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# ON ESTIMATES FOR FULLY NONLINEAR PARABOLIC EQUATIONS ON RIEMANNIAN MANIFOLDS

BO GUAN, SHUJUN SHI AND ZHENAN SUI

We present some new ideas to derive *a priori* second-order estimates for a wide class of fully nonlinear parabolic equations. Our methods, which produce new existence results for the initial-boundary value problems in $\mathbb{R}^n$, are powerful enough to work in general Riemannian manifolds.

## 1. Introduction

Let $M^n$ be a compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary $\partial M$, which may be empty (then $M$ is closed), and $f$ a smooth symmetric function of $n$ variables. We consider the fully nonlinear parabolic equation

$$f(\lambda(\nabla^2 u + \chi)) = e^{u_t + \psi} \quad \text{in } M \times \{t > 0\}, \tag{1-1}$$

where $\chi$ is a smooth $(0, 2)$-tensor on $\overline{M} = M \cup \partial M$, $\nabla^2 u$ denotes the spatial Hessian of $u$, $u_t = \partial u/\partial t$, and $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$ will be the eigenvalues of a $(0, 2)$-tensor $A$; throughout the paper we shall use $\nabla$ to denote the Levi-Civita connection of $(M^n, g)$ and assume $\psi \in C^\infty(\overline{M} \times \{t \geq 0\})$.

While most attention in previous work had been on the two canonical cases, $\chi = 0$ and $\chi = g$, both of which occur, for instance, in the classical Darboux equations in isometric embedding, there are many important quantities of the form $\nabla^2 u + \chi$ in differential geometry and other areas. A well-known example is the gradient Ricci soliton equation

$$\nabla^2 u + \text{Ric} = \lambda g,$$

which has been studied intensively, where Ric denotes the Ricci tensor of $(M^n, g)$. In a different context, $\nabla^2 u + \text{Ric}$ is known as the *Bakry–Emery Ricci tensor* of the Riemannian measure space $(M^n, g, e^{-u} d \text{Vol}_g)$, on which there are interesting recent results; see, e.g., [Wei and Wylie 2009] and references therein. When $\chi$ as well as $\psi$ is allowed to depend on $u$ and $\nabla u$, there are even more equations of the form (1-1) and their elliptic counterparts, which arise naturally in connection with important geometric problems, such as the generalized Minkowski and Christoffel–Minkowski problems in classical geometry, fully nonlinear versions of the Yamabe problem in conformal geometry, and in other applications including the Monge–Kantorovich optimal mass transport problem. From both the theoretic point of view and that of applications, it is important and highly desirable to establish a general existence and regularity theory

for (1-1) with as few technical assumptions as possible, so that it covers a wide range of applications in different areas.

In order to study (1-1) in the context of parabolic theory, we follow [Caffarelli et al. 1985] and assume that $f$ is defined in an open, symmetric, convex cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin, $\Gamma_n := \{\lambda \in \mathbb{R}^n : \lambda_i > 0 \text{ for all } 1 \leq i \leq n\} \subseteq \Gamma$, and satisfies

$$f_i = f_{\lambda_i} \equiv \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{in } \Gamma, \quad 1 \leq i \leq n, \tag{1-2}$$

$$f \text{ is a concave function in } \Gamma, \tag{1-3}$$

and

$$\sup_{\partial \Gamma} f := \sup_{\lambda_0 \in \partial \Gamma} \lim_{\lambda \to \lambda_0} f(\lambda) \leq 0. \tag{1-4}$$

Equation (1-1) is parabolic for solutions $u \in C^{2,1}(M_T)$ with $\lambda[u] := \lambda(\nabla^2 u + \chi) \in \Gamma$ for $x \in M$ and $t > 0$ (see [Caffarelli et al. 1985]); we shall call such functions *admissible*.

The structure conditions (1-2)–(1-4) are fundamental to the classical solvability of fully nonlinear elliptic and parabolic equations, and have been standard in the literature since the work of Caffarelli, Nirenberg and Spruck [Caffarelli et al. 1985]. Condition (1-4) prevents (1-1) from being degenerate, which may occur if $\lambda[u] \in \bar{\Gamma} = \Gamma \cup \partial \Gamma$. So both conditions (1-2) and (1-4) are natural for the nondegenerate parabolicity of (1-1), without which the $C^{2+\alpha, 1+\alpha/2}$ estimates may fail. An important fact is that conditions (1-2) and (1-4) ensure that (1-1) becomes uniformly parabolic once global a priori $C^{2,1}$ estimates are established for admissible solutions. Consequently, one may obtain $C^{2+\alpha, 1+\alpha/2}$ estimates by the Evans–Krylov theorem, which depends on the concavity condition (1-3).

The short-time existence of admissible solutions is well known from the classical theory of parabolic equations for given admissible initial data (and boundary data as well when $\partial M \neq \varnothing$) with suitable smoothness assumptions. The global (long-time) existence and behavior of solutions depend on the establishment of a priori estimates in $C^{2,1}(\overline{M_T})$. Our primary goal in this paper is to derive second-order estimates for fully nonlinear parabolic equations on Riemannian manifolds.

For fixed $T > 0$, let $M_T = M \times (0, T]$, $\overline{M}_T = \overline{M} \times (0, T]$, and let $\partial M_T := \partial_s M_T \cup \partial_b M_T$ be the parabolic boundary of $M_T$, where

$$\partial_s M_T = \partial M \times [0, T), \quad \partial_b M_T = \overline{M} \times \{t = 0\}.$$

Throughout the paper we assume $\varphi^b := \varphi|_{t=0} \in C^\infty(\overline{M})$ with

$$\lambda[\varphi^b] \in \Gamma, \quad f(\lambda[\varphi^b]) > 0 \quad \text{in } \overline{M}, \tag{1-5}$$

and $\varphi^s := \varphi|_{\partial M \times \{t \geq 0\}} \in C^\infty(\partial M \times \{t \geq 0\})$. Let $u \in C^{4,2}(M_T) \cap C^{2,1}(\overline{M_T})$ be an admissible solution of (1-1) satisfying the initial-boundary conditions

$$u|_{t=0} = \varphi^b \quad \text{in } \overline{M}, \qquad u = \varphi^s \quad \text{on } \partial_s M_T. \tag{1-6}$$

The main result of this paper is the following second-order estimates:

**Theorem 1.1.** *Suppose that there exists an admissible subsolution $\underline{u} \in C^{2,1}(\overline{M_T})$ satisfying*

$$f(\lambda[\underline{u}]) \geq e^{\underline{u}_t + \psi} \quad in \ M_T. \tag{1-7}$$

*Then, under conditions (1-2)–(1-4),*

$$\sup_{M_T} |\nabla^2 u| \leq C_1\big(1 + \max_{\partial M_T} |\nabla^2 u|\big). \tag{1-8}$$

*In particular, when $M$ is closed,*

$$|\nabla^2 u| \leq C_1 \quad in \ \overline{M_T}. \tag{1-9}$$

*Suppose in addition that*

$$\underline{u} \leq \varphi^b \quad on \ \partial_b M_T, \qquad \underline{u} = \varphi^s \quad on \ \partial_s M_T. \tag{1-10}$$

*Then*

$$\max_{\partial M_T} |\nabla^2 u| \leq C_2. \tag{1-11}$$

**Remark 1.2.** In Theorem 1.1 and the rest of this paper, unless otherwise indicated, the constant $C_1$ in (1-8) will depend on

$$|u|_{C^1(\overline{M_T})}, \quad |\psi|_{C^{2,1}(\overline{M_T})}, \quad |\underline{u}|_{C^{2,1}(\overline{M_T})}, \quad \inf_{M_T} \text{dist}(\lambda[\underline{u}], \partial\Gamma), \tag{1-12}$$

and

$$\Lambda := \sup_\Gamma f - \sup_{M_T} e^{u_t + \psi} \tag{1-13}$$

as well as geometric quantities of $M$, while $C_2$ in (1-11) will depend in addition on $|\varphi^b|_{C^2(\overline{M})}, |\varphi^s|_{C^{4,1}(\partial_s M_T)},$ $\inf_{M_T} e^{u_t + \psi}$ and geometric quantities of $\partial M$. If $f$ satisfies

$$\lim_{|\lambda| \to \infty} |\lambda|^2 \sum f_i = \infty, \tag{1-14}$$

then $C_1$ can be chosen independently of $\Lambda$ and $|u_t|_{C^0(\overline{M_T})}$; see Remark 2.4.

