Analysis & PDE

Volume 8 No. 5 2015

Shaoming Guo

HILBERT TRANSFORM ALONG MEASURABLE VECTOR FIELDS CONSTANT ON LIPSCHITZ CURVES: L^2 BOUNDEDNESS

msp

ANALYSIS AND PDE
Vol. 8, No. 5, 2015
dx.doi.org/10.2140/apde.2015.8.1263

msp

HILBERT TRANSFORM ALONG MEASURABLE VECTOR FIELDS CONSTANT ON LIPSCHITZ CURVES: L^2 BOUNDEDNESS

SHAOMING GUO

We prove the L^2 boundedness of the directional Hilbert transform in the plane relative to measurable vector fields which are constant on suitable Lipschitz curves. One novelty of our proof lies in the definition of the adapted Littlewood–Paley projection (see Definition 3.3). The other novelty is that we will use Jones' beta numbers to control certain commutator in the critical Lipschitz regularity (see Lemma 5.5).

1. Introduction and statement of the main result

On $\mathbb{R}^2$, a direction is given by vector $v_u = (1, u)$, where $u \in \mathbb{R}$. Below, we will suppress the dependence of v upon u. Consider the directional Hilbert transform in the plane defined for a fixed direction $v = (1, u)$ as

$$H_v f(x, y) := \text{p.v.} \int_{\mathbb{R}} f(x - t, y - ut) \frac{dt}{t} \tag{1-1}$$

for any test function f. By the dilation symmetry, the length of the vector v is irrelevant for the value of H_v, which explains our normalization of the first component. By an application of Fubini's theorem and the L^p bounds for the classical Hilbert transform, one obtains a priori L^p bounds for H_v. On the other hand, the corresponding maximal operator $\sup_u |H_v f(x, y)|$ for varying directions is well known to not satisfy any a priori L^p bounds; see the work of Karagulyan [2007].

Bateman and Thiele [2013] proved that

$$\Big\| \sup_{u \in \mathbb{R}} \|H_v f(x, y)\|_{L^p(y)} \Big\|_{L^p(x)} \le C_p \|f\|_p \tag{1-2}$$

for the range $\frac{3}{2} < p < \infty$. Note that the supremum falls between the computation of the norm in y and in x, compared to being completely inside or outside as in the first paragraph. The case $p = 2$ of (1-2) goes back to Coifman and El Kohen (see page 1578 of [Bateman and Thiele 2013] for a detailed discussion), who noticed that a Fourier transform in the y direction makes (1-2) for $p = 2$ equivalent to L^2 bounds for Carleson's operator.

Estimate (1-2) highlights a biparameter structure of the directional Hilbert transform. The biparameter structure arises since the kernel is a tensor product between a Hilbert kernel in direction v and a Dirac delta distribution in the perpendicular direction.

MSC2010: 42B20, 42B25.
Keywords: singular integrals, differentiation theory, Jones' beta numbers, Littlewood–Paley theory on Lipschitz curves, Carleson embedding theorem.

If one considers the linearized maximal operator

$$H_v f(x, y) := \text{p.v.} \int f(x-t, y-u(x, y)t) \frac{dt}{t} \tag{1-3}$$

for some function u, then inequality (1-2) can be rephrased as a bound for the linearized maximal operator under the assumption that u is constant on every vertical line $x = x_0$ for all $x_0 \in \mathbb{R}$. Such vector fields v of the form $(1, u(x_0))$ for some measurable function $u : \mathbb{R} \to \mathbb{R}$ are called one-variable vector fields in [Bateman and Thiele 2013].

The purpose of the present paper is to relax this rigid assumption on u, and prove an analogue of (1-2) for vector fields which are constant along suitable families of Lipschitz curves. To formulate such a result, we perturb (1-2) by a bi-Lipschitz horizontal distortion, that is,

$$(x, y) \mapsto (g(x, y), y) \tag{1-4}$$

with

$$(x'-x)/a_0 \le g(x', y) - g(x, y) \le a_0(x'-x) \tag{1-5}$$

for every $x < x'$ and every y, so that the transformation (1-4) maps vertical lines to near vertical Lipschitz curves:

$$|g(x, y) - g(x, y')| \le b_0 |y'-y| \tag{1-6}$$

for all x, y, y'. These two conditions can be rephrased as

$$1/a_0 \le \partial_1 g \le a_0 \quad \text{and} \quad |\partial_2 g| \le b_0 \quad \text{a.e.} \tag{1-7}$$

Under these assumptions, L^p norms are distorted boundedly under the transformation (1-4). Namely, (1-5) implies for every y that

$$a_0^{-1} \|f(x, y)\|^p_{L^p(x)} \le \|f(g(x, y), y)\|^p_{L^p(x)} \le a_0 \|f(x, y)\|^p_{L^p(x)} \tag{1-8}$$

and we may integrate this in the y direction to obtain equivalence of L^p norms in the plane. Hence the change of measure is not the main point of the following theorem, but rather the effect of the transformation on the linearizing function u, which is now constant along the family of Lipschitz curves which are the images of the lines $x = x_0$ under the map (1-4).

Theorem 1.1 (main theorem). *Let $g : \mathbb{R}^2 \to \mathbb{R}$ satisfy assumption* (1-5) *for some a_0 and assumption* (1-6) *for some b_0. Then, for any $c_0 \in (0, 1)$, we have*

$$\Big\| \sup_{|u| \le c_0/b_0} \|H_v f(g(x, y), y)\|_{L^2(y)} \Big\|_{L^2(x)} \le C \|f\|_2. \tag{1-9}$$

Here C is a constant depending only on a_0 and c_0.

Remark 1.2. The constant C is independent of b_0 due to the anisotropic scaling symmetry $x \mapsto x$, $y \mapsto \lambda y$.

In view of the implicit function theorem (see [Azzam and Schul 2012] for recent developments), our result covers a large class of vector fields which are of the critical Lipschitz regularity. Indeed, it implies the following:

Corollary 1.3. *For a measurable unit vector field* $v_0 : \mathbb{R}^2 \to S^1$, *suppose that*:

(i) *there exists a bi-Lipschitz map* $g_0 : \mathbb{R}^2 \to \mathbb{R}^2$ *such that*

$$v_0(g_0(x, y)) \text{ is constant in } y; \tag{1-10}$$

(ii) *there exists* $d_0 > 0$ *such that, for all* $x \in \mathbb{R}$,

$$\angle\big(\partial_2 g_0(x, y), \pm v_0(g_0(x, y))\big) \geq d_0 \quad y\text{-a.e. in } \mathbb{R}. \tag{1-11}$$

Then the associated Hilbert transform, which is defined as

$$H_{v_0} f(x, y) := \int_{\mathbb{R}} f((x, y) - t v_0(x, y)) \frac{dt}{t}, \tag{1-12}$$

is bounded in L^2, *with the operator norm depending only on* d_0 *and the bi-Lipschitz norm of* g_0.

Remark 1.4. The structure theorem for Lipschitz functions by Azzam and Schul [2012] states exactly that any Lipschitz function $u : \mathbb{R}^2 \to \mathbb{R}$ (any Lipschitz unit vector field v_0 in our case) can be precomposed with a bi-Lipschitz function $g_0 : \mathbb{R}^2 \to \mathbb{R}^2$ so that $u \circ g_0$ is Lipschitz in the first coordinate and constant in the second coordinate, when restricted to a "large" portion of the domain.

Remark 1.5. Without the assumption that $d_0 > 0$, the operator H_{v_0} might be unbounded in L^p for any $p > 1$. The counterexample is based on the Besicovitch–Kakeya set construction, which will be discussed at the end of the proof of the corollary.

To our knowledge, this is the first result in the context of the directional Hilbert transform with a Lipschitz assumption in the hypothesis. Lipschitz regularity is critical for the directional Hilbert transform, as we will elaborate shortly.

To use the assumption that v is constant along Lipschitz curves, we apply an adapted Littlewood–Paley theory along the level lines of v. This is a refinement of the analysis of Coifman and El Kohen, who use a Fourier transform in the y variable and the analysis of Bateman and Thiele [2013], who use a classical Littlewood–Paley theory in the y variable. This adapted Littlewood–Paley theory is the main novelty of the present paper. It is in the spirit of prior work on the Cauchy integral on Lipschitz curves, for example [Coifman et al. 1989], but it differs from this classical theme in that it is more of biparameter type as it is governed by a whole fibration into Lipschitz curves. We crucially use Jones' beta numbers as a tool to control the adapted Littlewood–Paley theory. To our knowledge this is also the first use of Jones' beta numbers in the context of the directional Hilbert transform.

In this paper we focus on the case L^2, since our goal here is to highlight the use of the adapted Littlewood–Paley theory and Jones' beta numbers in the technically most simple case. We expect to address the more general case L^p with a range of p, as in the Bateman–Thiele theorem, in forthcoming work.

While Coifman and El Kohen use the difficult bounds on Carleson's operator as a black box, Bateman and Thiele [2013] have to unravel this black box following the work of Lacey and Li [2006; 2010] and use time-frequency analysis to prove bounds for a suitable generalization of Carleson's operator. Luckily, in the present work we do not have to delve into time-frequency analysis as we can largely recycle the work of Bateman and Thiele for this aspect of the argument.

An upper bound such as $|u| \leq c_0/b_0$ is necessary in our theorem. By a limiting argument we may recover the theorem of Bateman and Thiele, using the scaling to tighten the Lipschitz constant b_0 at the same time as relaxing the condition $|u| \leq c_0/b_0$.

An interesting open question remains whether the same holds true for $c_0 = 1$. We do not know of a soft argument to achieve this relaxation. Our estimate of the norms become unbounded as c_0 approaches 1. This question suggests itself for further study.

Part of our motivation is a long history of studies of the linearized maximal operator (1-3) under various assumptions on the linearizing function u. If one truncates (1-3) as

$$H_{v,\epsilon_0} f(x, y) := \text{p.v.} \int_{-\epsilon_0}^{\epsilon_0} f(x-t, y-u(x, y)t) \,\frac{dt}{t},$$

then it is reasonable to ask for pure regularity assumptions on u to obtain boundedness of H_{v,ϵ_0}. It is known that Lipschitz regularity of u is critical, since a counterexample in [Lacey and Li 2010] based on a construction of the Besicovitch–Kakeya set shows that no bounds are possible for C^α regularity with $\alpha < 1$. However, it remains open whether Lipschitz regularity suffices for bounds for H_{v,ϵ_0}. On the regularity scale, the only known result is for real analytic vector fields v by Stein and Street [2012]. A prior partial result in this direction appears in [Christ et al. 1999].

It is our hope that our result corners some of the difficulties of approaching Lipschitz regularity in the classical problem. Further substantial progress (including the case $c_0 = 1$) is likely to use Lipschitz regularity not only of the level curves of u but also of u itself across the level curves. For example, one possibility would be to cut the plane into different pieces by the theorem of Azzam and Schul stated in Remark 1.4, and to analyze each piece separately using Theorem 1.1. We leave this for future study.

