ANALYSIS & PDE

Volume 8 No. 5 2015

msp

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2015 is US $205/year for the electronic version, and $390/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS AND PDE
Vol. 8, No. 5, 2015
dx.doi.org/10.2140/apde.2015.8.1025

PARTIAL COLLAPSING AND THE SPECTRUM OF THE HODGE–DE RHAM OPERATOR

COLETTE ANNÉ AND JUNYA TAKAHASHI

Our goal is to calculate the limit spectrum of the Hodge–de Rham operator under the perturbation of collapsing one part of a manifold obtained by gluing together two manifolds with the same boundary. It appears to take place in the general problem of blowing up conical singularities as introduced by Mazzeo and Rowlett.

1. Introduction

This work takes place in the context of the spectral studies of singular perturbations of the metrics, as a means to know what are the topological or metrical meanings carried by the spectrum of geometric operators. We can mention in this direction, without exhaustivity, studies on the adiabatic limits [Mazzeo and Melrose 1990; Rumin 2000], on collapsing [Fukaya 1987; Lott 2002a; 2002b; 2004], on resolution blowups of conical singularities [Mazzeo 2006; Rowlett 2006; 2008] and on shrinking handles [Anné and Colbois 1995; Anné et al. 2009].

The present study can be considered as a generalization of the results of [Anné and Takahashi 2012], where we studied the limit of the spectrum of the Hodge–de Rham (or the Hodge–Laplace) operator under collapsing of one part of a connected sum.

In our previous work, we restricted the submanifold Σ used to glue the two parts to be a sphere. In fact, this problem is quite related to resolution blowups of conical singularities: the point is to measure the influence of the topology of the part which disappears and of the conical singularity created at the limit of the "big part". If we look at the situation from the "small part", we understand the importance of the *quasiasymptotically conical space* obtained from rescaling the small part and gluing an infinite cone; see the definition below in (1).

When Σ is the sphere $\mathbb{S}^n$, the conical singularity is quite simple. There are no *half-bound states* — called extended solutions in the sequel — on the quasiasymptotically conical space. Our result presented here takes care of these new possibilities and gives a general answer to the problem studied by Mazzeo and Rowlett. Indeed, in [Mazzeo 2006; Rowlett 2006; 2008], it is supposed that the spectrum of the operator on the quasiasymptotically conical space does not meet 0. Our study relaxes this hypothesis. It is done only with the Hodge–de Rham operator, but can easily be generalized.

Let us fix some notations.

MSC2010: primary 58J50; secondary 35P15, 53C23, 58J32.
Keywords: Laplacian, Hodge–de Rham operator, differential form, eigenvalue, collapsing of Riemannian manifolds, conical singularity, elliptic boundary value problem.

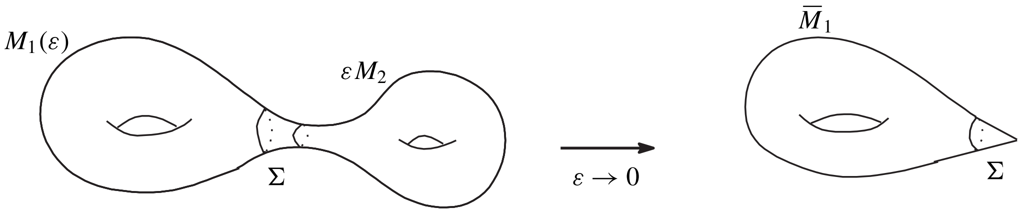

Figure 1. Partial collapsing of M_ε.

1.1. ***Set-up.*** Let M_1 and M_2 be two connected, oriented, compact manifolds with the same boundary Σ, a compact manifold of dimension $n \geq 2$. We denote by $m = n+1$ the dimension of M_1 and M_2. We endow Σ with a fixed metric h.

Let $\overline{M}_1$ be the manifold with conical singularity obtained from M_1 by gluing M_1 to a cone $\mathscr{C} = [0, 1) \times \Sigma$; we write (r, y) for points on $\mathscr{C}$, and there exists on $\overline{M}_1 = M_1 \cup \mathscr{C}$ a metric $\bar{g}_1$ which equals $dr^2 + r^2 h$ on the smooth part $r > 0$ of the cone.

We choose on M_2 a metric g_2 which is "trumpet-like", i.e., M_2 is isometric near the boundary to $\left[0, \frac{1}{2}\right) \times \Sigma$ with the conical metric which equals $ds^2 + (1-s)^2 h$ if s is the coordinate defining the boundary by $s = 0$.

For any ε with $0 \leq \varepsilon < 1$, we define

$$\mathscr{C}_{\varepsilon,1} = \{(r, y) \in \mathscr{C} \mid r > \varepsilon\} \quad \text{and} \quad M_1(\varepsilon) = M_1 \cup \mathscr{C}_{\varepsilon,1}.$$

The goal of the following calculus is to determine the limit spectrum of the Hodge–de Rham operator acting on the differential forms of the Riemannian manifold

$$M_\varepsilon = M_1(\varepsilon) \cup_{\varepsilon.\Sigma} \varepsilon.M_2,$$

which is obtained by gluing together $(M_1(\varepsilon), g_1)$ and $(M_2, \varepsilon^2 g_2)$. By construction, these two manifolds have isometric boundary and the metric g_ε obtained on M_ε is smooth.

Remark 1. The common boundary Σ of dimension n has some topological obstructions. In fact, since Σ is the boundary of the oriented, compact manifold M_1, Σ is oriented cobordant to zero. So, by Thom's cobordism theory, all the Stiefel–Whitney and all the Pontrjagin numbers vanish (see C. T. C. Wall [1960] or [Milnor and Stasheff 1974, §18, p. 217]). Furthermore, this condition is also sufficient; that is, the inverse does hold.

In particular, it is impossible to take Σ^{4k} as the complex projective spaces $\mathbb{CP}^{2k}$ $(k \geq 1)$ because the Pontrjagin number $p_k(\mathbb{CP}^{2k})$ is nonzero.

1.2. ***Results.*** We can describe the limit spectrum as follows; it has two parts. One part comes from the big part, namely $\overline{M}_1$, and is expressed by the spectrum of a good extension of the Hodge–de Rham operator on this manifold with the conical singularity. This extension is self-adjoint and comes from an extension of the Gauss–Bonnet operators. All these extensions are classified by subspaces W of the total eigenspaces corresponding to the eigenvalues within $\left(-\frac{1}{2}, \frac{1}{2}\right)$ of an operator A acting on the boundary Σ. This point is developed in Section 2.2. The other part comes from the collapsing part, namely M_2, where the limit Gauss–Bonnet operator is taken with boundary conditions of Atiyah–Patodi–Singer-type. This point is developed in Section 2.3. This operator, denoted $\mathscr{D}_2$ in the sequel, can also be seen on the quasiasymptotically conical space $\widetilde{M}_2$ already mentioned, namely

$$\widetilde{M}_2 = M_2 \cup ([1, \infty) \times \Sigma) \tag{1}$$

with the metric $dr^2 + r^2 h$ on the conical part. Only the zero eigenvalue is concerned with this part. In fact, the manifold M_ε has small eigenvalues, in contrast to [Anné and Takahashi 2012], and the multiplicity of 0 at the limit corresponds to the total eigenspaces of these small and null eigenvalues. Thus, our main theorem, which asserts the convergence of the spectrum, has two components.

Theorem A. *The set of all positive limit values is just equal to that of all positive spectrum of the Hodge–de Rham operator $\Delta_{1,W}$ on $\overline{M}_1$, where*

$$W \subset \bigoplus_{|\gamma|<\frac{1}{2}} \operatorname{Ker}(A - \gamma)$$

is the space of the elements that generate extended solutions on $\widetilde{M}_2$. A precise definition is given in (7).

Theorem B. *The multiplicity of* 0 *in the limit spectrum is given by the sum*

$$\dim \operatorname{Ker}(\Delta_{1,W}) + \dim \operatorname{Ker}(\mathscr{D}_2) + i_{1/2},$$

where $i_{1/2}$ denotes the dimension of the vector space $\mathscr{I}_{1/2}$ — see (8) — of extended solutions ω on $\widetilde{M}_2$ introduced by Carron [2001b], admitting on restriction to $r = 1$ a nontrivial component in $\operatorname{Ker}\big(A - \frac{1}{2}\big)$.

1.3. *Comments.*

1.3.1. This result is also valid in dimension 2. In order to understand it, look at the following example. Let $I = [0, 1]$ and $M_1 = M_2 = \mathbb{S}^1 \times I$. We can shrink half of a torus: $\mathbb{S}^1 \times \mathbb{S}^1 = M_1 \cup_\Sigma M_1$ for $\Sigma = \mathbb{S}^1 \sqcup \mathbb{S}^1$. Then $\overline{M}_1$ is a 2-sphere with no harmonic 1-forms and $\widetilde{M}_2$ has no L^2-harmonic 1-forms. But $i_{1/2} = 2$. Indeed $\widetilde{M}_2$ is a cylinder with flat ends. With obvious coordinates (r, θ), $d\theta$ and $*(d\theta) \sim dr/r$ near ∞ give a base for extended solutions.

1.3.2. We choose, in our study, a simple metric to make explicit computations. This fact is not a restriction, as already explained in [Anné and Takahashi 2012], because of the result of Dodziuk [1982] which assures uniform control of the eigenvalues of geometric operators with regard to variations of the metric.

1.3.3. More examples are given in the last section of the paper.

2. Gauss–Bonnet operator

On a Riemannian manifold, the Gauss–Bonnet operator is defined as the operator $D = d + d^*$ acting on differential forms. It is symmetric and can have some closed extensions on manifolds with boundary or with conical singularities. We review these extensions in the cases involved in our study.

2.1. ***Gauss–Bonnet operator on M_ε.*** We recall that, on M_ε, a Gauss–Bonnet operator D_ε, Sobolev spaces and also a Hodge–de Rham operator Δ_ε can be defined as a general construction on any manifold $X = X_1 \cup X_2$, which is the union of two Riemannian manifolds with isometric boundaries (the details are given in [Anné and Colbois 1995]): if D_1 and D_2 are the Gauss–Bonnet operators "$d + d^*$" acting on the differential forms of each part, the quadratic form

$$q(\phi) = \int_{X_1} |D_1(\phi\restriction_{X_1})|^2 \, d\mu_{X_1} + \int_{X_2} |D_2(\phi\restriction_{X_2})|^2 \, d\mu_{X_2} \tag{2}$$

is well-defined and closed on the domain

$$\mathrm{Dom}(q) = \{\phi = (\phi_1, \phi_2) \in H^1(\Lambda T^* X_1) \times H^1(\Lambda T^* X_2) \mid \phi_1\restriction_{\partial X_1} =_{L^2} \phi_2\restriction_{\partial X_2}\}.$$

On this space, the total Gauss–Bonnet operator $D(\phi) = (D_1(\phi_1), D_2(\phi_2))$ is defined and self-adjoint. For this definition, we have to, in particular, identify $(\Lambda T^* X_1)\restriction_{\partial X_1}$ and $(\Lambda T^* X_2)\restriction_{\partial X_2}$. This can be done by decomposing the forms into tangential and normal parts (with inner normal); the equality above means then that the tangential parts are equal and the normal parts opposite. This definition generalizes the definition in the smooth case.

The Hodge–de Rham operator $(d+d^*)^2$ of X is then defined as the operator obtained by the polarization of the quadratic form q. This gives compatibility conditions between ϕ_1 and ϕ_2 on the common boundary. We do not give details on these facts, because our manifold is smooth. But we shall use this presentation for the quadratic form.

2.2. ***Gauss–Bonnet operator on $\overline{M}_1$.*** Let $D_{1,\min}$ be the closure of the Gauss–Bonnet operator defined on the smooth forms with compact support in the smooth part $M_1(0)$. For any such form ϕ_1, following [Brüning and Seeley 1988; Anné et al. 2009], on the cone $\mathscr{C}$ we write

$$\phi_1 = dr \wedge r^{-(n/2-p+1)} \beta_{1,\varepsilon} + r^{-(n/2-p)} \alpha_{1,\varepsilon}$$

and define $\sigma_1 = (\beta_1, \alpha_1) = U(\phi_1)$. The operator has, on the cone $\mathscr{C}$, the expression

$$U D_1 U^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\partial_r + \frac{1}{r} A\right) \quad \text{with} \quad A = \begin{pmatrix} \frac{1}{2}n - P & -D_0 \\ -D_0 & P - \frac{1}{2}n \end{pmatrix},$$

where P is the operator of degree, that is, $P\omega = p \cdot \omega$ for a p-form ω, and $D_0 = d_0 + d_0^*$ is the Gauss–Bonnet operator on the manifold (Σ, h), while the Hodge–de Rham operator has, in these coordinates, the expression

$$U \Delta_1 U^* = -\partial_r^2 + \frac{1}{r^2} A(A+1). \tag{3}$$

The closed extensions of the operator $D_1 = d + d^*$ on the manifold with conical singularity $\overline{M}_1$ have been studied in [Brüning and Seeley 1988; Lesch 1997]. They are classified by the spectrum of its *Mellin symbol*, which is here the operator with parameter $A + z$.

Spectrum of A. The spectrum of A was calculated in [Brüning and Seeley 1988, p. 703]. By their result, the spectrum of A is given by the values

$$\begin{cases} \pm\left(p-\frac{1}{2}n\right) & \\ (-1)^{p+1}\frac{1}{2}\pm\sqrt{\mu^2+\left(\frac{1}{2}(n-1)-p\right)^2}, & \end{cases} \quad \text{with multiplicity } \dim H^p(\Sigma), \tag{4}$$

where p is any integer, $0 \le p \le n$, and μ^2 runs over the spectrum of the Hodge–de Rham operator on (Σ, h) acting on the coexact p-forms.

Indeed, looking at the Gauss–Bonnet operator acting on even forms, they identify even forms on the cone with the sections $(\phi_0, \dots, \phi_n)$ of the total bundle $\Lambda T^*(\Sigma)$ by $\phi_0 + \phi_1 \wedge dr + \phi_2 + \phi_3 \wedge dr + \cdots$. These sections can also represent odd forms on the cone by $\phi_0 \wedge dr + \phi_1 + \phi_2 \wedge dr + \phi_3 + \cdots$. With these identifications, they have to study the spectrum of the following operator acting on sections of $\Lambda T^*(\Sigma)$:

$$S_0 = \begin{pmatrix} c_0 & d_0^* & 0 & \cdots & 0 \\ d_0 & c_1 & d_0^* & \ddots & \vdots \\ 0 & d_0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & c_{n-1} & d_0^* \\ 0 & \cdots & 0 & d_0 & c_n \end{pmatrix}$$

if $c_p = (-1)^{p+1}\left(p-\frac{1}{2}n\right)$. With the same identification, if we introduce the operator $\widetilde{S}_0$ having the same formula but on the diagonal the terms $\tilde{c}_p = (-1)^p\left(p-\frac{1}{2}n\right) = -c_p$, then the operator A can be written as

$$A = -(S_0 \oplus \widetilde{S}_0).$$

The expression of the spectrum of A is then a direct consequence of the computations of [Brüning and Seeley 1988].

Closed extensions of D_1. Let $D_{1,\max}$ be the maximal closed extension of D_1, with the domain

$$\operatorname{Dom}(D_{1,\max}) = \{\phi \in L^2(\overline{M}_1) \mid D_1\phi \in L^2(\overline{M}_1)\}.$$

If $\operatorname{Spec}(A) \cap \left(-\frac{1}{2}, \frac{1}{2}\right) = \varnothing$, then $D_{1,\max} = D_{1,\min}$. In particular, D_1 is essentially self-adjoint on the space of smooth forms with compact support away from the conical singularity.

Otherwise, the quotient $\operatorname{Dom}(D_{1,\max}) / \operatorname{Dom}(D_{1,\min})$ is isomorphic to

$$B := \bigoplus_{|\gamma|<\frac{1}{2}} \operatorname{Ker}(A-\gamma).$$

More precisely, by Lemma 3.2 of [Brüning and Seeley 1988], there exists a surjective linear map

$$\mathscr{L} : \operatorname{Dom}(D_{1,\max}) \to B$$

with $\operatorname{Ker}(\mathscr{L}) = \operatorname{Dom}(D_{1,\min})$. Furthermore, we have the estimate

$$\|u(r) - r^{-A}\mathscr{L}(\phi)\|^2_{L^2(\Sigma)} \le C(\phi)|r \log r|$$

for $\phi \in \operatorname{Dom}(D_{1,\max})$ and $u = U(\phi)$.

Now, for any subspace $W \subset B$, we can associate the operator $D_{1,W}$ with $\mathrm{Dom}(D_{1,W}) := \mathcal{L}^{-1}(W)$. As a result of [Brüning and Seeley 1988], all closed extensions of $D_{1,\min}$ are obtained by this way. Note that each $D_{1,W}$ defines a self-adjoint extension $\Delta_{1,W} = (D_{1,W})^* \circ D_{1,W}$ of the Hodge–de Rham operator, and, as a result, we have $(D_{1,W})^* = D_{1,\mathbb{I}(W^\perp)}$, where

$$\mathbb{I} = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}, \quad \text{so} \quad \mathbb{I}\begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}.$$

This extension is associated with the quadratic form $\phi \mapsto \|D\phi\|^2_{L^2}$ on the domain $\mathrm{Dom}(D_{1,W})$.

Finally, we recall the results of [Lesch 1997]. The operators $D_{1,W}$, and in particular $D_{1,\min}$ and $D_{1,\max}$, are elliptic and satisfy the singular estimate (SE) — see [Lesch 1997, p. 54] — so by Proposition 1.4.6 of [Lesch 1997] and the compactness of $\overline{M}_1$, they satisfy the *Rellich property*: the inclusion of $\mathrm{Dom}(D_{1,W})$ into $L^2(\overline{M}_1)$ is compact.

2.3. ***Gauss–Bonnet operator on M_2.*** We know, by the works of Carron [2001a; 2001b], following Atiyah, Patodi and Singer [Atiyah et al. 1975], that the operator D_2 admits a closed extension $\mathfrak{D}_2$ with the domain defined by the global boundary condition

$$\Pi_{\le 1/2} \circ U = 0$$

if Π_I is the spectral projection of A relative to the interval I, and $\le \frac{1}{2}$ denotes the interval $\left(-\infty, \frac{1}{2}\right]$. Moreover, this extension is elliptic in the sense that the H^1-norm of elements of the domain is controlled by the norm of the graph. Indeed, this boundary condition is related to a problem on a complete unbounded manifold as follows:

Let $\widetilde{M}_2$ denote the large manifold obtained from M_2 by gluing a conical cylinder $\mathscr{C}_{1,\infty} = [1, \infty) \times \Sigma$ with metric $dr^2 + r^2 h$ and $\widetilde{D}_2$ its Gauss–Bonnet operator. A differential form on M_2 admits an L^2-harmonic extension on $\widetilde{M}_2$ precisely when the restriction on the boundary satisfies $\Pi_{\le 1/2} \circ U = 0$.

Indeed, from the harmonicity, these L^2-forms must satisfy $(\partial_r + (1/r)A)\sigma = 0$, or, if we decompose the form associated with the eigenspaces of A as $\sigma = \sum_{\gamma \in \mathrm{Spec}(A)} \sigma_\gamma$, then the equation imposes that for all $\gamma \in \mathrm{Spec}(A)$ there exists $\sigma^0_\gamma \in \mathrm{Ker}(A - \gamma)$ such that $\sigma_\gamma = r^{-\gamma}\sigma^0_\gamma$. This expression is in $L^2(\mathscr{C}_{1,\infty})$ if and only if $\gamma > \frac{1}{2}$ or $\sigma^0_\gamma = 0$.

It will be convenient to introduce the L^2-harmonic extension operator

$$P_2 : \Pi_{>1/2}(H^{1/2}(\Sigma)) \to L^2(\Lambda T^* \mathscr{C}_{1,\infty})$$
$$\sigma = \sum_{\substack{\gamma \in \mathrm{Spec}(A) \\ \gamma > \frac{1}{2}}} \sigma_\gamma \mapsto P_2(\sigma) = U^*\Bigg(\sum_{\substack{\gamma \in \mathrm{Spec}(A) \\ \gamma > \frac{1}{2}}} r^{-\gamma} \sigma_\gamma \Bigg).$$

This limit problem is of the category *nonparabolic at infinity* in the terminology of Carron — see particularly Theorem 2.2 of [Carron 2001b] and Proposition 5.1 of [Carron 2001a] — then, as a consequence of Theorem 0.4 of [Carron 2001b], we know that the kernel of $\mathfrak{D}_2$ is of finite dimension and that the graph norm of the operator controls the H^1-norm (Theorem 2.1 of [Carron 2001b]).

Proposition 2. *There exists a constant $C > 0$ such that, for each differential form $\phi \in H^1(\Lambda T^*M_2)$ satisfying the boundary condition $\Pi_{\leq 1/2} \circ U(\phi) = 0$,*

$$\|\phi\|^2_{H^1(M_2)} \leq C\{\|\phi\|^2_{L^2(M_2)} + \|D_2\phi\|^2_{L^2(M_2)}\}.$$

As a consequence, the kernel of $\mathfrak{D}_2$, which is isomorphic to $\operatorname{Ker}(\widetilde{D}_2)$, is of finite dimension and can be mapped into the total space $\sum_p H^p(M_2)$ of the absolute cohomology.

A proof of this proposition can be obtained by the same way as Proposition 5 in [Anné and Takahashi 2012].

Extended solutions. Recall that for this type of operator, behind the L^2-solutions of $\widetilde{D}_2(\phi) = 0$ which correspond to the solutions of the elliptic operator of Proposition 2, Carron defined *extended* solutions which are included in the bigger space $\mathcal{W}$, defined as the closure of the space of smooth p-forms with compact support in $\widetilde{M}_2$ for the norm

$$\|\phi\|^2_{\mathcal{W}} := \|\phi\|^2_{L^2(M_2)} + \|D_2\phi\|^2_{L^2(\widetilde{M}_2)}.$$

A Hardy-type inequality describes the growth at infinity of an extended solution:

Lemma 3. *For a function $v \in C_0^\infty(e, \infty)$ and a real number λ, we have*

$$\left(\lambda + \tfrac{1}{2}\right)^2 \int_e^\infty \frac{v^2}{r^2}\, dr \leq \int_e^\infty \frac{1}{r^{2\lambda}} |\partial_r(r^\lambda v)|^2\, dr \quad \text{if } \lambda \neq -\tfrac{1}{2},$$

$$\frac{1}{4}\int_e^\infty \frac{v^2}{r^2|\log r|^2}\, dr \leq \int_e^\infty r|\partial_r(r^{-1/2}v)|^2\, dr \quad \text{if } \lambda = -\tfrac{1}{2}.$$

We remark now that, for a p-form ϕ with support in the infinite cone $\mathscr{C}_{e,\infty}$, we can write

$$\begin{aligned}\|D_2\phi\|^2_{L^2(\widetilde{M}_2)} &= \sum_{\lambda \in \operatorname{Spec}(A)} \int_e^\infty \left\|\left(\partial_r + \frac{\lambda}{r}\right)\sigma_\lambda\right\|^2_{L^2(\Sigma)} dr \\ &= \sum_{\lambda \in \operatorname{Spec}(A)} \int_e^\infty \frac{1}{r^{2\lambda}} \|\partial_r(r^\lambda \sigma_\lambda)\|^2_{L^2(\Sigma)}\, dr.\end{aligned}$$

Thus, as an application of Lemma 3, we see that a kernel of $\widetilde{D}_2$, which must be $\sigma_\lambda(r) = r^{-\lambda}\sigma_\lambda(1)$ on the infinite cone, satisfies the condition of growth at infinity of Lemma 3. For $\lambda > -\frac{1}{2}$ there is no restriction, since $r^{-2\lambda-2}$ is integrable near ∞ as well as for $\lambda = -\frac{1}{2}$: if $v = r^{1/2}v_0$ for large r then the integral $\int v^2/|r \log r|^2\, dr$ is convergent, so, if we require that $(1/r)\phi$ is in L^2 then, for any $\lambda < -\frac{1}{2}$,

$$\sigma_\lambda(1) = 0.$$

While the L^2-solutions correspond to the condition $\sigma_\lambda(1) = 0$ for any $\lambda \leq \frac{1}{2}$. As a consequence, the extended solutions which are not in L^2 correspond to boundary terms with components in the total eigenspaces related to the eigenvalues of A in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. In the case studied in [Anné and Takahashi 2012], there do not exist such eigenvalues and we had not to take care of extended solutions.

More precisely, we must introduce the Dirac–Neumann operator (see [Carron 2001a, paragraphe 2.a])

$$\begin{aligned} T : H^{k+1/2}(\Sigma) &\to H^{k-1/2}(\Sigma) \\ \sigma &\mapsto U \circ D_2(\mathscr{E}(\sigma))\restriction_\Sigma, \end{aligned} \tag{5}$$

where $\mathscr{E}(\sigma)$ is the solution of the Poisson problem

$$(D_2)^2(\mathscr{E}(\sigma)) = 0 \quad \text{on } M_2 \qquad \text{and} \qquad U \circ \mathscr{E}(\sigma)\restriction_\Sigma = \sigma \quad \text{on } \Sigma.$$

In the same way, one can define

$$\begin{aligned} T_{\mathscr{C}} : H^{k+1/2}(\Sigma) &\to H^{k-1/2}(\Sigma) \\ \sigma &\mapsto U \circ D_2(\widetilde{\mathscr{E}}(\sigma))\restriction_\Sigma, \end{aligned} \tag{6}$$

where $\widetilde{\mathscr{E}}(\sigma)$ is the solution of the Poisson problem

$$(D_2)^2(\widetilde{\mathscr{E}}(\sigma)) = 0 \quad \text{on } \mathscr{C}_{1,\infty} \qquad \text{and} \qquad U \circ \widetilde{\mathscr{E}}(\sigma)\restriction_\Sigma = \sigma \quad \text{on } \Sigma.$$

Then $\operatorname{Im}(T_{\mathscr{C}}) = \operatorname{Im}(\Pi_{>1/2})$ is a subspace of $\operatorname{Ker}(T_{\mathscr{C}}) = \operatorname{Im}(\Pi_{\geq -1/2})$. Carron [2001a] proved that this operator is continuous for $k \geq 0$. The L^2-solutions correspond to the boundary values in $\operatorname{Im}(T) \cap \operatorname{Im}(\Pi_{>1/2})$, while extended solutions correspond to the space $\operatorname{Ker}(T) \cap \operatorname{Im}(\Pi_{\geq -1/2})$. Carron also proved that, in the compact case, $\operatorname{Ker}(T) = \operatorname{Im}(T)$. We can now define the space W that appears in Theorem A:

$$W = \bigoplus_{|\gamma|<\frac{1}{2}} W_\gamma, \quad \text{where} \quad W_\gamma = \{\phi \in \operatorname{Ker}(A-\gamma) \mid \exists \eta \in \operatorname{Im}(\Pi_{>\gamma}) \;\; T(\phi+\eta) = 0\}. \tag{7}$$

Let us denote by

$$\mathscr{I}_{1/2} := (\operatorname{Ker}(T) \cap \operatorname{Im}(\Pi_{\geq 1/2}))/(\operatorname{Ker}(T) \cap \operatorname{Im}(\Pi_{>1/2})) \tag{8}$$

the space of extended solutions with nontrivial component on $\operatorname{Ker}\big(A - \frac{1}{2}\big)$.

Proof of Lemma 3. Let $v \in C_0^\infty(e, \infty)$; by integration by parts and the Cauchy–Schwarz inequality, we obtain, for $\lambda \neq -\frac{1}{2}$,

$$\begin{aligned} \int_e^\infty \frac{v^2}{r^2}\,dr &= \int_e^\infty \frac{1}{r^{2\lambda+2}}|r^\lambda v|^2\,dr = \int_e^\infty \partial_r\left\{\frac{-1}{(2\lambda+1)r^{2\lambda+1}}\right\}|r^\lambda v|^2\,dr \\ &= \int_e^\infty \left\{\frac{1}{(2\lambda+1)r^{2\lambda+1}}\right\} 2(r^\lambda v)\partial_r(r^\lambda v)\,dr = \int_e^\infty \frac{2}{(2\lambda+1)}\frac{v}{r}\cdot r^{-\lambda}\partial_r(r^\lambda v)\,dr \\ &\leq \frac{2}{|2\lambda+1|}\sqrt{\int_e^\infty \frac{v^2}{r^2}\,dr}\cdot\sqrt{\int_e^\infty |r^{-\lambda}\partial_r(r^\lambda v)|^2\,dr}, \end{aligned}$$

which gives directly the first result of Lemma 3.

The second one is obtained in the same way:

$$\begin{aligned}\int_e^\infty \frac{v^2}{r^2|\log r|^2}\,dr &= \int_e^\infty \left(\frac{v}{\sqrt{r}}\right)^2 \frac{1}{r|\log r|^2}\,dr = \int_e^\infty \left(\frac{v}{\sqrt{r}}\right)^2 \partial_r\left(\frac{-1}{\log r}\right) dr \\ &= \int_e^\infty \frac{2v}{\sqrt{r}}\partial_r\left(\frac{v}{\sqrt{r}}\right)\cdot\frac{1}{\log r}\,dr = \int_e^\infty \frac{2v}{r\log r}\cdot\sqrt{r}\partial_r\left(\frac{v}{\sqrt{r}}\right) dr \\ &\le 2\sqrt{\int_e^\infty \frac{v^2}{r^2|\log r|^2}\,dr}\cdot\sqrt{\int_e^\infty \left|\sqrt{r}\partial_r\left(\frac{v}{\sqrt{r}}\right)\right|^2 dr}. \end{aligned}$$ □

3. Notations and tools

Let q_ε be the quadratic form defined on M_ε by the formula (2); to write a form ϕ_ε in $\mathrm{Dom}(q_\varepsilon)$, we use, as in [Anné et al. 2009], the following change of scales:

$$\phi_{1,\varepsilon} := \phi_\varepsilon\restriction_{M_1(\varepsilon)} \quad \text{and} \quad \phi_{2,\varepsilon} := \varepsilon^{m/2-p}\phi_\varepsilon\restriction_{M_2}.$$

We write, on the cone $\mathscr{C}_{\varepsilon,1}$,

$$\phi_{1,\varepsilon} = dr\wedge r^{-(n/2-p+1)}\beta_{1,\varepsilon} + r^{-(n/2-p)}\alpha_{1,\varepsilon}$$

and define $\sigma_{1,\varepsilon} = (\beta_{1,\varepsilon},\alpha_{1,\varepsilon}) = U(\phi_{1,\varepsilon})$.

On the other part, it is more convenient to define $r = 1-s$ for $s\in\left[0,\frac{1}{2}\right]$ and write $\phi_{2,\varepsilon} = dr\wedge r^{-(n/2-p+1)}\beta_{2,\varepsilon} + r^{-(n/2-p)}\alpha_{2,\varepsilon}$ near the boundary. Then we can define, for $r\in\left[\frac{1}{2},1\right]$ (the boundary of M_2 corresponds to $r=1$),

$$\sigma_{2,\varepsilon}(r) = (\beta_{2,\varepsilon}(r),\alpha_{2,\varepsilon}(r)) = U(\phi_{2,\varepsilon})(r).$$

The L^2-norm, for a p-form on M_1 supported in the cone $\mathscr{C}_{\varepsilon,1}$, has the expression

$$\|\phi_\varepsilon\|^2_{L^2(M_\varepsilon)} = \int_{M_1(\varepsilon)} |\sigma_{1,\varepsilon}|^2\,d\mu_{g_1} + \int_{M_2} |\phi_{2,\varepsilon}|^2\,d\mu_{g_2}$$

and the quadratic form in our study is

$$q_\varepsilon(\phi_\varepsilon) = \int_{M_\varepsilon} |(d+d^*)\phi_\varepsilon|^2_{g_\varepsilon}\,d\mu_{g_\varepsilon} = \int_{M_1(\varepsilon)} |UD_1U^*(\sigma_{1,\varepsilon})|^2\,d\mu_{g_1} + \frac{1}{\varepsilon^2}\int_{M_2} |D_2(\phi_{2,\varepsilon})|^2\,d\mu_{g_2}. \tag{9}$$

The compatibility condition for the quadratic form is $\varepsilon^{1/2}\alpha_{1,\varepsilon}(\varepsilon) = \alpha_{2,\varepsilon}(1)$ and $\varepsilon^{1/2}\beta_{1,\varepsilon}(\varepsilon) = \beta_{2,\varepsilon}(1)$, or

$$\sigma_{2,\varepsilon}(1) = \varepsilon^{1/2}\sigma_{1,\varepsilon}(\varepsilon). \tag{10}$$

The compatibility condition for the Hodge–de Rham operator, of the first order, is obtained by expressing that $D\phi_\varepsilon \sim (UD_1U^*\sigma_{1,\varepsilon}, \varepsilon^{-1}UD_2U^*\sigma_{2,\varepsilon})$ belongs to the domain of D. In terms of σ, it gives

$$\sigma'_{2,\varepsilon}(1) = \varepsilon^{3/2}\sigma'_{1,\varepsilon}(\varepsilon). \tag{11}$$

To understand the limit problem, we proceed to several estimates.

3.1. ***Expression of the quadratic form.*** For any ϕ such that the component ϕ_1 is supported in the cone $\mathscr{C}_{\varepsilon,1}$, one has, with $\sigma_1 = U(\phi_1)$ and by the same calculus as in [Anné et al. 2009]:

$$\int_{\mathscr{C}_{\varepsilon,1}} |D_1\phi|^2\, d\mu_{g_\varepsilon} = \int_\varepsilon^1 \left\| \left(\partial_r + \frac{1}{r}A\right)\sigma_1 \right\|^2_{L^2(\Sigma)} dr = \int_\varepsilon^1 \left[\|\sigma_1'\|^2_{L^2(\Sigma)} + \frac{2}{r}(\sigma_1', A\sigma_1)_{L^2(\Sigma)} + \frac{1}{r^2}\|A\sigma_1\|^2_{L^2(\Sigma)} \right] dr.$$

3.2. ***Limit problem.*** As a Hilbert space, we introduce

$$\mathscr{H}_\infty := L^2(\overline{M}_1) \oplus \mathrm{Ker}(\widetilde{D}_2) \oplus \mathscr{I}_{1/2} \tag{12}$$

with the space $\mathscr{I}_{1/2}$ as defined in (8), and the limit operator

$$\Delta_{1,W} \oplus 0 \oplus 0$$

with W as defined in (7).

Finally, let us define:

- *A cut-off function* ξ_1 on M_1 around the conical singularity,

$$\xi_1(r) = \begin{cases} 1 & \text{if } 0 \le r \le \frac{1}{2}, \\ 0 & \text{if } 1 \le r. \end{cases} \tag{13}$$

- *The prolongation operator*

$$\begin{gathered} P_\varepsilon : H^{1/2}(\Sigma) \to H^1(\mathscr{C}_{\varepsilon,1}) \\ \sigma = \sum_{\gamma \in \mathrm{Spec}(A)} \sigma_\gamma \mapsto P_\varepsilon(\sigma) = U^*\Bigg(\sum_{\gamma \in \mathrm{Spec}(A)} \varepsilon^{\gamma - 1/2} r^{-\gamma} \sigma_\gamma \Bigg). \end{gathered} \tag{14}$$

We remark that, restricted to $\mathrm{Im}(\Pi_{>1/2})$, $P_\varepsilon(\sigma)$ is the transplant on $M_1(\varepsilon)$ of $P_2(\sigma)$ (see Section 2.3); then there exists a constant $C > 0$ such that, for all $\sigma \in \mathrm{Im}(\Pi_{>1/2})$,

$$\|P_2(\sigma)\|^2_{L^2(\mathscr{C}_{1,1/\varepsilon})} = \|P_\varepsilon(\sigma)\|^2_{L^2(\mathscr{C}_{\varepsilon,1})} \le C \sum_{\gamma > \frac{1}{2}} \|\sigma_\gamma\|^2_{L^2(\Sigma)} = C\|\sigma\|^2_{L^2(\Sigma)}, \tag{15}$$

and also that, if $\psi_2 \in \mathrm{Dom}(\mathscr{D}_2)$, then $\big(\xi_1 P_\varepsilon(U(\psi_2 \restriction_\Sigma)), \psi_2\big)$ defines an element of $H^1(M_\varepsilon)$.

4. Proof of the spectral convergence

We denote by $\lambda_N(\varepsilon)$, $N \ge 1$, the spectrum of the total Hodge–de Rham operator of M_ε and by λ_N, $N \ge 1$, the spectrum of the limit operator defined in Section 3.2.

4.1. ***Upper bound:*** $\boldsymbol{\limsup_{\varepsilon\to 0} \lambda_N(\varepsilon) \le \lambda_N}$. With the min–max formula, which says that

$$\lambda_N(\varepsilon) = \inf_{\substack{E \subset \mathrm{Dom}(D_\varepsilon) \\ \dim E = N}} \inf \left\{ \sup_{\substack{\phi \in E \\ \|\phi\| = 1}} \int_{M_\varepsilon} |D_\varepsilon \phi|^2_{g_\varepsilon}\, d\mu_{g_\varepsilon} \right\},$$

we have to describe how to transplant eigenforms of the limit problem on M_ε.

We describe this transplantation term by term. For the first term, we use the same ideas as in [Anné et al. 2009].

For an eigenform ϕ of $\Delta_{1,W}$ corresponding to the eigenvalue λ, $U(\phi)$ can be decomposed on an orthonormal base $\{\sigma_\gamma\}_\gamma$ of eigenforms of A and each component can be expressed by the Bessel functions. For $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, it has the form

$$\{c_\gamma r^{\gamma+1} F_\gamma(\lambda r^2) + d_\gamma r^{-\gamma} G_\gamma(\lambda r^2)\}\sigma_\gamma,$$

where F_γ, G_γ are entire functions satisfying $F_\gamma(0) = G_\gamma(0) = 1$ and c_γ, d_γ are constants.

We remark that $c_\gamma r^{\gamma+1} F_\gamma(\lambda r^2)\sigma_\gamma \in \mathrm{Dom}(D_{1,\min})$ and $d_\gamma r^{-\gamma}(G_\gamma(\lambda r^2) - G_\gamma(0))\sigma_\gamma \in \mathrm{Dom}(D_{1,\min})$. So we can write $\phi = \phi_0 + \bar{\phi}$ with

$$\phi_0 \in \mathrm{Dom}(D_{1,\min}) \quad \text{and} \quad U(\bar{\phi})(r) = \xi_1(r) \sum_{\substack{\gamma \in \mathrm{Spec}(A) \\ |\gamma| < \frac{1}{2}}} d_\gamma r^{-\gamma} \sigma_\gamma.$$

By the definition of $D_{1,\min}$, ϕ_0 can be approached, with the operator norm, by a sequence of smooth forms $\phi_{0,\varepsilon}$ with compact support in $M_1(\varepsilon)$.

By the definition of W, we know that $\sum_{|\gamma|<1/2} d_\gamma \sigma_\gamma \in W$. So there exists $\phi_{2,\gamma} \in \mathrm{Ker}(D_2)$ such that $U(\phi_{2,\gamma}(1)) - d_\gamma \sigma_\gamma \in \mathrm{Im}(\Pi_{>\gamma})$. We remark finally that, by the definition (14), we can write $U(\bar{\phi})(r) = \xi_1(r) \sum_{|\gamma|<1/2} \varepsilon^{1/2-\gamma} P_\varepsilon(d_\gamma \sigma_\gamma)$.

Let $\phi_{2,\varepsilon} = \sum_{|\gamma|<1/2} \varepsilon^{1/2-\gamma} \phi_{2,\gamma}$ and

$$\phi_\varepsilon = \left(\phi_{0,\varepsilon} + \xi_1 P_\varepsilon\left(\sum_{\substack{\gamma \in \mathrm{Spec}(A) \\ |\gamma| < \frac{1}{2}}} \varepsilon^{1/2-\gamma} U(\phi_{2,\gamma}(1))\right), \phi_{2,\varepsilon}\right) \in H^1(M_\varepsilon).$$

It is a good transplantation: $\|\phi_{2,\varepsilon}\| \to 0$ as the term added on $M_1(\varepsilon)$ (indeed, a term of the sum $\xi_1 \varepsilon^{1/2-\gamma} P_\varepsilon(U\phi_{2,\gamma}(1) - d_\gamma \sigma_\gamma)$ corresponds to some $\gamma' > \gamma$; if $\gamma' > \frac{1}{2}$ it is $O(\varepsilon^{1/2-\gamma})$ by (15), if $\gamma' < \frac{1}{2}$ it is $O(\varepsilon^{\gamma'-\gamma})$, and if $\gamma' = \frac{1}{2}$ it is $O(\varepsilon^{1/2-\gamma}\sqrt{|\log \varepsilon|})$). Moreover, they are harmonic, up to ξ_1.

For the two last ones, we shrink the infinite cone on M_1 and cut with the function ξ_1, already defined in (13).

Finally, if $\mathrm{Ker}\left(A - \frac{1}{2}\right) \neq \{0\}$, then, for each nonzero element $[\bar{\sigma}^{1/2}] \in \mathcal{I}_{1/2}$, there exists ψ_2 with $D_2(\psi_2) = 0$ on M_2 that has the boundary value $\bar{\sigma}^{1/2}$ modulo $\mathrm{Im}(\Pi_{>1/2})$. Then, we can construct a *quasimode* as follows:

$$\psi_\varepsilon := |\log \varepsilon|^{-1/2}\left(\xi_1.\left\{r^{-1/2} U^*(\bar{\sigma}^{1/2}) + P_\varepsilon(U(\psi_2)\restriction_\Sigma - \bar{\sigma}^{1/2})\right\}, \psi_2\right). \tag{16}$$

The L^2-norm of this element is uniformly bounded from above and below, and

$$\lim_{\varepsilon \to 0} \|\psi_\varepsilon\|_{L^2(M_\varepsilon)} = \|\bar{\sigma}^{1/2}\|_{L^2(\Sigma)}.$$

Moreover, it satisfies $q(\psi_\varepsilon) = O(|\log \varepsilon|^{-1})$, giving then a "small eigenvalue", as well as the elements of $\mathrm{Ker}(\mathcal{D}_2)$ and of $\mathrm{Ker}(\Delta_{1,W})$.

Note, as an aside, that it is remarkable that the same construction, for an extended solution with corresponding boundary value in $\mathrm{Ker}(A - \gamma)$, $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, does not give a quasimode: indeed, if ψ_2 is such a solution, the transplanted element will be

$$\psi_\varepsilon = \big(\xi_1.\big\{r^{-\gamma}U^*(\bar{\sigma}^\gamma) + \varepsilon^{1/2-\gamma}P_\varepsilon(U(\psi_2)\restriction_\Sigma - \bar{\sigma}^\gamma)\big\}, \varepsilon^{1/2-\gamma}\psi_2\big),$$

for which $q(\psi_\varepsilon)$ does not converge to 0 as $\varepsilon \to 0$.

To conclude the estimate of the upper bounds, we have only to verify that these transplanted forms have a Rayleigh–Ritz quotient comparable to the initial one and that the orthogonality is almost conserved by transplantation.

4.2. ***Lower bound:*** $\boldsymbol{\liminf_{\varepsilon\to 0}\lambda_N(\varepsilon) \geq \lambda_N}$**.** We first proceed for one index. We know, by Section 4.1, that for each N the family $\{\lambda_N(\varepsilon)\}_{\varepsilon>0}$ is bounded; set

$$\lambda := \liminf_{\varepsilon\to 0}\lambda_N(\varepsilon).$$

There exists a sequence $\{\varepsilon_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty}\lambda_N(\varepsilon_i) = \lambda$. For each i, let ϕ_i be a normalized eigenform relative to $\lambda_i = \lambda_N(\varepsilon_i)$.

4.2.1. *On the regular part of* $\overline{M}_1$.

Lemma 4. *For our given family* ϕ_i, *the family* $\{(1-\xi_1).\phi_{1,i}\}_{i\in\mathbb{N}}$ *is bounded in* $H_0^1(M_1(0), g_1)$.

Then it remains to study $\xi_1.\phi_{1,i}$, which can be expressed with the polar coordinates. We remark that the quadratic form of these forms is uniformly bounded.

4.2.2. *Estimates of the boundary term.* The expression above can be decomposed with respect to the eigenspaces of A; in the following calculus, we suppose that $\sigma_1(1) = 0$:

$$\begin{aligned}
&\int_\varepsilon^1 \left[\|\sigma_1'\|^2_{L^2(\Sigma)} + \frac{2}{r}(\sigma_1', A\sigma_1)_{L^2(\Sigma)} + \frac{1}{r^2}\|A\sigma_1\|^2_{L^2(\Sigma)}\right] dr \\
&\qquad = \int_\varepsilon^1 \left[\|\sigma_1'\|^2_{L^2(\Sigma)} + \partial_r\left(\frac{1}{r}(\sigma_1, A\sigma_1)_{L^2(\Sigma)}\right) + \frac{1}{r^2}\big\{(\sigma_1, A\sigma_1)_{L^2(\Sigma)} + \|A\sigma_1\|^2_{L^2(\Sigma)}\big\}\right] dr \\
&\qquad = \int_\varepsilon^1 \left[\|\sigma_1'\|^2_{L^2(\Sigma)} + \frac{1}{r^2}(\sigma_1, (A+A^2)\sigma_1)_{L^2(\Sigma)}\right] dr - \frac{1}{\varepsilon}(\sigma_1(\varepsilon), A\sigma_1(\varepsilon))_{L^2(\Sigma)}.
\end{aligned}$$

This shows that the quadratic form controls the boundary term if the operator A is negative but $(A + A^2)$ is nonnegative. The latter condition is satisfied exactly on the orthogonal complement of the spectral space corresponding to the interval $(-1, 0)$. By applying $\xi_1.\phi_{1,i}$ to this fact, we obtain the following lemma:

Lemma 5. *Let* $\Pi_{\leq -1}$ *be the spectral projection of the operator* A *relative to the interval* $(-\infty, -1]$. *There exists a constant* $C > 0$ *such that, for any* $i \in \mathbb{N}$,

$$\|\Pi_{\leq -1} \circ U(\phi_{1,i}(\varepsilon_i))\|_{H^{1/2}(\Sigma)} \leq C\sqrt{\varepsilon_i}.$$

In view of Proposition 2, we also want a control of the components of σ_1 associated with the eigenvalues of A in $\left(-1, \frac{1}{2}\right]$. The number of these components is finite and we can work term by term. So we write, on $\mathscr{C}_{\varepsilon,1}$,

$$\sigma_1(r) = \sum_{\gamma \in \mathrm{Spec}(A)} \sigma_1{}^\gamma(r) \quad \text{with} \quad A\sigma_1{}^\gamma(r) = \gamma\sigma_1{}^\gamma(r)$$

and we suppose again $\sigma_1(1) = 0$. From the equation $(\partial_r + A/r)\sigma_1^\gamma = r^{-\gamma}\partial_r(r^\gamma\sigma_1^\gamma)$ and the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}\|\varepsilon^\gamma\sigma_1^\gamma(\varepsilon)\|^2_{L^2(\Sigma)} &= \left\|\int_\varepsilon^1 \partial_r(r^\gamma\sigma_1^\gamma)\,dr\right\|^2_{L^2(\Sigma)}\\ &\le \left\{\int_\varepsilon^1 \left\|r^\gamma\cdot\left(\partial_r + \frac{1}{r}A\right)\sigma_1^\gamma(r)\right\|_{L^2(\Sigma)} dr\right\}^2\\ &\le \int_\varepsilon^1 r^{2\gamma}\,dr\cdot\int_{\varepsilon_i}^1 \left\|\partial_r(\sigma_1^\gamma) + \frac{\gamma}{r}(\sigma_1^\gamma)\right\|^2_{L^2(\Sigma)} dr.\end{aligned}$$

Thus, if the quadratic form is bounded, there exists a constant $C > 0$ such that

$$\|\sigma_1^\gamma(\varepsilon)\|^2_{L^2(\Sigma)} \le \begin{cases} C\varepsilon^{-2\gamma}(1-\varepsilon^{2\gamma+1})/(2\gamma+1) & \text{if } \gamma \ne -\frac{1}{2},\\ C\varepsilon|\log\varepsilon| & \text{if } \gamma = -\frac{1}{2}.\end{cases} \tag{17}$$

This gives:

Lemma 6. *Let Π_I be the spectral projection of the operator A relative to the interval I. There exist constants α, $C > 0$ such that, for any $i \in \mathbb{N}$,*

$$\|\Pi_{(-1,0)} \circ U(\phi_{1,i}(\varepsilon_i))\|_{H^{1/2}(\Sigma)} \le C\varepsilon_i^\alpha.$$

Here, $0 < \alpha < \frac{1}{2}$ satisfies that $-\alpha$ is larger than any negative eigenvalue of A.

With the compatibility condition (10) and the ellipticity of A, the estimate above gives also:

Lemma 7. *With the same notation, there exist constants β, $C > 0$ such that, for any $i \in \mathbb{N}$*

$$\|\Pi_{[0,1/2)} \circ U(\phi_{2,i}(1))\|_{H^{1/2}(\Sigma)} \le C\varepsilon_i^\beta.$$

Here, $\frac{1}{2} - \beta$ is the largest nonnegative eigenvalue of A strictly smaller than $\frac{1}{2}$ (if there is no such eigenvalue, we put $\beta = \frac{1}{2}$).

Finally, we study $\sigma_1^{1/2}$ for our family of forms (the parameter i is omitted in the notation). It satisfies, for $\varepsilon_i < r < \frac{1}{2}$, the equation

$$\left(-\partial_r^2 + \frac{3}{4r^2}\right)\sigma_1^{1/2} = \lambda_i\sigma_1^{1/2}.$$

The solutions of this equation can be expressed in terms of the Bessel and the Neumann functions: there exist entire functions F, G with $F(0) = G(0) = 1$ and differential forms c_i, d_i in $\mathrm{Ker}\big(A - \frac{1}{2}\big)$ such that

$$\sigma_1^{1/2}(r) = c_i r^{3/2}F(\lambda_i r^2) + d_i\left\{r^{-1/2}G(\lambda_i r^2) + \frac{2}{\pi}\log(r)r^{3/2}F(\lambda_i r^2)\right\} \tag{18}$$

(see [Anné et al. 2009, Lemma 4]). The fact that the L^2-norm is bounded gives that $\|c_i\|^2_{L^2}+|\log\varepsilon_i|\,\|d_i\|^2_{L^2}$ is bounded. Finally, by substituting this estimate in the expression above, we have

$$\|\sigma_1^{1/2}(\varepsilon_i)\|^2_{L^2(\Sigma)} = O\left(\frac{1}{\varepsilon_i|\log\varepsilon_i|}\right).$$

With the compatibility condition (10), we obtain:

Lemma 8. *There exists a constant $C>0$ such that, for any $i\in\mathbb{N}$,*

$$\|\Pi_{\{1/2\}}\circ U(\phi_{2,i})(1)\|_{H^{1/2}(\Sigma)} \le \frac{C}{\sqrt{|\log\varepsilon_i|}}.$$

4.2.3. *Convergence of $\phi_{2,i}$.* Let us now define, in general, $\tilde\phi_{2,\varepsilon}$ as the form obtained by the prolongation of $\phi_{2,\varepsilon}$ by $\sqrt{\varepsilon}\xi_1(\varepsilon r)\phi_{1,\varepsilon}(\varepsilon r)$ on the infinite cone $\mathscr{C}_{1,\infty}$. A change of variables gives that

$$\|\tilde\phi_{2,\varepsilon}\|_{L^2(\mathscr{C}_{1,\infty})} = \|\xi_1\phi_{1,\varepsilon}\|_{L^2(\mathscr{C}_{\varepsilon,1})},$$

while

$$\int_{\widetilde M_2}|\widetilde D_2(\tilde\phi_{2,\varepsilon})|^2\,d\mu = \varepsilon^2\int_{\mathscr{C}_{\varepsilon,1}}|D_1(\xi_1\phi_{1,\varepsilon})|^2\,d\mu_{g_1} + \int_{M_2}|D_2(\phi_{2,\varepsilon})|^2\,d\mu_{g_2}.$$

Thus, by the definition of ϕ_i, the family $\{\tilde\phi_{2,i}\}_{i\in\mathbb{N}}$ is bounded in $\mathscr{W}$ and $\int_{\mathscr{C}_{1,\infty}}|\widetilde D_2(\tilde\phi_{2,i})|^2\,d\mu = O(\varepsilon_i^2)$. The work of Carron [2001b] gives us that $\|\tilde\phi_{2,i}(1)\|_{H^{1/2}(\Sigma)}$ is bounded and the following:

Proposition 9. *There exists a subfamily of the family $\{\tilde\phi_{2,i}\}_{i\in\mathbb{N}}$ which converges in $L^2(M_2,g_2)$. Its limit $\tilde\phi_2$ defines an extended solution on $\widetilde M_2$, i.e., $\widetilde D_2(\tilde\phi_2)=0$ and $\tilde\phi_2\restriction_\Sigma\in\operatorname{Ker}(T)\cap\operatorname{Im}(\Pi_{\ge -1/2})$.*

We still denote by $\tilde\phi_{2,i}$ the subfamily obtained.

4.2.4. *Convergence near the singularity.* Now we use the fact that eigenforms satisfy an equation which imposes a local form. We concentrate on $\gamma\in\left[-\frac12,\frac12\right]$. If we write

$$\phi_{1,i}^{[-1/2,1/2]} = \sum_{\gamma\in[-1/2,1/2]} U^*\sigma_1^\gamma(r),$$

the terms σ_1^γ satisfy the equations

$$\left(-\partial_r^2+\frac{\gamma(1+\gamma)}{r^2}\right)\sigma_1^\gamma = \lambda_i\sigma_1^\gamma.$$

The solutions of this equation can be expressed in term of the Bessel functions: there exist entire functions F, G with $F(0)=G(0)=1$ and differential forms $c_{\gamma,i}$, $d_{\gamma,i}$ in $\operatorname{Ker}(A-\gamma)$ such that

$$\sigma_1^\gamma(r) = \begin{cases} c_{\gamma,i}r^{\gamma+1}F_\gamma(\lambda_ir^2)+d_{\gamma,i}(r^{-\gamma}G_\gamma(\lambda_ir^2)), & |\gamma|<\frac12,\\ c_{1/2,i}r^{3/2}F_{1/2}(\lambda_ir^2)+d_{1/2,i}(r^{-1/2}G_{1/2}(\lambda_ir^2)+\dfrac{2}{\pi}\log(r)r^{3/2}F_{1/2}(\lambda_ir^2)), & \gamma=\frac12,\\ c_{-1/2,i}r^{1/2}F_{-1/2}(\lambda_ir^2)+d_{-1/2,i}(r^{1/2}\log(r)G_{-1/2}(\lambda_ir^2)), & \gamma=-\frac12.\end{cases} \tag{19}$$

The lemmas of the previous subsections give us the result that the families $c_{\gamma,i}$ and $d_{\gamma,i}$ are bounded and, by extraction, we can suppose that they converge. In the case of $\gamma=\frac12$, we have more: $\|d_{1/2,i}\|_{L^2(\Sigma)} = O(|\log\varepsilon_i|^{-1/2})$.

But, turning back to the family of the last proposition, we also know that the family $\sqrt{\varepsilon_i}\xi_1(\varepsilon_i r)\phi_{1,i}(\varepsilon_i r)$ converges to 0 on any sector $1 \le r \le R$, according to the explicit form of $\sigma_1^\gamma(r)$. As a consequence, the form $\tilde{\phi}_2$ has no component for $\gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and is indeed an L^2-solution. We have proved:

Proposition 10. *The form $\tilde{\phi}_2$ in Proposition 9 has no component for $\gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. If we set $\phi_2 := \tilde{\phi}_2 \restriction_{M_2}$, there exists a subfamily of $\{\phi_{2,i}\}_i$ which converges to ϕ_2 as $i \to \infty$, and it satisfies*

$$\phi_2 \in \mathrm{Dom}(\mathfrak{D}_2), \quad \|\phi_2\|_{L^2(M_2,g_2)} \le 1 \quad \textit{and} \quad D_2(\phi_2) = 0.$$

Moreover, the harmonic prolongation of $\sqrt{\varepsilon_i}\xi_1(\varepsilon_i r)\phi_{1,i}(\varepsilon_i r)$,

$$\bar{\phi}_{2,i} = \mathscr{E}(\sqrt{\varepsilon_i}\xi_1(\varepsilon_i r)\phi_{1,i}(\varepsilon_i r)),$$

minimizes the norm of $D_2(\phi_2)$. As a consequence, $\|D_2(\bar{\phi}_{2,i})\|_{L^2(M_2)} = O(\varepsilon_i)$ implies

$$\|T(\sqrt{\varepsilon_i}\phi_{1,i}(\varepsilon_i))\|_{H^{-1/2}(\Sigma)} = O(\varepsilon_i)$$

with the Dirac–Neumann operator T defined in (5).

But, by Lemmas 5 and 6, we know that $\|\Pi_{<-1/2}(\phi_{1,i}(\varepsilon))\|_{H^{1/2}(\Sigma)} = O(\sqrt{\varepsilon})$. The continuity of T thus gives $\|T \circ \Pi_{\ge -1/2}(\phi_{1,i}(\varepsilon_i))\|_{H^{-1/2}(\Sigma)} = O(\sqrt{\varepsilon_i})$. To obtain consequences of this result for the term $\Pi_{[-1/2,1/2]}(\phi_{1,i}(\varepsilon_i))$, we must make sense of the possibility of working modulo $\mathrm{Im}(T)$. In the following, for simplicity of notation, we identify the spectral projection Π_I of A for the interval I with $U^*\Pi_I U$.

Proposition 11. *The space $T(\mathrm{Im}(\Pi_{>1/2}) \cap H^{1/2}(\Sigma))$ is closed in $H^{-1/2}(\Sigma)$, as a consequence of the work of Carron. Let us define $B(\phi)$ for $\phi \in \mathrm{Im}(\Pi_{[-1/2,1/2]})$ as the orthogonal projection of $T(\phi)$ onto the orthogonal complement of this space. Then B is linear and satisfies*:

- $\|B\phi\|_{H^{-1/2}(\Sigma)} \le \|T\phi\|_{H^{-1/2}(\Sigma)}$.
- *If $B(\phi) = 0$, there exists an $\eta \in \mathrm{Im}(\Pi_{>1/2})$ such that $T(\phi + \eta) = 0$.*

Proof. To prove that $T(\mathrm{Im}(\Pi_{>1/2}) \cap H^{1/2}(\Sigma))$ is closed, we must recall some facts contained in [Carron 2001a]. Let us denote here $T_{\mathscr{C}}$ the operator constructed as T, but for the infinite part $\mathscr{C}_{1,\infty}$. Then $\mathrm{Im}(T_{\mathscr{C}}) = \mathrm{Im}(\Pi_{>1/2})$ is a subspace of $\mathrm{Ker}(T_{\mathscr{C}}) = \mathrm{Im}(\Pi_{\ge -1/2})$. We know that $T + T_{\mathscr{C}}$ is an elliptic operator of order 1 on Σ which is compact. As a consequence, $\mathrm{Ker}(T + T_{\mathscr{C}})$ is finite-dimensional, $(T + T_{\mathscr{C}})(H^{1/2}(\Sigma))$ is a closed subspace of $H^{-1/2}(\Sigma)$ and $T + T_{\mathscr{C}}$ admits a continuous parametrix $Q : H^{-1/2}(\Sigma) \to H^{1/2}(\Sigma)$ such that

$$Q \circ (T + T_{\mathscr{C}}) = \mathrm{Id} - \Pi_{\mathrm{Ker}(T+T_{\mathscr{C}})},$$

where $\Pi_{\mathrm{Ker}(T+T_{\mathscr{C}})}$ denotes the orthogonal projection onto $\mathrm{Ker}(T + T_{\mathscr{C}})$ for the inner product of $H^{1/2}(\Sigma)$. We can now prove that $T(\mathrm{Im}\,\Pi_{>1/2} \cap H^{1/2}(\Sigma))$ is closed.

Let $\{\sigma_i\}_i$ be a sequence of elements in $\mathrm{Im}(\Pi_{>1/2}) \cap H^{1/2}(\Sigma)$ such that $T(\sigma_i)$ converges, and let $\psi = \lim_{i\to\infty} T(\sigma_i)$. We can suppose that

$$\sigma_i \in (\mathrm{Ker}(T) \cap \mathrm{Im}(\Pi_{>1/2}) \cap H^{1/2}(\Sigma))^\perp.$$

We have $\operatorname{Im}(\Pi_{>1/2})\cap H^{1/2}(\Sigma)\subset \operatorname{Ker}(T_{\mathscr{C}})$. Then $(T+T_{\mathscr{C}})\sigma_i = T(\sigma_i)$ converges and $\tau_i = Q\circ(T+T_{\mathscr{C}})\sigma_i$ converges; let $\tau=\lim_{i\to\infty}\tau_i$. Thus,

$$\sigma_i=\tau_i+e_i \quad\text{with}\quad \tau_i\in\operatorname{Ker}(T+T_{\mathscr{C}})^{\perp},\ e_i\in\operatorname{Ker}(T+T_{\mathscr{C}}).$$

The sequence $\{e_i\}_i$ must be bounded, unless we can extract a subsequence $\|e_i\|\to\infty$, so it is true also for $\|\sigma_i\|$ and, by extraction, we can suppose that the bounded sequence $e_i/\|\sigma_i\|$ converges, since it lives in a finite-dimensional space. Let e' be this limit; then $e'=\lim e_i/\|\sigma_i\|$ also and $e'\in\operatorname{Im}(\Pi_{>1/2})\cap H^{1/2}(\Sigma)$.

Finally, e' satisfies $\|e'\|=1$, and

$$e'\in\operatorname{Ker}(T+T_{\mathscr{C}}) \quad\text{and}\quad e'\in\operatorname{Ker}(T_{\mathscr{C}}),$$

as well as e_i and σ_i, which implies $T(e')=0$. Thus, $e'=\lim\sigma_i/\|\sigma_i\|\in\operatorname{Im}(\Pi_{>1/2})\cap H^{1/2}(\Sigma)\cap\operatorname{Ker}(T)$. But, by the assumption on σ_i, e' must be orthogonal to this space, which is a contradiction.

So, e_i is a bounded sequence in a finite-dimensional space; by extraction, we can suppose that it converges. Then σ_i admits a convergent subsequence, and let σ denote its limit; then

$$\sigma\in\operatorname{Im}(\Pi_{>1/2})\cap H^{1/2}(\Sigma) \quad\text{and}\quad \psi=T(\sigma). \qquad \square$$

As an application of Proposition 11, we have

$$\|B\circ\Pi_{[-1/2,1/2]}(\phi_{1,i}(\varepsilon_i))\|_{H^{-1/2}(\Sigma)}=O(\sqrt{\varepsilon_i}).$$

This is the sum of few terms. We remark that the term with $c_{\gamma,i}$ is in fact always $O(\sqrt{\varepsilon_i})$. For the same reason, we can freeze the function G at 0, where its value is 1. So we can say

$$\left\|\varepsilon_i^{1/2}\log(\varepsilon_i)B\circ U^*(d_{-1/2,i})+\sum_{|\gamma|<\frac12}\varepsilon_i^{-\gamma}B\circ U^*(d_{\gamma,i})+\varepsilon_i^{-1/2}B\circ U^*(d_{1/2,i})\right\|_{H^{-1/2}(\Sigma)}=O(\sqrt{\varepsilon_i}), \quad (20)$$

while all the other terms, which behave like r^{δ} with $\delta>\frac12$, occur in an expression belonging to $\operatorname{Dom}(D_{1,\min})$.

In fact, we have the following result:

Proposition 12. *One can write* $\Pi_{(-1/2,1/2]}\circ U(\xi_1\phi_{1,i})=\bar\sigma_{1,i}+\sigma_{0,i}$ *with the bounded sequence* $U^*(\sigma_{0,i})$ *in* $\operatorname{Dom}(D_{1,\min})$ *and* $\bar\sigma_{1,i}=\bar\sigma_{1,i}^{<1/2}+\bar\sigma_{1,i}^{1/2}$ *satisfies that there exists a subfamily of* $\bar\sigma_{1,i}^{<1/2}$ *which converges to* $\sum_{\gamma\in(-1/2,1/2)}r^{-\gamma}\sigma_\gamma$ *as* $i\to\infty$ *with* $\sum_{\gamma\in(-1/2,1/2)}\sigma_\gamma\in W$*, while*

$$\bar\sigma_{1,i}^{1/2}\sim\frac{1}{\sqrt{|\log\varepsilon_i|}}r^{-1/2}\bar\sigma_{1/2}\quad\textit{for some }\bar\sigma_{1/2}\in\operatorname{Ker}\big(A-\tfrac12\big).$$

Thus, $\bar\sigma_{1,i}^{1/2}$ *concentrates on the singularity.*

Proof. The term $\bar\sigma_{1,i}$ comes from the expression obtained in (20), while $\sigma_{0,i}$ is the sum of all the other terms.

We then concentrate on (20). First, we gather the terms concerning the same eigenvalue and still denote by $d_{\gamma,i}$ the sum of all the terms with the same eigenvalue. Let $-\frac12\le\gamma_p<\cdots<\gamma_0\le\frac12$ be the eigenvalues of A in $\left[-\frac12,\frac12\right]$.

We then define the limit d_γ as

$$d_\gamma := \begin{cases} \lim_{i\to\infty} d_{\gamma,i}, & \gamma \neq \frac{1}{2}, \\ \lim_{i\to\infty} \sqrt{|\log \varepsilon_i|}\, d_{1/2,i}, & \gamma = \frac{1}{2}, \end{cases}$$

and put $E_\gamma = \operatorname{Ker}(A-\gamma)$.

Indeed, we can, step by step, decompose $d_{\gamma,i}$ into a part in $\operatorname{Ker}(B\circ U^*)$ and a part which exhibits a smaller behavior in ε_i.

• *First step: in $E_{1/2}$.* Multiplying (20) by $\sqrt{\varepsilon_i}$, we obtain that $\|B\circ U^*(d_{1/2,i})\|_{H^{-1/2}(\Sigma)} = O(\varepsilon_i^{1/2-\gamma_1})$. We decompose $d_{1/2,i} = (1/\sqrt{|\log \varepsilon_i|})d^{(0)}_{1/2,i} + d^{\perp}_{1/2,i}$ along $\operatorname{Ker}(B\circ U^*\restriction_{E_{1/2}})$ and its orthogonal complement in $E_{1/2}$. Then, $\|B\circ U^*(d_{1/2,i})\|_{H^{-1/2}(\Sigma)} = O(\varepsilon_i^{1/2-\gamma_1})$ implies $\|d^{\perp}_{1/2,i}\|_{H^{1/2}(\Sigma)} = O(\varepsilon_i^{1/2-\gamma_1})$. So,

$$d_{1/2} = \lim_{i\to\infty} \sqrt{|\log \varepsilon_i|}\, d_{1/2,i} = \lim_{i\to\infty} d^{(0)}_{1/2,i} \in \operatorname{Ker}(B\circ U^*)$$

and, if we write $d^{\perp}_{1/2,i} = \varepsilon_i^{1/2-\gamma_1} d_i^{(1)}$ and reintroduce this in (20), then it has the new expression

$$\left\| \varepsilon_i^{1/2} \log(\varepsilon_i) B\circ U^*(d_{-1/2,i}) + \sum_{j=2}^{p} \varepsilon_i^{-\gamma_j} B\circ U^*(d_{\gamma_j,i}) + \varepsilon_i^{-\gamma_1} B\circ U^*(d_i^{(1)} + d_{\gamma_1,i}) \right\|_{H^{-1/2}(\Sigma)} = O(\sqrt{\varepsilon_i}).$$

• *Second step: in $E_{1/2}\oplus E_{\gamma_1}$.* Multiplying by $\varepsilon_i^{\gamma_1}$ in the above, we obtain that

$$\|B\circ U^*(d_i^{(1)} + d_{\gamma_1,i})\|_{H^{-1/2}(\Sigma)} = O(\varepsilon_i^{\gamma_1-\gamma_2}). \tag{21}$$

We decompose $d_i^{(1)} + d_{\gamma_1,i} = d^{(0)}_{\gamma_1,i} + d^{\perp}_{\gamma_1,i}$ along $\operatorname{Ker}(B\circ U^*\restriction_{E_{1/2}\oplus E_{\gamma_1}})$ and its orthogonal complement in $E_{1/2}\oplus E_{\gamma_1}$.