**Remark 1.3.** The assumption $u \in C^{4,2}(M_T) \cap C^{2,1}(\overline{M_T})$ does not restrict the applications of Theorem 1.1. This can be seen as follows. By the short-time existence theorem, (1-1) admits a unique admissible solution $u \in C^\infty(\overline{M} \times (0, t_0]) \cap C^0(M \times [0, t_0])$ satisfying the initial-boundary condition (1-10) for some $t_0 > 0$. We can then consider a new initial time, say $t = t_0/2$, in place of $t = 0$, and may therefore assume the compatibility condition

$$f(\lambda[\varphi^b]) = e^{\varphi_t^s + \psi} \quad on \ \overline{M} \qquad and \qquad \varphi^s = \varphi^b \quad on \ \partial M \times \{t = 0\}. \tag{1-15}$$

Theorem 1.1 is an important step towards solving the initial-boundary problem (1-1) and (1-6) under optimal structure conditions. It can be applied in many interesting cases to prove new long-time existence results. Let us give a few examples here.

First, for a bounded smooth domain (with boundary of arbitrary geometric shape) in $\mathbb{R}^n$ we have the following result, which is essentially optimal, both in terms of the generality of $f$ and that of the underlying domain:

**Theorem 1.4.** *Let $M$ be a bounded smooth domain in $\mathbb{R}^n$, $0 < T \leq \infty$, and $f$ satisfy (1-2)–(1-5). There exists a unique admissible solution $u \in C^\infty(\overline{M}_T) \cap C^0(\overline{M_T})$ of (1-1) satisfying (1-6) provided that there exists an admissible subsolution $\underline{u} \in C^{2,1}(\overline{M_T})$ satisfying (1-7) and (1-10).*

The first initial-boundary value problem for (1-1), or (1-20) below, in $\mathbb{R}^n$ was treated, among many others, by Ivochkina and Ladyzhenskaya [1995], who used essentially the same assumptions as in the elliptic case introduced in [Caffarelli et al. 1985]; see [Lieberman 1996] for further improvements and references. Jiao and Sui [2015] studied (1-20) on Riemannian manifolds under additional assumptions. To the best of our knowledge, Theorem 1.4 had not been proved before in the current generality.

We remark that since there are no geometric restrictions on $\partial M$, (1-1) and (1-6) may fail to admit a long-term admissible solution without the subsolution assumption. This is well known and may be seen from simple examples.

**Theorem 1.5.** *When $\Gamma = \Gamma_n$, Theorem 1.4 holds for compact Riemannian manifolds.*

Theorem 1.1 applies to a very general class of equations, including $f = \sigma_k^{1/k}$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$, $1 \leq l < k \leq n$, where $\sigma_k$ is the $k$-th elementary symmetric function defined on the cone $\Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \text{ for all } 1 \leq j \leq k\}$. Another interesting example is $f = \log P_k$, to which Theorem 1.10 applies, where

$$P_k(\lambda) := \prod_{i_1 < \cdots < i_k} (\lambda_{i_1} + \cdots + \lambda_{i_k}), \quad 1 \leq k \leq n,$$

defined in the cone

$$\mathscr{P}_k := \{\lambda \in \mathbb{R}^n : \lambda_{i_1} + \cdots + \lambda_{i_k} > 0 \text{ for all } 1 \leq i_1 < \cdots < i_k \leq n\}.$$

**Theorem 1.6.** *Let $f = (\sigma_k/\sigma_l)^{1/(k-l)}$ and $\Gamma = \Gamma_k$ for $0 \leq l < k \leq n$, with $\sigma_0 = 1$, or $f = \log P_k$ and $\Gamma = \mathscr{P}_k$. The parabolic problem (1-1) and (1-6) with smooth data has a unique admissible solution $u \in C^\infty(\overline{M}_T) \cap C^0(\overline{M_T})$ provided that there exists an admissible subsolution $\underline{u} \in C^{2,1}(\overline{M_T})$ satisfying (1-7) and (1-10).*

Theorem 1.6 is known for $f = \sigma_k^{1/k}$, but seems to be new for $f = (\sigma_k/\sigma_l)^{1/(k-l)}$ or $f = \log P_k$, even when $M$ is a bounded smooth domain in $\mathbb{R}^n$; see also [Jiao and Sui 2015].

**Remark 1.7.** In Theorem 1.1, the constants $C_1$ and $C_2$ depend on $T$ only implicitly. For instance, if the quantities listed in (1-12) are all independent of $T$, then so is $C_1$. The independence of $T$ from the estimates is important to understanding the asymptotic behaviors of solutions as $t$ goes to infinity. If one allows $C_1$ to depend on $T$ (explicitly), (1-8) can be derived under much weaker conditions, and more easily.

**Theorem 1.8.** *Under assumptions (1-2), (1-3) and (1-5),*

$$|\nabla^2 u(x, t)| \leq C e^{Bt} \left(1 + \max_{\partial M_T} |\nabla^2 u|\right) \quad \text{for all } (x, t) \in M_T, \tag{1-16}$$

*where $C$ and $B$ depend on $|\nabla u|_{C^0(\overline{M_T})}$, $|\varphi^b|_{C^2(\overline{M})}$ and other known data. In particular, if $M$ is closed then $|\nabla^2 u(x, t)| \leq C e^{Bt}$.*

Note that, by (1-5), the function

$$\underline{u} := \varphi^b + t \min_{\overline{M}} \left\{ \log f(\lambda[\varphi^b]) - \psi \right\}$$

is admissible and satisfies (1-7).

An immediate consequence of Theorem 1.8 is the following characterization of finite-time blow-up solutions on closed manifolds:

**Corollary 1.9.** *Assume $M$ is closed and $f$ satisfies* (1-2)–(1-4). *Then* (1-1) *admits a unique admissible solution $u \in C^\infty(M \times \mathbb{R}^+)$ with initial value function $\varphi^b$ satisfying* (1-5) *provided that the a priori gradient estimate*

$$\sup_{M_T} |\nabla u| \le C \quad \text{for all } T > 0 \tag{1-17}$$

*holds, where $C$ may depend on $T$. In other words, if $u$ has a finite-time blow-up at $T < \infty$, then*

$$\lim_{t \to T^-} \max_{x \in M} |\nabla u(x, t)| = \infty.$$

So, the long-time existence of solutions in $0 \le t < \infty$ reduces to establishing the gradient estimate (1-17). This is also true when $\partial M \neq \varnothing$. Using Theorem 1.1, we can prove the following existence results:

**Theorem 1.10.** *Assume that* (1-2)–(1-5), (1-7), *and* (1-10) *hold for $T \in (0, \infty]$. There exists a unique admissible solution $u \in C^\infty(\overline{M}_T) \cap C^0(\overline{M}_T)$ of* (1-1) *satisfying* (1-6) *provided that any one of the following conditions holds*:

(i) $\Gamma = \Gamma_n$;

(ii) $(M, g)$ *has nonnegative sectional curvature*;

(iii) *there is $\delta_0 > 0$ such that, if $\lambda_j < 0$,*

$$f_j \ge \delta_0 \sum f_i \quad \text{on } \partial\Gamma^\sigma \quad \text{for all } \sigma > 0; \tag{1-18}$$

(iv) $\nabla^2 w \ge \chi$ *for some function $w \in C^2(\overline{M})$ and*

$$\sum f_i \lambda_i \ge 0 \quad \text{in } \Gamma. \tag{1-19}$$

The assumptions (i)–(iv) are only needed in deriving the gradient estimates. It would be interesting to remove these assumptions. When $\partial M = \varnothing$, Theorem 1.10 holds without the subsolution assumption.

The rest of the article is divided into three sections. In Sections 2 and 3, we derive (1-8) and (1-11), respectively, completing the proofs of Theorems 1.1 and 1.8. Instead of (1-1), we shall deal with the equation

$$f(\lambda(\nabla^2 u + \chi)) = u_t + \psi \tag{1-20}$$

under essentially the same assumptions on $f$, with the exception that (1-4) is replaced by

$$\inf_{\partial_s M_T} (\varphi_t + \psi) - \sup_{\partial\Gamma} f > 0, \tag{1-21}$$

which is needed in the proof of (1-11) . Accordingly, the functions $\varphi^b$ and $\underline{u} \in C^{2,1}(\overline{M_T})$ are assumed to satisfy $\lambda[\varphi^b] \in \Gamma$ in $\overline{M}$ and, respectively,

$$f(\lambda[\underline{u}]) \geq \underline{u}_t + \psi \quad \text{in } M_T \tag{1-22}$$

in place of (1-7). Note that if $f > 0$ in $\Gamma$ and $f$ satisfies (1-2), (1-3), and (1-19), then the function $\log f$ still satisfies theses assumptions. So (1-1) is covered by (1-20) in most cases, and we shall derive the estimates for (1-20). In Section 4 we briefly discuss the proof of the existence results and the preliminary estimates needed in the proof.

