admissible and $T < T^*$,

$$\|w\|_{L^\infty_T L^2 \cap L^p_T L^q} \lesssim \|w\|_{L^1_T L^2} + \big\|(|u_1| + |u_2|)^\alpha w\big\|_{L^r_T L^s} \le T\|w\|_{L^\infty_T L^2} + + \big\|(|u_1| + |u_2|)^\alpha w\big\|_{L^r_T L^s}.$$

If we can choose (r, s, p_1, q_1, p, q) satisfying

$$\begin{cases} \dfrac{1}{p} + \dfrac{d}{q} = \dfrac{d}{2}, \quad \dfrac{1}{r} + \dfrac{d}{s} = 1 + \dfrac{d}{2}, \\ \dfrac{1}{p_1} + \dfrac{1}{2} = \dfrac{1}{r}, \quad \dfrac{1}{q_1} + \dfrac{1}{q} = \dfrac{1}{s}, \\ \dfrac{\alpha(d-2)}{2d} < \dfrac{1}{q_1} < \dfrac{\alpha}{2}, \quad \dfrac{1}{p} < \dfrac{1}{2}, \quad 0 < \dfrac{1}{p_1} < \alpha, \end{cases} \tag{4-5}$$

we get from the Sobolev embedding and Hölder estimate in time that

$$\begin{aligned}\big\|(|u_1| + |u_2|)^\alpha w\big\|_{L^r_T L^s} &\lesssim \big\||u_1| + |u_2|\big\|^\alpha_{L^{\alpha p_1} L^{\alpha q_1}} \|w\|_{L^2 L^q} \\ &\lesssim T^{1/2 - 1/p}(\|u_1\|_{L^\infty_T H^1} + \|u_2\|_{L^\infty_T H^1})^\alpha \|w\|_{L^p L^q},\end{aligned}$$

and thus $w = 0$ for $0 \le t \le T$, T small enough only depending on $\|u_1\|_{L^\infty H^1} + \|u_2\|_{L^\infty H^1}$. Iterating the argument implies $u = v$ on $[0, T^*[$. The system (4-5) implies

$$1 + \frac{d}{2} = \frac{1}{r} + \frac{d}{s} = \frac{1}{p_1} + \frac{1}{2} + \frac{d}{q_1} + \frac{d}{q} > \frac{1}{p_1} + \frac{d}{2} + \frac{\alpha(d-2)}{2} + \left(\frac{1}{2} - \frac{1}{p}\right), \tag{4-6}$$

which can be solved since $\frac{1}{2}\alpha(d-2) < 1$: we first choose $p > 2$ close enough to 2 that $\frac{1}{2}\alpha(d-2) + \frac{1}{2} - \frac{1}{p} < 1$, then it is possible to choose p_1 that satisfies (4-6) and $0 < \frac{1}{p_1} < \alpha$; up to increasing p we can assume $\frac{1}{p_1} < \frac{1}{2}$. The choice of p determines the value of $q > 2$, the choice of p_1 determines the value of $1 < r < 2$, and then of $1 < s < 2$. The only equation left is $\frac{1}{q_1} = \frac{1}{s} - \frac{1}{q}$; its solution $\frac{1}{q_1}$ belongs to $]0, 1[$, and thus is an acceptable Hölder index.

Causality: This can be proved as for uniqueness, since if g_1, g_2 coincide on $[0,t]$, the uniqueness argument can be applied on $[0,t]$ and implies the associated solutions satisfy $u_1|_{[0,t]} = u_2|_{[0,t]}$.

Local existence: We recall that $(2,q_0)$ is assumed to be weakly admissible. According to Lemma 4.2 and Theorems 2.2 and 3.7, $\tilde{u}\in C_TH^1\cap L^2_TW^{1,q_0}$, since $F(\tilde{g})\in H^{1,2}\subset H^{1/2,2}$. Setting $w=u-\tilde{u}$, the local existence will be a consequence of the existence of a local solution to

$$\begin{cases} i\partial_t w+\Delta w=F(\tilde{u}+w)-F(\tilde{g}),\\ w|_{t=0}=0,\\ w|_{\partial\Omega\times[0,T]}=0.\end{cases}$$

This is a nonlinear homogeneous boundary value problem; the existence of a solution is essentially a consequence of (the proof of) Theorem 1 in [Burq et al. 2004]. As it does not strictly cover the case of our nonlinearity, we briefly sketch the argument. Let us define the map L as

$$\begin{aligned} L:X_T=C_TH^1_0\cap L^p_TW^{1,q}&\to C_TH^1_0\cap L^p_TW^{1,q},\\ w\mapsto L(w)&=\int_0^t e^{i(t-s)\Delta_D}(F(\tilde{u}+w)-F(\tilde{g}))\,ds;\end{aligned}$$

we will check that it has a fixed point for T small enough. Burq et al. [2004] prove that, for a convenient choice of weakly admissible pairs (p,q), (p_1,q_1) (depending on $\alpha<2/(d-2)$ and d), the map $\tilde{L}(w)=\int_0^t e^{i(t-s)\Delta_D}F(w)\,ds$ satisfies

$$\begin{aligned} \|\tilde{L}w\|_{X_T}&\lesssim T^{\theta}(\|w\|_{X_T}+\|w\|_{X_T}^{1+\alpha}),\\ \|\tilde{L}w_1-\tilde{L}w_2\|_{X_T}&\lesssim T^{\theta'}\|w_1-w_2\|_{X_T}(1+\|w_1\|^{\alpha}_{X_T}+\|w_2\|^{\alpha}_{X_T}) && \text{if } d<4,\\ \|\tilde{L}w_1-\tilde{L}w_2\|_{C_TL^2\cap L^{p_1}L^{q_1}}&\lesssim T^{\theta''}\|w_1-w_2\|_{C_TL^2\cap L^{p_1}L^{q_1}}(1+\|w_1\|^{\alpha}_{X_T}+\|w_2\|^{\alpha}_{X_T}) && \text{if } d\ge 4,\end{aligned}$$

where θ, θ', θ'' are positive, and the second inequality ($d<4$) also requires the assumption (4-3) on F (this is Propositions 3.1, 3.3, 3.4 and equations (3.9)–(3.10) from [Burq et al. 2004]).