Now, (21) says that $\|d^{\perp}_{\gamma_1,i}\|_{H^{1/2}(\Sigma)} = O(\varepsilon_i^{\gamma_1-\gamma_2})$, so $d_{\gamma_1} = \lim_{i\to\infty} d_{\gamma_1,i} = \lim_{i\to\infty} \Pi_{\{\gamma_1\}}(d^{(0)}_{\gamma_1,i})$ and, as $d^{(0)}_{\gamma_1,i} \in \operatorname{Ker}(B\circ U^*\restriction_{E_{1/2}\oplus E_{\gamma_1}})$, extracting from $\Pi_{\{1/2\}}(d^{(0)}_{\gamma_1,i})$ a convergent subsequence, we can say that there exists an $e_{1/2}\in E_{1/2}$ such that

$$d_{\gamma_1} + e_{1/2} \in \operatorname{Ker}(B\circ U^*).$$

On the other hand, if we can write

$$d^{\perp}_{\gamma_1,i} = \varepsilon_i^{\gamma_1-\gamma_2} d_i^{(2)},$$

then the new expression of (20) is

$$\left\| \varepsilon_i^{1/2} \log(\varepsilon_i) B\circ U^*(d_{-1/2,i}) + \sum_{j=3}^{p} \varepsilon_i^{-\gamma_j} B\circ U^*(d_{\gamma_j,i}) + \varepsilon_i^{-\gamma_2} B\circ U^*(d_i^{(2)} + d_{\gamma_2,i}) \right\|_{H^{-1/2}(\Sigma)} = O(\sqrt{\varepsilon_i}).$$

We can continue in this way until the term concerning γ_p. It constructs terms

$$\begin{aligned} d^{(0)}_{\gamma_k,i} &\in (E_{1/2}\oplus\cdots\oplus E_{\gamma_k}) \cap \operatorname{Ker}(B\circ U^*), \\ d_i^{(k+1)} &\in E_{1/2}\oplus\cdots\oplus E_{\gamma_k} \end{aligned}$$

with $0 \le k \le p$. If we decompose $d^{(0)}_{\gamma_k,i} = \sum_{j=0}^{k} d^{\gamma_k(0)}_{\gamma_j,i}$ and $d^{(k+1)}_i = \sum_{j=0}^{k} d^{(k+1)}_{\gamma_j,i}$, then

$$d_{1/2,i} = \frac{1}{\sqrt{|\log \varepsilon_i|}} d^{(0)}_{1/2,i} + \varepsilon_i^{1/2-\gamma_1} d^{\gamma_1(0)}_{1/2,i} + \varepsilon_i^{1/2-\gamma_2} d^{\gamma_2(0)}_{1/2,i} + \cdots + \varepsilon_i \log(\varepsilon_i) d^{(p+1)}_{1/2,i},$$
$$d_{\gamma_1,i} = \Pi_{\{\gamma_1\}}(d^{(0)}_{\gamma_1,i}) + \varepsilon_i^{\gamma_1-\gamma_2} d^{\gamma_2(0)}_{\gamma_1,i} + \varepsilon_i^{\gamma_1-\gamma_3} d^{\gamma_3(0)}_{\gamma_1,i} + \cdots.$$

Now, because all the families involved here (finite in number) are bounded in a finite-dimensional space, we can suppose, by successive extractions, that they converge. We have

$$d_\gamma = \lim_{\varepsilon_i \to 0} \Pi_{\{\gamma\}}\big(d^{(0)}_{\gamma,i}\big).$$

This means that there exist elements $\bar\sigma_\gamma = d_\gamma \in \operatorname{Ker}(A-\gamma)$, $|\gamma| \le \frac{1}{2}$, such that there exists an $\eta_\gamma \in \operatorname{Im}(\Pi_{>\gamma})$ with

$$(T \circ U^*)(\bar\sigma_\gamma + \eta_\gamma) = 0,$$

and, if we denote

$$\Pi_{(\gamma,1/2]}(\eta_\gamma) = \sum_{\mu>\gamma} \eta^\mu_\gamma,$$

then we obtain

$$\Pi_{(-1/2,1/2]} \circ U(\phi_{1,i}(r)) \sim \sum_{-\frac{1}{2} \le \mu < \gamma < \frac{1}{2}} r^{-\gamma}(\bar\sigma_\gamma + \varepsilon_i^{\gamma-\mu} \eta^\gamma_\mu) + r^{-1/2}\Big\{ |\log \varepsilon_i|^{-1/2} \bar\sigma_{1/2} + \sum_{-\frac{1}{2} \le \mu < \frac{1}{2}} \varepsilon_i^{1/2-\mu} \eta^{1/2}_\mu \Big\}.$$

Here, the term $\varepsilon_i^{-\mu}$ has to be replaced by $\varepsilon_i^{1/2} \log \varepsilon_i$ in the case of $\mu = -\frac{1}{2}$. □

4.2.5. *Conclusions on the side of M_1.* We now decompose $\phi_{1,i} = \phi_{1,\varepsilon_i}$ near the singularity as follows: Let

$$\xi_1 \phi_{1,\varepsilon_i} = \xi_1 \big\{ \phi^{\le -1/2}_{1,i} + \phi^{(-1/2,1/2]}_{1,i} + \phi^{>1/2}_{1,i} \big\}$$

according to the decomposition, on the cone, of σ_1 along the eigenvalues of A respectively less than $-\frac{1}{2}$, in $\big(-\frac{1}{2}, \frac{1}{2}\big]$ and greater than $\frac{1}{2}$.

We first remark that the expression and the convergence of $\phi^{(-1/2,1/2]}_{1,i}$ are given by the preceding Proposition 12.

Now $\phi^{>1/2}_{1,i}$ and $\widetilde\psi_{1,i} = \xi_1 P_{\varepsilon_i}\big(\Pi_{>1/2} \circ U(\phi_{2,i}(1))\big)$ have the same boundary value. But, by Propositions 9 and 10, we have

$$\lim_{i\to\infty} U(\phi_{2,i}(1)) = U(\phi_2(1)) \in \operatorname{Im}(\Pi_{>1/2}) \quad \text{for the norm of } H^{1/2}(\Sigma).$$

So, $\xi_1 \phi^{>1/2}_{1,i} - \widetilde\psi_{1,i}$ can be considered in $H^1(M_1(0))$ by a prolongation by 0 and:

Proposition 13. *By uniform continuity of P_{ε_i}, and the convergence property just recalled,*

$$\lim_{i\to\infty} \|\widetilde\psi_{1,i} - \xi_1 P_{\varepsilon_i}(U(\phi_2 \restriction_\Sigma))\|_{L^2(M_1(\varepsilon_i))} = 0.$$

On the other hand, $\xi_1 P_{\varepsilon_i}(U(\phi_2\restriction_\Sigma))$ *converges weakly to* 0 *on the open manifold* $M_1(0)$; *more precisely, for any fixed* η *with* $0<\eta<1$,

$$\lim_{i\to\infty}\|\xi_1 P_{\varepsilon_i}(U(\phi_2\restriction_\Sigma))\|_{L^2(M_1(\eta))}=0.$$

We remark finally that the boundary value of $\phi_{1,i}^{\le -1/2}$ is small. For this term we introduce the cut-off function taken in [Anné et al. 2009],

$$\xi_{\varepsilon_i}(r)=\begin{cases}1 & \text{if } 2\sqrt{\varepsilon_i}\le r,\\ (1/\log\sqrt{\varepsilon_i})\log(2\varepsilon_i/r) & \text{if } 2\varepsilon_i\le r\le 2\sqrt{\varepsilon_i},\\ 0 & \text{if } r\le 2\varepsilon_i.\end{cases}$$

Proposition 14. $$\lim_{i\to\infty}\|(1-\xi_{\varepsilon_i})\xi_1\phi_{1,i}^{\le -1/2}\|_{L^2(M_1(\varepsilon_i))}=0.$$

This is a consequence of the estimates of Lemmas 5 and 6; we remark that, by the same argument, we obtain also $\|\xi_1\phi_{1,i}^{\le -1/2}(r)\|_{L^2(\Sigma)}\le C\sqrt{r}$, so

$$\|(1-\xi_{\varepsilon_i})\xi_1\phi_{1,i}^{\le -1/2}\|_{L^2(M_1(\varepsilon_i))}=O(\varepsilon_i^{1/4}).$$

Proposition 15. *The forms*

$$\psi_{1,i}=(1-\xi_1)\phi_{1,i}+(\xi_1\phi_{1,i}^{>1/2}-\widetilde{\psi}_{1,i})+\xi_{\varepsilon_i}\xi_1\phi_{1,i}^{\le -1/2}+\xi_1U^*(\overline{\sigma}_{0,i}^{1/2})$$

belong to $\mathrm{Dom}(D_{1,\min})$ *and define a bounded family.*

Proof. We will show that each term is bounded. For the last one, it is a consequence of Proposition 12. For the first one, it is already done in Lemma 4. For the second one, we note that

$$\begin{aligned} f_i &:= \Big(\partial_r+\frac{A}{r}\Big)U(\xi_1\phi_{1,i}^{>1/2}-\widetilde{\psi}_{1,i})\\ &=\xi_1\Big(\partial_r+\frac{A}{r}\Big)(U\phi_{1,i}^{>1/2})+\partial_r(\xi_1)U\big(\phi_{1,i}^{>1/2}-P_{\varepsilon_i}(\Pi_{>1/2}\phi_{2,i}(1))\big) \end{aligned} \tag{22}$$

is uniformly bounded in $L^2(\overline{M}_1)$, because of (15). This estimate (15) shows also that the L^2-norm of $\xi_1\phi_{1,i}^{>1/2}-\tilde{\psi}_{1,i}$ is bounded.

For the third one, we use the estimate due to the expression of the quadratic form. The estimate that $\int_{\mathscr{C}_{r,1}}|D_1(\xi_1\phi^{\le -1/2})|^2\,d\mu\le\Lambda$ gives that

$$\|\sigma_1^{\le -1/2}(r)\|^2_{L^2(\Sigma)}\le \Lambda r|\log r| \tag{23}$$

by the same argument as in Lemmas 5 and 6. Now

$$\begin{aligned}\|D_1(\xi_{\varepsilon_i}\xi_1\phi_{1,i}^{\le -1/2})\|_{L^2(\overline{M}_1)} &\le \|\xi_{\varepsilon_i}D_1(\xi_1\phi_{1,i}^{\le -1/2})\|_{L^2(\overline{M}_1)}+\||d\xi_{\varepsilon_i}|\cdot\xi_1\phi_{1,i}^{\le -1/2}\|_{L^2(\overline{M}_1)}\\ &\le \|D_1(\xi_1\phi_{1,i}^{\le -1/2})\|_{L^2(\mathscr{C}_{\varepsilon_i,1})}+\||d\xi_{\varepsilon_i}|\cdot\xi_1\phi_{1,i}^{\le -1/2}\|_{L^2(\mathscr{C}_{\varepsilon_i,\sqrt{\varepsilon_i}})}.\end{aligned}$$

The first term is bounded and, with $|A| \geq \frac{1}{2}$ for this term, and the estimate (23), we have

$$\| |d\xi_{\varepsilon_i}| \xi_1 \phi_{1,i}^{\leq -1/2} \|^2_{L^2(\mathscr{C}_{\varepsilon_i,\sqrt{\varepsilon_i}})} \leq \frac{4\Lambda}{|\log \varepsilon_i|^2} \int_{\varepsilon_i}^{\sqrt{\varepsilon_i}} \frac{\log r}{r} \, dr \leq \tfrac{3}{2}\Lambda.$$

This completes the proof. □

In fact, the decomposition used here is almost orthogonal:

Lemma 16. *There exists $\beta > 0$ such that*

$$(\phi_{1,i}^{>1/2} - \widetilde{\psi}_{1,i}, \widetilde{\psi}_{1,i})_{L^2(M_1(\varepsilon_i))} = O(\varepsilon_i^\beta).$$

Proof. If we decompose the terms into the eigenspaces of A, we see that only the eigenvalues in $\left(\frac{1}{2}, \infty\right)$ are involved. With $f_i = \sum_{\gamma > \frac{1}{2}} f^\gamma$ and $U(\phi_{1,i}^{>1/2} - \widetilde{\psi}_{1,i}) = \sum_{\gamma > \frac{1}{2}} \phi_0^\gamma$, equation (22) and the fact that $(\phi_{1,i}^{>1/2} - \widetilde{\psi}_{1,i})(\varepsilon_i) = 0$ imply

$$\phi_0^\gamma(r) = r^{-\gamma} \int_{\varepsilon_i}^r \rho^\gamma f^\gamma(\rho) \, d\rho.$$

Then, for each eigenvalue $\gamma > \frac{1}{2}$ of A,

$$\begin{aligned}(\phi_0^\gamma, \tilde{\psi}_{1,i}^\gamma)_{L^2(\mathscr{C}_{\varepsilon_i,1})} &= \varepsilon_i^{\gamma-1/2} \int_{\varepsilon_i}^1 r^{-2\gamma} \int_{\varepsilon_i}^r \rho^\gamma (\sigma_\gamma, f^\gamma(\rho))_{L^2(\Sigma)} \, d\rho \, dr \\ &= \varepsilon_i^{\gamma-1/2} \int_{\varepsilon_i}^1 \frac{r^{-2\gamma+1}}{2\gamma - 1} \cdot r^\gamma \cdot (\sigma_\gamma, f^\gamma(r))_{L^2(\Sigma)} \, dr + \frac{\varepsilon_i^{\gamma-1/2}}{2\gamma-1} \int_{\varepsilon_i}^1 \rho^\gamma (\sigma_\gamma, f^\gamma(\rho))_{L^2(\Sigma)} \, d\rho.\end{aligned}$$

Thus, if $\gamma > \frac{3}{2}$, we have the upper bound

$$\begin{aligned}&|(\phi_0^\gamma, \widetilde{\psi}_{1,i}^\gamma)_{L^2(\mathscr{C}_{\varepsilon_i,1})}| \\ &\quad \leq \varepsilon_i^{\gamma-1/2} \int_{\varepsilon_i}^1 \frac{r^{-\gamma+1}}{2\gamma-1} |(\sigma_\gamma, f^\gamma(r))_{L^2(\Sigma)}| \, dr + \frac{\varepsilon_i^{\gamma-1/2}}{(2\gamma-1)\sqrt{2\gamma+1}} \|\sigma_\gamma\|_{L^2(\Sigma)} \cdot \|f^\gamma\|_{L^2(\mathscr{C}_{\varepsilon_i,1})} \\ &\quad \leq C \varepsilon_i^{\gamma - \frac{1}{2}} \|\sigma_\gamma\|_{L^2(\Sigma)} \frac{\varepsilon_i^{(-2\gamma+3)/2}}{(2\gamma-1)\sqrt{2\gamma-3}} \|f^\gamma\|_{L^2(\mathscr{C}_{\varepsilon_i,1})} + \frac{\varepsilon_i^{\gamma-1/2}}{(2\gamma-1)\sqrt{2\gamma+1}} \|\sigma_\gamma\|_{L^2(\Sigma)} \cdot \|f^\gamma\|_{L^2(\mathscr{C}_{\varepsilon_i,1})},\end{aligned}$$

while, for $\gamma = \frac{3}{2}$, the first term is $O(\varepsilon_i \sqrt{|\log \varepsilon_i|})$ and, for $\frac{1}{2} < \gamma < \frac{3}{2}$, it is $O(\varepsilon_i^{\gamma-1/2})$. In short, we have

$$|(\phi_0^\gamma, \widetilde{\psi}_{1,i}^\gamma)_{L^2(\mathscr{C}_{\varepsilon_i,1})}| \leq C \varepsilon_i^\beta \|\sigma_\gamma\|_{L^2(\Sigma)} \cdot \|f^\gamma\|_{L^2(\mathscr{C}_{\varepsilon_i,1})}$$

if $\beta > 0$ satisfies $\gamma \geq \beta + \frac{1}{2}$ for all eigenvalues γ of A in $\left(\frac{1}{2}, \infty\right)$. This estimate gives Lemma 16. □

Corollary 17. *There exists in $\{\psi_{1,i} + \phi_{1,i}^{(-1/2,1/2)}\}_i$ a subfamily which converges in L^2 to a form ϕ_1 in* $\mathrm{Dom}(D_{1,W})$ *that satisfies on the open manifold $M_1(0)$ the equation $\Delta \phi_1 = \lambda \phi_1$. Moreover,*

$$\|\phi_1\|^2_{L^2(M_1(0))} + \|\tilde{\phi}_2\|^2_{L^2(\widetilde{M}_2)} + \|\overline{\sigma}_{1/2}\|^2_{L^2(\Sigma)} = 1, \tag{24}$$

where $\tilde{\phi}_2$ is the prolongation of ϕ_2 by $P_2(\phi_2 \restriction_\Sigma)$ on $\widetilde{M}_2$, and $\overline{\sigma}_{1/2}$ is given by Proposition 12.

Proof. Indeed, the family $\{\psi_{1,i}+\phi_{1,i}^{(-1/2,1/2)}\}_i$ is bounded in $\mathrm{Dom}(D_{1,\max})$; one can then extract a subfamily which converges in $L^2(\overline{M}_1, \bar{g}_1)$. But we know that $\widetilde{\psi}_{1,i}$ converges to 0 in any $M_1(\eta)$; the conclusion follows. We obtain also, with the help of Lemma 16, that

$$1-\{\|\phi_1\|^2_{L^2(M_1(0))}+\|\phi_2\|^2_{L^2(M_2)}\} = \lim_{i\to\infty}\left\{\|\widetilde{\psi}_{1,i}\|^2_{L^2(M_1(\varepsilon_i))}+\left\|\xi_1 U^*\left(\frac{1}{\sqrt{|\log\varepsilon_i|}}r^{-1/2}\bar{\sigma}_{1/2}\right)\right\|^2_{L^2(M_1(\varepsilon_i))}\right\}.$$

We remark that, by Proposition 13, $\phi_2=0$ implies $\lim_{i\to\infty}\|\widetilde{\psi}_{1,i}\|_{L^2(M_1(\varepsilon_i))}=0$. In fact, one has, by (15),

$$\lim_{i\to\infty}\|\widetilde{\psi}_{1,i}\|_{L^2(M_1(\varepsilon_i))}=\|P_2(U\phi_2\restriction_\Sigma)\|_{L^2(\widetilde{M}_2)}. \tag{25}$$

Finally, one has

$$\lim_{i\to\infty}\left\|\xi_1 U^*\left(\frac{1}{\sqrt{|\log\varepsilon_i|}}r^{-1/2}\bar{\sigma}_{1/2}\right)\right\|_{L^2(M_1(\varepsilon_i))}=\|\bar{\sigma}_{1/2}\|_{L^2(\Sigma)}. \qquad \square$$

4.3. *Lower bound, the end.* Now let $\{\phi_1(\varepsilon),\dots,\phi_N(\varepsilon)\}$ be an orthonormal family of eigenforms of the Hodge–de Rham operator associated with the eigenvalues $\lambda_1(\varepsilon),\dots,\lambda_N(\varepsilon)$. We can use the same procedure of extraction for all the families. This gives, in the limit domain, a family $\{(\phi_1^j,\phi_2^j,\bar{\sigma}_{1/2}^j)\}_{1\le j\le N}$. We already know, by Corollary 17, that each element has norm 1. If we show that they are orthogonal, then we are done, by applying the min–max formula to the limit problem (12).

Lemma 18. *The limit family is orthonormal in* $\mathscr{H}_\infty$.

Proof. If we follow the procedure for one index, up to terms converging to zero, we have decomposed the eigenforms $\phi_j(\varepsilon)$ on M_ε into three terms:

$$\Phi_\varepsilon^j=\psi_{1,i}+\phi_{1,i}^{(-1/2,1/2)},\qquad \widetilde{\Phi}_\varepsilon^j=\widetilde{\psi}_{1,i},\qquad\text{and}\qquad \overline{\Phi}_\varepsilon^j=U^*\left(\frac{1}{\sqrt{|\log\varepsilon|}}r^{-1/2}\bar{\sigma}_{1/2}^j\right). \tag{26}$$

Let $a\neq b$ be two indices. If we apply Lemma 16 to any linear combination of $\phi_a(\varepsilon)$ and $\phi_b(\varepsilon)$, we obtain that

$$\lim_{i\to\infty}\{(\Phi^a_{\varepsilon_i},\widetilde{\Phi}^b_{\varepsilon_i})_{L^2(M_1(\varepsilon_i))}+(\Phi^b_{\varepsilon_i},\widetilde{\Phi}^a_{\varepsilon_i})_{L^2(M_1(\varepsilon_i))}\}=0.$$

If we apply (25), we obtain

$$\lim_{i\to\infty}\{(\widetilde{\Phi}^a_{\varepsilon_i},\widetilde{\Phi}^b_{\varepsilon_i})_{L^2(M_1(\varepsilon_i))}+(\phi^a_{2,\varepsilon},\phi^b_{2,\varepsilon})_{L^2(M_2)}\}=(\tilde{\phi}^a_2,\tilde{\phi}^b_2)_{L^2(\widetilde{M}_2)}.$$

Then finally, from $(\phi_a(\varepsilon),\phi_b(\varepsilon))_{L^2(M_\varepsilon)}=0$, we conclude that

$$(\phi_1^a,\phi_1^b)_{L^2(\overline{M}_1)}+(\phi_2^a,\phi_2^b)_{L^2(\widetilde{M}_2)}+(\bar{\sigma}^a_{1/2},\bar{\sigma}^b_{1/2})_{L^2(\Sigma)}=0. \qquad \square$$

Proposition 19. *The multiplicity of* 0 *in the limit spectrum is given by the sum*

$$\dim\mathrm{Ker}(\Delta_{1,W})+\dim\mathrm{Ker}(\mathfrak{D}_2)+i_{1/2},$$

where $i_{1/2}$ *denotes the dimension of the vector space* $\mathcal{I}_{1/2}$ *— see* (8) *— of extended solutions* ω *on* $\widetilde{M}_2$ *introduced by Carron* [2001b], *corresponding to a boundary term on restriction to* $r=1$ *with nontrivial component in* $\mathrm{Ker}\big(A-\frac{1}{2}\big)$.

If the limit value λ is nonzero, then it belongs to the positive spectrum of the Hodge–de Rham operator $\Delta_{1,W}$ on $\overline{M}_1$, with the space W as defined in (7).

Proof. The last process, with, in particular, (25) and (16), in fact constructs an element in the limit Hilbert space

$$\mathcal{H}_\infty := L^2(\overline{M}_1) \oplus \operatorname{Ker}(\widetilde{D}_2) \oplus \mathcal{I}_{1/2}.$$

This process is clearly *isometric* in the sense that, if we have an orthonormal family $\{\phi_j(\varepsilon_i)\}_j$ $(1 \le j \le N)$, we obtain at the limit an orthonormal family, where $\mathcal{H}_\infty$ is defined as an orthogonal sum of the Hilbert spaces. And, if we begin with eigenforms of Δ_{ε_i}, we obtain at the limit eigenforms of $\Delta_{1,W} \oplus \{0\} \oplus \{0\}$. The last calculus implies that $\liminf_{i\to\infty} \lambda_N(\varepsilon_i) \ge \lambda_N$. □

Remark 20. In order to understand this result, it is important to remember when the eigenvalue $\frac{1}{2}$ occurs in the spectrum of A. By the expression (4), we find that it occurs exactly:

- For n even, if $\frac{3}{4}$ is an eigenvalue of the Hodge–de Rham operator Δ_Σ acting on coexact forms of degree $\frac{1}{2}n$ or $\frac{1}{2}n - 1$ of the submanifold Σ.
- For n odd, if 0 is an eigenvalue of Δ_Σ on forms of degree $\frac{1}{2}(n-1)$ or $\frac{1}{2}(n+1)$, but also if 1 is an eigenvalue on coexact forms of degree $\frac{1}{2}(n-1)$ on Σ.

A dilation of the metric on Σ allows us to avoid positive eigenvalues, but harmonic forms of degree $\frac{1}{2}(n-1)$ or $\frac{1}{2}(n+1)$ on Σ can not be avoided.

Moreover, Carron [2001a, Theorem 0.6] has proved that the extended index depends only on geometry at infinity: these harmonic forms on Σ will indeed create half-bound states, and then small eigenvalues will always appear.

5. Harmonic forms and small eigenvalues

It would be interesting to know how many small (but nonzero) eigenvalues appear. For this purpose, we can use the topological meaning of harmonic forms.

5.1. *Cohomology groups.* The topology of M_ε is independent of $\varepsilon \ne 0$ and can be understood by the Mayer–Vietoris exact sequence:

$$\cdots \longrightarrow H^p(M_\varepsilon) \xrightarrow{\text{res}} H^p(M_1(\varepsilon)) \oplus H^p(M_2) \xrightarrow{\text{dif}} H^p(\Sigma) \xrightarrow{\text{ext}} H^{p+1}(M_\varepsilon) \longrightarrow \cdots.$$

As already mentioned, the space $\operatorname{Ker}(\mathcal{D}_2) \oplus \mathcal{I}_{1/2}$ can be mapped into $H^*(M_2)$. More precisely, Hausel, Hunsicker and Mazzeo [Hausel et al. 2004, Theorem 1.A, p. 490] have proved that the space of the L^2-harmonic forms $\mathcal{H}^k_{L^2}(\widetilde{M}_2)$ on $\widetilde{M}_2$ is given by

$$\mathcal{H}^k_{L^2}(\widetilde{M}_2) \cong \begin{cases} H^k(M_2, \Sigma) & \text{if } k < \frac{1}{2}(n+1), \\ \operatorname{Im}(H^{(n+1)/2}(M_2, \Sigma) \to H^{(n+1)/2}(M_2)) & \text{if } k = \frac{1}{2}(n+1), \\ H^k(M_2) & \text{if } k > \frac{1}{2}(n+1). \end{cases} \tag{27}$$

We note that the space of L^2-harmonic forms is equal to that of L^2-harmonic fields, or the Hodge cohomology group, since $\widetilde{M}_2$ is complete.

For $\overline{M}_1$, we can use the results of Cheeger [1980; 1983]. Following his work, we know that the intersection cohomology groups $IH^*(\overline{M}_1)$ of $\overline{M}_1$ coincide with $\mathrm{Ker}(D_{1,\max} \circ D_{1,\min})$ if $H^{n/2}(\Sigma) = 0$. We also know that

$$IH^p(\overline{M}_1) \cong \begin{cases} H^p(M_1(\varepsilon)) & \text{if } p \le \frac{1}{2}n, \\ H^p_c(M_1(\varepsilon)) & \text{if } p \ge \frac{1}{2}n + 1. \end{cases} \tag{28}$$

These results can be used for our study only if $D_{1,\max}$ and $D_{1,\min}$ coincide. This occurs if and only if A has no eigenvalues in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$. As a consequence of the expression of the eigenvalues of A, recalled in (4), this is the case if and only if:

- *for n odd*, the operator Δ_Σ has no eigenvalues in $(0, 1)$ on coexact forms of degree $\frac{1}{2}(n-1)$;
- *for n even*, the operator Δ_Σ has no eigenvalues in $\left(0, \frac{3}{4}\right)$ on coexact forms of degree $\frac{1}{2}n$ or $\frac{1}{2}n - 1$, and $H^{n/2}(\Sigma) = 0$.

Thus, if $D_{1,\max} = D_{1,\min}$, which implies $H^{n/2}(\Sigma) = 0$ in the case where n is even, then the map

$$H^{n/2}(M_\varepsilon) \xrightarrow{\text{res}} H^{n/2}(M_1(\varepsilon)) \oplus H^{n/2}(M_2)$$

is surjective, and then any small eigenvalue in this degree must come from an element of $\mathrm{Ker}(\mathscr{D}_2) \oplus \mathscr{I}_{1/2}$ sent to 0 in $H^{n/2}(M_2)$. In this case also, the map

$$H^{n/2+1}(M_\varepsilon) \xrightarrow{\text{res}} H^{n/2+1}(M_1(\varepsilon)) \oplus H^{n/2+1}(M_2)$$

is injective, so there may exist small eigenvalues in this degree.

5.2. ***Some examples.*** We exhibit a general procedure to construct new examples as follows: Let W_i, $i = 1, 2$, be two compact Riemannian manifolds with boundary Σ_i and dimension $n_i + 1$ such that $n_1 + n_2 = n \ge 2$. We can apply our result to $M_1 := W_1 \times \Sigma_2$ and $M_2 := \Sigma_1 \times W_2$. The manifold M_ε is always diffeomorphic to $M = M_1 \cup M_2$.

For instance, let v_2 be the volume form of (Σ_2, h_2). It defines a harmonic form on M_1, and this form will appear in the limit spectrum if, transplanted onto $\overline{M}_1$, it defines an element in the domain of the operator $\Delta_{1,W}$.

In the notation introduced in Section 2.2, this element corresponds to $\beta = 0$ and $\alpha = r^{n/2-n_2} v_2$, and the expression of A gives that

$$A(\beta, \alpha) = \left(n_2 - \tfrac{1}{2}n\right)(\beta, \alpha).$$

If $\frac{1}{2}n - n_2 > 0$, then (β, α) is in the domain of $D_{1,\max} \circ D_{1,\min}$, and, if $n_2 = \frac{1}{2}n$, it is in the domain of $\Delta_{1,W}$ for the eigenvalue 0 of A.

So, if we know that $H^{n_2}(M) = 0$ or, more generally, $\dim H^{n_2}(M) < \dim H^{n_2}(\Sigma_2)$ in the case where Σ_2 is not connected, then this element will create a small eigenvalue on M_ε. If D^k denotes the unit ball in $\mathbb{R}^k$, this is the case for

$$W_1 = D^{n_1+1} \quad \text{and} \quad W_2 = D^{n_2+1} \quad \text{for } n_2 \le n_1.$$

Then, $M = \mathbb{S}^{n_1+n_2+1}$ and we obtain:

Corollary 21. *For any degree k and any $\varepsilon > 0$, there exists a metric on $\mathbb{S}^m$ such that the Hodge–de Rham operator acting on k-forms admits an eigenvalue smaller than ε. We can see that, for $k < \frac{1}{2}m$, it is in the spectrum of coexact forms, and, by duality, for $k \geq \frac{1}{2}m$ it is in the spectrum of exact k-forms.*

Indeed, the case $k < \frac{1}{2}m$ is a direct application, as explained above. We see that our quasimode is coclosed. Thus, in the case where m is even, if ω is an eigenform of degree $\frac{1}{2}m-1$ with small eigenvalue, then $d\omega$ is a closed eigenform with the same eigenvalue and degree $\frac{1}{2}m$. Finally, the case $k > \frac{1}{2}m$ is obtained by Hodge duality. We remark that in the case $k = 0$ we recover *Cheeger's dumbbell*, and also that this result has been proved by Guerini [2004] with another deformation, although he did not give the convergence of the spectrum.

By the surgery of the previous case, we obtain, for

$$W_1 := \mathbb{S}^{n_1} \times [0,1] \quad \text{and} \quad W_2 := D^{n_2+1} \quad \text{for } 0 \leq n_2 < n_1 \text{ and } n = n_1 + n_2 \geq 2,$$

that $\Sigma_1 = \mathbb{S}^{n_1} \sqcup \mathbb{S}^{n_1}$, $\Sigma_2 = \mathbb{S}^{n_2}$ and $M = \mathbb{S}^{n_1} \times \mathbb{S}^{n_2+1}$. The volume form $v_2 \in H^{n_2}(\Sigma_2)$ again defines a harmonic form on $\bar{M}_1$ and, since $H^{n_2}(\mathbb{S}^{n_1} \times \mathbb{S}^{n_2+1}) = 0$, if $n_2 < n_1$, then v_2 defines a small eigenvalue on n_2-forms of M_ε.

Thus, by the duality, we obtain:

Corollary 22. *For any $k, l \geq 0$ with $0 \leq k-1 < l$ and any $\varepsilon > 0$, there exists a metric on $\mathbb{S}^l \times \mathbb{S}^k$ such that the Hodge–de Rham operator acting on $(k-1)$-forms and on $(l+1)$-forms admits an eigenvalue smaller than ε.*

This corollary is also a consequence of the previous one: we know that there exists a metric on $\mathbb{S}^k$ whose Hodge–de Rham operator admits a small eigenvalue on $(k-1)$-forms, and this property is maintained on $\mathbb{S}^l \times \mathbb{S}^{k+1}$.

With the same construction, we can exchange the roles of M_1 and M_2: the two volume forms of $\mathbb{S}^{n_1} \sqcup \mathbb{S}^{n_1}$ create one n_1-form with small but nonzero eigenvalue on $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2+1}$ if $n_1 \leq n_2 + 1$. By the duality, we obtain an (n_2+1)-form with small eigenvalue. So, with new notations, we have obtained:

Corollary 23. *For any $k < l$ with $k + l \geq 3$ and any $\varepsilon > 0$, there exists a metric on $\mathbb{S}^l \times \mathbb{S}^k$ such that the Hodge–de Rham operator acting on l-forms and on k-forms admits a positive eigenvalue smaller than ε.*

More generally, by repeating the $(k-1)$-dimensional surgery L times, we obtain the following:

Proposition 24 [Sha and Yang 1991]. *The connected sum of L copies of the product spheres, $\sharp_{i=1}^{L}(\mathbb{S}^k \times \mathbb{S}^l)$, can be decomposed as follows:*

$$\mathop{\sharp}_{i=1}^{L} (\mathbb{S}^k \times \mathbb{S}^l) \cong \left(\mathbb{S}^{k-1} \times \left(\mathbb{S}^{l+1} \setminus \coprod_{i=0}^{L} D_i^{l+1}\right)\right) \cup_\partial \left(D^k \times \coprod_{i=0}^{L} \mathbb{S}_i^l\right).$$

Remark 25. J.-P. Sha and D. Yang [1991] constructed a Riemannian metric of positive Ricci curvature on this manifold. More generally, see also [Wraith 2007].

In a similar way, using Proposition 24, we can obtain the small positive eigenvalues on the connected sum of L copies of the product spheres $\sharp_{i=1}^{L}(\mathbb{S}^k \times \mathbb{S}^l)$.

All these examples use the spectrum of $\overline{M}_1$. We can obtain also examples using the reduced L^2-cohomology group of $\widetilde{M}_2$, which is given by (27) [Hausel et al. 2004].

Suppose now that $n = \dim \Sigma$ is odd. Then, we have the long exact sequence

$$\cdots \to H^k(M_2, \Sigma) \to H^k(M_2) \to H^k(\Sigma) \to H^{k+1}(M_2, \Sigma) \to \cdots .$$

For $k = \frac{1}{2}(n-1)$, the space $H^k(M_2, \Sigma)$ is isomorphic to the reduced L^2-cohomology group of $\widetilde{M}_2$. If $H^{(n-1)/2}(\Sigma)$ is nontrivial, then any nontrivial harmonic k-form on Σ will create an extended solution, corresponding to an eigenvector of A with eigenvalue $\frac{1}{2}$.

For example, take $\Sigma = \mathbb{S}^k \times \mathbb{S}^{k+1}$ for $k = \frac{1}{2}(n-1)$; then $H^k(\Sigma)$ is nontrivial. Any nontrivial form $\omega \in H^k(\Sigma)$ sent to $0 \in H^{k+1}(M_2, \Sigma)$ comes from an element $\tilde{\omega} \in H^k(M_2)$ which is not in the reduced L^2-cohomology group of $\widetilde{M}_2$, by (27).

Acknowledgement

Takahashi was partially supported by Grant-in-Aid for Young Scientists (B) 24740034.

References

[Anné and Colbois 1995] C. Anné and B. Colbois, "Spectre du laplacien agissant sur les p-formes différentielles et écrasement d'anses", *Math. Ann.* **303**:3 (1995), 545–573. MR 97b:58141 Zbl 0909.58054

[Anné and Takahashi 2012] C. Anné and J. Takahashi, "p-spectrum and collapsing of connected sums", *Trans. Amer. Math. Soc.* **364**:4 (2012), 1711–1735. MR 2869189 Zbl 1254.58012

[Anné et al. 2009] C. Anné, G. Carron, and O. Post, "Gaps in the differential forms spectrum on cyclic coverings", *Math. Z.* **262**:1 (2009), 57–90. MR 2010a:58037 Zbl 1166.58013

[Atiyah et al. 1975] M. F. Atiyah, V. K. Patodi, and I. M. Singer, "Spectral asymmetry and Riemannian geometry, I", *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 43–69. MR 53 #1655a Zbl 0297.58008

[Brüning and Seeley 1988] J. Brüning and R. Seeley, "An index theorem for first order regular singular operators", *Amer. J. Math.* **110**:4 (1988), 659–714. MR 89k:58271 Zbl 0664.58035

[Carron 2001a] G. Carron, "Théorèmes de l'indice sur les variétés non-compactes", *J. Reine Angew. Math.* **541** (2001), 81–115. MR 2002i:58020 Zbl 1014.58012

[Carron 2001b] G. Carron, "Un théorème de l'indice relatif", *Pacific J. Math.* **198**:1 (2001), 81–107. MR 2002e:58038 Zbl 1083.58502

[Cheeger 1980] J. Cheeger, "On the Hodge theory of Riemannian pseudomanifolds", pp. 91–146 in *Geometry of the Laplace operator* (Honolulu, HI, 1979)), edited by R. Osserman and A. Weinstein, Proc. Sympos. Pure Math. **XXXVI**, Amer. Math. Soc., Providence, RI, 1980. MR 83a:58081 Zbl 0461.58002

[Cheeger 1983] J. Cheeger, "Spectral geometry of singular Riemannian spaces", *J. Differential Geom.* **18**:4 (1983), 575–657. MR 85d:58083 Zbl 0529.58034

[Dodziuk 1982] J. Dodziuk, "Eigenvalues of the Laplacian on forms", *Proc. Amer. Math. Soc.* **85** (1982), 437–443. MR 84k:58223 Zbl 0502.58038

[Fukaya 1987] K. Fukaya, "Collapsing of Riemannian manifolds and eigenvalues of Laplace operator", *Invent. Math.* **87**:3 (1987), 517–547. MR 88d:58125 Zbl 0589.58034

[Guerini 2004] P. Guerini, "Prescription du spectre du laplacien de Hodge–de Rham", *Ann. Sci. École Norm. Sup.* (4) **37**:2 (2004), 270–303. MR 2005k:58063 Zbl 1068.58016

[Hausel et al. 2004] T. Hausel, E. Hunsicker, and R. Mazzeo, "Hodge cohomology of gravitational instantons", *Duke Math. J.* **122**:3 (2004), 485–548. MR 2005d:58039 Zbl 1062.58002

[Lesch 1997] M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte zur Mathematik **136**, B. G. Teubner, Stuttgart, 1997. MR 98d:58174 Zbl 1156.58302

[Lott 2002a] J. Lott, "Collapsing and Dirac-type operators", pp. 175–196 in *Proceedings of the Euroconference on Partial Differential Equations and their Applications to Geometry and Physics* (Castelvecchio Pascoli, 2000), vol. 91, 2002. MR 2003h:58042 Zbl 1001.58020

[Lott 2002b] J. Lott, "Collapsing and the differential form Laplacian: the case of a smooth limit space", *Duke Math. J.* **114**:2 (2002), 267–306. MR 2003e:58047 Zbl 1072.58023

[Lott 2004] J. Lott, "Remark about the spectrum of the p-form Laplacian under a collapse with curvature bounded below", *Proc. Amer. Math. Soc.* **132**:3 (2004), 911–918. MR 2004h:58049 Zbl 1042.58018

[Mazzeo 2006] R. Mazzeo, "Resolution blowups, spectral convergence and quasi-asymptotically conical spaces", Exposé No. 8 in *Journées équations aux dérivées partielles* (Évian-les-Bains, 2006), CEDRAM, Grenoble, 2006.

[Mazzeo and Melrose 1990] R. R. Mazzeo and R. B. Melrose, "The adiabatic limit, Hodge cohomology and Leray's spectral sequence for a fibration", *J. Differential Geom.* **31**:1 (1990), 185–213. MR 90m:58004 Zbl 0702.58007

[Milnor and Stasheff 1974] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies **76**, Princeton University Press, 1974. MR 55 #13428 Zbl 0298.57008

[Rowlett 2006] J. M. Rowlett, *Spectral geometry and asymptotically conic convergence*, Ph.D. thesis, Stanford University, 2006, available at http://search.proquest.com/docview/304980386. MR 2708809

[Rowlett 2008] J. Rowlett, "Spectral geometry and asymptotically conic convergence", *Comm. Anal. Geom.* **16**:4 (2008), 735–798. MR 2010e:58028 Zbl 1160.58307

[Rumin 2000] M. Rumin, "Sub-Riemannian limit of the differential form spectrum of contact manifolds", *Geom. Funct. Anal.* **10**:2 (2000), 407–452. MR 2002f:53044 Zbl 1008.53033

[Sha and Yang 1991] J.-P. Sha and D. Yang, "Positive Ricci curvature on the connected sums of $S^n \times S^m$", *J. Differential Geom.* **33**:1 (1991), 127–137. MR 92f:53048 Zbl 0728.53027

[Wall 1960] C. T. C. Wall, "Determination of the cobordism ring", *Ann. of Math.* (2) **72** (1960), 292–311. MR 22 #11403 Zbl 0097.38801

[Wraith 2007] D. J. Wraith, "New connected sums with positive Ricci curvature", *Ann. Global Anal. Geom.* **32**:4 (2007), 343–360. MR 2008h:53058 Zbl 1132.53018

Received 13 Jun 2014. Accepted 25 Mar 2015.

COLETTE ANNÉ: colette.anne@univ-nantes.fr
Laboratoire de Mathématiques Jean Leray, Université de Nantes, CNRS, Faculté des Sciences, BP 92208, 44322 Nantes, France

JUNYA TAKAHASHI: t-junya@math.is.tohoku.ac.jp
Research Center for Pure and Applied Mathematics, Tôhoku University, Graduate School of Information Sciences, 6-3-09 Aoba, Sendai 980-8579, Japan

ANALYSIS AND PDE
Vol. 8, No. 5, 2015
dx.doi.org/10.2140/apde.2015.8.1051

SHARP L^p BOUNDS FOR THE WAVE EQUATION ON GROUPS OF HEISENBERG TYPE

DETLEF MÜLLER AND ANDREAS SEEGER

Consider the wave equation associated with the sub-Laplacian on groups of Heisenberg type. We construct parametrices using oscillatory integral representations and use them to prove sharp L^p and Hardy space regularity results.

Introduction

Given a second-order differential operator L on a suitable manifold, we consider the Cauchy problem for the associated wave equation

$$(\partial_\tau^2 + L)u = 0, \quad u\big|_{\tau=0} = f, \quad \partial_\tau u\big|_{\tau=0} = g. \tag{1}$$

This paper is a contribution to the problem of L^p bounds of the solutions at fixed time τ in terms of L^p-Sobolev norms of the initial data f and g. This problem is well understood if L is the standard Laplacian $-\Delta$ (i.e., defined as a positive operator) in $\mathbb{R}^d$ [Miyachi 1980; Peral 1980], or the Laplace–Beltrami operator on a compact manifold [Seeger et al. 1991] of dimension d. In this case, (1) is a strictly hyperbolic problem and reduces to estimates for Fourier integral operators associated to a local canonical

Research partially supported by NSF grant DMS 1200261. The authors thank the Hausdorff Research Institute for Mathematics in Bonn for support. The paper was completed in the summer of 2014 when they participated in the Hausdorff Trimester Program in Harmonic Analysis and Partial Differential Equations.
MSC2010: 35L05, 35S30, 42B15, 43A80.
Keywords: wave equation, subelliptic Laplacian, Heisenberg group.

graph. The known sharp regularity results in this case say that, if $\gamma(p) = (d-1)\left|\frac{1}{p}-\frac{1}{2}\right|$ and the initial data f and g belong to the L^p-Sobolev spaces $L^p_{\gamma(p)}$ and $L^p_{\gamma(p)-1}$, respectively, then the solution $u(\,\cdot\,,\tau)$ at fixed time τ (say $\tau = \pm 1$) belongs to L^p.

In the absence of strict hyperbolicity, the classical Fourier integral operator techniques do not seem available anymore and it is not even clear how to efficiently construct parametrices for the solutions; consequently, the L^p regularity problem is largely open. However, some considerable progress has been made for the specific case of an invariant operator on the Heisenberg group $\mathbb{H}_m$, which is often considered as a model case for more general situations. Recall that coordinates on $\mathbb{H}_m$ are given by (z,u) with $z = x+iy \in \mathbb{C}^m$, $u \in \mathbb{R}$, and the group law is given by $(z,u)\cdot(z',u') = \big(z+z', u+u' - \frac{1}{2}\Im(z\cdot\overline{z'})\big)$. A basis of left-invariant vector fields is given by $X_j = \partial/\partial x_j - \frac{1}{2}y_j\partial/\partial u$, $Y_j = \partial/\partial y_j + \frac{1}{2}x_j\partial/\partial u$, and we consider the sub-Laplacian

$$L = -\sum_{j=1}^{m}(X_j^2+Y_j^2).$$

This operator is perhaps the simplest example of a nonelliptic sum-of-squares operator in the sense of [Hörmander 1967]. In view of the Heisenberg group structure, it is natural to analyze the corresponding wave group using tools from noncommutative Fourier analysis. The operator L is essentially selfadjoint on $C_0^\infty(G)$ (this follows from the methods used in [Nelson and Stinespring 1959]) and the solution of (1) can be expressed using the spectral theorem in terms of functional calculus; it is given by

$$u(\,\cdot\,,\tau) = \cos(\tau\sqrt{L})f + \frac{\sin(\tau\sqrt{L})}{\sqrt{L}}g.$$

We are then aiming to prove estimates of the form

$$\|u(\,\cdot\,,\tau)\|_p \lesssim \|(I+\tau^2L)^{\gamma/2}f\|_p + \|\tau(I+\tau^2L)^{\gamma/2-1}g\|_p \tag{2}$$

involving versions of L^p-Sobolev spaces defined by the subelliptic operator L. Alternatively, one can consider equivalent uniform $L^p \to L^p$ bounds for operators $a(\tau\sqrt{L})e^{\pm i\tau\sqrt{L}}$, where a is a standard (constant coefficient) symbol of order $-\gamma$. Note that it suffices to prove those bounds for times $\tau = \pm 1$, after a scaling using the automorphic dilations $(z,u) \mapsto (rz, r^2u)$, $r > 0$.

A first study about the solutions to (1) was undertaken by Nachman [1982], who showed that the wave operator on $\mathbb{H}_m$ has a fundamental solution whose singularities lie on the cone Γ formed by the characteristics through the origin. He showed that the singularity set Γ has a far more complicated structure for $\mathbb{H}_m$ than the corresponding cone in the Euclidean case. The fundamental solution is given by a series involving Laguerre polynomials and Nachman was able to examine the asymptotic behavior as one approaches a generic singular point on Γ. However, his method does not seem to yield uniform estimates in a neighborhood of the singular set, which are crucial for obtaining L^p-Sobolev estimates for solutions to (1).

D. Müller and E. M. Stein [1999] were able to derive nearly sharp L^1 estimates (and, by interpolation, also L^p estimates, leaving open the interesting endpoint bounds). Their approach relied on explicit calculations using Gelfand transforms for the algebra of radial L^1 functions on the Heisenberg group, and the geometry of the singular support remained hidden in this approach. Later, Greiner, Holcman and Kannai [Greiner et al. 2002] used contour integrals and an explicit formula for the heat kernel on the Heisenberg group to derive an integral formula for the fundamental solution of the wave equation on $\mathbb{H}^m$ which exhibits the singularities of the wave kernel. We shall follow a somewhat different approach, which allows us to link the geometrical picture to a decomposition of the joint spectrum of L and the operator U of differentiation in the central direction (see also [Strichartz 1991]); this linkage is crucial to prove optimal L^p regularity estimates.

In order to derive parametrices we will use a subordination argument based on stationary phase calculations to write the wave operator as an integral involving Schrödinger operators for which explicit formulas are available [Gaveau 1977; Hulanicki 1984]. This will yield a type of oscillatory integral representation of the kernels, as in the theory of Fourier integral operators, which will be amenable to proving L^p estimates. Unlike in the classical theory of Fourier integral operators [Hörmander 1971], our phase functions are not smooth everywhere and have substantial singularities; this leads to considerable complications. Finally, an important point in our proof is the identification of a suitable Hardy space for the problem, so that L^p bounds can be proved by interpolation of L^2 and Hardy space estimates. We then obtain the following sharp L^p regularity result, which is a direct analogue of the result by Peral [1980] and Miyachi [1980] on the wave equation in the Euclidean setting.

Theorem. *Let $d = 2m+1$, $1 < p < \infty$, and $\gamma \geq (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$. Then the operators*

$$(I + \tau^2 L)^{-\gamma/2} \exp(\pm i\tau\sqrt{L})$$

extend to bounded operators on $L^p(\mathbb{H}^m)$. The solutions u to the initial value problem (1) satisfy the Sobolev-type inequalities (2).

Throughout the paper we shall in fact consider the more general situation of *groups of Heisenberg type*, introduced by Kaplan [1980]. These include groups with center of dimension greater than 1. The extension of the above result for the wave operator to groups of Heisenberg type and further results will be formulated in the next section.

1. The results for groups of Heisenberg type

Groups of Heisenberg type. Let d_1, d_2 be positive integers, with d_1 *even*, and consider a Lie algebra $\mathfrak{g}$ of Heisenberg type, where $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, with $\dim \mathfrak{g}_1 = d_1$ and $\dim \mathfrak{g}_2 = d_2$, and

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}_2 \subset \mathfrak{z}(\mathfrak{g}),$$

$\mathfrak{z}(\mathfrak{g})$ being the center of $\mathfrak{g}$. Now $\mathfrak{g}$ is endowed with an inner product $\langle\ ,\ \rangle$ such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ or orthogonal subspaces. For $\mu \in \mathfrak{g}_2^* \setminus \{0\}$, we define the symplectic form ω_μ on $\mathfrak{g}_1$ by

$$\omega_\mu(V, W) := \mu([V, W]), \tag{3}$$

then there is a unique skew-symmetric linear endomorphism J_μ of $\mathfrak{g}_1$ such that

$$\omega_\mu(V, W) = \langle J_\mu(V), W\rangle \tag{4}$$

(here, we also used the natural identification of $\mathfrak{g}_2^*$ with $\mathfrak{g}_2$ via the inner product). Then, on a Lie algebra of Heisenberg type,

$$J_\mu^2 = -|\mu|^2 I \tag{5}$$

for every $\mu \in \mathfrak{g}_2^*$. As the corresponding connected, simply connected Lie group G we then choose the linear manifold $\mathfrak{g}$, endowed with the Baker–Campbell–Hausdorff product

$$(V_1, U_1) \cdot (V_2, U_2) := \big(V_1 + V_2, U_1 + U_2 + \tfrac{1}{2}[V_1, V_2]\big).$$

As usual, we identify $X \in \mathfrak{g}$ with the corresponding left-invariant vector field on G given by the Lie derivative:

$$Xf(g) := \frac{d}{dt} f(g \exp(tX))\Big|_{t=0},$$

where $\exp : \mathfrak{g} \to G$ denotes the exponential mapping, which agrees with the identity mapping in our case.

Let us next fix an orthonormal basis $X_1, \dots, X_{d_1}$ of $\mathfrak{g}_1$, as well as an orthonormal basis $U_1, \dots, U_{d_2}$ of $\mathfrak{g}_2$. We may then identify $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ and G with $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ by means of the basis $X_1, \dots, X_{d_1}$, $U_1, \dots, U_{d_2}$ of $\mathfrak{g}$. Then our inner product on $\mathfrak{g}$ will agree with the canonical Euclidean product $v \cdot w = \sum_{j=1}^{d_1+d_2} v_j w_j$ on $\mathbb{R}^{d_1+d_2}$, and J_μ will be identified with a skew-symmetric $d_1 \times d_1$ matrix. We shall also identify the dual spaces of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ with $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively, by means of this inner product. Moreover, the Lebesgue measure $dx\,du$ on $\mathbb{R}^{d_1+d_2}$ is a biinvariant Haar measure on G. By

$$d := d_1 + d_2 \tag{6}$$

we denote the topological dimension of G. The group law on G is then given by

$$(x, u) \cdot (x', u') = \big(x + x', u + u' + \tfrac{1}{2}\langle \vec{J} x, x' \rangle\big), \tag{7}$$

where $\langle \vec{J} x, x' \rangle$ denotes the vector in $\mathbb{R}^{d_2}$ with components $\langle J_{U_i} x, x' \rangle$.

Let

$$L := -\sum_{j=1}^{d_1} X_j^2 \tag{8}$$

denote the sub-Laplacian corresponding to the basis $X_1, \dots, X_{d_1}$ of $\mathfrak{g}_1$.

In the special case $d_2 = 1$, we may assume that $J_\mu = \mu J$, $\mu \in \mathbb{R}$, where

$$J := \begin{pmatrix} 0 & I_{d_1/2} \\ -I_{d_1/2} & 0 \end{pmatrix} \tag{9}$$

and $I_{d_1/2}$ is the identity matrix on $\mathbb{R}^{d_1/2}$. In this case G is the *Heisenberg group* $\mathbb{H}_{d_1/2}$, discussed in the introduction.

Finally, some dilation structures and the corresponding metrics will play an important role in our proofs; we shall work with both isotropic and nonisotropic dilations. First, the natural dilations on the Heisenberg-type groups are the automorphic dilations

$$\delta_r(x, u) := (rx, r^2 u), \quad r > 0, \tag{10}$$

on G. We work with the *Koranyi norm*

$$\|(x,u)\|_{\mathrm{Ko}} := (|x|^4 + |4u|^2)^{1/4},$$

which is a homogeneous norm with respect to the dilations δ_r. Moreover, if we denote the corresponding balls by

$$Q_r(x,u) := \{(y,v) \in G : \|(y,v)^{-1} \cdot (x,u)\|_{\mathrm{Ko}} < r\}, \quad (x,u) \in G,\ r > 0,$$

then the volume $|Q_r(x,u)|$ is given by

$$|Q_r(x,u)| = |Q_1(0,0)|\, r^{d_1+2d_2}.$$

Recall that $d_1 + 2d_2 = d + d_2$ is the *homogeneous dimension* of G.

We will also have to work with a variant of the "Euclidean" balls, i.e., "isotropic balls" skewed by the Heisenberg translation, denoted by $Q_{r,E}(x,u)$:

$$\begin{aligned} Q_{r,E}(x,u) &:= \{(y,v) \in G : |(y,v)^{-1}(x,u)|_E < r\} \\ &= \Big\{(y,v) \in G : |x-y| + \big|u - v + \tfrac{1}{2}\langle \vec{J}x, y\rangle\big| < r\Big\}; \end{aligned} \tag{11}$$

here

$$|(x,u)|_E := |x| + |u|$$

is comparable with the standard Euclidean norm $(|x|^2 + |u|^2)^{1/2}$. Observe that the balls $Q_r(x,u)$ and $Q_{r,E}(x,u)$ are the left translates by (x,u) of the corresponding balls centered at the origin.

The main results. We consider symbols a of class $S^{-\gamma}$, i.e., satisfying the estimates

$$\left|\frac{d^j}{(ds)^j} a(s)\right| \le c_j (1+|s|)^{-\gamma-j} \tag{12}$$

for all $j = 0, 1, 2, \dots$. Our main boundedness result is:

Theorem 1.1. *Let* $1 < p < \infty$, $\gamma(p) := (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$ *and* $a \in S^{-\gamma(p)}$. *Then, for* $-\infty < \tau < \infty$, *the operators* $a(\tau\sqrt{L})e^{i\tau\sqrt{L}}$ *extend to bounded operators on* $L^p(G)$.

The solutions u to the initial value problem (1) satisfy the Sobolev-type inequalities (2) for $\gamma \ge \gamma(p)$.

Our proof also gives sharp L^1 estimates for operators with symbols supported in dyadic intervals.

Theorem 1.2. *Let* $\chi \in C_c^\infty$ *supported in* $\left(\frac{1}{2}, 2\right)$ *and let* $\lambda \ge 1$. *Then the operators* $\chi(\lambda^{-1}\tau\sqrt{L})e^{\pm i\tau\sqrt{L}}$ *extend to bounded operators on* $L^1(G)$, *with operator norms* $O(\lambda^{(d-1)/2})$.

In view of the invariance under automorphic dilations it suffices to prove these results for $\tau = \pm 1$, and, by symmetry considerations, we only need to consider $\tau = 1$.

An interesting question posed in [Müller and Stein 1999] concerns the validity of an appropriate result in the limiting case $p = 1$ (such as a Hardy space bound). Here the situation is more complicated than in the Euclidean case because of the interplay of isotropic and nonisotropic dilations. The usual Hardy spaces $H^1(G)$ are defined using the nonisotropic automorphic dilations (10) together with the Koranyi balls. This geometry is not appropriate for our problem; instead, the estimates for our kernels require a

Hardy space that is defined using isotropic dilations (just as in the Euclidean case) and yet is compatible with the Heisenberg group structure. On the other hand, we shall use a dyadic decomposition of the spectrum of L, which corresponds to a Littlewood–Paley decomposition using nonisotropic dilations.

This space $h^1_{\mathrm{iso}}(G)$ is a variant of the isotropic local or (nonhomogeneous) Hardy space in the Euclidean setting [Goldberg 1979]. To define it we first introduce the appropriate notion of atoms. For $0 < r \le 1$, we define a (P, r) atom as a function b supported in the isotropic Heisenberg ball $Q_{r,E}(P)$ with radius r centered at P (see (11)) such that $\|b\|_2 \le r^{-d/2}$, and $\int b = 0$ if $r \le \frac{1}{2}$. A function f belongs to $h^1_{\mathrm{iso}}(G)$ if $f = \sum c_\nu b_\nu$, where b_ν is a (P_ν, r^ν) atom for some point P_ν and some radius $r_\nu \le 1$, and the sequence $\{c_\nu\}$ is absolutely convergent. The norm on $h^1_{\mathrm{iso}}(G)$ is given by

$$\inf \sum_\nu |c_\nu|,$$

where the infimum is taken over representations of f as a sum $f = \sum_\nu c_\nu b_\nu$ where the b_ν are atoms. It is easy to see that $h^1_{\mathrm{iso}}(G)$ is a closed subspace of $L^1(G)$. The spaces $L^p(G)$, $1 < p < 2$, are complex interpolation spaces for the couple $(h^1_{\mathrm{iso}}(G), L^2(G))$ (see Section 10) and by an analytic interpolation argument Theorem 1.1 can be deduced from an L^2 estimate and the following $h^1_{\mathrm{iso}} \to L^1$ result:

Theorem 1.3. *Let $a \in S^{-(d-1)/2}$. Then the operators $a(\sqrt{L})e^{\pm i\sqrt{L}}$ map the isotropic Hardy space $h^1_{\mathrm{iso}}(G)$ boundedly to $L^1(G)$.*

The norm in the Hardy space $h^1_{\mathrm{iso}}(G)$ is not invariant under the automorphic dilations (10). It is not currently known whether there is a suitable Hardy space result which can be used for interpolation and works for all $a(\tau\sqrt{L})e^{i\tau\sqrt{L}}$ with bounds uniform in τ.

Spectral multipliers. If m is a bounded spectral multiplier, then clearly the operator $m(L)$ is bounded on $L^2(G)$. An important question is then under which additional conditions on the spectral multiplier m the operator $m(L)$ extends from $L^2 \cap L^p(M)$ to an L^p bounded operator for a given $p \neq 2$.

Fix a nontrivial cutoff function $\chi \in C^\infty_0(\mathbb{R})$ supported in the interval $[1, 2]$; it is convenient to assume that $\sum_{k\in\mathbb{Z}} \chi(2^k s) = 1$ for all $s > 0$. Let $L^2_\alpha(\mathbb{R})$ denote the classical Sobolev space of order α. Hulanicki and Stein (see Theorem 6.25 in [Folland and Stein 1982]) proved analogs of the classical Mikhlin–Hörmander multiplier theorem on stratified groups, namely the inequality

$$\|m(L)\|_{L^p\to L^p} \le C_{p,\alpha} \sup_{t>0} \|\chi m(t\,\cdot)\|_{L^2_\alpha} \tag{13}$$

for sufficiently large α. By the work of M. Christ [1991], and also Mauceri and Meda [1990], the inequality (13) holds true for $\alpha > (d + d_2)/2$; in fact, they established a more general result for all stratified groups. Observe that, in comparison to the classical case $G = \mathbb{R}^d$, the homogeneous dimension $d + d_2$ takes over the role of the Euclidean dimension d. However, for the special case of the Heisenberg groups, it was shown by [Müller and Stein 1994] that (13) holds for the larger range $\alpha > d/2$. This result, as well as an extension to Heisenberg-type groups has been proved independently by Hebisch [1993], and Martini [2012] showed that Hebisch's argument can be used to prove a similar result on Métivier groups.

Here we use our estimate on the wave equation to prove, only for Heisenberg-type groups, a result that covers a larger class of multipliers:

Theorem 1.4. *Let G be a group of Heisenberg type with topological dimension d. Let $m \in L^\infty(\mathbb{R})$, let $\chi \in C_0^\infty$ be as above, let*

$$\mathfrak{A}_R := \sup_{t>0} \int_{|s|\geq R} \left|\mathcal{F}_{\mathbb{R}}^{-1}[\chi m(t\,\cdot\,)](s)\right| s^{(d-1)/2}\, ds$$

and assume

$$\|m\|_\infty + \int_2^\infty \mathfrak{A}_R \frac{dR}{R} < \infty. \tag{14}$$

Then the operator $m(\sqrt{L})$ is of weak type $(1,1)$ and bounded on $L^p(G)$, $1 < p < \infty$.

Remarks. (i) Let $H^1(G)$ be the standard Hardy space defined using the automorphic dilations (10). Our proof shows that, under condition (14), $m(\sqrt{L})$ maps $H^1(G)$ to $L^1(G)$.

(ii) By an application of the Cauchy–Schwarz inequality and Plancherel's theorem, the condition

$$\sup_{t>0} \|\chi m(t\,\cdot\,)\|_{L^2_\beta} < \infty \quad \text{for some } \beta > \frac{d}{2}$$

implies $\mathfrak{A}_R \lesssim_\gamma R^{d/2-\beta}$ for $R \geq 2$, and thus Theorem 1.4 covers and extends the above-mentioned multiplier results in [Müller and Stein 1994; Hebisch 1993].

(iii) More refined results for fixed $p > 1$ could be deduced by interpolation, but such results would likely not be sharp.

2. Some notation

Smooth cutoff functions. We denote by ζ_0 an even C^∞ function supported in $(-1, 1)$ and assume that $\zeta_0(s) = 1$ for $|s| \leq \frac{9}{16}$. Let $\zeta_1(s) = \zeta_0(s/2) - \zeta_0(s)$, so that ζ_1 is supported in $\left(-2, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 2\right)$. If we set $\zeta_j(s) = \zeta_1(2^{1-j}s)$, then ζ_j is supported in $(-2^j, -2^{j-2}) \cup (2^{j-2}, 2^j)$ and we have $1 = \sum_{j=0}^\infty \zeta_j(s)$ for all $s \in \mathbb{R}$.

Let η_0 be a C^∞ function supported in $\left(-\frac{5}{8}\pi, \frac{5}{8}\pi\right)$ which has the property that $\eta_0(s) = 1$ for $|s| \leq \frac{3}{8}\pi$ and satisfies $\sum_{k\in\mathbb{Z}} \eta_0(t - k\pi) = 1$ for all $t \in \mathbb{R}$. For $l = 1, 2, \ldots$, let $\eta_l(s) = \eta(2^{l-1}s) - \eta_0(2^l s)$, so that $\eta_0(s) = \sum_{l=1}^\infty \eta_l(s)$ for $s \neq 0$.

Inequalities. We use the notation $A \lesssim B$ to indicate $A \leq CB$ for some constant C. We sometimes use the notation $A \lesssim_\kappa B$ to emphasize that the implicit constant depends on the parameter κ. We use $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

Other notation. We use the definition

$$\hat{f}(\xi) \equiv \mathcal{F} f(\xi) = \int f(y) e^{-2\pi i \langle \xi, y\rangle}\, dy$$

for the Fourier transform in Euclidean space $\mathbb{R}^d$.

The convolution on G is given by

$$f * g(x, u) = \int f(y, v) g\big(x - y, u - v + \tfrac{1}{2}\langle \vec{J} x, y\rangle\big)\, dy\, dv.$$

3. Background on groups of Heisenberg type and the Schrödinger group

For more on the material reviewed here, see, e.g., [Folland 1989; Müller 1999; Müller and Ricci 1996].

The Fourier transform on a group of Heisenberg type. Let us first briefly recall some facts about the unitary representation theory of a Heisenberg-type group G. In many contexts, it is useful to establish analogues of the Bargmann–Fock representations of the Heisenberg group for such groups [Kaplan and Ricci 1983] (compare also [Ricci 1982; Damek and Ricci 1992]). For our purposes, it will be more convenient to work with Schrödinger-type representations. It is well known that these can be reduced to the case of the Heisenberg group $\mathbb{H}_{d_1/2}$, whose product is given by $(z, t) \cdot (z', t') = \big(z + z', t + t' + \tfrac{1}{2}\omega(z, z')\big)$, where ω denotes the *canonical symplectic form* $\omega(z, w) := \langle Jz, w\rangle$, with J as in (9). For the convenience of the reader, we shall outline this reduction to the Heisenberg group.

Let us split coordinates $z = (x, y) \in \mathbb{R}^{d_1/2} \times \mathbb{R}^{d_1/2}$ in $\mathbb{R}^{d_1}$, and consider the associated natural basis of left-invariant vector fields of the Lie algebra of $\mathbb{H}_{d_1/2}$,

$$\tilde{X}_j := \partial_{x_j} - \tfrac{1}{2} y_j \partial_t, \quad \tilde{Y}_j := \partial_{y_j} + \tfrac{1}{2} x_j \partial_t, \quad j = 1, \dots, \tfrac{1}{2} d_1, \qquad \text{and} \qquad T := \partial_t.$$

For $\tau \in \mathbb{R} \setminus \{0\}$, the *Schrödinger representation* ρ_τ of $\mathbb{H}_{d_1/2}$ acts on the Hilbert space $L^2(\mathbb{R}^{d_1/2})$ as follows:

$$[\rho_\tau(x, y, t)h](\xi) := e^{2\pi i \tau (t + y\cdot\xi + y\cdot x/2)} h(\xi + x), \quad h \in L^2(\mathbb{R}^{d_1/2}).$$

This is an irreducible, unitary representation, and every irreducible, unitary representation of $\mathbb{H}_{d_1/2}$ which acts nontrivially on the center is in fact unitarily equivalent to exactly one of these, by the Stone–von Neumann theorem (a good reference for these and related results is [Folland 1989]; see also [Müller 1999]).

Next, if π is any unitary representation, say, of a Heisenberg-type group G, we denote by

$$\pi(f) := \int_G f(g)\pi(g)\, dg, \quad f \in L^1(G),$$

the associated representation of the group algebra $L^1(G)$. For $f \in L^1(G)$ and $\mu \in \mathfrak{g}_2^* = \mathbb{R}^{d_2}$, it will also be useful to define the partial Fourier transform f^μ of f along the center by

$$f^\mu(x) \equiv \mathcal{F}_2 f(x, \mu) := \int_{\mathbb{R}^{d_2}} f(x, u) e^{-2\pi i \mu \cdot u}\, du, \quad x \in \mathbb{R}^{d_1}. \tag{15}$$

Going back to the Heisenberg group (where $\mathfrak{g}_2^* = \mathbb{R}$), if $f \in \mathcal{S}(\mathbb{H}_{d_1/2})$, then it is well known and easily seen that

$$\rho_\tau(f) = \int_{\mathbb{R}^{d_1}} f^{-\tau}(z)\rho_\tau(z, 0)\, dz$$

defines a trace class operator on $L^2(\mathbb{R}^{d_1/2})$, and its trace is given by

$$\operatorname{tr}(\rho_\tau(f)) = |\tau|^{-d_1/2} \int_{\mathbb{R}} f(0,0,t) e^{2\pi i \tau t}\, dt = |\tau|^{-d_1/2} f^{-\tau}(0,0) \tag{16}$$

for every $\tau \in \mathbb{R} \setminus 0$.

From these facts, one derives the Plancherel formula for our Heisenberg-type group G. Given $\mu \in \mathfrak{g}_2^* = \mathbb{R}^{d_2}$, $\mu \neq 0$, consider the matrix J_μ as in (4). By (5) we have $J_\mu^2 = -I$ if $|\mu| = 1$, and J_μ has only eigenvalues $\pm i$. Since it is orthogonal, there exists an orthonormal basis

$$X_{\mu,1}, \ldots, X_{\mu,d_1/2}, Y_{\mu,1}, \ldots, Y_{\mu,d_1/2}$$

of $\mathfrak{g}_1 = \mathbb{R}^{d_1}$ which is symplectic with respect to the form ω_μ, i.e., ω_μ is represented by the standard symplectic matrix J in (9) with respect to this basis.

This means that, for every $\mu \in \mathbb{R}^{d_2} \setminus \{0\}$, there is an orthogonal matrix $R_\mu = R_{\mu/|\mu|} \in O(d_1, \mathbb{R})$ such that

$$J_\mu = |\mu| R_\mu J\, {}^t R_\mu. \tag{17}$$

Condition (17) is in fact equivalent to G being of Heisenberg type.

Now consider the subalgebra $L^1_{\mathrm{rad}}(G)$ of $L^1(G)$ consisting of all "radial" functions $f(x,u)$ in the sense that they depend only on $|x|$ and u. As for Heisenberg groups [Folland 1989; Müller 1999], this algebra is commutative for arbitrary Heisenberg-type groups [Ricci 1982], i.e.,

$$f * g = g * f \quad \text{for every } f,\ g \in L^1_{\mathrm{rad}}(G). \tag{18}$$

This can indeed be reduced to the corresponding result on Heisenberg groups by applying the partial Fourier transform in the central variables.

The following lemma is easy to check and establishes a useful link between representations of G and those of $\mathbb{H}_{d_1/2}$.

Lemma 3.1. *The mapping $\alpha_\mu : G \to \mathbb{H}_{d_1/2}$ given by*

$$\alpha_\mu(z,u) := \left({}^t R_\mu z, \frac{\mu \cdot u}{|\mu|} \right), \qquad (z,u) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2},$$

is an epimorphism of Lie groups. In particular, $G/\ker \alpha_\mu$ is isomorphic to $\mathbb{H}_{d_1/2}$, where $\ker \alpha_\mu = \mu^\perp$ is the orthogonal complement of μ in the center $\mathbb{R}^{d_2}$ of G.

Given $\mu \in \mathbb{R}^{d_2} \setminus \{0\}$, we can now define an irreducible unitary representation π_μ of G on $L^2(\mathbb{R}^{d_1})$ by putting

$$\pi_\mu := \rho_{|\mu|} \circ \alpha_\mu.$$

Observe that then $\pi_\mu(0,u) = e^{2\pi i \mu \cdot u} I$. In fact, any irreducible representation of G with central character $e^{2\pi i \mu \cdot u}$ factors through the kernel of α_μ and hence, by the Stone–von Neumann theorem, must be equivalent to π_μ.

One then computes that, for $f \in \mathcal{S}(G)$,

$$\pi_\mu(f) = \int_{\mathbb{R}^{d_1}} f^{-\mu}(R_\mu z)\rho_{|\mu|}(z,0)\,dz,$$

so that the trace formula (16) yields the analogous trace formula

$$\operatorname{tr}\pi_\mu(f) = |\mu|^{-d_1/2} f^{-\mu}(0)$$

on G. The Fourier inversion formula in $\mathbb{R}^{d_2}$ then leads to

$$f(0,0) = \int_{\mu\in\mathbb{R}^{d_2}\setminus\{0\}} \operatorname{tr}\pi_\mu(f)|\mu|^{d_1/2}\,d\mu.$$

When applied to $\delta_{g^{-1}} * f$, we arrive at the Fourier inversion formula

$$f(g) = \int_{\mu\in\mathbb{R}^{d_2}\setminus\{0\}} \operatorname{tr}(\pi_\mu(g)^*\pi_\mu(f))|\mu|^{d_1/2}\,d\mu, \quad g \in G. \tag{19}$$

Applying this to $f^* * f$ at $g = 0$, where $f^*(g) := \overline{f(g^{-1})}$, we obtain the Plancherel formula

$$\|f\|_2^2 = \int_{\mu\in\mathbb{R}^n\setminus\{0\}} \|\pi_\mu(f)\|_{HS}^2|\mu|^{d_1/2}\,d\mu, \tag{20}$$

where $\|T\|_{HS} = (\operatorname{tr}(T^*T))^{1/2}$ denotes the Hilbert–Schmidt norm.

The sub-Laplacian and the group Fourier transform. Let us next consider the group Fourier transform of our sub-Laplacian L on G.