Related to the directional Hilbert transform, and thus additional motivation for the present work, is the directional maximal operator

$$M_{v,\epsilon_0} f(x) := \sup_{0<\epsilon<\epsilon_0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |f(x-t, y-u(x, y)t)| \,dt, \tag{1-13}$$

which arises for example in Lebesgue-type differentiation questions and has an even longer history of interest than the directional Hilbert transform. Hilbert transforms and maximal operators share many features; in particular, they have the same scaling and thus share the same potential L^p bounds. The maximal operator is in some ways easier as it is positive and does not have a singular kernel. For example, bounds for the maximal operator under the assumption of real analytic vector fields were proved much earlier by Bourgain [1989].

An instance of bounds satisfied by the maximal operator but not the Hilbert transform arises when one restricts the range of the function u instead of the regularity. For certain sets of directions characterized by Bateman [2009a] there are bounds for the maximal operator (for example for the set of lacunary directions), while Karagulyan [2007] proves that no such bounds are possible for the Hilbert transform.

On the other hand, the Hilbert transform is easier in some other aspects; most notably it is a linear operator. For example, bounds for the bilinear Hilbert transform mapping into L^1 were known [Lacey and Thiele 1997; 1998] before the corresponding maximal operator bounds [Lacey 2000], due to the fact that orthogonality between different tiles is preserved under the Hilbert transform but not the maximal operator. In particular we do not know at the moment whether the analogue of our main theorem holds for the directional maximal operator. This may be an interesting subject for further investigation.

Outline of paper. In Section 2 we will prove Corollary 1.3 by reducing it to the main theorem. The reduction will also be used later in the proof of the main theorem.

In Section 3 we will state the strategy of the proof for the main theorem. As it appears that our result is a Lipschitz perturbation of the one by Bateman and Thiele, this turns out also to be the case for the proof: if we denote by P_k a Littlewood–Paley operator in the y-variable, the main observation in Bateman and Thiele's proof is that H_v commutes with P_k. In our case, this is no longer true. However, we can make use of an adapted version of the Littlewood–Paley projection operator $\tilde{P}_k$ (see Definition 3.3) to partially recover the orthogonality. We split the operator H_v into a main term and a commutator term

$$\sum_{k\in\mathbb{Z}} H_v P_k(f) = \sum_{k\in\mathbb{Z}} (H_v P_k(f) - \tilde{P}_k H_v P_k(f) + \tilde{P}_k H_v P_k(f)). \tag{1-14}$$

The boundedness of the main term $\sum_{k\in\mathbb{Z}} \tilde{P}_k H_v P_k(f)$ is essentially due to Lacey and Li [2010], with conditionality on certain maximal operator estimate. In Section 4 we modify Bateman's argument [2009b; 2013] to the case of vector fields constant on Lipschitz curves and remove the conditionality on that maximal operator.

The main novelty is the boundedness of the commutator term

$$\sum_{k\in\mathbb{Z}} (H_v P_k(f) - \tilde{P}_k H_v P_k(f)), \tag{1-15}$$

which will be presented in Section 5. To achieve this, we will view Lipschitz curves as perturbations of straight lines and use Jones' beta number condition for Lipschitz curves and the Carleson embedding theorem to control the commutator. Here we shall emphasize again that the commutator estimate is free of time-frequency analysis.

Notations. Throughout this paper, we will write $x \ll y$ to mean that $x \le y/10$, $x \lesssim y$ to mean that there exists a universal constant C such that $x \le Cy$, and $x \sim y$ to mean that $x \lesssim y$ and $y \lesssim x$. Lastly, $\mathbb{1}_E$ will always denote the characteristic function of the set E.

2. Proof of Corollary 1.3

In this section we prove Corollary 1.3, by reducing it to the main theorem. The reduction is based on a cutting and pasting argument. Some parts of the reduction will also be used in the proof of the main theorem in the rest of the paper.

We first divide the unit circle S^1 into N arcs of equal length, with the angle of each arc being $2\pi/N$. Choose

$$N > 6\pi/d_0, \tag{2-1}$$

so that $2\pi/N < d_0/3$. Denote these arcs as $\Omega_1, \Omega_2, \dots, \Omega_N$. For each Ω_i, define

$$H_{v_0,\Omega_i} f(x, y) := \begin{cases} H_{v_0} f(x, y) & \text{if } v_0(x, y) \in \Omega_i, \\ 0 & \text{else.} \end{cases}$$

If we were able to prove that $\|H_{v_0,\Omega_i}\|_{2\to 2}$ is bounded by a constant C which is independent of $i \in \{1, 2, \dots, N\}$, then we could conclude that

$$\|H_{v_0}\|_{2\to 2} \le C N(d_0). \tag{2-2}$$

Now fix one Ω_i; we want to show the boundedness of H_{v_0,Ω_i}. Choose a new coordinate so that the x-axis passes through Ω_i and bisects it. Then all the vectors in Ω_i form an angle less than $d_0/6$ with the x-axis. As we assume that

$$\angle(\partial_2 g_0, \pm v_0(g_0)) \ge d_0 > 0, \tag{2-3}$$

we see that the vector $\partial_2 g_0$ forms an angle less than $(\pi - d_0)/2$ with the y-axis.

Renormalize the unit vector v_0 so that the first component is 1, and write $v_0 = (1, u_0)$; then, by the fact that v_0 forms an angle less than $d_0/6$ with the x-axis, we obtain

$$|u_0| \le \tan(d_0/6). \tag{2-4}$$

Next we construct the Lipschitz function g in the main theorem from the bi-Lipschitz map g_0, and the coordinate we will use here is still the one associated to Ω_i as above. Under this linear change of variables, we know that g_0 is still bi-Lipschitz. We renormalize the bi-Lipschitz map in such a way that

$$g_0(x, 0) = (x, 0) \quad \text{for all } x \in \mathbb{R}. \tag{2-5}$$

Fix $x \in \mathbb{R}$, the map g_0, when restricted on the vertical line $\{(x, y) : y \in \mathbb{R}\}$, is still bi-Lipschitz. We denote by Γ_x the image of this bi-Lipschitz map, i.e.,

$$\Gamma_x := \{g_0(x, y) : y \in \mathbb{R}\}. \tag{2-6}$$

Define the function g by the relation

$$(g(x, y), y) = g_0(x, y\prime), \tag{2-7}$$

for some $y\prime$. By the fact that g_0 is bi-Lipschitz, we know that such $y\prime$ exists and is unique.

From the above construction and the fact that $\partial_2 g_0$ forms an angle less than $(\pi - d_0)/2$ with the y-axis, we see easily that

$$|g(x, y_1) - g(x, y_2)| \le \cot(d_0/2)|y_1 - y_2| \quad \text{for all } x, y_1, y_2 \in \mathbb{R}. \tag{2-8}$$

Hence, it remains to show that condition (1-5) is also satisfied with a constant a_0 depending only on d_0 and the bi-Lipschitz constant of g_0. One side of the equivalence, $(x_1 - x_2)/a_0 \le g(x_1, y) - g(x_2, y)$, is

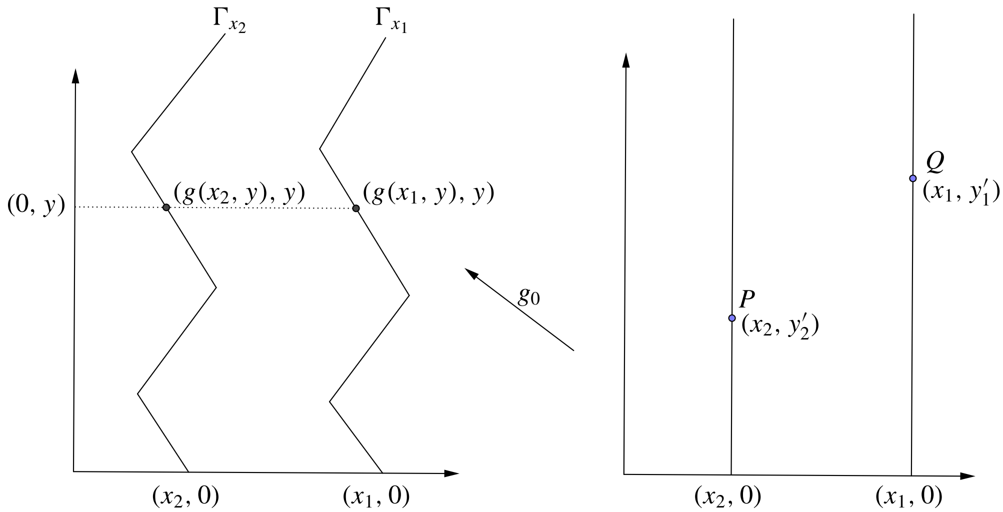

Figure 1. Illustration of the bi-Lipschitz map g_0.

quite clear from Figure 1: the bi-Lipschitz map g_0 sends the points P, Q to $(g(x_1, y), y)$, $(g(x_2, y), y)$ separately, then, by definition of a bi-Lipschitz map, there exists constant a_0 such that

$$g(x_1, y) - g(x_2, y) \geq \frac{1}{a_0}|P - Q| \geq \frac{1}{a_0}(x_1 - x_2). \tag{2-9}$$

For the other side, we argue by contradiction. If, for any $M \in \mathbb{N}$ large, there exists $x_1, x_2, y \in \mathbb{R}$ such that

$$g(x_1, y) - g(x_2, y) \geq M(x_1 - x_2), \tag{2-10}$$

then, together with (2-8), this implies that

$$\operatorname{dist}(K, \Gamma_{x_1}) \geq M \sin(d_0/2)(x_1 - x_2). \tag{2-11}$$

But this is not allowed as, by the definition of the bi-Lipschitz map g_0 and the Lipschitz function g, $\operatorname{dist}(K, \Gamma_{x_1})$ must be comparable to $|x_1 - x_2|$.

So far, we have verified all the conditions in the main theorem with

$$b_0 = \cot(d_0/2) \quad \text{and} \quad c_0 = \frac{\tan(d_0/6)}{\cot(d_0/2)} < 1. \tag{2-12}$$

Hence we can apply the main theorem to obtain the boundedness of H_{v_0,Ω_i}.

In the end, as claimed in the corollary, we still need to show that the operator norm in L^p (for all $p > 1$) blows up without the assumption that $d_0 > 0$. For the range $p \leq 2$, the counterexample is simply a Knapp example: let $B_1(0)$ denote the ball of radius one centered at origin, take the function $f(x) = \mathbb{1}_{B_1(0)}(x)$, let Γ be the upper cone which forms an angle less than $\pi/4$ with the vertical axis. First define the vector field $v(x) = x/|x|$ for $x \in \Gamma \setminus B_1(0)$, then extend the definition to the whole plane properly such that v

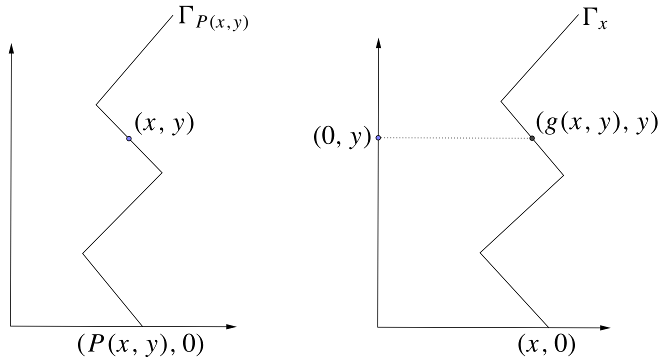

Figure 2. The projection $P(x, y)$.

satisfies the condition (1-10). It is then easy to see that

$$|H_v f(x)| \sim \frac{1}{|x|} \quad \text{for all } x \in \Gamma \setminus B_1(0), \tag{2-13}$$

which does not belong to $L^p(\mathbb{R}^2)$ for $p \leq 2$. For the range $p > 2$, the counterexample is given by the standard Besicovitch–Kakeya set construction, which can be found in [Bateman 2013, page 1022] and [Lacey and Li 2010, page 7].