At the end of this introduction we recall the following commonly used notations:

$$|u|_{C^{k,l}(\overline{M_T})} = \sum_{j=0}^{k} |\nabla^j u|_{C^0(\overline{M_T})} + \sum_{j=1}^{l} \left| \frac{\partial^j u}{\partial t^j} \right|_{C^0(\overline{M_T})},$$

$$|u|_{C^{k+\alpha,l+\beta}(\overline{M_T})} = |u|_{C^{k,l}(\overline{M_T})} + |\nabla^k u|_{C^\alpha(\overline{M_T})} + \left| \frac{\partial^l u}{\partial t^l} \right|_{C^\beta(\overline{M_T})},$$

where $0 < \alpha, \beta < 1$ and $k, l = 1, 2, \ldots$, for a function $u$ sufficiently smooth on $\overline{M_T}$. We shall also write $|u|_{C^k(\overline{M_T})} = |u|_{C^{k,k}(\overline{M_T})}$.

## 2. Global estimates for second derivatives

A substantial difficulty in deriving the global estimate (1-8), which is our primary goal in this section, is caused by curvature of $M$; another is the lack of (globally defined) functions or geometric quantities with desirable properties. In our proof, the use of the admissible subsolution $\underline{u}$ is critical. We shall consider (1-20) in place of (1-1).

Let $u \in C^{4,2}(M_T) \cap C^{2,1}(\overline{M_T})$ be an admissible solution of (1-20) and $\underline{u} \in C^{2,1}(\overline{M_T})$ an admissible function. We assume that $u$ admits an a priori $C^1$ bound

$$|u|_{C^1(\overline{M_T})} \leq C. \tag{2-1}$$

Let $\phi(s) = -\log(1 - bs^2)$ and

$$\eta = \phi(1 + |\nabla(u - \underline{u})|^2) + a(\underline{u} - u - \delta t), \tag{2-2}$$

where $a, b, \delta > 0$ are constants and $\underline{u} \in C^{2,1}(\overline{M_T})$ is an admissible function; we shall choose $\delta = 1$ or $0$, $a$ sufficiently large, and $b$ small enough, namely

$$b \leq \frac{1}{8b_1^2}, \quad b_1 = 1 + \sup_{M_T} |\nabla(u - \underline{u})|^2. \tag{2-3}$$

Consider the quantity

$$W = \sup_{(x,t) \in M_T} \max_{\xi \in T_x M^n, |\xi|=1} (\nabla_{\xi\xi} u + \chi(\xi, \xi)) e^\eta.$$

Suppose $W$ is achieved at an interior point $(x_0, t_0) \in M_T$ for a unit vector $\xi \in T_{x_0} M^n$. Let $e_1, \ldots, e_n$ be smooth orthonormal local frames about $x_0$ such that $e_1 = \xi$, $\nabla_i e_j = 0$ and the $U_{ij} := \nabla_{ij} u + \chi_{ij}$ are

diagonal at $(x_0, t_0)$. So $W = U_{11}(x_0, t_0)e^{\eta(x_0,t_0)}$. We wish to derive a bound

$$U_{11}(x_0, t_0) \leq C. \tag{2-4}$$

Write (1-20) in the form

$$u_t = F(U) - \psi, \quad U = \{U_{ij}\}, \tag{2-5}$$

where $F$ is defined by

$$F(A) \equiv f(\lambda[A])$$

for an $n \times n$ symmetric matrices $A = \{A_{ij}\}$ with eigenvalues $\lambda[A] \in \Gamma$. Differentiating (2-5) gives

$$
\begin{aligned}
u_{tt} &= F^{ij}U_{ijt} - \psi_t, \\
\nabla_k u_t &= F^{ij}\nabla_k U_{ij} - \nabla_k \psi \quad \text{for all } k, \\
\nabla_{11} u_t &= F^{ij}\nabla_{11}U_{ij} + F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} - \nabla_{11}\psi.
\end{aligned}
\tag{2-6}
$$

Throughout the paper we denote

$$F^{ij} = \frac{\partial F}{\partial A_{ij}}(U), \quad F^{ij,kl} = \frac{\partial^2 F}{\partial A_{ij}\partial A_{kl}}(U).$$

The matrix $\{F^{ij}\}$ has eigenvalues $f_1, \ldots, f_n$, and therefore is positive-definite when $f$ satisfies (1-2), while (1-3) implies that $F$ is a concave function; see [Caffarelli et al. 1985]. Moreover, the following identities hold:

$$F^{ij}U_{ij} = \sum f_i \lambda_i, \quad F^{ij}U_{ik}U_{kj} = \sum f_i \lambda_i^2.$$

We also note that the $F^{ij}$ are diagonal at $(x_0, t_0)$.

**Proposition 2.1.** *For any $a, C_1 > 0$, there exists a constant $b > 0$ satisfying (2-3) such that, at $(x_0, t_0)$, if $U_{11} \geq C_1 a/\sqrt{b}$ then*

$$\frac{b}{2}F^{ii}U_{ii}^2 + aF^{ii}\nabla_{ii}(\underline{u} - u) - a(\underline{u}_t - u_t) + a\delta \leq C\sum F^{ii} + C. \tag{2-7}$$

*Proof.* We shall assume $U_{11}(x_0, t_0) \geq 1$. At $(x_0, t_0)$, where the function $\log U_{11} + \eta$ has its maximum,

$$\frac{(\nabla_{11}u)_t}{U_{11}} + \eta_t \geq 0, \quad \frac{\nabla_i U_{11}}{U_{11}} + \nabla_i \eta = 0, \quad 1 \leq i \leq n, \tag{2-8}$$

and

$$\frac{1}{U_{11}}F^{ii}\nabla_{ii}U_{11} - \frac{1}{U_{11}^2}F^{ii}(\nabla_i U_{11})^2 + F^{ii}\nabla_{ii}\eta \leq 0. \tag{2-9}$$

We recall the identities, on a Riemannian manifold,

$$\nabla_{ijk}v - \nabla_{jik}v = R^l_{kij}\nabla_l v, \tag{2-10}$$

$$\nabla_{ijkl}v - \nabla_{klij}v = R^m_{ljk}\nabla_{im}v + \nabla_i R^m_{ljk}\nabla_m v + R^m_{lik}\nabla_{jm}v + R^m_{jik}\nabla_{lm}v + R^m_{jil}\nabla_{km}v + \nabla_k R^m_{jil}\nabla_m v. \tag{2-11}$$

It follows that

$$F^{ii}\nabla_{ii}U_{11} \geq F^{ii}\nabla_{11}U_{ii} - CU_{11}\sum F^{ii}, \tag{2-12}$$

where $C$ depends on $|\nabla u|_{C^0(\overline{M}_T)}$ and geometric quantities of $M$. By (2-8), (2-9), (2-12), and (2-6), we obtain

$$F^{ii}\nabla_{ii}\eta - \eta_t \leq \frac{1}{U_{11}}F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} + \frac{1}{U_{11}^2}F^{ii}(\nabla_i U_{11})^2 - \frac{\nabla_{11}\psi}{U_{11}} + C\sum F^{ii}. \qquad (2\text{-}13)$$

Let

$$J = \{i : 3U_{ii} \leq -U_{11}\}, \quad K = \{i > 1 : 3U_{ii} > -U_{11}\}.$$

As in [Guan 2014b], which uses an idea of Urbas [2002], one derives

$$F^{ii}\nabla_{ii}\eta - \eta_t \leq \sum_{i\in J}F^{ii}(\nabla_i\eta)^2 + CF^{11}\sum_{i\notin J}(\nabla_i\eta)^2 - \frac{\nabla_{11}\psi}{U_{11}} + C\sum F^{ii}. \qquad (2\text{-}14)$$

For convenience, we write $w = \underline{u} - u$, $s = 1 + |\nabla w|^2$, and calculate

$$\nabla_i\eta = 2\phi'\nabla_k w\nabla_{ik}w + a\nabla_i w,$$

$$\eta_t = 2\phi'\nabla_k w(\nabla_k w)_t + aw_t - a\delta,$$

$$\nabla_{ii}\eta = 2\phi'(\nabla_{ik}w\nabla_{ik}w + \nabla_k w\nabla_{iik}w) + 4\phi''(\nabla_k w\nabla_{ik}w)^2 + a\nabla_{ii}w,$$

while

$$\phi'(s) = \frac{2bs}{1-bs^2}, \quad \phi''(s) = \frac{2b+2b^2s^2}{(1-bs^2)^2} > 4(\phi')^2.$$

Hence,

$$\sum_{i\in J}F^{ii}(\nabla_i\eta)^2 \leq 8(\phi')^2\sum_{i\in J}F^{ii}(\nabla_k w\nabla_{ik}w)^2 + 2|\nabla w|^2 a^2\sum_{i\in J}F^{ii} \qquad (2\text{-}15)$$

and

$$\sum_{i\notin J}(\nabla_i\eta)^2 \leq Ca^2 + C(\phi')^2 U_{11}^2. \qquad (2\text{-}16)$$