We first observe that $d\alpha_\mu(X) = {}^tR_\mu X$ for every $X \in \mathfrak{g}_1 = \mathbb{R}^{d_1}$ if we view, for the time being, elements of the Lie algebra as tangential vectors at the identity element. Moreover, by (17), we see that

$${}^tR_\mu X_{\mu,1}, \dots, {}^tR_\mu X_{\mu,d_1/2}, {}^tR_\mu Y_{\mu,1}, \dots, {}^tR_\mu Y_{\mu,d_1/2}$$

forms a symplectic basis with respect to the canonical symplectic form ω on $\mathbb{R}^{d_1}$. We may thus assume without loss of generality that this basis agrees with our basis $\tilde{X}_1, \dots, \tilde{X}_{d_1/2}, \tilde{Y}_1, \dots, \tilde{Y}_{d_1/2}$ of $\mathbb{R}^{d_1}$, so that

$$d\alpha_\mu(X_{\mu,j}) = \tilde{X}_j, \quad d\alpha_\mu(Y_{\mu,j}) = \tilde{Y}_j, \quad j = 1, \dots, \frac{d_1}{2}.$$

By our construction of the representation π_μ, we thus obtain for the derived representation $d\pi_\mu$ of $\mathfrak{g}$ that

$$d\pi_\mu(X_{\mu,j}) = d\rho_{|\mu|}(\tilde{X}_j), \quad d\pi_\mu(Y_{\mu,j}) = d\rho_{|\mu|}(\tilde{Y}_j), \quad j = 1, \dots, \frac{d_1}{2}. \tag{21}$$

Let us define the sub-Laplacians L_μ on G and $\tilde{L}$ on $\mathbb{H}_{d_1/2}$ by

$$L_\mu := -\sum_{j=1}^{d_1/2}(X_{\mu,j}^2 + Y_{\mu,j}^2), \quad \tilde{L} := -\sum_{j=1}^{d_1/2}(\tilde{X}_j^2 + \tilde{Y}_j^2),$$

where from now on we consider elements of the Lie algebra again as left-invariant differential operators. Then, by (21),

$$d\pi_\mu(L_\mu) = d\rho_{|\mu|}(\tilde{L}).$$

Moreover, since the basis $X_{\mu,1}, \dots, X_{\mu,d_1/2}, Y_{\mu,1}, \dots, Y_{\mu,d_1/2}$ and our original basis $X_1, \dots, X_{d_1}$ of $\mathfrak{g}_1$ are both orthonormal bases, it is easy to verify that the distributions $L\delta_0$ and $L_\mu\delta_0$ agree. Since $Af = f*(A\delta_0)$ for every left-invariant differential operator A, we thus have $L = L_\mu$; hence

$$d\pi_\mu(L) = d\rho_{|\mu|}(\tilde{L}). \tag{22}$$

But, it follows immediately from our definition of Schrödinger representation ρ_τ that $d\rho_\tau(\tilde{X}_j) = \partial_{\xi_j}$ and $d\rho_\tau(\tilde{Y}_j) = 2\pi i\tau\xi_j$, so that $d\rho_{|\mu|}(\tilde{L}) = -\Delta_\xi + (2\pi|\mu|)^2|\xi|^2$ is a rescaled Hermite operator (see also [Folland 1989]), and an orthonormal basis of $L^2(\mathbb{R}^{d_1/2})$ is given by the tensor products

$$h_\alpha^{|\mu|} := h_{\alpha_1}^{|\mu|} \otimes \cdots \otimes h_{\alpha_{d_1/2}}^{|\mu|}, \quad \alpha \in \mathbb{N}^{d_1/2},$$

where $h_k^\mu(x) := (2\pi|\mu|)^{1/4} h_k((2\pi|\mu|)^{1/2}x)$, and

$$h_k(x) = c_k(-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2}$$

denotes the L^2-normalized Hermite function of order k on $\mathbb{R}$. Consequently,

$$d\pi_\mu(L) h_\alpha^{|\mu|} = 2\pi|\mu|\left(\frac{d_1}{2} + 2|\alpha|\right) h_\alpha^{|\mu|}, \quad \alpha \in \mathbb{N}^{d_1/2}. \tag{23}$$

It is also easy to see that

$$d\pi_\mu(U_j) = 2\pi i \mu_j I, \quad j = 1, \dots, d_2. \tag{24}$$

Now, the operators $L, -iU_1, \dots, -iU_{d_2}$ form a set of pairwise strongly commuting self-adjoint operators with joint core $\mathcal{S}(G)$, so that they admit a joint spectral resolution, and we can thus give meaning to expressions like $\varphi(L, -iU_1, \dots, -iU_{d_2})$ for each continuous function φ defined on the corresponding joint spectrum. For simplicity of notation we write

$$U := (-iU_1, \dots, -iU_{d_2}).$$

If φ is bounded, then $\varphi(L, U)$ is a bounded, left-invariant operator on $L^2(G)$, so that it is a convolution operator

$$\varphi(L, U)f = f * K_\varphi, \quad f \in \mathcal{S}(G),$$

with a convolution kernel $K_\varphi \in \mathcal{S}'(G)$ which will also be denoted by $\varphi(L, U)\delta$. Moreover, if $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^{d_2})$, then $\varphi(L, U)\delta \in \mathcal{S}(G)$ (see [Müller et al. 1996]). Since functional calculus is compatible with unitary representation theory, we obtain in this case, from (23) and (24), that

$$\pi_\mu(\varphi(L, U)\delta) h_\alpha^{|\mu|} = \varphi\left(2\pi|\mu|\left(\frac{d_1}{2} + 2|\alpha|\right), 2\pi\mu\right) h_\alpha^{|\mu|} \tag{25}$$

(this identity in combination with the Fourier inversion formula could in fact be taken as the definition of $\varphi(L, U)\delta$). In particular, the Plancherel theorem then implies that the operator norm on $L^2(G)$ is given by

$$\|\varphi(L, U)\| = \sup\left\{\left|\varphi\left(|\mu|\left(\frac{d_1}{2} + 2q\right), \mu\right)\right| : \mu \in \mathbb{R}^{d_2},\ q \in \mathbb{N}\right\}. \tag{26}$$

Finally, observe that

$$K_\varphi^\mu = \varphi(L^\mu, 2\pi\mu)\delta; \tag{27}$$

this follows, for instance, by applying the unitary representation induced from the character $e^{2\pi i\mu\cdot u}$ on the center of G to K_φ.

We shall in fact only work with functions of L and $|U|$, defined by

$$\pi_\mu\big(\varphi(L, |U|)\delta\big)h_\alpha^{|\mu|} = \varphi\bigg(2\pi|\mu|\bigg(\frac{d_1}{2} + 2|\alpha|\bigg), 2\pi|\mu|\bigg)h_\alpha^{|\mu|}.$$

Observe that, if φ depends only on the second variable, then $\varphi(|U|)$ is just the radial convolution operator acting only in the central variables, given by

$$\mathcal{F}_{\mathbb{R}^{d_2}}\big[\varphi(|U|)f\big](x,\mu) = \varphi(2\pi|\mu|)\mathcal{F}_{\mathbb{R}^{d_2}} f(x,\mu) \quad \text{for all } \mu \in (\mathbb{R}^{d_2})^*. \tag{28}$$

Partial Fourier transforms and twisted convolution. For $\mu \in \mathfrak{g}_2^*$, one defines the *μ-twisted convolution* of two suitable functions (or distributions) φ and ψ on $\mathfrak{g}_1 = \mathbb{R}^{d_1}$ by

$$(\varphi *_\mu \psi)(x) := \int_{\mathbb{R}^{d_1}} \varphi(x-y)\psi(y)e^{-i\pi\omega_\mu(x,y)}\,dy,$$

where ω_μ is as in (3). Then, with f^μ as in (15),

$$(f * g)^\mu = f^\mu *_\mu g^\mu,$$

where $f * g$ denotes the convolution product of the two functions $f, g \in L^1(G)$. Accordingly, the vector fields X_j are transformed into the μ-twisted first-order differential operators X_j^μ so that $(X_j f)^\mu = X_j^\mu f^\mu$, and the sub-Laplacian is transformed into the μ-twisted Laplacian L^μ, i.e.,

$$(Lf)^\mu = L^\mu f^\mu = -\sum_{j=1}^{d_1}(X_j^\mu)^2,$$

say for $f \in \mathcal{S}(G)$. A computation shows that, explicitly,

$$X_j^\mu = \partial_{x_j} + i\pi\omega_\mu(\,\cdot\,, X_j). \tag{29}$$

The Schrödinger group $\{e^{itL^\mu}\}$. It will be important for us that the Schrödinger operators e^{itL^μ}, $t \in \mathbb{R}$, generated by L^μ, can be computed explicitly.

Proposition 3.2. (i) *For $f \in \mathcal{S}(G)$,*

$$e^{itL^\mu} f = f *_\mu \gamma_t^\mu, \quad t \geq 0, \tag{30}$$

where $\gamma_t^\mu \in \mathcal{S}'(\mathbb{R}^{d_1})$ is a tempered distribution.

(ii) *For all t such that $\sin(2\pi t|\mu|) \neq 0$, the distribution γ_t^μ is given by*

$$\gamma_t^\mu(x) = 2^{-d_1/2}\bigg(\frac{|\mu|}{\sin(2\pi t|\mu|)}\bigg)^{\frac{d_1}{2}} e^{-(i\pi/2)|\mu|\cot(2\pi t|\mu|)|x|^2}. \tag{31}$$

(*iii*) *For all t such that* $\cos(2\pi t|\mu|) \neq 0$, *the Fourier transform of* γ_t^μ *is given by*

$$\hat{\gamma}_t^\mu(\xi) = \frac{1}{(\cos(2\pi t|\mu|))^{d_1/2}} \, e^{(2\pi i/|\mu|)\tan(2\pi t|\mu|)|\xi|^2}. \tag{32}$$

Indeed, for $\mu \neq 0$, let us consider the symplectic vector space $V := \mathfrak{g}_1$, endowed with the symplectic form $\sigma := (1/|\mu|)\omega_\mu$. Notice first that, because of (5), the volume form $\sigma^{\wedge(d_1/2)}$, i.e., the $(d_1/2)$-fold exterior product of σ with itself, can be identified with Lebesgue measure on $\mathbb{R}^{d_1}$. As in [Müller 2007], we then associate to the pair (V, σ) the Heisenberg group $\mathbb{H}_V$, with underlying manifold $V \times \mathbb{R}$ and endowed with the product

$$(v, u)(v', u') := \big(v + v', u + u' + \tfrac{1}{2}\sigma(v, v')\big).$$

It is then common to denote, for $\tau \in \mathbb{R}$, the τ-twisted convolution by $\times_\tau$ in place of $*_\tau$ (compare §5 in [Müller 2007]). The μ-twisted convolution associated to the group G will then agree with the $|\mu|$-twisted convolution $\times_{|\mu|}$ defined on the symplectic vector space (V, σ). Moreover, if we identify the $X_j \in V$ also with left-invariant vector fields on $\mathbb{H}_V$, then (29) shows that

$$X_j^\mu = \partial_{x_j} + i\pi|\mu|\sigma(\,\cdot\,, X_j)$$

agrees with the corresponding $|\mu|$-twisted differential operators $\tilde{X}_j^{|\mu|}$ defined in [Müller 2007].

Accordingly, our μ-twisted Laplacian L^μ will agree with the $|\mu|$-twisted Laplacian

$$\tilde{L}_S^{|\mu|} = \widetilde{\mathscr{L}}^\mu_{-I} = \sum_{j=1}^{d_1} (\tilde{X}_j^{|\mu|})^2$$

associated to the symmetric matrix $A := -I$ in [Müller 2007]. Here,

$$S = -A\frac{1}{|\mu|}J_\mu = \frac{1}{|\mu|}J_\mu.$$

Consequently,

$$e^{itL^\mu} = e^{it\tilde{L}_S^{|\mu|}}.$$

From Theorem 5.5 in [Müller 2007] we therefore obtain that, for $f \in L^2(V)$,

$$\exp\left(\frac{it}{|\mu|}\tilde{L}_S^{|\mu|}\right) f = f \times_{|\mu|} \Gamma_{t,iS}^{|\mu|}, \quad t \geq 0,$$

where $\Gamma_{t,iS}^{|\mu|}$ is a tempered distribution whose Fourier transform is given by

$$\widehat{\Gamma}_{t,iS}^{|\mu|}(\xi) = \frac{1}{\sqrt{\det\cos(2\pi itS)}} e^{-(2\pi/|\mu|)\sigma(\xi, \tan(2\pi itS)\xi)}$$

whenever $\det\cos(2\pi itS) \neq 0$. Since $S^2 = -I$ because of (5), one sees that

$$\sin(2\pi itS) = i\sin(2\pi t)S, \quad \cos(2\pi itS) = \cos(2\pi t)I.$$

Note also that $\sigma(\xi,\eta)=\langle S\xi,\eta\rangle$. We thus see that (30) and (32) hold true, and the formula (31) follows by Fourier inversion (see Lemma 1.1 in [Müller and Ricci 1990]).

4. An approximate subordination formula

We shall use Proposition 3.2 and the following subordination formula to obtain manageable expressions for the wave operators.

Proposition 4.1. *Choose $\chi_1\in C^\infty$ so that $\chi_1(s)=1$ for $s\in\left[\frac14,4\right]$. Let g be a C^∞ function supported in $\left(\frac12,2\right)$. Then there are C^∞ functions a_λ and ρ_λ, depending linearly on g, with a_λ supported in $\left[\frac{1}{16},4\right]$ and ρ_λ supported in $\left[\frac14,4\right]$, such that, for all $K=2,3,\dots$, all $N_1,N_2\ge 0$, and all $\lambda\ge 1$,*

$$\sup_s|\partial_s^{N_1}\partial_\lambda^{N_2}a_\lambda(s)|\le c(K)\lambda^{-N_2}\sum_{\nu=0}^{K}\|g^{(\nu)}\|_\infty,\qquad N_1+N_2<\frac{K-1}{2},\tag{33}$$

$$\sup_s|\partial_s^{N_1}\partial_\lambda^{N_2}\rho_\lambda(s)|\le c(K,N_2)\lambda^{N_1+1-K}\sum_{\nu=0}^{K}\|g^{(\nu)}\|_\infty,\quad N_1\le K-2.\tag{34}$$

and the formula

$$g(\lambda^{-1}\sqrt{L})e^{i\sqrt{L}}=\chi_1(\lambda^{-2}L)\sqrt{\lambda}\int e^{i\lambda/4s}a_\lambda(s)e^{isL/\lambda}\,ds+\rho_\lambda(\lambda^{-2}L)\tag{35}$$

holds. For any $N\in\mathbb{N}$, the functions $\lambda^N\rho_\lambda$ are uniformly bounded in the topology of the Schwartz space and the operators $\rho_\lambda(\lambda^{-2}L)$ are bounded on $L^p(G)$, $1\le p\le\infty$, with operator norm $O(\lambda^{-N})$.

For explicit formulas for a_λ and ρ_λ, see Lemma 4.3 below. The proposition is essentially an application of the method of stationary phase where we keep track on how a_λ and ρ_λ depend on g. We shall need an auxiliary lemma:

Lemma 4.2. *Let $K\in\mathbb{N}$ and $g\in C^K(\mathbb{R})$. Let $\zeta_1\in C^\infty(\mathbb{R})$ be supported in $\left(\frac12,2\right)\cup\left(-2,-\frac12\right)$ and let $\Lambda\ge 1$ and $\ell\ge 0$. Then, for all nonnegative integers M,*

$$\left|\int y^{2M}g(y)\zeta_1(\Lambda^{1/2}2^{-\ell}y)e^{i\Lambda y^2}\,dy\right|\le C_{M,K}2^{-2\ell K}(2^\ell\Lambda^{-1/2})^{1+2M}\sum_{j=0}^{K}(2^\ell\Lambda^{-1/2})^j\|g^{(j)}\|_\infty.\tag{36}$$

Moreover, for $0\le m<(K-1)/2$,

$$\left|\left(\frac{d}{d\Lambda}\right)^m\int g(y)e^{i\Lambda y^2}\,dy\right|\le C_K\Lambda^{-m-1/2}\sum_{j=0}^{K}\Lambda^{-j/2}\|g^{(j)}\|_\infty.\tag{37}$$

Proof. By induction on K we prove the following assertion:

($\mathscr{A}_K$): If $g\in C^K$ then

$$\int y^{2M}g(y)\zeta_1(\Lambda^{1/2}2^{-\ell}y)e^{i\Lambda y^2}\,dy=\Lambda^{-K}\sum_{j=0}^{K}\int g^{(j)}(y)\zeta_{j,K,M,\Lambda}(y)e^{i\Lambda y^2}\,dy,\tag{38}$$

where $\zeta_{j,K,M,\Lambda}$ is supported on $\{y : |y| \in [2^{\ell-1}\Lambda^{-1/2}, 2^{\ell+1}\Lambda^{-1/2}]\}$ and, for $0 \le j \le K$, satisfies the differential inequalities

$$|\zeta^{(n)}_{j,K,M,\Lambda}(y)| \le C(j, K, M, n)(2^{-\ell}\Lambda^{1/2})^{n-2M}2^{-\ell(2K-j)}\Lambda^{K-j/2}. \tag{39}$$

Clearly this assertion implies (36).

We set $\zeta_{0,0,M,\Lambda}(y) = y^{2M}\zeta_1(\Lambda^{1/2}2^{-\ell}y)$ and the claim $(\mathscr{A}_K)$ is immediate for $K = 0$. It remains to show that $(\mathscr{A}_K)$ implies $(\mathscr{A}_{K+1})$ for all $K \ge 0$.

Assume $(\mathscr{A}_K)$ for some $K \ge 0$ and let $g \in C^{K+1}$. We let $0 \le j \le K$ and examine the j-th term in the sum in (38). Integration by parts yields

$$\int g^{(j)}(y)\zeta_{j,K,M,\Lambda}(y)e^{i\Lambda y^2}\,dy = i\int\left[\frac{g^{(j+1)}(y)}{2y\Lambda}\zeta_{j,K,M,\Lambda}(y) + g^{(j)}(y)\frac{d}{dy}\left(\frac{\zeta_{j,K,M,\Lambda}(y)}{2y\Lambda}\right)\right]e^{i\Lambda y^2}\,dy.$$

The sum $\Lambda^{-K}\sum_{j=0}^{K}\int g^{(j)}(y)\zeta_{j,K,M,\Lambda}(y)e^{i\Lambda y^2}\,dy$ can now be rewritten as

$$\Lambda^{-K-1}\sum_{\nu=0}^{K+1}\int g^{(\nu)}(y)\zeta_{\nu,K+1,M,\Lambda}(y)e^{i\Lambda y^2}\,dy,$$

where

$$\begin{aligned}
\zeta_{0,K+1,M,\Lambda}(y) &= i\frac{d}{dy}\left(\frac{\zeta_{0,K,M,\Lambda}(y)}{2y}\right),\\
\zeta_{\nu,K+1,M,\Lambda}(y) &= i\frac{d}{dy}\left(\frac{\zeta_{\nu,K,M,\Lambda}(y)}{2y}\right) + i\frac{\zeta_{\nu-1,K,M,\Lambda}(y)}{2y}, \qquad 1 \le \nu \le K,\\
\zeta_{K+1,K+1,M,\Lambda}(y) &= i\frac{\zeta_{K,K,M,\Lambda}(y)}{2y}.
\end{aligned}$$

On the support of the cutoff functions, we have $|y| \ge 2^{\ell-1}\Lambda^{-1/2}$ and the asserted differential inequalities for the functions $\zeta_{\nu,K+1,M,\Lambda}$ can be verified using the Leibniz rule. This finishes the proof that $(\mathscr{A}_K)$ implies $(\mathscr{A}_{K+1})$, and thus the proof of (36).

We now prove (37). Let ζ_0 be an even C^∞ function supported in $(-1, 1)$ and assume that $\zeta_0(s) = 1$ for $|s| \le \frac{1}{2}$. Let $\zeta_1(s) = \zeta_0(s/2) - \zeta_0(s)$, so that ζ_1 is supported in $\left[-2, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 2\right]$, as in the statement of (36). We split the left-hand side of (37) as $\sum_{\ell=0}^{\infty} I_{\ell,m}$, where

$$I_{\ell,m} = \int (iy^2)^m g(y)\zeta_1(\Lambda^{1/2}2^{-\ell}y)e^{i\Lambda y^2}\,dy \quad \text{for } \ell > 0,$$

and $I_{0,m}$ is defined similarly with $\zeta_0(\Lambda^{1/2}y)$ in place of $\zeta_1(\Lambda^{1/2}2^{-\ell}y)$. Clearly $|I_{0,m}| \lesssim \Lambda^{-m-1/2}\|g\|_\infty$ and, by (36),

$$I_{\ell,m} \lesssim_{m,K} \sum_{j=0}^{K} 2^{-\ell(2K-2m-j-1)}\Lambda^{-(1+2m+j)/2}\|g^{(j)}\|_\infty.$$

Since $j \le K$ we can sum in ℓ if $m < (K-1)/2$ and the assertion (37) follows. □

Lemma 4.3. *Let $K \in \mathbb{N}$ and let $g \in C^K(\mathbb{R})$ be supported in $\left(\frac{1}{2}, 2\right)$. Let $\chi_1 \in C_c^\infty(\mathbb{R})$ be such that $\chi_1(x) = 1$ on $\left(\frac{1}{4}, 4\right)$. Also let ς be a $C_0^\infty(\mathbb{R})$ function supported in $\left[\frac{1}{9}, 3\right]$ with the property that $\varsigma(s) = 1$ on $\left[\frac{1}{8}, 2\right]$. Then*

$$g(\sqrt{x})e^{i\lambda\sqrt{x}} = \chi_1(x)\left[\sqrt{\lambda}\int e^{i\lambda/4s}a_\lambda(s)e^{i\lambda sx}\,ds + \tilde{\rho}_\lambda(x)\right], \tag{40}$$

where a_λ is supported in $\left[\frac{1}{16}, 4\right]$ and

$$a_\lambda(s) = \pi^{-1}\sqrt{\lambda}\varsigma(s)\int\left(y + \frac{1}{2s}\right)g\left(y + \frac{1}{2s}\right)e^{-i\lambda sy^2}\,dy \tag{41}$$

and

$$\mathcal{F}[\tilde{\rho}_\lambda](\xi) = \left(1 - \varsigma\left(\frac{2\pi\xi}{\lambda}\right)\right)\mathcal{F}[g(\sqrt{\cdot})e^{i\lambda\sqrt{\cdot}}](\xi). \tag{42}$$

Let $\rho_\lambda = \chi_1\tilde{\rho}_\lambda$. Then the estimates (33) and (34) hold for all $\lambda \geq 1$.

Proof. Let Ψ_λ be the Fourier transform of $x \mapsto g(\sqrt{x})e^{i\lambda\sqrt{x}}$, i.e.,

$$\Psi_\lambda(\xi) = \int g(\sqrt{x})e^{i\lambda\sqrt{x}}e^{-2\pi i\xi x}\,dx = \int 2sg(s)e^{i(\lambda s - 2\pi\xi s^2)}\,ds. \tag{43}$$

Observe that $g(\sqrt{x}) = 0$ for $x \notin \left(\frac{1}{4}, 4\right)$, thus $g(\sqrt{x}) = \chi_1(x)g(\sqrt{x})$. By the Fourier inversion formula, we have

$$g(\sqrt{x})e^{i\lambda\sqrt{x}} = \chi_1(x)(\upsilon_\lambda(x) + \rho_\lambda(x)),$$

where

$$\begin{aligned}\upsilon_\lambda(x) &= \int \varsigma\left(\frac{2\pi\xi}{\lambda}\right)\Psi_\lambda(\xi)e^{2\pi ix\xi}\,d\xi,\\ \tilde{\rho}_\lambda(x) &= \int\left(1 - \varsigma\left(\frac{2\pi\xi}{\lambda}\right)\right)\Psi_\lambda(\xi)e^{2\pi ix\xi}\,d\xi,\end{aligned} \tag{44}$$

so that $\tilde{\rho}_\lambda$ is as in (42).

We first consider $\tilde{\rho}_\lambda$ and verify that the inequalities (34) hold. On the support of $1 - \varsigma(2\pi\xi/\lambda)$, we have either $|2\pi\xi| \leq \lambda/8$ or $|2\pi\xi| \geq 2\lambda$. Clearly, on the support of g we have $|\partial_s(\lambda s - 2\pi\xi s^2)| \geq \lambda/2$ if $|2\pi\xi| \leq \lambda/8$, and $|\partial_s(\lambda s - 2\pi\xi s^2)| \geq |2\pi\xi|/2$ if $|2\pi\xi| \geq 2\lambda$. Integration by parts in (43) yields

$$\left|\partial_\xi^{M_1}\partial_\lambda^{M_2}\left[(1 - \varsigma(2\pi\xi/\lambda))\Psi_\lambda(\xi)\right]\right| \leq C_{M_1,M_2,K}\|g\|_{C_K}(1 + |\xi| + |\lambda|)^{-K}.$$

Thus, if $N_1 \leq K - 2$,

$$\begin{aligned}\left|\left(\frac{d}{dx}\right)^{N_1}\tilde{\rho}_\lambda(x)\right| &= \left|\int(2\pi\xi)^{N_1}\left(1 - \varsigma\left(\frac{2\pi\xi}{\lambda}\right)\right)\Psi_\lambda(\xi)e^{2\pi ix\xi}\,d\xi\right|\\ &\leq C_{N_1,K}\|g\|_{C^K}\int\frac{(1 + |\xi|)^{N_1}}{(1 + |\xi| + |\lambda|)^K}\,d\xi \leq C'_{N_1,K}\|g\|_{C^K}\lambda^{-K+N_1+1}.\end{aligned}$$

This yields (34) for $N_2 = 0$, and the same argument applies to the λ-derivatives.

It remains to represent the function $\lambda^{-1/2}\upsilon_\lambda$ as the integral in (40). Let

$$\tilde{g}(s) = 2sg(s). \tag{45}$$

By a change of variable, we may write

$$\Psi_\lambda(\xi) = e^{i\lambda^2/(8\pi\xi)} \int \tilde{g}\Big(y + \frac{\lambda}{4\pi\xi}\Big) e^{-2\pi i\xi y^2}\, dy. \tag{46}$$

We compute, from (44), (46),

$$\upsilon_\lambda(x) = \lambda \int \varsigma(s) e^{i\lambda/(4s)+i\lambda s x} \lambda^{-1/2} a_\lambda(s)\, ds,$$

where

$$a_\lambda(s) = (2\pi)^{-1}\sqrt{\lambda}\varsigma(s) \int \tilde{g}\Big(y + \frac{1}{2s}\Big) e^{-i\lambda s y^2}\, dy,$$

i.e., a_λ is as in (41). In order to show the estimate (33), observe

$$2\pi\, \partial_\lambda^{N_2}(\lambda^{-1/2} a_\lambda(s)) = \varsigma(s) \int \tilde{g}\Big(y + \frac{1}{2s}\Big)(-isy^2)^{N_2} e^{-i\lambda s y^2}\, dy$$

and then, by the Leibniz rule, $\partial_s^{N_1}\partial_\lambda^{N_2}[\lambda^{-1/2}a_\lambda(s)]$ is a linear combination of terms of the form

$$\Big(\frac{d}{ds}\Big)^{N_3}[\varsigma(s)s^{N_2}] \int y^{2N_2}(\lambda y^2)^{N_5}\Big(\frac{d}{ds}\Big)^{N_4}\Big[\tilde{g}\Big(y + \frac{1}{2s}\Big)\Big] e^{i\lambda s y^2}\, dy, \tag{47}$$

where $N_3 + N_4 + N_5 = N_1$. By the estimate (37) in Lemma 4.2, we see that the term (47) is bounded (uniformly in $s \in [\frac{1}{9}, 3]$) by a constant times

$$\lambda^{-N_2-1/2} \Big\| \Big(\frac{d}{ds}\Big)^{N_4}\Big[\tilde{g}\Big(\cdot + \frac{1}{2s}\Big)\Big]\Big\|_{C^{K-N_4}}$$

provided that $N_2 + N_5 < (K - N_4 - 1)/2$. This condition is satisfied if $N_1 + N_2 < (K-1)/2$, and under this condition we get

$$\sup_s |\partial_s^{N_1}\partial_\lambda^{N_2}[\lambda^{-1/2}a_\lambda(s)]| \lesssim \lambda^{-N_2-1/2}\|g\|_{C^K}.$$

Now (33) is a straightforward consequence. □

Proof of Proposition 4.1. The identity (35) is an immediate consequence of the spectral resolution $L = \int_{\mathbb{R}^+} x\, dE_x$, Lemma 4.3 (applied with x/λ in place of x) and Fubini's theorem. Note that in view of the symbol estimates (34), any Schwartz norm of $\rho_\lambda(\lambda^{-2}\,\cdot\,)$ is $O(\lambda^{-N})$ for every $N \in \mathbb{N}$. The statement on the operator norms of $\rho_\lambda(\lambda^{-2}L)$ then follows from the known multiplier theorems (such as the original one by Hulanicki and Stein; see [Hulanicki 1984; Folland and Stein 1982]). □

Thus, in order to get manageable formulas for our wave operators, it will be important to get explicit formulas for the Schrödinger group e^{isL}, $s \in \mathbb{R}$.

5. Basic decompositions of the wave operator and statements of refined results

We consider operators $a(\sqrt{L})e^{i\sqrt{L}}$, where $a \in S^{(d-1)/2}$ (satisfying (12) with $\gamma = (d-1)/2$). We split off the part of the symbol supported near 0. Let $\chi_0 \in C_c^\infty(\mathbb{R})$ be supported in $[-1, 1]$; then we observe that the operator $\chi_0(\sqrt{L})\exp(i\sqrt{L})$ extends to a bounded operator on $L^p(G)$ for $1 \le p \le \infty$. To see this, we decompose $\chi_0(\sqrt{\tau})e^{i\sqrt{\tau}} = \chi_0(\sqrt{\tau}) + \sum_{n=0}^{\infty} \alpha_n(\tau)$, $\tau > 0$, where

$$\alpha_n(\tau) = \chi_0(\sqrt{\tau})(e^{i\sqrt{\tau}} - 1)(\zeta_0(2^{n-1}\tau) - \zeta_0(2^n\tau))$$

with ζ_0 as in Section 2. Clearly $\chi_0(\sqrt{\cdot}) \in C_0^\infty$. Thus, by Hulanicki's theorem [1984], the convolution kernel of $\chi_0(\sqrt{L})$ is a Schwartz function and hence $\chi_0(\sqrt{L})$ is bounded on $L^1(G)$. Moreover, the functions $2^{n/2}\alpha_n(2^{-n}\,\cdot)$ belong to a bounded set of Schwartz functions supported in $[-2, 2]$. By dilation invariance and Hulanicki's theorem again, the convolution kernels of $2^{n/2}\alpha_n(2^{-n}L)$ are Schwartz functions and form a bounded subset of the Schwartz space $\mathcal{S}(G)$. Thus, by rescaling, the operator $\alpha_n(L)$ is bounded on $L^1(G)$ with operator norm $O(2^{-n/2})$. We may sum in n and obtain the desired bounds for $\chi_0(\sqrt{\tau})e^{i\sqrt{\tau}}$.

The above also implies that, for any λ, the operator $\chi(\lambda^{-1}\sqrt{L})\exp(i\sqrt{L})$ is bounded on L^1 (with a polynomial and nonoptimal growth in λ). Thus, in what follows, it suffices to consider symbols $a \in S^{-(d-1)/2}$ with the property that $a(s) = 0$ in a neighborhood of 0. Then

$$a(\sqrt{L})e^{i\sqrt{L}} = \sum_{j>C} 2^{-j(d-1)/2} g_j(\sqrt{2^{-2j}L})e^{i\sqrt{L}}, \tag{48}$$

where the g_j form a family of smooth functions supported in $\left(\frac{1}{2}, 2\right)$ and bounded in the C_0^∞ topology. In many calculations below, when j is fixed, we shall also use the parameter λ for 2^j.

Let χ_1 be a smooth function such that

$$\operatorname{supp}(\chi_1) \subset (2^{-10}, 2^{10}), \tag{49a}$$

$$\chi_1(s) = 1 \quad \text{for } s \in (2^{-9}, 2^9). \tag{49b}$$

By Proposition 4.1 and Lemma 4.3 we may thus write

$$a(\sqrt{L})e^{i\sqrt{L}} = m_{\text{negl}}(L) + \sum_{j>100} 2^{-j(d-1)/2}\chi_1(2^{-2j}L)m_{2^j}(L), \tag{50}$$

where the "negligible" operator $m_{\text{negl}}(L)$ is a convolution with a Schwartz kernel,

$$m_\lambda(\rho) = \sqrt{\lambda}\int e^{i\lambda/(4\tau)}a_\lambda(\tau)e^{i\tau\rho/\lambda}\,d\tau \quad \text{with} \quad \lambda = 2^j, \tag{51}$$

and the a_λ form a family of smooth functions supported in $\left(\frac{1}{16}, 4\right)$ and bounded in the C_0^∞ topology.

We shall use the formulas (31), which give explicit expressions for the partial Fourier transform in the central variables of the Schwartz kernel of e^{itL}. In undoing this partial Fourier transform, it will be useful to recall from Section 3 that, if ρ_1 denotes the spectral parameter for L, then the joint spectrum of the operators L and $|U|$ is contained in the closure of

$$\left\{(\rho_1, \rho_2) : \rho_2 \ge 0,\ \rho_1 = \left(\tfrac{1}{2}d_1 + 2q\right)\rho_2 \text{ for some nonnegative integer } q\right\}. \tag{52}$$

As the phase in (31) exhibits periodic singularities, it is natural to introduce an equally spaced decomposition in the central Fourier variable (i.e., in the spectrum of the operator $|U|$). Let η_0 be a C^∞

function such that

$$\operatorname{supp}(\eta_0) \subset \left(-\tfrac{5}{8}\pi, \tfrac{5}{8}\pi\right), \tag{53a}$$

$$\eta_0(s) = 1 \quad \text{for } s \in \left(-\tfrac{3}{8}\pi, \tfrac{3}{8}\pi\right), \tag{53b}$$

$$\sum_{k\in\mathbb{Z}} \eta_0(t - k\pi) = 1 \quad \text{for } t \in \mathbb{R}. \tag{53c}$$

We decompose

$$\chi_1(\lambda^{-2}L)m_\lambda(L) = \sum_{k=0}^{\infty} \chi_1(\lambda^{-2}L)T_\lambda^k, \tag{54}$$

where

$$T_\lambda^k = \lambda^{1/2} \int e^{i\lambda/(4\tau)} a_\lambda(\tau)\eta_0\Big(\frac{\tau}{\lambda}|U| - k\pi\Big)e^{i\tau L/\lambda}\, d\tau. \tag{55}$$

The description (52) of the joint spectrum of L and $|U|$ gives a restriction on the summation in k. Namely the operator $\eta_0((\tau/\lambda)|U| - k\pi)\chi_1(\lambda^{-2}L)$ is identically zero unless there exist positive ρ_1 and ρ_2 with $\rho_1 \geq \rho_2 d_1/2$ such that $\lambda^2/5 < \rho_1 < 5\lambda^2$ and $\big(k\pi - \frac{5}{8}\pi\big)\lambda/\tau < \rho_2 < \big(k\pi + \frac{5}{8}\pi\big)\lambda/\tau$ for some $\tau \in \big(\frac{1}{16}, 4\big)$. A necessary condition for these two conditions to hold simultaneously is, of course, $\frac{1}{2}d_1\big(k\pi - \frac{5}{8}\pi\big)\lambda/4 \leq 5\lambda^2$ and, since $d_1 \geq 2$ and $\lambda \geq 1$, we see that the sum in (54) extends only over k with

$$0 \leq k < 8\lambda. \tag{56}$$

We now derive formulas for the convolution kernels of T_λ^k, which we denote by K_λ^k. The identity (31) first gives formulas for the partial Fourier transforms $\mathcal{F}_{\mathbb{R}^{d_2}} K_\lambda^k$. Applying the Fourier inversion formula, we get

$$K_\lambda^k(x, u) = \lambda^{1/2} \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}} e^{i\lambda/(4\tau)} a_\lambda(\tau)\eta_0\Big(2\pi|\mu|\frac{\tau}{\lambda} - k\pi\Big)\Big(\frac{|\mu|}{2\sin(2\pi|\mu|\tau/\lambda)}\Big)^{\frac{d_1}{2}} \times e^{-(i\pi/2)|x|^2|\mu|\cot(2\pi|\mu|\tau/\lambda)}\, d\tau\, e^{2\pi i\langle u,\mu\rangle}\, d\mu. \tag{57}$$

We note that the term $|\mu|\cot(2\pi t|\mu|)$ in (57) is singular for $2t|\mu| \in \mathbb{Z}\setminus\{0\}$, and therefore we shall treat the operator T_λ^0 separately from T_λ^k for $k > 0$. We shall see that T_λ^0 and the operators $\sum_j \chi(2^{-2j}L)T_{2^j}^0$ can be handled using known results about Fourier integral operators, while the operators $T_{2^j}^k$ need a more careful treatment due to the singularities of the phase function. We shall see that the decomposition into the operators $T_{2^j}^k$ encodes useful information on the singularities of the wave kernels.

In Sections 7 and 8, we shall prove the following L^1 estimates:

Theorem 5.1. *(i) For* $\lambda \geq 2^{10}$,

$$\|T_\lambda^0\|_{L^1\to L^1} \lesssim \lambda^{(d-1)/2}. \tag{58}$$

(ii) For $\lambda \geq 2^{10}$, $k = 1, 2, \ldots,$

$$\|T_\lambda^k\|_{L^1\to L^1} \lesssim k^{-(d_1+1)/2}\lambda^{(d-1)/2}. \tag{59}$$

Note that $d_1 \geq 2$ and thus the estimates (59) can be summed in k. Hence Theorem 1.2 is an immediate consequence of Theorem 5.1.

Dyadic decompositions. For the Hardy space bounds, we shall need to combine the dyadic pieces in j and also refine the dyadic decomposition in (50).

Define

$$V_j = 2^{-j(d-1)/2}\chi_1(2^{-2j}L)T^0_{2^j}, \tag{60}$$
$$W_j = 2^{-j(d-1)/2}\chi_1(2^{-2j}L)(m_{2^j}(L) - T^0_{2^j}). \tag{61}$$

In Section 6 we shall use standard estimates on Fourier integral operators to prove:

Theorem 5.2. *The operator* $\mathcal{V} = \sum_{j>100} V_j$ *extends to a bounded operator from* h^1_{iso} *to* L^1.

We further decompose the pieces W_j in (61) and let

$$\begin{aligned} W_{j,0} &= \zeta_0(2^{-j}|U|)W_j,\\ W_{j,n} &= \zeta_1(2^{-j-n}|U|)W_j; \end{aligned} \tag{62}$$

here again, ζ_0 and ζ_1 are as in Section 2, i.e., ζ_0 is supported in $(-1, 1)$ and ζ_1 is supported in $\pm\big(\frac{1}{2}, 2\big)$ with $\zeta_0 + \sum_j \zeta_1(2^{1-j}\,\cdot\,) \equiv 1$.

By the description (52) of the joint spectrum of L and $|U|$ and the support property (49a), we also have

$$\chi_1(2^{-2j}L)\zeta_1(2^{-j-n}|U|) = 0 \quad \text{when } 2^{2j+10} \le 2^{j+n-1},$$

i.e., when $j \le n - 11$, and thus

$$W_{j,n} = 0 \quad \text{when } n \ge j + 11. \tag{63}$$

Observe from (52), as in the discussion following (55), that, for $k = 1, 2, \dots,$

$$\zeta_0(2^{-j}\rho_2)\eta_0\Big(\frac{\tau}{2^j}\rho_2 - k\pi\Big) = 0 \quad \text{for } \tau \in \big(\tfrac{1}{16}, 4\big),\ \rho_2 \ge 0, \quad \text{if } 2^j \le \big(k - \tfrac{5}{8}\big)\pi\frac{2^j}{4},$$

and

$$\zeta_1(2^{j-n}\rho_2)\eta_0\Big(\frac{\tau}{2^j}\rho_2 - k\pi\Big) = 0 \quad \text{for } \tau \in \big(\tfrac{1}{16}, 4\big),\ \rho_2 \ge 0,$$
$$\text{if } 2^{j+n+1} \le 2^j\big(k - \tfrac{5}{8}\big)\frac{\pi}{4} \text{ or } 16\cdot 2^j\big(k + \tfrac{5}{8}\big)\pi \le 2^{j+n-1}.$$

Thus we have, for $k = 1, 2, \dots,$

$$\begin{aligned} \zeta_0(2^{-j}|U|)T^k_{2^j} &= 0 \quad \text{when } k \ge 2,\\ \zeta_1(2^{-j-n}|U|)T^k_{2^j} &= 0 \quad \text{when } k \notin [2^{n-8}, 2^{n+2}]. \end{aligned}$$

Let

$$\mathcal{J}_n = \begin{cases} \{1\} & \text{if } n = 0,\\ \{k : 2^{n-8} \le k \le 2^{n+2}\} & \text{if } n \ge 1. \end{cases} \tag{64}$$

Then, by (54), we have $m_{2^j}(L) - T^0_{2^j} = \sum_{k=1}^{\infty} T^k_{2^j}$ and therefore we get

$$W_{j,0} = 2^{-j(d-1)/2}\chi_1(2^{-2j}L)\zeta_0(2^{-j}|U|)\sum_{k\in\mathcal{J}_0} T^k_{2^j}, \tag{65a}$$

$$W_{j,n} = 2^{-j(d-1)/2}\chi_1(2^{-2j}L)\zeta_1(2^{-j-n}|U|)\sum_{k\in\mathcal{J}_n} T^k_{2^j}. \tag{65b}$$

Observe that Theorem 5.1 implies

$$\|W_{j,n}\|_{L^1\to L^1} \lesssim 2^{-n(d_1-1)/2} \tag{66}$$

uniformly in j.

Define, for $n = 0, 1, 2, \dots,$

$$\mathcal{W}_n = \sum_{j>100} W_{j,n}. \tag{67}$$

Theorem 1.3 will then be a consequence of Theorem 5.2 and:

Theorem 5.3. *The operators $\mathcal{V}$ and $\mathcal{W}_n$ are bounded from h^1_{iso} to L^1; moreover,*

$$\|\mathcal{W}_n\|_{h^1_{\mathrm{iso}}\to L^1} \lesssim (1+n)2^{-n(d_1-1)/2} \tag{68}$$

The proofs will be given in Section 6 and Section 9.

6. Fourier integral estimates

In this section we shall reduce the proof of the estimates for T^0_λ and $\mathcal{V}$ in Theorems 5.1 and 5.3 to standard bounds for Fourier integral operators in [Seeger et al. 1991] or [Beals 1982].

We will prove a preliminary lemma that allows us to add or suppress $\chi_1(\lambda^{-2}L)$ from the definition of T^0_λ.

Lemma 6.1. *For $\lambda > 2^{10}$, we have*

$$\|T^0_\lambda - \chi_1(\lambda^{-2}L)T^0_\lambda\|_{L^1\to L^1} \lesssim C_N\lambda^{-N}$$

for any N.

Proof. The operator $T^0_\lambda - \chi_1(\lambda^{-2}L)T^0_\lambda$ can be written as $b_\lambda(|L|, |U|)$, where

$$b_\lambda(\rho_1,\rho_2) = \lambda^{1/2}(1-\chi_1(\lambda^{-2}\rho_1))\lambda^{1/2}\int a_\lambda(\tau)e^{i\varphi(\tau,\rho_1,\lambda)}\eta_0\left(\frac{\tau\rho_2}{\lambda}\right)d\tau$$

with

$$\varphi(\tau,\rho_1,\lambda) = \frac{\lambda}{4\tau} + \frac{\tau\rho_1}{\lambda}.$$

Only the values where $\rho_1 \le \lambda^2 2^{-9}$ and $\rho_1 \ge 2^9\lambda^2$ are relevant. Now

$$\frac{\partial\varphi}{\partial\tau} = -\frac{\lambda}{4\tau^2} + \frac{\rho_1}{\lambda}$$

and $(\partial/\partial\tau)^n\varphi = c_n\lambda\tau^{-n-1}$ for $n \ge 2$. Note that, for $\rho_1 \ge 2^9\lambda^2$, we have $|\varphi'_\tau| \ge \rho_1/\lambda - \frac{16^2}{4}\lambda \ge \rho_1\lambda^{-1}(1-2^{-9}2^6) \ge \rho_1/(2\lambda)$. Similarly, for $\rho_1 \le 2^{-9}\lambda^2$, we have $|\varphi'_\tau| \ge \lambda/16 - 16\cdot 2^{-9}\lambda \ge 2^{-5}\lambda$. We use integrations by parts to conclude that

$$\left|\frac{\partial^{n_1+n_2}[b_\lambda(\lambda^2\cdot,\lambda\cdot)]}{(\partial\rho_1)^{n_1}(\partial\rho_2)^{n_2}}(\rho_1,\rho_2)\right| \le C_{n_1,n_2,N}\lambda^{-N}$$

and, in view of the compact support of $b_\lambda(\lambda^2\rho_1, \lambda\rho_2)$, the assertion can be deduced from a result in [Müller et al. 1996] (or alternatively from Hulanicki's result [1984] and a Fourier expansion in ρ_2). □

The convolution kernel for T^0_λ. This is given by

$$K^0_\lambda(x,u) = \lambda^{1/2}\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}} e^{i\lambda/(4s)}a_\lambda(s)\eta_0\Big(2\pi|\mu|\frac{s}{\lambda}\Big)\Big(\frac{|\mu|}{2\sin(2\pi|\mu|s/\lambda)}\Big)^{\frac{d_1}{2}} \times e^{-(i\pi/2)|x|^2|\mu|\cot(2\pi|\mu|s/\lambda)}\,ds\,e^{2\pi i\langle u,\mu\rangle}\,d\mu.$$

We introduce frequency variables $\theta = (\omega,\sigma)$ on the cone

$$\Gamma_\delta = \{\theta = (\omega,\sigma) \in \mathbb{R}^{d_2}\times\mathbb{R} : |\omega| \le (\pi-\delta)\sigma,\ \sigma > 0\}. \tag{69}$$

Set

$$\omega = \frac{\pi\mu}{2}, \qquad \sigma = \frac{\lambda}{4s}.$$

Note that $\sigma \approx \lambda$ for $s \in \mathrm{supp}(a_\lambda)$. We will consider the case $\delta = \frac14\pi$ in view of the support of η_0, but any choice of $\delta \in \big(0, \frac14\pi\big)$ is permissible with some constants below depending on δ.

If we set

$$g(\tau) := \tau\cot\tau, \tag{70}$$

the above integral becomes

$$K^0_\lambda(x,u) = \iint e^{i\Psi(x,u,\omega,\sigma)}\beta_\lambda(\omega,\sigma)\,d\omega\,d\sigma \tag{71}$$

with

$$\Psi(x,u,\omega,\sigma) = \sigma\Big(1 - |x|^2 g\Big(\frac{|\omega|}{\sigma}\Big)\Big) + \langle 4u,\omega\rangle$$

and

$$\beta_\lambda(\omega,\sigma) = 4^{-1}\Big(\frac{2}{\pi}\Big)^{\frac{d_1}{2}+d_2}\lambda^{3/2}\sigma^{d_1/2-2}a_\lambda\Big(\frac{\lambda}{4\sigma}\Big)\eta_0\Big(\frac{|\omega|}{\sigma}\Big)\Big(\frac{|\omega|/\sigma}{2\sin(|\omega|/\sigma)}\Big)^{\frac{d_1}{2}}.$$

The β_λ are symbols of order $(d_1-1)/2$ uniformly in λ and supported in Γ. The same applies to $\sum_{k>10}\beta_{2^k}$.

We will need formulas for the derivatives of Ψ with respect to the frequency variables $\theta = (\omega,\sigma)$:

$$\begin{aligned}\frac{\partial\Psi}{\partial\omega_i} &= 4u_i - |x|^2\frac{\omega_i}{\sigma}\frac{g'(|\omega|/\sigma)}{|\omega|/\sigma},\\ \frac{\partial\Psi}{\partial\sigma} &= 1 - |x|^2\Big(g\Big(\frac{|\omega|}{\sigma}\Big) - \frac{|\omega|}{\sigma}g'\Big(\frac{|\omega|}{\sigma}\Big)\Big).\end{aligned} \tag{72}$$

Now, g is analytic for $|\tau| < 2\pi$ and we have

$$g'(\tau) = \frac{\sin(2\tau) - 2\tau}{2\sin^2\tau}, \tag{73a}$$

$$g''(\tau) = \frac{2(\tau\cos\tau - \sin\tau)}{\sin^3\tau}. \tag{73b}$$

Observe that

$$g'(\tau) < 0 \quad \text{and} \quad g''(\tau) < 0 \quad \text{for } 0 < \tau < \pi.$$

Moreover, as $\tau \to 0$,

$$g(\tau) = 1 - \tfrac{1}{3}\tau^2 + O(\tau^4),$$

and hence $g'(0) = 0$ and $g''(0) = -\frac{2}{3}$. The even expression

$$g(\tau) - \tau g'(\tau) = 1 + \int_0^\tau (-s g''(s))\, ds$$

will frequently occur; from the above, we get

$$\begin{aligned} g(\tau) - \tau g'(\tau) &\geq 1 \quad &&\text{for } 0 \leq |\tau| < \pi, \\ |g(\tau) - \tau g'(\tau)| &\leq 10 \quad &&\text{for } 0 \leq |\tau| < \tfrac{3}{4}\pi. \end{aligned} \tag{74}$$

Lemma 6.2. *We have*

$$|K_\lambda^0(x,u)| \lesssim \lambda^{(d_1+2d_2+1)/2-N}(|x|^2 + |u|)^{-N}, \quad |x|^2 + 4|u| > 2, \tag{75}$$

and

$$|K_\lambda^0(x,u)| \lesssim \lambda^{(d_1+2d_2+1)/2-N}(1 + |u|)^{-N}, \quad |x|^2 \leq \tfrac{1}{20}. \tag{76}$$

Proof. If $|x| \geq \sqrt{2}$ we may integrate by parts with respect to σ (using (74)), and obtain

$$|K_\lambda^0(x,u)| \lesssim_N \lambda^{(d_1+2d_2+1)/2-N}|x|^{-N}, \quad |x| \geq \sqrt{2}.$$

If $|u| \leq 10|x|^2$ this also yields (75). Since $\max_{|\tau|\leq 3\pi/4}|g'(\tau)| \leq \frac{3}{2}\pi$, we have $|\nabla_\omega\Psi| \geq 4|u| - \frac{3}{2}\pi|x|^2$, and hence $|\nabla_\omega\Psi| \geq |u|$ when $|u| \geq 10|x|^2$. Thus, integration by parts in ω yields

$$|K_\lambda^0(x,u)| \lesssim_N \lambda^{(d_1+2d_2+1)/2-N}|u|^{-N}, \quad |u| \geq 10|x|^2.$$

This proves (75).

Since $|g'(\tau)| \leq 3\pi$ for $|\tau| \leq \frac{3}{2}\pi$, we have $|\nabla_\omega\Psi| \geq 2|u|$ if $|x|^2 \leq 2|u|/(3\pi)$, and $|\Psi_\sigma| \geq \frac{1}{2}$ if $|x|^2 \leq \frac{1}{20}$. Integrations by parts imply (76). □

Fourier integral operators. Let $\rho \ll 10^{-2}$. Choose $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ so that

$$\chi(x,u,y,v) = 0 \quad \text{for} \begin{cases} |y| + |v| \geq \rho, \\ |x - y| < \frac{1}{20}, \\ |x-y|^2 + |u - v| \geq 4. \end{cases}$$

Let

$$b_\lambda(x,y,u,v,\omega,\sigma) = \chi(x,u,y,v)\beta_\lambda(\omega,\sigma),$$

as before let $g(\tau)=\tau\cot\tau$, and let

$$\begin{aligned}\Phi(x,u,y,v,\omega,\sigma)&=\Psi\big(x-y,u-v+\tfrac12\langle\vec{J}x,y\rangle,\omega,\sigma\big)\\&=\sigma\Big(1-|x-y|^2g\Big(\frac{|\omega|}{\sigma}\Big)\Big)+\sum_{i=1}^{d_2}(4u_i-4v_i-2x^\intercal J_iy)\omega_i.\end{aligned}\tag{77}$$

Let $\mathfrak{F}_\lambda$ be the Fourier integral operator with Schwartz kernel

$$\mathcal{K}_\lambda(x,u,y,v)=\iint e^{i\Phi(x,u,y,v,\omega,\sigma)}b_\lambda(\omega,\sigma)\,d\omega\,d\sigma.\tag{78}$$

Given Lemma 6.2, it suffices to prove the inequalities

$$\|\mathfrak{F}_\lambda\|_{L^1\to L^1}\lesssim\lambda^{(d-1)/2}\tag{79}$$

and

$$\Big\|\sum_{k>C}2^{-k(d-1)/2}\mathfrak{F}_{2^k}\Big\|_{h^1\to L^1}<\infty.\tag{80}$$

To this end we apply results in [Seeger et al. 1991] on Fourier integral operators associated with canonical graphs and now check the required hypotheses.

***Analysis of the phase function* Φ.** We compute the first derivatives:

$$\begin{aligned}\Phi_{x_j}&=-2\sigma(x_j-y_j)g\Big(\frac{|\omega|}{\sigma}\Big)-2\sum_{i=1}^{d_2}\omega_ie_j^\intercal J_iy,\\ \Phi_{u_i}&=4\omega_i,\\ \Phi_{\omega_i}&=-|x-y|^2g'\Big(\frac{|\omega|}{\sigma}\Big)\frac{\omega_i}{|\omega|}+4u_i-4v_i-2x^\intercal J_iy,\\ \Phi_\sigma&=\Big(1-|x-y|^2g\Big(\frac{|\omega|}{\sigma}\Big)\Big)+|x-y|^2\frac{|\omega|}{\sigma}g'\Big(\frac{|\omega|}{\sigma}\Big).\end{aligned}$$

For the second derivatives we have, with δ_{jk} denoting the Kronecker delta and $J^\omega=\sum_{i=1}^{d_2}\omega_iJ_i$,

$$\begin{aligned}\Phi_{x_jy_k}&=2\sigma g\Big(\frac{|\omega|}{\sigma}\Big)\delta_{jk}-2e_j^\intercal J^\omega e_k,\\ \Phi_{x_jv_l}&=0,\\ \Phi_{x_j\omega_l}&=-2(x_j-y_j)g'\Big(\frac{|\omega|}{\sigma}\Big)\frac{\omega_l}{|\omega|}-2e_j^\intercal J_ly,\\ \Phi_{x_j\sigma}&=2(x_j-y_j)\Big(-g\Big(\frac{|\omega|}{\sigma}\Big)+\frac{|\omega|}{\sigma}g'\Big(\frac{|\omega|}{\sigma}\Big)\Big),\end{aligned}$$

and

$$\Phi_{u_iy_k}=0,\qquad\Phi_{u_iv_l}=0,\qquad\Phi_{u_i\omega_l}=4\delta_{il},\qquad\Phi_{u_i\sigma}=0.$$

Moreover,

$$\begin{aligned}
\Phi_{\omega_i y_k} &= 2(x_k - y_k) g'\Big(\frac{|\omega|}{\sigma}\Big)\frac{\omega_i}{|\omega|} - 2x^{\intercal} J_i e_k,\\
\Phi_{\omega_i v_l} &= -4\delta_{il},\\
\Phi_{\omega_i \omega_l} &= -|x-y|^2\Big(g'\Big(\frac{|\omega|}{\sigma}\Big)\frac{\delta_{il}|\omega|^2 - \omega_i\omega_l}{|\omega|^3} + g''\Big(\frac{|\omega|}{\sigma}\Big)\frac{\omega_i\omega_l}{\sigma|\omega|^2}\Big),\\
\Phi_{\omega_i \sigma} &= |x-y|^2 \frac{\omega_i}{\sigma^2} g''\Big(\frac{|\omega|}{\sigma}\Big),
\end{aligned}$$

and

$$\begin{aligned}
\Phi_{\sigma y_k} &= 2(x_k - y_k)\Big(g\Big(\frac{|\omega|}{\sigma}\Big) - \frac{|\omega|}{\sigma} g'\Big(\frac{|\omega|}{\sigma}\Big)\Big),\\
\Phi_{\sigma v_l} &= 0,\\
\Phi_{\sigma \omega_l} &= |x-y|^2 \frac{\omega_l}{\sigma^2} g''\Big(\frac{|\omega|}{\sigma}\Big),\\
\Phi_{\sigma\sigma} &= -|x-y|^2 \frac{|\omega|^2}{\sigma^3} g''\Big(\frac{|\omega|}{\sigma}\Big).
\end{aligned}$$

The required L^2 boundedness properties follow if we can show that the associated canonical relation is locally the graph of a canonical transformation; this follows from the invertibility of the matrix

$$\begin{pmatrix}
\Phi_{xy} & \Phi_{xv} & \Phi_{x\omega} & \Phi_{x\sigma}\\
\Phi_{uy} & \Phi_{uv} & \Phi_{u\omega} & \Phi_{u\sigma}\\
\Phi_{\omega y} & \Phi_{\omega v} & \Phi_{\omega\omega} & \Phi_{\omega\sigma}\\
\Phi_{\sigma y} & \Phi_{\sigma v} & \Phi_{\sigma\omega} & \Phi_{\sigma\sigma}
\end{pmatrix}; \tag{81}$$

see [Hörmander 1971]. This matrix is given by

$$\begin{pmatrix}
2\sigma g I_{d_1} - 2J^{\omega} & 0 & (*)_{13} & 2(x-y)(\tau g' - g)\\
0 & 0 & 4I_{d_2} & 0\\
(*)_{31} & -4I_{d_2} & (*)_{33} & (*)_{34}\\
2(x-y)^{\intercal}(g - \tau g') & 0 & (*)_{43} & -|x-y|^2\sigma^{-1}\tau^2 g''
\end{pmatrix},$$

where $\tau = |\omega|/\sigma$, g, g', g'' are evaluated at $\tau = |\omega|/\sigma$, $x-y$ is considered a $d_1 \times 1$ matrix, $(*)_{13}$ is a $d_1 \times d_2$ matrix, $(*)_{31}$ is a $d_2 \times d_1$ matrix, $(*)_{33}$ is a $d_2 \times d_2$ matrix, $(*)_{34}$ is a $d_2 \times 1$ matrix, and $(*)_{43} = (*)_{34}^{\intercal}$.

The determinant D of the displayed matrix is equal to

$$D = 16^{d_2} \det\begin{pmatrix}
2\sigma g I_{d_1} - 2J^{\omega} & 2(x-y)(\tau g' - g)\\
2(x-y)^{\intercal}(g - \tau g') & -|x-y|^2\sigma^{-1}\tau^2 g''
\end{pmatrix}. \tag{82}$$

To compute this, we use the formula

$$\begin{pmatrix} I & 0\\ a^{\intercal} & 1\end{pmatrix}\begin{pmatrix} A & -b\\ b^{\intercal} & \gamma\end{pmatrix}\begin{pmatrix} I & -a\\ 0 & 1\end{pmatrix} = \begin{pmatrix} A & -Aa - b\\ a^{\intercal}A + b^{\intercal} & -a^{\intercal}Aa - 2a^{\intercal}b + \gamma\end{pmatrix}.$$

If A is invertible, we can choose $a = -A^{-1}b$. Since $b^\intercal S b = 0$ for the skew-symmetric matrix $S = (A^{-1})^\intercal - A^{-1}$, this choice of a yields the matrix

$$\begin{pmatrix} A & 0 \\ -b^\intercal (A^{-1})^\intercal A + b^\intercal & -b^\intercal (A^{-1})^\intercal b - 2b^\intercal A^{-1} b + \gamma \end{pmatrix} = \begin{pmatrix} A & 0 \\ * & \gamma + b^\intercal A^{-1} b \end{pmatrix},$$

and hence

$$\det \begin{pmatrix} A & -b \\ b^\intercal & \gamma \end{pmatrix} = (\gamma + b^\intercal A^{-1} b) \det A. \tag{83}$$

Lemma 6.3. *Let $c, \Lambda \in \mathbb{R}$, $c^2 + \Lambda^2 \neq 0$. Let S be a skew-symmetric $d_1 \times d_1$ matrix satisfying $S^2 = -\Lambda^2 I$. Then $cI + S$ is invertible with*

$$(cI + S)^{-1} = \frac{c}{c^2 + \Lambda^2} I - \frac{1}{c^2 + \Lambda^2} S,$$

and $\det(cI + S) = (c^2 + \Lambda^2)^{d_1/2}$.

Proof. $(cI + S)(cI + S)^* = (cI + S)(cI - S) = c^2 I - S^2 = (c^2 + \Lambda^2) I$. □

In our situation (82), we have $A = cI + S$ with

$$c = 2\sigma g\Big(\frac{|\omega|}{\sigma}\Big), \qquad S = -2J^\omega;$$

moreover,

$$\begin{aligned} \Lambda &= 2|\omega|, \\ \gamma &= -|x-y|^2 \sigma^{-1} \Big(\frac{|\omega|}{\sigma}\Big)^2 g''\Big(\frac{|\omega|}{\sigma}\Big), \\ b &= 2(x-y)\Big(g\Big(\frac{|\omega|}{\sigma}\Big) - \frac{|\omega|}{\sigma} g'\Big(\frac{|\omega|}{\sigma}\Big)\Big). \end{aligned}$$

In particular, if we recall that $\tau = |\omega|/\sigma$, we see that

$$\det A = \big((2\sigma g(\tau))^2 + (2|\omega|)^2\big)^{d_1/2} = (2\sigma)^{d_1} \Big(\frac{\tau}{\sin\tau}\Big)^{d_1}.$$

Moreover,

$$\begin{aligned} \gamma + b^\intercal A^{-1} b &= |x-y|^2 \Big(-\frac{|\omega|^2}{\sigma^3} g''\Big(\frac{|\omega|}{\sigma}\Big) + 4\Big(g\Big(\frac{|\omega|}{\sigma}\Big) - \frac{|\omega|}{\sigma} g'\Big(\frac{|\omega|}{\sigma}\Big)\Big)^2 \frac{2\sigma g(|\omega|/\sigma)}{4\sigma^2 g(|\omega|/\sigma)^2 + 4|\omega|^2} \Big) \\ &= \frac{|x-y|^2}{\sigma} \Big(-\tau^2 g''(\tau) + 2(g(\tau) - \tau g'(\tau))^2 \frac{g(\tau)}{g(\tau)^2 + \tau^2} \Big). \end{aligned}$$

From (73a), we get

$$g(\tau) - \tau g'(\tau) = \Big(\frac{\tau}{\sin\tau}\Big)^2,$$

and, in combination with (73b), this implies after a calculation that

$$\gamma + b^\intercal A^{-1} b = \frac{|x-y|^2}{\sigma} 2 \Big(\frac{\tau}{\sin\tau}\Big)^2.$$

Thus we see from (83) that the determinant of the matrix (81) is given by

$$D = 2^{d_1+4d_2+1}\sigma^{d_1-1}\left(\frac{|\omega|/\sigma}{\sin(|\omega|/\sigma)}\right)^{d_1+2}. \tag{84}$$

This shows that $D > 0$ for $|\omega|/\sigma \in [0, \pi)$, and $D \sim \sigma^{d_1-1}$ for $|\omega|/\sigma \in [0, \pi-\delta]$ for every sufficiently small $\delta > 0$. In particular, the matrix (81) is invertible for $|\omega|/\sigma \in [0, \pi-\delta]$.

We now write

$$\mathfrak{F}_\lambda f(x) = \int K_\lambda(x, y) f(y)\, dy,$$

where K_λ is given by our oscillatory integral representation (78). In that formula, we have d_2+1 frequency variables, and thus, given any $\alpha \in \mathbb{R}$, the operator convolution with $\sum_{k>C} \mathfrak{F}_{2^k} 2^{-k\alpha}$ is a Fourier integral operator of order

$$\frac{d_1-1}{2} - \alpha - \frac{d-(d_2+1)}{2} = -\alpha.$$

With these observations, we can now apply the boundedness result of [Seeger et al. 1991] and deduce that

$$\|\mathfrak{F}_\lambda f\|_1 \lesssim \lambda^{(d-1)/2}\|f\|_1 \quad \text{and} \quad \Big\|\sum_{k>C} 2^{-k(d-1)/2}\mathfrak{F}_{2^k} f\Big\|_1 \lesssim 1$$

for atoms supported in B_ρ, in the standard Euclidean Hardy space h_1. But atoms associated to balls centered at the origin are also atoms in our Heisenberg Hardy space h^1_{iso}. Thus, if we also take into account Lemma 6.2 and use invariance under Heisenberg translations, we get

$$\Big\|\sum_{k\ge 0} T^0_{2^k} f\Big\|_1 \lesssim \|f\|_{h^1_{\text{iso}}}.$$

Remark. We also have

$$\begin{pmatrix}\Phi_{\omega\omega} & \Phi_{\omega\sigma}\\ \Phi_{\sigma\omega} & \Phi_{\sigma\sigma}\end{pmatrix} = |x-y|^2 \begin{pmatrix} -\left(g'\left(\frac{|\omega|}{\sigma}\right)\frac{I_{d_2}|\omega|^2-\omega\omega^\intercal}{|\omega|^3} + g''\left(\frac{|\omega|}{\sigma}\right)\frac{\omega\omega^\intercal}{\sigma|\omega|^2}\right) & \frac{\omega}{\sigma^2} g''\left(\frac{|\omega|}{\sigma}\right)\\ \frac{\omega^\intercal}{\sigma^2} g''\left(\frac{|\omega|}{\sigma}\right) & -\frac{|\omega|^2}{\sigma^3} g''\left(\frac{|\omega|}{\sigma}\right)\end{pmatrix},$$

which has maximal rank $d_2+1-1 = d_2$. Thus the above result can also be deduced from [Beals 1982], via the equivalence of phase functions theorem.

7. The operators T^k_λ

We now consider the operators T^λ_k for $k \ge 1$, as defined in (55). In view of the singularities of cot we need a further decomposition in terms of the distance to the singularities. For $l = 1, 2, \ldots$, let $\eta_l(s) = \eta_0(2^{l-1}s) - \eta_0(2^l s)$, so that

$$\eta_0(s) = \sum_{l=1}^{\infty} \eta_l(s) \quad \text{for } s \ne 0.$$

Define

$$T_\lambda^{k,l} = \lambda^{1/2} \int e^{i\lambda/(4\tau)} a_\lambda(\tau)\eta_l\Big(\frac{\tau}{\lambda}|U| - k\pi\Big) e^{i\tau L/\lambda}\, d\tau; \tag{85}$$

then

$$T_\lambda^k = \sum_{l=1}^{\infty} T_\lambda^{k,l}. \tag{86}$$

From the formula (57) for the kernels K_λ^k we get a corresponding formula for the kernels $K_\lambda^{k,l}$, namely

$$K_\lambda^{k,l}(x,u) = \lambda^{1/2} \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}} e^{i\lambda/(4\tau)} a_\lambda(\tau)\eta_l\Big(2\pi|\mu|\frac{\tau}{\lambda} - k\pi\Big)\Big(\frac{|\mu|}{2\sin(2\pi|\mu|\tau/\lambda)}\Big)^{\frac{d_1}{2}} \times e^{-(i\pi/2)|x|^2|\mu|\cot(2\pi|\mu|\tau/\lambda)}\, d\tau\, e^{2\pi i\langle u,\mu\rangle}\, d\mu.$$

Now we use polar coordinates in $\mathbb{R}^{d_2}$ and the fact that the Fourier transform of the surface-carried measure on the unit sphere in $\mathbb{R}^{d_2}$ is given by

$$(2\pi)^{d_2/2}\mathcal{J}_{d_2}(2\pi|u|) \quad \text{with} \quad \mathcal{J}_{d_2}(\sigma) := \sigma^{-(d_2-2)/2} J_{(d_2-2)/2}(\sigma),$$

the standard Bessel function formula (see [Stein and Weiss 1971, p. 154]). Thus,

$$K_\lambda^{k,l}(x,u) = \lambda^{1/2} \int_0^{\infty}\int_{\mathbb{R}} e^{i\lambda/(4\tau)} a_\lambda(\tau)\eta_l\Big(\frac{2\pi\tau\rho}{\lambda} - k\pi\Big)\Big(\frac{\rho}{2\sin(2\pi\tau\rho/\lambda)}\Big)^{\frac{d_1}{2}} \times e^{-(i\pi/2)|x|^2\rho\cot(2\pi\rho\tau/\lambda)}\, d\tau\, (2\pi)^{d_2/2}\mathcal{J}_{d_2}(2\pi\rho|u|)\rho^{d_2-1}\, d\rho.$$

In this integral we introduce new variables

$$(s,t) = \Big(\frac{1}{4\tau}, \frac{2\pi\tau\rho}{\lambda}\Big), \tag{87}$$

so that $(\tau,\rho) = ((4s)^{-1}, 2\lambda ts/\pi)$ with $d\tau\, d\rho = \lambda(2\pi s)^{-1}\, ds\, dt$. Then we obtain, for $k \geq 1$,

$$K_\lambda^{k,l}(x,u) = \lambda^{d_2+(d_1+1)/2} \iint \beta_\lambda(s)\eta_l(t-k\pi)\Big(\frac{t}{\sin t}\Big)^{\frac{d_1}{2}} t^{d_2-1} e^{i\lambda s\psi(t,|x|)}\mathcal{J}_{d_2}(4s\lambda t|u|)\, ds\, dt, \tag{88}$$

where

$$\psi(t,r) = 1 - r^2 t\cot t \tag{89}$$

and

$$\beta_\lambda(s) = 2^{3d_2/2-2}\pi^{-(d_1+d_2)/2} a_\lambda\Big(\frac{1}{4s}\Big)s^{d_1/2+d_2-2}; \tag{90}$$

thus β_λ is C^∞ with bounds uniform in λ, and β_λ is also supported in $\big[\frac{1}{16}, 4\big]$.

In the next two sections we shall prove the L^1 estimates

$$\sum_{k<8\lambda}\sum_{l=0}^{\infty} \iint \lambda^{-(d-1)/2}|K_\lambda^{k,l}(x,u)|\, dx\, du = O(1), \tag{91}$$

and Theorem 5.1 and then also Theorem 1.2 will follow by summing the pieces. Moreover, we shall give some refined estimates which will be used in the proof of Theorem 5.3.

An L^∞ bound for the kernels. The expression

$$\mathfrak{C}_{\lambda,k,l} = \lambda^{1+d_2/2} k^{d_2-1} (2^l k)^{d_1/2} \tag{92}$$

will frequently appear in pointwise estimates, namely as upper bounds for the integrand in the integral defining $\lambda^{-(d-1)/2} K_\lambda^{k,l}$. Note that

$$\|\lambda^{-(d-1)/2} K_\lambda^{k,l}\|_\infty \lesssim 2^{-l} \mathfrak{C}_{\lambda,k,l}; \tag{93}$$

the additional factor of 2^{-l} occurs since the integration in t is over the union of two intervals of length comparable to 2^{-l}.

Formulas for the phase functions. For later reference, we gather some formulas for the t-derivatives of the phase $\psi(t,r) = 1 - r^2 t \cot t$:

$$\psi_t(t,r) = r^2 \left(\frac{t}{\sin^2 t} - \cot t \right) \tag{94a}$$

$$= r^2 \left(\frac{2t - \sin 2t}{2 \sin^2 t} \right); \tag{94b}$$

moreover,

$$\psi_{tt}(t,r) = \frac{2r^2}{\sin^3 t} (\sin t - t \cos t) = \frac{2r^2}{\sin^3 t} \int_0^t \tau \sin \tau \, d\tau. \tag{95}$$

Observe that $\psi_{tt} = 0$ when $\tan t = t$ and $t \neq 0$, and thus $\psi_{tt}(t,r) \approx r^2$ for $0 \le t \le \frac{3}{4}\pi$; namely, we use $(2\sqrt{2}/(3\pi))t \le \sin t \le t$ to get the crude estimate

$$\pi^{-1} r^2 < \psi_{tt}(t,r) < \pi^3 r^2, \quad 0 < t \le \tfrac{3}{4}\pi. \tag{96}$$

It is also straightforward to establish estimates for the higher derivatives:

$$|\partial_t^n \psi(t,r)| \lesssim r^2, \quad |t| \le \tfrac{3}{4}\pi, \tag{97}$$

and

$$\partial_t^n \psi(t,r) = O\left(\frac{r^2 |t|}{|\sin t|^{n+1}} \right) \tag{98}$$

for all t.