3. Strategy of the proof of the main theorem

If we linearize the maximal operator in the main theorem, what we need to prove turns to be the following

$$\left\| \int_{\mathbb{R}} f(g(x, y) - t, y - tu(x)) \frac{dt}{t} \right\|_2 \lesssim \|f\|_2, \tag{3-1}$$

where $u : \mathbb{R} \to \mathbb{R}$ is a measurable function such that $\|u\|_\infty \leq c_0/b_0$. The change of coordinates

$$(x, y) \mapsto (g(x, y), y) \tag{3-2}$$

in (1-4) also changes the measure on the plane. However, we still want to use the original Lebesgue measure for the Littlewood–Paley decomposition. Hence we invert (1-4) and denote the inversion by

$$(x, y) \mapsto (P(x, y), y), \tag{3-3}$$

where "P" stands for "projection". Figure 2 illustrates why we call the map (3-3) a projection.

The change of coordinates in (3-3) turns the estimate (3-1) into the equivalent form

$$\left\| \int_{\mathbb{R}} f(x - t, y - tu(P(x, y))) \frac{dt}{t} \right\|_2 \lesssim \|f\|_2. \tag{3-4}$$

Moreover, we will denote

$$H_v f(x, y) := \int_{\mathbb{R}} f(x - t, y - tu(P(x, y))) \frac{dt}{t}. \tag{3-5}$$

In the rest of the paper, we want to make the convention that whenever H_v appears it denotes the Hilbert transform along the vector field $v(x, y) = (1, u(P(x, y)))$, that is, the above (3-5), to distinguish it from the various H_v that have appeared in the introduction.

To prove the above estimate, we first make several reductions: by the anisotropic scaling

$$x \mapsto x, \quad y \mapsto \lambda y, \tag{3-6}$$

we can without loss of generality assume that $b_0 = 10^{-2}$. By a similar cutting and pasting argument to that in the proof of Corollary 1.3, we can assume that $c_0 \ll 10^{-2}$, that is, the vector field v is of the form $(1, u)$ with $|u| \ll 1$.

Now we start the proof. It was already observed in [Bateman 2013, page 1024] that, under the assumption $|u| \ll 1$, we can without loss of generality assume that supp $\hat{f}$ lies in a two-ended cone which forms an angle less than $\pi/4$ with the vertical axis, as, for functions f with frequency supported on the cone near the horizontal axis, we have that

$$H_v f(x, y) = H_{(1,0)} f(x, y), \tag{3-7}$$

which is the Hilbert transform along the constant vector field $(1, 0)$. But $H_{(1,0)}$ is bounded by Fubini's theorem and the L^2 boundedness of the Hilbert transform.

For the frequencies outside the cone near the horizontal axis, the proof consists of two steps. In the first step we will prove the boundedness of H_v when acting on functions with frequency supported in a single annulus. To be precise, let Γ be the cone which forms an angle less than $\pi/4$ with the vertical axis and Π_Γ be the projection operator on Γ, i.e.,

$$\Pi_\Gamma f := \mathscr{F}^{-1} \mathbb{1}_\Gamma \mathscr{F} f, \tag{3-8}$$

where $\mathscr{F}$ stands for the Fourier transform and $\mathscr{F}^{-1}$ the inverse transform. Let P_k be the k-th Littlewood–Paley projection operator in the vertical direction; as we are always concerned with the frequency in Γ, later for simplicity we will just write P_k instead of $P_k \Pi_\Gamma$ for short. Then what we will prove first is:

Proposition 3.1. *Under the same assumptions as in the main theorem, we have for $p \in (1, \infty)$ that*

$$\|H_v P_k(f)\|_p \lesssim \|P_k(f)\|_p, \tag{3-9}$$

with the constant being independent of $k \in \mathbb{Z}$.

In order to prove the boundedness of H_v, we need to put all the frequency pieces together. In the case of $C^{1+\alpha}$ vector fields for any $\alpha > 0$, Lacey and Li's idea [2010] is to prove the almost orthogonality between different frequency annuli. In the case where the vector field is constant along vertical lines, an important observation in [Bateman and Thiele 2013] is that H_v and P_k commute, which then makes it possible to apply a Littlewood–Paley square function estimate.

In our case, Bateman and Thiele's observation is no longer true. We need to take into account that the vector field is constant along Lipschitz curves, which gives rise to an adapted Littlewood–Paley projection operator (Definition 3.3).

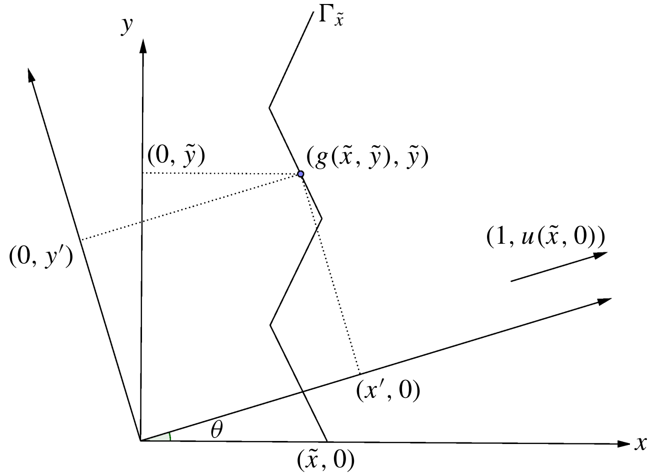

Figure 3. The setting of Lemma 3.2.

Before defining this operator, we first need to make some preparation. Fix one $\tilde{x} \in \mathbb{R}$, take the curve $\Gamma_{\tilde{x}}$ which passes through $(\tilde{x}, 0)$; recall that $\Gamma_{\tilde{x}}$ is given by the set $\{(g(\tilde{x}, \tilde{y}), \tilde{y}) : \tilde{y} \in \mathbb{R}\}$, where g is the Lipschitz function in the main theorem. By the definition of the operator H_v we know that the vector field v is equal to the constant vector $(1, u(\tilde{x}))$ along $\Gamma_{\tilde{x}}$. Change the coordinate so that the horizontal x'-axis is parallel to $(1, u(\tilde{x}))$. The following lemma says that, in the new coordinate, the curve $\Gamma_{\tilde{x}}$ can still be realized as the graph of a Lipschitz function.

Lemma 3.2. *For any fixed $\tilde{x} \in \mathbb{R}$, there exists a Lipschitz function $x' = g_{\tilde{x}}(y')$ such that $\Gamma_{\tilde{x}}$ can be reparametrized as $\{(g_{\tilde{x}}(y'), y') : y' \in \mathbb{R}\}$. Moreover, we have that $\|g_{\tilde{x}}\|_{\mathrm{Lip}} \le (1+b_0)/(1-b_0)$, where b_0 is the constant in the main theorem.*

Proof. Denote by θ the angle between the vector $(1, u(\tilde{x}))$ and the x-axis as in Figure 3.

The new coordinate of the point $(g(\tilde{x}, \tilde{y}), \tilde{y})$ will be given by

$$(x', y') = \left(\tilde{y}\sin\theta + g(\tilde{x}, \tilde{y})\frac{1+\sin^2\theta}{\cos\theta},\ \tilde{y}\cos\theta - g(\tilde{x}, \tilde{y})\sin\theta\right). \tag{3-10}$$

Looking at the identity for the second component,

$$y' = \tilde{y}\cos\theta - g(\tilde{x}, \tilde{y})\sin\theta, \tag{3-11}$$

we want to solve $\tilde{y}$ in terms of y' by using the implicit function theorem. As

$$\frac{dy'}{d\tilde{y}} = \cos\theta - \frac{\partial g}{\partial \tilde{y}}\sin\theta, \tag{3-12}$$

by the fact that $|u| \ll 1$ and $|\partial g/\partial \tilde{y}| \le b_0 \le 10^{-2}$ we obtain that

$$\frac{1-b_0}{\sqrt{2}} \le \frac{dy'}{d\tilde{y}} \le \frac{1+b_0}{\sqrt{2}}, \tag{3-13}$$

from which it is clear that the implicit function theorem is applicable.

After solving $\tilde{y}$ in terms of y', we just need to substitute $\tilde{y}$ into the identity for the first component in (3-10), which is

$$x' = \tilde{y}\sin\theta + g(\tilde{x}, \tilde{y})\frac{1+\sin^2\theta}{\cos\theta}, \tag{3-14}$$

to get an implicit expression of x' in terms of y', which we will denote as $x' = g_{\tilde{x}}(y')$.

To estimate the Lipschitz norm of the function $g_{\tilde{x}}$, we just need to observe that, when doing the above change of variables, we have rotated the axis by an angle θ which satisfies $|\theta| \le \pi/4$. Together with the fact that $|\partial g/\partial \tilde{y}| \le b_0$, we can then derive that

$$\left|\frac{\partial g_{\tilde{x}}}{\partial y'}\right| \le \frac{1+b_0}{1-b_0}, \tag{3-15}$$

which finishes the proof of Lemma 3.2. □

Definition 3.3 (adapted Littlewood–Paley projection). Select a Schwartz function ψ_0 with support on $\left[\frac{1}{2}, \frac{5}{2}\right] \cup \left[-\frac{5}{2}, -\frac{1}{2}\right]$ such that

$$\sum_{k\in\mathbb{Z}} \psi_0(2^{-k}t) = 1 \quad \text{for all } t \neq 0. \tag{3-16}$$

For $f : \mathbb{R}^2 \to \mathbb{R}$ and for every fixed $\tilde{x} \in \mathbb{R}$, define the adapted (one-dimensional) Littlewood–Paley projection on $\Gamma_{\tilde{x}}$ by

$$\tilde{P}_k(f)(x', y') := \int_{\mathbb{R}} f(g_{\tilde{x}}(z), z)\check{\psi}_k(y'-z)\,dz = P_k(\tilde{f})(y'), \tag{3-17}$$

where $(x', y') = (g_{\tilde{x}}(y'), y')$ denotes one point in $\Gamma_{\tilde{x}}$, $\psi_k(\,\cdot\,) := \psi_0(2^{-k}\cdot)$ and we use $\tilde{f}(\,\cdot\,)$ to denote the function $f(g_{\tilde{x}}(\,\cdot\,), \cdot\,)$, and P_k the one-dimensional Littlewood–Paley projection operator.

Now it is instructive to regard the Lipschitz curves as perturbations of the straight lines, or, equivalently, to think that $H_v P_k f$ still has frequency supported near the k-th frequency band, which has already been used by Lacey and Li [2010] in their almost orthogonality estimate for $C^{1+\alpha}$ vector fields. We then subtract the term $\tilde{P}_k H_v P_k(f)$ from $H_v P_k(f)$, and estimate the commutator.