Since $F(\tilde{u}+w)-F(\tilde{g})$ has trace 0 on $\partial\Omega\times[0,T]$, we can use these estimates. We recall $\tilde{g}$ is in $H^{2,2}\hookrightarrow L^\infty_TH^1\cap L^2_TW^{1,q_0}$; therefore, setting $M(w)=\|w\|_{X_T}+\|\tilde{u}\|_{X_T}+\|g\|_{H^{3/2,2}}$ the estimates give, directly in our case,

$$\|Lw\|_{X_T}\lesssim T^{\theta}(M+(M)^{1+\alpha}), \tag{4-7}$$

$$\|Lw_1-Lw_2\|_{X_T}\lesssim T^{\theta'}\|w_1-w_2\|_{X_T}\big(1+(M(w_1)+M(w_2))^{\alpha}\big) \quad \text{if } d<4, \tag{4-8}$$

$$\begin{aligned}&\|\tilde{L}w_1-\tilde{L}w_2\|_{C_TL^2\cap L^{p_1}L^{q_1}}\\ &\qquad\lesssim T^{\theta''}\|w_1-w_2\|_{C_TL^2\cap L^{p_1}L^{q_1}}\big(1+(M(w_1)+M(w_2))^{\alpha}\big) \quad \text{if } d\ge 4.\end{aligned} \tag{4-9}$$

If $d<4$, from (4-7)–(4-8) we can apply the Picard–Banach fixed-point theorem in $C_TH^1\cap L^p_TW^{1,q}$ for some $T(\|u_0\|_{H^1}+\|g\|_{H^{3/2+\varepsilon,2}(\partial\Omega\times[0,T])})$ and it also implies that the flow is Lipschitz. If $d\ge 4$, (4-7) implies that L sends some ball of X_T to itself, and from (4-9) it is contractive in the weaker space $C_TL^2\cap L^{p_1}_TL^{q_1}$. By a standard argument, the metric space $\{u:\|u\|_{X_T}\le M\}$ with distance $d(u,v)=\|u-v\|_{L^\infty_TL^2\cap L^p_TL^q}$ is complete (e.g., [Cazenave 2003, Theorem 1.2.5]), so that the existence of a solution is again a consequence of the Picard–Banach fixed point theorem.

Blow-up alternative: This is a direct consequence of the fact that the time of local existence only depends on $\|u_0\|_{H^1} + \|g\|_{H^{3/2+\varepsilon}}$. Let u be a solution on $[0, T^*[$ such that $\underline{\lim}\|u(t)\|_{H^1} = C < \infty$ and let δ be such that $T(2C + \|g\|_{H^{3/2+\varepsilon,2}([T^*-1,T^*+1]\times\Omega)}) \geq 2\delta$. Up to decreasing δ, we can assume $\|u(T^*-\delta)\|_{H^1} \leq 2C$. Since $u \in C_T H^1$ and $u|_{\partial\Omega} = g$ the pair $u(T^*-\delta)$, $g|_{[T^*-\delta,+\infty[}$ satisfies (CC0) on $\partial\Omega \times \{T^*-\delta\}$, thus (NLS) has a local solution on the time interval $[T^*-\delta, T^*+\delta]$. Thanks to the uniqueness on $[T^*-\delta, T^*[$, this allows us to extend the solution on $[0, T^*+\delta]$. □

Remark 4.3. If one chooses to use instead the Strichartz estimate from Remark 3.8, namely

$$\|u\|_{L^p_T W^{1,q}} \lesssim \|u_0\|_{H^1} + \|g\|_{H^{3/2}} + \|f\|_{H^{1/2,1/4}} \quad \text{when } \frac{1}{p} + \frac{d}{q} = \frac{d}{2} + \frac{2\varepsilon}{p},$$

the restriction on α becomes (supposedly) $\alpha < (2-4\varepsilon)/(d-2)$. Consequently, well-posedness for the whole range $\alpha < 2/(d-2)$ and boundary data in the optimal space $H^{3/2,2}$ can most likely be obtained, up to more involved estimates with some ε in all indices.

Since our Strichartz estimates for the IBVP only give a gain of half a derivative, the natural limitation on the nonlinearity is $\alpha < 2/(d-2)$ (as in [Burq et al. 2004]). However better (scale-invariant) estimates are available for the homogeneous boundary value problem, and they can be combined with our estimates to improve the range of α. The following theorem illustrates this idea.

Theorem 4.4. *If Ω is the exterior of a smooth strictly convex obstacle, then Theorem 4.1 is true for $\alpha < 3/(d-2)$.*

Proof. From [Ivanovici 2010], the usual Strichartz estimates with (p,q) such that $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$, $p > 2$, are true for the semigroup $e^{it\Delta_D}$. The uniqueness in $L^\infty_T H^1$ follows from standard arguments; see, e.g., [Cazenave 2003, Section 4.2]. The existence part is again an application of the Picard–Banach fixed point theorem: let (p,q) be weakly admissible, $p > 2$, such that

$$\alpha < \frac{2}{d-2}\Big(1 + \frac{1}{p}\Big). \tag{4-10}$$

We set $X_T = C_T H^1 \cap L^p_T W^{1,q}$ and, as in Theorem 4.1,

$$L : w \mapsto L(w) = \int_0^t e^{i(t-s)\Delta_D}(F(\tilde{u}+w) - F(\tilde{g}))\,ds.$$

From the Sobolev embedding, $\tilde{g} \in H^{2,2} \hookrightarrow L^2 H^2 \cap C_T H^1 \hookrightarrow X_T$. Let q_1 be such that $\frac{2}{p} + \frac{d}{q_1} = \frac{d}{2}$. From the scale-invariant Strichartz estimates we have

$$\|Lw\|_{X_T} \lesssim \|Lw\|_{L^\infty_T H^1 \cap L^p_T W^{1,q_1}} \lesssim \|F(\tilde{u}+w) - F(\tilde{g})\|_{L^{p'}_T W^{1,q_1'} + L^1_T H^1},$$

and we will prove that there exists $\theta > 0$ such that

$$\|F(v)\|_{L^{p'}_T W^{1,q_1'} + L^1_T H^1} \lesssim T^\theta (1 + \|v\|_{X_T}^{1+3/(d-2)}). \tag{4-11}$$

Let $\psi \in C^\infty(\mathbb{R}^+)$ with $\psi \equiv 1$ for $x \geq 1$ and $\psi \equiv 0$ for $x \leq \frac{1}{2}$. Since $\operatorname{supp}(1-\psi(|v|^2)) \subset \{|v| \leq 1\}$, we have

$$\big\|1 - \psi(|v|^2)F(v)\big\|_{L^1_T H^1} \lesssim \|v\|_{L^1_T H^1} \leq T\|v\|_{X_T}.$$

On the other hand, for any $\beta \geq \alpha$,

$$\big|\psi(|v|^2)F(v)\big| \lesssim |v|^{1+\beta}, \quad \big|\nabla(\psi(|v|^2)F(v))\big| \lesssim |v|^{\beta}|\nabla v|.$$

Since

$$(1+\alpha)q_1' \leq \Big(1+\frac{2}{d-2}\Big(1+\frac{1}{p}\Big)\Big)\Big(\frac{1}{2}+\frac{2}{dp}\Big)^{-1} = \frac{2d}{d-2}\frac{dp+2}{dp+4} < \frac{2d}{d-2},$$

there exists $\beta \geq \alpha$ such that $2 \leq (1+\beta)q_1' \leq 2d/(d-2)$, and this choice leads to

$$\big\||v|^{1+\beta}\big\|_{L^{p'}L^{q_1'}} \lesssim \|v\|^{1+\beta}_{L^{(1+\beta)p'}L^{(1+\beta)q_1'}} \lesssim T^{1/p'}\|v\|^{1+\beta}_{L^\infty H^1}.$$