Asymptotics in the main case $|u| \gg (k\lambda)^{-1}$. We shall see in the next section that there are straightforward L^1 bounds in the region where $|u| \lesssim (k+1)^{-1}\lambda^{-1}$. We therefore concentrate on the region

$$\{(x,u) : |u| \ge C(k+1)^{-1}\lambda^{-1}\},$$

where we have to take into account the oscillation of the terms $\mathcal{J}_{d_2}(4s\lambda t|u|)$. The standard asymptotics for Bessel functions imply that

$$\mathcal{J}_{d_2}(\sigma) = e^{-i|\sigma|} \varpi_1(|\sigma|) + e^{i|\sigma|} \varpi_2(|\sigma|), \quad |\sigma| \ge 2, \tag{99}$$

where $\varpi_1, \varpi_2 \in S^{-(d_2-1)/2}$ are supported in $\mathbb{R} \setminus [-1, 1]$.

Thus, we may split, for $|u| \gg (k+1)^{-1}\lambda^{-1}$,

$$\lambda^{-(d-1)/2} K_\lambda^{k,l}(x,u) = A_\lambda^{k,l}(x,u) + B_\lambda^{k,l}(x,u), \tag{100}$$

where, with $\mathfrak{C}_{\lambda,k,l}$ as defined in (92),

$$A_\lambda^{k,l}(x,u) = \mathfrak{C}_{\lambda,k,l} \iint \eta_{\lambda,k,l}(s,t) e^{i\lambda s(\psi(t,|x|)-4t|u|)} \varpi_1(4\lambda st|u|)\, dt\, ds, \tag{101}$$

and

$$B_\lambda^{k,l}(x,u) = \mathfrak{C}_{\lambda,k,l} \iint \eta_{\lambda,k,l}(s,t) e^{i\lambda s(\psi(t,|x|)+4t|u|)} \varpi_2(4\lambda st|u|)\, dt\, ds; \tag{102}$$

here, as before, $\psi(t,r) = 1 - r^2 t \cot t$ and, with β_λ as in (90),

$$\eta_{\lambda,0}(s,t) = \beta_\lambda(s)\eta_0(t)\Big(\frac{t}{\sin t}\Big)^{\frac{d_1}{2}} t^{d_2-1}, \tag{103a}$$

$$\eta_{\lambda,k,l}(s,t) = \beta_\lambda(s)\eta_l(t-k\pi)\Big(\frac{t/k}{2^l \sin t}\Big)^{\frac{d_1}{2}} \Big(\frac{t}{k}\Big)^{d_2-1}. \tag{103b}$$

Note that $\|\partial_s^{N_1}\partial_t^{N_2}\eta_{\lambda,k,l}\|_\infty \le C_{N_1,N_2} 2^{lN_2}$. Moreover, if

$$J_{k,l} := \big(k\pi - 2^{-l}\tfrac{5}{4}\pi,\, k\pi - 2^{-l}\tfrac{3}{8}\pi\big] \cup \big[k\pi + 2^{-l}\tfrac{3}{8}\pi,\, k\pi + 2^{-l}\tfrac{5}{4}\pi\big), \tag{104}$$

then

$$\eta_{\lambda,k,l}(s,t) \ne 0 \implies t \in J_{k,l}. \tag{105}$$

The main contribution in our estimates comes from the kernels $A_\lambda^{k,l}$, while the kernels $B_\lambda^{k,l}$ are negligible terms with rather small L^1 norm. The latter will follow from the support properties of $\eta_{\lambda,k,l}$ and the observation that

$$\partial_t(\psi(t,|x|) + 4t|u|) \ne 0, \qquad (x,u) \ne (0,0);$$

see (94b). As a consequence, only the kernels $A_\lambda^{k,l}$ will exhibit the singularities of the kernel away from the origin.

The phase functions and the singular support. We introduce polar coordinates in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ (scaled by a factor of 4 in the latter) and set

$$r = |x|, \qquad v = 4|u|.$$

We define, for *all* $v \in \mathbb{R}$,

$$\phi(t,r,v) := \psi(t,r) - tv = 1 - r^2 t \cot t - tv. \tag{106}$$

Then, from (94b) and (94a),

$$\phi_t(t,r,v) = r^2\Big(\frac{2t - \sin 2t}{2\sin^2 t}\Big) - v = \frac{r^2 t}{\sin^2 t} - \frac{1}{t} + \frac{\phi(t,r,v)}{t}. \tag{107}$$

Moreover, $\phi_{tt} = \psi_{tt}$, and we will use the formulas (95) and (98) for the derivatives of ϕ_t.

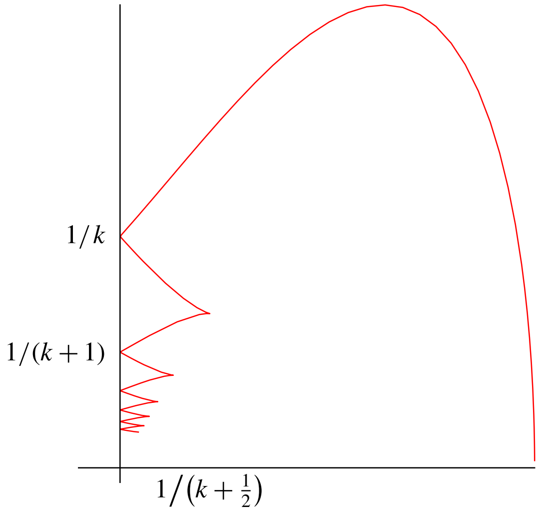

Figure 1. The set $\{\pi(r(t), v(t)) : t > 0\}$.

If we set

$$r(t) = \left|\frac{\sin t}{t}\right|, \qquad r(0) = 1,$$
$$v(t) = \frac{1}{t} - \frac{\sin(2t)}{2t^2}, \quad v(0) = 0, \tag{108}$$

then we have

$$\phi_t(t, r, v) = \frac{v(t)}{r^2(t)}r^2 - v = -\left(v - v(t) - v(t)\frac{r^2 - r(t)^2}{r(t)^2}\right), \tag{109a}$$

$$\phi(t, r, v) = \frac{r(t)^2 - r^2}{r(t)^2} + t\phi_t(t, r, v). \tag{109b}$$

Thus,

$$\phi(t, r, v) = \phi_t(t, r, v) = 0 \iff (r, v) = (r(t), v(t)). \tag{110}$$

Only the points (r, v) for which there exists a t satisfying (110) may contribute to the singular support Γ of $e^{i\sqrt{L}}\delta_0$. One recognizes the result by Nachman [1982], who showed for the Heisenberg group that the singular support of the convolution kernel of $e^{i\sqrt{L}}$ consists of those (x, u) for which there is a $t > 0$ with $(|x|, 4|u|) = (r(t), v(t))$.

Figure 1 illustrates the singular support, including the contribution with $|u|$ near 0 and $|x|$ near 1. However, we have taken care of the corresponding estimates in Section 6, and thus we are only interested in the above formulas for $t > \frac{3}{8}\pi$.

For later reference we gather some formulas and estimates for the derivatives of $r(t)$ and $v(t)$. For the vector of first derivatives we get, for $t \notin \pi\mathbb{Z}$,

$$\begin{pmatrix} r'(t) \\ v'(t) \end{pmatrix} = \frac{\sin t - t\cos t}{t^2}\begin{pmatrix} -\mathrm{sign}(\sin(t)/t) \\ 2t^{-1}\cos t \end{pmatrix} \tag{111}$$

with $r'(t) = O(t)$ and $v'(t) - \frac{2}{3} = O(t)$ as $t \to 0$. Hence, for $t \notin \pi\mathbb{Z}$,

$$\frac{v'(t)}{r'(t)} = -\operatorname{sign}\Big(\frac{\sin t}{t}\Big)\frac{2\cos t}{t} = -2r(t)\cot t. \tag{112}$$

Clearly, all derivatives of t and v extend to functions continuous at $t = 0$. Further computation yields, for positive $t \notin \pi\mathbb{Z}$, $\nu \geq 1$,

$$\operatorname{sign}\Big(\frac{\sin t}{t}\Big)r^{(\nu)}(t) = \sum_{n=1}^{\nu+1} a_{n,\nu}t^{-n}\sin t + \sum_{n=1}^{\nu} b_{n,\nu}t^{-n}\cos t \tag{113a}$$

and

$$v^{(\nu)}(t) = \gamma_\nu t^{-\nu-1} + \sum_{n=1}^{\nu+1} c_{n,\nu}t^{-n-1}\sin 2t + \sum_{n=1}^{\nu} d_{n,\nu}t^{-n-1}\cos 2t\,; \tag{113b}$$

here $a_{n,\nu} = c_{n,\nu} = 0$ if $n-\nu$ is even, and $b_{n,\nu} = d_{n,\nu} = 0$ if $n-\nu$ is odd; moreover, $\gamma_\nu = (-1)^\nu(\nu-1)!$ and $a_{1,\nu} = (-1)^{\nu/2}$ for $\nu = 2, 4, \ldots$. For the coefficients in the first derivatives formula, we get $b_{1,1} = 1$, $a_{2,1} = -1$, $d_{1,1} = -1$, and $c_{2,1} = 1$. For the second derivatives, we have the coefficients $a_{1,2} = -1$, $b_{2,2} = -2$, $a_{3,2} = 2$, $c_{1,2} = 2$, $d_{2,2} = 4$ and $c_{3,2} = -3$. Consequently, for the second derivatives we get the estimates

$$|r''(t)| \lesssim t^{-1}|\sin t| + (1+t)^{-2}, \quad |v''(t)| \lesssim t^{-2}|\sin 2t| + (1+t)^{-3}. \tag{114}$$

Also, $|r^{(\nu)}(t)| \lesssim_\nu (1+t)^{-1}$, and $|v^{(\nu)}(t)| \lesssim_\nu (1+t)^{-2}$ for all $t > 0$.

8. L^1 estimates

In this section we prove the essential L^1 bounds needed for the proof of Theorem 1.2. We may assume that λ is large.

In what follows, we frequently need to perform repeated integrations by parts in the presence of oscillatory terms with nonlinear phase functions, and we start with a standard calculus lemma which will be used several times.

Two preliminary lemmata. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ and choose $\Phi \in C^\infty$ so that $\nabla\Phi \neq 0$ in the support of η. Then, after repeated integration by parts,

$$\int e^{i\lambda\Phi(y)}\eta(y)\,dy = \Big(\frac{i}{\lambda}\Big)^N \int e^{i\lambda\Phi(y)}\mathscr{L}^N\eta(y)\,dy, \tag{115}$$

where the operator $\mathscr{L}$ is defined by

$$\mathscr{L}a = \operatorname{div}\Big(\frac{a\nabla\Phi}{|\nabla\Phi|^2}\Big). \tag{116}$$

In order to analyze the behavior of $\mathscr{L}^N$ we shall need a lemma. We use multiindex notation, i.e., for $\beta = (\beta^1, \ldots, \beta^n) \in (\mathbb{N}\cup\{0\})^n$ we write $\partial^\beta = \partial_{y_1}^{\beta^1}\cdots\partial_{y_n}^{\beta^n}$ and let $|\beta| = \sum_{i=1}^n \beta^i$ be the order of the multiindex.

Lemma 8.1. *Let $\mathscr{L}$ be as in (116). Then $\mathscr{L}^N a$ is a linear combination of $C(N, n)$ terms of the form*

$$\frac{\partial^\alpha a \prod_{\nu=1}^{j} \partial^{\beta_\nu} \Phi}{|\nabla \Phi|^{4N}},$$

where $2N \le j \le 4N-1$ and $\alpha, \beta_1, \ldots, \beta_j$ are multiindices in $(\mathbb{N} \cup \{0\})^n$ with $1 \le |\beta_\nu| \le |\beta_{\nu+1}|$ satisfying:

(1) $0 \le |\alpha| \le N$;

(2) $|\beta_\nu| = 1$ *for* $\nu = 1, \ldots, 2N$;

(3) $|\alpha| + \sum_{\nu=1}^{j} |\beta_\nu| = 4N$;

(4) $\sum_{\nu=1}^{j} (|\beta_\nu| - 1) = N - |\alpha|$.

Proof. Use induction on N. We omit the straightforward details. □

Remark. In dimension $n = 1$, we see that $\mathscr{L}^N a$ is a linear combination of $C(N, 1)$ terms of the form

$$\frac{a^{(\alpha)}}{(\Phi')^\alpha} \prod_{\beta \in \mathfrak{I}} \frac{\Phi^{(\beta)}}{(\Phi')^\beta},$$

where $\mathfrak{I}$ is a set of integers $\beta \in \{2, \ldots, N+1\}$ with the property that $\sum_{\beta \in \mathfrak{I}} (\beta - 1) = N - \alpha$. If $\mathfrak{I}$ is the empty set, then we interpret the product as 1.

In what follows we shall often use:

Lemma 8.2. *Let $\Lambda > 0$, $\rho > 0$, $n \ge 1$ and $N > (n+1)/2$. Then*

$$\int_{-\infty}^{\infty} \frac{(1 + \Lambda |v|)^{-(n-1)/2} |v|^{n-1}}{(1 + \Lambda |\rho - v|)^N} \, dv \lesssim_n \begin{cases} \Lambda^{-(n+1)/2} \rho^{(n-1)/2} & \text{if } \Lambda\rho \ge 1, \\ \Lambda^{-n} & \text{if } \Lambda\rho \le 1. \end{cases}$$

We omit the proof. Lemma 8.2 will usually be applied after using integration by parts with respect to the s variable, with the parameters $n = d_2$ and $\Lambda = \lambda k$.

Estimates for $|u| \lesssim (k+1)^{-1}\lambda^{-1}$. We begin by proving an L^1 bound for the part of the kernels $K_\lambda^{k,l}$ for which the terms $\mathscr{J}_{d_2}(4s\lambda t|u|)$ have no significant oscillation, i.e., for the region where $|u| \le C(\lambda k)^{-1}$ (or $|u| \lesssim \lambda^{-1}$ if $k = 0$).

Lemma 8.3. *Let $\lambda \ge 1$, $k \ge 1$, $l \ge 1$. Then*

$$\iint_{|u| \lesssim (\lambda k)^{-1}} |\lambda^{-(d-1)/2} K_\lambda^{k,l}(x, u)| \, dx \, du \lesssim (2^l k)^{-1} \lambda^{1-d/2}. \tag{117}$$

Proof. First we integrate the pointwise bound (93) over the region where $|x| \le (\lambda k 2^l)^{-1/2}$, $|u| \le (\lambda k)^{-1}$, and obtain

$$\iint\limits_{\substack{|x| \le C(\lambda k 2^l)^{-1/2} \\ |u| \le C(\lambda k)^{-1}}} |\lambda^{-(d-1)/2} K_\lambda^{k,l}(x, u)| \, dx \, du \lesssim 2^{-l} \mathfrak{C}_{\lambda,k,l} (\lambda k 2^l)^{-d_1/2} (\lambda k)^{-d_2} = (2^l k)^{-1} \lambda^{1-(d_1+d_2)/2}.$$

If $|x| \ge C(\lambda k 2^l)^{-1/2}$ then, from (94b) and (98), we get that $|\psi_t(t,|x|)| \gtrsim 2^{2l} k|x|^2$ on the support of $\eta_l(t-k\pi)$; moreover, $(\partial/\partial t)^{(n)}\psi(t,|x|) = O(|x|^2 k 2^{l(n+1)})$. The n-th t-derivative of $\eta_l(t-k\pi)\mathcal{J}_{d_2}(4s\lambda t|u|)$ is $O(2^{ln})$. Thus an integration by parts gives

$$\lambda^{-(d-1)/2}|K_\lambda^{k,l}(x,u)| \le C_N 2^{-l}\mathfrak{C}_{\lambda,k,l}(\lambda 2^l k|x|^2)^{-N}$$

for $|x| \ge (\lambda k 2^l)^{-1/2}$ and $|u| \le (\lambda k)^{-1}$. The bound $O((2^l k)^{-1}\lambda^{1-d/2})$ follows by integration. □

Estimates for $|u| \gg (k+1)^{-1}\lambda^{-1}$. We now proceed to give L^1 estimates for the kernels $A_\lambda^{k,l}$ and $B_\lambda^{k,l}$ for $k \ge 1$ in the region where $|u| \gg (k\lambda)^{-1}$.

An estimate for small x. As a first application we prove L^1 estimates for $|x| \lesssim (2^l\lambda k)^{-1/2}$, $k \ge 1$.

Lemma 8.4. *Let $C \ge 1$. Then*

$$\iint\limits_{\substack{(x,u):\\ |x|\le C(2^l\lambda k)^{-1/2}}} [|A_\lambda^{k,l}(x,u)| + |B_\lambda^{k,l}(x,u)|]\,dx\,du \lesssim_C (2^l k)^{-1}\lambda^{-(d_1-1)/2}. \tag{118}$$

Proof. Integration by parts with respect to s yields

$$|A_\lambda^{k,l}(x,u)| + |B_\lambda^{k,l}(x,u)| \lesssim_N \sum_{\pm} \frac{\mathfrak{C}_{\lambda,k,l}}{(1+\lambda k|u|)^{(d_2-1)/2}} \int_{|t-k\pi|\lesssim 2^{-l}} \big(1+\lambda k\big|\pm|4u| - |x|^2\cot t + t^{-1}\big|\big)^{-N}\,dt. \tag{119}$$

We first integrate in u. Notice that by Lemma 8.2 we have, for fixed t and fixed $r \le (2^l\lambda k)^{-1/2}$,

$$\int_0^\infty \frac{(1+\lambda k v)^{-(d_2-1)/2} v^{d_2-1}}{\big(1+\lambda k\big|\pm|v| - r^2\cot t + t^{-1}\big|\big)^N}\,dv \lesssim \lambda^{-(d_2+1)/2}k^{-d_2}.$$

We integrate in x over a set of measure $\lesssim (2^l k\lambda)^{-d_1/2}$ and then in t (over an interval of length $\approx 2^{-l}$) and (118) follows. □

L^1 bounds for $B_\lambda^{k,l}$.

Lemma 8.5. *For $\lambda \ge 1$, $0 < k \le 8\lambda$,*

$$\|B_\lambda^{k,l}\|_1 \lesssim (2^l k)^{-1}\lambda^{-(d_1-1)/2}. \tag{120}$$

Proof. The bound for the region with $|x| \lesssim (2^l\lambda k)^{-1/2}$ (for which there is no significant oscillation in the t-integral) is proved in Lemma 8.4.

Consider the region where $|x| \approx 2^m(2^l\lambda k)^{-1/2}$. We perform N_1 integrations by parts in t followed by N_2 integrations by parts with respect to s. Denote by $\mathcal{L}_t$ the operator defined by $\mathcal{L}_t g = \partial_t\big(g(t)/(\psi_t(t,|x|)+4|u|)\big)$. Then

$$B_\lambda^{k,l}(x,u) = \mathfrak{C}_{\lambda,k,l}(i/\lambda)^{N_1}\iint e^{i\lambda s(\psi(t,|x|)+4t|u|)}\frac{(I-\partial_s^2)^{N_2}\big[s^{-N_1}\mathcal{L}_t^{N_1}\{\eta_{\lambda,k,l}(s,t)\varpi_2(4\lambda st|u|)\}\big]}{(1+\lambda^2|\psi(t,|x|)+4t|u||^2)^{N_2}}\,dt\,ds$$

From (94b),

$$\left|\partial_t(\psi(t,|x|)+4t|u|)\right| \gtrsim 2^{2l}k|x|^2+4|u| \gtrsim 2^{2m+l}\lambda^{-1}.$$

Moreover, for $\nu \geq 2$, $\partial_t^\nu \psi = O(2^{2m+l\nu}\lambda^{-1})$ and ν differentiations of the amplitude produce factors of $2^{l\nu}$. Thus we obtain the bound

$$|B_\lambda^{k,l}(x,u)| \lesssim \frac{\mathfrak{C}_{\lambda,l,k}}{(1+4\lambda k|u|)^{(d_2-1)/2}} 2^{-2mN_1} \int_{|t-k\pi|\lesssim 2^{-l}} \left(1+\lambda k\left||t^{-1}-|x|^2\cot t+4|u|\right|\right)^{-2N_2} dt.$$

From Lemma 8.2 (with $n=d_2$, $\Lambda=\lambda k$ and $\rho \lesssim k^{-1}\max\{1, 2^{2m}\lambda^{-1}\}$)

$$\int_{v=0}^\infty \frac{(1+\lambda k v)^{-(d_2-1)/2} v^{d_2-1}}{\left(1+\lambda k\left|v-|x|^2\cot t+t^{-1}\right|\right)^N} dv \lesssim \lambda^{-(d_2+1)/2}k^{-d_2}\max\{1, (2^{2m}\lambda^{-1})^{(d_2-1)/2}\}. \tag{121}$$

We integrate in t over an interval of length $O(2^{-l})$ and in x over the annulus $\{x : |x| \approx 2^m(2^l\lambda k)^{-1/2}\}$. This gives

$$\begin{aligned}\iint_{\substack{(x,u):\\ |x|\approx 2^m(2^l\lambda k)^{-1/2}}} |B_\lambda^{k,l}(x,u)|\,dx\,du &\lesssim 2^{-2mN}2^{-l}\left(\frac{2^m}{\sqrt{2^l\lambda k}}\right)^{d_1} \mathfrak{C}_{\lambda,k,l}\lambda^{-(d_2+1)/2}k^{-d_2}\max\{1, (2^{2m}\lambda^{-1})^{(d_2-1)/2}\} \\ &\lesssim (2^l k)^{-1}\lambda^{-(d_1-1)/2}2^{-m(2N-d_1)}\max\{1, (2^{2m}\lambda^{-1})^{(d_2-1)/2}\}, \end{aligned} \tag{122}$$

and, choosing N sufficiently large, the lemma follows by summation in m. □

L^1 bounds for $A_\lambda^{k,l}$, $2^l k \geq 10^5\lambda$.

Lemma 8.6. *For $k \leq 8\lambda$, $2^l \geq 10^5\lambda/k$,*

$$\|A_\lambda^{k,l}\|_1 \lesssim (2^l k)^{-1}\lambda^{-(d_1-1)/2}. \tag{123}$$

Proof. We use Lemma 8.4 to obtain the appropriate L^1 bound in the region $\{(x,u) : |x| \leq C_0(2^l\lambda k)^{-1/2}\}$. Next, consider the region where

$$2^m(2^l\lambda k)^{-1/2} \leq |x| \leq 2^{m+1}(2^l\lambda k)^{-1/2} \tag{124}$$

for large m. This region is then split into two subregions, one where $4|u| = v \leq 10^{-2}2^{2m+l}\lambda^{-1}$ and the complementary region.

For the region with small v, we proceed as in Lemma 8.5. From (94b), we have $|\psi_t| \geq kr^2 2^{2l}/20$ and hence $|\psi_t| \geq 2^{2m+l-5}\lambda^{-1}$. Thus, if $v \leq 10^{-2}2^{2m+l}\lambda^{-1}$ then $|\phi_t| \approx k2^{2l}r^2 \approx 2^{2m+l}\lambda^{-1}$. Moreover, $\partial_t^\nu\phi = O(2^{2m+l\nu}\lambda^{-1})$ for $\nu \geq 2$. Therefore, if we perform integration by parts in t several times, followed by integrations by parts on s, we obtain the bound

$$|A_\lambda^{k,l}(x,u)| \lesssim \frac{\mathfrak{C}_{\lambda,l,k}}{(1+\lambda k|u|)^{(d_2-1)/2}} 2^{-2mN} \int_{|t-k\pi|\lesssim 2^{-l}} \left(1+\lambda k\left||x|^2\cot t-t^{-1}-4|u|\right|\right)^{-N} dt.$$

In the present range, $|x|^2|\cot t| \approx 2^{2m}(\lambda k)^{-1}$ and $t^{-1} \approx k^{-1}$, and thus we see from Lemma 8.2 that inequality (121) in the proof of Lemma 8.5 holds. From this we proceed as in (122) to bound

$$\iint_{\substack{|x|\approx 2^m(2^l\lambda k)^{-1/2}\\ 4|u|\le 10^{-2}2^{2m+l}\lambda^{-1}}} |A_\lambda^{k,l}(x,u)|\,dx\,du \lesssim (2^l k)^{-1}\lambda^{-(d_1-1)/2}2^{-m(2N-d_1)}\max\{1,(2^{2m}\lambda^{-1})^{(d_2-1)/2}\}.$$

For large N_1, we can sum in m and obtain the bound $C(2^l k)^{-1}\lambda^{-(d_1-1)/2}$.

Next, assume that $v \ge 2^{2m+l}\lambda^{-1}/100$ (and still keep (124)). Then

$$|tv + r^2 t\cot t - 1| \ge k|v| \quad \text{for } t \in \operatorname{supp}(\eta_{\lambda,k,l}). \tag{125}$$

Indeed, we have $tv \ge 2^{2m}2^l k\lambda^{-1}/100 \ge 10^3$ and

$$r^2 t|\cot t| \le 2^{2m+2}(2^l\lambda k)^{-1} t\Big[\sin\Big(\frac{3\pi}{8}2^{-l}\Big)\Big]^{-1} \le 2^{2m+6}\lambda^{-1} \le \frac{2^{2m+l}}{100\lambda}2^5 10^2 2^{-l},$$

where we used (124) and that $\sin\alpha > 2\alpha/\pi$ for $0 \le \alpha \le \frac{1}{2}\pi$. By our assumptions, $2^l \ge 10^5\lambda/k > 10^4$, and thus the right-hand side of the display is at most $v/10$. Now (125) is immediate by the triangle inequality.

We use (125) to get, from an N_1-fold integration by parts in s,

$$|A_\lambda^{k,l}(x,u)| \lesssim 2^{-l}\mathfrak{C}_{\lambda,l,k}(\lambda k v)^{-N_1-(d_2-1)/2}.$$

Then

$$\begin{aligned}\iint_{\substack{|x|\approx 2^m(2^l\lambda k)^{-1/2}\\ 4|u|\ge 10^{-2}2^{2m+l}\lambda^{-1}}} |A_\lambda^{k,l}(x,u)|\,du\,dx &\lesssim 2^{-l}\mathfrak{C}_{\lambda,l,k}\Big(\frac{2^m}{\sqrt{\lambda 2^l k}}\Big)^{d_1}(\lambda k)^{-N_1-(d_2-1)/2}\Big(\frac{2^{2m+l}}{\lambda}\Big)^{-N_1+\frac{d_2+1}{2}}\\ &\lesssim \lambda^{1-d_1/2-d_2/2}2^{-l(N_1-(d_2-1)/2)}k^{(d_2-1)/2-N_1}2^{m(d_1+d_2+1-2N_1)}.\end{aligned}$$

For N_1 large we may sum in m to finish the proof. □

Estimates for $A_\lambda^{k,l}$, $2^l \lesssim \lambda/k$. In the early approaches, to prove L^p boundedness for Fourier integral operators, the oscillatory integrals were analyzed using the method of stationary phase [Peral 1980; Miyachi 1980; Beals 1982]. This creates some difficulties in our case at points where ϕ, ϕ_t and ϕ_{tt} vanish simultaneously, namely at positive t satisfying $\tan t = t$. To avoid this difficulty we use a decomposition in the spirit of [Seeger et al. 1991].

In what follows we assume $k \le 8\lambda$ and $2^l \le C_0\lambda/k$ for large C_0 chosen independently of λ, k and l. The choice $C_0 = 10^{10}$ is suitable. We decompose the interval $J_{k,l}$ into smaller subintervals of length $\varepsilon\sqrt{k/(2^l\lambda)}$ (which is $\lesssim 2^{-l}$ in the range under consideration); here $\varepsilon \ll 10^{-100}$ (to be chosen sufficiently small but independent of λ, k and l).

This decomposition is motivated by the following considerations: According to (130), $\lambda\phi(t,r,v)$ contains the term $-\lambda(r-r(t))^2 t\cot t$ depending entirely on r and t. For $t \in J_{k,l}$, this is of size $\lambda k 2^l|r-r(t)|^2$, hence of order $O(1)$ if $|r-r(t)| \lesssim (\lambda k 2^l)^{-1/2}$. Moreover, on a subinterval I of $J_{k,l}$ on which $r(t)$ varies by at most a small fraction of the same size, the term $-\lambda(r-r(t))^2 t\cot t$ is still $O(1)$ and contributes to no oscillation in the integration with respect to s. Since $|r'(t)| \sim 1/k$ by (111), this suggests we choose intervals I of length $\ll k(\lambda k 2^l)^{-1/2} = \sqrt{k 2^{-l}\lambda^{-1}}$. Similarly, the first term of $\lambda\phi(t,r,v)$ in (130) is of size $\lambda k|w(t,r,v)|$ and does not contribute to any oscillation in the integration with respect to s if $|w(t,r,v)| \lesssim (\lambda k)^{-1}$. These considerations also motivate our later definitions of the set $\mathcal{P}_0$ and the sets $\mathcal{P}_m$, $m \geq 1$; see (133).

As before, we denote by η_0 a $C_0^\infty(\mathbb{R})$ function such that $\sum_{n\in\mathbb{Z}} \eta_0(t-\pi n) = 1$ and $\operatorname{supp}(\eta_0) \subset (-\pi,\pi)$. Define, for $b \in \pi\varepsilon\sqrt{k2^{-l}\lambda^{-1}}\,\mathbb{Z}$,

$$\eta_{\lambda,k,l,b}(s,t) = \eta_{\lambda,k,l}(s,t)\eta_0\left(\varepsilon^{-1}\sqrt{\frac{\lambda 2^l}{k}}(t-b)\right). \tag{126}$$

Then we may split

$$A_\lambda^{k,l} = \sum_{b\in\mathcal{T}_{\lambda,k,l}} A_{\lambda,b}^{k,l}, \tag{127}$$

where $\mathcal{T}_{\lambda,k,l} \subset \pi\varepsilon\sqrt{k2^{-l}\lambda^{-1}}\,\mathbb{Z} \cap J_{k,l}$ (see (104)), $\#\mathcal{T}_{\lambda,k,l} = O(\varepsilon^{-1}\sqrt{\lambda 2^{-l}k^{-1}})$, and

$$A_{\lambda,b}^{k,l}(x,u) = \mathfrak{C}_{\lambda,l,k} \iint \chi(s)\eta_{\lambda,k,l,b}(t) e^{i\lambda s(1-|x|^2 t\cot t - t|4u|)} \varpi_1(\lambda s t|4u|)\,dt\,ds. \tag{128}$$

We now give some formulas relating the phase $\phi(t,r,v) = 1 - r^2 t\cot t - tv$ to the geometry of the curve $(r(t), v(t))$ (see (108)). By (110) and (112),

$$\begin{aligned}\frac{\phi(t,r,v)}{t} &= \frac{\phi(t,r,v) - \phi(t,r(t),v(t))}{t}\\ &= (r(t)^2 - r^2)\cot t + v(t) - v\\ &= v(t) - v - (r - r(t))2r(t)\cot t - (r-r(t))^2\cot t\end{aligned}$$

and, setting

$$w(t,r,v) = v - v(t) - \frac{v'(t)}{r'(t)}(r - r(t)), \tag{129}$$

we get

$$\frac{\phi(t,r,v)}{t} = -w(t,r,v) - (r-r(t))^2\cot t. \tag{130}$$

Moreover,

$$\begin{aligned}\phi_t(t,r,v) &= \frac{\phi(t,r,v)}{t} + \frac{r^2 t}{\sin^2 t} - \frac{1}{t}\\ &= \frac{\phi(t,r,v)}{t} + \frac{t}{\sin^2 t}(r + r(t))(r - r(t)).\end{aligned} \tag{131}$$

We shall need estimates describing how $w(t,r,v)$ changes in t. Use (130) and the expansion

$$\begin{aligned}&w(t,r,v) - w(b,r,v)\\ &= -\left[v(t) - v(b) - \frac{v'(b)}{r'(b)}(r(t) - r(b))\right] - \left[\frac{v'(t)}{r'(t)} - \frac{v'(b)}{r'(b)}\right](r - r(b)) + \left[\frac{v'(t)}{r'(t)} - \frac{v'(b)}{r'(b)}\right](r(t) - r(b)).\end{aligned}$$

From (114), we get $|r''| + k|v''| \lesssim 2^{-l}k^{-1} + k^{-2}$ on $J_{k,l}$, thus the first term in the displayed formula is $\lesssim (2^{-l}k^{-2} + k^{-3})|t-b|^2$. Differentiating in (112) we also get $(v'/r')' = O(2^{-l}k + k^{-2})$ on $J_{k,l}$, and see that the second term in the display is $\lesssim (2^{-l}k^{-1}+k^{-2})|t-b||r-r(b)|$ and the third is $\lesssim (2^{-l}+k^{-1})k^{-2}(t-b)^2$. Hence,

$$|w(t,r,v)-w(b,r,v)| \lesssim (2^{-l}+k^{-1})|t-b|\left(\frac{|t-b|}{k^2}+\frac{|r-r(b)|}{k}\right). \tag{132}$$

We now turn to the estimation of $A^{k,l}_{\lambda,b}$ with $k \geq 1$ and $b \in \mathcal{T}_{\lambda,k,l}$. Let, for $b > \frac{1}{2}$, $l = 1, 2, \ldots,$ and $m = 0, 1, 2, \ldots,$

$$\begin{aligned}\mathcal{P}_m &\equiv \mathcal{P}_m(\lambda, l, k; b)\\ &:= \big\{(r,v)\in(0,\infty)\times(0,\infty) : v \geq (\lambda k)^{-1},\ |r-r(b)| \leq 2^m(\lambda k 2^l)^{-1/2},\ |w(b,r,v)| \leq 2^{2m}(\lambda k)^{-1}\big\}\end{aligned} \tag{133}$$

and let

$$\Omega_m \equiv \Omega_m(\lambda,l,k;b) := \begin{cases}\{(x,u) : (|x|,4|u|) \in \mathcal{P}_0\} & \text{if } m=0,\\ \{(x,u) : (|x|,4|u|) \in \mathcal{P}_m \setminus \mathcal{P}_{m-1}\} & \text{if } m>0.\end{cases} \tag{134}$$

For later reference we note that, in view of $2^l \leq \lambda/k$, $|t-b| \leq \varepsilon\sqrt{k/(\lambda 2^l)}$ and the upper bound $|r'(t)| \leq 2t^{-1}$, we have $r(t)-r(b) = O(\varepsilon/\sqrt{k\lambda 2^l})$, and, by (132),

$$|w(t,r,v)-w(b,r,v)| \lesssim \varepsilon 2^m(\lambda k)^{-1}, \quad (r,v) \in \mathcal{P}_m. \tag{135}$$

Moreover, it is easy to check that, still for $|t-b| \leq \varepsilon\sqrt{k/(\lambda 2^l)}$,

$$\big|(r-r(t))^2\cot t-(r-r(b))^2\cot b\big| \lesssim \varepsilon 2^{2m}(\lambda k)^{-1}. \tag{136}$$

Proposition 8.7. *Assume that* $1 \leq k \leq 8\lambda$, $l = 1, 2, \ldots,$ *and* $2^l \leq C_0\lambda/k$ (*and let* ε *in the definition* (126) *be at most* $C_0^{-1}10^{-100}$). *Let* $b \geq 1$ *and* $b \in \mathcal{T}_{\lambda,k,l}$. *Then*

$$\iint_{\Omega_0(\lambda,l,k;b)} |A^{k,l}_{\lambda,b}(x,u)|\,dx\,du \lesssim (2^l k)^{-(d_1+1)/2}\sqrt{\frac{2^l k}{\lambda}}, \tag{137}$$

$$\iint_{\Omega_m(\lambda,l,k;b)} |A^{k,l}_{\lambda,b}(x,u)|\,dx\,du \lesssim_N 2^{-mN}(2^l k)^{-(d_1+1)/2}\sqrt{\frac{2^l k}{\lambda}}. \tag{138}$$

Proof. Note that, for fixed $k \geq 1$, $l \geq 1$, $b \in \mathcal{T}_{\lambda,k,l}$,

$$(r,v) \in \mathcal{P}_m \quad\Longrightarrow\quad r \lesssim 2^m(2^l k)^{-1} \quad\text{and}\quad v \lesssim 2^{2m}k^{-1}. \tag{139}$$

This is immediate in view of $2^l k \lesssim \lambda$, $r(b) \approx (2^l k)^{-1}$ and $v(b) \approx k^{-1}$, and thus

$$\begin{aligned} r &\lesssim (2^l k)^{-1}\left(1+2^m\sqrt{\frac{k2^l}{\lambda}}\right) \lesssim 2^m(2^l k)^{-1},\\ v &\lesssim k^{-1}(1+2^{2m}\lambda^{-1}) \lesssim 2^{2m}k^{-1}.\end{aligned} \tag{140}$$

Also recall that $v = 4|u| \geq (\lambda k)^{-1}$ for $(x,u) \in \Omega_m(\lambda,l,k;b)$.

A crude size estimate yields

$$\iint_{(|x|,4|u|)\in\mathcal{P}_m} |A^{k,l}_{\lambda,b}(x,u)|\,dx\,du \lesssim 2^{m(d_1+d_2+1)}(2^l k)^{-(d_1+1)/2}\sqrt{\frac{2^l k}{\lambda}}. \tag{141}$$

Indeed, the left-hand side is $\lesssim \varepsilon\sqrt{k/(2^l\lambda)}\,\mathfrak{C}_{\lambda,k,l}\mathcal{I}$, where

$$\mathcal{I} := \iint_{\substack{|r-r(b)|\lesssim 2^m(2^l\lambda k)^{-1/2}\\ |w(b,r,v)|\lesssim 2^{2m}(\lambda k)^{-1}}} (\lambda k v)^{-(d_2-1)/2} v^{d_2-1} r^{d_1-1}\,dv\,dr \lesssim \frac{2^m}{\sqrt{\lambda 2^l k}}\Big(\frac{2^m}{2^l k}\Big)^{d_1-1}\frac{2^{2m}}{\lambda k}\Big(\frac{2^{2m}k^{-1}}{\lambda k}\Big)^{\frac{d_2-1}{2}},$$

in view of (129) and (140). This yields (141). In regard to its dependence on m, this bound is nonoptimal and will be used for $2^m \le C(\varepsilon)$.

We now derive an improved L^1 bound for the region Ω_m when m is large. For $(r, v) \in \mathcal{P}_m \setminus \mathcal{P}_{m-1}$ we distinguish two cases, I and II, depending on the size of $|\phi(b, r, v)|$, and define for $m > 0$, and fixed k, l and b,

$$\mathcal{R}^I_m = \big\{(r, v) \in \mathcal{P}_m \setminus \mathcal{P}_{m-1} : |\phi(b, r, v)| > 2^{l-100}(r - r(b))^2\big\},$$
$$\mathcal{R}^{II}_m = \big\{(r, v) \in \mathcal{P}_m \setminus \mathcal{P}_{m-1} : |\phi(b, r, v)| \le 2^{l-100}(r - r(b))^2\big\}.$$

We also have the corresponding decomposition $\Omega_m = \Omega^I_m + \Omega^{II}_m$, where Ω^I_m and Ω^{II}_m consist of those (x, u) with $(|x|, 4|u|) \in \mathcal{R}^I_m$ and $(|x|, 4|u|) \in \mathcal{R}^{II}_m$, respectively.

Case I: $|\phi(b, r, v)| \ge 2^{l-100}k(r - r(b))^2$. We shall show that

$$|\phi(t, r, v)| \gtrsim c2^{2m}\lambda^{-1} \quad \text{for } (r, v) \in \mathcal{R}^I_m,\ |t - b| \le \varepsilon\sqrt{\frac{k}{2^l\lambda}}, \tag{142}$$

with $c > 0$ if $0 < \varepsilon \ll 10^{-100}$ is chosen sufficiently small. Given (142) we can use an N_2-fold integration by parts in s to obtain a gain of 2^{-2mN_2} over the above straightforward size estimate (141), which leads to

$$\iint_{\Omega^I_m} |A^{k,l}_{\lambda,b}(x, u)|\,dx\,du \lesssim_{\varepsilon,N_2} 2^{m(d_1+d_2+1-2N_2)}(2^l k)^{-(d_1+1)/2}\sqrt{\frac{2^l k}{\lambda}}. \tag{143}$$

It remains to show (142). We distinguish between two subcases. First, if $|r - r(b)| \ge 2^{m-5}(\lambda k 2^l)^{-1/2}$, then by the case I assumption we have $|\phi(b, r, v)| \ge 2^{l-100}k2^{2m-10}(\lambda k 2^l)^{-1} = 2^{2m-110}\lambda^{-1}$, and, by (130), (135) and (136), we also get (142) provided that $\varepsilon \ll 2^{-200}$.

For the second subcase we have $|r - r(b)| \le 2^{m-5}(\lambda k 2^l)^{-1/2}$. Since $(r, v) \notin \mathcal{P}_{m-1}$, this implies that $|w(b, r, v)| \ge 2^{2m-2}(\lambda k)^{-1}$, and since the quantity $b(r - r(b))^2|\cot b|$ is bounded by $2^{l+4}b(r - r(b))^2 \le 2^{2m-6}(b/k)\lambda^{-1}$, we also get $|\phi(b, r, v)| \ge 2^{2m-3}\lambda^{-1}$, by (130). Now, by (130), (135) and (136), we also get $|\phi(t, r, v)| \ge 2^{2m-4}\lambda^{-1}$ if ε is sufficiently small. Thus (142) is verified and (143) is proved.

Case II: $|\phi(b, r, v)| \le 2^{l-100}k(r - r(b))^2$. We show

$$|\phi_t(t, r, v)| \ge 2^{m-20}2^{3l/2}k^{1/2}(r + r(b))\lambda^{-1/2} \quad \text{if } (r, v) \in \mathcal{R}^{II}_m,\ |t - b| \le \varepsilon\sqrt{\frac{\lambda 2^l}{k}}, \tag{144}$$

and this will enable us to get a gain when integrating by parts in t. To prove (144), we first establish

$$|r - r(b)| \ge 2^{m-10}(\lambda k 2^l)^{-1/2} \quad \text{for } (r, v) \in \mathcal{R}^{II}_m. \tag{145}$$

Note that if $|w(b,r,v)| \le 2^{2m-3}(\lambda k)^{-1}$ then $|r-r(b)| \ge 2^{m-1}(\lambda k 2^l)^{-1/2}$, since $\mathcal{R}^{II}_m \subset \mathcal{P}^{\complement}_{m-1}$. Thus, to verify (145), we may assume $|w(b,r,v)| \ge 2^{2m-3}(\lambda k)^{-1}$. In this case we get, from (130), $(r,v) \in \mathcal{P}_m$ and the case II assumption,

$$(r-r(b))^2|\cot b| \ge |w(b,r,v)| - b^{-1}|\phi(b,r,v)| \ge 2^{2m-3}(\lambda k)^{-1} - b^{-1}k2^{l-100}2^{2m}(\lambda k 2^l)^{-1} \ge 2^{2m-4}(\lambda k)^{-1}$$

and hence $(r-r(b))^2 2^{l+4} \ge 2^{2m-4}(\lambda k)^{-1}$, which implies (145). In order to prove (144), we use (131) and (145) to estimate

$$\begin{aligned}
|\phi_t(b,r,v)| &\ge \frac{b}{\sin^2 b}(r+r(b))|r-r(b)| - 2^{l-100}\frac{k}{b}(r-r(b))^2 \\
&\ge \frac{|r-r(b)|}{b}\left(\frac{r+r(b)}{r(b)^2} - \frac{2^l k}{2^{100}}|r-r(b)|\right) \ge \frac{(r+r(b))|r-r(b)|}{2br(b)^2} \\
&\ge 2^{2l-4}k(r+r(b))\frac{2^{m-10}}{\sqrt{\lambda k 2^l}} \ge 2^{m-15}k^{1/2}2^{3l/2}(r+r(b))\lambda^{-1/2},
\end{aligned}$$

which yields (144) for $t=b$. We need to show the lower bound for $|t-b| \le \varepsilon\sqrt{k/(2^l\lambda)}$. By (95) we have $|\phi_{tt}(t',r,v)| \le r^2 b 2^{3l+4}$ for $|t'-b| \le \varepsilon\sqrt{b/(2^l\lambda)}$, and thus

$$|\phi_t(t,r,v) - \phi_t(b,r,v)| \le 2^6 r^2 2^{3l} k\varepsilon\sqrt{\frac{k}{2^l\lambda}} \le 2^{m-30}2^{3l/2}k^{1/2}\lambda^{-1/2}(r+r(b))$$

if $\varepsilon \ll 2^{-100}$. The second inequality in the last display is easy to check. If $r \le 2r(b)$, then use $r \lesssim (2^l k)^{-1} \approx r+r(b)$, and, if $r > 2r(b)$, then use $r-r(b) \approx r+r(b) \approx r$. In both cases the asserted inequality holds for small ε and thus (144) holds for $|t-b| \le \varepsilon\sqrt{k/(2^l\lambda)}$. We note that, under the condition (145), the range $r \le 2r(b)$ corresponds to $2^m \lesssim \sqrt{\lambda(2^l k)^{-1}}$ and the range $r \ge 2r(b)$ corresponds to $2^m \gtrsim \sqrt{\lambda(2^l k)^{-1}}$.

We now estimate the L^1 norm over the region where $(r,v) \in \mathcal{R}^{II}_m$. Let $\mathcal{L}_t$ be the differential operator defined by $\mathcal{L}_t g = \partial(g/\phi_t)/\partial t$. By N_1 integration by parts in t we get (with $|x|=r$, $4|u|=v$)

$$A^{k,l}_{\lambda,b}(x,u) = i^{N_1}\lambda^{-N_1}\mathfrak{C}_{\lambda,k,l}\iint e^{i\lambda s\phi(t,|x|,4|u|)}s^{-N_1}\mathcal{L}_t^{N_1}[\eta_{\lambda,k,l,b}(s,t)\varpi_1(\lambda stv)]\,dt\,ds.$$

To estimate the integrand use the lower bound on $|\phi_t|$, (144). Moreover, we have the upper bounds (98) for the higher derivatives of ψ (and then ϕ), which give $\partial_t^n\phi = O(2^{l(n+1)}br^2)$ for $n \ge 2$. Each differentiation of the cutoff function produces a factor of $(\lambda 2^l k^{-1})^{1/2}$. By the one-dimensional version of Lemma 8.1 described in the following remark, the expression $\lambda^{-N_1}(\lambda bv)^{(d_2-1)/2}\big|\mathcal{L}_t^{N_1}[\eta_{\lambda,k,l,b}(s,t)\varpi_1(\lambda stv)]\big|$ can be estimated by a sum of $C(N_1)$ terms of the form

$$\lambda^{-N_1}\frac{(\lambda 2^l/k)^{\alpha/2}}{(2^m 2^{3l/2}k^{1/2}(r+r(b))\lambda^{-1/2})^\alpha}\prod_{\beta\in\mathfrak{I}}\frac{2^{l(\beta+1)}kr^2}{(2^m 2^{3l/2}k^{1/2}(r+r(b))\lambda^{-1/2})^\beta}, \tag{146}$$

where $\alpha \in \{0,\dots,N_1\}$, $\mathfrak{I}$ is a set of integers $\beta \in \{2,\dots,N_1+1\}$ with the property that $\sum_{\beta\in\mathfrak{I}}(\beta-1) = N_1-\alpha$. If $\mathfrak{I}$ is the empty set then we interpret the product as 1. We observe that, for $(r,v) \in \mathcal{R}^{II}_m$, we have

$|r-r(b)| \approx 2^m(\lambda k 2^l)^{-1/2}$. Thus, if $2^m \le \sqrt{\lambda(2^l k)^{-1}}$, we have $r \lesssim (2^l k)^{-1}$ and $r+r(b) \approx (2^l k)^{-1}$, while for $2^m > \sqrt{\lambda(2^l k)^{-1}}$ we have $r \approx r-r(b) \approx r+r(b) \approx 2^m(\lambda k 2^l)^{-1/2}$.

A short computation which uses these observations shows that, in the case $2^m \le \sqrt{\lambda(2^l k)^{-1}}$, the terms (146) are $\lesssim 2^{-m\alpha}\prod_{\beta\in\mathfrak{I}}[2^{-m\beta}(2^l k/\lambda)^{\beta/2-1}]$. In the case $2^m > \sqrt{\lambda(2^l k)^{-1}}$, the terms (146) are dominated by a constant times $(\lambda 2^{-l}k^{-1})^{\alpha/2}2^{-2m\alpha}\prod_{\beta\in\mathfrak{I}}2^{-m(\beta-1)}$. In either case, the terms (146) are $\lesssim 2^{-mN_1}$, since $\alpha+\sum_{\beta\in\mathfrak{I}}\beta \ge N_1$. This means that we gain a factor of 2^{-mN_1} over the size estimate (141). Consequently,

$$\iint_{\Omega_m^{II}} |A^{k,l}_{\lambda,b}(x,u)|\,dx\,du \lesssim 2^{m(d_1+d_2+1-N_1)}(2^l k)^{-(d_1+1)/2}\sqrt{\frac{2^l k}{\lambda}}. \tag{147}$$

The assertion of the proposition then follows from (143) and (147). □

L^1 estimates for T^k_λ and $W_{j,n}$.

Proof of (59). Let us recall that $k \le 8\lambda$. If we sum the bounds in Proposition 8.7 over $b \in \mathcal{T}_{2^j,k,l}$, we get

$$\|A^{k,l}_{2^j}\|_{L^1} \lesssim (2^l k)^{-(d_1+1)/2}, \quad 2^l \lesssim \frac{2^j}{k}.$$

We also have

$$\|2^{-j(d-1)/2}K^{k,l}_{2^j} - A^{k,l}_{2^j}\|_1 \lesssim (2^l k)^{-1}2^{-j(d_1-1)/2}; \tag{148}$$

for the part of $K^{k,l}_{2^j}$ where $|u| \lesssim 1/(k\lambda)$, this follows from Lemma 8.3 and, for the remaining part, this follows from Lemma 8.5. Combining these two estimates, we find that

$$\|2^{-j(d-1)/2}K^{k,l}_{2^j}\|_1 \lesssim (2^l k)^{-(d_1+1)/2}, \quad 2^l \lesssim \frac{2^j}{k}. \tag{149}$$

Moreover, by Lemma 8.5 and Lemma 8.6, we have

$$\|2^{-j(d-1)/2}K^{k,l}_{2^j}\|_1 \lesssim (2^l k)^{-1}2^{-j(d_1-1)/2}, \quad 2^l \ge 10^6\frac{2^j}{k}. \tag{150}$$

Altogether this leads to

$$2^{-j(d-1)/2}\|T^{k,l}_{2^j}\|_{L^1\to L^1} \lesssim (2^l k)^{-(d_1+1)/2}. \tag{151}$$

and (59) follows if we sum in l. □

An estimate away from the singular support. For later use in the proof of Theorem 1.4, we need the following observation:

Proposition 8.8. *Let $\lambda \ge 1$, let K_λ be the convolution kernel for the operator $\chi(\lambda^{-1}\sqrt{L})e^{i\sqrt{L}}$, where $\chi \in \mathcal{S}(\mathbb{R})$, and let $R \ge 10$. Then, for every $N \in \mathbb{N}$,*

$$\int_{\max\{|x|,|u|\}\ge R} |K_\lambda(x,u)|\,dx\,du \le C_N(\lambda R)^{-N}.$$

Moreover, the constants C_N depend only on N and a suitable Schwartz norm of χ.

Proof. This estimate is implicit in our arguments above, but it is easier to establish it as a direct consequence of the finite propagation speed of solutions to the wave equation [Melrose 1986]. Indeed, write

$$\chi(\lambda^{-1}\sqrt{L})e^{i\sqrt{L}} = \chi(\lambda^{-1}\sqrt{L})\cos\sqrt{L} + i\lambda\tilde{\chi}(\lambda^{-1}\sqrt{L})\frac{\sin\sqrt{L}}{\sqrt{L}}$$

with $\tilde{\chi}(s) = s\chi(s)$, and denote by φ_λ and $\mathcal{P}$ the convolution kernels for the operators $\chi(\lambda^{-1}\sqrt{L})$ and $\cos\sqrt{L}$, respectively. Then $\mathcal{P}$ is a compactly supported distribution (of finite order). Indeed, $\mathcal{P}$ is supported in the unit ball with respect to the optimal control distance associated to the Hörmander system of vector fields $X_1, \dots, X_{d_1}$, which is contained in the Euclidean ball of radius 10. Moreover, by homogeneity, $\varphi_\lambda(x, u) = \lambda^{d_1+2d_2}\varphi(\lambda x, \lambda^2 u)$, with a fixed Schwartz function φ. Note also that, by Hulanicki's theorem [1984], the mapping taking χ to φ is continuous in the Schwartz topologies. Since the convolution kernel K^c_λ for the operator $\chi(\lambda^{-1}\sqrt{L})\cos\sqrt{L}$ is given by $\varphi_\lambda * \mathcal{P}$, it is then easily seen that $K^c_\lambda(x, u)$ can be estimated by $C_N\lambda^M(\lambda|x| + \lambda^2|u|)^{-N}$ for every $N \in \mathbb{N}$, with a fixed constant M. A very similar argument applies to $\tilde{\chi}(\lambda^{-1}\sqrt{L})\sin(\sqrt{L})/\sqrt{L}$, and thus we obtain the above integral estimate for K_λ. ☐

9. Controlling the $h^1_{\mathrm{iso}} \to L^1$ bounds for the operators $\mathcal{W}_n$

In this section, we consider the operators $\mathcal{W}_n = \sum_j W_{j,n}$ and prove the relevant estimate in Theorem 5.3. In the proof we shall use a simple L^2 bound which follows from the spectral theorem, namely, for $j_0 > 0$,

$$\Big\| \sum_{j \ge j_0} W_{j,n} \Big\|_{L^2 \to L^2} \lesssim 2^{-j_0(d-1)/2}. \tag{152}$$

Preliminary considerations. Let $\rho \le 1$ and let f_ρ be an L^2 function satisfying

$$\|f_\rho\|_2 \le \rho^{-d/2}, \quad \mathrm{supp}(f_\rho) \subset Q_{\rho,E} := \{(x, u) : \max\{|x|, |u|\} \le \rho\}, \tag{153}$$

and we also assume that

$$\iint f_\rho(x, u)\, dx\, du = 0 \quad \text{if } \rho \le \tfrac{1}{2}. \tag{154}$$

In what follows we also need notation for the expanded Euclidean "ball"

$$Q_{\rho,E,*} = \{(x, u) : \max\{|x|, |u|\} \le C_*\rho\}, \tag{155}$$

where $C_* = 10(1 + d_2 \max_i \|J_i\|)$.

We begin with the situation given by (154). By translation invariance and the definition of h^1_{iso}, it will suffice to check that

$$\|\mathcal{W}_n f_\rho\|_{L^1} \lesssim (1 + n)2^{-n(d_1-1)/2}. \tag{156}$$

We work with dyadic spectral decompositions for the operators $|U|$ and $\sqrt{L}$, and need to discuss how they act on the atom f_ρ.

For $j > 0$ and $n \ge 0$, let $H_{j,n}$ be the convolution kernel defined by

$$\chi_1(2^{-2j}L)\zeta_1(2^{-j-n}|U|)f = f * H_{j,n}.$$

From (52) we see that

$$H_{j,n} = 0 \quad \text{when } n > j + 11.$$

Lemma 9.1. *Let $\rho \le 1$, and let f_ρ be as in (153). Then:*

(i) $\|f_\rho * H_{j,n}\|_1 \lesssim 1$ *and*

$$\|f_\rho * H_{j,n}\|_{L^1(Q^\complement_{\rho,E,*})} \lesssim_N (2^j\rho)^{-N}. \tag{157}$$

(ii) *If f_ρ satisfies (154) then*

$$\|f_\rho * H_{j,n}\|_1 \lesssim \min\{1, 2^{j+n}\rho\}. \tag{158}$$

Proof. By Hulanicki's theorem [1984] the convolution kernel of $\chi_1(L)$ is a Schwartz function g_1 on $\mathbb{R}^{d_1+d_2}$. The convolution kernel of $\zeta_1(|U|)$ is $\delta \otimes g_2$, where δ is the Dirac measure in $\mathbb{R}^{d_1}$ and g_2 is a Schwartz function on $\mathbb{R}^{d_2}$. Then

$$H_{j,n}(x,u) = \int 2^{j(d_1+2d_2)} g_1(2^j x, 2^{2j} w) 2^{(j+n)d_2} g_2(2^{j+n}(u-w))\, dw. \tag{159}$$

Clearly $\|H_{j,n}\|_1 = O(1)$ uniformly in j and n and, since $\|f_\rho\|_1 \lesssim 1$, we get from Minkowski's inequality that $\|f_\rho * H_{j,n}\|_1 \lesssim 1$.

For the proof of (157) we may thus assume $2^j \ge 1/\rho$, and it suffices to verify that, for every $(y,v) \in Q_{\rho,E}$, the $L^1(Q^\complement_{A\rho,E})$ norm of the function

$$(x,u) \mapsto \int \frac{2^{j(d_1+2d_2)}}{(1+2^j|x-y|+2^{2j}|w|)^{N_1}} \frac{2^{(j+n)d_2}}{\big(1+2^{j+n}\big|u-v-w+\frac{1}{2}\langle \vec{J}x, y\rangle\big|\big)^{N_1}}\, dw$$

is bounded by $C(2^j\rho)^{-N}$ if $N_1 \gg N + d_1 + 2d_2$. This is straightforward. For the proof of (158), we observe that (159) implies

$$2^{-j}\|\nabla_x H_{j,n}\|_1 + 2^{-j-n}\|\nabla_u H_{j,n}\|_1 = O(1).$$

Moreover, $2^{-n}\||x|\nabla_u H_{j,n}\|_1 = O(1)$. By the cancellation condition (154),

$$\begin{aligned} f * H_{j,n}(x,u) &= \int f_\rho(y,v)\big[H_{j,n}\big(x-y, u-v+\tfrac{1}{2}\langle \vec{J}x, y\rangle\big) - H_{j,n}(x,u)\big]\, dy\, dv \\ &= -\int f_\rho(y,v)\bigg(\int_0^1 \big\langle y, \nabla_x H_{j,n}\big(x-sy, u-sv+\tfrac{1}{2}s\langle \vec{J}x, y\rangle\big)\big\rangle \\ &\qquad + \big\langle v + \tfrac{1}{2}\langle \vec{J}x, y\rangle, \nabla_u H\big(x-sy, u-sv+\tfrac{1}{2}s\langle \vec{J}x, y\rangle\big)\big\rangle\, ds\bigg)\, dy\, dv. \end{aligned}$$

We also use $\langle \vec{J}x, y\rangle = \langle \vec{J}(x-sy), y\rangle$ and a change of variable to estimate

$$\|f_\rho * H_{j,n}\| \lesssim \|f_\rho\|_1\, \rho\, [\|\nabla_x H_{j,n}\|_1 + \|\nabla_u H_{j,n}\|_1 + \||x|\nabla_u H_{j,n}\|_1],$$

and (158) follows. □

Proof of (156). For $n > 0$, split

$$\mathcal{W}_n f_\rho = \sum_{\substack{j\ge n-11\\ 2^j\rho<2^{-10n}}} W_{j,n} f_\rho + \sum_{\substack{j\ge n-11\\ 2^{-10n}\le 2^j\rho\le 2^{10n}}} W_{j,n} f_\rho + \sum_{2^{10n}<2^j\rho} W_{j,n} f_\rho =: I_{n,\rho} + II_{n,\rho} + III_{n,\rho}.$$

The main contribution comes from the middle term and, by (66) and the estimate $\|f_\rho\|_1 \lesssim 1$, we immediately get

$$\|II_{n,\rho}\|_1 \lesssim (1+n)2^{-n(d_1-1)/2}. \tag{160}$$

Let $\mathcal{J}_n$ be as in (64), so that $\#(\mathcal{J}_n) = O(2^n)$. We use the estimate (151) in conjunction with (158), and estimate

$$\begin{aligned}\|I_{n,\rho}\|_1 &\le \sum_{2^j\rho<2^{-10n}} \sum_{k\in\mathcal{J}_n} \sum_{l=1}^{\infty} \|2^{-j(d-1)/2} T^{k,l}_{2^j}(f_\rho * H_{j,n})\|_1 \\ &\lesssim \sum_{2^j\rho<2^{-10n}} \sum_{k\in\mathcal{J}_n} \sum_{l=1}^{\infty} (2^l k)^{-(d_1+1)/2} 2^{n+j}\rho \lesssim 2^{-n(9+(d_1-1)/2)}.\end{aligned}$$

We turn to the estimation of the term $III_{n,\rho}$. Let $\mathfrak{T}_{\rho,n}$ be a maximal $\sqrt{\varepsilon\rho}$-separated set of $[2^{n-6}, 2^{n+6}]$. For each $\beta \in \mathfrak{T}_{\rho,n}$, let, for large $C_1 \gg 1$,

$$\mathcal{N}_{n,\rho}(\beta) = \{(x,u) : \big||x| - r(\beta)\big| \le \sqrt{C_1\rho},\ \big|w(\beta, x, 4|u|)\big| \le C_1\rho\} \tag{161}$$

and

$$\mathcal{N}_{n,\rho} = \bigcup_{\beta\in\mathfrak{T}_{\rho,n}} \mathcal{N}_{n,\rho}(\beta).$$

Observe that $\mathrm{meas}(\mathcal{N}_{n,\rho}(\beta)) \lesssim_{C_1} 2^{-n(d_1+d_2-2)}\rho^{3/2}$ (by (108) and (112)), and thus $\mathrm{meas}(\mathcal{N}_{n,\rho}) \lesssim_{C_1} \rho$. We separately estimate the quantity $III_{n,\rho}$ on $\mathcal{N}_{n,\rho}$ and its complement. First, by the Cauchy–Schwarz inequality and (152) (with $2^{j_0} \approx 2^{10n}\rho^{-1}$),

$$\|III_{n,\rho}\|_{L^1(\mathcal{N}_{n,\rho})} \lesssim \rho^{1/2}\|III_{n,\rho}\|_2 \lesssim (2^{-10n}\rho)^{(d-1)/2}\rho^{1/2}\|f_\rho\|_2$$

and, since $\rho^{d/2}\|f_\rho\|_2 \lesssim 1$,

$$\|III_{n,\rho}\|_{L^1(\mathcal{N}_{n,\rho})} \lesssim 2^{-5(d-1)n}. \tag{162}$$

In the complement of the exceptional set $\mathcal{N}_{n,\rho}$, we split the term $III_{n,\rho}$ as

$$III_{n,\rho} = \sum_{2^j\rho>2^{10n}} \sum_{k\in\mathcal{J}_n} \sum_{l=1}^{\infty} (III^{k,l}_{n,\rho,j} + IV^{k,l}_{n,\rho,j}),$$

where

$$\begin{aligned} III^{k,l}_{n,\rho,j} &= 2^{-j(d-1)/2} T^{k,l}_{2^j}[(f_\rho * H_{j,n})\chi_{Q_{\rho,E,*}}], \\ IV^{k,l}_{n,\rho,j} &= 2^{-j(d-1)/2} T^{k,l}_{2^j}[(f_\rho * H_{j,n})\chi_{Q^{\complement}_{\rho,E,*}}], \end{aligned}$$

and $Q_{\rho,E,*}$ is as in (155). From (157) and (151) we immediately get $\|IV^{k,l}_{n,\rho,j}\|_1 \lesssim_N (2^l k)^{-(d_1+1)/2}(2^j\rho)^{-N}$, and thus

$$\sum_{2^j\rho>2^{10n}} \sum_{l=1}^{\infty} \sum_{k\in\mathcal{J}_n} \|IV^{k,l}_{n,\rho,j}\|_1 \lesssim 2^{-10nN}.$$

It remains to show that

$$\sum_{l=1}^{\infty} \sum_{k\in\mathcal{J}_n} \sum_{2^j\rho>2^{10n}} \|III^{k,l}_{n,\rho,j}\|_{L^1(\mathcal{N}^{\complement}_{n,\rho})} \lesssim 2^{-n(d_1-1)/2}. \tag{163}$$

Let $F_{j,n,\rho} = (f_\rho * H_{j,n})\chi_{Q_{\rho,E,*}}$, so that $\|F_{j,n,\rho}\|_1 \lesssim 1$. We shall show that, for $k \approx 2^n$,

$$\|F_{j,n,\rho} * A^{k,l}_{2^j}\|_{L^1(\mathcal{N}^{\complement}_{n,\rho})} \lesssim_N (2^{j-n}\rho)^{-N} 2^{-(l+n)d_1/2}, \quad 2^l \le 10^8 2^{j-n}, \tag{164}$$

and (163) follows by combining (164) with the estimates (148) and (150).

Proof of (164). We split $A^{k,l}_{2^j} = \sum_{b\in\mathcal{T}_{2^j,k,l}} A^{k,l}_{2^j,b}$ as in (127). For each $b \in \mathcal{T}_{2^j,k,l}$, we may assign a $\beta(b) \in \mathfrak{T}_{\rho,n}$ such that $|\beta(b) - b| \le \sqrt{\varepsilon\rho}$. Let $\mathcal{T}^{\beta}_{2^j,k,l}$ be the set of $b \in \mathcal{T}_{2^j,k,l}$ with $\beta(b) = \beta$. Then $\#\mathcal{T}^{\beta}_{2^j,k,l} \lesssim 2^{-n/2}\sqrt{2^{l+j}\rho}$. In order to see (164) it thus suffices to show that, for $2^l \le 10^8 2^{j-n}$, $|\beta - b| \le \rho$,

$$\|F_{j,n,\rho} * A^{k,l}_{2^j,b}\|_{L^1((\mathcal{N}_{n,\rho}(\beta))^{\complement})} \lesssim_{N_1} (2^{j-n}\rho)^{-N_1} 2^{-(l+n)(d_1+1)/2} 2^{(n+l-j)/2}. \tag{165}$$

To prove this we verify the following claim: if $(\tilde{x},\tilde{u}) \in Q_{\rho,E,*}$, $(x,u) \in (\mathcal{N}_{n,\rho}(\beta))^{\complement}$ and $2^{2m-j+n} \le \rho$, then

$$\big(|x-\tilde{x}|, 4\big|u-\tilde{u}+\tfrac{1}{2}\langle\vec{J}x,\tilde{x}\rangle\big|\big) \notin \mathcal{P}_m(2^j,l,k;b); \tag{166}$$

$\mathcal{P}_m(2^j,l,k;b)$ was defined in (133). Indeed the claim implies

$$\|F_{j,n,\rho} * A^{k,l}_{2^j,b}\|_{L^1((\mathcal{N}_{n,\rho}(\beta))^{\complement})} \lesssim \int_{(|x|,4|u|)\notin\mathcal{P}_m(2^j,l,k;b)} |A^{k,l}_{2^j,b}(x,u)|\,dx\,du,$$

since $\|F_{j,n,\rho}\|_1 = O(1)$, and (165) follows from Proposition 8.7.

To verify the claim (166), we pick $(x,u) \notin \mathcal{N}_{n,\rho}(\beta)$ and distinguish two cases:

(1) $\big||x| - r(\beta)\big| \ge \sqrt{C_1\rho}$.

(2) $\big|w(\beta,|x|,4|u|)\big| \ge C_1\rho$ and $\big||x| - r(\beta)\big| \le \sqrt{C_1\rho}$.

It is clear that the conclusion of the claim holds if we can show that, under the assumption that C_1 in the definition (161) is chosen sufficiently large (depending only on $\vec{J}$ and the dimension d), we have, for all $(\tilde{x},\tilde{u}) \in Q_{\rho,E,*}$,

$$\big||x-\tilde{x}| - r(b)\big| \ge \frac{\sqrt{C_1\rho}}{2} \qquad \text{in case (1),} \tag{167}$$

$$\big|w\big(b, |x-\tilde{x}|, 4\big|u-\tilde{u}+\tfrac{1}{2}\langle\vec{J}x,\tilde{x}\rangle\big|\big)\big| \ge \frac{C_1\rho}{2} \qquad \text{in case (2).} \tag{168}$$

The case (1) assumption implies, for $(\tilde{x}, \tilde{u}) \in Q_{\rho,E,*}$ (and sufficiently large C_1),

$$\big||x-\tilde{x}|-r(b)\big| \geq \big||x|-r(\beta)\big| - |\tilde{x}| - |r(b)-r(\beta)| \geq C_1\rho^{1/2} - C_*\rho - C|b-\beta|2^{-n} \geq \frac{\sqrt{C_1\rho}}{2},$$

which is (167).