By (2-6) and (2-10), we obtain

$$F^{ii}\nabla_{ii}\eta - \eta_t \geq \phi'F^{ii}U_{ii}^2 + 2\phi''F^{ii}(\nabla_k w\nabla_{ik}w)^2 + aF^{ii}\nabla_{ii}w - aw_t + a\delta - C\phi'\Big(1+\sum F^{ii}\Big). \qquad (2\text{-}17)$$

It follows from (2-14)–(2-17) that

$$\phi'F^{ii}U_{ii}^2 + aF^{ii}\nabla_{ii}w - aw_t + a\delta \leq Ca^2\sum_{i\in J}F^{ii} + C(a^2+(\phi')^2 U_{11}^2)F^{11} - \frac{\nabla_{11}\psi}{U_{11}} + C\Big(\phi'+\sum F^{ii}\Big). \qquad (2\text{-}18)$$

Note that

$$F^{ii}U_{ii}^2 \geq F^{11}U_{11}^2 + \sum_{i\in J}F^{ii}U_{ii}^2 \geq F^{11}U_{11}^2 + \frac{U_{11}^2}{9}\sum_{i\in J}F^{ii}. \qquad (2\text{-}19)$$

We may fix $b$ small to derive (2-7) when $U_{11} \geq Ca/\sqrt{b}$. $\hfill\square$

To proceed, we need the following lemma, which is key to the proof of Theorem 1.1, both for (1-8) in this section and (1-11) in the next section; compare with Lemma 2.1 in [Guan 2014a]. Let $\nu_\lambda = Df(\lambda)/|Df(\lambda)|$ denote the unit normal vector to the level surface of $f$ through $\lambda$.

**Lemma 2.2.** *Let $K$ be a compact subset of $\Gamma$ and $\beta > 0$. There is a constant $\varepsilon > 0$ such that, for any $\mu \in K$ and $\lambda \in \Gamma$ with $|\nu_\mu - \nu_\lambda| \geq \beta$,*

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda) + \varepsilon\Big(1 + \sum f_i(\lambda)\Big). \tag{2-20}$$

*Proof.* Since $\nu_\mu$ is smooth in $\mu \in \Gamma$ and $K$ is compact, there is $\epsilon_0 > 0$ such that, for any $0 \leq \epsilon \leq \epsilon_0$,

$$K^\epsilon := \{\mu^\epsilon := \mu - \epsilon \mathbf{1} : \mu \in K\}$$

is still a compact subset of $\Gamma$ and

$$|\nu_\mu - \nu_{\mu^\epsilon}| \leq \frac{\beta}{2} \quad \text{for all } \mu \in K.$$

Consequently, if $\mu \in K$ and $\lambda \in \Gamma$ satisfy $|\nu_\mu - \nu_\lambda| \geq \beta$ then $|\nu_{\mu^\epsilon} - \nu_\lambda| \geq \beta/2$.

By the smoothness of the level surfaces of $f$, there exists $\delta > 0$ (which depends on $\beta$ but is uniform in $\epsilon \in [0, \epsilon_0]$) such that

$$\min_{\mu \in K} \min_{0 \leq \epsilon \leq \epsilon_0} \operatorname{dist}(\partial B_\delta^{\beta/2}(\mu^\epsilon), \partial \Gamma^{f(\mu^\epsilon)}) > 0,$$

where $\partial B_\delta^{\beta/2}(\mu^\epsilon)$ denotes the spherical cap

$$\partial B_\delta^{\beta/2}(\mu^\epsilon) = \left\{ \zeta \in \partial B_\delta(\mu^\epsilon) : \nu_{\mu^\epsilon} \cdot \frac{\zeta - \mu^\epsilon}{\delta} \geq \frac{\beta}{2}\sqrt{1 - \frac{\beta^2}{16}} \right\}.$$

Therefore,

$$\theta \equiv \min_{\mu \in K} \min_{0 \leq \epsilon \leq \epsilon_0} \min_{\zeta \in \partial B_\delta^{\beta/2}(\mu^\epsilon)} \{f(\zeta) - f(\mu^\epsilon)\} > 0. \tag{2-21}$$

Let $P$ be the two-plane through $\mu^\epsilon$ spanned by $\nu_{\mu^\epsilon}$ and $\nu_\lambda$ (translated to $\mu^\epsilon$), and $L$ the line on $P$ through $\mu^\epsilon$ and perpendicular to $\nu_\lambda$. Since $0 < \nu_{\mu^\epsilon} \cdot \nu_\lambda \leq 1 - \beta^2/8$, $L$ intersects $\partial B_\delta^{\beta/2}(\mu^\epsilon)$ at a unique point $\zeta$. By the concavity of $f$, we see that

$$\begin{aligned}
\sum f_i(\lambda)(\mu_i^\epsilon - \lambda_i) &= \sum f_i(\lambda)(\zeta_i - \lambda_i) \\
&\geq f(\zeta) - f(\lambda) \\
&\geq \theta + f(\mu^\epsilon) - f(\lambda) \quad \text{for all } 0 \leq \epsilon \leq \epsilon_0.
\end{aligned} \tag{2-22}$$

Next, by the continuity of $f$ we may choose $0 < \epsilon_1 \leq \epsilon_0$ with $|f(\mu^{\epsilon_1}) - f(\mu)| \leq \frac{1}{2}\theta$. Hence

$$\sum f_i(\lambda)(\mu_i - \epsilon_1 - \lambda_i) \geq f(\mu) - f(\lambda) + \tfrac{1}{2}\theta. \tag{2-23}$$

This proves (2-20) with $\varepsilon = \min\{\theta/2, \epsilon_1\}$.   $\square$

**Remark 2.3.** Alternatively, one can first prove

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \geq \theta + f(\mu) - f(\lambda).$$

Then choose $\epsilon > 0$ small such that $0 \leq f(\mu) - f(\mu^\epsilon) \leq \theta/2$. By the concavity of $f$,

$$\sum f_i(\lambda)(\mu_i^\epsilon - \lambda_i) \geq f(\mu^\epsilon) - f(\lambda) \geq f(\mu) - f(\lambda) - \frac{\theta}{2}. \tag{2-24}$$

Now add these two inequalities to obtain (2-20).

We now continue to prove (2-4). Assume first that $\underline{u}$ is a subsolution, i.e., $\underline{u}$ satisfies (1-22). Since $\lambda[\underline{u}]$ falls in a compact subset of $\Gamma$,

$$\beta := \tfrac{1}{2} \min_{\overline{M_T}} \operatorname{dist}(\nu_{\lambda[\underline{u}]}, \partial \Gamma_n) > 0. \tag{2-25}$$

Let $\lambda = \lambda[u](x_0, t_0)$ and $\mu = \lambda[\underline{u}](x_0, t_0)$. If $|\nu_\mu - \nu_\lambda| \geq \beta$ then, by Lemma 2.2,

$$F^{ii} \nabla_{ii} w - w_t \geq \sum f_i(\lambda)(\mu_i - \lambda_i) - f(\mu) + f(\lambda) \geq \varepsilon \Big( 1 + \sum F^{ii} \Big). \tag{2-26}$$

The first inequality follows from Lemma 6.2 in [Caffarelli et al. 1985]; see [Guan 2014b]. We may fix $a$ sufficiently large to derive a bound $U_{11}(x_0, t_0) \leq C$ by (2-7).

Suppose now that $|\nu_\mu - \nu_\lambda| < \beta$ and therefore $\nu_\lambda - \beta \mathbf{1} \in \Gamma_n$. It follows that

$$F^{ii} \geq \frac{\beta}{\sqrt{n}} \sum F^{kk} \quad \text{for all } 1 \leq i \leq n. \tag{2-27}$$

Since $\underline{u}$ is a subsolution, $F^{ii} \nabla_{ii} w - w_t \geq 0$ by the concavity of $f$. By (2-7) and (2-27), we obtain

$$\frac{b\beta}{2\sqrt{n}} U_{11}^2 \sum F^{ii} + a\delta \leq C \sum F^{ii} + C. \tag{2-28}$$

If we allow $\delta = 1$, a bound $U_{11}(x_0, t_0) \leq C$ would follow when $a$ is sufficiently large. This gives (1-16) in Theorem 1.8.

We now consider the case $\delta = 0$. First, by the concavity of $f$,

$$|\lambda| \sum f_i \geq f(|\lambda|\mathbf{1}) - f(\lambda) + \sum f_i \lambda_i \geq f(|\lambda|\mathbf{1}) - f(\lambda) - \frac{1}{4|\lambda|} \sum f_i \lambda_i^2 - |\lambda| \sum f_i. \tag{2-29}$$

Hence,

$$U_{11}^2 \sum F^{ii} \geq \frac{U_{11}}{2n}(f(U_{11}\mathbf{1}) - u_t - \psi) - \frac{1}{8} \sum F^{ii} U_{ii}^2 \geq \frac{\Lambda U_{11}}{4n} - \frac{U_{11}^2}{8} \sum F^{ii} \tag{2-30}$$

when $U_{11}$ is sufficiently large. A bound $U_{11}(x_0, t_0) \leq C$ therefore follows from (2-28). The proof of (1-8) in Theorem 1.1 is complete.