To be precise, we first write

$$\sum_k H_v P_k(f) = \sum_k (H_v P_k(f) - \tilde{P}_k H_v P_k(f) + \tilde{P}_k H_v P_k(f)), \tag{3-18}$$

then, by the triangle inequality, we have

$$\Big\|\sum_k H_v P_k(f)\Big\|_2 \lesssim \Big\|\sum_k (H_v P_k(f) - \tilde{P}_k H_v P_k(f))\Big\|_2 + \Big\|\sum_k \tilde{P}_k H_v P_k(f)\Big\|_2. \tag{3-19}$$

We call the second term the main term, and the first term the commutator term. The L^2 boundedness of the main term will follow from an orthogonality argument, which is an adapted Littlewood–Paley theorem:

Lemma 3.4. *For $p \in (1, +\infty)$, we have the following variants of the Littlewood–Paley estimates*:

$$\Big\| \Big(\sum_{k \in \mathbb{Z}} |\tilde{P}_k(f)|^2 \Big)^{\frac{1}{2}} \Big\|_p \sim \|f\|_p, \tag{3-20}$$

$$\Big\| \Big(\sum_{k \in \mathbb{Z}} |\tilde{P}_k^*(f)|^2 \Big)^{\frac{1}{2}} \Big\|_p \sim \|f\|_p, \tag{3-21}$$

with constants depending only on a_0.

Proof. In (1-8) from the introduction, we have already explained the coarea formula

$$\int_{\mathbb{R}^2} |f(x, y)|\, dx\, dy \sim \int_{\mathbb{R}} \Big[\int_{\Gamma_{\tilde{x}}} |f|\, ds_{\tilde{x}} \Big]\, d\tilde{x}. \tag{3-22}$$

We apply this formula to the left-hand side of (3-20) to obtain

$$\Big\| \Big(\sum_{k \in \mathbb{Z}} |\tilde{P}_k(f)|^2 \Big)^{\frac{1}{2}} \Big\|_p^p \sim \int_{\mathbb{R}} \int_{\Gamma_{\tilde{x}}} \Big(\sum_{k \in \mathbb{Z}} |\tilde{P}_k(f)|^2 \Big)^{\frac{p}{2}} ds_{\tilde{x}}\, d\tilde{x}. \tag{3-23}$$

For every fixed $\tilde{x}$, by Definition 3.3, the right-hand side of (3-23) becomes

$$\int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \Big(\sum_k |P_k(\tilde{f}_{\tilde{x}})(y')|^2 \Big)^{\frac{p}{2}} dy' \Big]\, d\tilde{x}, \tag{3-24}$$

where $\tilde{f}_{\tilde{x}}(y') = f(g_{\tilde{x}}(y'), y')$. Then the classical Littlewood–Paley theory applies and we can bound the last expression by

$$\int_{\mathbb{R}} \|f\|^p_{L^p(\Gamma_{\tilde{x}})}\, d\tilde{x} \lesssim \|f\|^p_{L^p}. \tag{3-25}$$

For the boundedness of the adjoint operator, it suffices to prove that

$$\sum_{k \in \mathbb{Z}} \langle \tilde{P}_k^*(f), f_k \rangle \lesssim \|f\|_{L^p} \Big\| \Big(\sum_{k \in \mathbb{Z}} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p'}}. \tag{3-26}$$

First, by linearity and Hölder's inequality, we derive

$$\sum_{k \in \mathbb{Z}} \langle \tilde{P}_k^*(f), f_k \rangle = \Big\langle f, \sum_{k \in \mathbb{Z}} \tilde{P}_k(f_k) \Big\rangle \lesssim \|f\|_{L^p} \Big\| \sum_{k \in \mathbb{Z}} \tilde{P}_k(f_k) \Big\|_{L^{p'}}. \tag{3-27}$$

Applying the coarea formula (3-22), we obtain

$$\Big\| \sum_{k \in \mathbb{Z}} \tilde{P}_k(f_k) \Big\|_{L^{p'}} \sim \Big(\int_{\mathbb{R}} \Big(\int_{\Gamma_{\tilde{x}}} \Big| \sum_{k \in \mathbb{Z}} \tilde{P}_k(f_k) \Big|^{p'} ds_{\tilde{x}} \Big)\, d\tilde{x} \Big)^{\frac{1}{p'}}. \tag{3-28}$$

By Definition 3.3, for every fixed $\tilde{x} \in \mathbb{R}$, the inner integration in the last expression becomes

$$\int_{\mathbb{R}} \Big| \sum_{k \in \mathbb{Z}} P_k(\tilde{f}_{k,\tilde{x}})(y') \Big|^{p'} dy', \tag{3-29}$$

where $\tilde{f}_{k,\tilde{x}}(y') := f_k(g_{\tilde{x}}(y'), y')$. Now the classical Littlewood–Paley theory applies and we bound the term in (3-29) by

$$\int_{\mathbb{R}} \Big(\sum_{k\in\mathbb{Z}} |\tilde{f}_{k,\tilde{x}}(y')|^2 \Big)^{\frac{p'}{2}} dy' \lesssim \int_{\Gamma_{\tilde{x}}} \Big(\sum_{k\in\mathbb{Z}} |f_k|^2 \Big)^{\frac{p'}{2}} ds_{\tilde{x}} \lesssim \Big\| \Big(\sum_{k\in\mathbb{Z}} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p'}(\Gamma_{\tilde{x}})}^{p'}. \tag{3-30}$$

Then, to prove (3-26), we just need to integrate $d\tilde{x}$ in (3-30) and apply the coarea formula (3-22) to derive

$$\Big\| \sum_{k\in\mathbb{Z}} \tilde{P}_k(f_k) \Big\|_{L^{p'}} \lesssim \Big(\int_{\mathbb{R}} \Big\| \Big(\sum_{k\in\mathbb{Z}} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p'}(\Gamma_{\tilde{x}})}^{p'} d\tilde{x} \Big)^{\frac{1}{p'}} \lesssim \Big\| \Big(\sum_{k\in\mathbb{Z}} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p'}}.$$

Thus we have finished the proof of Lemma 3.4. □

Now we will show how to prove the L^2 boundedness of the main term using Lemma 3.4 and Proposition 3.1: first by duality, we have

$$\Big\| \sum_k \tilde{P}_k H_v P_k(f) \Big\|_2 = \sup_{\|g\|_2=1} \Big| \Big\langle \sum_k \tilde{P}_k H_v P_k(f), g \Big\rangle \Big| = \sup_{\|g\|_2=1} \Big| \Big\langle \sum_k H_v P_k(f), \tilde{P}_k^*(g) \Big\rangle \Big|.$$

Applying the Cauchy–Schwartz inequality and Hölder's inequality, we can bound the last term by

$$\sup_{\|g\|_2=1} \Big\| \Big(\sum_k |H_v P_k(f)|^2 \Big)^{\frac{1}{2}} \Big\|_2 \Big\| \Big(\sum_k |\tilde{P}_k^*(g)|^2 \Big)^{\frac{1}{2}} \Big\|_2. \tag{3-31}$$

For the former term, Proposition 3.1 implies that

$$\Big\| \Big(\sum_k |H_v P_k(f)|^2 \Big)^{\frac{1}{2}} \Big\|_2 \le \Big(\sum_{k\in Z} \|H_v P_k(f)\|_2^2 \Big)^{\frac{1}{2}} \lesssim \Big(\sum_{k\in\mathbb{Z}} \|P_k(f)\|_2^2 \Big)^{\frac{1}{2}} \lesssim \|f\|_2.$$

For the latter term, Lemma 3.4 implies that

$$\Big\| \Big(\sum_k |\tilde{P}_k^*(g)|^2 \Big)^{\frac{1}{2}} \Big\|_2 \lesssim \|g\|_2. \tag{3-32}$$

Thus we have proved the L^2 boundedness the main term, modulo Proposition 3.1.

As the second step, we will prove the L^2 boundedness of the commutator, which is

$$\Big\| \sum_k (H_v P_k(f) - \tilde{P}_k H_v P_k(f)) \Big\|_2 \lesssim \|f\|_2. \tag{3-33}$$

To do this, we first split the operator H_v into a dyadic sum: Select a Schwartz function ψ_0 such that ψ_0 is supported on $\big[\frac{1}{2}, \frac{5}{2}\big]$, let

$$\psi_l(t) := \psi_0(2^{-l}t); \tag{3-34}$$

by choosing ψ_0 properly, we can construct a partition of unity for $\mathbb{R}^+$, i.e.,

$$\mathbb{1}_{(0,\infty)} = \sum_{l\in\mathbb{Z}} \psi_l. \tag{3-35}$$

Let

$$H_l h(x, y) := \int \check{\psi}_l(t) h(x - t, y - tu(P(x, y)))\, dt; \tag{3-36}$$

then the operator H_v can be decomposed into the sum

$$H_v = -\mathbb{1} + 2 \sum_{l \in \mathbb{Z}} H_l. \tag{3-37}$$

Hence, to bound the commutator, it is equivalent to bound

$$\sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} (H_l P_k f - \tilde{P}_k H_l P_k f). \tag{3-38}$$

Notice that, by definition, $H_l P_k f$ vanishes for $l > k$, which simplifies the last expression to

$$\sum_{l \geq 0} \sum_{k \in \mathbb{Z}} (H_{k-l} P_k f - \tilde{P}_k H_{k-l} P_k f). \tag{3-39}$$

By the triangle inequality, it suffices to prove:

Proposition 3.5. *Under the same assumption as in the main theorem, there exists $\gamma > 0$ such that*

$$\Big\| \sum_{k \in \mathbb{Z}} (H_{k-l} P_k f - \tilde{P}_k H_{k-l} P_k f) \Big\|_2 \lesssim 2^{-\gamma l} \|f\|_2, \tag{3-40}$$

with the constant independent of $l \in \mathbb{N}$.

So far, we have reduced the proof of the main theorem to that of Proposition 3.1 and Proposition 3.5, which we will present separately in the following sections.

4. Boundedness of the Lipschitz–Kakeya maximal function and proof of Proposition 3.1

In their prominent work, Lacey and Li [2010] have reduced the L^2 boundedness of the operator H_{v,ϵ_0} to the boundedness of an operator they introduced, the so called Lipschitz–Kakeya maximal operator. As soon as this operator is bounded, we can then repeat the argument in Chapter 4 of [Lacey and Li 2010] to obtain Proposition 3.1 as a corollary.

Here we follow [Bateman 2013], where a slightly different version of the Lipschitz–Kakeya maximal operator is used; see Lemma 4.3. The only place in [Bateman 2013] where the one-variable vector field plays a special role is Lemma 6.2 on page 1037. Hence, to prove Proposition 3.1, we just need to replace this lemma by Lemma 4.3, and leave the rest of the argument unchanged.

In this section, we make an observation that both the boundedness of the Lipschitz–Kakeya maximal operator (Corollary 4.4) and its variant (Lemma 4.3) can be proved by adapting Bateman's argument [2009b] to our case, where the vector fields are constant only on Lipschitz curves.