To estimate $\nabla(\psi(|v|^2)F(v))$, we use Hölder's inequality on $|v|^\beta\nabla v$ combined with the Sobolev embedding $W^{1,r} \hookrightarrow L^s$, $\frac{1}{s}=\frac{1}{r}-\frac{1}{d}$:

$$\big\||v|^\beta\nabla v\big\|_{L^{p'}L^{q_1'}} \lesssim \|v\|^\beta_{L_T^{\hat{p}}W^{1,\hat{q}}}\|\nabla v\|_{L^pL^q}, \tag{4-12}$$

where

$$\begin{aligned}
\frac{1}{\hat{p}} &= \frac{1}{\beta}\Big(\frac{1}{p'}-\frac{1}{p}\Big) = \frac{1}{\beta}\Big(1-\frac{2}{p}\Big) && \text{(Hölder in time)},\\
\frac{1}{\hat{q}} &= \frac{1}{\beta}\Big(\frac{1}{q_1'}-\frac{1}{q}\Big)+\frac{1}{d} = \frac{1}{d}\Big(1+\frac{3}{\beta p}\Big) && \text{(Hölder in space and Sobolev embedding)}.
\end{aligned}$$

Note that q, $\hat{p}$, $\hat{q}$ are defined by p and β. If we can choose $p>2$ and $\beta \geq \alpha$ such that

$$\frac{1}{\hat{p}}+\frac{d}{\hat{q}} > \frac{d}{2}, \quad \frac{1}{\hat{p}} < \frac{1}{2}, \quad \frac{1}{q} \leq \frac{1}{\hat{q}} \leq \frac{1}{2}, \tag{4-13}$$

this gives (4-11); indeed, for such p, β, if $1/p_1+d/\hat{q}=d/2$ we have $L_T^{p_1}W^{1,\hat{q}} \subset X_T$, $1/p_1 < 1/\hat{p}$, and (4-12) gives

$$\|v\|^\beta_{L_T^{\hat{p}}W^{1,\hat{q}}}\|\nabla v\|_{L^pL^q} \lesssim T^{\beta(1/\hat{p}-1/p_1)}\|v\|^\beta_{L^{p_1}W^{1,\hat{q}}}\|\nabla v\|_{L^pL^q} \lesssim T^{\beta(1/\hat{p}-1/p_1)}\|v\|^{1+\beta}_{X_T}. \tag{4-14}$$

Let us now check that there exists a choice of β and p for which (4-13) holds. The first two conditions become

$$\frac{1}{\beta}\Big(1-\frac{2}{p}\Big)+\Big(1+\frac{3}{\beta p}\Big) > \frac{d}{2} \iff \frac{1}{p} > \beta\Big(\frac{d}{2}-1\Big)-1,$$

$$\frac{1}{\beta}\Big(1-\frac{2}{p}\Big) < \frac{1}{2} \iff \frac{1}{p} > \frac{1}{2}-\frac{\beta}{4}.$$

Or, more compactly,

$$\frac{1}{2} > \frac{1}{p} > \max\Big(\frac{1}{2}-\frac{\beta}{4},\ \beta\Big(\frac{d}{2}-1\Big)-1\Big)$$

The condition $\frac{1}{2}-\frac{\beta}{4} < \frac{1}{2}$ is automatically satisfied. To ensure $1/q \leq 1/\hat{q} \leq \frac{1}{2}$, we must have

$$\frac{1}{\beta} \leq \frac{p(d-2)}{6} \quad \text{and} \quad \frac{1}{\beta} \geq \frac{p(d-2)}{6}-\frac{1}{3},$$

so that the condition is finally equivalent to

$$\beta\Big(\frac{d}{2}-1\Big)-1<\frac{1}{p}\le\frac{\beta(d-2)}{6},$$

and there exist solutions $p>2$, $\beta\ge\alpha$ if and only if $\beta<3/(d-2)$, which is always compatible with $\beta\ge\alpha$ and the initial assumption (4-10).

From (4-11), we infer

$$\|Lw\|_{X_T}\lesssim T^{\theta}\big(1+(\|\tilde u\|_{X_T}+\|w\|_{X_T}+\|\tilde g\|_{X_T})^{3/(d-2)}\big),$$

so that for T small enough, L maps the ball of radius one in X_T to itself. It is not clear if L is contractive in X_T even for smaller T, however contractivity for the weaker topology induced by $L^\infty_T L^2\cap L^pL^q$ is an easy consequence of the previous estimates and the assumptions on F:

$$|F(\tilde u+w_1)-F(\tilde u+w_2)|\lesssim|w_1-w_2|+(|w_1|+|w_2|+|\tilde u|)^\beta|w_1-w_2|,$$

and (4-14) gives

$$\begin{aligned}\|Lw_1-Lw_2\|_{X_T}&\\ &\lesssim\|w_1-w_2\|_{L^1_TL^2}+\big\||w_1-w_2|+(|w_1|+|w_2|+|\tilde u|)^\beta|w_1-w_2|\big\|_{L^{p'}_TL^{q'_1}}\\ &\lesssim T^{\beta(1/\hat p-1/p_1)}(\|\tilde u\|_{X_T}+\|w_1\|_{X_T}+\|w_2\|_{X_T})^\beta\|w_1-w_2\|_{L^p_TL^q}+T\|w_1-w_2\|_{L^\infty_TL^2}.\end{aligned}\tag{4-15}$$

As for Theorem 4.1, the contractivity of L for the $L^p_TL^q\cap L^\infty_TL^2$ topology and the mapping of a ball of X_T to itself gives the existence of a solution as a fixed point. □

Remark 4.5. The only thing limiting us to $\alpha<3/(d-2)$ is that $\tilde u$ only belongs to $C_TH^1\cap L^2W^{1,q_0}$ with $\frac{1}{2}+\frac{d}{q_0}=\frac{d}{2}$. If this limitation was lifted the fixed point argument on w could be performed in the usual scale-invariant spaces.

Remark 4.6. Theorem 4.4 is only an example of how one may mix optimal and nonoptimal Strichartz estimates. If Ω is only assumed to be the exterior of a nontrapping obstacle, [Blair et al. 2012] proved scale-invariant Strichartz estimates with loss of derivatives, namely

$$\|e^{it\Delta_D}u_0\|_{L^pL^q}\lesssim\|u_0\|_{H^\sigma}\quad\text{with}\quad\frac{2}{p}+\frac{d}{q}=\frac{d}{2}-\sigma,\ \frac{1}{p}+\frac{1}{q}\le\frac{1}{2}.$$

Such estimates could probably be used to improve the range of α if Ω^c is only star-shaped. Since the method seems similar and with numerous specific cases, we chose not to develop this issue.

Global well-posedness. In order to obtain global well-posedness for the defocusing nonlinear Schrödinger equation

$$\begin{cases}i\partial_tu+\Delta u=|u|^\alpha u, & (x,t)\in\Omega\times[0,T[,\\ u|_{t=0}=u_0, & x\in\Omega,\\ u|_{\partial\Omega\times[0,T]}=g, & (x,t)\in\partial\Omega\times[0,T[,\end{cases}\tag{NLSD}$$

the argument based on local well-posedness and conservation of energy cannot be trivially applied. Indeed we only have the formal identities

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|u|^2\,dx = -\operatorname{Im}\int_{\partial\Omega}\partial_n u\bar{g}\,dS, \tag{4-16}$$

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla u|^2+\frac{1}{\alpha+2}|u|^{\alpha+2}\,dx = \operatorname{Re}\int_{\partial\Omega}\partial_n u\partial_t\bar{g}\,dS \tag{4-17}$$

If $g\in H^{s,2}$, the control of $\|u\|_{C_TH^1}$ requires us to control $\|\partial_n u\|_{H^{2-s,2}}$. In particular, for the almost optimal regularity $s=\frac{3}{2}+\varepsilon$, we must have some control on $\partial_n u\in H^{1/2-\varepsilon,2}(\partial\Omega\times[0,T])$, which is its (almost) optimal space of regularity.