Now assume that (x, u) satisfies the case (2) assumption. We then have, for all $(\tilde{x}, \tilde{u}) \in Q_{\rho,E,*}$,

$$\begin{aligned}&\Big|w\big(b, |x-\tilde{x}|, 4\big|u-\tilde{u}+\tfrac{1}{2}\langle\vec{J}x, \tilde{x}\rangle\big|\big) - w(\beta, |x|, 4|u|)\Big| \\ &\qquad \leq \big|w(b, |x|, 4|u|) - w(\beta, |x|, 4|u|)\big| + \Big|w\big(b, |x-\tilde{x}|, 4\big|u-\tilde{u}+\tfrac{1}{2}\langle\vec{J}x, \tilde{x}\rangle\big|\big) - w(b, |x|, 4|u|)\Big|\end{aligned}$$

The first term on the right-hand side can be estimated using (132) (with (t, b) replaced by (b, β)), and we see that it is at most $(C + \sqrt{C_1})\rho$ under the present case (2) assumption. The second term on the right-hand side is equal to

$$\left|4|u| - 4\big|u-\tilde{u}+\tfrac{1}{2}\langle\vec{J}x, \tilde{x}\rangle\big| - \frac{v'(b)}{r'(b)}(|x|-|x-\tilde{x}|)\right|,$$

and, since the case (2) assumption implies $|x| = O(1)$, we see that the displayed expression is $O(\rho)$. Thus, if C_1 in the definition is sufficiently large, we obtain (168). This concludes the proof of the claim (166) and thus the estimate (164). □

We finally consider the case where $\frac{1}{2} < \rho \leq 1$, in which condition (154) is not required. This case can easily be handled by means of Proposition 8.8. To this end, we decompose

$$a(\sqrt{L})e^{i\sqrt{L}} = \sum_{j\geq 10} 2^{-j(d-1)/2} g_j(2^{-j}\sqrt{L})e^{i\sqrt{L}}$$

with $g_j(s) = 2^{j(d-1)/2}a(2^j s)\chi_1(s)$. The family of functions g_j is uniformly bounded in the Schwartz space. If K_j denotes the convolution kernel for the operator $g_j(2^{-j}\sqrt{L})e^{i\sqrt{L}}$, we thus obtain from Proposition 8.8 the uniform estimates

$$\int_{\max\{|x|,|u|\}\geq 100} |K_j(x, u)|\,dx\,du \leq C_N 2^{-jN}.$$

This implies that

$$\int_{\max\{|x|,|u|\}\geq 200} |(a(\sqrt{L})e^{i\sqrt{L}} f_\rho)(x)|\,dx\,du \lesssim \|f\|_1 \lesssim 1.$$

And, by Hölder's inequality,

$$\int_{\max\{|x|,|u|\}\leq 200} |(a(\sqrt{L})e^{i\sqrt{L}} f_\rho)(x)|\,dx \lesssim \|(a(\sqrt{L})e^{i\sqrt{L}} f_\rho)\|_2 \lesssim \|f_\rho\|_2 \lesssim 1.$$

This concludes the proof of Theorem 5.3. □

10. Interpolation and proof of Theorem 1.1

Using interpolation for analytic families, one can deduce Theorem 1.1 from the Hardy space estimate by noticing that $L^p(G)$ is an interpolation space for the couple $(h^1_{\rm iso}(G), L^2(G))$ with respect to Calderón's complex $[\,\cdot\,,\cdot\,]_\vartheta$ method. One has

$$[h^1_{\rm iso}(G), L^2(G)]_\vartheta = L^p(G), \quad \vartheta = 2-\frac{2}{p},\ 1 < p \le 2, \tag{169}$$

with equivalence of norms. It is straightforward to deduce (169) using the method of retractions and coretractions [Triebel 1995] from an analogous formula for the Euclidean local Hardy spaces h^1_E; more precisely, from a vector-valued extension for the spaces $[\ell^1(h^1_E), \ell^2(L^2)]_\vartheta = \ell^p(L^p)$, $\vartheta = 2-2/p$. For a direct proof see the preprint version, arXiv:1408.3051, of this paper. However, (169) can also be seen as a special case of a more general interpolation result by M. Taylor [2009], since $h^1_{\rm iso}$ can be identified with the local Hardy space associated with a left-invariant Riemannian metric on the group. We would like to thank the referee for pointing out this reference.

Proof of Theorem 1.1. By duality we may assume $1 < p < 2$. By scaling and symmetry we may assume $\tau = 1$. Let $a \in S^{-(d-1)(1/p-1/2)}$. Consider the analytic family of operators

$$\mathscr{A}_z = e^{z^2} \sum_{j=0}^{\infty} 2^{-jz(d-1)/2} 2^{j(d-1)(1/p-1/2)} \zeta_j(\sqrt{L}) a(\sqrt{L}) e^{i\sqrt{L}}.$$

We need to check that $\mathscr{A}_z$ is bounded on L^p for $z = 2/p-1$. But, for $\Re(z) = 0$, the operators $\mathscr{A}_z$ are bounded on L^2; and for $\Re(z) = 1$ we have shown that $\mathscr{A}_z$ maps $h^1_{\rm iso}$ boundedly to L^1, by Theorem 1.3. We apply the abstract version of the interpolation theorem for analytic families in conjunction with (169) and the corresponding standard version interpolation result for L^p spaces; the result is that $\mathscr{A}_\vartheta$ is bounded on L^p for $\vartheta = 2/p-1$. This proves Theorem 1.1. □

11. Proof of Theorem 1.4

We decompose $m = \sum_{k\in\mathbb{Z}} m_k$, where m_k is supported in $(2^{k-1}, 2^{k+1})$ and where $h_k = m_k(2^k\,\cdot\,)$ satisfies

$$\sum_{\ell>1} \sup_k \int_{2^\ell}^\infty |\hat{h}_k(\tau)| \tau^{(d-1)/2}\, d\tau \le A.$$

By the translation invariance and the usual Calderón–Zygmund arguments (see, e.g., [Stein 1993]) it suffices to prove that, for all $\rho > 0$ and for all L^1 functions f_ρ supported in the Koranyi ball $Q_\rho := Q_\rho(0,0)$ and satisfying $\int f_\rho\, dx = 0$, we have

$$\sum_k \iint_{Q^\complement_{10\rho}} |m_k(\sqrt{L}) f_\rho|\, dx \lesssim A + \|m\|_\infty. \tag{170}$$

Let $\chi_1 \in C^\infty_0$ be supported in $\left(\frac{1}{5}, 5\right)$ and such that $\chi_1(s) = 1$ for $s \in \left[\frac{1}{4}, 4\right]$. Then, for each k, write

$$m_k(\sqrt{L}) = h_k(2^{-k}\sqrt{L})\chi_1(2^{-k}\sqrt{L}) = \int \hat{h}_k(\tau)\chi_1(2^{-k}\sqrt{L}) e^{i2^{-k}\tau\sqrt{L}}\, d\tau.$$

By scale invariance and Theorem 1.2, the L^1 operator norm of the operator $\chi_1(2^{-k}\sqrt{L})e^{i2^{-k}\tau\sqrt{L}}$ is $O(1+|\tau|)^{(d-1)/2}$, and thus

$$\|m_k(\sqrt{L})\|_{L^1\to L^1} \lesssim \int_{-\infty}^{\infty} |\hat{h}_k(\tau)|(1+|\tau|)^{(d-1)/2}\,d\tau.$$

Also observe that, since the convolution kernel of $\chi_1(\sqrt{L})$ is a Schwartz kernel, we can use the cancellation and support properties of f_ρ to get, for some $\varepsilon > 0$,

$$\|\chi_1(2^{-k}\sqrt{L})f_\rho\|_1 \lesssim \min\{1, (2^k\rho)^\varepsilon\}\|f_\rho\|_1.$$

Thus, the two preceding displayed inequalities yield

$$\begin{aligned}\sum_{k:2^k\rho\le M} \|m_k(\sqrt{L})f_\rho\|_1 &\le C_M \sup_k \int_{-\infty}^{\infty} |\hat{h}_k(\tau)|(1+|\tau|)^{(d-1)/2}\,d\tau\,\|f_\rho\|_1\\ &\lesssim_M (\|m\|_\infty + \mathfrak{A}_2)\|f_\rho\|_1,\end{aligned}\tag{171}$$

where for the last estimate we use $|\hat{h}_k(\tau)| \le \|h_k\|_\infty \lesssim \|m\|_\infty$ when $|\tau| \le 2$.

We now consider the terms for $2^k\rho \ge M$ and M large, in the complement of the expanded Koranyi ball $Q_{\rho,*} = Q_{C\rho}$ (for suitable large $C \gg 2$). By a change of variable and an application of Proposition 8.8,

$$\|e^{i2^{-k}\tau\sqrt{L}}\chi_1(2^{-k}\sqrt{L})f_\rho\|_{L^1(Q^{\complement}_{\rho,*})} = \|e^{i\sqrt{L}}\chi_1(\tau^{-1}\sqrt{L})f_\rho^{2^k/\tau}\|_{L^1(Q^{\complement}_{C_*\tau^{-1}2^k\rho})} \lesssim (2^k\rho\tau^{-1})^{-N} \quad \text{if } 2^k\rho \gg \tau,$$

where $f_\rho^{2^k/\tau}$ is a rescaling of f_ρ such that $\|f_\rho^{2^k/\tau}\|_1 = \|f_\rho\|_1 \lesssim 1$.

Hence, if M is sufficiently large then, for $2^k\rho > M$,

$$\begin{aligned}&\|m_k(\sqrt{L})f_\rho\|_{L^1(Q^{\complement}_{\rho,*})}\\ &\qquad\lesssim_N \|f_\rho\|_1\Big[\int_{|\tau|>2^k\rho} |\hat{h}_k(\tau)|(1+|\tau|)^{(d-1)/2}\,d\tau + (2^k\rho)^{-N}\int_{|\tau|\le 2^k\rho} |\hat{h}_k(\tau)|(1+|\tau|)^{-N}\,d\tau\Big],\end{aligned}$$

and thus

$$\sum_{2^k\rho>M} \|m_k(\sqrt{L})f_\rho\|_{L^1(Q^{\complement}_{\rho,*})} \lesssim \|m\|_\infty + \sum_{k:2^k\rho>M} \mathfrak{A}_{2^k\rho}.\tag{172}$$

The theorem follows from (171) and (172).

References

[Beals 1982] R. M. Beals, *L^p boundedness of Fourier integral operators*, Mem. Amer. Math. Soc. **264**, American Mathematical Society, Providence, RI, 1982. MR 84m:42026 Zbl 0508.42020

[Christ 1991] M. Christ, "L^p bounds for spectral multipliers on nilpotent groups", *Trans. Amer. Math. Soc.* **328**:1 (1991), 73–81. MR 92k:42017 Zbl 0739.42010

[Damek and Ricci 1992] E. Damek and F. Ricci, "Harmonic analysis on solvable extensions of H-type groups", *J. Geom. Anal.* **2**:3 (1992), 213–248. MR 93d:43006 Zbl 0788.43008

[Folland 1989] G. B. Folland, *Harmonic analysis in phase space*, Annals of Mathematics Studies **122**, Princeton University Press, NJ, 1989. MR 92k:22017 Zbl 0682.43001

[Folland and Stein 1982] G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes **28**, Princeton University Press, NJ, 1982. MR 84h:43027 Zbl 0508.42025

[Gaveau 1977] B. Gaveau, "Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents", *Acta Math.* **139**:1–2 (1977), 95–153. MR 57 #1574 Zbl 0366.22010

[Goldberg 1979] D. Goldberg, "A local version of real Hardy spaces", *Duke Math. J.* **46**:1 (1979), 27–42. MR 80h:46052 Zbl 0409.46060

[Greiner et al. 2002] P. C. Greiner, D. Holcman, and Y. Kannai, "Wave kernels related to second-order operators", *Duke Math. J.* **114**:2 (2002), 329–386. MR 2003i:35204 Zbl 1072.35130

[Hebisch 1993] W. Hebisch, "Multiplier theorem on generalized Heisenberg groups", *Colloq. Math.* **65**:2 (1993), 231–239. MR 94m:43013 Zbl 0841.43009

[Hörmander 1967] L. Hörmander, "Hypoelliptic second order differential equations", *Acta Math.* **119** (1967), 147–171. MR 36 #5526 Zbl 0156.10701

[Hörmander 1971] L. Hörmander, "Fourier integral operators, I", *Acta Math.* **127**:1–2 (1971), 79–183. MR 52 #9299 Zbl 0212.46601

[Hulanicki 1984] A. Hulanicki, "A functional calculus for Rockland operators on nilpotent Lie groups", *Studia Math.* **78**:3 (1984), 253–266. MR 86g:22009 Zbl 0595.43007

[Kaplan 1980] A. Kaplan, "Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms", *Trans. Amer. Math. Soc.* **258**:1 (1980), 147–153. MR 81c:58059 Zbl 0393.35015

[Kaplan and Ricci 1983] A. Kaplan and F. Ricci, "Harmonic analysis on groups of Heisenberg type", pp. 416–435 in *Harmonic analysis* (Cortona, 1982), edited by F. Ricci and G. Weiss, Lecture Notes in Math. **992**, Springer, Berlin, 1983. MR 85h:22017 Zbl 0521.43008

[Martini 2012] A. Martini, "Analysis of joint spectral multipliers on Lie groups of polynomial growth", *Ann. Inst. Fourier (Grenoble)* **62**:4 (2012), 1215–1263. MR 3025742 Zbl 1255.43003

[Mauceri and Meda 1990] G. Mauceri and S. Meda, "Vector-valued multipliers on stratified groups", *Rev. Mat. Iberoamericana* **6**:3–4 (1990), 141–154. MR 92j:42014 Zbl 0763.43005

[Melrose 1986] R. Melrose, "Propagation for the wave group of a positive subelliptic second-order differential operator", pp. 181–192 in *Hyperbolic equations and related topics* (Katata/Kyoto, 1984), edited by S. Mizohata, Academic Press, Boston, 1986. MR 89h:35177 Zbl 0696.35064

[Miyachi 1980] A. Miyachi, "On some estimates for the wave equation in L^p and H^p", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27**:2 (1980), 331–354. MR 83g:35060 Zbl 0437.35042

[Müller 1999] D. Müller, "Analysis of invariant PDO's on the Heisenberg group", lecture notes, 1999. arXiv 1408.2634

[Müller 2007] D. Müller, "Local solvability of linear differential operators with double characteristics, II: Sufficient conditions for left-invariant differential operators of order two on the Heisenberg group", *J. Reine Angew. Math.* **607** (2007), 1–46. MR 2008f:35047 Zbl 1215.35014

[Müller and Ricci 1990] D. Müller and F. Ricci, "Analysis of second order differential operators on Heisenberg groups, I", *Invent. Math.* **101**:3 (1990), 545–582. MR 92d:22011 Zbl 0742.43006

[Müller and Ricci 1996] D. Müller and F. Ricci, "Solvability for a class of doubly characteristic differential operators on 2-step nilpotent groups", *Ann. of Math.* (2) **143**:1 (1996), 1–49. MR 96m:22018 Zbl 0841.22006

[Müller and Stein 1994] D. Müller and E. M. Stein, "On spectral multipliers for Heisenberg and related groups", *J. Math. Pures Appl.* (9) **73**:4 (1994), 413–440. MR 96m:43004 Zbl 0838.43011

[Müller and Stein 1999] D. Müller and E. M. Stein, "L^p-estimates for the wave equation on the Heisenberg group", *Rev. Mat. Iberoamericana* **15**:2 (1999), 297–334. MR 2001c:43009 Zbl 0935.43003

[Müller et al. 1996] D. Müller, F. Ricci, and E. M. Stein, "Marcinkiewicz multipliers and multi-parameter structure on Heisenberg (-type) groups, II", *Math. Z.* **221**:2 (1996), 267–291. MR 97c:43007 Zbl 0863.43001

[Nachman 1982] A. I. Nachman, "The wave equation on the Heisenberg group", *Comm. Partial Differential Equations* **7**:6 (1982), 675–714. MR 84e:58074 Zbl 0524.35065

[Nelson and Stinespring 1959] E. Nelson and W. F. Stinespring, "Representation of elliptic operators in an enveloping algebra", *Amer. J. Math.* **81** (1959), 547–560. MR 22 #907 Zbl 0092.32103

[Peral 1980] J. C. Peral, "L^p estimates for the wave equation", *J. Funct. Anal.* **36**:1 (1980), 114–145. MR 81k:35089 Zbl 0442.35017

[Ricci 1982] F. Ricci, "Harmonic analysis on generalized Heisenberg groups", preprint, Politecnico di Torino, 1982.

[Seeger et al. 1991] A. Seeger, C. D. Sogge, and E. M. Stein, "Regularity properties of Fourier integral operators", *Ann. of Math.* (2) **134**:2 (1991), 231–251. MR 92g:35252 Zbl 0754.58037

[Stein 1993] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series **43**, Princeton University Press, NJ, 1993. MR 95c:42002 Zbl 0821.42001

[Stein and Weiss 1971] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series **32**, Princeton University Press, NJ, 1971. MR 46 #4102 Zbl 0232.42007

[Strichartz 1991] R. S. Strichartz, "L^p harmonic analysis and Radon transforms on the Heisenberg group", *J. Funct. Anal.* **96**:2 (1991), 350–406. MR 92d:22015 Zbl 0734.43004

[Taylor 2009] M. Taylor, "Hardy spaces and BMO on manifolds with bounded geometry", *J. Geom. Anal.* **19**:1 (2009), 137–190. MR 2009k:58053 Zbl 1189.46030

[Triebel 1995] H. Triebel, *Interpolation theory, function spaces, differential operators*, 2nd ed., Johann Ambrosius Barth, Heidelberg, 1995. MR 96f:46001 Zbl 0830.46028

Received 13 Aug 2014. Revised 17 Dec 2014. Accepted 9 Feb 2015.

DETLEF MÜLLER: mueller@math.uni-kiel.de
Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, Ludewig-Meyn-Str. 4, D-24098 Kiel, Germany

ANDREAS SEEGER: seeger@math.wisc.edu
Department of Mathematics, University of Wisconsin, Van Vleck Hall, 480 Lincoln Drive, Madison, WI 53706, United States

ANALYSIS AND PDE
Vol. 8, No. 5, 2015
dx.doi.org/10.2140/apde.2015.8.1101

GLOBAL WELL-POSEDNESS ON THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

YIFEI WU

As a continuation of our previous work, we consider the global well-posedness for the derivative nonlinear Schrödinger equation. We prove that it is globally well posed in the energy space, provided that the initial data $u_0 \in H^1(\mathbb{R})$ with $\|u_0\|_{L^2} < 2\sqrt{\pi}$.

1. Introduction

We study the following Cauchy problem of the nonlinear Schrödinger equation with derivative (DNLS):

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x(|u|^2 u), & t \in \mathbb{R},\ x \in \mathbb{R}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}). \end{cases} \tag{1-1}$$

It arises from studying the propagation of circularly polarized Alfvén waves in magnetized plasma with a constant magnetic field; see [Mio et al. 1976; Mjolhus 1976; Sulem and Sulem 1999] and the references therein. The equation in (1-1) is L^2-critical and completely integrable. The H^1-solution of (1-1) obeys the following mass, energy, and momentum conservation laws:

$$M(u(t)) := \int_{\mathbb{R}} |u(t,x)|^2\,dx = M(u_0), \tag{1-2}$$

$$E_D(u(t)) := \int_{\mathbb{R}} \Big(|u_x(t,x)|^2 + \tfrac{3}{2}\operatorname{Im}|u(t,x)|^2 u(t,x)\overline{u_x(t,x)} + \tfrac{1}{2}|u(t,x)|^6\Big)\,dx = E_D(u_0), \tag{1-3}$$

$$P_D(u(t)) := \operatorname{Im}\int_{\mathbb{R}} \overline{u(t,x)}u_x(t,x)\,dx - \frac{1}{2}\int_{\mathbb{R}} |u(t,x)|^4\,dx = P_D(u_0). \tag{1-4}$$

Local well-posedness for the Cauchy problem (1-1) is well understood. It was proved in the energy space $H^1(\mathbb{R})$ in [Hayashi 1993; Hayashi and Ozawa 1992; 1994], and earlier by Guo and Tan [1991] and Tsutsumi and Fukuda [1980; 1981] in smooth spaces. See [Biagioni and Linares 2001; Takaoka 1999; 2001] for local well-posedness and ill-posedness results for rough data below the energy space.

The global well-posedness for (1-1) has also been widely studied. By using mass and energy conservation laws, and the gauge transformations, Hayashi and Ozawa [1994; Ozawa 1996] proved that (1-1) is globally well-posed in the energy space $H^1(\mathbb{R})$ under the condition

$$\|u_0\|_{L^2} < \sqrt{2\pi}. \tag{1-5}$$

The author was partially supported by the NSF of China (No. 11101042), and the Fundamental Research Funds for the Central Universities of China.
MSC2010: primary 35Q55; secondary 35A01.
Keywords: nonlinear Schrödinger equation with derivative, global well-posedness, energy space.

Here 2π is the mass of the ground state Q, which is the unique (up to some symmetries) positive solution of the elliptic equation

$$-Q_{xx}+Q-\tfrac{3}{16}Q^5=0. \tag{1-6}$$

As shown in [Weinstein 1983], $Q=2[\cosh(2x)]^{-1/2}$. Since Q is an optimizer for the Gagliardo–Nirenberg inequality (1-12), any function with mass strictly less than the mass of Q has positive energy.

Condition (1-5) was improved recently in [Wu 2013]. We proved that there exists a small constant $\varepsilon_* > 0$ such that (1-1) is still globally well-posed in the energy space when the initial data satisfies $\|u_0\|_{L^2} < \sqrt{2\pi}+\varepsilon_*$. The result implies that, for (1-1), the ground state mass 2π is not the threshold of the global well-posedness and blow-up. This is different from the L^2-critical power-type Schrödinger equation (the nonlinearity $i\partial_x(|u|^2u)$ in (1-1) is replaced by $-\frac{3}{16}|u|^4u$); see [Wu 2013] for further discussion.

For related results on the well-posedness and stability theory for the derivative nonlinear Schrödinger equation (1-1), see [Colin and Ohta 2006; Colliander et al. 2001; 2002; Grünrock and Herr 2008; Guo and Wu 1995; Herr 2006; Miao et al. 2011; Nahmod et al. 2012; Takaoka 2001; Thomann and Tzvetkov 2010; Win 2010].

In this paper, we continue to consider the L^2-assumption on initial data and obtain the global well-posedness as follows:

Theorem 1.1. *For any $u_0 \in H^1(\mathbb{R})$ with*

$$\int_{\mathbb{R}} |u_0(x)|^2\,dx < 4\pi, \tag{1-7}$$

the Cauchy problem (1-1) is globally well-posed in $H^1(\mathbb{R})$ and the solution u satisfies

$$\|u\|_{L_t^\infty H_x^1} \le C(\|u_0\|_{H^1}).$$

As $2\pi = \|Q\|_{L^2}^2$, we notice that there is also a solitary wave solution whose mass is 4π, given by

$$u(t,x)=e^{3i/4\int_{-\infty}^{x+t}|W(y)|^2\,dy}e^{-it/4-ix/2}W(x+t), \tag{1-8}$$

where W is the ground state of the elliptic equation

$$-W_{xx}+\tfrac{1}{2}W^3-\tfrac{3}{16}W^5=0. \tag{1-9}$$

Up to some symmetries,

$$W(x)=2(x^2+1)^{-1/2}. \tag{1-10}$$

Therefore, Theorem 1.1 indicates that the Cauchy problem (1-1) is globally well-posed in $H^1(\mathbb{R})$ when $\|u_0\|_{L^2} < \|W\|_{L^2}$.

Compared to Q, W is polynomial decaying at infinity. Furthermore, W is an optimal function of the sharp Gagliardo–Nirenberg inequality (see [Agueh 2006])

$$\|f\|_{L^6} \le C_{\mathrm{GN}}\|f\|_{L^4}^{8/9}\|f_x\|_{L^2}^{1/9}, \tag{1-11}$$

where we wrote C_{GN} for the sharp constant $C_{\mathrm{GN}} = 3^{1/6}(2\pi)^{-1/9}$. This inequality plays an important role in the proof of our main theorem. There is also a comparison with another sharp Gagliardo–Nirenberg inequality (see [Weinstein 1983]),

$$\|f\|_{L^6}^6 \le \frac{4}{\pi^2}\|f\|_{L^2}^4\|f_x\|_{L^2}^2, \tag{1-12}$$

in which the equality is attained by Q, which was applied previously to prove the global well-posedness when $\|u_0\|_{L^2} < \sqrt{2\pi}$.

So there is an interesting problem of whether $\|W\|_{L^2}^2 = 4\pi$ is the mass threshold of the global well-posedness and blowup for (1-1). See Section 3 below for further discussion.

Now let us have a look at the strategy of the proof of Theorem 1.1. Developed by Hayashi and Ozawa, the gauge transformation is an important tool to study the derivative nonlinear Schrödinger equation. Let

$$v(t,x) := e^{-3i/4\int_{-\infty}^{x}|u(t,y)|^2\,dy}u(t,x); \tag{1-13}$$

then, from (1-1), v is the solution of

$$i\partial_t v + \partial_x^2 v = \tfrac{1}{2}i|v|^2 v_x - \tfrac{1}{2}iv^2\bar{v}_x - \tfrac{3}{16}|v|^4 v \tag{1-14}$$

with the initial data $v_0 = \exp\bigl(-\frac{3}{4}i\int_{-\infty}^{x}|u_0(y)|^2\,dy\bigr)u_0$. Moreover, v obeys the same mass conservation law as (1-2), the energy conservation law (1-3) becomes

$$E(v(t)) := \|v_x(t)\|_{L_x^2}^2 - \tfrac{1}{16}\|v(t)\|_{L_x^6}^6 = E(v_0), \tag{1-15}$$

and the momentum conservation law (1-4) becomes

$$P(v(t)) := \operatorname{Im}\int_{\mathbb{R}}\overline{v(t,x)}v_x(t,x)\,dx + \frac{1}{4}\int_{\mathbb{R}}|v(t,x)|^4\,dx = P(v_0). \tag{1-16}$$

From the argument used in [Wu 2013] to prove the global well-posedness for the DNLS, an important consideration is the usage of the momentum conservation law. We observe that the key point is to give a small control of the following term from (1-16):

$$\operatorname{Im}\int_{\mathbb{R}}\overline{v(t,x)}v_x(t,x)\,dx. \tag{1-17}$$

To be more precise, one may prove that

$$-\operatorname{Im}\int_{\mathbb{R}}\overline{v(t,x)}v_x(t,x)\,dx \le c\|v_x(t)\|_{L^2}\|v(t)\|_{L^2}, \tag{1-18}$$

where c is a positive constant. This is trivial for $c = 1$ by Hölder's inequality. Suppose that one can obtain the inequality with a suitable small constant c. Then the global well-posedness will follow. In [Wu 2013], by using the rigidity of the ground state Q, we proved that, if the mass is larger but close to 2π and there is a time sequence $\{t_n\}$ such that $\|v(t_n)\|_{H^1}$ tends to infinity, then $v(t_n)$ is close to Q up to some symmetries. Since Q is real-valued, (1-18) can be given for small $c > 0$.

In this paper, we present a different argument to prove the bound (1-18) under the suitable but explicit assumption of L^2-norm of the initial data. Our method here does not need to use the property of the ground state Q of (1-6). As was previously mentioned, it depends heavily on the sharp Gagliardo–Nirenberg inequality (1-11). This is to be expected, since the norms involved in the inequality (1-11) are strongly related to the energy and momentum conservation laws.

Let us expand our argument. If $\|v(t)\|_{H^1}$ tends to infinity, then, by the momentum and energy conservation laws, (1-18) is approximately

$$\tfrac{1}{4}\|v(t)\|_{L^4}^4 \approx -\operatorname{Im}\int_{\mathbb{R}} \overline{v(t,x)}v_x(t,x)\,dx \le c\|v_x(t)\|_{L^2}\|v(t)\|_{L^2} \approx c\|v_0\|_{L^2}\|v(t)\|_{L^6}^3.$$

So, to obtain the small bound c, we turn to consider the quantity

$$f(t) := \frac{\|v(t)\|_{L^4}^4}{\|v(t)\|_{L^6}^3}.$$

Indeed, we shall prove that f^2 obeys some cubic inequality. Thus, the condition for global well-posedness is transformed to finding the solution to an elementary cubic equation.

This paper is organized as follows. In Section 2, we present the proof of Theorem 1.1. In Section 3, we discuss some related problems.

2. The proof of Theorem 1.1

Let v be the function in (1-13), which is the solution of the equation (1-14). Note that

$$u_x = e^{3i/4\int_{-\infty}^{x}|v(t,y)|^2\,dy}\left(\tfrac{3}{4}i|v|^2v + v_x\right).$$

Therefore, by the sharp Gagliardo–Nirenberg inequality (1-12) and mass conservation law, for any $t \in \mathbb{R}$,

$$\begin{aligned}\|u_x(t)\|_{L^2} &\le \|v_x(t)\|_{L^2} + \frac{3}{4}\|v(t)\|_{L^6}^3 \le \|v_x(t)\|_{L^2} + \frac{3}{2\pi}\|v(t)\|_{L^2}^2\|v_x(t)\|_{L^2} \\ &\le \left(1 + \frac{3}{2\pi}\|u_0\|_{L^2}^2\right)\|v_x(t)\|_{L^2}.\end{aligned}$$

That is, the boundedness of v in H^1-norm implies the boundedness of u in H^1-norm. Therefore, to prove the theorem, we may consider the function v in (1-13) instead. To simplify the notations, we set

$$E_0 = E(v_0), \quad P_0 = P(v_0), \quad m_0 = M(v_0).$$

Furthermore, we assume $m_0 > 2\pi$. Otherwise, the global well-posedness has been proved in [Hayashi and Ozawa 1994; Wu 2013].

Let $(-T_-(v_0), T_+(v_0))$ be the maximal lifespan of the solution v of (1-14). To prove Theorem 1.1, it is sufficient to obtain the (indeed uniformly) a priori estimate of the solutions in H^1-norm. That is,

$$\sup_{t\in(-T_-(v_0),T_+(v_0))} \|v_x(t)\|_{L^2} < +\infty.$$

As in [Wu 2013], we argue by contradiction. Suppose that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ with limit $-T_-(v_0)$ or $T_+(v_0)$ such that

$$\|v_x(t_n)\|_{L^2} \to +\infty \quad \text{as } n \to \infty. \tag{2-1}$$

Then, from the energy conservation law, we also have

$$\|v(t_n)\|_{L^6} \to +\infty \quad \text{as } n \to \infty.$$

Let us define the sequence $\{f_n\}_{n=1}^{\infty}$ by

$$f_n = \frac{\|v(t_n)\|_{L^4}^4}{\|v(t_n)\|_{L^6}^3};$$

then we have both the lower and upper bounds of f_n as follows:

Lemma 2.1. *There exists a sequence ε_n, with $\varepsilon_n \to 0$ as $n \to \infty$, such that*

$$2C_{\mathrm{GN}}^{-9/2} + \varepsilon_n \le f_n \le \sqrt{m_0}. \tag{2-2}$$

Proof of Lemma 2.1. From Hölder's inequality, we have

$$\|v(t_n)\|_{L^4}^4 \le \|v(t_n)\|_{L^2} \|v(t_n)\|_{L^6}^3 = \sqrt{m_0}\|v(t_n)\|_{L^6}^3,$$

and thus

$$f_n \le \sqrt{m_0}.$$

On the other hand, from the sharp Gagliardo–Nirenberg inequality (1-11) and the energy conservation law (1-15), we have

$$\begin{aligned} f_n \ge \frac{\big(C_{\mathrm{GN}}^{-6}\|v(t_n)\|_{L^6}^6 \|v_x(t_n)\|_{L^2}^{-2/3}\big)^{3/4}}{\|v(t_n)\|_{L^6}^3} &= C_{\mathrm{GN}}^{-9/2}\frac{\|v(t_n)\|_{L^6}^{3/2}}{\|v_x(t_n)\|_{L^2}^{1/2}} \\ &= 2C_{\mathrm{GN}}^{-9/2}\frac{\|v(t_n)\|_{L^6}^{3/2}}{\big(\|v(t_n)\|_{L^6}^6 + 16E_0\big)^{1/4}} \\ &= 2C_{\mathrm{GN}}^{-9/2} + \varepsilon_n, \end{aligned}$$

where

$$\varepsilon_n := 2C_{\mathrm{GN}}^{-9/2}\frac{\|v(t_n)\|_{L^6}^{3/2} - \big(\|v(t_n)\|_{L^6}^6 + 16E_0\big)^{1/4}}{\big(\|v(t_n)\|_{L^6}^6 + 16E_0\big)^{1/4}}.$$

By the mean value theorem, we have

$$\varepsilon_n = O\big(\|v(t_n)\|_{L^6}^{-6}\big) \to 0.$$

This proves the lemma. □

By Lemma 2.1, and $\|v(t_n)\|_{L^4}^4 = f_n\|v(t_n)\|_{L^6}^3$, we have

$$\|v(t_n)\|_{L^4} \to +\infty \quad \text{as } n \to \infty.$$

In the spirit of [Banica 2004], we define

$$\phi(t,x) = e^{i\alpha x} v(t,x),$$

where the parameter α depends on t and is given below. Then $\phi_x(t,x) = e^{i\alpha x}(i\alpha v(t,x) + v_x(t,x))$, and thus

$$\|\phi_x\|_{L^2}^2 = \|v_x\|_{L^2}^2 + 2\alpha \operatorname{Im} \int \bar{v} v_x \, dx + \alpha^2 \|v\|_{L^2}^2.$$

Subtracting $\frac{1}{16}\|\phi\|_{L^6}^6 = \frac{1}{16}\|v\|_{L^6}^6$ from both sides yields

$$E(\phi) = E(v) + 2\alpha \operatorname{Im} \int \bar{v} v_x \, dx + \alpha^2 \|v\|_{L^2}^2.$$

By the mass and energy conservation laws (1-2) and (1-15), this gives

$$-2\alpha \operatorname{Im} \int \overline{v(t,x)} v_x(t,x) \, dx = -E(\phi(t)) + \alpha^2 m_0 + E_0. \tag{2-3}$$

On the other hand, using (1-11), we have

$$\begin{aligned} E(\phi(t_n)) &= \|\phi_x(t_n)\|_{L^2}^2 - \tfrac{1}{16}\|\phi(t_n)\|_{L^6}^6 \\ &\geq C_{\mathrm{GN}}^{-18} \|\phi(t_n)\|_{L^6}^{18} \|\phi(t_n)\|_{L^4}^{-16} - \tfrac{1}{16}\|\phi(t_n)\|_{L^6}^6 \\ &= \big(C_{\mathrm{GN}}^{-18} \|v(t_n)\|_{L^6}^{12} \|v(t_n)\|_{L^4}^{-16} - \tfrac{1}{16}\big) \|\phi(t_n)\|_{L^6}^6 \\ &= \big(C_{\mathrm{GN}}^{-18} f_n^{-4} - \tfrac{1}{16}\big) \|v(t_n)\|_{L^6}^6. \end{aligned}$$

Combining this with (2-3) gives

$$-2\alpha \operatorname{Im} \int \overline{v(t_n,x)} v_x(t_n,x) \, dx \leq \big(\tfrac{1}{16} - C_{\mathrm{GN}}^{-18} f_n^{-4}\big) \|v(t_n)\|_{L^6}^6 + \alpha^2 m_0 + E_0,$$

which implies, for $\alpha > 0$,

$$-\operatorname{Im} \int \overline{v(t_n,x)} v_x(t_n,x) \, dx \leq \frac{1}{2\alpha}\big(\tfrac{1}{16} - C_{\mathrm{GN}}^{-18} f_n^{-4}\big) \|v(t_n)\|_{L^6}^6 + \frac{1}{2}\alpha m_0 + \frac{1}{2\alpha} E_0. \tag{2-4}$$

For convenience, we define β_n as

$$\beta_n := m_0^{-1}\big(\tfrac{1}{16} - C_{\mathrm{GN}}^{-18} f_n^{-4}\big) \|v(t_n)\|_{L^6}^6.$$

We split this into two cases:

Case 1: $\beta_n < 1$ for infinitely many n. This implies that, for such n,

$$\big(\tfrac{1}{16} - C_{\mathrm{GN}}^{-18} f_n^{-4}\big) \|v(t_n)\|_{L^6}^6 < m_0.$$

Therefore, from (2-4), we have

$$-\operatorname{Im} \int \overline{v(t_n,x)} v_x(t_n,x) \, dx \leq \frac{1}{2\alpha} m_0 + \frac{1}{2}\alpha m_0 + \frac{1}{2\alpha} E_0. \tag{2-5}$$

In particular, choosing $\alpha = 1$, we obtain

$$-\operatorname{Im} \int \overline{v(t_n,x)} v_x(t_n,x) \, dx \leq m_0 + \tfrac{1}{2} E_0. \tag{2-6}$$

By the momentum conservation law (1-16), we have

$$\tfrac{1}{4}\|v(t_n)\|_{L^4}^4 = -\operatorname{Im}\int \overline{v(t_n,x)}v_x(t_n,x)\,dx + P_0. \tag{2-7}$$

Hence, combining this with (2-6) and (2-7), we obtain

$$\|v(t_n)\|_{L^4}^4 \le 2(2m_0 + E_0 + 2P_0).$$

This contradicts $\|v(t_n)\|_{L^4} \to +\infty$, and thus we can rule out this case.

Case 2: $\beta_n \ge 1$ for all sufficiently large n. In this case, we set $\alpha = \alpha(t_n) = \sqrt{\beta_n}$. Then (2-4) becomes

$$-\operatorname{Im}\int \overline{v(t_n,x)}v_x(t_n,x)\,dx \le \tfrac{1}{4}\sqrt{m_0(1-16C_{\mathrm{GN}}^{-18}f_n^{-4})}\,\|v(t_n)\|_{L^6}^3 + \tfrac{1}{2}\beta_n^{-1/2}E_0. \tag{2-8}$$

By (2-7) and (2-8),

$$\|v(t_n)\|_{L^4}^4 \le \sqrt{m_0(1-16C_{\mathrm{GN}}^{-18}f_n^{-4})}\,\|v(t_n)\|_{L^6}^3 + 2\beta_n^{-1/2}E_0 + 4P_0,$$

which implies that

$$f_n \le \sqrt{m_0(1-16C_{\mathrm{GN}}^{-18}f_n^{-4})} + (2\beta_n^{-1/2}E_0 + 4P_0)\|v(t_n)\|_{L^6}^{-3}.$$

This provides the inequality

$$f_n^6 \le m_0 f_n^4 - 16m_0C_{\mathrm{GN}}^{-18} + f_n^4\mathcal{R}_n, \tag{2-9}$$

where

$$\mathcal{R}_n = 2\sqrt{m_0(1-16C_{\mathrm{GN}}^{-18}f_n^{-4})}\,(2\beta_n^{-1/2}E_0+4P_0)\|v(t_n)\|_{L^6}^{-3} + (2\beta_n^{-1/2}E_0+4P_0)^2\|v(t_n)\|_{L^6}^{-6}.$$

Since $\beta_n \ge 1$ and $0 \le 1-16C_{\mathrm{GN}}^{-18}f_n^{-4} \le 1$, we have

$$\mathcal{R}_n \le 2\sqrt{m_0}(2E_0+4P_0)\|v(t_n)\|_{L^6}^{-3} + (2E_0+4P_0)^2\|v(t_n)\|_{L^6}^{-6} = O\big(\|v(t_n)\|_{L^6}^{-3}\big).$$

From Lemma 2.1, we have

$$f_n^4\mathcal{R}_n = O\big(\|v(t_n)\|_{L^6}^{-3}\big) \to 0 \quad \text{as } n \to \infty.$$

Thus, for any small fixed $\epsilon > 0$, by choosing n large enough we have $f_n^4\mathcal{R}_n \le \epsilon$. Hence (2-9) becomes

$$f_n^6 \le m_0 f_n^4 - 16m_0C_{\mathrm{GN}}^{-18} + \epsilon. \tag{2-10}$$

Let $X = f_n^2$; then (2-10) becomes the inequality

$$X^3 - m_0X^2 + b \le 0, \tag{2-11}$$

where $b = 16m_0C_{\mathrm{GN}}^{-18} - \epsilon > 0$. Let

$$F(X) = X^3 - m_0X^2 + b;$$

then $F(X)$ attains its minimum value at $\frac{2}{3}m_0$ in the region $[0,\infty)$. Therefore, there are two positive solutions X_1 and X_2 of the equation

$$X^3 - m_0X^2 + b = 0 \tag{2-12}$$

if and only if $F\left(\frac{2}{3}m_0\right) < 0$. In other words, the inequality (2-11) has no solution in the region $[0,+\infty)$ if and only if

$$F\left(\tfrac{2}{3}m_0\right) > 0. \tag{2-13}$$

Hence, this leads to a contradiction under the condition (2-13).

Condition (2-13) is equivalent to

$$\tfrac{8}{27}m_0^3 - \tfrac{4}{9}m_0^3 + b > 0.$$

Since ϵ is arbitrarily small, this reduces to

$$\tfrac{8}{27}m_0^3 - \tfrac{4}{9}m_0^3 + 16m_0C_{\mathrm{GN}}^{-18} > 0,$$

which yields

$$m_0 < 6\sqrt{3}C_{\mathrm{GN}}^{-9} = 6\sqrt{3}\frac{1}{3^{3/2}(2\pi)^{-1}} = 4\pi.$$

Therefore, (1-14) is globally well-posed when $m_0 < 4\pi$. This proves the theorem.

One may expect to get some profit from the restriction $X \in (4C_{\mathrm{GN}}^{-9}, m_0)$ (rather than $[0,+\infty)$) given by Lemma 2.1. However, we cannot get any more from it. To see this, we note that, in the case $m_0 \ge 4\pi$, (2-11) is solved in the region $[0,+\infty)$ by

$$X_1 < X < X_2,$$

and we claim that

$$4C_{\mathrm{GN}}^{-9} < X_1 < X_2 < m_0. \tag{2-14}$$

Indeed, when $m_0 \ge 4\pi$,

$$\tfrac{2}{3}m_0 \ge \tfrac{8}{3}\pi > 4C_{\mathrm{GN}}^{-9} = \frac{8}{3\sqrt{3}}\pi,$$

and, for small ϵ, we have

$$F(4C_{\mathrm{GN}}^{-9}) = 64C_{\mathrm{GN}}^{-27} - \epsilon > 0,$$

which together imply that $4C_{\mathrm{GN}}^{-9} < X_1$. Similarly, since

$$\tfrac{2}{3}m_0 < m_0 \quad \text{and} \quad F(m_0) = b > 0,$$

we have $X_2 < m_0$. In conclusion, we have (2-14). Therefore, the inequality (2-11) is always solvable in the region of $(4C_{\mathrm{GN}}^{-9}, m_0)$ when $m_0 \ge 4\pi$, and so we can not obtain the contradiction from the restriction of $(4C_{\mathrm{GN}}^{-9}, m_0)$. We show this case graphically in Figure 1.

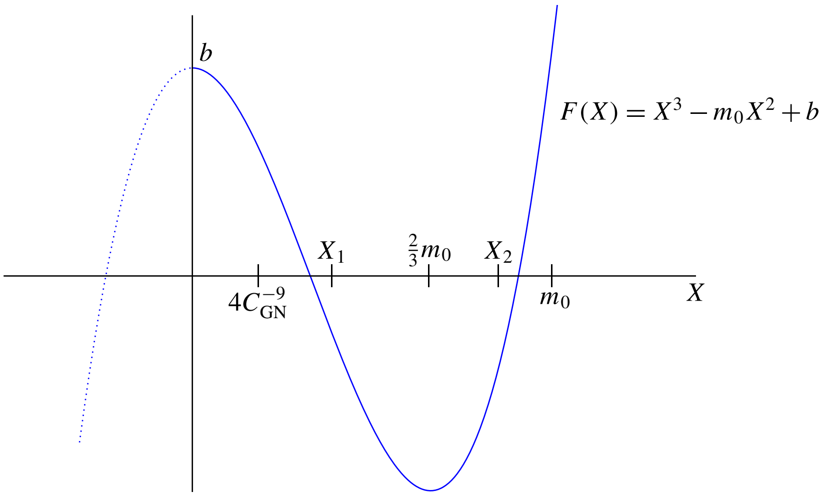

Figure 1. Graph of $F(X)$.

3. Further discussion

In this section, we would like to make a few remarks and indicate some related problems which remain open.[1]

First of all, whether or not the mass $M(W) = 4\pi$ is the mass threshold for global well-posedness of (1-1) is not resolved in this paper. To understand the problem, we make some remarks on W and the equation (1-9) in the following.

As shown in [Colin and Ohta 2006; Guo and Wu 1995], (1-14) has a two-parameter family of solitary wave solutions,

$$v_{\omega,c} = \phi_{\omega,c}(x+ct)e^{i\omega t-(ic/2)(x+ct)}, \tag{3-1}$$

where $(\omega, c) \in \mathbb{R}^2$ and $\phi_{\omega,c}$ is a positive solution of the elliptic equation

$$-\partial_{xx}\phi + \left(\omega - \tfrac{1}{4}c^2\right)\phi + \tfrac{1}{2}c\phi^3 - \tfrac{3}{16}\phi^5 = 0. \tag{3-2}$$

When $c^2 < 4\omega$, $\phi_{\omega,c}$ can be written as

$$\phi_{\omega,c}(x) = \left\{\frac{\sqrt{\omega}}{4\omega - c^2}\left[\cosh(\sqrt{4\omega - c^2}x) - \frac{c}{2\sqrt{\omega}}\right]\right\}^{-\frac{1}{2}}.$$

Guo and Wu [1995] proved that the solitary wave solutions (3-1) are orbitally stable when $c < 0$ and $c^2 < 4\omega$. This was extended by Colin and Ohta [2006], who proved the orbital stability for any $c^2 < 4\omega$.

Now we consider the other cases. From Pohožaev's identity, there is no solution for (3-2) when $4\omega \le c^2$ and $c \le 0$, and, from [Berestycki and Lions 1983] (see Section 6, Theorem 5), when $c^2 > 4\omega$ (3-2) has no positive solution which vanishes at infinity. Hence, the only remaining case is the "zero mass" case,

[1]Part of the contents in this section are from discussions with Soonsik Kwon.

$c^2 = 4\omega$ and $c > 0$. Thus, the "zero mass" case can be regarded as the endpoint case in the family of the solitary wave solutions (3-1).

For the endpoint case $c^2 = 4\omega$ and $c > 0$,

$$-\partial_{xx}\phi + \tfrac{1}{2}c\phi^3 - \tfrac{3}{16}\phi^5 = 0$$

is exactly solved by

$$W_c(x) = c^{1/2}W(cx),$$

where W is as defined in (1-10). Moreover,

$$\|W_c\|_{L^2}^2 = \|W\|_{L^2}^2 = 4\pi.$$

So it is an interesting problem whether the solitary wave solution (1-8) is orbitally stable or unstable, which is not covered in [Colin and Ohta 2006; Guo and Wu 1995]. See [Ohta 2014] for related studies.

The existence of the finite-time blow-up solution is also an open problem for (1-1). There are some related results on the generalized derivative nonlinear Schrödinger equation,

$$i\partial_t u + \partial_x^2 u = i|u|^{2\sigma}\partial_x u, \quad \sigma > 1. \tag{3-3}$$

This is a mass supercritical equation. See [Ambrose and Simpson 2014; Hao 2007; Liu et al. 2013b] for local and stability theories. Numerical simulations by Liu, Simpson and Sulem [Liu et al. 2013a] suggest the existence of finite-time blow-up solutions for (3-3). However, a rigorous proof remains to be found.

Acknowledgements

The author would like to thank Professor Martial Agueh for useful discussions on the sharp Gagliardo–Nirenberg inequality. The author is also grateful to Soonsik Kwon who kindly suggested incorporating some private discussions in this paper, and to the anonymous referee who carefully read the paper and gave many helpful comments and suggestions.

References

[Agueh 2006] M. Agueh, "Sharp Gagliardo–Nirenberg inequalities and mass transport theory", *J. Dynam. Differential Equations* **18**:4 (2006), 1069–1093. MR 2008j:49108 Zbl 1155.35320

[Ambrose and Simpson 2014] D. Ambrose and G. Simpson, "Local existence theory for derivative nonlinear Schrödinger equations with non-integer power nonlinearities", preprint, 2014. To appear in *SIAM J. Math. Anal.* arXiv 1401.7060

[Banica 2004] V. Banica, "Remarks on the blow-up for the Schrödinger equation with critical mass on a plane domain", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **3**:1 (2004), 139–170. MR 2005e:35209 Zbl 1170.35528

[Berestycki and Lions 1983] H. Berestycki and P.-L. Lions, "Nonlinear scalar field equations, I: Existence of a ground state", *Arch. Rational Mech. Anal.* **82**:4 (1983), 313–345. MR 84h:35054a Zbl 0533.35029

[Biagioni and Linares 2001] H. A. Biagioni and F. Linares, "Ill-posedness for the derivative Schrödinger and generalized Benjamin–Ono equations", *Trans. Amer. Math. Soc.* **353**:9 (2001), 3649–3659. MR 2002e:35215 Zbl 0970.35154

[Colin and Ohta 2006] M. Colin and M. Ohta, "Stability of solitary waves for derivative nonlinear Schrödinger equation", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **23**:5 (2006), 753–764. MR 2007e:35255 Zbl 1104.35050

[Colliander et al. 2001] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "Global well-posedness for Schrödinger equations with derivative", *SIAM J. Math. Anal.* **33**:3 (2001), 649–669. MR 2002j:35278 Zbl 1002.35113

[Colliander et al. 2002] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "A refined global well-posedness result for Schrödinger equations with derivative", *SIAM J. Math. Anal.* **34**:1 (2002), 64–86. MR 2004c:35381 Zbl 1034.35120

[Grünrock and Herr 2008] A. Grünrock and S. Herr, "Low regularity local well-posedness of the derivative nonlinear Schrödinger equation with periodic initial data", *SIAM J. Math. Anal.* **39**:6 (2008), 1890–1920. MR 2009a:35233 Zbl 1156.35471

[Guo and Tan 1991] B. L. Guo and S. B. Tan, "On smooth solutions to the initial value problem for the mixed nonlinear Schrödinger equations", *Proc. Roy. Soc. Edinburgh Sect. A* **119**:1–2 (1991), 31–45. MR 92i:35114 Zbl 0766.35051

[Guo and Wu 1995] B. L. Guo and Y. P. Wu, "Orbital stability of solitary waves for the nonlinear derivative Schrödinger equation", *J. Differential Equations* **123**:1 (1995), 35–55. MR 96k:35166 Zbl 0844.35116

[Hao 2007] C. Hao, "Well-posedness for one-dimensional derivative nonlinear Schrödinger equations", *Commun. Pure Appl. Anal.* **6**:4 (2007), 997–1021. MR 2008k:35443 Zbl 1152.35104

[Hayashi 1993] N. Hayashi, "The initial value problem for the derivative nonlinear Schrödinger equation in the energy space", *Nonlinear Anal.* **20**:7 (1993), 823–833. MR 94c:35007 Zbl 0787.35099

[Hayashi and Ozawa 1992] N. Hayashi and T. Ozawa, "On the derivative nonlinear Schrödinger equation", *Phys. D* **55**:1–2 (1992), 14–36. MR 93h:35190 Zbl 0741.35081

[Hayashi and Ozawa 1994] N. Hayashi and T. Ozawa, "Finite energy solutions of nonlinear Schrödinger equations of derivative type", *SIAM J. Math. Anal.* **25**:6 (1994), 1488–1503. MR 95i:35272 Zbl 0809.35124

[Herr 2006] S. Herr, "On the Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition", *Int. Math. Res. Not.* **2006** (2006), Art. ID 96763, 33. MR 2007e:35258 Zbl 1149.35074

[Liu et al. 2013a] X. Liu, G. Simpson, and C. Sulem, "Focusing singularity in a derivative nonlinear Schrödinger equation", *Phys. D* **262** (2013), 48–58. MR 3144018

[Liu et al. 2013b] X. Liu, G. Simpson, and C. Sulem, "Stability of solitary waves for a generalized derivative nonlinear Schrödinger equation", *J. Nonlinear Sci.* **23**:4 (2013), 557–583. MR 3079669 Zbl 1271.35004

[Miao et al. 2011] C. Miao, Y. Wu, and G. Xu, "Global well-posedness for Schrödinger equation with derivative in $H^{1/2}(\mathbb{R})$", *J. Differential Equations* **251**:8 (2011), 2164–2195. MR 2012i:35372 Zbl 1227.35236

[Mio et al. 1976] K. Mio, T. Ogino, K. Minami, and S. Takeda, "Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas", *J. Phys. Soc. Japan* **41**:1 (1976), 265–271. MR 57 #2116

[Mjolhus 1976] E. Mjolhus, "On the modulational instability of hydromagnetic waves parallel to the magnetic field", *J. Plasma Phys.* **16** (1976), 321–334.

[Nahmod et al. 2012] A. R. Nahmod, T. Oh, L. Rey-Bellet, and G. Staffilani, "Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS", *J. Eur. Math. Soc.* (*JEMS*) **14**:4 (2012), 1275–1330. MR 2928851 Zbl 1251.35151

[Ohta 2014] M. Ohta, "Instability of solitary waves for nonlinear Schrödinger equations of derivative type", preprint, 2014. arXiv 1408.5537

[Ozawa 1996] T. Ozawa, "On the nonlinear Schrödinger equations of derivative type", *Indiana Univ. Math. J.* **45**:1 (1996), 137–163. MR 98b:35186 Zbl 0859.35117

[Sulem and Sulem 1999] C. Sulem and P.-L. Sulem, *The nonlinear Schrödinger equation: self-focusing and wave collapse*, Applied Mathematical Sciences **139**, Springer, New York, 1999. MR 2000f:35139 Zbl 0928.35157

[Takaoka 1999] H. Takaoka, "Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity", *Adv. Differential Equations* **4**:4 (1999), 561–580. MR 2000e:35221 Zbl 0951.35125

[Takaoka 2001] H. Takaoka, "Global well-posedness for Schrödinger equations with derivative in a nonlinear term and data in low-order Sobolev spaces", *Electron. J. Differential Equations* **42** (2001), 1–23. MR 2002f:35033

[Thomann and Tzvetkov 2010] L. Thomann and N. Tzvetkov, "Gibbs measure for the periodic derivative nonlinear Schrödinger equation", *Nonlinearity* **23**:11 (2010), 2771–2791. MR 2011j:35228 Zbl 1204.35154

[Tsutsumi and Fukuda 1980] M. Tsutsumi and I. Fukuda, "On solutions of the derivative nonlinear Schrödinger equation: existence and uniqueness theorem", *Funkcial. Ekvac.* **23**:3 (1980), 259–277. MR 83c:35108a Zbl 0478.35032

[Tsutsumi and Fukuda 1981] M. Tsutsumi and I. Fukuda, "On solutions of the derivative nonlinear Schrödinger equation, II", *Funkcial. Ekvac.* **24**:1 (1981), 85–94. MR 83c:35108b Zbl 0491.35016

[Weinstein 1983] M. I. Weinstein, "Nonlinear Schrödinger equations and sharp interpolation estimates", *Comm. Math. Phys.* **87**:4 (1983), 567–576. MR 84d:35140 Zbl 0527.35023

[Win 2010] Y. Y. S. Win, "Global well-posedness of the derivative nonlinear Schrödinger equations on T", *Funkcial. Ekvac.* **53**:1 (2010), 51–88. MR 2011j:35229 Zbl 1194.35433

[Wu 2013] Y. Wu, "Global well-posedness for the nonlinear Schrödinger equation with derivative in energy space", *Anal. PDE* **6**:8 (2013), 1989–2002. MR 3198590 Zbl 1298.35207

Received 14 Sep 2014. Revised 4 Feb 2015. Accepted 6 Mar 2015.

YIFEI WU: yifei@bnu.edu.cn
Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

and

School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Ministry of Education, Beijing, 100875, China

ANALYSIS AND PDE
Vol. 8, No. 5, 2015
dx.doi.org/10.2140/apde.2015.8.1113

ON THE BOUNDARY VALUE PROBLEM FOR THE SCHRÖDINGER EQUATION COMPATIBILITY CONDITIONS AND GLOBAL EXISTENCE

CORENTIN AUDIARD

We consider linear and nonlinear Schrödinger equations on a domain Ω with nonzero Dirichlet boundary conditions and initial data. We first study the linear boundary value problem with boundary data of optimal regularity (in anisotropic Sobolev spaces) with respect to the initial data. We prove well-posedness under natural compatibility conditions. This is essential for the second part, where we prove the existence and uniqueness of maximal solutions for nonlinear Schrödinger equations. Despite the nonconservation of energy, we also obtain global existence in several (defocusing) cases.

On étudie des équations de Schrödinger linéaires et non linéaires sur un domaine Ω avec donnée initiale et condition au bord de Dirichlet non nulles. Dans une première partie on étudie le problème linéaire pour des données au bord dans des espaces de Sobolev anisotropes de régularité optimale par rapport aux données de Cauchy. On démontre la nature bien posée du problème avec les conditions de compatibilité naturelles à tout ordre de régularité. Ces résultats sont essentiels pour établir des résultats de type Cauchy–Lipschitz pour le problème non linéaire, ceux ci font l'objet de la deuxième partie. Malgré la non conservation de l'énergie, on obtient des solutions globales en dimension 2.

Introduction

This article is a continuation of [Audiard 2013] on the initial boundary value problem for the (linear and nonlinear) Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = f, & (x,t) \in \Omega \times [0,T[, \\ u|_{t=0} = u_0, & x \in \Omega, \\ u|_{\partial\Omega\times[0,T]} = g, & (x,t) \in \partial\Omega \times [0,T[, \end{cases} \tag{IBVP}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a smooth open set. Our main purpose is to deal with boundary data of arguably optimal regularity, and in particular too rough to be dealt with by lifting arguments. When f depends on u we generically refer to the nonlinear Schrödinger equation as NLS. We will study nonlinearities that are essentially similar to $\lambda|u|^\alpha u$.

A classical tool to deal with the well-posedness of NLS is Strichartz estimates. It is well known that if $\Omega = \mathbb{R}^d$, the semigroup $e^{it\Delta}$ satisfies

$$\|e^{it\Delta}u_0\|_{L^p(\mathbb{R},\, L^q(\mathbb{R}^d))} \lesssim \|u_0\|_{L^2} \quad \text{when } \frac{2}{p} + \frac{d}{q} = \frac{d}{2},$$

MSC2010: primary 35A01, 35A02, 35Q41; secondary 35B45.
Keywords: boundary value problems, Schrödinger equation, global well-posedness, boundary data with sharp regularity.

for $p, q \geq 2$, $q < \infty$ for $d = 2$ (see [Cazenave 2003] and [Keel and Tao 1998] for the endpoint), and more generally the *scale-invariant* estimates

$$\|e^{it\Delta}u_0\|_{L^p(\mathbb{R},\, L^q(\mathbb{R}^d))} \lesssim \|u_0\|_{H^s} \quad \text{when } \frac{2}{p}+\frac{d}{q}=\frac{d}{2}-s.$$

Similar estimates with $\frac{2}{p}+\frac{d}{q}\geq\frac{d}{2}-s$ are true on *bounded* time intervals and simple scaling considerations show that the condition $\frac{2}{p}+\frac{d}{q}\geq\frac{d}{2}-s$ is necessary. When $\frac{2}{p}+\frac{d}{q}-\frac{d}{2}+s=r>0$, they are often called Strichartz estimates with loss of r derivatives. The derivation of such estimates (and the associated well-posedness results) for NLS on a domain with the Dirichlet (or Neumann) laplacian has been intensively studied over the last decade in various geometric settings. We will only cite results in the case where Ω is the exterior of a nontrapping obstacle, since it is the one studied here. Roughly speaking, a nontrapping obstacle is an obstacle such that any ray propagating according to the laws of geometric optic leaves any compact set in finite time (for a mathematical definition of the rays, see [Melrose and Sjöstrand 1978]). In seminal work, Burq, Gérard and Tzvetkov [Burq et al. 2004] proved a local smoothing property similar to the one on $\mathbb{R}^d$ (see [Constantin and Saut 1988]) and deduced Strichartz estimates with loss of $\frac{1}{p}$ derivative. Since then numerous improvements were obtained [Anton 2008a; 2008b; Blair et al. 2008] and eventually led to scale-invariant Strichartz estimates: see Blair, Smith and Sogge [Blair et al. 2012] in the general nontrapping case ($s > 0$ and limited range of exponents), [Ivanovici 2010] for the exterior of a convex obstacle ($s = 0$, all exponents except endpoints). The methods used relied heavily on spectral localization and construction of parametrices. As such they are not very convenient for the study of nonhomogeneous boundary value problems when the boundary data are not smooth enough to reduce the problem to a homogeneous one.

On the other hand, Morawetz and virial identities have proved to be very robust tools to study linear and nonlinear Schrödinger equations. One of their first applications goes back to [Glassey 1977], and it has since been massively refined (as a tool of a much larger machinery) to the point where exhaustive attribution is now impossible (we may cite, at least, [Kenig and Merle 2006; Planchon and Vega 2009; Colliander et al. 2008]). Such tools only rely on differentiation and integration by parts; this makes them flexible enough to be used even with nonzero boundary data and part of our results rely on this approach.

As already mentioned, our aim is to treat Schrödinger equations on a domain with nonzero Dirichlet conditions. The case of dimension one is by now relatively well understood: the local Cauchy theory on intervals is essentially on par with the theory on $\mathbb{R}$ (see [Holmer 2005] for local existence in H^s, $0 \leq s \leq 1$, subcritical and critical nonlinearities). For $d \geq 2$, there are many fewer results. We might mention the classical linear results of [Lions and Magenes 1968b], which were based on lifting arguments and thus prevented boundary data of very low regularity. Indeed, if one takes a lifting Lg of the boundary data, then $u - Lg$ satisfies

$$\begin{cases} i\partial_t(u-Lg)+\Delta(u-Lg)=f-(i\partial_t+\Delta)Lg, \\ (u-Lg)|_{\partial\Omega}=0, \\ (u-Lg)|_{t=0}=u_0-Lg|_{t=0}, \end{cases}$$

so that $u \in C_T L^2$ would require $(i\partial_t + \Delta)Lg \in L^1_T L^2$. For a general L this would require $g \in L^1 H^{3/2}$, which is a loss of one derivative in space compared to our result (see below).

Bu and Strauss [2001] obtained the existence of global weak H^1 solutions for defocusing nonlinear Schrödinger equations with smooth (C^3) boundary data. In the important field of control theory, linear well-posedness and controllability in H^{-1} was obtained for Dirichlet data in L^2 when Ω is a smooth *bounded* domain. While optimal on bounded domains, this "loss" of one derivative on the boundary data is not natural in general. On the half line, it is generally believed that, for initial data $u_0 \in H^s(R^+)$, optimally $g \in H^{s/2+1/4}(\mathbb{R}^+)$ (see [Holmer 2005] for a discussion on this). This pair of spaces is considered to be optimal for at least two reasons: if one rescales solutions as $u(\lambda x, \lambda^2 t)$ both spaces scale as $\lambda^{s-1/2}$, and the space also appears in the famous Kato smoothing property for the Cauchy problem, $\|e^{it\partial_x^2}u_0\|_{L^\infty_x H_t^{s/2+1/4}} \lesssim \|u_0\|_{H^s}$ (see [Kenig et al. 1991]), which can be read as a trace estimate.

The natural generalization of $H^{s/2+1/4}(\mathbb{R}^+)$ in larger dimension is the *anisotropic* Sobolev space $H^{s+1/2,2}(\partial\Omega \times [0,T]) = L^2_T H^{s+1/2} \cap H_T^{s/2+1/4} L^2$ of functions that, roughly speaking, have twice more regularity in space than in time. We obtained in [Audiard 2012] well-posedness for the linear Schrödinger equation on the half space with boundary conditions having this regularity (and satisfying some Kreiss–Lopatinskii condition). However, the method relied quite heavily on the simple geometry of Ω. When Ω is the exterior of a nontrapping obstacle, a simple duality argument was used to obtain the following linear result:

Theorem 0.1 [Audiard 2013]. *For $f \in L^2_T H^{s-1/2}$ compactly supported, $g \in H_0^{s+1/2,2}(\partial\Omega \times [0,T])$, $u_0 \in H^s_D$, $-\frac{1}{2} < s \le \frac{3}{2}$, the initial boundary value problem (IBVP) has a unique transposition solution. It satisfies*

$$\|u\|_{C_T H^s} \lesssim \|f\|_{L^2_T H^{s-1/2}} + \|g\|_{H_0^{s+1/2}} + \|u_0\|_{H_0^s}.$$

In the case $s = -\frac{1}{2}$, the result is true if $H^{-1/2}$ is replaced by $(H_D^{1/2})'$.

Thanks to a virial identity, we also obtained a local smoothing property similar to the one in [Burq et al. 2004], which allowed us to derive Strichartz estimates with a loss of $\frac{1}{p}$ derivative. Well-posedness in $H^{1/2}$ for the expected range of nonlinearities followed by the usual fixed-point argument.

This work contained, however, a number of important limitations:

- The virial estimate was derived when Ω is the exterior of a strictly convex obstacle.
- Since the natural space for our virial estimate is $H^{1/2}$, the local well-posedness theorem was stated for $u_0 \in H_D^{1/2}$ rather than the energy space H^1.
- The linear well-posedness theorem was obtained for trivial compatibility conditions, $u_0 \in H_D^{1/2}(\Omega)$ and $g \in H_0^{1,2}(\partial\Omega \times [0,T])$.
- Since such conditions are certainly not preserved by the flow, continuation arguments were not available, so the existence of a maximal solution (let alone global solution) was out of reach.

The main purpose of this article is to lift most of the previous restrictions to provide a good local and global Cauchy theory in the energy space. Rather than the exterior of a convex compact obstacle, we will only assume that Ω is the exterior of a compact star-shaped obstacle. On the other hand, we do not improve the loss in the Strichartz estimates, so that we obtain local well-posedness for a range of nonlinearities essentially similar to $|u|^\alpha u$ with the limitation $\alpha < 2/(d-2)$ (the whole subcritical range is $\alpha < 4/(d-2)$). In the case where Ω^c is strictly convex, however, we improve it to $\alpha < 3/(d-2)$. These results are true for boundary data in the almost optimal space $H^{3/2+\varepsilon,2}$ and a discussion is included on the possibility to replace it by the optimal space. If one takes slightly smoother boundary data in $H^{2+\varepsilon,2}(\partial\Omega \times [0,T])$, we obtain global well-posedness for $\alpha < 2/(d-2)$ if Ω^c is star-shaped, and for the whole subcritical range $\alpha < 4/(d-2)$ if Ω^c is strictly convex. The existence of global solutions for $g \in H^{3/2+\varepsilon,2}$ is much more intricate, and is only obtained in dimension 2 with a quite technical limitation on α.

The presence of ε in the trace spaces can most likely be avoided up to lengthier computations that we chose to avoid for simplicity of the proofs (see Remarks 3.5, 3.8, 4.3).

Structure of the article.

- The functional spaces that we use are defined in Section 1, which also provide some useful trace and interpolation results.
- In Section 2 we define the natural compatibility conditions and we prove well-posedness for the linear IBVP when such conditions are met.
- In Section 3 we provide the basic modifications to the proof in [Audiard 2013] that give local smoothing through a virial estimate when Ω is star-shaped. The boundary data is assumed to be in the almost optimal space $H^{3/2+\varepsilon,2}$. We deduce Strichartz estimates at the H^1 level thanks to an interpolation argument; this section also includes a smoothing property on $\partial_n u$ that is essential for global existence issues.
- In Section 4 we prove the nonlinear well-posedness results stated above.
- The Appendix contains two elementary interpolation results.

1. Functional spaces and Strichartz estimates

Functional spaces. For $p \geq 1$ we denote by $L^p(\Omega)$ the usual Lebesgue spaces. If there is no ambiguity, when X is a Banach space we write

$$L^p([0,T],X) = L^p_T X, \quad L^p(\mathbb{R}^+,X) = L^p_t X.$$

For integer m we denote by $W^{m,p}(\Omega)$ the usual Sobolev spaces; $W_0^{m,p}$ is the closure of $C_c^\infty(\Omega)$ for the $W^{m,p}$ topology.

For $s \geq 0$, the space $W^{s,p}(\Omega)$ is defined by real interpolation; see [Tartar 2007, Sections 32 and 34]. When $p = 2$, the Sobolev spaces are denoted by H^s, H_0^s. For $s > 0$, we set $H^{-s}(\Omega) = (H_0^s(\Omega))'$.

For $s \geq 0$ and Δ_D the Dirichlet laplacian on Ω, the space H_D^s is the domain of $(1-\Delta_D)^{s/2}$. When $\frac{1}{2} < s \leq 1$, $H_D^s = H_0^s$, and when $0 \leq s < \frac{1}{2}$, $H_D^s = H^s$. The space $H_D^{1/2}$ does not coincide with $H_0^{1/2} = H^{1/2}$ (it is the Lions–Magenes space $H_{00}^{1/2}$ but we will use the notation $H_D^{1/2}$).

The Besov spaces $B^s_{p,q}(\Omega)$ are the restrictions to Ω of functions in $B^s_{p,q}(\mathbb{R}^d)$ [Tartar 2007, Sections 32 and 34]. For $s \geq 0$, $s \notin \mathbb{N}$, we have $B^s_{p,p} = W^{s,p}$ (see [Bergh and Löfström 1976; Tartar 2007]). The spaces $B^s_{p,q,0}$ are defined as the closure of $C^\infty_c(\Omega)$ in $B^s_{p,q}$.

The anisotropic Sobolev spaces on $[0,T]\times\Omega$ are defined as

$$H^{s,2} = L^2([0,T], H^s(\Omega)) \cap H^{s/2}([0,T], L^2(\Omega)).$$

Anisotropic Besov spaces can be defined in a similar way (see [Amann 2009]):

$$B^{s,2}_{p,q,0} = L^p_T B^s_{p,q,0} \cap B^{s/2}_{p,q}([0,T], L^p(\Omega)).$$

Finally, we use the same definitions for functions defined on $\partial\Omega$ or $\partial\Omega\times[0,T]$ using local maps.

We recall in the following proposition the classical rules on embeddings and traces of functional spaces:

Proposition 1.1 (Sobolev embeddings and traces [Lions and Magenes 1968b; Triebel 1983]).

- *If* $0 \leq sp < d$, $t \geq 0$, *we have* $B^{t+s}_{p,q}(\Omega) \hookrightarrow B^s_{p_1,q}$ *when* $\frac{1}{p_1} = \frac{1}{p} - \frac{s}{d}$.
- *If* $sp > d$, $W^{s,p} \hookrightarrow C^0(\overline{\Omega})$ *and, if* $sp < d$, *then* $W^{s,p} \hookrightarrow L^q(\Omega)$ *when* $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$.
- *If* $sp > 1$, *the trace operator* $C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$ *extends continuously to*

$$W^{s,p}(\Omega) \to W^{s-1/p,p}(\partial\Omega).$$

- *For* $0 \leq s' \leq \frac{s}{2}$, *the anisotropic spaces* $H^{s,2}(\Omega\times[0,T])$ *are embedded in* $H^{s'}_T H^{s-2s'}$.
- *For* $s > \frac{1}{2}$, *the trace operator* $H^{s,2}(\Omega\times[0,T]) \to H^{s-1/2,2}(\partial\Omega\times[0,T])$ *is continuous.*
- *For* $s > 1$, $\mathbb{O} = \Omega$ *or* $\partial\Omega$, *there is a time-trace operator from the embedding*

$$H^{s,2}([0,T]\times\mathbb{O}) \hookrightarrow C([0,T], H^{s-1}(\mathbb{O})).$$

For $s_0, s_1 \geq 0$, we have the interpolation identity (see [Triebel 1983])

$$[B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1}]_{\theta,q} = B^{\theta s_0+(1-\theta)s_1}_{p,q}.$$

Similar interpolation results are true for anisotropic Sobolev spaces. In [Lions and Magenes 1968b] it is proved that for $s > 0$, $\mathbb{O} = \Omega$ or $\partial\Omega$, $0 \leq \theta \leq 1$ and $t = \theta s$, $H^{t,2}([0,T]\times\mathbb{O}) = [L^2, H^{s,2}]_\theta$.

In addition to their nice interpolation properties, composition rules in Besov spaces are relatively simple: if $F(0) = 0$ and $|\nabla F(z)| \lesssim |z|^\alpha$, then for $0 < s < 1$, $1 \leq q \leq \infty$, $1 \leq p \leq r \leq \infty$, $\frac{1}{\sigma} + \frac{1}{r} = \frac{1}{p}$, we have

$$\|F(u)\|_{B^s_{p,q}} \lesssim \|u\|^\alpha_{L^{\alpha\sigma}} \|u\|_{B^s_{r,q}}; \tag{1-1}$$

this is Proposition 4.9.4 in [Cazenave 2003] when $\Omega = \mathbb{R}^d$, and it follows from the existence of a (universal) extension operator when Ω is an exterior domain; see [Amann 2009, Sections 4.1, 4.4].