Before defining the Lipschitz–Kakeya maximal operator, we first need to introduce several definitions.

Definition 4.1 (popularity). For a rectangle $R \subset \mathbb{R}^2$, with $l(R)$ its length and $w(R)$ its width, we define its uncertainty interval $EX(R) \subset \mathbb{R}$ to be the interval of width $w(R)/l(R)$ and centered at slope(R). Then the popularity of the rectangle R is defined to be

$$\mathrm{pop}_R := \left|\left\{(x, y) \in \mathbb{R}^2 : u(P(x, y)) \in EX(R)\right\}\right| \,/\, |R|. \tag{4-1}$$

Definition 4.2. Given two rectangles R_1 and R_2 in $\mathbb{R}^2$, we write $R_1 \le R_2$ whenever $R_1 \subset CR_2$ and $EX(R_2) \subset EX(R_1)$, where C is some properly chosen large constant and CR_2 is the rectangle with the same center as R_2 but dilated by the factor C.

Denote $\mathcal{R}_{\delta,\omega} := \{R \in \mathcal{R} : \mathrm{slope}(R) \in [-1, 1], \mathrm{pop}_R \ge \delta, w(R) = \omega\}$, where $\mathcal{R}$ is the collection of all the rectangles in $\mathbb{R}^2$. Then the Lipschitz–Kakeya maximal function is defined as

$$M_{\mathcal{R}_{\delta,\omega}}(f)(x) := \sup_{x \in R \in \mathcal{R}_{\delta,\omega}} \frac{1}{|R|} \int_R |f|. \tag{4-2}$$

Lemma 4.3. *Let u and P be the functions given in the definition of the operator H_v in* (3-5). *Suppose $\mathcal{R}_0$ is a collection of pairwise incomparable* (*under "$\le$"*) *rectangles of uniform width such that, for each $R \in \mathcal{R}_0$, we have*

$$\frac{|(u \circ P)^{-1}(EX(R)) \cap R|}{R} \ge \delta \tag{4-3}$$

(*i.e.*, $\mathrm{pop}_R \ge \delta$) *and*

$$\frac{1}{|R|} \int_R \mathbb{1}_F \ge \lambda. \tag{4-4}$$

Then, for each $p > 1$,

$$\sum_{R \in \mathcal{R}_0} |R| \lesssim \frac{|F|}{\delta \lambda^p}. \tag{4-5}$$

The same covering lemma argument as in Lemma 3.1 of [Bateman 2009b] shows the boundedness of Lacey and Li's Lipschitz–Kakeya maximal operator as a corollary of Lemma 4.3.

Corollary 4.4. *For all $p \in (1, \infty)$ we have the bound*

$$\|M_{\mathcal{R}_{\delta,\omega}}\|_{L^p \to L^p} \le C(p, a_0)\frac{1}{\delta}. \tag{4-6}$$

Proof of Lemma 4.3. The proof is essentially due to Bateman [2009b], with just one minor modification in order to adapt to the family of Lipschitz curves on with the vector field is constant.

Definition 4.5 (rectangles adapted to the vector field). For a rectangle $R \in \mathcal{R}_{\delta,\omega}$, with its two long sides lying on the parallel lines $y = kx + b_1$ and $y = kx + b_2$ for some $k \in [-1, 1]$ and $b_1, b_2 \in \mathbb{R}$, define $\tilde{R}$ to be the adapted version of R, which is given by the set

$$\{(x, y) : P(x, y) \in P(R)\} \cap \{(x, kx + b) : x \in \mathbb{R}, b \in [b_1, b_2]\}, \tag{4-7}$$

where P is the projection operator in (3-3).

What we need to do is just to replace the rectangles R in [Bateman 2009b] by $\tilde{R}$, and observe that the two key quantities — length and popularity of rectangles — are both preserved under the projection operator P up to a constant depending on the constant a_0 in the main theorem. Hence, we leave out the details. □

5. Proof of Proposition 3.5

This section consists of two subsections. In Section 5A we will introduce some notations, most of which we adopt from Bateman [2013] with minor changes for our purpose. In Section 5B we will use Jones' beta numbers and the Carleson embedding theorem to prove Proposition 3.5.

5A. *Discretization.* The content of this subsection is basically taken from Bateman [2013], with minor changes as we are now dealing with all frequencies instead of a single frequency annulus.

Discretizing the functions. Fix $l \geq 0$; we write $\mathcal{D}_l$ as the collection of the dyadic intervals of length 2^{-l} contained in $[-2, 2]$. Fix a smooth positive function $\beta : \mathbb{R} \to \mathbb{R}$ such that

$$\beta(x) = 1 \quad \text{for all } |x| \leq 1 \qquad \text{and} \qquad \beta(x) = 0 \quad \text{for all } |x| \geq 2. \tag{5-1}$$

Also choose β so that $\sqrt{\beta}$ is a smooth function. Then fix an integer c (whose exact value is unimportant), and, for each $\omega \in \mathcal{D}_l$, define

$$\beta_\omega(x) = \beta(2^{l+c}(x - c_{\omega_1})), \tag{5-2}$$

where ω_1 is the right half of ω and c_{ω_1} is its center.

Define

$$\beta_l(x) = \sum_{\omega \in \mathcal{D}_l} \beta_\omega(x); \tag{5-3}$$

note that

$$\beta_l(x + 2^{-l}) = \beta_l(x) \quad \text{for all } x \in [-2, 2 - 2^{-l}]. \tag{5-4}$$

Define

$$\gamma_l = \frac{1}{2} \int_{-1}^{1} \beta_l(x + t)\, dt; \tag{5-5}$$

because of the above periodicity, we know that γ_l is constant for $x \in [-1, 1]$, independent of l. Say $\gamma_l(x) = \delta > 0$; hence,

$$\frac{1}{\delta} \gamma_l(x) \mathbb{1}_{[-1,1]}(x) = \mathbb{1}_{[-1,1]}(x). \tag{5-6}$$

Define another multiplier $\tilde{\beta} : \mathbb{R} \to \mathbb{R}$ with support in $\left[\frac{1}{2}, \frac{5}{2}\right]$ and $\tilde{\beta}(x) = 1$ for $x \in [1, 2]$. We define the corresponding multiplier on $\mathbb{R}^2$,

$$\hat{m}_{k,\omega}(\xi, \eta) = \tilde{\beta}(2^{-k}\eta)\beta_\omega\left(\frac{\xi}{\eta}\right) \hat{m}_{k,l,t}(\xi, \eta) = \tilde{\beta}(2^{-k}\eta)\beta_l\left(t + \frac{\xi}{\eta}\right) \hat{m}_{k,l}(\xi, \eta) = \tilde{\beta}(2^{-k}\eta)\gamma_l\left(\frac{\xi}{\eta}\right).$$

Then what we need to bound can be written as

$$\left\| \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} H_l P_k(f) \right\|_p = \left\| \int_{-1}^{1} \sum_{k \in \mathbb{Z}} \sum_{l \geq 0} H_{k-l}\left(\frac{1}{\delta} m_{k,l} * f\right) dt \right\|_p \leq \int_{-1}^{1} \left\| \sum_{k \in \mathbb{Z}} \sum_{l \geq 0} H_{k-l}\left(\frac{1}{\delta} m_{k,l,t} * f\right) \right\|_p dt,$$

where the terms $H_l P_k$ for $l > k$ in the sum vanish as explained before.

So it suffices to prove a uniform bound on $t \in [-1, 1]$; without loss of generality we will just consider the case $t = 0$, which is

$$\sum_{k\in\mathbb{Z}}\sum_{l\geq 0} H_{k-l}(m_{k,l,0} * f) = \sum_{k\in\mathbb{Z}}\sum_{l\geq 0} H_{k-l}\left(\left[\tilde{\beta}(2^{-k}\eta)\beta_l\left(\frac{\xi}{\eta}\right)\right] * f\right). \tag{5-7}$$

Constructing the tiles. For each $k \in \mathbb{Z}$ and $\omega \in \mathcal{D}_l$ with $l \geq 0$, let $\mathcal{U}_{k,\omega}$ be a partition of $\mathbb{R}^2$ by rectangles of width 2^{-k} and length 2^{-k+l} whose long sides have slope θ, where $\tan\theta = -c(\omega)$, which is the center of the interval ω. If $s \in \mathcal{U}_{k,\omega}$, we will write $\omega_s := \omega$, and $\omega_{s,1}$ to be the right half of ω and $\omega_{s,2}$ the left half.

An element of $\mathcal{U}_{k,\omega}$ for some $\omega \in \mathcal{D}_l$ is called a "tile". Choose $\varphi_{k,\omega}$ such that

$$|\hat{\varphi}_{k,\omega}|^2 = \hat{m}_{k,\omega}; \tag{5-8}$$

then $\varphi_{k,\omega}$ is smooth by our assumption on β mentioned above.

For a tile $s \in \mathcal{U}_{k,\omega}$, define

$$\varphi_s(p) := \sqrt{|s|}\varphi_{k,\omega}(p - c(s)), \tag{5-9}$$

where $c(s)$ is the center of s. Notice that

$$\|\varphi_s\|_2^2 = \int_{\mathbb{R}^2} |s|\varphi_{k,\omega}^2 = |s| \int_{\mathbb{R}^2} \hat{m}_{k,\omega} = 1, \tag{5-10}$$

i.e., φ_s is L^2 normalized.

The purpose of constructing of the tiles above, by the uncertainty principle, is to localize the function further in space, which is realized through:

Lemma 5.1 [Bateman 2013, page 1030]. *Under the above notations, for the frequency-localized function $f * m_{k,\omega}$, we have the representation*

$$f * m_{k,\omega}(x) = \lim_{N\to\infty} \frac{1}{4N^2} \int_{[-N,N]^2} \sum_{s\in\mathcal{U}_{k,\omega}} \langle f, \varphi_s(p + \cdot)\rangle \varphi_s(p + x)\, dp \tag{5-11}$$

The above lemma allows us to pass to the model sum

$$\sum_{k\in\mathbb{Z}}\sum_{l\geq 0} H_{k-l}(f * m_{k,l,0}) = \sum_{k\in\mathbb{Z}}\sum_{l\geq 0}\sum_{\omega\in\mathcal{D}_l}\sum_{s\in\mathcal{U}_{k,\omega}} \langle f, \varphi_s\rangle H_{k-l}(\varphi_s),$$

define

$$\psi_s = \psi_{-\log(\text{length}(s))}, \tag{5-12}$$

and

$$\phi_s(x, y) := \int \check{\psi}_s(t)\varphi_s(x - t, y - tu(P(x, y)))\, dt; \tag{5-13}$$

then the model sum becomes

$$\sum_{k\in\mathbb{Z}}\sum_{l\geq 0}\sum_{\omega\in\mathcal{D}_l}\sum_{s\in\mathcal{U}_{k,\omega}} \langle f, \varphi_s\rangle \phi_s. \tag{5-14}$$

Lemma 5.2. *We have that* $\phi_s(x, y) = 0$ *unless* $-u(P(x, y)) \in \omega_{s,2}$.