We will first deal with the simpler case $g\in H^{2,2}$; in this case we only need to control $\|\partial_n u\|_{L^2}$. This can be done thanks to a nonlinear variation of the virial identity from Proposition 3.1.

Theorem 4.7. (1) *For any $0<\alpha<2/(d-2)$, if $(u_0,g)\in H^1(\Omega)\times H^{2,2}_{\mathrm{loc}}(\mathbb{R}^+\times\partial\Omega)$ satisfy (CC0), then (NLSD) has a unique global solution $u\in C(\mathbb{R}^+,H^1)$.*

(2) *If Ω^c is strictly convex and there exists $\varepsilon>0$ such that $g\in H^{2+\varepsilon,2}$, then the theorem is true for $\alpha<4/(d-2)$.*

Proof. The case (1) is a simple consequence of the virial identity and the blow-up alternative, indeed the (nonlinear) virial identity writes

$$\begin{aligned}
\frac{d}{dt}I(u(t))
&= 4\operatorname{Re}\int_\Omega \operatorname{Hess}(h)(\nabla u,\overline{\nabla u})-\tfrac{1}{4}|u|^2\Delta^2h+\nabla h\cdot\nabla u|u|^\alpha\bar{u}+\tfrac{1}{2}\bar{u}\Delta h|u|^\alpha u\,dx\\
&\qquad+\operatorname{Re}\int_{\partial\Omega}2\partial_nh|\nabla_\tau g|^2-2\partial_nh|\partial_nu|^2-2i\,\partial_nh\partial_tg\bar{g}\,dS+\operatorname{Re}\int_{\partial\Omega}-2\bar{g}\Delta h\partial_nu+|g|^2\partial_n\Delta h\,dS\\
&= 4\operatorname{Re}\int_\Omega \operatorname{Hess}(h)(\nabla u,\overline{\nabla u})-\tfrac{1}{4}|u|^2\Delta^2h+|u|^{\alpha+2}\Delta h\Big(\frac{1}{2}-\frac{1}{\alpha+2}\Big)\,dx\\
&\qquad+\operatorname{Re}\int_{\partial\Omega}2\partial_nh|\nabla_\tau g|^2-2\partial_nh|\partial_nu|^2-2i\,\partial_nh\partial_tg\bar{u}\,dS\\
&\qquad\quad+\operatorname{Re}\int_{\partial\Omega}-2\bar{g}\Delta h\partial_nu+|g|^2\partial_n\Delta h+\frac{|g|^{\alpha+2}}{\alpha+2}\partial_nh\,dS.
\end{aligned}$$

As for Proposition 3.2, we choose $h=\sqrt{1+|x|^2}$ so that $\operatorname{Hess}(h)$, $\Delta h>0$, $\partial_nh\le 0$ and integrate in time. From the embedding $H^{2,2}(\partial\Omega\times[0,T])\hookrightarrow H_T^{2/(d+1)}H^{(2d-2)/(d+1)}\hookrightarrow L^{2(d+1)/(d-3)}(\partial\Omega\times[0,T])$ (or L^∞ if $d=2$, L^p for any $2\le p<\infty$ if $d=3$) we have

$$\int_0^T\int_{\partial\Omega}|g|^{\alpha+2}\,dS\,dt\lesssim\|g\|^{\alpha+2}_{H^{2,2}(\partial\Omega\times[0,T])}.$$

If K is a compact neighbourhood of $\partial\Omega$, we deduce

$$\int_{K\times[0,T]}|\nabla u|^2+|u|^{\alpha+2}\,dx\,dt-\int_{\partial\Omega\times[0,T]}|\partial_nu|^2x\cdot n\,dS\,dt\le M(T)(1+\|u\|^2_{C_TH^1}+\|g\|^{\alpha+2}_{H^{2,2}}).$$

If $x \cdot n < 0$ on $\partial\Omega$, this gives directly a control of $\|\partial_n u\|_{L^2}$; if not then we can argue as in Proposition 3.2 by using some function h compactly supported in K such that $\partial_n h < 0$. For this choice, Δh and $\mathrm{Hess}(h)$ are no longer signed, but using the estimate $\|u\|^{\alpha+2}_{L^{\alpha+2}([0,T]\times K)} \lesssim 1 + \|u\|^2_{C_T H^1} + \|g\|^{\alpha+2}_{H^{2,2}}$ we get

$$\|\partial_n u\|_{L^2} \leq M(T)(1 + \|u\|_{C_T H^1} + \|g\|^{\alpha/2+1}_{H^{2,2}}).$$

Plugging this in the "conservation" laws (4-16)–(4-17) implies

$$\|u\|^2_{C_T H^1} \leq \|u_0\|^2_{H^1} + \|\partial_n u\|_{L^2}\|g\|_{H^{2,2}} \lesssim 1 + \|u_0\|^2_{H^1} + (\|u\|_{C_T H^1} + \|g\|^{\alpha/2+1}_{H^{2,2}})\|g\|_{H^{2,2}}$$

and thus

$$\tfrac{1}{2}\|u\|^2_{C_T H^1} \lesssim 1 + \|u_0\|^2_{H^1} + \|g\|^{\alpha/2+2}_{H^{2,2}(\partial\Omega\times[0,T])}.$$

As a consequence, u remains locally bounded in H^1 and the solution must be global.

The case (2) is a bit more intricate, indeed even the local existence of a solution for $3/(d-2) \leq \alpha < 4/(d-2)$ has not been covered yet. The main argument is that we can modify $\tilde{u}$ from problem (4-4) so that it belongs to $C_T H^1 \cap L^2_T W^{1,q_0}$, $1 + d/q_0 = d/2$: since $g \in H^{2+\varepsilon,2}$, we have from (CC0) that $u_0|_{\partial\Omega} = g|_{t=0} \in H^{1+\varepsilon,2}$. Let $v_0 \in H^{3/2+\varepsilon}(\Omega)$ be a lifting of $u_0|_{\partial\Omega}$; we define $\tilde{v}$ as the solution of the linear IBVP

$$\begin{cases} i\partial_t \tilde{v} + \Delta\tilde{v} = F(\tilde{g}), \\ \tilde{v}|_{t=0} = v_0, \\ \tilde{v}|_{\partial\Omega\times[0,T]} = g. \end{cases}$$

Since $F(\tilde{g}) \in H^{1,2}$ (see Lemma 4.2), $g \in H^{2+\varepsilon,2}$, $v_0 \in H^{3/2}$, the Strichartz estimates imply $\tilde{v}$ is in $L^2_T W^{3/2,q} \hookrightarrow L^2_T W^{1,q_0}$, where $1 + d/q_0 = d/2$. We are now left to solve the homogeneous boundary value problem

$$\begin{cases} i\partial_t w + \Delta w = F(\tilde{v} + w) - F(\tilde{g}), \\ w|_{t=0} = u_0 - v_0 \in H^1_0, \\ w|_{\partial\Omega\times[0,T]} = 0, \end{cases}$$

or equivalently obtain a fixed point to the map

$$Lw = e^{it\Delta_D}(u_0 - v_0) + \int_0^t e^{i(t-s)\Delta_D}(F(\tilde{v}+w) - F(\tilde{g}))\,ds.$$