**Remark 2.4.** If (1-14) holds, a bound $U_{11}(x_0, t_0) \leq C$ follows from (2-28) directly and is independent of $|u_t|_{C^0(\overline{M_T})}$.

**Remark 2.5.** If $\underline{u}$ is an admissible strict subsolution, i.e.,

$$f(\lambda[\underline{u}]) \geq \underline{u}_t + \psi + \delta \quad \text{in } M_T \tag{2-31}$$

for some $\delta > 0$, then we can choose $\epsilon > 0$ such that $\lambda^\epsilon[\underline{u}] := \lambda[\underline{u}] - \epsilon \mathbf{1} \in \Gamma$ and

$$f(\lambda^\epsilon[\underline{u}]) \geq \underline{u}_t + \psi + \frac{\delta}{2} \quad \text{in } M_T. \tag{2-32}$$

By the concavity of $f$, we see that

$$\sum f_i(\lambda[u])(\lambda_i^\epsilon[\underline{u}] - \lambda_i[u]) \geq f(\lambda^\epsilon[\underline{u}]) - f(\lambda[u]) \geq \underline{u}_t - u_t + \frac{\delta}{2}.$$

Therefore, one can derive (2-4) directly from Proposition 2.1. This can be used to prove Theorem 1.8 as $\underline{u} = \varphi^b + At$ is a strict subsolution of (1-20) for any constant $A < \inf_M f(\lambda[\varphi^b]) - \sup_{M_T} \psi$.

## 3. Second-order boundary estimates

Let $u \in C^{3,1}(\overline{M_T})$ be an admissible solution of (1-20) and (1-6), and $\underline{u} \in C^{2,1}(\overline{M_T})$ an admissible subsolution satisfying (1-22) and (1-10). In this section, we derive (1-11) under conditions (1-2), (1-3) and (1-21) on $f$. Clearly we only need to focus on $\partial_s M_T$.

For a point $x_0 \in \partial M$ we shall choose smooth orthonormal local frames $e_1, \ldots, e_n$ around $x_0$ such that $e_n$, when restricted to $\partial M$, is the interior unit normal to $\partial M$. By the boundary condition $u = \varphi^s$ on $\partial_s M_T$, we obtain

$$|\nabla_{\alpha\beta} u(x_0, t)| \leq C \quad \text{for all } 1 \leq \alpha, \beta < n, \ 0 \leq t \leq T. \tag{3-1}$$

Let $\rho(x)$ and $d(x)$ denote the distances from $x \in \overline{M}$ to $x_0$ and $\partial M$, respectively. Let $M_T^\delta = \{(x, t) \in M_T : \rho(x) < \delta\}$, and $\partial M_T^\delta$ be the parabolic boundary of $M_T^\delta$,

$$\partial M_T^\delta = \overline{M_T^\delta} \setminus M_T^\delta.$$

We fix $\delta_0 > 0$ sufficiently small that both $\rho$ and $d$ are smooth in $M_T^{\delta_0}$. Let $\mathscr{L}$ denote the linear parabolic operator

$$\mathscr{L} w = F^{ij} \nabla_{ij} w - w_t.$$

We construct a barrier function of the form

$$\Psi = A_1 v + A_2 \rho^2 - A_3 \sum_{l<n} |\nabla_l (u - \varphi)|^2, \tag{3-2}$$

where

$$v = u - \underline{u} + sd - \frac{Nd^2}{2}. \tag{3-3}$$

**Lemma 3.1.** *Assume that (1-2), (1-3) and (1-21) hold. For constant $K > 0$, there exist uniform positive constants $s$, $\delta$ sufficiently small, and $A_1$, $A_2$, $A_3$, $N$ sufficiently large, such that $\Psi \geq K(d + \rho^2)$ in $\overline{M_T^\delta}$ and*

$$\mathscr{L}\Psi \leq -K\left(1 + \sum f_i |\lambda_i| + \sum f_i\right) \quad \text{in } M_T^\delta. \tag{3-4}$$

*Proof.* This is a parabolic version of Lemma 3.1 in [Guan 2014a]. Since there are some substantial differences in several places, for completeness we include a detailed proof.

First we note that, since $\underline{u}$ is a subsolution, $\mathscr{L}(u - \underline{u}) \leq 0$ by the concavity of $f$, and, by (2-6),

$$|\mathscr{L}\nabla_k(u - \varphi)| \leq C\left(1 + \sum f_i |\lambda_i| + \sum f_i\right) \quad \text{for all } 1 \leq k \leq n. \tag{3-5}$$

It follows that

$$\sum_{l<n} \mathscr{L}|\nabla_l(u - \varphi)|^2 \geq 2 \sum_{l<n} F^{ij} U_{il} U_{jl} - C\left(1 + \sum f_i |\lambda_i| + \sum f_i\right). \tag{3-6}$$

By Proposition 2.19 in [Guan 2014b], there exists an index $r$ such that

$$\sum_{l<n} F^{ij} U_{il} U_{jl} \geq \frac{1}{2} \sum_{i \neq r} f_i \lambda_i^2. \tag{3-7}$$

At a fixed point $(x, t)$, denote $\mu = \lambda(\nabla^2 \underline{u} + \chi)$ and $\lambda = \lambda(\nabla^2 u + \chi)$. As in Section 2 we consider two cases separately: (a) $|\nu_\mu - \nu_\lambda| < \beta$, and (b) $|\nu_\mu - \nu_\lambda| \geq \beta$, where $\beta$ is as given in (2-25).

*Case (a)*: $|\nu_\mu - \nu_\lambda| < \beta$. We have, by (2-27),

$$f_i \geq \frac{\beta}{\sqrt{n}} \sum f_k \quad \text{for all } 1 \leq i \leq n. \tag{3-8}$$

We next show that this implies the following inequality for any index $r$:

$$\sum_{i \neq r} f_i \lambda_i^2 \geq c_0 \sum f_i \lambda_i^2 - C_0 \sum f_i \tag{3-9}$$

for some $c_0, C_0 > 0$.

Since $\sum \lambda_i \geq 0$, we see that

$$\sum_{\lambda_i < 0} \lambda_i^2 \leq \left(-\sum_{\lambda_i < 0} \lambda_i\right)^2 \leq n \sum_{\lambda_i > 0} \lambda_i^2. \tag{3-10}$$

Therefore, by (3-8) and (3-10), we obtain, if $\lambda_r < 0$,

$$f_r \lambda_r^2 \leq n f_r \sum_{\lambda_i > 0} \lambda_i^2 \leq \frac{n\sqrt{n}}{\beta} \sum_{\lambda_i > 0} f_i \lambda_i^2.$$

On the other hand, by the concavity of $f$,

$$\sum f_i (b - \lambda_i) \geq f(b\mathbf{1}) - f(\lambda) = f(b\mathbf{1}) - u_t - \psi \geq \frac{\Lambda}{2} \tag{3-11}$$

for $b > 0$ sufficiently large. It follows that, if $\lambda_r > 0$,

$$f_r \lambda_r \leq b \sum f_i - \sum_{\lambda_i < 0} f_i \lambda_i.$$

By (3-8) and the Schwarz inequality,

$$\frac{\beta f_r \lambda_r^2}{\sqrt{n}} \sum f_k \leq f_r^2 \lambda_r^2 \leq 2b^2 \left(\sum f_i\right)^2 + 2 \sum_{\lambda_k < 0} f_k \sum_{\lambda_i < 0} f_i \lambda_i^2 \leq 2\left(\sum_{\lambda_i < 0} f_i \lambda_i^2 + b^2 \sum f_i\right) \sum f_k.$$

This finishes the proof of (3-9).

Letting $b = n|\lambda|$ in (3-11), we see that

$$(n+1)|\lambda| \sum f_i \geq \sum f_i(n|\lambda| - \lambda_i) \geq f(n|\lambda|\mathbf{1}) - f(\lambda) \geq \frac{\Lambda}{2}, \tag{3-12}$$

and consequently, by (3-8),

$$\sum f_i \lambda_i^2 \geq \frac{\beta |\lambda|^2}{\sqrt{n}} \sum f_i \geq \frac{\beta |\lambda|}{(n+1)\sqrt{n}} \frac{\Lambda}{2} \tag{3-13}$$

provided that $|\lambda| \geq R$ for $R$ sufficiently large.

It now follows from (3-6), (3-7), (3-9), (3-13) and the Schwarz inequality that, when $|\lambda| \geq R$,

$$\sum_{l<n} \mathcal{L}|\nabla_l(u-\varphi)|^2 \geq c_1 \sum f_i \lambda_i^2 + 2c_1|\lambda| - C - C_1 \sum f_i \qquad (3\text{-}14)$$

for some $c_1, C_1 > 0$. We now fix $R \geq C/c_1$.