Since anisotropic Besov spaces are more intricate and scarcely used in the article, we will cite their properties we need when relevant, pointing to the reference [Amann 2009].

Finally, we recall some Strichartz estimates known for the boundary value problem with homogeneous boundary condition.

Theorem 1.2 [Burq et al. 2004; Ivanovici 2010]. *If Ω is the exterior of a nontrapping obstacle, then for any $T > 0$,*

$$\|e^{it\Delta_D}u_0\|_{L^p_tL^q} \lesssim \|u_0\|_{L^2} \quad \textit{when } \frac{1}{p}+\frac{d}{q}=\frac{d}{2},\ p\geq 2. \tag{1-2}$$

If Ω is the exterior of a strictly convex obstacle then

$$\|e^{it\Delta_D}u_0\|_{L^p_TL^q} \lesssim \|u_0\|_{L^2} \quad \textit{when } \frac{2}{p}+\frac{d}{q}=\frac{d}{2},\ p>2. \tag{1-3}$$

2. Linear well-posedness

In this section, we assume that Ω is the exterior of a compact nontrapping obstacle. We recall what we meant by "transposition solution" in Theorem 0.1:

Definition 2.1. Let $\chi \in C_c^\infty(\mathbb{R}^d)$, $f \in L^2_TH^{-1}(\Omega)$. We say that u is a *transposition solution* of the problem

$$\begin{cases} i\partial_t u + \Delta u = \chi f \in L^2_TH^{-1}, \\ u|_{t=0} = u_0 \in (H_D^{1/2}(\Omega))', \\ u|_{\partial\Omega\times[0,T]} = g \in L^2([0,T]\times\partial\Omega) \end{cases} \tag{2-1}$$

when $u \in C_T(H_D^{1/2})'$ and, for any $f_1 \in L^1_TH_D^{1/2}$, if v is the solution of

$$\begin{cases} i\partial_t v + \Delta v = f_1, \\ v|_{t=T} = 0, \\ v|_{\partial\Omega\times[0,T]} = 0, \end{cases} \tag{2-2}$$

then we have the identity

$$\int_0^T \langle u, f_1\rangle_{(H_D^{1/2})',H_D^{1/2}}\,dt = \int_0^T \langle f, \chi v\rangle_{H^{-1},H_0^1}\,dt + \int_0^T (g,\partial_n v)_{L^2(\partial\Omega)}\,dt + i\langle u_0, v(0)\rangle_{(H_D^{1/2})',H_D^{1/2}}, \tag{2-3}$$

where $\langle\cdot,\cdot\rangle_{X,X'}$ is the duality bracket.

In [Audiard 2013] we obtained by derivation/interpolation arguments well-posedness for (u_0, g) in $H_D^s \times H_0^{s+1/2,2}$; the aim of this section is to extend it to $(u_0, f, g) \in H^s \times H^{s-1/2,2} \times H^{s+1/2,2}$ for any $s \geq -\frac{1}{2}$, under natural compatibility conditions that we derive now.

Compatibility conditions. We consider the linear initial boundary value problem (IBVP)

$$\begin{cases} i\partial_t u + \Delta u = f, & (x,t) \in \Omega\times[0,T[, \\ u|_{t=0} = u_0, & x \in \Omega, \\ u|_{\partial\Omega\times[0,T]} = g, & (x,t) \in \partial\Omega\times[0,T[. \end{cases} \tag{2-4}$$

Local compatibility. If $u_0 \in H^s$, $g \in H^{s+1/2,2}$, $s > \frac{1}{2}$, then u_0 has a trace on $\partial\Omega$ and g has a trace at $t = 0$; the identity $u|_{t=0}|_{\partial\Omega} = u|_{\partial\Omega}|_{t=0}$ imposes the zeroth-order compatibility condition

$$u_0|_{\partial\Omega} = g|_{t=0}. \tag{CC0}$$

The next compatibility conditions are defined inductively: set $\varphi_0 = u_0$, $\varphi_{n+1} = \frac{1}{i}(\partial_t^n f|_{t=0} - \Delta\varphi_n)$; the k-th order compatibility condition is

$$\partial_t^k g|_{t=0} = \varphi_k|_{\partial\Omega}, \tag{CCk}$$

which must be satisfied if $u_0 \in H^s(\Omega)$, $g \in H^{s+1/2,2}(\partial\Omega\times[0,T])$, $f \in H^{s-1/2,2}(\Omega\times[0,T])$, $s > 2k+\frac{1}{2}$.

Global compatibility. If $s = \frac{1}{2}$, there is a more subtle compatibility condition, the so-called "global compatibility condition": thanks to local maps, we can assume that u_0, g are defined by a collection of $(u_0^j, f^j g^j)_{1\le j\le J}$ defined on $\mathbb{R}^{d-1}\times\mathbb{R}^+$ ($\mathbb{R}^+$ corresponds to the t-variable for g^j and normal space variable for u_0^j, f^j); we say that (u_0, g) satisfy the zeroth-order global compatibility condition when

$$\forall 1 \le j \le J \quad \int_0^\infty \int_{\mathbb{R}^{d-1}} |u_0^j(x', h) - g^j(x', h^2)|^2 \, dx' \, \frac{dh}{h} < \infty; \tag{CCG0}$$

similarly, we define the global compatibility conditions of order k for $s = \frac{1}{2} + 2k$ as

$$\forall 1 \le j \le J \quad \int_0^\infty \int_{\mathbb{R}^{d-1}} |\varphi_k^j(x', h) - \partial_t^k g^j(x', h^2)|^2 \, dx' \, \frac{dh}{h} < \infty, \tag{CCGk}$$

It is standard [Lions and Magenes 1968a] that (CCk) is stronger than (CCGk).

In what follows, we say that $(u_0, f, g) \in H^s \times H^{s-1/2,2} \times H^{s+1/2,2}$ "satisfy the compatibility conditions" when all conditions that make sense are satisfied, namely (CCk) holds for $k < \frac{s}{2} - \frac{1}{4}$, and also (CCG$k$) if $s = \frac{1}{2} + 2k$.

Theorem 2.2. *For $-\frac{1}{2} < s \le \frac{3}{2}$, let $(u_0, f, g) \in H^s \times L_T^2 H^{s-1/2} \times H^{s+1/2,2}$ be such that f is compactly supported and (u_0, f, g) satisfy the compatibility conditions; then the solution of (IBVP) is in $C_T H^s$.*

For $s > \frac{3}{2}$ and $(u_0, f, g) \in H^s \times H^{s-1/2,2} \times H^{s+1/2,2}$ satisfying the compatibility conditions, $u \in C_T H^s$.

The spirit of the proof is relatively similar to the classical argument of [Rauch and Massey 1974] for hyperbolic boundary value problems. Let us describe it and where the difficulty lies: the natural idea is to consider Δu, which is formally a solution of a similar boundary value problem; the low regularity theorem implies $\Delta u \in C_T(H_D^{1/2})'$, and we conclude, by an elliptic regularity argument, that $u \in C_T H^{3/2}$. However, due to the weak setting it is not clear that Δu is actually a solution of the expected boundary value problem. For "trivial" compatibility conditions it is sufficient to approximate the initial data by $(u_{0,n}, g_n, f_n) \in C_c^\infty(\Omega) \times C_c^\infty(\partial\Omega\times]0,T]) \times C_c^\infty(\overline{\Omega}\times]0,T])$ that automatically satisfy the compatibility conditions at any order. In general, the existence of smooth data that satisfy the compatibility conditions at a sufficient order will be done in Lemma 2.4.

Lemma 2.3. *If $(u_0, f, g) \in H^{3/2} \times L_T^2 H^1 \times H^{2,2}$ with f compactly supported and (CC0) satisfied, the unique transposition solution of (IBVP) belongs to $C_T H^{3/2}$.*

For $k \ge 2$, if $(u_0, f, g) \in H^{2k-1/2} \times H^{2k-1,2} \times H^{2k,2}$, f compactly supported and (CCj), $0 \le j \le k-1$ satisfied, the unique transposition solution of (IBVP) belongs to $C_T H^{2k-1/2}$.

The proof is postponed until after the following approximation lemma:

Lemma 2.4. *For* $(u_0, f, g) \in H^{3/2}(\Omega) \times L^2([0,T], H^1(\Omega)) \times H^{2,2}([0,T]\times\partial\Omega)$ *satisfying* (CC0), *there exists a sequence* $(u_{0,k}, f_k, g_k) \in H^2 \times H^{2,2} \times H^{5/2,2}$ *satisfying* (CC0) *such that*

$$\|(u_0, f, g) - (u_{0,k}, f_k, g_k)\|_{H^{3/2}\times L^2_T H^1 \times H^{2,2}} \to_k 0.$$

Proof. By density of smooth functions in Sobolev spaces, there exists (v_k, f_k, g_k) smooth such that $(v_k, f_k, g_k) \to_k (u_0, f, g)$ $(H^{3/2} \times L^2_T H^1 \times H^{2,2})$; however, the sequence a priori does not satisfy (CC0). Let us modify $u_{0,k} = v_k + \varphi_k$; it is sufficient to construct $\varphi_k \in H^2(\Omega)$ such that $\|\varphi_k\|_{H^{3/2}} \to_k 0$ and

$$\varphi_k|_{\partial\Omega} = g_k|_{t=0} - v_k|_{\partial\Omega}. \tag{2-5}$$

This is an underdetermined system on $(\partial_n^j \varphi_k)_{0\le j\le 1}$ that we close by imposing $\partial_k\varphi_k = 0$: we define $\varphi_k \in H^2$ as the lifting of $(g_k|_{t=0} - v_k|_{\partial\Omega}, 0)$. From standard trace theory, there exists a lifting operator

$$\begin{aligned} L:\ H^{3/2}(\partial\Omega) &\to H^2(\Omega) \\ b &\mapsto v \quad \text{such that } v|_{\partial\Omega} = b,\ \partial_n v = 0, \end{aligned}$$

that extends continuously as a lifting operator $H^1 \to H^{3/2}$ (on the half space in Fourier variables $\xi = (\xi', \xi_d)$ one may take $\widehat{Lb} = \hat{b}(\xi')h(\xi_d/\sqrt{1+|\xi'|^2})/\sqrt{1+|\xi'|^2}$ with h smooth and compactly supported, $\int h\, d\xi_1 = 1$, $\int \xi_1 h\, d\xi_1 = 0$; see [Lions and Magenes 1968a] for more details). In particular, we have $\|g_k|_{t=0} - v_k|_{\partial\Omega}\|_{H^1} \to \|g|_{t=0} - u_0|_{\partial\Omega}\|_{H^1} = 0$, which implies $\|\varphi_k\|_{H^{3/2}} \to 0$. □

Proof of Lemma 2.3. We first detail the case $s = \frac{3}{2}$ and will deal with $s = -\frac{1}{2} + 2k$, $k \in \mathbb{N}$ by induction. Let u be the solution of (IBVP). If (CC0) is satisfied, then there exists $(u_{0,k}, g_k, f_k)$ as in Lemma 2.4, and we call the associated solutions u_k. Since $\|u_k - u\|_{C_T(H_D^{1/2})'} \to_k 0$, it is sufficient to prove the convergence of u_k in $C_T H^{3/2}$. We first check that $u_k \in C_T H^2$. Let $\tilde{g}_k \in H^{3,2}(\Omega \times [0,T])$ be a lifting (for its existence, see [Lions and Magenes 1968b, chapitre 4, section 2]) such that

$$\begin{cases} \tilde{g}_k|_{\partial\Omega\times[0,T]} = g_k, \\ \Delta\tilde{g}_k|_{\partial\Omega\times[0,T]} = f_k|_{\partial\Omega\times[0,T]} - i\partial_t g_k. \end{cases}$$

We define

$$w_k = e^{it\Delta_D}(u_{0,k} - \tilde{g}_k|_{t=0}) + \int_0^t e^{i(t-s)\Delta_D}(f_k - i\partial_t\tilde{g}_k - \Delta\tilde{g}_k)\, ds,$$

the solution of the homogeneous IBVP with initial data $u_{0,k} - \tilde{g}_k|_{t=0}$ and forcing term $f_k - i\partial_t\tilde{g}_k - \Delta\tilde{g}_k$, so that $u_k = w_k + \tilde{g}_k$. The embedding $H^{3,2} \hookrightarrow C_T H^2$ and (CC0) then imply $u_{0,k} - \tilde{g}_k|_{t=0} \in H^2_D$ and $f_k - i\partial_t\tilde{g}_k - \Delta\tilde{g}_k \in L^1_T H^2_D$, thus $w_k \in C_T H^2_D$ and $u_k = w_k + \tilde{g}_k \in C_T H^2$. In particular, $\Delta u_k \in C_T L^2$ and we can now check that it is the transposition solution of the IBVP

$$\begin{cases} i\partial_t v_k + \Delta v_k = \Delta f_k, & (x,t) \in \Omega \times [0, T[, \\ v_k|_{t=0} = \Delta u_{0,k}, & x \in \Omega, \\ v_k|_{\partial\Omega\times[0,T]} = -i\partial_t g_k + f_k|_{\partial\Omega\times[0,T]}; \end{cases} \tag{2-6}$$

that is to say (2-3) is satisfied with data $(\Delta u_{0,k}, \Delta f_k, -i\partial_t g_k + f_k|_{\partial\Omega\times[0,T]})$.

Let $\varphi \in C^\infty([0,T], C_c^\infty(\Omega))$; we set $w = \int_T^t e^{i(t-s)\Delta_D} \Delta\varphi \, ds$ the solution of the dual boundary value problem with data $\Delta\varphi$. By definition of u_k,

$$\iint_{\Omega\times[0,T]} \Delta u_k \overline{\varphi} \, dx \, dt = \iint_{\Omega\times[0,T]} u_k \overline{\Delta\varphi} \, dx \, dt$$
$$= \iint_{\Omega\times[0,T]} f_k \overline{w} \, dx \, dt + i \int_\Omega u_{0,k} \overline{w(0)} \, dx + \iint_{\partial\Omega\times[0,T]} g_k \overline{\partial_n w} \, dS \, dt.$$

Now, since $w = \Delta \int_T^t e^{i(t-s)\Delta_D} \varphi \, ds := \Delta v$, where $v \in C^1 H_D^2$, we can write

$$\iint_{\Omega\times[0,T]} \Delta u_k \overline{\varphi} \, dx \, dt$$
$$= \iint_{[0,T]\times\Omega} f_k \overline{\Delta v} \, dx \, dt + i \int_\Omega u_{0,n} \overline{\Delta v(0)} dx + \iint_{\partial\Omega\times[0,T]} g_k \overline{\partial_n \Delta v} \, dS \, dt$$
$$= \iint_{\Omega\times[0,T]} \Delta f_k \bar{v} \, dx \, dt + i \int_\Omega \Delta u_{0,k} \overline{v(0)} \, dx + i \int_{\partial\Omega} u_{0,k} \partial_n \overline{v(0)} \, dx$$
$$+ \iint_{\partial\Omega\times[0,T]} g_k \partial_n (\overline{-i\partial_t v + \varphi}) + f_k \overline{\partial_n v} \, dS \, dt$$
$$= \iint_{\Omega\times[0,T]} \Delta f_k \bar{v} \, dx \, dt + \iint_{\partial\Omega\times[0,T]} (f_k - i\partial_t g_k) \partial_n \bar{v} \, dS \, dt + i \int_\Omega \Delta u_{0,k} \overline{v(0)} \, dx$$
$$+ i \int_{\partial\Omega} u_{0,k} \partial_n \overline{v(0)} \, dS + i \Big[\int_{\partial\Omega} g_k \overline{\partial_n v} \, dS\Big]_0^T$$
$$= \iint_{\Omega\times[0,T]} \Delta f_k \bar{v} \, dx \, dt + \iint_{\partial\Omega\times[0,T]} (f_k - i\partial_t g_k) \partial_n \bar{v} \, dS \, dt + i \int_\Omega \Delta u_{0,k} \overline{v(0)} \, dx,$$

where in the last equality we used (CC0) and the cancellation of $v|_{t=T}$. Since the equality is true for arbitrary φ, by density of $C^\infty([0,T], C_c^\infty(\Omega))$ in $L_T^1 H_D^{1/2}$ we obtain that Δu_k is the transposition solution of (2-6), and Δu_k converges in $C_T(H_D^{1/2})'$ since $\Delta u_{0,k}, \Delta f_k, i\partial_t g_k - f_k|_{\partial\Omega\times[0,T]}$ converge in $(H^{1/2})'_D \times L_T^2 H^{-1} \times L^2$. Arguing as in the end of proof of [Audiard 2013, Proposition 6], we obtain the convergence of u_k in $C_T H^{3/2}$ and its limit is u by uniqueness of the limit. This settles the case $s = \frac{3}{2}$.

For $s = -\frac{1}{2} + 2k$, $k \geq 2$, we argue by induction. Let us introduce the boundary value problems

$$\begin{cases} i\partial_t v + \Delta v = \Delta^m f, & (x,t) \in \Omega \times [0,T[, \\ v|_{t=0} = \Delta^m u_0, & x \in \Omega, \\ v|_{\partial\Omega\times[0,T]} = \psi_m|_{\partial\Omega\times[0,T]}, \end{cases} \tag{IBVPm}$$

where ψ_m is defined inductively by $\psi_0 = g$, $\psi_{j+1} = \Delta^j f|_{\partial\Omega\times[0,T]} - i\partial_t \psi_j$. We assume that (u_0, f, g) in $H^{-1/2+2k} \times H^{-1+2k,2} \times H^{2k,2}$ satisfy (CCj), $0 \leq j \leq k-1$, and $\Delta^j u$ is a solution of (IBVPj) for $0 \leq j \leq k-1$. In particular, $\Delta^{k-1} u$ is a solution of (IBVP$k-1$) and the previous argument implies that $\Delta^{k-1} u \in C_T H^{3/2}$ if $(\Delta^{k-1} u_0, \Delta^{k-1} f, \psi_{k-1})$ belong to $H^{3/2} \times L_T^2 H^1 \times H^{2,2}$ and satisfy the compatibility condition $\psi_{k-1}|_{t=0} = \Delta^{k-1} u_0|_{\partial\Omega}$. The first condition is clear, since[1] $\psi_j \in H^{2k-j}(\partial\Omega \times [0,T])$,

[1] Actually, the careful reader may note that the regularity of the boundary data only requires $f \in H^{2m-3/2+\varepsilon,2}$, $\varepsilon > 0$, rather than $H^{2m-1,2}$. This is not important as the dispersive estimates in next section require the full regularity $f \in H^{2m-1,2}$.

and for the compatibility condition we may note that

$$\forall j \geq 1 \quad \begin{aligned} \psi_j &= (-i\partial_t)^j g + \sum_{p=0}^{j-1} (-i\partial_t)^p \Delta^{j-1-p} \frac{f}{i}\Big|_{\partial\Omega\times[0,T]}, \\ \varphi_j &= (i\Delta)^j u_0 + \sum_{p=0}^{j-1} \partial_t^{j-1-p} (i\Delta)^p f|_{t=0}, \end{aligned}$$

so that $\psi_{k-1}|_{t=0} = \Delta^{k-1} u_0$ is equivalent to (CC$k-1$). Thus

$$\Delta^{k-1} u \in C_T H^{3/2} \qquad \text{and} \qquad \Delta^j u|_{\partial\Omega} = \psi_j \in H^{2(k-j)} \hookrightarrow C_T H^{2(k-j)-1}, \quad 0 \leq j \leq k-2,$$

so that, by elliptic regularity, $u \in C_T H^{2k-1/2}$. □

We can now conclude this section:

Proof of Theorem 2.2. We have obtained well-posedness for $s = -\frac{1}{2}, \frac{3}{2}$. The case $-\frac{1}{2} \leq s \leq \frac{3}{2}$ follows by interpolation if we check that $H^s \times H^{s+1/2,2} \times L^2_T H^{s-1/2}$ with compatibility condition is the interpolated space between $(H_D^{1/2})' \times L^2 \times L^2_T H^{-1/2}$ and $H^{3/2} \times H^{2,2} \times L^2_T H^1$ with compatibility condition; this is proved in Lemma A.2 in the Appendix.

For $s \geq \frac{3}{2}$, let $m \in \mathbb{N}$ be such that $-\frac{1}{2} + 2m \leq s < -\frac{1}{2} + 2(m+1)$. The case of equality is Lemma 2.3; in the case of strict inequality we recall that $\Delta^m u$ is a solution of (IBVPm), where it is easily seen that if $(f, g) \in H^{s-1/2,2}(\Omega\times[0,T]) \times H^{s+1/2}(\partial\Omega\times[0,T])$ then $\psi_m \in H^{s+1/2-2m}$. Since $-\frac{1}{2} \leq s-2m \leq \frac{3}{2}$, we have from the previous case that $\Delta^m u \in C_T H^{s-2m}$; the regularity of u follows by elliptic regularity. □

3. Dispersive estimates

From now on we assume that Ω^c is star-shaped; up to translation we can also assume that it is star-shaped with respect to 0.

Local smoothing. Let us first recall the key virial identity:

Proposition 3.1 [Audiard 2013]. *If u is a smooth solution of (IBVP), $h \in C^k(\Omega)$, $\nabla^k h$ bounded for $1 \leq k \leq 4$, and $I(u) = 2\,\mathrm{Im}\int_\Omega \nabla h \cdot \nabla u \bar{u}\, dx$, then, setting $\nabla_\tau = \nabla - n\partial_n$,*

$$\frac{d}{dt} I(u(t)) = 4\,\mathrm{Re} \int_\Omega \mathrm{Hess}(h)(\nabla u, \overline{\nabla u}) - \tfrac{1}{4}|u|^2 \Delta^2 h + \nabla h \cdot \nabla u \bar{f} + \tfrac{1}{2}\bar{u}\Delta h f\, dx$$
$$+ \mathrm{Re} \int_{\partial\Omega} 2\partial_n h |\nabla_\tau u|^2 - 2\partial_n h|\partial_n u|^2 - 2i\partial_n h \partial_t u \bar{u}\, dS + \mathrm{Re}\int_{\partial\Omega} -2\bar{u}\Delta h \partial_n u + |u|^2 \partial_n \Delta h\, dS.$$

For the choice $h(x) = \sqrt{1+|x|^2}$, we have $\mathrm{Hess}(h) \geq 1/(1+|x|^2)^{3/2}$, $\partial_n h \leq 0$ (because Ω is star-shaped); this leads to the following result:

Proposition 3.2. *For any $\varepsilon > 0$, $(u_0, f, g) \in H^{1/2}(\Omega) \times L^2(\Omega\times[0,T]) \times H^{1+\varepsilon,(1+\varepsilon)/2}(\partial\Omega\times[0,T])$ that satisfy (CCG0), f compactly supported, we have*

$$\left\| \frac{\nabla u}{(1+|x|^2)^{3/4}} \right\|_{L^2([0,T],L^2(\Omega))} + \|\partial_n u\|_{L^2(\partial\Omega\times[0,T])} \lesssim (\|u_0\|_{H^{1/2}} + \|f\|_{L^2} + \|g\|_{H^{1+\varepsilon,2}}).$$

Remark 3.3. The constant in $\lesssim$ depends on ε, T and the size of $\operatorname{supp}(f)$, and blows up if $\varepsilon \to 0$, $T \to \infty$ or $\operatorname{supp}(f) \to \Omega$. We chose not to emphasize this as it will not matter in the rest of the article.

Proof. The proof was essentially done in [Audiard 2013] for a strictly convex obstacle; we write it out since it must be slightly modified for the case of a star-shaped obstacle. We use that f is compactly supported to absorb the term $\int \nabla h \nabla u \overline{f}\, dx$ in $\int \operatorname{Hess}(h)(\nabla u \overline{\nabla u})\, dx$, and Ω^c is star-shaped thus $\partial_n h \le 0$ (n is the outer normal of Ω), so integration in time gives

$$\left\| \frac{\nabla u}{(1+|x|^2)^{3/4}} \right\|^2_{L^2(\Omega\times[0,T])} \lesssim \|u\|^2_{L^2(\Omega\times[0,T])} + \|f\|^2_{L^2(\Omega\times[0,T])} + \|g\|^2_{H^{1+\varepsilon,2}(\partial\Omega\times[0,T])} + |I(u(T))| + |I u_0|.$$

To estimate $|I(u(T))|+|I(u(0))|$ the main issue is that $\nabla u \in (H_D^{1/2})'$, which is slightly larger than $H^{-1/2}$. Following the notations of Lemma A.2, we first remark that the assumptions of the lemma imply $(u_0, g) \in X^{1/2}$ and we use the lifting operator $H^{s,s/2} \to H^{s+1/2,s/2+1/4}(\Omega \times [0,T])$, $g \mapsto R_1 g$. If $(u_0, g) \in X^{3/4}$, then $(u_0 - R_1 g|_{t=0}, u(T) - R_1 g|_{t=T}) \in (H_0^1(\Omega))^2$, while, if $(u_0, g) \in X^{1/3}$, then $(u_0 - R_1 g|_{t=0}, u(T) - R_1 g|_{t=T}) \in (H^{1/6}(\Omega))^2$, thus by interpolation

$$(u_0, g) \in X^{1/2} \implies (u_0 - \tilde{g}|_{t=0}, u(T) - \tilde{g}|_{t=T}) \in (H_D^{1/2}(\Omega))^2.$$

This implies for $t \in [0,T]$

$$\left| \int_\Omega \overline{u(t) - R_1 g(t)} \nabla u \cdot \nabla h\, dx \right| \lesssim \|u\|_{C([0,T],H^{1/2})} \|g\|_{H^{1,2}}$$

On the other hand, an integration by parts formally gives

$$\begin{aligned}\left| \int_\Omega \overline{R_1 g(t)} \nabla u \cdot \nabla h\, dx \right| &\le \left| \int_\Omega u \operatorname{div}(\overline{R_1 g(t)} \nabla h)\, dx \right| + \left| \int_{\partial\Omega} g \overline{R_1 g(t)} \partial_n h\, dx \right| \\ &\le C_\varepsilon (\|u(t)\|_{H^{1/2-\varepsilon}} \|R_1 g(t)\|_{H^{1/2+\varepsilon}} + \|g(t)\|^2_{L^2}) \\ &\le C_\varepsilon (\|u\|_{C_T H^{1/2}} \|g\|_{H^{1+\varepsilon,2}} + \|g\|^2_{H^{1+\varepsilon,2}}),\end{aligned}$$

so that by a density argument we obtain

$$\begin{aligned}\left\| \frac{\nabla u}{(1+|x|^2)^{3/4}} \right\|_{L^2(\Omega\times[0,T])} &\le C_{\varepsilon,T} (\|u\|_{C_T\, H^{1/2}} + \|g\|_{H^{1+\varepsilon,2}} + \|f\|_{L^2}) \\ &\le C_{\varepsilon,T} (\|u_0\|_{H^{1/2}} + \|f\|_{L^2} + \|g\|_{H^{1+\varepsilon,2}}).\end{aligned} \tag{3-1}$$

The estimate on $\|\partial_n u\|_{L^2}$ cannot in general be obtained directly from the virial identity with $h = \sqrt{1+|x|^2}$ since we may have, for some $x \in \partial\Omega$, $\partial_n h = x \cdot n / \sqrt{1+|x|^2} = 0$. However, once local smoothing has been obtained it is quite simple to derive an estimate on $\partial_n u$. The argument that we give now is essentially the same as the one from [Planchon and Vega 2009] for the homogeneous case. Using the identity from Proposition 3.1 with some h smooth and compactly supported such that $\partial_n h < 0$, we obtain

$$\|\partial_n u\|^2_{L^2} \lesssim |I(u(T))| + |I(u_0)| + \|u\|^2_{L^2} + \|f\|^2_{L^2} + \|g\|_{H^{1+\varepsilon,2}} + \int_0^T \int_\Omega \operatorname{Hess}(h)(\nabla u, \overline{\nabla u})\, dx\, dt.$$

The integral of $\mathrm{Hess}(h)(\nabla u, \nabla u)\,dx$ is no longer positive; however, since h is compactly supported, it is controlled thanks to (3-1). □

We can now state the local smoothing property for more general regularity:

Corollary 3.4. *Let* $\varepsilon>0$, $\frac{1}{2}\le s<2$, $(u_0, f, g)\in H^s(\Omega)\times H^{s-1/2,2}(\Omega\times[0,T])\times H^{s+1/2+\varepsilon,2}(\partial\Omega\times[0,T])$ *satisfying the compatibility conditions*, f *compactly supported*, $\varepsilon>0$; *then the solution* $u\in C_T H^s$ *of* (IBVP) *has the local smoothing property*

$$\left\|\frac{u}{(1+|x|^2)^{3/4}}\right\|_{L^2_T H^{s+1/2}}+\|\partial_n u\|_{H^{s-1/2,2}}\lesssim \|u_0\|_{H^s}+\|g\|_{H^{s+\varepsilon+1/2,2}}+\|f\|_{H^{s-1/2,2}}.$$

Proof. The case $s=\frac{1}{2}$ is Proposition 3.2. For $s=\frac{5}{2}$, we have already seen that Δu is a solution of the IBVP with forcing term Δf, initial conditions Δu_0 and boundary data $-i\partial_t g+f|_{\partial\Omega\times[0,T]}$, thus the local smoothing implies

$$\begin{aligned}\left\|\frac{\nabla\Delta u}{(1+|x|^2)^{3/4}}\right\|_{L^2(\Omega\times[0,T])}&\lesssim \|u_0\|_{H^{5/2}}+\|f\|_{L^2_T H^2}+\|g\|_{H^{3+\varepsilon,2}}+\|f\|_{H^{1+\varepsilon,2}(\partial\Omega\times[0,T])}\\&\lesssim \|u_0\|_{H^{5/2}}+\|f\|_{L^2_T H^2}+\|g\|_{H^{3+\varepsilon,2}}+\|f\|_{H^{3/2+\varepsilon,2}(\Omega\times[0,T])}\\&\lesssim \|u_0\|_{H^{5/2}}+\|f\|_{H^{2,2}(\Omega\times[0,T])}+\|g\|_{H^{3+\varepsilon,(3+\varepsilon)/2}}.\end{aligned}$$

Elliptic regularity then implies the estimate on $\|u/(1+|x|^2)^{3/4}\|_{H^3}$. The control of $\|\partial_n u\|_{H^{2,2}}$ requires a bit more care, since we cannot directly use the estimate on $\partial_n\Delta u$: for $x_0\in\partial\Omega$, we use local coordinates $(y_1,\dots,y_d)$ such that, on a neighbourhood U of x_0, $\partial\Omega\cap U=\{y_d=0\}$ and $\Omega\cap U\subset\{y_d>0\}$, and we define the differential operators $D_k=\varphi(y_1,\dots,y_{d-1})\psi(y_d)\partial_{y_k}$, $1\le k\le d-1$, with φ, ψ such that $\mathrm{supp}(\varphi\psi)\subset U$ and $\psi=1$ on a neighbourhood of 0. Setting $D_k=0$ outside U, the D_k define second-order differential operators on Ω and, by restriction, on $\partial\Omega$. For $1\le k,\ p\le d-1$, it can be checked as for Δu that $u_{kp}=D_kD_pu$ is the transposition solution of

$$\begin{cases} i\partial_t w+\Delta w=D_kD_pf+[\Delta, D_kD_p]u,\\ w|_{t=0}=D_kD_pu_0,\\ w|_{\partial\Omega}=D_kD_pg,\end{cases}$$

where the commutator $[\Delta,\ D_kD_p]$ is a third-order differential operator. The virial identity gives

$$\begin{aligned}\frac{dI(u_{kp})}{dt}=4\,\mathrm{Re}\int_\Omega \mathrm{Hess}(h)(\nabla u_{kp},\nabla\bar u_{kp})-\tfrac{1}{4}|u_{kp}|^2\Delta^2h+\nabla h\cdot\nabla u_{kp}(\overline{D_kD_pf+[\Delta,D_kD_p]u})\,dx\\ +2\,\mathrm{Re}\int_\Omega \bar u_{kp}\Delta h(D_kD_pf+[\Delta,D_kD_p]u)\,dx\\ +\mathrm{Re}\int_{\partial\Omega}2\partial_nh|\nabla_\tau u_{kp}|^2-2\partial_nh|\partial_nu_{kp}|^2-2i\partial_nh\partial_tu_{kp}\bar u_{kp}\,dS\\ +\mathrm{Re}\int_{\partial\Omega}-2\bar u_{kp}\Delta h\partial_nu_{kp}+|u_{kp}|^2\partial_n\Delta h\,dS,\end{aligned}$$

Choosing h compactly supported such that $\partial_n h < 0$ on $\operatorname{supp} D_k$ as in the proof of Proposition 3.2 gives an estimate on $\|\partial_n u_{kp}\|_{L^2(\partial\Omega\times[0,T])}$, provided the new terms induced by $[\Delta,\ D_k D_p]u$ are controlled; this last point is a consequence of the local smoothing

$$\left|4\int_0^T\int_\Omega \nabla h\cdot\nabla u_{kp}\overline{[\Delta, D_k D_p]u}+\tfrac{1}{2}\bar{u}_{kp}\Delta h[\Delta, D_k D_p]u\,dx\right|dt \lesssim \|u_{kp}\|_{L^2_T H^1}\|u\|_{L^2_T H^3}$$
$$\lesssim \|u_0\|^2_{H^{5/2}}+\|f\|^2_{H^{2,2}}+\|g\|^2_{H^{3+\varepsilon,2}}.$$

This gives $\|\partial_n u_{kp}\|_{L^2} \lesssim \|u_0\|_{H^{5/2}}+\|f\|_{H^{2,2}}+\|g\|_{H^{3+\varepsilon,2}}$. Since $\psi = 1$ on a neighbourhood of 0 and $\partial_n = \partial_{y_d}$ on U, we have $\partial_n D_k D_p = D_k D_p \partial_n$, so that

$$\|D_k D_p \partial_n u\|_{L^2(\partial\Omega\times[0,T])} \lesssim \|u_0\|_{H^{5/2}}+\|f\|_{H^{2,2}}+\|g\|_{H^{3+\varepsilon,2}}.$$

Finally, since $\|\partial_n u(t)\|_{H^1} \lesssim \|u(t)\|_{H^{5/2}}$ and using a partition of unity, we get

$$\|\partial_n u\|_{L^2_T H^2} \lesssim \|u_0\|_{H^{5/2}}+\|f\|_{H^{2,2}}+\|g\|_{H^{3+\varepsilon,2}}.$$

The time regularity of $\partial_n u$ can be obtained in a similar way by considering the IBVP satisfied by $\partial_t u$; the application of Proposition 3.2 requires $\partial_t f \in L^2(\Omega\times[0,T])$ and $\partial_t u|_{t=0} = i\Delta u_0 - if|_{t=0} \in H^{1/2}$, both of which are ensured by $f \in H^{2,2}$. Since $\partial_t\partial_n = \partial_n\partial_t$, the local smoothing property gives directly

$$\|\partial_t\partial_n u\|_{L^2(\partial\Omega\times[0,T])} \lesssim \|u_0\|_{H^{5/2}}+\|f\|_{H^{2,2}}+\|g\|_{H^{3,2}}.$$

The result for $\frac{1}{2} \le s < 2$ then follows by a (nontrivial) interpolation argument similar to Lemma A.2 that we sketch now: Setting

$$Y^\alpha = \{(u_0, f, g) \in H^\alpha \times H^{\alpha-1/2,2}\times H^{\alpha+1/2,2} \text{ that satisfy the compatibility conditions}\},$$

it is sufficient to prove $[Y^{1/2}, Y^{5/2}]_\theta \supset Y^{2\theta+1/2}$ for $\theta < \frac{3}{4}$. To get rid of the link between u_0, f and g, let us define $H^{2,2}_{(0)}(\Omega\times[0,T]) = \{f \in H^{2,2} : f|_{\partial\Omega\times\{0\}} = 0\}$. Clearly

$$Y^{5/2} \supset \{(u_0, f, g) \in H^{5/2}\times H^{2,2}_{(0)}\times H^{3,2} \text{ with (CC0), (CCG1)}\} := Y^{5/2}_{(0)}.$$

The key point of $Y^{5/2}_{(0)}$ is that $f|_{t=0} \in H^1_0$, so that the $(f^j)_{1\le j\le J}$ introduced in the description of global compatibility conditions automatically satisfy $\int_0^\infty\int_{\mathbb{R}^{d-1}} |f^j(x', h)|^2\,dx'\,dh/h < \infty$. Therefore the conditions (CC0), (CCG1) only involve u_0 and g, and

$$Y^{5/2}_{(0)} = \{(u_0, g) \in H^{5/2}\times H^{3,2} \text{ with (CC0), (CCG1)}\}\times H^{2,2}_{(0)}.$$

For $\theta < \frac{3}{4}$, we have, from Proposition A.4, $[L^2, H^{2,2}_{(0)}\}]_\theta = H^{2\theta,2}(\Omega\times[0,T])$. As a consequence, setting $X^{3/2} = \{(u_0, g) \in H^{5/2}\times H^{3,2}$ with (CC0), (CCG1)$\}$ (as in Lemma A.2), we are reduced to checking that $[X^{1/2}, X^{3/2}]_\theta = X^{1/2+\theta}$, which can be done as in Lemma A.2. □

Remark 3.5. The loss of regularity on the boundary data can be avoided up to an arbitrary loss on the local smoothing. Indeed for $(u_0, f, g) \in H^{1/2+\varepsilon}\times H^{\varepsilon,2}\times H^{1+\varepsilon,2}$, the virial estimate implies $u \in L^2_T H^1$, and from an argument similar to Corollary 3.4 we find that, for $\frac{1}{2}+\varepsilon \le s < 2$, we have $(u_0, f, g) \in H^s\times H^{s-1/2,2}\times H^{s+1/2,2}$, then $u \in L^2_T H^{s+1/2-\varepsilon}$.

We choose to focus on the case where we lose some regularity on the boundary data because it avoids the use of peculiar numerology for the Strichartz estimates and well-posedness theorems in the rest of the article; however, we will continue to discuss this alternative approach in Remarks 3.8 and 4.3.

The estimate is restricted to functions f compactly supported near $\partial\Omega$. For the well-posedness results of next section we will also need smoothing of the normal derivative when f is supported "away from $\partial\Omega$":

Proposition 3.6. *Let w be the solution of the homogeneous boundary value problem*

$$\begin{cases} i\partial_t w + \Delta_D w = f, \\ w|_{t=0} = 0, \\ w|_{\partial\Omega} = 0; \end{cases}$$

then w satisfies the estimate

$$\|\partial_n w\|_{H^{1/2,2}(\partial\Omega\times[0,T])} \lesssim \|f\|_{B^{1,2}_{3/2,2,0}}.$$

Proof. From the Strichartz estimate in [Burq et al. 2004], we have

$$\|w\|_{C_T H_D^{1/2}\cap L^3 W_0^{1/2,3}} \lesssim \|f\|_{L_T^{3/2} W_0^{1/2,3/2}}.$$

The virial identity gives

$$\|\partial_n w\|^2_{L^2(\partial\Omega\times[0,T])} \lesssim \|u\|_{C_T H_D^{1/2}} + \|u\|_{L_T^3 W_0^{1/2,3}} \|f\|_{L^{3/2} W_0^{1/2,3/2}} \lesssim \|f\|^2_{L^{3/2} W_0^{1/2,3/2}},$$

and similarly, using the same differentiation arguments as in Corollary 3.4, we get[2]

$$\|\partial_n w\|_{H^{2,2}(\partial\Omega\times[0,T])} \lesssim \|f\|_{L_T^{3/2} W_0^{5/2,3/2,2,2}\cap W_T^{5/4,3/2} L^{3/2}}.$$

Let us recall that, for $s \geq 0$, $s \notin \mathbb{N}$, $B^{s,2}_{3/2,3/2,0}(\Omega\times[0,T]) = W_T^{s/2,3/2} L^{3/2} \cap L_T^{3/2} W_0^{s,3/2}$. Using real interpolation with parameter $\theta = \frac{1}{4}$ and $q = 2$ gives the expected result, as a consequence of

$$[L^{3/2} W_0^{1/2,3/2}, L_T^{3/2} W_0^{5/2,3/2} \cap W_T^{5/4,3/2} L^{3/2}]_{1/4,2} \supset [B^{1/2,2}_{3/2,3/2,0}, B^{5/2,2}_{3/2,3/2,0}]_{1/4,2} = B^{1,2}_{3/2,2,0}.$$

The first inclusion is clear, and the equality follows from the interpolation of anisotropic Sobolev spaces; see the book of H. Amann [2009], Section 3.3 for the interpolation of anisotropic spaces on $\mathbb{R}^d$ and Section 4.4 for domains with corner. □

Strichartz estimates. We deduce in this section Strichartz estimates (with loss of derivatives) from the local smoothing. Following the terminology of admissible pair (those (p,q) such that $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$), we say that (p,q) is a weakly admissible pair if

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2}. \tag{3-2}$$

[2]When differentiating in time, we obtain $\partial_t u|_{t=0} = -if|_{t=0} \in W_0^{7/6,3/2} \hookrightarrow H_0^1 \hookrightarrow H_D^{1/2}$, thus the initial data for the problem satisfied by $\partial_t u$ is smooth enough to use the virial identity.

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, such that Ω^c is star-shaped with respect to 0. For $\varepsilon > 0$, $T < \infty$, $\frac{1}{2} \leq s < 2$, $(u_0, f, g) \in H^s \times H^{s-1/2,1/4} \times H^{s+1/2+\varepsilon,2}$ satisfying the compatibility conditions, f compactly supported, and any weakly admissible (p, q) with $p, q > 2$, the solution $u \in C_T H^1$ satisfies*

$$\|u\|_{L^p([0,T],W^{s,q}(\Omega))} \lesssim \|u_0\|_{H^s} + \|g\|_{H^{s+1/2+\varepsilon}} + \|f\|_{H^{s-1/2,1/4}}.$$

Proof. The argument from [Burq et al. 2004, Proposition 2.14] can be used with no meaningful modification (see also [Audiard 2013, Corollary 1]). Let us sketch it briefly: we decompose $u = \chi u + (1-\chi u)$, χ compactly supported, $\chi = 1$ near $\partial\Omega \cup \operatorname{supp}(f)$. From the local smoothing property, $\chi u \in L^2_T H^{s+1/2} \cap L^\infty_T H^s$, we have by (complex) interpolation that $u \in L^p_T H^{s+1/p}$. The Sobolev embedding $H^{s+1/p} \hookrightarrow W^{s,q}$ with $\frac{1}{q} = \frac{1}{2} - \frac{1}{dp}$ and the local smoothing property from Corollary 3.4 imply $\chi u \in L^p_T W^{s,q}$.

The function $(1-\chi)u$ extended by 0 outside $\operatorname{supp}(1-\chi)$ satisfies a Schrödinger equation on $\mathbb{R}^d$, and the usual Strichartz estimates on $\mathbb{R}^d$ imply (by a standard but nontrivial argument that originates in [Staffilani and Tataru 2002])

$$\|(1-\chi)u\|_{L^{2p}([0,T],W^{s,q})} \lesssim \|u_0\|_{H^s} + \|g\|_{H^{s+1/2+\varepsilon,2}} + \|f\|_{H^{s-1/2,1/4}}.$$

From $L^{2p}([0,T]) \subset L^p([0,T])$ we obtain the expected estimate. □

Remark 3.8. Following the observations of Remark 3.5, we could also prove an alternate Strichartz estimate with optimal boundary data in $H^{s+1/2,2}$ but $\frac{1}{p} + \frac{d}{q} = \frac{d}{2} + \frac{2\varepsilon}{p}$, simply by using the embedding $H^{s+1/2-\varepsilon} \hookrightarrow W^{s,q_1}$, $1/q_1 = \frac{1}{2} - \left(\frac{1}{2} - \varepsilon\right)/d$.

4. Nonlinear well-posedness

We consider here nonlinear IBVPs of the form

$$\begin{cases} i\partial_t u + \Delta u = F(u), & (x,t) \in \Omega \times [0,T[, \\ u|_{t=0} = u_0, & x \in \Omega, \\ u|_{\partial\Omega\times[0,T]} = g, & (x,t) \in \partial\Omega \times [0,T[, \end{cases} \tag{NLS}$$

with the following assumptions on $F \in C^1(\mathbb{C})$: there exists $\alpha > 0$ such that

$$|F(z)| \lesssim |z|(1+|z|^\alpha), \tag{4-1}$$

$$|\nabla F(z)| \lesssim (1+|z|)^\alpha. \tag{4-2}$$

For the smoothness of the flow we will assume $F \in C^2(\mathbb{C})$ and

$$|\nabla^2 F(z)| \lesssim (1+|z|)^{\max(\alpha-1,0)} \tag{4-3}$$

Local well-posedness. Since our first result is local in time, we define

$$H^{3/2+\varepsilon,2}_{\text{loc}}(\mathbb{R}^+ \times \partial\Omega) = \{g : \chi(t)g \in H^{3/2+\varepsilon,2}(\mathbb{R}^+_t \times \partial\Omega) \text{ for all } \chi \in C^\infty_c(\mathbb{R}^+)\}.$$

We say that $u \in C_T H^1$ is a local solution to (NLS) if it satisfies $i\partial_t u + \Delta u = F(u)$ in the sense of distributions (for $u \in C_T H^1$ all quantities in the equality make sense), $u|_{\partial\Omega\times[0,T]} = g$ in the usual sense of traces and $u|_{t=0} = u_0$.

Theorem 4.1. *If F satisfies (4-1)–(4-2), then for any $(u_0, g) \in H^1(\Omega) \times H^{3/2+\varepsilon,2}_{\mathrm{loc}}(\mathbb{R}^+ \times \Omega)$ satisfying (CC0) and $\alpha < 2/(d-2)$, there exists a unique maximal solution $u \in C_{T^*}H^1$ of (NLS).*

The solution is causal in the sense that $u(t)$ only depends of u_0 and $g|_{s\le t}$, and, if $T^* < \infty$, then $\lim_{t\to T^*} \|u(t)\|_{H^1} = +\infty$.

If F satisfies (4-3) and $d \le 3$, then for any $T < T^*$ the solution map is Lipschitz from bounded sets of $H^1(\Omega) \times H^{3/2+\varepsilon,2}(\mathbb{R}^+ \times \Omega)$ to $C([0,T], H^1)$.

It will be convenient to introduce $\tilde{u}$, the solution of

$$\begin{cases} i\partial_t \tilde{u} + \Delta \tilde{u} = F(\tilde{g}), & (x,t) \in \Omega \times [0,T[, \\ \tilde{u}|_{t=0} = u_0, & x \in \Omega, \\ \tilde{u}|_{\partial\Omega\times[0,T]} = g, \ (x,t) \in \partial\Omega \times [0,T[, & \end{cases} \tag{4-4}$$

where $\tilde{g} \in H^{2,2}(\Omega \times [0,T])$ is a compactly supported lifting of g. Thus u must satisfy

$$u = \tilde{u} + \int_0^t e^{i(t-s)\Delta_D}(F(u) - F(\tilde{g}))(s)\,ds \quad \text{for all } t \in [0,T].$$

Choose q_0 such that $(2, q_0)$ is weakly admissible. According to Theorems 2.2 and 3.7, we have $\tilde{u} \in C_T H^1 \cap L^2_T W^{1,q_0}$ if $F(\tilde{g}) \in H^{1/2,2}$. Actually $F(\tilde{g})$ is smoother than needed:

Lemma 4.2. *For $\varphi \in H^{2,2}(\Omega \times [0,T])$ and F satisfying (4-1)–(4-2), $F(\varphi) \in H^{1,2}$.*

Proof. It is clear that $F(\varphi) \in L^2_T L^2$; indeed

$$\|F(\varphi)\|_{L^2_T L^2} \lesssim \|\varphi\|_{L^2_T L^2} + \|\varphi\|^{1+\alpha}_{L^{2(1+\alpha)}} \lesssim \|\varphi\|_{L^2_T H^1}(1 + \|\varphi\|^{\alpha}_{L^2_T H^1}).$$

Since $\alpha < 2/(d-2)$, there exist p, q satisfying

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad \min\Big(\frac{\alpha}{2}, \frac{1}{d}\Big) \ge \frac{1}{p} > \frac{\alpha(d-2)}{2d}, \quad \frac{1}{q} > \frac{d-2}{2d},$$

and Hölder's inequality gives, for any $t \in [0,T]$,

$$\begin{aligned} \|\nabla F(\varphi)(t)\|_{L^2(\Omega)} &\lesssim \|(1 + |\varphi|^\alpha)\nabla\varphi\|_{L^2} \\ &\lesssim \|\varphi\|_{H^1} + \|\varphi\|^{\alpha}_{L^{\alpha p}} \|\nabla\varphi\|_{L^q} \\ &\lesssim \|\varphi\|_{H^1} + \|\varphi\|^{\alpha}_{H^1}\|\varphi\|_{H^2}, \end{aligned}$$

where we used the Sobolev embedding $H^1 \hookrightarrow L^q$, $2 \le q \le 2d/(d-2)$ (or $q < \infty$ if $d = 2$). From the embedding $H^{2,2} \hookrightarrow C_T H^1$ we deduce, by taking the L^2_T norm,

$$\|\nabla F(\varphi)\|_{L^2_T H^1} \lesssim \|\varphi\|_{L^2_T H^1} + \|\varphi\|^{\alpha}_{L^\infty_T H^1}\|\varphi\|_{L^2_T H^2} \lesssim \|\varphi\|_{H^{2,2}}(1 + \|\varphi\|^{\alpha}_{H^{2,2}}).$$

For the time regularity we have, using Hölder's inequalities again,

$$\begin{aligned} \|F(\varphi(t)) - F(\varphi(s))\|_{L^2(\Omega)} &\lesssim \|\varphi(t) - \varphi(s)\|_{L^2} + \big\||\varphi(t)| + |\varphi(s)|\big\|^{\alpha}_{L^{\alpha p}}\|\varphi(t) - \varphi(s)\|_{L^q} \\ &\lesssim \|\varphi(t) - \varphi(s)\|_{L^2} + \big\||\varphi(t)| + |\varphi(s)|\big\|^{\alpha}_{H^1}\|\varphi(t) - \varphi(s)\|_{H^1}, \end{aligned}$$

thus the embedding $H^{2,2} \hookrightarrow H^{1/2}([0,T], H^1(\Omega))$ gives

$$\begin{aligned}\|F(\varphi)\|^2_{\dot H^{1/2}_T L^2} &= \iint_{[0,T]^2} \frac{\|F(\varphi(t)) - F(\varphi(s))\|^2_{L^2}}{|t-s|^2}\, ds\, dt \\ &\lesssim \|\varphi\|^2_{H^{1/2}L^2} + \|\varphi\|^{2\alpha}_{L^\infty_T H^1} \|\varphi\|^2_{\dot H^{1/2}_T H^1} \\ &\lesssim \|\varphi\|^2_{H^{2,2}}(1 + \|\varphi\|^{2\alpha}_{H^{2,2}}).\end{aligned}$$

□

Proof of Theorem 4.1. Uniqueness: The uniqueness can be done as in the case of homogeneous Dirichlet boundary conditions from [Burq et al. 2004]. If u_1 and u_2 are two solutions in $C_{T^*}H^1$, then $w = u_1 - u_2$ is a solution of

$$\begin{cases} i\partial_t w + \Delta w = F(u_1) - F(u_2), & (x,t) \in \Omega \times [0,T[, \\ w|_{t=0} = 0, & x \in \Omega, \\ \tilde w|_{\partial\Omega \times [0,T]} = 0, & (x,t) \in \partial\Omega \times [0,T[. \end{cases}$$

This is a homogeneous boundary value problem for which the Strichartz estimates (1-2) give, for (p,q) weakly admissible as in (3-2), (r', s') weakly admissible and $T < T^*$,

$$\|w\|_{L^\infty_T L^2 \cap L^p_T L^q} \lesssim \|w\|_{L^1_T L^2} + \big\|(|u_1| + |u_2|)^\alpha w\big\|_{L^r_T L^s} \le T\|w\|_{L^\infty_T L^2} + + \big\|(|u_1| + |u_2|)^\alpha w\big\|_{L^r_T L^s}.$$

If we can choose (r, s, p_1, q_1, p, q) satisfying

$$\begin{cases} \dfrac{1}{p} + \dfrac{d}{q} = \dfrac{d}{2}, & \dfrac{1}{r} + \dfrac{d}{s} = 1 + \dfrac{d}{2}, \\ \dfrac{1}{p_1} + \dfrac{1}{2} = \dfrac{1}{r}, & \dfrac{1}{q_1} + \dfrac{1}{q} = \dfrac{1}{s}, \\ \dfrac{\alpha(d-2)}{2d} < \dfrac{1}{q_1} < \dfrac{\alpha}{2}, & \dfrac{1}{p} < \dfrac{1}{2}, \quad 0 < \dfrac{1}{p_1} < \alpha, \end{cases} \tag{4-5}$$

we get from the Sobolev embedding and Hölder estimate in time that

$$\begin{aligned}\big\|(|u_1| + |u_2|)^\alpha w\big\|_{L^r_T L^s} &\lesssim \big\||u_1| + |u_2|\big\|^\alpha_{L^{\alpha p_1} L^{\alpha q_1}} \|w\|_{L^2 L^q} \\ &\lesssim T^{1/2 - 1/p}(\|u_1\|_{L^\infty_T H^1} + \|u_2\|_{L^\infty_T H^1})^\alpha \|w\|_{L^p L^q},\end{aligned}$$

and thus $w = 0$ for $0 \le t \le T$, T small enough only depending on $\|u_1\|_{L^\infty H^1} + \|u_2\|_{L^\infty H^1}$. Iterating the argument implies $u = v$ on $[0, T^*[$. The system (4-5) implies

$$1 + \frac{d}{2} = \frac{1}{r} + \frac{d}{s} = \frac{1}{p_1} + \frac{1}{2} + \frac{d}{q_1} + \frac{d}{q} > \frac{1}{p_1} + \frac{d}{2} + \frac{\alpha(d-2)}{2} + \Big(\frac{1}{2} - \frac{1}{p}\Big), \tag{4-6}$$

which can be solved since $\frac{1}{2}\alpha(d-2) < 1$: we first choose $p > 2$ close enough to 2 that $\frac{1}{2}\alpha(d-2) + \frac{1}{2} - \frac{1}{p} < 1$, then it is possible to choose p_1 that satisfies (4-6) and $0 < \frac{1}{p_1} < \alpha$; up to increasing p we can assume $\frac{1}{p_1} < \frac{1}{2}$. The choice of p determines the value of $q > 2$, the choice of p_1 determines the value of $1 < r < 2$, and then of $1 < s < 2$. The only equation left is $\frac{1}{q_1} = \frac{1}{s} - \frac{1}{q}$; its solution $\frac{1}{q_1}$ belongs to $]0,1[$, and thus is an acceptable Hölder index.

Causality: This can be proved as for uniqueness, since if g_1, g_2 coincide on $[0,t]$, the uniqueness argument can be applied on $[0,t]$ and implies the associated solutions satisfy $u_1|_{[0,t]} = u_2|_{[0,t]}$.

Local existence: We recall that $(2, q_0)$ is assumed to be weakly admissible. According to Lemma 4.2 and Theorems 2.2 and 3.7, $\tilde{u} \in C_T H^1 \cap L^2_T W^{1,q_0}$, since $F(\tilde{g}) \in H^{1,2} \subset H^{1/2,2}$. Setting $w = u - \tilde{u}$, the local existence will be a consequence of the existence of a local solution to

$$\begin{cases} i\partial_t w + \Delta w = F(\tilde{u} + w) - F(\tilde{g}), \\ w|_{t=0} = 0, \\ w|_{\partial\Omega\times[0,T]} = 0. \end{cases}$$

This is a nonlinear homogeneous boundary value problem; the existence of a solution is essentially a consequence of (the proof of) Theorem 1 in [Burq et al. 2004]. As it does not strictly cover the case of our nonlinearity, we briefly sketch the argument. Let us define the map L as

$$L : X_T = C_T H^1_0 \cap L^p_T W^{1,q} \to C_T H^1_0 \cap L^p_T W^{1,q},$$
$$w \mapsto L(w) = \int_0^t e^{i(t-s)\Delta_D}(F(\tilde{u} + w) - F(\tilde{g}))\, ds;$$

we will check that it has a fixed point for T small enough. Burq et al. [2004] prove that, for a convenient choice of weakly admissible pairs (p, q), (p_1, q_1) (depending on $\alpha < 2/(d-2)$ and d), the map $\tilde{L}(w) = \int_0^t e^{i(t-s)\Delta_D} F(w)\, ds$ satisfies

$$\|\tilde{L}w\|_{X_T} \lesssim T^\theta(\|w\|_{X_T} + \|w\|^{1+\alpha}_{X_T}),$$
$$\|\tilde{L}w_1 - \tilde{L}w_2\|_{X_T} \lesssim T^{\theta'}\|w_1 - w_2\|_{X_T}(1 + \|w_1\|^\alpha_{X_T} + \|w_2\|^\alpha_{X_T}) \quad \text{if } d < 4,$$
$$\|\tilde{L}w_1 - \tilde{L}w_2\|_{C_T L^2 \cap L^{p_1} L^{q_1}} \lesssim T^{\theta''}\|w_1 - w_2\|_{C_T L^2 \cap L^{p_1} L^{q_1}}(1 + \|w_1\|^\alpha_{X_T} + \|w_2\|^\alpha_{X_T}) \quad \text{if } d \geq 4,$$

where θ, θ', θ'' are positive, and the second inequality ($d < 4$) also requires the assumption (4-3) on F (this is Propositions 3.1, 3.3, 3.4 and equations (3.9)–(3.10) from [Burq et al. 2004]).

Since $F(\tilde{u} + w) - F(\tilde{g})$ has trace 0 on $\partial\Omega \times [0, T]$, we can use these estimates. We recall $\tilde{g}$ is in $H^{2,2} \hookrightarrow L^\infty_T H^1 \cap L^2_T W^{1,q_0}$; therefore, setting $M(w) = \|w\|_{X_T} + \|\tilde{u}\|_{X_T} + \|g\|_{H^{3/2,2}}$ the estimates give, directly in our case,

$$\|Lw\|_{X_T} \lesssim T^\theta(M + (M)^{1+\alpha}), \tag{4-7}$$
$$\|Lw_1 - Lw_2\|_{X_T} \lesssim T^{\theta'}\|w_1 - w_2\|_{X_T}\big(1 + (M(w_1) + M(w_2))^\alpha\big) \quad \text{if } d < 4, \tag{4-8}$$
$$\|\tilde{L}w_1 - \tilde{L}w_2\|_{C_T L^2 \cap L^{p_1} L^{q_1}} \lesssim T^{\theta''}\|w_1 - w_2\|_{C_T L^2 \cap L^{p_1} L^{q_1}}\big(1 + (M(w_1) + M(w_2))^\alpha\big) \quad \text{if } d \geq 4. \tag{4-9}$$

If $d < 4$, from (4-7)–(4-8) we can apply the Picard–Banach fixed-point theorem in $C_T H^1 \cap L^p_T W^{1,q}$ for some $T(\|u_0\|_{H^1} + \|g\|_{H^{3/2+\varepsilon,2}(\partial\Omega\times[0,T])})$ and it also implies that the flow is Lipschitz. If $d \geq 4$, (4-7) implies that L sends some ball of X_T to itself, and from (4-9) it is contractive in the weaker space $C_T L^2 \cap L^{p_1}_T L^{q_1}$. By a standard argument, the metric space $\{u : \|u\|_{X_T} \leq M\}$ with distance $d(u, v) = \|u - v\|_{L^\infty_T L^2 \cap L^p_T L^q}$ is complete (e.g., [Cazenave 2003, Theorem 1.2.5]), so that the existence of a solution is again a consequence of the Picard–Banach fixed point theorem.

Blow-up alternative: This is a direct consequence of the fact that the time of local existence only depends on $\|u_0\|_{H^1} + \|g\|_{H^{3/2+\varepsilon}}$. Let u be a solution on $[0, T^*[$ such that $\underline{\lim}\|u(t)\|_{H^1} = C < \infty$ and let δ be such that $T(2C + \|g\|_{H^{3/2+\varepsilon,2}([T^*-1,T^*+1]\times\Omega)}) \geq 2\delta$. Up to decreasing δ, we can assume $\|u(T^*-\delta)\|_{H^1} \leq 2C$. Since $u \in C_T H^1$ and $u|_{\partial\Omega} = g$ the pair $u(T^*-\delta)$, $g|_{[T^*-\delta,+\infty[}$ satisfies (CC0) on $\partial\Omega \times \{T^*-\delta\}$, thus (NLS) has a local solution on the time interval $[T^*-\delta, T^*+\delta]$. Thanks to the uniqueness on $[T^*-\delta, T^*[$, this allows us to extend the solution on $[0, T^*+\delta]$. □

Remark 4.3. If one chooses to use instead the Strichartz estimate from Remark 3.8, namely

$$\|u\|_{L^p_T W^{1,q}} \lesssim \|u_0\|_{H^1} + \|g\|_{H^{3/2}} + \|f\|_{H^{1/2,1/4}} \quad \text{when } \frac{1}{p} + \frac{d}{q} = \frac{d}{2} + \frac{2\varepsilon}{p},$$

the restriction on α becomes (supposedly) $\alpha < (2-4\varepsilon)/(d-2)$. Consequently, well-posedness for the whole range $\alpha < 2/(d-2)$ and boundary data in the optimal space $H^{3/2,2}$ can most likely be obtained, up to more involved estimates with some ε in all indices.

Since our Strichartz estimates for the IBVP only give a gain of half a derivative, the natural limitation on the nonlinearity is $\alpha < 2/(d-2)$ (as in [Burq et al. 2004]). However better (scale-invariant) estimates are available for the homogeneous boundary value problem, and they can be combined with our estimates to improve the range of α. The following theorem illustrates this idea.

Theorem 4.4. *If Ω is the exterior of a smooth strictly convex obstacle, then Theorem 4.1 is true for $\alpha < 3/(d-2)$.*

Proof. From [Ivanovici 2010], the usual Strichartz estimates with (p,q) such that $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$, $p > 2$, are true for the semigroup $e^{it\Delta_D}$. The uniqueness in $L^\infty_T H^1$ follows from standard arguments; see, e.g., [Cazenave 2003, Section 4.2]. The existence part is again an application of the Picard–Banach fixed point theorem: let (p,q) be weakly admissible, $p > 2$, such that

$$\alpha < \frac{2}{d-2}\Big(1 + \frac{1}{p}\Big). \tag{4-10}$$

We set $X_T = C_T H^1 \cap L^p_T W^{1,q}$ and, as in Theorem 4.1,

$$L : w \mapsto L(w) = \int_0^t e^{i(t-s)\Delta_D}(F(\tilde{u}+w) - F(\tilde{g}))\,ds.$$

From the Sobolev embedding, $\tilde{g} \in H^{2,2} \hookrightarrow L^2 H^2 \cap C_T H^1 \hookrightarrow X_T$. Let q_1 be such that $\frac{2}{p} + \frac{d}{q_1} = \frac{d}{2}$. From the scale-invariant Strichartz estimates we have

$$\|Lw\|_{X_T} \lesssim \|Lw\|_{L^\infty_T H^1 \cap L^p_T W^{1,q_1}} \lesssim \|F(\tilde{u}+w) - F(\tilde{g})\|_{L^{p'}_T W^{1,q_1'} + L^1_T H^1},$$

and we will prove that there exists $\theta > 0$ such that

$$\|F(v)\|_{L^{p'}_T W^{1,q_1'} + L^1_T H^1} \lesssim T^\theta (1 + \|v\|_{X_T}^{1+3/(d-2)}). \tag{4-11}$$

Let $\psi \in C^\infty(\mathbb{R}^+)$ with $\psi \equiv 1$ for $x \geq 1$ and $\psi \equiv 0$ for $x \leq \frac{1}{2}$. Since $\operatorname{supp}(1-\psi(|v|^2)) \subset \{|v| \leq 1\}$, we have

$$\big\|1 - \psi(|v|^2)F(v)\big\|_{L^1_T H^1} \lesssim \|v\|_{L^1_T H^1} \leq T\|v\|_{X_T}.$$

On the other hand, for any $\beta \geq \alpha$,

$$\left|\psi(|v|^2)F(v)\right| \lesssim |v|^{1+\beta}, \quad \left|\nabla(\psi(|v|^2)F(v))\right| \lesssim |v|^{\beta}|\nabla v|.$$

Since

$$(1+\alpha)q_1' \leq \left(1+\frac{2}{d-2}\left(1+\frac{1}{p}\right)\right)\left(\frac{1}{2}+\frac{2}{dp}\right)^{-1} = \frac{2d}{d-2}\frac{dp+2}{dp+4} < \frac{2d}{d-2},$$

there exists $\beta \geq \alpha$ such that $2 \leq (1+\beta)q_1' \leq 2d/(d-2)$, and this choice leads to

$$\left\||v|^{1+\beta}\right\|_{L^{p'}L^{q_1'}} \lesssim \|v\|^{1+\beta}_{L^{(1+\beta)p'}L^{(1+\beta)q_1'}} \lesssim T^{1/p'}\|v\|^{1+\beta}_{L^\infty H^1}.$$

To estimate $\nabla(\psi(|v|^2)F(v))$, we use Hölder's inequality on $|v|^\beta \nabla v$ combined with the Sobolev embedding $W^{1,r} \hookrightarrow L^s$, $\frac{1}{s} = \frac{1}{r} - \frac{1}{d}$:

$$\left\||v|^\beta \nabla v\right\|_{L^{p'}L^{q_1'}} \lesssim \|v\|^{\beta}_{L^{\hat{p}}_T W^{1,\hat{q}}}\|\nabla v\|_{L^pL^q}, \tag{4-12}$$

where

$$\frac{1}{\hat{p}} = \frac{1}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right) = \frac{1}{\beta}\left(1-\frac{2}{p}\right) \qquad \text{(Hölder in time)},$$

$$\frac{1}{\hat{q}} = \frac{1}{\beta}\left(\frac{1}{q_1'}-\frac{1}{q}\right)+\frac{1}{d} = \frac{1}{d}\left(1+\frac{3}{\beta p}\right) \qquad \text{(Hölder in space and Sobolev embedding)}.$$

Note that q, $\hat{p}$, $\hat{q}$ are defined by p and β. If we can choose $p > 2$ and $\beta \geq \alpha$ such that

$$\frac{1}{\hat{p}}+\frac{d}{\hat{q}} > \frac{d}{2}, \quad \frac{1}{\hat{p}} < \frac{1}{2}, \quad \frac{1}{q} \leq \frac{1}{\hat{q}} \leq \frac{1}{2}, \tag{4-13}$$

this gives (4-11); indeed, for such p, β, if $1/p_1 + d/\hat{q} = d/2$ we have $L^{p_1}_T W^{1,\hat{q}} \subset X_T$, $1/p_1 < 1/\hat{p}$, and (4-12) gives

$$\|v\|^{\beta}_{L^{\hat{p}}_T W^{1,\hat{q}}}\|\nabla v\|_{L^pL^q} \lesssim T^{\beta(1/\hat{p}-1/p_1)}\|v\|^{\beta}_{L^{p_1}W^{1,\hat{q}}}\|\nabla v\|_{L^pL^q} \lesssim T^{\beta(1/\hat{p}-1/p_1)}\|v\|^{1+\beta}_{X_T}. \tag{4-14}$$

Let us now check that there exists a choice of β and p for which (4-13) holds. The first two conditions become

$$\frac{1}{\beta}\left(1-\frac{2}{p}\right)+\left(1+\frac{3}{\beta p}\right) > \frac{d}{2} \iff \frac{1}{p} > \beta\left(\frac{d}{2}-1\right)-1,$$

$$\frac{1}{\beta}\left(1-\frac{2}{p}\right) < \frac{1}{2} \iff \frac{1}{p} > \frac{1}{2}-\frac{\beta}{4}.$$

Or, more compactly,

$$\frac{1}{2} > \frac{1}{p} > \max\left(\frac{1}{2}-\frac{\beta}{4}, \beta\left(\frac{d}{2}-1\right)-1\right)$$

The condition $\frac{1}{2}-\frac{\beta}{4} < \frac{1}{2}$ is automatically satisfied. To ensure $1/q \leq 1/\hat{q} \leq \frac{1}{2}$, we must have

$$\frac{1}{\beta} \leq \frac{p(d-2)}{6} \quad \text{and} \quad \frac{1}{\beta} \geq \frac{p(d-2)}{6}-\frac{1}{3},$$

so that the condition is finally equivalent to

$$\beta\Big(\frac{d}{2}-1\Big)-1<\frac{1}{p}\le\frac{\beta(d-2)}{6},$$

and there exist solutions $p>2$, $\beta\ge\alpha$ if and only if $\beta<3/(d-2)$, which is always compatible with $\beta\ge\alpha$ and the initial assumption (4-10).

From (4-11), we infer

$$\|Lw\|_{X_T}\lesssim T^\theta\big(1+(\|\tilde u\|_{X_T}+\|w\|_{X_T}+\|\tilde g\|_{X_T})^{3/(d-2)}\big),$$

so that for T small enough, L maps the ball of radius one in X_T to itself. It is not clear if L is contractive in X_T even for smaller T, however contractivity for the weaker topology induced by $L^\infty_T L^2\cap L^pL^q$ is an easy consequence of the previous estimates and the assumptions on F:

$$|F(\tilde u+w_1)-F(\tilde u+w_2)|\lesssim|w_1-w_2|+(|w_1|+|w_2|+|\tilde u|)^\beta|w_1-w_2|,$$

and (4-14) gives

$$\begin{aligned}\|Lw_1-Lw_2\|_{X_T}&\\ &\lesssim\|w_1-w_2\|_{L^1_TL^2}+\big\||w_1-w_2|+(|w_1|+|w_2|+|\tilde u|)^\beta|w_1-w_2|\big\|_{L^{p'}_TL^{q'_1}}\\ &\lesssim T^{\beta(1/\hat p-1/p_1)}(\|\tilde u\|_{X_T}+\|w_1\|_{X_T}+\|w_2\|_{X_T})^\beta\|w_1-w_2\|_{L^p_TL^q}+T\|w_1-w_2\|_{L^\infty_TL^2}.\end{aligned}\tag{4-15}$$

As for Theorem 4.1, the contractivity of L for the $L^p_TL^q\cap L^\infty_TL^2$ topology and the mapping of a ball of X_T to itself gives the existence of a solution as a fixed point. □

Remark 4.5. The only thing limiting us to $\alpha<3/(d-2)$ is that $\tilde u$ only belongs to $C_TH^1\cap L^2W^{1,q_0}$ with $\frac{1}{2}+\frac{d}{q_0}=\frac{d}{2}$. If this limitation was lifted the fixed point argument on w could be performed in the usual scale-invariant spaces.