The proof of Lemma 5.2 is by the Plancherel theorem; we just need to observe that the frequency support of ψ_s and $\hat{\varphi}_s$ will be disjoint at the point (x, y) unless $-u(P(x, y)) \in \omega_{s,2}$.

5B. ***Boundedness of the commutator and proof of Proposition 3.5.*** This subsection is devoted to the proof of Proposition 3.5, which is largely motivated by the proof of the $T(b)$ theorem and the boundedness of the paraproduct; see [Auscher et al. 2002; Coifman et al. 1989], for example.

In our case, unlike Bateman and Thiele's proof for the one-variable vector fields, it's no longer true that $H_v P_k f$ still has frequency in the k-th annulus. In order to get enough orthogonality for the term $H_v P_k f$ to apply the Littlewood–Paley theory, we need to subtract the term $H_v P_k f - \tilde{P}_k H_v P_k f$, which should be viewed as a family of paraproducts.

We proceed with the details of the proof. If we expand the summation on the left-hand side of Proposition 3.5 with (5-14), what we need to bound can be rewritten as

$$\Big\| \sum_k \sum_{\omega \in \mathcal{D}_l} \sum_{s \in \mathcal{U}_{k,\omega}} \langle f, \varphi_s \rangle (\phi_s - \tilde{P}_k \phi_s) \Big\|_2 \lesssim 2^{-\gamma l} \| f \|_2. \tag{5-15}$$

In order to use the orthogonality of different wave packets, we will prove the L^2 bound for the dual operator, which is

$$\sum_k \sum_{\omega \in \mathcal{D}_l} \sum_{s \in \mathcal{U}_{k,\omega}} \langle h, \phi_s - \tilde{P}_k \phi_s \rangle \varphi_s. \tag{5-16}$$

Notice that, for $s_1 \in \mathcal{U}_{k_1,\omega_1}$ and $s_2 \in \mathcal{U}_{k_2,\omega_2}$ with $(k_1, \omega_1) \neq (k_1, \omega_2)$, we have

$$\langle \varphi_{s_1}, \varphi_{s_2} \rangle = 0 \tag{5-17}$$

by the definition of the wavelet function φ_s in (5-9). Also, if we know that s_1 and s_2 are in the same $\mathcal{U}_{k,\omega}$ for some k and ω, then we can find $m_0, n_0 \in \mathbb{Z}$ such that

$$c(s_2) = c(s_1) + (m_0 \cdot l(s_1), n_0 \cdot w(s_1)), \tag{5-18}$$

where $c(s)$ is the center of the tile s, $l(s)$ its length and $w(s)$ its width. Then, by the nonstationary phase method, for any $N \in \mathbb{N}$, there exists a constant C_N depending only on N such that

$$|\langle \varphi_{s_1}, \varphi_{s_2} \rangle| \le \frac{C_N}{(|m_0| + |n_0| + 1)^N}. \tag{5-19}$$

Here we want to make a remark that the exact value of N is not important, it just denotes some large number which might vary from line to line if we use the same notation later.

Applying the above two estimates, (5-17) and (5-19), we obtain

$$\Big\| \sum_k \sum_{\omega \in \mathcal{D}_l} \sum_{s \in \mathcal{U}_{k,\omega}} \langle h, \phi_s - \tilde{P}_k \phi_s \rangle \varphi_s \Big\|_2^2 = \sum_k \sum_{\omega \in \mathcal{D}_l} \sum_{s_1 \in \mathcal{U}_{k,\omega}} \sum_{s_2 \in \mathcal{U}_{k,\omega}} \langle h, \phi_{s_1} - \tilde{P}_k \phi_{s_1} \rangle \langle \varphi_{s_1}, \varphi_{s_2} \rangle \langle h, \phi_{s_2} - \tilde{P}_k \phi_{s_2} \rangle.$$

For any $s_1, s_2 \in \mathcal{U}_{k,\omega}$ there exist $m_0, n_0 \in \mathbb{Z}$ such that

$$c(s_2) = c(s_1) + (m_0 \cdot l(s_1), n_0 \cdot w(s_1)), \tag{5-20}$$

so the above sum can be rewritten as

$$\sum_{m_0,n_0\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{\omega\in\mathcal{D}_l}\sum_{s_1\in\mathcal{U}_{k,\omega}}\langle h,\phi_{s_1}-\tilde{P}_k\phi_{s_1}\rangle\langle\varphi_{s_1},\varphi_{s_2}\rangle\langle h,\phi_{s_2}-\tilde{P}_k\phi_{s_2}\rangle \tag{5-21}$$

with s_1, s_2 satisfying the relation (5-20).

Now fix $m_0, n_0 \in \mathbb{Z}$; by the estimate in (5-19), we know that

$$\begin{aligned}\sum_{k}\sum_{\omega\in\mathcal{D}_l}\sum_{s_1\in\mathcal{U}_{k,\omega}}&|\langle h,\phi_{s_1}-\tilde{P}_k\phi_{s_1}\rangle\langle\varphi_{s_1},\varphi_{s_2}\rangle\langle h,\phi_{s_2}-\tilde{P}_k\phi_{s_2}\rangle|\\ &\lesssim\frac{1}{(|m_0|+|n_0|+1)^N}\sum_{k}\sum_{\omega\in\mathcal{D}_l}\sum_{s_1\in\mathcal{U}_{k,\omega}}|\langle h,\phi_{s_1}-\tilde{P}_k\phi_{s_1}\rangle\langle h,\phi_{s_2}-\tilde{P}_k\phi_{s_2}\rangle|,\end{aligned}$$

and, by the Cauchy–Schwarz inequality, the last term is bounded by

$$\frac{1}{(|m_0|+|n_0|+1)^N}\sum_{k}\sum_{\omega\in\mathcal{D}_l}\sum_{s\in\mathcal{U}_{k,\omega}}|\langle h,\phi_s-\tilde{P}_k\phi_s\rangle|^2, \tag{5-22}$$

so it suffices to prove that

$$\sum_{k}\sum_{\omega\in\mathcal{D}_l}\sum_{s\in\mathcal{U}_{k,\omega}}\langle h,\phi_s-\tilde{P}_k\phi_s\rangle^2\lesssim 2^{-\gamma l}\|h\|_2^2. \tag{5-23}$$

First we estimate every single term $\langle h, \phi_s - \tilde{P}_k\phi_s\rangle$ for a fixed tile s: let $s_{m,n}$ be the shift of s by (m, n) units, that is,

$$s_{m,n} := \{(x, y) \in \mathbb{R}^2 : (x - m\cdot l(s),\ y - n\cdot w(s)) \in s\}; \tag{5-24}$$

then, by the triangle inequality, we know that

$$|\langle h,\phi_s-\tilde{P}_k\phi_s\rangle|\le\sum_{m,n\in\mathbb{Z}}\left|\int_{s_{m,n}}h\cdot(\phi_s-\tilde{P}_k\phi_s)\,dy\,dx\right|. \tag{5-25}$$

Recall that in Definition 4.5 we use $\tilde{R}$ to denote the adapted version of the rectangle R to the family of Lipschitz curves; then clearly $\tilde{s}_{m,n} \supset s_{m,n}$. Thus

$$|\langle h,\phi_s-\tilde{P}_k\phi_s\rangle|\le\sum_{m,n\in\mathbb{Z}}\left|\int_{\tilde{s}_{m,n}}h\cdot(\phi_s-\tilde{P}_k\phi_s)\,dy\,dx\right|. \tag{5-26}$$

By the coarea formula (3-22), we obtain

$$\begin{aligned}|\langle h,\phi_s-\tilde{P}_k\phi_s\rangle|&\le\sum_{m,n\in\mathbb{Z}}\left|\int_{\tilde{s}_{m,n}}h\cdot(\phi_s-\tilde{P}_k\phi_s)\,dy\,dx\right|\\ &\lesssim\sum_{m,n\in\mathbb{Z}}\int_{P(s_{m,n})}\int_{\Gamma_x\cap\tilde{s}_{m,n}}|h\cdot(\phi_s-\tilde{P}_k\phi_s)|\,ds_x\,dx,\end{aligned}$$

where ds_x stands for the arc-length measure of the Lipschitz curve Γ_x.

Now, for the inner integration along the curve Γ_x, we do the same change of coordinates and the same parametrization of Γ_x as in Definition 3.3, i.e., we choose coordinates such that the horizontal axis is parallel to $(1, u(x))$, and represent the curve Γ_x by the Lipschitz function $g_x(\cdot)$. If we let $J(x, s_{m,n})$ denote the projection of $\Gamma_x \cap \tilde{s}_{m,n}$ on the new vertical axis, the last expression becomes

$$\sum_{m,n\in\mathbb{Z}} \int_{P(s_{m,n})} \int_{J(x,s_{m,n})} \bigl|h(g_x(y), y)\bigl(\phi_s(g_x(y), y) - P_k[\phi_s(g_x(y), y)]\bigr)\bigr|\, dy\, dx. \tag{5-27}$$

To bound the above term, Jones' beta number will play a crucial role.

Definition 5.3 [Jones 1989]. For a Lipschitz function $A : \mathbb{R} \to \mathbb{R}$, we first take the Calderón decomposition of $a(x) = A'(x)$, which yields the representation

$$a(x) = \sum_{I \text{ dyadic}} a_I \psi_I(x), \tag{5-28}$$

where ψ_I is some mean-zero function supported on $3I$ with $|\psi_I'(x)| \le |I|^{-1}$. For each dyadic interval I, let

$$\alpha_I = \sum_{|J|\ge|I|} a_I \psi_J(c_I) \tag{5-29}$$

denote the "average slope" of the Lipschitz curve near I, where c_I stands for the center of I, and define the beta number

$$\beta_0(I) := \sup_{x\in 3I} \frac{|A(x) - A(c_I) - \alpha_I(x - c_I)|}{|I|}, \tag{5-30}$$

and the j_0-th beta number

$$\beta_{j_0}(I) := \sup_{x\in 3 j_0 I} \frac{|A(x) - A(c_I) - \alpha_I(x - c_I)|}{|I|}. \tag{5-31}$$

For beta numbers, we have the following Carleson condition:

Lemma 5.4 [Jones 1989]. *For any Lipschitz function A, we have*

$$\sup_J \frac{1}{|J|} \sum_{I\subset J} \beta_0^2(I)|I| \lesssim \|A\|_{\mathrm{Lip}}^2, \tag{5-32}$$

and also, for any $j_0 \in \mathbb{N}$,

$$\sup_J \frac{1}{|J|} \sum_{I\subset J} \beta_{j_0}^2(I)|I| \lesssim j_0^3 \|A\|_{\mathrm{Lip}}^2. \tag{5-33}$$

After introducing Jones' beta number, we are ready to state:

Lemma 5.5. *for $x \in P(s_{m,n})$, we have the estimate*

$$\int_{J(x,s_{m,n})} \bigl|h(g_x(y), y)\bigl(\phi_s(g_x(y), y) - P_k[\phi_s(g_x(y), y)]\bigr)\bigr|\, dy \\ \lesssim \sum_{j_0\in\mathbb{N}} \frac{2^{-3l/2}}{(|j_0| + |m| + |n| + 1)^N} \beta_{j_0}(x, s_{m,n}) [h]_{x,s_{m,n}} \mathbb{1}_{\{-u(x)\in\omega_{s,2}\}}(x),$$

where $\beta_{j_0}(x, s_{m,n})$ is the j_0-th beta number for the Lipschitz curve $g_x(\cdot)$ on the interval $J(x, s_{m,n})$ and $[h]_{x,s_{m,n}}$ is the average of the function h on the interval $J(x, s_{m,n})$,

$$[h]_{x,s_{m,n}} := \frac{1}{w(s)} \int_{J(x,s_{m,n})} |h(g_x(y), y)|\, dy. \tag{5-34}$$

The proof of Lemma 5.5 will be postponed to the end. Substitute the estimate in Lemma 5.5 into the estimate for the term $\langle h, \phi_s - \tilde{P}_k\phi_s\rangle$; we then have that

$$\begin{aligned} |\langle h, \phi_s - \tilde{P}_k\phi_s\rangle| &\lesssim \sum_{m,n} \int_{P(s_{m,n})} \int_{J(x,s_{m,n})} \big| h(g_x(y), y)\big(\phi_s(g_x(y), y) - P_k[\phi_s(g_x(y), y)]\big)\big|\, dy\, dx \\ &\lesssim \sum_{m,n} \int_{P(s_{m,n})} \sum_{j_0\in\mathbb{N}} \frac{2^{-3l/2}}{(|j_0|+|m|+|n|+1)^N} \beta_{j_0}(x, s_{m,n})[h]_{x,s_{m,n}} \mathbb{1}_{\{-u(x)\in\omega_{s,2}\}}(x)\, dx, \end{aligned}$$

hence

$$\begin{aligned} &\sum_k \sum_{\omega\in\mathfrak{D}_l} \sum_{s\in\mathfrak{U}_{k,\omega}} |\langle h, \phi_s - \tilde{P}_k\phi_s\rangle|^2 \\ &\quad\lesssim \sum_k \sum_{\omega\in\mathfrak{D}_l} \sum_{s\in\mathfrak{U}_{k,\omega}} \sum_{m,n,j_0} \frac{2^{-3l}}{(|j_0|+|m|+|n|+1)^N} \left| \int_{P(s_{m,n})} \beta_{j_0}(x, s_{m,n})[h]_{x,s_{m,n}} \mathbb{1}_{\{-u(x)\in\omega_{s,2}\}}(x)\, dx \right|^2 \\ &\quad\lesssim \sum_{m,n,j_0} \frac{2^{-2l}}{(|j_0|+|m|+|n|+1)^N} \sum_k \sum_{\omega\in\mathfrak{D}_l} \sum_{s\in\mathfrak{U}_{k,\omega}} w(s) \int_{P(s_{m,n})} \beta^2_{j_0}(x, s_{m,n})[h]^2_{x,s_{m,n}} \mathbb{1}_{\{-u(x)\in\omega_{s,2}\}}(x)\, dx. \end{aligned}$$

Lemma 5.6. *For any fixed x, m, n, j_0,*

$$\sum_k \sum_{\omega\in\mathfrak{D}_l} \sum_{s\in\mathfrak{U}_{k,\omega}} w(s)\mathbb{1}_{P(s_{m,n})}(x)\beta^2_{j_0}(x, s_{m,n})[h]^2_{x,s_{m,n}} \mathbb{1}_{\{-u(x)\in\omega_{s,2}\}}(x) \lesssim j_0^3 \|h\|^2_{L^2(\Gamma_x)}. \tag{5-35}$$

Proof. This lemma is akin to the Carleson embedding theorem, as we have the Carleson-type condition

$$\sup_{s_{m,n}} \frac{1}{|J(x, s_{m,n})|} \sum_{s'_{m,n}: J(x,s'_{m,n})\subset J(x,s_{m,n})} \beta^2_{j_0}(J(x, s'_{m,n}))w(s'_{m,n}) \lesssim j_0^3 \operatorname{Lip}^2(\Gamma_x), \tag{5-36}$$

where the term $\mathbb{1}_{\{-u(x)\in\omega_{s,2}\}}$ has the following purpose: originally there are 2^l groups of dyadic rectangles

$$\bigcup_k \bigcup_{\omega\in\mathfrak{D}_l} \bigcup_{s\in\mathfrak{U}_{k,\omega}} \{s_{m,n}\} \tag{5-37}$$

in the summation $\sum_k \sum_{\omega\in\mathfrak{D}_l} \sum_{s\in\mathfrak{U}_{k,\omega}}$, which means that there are also 2^l groups of dyadic intervals

$$\bigcup_k \bigcup_{\omega\in\mathfrak{D}_l} \bigcup_{s\in\mathfrak{U}_{k,\omega}} \{J(x, s_{m,n})\} \tag{5-38}$$

which are the projections of the intersection of the dyadic rectangles with Γ_x on the vertical axis; the term $\mathbb{1}_{\{-u(x)\in\omega_{s,2}\}}$ just guarantees that there is just one such collection that contributes, i.e., which has the right orientation in the sense of Lemma 5.2.

Then the desired estimate will just follow from the Carleson embedding theorem, for which we refer to Lemma 5.1 in [Auscher et al. 2002]. □

Continuing the calculation before the above lemma,

$$\sum_k \sum_{\omega\in\mathfrak{D}_l} \sum_{s\in\mathfrak{U}_{k,\omega}} |\langle h, \phi_s - \tilde{P}_k\phi_s\rangle|^2 \lesssim \sum_{m,n,j_0} \frac{2^{-2l} j_0^3}{(|j_0|+|m|+|n|+1)^N} \int_{\mathbb{R}} \|h\|^2_{L^2(\Gamma_x)}\, dx \lesssim 2^{-2l}\|h\|_2^2.$$

This finishes the proof for (5-23) and then Proposition 3.5, modulo Lemma 5.5, which we will present now.

Proof of Lemma 5.5. We assume that $-u(x)\in\omega_{s,2}$, which means the vector $(1, u(x))$ is roughly parallel to the long side of $s_{m,n}$, otherwise the left-hand side in Lemma 5.5 will also vanish due to Lemma 5.2. After the change of variables in (5-27), the vector $(1, u(x))$ becomes $(1, 0)$.

Proof by ignoring the tails. In order to explain how Jones' beta number appears, we first sketch the proof by ignoring the tails of the wavelet functions and the tail of the kernel of the Littlewood–Paley projection operator P_k.

By the above simplification, we only need to consider the case $m = n = 0$. What we need to "prove" becomes

$$\int_{J(x,s)} \big|h(g_x(y), y)\big(\phi_s(g_x(y), y) - P_k[\phi_s(g_x(y), y)]\big)\big|\, dy \lesssim 2^{-3l/2}\beta_0(J(x,s))[h]_{x,s}. \tag{5-39}$$

For fixed x, we denote by $\tau_{x,s}y + b$ the line of "average slope" we picked in the definition of the beta number for the Lipschitz curve $g_x(\cdot)$ on the interval $J(x,s)$; for the sake of simplicity we assume $b = 0$. Moreover, as both x and s are fixed, we will also just write τ instead of $\tau_{x,s}$. Then we make the crucial observation that

$$P_k[\phi_s^x(\tau y, y)] = \phi_s^x(\tau y, y), \tag{5-40}$$

where

$$\phi_s^x(\tau y, y) := \int_{\mathbb{R}} \check{\psi}_s(t)\varphi_s(\tau y - t, y)\, dt, \tag{5-41}$$

due to the fact that, for any function φ_s with frequency supported on the k-th annulus, if we restrict the function to a straight line, it will still have frequency supported on the k-th annulus (with one dimension less).

In comparison with the definition of ϕ_s in (5-13), $\phi_s^x(\tau y, y)$ is defined as the Hilbert transform along the vector $(1, u(x))$ (which is $(1, 0)$ after the change of the variables we made in Lemma 3.2 and in the expression (5-27)) instead of the direction of the vector field v at the point $(\tau y, y)$.

Hence, from the identity in (5-40), we obtain

$$\begin{aligned}\phi_s(g_x(y), y) - P_k[\phi_s(g_x(y), y)] &= \phi_s(g_x(y), y) - P_k[\phi_s(g_x(y), y) - \phi_s^x(\tau y, y) + \phi_s^x(\tau y, y)]\\ &= \phi_s(g_x(y), y) - \phi_s^x(\tau y, y) - P_k[\phi_s(g_x(y), y) - \phi_s^x(\tau y, y)].\end{aligned} \tag{5-42}$$

As we have also ignored the tails of the kernel of P_k, it is easy to see that the former and the latter terms in the last expression can essentially be handled in the same way. Hence in the following we will only consider the former term, which corresponds to the term

$$\int_{J(x,s)} \big|h(g_x(y), y)\big(\phi_s(g_x(y), y) - \phi_s^x(\tau y, y)\big)\big|\, dy. \tag{5-43}$$

By the definitions of ϕ_s and ϕ_s^x, we have

$$\begin{aligned}|\phi_s(g_x(y), y) - \phi_s^x(\tau y, y)| &= \left|\int_{\mathbb{R}} \check{\psi}_{k-l}(t)\varphi_s(g_x(y)-t, y)\, dt - \int_{\mathbb{R}} \check{\psi}_{k-l}(t)\varphi_s(\tau y - t, y)\, dt\right| \\ &= 2^{k-l}\left|\int_{\mathbb{R}} \check{\psi}_0(2^{k-l}t)\varphi_s(g_x(y)-t, z)\, dt - \int_{\mathbb{R}} \check{\psi}_0(2^{k-l}t)\varphi_s(\tau y - t, y)\, dt\right| \\ &= 2^{k-l}\left|\int_{\mathbb{R}} \big[\check{\psi}_0(2^{k-l}(t+g_x(y)-\tau y)) - \check{\psi}_0(2^{k-l}t)\big]\varphi_s(\tau y - t, z)\, dt\right|. \end{aligned} \tag{5-44}$$

By the definition of the beta numbers, we have that

$$|g_x(y) - \tau y| \lesssim \beta_0(x, s)2^{-k}, \tag{5-45}$$

which implies that

$$|\check{\psi}_0(2^{k-l}(t+g_x(y)-\tau y)) - \check{\psi}_0(2^{k-l}t)| \lesssim 2^{-l}\beta_0(x, s) \tag{5-46}$$

by the fundamental theorem. In the end, by substituting the above estimate into (5-44) and (5-43), we obtain the desired estimate (5-39).

The full proof. The main idea is still the same, and the difference is that we need to be more careful with the tails of the wavelet functions and the kernel of P_k.