Turning to the function $v$, we note that, by (3-8),

$$\mathcal{L}v \leq \mathcal{L}(u-\underline{u}) + C(s+Nd) \sum F^{ii} - N F^{ij} \nabla_i d \nabla_j d \leq \left(C(s+Nd) - \frac{\beta N}{\sqrt{n}}\right) \sum F^{ii}, \qquad (3\text{-}15)$$

since $\mathcal{L}(u-\underline{u}) \leq 0$ and $|\nabla d| \equiv 1$. For $N$ sufficiently large, we have

$$\mathcal{L}v \leq -\sum f_i \quad \text{in } M_T^\delta, \qquad (3\text{-}16)$$

and therefore, in view of (3-14) and (3-16),

$$\mathcal{L}\Psi \leq -A_3 c_1 \left(|\lambda| + \sum f_i \lambda_i^2\right) + (-A_1 + CA_2 + C_1 A_3) \sum f_i \qquad (3\text{-}17)$$

when $|\lambda| \geq R$ for any $s \in (0, 1]$ as long as $\delta$ is sufficiently small. From now on $A_3$ is fixed such that $A_3 c_1 R \geq K$, so $A_3 \geq CK/c_1^2$.

Suppose now that $|\lambda| \leq R$. By (1-2) and (1-3), we have

$$2R \sum f_i \geq \sum f_i \lambda_i + f(2R\mathbf{1}) - f(\lambda) \geq -R \sum f_i + f(2R\mathbf{1}) - f(R\mathbf{1}). \qquad (3\text{-}18)$$

Therefore,

$$\sum f_i \geq \frac{f(2R\mathbf{1}) - f(R\mathbf{1})}{3R} \equiv C_R > 0.$$

It follows from (2-27) that there is a uniform lower bound

$$f_i \geq \frac{\beta}{\sqrt{n}} \sum f_k \geq \frac{\beta C_R}{\sqrt{n}} \quad \text{for all } 1 \leq i \leq n. \qquad (3\text{-}19)$$

Consequently, since $|\nabla d| = 1$,

$$F^{ij} \nabla_i d \nabla_j d \geq \frac{\beta}{2\sqrt{n}}\left(C_R + \sum f_i\right).$$

From (3-15) we see that, when $\delta$ is sufficiently small and $N$ sufficiently large,

$$\mathcal{L}v \leq -\left(1 + \sum f_i\right) \quad \text{in } M_T^\delta. \qquad (3\text{-}20)$$

Combining (3-6), (3-7), (3-9), and (3-20) yields

$$\mathcal{L}\Psi \leq -A_3 c_1 \sum f_i \lambda_i^2 + (-A_1 + CA_2 + CA_3) \sum f_i - A_1 + CA_3 \qquad (3\text{-}21)$$

We now fix $N$ such that (3-16) holds when $|\lambda| > R$, while (3-20) holds when $|\lambda| \leq R$, for any $s$ and $\delta$ sufficiently small.

*Case (b)*: $|v_\mu - v_\lambda| \geq \beta$. It follows from Lemma 2.2 that, for some $\varepsilon > 0$,

$$\mathcal{L}(\underline{u} - u) \geq \sum f_i(\mu_i - \lambda_i) - (\underline{u} - u)_t \geq \varepsilon\Big(1 + \sum f_i\Big).$$

By (3-15), we may fix $s$ and $\delta$ sufficiently small such that $v \geq 0$ on $\overline{M_T^\delta}$ and

$$\mathcal{L}v \leq -\frac{\varepsilon}{2}\Big(1 + \sum f_i\Big) \quad \text{in } M_T^\delta. \tag{3-22}$$

Finally, we choose $A_2$ large such that

$$(A_2 - K)\rho^2 \geq A_3 \sum_{l<n} |\nabla_l(u - \varphi)|^2 \quad \text{on } \partial M_T^\delta,$$

and then fix $A_1$ sufficiently large so that (3-4) holds. In case (a) this follows from (3-17) when $|\lambda| > R$, and from (3-21) when $|\lambda| \leq R$. In case (b) we note that, from (3-6) and (3-7),

$$\mathcal{L}\Psi \leq A_1 \mathcal{L}v + A_2 \mathcal{L}\rho^2 - A_3 \sum_{i \neq r} f_i \lambda_i^2 + CA_3\Big(1 + \sum f_i|\lambda_i| + \sum f_i\Big)$$

$$\leq A_1 \mathcal{L}v - A_3 \sum_{i \neq r} f_i \lambda_i^2 + CA_3\Big(1 + \sum f_i|\lambda_i|\Big) + C(A_2 + A_3) \sum f_i.$$

Suppose now that $\lambda_r < 0$. Then,

$$\sum f_i|\lambda_i| = 2 \sum_{\lambda_i > 0} f_i \lambda_i - \sum f_i \lambda_i \leq \epsilon \sum_{\lambda_i > 0} f_i \lambda_i^2 - \mathcal{L}v + C \sum f_i + C.$$

Similarly, if $\lambda_r > 0$,

$$\sum f_i|\lambda_i| = \sum f_i \lambda_i - 2 \sum_{\lambda_i < 0} f_i \lambda_i \leq \epsilon \sum_{\lambda_i < 0} f_i \lambda_i^2 + \mathcal{L}v + C \sum f_i + C.$$

By (3-22) we obtain (3-4) when $A_1$ is chosen sufficiently large. $\qquad\square$

Applying Lemma 3.1, by (3-5) we immediately derive a bound for the mixed tangential–normal derivatives at any point $(x_0, t_0) \in \partial M_T$,

$$|\nabla_{n\alpha} u(x_0, t_0)| \leq C \quad \text{for all } \alpha < n. \tag{3-23}$$

It remains to establish the double normal derivative estimate

$$|\nabla_{nn} u(x_0, t_0)| \leq C. \tag{3-24}$$

As in [Guan 2014a; 2014b], we use an idea originally due to Trudinger [1995].

For $(x, t) \in \partial_s M_T$, let $\tilde{U}(x, t)$ be the restriction to $T_x \partial M$ of $U(x, t)$, viewed as a bilinear map on the tangent space of $M$ at $x$, and let $\lambda'(\tilde{U})$ denote the eigenvalues of $\tilde{U}$ with respect to the induced metric on $\partial M$. We next show that there are uniform positive constants $c_0$, $R_0$ such that, for all $R > R_0$, $\big(\lambda'(\tilde{U}(x, t)), R\big) \in \Gamma$ and

$$f\big(\lambda'(\tilde{U}(x, t)), R\big) \geq f\big(\lambda(U(x, t))\big) + c_0, \quad \text{for all } 0 \leq t \leq T, \ x \in \partial M. \tag{3-25}$$

It is known that (3-25) implies (3-24); see, e.g., [Guan 2014b].

For $R > 0$ sufficiently large, let

$$m_R := \min_{\partial_s M_T}[f(\lambda'(\tilde{U}), R) - f(\lambda(U))],$$

$$c_R := \min_{\partial_s M_T}[f(\lambda'(\tilde{\underline{U}}), R) - f(\lambda(\underline{U}))].$$

Note that $\big(\lambda'(\tilde{U}(x,t)), R\big) \in \Gamma$ and $\big(\lambda'(\tilde{U}(x,t)), R\big) \in \Gamma$ for all $(x,t) \in \partial_s M_T$ and all $R$ large, and it is clear that both $m_R$ and $c_R$ are increasing in $R$. We wish to show that, for some uniform $c_0 > 0$,

$$\tilde{m} := \lim_{R \to \infty} m_R \geq c_0.$$

Assume $\tilde{m} < \infty$ (otherwise we are done) and fix $R > 0$ such that $c_R > 0$ and $m_R \geq \tilde{m}/2$. Let $(x_0, t_0) \in \partial_s M_T$ be such that $m_R = f\big(\lambda'(\tilde{U}(x_0, t_0)), R\big)$. Choose local orthonormal frames $e_1, \ldots, e_n$ around $x_0$ as before such that $e_n$ is the interior normal to $\partial M$ along the boundary and $U_{\alpha\beta}(x_0, t_0)$ $(1 \leq \alpha, \beta \leq n-1)$ is diagonal. Since $u - \underline{u} = 0$ on $\partial_s M_T$, we have

$$U_{\alpha\beta} - \underline{U}_{\alpha\beta} = -\nabla_n(u - \underline{u})\sigma_{\alpha\beta} \quad \text{on } \partial_s M_T, \tag{3-26}$$

where $\sigma_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n \rangle$. Similarly,

$$U_{\alpha\beta} - \nabla_{\alpha\beta}\varphi - \chi_{\alpha\beta}\varphi = -\nabla_n(u - \varphi)\sigma_{\alpha\beta} \quad \text{on } \partial_s M_T. \tag{3-27}$$