Since $\tilde{v}, \tilde{g} \in L^\infty_T H^1 \cap L^2_T W^{1,q_0}$, the fixed point argument can be done as in the $\mathbb{R}^d$ case, e.g., [Cazenave 2003, Section 4.4], leading to local existence. We can still use the virial identity as in case (1) since $\alpha + 2 < (d+2)/(d-2) < 2(d+1)/(d-3)$, and the energy argument is ended in the same way. □

If we only assume $\tilde{g} \in H^{3/2+\varepsilon,2}$, global existence becomes a much more delicate issue since we need to control $\|\partial_n u\|_{H^{1/2,2}}$. Let us sketch the main issue: the linear smoothing gives a control $\|\partial_n u\|_{H^{1/2,2}} \lesssim \|u_0\|_{H^1} + \|g\|_{3/2+\varepsilon,2} + \|f\|_{H^{1/2,2}}$, where $f = |u|^\alpha u$ has scaling $1+\alpha$. In order to estimate the time regularity of f we need to again use the equation, which adds another power α to the scaling. Using various chain rules, the conservation laws (4-16)–(4-17) should give at best $\|u\|^2_{C_T H^1} \lesssim \prod \|u\|^{\alpha_j}_{X_j}$, where $\sum \alpha_j = 1 + 2\alpha$ and, for all j, $X_j \hookrightarrow C_T H^1$. Eventually, $\|u\|^2_{C_T H^1} \lesssim \|u\|^\beta_{C_T H^1}$ for some β depending on α, and this allows us to close the estimate if $\beta < 2$. It is clear that such an approach will be limited to small values of α. Nevertheless, this is the method used in the following theorem, where the restriction on α is of course purely technical.

Theorem 4.8. *For $d = 2$, $\frac{1}{2} \le \alpha < \frac{11}{9}$, and $(u_0, g) \in H^1 \times H^{3/2+\varepsilon,2}$ satisfying the compatibility conditions, the problem (NLSD) has a unique global solution in $C(\mathbb{R}^+, H^1)$.*

Proof. The existence of a maximal solution is Theorem 4.1; it remains to prove that u is locally bounded in H^1. In this proof, $\lesssim$ means that the inequality is true up to a multiplicative constant that may depend on T, g *and an additive constant* that may depend on T, g and u_0. We use δ as a placeholder for some positive quantity that can be chosen arbitrarily small.

As in Theorem 4.7, we can use the nonlinear virial identity provided $g \in L^{\alpha+2}(\partial\Omega \times [0, T])$, which is ensured by $H^{3/2,2} \hookrightarrow H_T^{1/2} H^{1/2}(\partial\Omega) \hookrightarrow L^p(\partial\Omega \times [0, T])$ for any $2 \le p < \infty$. From the nonlinear virial identity we obtain

$$\|\partial_n u\|_{L^2_T L^2} + \|\nabla u\|_{L^2_T L^2} \lesssim \|u\|^{1/2}_{C_T H^1} \|u\|^{1/2}_{C_T L^2} + \|g\|^{1+\alpha/2}_{H^{3/2+\varepsilon,2}} \lesssim \|u\|^{1/2}_{C_T H^1} \|u\|^{1/2}_{C_T L^2}; \tag{4-18}$$

plugging this in (4-16) gives

$$\|u\|^2_{C_T L^2} \lesssim \|\partial_n u\|_{L^2_T L^2} \|g\|_{L^2_T L^2} \lesssim (\|u\|^{1/2}_{C_T H^1} \|u\|^{1/2}_{C_T L^2} + \|g\|_{H^{3/2+\varepsilon,2}}) \|g\|_{L^2_T L^2},$$

thus

$$\|u\|_{C_T L^2} \lesssim \|u\|^{1/3}_{C_T H^1}, \tag{4-19}$$

$$\text{and} \quad \|u\|_{L^2_T H^1_{\mathrm{loc}}} \lesssim \|u\|^{1/2+1/6}_{C_T H^1} = \|u\|^{2/3}_{C_T H^1}. \tag{4-20}$$

For later use, let us note that Hölder's inequality and the Sobolev embedding $H^1 \hookrightarrow L^r$ for $2 \le r < \infty$ imply

$$\|u\|_{L^q} \lesssim \|u\|^{1-2/q+\delta}_{H^1} \|u\|^{2/q-\delta}_{L^2} \quad \text{for all } q > 2,\ 0 < \delta < 2/q. \tag{4-21}$$

On the other hand, (4-16)–(4-17) give

$$\|u\|^2_{C_T H^1} + \|u\|^{\alpha+2}_{L^{\alpha+2}} \lesssim \|u_0\|^2_{H^1} + \|g\|_{H^{3/2+\varepsilon,2}} \|\partial_n u\|_{H^{1/2,2}}. \tag{4-22}$$

To estimate $\partial_n u$, we fix $\chi \in C_c^\infty(\overline{\Omega})$ such that $\chi \equiv 1$ on a neighbourhood of $\partial\Omega$, and split $u = u_1 + u_2$, where u_1 and u_2 are solutions of

$$\begin{cases} i\partial_t u_1 + \Delta u_1 = \chi |u|^\alpha u, \\ u_1|_{t=0} = u_0, \\ u_1|_{\partial\Omega\times[0,T]} = g, \end{cases} \quad \text{and} \quad \begin{cases} i\partial_t u_2 + \Delta u_2 = (1-\chi)|u|^\alpha u, \\ u_2|_{t=0} = 0, \\ u_2|_{\partial\Omega\times[0,T]} = 0. \end{cases}$$

Corollary 3.4 gives

$$\|\partial_n u_1\|_{H^{1/2,2}} \lesssim \|u_0\|_{H^1} + \|g\|_{H^{3/2+\varepsilon,2}} + \|\chi |u|^\alpha u\|_{H^{1/2,2}}.$$

We estimate the nonlinear term using $H^1 \hookrightarrow B^{1/2}_{4,2}$ [Triebel 1983, Section 3.3] and (4-19)–(4-20):

$$\|\chi|u|^\alpha u\|_{L^2_T H^{1/2}} \lesssim \big\||u|^\alpha\big\|_{L^\infty_T L^4_{\mathrm{loc}}} \|u\|_{L^2_T B^{1/2,4}_{2,\mathrm{loc}}} \lesssim \|u\|^{\alpha-1/2+\delta}_{L^\infty H^1} \|u\|^{1/2-\delta}_{L^\infty_T L^2} \|u\|_{L^2_T H^1_{\mathrm{loc}}}$$
$$\lesssim \|u\|^{\alpha+1/3+\delta}_{C_T H^1}. \tag{4-23}$$