For fixed x, m and n, denote by $\tau(x, s_{m,n})y + b$ the line of "average slope" for the Lipschitz curve $g_x(\cdot)$ on the interval $J(x, s_{m,n})$; for the sake of simplicity we assume $b = 0$. Then the crucial observation (5-40) becomes

$$P_k[\phi_s^x(\tau(x, s_{m,n})y, y)] = \phi_s^x(\tau(x, s_{m,n})y, y). \tag{5-47}$$

Hence, similar to (5-42), we obtain from (5-47) that

$$\begin{aligned}&\phi_s(g_x(y), y) - P_k[\phi_s(g_x(y), y)] \\ &\qquad = \phi_s(g_x(y), y) - \phi_s^x(\tau(x, s_{m,n})y, y) - P_k[\phi_s(g_x(y), y) - \phi_s^x(\tau(x, s_{m,n})y, y)].\end{aligned}$$

Denote

$$I_{s_{m,n}} = \left|\int_{J(x,s_{m,n})} h(g_x(y), y)\cdot\big(\phi_s(g_x(y), y) - \phi_s^x(\tau(x, s_{m,n})y, y)\big)\, dy\right| \tag{5-48}$$

and also

$$II_{s_{m,n}} = \left|\int_{J(x,s_{m,n})} h(g_x(y), y)\cdot P_k\big[\phi_s(g_x(y), y) - \phi_s^x(\tau(x, s_{m,n})y, y)\big]\, dy\right|. \tag{5-49}$$

Lemma 5.7. *Under the above notations, for $z \in J(x, s_{m,n}) + j_0 2^{-k}$ with $j_0 \in \mathbb{Z}$, we have the pointwise estimate*

$$|\phi_s(g_x(z), z) - \phi_s^x(\tau(x, s_{m,n})z, z)| \lesssim \frac{\beta_{|j_0|}(x, s_{m,n})2^k 2^{-3l/2}}{(\min\{|m|+|n|, |m|+|n|-|j_0|\}+1)^N}. \tag{5-50}$$

Let us first complete the proof of Lemma 5.5: For the first term $I_{s_{m,n}}$, we take j_0 in Lemma 5.7 to be zero, then

$$|\phi_s(g_x(z), z) - \phi_s^x(\tau(x, s_{m,n})z, z)| \lesssim \frac{\beta_0(x, s_{m,n})2^k 2^{-3l/2}}{(|m|+|n|+1)^N}, \tag{5-51}$$

which implies that

$$I_{s_{m,n}} \lesssim \frac{2^{-3l/2}}{(|m|+|n|+1)^N} \beta_0(x, s_{m,n})[h]_{x,s_{m,n}}. \tag{5-52}$$

For the second term $II_{s_{m,n}}$, by the definition of P_k,

$$\begin{aligned}\big|P_k[\phi_s(g_x(y), y) - \phi_s^x(\tau(x, s_{m,n})y, y)]\big| \\ &= \left|\int_{\mathbb{R}} \big(\phi_s(g_x(z), z) - \phi_s^x(\tau(x, s_{m,n})z, z)\big)2^k \check{\psi}_0(2^k(y-z))\, dz\right| \\ &\le \left|\sum_{j_0\in\mathbb{Z}} \int_{J(x,s_{m,n})+j_0 2^{-k}} \big(\phi_s(g_x(z), z) - \phi_s^x(\tau(x, s_{m,n})z, z)\big)2^k \check{\psi}_0(2^k(y-z))\, dz\right|.\end{aligned}$$

For $y \in J(x, s_{m,n})$ and $z \in J(x, s_{m,n}) + j_0 2^{-k}$, by the nonstationary phase method we have that

$$|\check{\psi}_0(2^k(y-z))| \lesssim \frac{1}{(j_0+1)^N}. \tag{5-53}$$

Together with the estimate in Lemma 5.7, we arrive at

$$\begin{aligned}\big|P_k[\phi_s(g_x(y), y) - \phi_s^x(\tau(x, s_{m,n})y, y)]\big| &\lesssim \sum_{j_0\in\mathbb{Z}} \frac{\beta_{|j_0|}(x, s_{m,n})2^k 2^{-3l/2}}{(\min\{|m|+|n|, |m|+|n|-|j_0|\}+1)^N} \frac{1}{(j_0+1)^N} \\ &\lesssim \sum_{j_0\in\mathbb{Z}} \frac{\beta_{|j_0|}(x, s_{m,n})2^k 2^{-3l/2}}{(|m|+|n|+|j_0|+1)^N}.\end{aligned}$$

Substituting the last expression into the estimate for $II_{s_{m,n}}$, we get the desired estimate.

So far we have finished the proof of Lemma 5.5 except for Lemma 5.7, which we will do now.

Proof of Lemma 5.7. As x and $s_{m,n}$ are fixed now, for simplicity we will just write τ instead of $\tau_{x,s_{m,n}}$. Notice that in the new coordinate we chose for Γ_x, the vector field along Γ_x points in the direction of $(1, 0)$. Then, by the definition of ϕ_s and ϕ_s^x, we have

$$|\phi_s(g_x(z), z) - \phi_s^x(\tau z, z)| = 2^{k-l}\left|\int_{\mathbb{R}} \big[\check{\psi}_0(2^{k-l}(t + g_x(z) - \tau z)) - \check{\psi}_0(2^{k-l}t)\big]\varphi_s(\tau z - t, z)\, dt\right|.$$

By the definition of the beta numbers, we have that

$$|g_x(z) - \tau z| \lesssim \beta_{|j_0|}(x, s_{m,n})2^{-k}, \tag{5-54}$$

which implies that

$$|\check{\psi}_0(2^{k-l}(t + g_x(z) - \tau z)) - \check{\psi}_0(2^{k-l}t)| \lesssim 2^{-l}\beta_{|j_0|}(x, s_{m,n}) \tag{5-55}$$

by the fundamental theorem of calculus. The nonstationary phase method leads to the final estimate:

$$2^{k-l}\left|\int_{\mathbb{R}}\left[\check{\psi}_0(2^{k-l}(t+g_x(z)-\tau z))-\check{\psi}_0(2^{k-l}t)\right]\varphi_s(\tau z-t,\,z)\,dt\right| \lesssim \frac{2^{-l}\beta_{|j_0|}(x,\,s_{m,n})2^{k/2}2^{(k-l)/2}}{(\min\{|m|+|n|,\,|m|+|n|-|j_0|\}+1)^N}.$$

Thus we have finished the proof of Lemma 5.7, and hence Lemma 5.5. □

Acknowledgements

This work was done under the supervision of Prof. Christoph Thiele, to whom the author would like to express his most sincere gratitude for suggesting to him such an interesting topic, for sharing with him his deep insight into this problem, and for numerous suggestions on the exposition of this paper. The author would also like to thank the anonymous referee for many helpful comments.

References

[Auscher et al. 2002] P. Auscher, S. Hofmann, C. Muscalu, T. Tao, and C. Thiele, "Carleson measures, trees, extrapolation, and $T(b)$ theorems", *Publ. Mat.* **46**:2 (2002), 257–325. MR 2003f:42019 Zbl 1027.42009

[Azzam and Schul 2012] J. Azzam and R. Schul, "Hard Sard: quantitative implicit function and extension theorems for Lipschitz maps", *Geom. Funct. Anal.* **22**:5 (2012), 1062–1123. MR 2989430 Zbl 1271.26004

[Bateman 2009a] M. Bateman, "Kakeya sets and directional maximal operators in the plane", *Duke Math. J.* **147**:1 (2009), 55–77. MR 2009m:42029 Zbl 1165.42005

[Bateman 2009b] M. Bateman, "L^p estimates for maximal averages along one-variable vector fields in $\mathbb{R}^2$", *Proc. Amer. Math. Soc.* **137**:3 (2009), 955–963. MR 2010e:42023 Zbl 1173.42008

[Bateman 2013] M. Bateman, "Single annulus L^p estimates for Hilbert transforms along vector fields", *Rev. Mat. Iberoam.* **29**:3 (2013), 1021–1069. MR 3090145 Zbl 1283.42019

[Bateman and Thiele 2013] M. Bateman and C. Thiele, "L^p estimates for the Hilbert transforms along a one-variable vector field", *Anal. PDE* **6**:7 (2013), 1577–1600. MR 3148061 Zbl 1285.42014

[Bourgain 1989] J. Bourgain, "A remark on the maximal function associated to an analytic vector field", pp. 111–132 in *Analysis at Urbana, I* (Urbana, IL, 1986–1987), edited by E. Berkson and T. Peck, London Math. Soc. Lecture Note Ser. **137**, Cambridge Univ. Press, Cambridge, 1989. MR 90h:42028 Zbl 0692.42006

[Christ et al. 1999] M. Christ, A. Nagel, E. M. Stein, and S. Wainger, "Singular and maximal Radon transforms: analysis and geometry", *Ann. of Math.* (2) **150**:2 (1999), 489–577. MR 2000j:42023 Zbl 1285.42014

[Coifman et al. 1989] R. R. Coifman, P. W. Jones, and S. Semmes, "Two elementary proofs of the L^2 boundedness of Cauchy integrals on Lipschitz curves", *J. Amer. Math. Soc.* **2**:3 (1989), 553–564. MR 90k:42017 Zbl 0713.42010

[Jones 1989] P. W. Jones, "Square functions, Cauchy integrals, analytic capacity, and harmonic measure", pp. 24–68 in *Harmonic analysis and partial differential equations* (El Escorial, 1987), edited by J. García-Cuerva, Lecture Notes in Math. **1384**, Springer, Berlin, 1989. MR 91b:42032 Zbl 0675.30029

[Karagulyan 2007] G. A. Karagulyan, "On unboundedness of maximal operators for directional Hilbert transforms", *Proc. Amer. Math. Soc.* **135**:10 (2007), 3133–3141. MR 2008e:42044 Zbl 1162.42009

[Lacey 2000] M. T. Lacey, "The bilinear maximal functions map into L^p for $\frac{2}{3} < p \leq 1$", *Ann. of Math.* (2) **151**:1 (2000), 35–57. MR 2001b:42015 Zbl 0967.47031

[Lacey and Li 2006] M. T. Lacey and X. Li, "Maximal theorems for the directional Hilbert transform on the plane", *Trans. Amer. Math. Soc.* **358**:9 (2006), 4099–4117. MR 2006k:42018 Zbl 1095.42010

[Lacey and Li 2010] M. Lacey and X. Li, *On a conjecture of E. M. Stein on the Hilbert transform on vector fields*, vol. 205, Mem. Amer. Math. Soc. **965**, American Mathematical Society, Providence, RI, 2010. MR 2011c:42019 Zbl 1190.42005

[Lacey and Thiele 1997] M. Lacey and C. Thiele, "L^p estimates for the bilinear Hilbert transform", *Proc. Nat. Acad. Sci. U.S.A.* **94**:1 (1997), 33–35. MR 98e:44001 Zbl 1027.42009

[Lacey and Thiele 1998] M. T. Lacey and C. M. Thiele, "On Calderón's conjecture for the bilinear Hilbert transform", *Proc. Natl. Acad. Sci. USA* **95**:9 (1998), 4828–4830. MR 99e:42013 Zbl 0915.42011

[Stein and Street 2012] E. M. Stein and B. Street, "Multi-parameter singular Radon transforms, III: Real analytic surfaces", *Adv. Math.* **229**:4 (2012), 2210–2238. MR 2880220 Zbl 1242.42010

Received 10 Jan 2015. Revised 8 Mar 2015. Accepted 15 Apr 2015.

SHAOMING GUO: shaoming@math.uni-bonn.de
Mathematisches Institut, University of Bonn, Endenicher Allee 60, D-53115 Bonn, Germany

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2015 is US $205/year for the electronic version, and $390/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 8 No. 5 2015

2157-5045(2015)8:5;1-9