For an $(n-1) \times (n-1)$ symmetric matrix $\{r_{\alpha,\beta}\}$ with $\big(\lambda'(\{r_{\alpha,\beta}\}), R\big) \in \Gamma$, define

$$\tilde{F}[r_{\alpha\beta}] := f\big(\lambda'(\{r_{\alpha,\beta}\}), R\big)$$

and

$$\tilde{F}_0^{\alpha\beta} = \frac{\partial \tilde{F}}{\partial r_{\alpha\beta}}[U_{\alpha\beta}(x_0, t_0)].$$

We see that $\tilde{F}$ is concave since $f$ is, and therefore, by (3-26),

$$\nabla_n(u - \underline{u})(x_0, t_0)\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}(x_0) \geq \tilde{F}[\underline{U}_{\alpha\beta}(x_0, t_0)] - \tilde{F}[U_{\alpha\beta}(x_0, t_0)] \geq c_R - m_R.$$

Suppose that

$$\nabla_n(u - \underline{u})(x_0, t_0)\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}(x_0) \leq \frac{c_R}{2};$$

then $m_R \geq c_R/2$ and we are done. So we shall assume

$$\nabla_n(u - \underline{u})(x_0, t_0)\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}(x_0) > \frac{c_R}{2}.$$

Consequently,

$$\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}(x_0) \geq \frac{c_R}{2\nabla_n(u - \underline{u})(x_0, t_0)} \geq 2\epsilon_1 c_R \tag{3-28}$$

for some constant $\epsilon_1 > 0$ depending on $\max_{\partial_s M_T} |\nabla u|$. By continuity, we may assume $\eta := \tilde{F}_0^{\alpha\beta} \sigma_{\alpha\beta} \geq \epsilon_1 c_R$ on $\overline{M_T^\delta}$ by requiring $\delta$ to be small (which may depend on the fixed $R$). Define, in $M_T^\delta$,

$$\Phi = -\nabla_n(u - \varphi) + \frac{Q}{\eta}, \tag{3-29}$$

where

$$Q = \tilde{F}_0^{\alpha\beta}(\nabla_{\alpha\beta}\varphi + \chi_{\alpha\beta} - U_{\alpha\beta}(x_0, t_0)) - \underline{u}_t - \psi + u_t(x_0, t_0) + \psi(x_0, t_0)$$

is smooth in $M_T^\delta$. By (3-5), we have

$$\mathscr{L}\Phi \leq -\mathscr{L}\nabla_n u + C\left(1 + \sum F^{ii}\right) \leq C\left(1 + \sum f_i |\lambda_i| + \sum f_i\right). \tag{3-30}$$

From (3-27), we see that $\Phi(x_0, t_0) = 0$ and

$$\Phi \geq 0 \quad \text{on } \overline{M_T^\delta} \cap \partial_s M_T, \tag{3-31}$$

since, for $(x, t) \in \partial_s M_T$, by the concavity of $\tilde{F}$,

$$\tilde{F}_0^{\alpha\beta}(U_{\alpha\beta}(x, t) - U_{\alpha\beta}(x_0, t_0)) \geq \tilde{F}(\tilde{U}(x, t)) - \tilde{F}(\tilde{U}(x_0, t_0))$$
$$= \tilde{F}(\tilde{U}(x, t)) - m_R - u_t(x_0, t_0) - \psi(x_0, t_0)$$
$$\geq \psi(x, t) + u_t(x, t) - u_t(x_0, t_0) - \psi(x_0, t_0).$$

On the other hand, on $\partial_b M_T^\delta$ we have $\nabla_n(u - \varphi) = 0$ and therefore, by (3-31),

$$\Phi(x, 0) \geq \Phi(\hat{x}, 0) - Cd(x) \geq -Cd(x), \tag{3-32}$$

where $C$ depends on $C^1$ bounds of $\nabla^2\varphi(\cdot, 0)$, $\underline{u}_t(\cdot, 0)$, and $\psi(\cdot, 0)$ on $\overline{M}$, and $\hat{x} \in \partial M$ satisfies $d(x) = \text{dist}(x, \hat{x})$ for $x \in M$; when $d(x)$ is sufficiently small, $\hat{x}$ is unique.

Finally, note that $|\Phi| \leq C$ in $M_T^\delta$. So we may apply Lemma 3.1 to derive $\Psi + \Phi \geq 0$ on $\partial M_T^\delta$ and

$$\mathscr{L}(\Psi + \Phi) \leq 0 \quad \text{in } M_T^\delta \tag{3-33}$$

for $A_1$, $A_2$, and $A_3$ sufficiently large. By the maximum principle, $\Psi + \Phi \geq 0$ in $M_T^\delta$. This gives $\nabla_n \Phi(x_0, t_0) \geq -\nabla_n \Psi(x_0, t_0) \geq -C$, since $\Phi + \Psi = 0$ at $(x_0, t_0)$, and, therefore, $\nabla_{nn} u(x_0, t_0) \leq C$.

Consequently, we have obtained a priori bounds for all second derivatives of $u$ at $(x_0, t_0)$. It follows that $\lambda(U(x_0, t_0))$ is contained in a compact subset of $\Gamma$ (independent of $u$) by assumption (1-4). Therefore,

$$c_0 \equiv \frac{f\big(\lambda(U(x_0, t_0)) + Re_n\big) - f\big(\lambda(U(x_0, t_0))\big)}{2} > 0,$$

where $e_n = (0, \ldots, 0, 1) \in \mathbb{R}^n$. By Lemma 1.2 in [Caffarelli et al. 1985], we have

$$\tilde{m} \geq m_{R'} \geq f\big(\lambda(U(x_0, t_0)) + R'e_n\big) - c_0 - f\big(\lambda(U(x_0, t_0))\big) \geq c_0$$

for $R' \geq R$ sufficiently large. The proof of (1-11) in Theorem 1.1 is complete.

**Remark 3.2.** When $M$ is a bounded smooth domain in $\mathbb{R}^n$, one can make use of an identity in [Caffarelli et al. 1985], and modify the operator $\mathscr{L}$, to derive the boundary estimates without using assumption (1-19). We omit the proof here since it is similar to the elliptic case in [Guan 2014a], which we refer the reader to for details.

## 4. Existence and $C^1$ estimates

In order to prove Theorem 1.10, it remains to derive the $C^1$ estimate

$$|u|_{C^0(\overline{M}_T)} + \max_{\overline{M} \times [t_0, T]} (|\nabla u| + |u_t|) \leq C \tag{4-1}$$

for any $t_0 \in (0, T)$, where $C$ may depend on $t_0$. Indeed, by assumption (1-4) we see that (1-1) becomes uniformly parabolic once the $C^{2,1}$ estimate

$$|u|_{C^{2,1}(\overline{M} \times [t_0, T])} \leq C$$

is established, which yields $|u|_{C^{2+\alpha, 1+\alpha/2}(\overline{M} \times [t_0, T])} \leq C$ by the Evans–Krylov theorem (see, e.g., [Lieberman 1996]). Higher-order estimates now follow from the classical Schauder theory of linear parabolic equations, and one obtains a smooth admissible solution in $0 \leq t \leq T$ by the short-time existence and continuation. We refer the reader to [Lieberman 1996] for details.

Let $h \in C^2(\overline{M}_T)$ be the solution of $\Delta h + \operatorname{tr} \chi = 0$ in $\overline{M}_T$ with $h = \varphi$ on $\partial M_T$. By the maximum principle we have $\underline{u} \leq u \leq h$, which gives a bound

$$|u|_{C^0(\overline{M}_T)} + \max_{\partial M_T} |\nabla u| \leq C. \tag{4-2}$$

For the bound of $u_t$, we have the following maximum principle:

**Lemma 4.1.** *We have*

$$|u_t(x, t)| \leq \max_{\partial M_T} |u_t| + t \sup_{M_T} |\psi_t| \quad \text{for all } (x, t) \in \overline{M}_T. \tag{4-3}$$

*Suppose moreover that there is a strictly convex function $h \in C^2(\overline{M})$ with $\nabla^2 h \geq c_0 g$ for some $c_0 > 0$. Then*

$$\sup_{M_T} |u_t| \leq \max_{\partial M_T} |u_t| + 2 \sup_{M_T} |\psi| + \frac{2|h|_{C^0(\overline{M})}}{c_0} \sup_{M_T} |\nabla^2 \psi|. \tag{4-4}$$

*Proof.* We have the identities $\mathscr{L} u_t = \psi_t$ and

$$|\mathscr{L}(u_t + \psi)| = |F^{ij} \nabla_{ij} \psi| \leq |\nabla^2 \psi| \sum F^{ii}.$$

Therefore,

$$\mathscr{L}(\pm u_t - Bt) = \pm \psi_t + B \geq 0$$

for $B \geq \sup_{M_T} |\psi_t|$. This gives (4-3), by the maximum principle. Similarly, (4-4) follows from

$$\mathscr{L}(\pm(u_t + \psi) + Bh) \geq (c_0 B - |\nabla^2 \psi|) \sum F^{ii} \geq 0$$

for $B \geq c_0^{-1} \sup_{M_T} |\nabla^2 \psi|$ and the maximum principle.                    □

It remains to derive the gradient estimate

$$\sup_{M_T} |\nabla u|^2 \leq C \big( |u|_{C^0(\overline{M_T})} + \sup_{\partial M_T} |\nabla u|^2 \big) \tag{4-5}$$

in each of the cases (i)–(iv) in Theorem 1.10. We shall omit case (i), which is trivial, and consider cases (ii)–(iv), following ideas from [Li 1990; Urbas 2002; Guan 2014b] in the elliptic case.