For the time regularity, we use the composition rules and interpolation of anisotropic Sobolev spaces [Lions and Magenes 1968b, chapitre 4, paragraphe 2.1]. For $\tilde{\chi}$ such that $\tilde{\chi} = 1$ on supp χ,

$$\|\chi|u|^\alpha u\|_{H^{1/4} L^2} \lesssim \big\||u|^\alpha\big\|_{L^\infty_T L^4} \|u\|_{H^{1/4}_T L^4_{\mathrm{loc}}} \lesssim \|u\|^\alpha_{L^\infty_T L^{4\alpha}} \|\tilde{\chi} u\|_{H^{1/4}_T H^{1/2}}$$
$$\lesssim \|u\|^\alpha_{L^\infty_T L^{4\alpha}} \|\tilde{\chi} u\|^{1/2}_{H^{1/2}_T L^2} \|\tilde{\chi} u\|^{1/2}_{L^2_T H^1}.$$

Since $i\partial_t \tilde{\chi} u + \Delta \tilde{\chi} u = \tilde{\chi}|u|^\alpha u + [\Delta, \tilde{\chi}]u$, we have

$$\|\partial_t \tilde{\chi} u\|_{L^2_T H^{-1}} \lesssim \|\tilde{\chi} u\|_{L^2_T H^1} + \|\tilde{\chi}|u|^\alpha u\|_{L^2_T H^{-1}} + \|u\|_{L^\infty_T L^2},$$

and since $H^{-1} \supset L^q$ for $1 < q \le 2$ we get

$$\|\partial_t \tilde{\chi} u\|_{L^2_T H^{-1}} \lesssim \|u\|^{2/3}_{L^\infty_T H^1} + \|\tilde{\chi}|u|^\alpha u\|_{L^2_T L^{2/(1+\alpha)}} \lesssim \|u\|^{2/3}_{L^\infty_T H^1} + \|u\|^{(1+\alpha)/3}_{L^\infty_T H^1}.$$

Next we use $\|\tilde{\chi} u\|_{H^{1/2}_T L^2} \lesssim \|\tilde{\chi} u\|^{1/2}_{H^1_T H^{-1}} \|\tilde{\chi} u\|^{1/2}_{L^2_T H^1}$, so that

$$\|\tilde{\chi} u\|_{H^{1/2}_T L^2} \lesssim \big(\|u\|^{2/3}_{L^\infty_T H^1} + \|u\|^{(1+\alpha)/3}_{L^\infty_T L H^1}\big)^{1/2} \|\tilde{\chi} u\|^{1/2}_{L^2_T H^1} \lesssim \|u\|^{2/3}_{L^\infty_T H^1} + \|u\|^{(3+\alpha)/6}_{L^\infty_T H^1}.$$

This implies, using (4-19)–(4-21),

$$\|\chi|u|^\alpha u\|_{H^{1/4} L^2} \lesssim \|u\|^\alpha_{L^\infty_T L^{4\alpha}} \big(\|u\|^{1/3}_{L^\infty_T H^1} + \|u\|^{(3+\alpha)/12}_{L^\infty_T H^1}\big) \|u\|^{1/3}_{L^\infty_T H^1}$$
$$\lesssim \|u\|^{1/3+\alpha+\delta}_{L^\infty_T H^1} + \|u\|^{13\alpha/12+1/4+\delta}_{L^\infty_T H^1}.$$

Combining the estimate above with (4-23) gives the following estimate on $\partial_n u_1$:

$$\|\partial_n u_1\|_{H^{1/2,2}} \lesssim \|u\|^{1/3+\alpha+\delta}_{C_T H^1} + \|u\|^{13\alpha/12+1/4+\delta}_{C_T H^1}. \tag{4-24}$$

We now treat $\partial_n u_2$. The situation is less favourable since we can not use the smoothing property $\|\chi u\|_{L^2_T H^1} \lesssim \|u\|^{2/3}_{L^\infty_T H^1}$. In particular we only have

$$\|(1-\chi)u\|_{H^{1/2}_T L^2} \lesssim \|u\|_{L^\infty_T H^1} + \|u\|^{(4+\alpha)/6}_{L^\infty_T H^1}. \tag{4-25}$$

Using Proposition 3.6, we have

$$\|\partial_n u_2\|_{H^{1/2,2}(\partial\Omega\times[0,T])} \lesssim \|(1-\chi)|u|^\alpha u\|_{L^{3/2} B^1_{3/2,2} \cap B^{1/2}_{3/2,2} L^{3/2}}.$$

For the first norm we write

$$\begin{aligned}\|(1-\chi)|u|^\alpha u\|_{L^{3/2}B^1_{3/2,2}} &\lesssim \|(1-\chi)|u|^\alpha u\|_{L^\infty_T W^{1,3/2}}\\ &\lesssim \|u\|^\alpha_{L^\infty_T L^{6\alpha}}\|u\|_{L^\infty_T H^1}\\ &\lesssim \|u\|^{1/3}_{L^\infty_T L^2}\|u\|^{\alpha-1/3+\delta}_{L^\infty_T H^1}\|u\|_{L^\infty_T H^1}\\ &\lesssim \|u\|^{\alpha+7/9+\delta}_{C_T H^1}.\end{aligned}$$

For the other norm, the composition rules and (4-25) give similarly

$$\begin{aligned}\|(1-\chi)|u|^\alpha u\|_{B^{1/2}_{3/2,2}L^{3/2}} &\lesssim \|u\|^\alpha_{L^{6\alpha}_T L^{6\alpha}}\|u\|_{H^{1/2}_T L^2}\\ &\lesssim \|u\|^{\alpha-2/9+\delta}_{C_T H^1}(\|u\|_{C_T H^1}+\|u\|^{(4+\alpha)/6}_{C_T H^1})\\ &= \|u\|^{\alpha+7/9+\delta}_{C_T H^1}+\|u\|^{7\alpha/6+4/9+\delta}_{C_T H^1},\end{aligned}$$

so that

$$\|\partial_n u_2\|_{H^{1/2,2}} \lesssim \|u\|^{\alpha+7/9+\delta}_{L^\infty_T H^1}+\|u\|^{7\alpha/6+4/9}_{L^\infty_T H^1}.$$

Combining this estimate with (4-24) in (4-22), we finally obtain (as previously, $\lesssim$ still means "up to multiplicative and additive quantities only depending on T and the data")

$$\|u\|^2_{C_T H^1} \lesssim \|u\|^\beta_{C_T H^1},$$

with $\beta = \max\left(\frac{1}{3}+\alpha, 13\alpha/12+\frac{1}{4}, \alpha+\frac{7}{9}, 7\alpha/6+\frac{4}{9}\right)+\delta$. If $\beta < 2$ then $\|u(t)\|_{H^1}$ is locally bounded, and hence the solution is global. The condition $\beta<2$ is equivalent to $\alpha < \frac{11}{9}$. □

Appendix: Two interpolation lemmas

In this section we give two results on the interpolation of Sobolev spaces. They do not seem standard as they involve compatibility conditions in some way. We do not claim that these results are new, however we did not find them in the literature, thus we decided to include reasonably self-contained proofs.