Remark 4.6. Theorem 4.4 is only an example of how one may mix optimal and nonoptimal Strichartz estimates. If Ω is only assumed to be the exterior of a nontrapping obstacle, [Blair et al. 2012] proved scale-invariant Strichartz estimates with loss of derivatives, namely

$$\|e^{it\Delta_D}u_0\|_{L^pL^q}\lesssim\|u_0\|_{H^\sigma}\quad\text{with}\quad\frac{2}{p}+\frac{d}{q}=\frac{d}{2}-\sigma,\ \frac{1}{p}+\frac{1}{q}\le\frac{1}{2}.$$

Such estimates could probably be used to improve the range of α if Ω^c is only star-shaped. Since the method seems similar and with numerous specific cases, we chose not to develop this issue.

Global well-posedness. In order to obtain global well-posedness for the defocusing nonlinear Schrödinger equation

$$\begin{cases}i\partial_tu+\Delta u=|u|^\alpha u, & (x,t)\in\Omega\times[0,T[,\\ u|_{t=0}=u_0, & x\in\Omega,\\ u|_{\partial\Omega\times[0,T]}=g, & (x,t)\in\partial\Omega\times[0,T[,\end{cases}\tag{NLSD}$$

the argument based on local well-posedness and conservation of energy cannot be trivially applied. Indeed we only have the formal identities

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|u|^2\,dx = -\operatorname{Im}\int_{\partial\Omega} \partial_n u\bar{g}\,dS, \tag{4-16}$$

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{\alpha+2}|u|^{\alpha+2}\,dx = \operatorname{Re}\int_{\partial\Omega} \partial_n u\partial_t\bar{g}\,dS \tag{4-17}$$

If $g\in H^{s,2}$, the control of $\|u\|_{C_TH^1}$ requires us to control $\|\partial_n u\|_{H^{2-s,2}}$. In particular, for the almost optimal regularity $s=\frac{3}{2}+\varepsilon$, we must have some control on $\partial_n u\in H^{1/2-\varepsilon,2}(\partial\Omega\times[0,T])$, which is its (almost) optimal space of regularity.

We will first deal with the simpler case $g\in H^{2,2}$; in this case we only need to control $\|\partial_n u\|_{L^2}$. This can be done thanks to a nonlinear variation of the virial identity from Proposition 3.1.

Theorem 4.7. (1) *For any $0<\alpha<2/(d-2)$, if $(u_0,g)\in H^1(\Omega)\times H^{2,2}_{\mathrm{loc}}(\mathbb{R}^+\times\partial\Omega)$ satisfy* (CC0), *then* (NLSD) *has a unique global solution $u\in C(\mathbb{R}^+,H^1)$.*

(2) *If Ω^c is strictly convex and there exists $\varepsilon>0$ such that $g\in H^{2+\varepsilon,2}$, then the theorem is true for $\alpha<4/(d-2)$.*

Proof. The case (1) is a simple consequence of the virial identity and the blow-up alternative, indeed the (nonlinear) virial identity writes

$$\begin{aligned}
&\frac{d}{dt}I(u(t))\\
&\quad= 4\operatorname{Re}\int_\Omega \operatorname{Hess}(h)(\nabla u,\overline{\nabla u})-\tfrac{1}{4}|u|^2\Delta^2 h+\nabla h\cdot\nabla u|u|^\alpha\bar{u}+\tfrac{1}{2}\bar{u}\Delta h|u|^\alpha u\,dx\\
&\qquad+\operatorname{Re}\int_{\partial\Omega} 2\partial_n h|\nabla_\tau g|^2-2\partial_n h|\partial_n u|^2-2i\,\partial_n h\partial_t g\bar{g}\,dS+\operatorname{Re}\int_{\partial\Omega} -2\bar{g}\Delta h\partial_n u+|g|^2\partial_n\Delta h\,dS\\
&\quad= 4\operatorname{Re}\int_\Omega \operatorname{Hess}(h)(\nabla u,\overline{\nabla u})-\tfrac{1}{4}|u|^2\Delta^2 h+|u|^{\alpha+2}\Delta h\Big(\frac{1}{2}-\frac{1}{\alpha+2}\Big)\,dx\\
&\qquad+\operatorname{Re}\int_{\partial\Omega} 2\partial_n h|\nabla_\tau g|^2-2\partial_n h|\partial_n u|^2-2i\,\partial_n h\partial_t g\bar{u}\,dS\\
&\qquad\quad+\operatorname{Re}\int_{\partial\Omega} -2\bar{g}\Delta h\partial_n u+|g|^2\partial_n\Delta h+\frac{|g|^{\alpha+2}}{\alpha+2}\partial_n h\,dS.
\end{aligned}$$

As for Proposition 3.2, we choose $h=\sqrt{1+|x|^2}$ so that $\operatorname{Hess}(h)$, $\Delta h>0$, $\partial_n h\le 0$ and integrate in time. From the embedding $H^{2,2}(\partial\Omega\times[0,T])\hookrightarrow H_T^{2/(d+1)}H^{(2d-2)/(d+1)}\hookrightarrow L^{2(d+1)/(d-3)}(\partial\Omega\times[0,T])$ (or L^∞ if $d=2$, L^p for any $2\le p<\infty$ if $d=3$) we have

$$\int_0^T\int_{\partial\Omega} |g|^{\alpha+2}\,dS\,dt \lesssim \|g\|^{\alpha+2}_{H^{2,2}(\partial\Omega\times[0,T])}.$$

If K is a compact neighbourhood of $\partial\Omega$, we deduce

$$\int_{K\times[0,T]} |\nabla u|^2+|u|^{\alpha+2}\,dx\,dt-\int_{\partial\Omega\times[0,T]} |\partial_n u|^2 x\cdot n\,dS\,dt \le M(T)(1+\|u\|^2_{C_TH^1}+\|g\|^{\alpha+2}_{H^{2,2}}).$$

If $x \cdot n < 0$ on $\partial\Omega$, this gives directly a control of $\|\partial_n u\|_{L^2}$; if not then we can argue as in Proposition 3.2 by using some function h compactly supported in K such that $\partial_n h < 0$. For this choice, Δh and $\mathrm{Hess}(h)$ are no longer signed, but using the estimate $\|u\|^{\alpha+2}_{L^{\alpha+2}([0,T]\times K)} \lesssim 1 + \|u\|^2_{C_T H^1} + \|g\|^{\alpha+2}_{H^{2,2}}$ we get

$$\|\partial_n u\|_{L^2} \le M(T)(1 + \|u\|_{C_T H^1} + \|g\|^{\alpha/2+1}_{H^{2,2}}).$$

Plugging this in the "conservation" laws (4-16)–(4-17) implies

$$\|u\|^2_{C_T H^1} \le \|u_0\|^2_{H^1} + \|\partial_n u\|_{L^2}\|g\|_{H^{2,2}} \lesssim 1 + \|u_0\|^2_{H^1} + (\|u\|_{C_T H^1} + \|g\|^{\alpha/2+1}_{H^{2,2}})\|g\|_{H^{2,2}}$$

and thus

$$\tfrac{1}{2}\|u\|^2_{C_T H^1} \lesssim 1 + \|u_0\|^2_{H^1} + \|g\|^{\alpha/2+2}_{H^{2,2}(\partial\Omega\times[0,T])}.$$

As a consequence, u remains locally bounded in H^1 and the solution must be global.

The case (2) is a bit more intricate, indeed even the local existence of a solution for $3/(d-2) \le \alpha < 4/(d-2)$ has not been covered yet. The main argument is that we can modify $\tilde{u}$ from problem (4-4) so that it belongs to $C_T H^1 \cap L^2_T W^{1,q_0}$, $1 + d/q_0 = d/2$: since $g \in H^{2+\varepsilon,2}$, we have from (CC0) that $u_0|_{\partial\Omega} = g|_{t=0} \in H^{1+\varepsilon,2}$. Let $v_0 \in H^{3/2+\varepsilon}(\Omega)$ be a lifting of $u_0|_{\partial\Omega}$; we define $\tilde{v}$ as the solution of the linear IBVP

$$\begin{cases} i\partial_t \tilde{v} + \Delta\tilde{v} = F(\tilde{g}), \\ \tilde{v}|_{t=0} = v_0, \\ \tilde{v}|_{\partial\Omega\times[0,T]} = g. \end{cases}$$

Since $F(\tilde{g}) \in H^{1,2}$ (see Lemma 4.2), $g \in H^{2+\varepsilon,2}$, $v_0 \in H^{3/2}$, the Strichartz estimates imply $\tilde{v}$ is in $L^2_T W^{3/2,q} \hookrightarrow L^2_T W^{1,q_0}$, where $1 + d/q_0 = d/2$. We are now left to solve the homogeneous boundary value problem

$$\begin{cases} i\partial_t w + \Delta w = F(\tilde{v} + w) - F(\tilde{g}), \\ w|_{t=0} = u_0 - v_0 \in H^1_0, \\ w|_{\partial\Omega\times[0,T]} = 0, \end{cases}$$

or equivalently obtain a fixed point to the map

$$Lw = e^{it\Delta_D}(u_0 - v_0) + \int_0^t e^{i(t-s)\Delta_D}(F(\tilde{v} + w) - F(\tilde{g}))\,ds.$$

Since $\tilde{v}, \tilde{g} \in L^\infty_T H^1 \cap L^2_T W^{1,q_0}$, the fixed point argument can be done as in the $\mathbb{R}^d$ case, e.g., [Cazenave 2003, Section 4.4], leading to local existence. We can still use the virial identity as in case (1) since $\alpha + 2 < (d+2)/(d-2) < 2(d+1)/(d-3)$, and the energy argument is ended in the same way. □

If we only assume $\tilde{g} \in H^{3/2+\varepsilon,2}$, global existence becomes a much more delicate issue since we need to control $\|\partial_n u\|_{H^{1/2,2}}$. Let us sketch the main issue: the linear smoothing gives a control $\|\partial_n u\|_{H^{1/2,2}} \lesssim \|u_0\|_{H^1} + \|g\|_{3/2+\varepsilon,2} + \|f\|_{H^{1/2,2}}$, where $f = |u|^\alpha u$ has scaling $1+\alpha$. In order to estimate the time regularity of f we need to again use the equation, which adds another power α to the scaling. Using various chain rules, the conservation laws (4-16)–(4-17) should give at best $\|u\|^2_{C_T H^1} \lesssim \prod \|u\|^{\alpha_j}_{X_j}$, where $\sum \alpha_j = 1 + 2\alpha$ and, for all j, $X_j \hookrightarrow C_T H^1$. Eventually, $\|u\|^2_{C_T H^1} \lesssim \|u\|^\beta_{C_T H^1}$ for some β depending on α, and this allows us to close the estimate if $\beta < 2$. It is clear that such an approach will be limited to small values of α. Nevertheless, this is the method used in the following theorem, where the restriction on α is of course purely technical.

Theorem 4.8. *For $d = 2$, $\frac{1}{2} \le \alpha < \frac{11}{9}$, and $(u_0, g) \in H^1 \times H^{3/2+\varepsilon,2}$ satisfying the compatibility conditions, the problem (NLSD) has a unique global solution in $C(\mathbb{R}^+, H^1)$.*

Proof. The existence of a maximal solution is Theorem 4.1; it remains to prove that u is locally bounded in H^1. In this proof, $\lesssim$ means that the inequality is true up to a multiplicative constant that may depend on T, g *and an additive constant* that may depend on T, g and u_0. We use δ as a placeholder for some positive quantity that can be chosen arbitrarily small.

As in Theorem 4.7, we can use the nonlinear virial identity provided $g \in L^{\alpha+2}(\partial\Omega \times [0,T])$, which is ensured by $H^{3/2,2} \hookrightarrow H_T^{1/2} H^{1/2}(\partial\Omega) \hookrightarrow L^p(\partial\Omega \times [0,T])$ for any $2 \le p < \infty$. From the nonlinear virial identity we obtain

$$\|\partial_n u\|_{L^2_T L^2} + \|\nabla u\|_{L^2_T L^2} \lesssim \|u\|^{1/2}_{C_T H^1} \|u\|^{1/2}_{C_T L^2} + \|g\|^{1+\alpha/2}_{H^{3/2+\varepsilon,2}} \lesssim \|u\|^{1/2}_{C_T H^1} \|u\|^{1/2}_{C_T L^2}; \tag{4-18}$$

plugging this in (4-16) gives

$$\|u\|^2_{C_T L^2} \lesssim \|\partial_n u\|_{L^2_T L^2} \|g\|_{L^2_T L^2} \lesssim (\|u\|^{1/2}_{C_T H^1} \|u\|^{1/2}_{C_T L^2} + \|g\|_{H^{3/2+\varepsilon,2}}) \|g\|_{L^2_T L^2},$$

thus

$$\|u\|_{C_T L^2} \lesssim \|u\|^{1/3}_{C_T H^1}, \tag{4-19}$$

$$\text{and} \quad \|u\|_{L^2_T H^1_{\text{loc}}} \lesssim \|u\|^{1/2+1/6}_{C_T H^1} = \|u\|^{2/3}_{C_T H^1}. \tag{4-20}$$

For later use, let us note that Hölder's inequality and the Sobolev embedding $H^1 \hookrightarrow L^r$ for $2 \le r < \infty$ imply

$$\|u\|_{L^q} \lesssim \|u\|^{1-2/q+\delta}_{H^1} \|u\|^{2/q-\delta}_{L^2} \quad \text{for all } q > 2,\ 0 < \delta < 2/q. \tag{4-21}$$

On the other hand, (4-16)–(4-17) give

$$\|u\|^2_{C_T H^1} + \|u\|^{\alpha+2}_{L^{\alpha+2}} \lesssim \|u_0\|^2_{H^1} + \|g\|_{H^{3/2+\varepsilon,2}} \|\partial_n u\|_{H^{1/2,2}}. \tag{4-22}$$

To estimate $\partial_n u$, we fix $\chi \in C_c^\infty(\overline{\Omega})$ such that $\chi \equiv 1$ on a neighbourhood of $\partial\Omega$, and split $u = u_1 + u_2$, where u_1 and u_2 are solutions of

$$\begin{cases} i\partial_t u_1 + \Delta u_1 = \chi |u|^\alpha u, \\ u_1|_{t=0} = u_0, \\ u_1|_{\partial\Omega\times[0,T]} = g, \end{cases} \quad \text{and} \quad \begin{cases} i\partial_t u_2 + \Delta u_2 = (1-\chi)|u|^\alpha u, \\ u_2|_{t=0} = 0, \\ u_2|_{\partial\Omega\times[0,T]} = 0. \end{cases}$$

Corollary 3.4 gives

$$\|\partial_n u_1\|_{H^{1/2,2}} \lesssim \|u_0\|_{H^1} + \|g\|_{H^{3/2+\varepsilon,2}} + \|\chi |u|^\alpha u\|_{H^{1/2,2}}.$$

We estimate the nonlinear term using $H^1 \hookrightarrow B^{1/2}_{4,2}$ [Triebel 1983, Section 3.3] and (4-19)–(4-20):

$$\|\chi|u|^\alpha u\|_{L^2_T H^{1/2}} \lesssim \big\||u|^\alpha\big\|_{L^\infty_T L^4_{\mathrm{loc}}}\|u\|_{L^2_T B^{1/2,4}_{2,\mathrm{loc}}} \lesssim \|u\|^{\alpha-1/2+\delta}_{L^\infty H^1}\|u\|^{1/2-\delta}_{L^\infty_T L^2}\|u\|_{L^2_T H^1_{\mathrm{loc}}} \lesssim \|u\|^{\alpha+1/3+\delta}_{C_T H^1}. \tag{4-23}$$

For the time regularity, we use the composition rules and interpolation of anisotropic Sobolev spaces [Lions and Magenes 1968b, chapitre 4, paragraphe 2.1]. For $\tilde\chi$ such that $\tilde\chi = 1$ on $\operatorname{supp}\chi$,

$$\|\chi|u|^\alpha u\|_{H^{1/4}L^2} \lesssim \big\||u|^\alpha\big\|_{L^\infty_T L^4}\|u\|_{H^{1/4}_T L^4_{\mathrm{loc}}} \lesssim \|u\|^\alpha_{L^\infty_T L^{4\alpha}}\|\tilde\chi u\|_{H^{1/4}_T H^{1/2}} \lesssim \|u\|^\alpha_{L^\infty_T L^{4\alpha}}\|\tilde\chi u\|^{1/2}_{H^{1/2}_T L^2}\|\tilde\chi u\|^{1/2}_{L^2_T H^1}.$$

Since $i\partial_t\tilde\chi u + \Delta\tilde\chi u = \tilde\chi|u|^\alpha u + [\Delta,\tilde\chi]u$, we have

$$\|\partial_t\tilde\chi u\|_{L^2_T H^{-1}} \lesssim \|\tilde\chi u\|_{L^2_T H^1} + \|\tilde\chi|u|^\alpha u\|_{L^2_T H^{-1}} + \|u\|_{L^\infty_T L^2},$$

and since $H^{-1} \supset L^q$ for $1 < q \le 2$ we get

$$\|\partial_t\tilde\chi u\|_{L^2_T H^{-1}} \lesssim \|u\|^{2/3}_{L^\infty_T H^1} + \|\tilde\chi|u|^\alpha u\|_{L^2_T L^{2/(1+\alpha)}} \lesssim \|u\|^{2/3}_{L^\infty_T H^1} + \|u\|^{(1+\alpha)/3}_{L^\infty_T H^1}.$$

Next we use $\|\tilde\chi u\|_{H^{1/2}_T L^2} \lesssim \|\tilde\chi u\|^{1/2}_{H^1_T H^{-1}}\|\tilde\chi u\|^{1/2}_{L^2_T H^1}$, so that

$$\|\tilde\chi u\|_{H^{1/2}_T L^2} \lesssim \big(\|u\|^{2/3}_{L^\infty_T H^1} + \|u\|^{(1+\alpha)/3}_{L^\infty_T LH^1}\big)^{1/2}\|\tilde\chi u\|^{1/2}_{L^2_T H^1} \lesssim \|u\|^{2/3}_{L^\infty_T H^1} + \|u\|^{(3+\alpha)/6}_{L^\infty_T H^1}.$$

This implies, using (4-19)–(4-21),

$$\|\chi|u|^\alpha u\|_{H^{1/4}L^2} \lesssim \|u\|^\alpha_{L^\infty_T L^{4\alpha}}\big(\|u\|^{1/3}_{L^\infty_T H^1} + \|u\|^{(3+\alpha)/12}_{L^\infty_T H^1}\big)\|u\|^{1/3}_{L^\infty_T H^1} \lesssim \|u\|^{1/3+\alpha+\delta}_{L^\infty_T H^1} + \|u\|^{13\alpha/12+1/4+\delta}_{L^\infty_T H^1}.$$

Combining the estimate above with (4-23) gives the following estimate on $\partial_n u_1$:

$$\|\partial_n u_1\|_{H^{1/2,2}} \lesssim \|u\|^{1/3+\alpha+\delta}_{C_T H^1} + \|u\|^{13\alpha/12+1/4+\delta}_{C_T H^1}. \tag{4-24}$$

We now treat $\partial_n u_2$. The situation is less favourable since we can not use the smoothing property $\|\chi u\|_{L^2_T H^1} \lesssim \|u\|^{2/3}_{L^\infty_T H^1}$. In particular we only have

$$\|(1-\chi)u\|_{H^{1/2}_T L^2} \lesssim \|u\|_{L^\infty_T H^1} + \|u\|^{(4+\alpha)/6}_{L^\infty_T H^1}. \tag{4-25}$$

Using Proposition 3.6, we have

$$\|\partial_n u_2\|_{H^{1/2,2}(\partial\Omega\times[0,T])} \lesssim \|(1-\chi)|u|^\alpha u\|_{L^{3/2}B^1_{3/2,2}\cap B^{1/2}_{3/2,2}L^{3/2}}.$$

For the first norm we write

$$\begin{aligned}\|(1-\chi)|u|^\alpha u\|_{L^{3/2}B^1_{3/2,2}} &\lesssim \|(1-\chi)|u|^\alpha u\|_{L^\infty_T W^{1,3/2}}\\ &\lesssim \|u\|^\alpha_{L^\infty_T L^{6\alpha}}\|u\|_{L^\infty_T H^1}\\ &\lesssim \|u\|^{1/3}_{L^\infty_T L^2}\|u\|^{\alpha-1/3+\delta}_{L^\infty_T H^1}\|u\|_{L^\infty_T H^1}\\ &\lesssim \|u\|^{\alpha+7/9+\delta}_{C_T H^1}.\end{aligned}$$

For the other norm, the composition rules and (4-25) give similarly

$$\begin{aligned}\|(1-\chi)|u|^\alpha u\|_{B^{1/2}_{3/2,2}L^{3/2}} &\lesssim \|u\|^\alpha_{L^{6\alpha}_T L^{6\alpha}}\|u\|_{H^{1/2}_T L^2}\\ &\lesssim \|u\|^{\alpha-2/9+\delta}_{C_T H^1}(\|u\|_{C_T H^1}+\|u\|^{(4+\alpha)/6}_{C_T H^1})\\ &= \|u\|^{\alpha+7/9+\delta}_{C_T H^1}+\|u\|^{7\alpha/6+4/9+\delta}_{C_T H^1},\end{aligned}$$

so that

$$\|\partial_n u_2\|_{H^{1/2,2}} \lesssim \|u\|^{\alpha+7/9+\delta}_{L^\infty_T H^1}+\|u\|^{7\alpha/6+4/9}_{L^\infty_T H^1}.$$

Combining this estimate with (4-24) in (4-22), we finally obtain (as previously, $\lesssim$ still means "up to multiplicative and additive quantities only depending on T and the data")

$$\|u\|^2_{C_T H^1} \lesssim \|u\|^\beta_{C_T H^1},$$

with $\beta = \max\big(\frac{1}{3}+\alpha, 13\alpha/12+\frac{1}{4}, \alpha+\frac{7}{9}, 7\alpha/6+\frac{4}{9}\big)+\delta$. If $\beta<2$ then $\|u(t)\|_{H^1}$ is locally bounded, and hence the solution is global. The condition $\beta<2$ is equivalent to $\alpha<\frac{11}{9}$. □

Appendix: Two interpolation lemmas

In this section we give two results on the interpolation of Sobolev spaces. They do not seem standard as they involve compatibility conditions in some way. We do not claim that these results are new, however we did not find them in the literature, thus we decided to include reasonably self-contained proofs.

Definition A.1 (real interpolation). If X_0, X_1 are two functional spaces embedded in $\mathcal{D}'(\Omega)$, we define, for $u \in X_0+X_1$,

$$K(t,u) = \inf_{u=u_0+u_1\in X_0+X_1} \|u_0\|_{X_0}+t\|u_1\|_{X_1}.$$

For $0<\theta<1$, the interpolated space $[X_0, X_1]_{\theta,q}$ is the set of functions such that

$$\int_0^\infty |K(t,u)^q| \frac{dt}{t^{1+\theta q}} < \infty.$$

Lemma A.2. *Let*

$$X^\theta = \big\{(u_0, g) \in H^{-1/2+2\theta}\times H^{2\theta,\theta}(\partial\Omega\times[0,T]) \text{ that satisfy the compatibility conditions}\big\},$$

where for $\theta = 0$ we take $(H_D^{1/2})'$ instead of $H^{-1/2}$. Then, for $0 \le \theta \le 1$,

$$[X^0, X^1]_\theta = X^\theta.$$

Remark A.3. While it is a bit tedious, the case $\theta = \frac{1}{2}$ really needs to be treated, as it corresponds to the natural space for the virial estimates.

Proof. We clearly have

$$H_0^{3/2}(\Omega) \times H_0^{2,2}(\partial\Omega \times [0,T]) \subset X^1 \subset H^{3/2}(\Omega) \times H^{2,2}(\partial\Omega \times [0,T]).$$

The interpolation of Sobolev spaces [Lions and Magenes 1968a; Lions and Magenes 1968b, chapitres 1, 4], gives, for $\theta < \frac{1}{2}$,

$$[(H_D^{1/2})'(\Omega), H_0^{3/2}]_\theta = H^{2\theta-1/2}, \quad [H^{0,0}(\partial\Omega \times [0,T]), H_0^{2,2}]_\theta = H^{2\theta,2},$$
$$[(H_D^{1/2})'(\Omega), H^{3/2}]_\theta = H^{2\theta-1/2}, \quad [H^{0,0}(\partial\Omega \times [0,T]), H^{2,2}]_\theta = H^{2\theta,2};$$

the two left-hand identities are not explicitly written in [Lions and Magenes 1968a], however $(H_D^{1/2})'$ does not cause any new difficulty since it can be bypassed using $(H_D^{1/2})' = [H^{-1}, H^2]_{1/6} = [H^{-1}, H_D^2]_{1/6}$ [Lions and Magenes 1968a, paragraphes 12.3, 12.4], and the reiteration theorem $[[X,Y]_{\theta_0}, [X,Y]_{\theta_1}]_\theta = [X,Y]_{(1-\theta)\theta_0+\theta\theta_1}$. We deduce that, for $0 < \theta < \frac{1}{2}$,

$$X^\theta = H^{2\theta-1/2} \times H^{2\theta,\theta} \subset [X^0, X^1]_\theta \subset X^\theta.$$

For $\theta \ge 1/2$ we first apply the Lions–Peetre reiteration theorem

$$[X^0, X^1]_\theta = [[X^0, X^1]_{3/8}, [X^0, X^1]_1]_{8\theta/5-3/5} = [X^{3/8}, X^1]_{8\theta/5-3/5},$$

so that we are reduced to proving $[X^{3/8}, X^1]_\theta = X^{(5\theta+3)/8}$ for $\frac{1}{5} < \theta < 1$. To this end, we use the existence of a lifting operator *independent of* $\frac{1}{4} < s \le 1$,[3]

$$R : X^s \to H^{2s+1/2, s+1/4}(\Omega \times [0,T]),$$
$$(u_0, g) \mapsto u \quad \text{such that } u|_{\Omega\times[0,T]} = g,\ u|_{t=0} = u_0,$$

Such an operator can be constructed as follows: for any $(g, u_0) \in X^s$, there exists a map

$$R_1 : H^{2s,s}(\partial\Omega \times [0,T]) \to H^{2s+1/2, s+1/4}(\Omega \times [0,T]),$$
$$g \mapsto R_1 g;$$

on the half space, $\mathcal{F}_{x',t} R_1 b = \hat{g}(\xi,\tau)\varphi(\sqrt{1+|\xi'|^2+|\tau|^2}x_d)$ with $\varphi(0) = 1$, φ smooth enough, works. There is also a map

$$R_2 : H_D^{2s-1/2}(\Omega) \to H_D^{2s+1/2, s+1/4}(\Omega \times \mathbb{R}),$$
$$u_0 \mapsto R_2 u_0;$$

[3] R is usually called a coretraction of the trace operator $u \mapsto (u|_{t=0}, u|_{\partial\Omega\times[0,T]})$.

in this case, one might take $R_2(u_0) = \varphi((1-\Delta_D)t)u_0$ (this is a very special case of [Lions and Magenes 1968a, chapitre 1, théorème 4.2]; see also [Lions and Magenes 1968b, chapitre 4, théorème 2.3]). With these two operators, we can now define

$$R(u_0, g) = R_2(u_0 - R_1(g)|_{t=0}) + R_1(g);$$

R is a continuous map $X^s \to H^{2s+\frac{1}{2},2}$ for $s > \frac{1}{4}$, since $u_0 - R_1 g|_{t=0} \in H_D^{2s-1/2}$. For $s > \frac{1}{2}$ this is a consequence of $H_D^s = H_0^s$ and (CC0), while for $s = \frac{1}{2}$ this comes from $H_D^{1/2} = H_{00}^{1/2}$ and (CCG0). We can conclude by introducing

$$T : H^{2s+1/2,2}(\Omega \times [0,T]) \to H^{2s-1/2}(\Omega) \times H^{2s,2}(\partial\Omega \times [0,T]),$$
$$u \mapsto (u|_{t=0}, u|_{\partial\Omega\times[0,T]}).$$

By construction, $T \circ R = \mathrm{Id}$ on $X^{3/8}$ and X^1, so that $[X^{3/8}, X^1]_\theta = T([H^{5/4,5/8}, H^{5/2,5/4}]_\theta)$. From basic results on anisotropic Sobolev spaces [Lions and Magenes 1968b, chapitre 4, proposition 2.1, théorème 2.3] we obtain, as expected,

$$T([H^{5/4,2}(\Omega \times [0,T]), H^{5/2,2}]_\theta) = T(H^{(5\theta+5)/4,2}) = X^{(5\theta+3)/8}. \qquad \square$$

Let $H^{2,2}_{(0)}(\Omega \times \mathbb{R}_t) = \{u \in H^{2,2}(\Omega \times [0,T]) : u|_{\partial\Omega\times\{0\}} = 0\}$.

Proposition A.4. *For $\theta < \frac{3}{4}$, $[L^2, H^{2,2}_{(0)}]_{\theta,2} = H^{2\theta,2}$.*

The result is to be expected, since the trace on $t = 0$ sends $H^{2\theta,2}(\partial\Omega \times [0,T])$ to $H^{2\theta-1}(\Omega)$, for which there is a trace on $\partial\Omega$ if and only if $2\theta - 1 > \frac{1}{2}$, or equivalently $\theta > \frac{3}{4}$.

Proof. The inclusion $\subset$ is obvious; we focus on the reverse inclusion.

Let R be the restriction operator $H^{2\theta,2}(\mathbb{R}^d \times [0,T]) \to H^{2\theta,2}(\Omega \times [0,T])$; since R is continuous for $0 \le \theta \le 1$ and surjective with value to $H^{2\theta,2}$, we only need to check that for $H^{2,2}_{(0),\partial\Omega}(\mathbb{R}^d \times \mathbb{R}_t) = \{u \in H^{2,2} : u|_{\partial\Omega\times\{0\}} = 0\}$ we have

$$[L^2, H^{2,2}_{(0),\partial\Omega}]_\theta = H^{2\theta,2}(\mathbb{R}^d \times \mathbb{R}_t) \quad \text{for all } \theta < \tfrac{3}{4} \tag{A-1}$$

Using a partition of the unity, we can reduce the problem to the case $\partial\Omega = \mathbb{R}^{d-1} \times \{0\}$ and for conciseness we write $H^{2,2}_{(0),\partial\Omega}(\mathbb{R}^d \times \mathbb{R}_t) = H^{2,2}_{(0)}$. Let $u \in H^{2\theta,2}(\mathbb{R}^d \times \mathbb{R}_t)$; then, since $L^2 \subset H^{2,2}$, it is easily seen from Definition A.1 that $u \in [L^2, H^{2,2}_{(0)}]_{\theta,2}$ if

$$\sum_{j=0}^{\infty} 2^{4\theta j} K(2^{-2j}, u)^2 < \infty, \quad \text{where} \quad K(t,u) = \inf_{u=u_0+u_1 \in L^2 + H^{2,2}_{(0)}} \|u_0\|_{L^2} + t\|u_1\|_{H^{2,2}_{(0)}}. \tag{A-2}$$

We define an anisotropic Littlewood–Paley decomposition as follows: the dual variables of x and t are $(\xi, \tau) = (\xi', \xi_d, \tau)$, and we set $u = \sum_{j\ge0} \Delta_j u(x,t)$, where, for $j \ge 1$, $\widehat{\Delta_j u}(\xi,\tau)$ is supported in $(|\xi|^2 + |\tau|)^{1/2} \sim 2^j$, $\widehat{\Delta_0 u}$ is supported in $|\xi|^2 + |\tau| \le 1$, and we set $S_j u = \sum_{k=0}^{j} \Delta_k u$, $R_j u = u - S_j u$. From the Plancherel theorem and $\int_{\mathbb{R}^d} \Delta_j u \Delta_l u = 0$ for $|j - l|$ large enough ("almost orthogonality"), we have

$$\|\Delta_j u\|_{H^{2,2}} \sim \|\Delta_j u\|_{L^2} 2^{2j} \implies \|u\|^2_{H^{2,2}} \sim \sum_{j\geq 0} 2^{4j} \|\Delta_j u\|^2_{L^2}. \tag{A-3}$$

Let us write

$$u = (R_j u + S_j u(x', 0, 0)\psi_j(x_d, t)) + (S_j u - S_j u(x', 0, 0)\psi_j(x_d, t)) = u_0 + u_1,$$

where $\widehat{\psi}_j = c_j 2^{-3j} 1_{(|\xi_d|^2+|\tau|)^{1/2} \sim 2^j}$ with c such that $\psi_j(0) = 1$. Since $\mathrm{vol}((|\xi_d|^2 + |\tau|)^{1/2} \sim 2^j) \sim 2^{3j}$, c_j is uniformly bounded in j. For this choice it is clear that $(u_0, u_1) \in L^2 \times H^{2,2}_{(0)}$. The decomposition $u = S_j u + R_j u$ would correspond to the standard interpolation $[L^2, H^{2,2}]_\theta$, thus we will only focus on how to estimate in (A-2)

$$\|S_j u(x', 0, 0)\psi_j(x_d, t)\|_{L^2} + 2^{-2j} \|S_j u(x', 0, 0)\psi_j(x_d, t)\|_{H^{2,2}}.$$

We first note that

$$\mathcal{F}(S_j u(x', 0, 0)\psi_j(x_d, t)) = \widehat{\psi}_j(\xi_d, \tau) \int_{\mathbb{R}^2} \widehat{S_j u}(\xi', \eta, \delta)\, d\eta\, d\delta,$$

so that $\mathcal{F}(S_j u(x', 0, 0)\psi_j(x_d, t))$ is supported in $(|\xi|^2 + |\tau|)^{1/2} \lesssim 2^j$. We deduce

$$\begin{aligned} 2^{-2j} \|S_j u(x', 0, 0)\psi_j(x_d, t)\|_{H^{2,2}} + \|S_j u(x', 0, 0)\psi_j(x_d, t)\|_{L^2} &\lesssim \|S_j u(x', 0, 0)\psi_j(x_d, t)\|_{L^2} \\ &\lesssim \|\psi_j\|_{L^2} \int_{\mathbb{R}^2} \|\widehat{S_j u}(\xi', \eta, \delta)\|_{L^2_{\xi'}}\, d\eta\, d\delta. \end{aligned}$$

Again using $\mathrm{vol}((|\xi_d|^2 + |\tau|)^{1/2} \sim 2^j) \sim 2^{3j}$, we have $\|\psi_j\|_{L^2} \sim 2^{-3j} 2^{3j/2} = 2^{-3j/2}$. Moreover, $\Delta_k u(\xi', \eta, \delta)$ is supported in $(|\eta|^2 + |\delta|)^{1/2} \lesssim 2^k$ independently of ξ', thus the Cauchy–Schwartz inequality implies

$$\int_{\mathbb{R}^2} \|\widehat{S_j u}(\xi', \eta, \delta)\|_{L^2_{\xi'}}\, d\eta\, d\delta \leq \int_{\mathbb{R}^2} \sum_{k=0}^{j} \|\Delta_k u(\xi', \eta, \delta)\|_{L^2_{\xi'}}\, d\eta\, d\delta \lesssim \sum_{k=0}^{j} \|\Delta_k u\|_{L^2} 2^{3k/2}.$$

Plugging this in (A-2) (and omitting the estimate on $S_j u$, $R_j u$),

$$\begin{aligned} \sum_{j=0}^{\infty} 2^{4\theta j} K(2^{-2j}, u)^2 &\lesssim \sum_{j=0}^{\infty} 2^{(4\theta-3)j} \left(\sum_{k=0}^{j} \|\Delta_k u\|_{L^2} 2^{2\theta k} 2^{(3/2-2\theta)k} \right)^2 \\ &\lesssim \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \|\Delta_k u\|_{L^2} 2^{2\theta k} 2^{(3/2-2\theta)(k-j)} \right)^2 \\ &= \|a * b\|^2_{l^2}, \end{aligned}$$

where $(a_k)_{k\geq 0} = (\|\Delta_k u\|_{L^2} 2^{2\theta k})_{k\geq 0} \in l^2$ and $(b_k)_{k\geq 0} = (2^{(2\theta-3/2)k})_{k\geq 0} \in l^1$, we can conclude by Young's inequality and (A-3) that

$$\sum_{0}^{\infty} 2^{4\theta j} K(2^{-2j,u})^2 \lesssim (\|a\|_{l^2} \|b\|_{l^1})^2 \lesssim \|u\|^2_{H^{2\theta,2}},$$

thus $H^{2\theta,2} \subset [L^2, H^{2,2}_{(0)}]_\theta$. □

Remark A.5. Using a similar argument, it is not difficult to check that $[L^2, H^{2,2}_{(0)}]_{\theta,2} = H^{2\theta,2}_{(0)}$ for $\theta > \frac{3}{4}$. Of course the identification in the case $\theta = \frac{3}{4}$ is less clear.

Acknowledgement

The author has been partially funded by the ANR project BoND ANR-13-BS01-0009-01.

References

[Amann 2009] H. Amann, *Anisotropic function spaces and maximal regularity for parabolic problems, I: Function spaces*, Jindřich Nečas Center for Mathematical Modeling Lecture Notes **6**, Matfyzpress, Prague, 2009. MR 2907677 Zbl 1216.35001

[Anton 2008a] R. Anton, "Global existence for defocusing cubic NLS and Gross–Pitaevskii equations in three dimensional exterior domains", *J. Math. Pures Appl.* (9) **89**:4 (2008), 335–354. MR 2009f:35312 Zbl 1148.35081

[Anton 2008b] R. Anton, "Strichartz inequalities for Lipschitz metrics on manifolds and nonlinear Schrödinger equation on domains", *Bull. Soc. Math. France* **136**:1 (2008), 27–65. MR 2009d:35303 Zbl 1157.35100

[Audiard 2012] C. Audiard, "Non-homogeneous boundary value problems for linear dispersive equations", *Comm. Partial Differential Equations* **37**:1 (2012), 1–37. MR 2864804 Zbl 1246.35048

[Audiard 2013] C. Audiard, "On the non-homogeneous boundary value problem for Schrödinger equations", *Discrete Contin. Dyn. Syst.* **33**:9 (2013), 3861–3884. MR 3038043 Zbl 1285.35104

[Bergh and Löfström 1976] J. Bergh and J. Löfström, *Interpolation spaces: an introduction*, Grundlehren der Math. Wissenschaften **223**, Springer, Berlin-New York, 1976. MR 58 #2349 Zbl 0344.46071

[Blair et al. 2008] M. D. Blair, H. F. Smith, and C. D. Sogge, "On Strichartz estimates for Schrödinger operators in compact manifolds with boundary", *Proc. Amer. Math. Soc.* **136**:1 (2008), 247–256. MR 2008k:35386 Zbl 1169.35012

[Blair et al. 2012] M. D. Blair, H. F. Smith, and C. D. Sogge, "Strichartz estimates and the nonlinear Schrödinger equation on manifolds with boundary", *Math. Ann.* **354**:4 (2012), 1397–1430. MR 2993000 Zbl 1262.58015

[Burq et al. 2004] N. Burq, P. Gérard, and N. Tzvetkov, "On nonlinear Schrödinger equations in exterior domains", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21**:3 (2004), 295–318. MR 2005g:35264 Zbl 1061.35126

[Cazenave 2003] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics **10**, Courant Institute of Mathematical Sciences, New York, 2003. MR 2004j:35266 Zbl 1055.35003

[Colliander et al. 2008] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$", *Ann. of Math.* (2) **167**:3 (2008), 767–865. MR 2009f:35315 Zbl 1178.35345

[Constantin and Saut 1988] P. Constantin and J.-C. Saut, "Local smoothing properties of dispersive equations", *J. Amer. Math. Soc.* **1**:2 (1988), 413–439. MR 89d:35150 Zbl 0667.35061

[Glassey 1977] R. T. Glassey, "On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations", *J. Math. Phys.* **18**:9 (1977), 1794–1797. MR 57 #842 Zbl 0372.35009

[Holmer 2005] J. Holmer, "The initial-boundary-value problem for the 1D nonlinear Schrödinger equation on the half-line", *Differential Integral Equations* **18**:6 (2005), 647–668. MR 2006d:35258 Zbl 1212.35448

[Ivanovici 2010] O. Ivanovici, "On the Schrödinger equation outside strictly convex obstacles", *Anal. PDE* **3**:3 (2010), 261–293. MR 2011j:58037 Zbl 1222.35186

[Keel and Tao 1998] M. Keel and T. Tao, "Endpoint Strichartz estimates", *Amer. J. Math.* **120** (1998), 955–980. MR 2000d:35018 Zbl 0922.35028

[Kenig and Merle 2006] C. E. Kenig and F. Merle, "Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case", *Invent. Math.* **166**:3 (2006), 645–675. MR 2007g:35232 Zbl 1115.35125

[Kenig et al. 1991] C. E. Kenig, G. Ponce, and L. Vega, "Oscillatory integrals and regularity of dispersive equations", *Indiana Univ. Math. J.* **40**:1 (1991), 33–69. MR 92d:35081 Zbl 0738.35022

[Lions and Magenes 1968a] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications, I*, Travaux et Recherches Mathématiques **17**, Dunod, Paris, 1968. MR 40 #512 Zbl 0165.10801

[Lions and Magenes 1968b] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications, II*, Travaux et Recherches Mathématiques **18**, Dunod, Paris, 1968. MR 40 #513 Zbl 0165.10801

[Melrose and Sjöstrand 1978] R. B. Melrose and J. Sjöstrand, "Singularities of boundary value problems, I", *Comm. Pure Appl. Math.* **31**:5 (1978), 593–617. MR 58 #11859 Zbl 0368.35020

[Planchon and Vega 2009] F. Planchon and L. Vega, "Bilinear virial identities and applications", *Ann. Sci. Éc. Norm. Supér.* (4) **42**:2 (2009), 261–290. MR 2010b:35441 Zbl 1192.35166

[Rauch and Massey 1974] J. B. Rauch and F. J. Massey, III, "Differentiability of solutions to hyperbolic initial-boundary value problems", *Trans. Amer. Math. Soc.* **189** (1974), 303–318. MR 49 #5582 Zbl 0282.35014

[Staffilani and Tataru 2002] G. Staffilani and D. Tataru, "Strichartz estimates for a Schrödinger operator with nonsmooth coefficients", *Comm. Partial Differential Equations* **27**:7-8 (2002), 1337–1372. MR 2003f:35248 Zbl 1010.35015

[Strauss and Bu 2001] W. Strauss and C. Bu, "An inhomogeneous boundary value problem for nonlinear Schrödinger equations", *J. Differential Equations* **173**:1 (2001), 79–91. MR 2002d:35196 Zbl 1055.35119

[Tartar 2007] L. Tartar, *An introduction to Sobolev spaces and interpolation spaces*, Lecture Notes of the Unione Matematica Italiana **3**, Springer, Berlin; UMI, Bologna, 2007. MR 2008g:46055 Zbl 1126.46001

[Triebel 1983] H. Triebel, *Theory of function spaces*, Monographs in Mathematics **78**, Birkhäuser, Basel, 1983. MR 86j:46026 Zbl 0546.46028

Received 9 Oct 2014. Revised 6 Mar 2015. Accepted 15 Apr 2015.

CORENTIN AUDIARD: corentin.audiard@upmc.fr
Laboratoire Jacques-Louis Lions, Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, F-75005 Paris, France

ANALYSIS AND PDE
Vol. 8, No. 5, 2015
dx.doi.org/10.2140/apde.2015.8.1145

ON ESTIMATES FOR FULLY NONLINEAR PARABOLIC EQUATIONS ON RIEMANNIAN MANIFOLDS

BO GUAN, SHUJUN SHI AND ZHENAN SUI

We present some new ideas to derive *a priori* second-order estimates for a wide class of fully nonlinear parabolic equations. Our methods, which produce new existence results for the initial-boundary value problems in $\mathbb{R}^n$, are powerful enough to work in general Riemannian manifolds.

1. Introduction

Let M^n be a compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary ∂M, which may be empty (then M is closed), and f a smooth symmetric function of n variables. We consider the fully nonlinear parabolic equation

$$f(\lambda(\nabla^2 u + \chi)) = e^{u_t + \psi} \quad \text{in } M \times \{t > 0\}, \tag{1-1}$$

where χ is a smooth $(0, 2)$-tensor on $\bar{M} = M \cup \partial M$, $\nabla^2 u$ denotes the spatial Hessian of u, $u_t = \partial u/\partial t$, and $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$ will be the eigenvalues of a $(0, 2)$-tensor A; throughout the paper we shall use ∇ to denote the Levi-Civita connection of (M^n, g) and assume $\psi \in C^\infty(\bar{M} \times \{t \geq 0\})$.

While most attention in previous work had been on the two canonical cases, $\chi = 0$ and $\chi = g$, both of which occur, for instance, in the classical Darboux equations in isometric embedding, there are many important quantities of the form $\nabla^2 u + \chi$ in differential geometry and other areas. A well-known example is the gradient Ricci soliton equation

$$\nabla^2 u + \mathrm{Ric} = \lambda g,$$

which has been studied intensively, where Ric denotes the Ricci tensor of (M^n, g). In a different context, $\nabla^2 u + \mathrm{Ric}$ is known as the *Bakry–Emery Ricci tensor* of the Riemannian measure space $(M^n, g, e^{-u} d\,\mathrm{Vol}_g)$, on which there are interesting recent results; see, e.g., [Wei and Wylie 2009] and references therein. When χ as well as ψ is allowed to depend on u and ∇u, there are even more equations of the form (1-1) and their elliptic counterparts, which arise naturally in connection with important geometric problems, such as the generalized Minkowski and Christoffel–Minkowski problems in classical geometry, fully nonlinear versions of the Yamabe problem in conformal geometry, and in other applications including the Monge–Kantorovich optimal mass transport problem. From both the theoretic point of view and that of applications, it is important and highly desirable to establish a general existence and regularity theory

Guan and Shi were supported in part by NSF grants and a scholarship from China Scholarship Council, respectively.
MSC2010: primary 35K55; secondary 35B45, 58J35.
Keywords: fully nonlinear parabolic equations, a priori estimates, subsolutions, concavity.

for (1-1) with as few technical assumptions as possible, so that it covers a wide range of applications in different areas.

In order to study (1-1) in the context of parabolic theory, we follow [Caffarelli et al. 1985] and assume that f is defined in an open, symmetric, convex cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin, $\Gamma_n := \{\lambda \in \mathbb{R}^n : \lambda_i > 0 \text{ for all } 1 \le i \le n\} \subseteq \Gamma$, and satisfies

$$f_i = f_{\lambda_i} \equiv \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{in } \Gamma, \quad 1 \le i \le n, \tag{1-2}$$

$$f \text{ is a concave function in } \Gamma, \tag{1-3}$$

and

$$\sup_{\partial \Gamma} f := \sup_{\lambda_0 \in \partial \Gamma} \lim_{\lambda \to \lambda_0} f(\lambda) \le 0. \tag{1-4}$$

Equation (1-1) is parabolic for solutions $u \in C^{2,1}(M_T)$ with $\lambda[u] := \lambda(\nabla^2 u + \chi) \in \Gamma$ for $x \in M$ and $t > 0$ (see [Caffarelli et al. 1985]); we shall call such functions *admissible*.

The structure conditions (1-2)–(1-4) are fundamental to the classical solvability of fully nonlinear elliptic and parabolic equations, and have been standard in the literature since the work of Caffarelli, Nirenberg and Spruck [Caffarelli et al. 1985]. Condition (1-4) prevents (1-1) from being degenerate, which may occur if $\lambda[u] \in \bar{\Gamma} = \Gamma \cup \partial \Gamma$. So both conditions (1-2) and (1-4) are natural for the nondegenerate parabolicity of (1-1), without which the $C^{2+\alpha,1+\alpha/2}$ estimates may fail. An important fact is that conditions (1-2) and (1-4) ensure that (1-1) becomes uniformly parabolic once global a priori $C^{2,1}$ estimates are established for admissible solutions. Consequently, one may obtain $C^{2+\alpha,1+\alpha/2}$ estimates by the Evans–Krylov theorem, which depends on the concavity condition (1-3).

The short-time existence of admissible solutions is well known from the classical theory of parabolic equations for given admissible initial data (and boundary data as well when $\partial M \neq \varnothing$) with suitable smoothness assumptions. The global (long-time) existence and behavior of solutions depend on the establishment of a priori estimates in $C^{2,1}(\overline{M_T})$. Our primary goal in this paper is to derive second-order estimates for fully nonlinear parabolic equations on Riemannian manifolds.

For fixed $T > 0$, let $M_T = M \times (0, T]$, $\overline{M}_T = \overline{M} \times (0, T]$, and let $\partial M_T := \partial_s M_T \cup \partial_b M_T$ be the parabolic boundary of M_T, where

$$\partial_s M_T = \partial M \times [0, T), \quad \partial_b M_T = \overline{M} \times \{t = 0\}.$$

Throughout the paper we assume $\varphi^b := \varphi|_{t=0} \in C^\infty(\overline{M})$ with

$$\lambda[\varphi^b] \in \Gamma, \quad f(\lambda[\varphi^b]) > 0 \quad \text{in } \overline{M}, \tag{1-5}$$

and $\varphi^s := \varphi|_{\partial M \times \{t \ge 0\}} \in C^\infty(\partial M \times \{t \ge 0\})$. Let $u \in C^{4,2}(M_T) \cap C^{2,1}(\overline{M_T})$ be an admissible solution of (1-1) satisfying the initial-boundary conditions

$$u|_{t=0} = \varphi^b \quad \text{in } \overline{M}, \qquad u = \varphi^s \quad \text{on } \partial_s M_T. \tag{1-6}$$

The main result of this paper is the following second-order estimates:

Theorem 1.1. *Suppose that there exists an admissible subsolution* $\underline{u} \in C^{2,1}(\overline{M_T})$ *satisfying*

$$f(\lambda[\underline{u}]) \geq e^{\underline{u}_t+\psi} \quad \text{in } M_T. \tag{1-7}$$

Then, under conditions (1-2)–(1-4),

$$\sup_{M_T} |\nabla^2 u| \leq C_1\bigl(1+\max_{\partial M_T} |\nabla^2 u|\bigr). \tag{1-8}$$

In particular, when M *is closed,*

$$|\nabla^2 u| \leq C_1 \quad \text{in } \overline{M_T}. \tag{1-9}$$

Suppose in addition that

$$\underline{u} \leq \varphi^b \quad \text{on } \partial_b M_T, \qquad \underline{u} = \varphi^s \quad \text{on } \partial_s M_T. \tag{1-10}$$

Then

$$\max_{\partial M_T} |\nabla^2 u| \leq C_2. \tag{1-11}$$

Remark 1.2. In Theorem 1.1 and the rest of this paper, unless otherwise indicated, the constant C_1 in (1-8) will depend on

$$|u|_{C^1(\overline{M_T})}, \quad |\psi|_{C^{2,1}(\overline{M_T})}, \quad |\underline{u}|_{C^{2,1}(\overline{M_T})}, \quad \inf_{M_T} \operatorname{dist}(\lambda[\underline{u}], \partial\Gamma), \tag{1-12}$$

and

$$\Lambda := \sup_{\Gamma} f - \sup_{M_T} e^{u_t+\psi} \tag{1-13}$$

as well as geometric quantities of M, while C_2 in (1-11) will depend in addition on $|\varphi^b|_{C^2(\overline{M})}$, $|\varphi^s|_{C^{4,1}(\partial_s M_T)}$, $\inf_{M_T} e^{u_t+\psi}$ and geometric quantities of ∂M. If f satisfies

$$\lim_{|\lambda|\to\infty} |\lambda|^2 \sum f_i = \infty, \tag{1-14}$$

then C_1 can be chosen independently of Λ and $|u_t|_{C^0(\overline{M_T})}$; see Remark 2.4.

Remark 1.3. The assumption $u \in C^{4,2}(M_T) \cap C^{2,1}(\overline{M_T})$ does not restrict the applications of Theorem 1.1. This can be seen as follows. By the short-time existence theorem, (1-1) admits a unique admissible solution $u \in C^\infty(\overline{M} \times (0, t_0]) \cap C^0(M \times [0, t_0])$ satisfying the initial-boundary condition (1-10) for some $t_0 > 0$. We can then consider a new initial time, say $t = t_0/2$, in place of $t = 0$, and may therefore assume the compatibility condition

$$f(\lambda[\varphi^b]) = e^{\varphi^s_t+\psi} \quad \text{on } \overline{M} \qquad \text{and} \qquad \varphi^s = \varphi^b \quad \text{on } \partial M \times \{t = 0\}. \tag{1-15}$$

Theorem 1.1 is an important step towards solving the initial-boundary problem (1-1) and (1-6) under optimal structure conditions. It can be applied in many interesting cases to prove new long-time existence results. Let us give a few examples here.

First, for a bounded smooth domain (with boundary of arbitrary geometric shape) in $\mathbb{R}^n$ we have the following result, which is essentially optimal, both in terms of the generality of f and that of the underlying domain:

Theorem 1.4. *Let M be a bounded smooth domain in $\mathbb{R}^n$, $0 < T \le \infty$, and f satisfy (1-2)–(1-5). There exists a unique admissible solution $u \in C^\infty(\bar{M}_T) \cap C^0(\overline{M_T})$ of (1-1) satisfying (1-6) provided that there exists an admissible subsolution $\underline{u} \in C^{2,1}(\overline{M_T})$ satisfying (1-7) and (1-10).*

The first initial-boundary value problem for (1-1), or (1-20) below, in $\mathbb{R}^n$ was treated, among many others, by Ivochkina and Ladyzhenskaya [1995], who used essentially the same assumptions as in the elliptic case introduced in [Caffarelli et al. 1985]; see [Lieberman 1996] for further improvements and references. Jiao and Sui [2015] studied (1-20) on Riemannian manifolds under additional assumptions. To the best of our knowledge, Theorem 1.4 had not been proved before in the current generality.

We remark that since there are no geometric restrictions on ∂M, (1-1) and (1-6) may fail to admit a long-term admissible solution without the subsolution assumption. This is well known and may be seen from simple examples.

Theorem 1.5. *When $\Gamma = \Gamma_n$, Theorem 1.4 holds for compact Riemannian manifolds.*

Theorem 1.1 applies to a very general class of equations, including $f = \sigma_k^{1/k}$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$, $1 \le l < k \le n$, where σ_k is the k-th elementary symmetric function defined on the cone $\Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \text{ for all } 1 \le j \le k\}$. Another interesting example is $f = \log P_k$, to which Theorem 1.10 applies, where

$$P_k(\lambda) := \prod_{i_1 < \cdots < i_k} (\lambda_{i_1} + \cdots + \lambda_{i_k}), \quad 1 \le k \le n,$$

defined in the cone

$$\mathcal{P}_k := \{\lambda \in \mathbb{R}^n : \lambda_{i_1} + \cdots + \lambda_{i_k} > 0 \text{ for all } 1 \le i_1 < \cdots < i_k \le n\}.$$

Theorem 1.6. *Let $f = (\sigma_k/\sigma_l)^{1/(k-l)}$ and $\Gamma = \Gamma_k$ for $0 \le l < k \le n$, with $\sigma_0 = 1$, or $f = \log P_k$ and $\Gamma = \mathcal{P}_k$. The parabolic problem (1-1) and (1-6) with smooth data has a unique admissible solution $u \in C^\infty(\bar{M}_T) \cap C^0(\overline{M_T})$ provided that there exists an admissible subsolution $\underline{u} \in C^{2,1}(\overline{M_T})$ satisfying (1-7) and (1-10).*

Theorem 1.6 is known for $f = \sigma_k^{1/k}$, but seems to be new for $f = (\sigma_k/\sigma_l)^{1/(k-l)}$ or $f = \log P_k$, even when M is a bounded smooth domain in $\mathbb{R}^n$; see also [Jiao and Sui 2015].

Remark 1.7. In Theorem 1.1, the constants C_1 and C_2 depend on T only implicitly. For instance, if the quantities listed in (1-12) are all independent of T, then so is C_1. The independence of T from the estimates is important to understanding the asymptotic behaviors of solutions as t goes to infinity. If one allows C_1 to depend on T (explicitly), (1-8) can be derived under much weaker conditions, and more easily.

Theorem 1.8. *Under assumptions (1-2), (1-3) and (1-5),*

$$|\nabla^2 u(x,t)| \le Ce^{Bt}\big(1 + \max_{\partial M_T} |\nabla^2 u|\big) \quad \text{for all } (x,t) \in M_T, \tag{1-16}$$

where C and B depend on $|\nabla u|_{C^0(\overline{M_T})}$, $|\varphi^b|_{C^2(\bar{M})}$ and other known data. In particular, if M is closed then $|\nabla^2 u(x,t)| \le Ce^{Bt}$.

Note that, by (1-5), the function

$$\underline{u} := \varphi^b + t \min_{\bar{M}} \{\log f(\lambda[\varphi^b]) - \psi\}$$

is admissible and satisfies (1-7).

An immediate consequence of Theorem 1.8 is the following characterization of finite-time blow-up solutions on closed manifolds:

Corollary 1.9. *Assume M is closed and f satisfies (1-2)–(1-4). Then (1-1) admits a unique admissible solution $u \in C^\infty(M \times \mathbb{R}^+)$ with initial value function φ^b satisfying (1-5) provided that the a priori gradient estimate*

$$\sup_{M_T} |\nabla u| \le C \quad \text{for all } T > 0 \tag{1-17}$$

holds, where C may depend on T. In other words, if u has a finite-time blow-up at $T < \infty$, then

$$\lim_{t \to T^-} \max_{x \in M} |\nabla u(x,t)| = \infty.$$

So, the long-time existence of solutions in $0 \le t < \infty$ reduces to establishing the gradient estimate (1-17). This is also true when $\partial M \ne \varnothing$. Using Theorem 1.1, we can prove the following existence results:

Theorem 1.10. *Assume that (1-2)–(1-5), (1-7), and (1-10) hold for $T \in (0, \infty]$. There exists a unique admissible solution $u \in C^\infty(\bar{M}_T) \cap C^0(\overline{M_T})$ of (1-1) satisfying (1-6) provided that any one of the following conditions holds:*

(i) $\Gamma = \Gamma_n$;

(ii) (M, g) *has nonnegative sectional curvature*;

(iii) *there is $\delta_0 > 0$ such that, if $\lambda_j < 0$,*

$$f_j \ge \delta_0 \sum f_i \quad \text{on } \partial\Gamma^\sigma \quad \text{for all } \sigma > 0; \tag{1-18}$$

(iv) $\nabla^2 w \ge \chi$ *for some function $w \in C^2(\bar{M})$ and*

$$\sum f_i \lambda_i \ge 0 \quad \text{in } \Gamma. \tag{1-19}$$

The assumptions (i)–(iv) are only needed in deriving the gradient estimates. It would be interesting to remove these assumptions. When $\partial M = \varnothing$, Theorem 1.10 holds without the subsolution assumption.

The rest of the article is divided into three sections. In Sections 2 and 3, we derive (1-8) and (1-11), respectively, completing the proofs of Theorems 1.1 and 1.8. Instead of (1-1), we shall deal with the equation

$$f(\lambda(\nabla^2 u + \chi)) = u_t + \psi \tag{1-20}$$

under essentially the same assumptions on f, with the exception that (1-4) is replaced by

$$\inf_{\partial_s M_T} (\varphi_t + \psi) - \sup_{\partial\Gamma} f > 0, \tag{1-21}$$

which is needed in the proof of (1-11) . Accordingly, the functions φ^b and $\underline{u} \in C^{2,1}(\overline{M_T})$ are assumed to satisfy $\lambda[\varphi^b] \in \Gamma$ in $\overline{M}$ and, respectively,

$$f(\lambda[\underline{u}]) \geq \underline{u}_t + \psi \quad \text{in } M_T \tag{1-22}$$

in place of (1-7). Note that if $f > 0$ in Γ and f satisfies (1-2), (1-3), and (1-19), then the function $\log f$ still satisfies theses assumptions. So (1-1) is covered by (1-20) in most cases, and we shall derive the estimates for (1-20). In Section 4 we briefly discuss the proof of the existence results and the preliminary estimates needed in the proof.

At the end of this introduction we recall the following commonly used notations:

$$|u|_{C^{k,l}(\overline{M_T})} = \sum_{j=0}^{k} |\nabla^j u|_{C^0(\overline{M_T})} + \sum_{j=1}^{l} \left|\frac{\partial^j u}{\partial t^j}\right|_{C^0(\overline{M_T})},$$

$$|u|_{C^{k+\alpha,l+\beta}(\overline{M_T})} = |u|_{C^{k,l}(\overline{M_T})} + |\nabla^k u|_{C^\alpha(\overline{M_T})} + \left|\frac{\partial^l u}{\partial t^l}\right|_{C^\beta(\overline{M_T})},$$

where $0 < \alpha, \beta < 1$ and $k, l = 1, 2, \ldots,$ for a function u sufficiently smooth on $\overline{M_T}$. We shall also write $|u|_{C^k(\overline{M_T})} = |u|_{C^{k,k}(\overline{M_T})}$.

2. Global estimates for second derivatives

A substantial difficulty in deriving the global estimate (1-8), which is our primary goal in this section, is caused by curvature of M; another is the lack of (globally defined) functions or geometric quantities with desirable properties. In our proof, the use of the admissible subsolution $\underline{u}$ is critical. We shall consider (1-20) in place of (1-1).

Let $u \in C^{4,2}(M_T) \cap C^{2,1}(\overline{M_T})$ be an admissible solution of (1-20) and $\underline{u} \in C^{2,1}(\overline{M_T})$ an admissible function. We assume that u admits an a priori C^1 bound

$$|u|_{C^1(\overline{M_T})} \leq C. \tag{2-1}$$

Let $\phi(s) = -\log(1 - bs^2)$ and

$$\eta = \phi(1 + |\nabla(u - \underline{u})|^2) + a(\underline{u} - u - \delta t), \tag{2-2}$$

where $a, b, \delta > 0$ are constants and $\underline{u} \in C^{2,1}(\overline{M_T})$ is an admissible function; we shall choose $\delta = 1$ or 0, a sufficiently large, and b small enough, namely

$$b \leq \frac{1}{8b_1^2}, \quad b_1 = 1 + \sup_{M_T} |\nabla(u - \underline{u})|^2. \tag{2-3}$$

Consider the quantity

$$W = \sup_{(x,t)\in M_T} \max_{\xi \in T_x M^n, |\xi|=1} (\nabla_{\xi\xi} u + \chi(\xi, \xi)) e^\eta.$$

Suppose W is achieved at an interior point $(x_0, t_0) \in M_T$ for a unit vector $\xi \in T_{x_0} M^n$. Let $e_1, \ldots, e_n$ be smooth orthonormal local frames about x_0 such that $e_1 = \xi$, $\nabla_i e_j = 0$ and the $U_{ij} := \nabla_{ij} u + \chi_{ij}$ are diagonal at (x_0, t_0). So $W = U_{11}(x_0, t_0)e^{\eta(x_0,t_0)}$. We wish to derive a bound

$$U_{11}(x_0, t_0) \le C. \tag{2-4}$$

Write (1-20) in the form

$$u_t = F(U) - \psi, \quad U = \{U_{ij}\}, \tag{2-5}$$

where F is defined by

$$F(A) \equiv f(\lambda[A])$$

for an $n \times n$ symmetric matrices $A = \{A_{ij}\}$ with eigenvalues $\lambda[A] \in \Gamma$. Differentiating (2-5) gives

$$\begin{aligned} u_{tt} &= F^{ij} U_{ijt} - \psi_t, \\ \nabla_k u_t &= F^{ij} \nabla_k U_{ij} - \nabla_k \psi \quad \text{for all } k, \\ \nabla_{11} u_t &= F^{ij} \nabla_{11} U_{ij} + F^{ij,kl} \nabla_1 U_{ij} \nabla_1 U_{kl} - \nabla_{11}\psi. \end{aligned} \tag{2-6}$$

Throughout the paper we denote

$$F^{ij} = \frac{\partial F}{\partial A_{ij}}(U), \quad F^{ij,kl} = \frac{\partial^2 F}{\partial A_{ij} \partial A_{kl}}(U).$$

The matrix $\{F^{ij}\}$ has eigenvalues $f_1, \dots, f_n$, and therefore is positive-definite when f satisfies (1-2), while (1-3) implies that F is a concave function; see [Caffarelli et al. 1985]. Moreover, the following identities hold:

$$F^{ij} U_{ij} = \sum f_i \lambda_i, \quad F^{ij} U_{ik} U_{kj} = \sum f_i \lambda_i^2.$$

We also note that the F^{ij} are diagonal at (x_0, t_0).

Proposition 2.1. *For any $a, C_1 > 0$, there exists a constant $b > 0$ satisfying (2-3) such that, at (x_0, t_0), if $U_{11} \ge C_1 a/\sqrt{b}$ then*

$$\frac{b}{2} F^{ii} U_{ii}^2 + a F^{ii} \nabla_{ii}(\underline{u} - u) - a(\underline{u}_t - u_t) + a\delta \le C \sum F^{ii} + C. \tag{2-7}$$

Proof. We shall assume $U_{11}(x_0, t_0) \ge 1$. At (x_0, t_0), where the function $\log U_{11} + \eta$ has its maximum,

$$\frac{(\nabla_{11} u)_t}{U_{11}} + \eta_t \ge 0, \quad \frac{\nabla_i U_{11}}{U_{11}} + \nabla_i \eta = 0, \quad 1 \le i \le n, \tag{2-8}$$

and

$$\frac{1}{U_{11}} F^{ii} \nabla_{ii} U_{11} - \frac{1}{U_{11}^2} F^{ii} (\nabla_i U_{11})^2 + F^{ii} \nabla_{ii} \eta \le 0. \tag{2-9}$$

We recall the identities, on a Riemannian manifold,

$$\nabla_{ijk} v - \nabla_{jik} v = R^l_{kij} \nabla_l v, \tag{2-10}$$

$$\nabla_{ijkl} v - \nabla_{klij} v = R^m_{ljk} \nabla_{im} v + \nabla_i R^m_{ljk} \nabla_m v + R^m_{lik} \nabla_{jm} v + R^m_{jik} \nabla_{lm} v + R^m_{jil} \nabla_{km} v + \nabla_k R^m_{jil} \nabla_m v. \tag{2-11}$$

It follows that

$$F^{ii} \nabla_{ii} U_{11} \ge F^{ii} \nabla_{11} U_{ii} - C U_{11} \sum F^{ii}, \tag{2-12}$$

where C depends on $|\nabla u|_{C^0(\overline{M}_T)}$ and geometric quantities of M. By (2-8), (2-9), (2-12), and (2-6), we obtain

$$F^{ii}\nabla_{ii}\eta-\eta_t\le\frac{1}{U_{11}}F^{ij,kl}\nabla_1U_{ij}\nabla_1U_{kl}+\frac{1}{U_{11}^2}F^{ii}(\nabla_iU_{11})^2-\frac{\nabla_{11}\psi}{U_{11}}+C\sum F^{ii}. \tag{2-13}$$

Let

$$J=\{i:3U_{ii}\le -U_{11}\},\qquad K=\{i>1:3U_{ii}>-U_{11}\}.$$

As in [Guan 2014b], which uses an idea of Urbas [2002], one derives

$$F^{ii}\nabla_{ii}\eta-\eta_t\le\sum_{i\in J}F^{ii}(\nabla_i\eta)^2+CF^{11}\sum_{i\notin J}(\nabla_i\eta)^2-\frac{\nabla_{11}\psi}{U_{11}}+C\sum F^{ii}. \tag{2-14}$$

For convenience, we write $w=\underline{u}-u$, $s=1+|\nabla w|^2$, and calculate

$$\begin{aligned}\nabla_i\eta&=2\phi'\nabla_kw\nabla_{ik}w+a\nabla_iw,\\ \eta_t&=2\phi'\nabla_kw(\nabla_kw)_t+aw_t-a\delta,\\ \nabla_{ii}\eta&=2\phi'(\nabla_{ik}w\nabla_{ik}w+\nabla_kw\nabla_{iik}w)+4\phi''(\nabla_kw\nabla_{ik}w)^2+a\nabla_{ii}w,\end{aligned}$$

while

$$\phi'(s)=\frac{2bs}{1-bs^2},\qquad \phi''(s)=\frac{2b+2b^2s^2}{(1-bs^2)^2}>4(\phi')^2.$$

Hence,

$$\sum_{i\in J}F^{ii}(\nabla_i\eta)^2\le 8(\phi')^2\sum_{i\in J}F^{ii}(\nabla_kw\nabla_{ik}w)^2+2|\nabla w|^2a^2\sum_{i\in J}F^{ii} \tag{2-15}$$

and

$$\sum_{i\notin J}(\nabla_i\eta)^2\le Ca^2+C(\phi')^2U_{11}^2. \tag{2-16}$$

By (2-6) and (2-10), we obtain

$$F^{ii}\nabla_{ii}\eta-\eta_t\ge\phi'F^{ii}U_{ii}^2+2\phi''F^{ii}(\nabla_kw\nabla_{ik}w)^2+aF^{ii}\nabla_{ii}w-aw_t+a\delta-C\phi'\Big(1+\sum F^{ii}\Big). \tag{2-17}$$

It follows from (2-14)–(2-17) that

$$\phi'F^{ii}U_{ii}^2+aF^{ii}\nabla_{ii}w-aw_t+a\delta\le Ca^2\sum_{i\in J}F^{ii}+C(a^2+(\phi')^2U_{11}^2)F^{11}-\frac{\nabla_{11}\psi}{U_{11}}+C\Big(\phi'+\sum F^{ii}\Big). \tag{2-18}$$

Note that

$$F^{ii}U_{ii}^2\ge F^{11}U_{11}^2+\sum_{i\in J}F^{ii}U_{ii}^2\ge F^{11}U_{11}^2+\frac{U_{11}^2}{9}\sum_{i\in J}F^{ii}. \tag{2-19}$$

We may fix b small to derive (2-7) when $U_{11}\ge Ca/\sqrt{b}$. □

To proceed, we need the following lemma, which is key to the proof of Theorem 1.1, both for (1-8) in this section and (1-11) in the next section; compare with Lemma 2.1 in [Guan 2014a]. Let $\nu_\lambda=Df(\lambda)/|Df(\lambda)|$ denote the unit normal vector to the level surface of f through λ.

Lemma 2.2. *Let K be a compact subset of Γ and $\beta > 0$. There is a constant $\varepsilon > 0$ such that, for any $\mu \in K$ and $\lambda \in \Gamma$ with $|\nu_\mu - \nu_\lambda| \geq \beta$,*

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda) + \varepsilon\Big(1 + \sum f_i(\lambda)\Big). \tag{2-20}$$

Proof. Since ν_μ is smooth in $\mu \in \Gamma$ and K is compact, there is $\epsilon_0 > 0$ such that, for any $0 \leq \epsilon \leq \epsilon_0$,

$$K^\epsilon := \{\mu^\epsilon := \mu - \epsilon\mathbf{1} : \mu \in K\}$$

is still a compact subset of Γ and

$$|\nu_\mu - \nu_{\mu^\epsilon}| \leq \frac{\beta}{2} \quad \text{for all } \mu \in K.$$

Consequently, if $\mu \in K$ and $\lambda \in \Gamma$ satisfy $|\nu_\mu - \nu_\lambda| \geq \beta$ then $|\nu_{\mu^\epsilon} - \nu_\lambda| \geq \beta/2$.