Let $\phi$ be a function to be chosen and assume that $|\nabla u| e^\phi$ achieves a maximum at an interior point $(x_0, t_0) \in M_T$. As before, we choose local orthonormal frames at $x_0$ such that both $U_{ij}$ and $F^{ij}$ are diagonal at $(x_0, t_0)$, where

$$\frac{\nabla_k u \nabla_k u_t}{|\nabla u|^2} + \phi_t \geq 0, \quad \frac{\nabla_k u \nabla_{ik} u}{|\nabla u|^2} + \nabla_i \phi = 0 \quad \text{for all } i = 1, \ldots, n, \tag{4-6}$$

$$F^{ii} \frac{\nabla_k u \nabla_{iik} u + \nabla_{ik} u \nabla_{ik} u}{|\nabla u|^2} - 2 F^{ii} \frac{(\nabla_k u \nabla_{ik} u)^2}{|\nabla u|^4} + F^{ii} \nabla_{ii} \phi \leq 0. \tag{4-7}$$

We have, for any $0 < \epsilon < 1$,

$$\sum_k (\nabla_{ik} u)^2 = \sum_k (U_{ik} - \chi_{ik})^2 \geq (1 - \epsilon) U_{ii}^2 - \frac{C}{\epsilon} \tag{4-8}$$

and

$$\left( \sum_k \nabla_k u \nabla_{ik} u \right)^2 \leq (1 + \epsilon) |\nabla_i u|^2 U_{ii}^2 + \frac{C}{\epsilon} |\nabla u|^2. \tag{4-9}$$

Let $\epsilon = \frac{1}{3}$ and $J = \{i : 2(n+2)|\nabla_i u|^2 > |\nabla u|^2\}$; note that $J \neq \varnothing$ and, by (4-8) and (4-9),

$$\sum_{i \notin J} F^{ii} (|\nabla u|^2 \nabla_{ik} u \nabla_{ik} u - 2(\nabla_k u \nabla_{ik} u)^2) \geq \sum_{i \notin J} F^{ii} (|\nabla u|^2 (1 - \epsilon) - 2(1 + \epsilon)|\nabla_i u|^2) U_{ii}^2 - \frac{C}{\epsilon} |\nabla u|^2$$

$$\geq -\frac{C}{\epsilon} |\nabla u|^2. \tag{4-10}$$

We derive, from (2-10), (4-6), (4-7) and (4-10),

$$\tfrac{1}{3} F^{ii} U_{ii}^2 - 2|\nabla u|^2 \sum_{i \in J} F^{ii} |\nabla_i \phi|^2 + |\nabla u|^2 (F^{ii} \nabla_{ii} \phi - \phi_t) \leq C(1 - K_0 |\nabla u|^2) \sum F^{ii} + C|\nabla u|, \tag{4-11}$$

where $K_0 = \inf_{k,l} R_{klkl}$.

Let

$$\phi = -\log(1 - bv^2) + A(\underline{u} + w - Bt),$$

where $v$ is a positive function, and $A$, $B$ and $b$ are constant, all to be determined; $b$ will be chosen sufficiently small such that $14bv^2 \leq 1$ in $\overline{M_T}$, while $A = 0$ in cases (ii) and (iii). By straightforward calculations,

$$\nabla_i \phi = \frac{2bv \nabla_i v}{1 - bv^2} + A \nabla_i (\underline{u} + w), \quad \phi_t = \frac{2bv v_t}{1 - bv^2} + A(\underline{u}_t - B)$$

and

$$\nabla_{ii}\phi = \frac{2bv\nabla_{ii}v + 2b|\nabla_i v|^2}{1 - bv^2} + \frac{4b^2v^2|\nabla_i v|^2}{(1 - bv^2)^2} + A\nabla_{ii}(\underline{u} + w)$$

$$= \frac{2bv\nabla_{ii}v}{1 - bv^2} + \frac{2b(1 + bv^2)|\nabla_i v|^2}{(1 - bv^2)^2} + A\nabla_{ii}(\underline{u} + w).$$

Plugging these into (4-11), we obtain

$$\tfrac{1}{3}F^{ii}U_{ii}^2 + |\nabla u|^2 \sum_{i \in J} F^{ii}\left(\frac{b(1 - 7bv^2)|\nabla_i v|^2}{(n + 2)(1 - bv^2)^2} - CA^2\right)$$

$$+ \frac{2bv|\nabla u|^2}{1 - bv^2}(F^{ii}\nabla_{ii}v - v_t) + A|\nabla u|^2(F^{ii}\nabla_{ii}(\underline{u} + w) - \underline{u}_t + B)$$

$$\leq C(1 - K_0|\nabla u|^2)\sum F^{ii} + C|\nabla u|. \quad (4\text{-}12)$$

In both cases (ii) and (iv), we take

$$v = \underline{u} - u + \sup_{\overline{M}_T}(u - \underline{u}) + 1 \geq 1.$$

Let $\mu = \lambda(\nabla^2\underline{u}(x_0, t_0) + \chi(x_0))$, $\lambda = \lambda(\nabla^2 u(x_0, t_0) + \chi(x_0))$, and $\beta$ as in (2-25). Suppose first that $|\nu_\mu - \nu_\lambda| \geq \beta$. By Lemma 2.2 and the assumptions that $\sum f_i\lambda_i \geq 0$ and $\nabla^2 w \geq \chi$, we see that

$$F^{ii}\nabla_{ii}(\underline{u} + w) - \underline{u}_t + B \geq F^{ii}\nabla_{ii}v - v_t + (B - u_t) \geq \varepsilon\sum F^{ii} + \varepsilon + (B - u_t)$$

for some $\varepsilon > 0$. Let $A = A_1 K_0^-/\varepsilon$, $K_0^- = \max\{-K_0, 0\}$, and fix $A_1$, $B$ sufficiently large. A bound $|\nabla u| \leq C$ follows from (4-12) in both cases (ii) and (iv).

We now consider the case $|\nu_\mu - \nu_\lambda| < \beta$. By (2-27) and (4-12), we see that, if $|\nabla u|$ is sufficiently large,

$$\frac{\beta}{\sqrt{n}}(|\lambda|^2 + c_1|\nabla u|^4)\sum F^{ii} \leq F^{ii}U_{ii}^2 + 2c_1|\nabla u|^4\sum_{i \in J} F^{ii} \leq C(1 - K_0|\nabla u|^2)\sum F^{ii} + C|\nabla u|, \quad (4\text{-}13)$$

where $c_1 > 0$.

Suppose $|\lambda| \geq R$ for $R$ sufficiently large. Then

$$\frac{\beta}{\sqrt{n}}(|\lambda|^2 + c_1|\nabla u|^4)\sum F^{ii} \geq \frac{2\beta|\lambda|\sqrt{c_1}}{\sqrt{n}}|\nabla u|^2\sum F^{ii} \geq c_2|\nabla u|^2 \quad (4\text{-}14)$$

for some uniform $c_2 > 0$. We obtain from (4-13) and (4-14) a bound for $|\nabla u(x_0, t_0)|$.

Suppose now that $|\lambda| \leq R$. Then $\sum F^{ii}$ has a positive lower bound, by (3-18) and (3-19). Therefore, a bound $|\nabla u(x_0, t_0)|$ follows from (4-13) again. This completes the proof of (4-5) in cases (ii) and (iv).

For case (iii) we choose $A = 0$ and $\phi = (u - \inf_{M_T} u + 1)^2$. By (4-12)

$$|\nabla u|^4\sum_{i \in J} F^{ii} \leq C(1 - K_0|\nabla u|^2)\sum F^{ii} + C|\nabla u|. \quad (4\text{-}15)$$

By (4-6) we see that $U_{ii} \leq 0$ for each $i \in J$ if $|\nabla u|$ is sufficiently large, and a bound for $|\nabla u(x_0, t_0)|$ therefore follows from (4-15) and assumption (1-18).

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