Definition A.1 (real interpolation). If X_0, X_1 are two functional spaces embedded in $\mathfrak{D}'(\Omega)$, we define, for $u \in X_0+X_1$,

$$K(t,u) = \inf_{u=u_0+u_1\in X_0+X_1} \|u_0\|_{X_0}+t\|u_1\|_{X_1}.$$

For $0<\theta<1$, the interpolated space $[X_0, X_1]_{\theta,q}$ is the set of functions such that

$$\int_0^\infty |K(t,u)^q|\,\frac{dt}{t^{1+\theta q}} < \infty.$$

Lemma A.2. *Let*

$$X^\theta = \left\{(u_0,g)\in H^{-1/2+2\theta}\times H^{2\theta,\theta}(\partial\Omega\times[0,T]) \text{ that satisfy the compatibility conditions}\right\},$$

where for $\theta = 0$ we take $(H_D^{1/2})'$ instead of $H^{-1/2}$. Then, for $0 \le \theta \le 1$,

$$[X^0, X^1]_\theta = X^\theta.$$

Remark A.3. While it is a bit tedious, the case $\theta = \frac{1}{2}$ really needs to be treated, as it corresponds to the natural space for the virial estimates.

Proof. We clearly have

$$H_0^{3/2}(\Omega) \times H_0^{2,2}(\partial\Omega \times [0,T]) \subset X^1 \subset H^{3/2}(\Omega) \times H^{2,2}(\partial\Omega \times [0,T]).$$

The interpolation of Sobolev spaces [Lions and Magenes 1968a; Lions and Magenes 1968b, chapitres 1, 4], gives, for $\theta < \frac{1}{2}$,

$$\begin{aligned} &[(H_D^{1/2})'(\Omega), H_0^{3/2}]_\theta = H^{2\theta-1/2}, \quad [H^{0,0}(\partial\Omega\times[0,T]), H_0^{2,2}]_\theta = H^{2\theta,2},\\ &[(H_D^{1/2})'(\Omega), H^{3/2}]_\theta = H^{2\theta-1/2}, \quad [H^{0,0}(\partial\Omega\times[0,T]), H^{2,2}]_\theta = H^{2\theta,2}; \end{aligned}$$

the two left-hand identities are not explicitly written in [Lions and Magenes 1968a], however $(H_D^{1/2})'$ does not cause any new difficulty since it can be bypassed using $(H_D^{1/2})' = [H^{-1}, H^2]_{1/6} = [H^{-1}, H_D^2]_{1/6}$ [Lions and Magenes 1968a, paragraphes 12.3, 12.4], and the reiteration theorem $[[X,Y]_{\theta_0}, [X,Y]_{\theta_1}]_\theta = [X,Y]_{(1-\theta)\theta_0+\theta\theta_1}$. We deduce that, for $0 < \theta < \frac{1}{2}$,

$$X^\theta = H^{2\theta-1/2} \times H^{2\theta,\theta} \subset [X^0, X^1]_\theta \subset X^\theta.$$

For $\theta \ge 1/2$ we first apply the Lions–Peetre reiteration theorem

$$[X^0, X^1]_\theta = [[X^0, X^1]_{3/8}, [X^0, X^1]_1]_{8\theta/5-3/5} = [X^{3/8}, X^1]_{8\theta/5-3/5},$$

so that we are reduced to proving $[X^{3/8}, X^1]_\theta = X^{(5\theta+3)/8}$ for $\frac{1}{5} < \theta < 1$. To this end, we use the existence of a lifting operator *independent of* $\frac{1}{4} < s \le 1$,[3]

$$\begin{aligned} R : X^s &\to H^{2s+1/2,s+1/4}(\Omega\times[0,T]),\\ (u_0, g) &\mapsto u \quad \text{ such that } u|_{\Omega\times[0,T]} = g,\ u|_{t=0} = u_0, \end{aligned}$$

Such an operator can be constructed as follows: for any $(g, u_0) \in X^s$, there exists a map

$$\begin{aligned} R_1 : H^{2s,s}(\partial\Omega\times[0,T]) &\to H^{2s+1/2,s+1/4}(\Omega\times[0,T]),\\ g &\mapsto R_1 g; \end{aligned}$$

on the half space, $\mathcal{F}_{x',t} R_1 b = \hat{g}(\xi,\tau)\varphi(\sqrt{1+|\xi'|^2+|\tau|^2}x_d)$ with $\varphi(0) = 1$, φ smooth enough, works. There is also a map

$$\begin{aligned} R_2 : H_D^{2s-1/2}(\Omega) &\to H_D^{2s+1/2,s+1/4}(\Omega\times\mathbb{R}),\\ u_0 &\mapsto R_2 u_0; \end{aligned}$$

[3] R is usually called a coretraction of the trace operator $u \mapsto (u|_{t=0}, u|_{\partial\Omega\times[0,T]})$.

in this case, one might take $R_2(u_0)=\varphi((1-\Delta_D)t)u_0$ (this is a very special case of [Lions and Magenes 1968a, chapitre 1, théorème 4.2]; see also [Lions and Magenes 1968b, chapitre 4, théorème 2.3]). With these two operators, we can now define

$$R(u_0,g)=R_2(u_0-R_1(g)|_{t=0})+R_1(g);$$

R is a continuous map $X^s\to H^{2s+\frac12,2}$ for $s>\frac14$, since $u_0-R_1g|_{t=0}\in H_D^{2s-1/2}$. For $s>\frac12$ this is a consequence of $H^s_D=H^s_0$ and (CC0), while for $s=\frac12$ this comes from $H_D^{1/2}=H_{00}^{1/2}$ and (CCG0). We can conclude by introducing

$$T: H^{2s+1/2,2}(\Omega\times[0,T])\to H^{2s-1/2}(\Omega)\times H^{2s,2}(\partial\Omega\times[0,T]),$$
$$u\mapsto (u|_{t=0},u|_{\partial\Omega\times[0,T]}).$$

By construction, $T\circ R=\mathrm{Id}$ on $X^{3/8}$ and X^1, so that $[X^{3/8},X^1]_\theta=T([H^{5/4,5/8},H^{5/2,5/4}]_\theta)$. From basic results on anisotropic Sobolev spaces [Lions and Magenes 1968b, chapitre 4, proposition 2.1, théorème 2.3] we obtain, as expected,

$$T([H^{5/4,2}(\Omega\times[0,T]),H^{5/2,2}]_\theta)=T(H^{(5\theta+5)/4,2})=X^{(5\theta+3)/8}. \qquad \square$$

Let $H^{2,2}_{(0)}(\Omega\times\mathbb{R}_t)=\{u\in H^{2,2}(\Omega\times[0,T]):u|_{\partial\Omega\times\{0\}}=0\}$.

Proposition A.4. *For $\theta<\frac34$, $[L^2,H^{2,2}_{(0)}]_{\theta,2}=H^{2\theta,2}$.*

The result is to be expected, since the trace on $t=0$ sends $H^{2\theta,2}(\partial\Omega\times[0,T])$ to $H^{2\theta-1}(\Omega)$, for which there is a trace on $\partial\Omega$ if and only if $2\theta-1>\frac12$, or equivalently $\theta>\frac34$.

Proof. The inclusion $\subset$ is obvious; we focus on the reverse inclusion.

Let R be the restriction operator $H^{2\theta,2}(\mathbb{R}^d\times[0,T])\to H^{2\theta,2}(\Omega\times[0,T])$; since R is continuous for $0\le\theta\le1$ and surjective with value to $H^{2\theta,2}$, we only need to check that for $H^{2,2}_{(0),\partial\Omega}(\mathbb{R}^d\times\mathbb{R}_t)=\{u\in H^{2,2}:u|_{\partial\Omega\times\{0\}}=0\}$ we have

$$[L^2,H^{2,2}_{(0),\partial\Omega}]_\theta=H^{2\theta,2}(\mathbb{R}^d\times\mathbb{R}_t)\quad\text{for all }\ \theta<\tfrac34 \tag{A-1}$$

Using a partition of the unity, we can reduce the problem to the case $\partial\Omega=\mathbb{R}^{d-1}\times\{0\}$ and for conciseness we write $H^{2,2}_{(0),\partial\Omega}(\mathbb{R}^d\times\mathbb{R}_t)=H^{2,2}_{(0)}$. Let $u\in H^{2\theta,2}(\mathbb{R}^d\times\mathbb{R}_t)$; then, since $L^2\subset H^{2,2}$, it is easily seen from Definition A.1 that $u\in[L^2,H^{2,2}_{(0)}]_{\theta,2}$ if

$$\sum_{j=0}^\infty 2^{4\theta j}K(2^{-2j},u)^2<\infty,\quad\text{where}\quad K(t,u)=\inf_{u=u_0+u_1\in L^2+H^{2,2}_{(0)}}\|u_0\|_{L^2}+t\|u_1\|_{H^{2,2}_{(0)}}. \tag{A-2}$$

We define an anisotropic Littlewood–Paley decomposition as follows: the dual variables of x and t are $(\xi,\tau)=(\xi',\xi_d,\tau)$, and we set $u=\sum_{j\ge0}\Delta_ju(x,t)$, where, for $j\ge1$, $\widehat{\Delta_ju}(\xi,\tau)$ is supported in $(|\xi|^2+|\tau|)^{1/2}\sim 2^j$, $\widehat{\Delta_0u}$ is supported in $|\xi|^2+|\tau|\le1$, and we set $S_ju=\sum_{k=0}^j\Delta_ku$, $R_ju=u-S_ju$. From the Plancherel theorem and $\int_{\mathbb{R}^d}\Delta_ju\Delta_lu=0$ for $|j-l|$ large enough ("almost orthogonality"), we have

$$\|\Delta_j u\|_{H^{2,2}} \sim \|\Delta_j u\|_{L^2} 2^{2j} \implies \|u\|_{H^{2,2}}^2 \sim \sum_{j\geq 0} 2^{4j} \|\Delta_j u\|_{L^2}^2. \tag{A-3}$$

Let us write

$$u = (R_j u + S_j u(x',0,0)\psi_j(x_d,t)) + (S_j u - S_j u(x',0,0)\psi_j(x_d,t)) = u_0 + u_1,$$

where $\widehat{\psi}_j = c_j 2^{-3j} 1_{(|\xi_d|^2+|\tau|)^{1/2} \sim 2^j}$ with c such that $\psi_j(0) = 1$. Since $\mathrm{vol}((|\xi_d|^2 + |\tau|)^{1/2} \sim 2^j) \sim 2^{3j}$, c_j is uniformly bounded in j. For this choice it is clear that $(u_0,\, u_1) \in L^2 \times H^{2,2}_{(0)}$. The decomposition $u = S_j u + R_j u$ would correspond to the standard interpolation $[L^2, H^{2,2}]_\theta$, thus we will only focus on how to estimate in (A-2)

$$\|S_j u(x',0,0)\psi_j(x_d,t)\|_{L^2} + 2^{-2j}\|S_j u(x',0,0)\psi_j(x_d,t)\|_{H^{2,2}}.$$

We first note that

$$\mathscr{F}(S_j u(x',0,0)\psi_j(x_d,t)) = \widehat{\psi}_j(\xi_d,\tau) \int_{\mathbb{R}^2} \widehat{S_j u}(\xi',\eta,\delta)\, d\eta\, d\delta,$$

so that $\mathscr{F}(S_j u(x',0,0)\psi_j(x_d,t))$ is supported in $(|\xi|^2 + |\tau|)^{1/2} \lesssim 2^j$. We deduce

$$\begin{aligned} 2^{-2j}\|S_j u(x',0,0)\psi_j(x_d,t)\|_{H^{2,2}} + \|S_j u(x',0,0)\psi_j(x_d,t)\|_{L^2} &\lesssim \|S_j u(x',0,0)\psi_j(x_d,t)\|_{L^2} \\ &\lesssim \|\psi_j\|_{L^2} \int_{\mathbb{R}^2} \|\widehat{S_j u}(\xi',\eta,\delta)\|_{L^2_{\xi'}}\, d\eta\, d\delta. \end{aligned}$$

Again using $\mathrm{vol}((|\xi_d|^2 + |\tau|)^{1/2} \sim 2^j) \sim 2^{3j}$, we have $\|\psi_j\|_{L^2} \sim 2^{-3j} 2^{3j/2} = 2^{-3j/2}$. Moreover, $\Delta_k u(\xi',\eta,\delta)$ is supported in $(|\eta|^2 + |\delta|)^{1/2} \lesssim 2^k$ independently of ξ', thus the Cauchy–Schwartz inequality implies

$$\int_{\mathbb{R}^2} \|\widehat{S_j u}(\xi',\eta,\delta)\|_{L^2_{\xi'}}\, d\eta\, d\delta \leq \int_{\mathbb{R}^2} \sum_{k=0}^{j} \|\Delta_k u(\xi',\eta,\delta)\|_{L^2_{\xi'}}\, d\eta\, d\delta \lesssim \sum_{k=0}^{j} \|\Delta_k u\|_{L^2} 2^{3k/2}.$$

Plugging this in (A-2) (and omitting the estimate on $S_j u$, $R_j u$),

$$\begin{aligned} \sum_{j=0}^{\infty} 2^{4\theta j} K(2^{-2j}, u)^2 &\lesssim \sum_{j=0}^{\infty} 2^{(4\theta-3)j} \left(\sum_{k=0}^{j} \|\Delta_k u\|_{L^2} 2^{2\theta k} 2^{(3/2-2\theta)k} \right)^2 \\ &\lesssim \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \|\Delta_k u\|_{L^2} 2^{2\theta k} 2^{(3/2-2\theta)(k-j)} \right)^2 \\ &= \|a * b\|_{l^2}^2, \end{aligned}$$

where $(a_k)_{k\geq 0} = (\|\Delta_k u\|_{L^2} 2^{2\theta k})_{k\geq 0} \in l^2$ and $(b_k)_{k\geq 0} = (2^{(2\theta-3/2)k})_{k\geq 0} \in l^1$, we can conclude by Young's inequality and (A-3) that

$$\sum_{0}^{\infty} 2^{4\theta j} K(2^{-2j,u})^2 \lesssim (\|a\|_{l^2} \|b\|_{l^1})^2 \lesssim \|u\|_{H^{2\theta,2}}^2,$$

thus $H^{2\theta,2} \subset [L^2, H^{2,2}_{(0)}]_\theta$. □

Remark A.5. Using a similar argument, it is not difficult to check that $[L^2, H^{2,2}_{(0)}]_{\theta,2} = H^{2\theta,2}_{(0)}$ for $\theta > \frac{3}{4}$. Of course the identification in the case $\theta = \frac{3}{4}$ is less clear.

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