By the smoothness of the level surfaces of f, there exists $\delta > 0$ (which depends on β but is uniform in $\epsilon \in [0, \epsilon_0]$) such that

$$\min_{\mu \in K} \min_{0 \leq \epsilon \leq \epsilon_0} \operatorname{dist}(\partial B_\delta^{\beta/2}(\mu^\epsilon), \partial\Gamma^{f(\mu^\epsilon)}) > 0,$$

where $\partial B_\delta^{\beta/2}(\mu^\epsilon)$ denotes the spherical cap

$$\partial B_\delta^{\beta/2}(\mu^\epsilon) = \left\{\zeta \in \partial B_\delta(\mu^\epsilon) : \nu_{\mu^\epsilon} \cdot \frac{\zeta - \mu^\epsilon}{\delta} \geq \frac{\beta}{2}\sqrt{1 - \frac{\beta^2}{16}}\right\}.$$

Therefore,

$$\theta \equiv \min_{\mu \in K} \min_{0 \leq \epsilon \leq \epsilon_0} \min_{\zeta \in \partial B_\delta^{\beta/2}(\mu^\epsilon)} \{f(\zeta) - f(\mu^\epsilon)\} > 0. \tag{2-21}$$

Let P be the two-plane through μ^ϵ spanned by ν_{μ^ϵ} and ν_λ (translated to μ^ϵ), and L the line on P through μ^ϵ and perpendicular to ν_λ. Since $0 < \nu_{\mu^\epsilon} \cdot \nu_\lambda \leq 1 - \beta^2/8$, L intersects $\partial B_\delta^{\beta/2}(\mu^\epsilon)$ at a unique point ζ. By the concavity of f, we see that

$$\begin{aligned}\sum f_i(\lambda)(\mu_i^\epsilon - \lambda_i) &= \sum f_i(\lambda)(\zeta_i - \lambda_i)\\ &\geq f(\zeta) - f(\lambda)\\ &\geq \theta + f(\mu^\epsilon) - f(\lambda) \quad \text{for all } 0 \leq \epsilon \leq \epsilon_0.\end{aligned} \tag{2-22}$$

Next, by the continuity of f we may choose $0 < \epsilon_1 \leq \epsilon_0$ with $|f(\mu^{\epsilon_1}) - f(\mu)| \leq \frac{1}{2}\theta$. Hence

$$\sum f_i(\lambda)(\mu_i - \epsilon_1 - \lambda_i) \geq f(\mu) - f(\lambda) + \tfrac{1}{2}\theta. \tag{2-23}$$

This proves (2-20) with $\varepsilon = \min\{\theta/2, \epsilon_1\}$. □

Remark 2.3. Alternatively, one can first prove

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \geq \theta + f(\mu) - f(\lambda).$$

Then choose $\epsilon > 0$ small such that $0 \leq f(\mu) - f(\mu^\epsilon) \leq \theta/2$. By the concavity of f,

$$\sum f_i(\lambda)(\mu_i^\epsilon - \lambda_i) \geq f(\mu^\epsilon) - f(\lambda) \geq f(\mu) - f(\lambda) - \frac{\theta}{2}. \tag{2-24}$$

Now add these two inequalities to obtain (2-20).

We now continue to prove (2-4). Assume first that $\underline{u}$ is a subsolution, i.e., $\underline{u}$ satisfies (1-22). Since $\lambda[\underline{u}]$ falls in a compact subset of Γ,

$$\beta := \tfrac{1}{2} \min_{\overline{M_T}} \operatorname{dist}(\nu_{\lambda[\underline{u}]}, \partial\Gamma_n) > 0. \tag{2-25}$$

Let $\lambda = \lambda[u](x_0, t_0)$ and $\mu = \lambda[\underline{u}](x_0, t_0)$. If $|\nu_\mu - \nu_\lambda| \geq \beta$ then, by Lemma 2.2,

$$F^{ii}\nabla_{ii} w - w_t \geq \sum f_i(\lambda)(\mu_i - \lambda_i) - f(\mu) + f(\lambda) \geq \varepsilon\Big(1 + \sum F^{ii}\Big). \tag{2-26}$$

The first inequality follows from Lemma 6.2 in [Caffarelli et al. 1985]; see [Guan 2014b]. We may fix a sufficiently large to derive a bound $U_{11}(x_0, t_0) \leq C$ by (2-7).

Suppose now that $|\nu_\mu - \nu_\lambda| < \beta$ and therefore $\nu_\lambda - \beta\mathbf{1} \in \Gamma_n$. It follows that

$$F^{ii} \geq \frac{\beta}{\sqrt{n}} \sum F^{kk} \quad \text{for all } 1 \leq i \leq n. \tag{2-27}$$

Since $\underline{u}$ is a subsolution, $F^{ii}\nabla_{ii} w - w_t \geq 0$ by the concavity of f. By (2-7) and (2-27), we obtain

$$\frac{b\beta}{2\sqrt{n}} U_{11}^2 \sum F^{ii} + a\delta \leq C \sum F^{ii} + C. \tag{2-28}$$

If we allow $\delta = 1$, a bound $U_{11}(x_0, t_0) \leq C$ would follow when a is sufficiently large. This gives (1-16) in Theorem 1.8.

We now consider the case $\delta = 0$. First, by the concavity of f,

$$|\lambda| \sum f_i \geq f(|\lambda|\mathbf{1}) - f(\lambda) + \sum f_i\lambda_i \geq f(|\lambda|\mathbf{1}) - f(\lambda) - \frac{1}{4|\lambda|}\sum f_i\lambda_i^2 - |\lambda|\sum f_i. \tag{2-29}$$

Hence,

$$U_{11}^2 \sum F^{ii} \geq \frac{U_{11}}{2n}(f(U_{11}\mathbf{1}) - u_t - \psi) - \frac{1}{8}\sum F^{ii}U_{ii}^2 \geq \frac{\Lambda U_{11}}{4n} - \frac{U_{11}^2}{8}\sum F^{ii} \tag{2-30}$$

when U_{11} is sufficiently large. A bound $U_{11}(x_0, t_0) \leq C$ therefore follows from (2-28). The proof of (1-8) in Theorem 1.1 is complete.

Remark 2.4. If (1-14) holds, a bound $U_{11}(x_0, t_0) \leq C$ follows from (2-28) directly and is independent of $|u_t|_{C^0(\overline{M_T})}$.

Remark 2.5. If $\underline{u}$ is an admissible strict subsolution, i.e.,

$$f(\lambda[\underline{u}]) \geq \underline{u}_t + \psi + \delta \quad \text{in } M_T \tag{2-31}$$

for some $\delta > 0$, then we can choose $\epsilon > 0$ such that $\lambda^\epsilon[\underline{u}] := \lambda[\underline{u}] - \epsilon\mathbf{1} \in \Gamma$ and

$$f(\lambda^\epsilon[\underline{u}]) \geq \underline{u}_t + \psi + \frac{\delta}{2} \quad \text{in } M_T. \tag{2-32}$$

By the concavity of f, we see that

$$\sum f_i(\lambda[u])(\lambda_i^\epsilon[\underline{u}] - \lambda_i[u]) \geq f(\lambda^\epsilon[\underline{u}]) - f(\lambda[u]) \geq \underline{u}_t - u_t + \frac{\delta}{2}.$$

Therefore, one can derive (2-4) directly from Proposition 2.1. This can be used to prove Theorem 1.8 as $\underline{u} = \varphi^b + At$ is a strict subsolution of (1-20) for any constant $A < \inf_M f(\lambda[\varphi^b]) - \sup_{M_T} \psi$.

3. Second-order boundary estimates

Let $u \in C^{3,1}(\overline{M_T})$ be an admissible solution of (1-20) and (1-6), and $\underline{u} \in C^{2,1}(\overline{M_T})$ an admissible subsolution satisfying (1-22) and (1-10). In this section, we derive (1-11) under conditions (1-2), (1-3) and (1-21) on f. Clearly we only need to focus on $\partial_s M_T$.

For a point $x_0 \in \partial M$ we shall choose smooth orthonormal local frames $e_1, \dots, e_n$ around x_0 such that e_n, when restricted to ∂M, is the interior unit normal to ∂M. By the boundary condition $u = \varphi^s$ on $\partial_s M_T$, we obtain

$$|\nabla_{\alpha\beta} u(x_0, t)| \le C \quad \text{for all } 1 \le \alpha, \beta < n, \ 0 \le t \le T. \tag{3-1}$$

Let $\rho(x)$ and $d(x)$ denote the distances from $x \in \overline{M}$ to x_0 and ∂M, respectively. Let $M_T^\delta = \{(x, t) \in M_T : \rho(x) < \delta\}$, and ∂M_T^δ be the parabolic boundary of M_T^δ,

$$\partial M_T^\delta = \overline{M_T^\delta} \setminus M_T^\delta.$$

We fix $\delta_0 > 0$ sufficiently small that both ρ and d are smooth in $M_T^{\delta_0}$. Let $\mathcal{L}$ denote the linear parabolic operator

$$\mathcal{L}w = F^{ij} \nabla_{ij} w - w_t.$$

We construct a barrier function of the form

$$\Psi = A_1 v + A_2 \rho^2 - A_3 \sum_{l<n} |\nabla_l (u - \varphi)|^2, \tag{3-2}$$

where

$$v = u - \underline{u} + sd - \frac{Nd^2}{2}. \tag{3-3}$$

Lemma 3.1. *Assume that (1-2), (1-3) and (1-21) hold. For constant $K > 0$, there exist uniform positive constants s, δ sufficiently small, and A_1, A_2, A_3, N sufficiently large, such that $\Psi \ge K(d + \rho^2)$ in $\overline{M_T^\delta}$ and*

$$\mathcal{L}\Psi \le -K\Big(1 + \sum f_i |\lambda_i| + \sum f_i\Big) \quad \text{in } M_T^\delta. \tag{3-4}$$

Proof. This is a parabolic version of Lemma 3.1 in [Guan 2014a]. Since there are some substantial differences in several places, for completeness we include a detailed proof.

First we note that, since $\underline{u}$ is a subsolution, $\mathcal{L}(u - \underline{u}) \le 0$ by the concavity of f, and, by (2-6),

$$|\mathcal{L}\nabla_k (u - \varphi)| \le C\Big(1 + \sum f_i |\lambda_i| + \sum f_i\Big) \quad \text{for all } 1 \le k \le n. \tag{3-5}$$

It follows that

$$\sum_{l<n} \mathcal{L} |\nabla_l (u - \varphi)|^2 \ge 2 \sum_{l<n} F^{ij} U_{il} U_{jl} - C\Big(1 + \sum f_i |\lambda_i| + \sum f_i\Big). \tag{3-6}$$

By Proposition 2.19 in [Guan 2014b], there exists an index r such that

$$\sum_{l<n} F^{ij} U_{il} U_{jl} \geq \frac{1}{2} \sum_{i \neq r} f_i \lambda_i^2. \tag{3-7}$$

At a fixed point (x, t), denote $\mu = \lambda(\nabla^2 \underline{u} + \chi)$ and $\lambda = \lambda(\nabla^2 u + \chi)$. As in Section 2 we consider two cases separately: (a) $|\nu_\mu - \nu_\lambda| < \beta$, and (b) $|\nu_\mu - \nu_\lambda| \geq \beta$, where β is as given in (2-25).

Case (a): $|\nu_\mu - \nu_\lambda| < \beta$. We have, by (2-27),

$$f_i \geq \frac{\beta}{\sqrt{n}} \sum f_k \quad \text{for all } 1 \leq i \leq n. \tag{3-8}$$

We next show that this implies the following inequality for any index r:

$$\sum_{i \neq r} f_i \lambda_i^2 \geq c_0 \sum f_i \lambda_i^2 - C_0 \sum f_i \tag{3-9}$$

for some $c_0, C_0 > 0$.

Since $\sum \lambda_i \geq 0$, we see that

$$\sum_{\lambda_i<0} \lambda_i^2 \leq \Big(-\sum_{\lambda_i<0} \lambda_i\Big)^2 \leq n \sum_{\lambda_i>0} \lambda_i^2. \tag{3-10}$$

Therefore, by (3-8) and (3-10), we obtain, if $\lambda_r < 0$,

$$f_r \lambda_r^2 \leq n f_r \sum_{\lambda_i>0} \lambda_i^2 \leq \frac{n\sqrt{n}}{\beta} \sum_{\lambda_i>0} f_i \lambda_i^2.$$

On the other hand, by the concavity of f,

$$\sum f_i (b - \lambda_i) \geq f(b\mathbf{1}) - f(\lambda) = f(b\mathbf{1}) - u_t - \psi \geq \frac{\Lambda}{2} \tag{3-11}$$

for $b > 0$ sufficiently large. It follows that, if $\lambda_r > 0$,

$$f_r \lambda_r \leq b \sum f_i - \sum_{\lambda_i<0} f_i \lambda_i.$$

By (3-8) and the Schwarz inequality,

$$\frac{\beta f_r \lambda_r^2}{\sqrt{n}} \sum f_k \leq f_r^2 \lambda_r^2 \leq 2b^2 \Big(\sum f_i\Big)^2 + 2 \sum_{\lambda_k<0} f_k \sum_{\lambda_i<0} f_i \lambda_i^2 \leq 2\Big(\sum_{\lambda_i<0} f_i \lambda_i^2 + b^2 \sum f_i\Big) \sum f_k.$$

This finishes the proof of (3-9).

Letting $b = n|\lambda|$ in (3-11), we see that

$$(n+1)|\lambda| \sum f_i \geq \sum f_i (n|\lambda| - \lambda_i) \geq f(n|\lambda|\mathbf{1}) - f(\lambda) \geq \frac{\Lambda}{2}, \tag{3-12}$$

and consequently, by (3-8),

$$\sum f_i \lambda_i^2 \geq \frac{\beta |\lambda|^2}{\sqrt{n}} \sum f_i \geq \frac{\beta |\lambda|}{(n+1)\sqrt{n}} \frac{\Lambda}{2} \tag{3-13}$$

provided that $|\lambda| \ge R$ for R sufficiently large.

It now follows from (3-6), (3-7), (3-9), (3-13) and the Schwarz inequality that, when $|\lambda| \ge R$,

$$\sum_{l<n} \mathcal{L}|\nabla_l(u-\varphi)|^2 \ge c_1 \sum f_i\lambda_i^2 + 2c_1|\lambda| - C - C_1 \sum f_i \tag{3-14}$$

for some c_1, $C_1 > 0$. We now fix $R \ge C/c_1$.

Turning to the function v, we note that, by (3-8),

$$\mathcal{L}v \le \mathcal{L}(u-\underline{u}) + C(s+Nd)\sum F^{ii} - NF^{ij}\nabla_i d\nabla_j d \le \left(C(s+Nd) - \frac{\beta N}{\sqrt{n}}\right)\sum F^{ii}, \tag{3-15}$$

since $\mathcal{L}(u-\underline{u}) \le 0$ and $|\nabla d| \equiv 1$. For N sufficiently large, we have

$$\mathcal{L}v \le -\sum f_i \quad \text{in } M_T^\delta, \tag{3-16}$$

and therefore, in view of (3-14) and (3-16),

$$\mathcal{L}\Psi \le -A_3c_1\left(|\lambda| + \sum f_i\lambda_i^2\right) + (-A_1 + CA_2 + C_1A_3)\sum f_i \tag{3-17}$$

when $|\lambda| \ge R$ for any $s \in (0, 1]$ as long as δ is sufficiently small. From now on A_3 is fixed such that $A_3c_1R \ge K$, so $A_3 \ge CK/c_1^2$.

Suppose now that $|\lambda| \le R$. By (1-2) and (1-3), we have

$$2R\sum f_i \ge \sum f_i\lambda_i + f(2R\mathbf{1}) - f(\lambda) \ge -R\sum f_i + f(2R\mathbf{1}) - f(R\mathbf{1}). \tag{3-18}$$

Therefore,

$$\sum f_i \ge \frac{f(2R\mathbf{1}) - f(R\mathbf{1})}{3R} \equiv C_R > 0.$$

It follows from (2-27) that there is a uniform lower bound

$$f_i \ge \frac{\beta}{\sqrt{n}}\sum f_k \ge \frac{\beta C_R}{\sqrt{n}} \quad \text{for all } 1 \le i \le n. \tag{3-19}$$

Consequently, since $|\nabla d| = 1$,

$$F^{ij}\nabla_i d\nabla_j d \ge \frac{\beta}{2\sqrt{n}}\left(C_R + \sum f_i\right).$$

From (3-15) we see that, when δ is sufficiently small and N sufficiently large,

$$\mathcal{L}v \le -\left(1 + \sum f_i\right) \quad \text{in } M_T^\delta. \tag{3-20}$$

Combining (3-6), (3-7), (3-9), and (3-20) yields

$$\mathcal{L}\Psi \le -A_3c_1\sum f_i\lambda_i^2 + (-A_1 + CA_2 + CA_3)\sum f_i - A_1 + CA_3 \tag{3-21}$$

We now fix N such that (3-16) holds when $|\lambda| > R$, while (3-20) holds when $|\lambda| \le R$, for any s and δ sufficiently small.

Case (b): $|\nu_\mu - \nu_\lambda| \ge \beta$. It follows from Lemma 2.2 that, for some $\varepsilon > 0$,

$$\mathscr{L}(\underline{u} - u) \ge \sum f_i(\mu_i - \lambda_i) - (\underline{u} - u)_t \ge \varepsilon\Big(1 + \sum f_i\Big).$$

By (3-15), we may fix s and δ sufficiently small such that $v \ge 0$ on $\overline{M_T^\delta}$ and

$$\mathscr{L}v \le -\frac{\varepsilon}{2}\Big(1 + \sum f_i\Big) \quad \text{in } M_T^\delta. \tag{3-22}$$

Finally, we choose A_2 large such that

$$(A_2 - K)\rho^2 \ge A_3 \sum_{l<n} |\nabla_l(u - \varphi)|^2 \quad \text{on } \partial M_T^\delta,$$

and then fix A_1 sufficiently large so that (3-4) holds. In case (a) this follows from (3-17) when $|\lambda| > R$, and from (3-21) when $|\lambda| \le R$. In case (b) we note that, from (3-6) and (3-7),

$$\begin{aligned}\mathscr{L}\Psi &\le A_1\mathscr{L}v + A_2\mathscr{L}\rho^2 - A_3\sum_{i\ne r} f_i\lambda_i^2 + CA_3\Big(1 + \sum f_i|\lambda_i| + \sum f_i\Big)\\ &\le A_1\mathscr{L}v - A_3\sum_{i\ne r} f_i\lambda_i^2 + CA_3\Big(1 + \sum f_i|\lambda_i|\Big) + C(A_2 + A_3)\sum f_i.\end{aligned}$$

Suppose now that $\lambda_r < 0$. Then,

$$\sum f_i|\lambda_i| = 2\sum_{\lambda_i>0} f_i\lambda_i - \sum f_i\lambda_i \le \epsilon \sum_{\lambda_i>0} f_i\lambda_i^2 - \mathscr{L}v + C\sum f_i + C.$$

Similarly, if $\lambda_r > 0$,

$$\sum f_i|\lambda_i| = \sum f_i\lambda_i - 2\sum_{\lambda_i<0} f_i\lambda_i \le \epsilon \sum_{\lambda_i<0} f_i\lambda_i^2 + \mathscr{L}v + C\sum f_i + C.$$

By (3-22) we obtain (3-4) when A_1 is chosen sufficiently large. □

Applying Lemma 3.1, by (3-5) we immediately derive a bound for the mixed tangential–normal derivatives at any point $(x_0, t_0) \in \partial M_T$,

$$|\nabla_{n\alpha} u(x_0, t_0)| \le C \quad \text{for all } \alpha < n. \tag{3-23}$$

It remains to establish the double normal derivative estimate

$$|\nabla_{nn} u(x_0, t_0)| \le C. \tag{3-24}$$

As in [Guan 2014a; 2014b], we use an idea originally due to Trudinger [1995].

For $(x, t) \in \partial_s M_T$, let $\tilde{U}(x, t)$ be the restriction to $T_x\partial M$ of $U(x, t)$, viewed as a bilinear map on the tangent space of M at x, and let $\lambda'(\tilde{U})$ denote the eigenvalues of $\tilde{U}$ with respect to the induced metric on ∂M. We next show that there are uniform positive constants c_0, R_0 such that, for all $R > R_0$, $\big(\lambda'(\tilde{U}(x, t)), R\big) \in \Gamma$ and

$$f\big(\lambda'(\tilde{U}(x, t)), R\big) \ge f\big(\lambda(U(x, t))\big) + c_0, \quad \text{for all } 0 \le t \le T,\ x \in \partial M. \tag{3-25}$$

It is known that (3-25) implies (3-24); see, e.g., [Guan 2014b].

For $R > 0$ sufficiently large, let

$$m_R := \min_{\partial_s M_T} [f(\lambda'(\tilde{U}), R) - f(\lambda(U))],$$
$$c_R := \min_{\partial_s M_T} [f(\lambda'(\tilde{\underline{U}}), R) - f(\lambda(\underline{U}))].$$

Note that $\big(\lambda'(\tilde{U}(x,t)), R\big) \in \Gamma$ and $\big(\lambda'(\tilde{\underline{U}}(x,t)), R\big) \in \Gamma$ for all $(x,t) \in \partial_s M_T$ and all R large, and it is clear that both m_R and c_R are increasing in R. We wish to show that, for some uniform $c_0 > 0$,

$$\tilde{m} := \lim_{R \to \infty} m_R \geq c_0.$$

Assume $\tilde{m} < \infty$ (otherwise we are done) and fix $R > 0$ such that $c_R > 0$ and $m_R \geq \tilde{m}/2$. Let $(x_0, t_0) \in \partial_s M_T$ be such that $m_R = f\big(\lambda'(\tilde{U}(x_0, t_0)), R\big)$. Choose local orthonormal frames $e_1, \dots, e_n$ around x_0 as before such that e_n is the interior normal to ∂M along the boundary and $U_{\alpha\beta}(x_0, t_0)$ $(1 \leq \alpha, \beta \leq n-1)$ is diagonal. Since $u - \underline{u} = 0$ on $\partial_s M_T$, we have

$$U_{\alpha\beta} - \underline{U}_{\alpha\beta} = -\nabla_n (u - \underline{u}) \sigma_{\alpha\beta} \quad \text{on } \partial_s M_T, \tag{3-26}$$

where $\sigma_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n \rangle$. Similarly,

$$U_{\alpha\beta} - \nabla_{\alpha\beta}\varphi - \chi_{\alpha\beta}\varphi = -\nabla_n (u - \varphi) \sigma_{\alpha\beta} \quad \text{on } \partial_s M_T. \tag{3-27}$$

For an $(n-1) \times (n-1)$ symmetric matrix $\{r_{\alpha,\beta}\}$ with $\big(\lambda'(\{r_{\alpha,\beta}\}), R\big) \in \Gamma$, define

$$\tilde{F}[r_{\alpha\beta}] := f\big(\lambda'(\{r_{\alpha,\beta}\}), R\big)$$

and

$$\tilde{F}_0^{\alpha\beta} = \frac{\partial \tilde{F}}{\partial r_{\alpha\beta}} [U_{\alpha\beta}(x_0, t_0)].$$

We see that $\tilde{F}$ is concave since f is, and therefore, by (3-26),

$$\nabla_n (u - \underline{u})(x_0, t_0) \tilde{F}_0^{\alpha\beta} \sigma_{\alpha\beta}(x_0) \geq \tilde{F}[\underline{U}_{\alpha\beta}(x_0, t_0)] - \tilde{F}[U_{\alpha\beta}(x_0, t_0)] \geq c_R - m_R.$$

Suppose that

$$\nabla_n (u - \underline{u})(x_0, t_0) \tilde{F}_0^{\alpha\beta} \sigma_{\alpha\beta}(x_0) \leq \frac{c_R}{2};$$

then $m_R \geq c_R/2$ and we are done. So we shall assume

$$\nabla_n (u - \underline{u})(x_0, t_0) \tilde{F}_0^{\alpha\beta} \sigma_{\alpha\beta}(x_0) > \frac{c_R}{2}.$$

Consequently,

$$\tilde{F}_0^{\alpha\beta} \sigma_{\alpha\beta}(x_0) \geq \frac{c_R}{2 \nabla_n (u - \underline{u})(x_0, t_0)} \geq 2\epsilon_1 c_R \tag{3-28}$$

for some constant $\epsilon_1 > 0$ depending on $\max_{\partial_s M_T} |\nabla u|$. By continuity, we may assume $\eta := \tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta} \geq \epsilon_1 c_R$ on $\overline{M_T^\delta}$ by requiring δ to be small (which may depend on the fixed R). Define, in M_T^δ,

$$\Phi = -\nabla_n(u-\varphi) + \frac{Q}{\eta}, \tag{3-29}$$

where

$$Q = \tilde{F}_0^{\alpha\beta}(\nabla_{\alpha\beta}\varphi + \chi_{\alpha\beta} - U_{\alpha\beta}(x_0,t_0)) - \underline{u}_t - \psi + u_t(x_0,t_0) + \psi(x_0,t_0)$$

is smooth in M_T^δ. By (3-5), we have

$$\mathscr{L}\Phi \leq -\mathscr{L}\nabla_n u + C\Big(1 + \sum F^{ii}\Big) \leq C\Big(1 + \sum f_i|\lambda_i| + \sum f_i\Big). \tag{3-30}$$

From (3-27), we see that $\Phi(x_0,t_0) = 0$ and

$$\Phi \geq 0 \quad \text{on } \overline{M_T^\delta} \cap \partial_s M_T, \tag{3-31}$$

since, for $(x,t) \in \partial_s M_T$, by the concavity of $\tilde{F}$,

$$\begin{aligned} \tilde{F}_0^{\alpha\beta}(U_{\alpha\beta}(x,t) - U_{\alpha\beta}(x_0,t_0)) &\geq \tilde{F}(\tilde{U}(x,t)) - \tilde{F}(\tilde{U}(x_0,t_0)) \\ &= \tilde{F}(\tilde{U}(x,t)) - m_R - u_t(x_0,t_0) - \psi(x_0,t_0) \\ &\geq \psi(x,t) + u_t(x,t) - u_t(x_0,t_0) - \psi(x_0,t_0). \end{aligned}$$

On the other hand, on $\partial_b M_T^\delta$ we have $\nabla_n(u-\varphi) = 0$ and therefore, by (3-31),

$$\Phi(x,0) \geq \Phi(\hat{x},0) - Cd(x) \geq -Cd(x), \tag{3-32}$$

where C depends on C^1 bounds of $\nabla^2\varphi(\,\cdot\,,0)$, $\underline{u}_t(\,\cdot\,,0)$, and $\psi(\,\cdot\,,0)$ on $\overline{M}$, and $\hat{x} \in \partial M$ satisfies $d(x) = \operatorname{dist}(x,\hat{x})$ for $x \in M$; when $d(x)$ is sufficiently small, $\hat{x}$ is unique.

Finally, note that $|\Phi| \leq C$ in M_T^δ. So we may apply Lemma 3.1 to derive $\Psi + \Phi \geq 0$ on ∂M_T^δ and

$$\mathscr{L}(\Psi + \Phi) \leq 0 \quad \text{in } M_T^\delta \tag{3-33}$$

for A_1, A_2, and A_3 sufficiently large. By the maximum principle, $\Psi + \Phi \geq 0$ in M_T^δ. This gives $\nabla_n\Phi(x_0,t_0) \geq -\nabla_n\Psi(x_0,t_0) \geq -C$, since $\Phi + \Psi = 0$ at (x_0,t_0), and, therefore, $\nabla_{nn}u(x_0,t_0) \leq C$.

Consequently, we have obtained a priori bounds for all second derivatives of u at (x_0,t_0). It follows that $\lambda(U(x_0,t_0))$ is contained in a compact subset of Γ (independent of u) by assumption (1-4). Therefore,

$$c_0 \equiv \frac{f\big(\lambda(U(x_0,t_0)) + R\boldsymbol{e}_n\big) - f\big(\lambda(U(x_0,t_0))\big)}{2} > 0,$$

where $\boldsymbol{e}_n = (0,\dots,0,1) \in \mathbb{R}^n$. By Lemma 1.2 in [Caffarelli et al. 1985], we have

$$\tilde{m} \geq m_{R'} \geq f\big(\lambda(U(x_0,t_0)) + R'\boldsymbol{e}_n\big) - c_0 - f\big(\lambda(U(x_0,t_0))\big) \geq c_0$$

for $R' \geq R$ sufficiently large. The proof of (1-11) in Theorem 1.1 is complete.

Remark 3.2. When M is a bounded smooth domain in $\mathbb{R}^n$, one can make use of an identity in [Caffarelli et al. 1985], and modify the operator $\mathscr{L}$, to derive the boundary estimates without using assumption (1-19). We omit the proof here since it is similar to the elliptic case in [Guan 2014a], which we refer the reader to for details.

4. Existence and C^1 estimates

In order to prove Theorem 1.10, it remains to derive the C^1 estimate

$$|u|_{C^0(\overline{M_T})} + \max_{\overline{M}\times[t_0,T]} (|\nabla u| + |u_t|) \le C \tag{4-1}$$

for any $t_0 \in (0, T)$, where C may depend on t_0. Indeed, by assumption (1-4) we see that (1-1) becomes uniformly parabolic once the $C^{2,1}$ estimate

$$|u|_{C^{2,1}(\overline{M}\times[t_0,T])} \le C$$

is established, which yields $|u|_{C^{2+\alpha,1+\alpha/2}(\overline{M}\times[t_0,T])} \le C$ by the Evans–Krylov theorem (see, e.g., [Lieberman 1996]). Higher-order estimates now follow from the classical Schauder theory of linear parabolic equations, and one obtains a smooth admissible solution in $0 \le t \le T$ by the short-time existence and continuation. We refer the reader to [Lieberman 1996] for details.

Let $h \in C^2(\overline{M}_T)$ be the solution of $\Delta h + \operatorname{tr}\chi = 0$ in $\overline{M}_T$ with $h = \varphi$ on ∂M_T. By the maximum principle we have $\underline{u} \le u \le h$, which gives a bound

$$|u|_{C^0(\overline{M_T})} + \max_{\partial M_T} |\nabla u| \le C. \tag{4-2}$$

For the bound of u_t, we have the following maximum principle:

Lemma 4.1. *We have*

$$|u_t(x,t)| \le \max_{\partial M_T} |u_t| + t \sup_{M_T} |\psi_t| \quad \textit{for all } (x,t) \in \overline{M}_T. \tag{4-3}$$

Suppose moreover that there is a strictly convex function $h \in C^2(\overline{M})$ with $\nabla^2 h \ge c_0 g$ for some $c_0 > 0$. Then

$$\sup_{M_T} |u_t| \le \max_{\partial M_T} |u_t| + 2 \sup_{M_T} |\psi| + \frac{2|h|_{C^0(\overline{M})}}{c_0} \sup_{M_T} |\nabla^2 \psi|. \tag{4-4}$$

Proof. We have the identities $\mathscr{L} u_t = \psi_t$ and

$$|\mathscr{L}(u_t + \psi)| = |F^{ij} \nabla_{ij} \psi| \le |\nabla^2 \psi| \sum F^{ii}.$$

Therefore,

$$\mathscr{L}(\pm u_t - Bt) = \pm \psi_t + B \ge 0$$

for $B \ge \sup_{M_T} |\psi_t|$. This gives (4-3), by the maximum principle. Similarly, (4-4) follows from

$$\mathscr{L}(\pm(u_t + \psi) + Bh) \ge (c_0 B - |\nabla^2 \psi|) \sum F^{ii} \ge 0$$

for $B \ge c_0^{-1} \sup_{M_T} |\nabla^2 \psi|$ and the maximum principle. □

It remains to derive the gradient estimate

$$\sup_{M_T} |\nabla u|^2 \le C\big(|u|_{C^0(\overline{M_T})} + \sup_{\partial M_T} |\nabla u|^2\big) \tag{4-5}$$

in each of the cases (i)–(iv) in Theorem 1.10. We shall omit case (i), which is trivial, and consider cases (ii)–(iv), following ideas from [Li 1990; Urbas 2002; Guan 2014b] in the elliptic case.

Let ϕ be a function to be chosen and assume that $|\nabla u| e^{\phi}$ achieves a maximum at an interior point $(x_0, t_0) \in M_T$. As before, we choose local orthonormal frames at x_0 such that both U_{ij} and F^{ij} are diagonal at (x_0, t_0), where

$$\frac{\nabla_k u \nabla_k u_t}{|\nabla u|^2} + \phi_t \ge 0, \qquad \frac{\nabla_k u \nabla_{ik} u}{|\nabla u|^2} + \nabla_i \phi = 0 \quad \text{for all } i = 1, \dots, n, \tag{4-6}$$

$$F^{ii} \frac{\nabla_k u \nabla_{iik} u + \nabla_{ik} u \nabla_{ik} u}{|\nabla u|^2} - 2F^{ii} \frac{(\nabla_k u \nabla_{ik} u)^2}{|\nabla u|^4} + F^{ii} \nabla_{ii} \phi \le 0. \tag{4-7}$$

We have, for any $0 < \epsilon < 1$,

$$\sum_k (\nabla_{ik} u)^2 = \sum_k (U_{ik} - \chi_{ik})^2 \ge (1-\epsilon) U_{ii}^2 - \frac{C}{\epsilon} \tag{4-8}$$

and

$$\Big(\sum_k \nabla_k u \nabla_{ik} u\Big)^2 \le (1+\epsilon) |\nabla_i u|^2 U_{ii}^2 + \frac{C}{\epsilon} |\nabla u|^2. \tag{4-9}$$

Let $\epsilon = \frac{1}{3}$ and $J = \{i : 2(n+2)|\nabla_i u|^2 > |\nabla u|^2\}$; note that $J \ne \varnothing$ and, by (4-8) and (4-9),

$$\begin{aligned}\sum_{i \notin J} F^{ii} (|\nabla u|^2 \nabla_{ik} u \nabla_{ik} u - 2(\nabla_k u \nabla_{ik} u)^2) &\ge \sum_{i \notin J} F^{ii} (|\nabla u|^2 (1-\epsilon) - 2(1+\epsilon)|\nabla_i u|^2) U_{ii}^2) - \frac{C}{\epsilon} |\nabla u|^2 \\ &\ge -\frac{C}{\epsilon} |\nabla u|^2.\end{aligned} \tag{4-10}$$

We derive, from (2-10), (4-6), (4-7) and (4-10),

$$\tfrac{1}{3} F^{ii} U_{ii}^2 - 2|\nabla u|^2 \sum_{i \in J} F^{ii} |\nabla_i \phi|^2 + |\nabla u|^2 (F^{ii} \nabla_{ii} \phi - \phi_t) \le C(1 - K_0 |\nabla u|^2) \sum F^{ii} + C|\nabla u|, \tag{4-11}$$

where $K_0 = \inf_{k,l} R_{klkl}$.

Let

$$\phi = -\log(1 - bv^2) + A(\underline{u} + w - Bt),$$

where v is a positive function, and A, B and b are constant, all to be determined; b will be chosen sufficiently small such that $14bv^2 \le 1$ in $\overline{M_T}$, while $A = 0$ in cases (ii) and (iii). By straightforward calculations,

$$\nabla_i \phi = \frac{2bv \nabla_i v}{1 - bv^2} + A \nabla_i (\underline{u} + w), \qquad \phi_t = \frac{2bv v_t}{1 - bv^2} + A(\underline{u}_t - B)$$

and

$$\nabla_{ii}\phi = \frac{2bv\nabla_{ii}v + 2b|\nabla_i v|^2}{1-bv^2} + \frac{4b^2v^2|\nabla_i v|^2}{(1-bv^2)^2} + A\nabla_{ii}(\underline{u}+w)$$
$$= \frac{2bv\nabla_{ii}v}{1-bv^2} + \frac{2b(1+bv^2)|\nabla_i v|^2}{(1-bv^2)^2} + A\nabla_{ii}(\underline{u}+w).$$

Plugging these into (4-11), we obtain

$$\tfrac{1}{3}F^{ii}U_{ii}^2 + |\nabla u|^2 \sum_{i\in J} F^{ii}\left(\frac{b(1-7bv^2)|\nabla_i v|^2}{(n+2)(1-bv^2)^2} - CA^2\right) + \frac{2bv|\nabla u|^2}{1-bv^2}(F^{ii}\nabla_{ii}v - v_t) + A|\nabla u|^2(F^{ii}\nabla_{ii}(\underline{u}+w) - \underline{u}_t + B) \le C(1-K_0|\nabla u|^2)\sum F^{ii} + C|\nabla u|. \quad (4\text{-}12)$$

In both cases (ii) and (iv), we take

$$v = \underline{u} - u + \sup_{\bar{M}_T}(u-\underline{u}) + 1 \ge 1.$$

Let $\mu = \lambda(\nabla^2\underline{u}(x_0,t_0) + \chi(x_0))$, $\lambda = \lambda(\nabla^2 u(x_0,t_0) + \chi(x_0))$, and β as in (2-25). Suppose first that $|\nu_\mu - \nu_\lambda| \ge \beta$. By Lemma 2.2 and the assumptions that $\sum f_i\lambda_i \ge 0$ and $\nabla^2 w \ge \chi$, we see that

$$F^{ii}\nabla_{ii}(\underline{u}+w) - \underline{u}_t + B \ge F^{ii}\nabla_{ii}v - v_t + (B-u_t) \ge \varepsilon\sum F^{ii} + \varepsilon + (B-u_t)$$

for some $\varepsilon > 0$. Let $A = A_1K_0^-/\varepsilon$, $K_0^- = \max\{-K_0, 0\}$, and fix A_1, B sufficiently large. A bound $|\nabla u| \le C$ follows from (4-12) in both cases (ii) and (iv).

We now consider the case $|\nu_\mu - \nu_\lambda| < \beta$. By (2-27) and (4-12), we see that, if $|\nabla u|$ is sufficiently large,

$$\frac{\beta}{\sqrt{n}}(|\lambda|^2 + c_1|\nabla u|^4)\sum F^{ii} \le F^{ii}U_{ii}^2 + 2c_1|\nabla u|^4\sum_{i\in J}F^{ii} \le C(1-K_0|\nabla u|^2)\sum F^{ii} + C|\nabla u|, \quad (4\text{-}13)$$

where $c_1 > 0$.

Suppose $|\lambda| \ge R$ for R sufficiently large. Then

$$\frac{\beta}{\sqrt{n}}(|\lambda|^2 + c_1|\nabla u|^4)\sum F^{ii} \ge \frac{2\beta|\lambda|\sqrt{c_1}}{\sqrt{n}}|\nabla u|^2\sum F^{ii} \ge c_2|\nabla u|^2 \quad (4\text{-}14)$$

for some uniform $c_2 > 0$. We obtain from (4-13) and (4-14) a bound for $|\nabla u(x_0,t_0)|$.

Suppose now that $|\lambda| \le R$. Then $\sum F^{ii}$ has a positive lower bound, by (3-18) and (3-19). Therefore, a bound $|\nabla u(x_0,t_0)|$ follows from (4-13) again. This completes the proof of (4-5) in cases (ii) and (iv).

For case (iii) we choose $A = 0$ and $\phi = (u - \inf_{M_T} u + 1)^2$. By (4-12)

$$|\nabla u|^4\sum_{i\in J}F^{ii} \le C(1-K_0|\nabla u|^2)\sum F^{ii} + C|\nabla u|. \quad (4\text{-}15)$$

By (4-6) we see that $U_{ii} \le 0$ for each $i \in J$ if $|\nabla u|$ is sufficiently large, and a bound for $|\nabla u(x_0,t_0)|$ therefore follows from (4-15) and assumption (1-18).

Acknowledgment

Part of this work was done while Shi was visiting the Department of Mathematics at Ohio State University. He wishes to thank the department and university for their hospitality.

References

[Caffarelli et al. 1985] L. Caffarelli, L. Nirenberg, and J. Spruck, "The Dirichlet problem for nonlinear second-order elliptic equations, III: Functions of the eigenvalues of the Hessian", *Acta Math.* **155**:3–4 (1985), 261–301. MR 87f:35098 Zbl 0654.35031

[Guan 2014a] B. Guan, "The Dirichlet problem for fully nonlinear elliptic equations on Riemannian manifolds", preprint, 2014. arXiv 1403.2133

[Guan 2014b] B. Guan, "Second-order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds", *Duke Math. J.* **163**:8 (2014), 1491–1524. MR 3284698 Zbl 1296.58012

[Ivochkina and Ladyzhenskaya 1995] N. Ivochkina and O. Ladyzhenskaya, "Flows generated by symmetric functions of the eigenvalues of the Hessian", *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) **221**:26 (1995), 127–144, 258. Reprinted in *J. Math. Sci.* **87**:2 (1997), 3353–3365. MR 96j:35128 Zbl 0888.35056

[Jiao and Sui 2015] H. Jiao and Z. Sui, "The first initial-boundary value problem for a class of fully nonlinear parabolic equations on Riemannian manifolds", *Int. Math. Res. Not.* **2015**:9 (2015), 2576–2595. MR 3344681 Zbl 06440989

[Li 1990] Y. Y. Li, "Some existence results for fully nonlinear elliptic equations of Monge–Ampère type", *Comm. Pure Appl. Math.* **43**:2 (1990), 233–271. MR 91b:58269 Zbl 0705.35038

[Lieberman 1996] G. M. Lieberman, *Second order parabolic differential equations*, World Scientific, River Edge, NJ, 1996. MR 98k:35003 Zbl 0884.35001

[Trudinger 1995] N. S. Trudinger, "On the Dirichlet problem for Hessian equations", *Acta Math.* **175**:2 (1995), 151–164. MR 96m:35113 Zbl 0887.35061

[Urbas 2002] J. Urbas, "Hessian equations on compact Riemannian manifolds", pp. 367–377 in *Nonlinear problems in mathematical physics and related topics, II*, edited by M. S. Birman et al., Int. Math. Ser. (N. Y.) **2**, Kluwer/Plenum, New York, 2002. MR 2004e:58035 Zbl 1052.58025

[Wei and Wylie 2009] G. Wei and W. Wylie, "Comparison geometry for the Bakry–Emery Ricci tensor", *J. Differential Geom.* **83**:2 (2009), 377–405. MR 2011a:53064 Zbl 1189.53036

Received 21 Oct 2014. Revised 13 Feb 2015. Accepted 30 Apr 2015.

BO GUAN: guan.19@osu.edu
Department of Mathematics, Ohio State University, 231 W 18th Street, Columbus, OH 43210, United States

and

School of Mathematical Sciences, Xiamen University, Xiamen 361005, China

SHUJUN SHI: shjshi@163.com
School of Mathematical Sciences, Harbin Normal University, Harbin 150025, China

ZHENAN SUI: sui@math.osu.edu
Department of Mathematics, Ohio State University, 231 W 18th Street, Columbus, OH 43210, United States

ANALYSIS AND PDE
Vol. 8, No. 5, 2015
dx.doi.org/10.2140/apde.2015.8.1165

CONCENTRATION PHENOMENA FOR THE NONLOCAL SCHRÖDINGER EQUATION WITH DIRICHLET DATUM

JUAN DÁVILA, MANUEL DEL PINO, SERENA DIPIERRO AND ENRICO VALDINOCI

For a smooth, bounded domain Ω, $s \in (0, 1)$, $p \in (1, (n+2s)/(n-2s))$ we consider the nonlocal equation

$$\varepsilon^{2s}(-\Delta)^s u + u = u^p \quad \text{in } \Omega$$

with zero Dirichlet datum and a small parameter $\varepsilon > 0$. We construct a family of solutions that concentrate as $\varepsilon \to 0$ at an interior point of the domain in the form of a scaling of the ground state in entire space. Unlike the classical case $s = 1$, the leading order of the associated reduced energy functional in a variational reduction procedure is of polynomial instead of exponential order on the distance from the boundary, due to the nonlocal effect. Delicate analysis is needed to overcome the lack of localization, in particular establishing the rather unexpected asymptotics for the Green function of $\varepsilon^{2s}(-\Delta)^s + 1$ in the expanding domain $\varepsilon^{-1}\Omega$ with zero exterior datum.

1. Introduction

Given $s \in (0, 1)$, $n \in \mathbb{N}$ with $n > 2s$, $p \in (1, (n+2s)/(n-2s))$ and a bounded smooth domain $\Omega \subset \mathbb{R}^n$, we consider the fractional Laplacian problem

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s U + U = U^p & \text{in } \Omega, \\ U = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1-1}$$

where $\varepsilon > 0$ is a small parameter.

MSC2010: 35R11.
Keywords: nonlocal quantum mechanics, Green functions, concentration phenomena.

As usual, the operator $(-\Delta)^s$ is the fractional Laplacian, defined at any point $x \in \mathbb{R}^n$ as

$$(-\Delta)^s U(x) := c(n,s) \int_{\mathbb{R}^n} \frac{2U(x) - U(x+y) - U(x-y)}{|y|^{n+2s}}\, dy$$

for a suitable positive normalizing constant $c(n,s)$. We refer to [Landkof 1972; Silvestre 2005; Di Nezza et al. 2012] for an introduction to the fractional Laplacian operator.

We provide in the Appendix a heuristic physical motivation of the problem considered and of the relevance of our results in the light of a nonlocal quantum mechanics theory.

The goal of this paper is to construct solutions of problem (1-1) that concentrate at interior points of the domain for sufficiently small values of ε. More precisely, we shall establish the existence of a solution U_ε that at main order looks like

$$U_\varepsilon(x) \approx w\left(\frac{x - \tilde{\xi}_\varepsilon}{\varepsilon}\right). \tag{1-2}$$

Here $\tilde{\xi}_\varepsilon$ is a point lying at a uniformly positive distance from the boundary $\partial\Omega$ and w designates the unique radial positive *least energy solution* of the problem

$$(-\Delta)^s w + w = w^p, \quad w \in H^s(\mathbb{R}^n). \tag{1-3}$$

See, for instance, [Felmer et al. 2012] for the existence of such a solution and its basic properties. See [Amick and Toland 1991; Frank and Lenzmann 2013; Fall and Valdinoci 2014] for the (delicate) proof of uniqueness in special situations and [Frank et al. 2015] for the general case. The solution w is smooth and has the asymptotic behavior

$$\alpha|x|^{-(n+2s)} \leqslant w(x) \leqslant \beta|x|^{-(n+2s)} \quad \text{for } |x| \geqslant 1 \tag{1-4}$$

for some positive constants α, β; see Theorem 1.5 of [Felmer et al. 2012] and the lower bound in formula (IV.6) of [Carmona et al. 1990].

Our main result is the following:

Theorem 1.1. *If ε is sufficiently small, there exist a point $\tilde{\xi}_\varepsilon \in \Omega$ and a solution U_ε of problem (1-1) such that*

$$\left| U_\varepsilon(x) - w\left(\frac{x - \tilde{\xi}_\varepsilon}{\varepsilon}\right) \right| \leqslant C\varepsilon^{n+2s} \tag{1-5}$$

and $\operatorname{dist}(\tilde{\xi}_\varepsilon, \partial\Omega) \geqslant c$. *Here, c and C are positive constants independent of ε and Ω.*

Further, the point $\xi_\varepsilon := \tilde{\xi}_\varepsilon/\varepsilon$ is such that

$$\mathcal{H}_\varepsilon(\xi_\varepsilon) = \min_{\xi \in \Omega_\varepsilon} \mathcal{H}_\varepsilon(\xi) + O(\varepsilon^{n+4s}) \tag{1-6}$$

for the functional $\mathcal{H}_\varepsilon(\xi)$ defined in (1-17) below, where

$$\Omega_\varepsilon := \frac{\Omega}{\varepsilon} = \left\{ \frac{x}{\varepsilon} \,\middle|\, x \in \Omega \right\}. \tag{1-7}$$

The basic idea of the proof, which also leads to the characterization (1-6) of the location of the point $\tilde{\xi}_\varepsilon$ goes as follows. Letting $u(x) := U(\varepsilon x)$, problem (1-1) becomes

$$\begin{cases} (-\Delta)^s u + u = u^p & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon, \end{cases} \tag{1-8}$$

where Ω_ε is as defined in (1-7).

For a given $\xi \in \Omega_\varepsilon$, a first approximation $\bar{u}_\xi$ for the solution of problem (1-8) consistent with the desired form (1-2) and the Dirichlet exterior condition can be taken as the solution of the linear problem

$$\begin{cases} (-\Delta)^s \bar{u}_\xi + \bar{u}_\xi = w_\xi^p & \text{in } \Omega_\varepsilon, \\ \bar{u}_\xi = 0 & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon, \end{cases} \tag{1-9}$$

where

$$w_\xi(x) := w(x - \xi).$$

The actual solution will be obtained as a small perturbation from $\bar{u}_\xi$ for a suitable point $\xi = \xi_\varepsilon$. Problem (1-8) is variational. It corresponds to the Euler–Lagrange equation for the functional

$$I_\varepsilon(u) = \frac{1}{2} \int_{\Omega_\varepsilon} ((-\Delta)^s u(x) u(x) + u^2(x))\, dx - \frac{1}{p+1} \int_{\Omega_\varepsilon} u^{p+1}(x)\, dx, \quad u \in H_0^s(\Omega_\varepsilon), \tag{1-10}$$

where

$$H_0^s(\Omega_\varepsilon) = \{u \in H^s(\mathbb{R}^n) \mid u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega_\varepsilon\}.$$

Since the solution we look for should be close to $\bar{u}_\xi$ for $\xi = \xi_\varepsilon$, the functional $\xi \mapsto I_\varepsilon(\bar{u}_\xi)$ should have a critical point near $\xi = \xi_\varepsilon$. We shall next argue that this functional actually has a global minimizer located at distance $\sim 1/\varepsilon$ from $\partial\Omega_\varepsilon$.

The expansion of $I_\varepsilon(\bar{u}_\xi)$ involves the regular part of the Green function for the operator $(-\Delta)^s + 1$ in Ω_ε, which we define next. In $\mathbb{R}^n$ the operator $(-\Delta)^s + 1$ has a unique decaying fundamental solution Γ, which solves

$$(-\Delta)^s \Gamma + \Gamma = \delta_0. \tag{1-11}$$

The function Γ is radially symmetric, positive and satisfies

$$\frac{\alpha}{|x|^{n+2s}} \leqslant \Gamma(x) \leqslant \frac{\beta}{|x|^{n+2s}} \tag{1-12}$$

for $|x| \geqslant 1$ and $\alpha, \beta > 0$; see for instance Lemma C.1 in [Frank et al. 2015].

The Green function G_ε for $(-\Delta)^s + 1$ in Ω_ε solves

$$\begin{cases} (-\Delta)^s G_\varepsilon(x, y) + G_\varepsilon(x, y) = \delta_y & \text{if } x \in \Omega_\varepsilon, \\ G_\varepsilon(x, y) = 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega_\varepsilon. \end{cases} \tag{1-13}$$

In other words,

$$G_\varepsilon(x, y) := \Gamma(x - y) - H_\varepsilon(x, y), \tag{1-14}$$

where $H_\varepsilon(x, y)$, the regular part, satisfies, for fixed $y \in \mathbb{R}^n$,

$$\begin{cases} (-\Delta)^s H_\varepsilon(x, y) + H_\varepsilon(x, y) = 0 & \text{if } x \in \Omega_\varepsilon, \\ H_\varepsilon(x, y) = \Gamma(x - y) & \text{if } x \in \mathbb{R}^n \setminus \Omega_\varepsilon. \end{cases} \tag{1-15}$$

We will show in Theorem 4.1 that, for $\operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant \delta/\varepsilon$, with $\delta > 0$ fixed and appropriately small, we have that

$$I_\varepsilon(\bar{u}_\xi) = I_0 + \tfrac{1}{2}\mathcal{H}_\varepsilon(\xi) + o(\varepsilon^{n+4s}), \tag{1-16}$$

where I_0 is the energy of w computed in $\mathbb{R}^n$ and $\mathcal{H}_\varepsilon(\xi)$ is given by

$$\mathcal{H}_\varepsilon(\xi) := \int_{\Omega_\varepsilon} \int_{\Omega_\varepsilon} H_\varepsilon(x, y) w_\xi^p(x) w_\xi^p(y)\, dx\, dy. \tag{1-17}$$

We will show that $\mathcal{H}_\varepsilon$ satisfies

$$\frac{\alpha}{\operatorname{dist}(\xi, \partial\Omega_\varepsilon)^{n+4s}} \leqslant \mathcal{H}_\varepsilon(\xi) \leqslant \frac{\beta}{\operatorname{dist}(\xi, \partial\Omega_\varepsilon)^{n+4s}}, \tag{1-18}$$

where $\alpha, \beta > 0$, for all points $\xi \in \Omega_\varepsilon$ such that $\operatorname{dist}(\xi, \partial\Omega_\varepsilon) \in [5, \bar{\delta}/\varepsilon]$ for $\bar{\delta} > 0$ fixed, suitably small, and $\varepsilon \ll \bar{\delta}$.

From (1-18) and estimate (1-16), we deduce the existence of a global minimizer ξ_ε for the functional $I_\varepsilon(\bar{u}_\xi)$ for all small $\varepsilon > 0$, which is located at distance $\sim 1/\varepsilon$ from $\partial\Omega_\varepsilon$. The actual proof *reduces* the problem of finding a solution close to w_ξ via a Lyapunov–Schmidt procedure to that of finding a critical point ξ_ε of a functional with a similar expansion to (1-16), as we will see in Section 7.

In the classical case (i.e., when $s = 1$ and the operator boils down to the classical Laplacian), there is a broad literature on concentration phenomena; we recall here the seminal papers [Ni and Wei 1995; Li and Nirenberg 1998] and we refer to [Ambrosetti and Malchiodi 2006] for detailed discussions and more precise references. In particular, we recall that [Ni and Wei 1995; Li and Nirenberg 1998; del Pino and Felmer 1999; del Pino et al. 2000] construct solutions of the classical Dirichlet problem that concentrate at points which maximize the distance from the boundary; in this sense, Theorem 1.1 may be seen as the nonlocal counterpart of these results. In our case, the determination of the concentrating point is less explicit than in the classical case, due to the nonlocal behavior of the energy expansion. More precisely, for $s = 1$ one gets the expansion parallel to (1-16),

$$I_\varepsilon(\bar{u}_\xi) = I_0 + \tfrac{1}{2}\mathcal{H}_\varepsilon(\xi) + O(e^{-(2+\sigma)\operatorname{dist}(\xi, \partial\Omega_\varepsilon)/\varepsilon}),$$

where now

$$\mathcal{H}_\varepsilon(\xi) \approx e^{-2\operatorname{dist}(\xi, \partial\Omega_\varepsilon)/\varepsilon}; \tag{1-19}$$

see for instance Y. Li and L. Nirenberg [1998] (compare (1-19) with (1-18)).

In the nonlocal case, much less is known. Multipeak solutions of a fractional Schrödinger equation set in the whole of $\mathbb{R}^n$ were considered recently in [Dávila et al. 2014]. The analysis needed in this paper is considerably more involved. Concentrating solutions for fractional problems involving critical or almost critical exponents were considered in [Choi et al. 2014]. See also [Chen and Zheng 2014] for some concentration phenomena in particular cases, and also [Secchi 2013] and references therein for related problems about Schrödinger-type equations in a fractional setting.

The paper is organized as follows. The rather delicate analysis of the behavior of the regular part of Green's function is contained in Section 2. We estimate the function $\bar{u}_\xi$ in Section 3, thus obtaining a first approximation of the energy expansion in Section 4.

The remainders of this expansion need to be carefully estimated; for this, we provide some decay and regularity estimates in Sections 5 and 6.

The Lyapunov–Schmidt method will be resumed in Section 7, where we discuss the linear theory and the bifurcation from it. A key ingredient is the linear nondegeneracy of the least energy solution w; this is an important result that was completely achieved only recently in [Frank et al. 2015], after preliminary works in particular cases discussed in [Amick and Toland 1991; Frank and Lenzmann 2013; Fall and Valdinoci 2014]. Then we complete the proof of Theorem 1.1 in Section 8.

2. Estimates on the Robin function H_ε and on the leading term of the energy functional

Given $\xi \in \Omega_\varepsilon$ and $x \in \mathbb{R}^n$, we define

$$\beta_\xi(x) := \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} \Gamma(z-\xi)\Gamma(x-z)\, dz.$$

Notice that, for any $x \in \Omega_\varepsilon$ and $z \in \mathbb{R}^n \setminus \Omega_\varepsilon$, we have

$$((-\Delta)^s + 1)\Gamma(x-z) = \delta_0(x-z) = 0, \tag{2-1}$$

and so

$$((-\Delta)^s + 1)\beta_\xi(x) = \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} \Gamma(z-\xi)((-\Delta)^s + 1)\Gamma(x-z)\, dz = 0 \tag{2-2}$$

for any $x \in \Omega_\varepsilon$. Our purpose is to use $\beta_\xi(x)$ as a barrier, from above and below, for the Robin function $H_\varepsilon(x,\xi)$, using (1-15), (2-2) and the comparison principle. For this scope, we estimate the behavior of β_ξ outside Ω_ε:

Lemma 2.1. *There exists $c \in (0,1)$ such that*

$$c H_\varepsilon(x,\xi) \leqslant \beta_\xi(x) \leqslant c^{-1} H_\varepsilon(x,\xi) \tag{2-3}$$

for any $x \in \mathbb{R}^n \setminus \Omega_\varepsilon$ and $\xi \in \Omega_\varepsilon$ with

$$\operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 1. \tag{2-4}$$

Proof. First we observe that, for any $x \in \mathbb{R}^n \setminus \Omega_\varepsilon$,

$$|B_{1/2}(x) \setminus \Omega_\varepsilon| \geqslant c_\star \tag{2-5}$$

for a suitable $c_\star > 0$. For concreteness, one can take $c_\star$ as the measure of the spherical segment

$$\Sigma := \left\{ z = (z', z_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \;\middle|\; |z| < \tfrac{1}{2} \text{ and } z_n \geqslant \tfrac{1}{4} \right\}.$$

To prove (2-5), we argue as follows. If $B_{1/2}(x) \subseteq (\mathbb{R}^n \setminus \Omega_\varepsilon)$ we are done. If not, let $p \in (\partial\Omega_\varepsilon) \cap B_{1/2}(x)$ with $\operatorname{dist}(x, \partial\Omega_\varepsilon) = |x-p|$. Notice that the ball centered at x of radius $|x-p|$ is tangent to Ω_ε from the outside at p, and $|x-p| \leqslant \frac{1}{2}$.

Up to a rigid motion, we suppose that $p = 0$ and $x = |x|e_n$. By scaling back, the ball of radius $|\hat{x}|$ centered at $\hat{x} := \varepsilon x = \varepsilon|x|e_n$ is tangent to Ω from the outside at the origin, and $|\hat{x}| = \varepsilon|x| = \varepsilon|x-p| \leqslant \varepsilon/2$.

From the regularity of Ω, we have that there exists a ball of universal radius $r_o > 0$ touching Ω from the outside at any point, so in particular $B_{r_o}(r_o e_n)$ touches Ω from the outside at the origin, hence

$$B_{r_o}(r_o e_n) \subseteq \mathbb{R}^n \setminus \Omega. \tag{2-6}$$

We observe that

$$\hat{x} + \varepsilon\Sigma \subseteq B_{r_o}(r_o e_n). \tag{2-7}$$

Indeed, if $z = (z', z_n) \in \hat{x} + \varepsilon\Sigma$ then $\varepsilon\Sigma \ni z - \hat{x} = (z', z_n - |\hat{x}|)$ and so $z_n - |\hat{x}| \in [\varepsilon/4, \varepsilon/2]$ and $|z'| \leqslant |z| \leqslant \varepsilon/2$. Hence, for small ε, we have that $r_o - z_n \geqslant r_o - |\hat{x}| - \varepsilon/2 \geqslant r_o - \varepsilon \geqslant 0$ and $r_o - z_n \leqslant r_o - |\hat{x}| - \varepsilon/4 \leqslant r_o - \varepsilon/4$, and so

$$|z_n - r_o| = r_o - z_n \leqslant r_o - \frac{\varepsilon}{4},$$

which gives

$$|z - r_o e_n|^2 = |z'|^2 + |z_n - r_o|^2 \leqslant \left(\frac{\varepsilon}{2}\right)^2 + \left(r_o - \frac{\varepsilon}{4}\right)^2 = r_o^2 + \frac{\varepsilon^2}{16} + \frac{\varepsilon^2}{4} - \frac{2r_o\varepsilon}{2} < r_o$$

if ε is sufficiently small. This proves (2-7).

As a consequence of (2-6) and (2-7), we conclude that $\hat{x} + \varepsilon\Sigma \subseteq \mathbb{R}^n \setminus \Omega$, that is, by scaling back, $x + \Sigma \subseteq \mathbb{R}^n \setminus \Omega_\varepsilon$. Accordingly,

$$(B_{1/2}(x) \setminus \Omega_\varepsilon) \supseteq B_{1/2}(x) \cap (x + \Sigma) = x + \Sigma$$

and this ends the proof of (2-5).

Now we observe that if $a, b \in \mathbb{R}^n$ satisfy $|a-b| \leqslant |b-\xi|/2$ and $\min\{|a-\xi|, |b-\xi|\} \geqslant 1$ then

$$\Gamma(a-\xi) \leqslant C\Gamma(b-\xi) \tag{2-8}$$

for some $C > 0$. Indeed,

$$|a-\xi| \geqslant |b-\xi| - |a-b| \geqslant \frac{|b-\xi|}{2}$$

and so, from (1-12),

$$\Gamma(a-\xi) \leqslant \frac{C}{|a-\xi|^{n+2s}} \leqslant \frac{2^{n+2s}C}{|b-\xi|^{n+2s}} \leqslant 2^{n+2s}C^2\Gamma(b-\xi).$$

This proves (2-8), up to relabeling the constants. As a consequence, given $x \in \mathbb{R}^n \setminus \Omega_\varepsilon$, we apply (2-8) with $a := x$ and $b := z \in B_{1/2}(x) \setminus \Omega_\varepsilon$, we recall (2-4) and (2-5), and we obtain that

$$\begin{aligned} \beta_\xi(x) &\geqslant \int_{B_{1/2}(x)\setminus\Omega_\varepsilon} \Gamma(z-\xi)\Gamma(x-z)\,dz \geqslant C^{-1}\int_{B_{1/2}(x)\setminus\Omega_\varepsilon} \Gamma(x-\xi)\Gamma(x-z)\,dz \\ &\geqslant C^{-1}\Gamma(x-\xi)\inf_{y\in B_{1/2}}\Gamma(y)\big|B_{1/2}(x)\setminus\Omega_\varepsilon\big| \geqslant c_\star C^{-1}\Gamma(x-\xi)\inf_{y\in B_{1/2}}\Gamma(y). \end{aligned}$$

This proves the first inequality in (2-3), since $x \in \mathbb{R}^n \setminus \Omega_\varepsilon$ and so

$$\Gamma(x-\xi) = H_\varepsilon(x,\xi). \tag{2-9}$$

Now we prove the second inequality in (2-3). For this, we use (2-8) once again (applied here with $a := z$, $b := x$ and recalling (2-4)) and the fact that

$$\int_{\mathbb{R}^n} \Gamma(z)\,dz = 1 \tag{2-10}$$

to see that

$$I_1 := \int_{B_{|x-\xi|/2}(x)\setminus\Omega_\varepsilon} \Gamma(z-\xi)\Gamma(x-z)\,dz \leqslant C \int_{B_{|x-\xi|/2}(x)} \Gamma(x-\xi)\Gamma(x-z)\,dz \leqslant C\Gamma(x-\xi). \tag{2-11}$$

On the other hand, if $z \notin B_{|x-\xi|/2}(x)$, we have that $|x-z| \geqslant |x-\xi|/2$ and so, by (1-12),

$$\Gamma(x-z) \leqslant \frac{C}{|x-z|^{n+2s}} \leqslant \frac{2^{n+2s}C}{|x-\xi|^{n+2s}} \leqslant 2^{n+2s}C^2\Gamma(x-\xi).$$

Consequently,

$$I_2 := \int_{\mathbb{R}^n\setminus B_{|x-\xi|/2}(x)} \Gamma(z-\xi)\Gamma(x-z)\,dz \leqslant C' \int_{\mathbb{R}^n\setminus B_{|x-\xi|/2}(x)} \Gamma(z-\xi)\Gamma(x-\xi)\,dz \leqslant C'\Gamma(x-\xi)$$

for some $C' > 0$, thanks to (2-10). From this and (2-11) we obtain that

$$\beta_\xi(x) \leqslant I_1 + I_2 \leqslant C''\Gamma(x-\xi)$$

for some $C'' > 0$. This, together with (2-9), completes the proof of (2-3). □

Corollary 2.2. *There exists $c \in (0,1)$ such that*

$$cH_\varepsilon(x,\xi) \leqslant \beta_\xi(x) \leqslant c^{-1}H_\varepsilon(x,\xi)$$

for any $x \in \mathbb{R}^n$ and $\xi \in \Omega_\varepsilon$ with $\operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 1$.

Proof. The desired estimate holds true outside Ω_ε, thanks to (2-3). Then it holds true inside Ω_ε as well, in virtue of (2-2), (1-15) and the comparison principle. □

The above result implies an interesting lower bound on the symmetric version of the Robin function $H_\varepsilon(\xi,\xi)$, and in general for the values of the Robin function sufficiently close to the diagonal, according to the following:

Proposition 2.3. *Let $\delta \in (0,1)$. Let $\xi \in \Omega_\varepsilon$ with*

$$d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \in \left[2, \frac{\delta}{\varepsilon}\right].$$

Let $x, y \in B_{d/2}(\xi)$. Then

$$H_\varepsilon(x,y) \geqslant \frac{c_o}{d^{n+4s}}$$

for a suitable $c_o \in (0,1)$, as long as δ is sufficiently small.

Proof. Let $z \in \mathbb{R}^n \setminus \Omega_\varepsilon$. Notice that

$$|\xi - y| \leqslant \frac{d}{2} \leqslant \frac{|z-\xi|}{2} \tag{2-12}$$

and so

$$|z-y| \leqslant |z-\xi| + |\xi - y| \leqslant \tfrac{3}{2}|z-\xi|. \tag{2-13}$$

Similarly,

$$|z-x| \leqslant \tfrac{3}{2}|z-\xi|. \tag{2-14}$$

Another consequence of (2-12) is that

$$|z-y| \geqslant |z-\xi| - |\xi - y| \geqslant \frac{|z-\xi|}{2} \geqslant \frac{d}{2} \geqslant 1, \tag{2-15}$$

hence $\operatorname{dist}(y, \partial\Omega_\varepsilon) \geqslant 1$ (as a matter of fact, till now we only exploited that $d \geqslant 2$). Notice that in the same way, one has that

$$|z-x| \geqslant 1. \tag{2-16}$$

Therefore we can use Corollary 2.2 with ξ replaced by y and so, recalling (1-12), (2-15), (2-16), (2-13) and (2-14), we conclude that

$$\begin{aligned} H_\varepsilon(x, y) &\geqslant c\beta_y(x) \\ &= c \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} \Gamma(z-y)\Gamma(x-z)\, dz \\ &\geqslant \frac{c}{C^2} \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} \frac{1}{|y-z|^{n+2s}|x-z|^{n+2s}}\, dz \\ &\geqslant c' \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} \frac{1}{|z-\xi|^{2n+4s}}\, dz \end{aligned} \tag{2-17}$$

for a suitable $c' > 0$.

Now we introduce some geometric considerations. By the smoothness of the domain, we can touch Ω from the outside at any point with balls of universal radius, say $r_o > 0$. By scaling, we can touch Ω_ε from the exterior by balls of radius $r_o\varepsilon^{-1}$, and so of radius d (notice indeed that $d \leqslant \delta\varepsilon^{-1} \leqslant r_o\varepsilon^{-1}$ if δ is small enough). Let $\eta \in \partial\Omega_\varepsilon$ be such that $|\xi - \eta| = d$. By the above considerations, we can touch Ω_ε from the outside at η with a ball $\mathcal{B}$ of radius d (i.e., of diameter $2d$). We stress that $\mathcal{B} \subseteq \mathbb{R}^n \setminus \Omega_\varepsilon$, that $|\mathcal{B}| \geqslant \bar{c}d^n$ for some $\bar{c} > 0$, and that if $z \in \mathcal{B}$ then

$$|z-\xi| \leqslant |z-\eta| + |\eta - \xi| \leqslant 2d + d = 3d.$$

These observations and (2-17) yield that

$$H_\varepsilon(x, y) \geqslant c' \int_{\mathcal{B}} \frac{1}{|z-\xi|^{2n+4s}}\, dz \geqslant c'(3d)^{-(2n+4s)}|\mathcal{B}| = c' 3^{-(2n+4s)}\bar{c}\, d^{-(n+4s)},$$

as desired. □

There is also an upper bound similar to the lower bound obtained in Proposition 2.3:

Proposition 2.4. *Let $\xi \in \Omega_\varepsilon$ with $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 2$, and $x, y \in B_{d/2}(\xi)$. Then*

$$H_\varepsilon(x, y) \leqslant \frac{C_o}{d^{n+4s}}$$

for a suitable $C_o > 0$.

Proof. As noticed in the proof of Proposition 2.3, we can use Corollary 2.2 with ξ replaced by y. Then, since $B_d(\xi) \subseteq \Omega_\varepsilon$, we have $(\mathbb{R}^n \setminus \Omega_\varepsilon) \subseteq (\mathbb{R}^n \setminus B_d(\xi))$, and therefore we obtain that

$$H_\varepsilon(x, y) \leqslant c^{-1}\beta_y(x) \leqslant c^{-1} \int_{\mathbb{R}^n \setminus B_d(\xi)} \Gamma(z-\xi)\Gamma(x-z)\, dz.$$

Also, if $z \in \mathbb{R}^n \setminus B_d(\xi)$, we have that

$$|z-x| \geqslant |z-\xi| - |\xi - x| \geqslant d - |\xi - x| \geqslant \frac{d}{2},$$

hence, by (1-12),

$$H_\varepsilon(x, y) \leqslant c^{-1}C^2 \int_{\mathbb{R}^n \setminus B_d(\xi)} \frac{1}{|z-\xi|^{n+2s}|z-x|^{n+2s}}\, dz \leqslant c^{-1}C^2 \left(\frac{2}{d}\right)^{n+2s} \int_{\mathbb{R}^n \setminus B_d(\xi)} \frac{1}{|z-\xi|^{n+2s}}\, dz.$$

By computing the latter integral in polar coordinates, we obtain the desired result. □

It will be convenient to define, for any $\xi \in \Omega_\varepsilon$,

$$\Pi_\varepsilon(x, \xi) := \int_{\Omega_\varepsilon} H_\varepsilon(x, y) w_\xi^p(y)\, dy. \tag{2-18}$$

As a consequence of Proposition 2.4, we have:

Lemma 2.5. *Let $\xi \in \Omega_\varepsilon$ with $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 2$. Let $x \in B_{d/8}(\xi)$. Then*

$$\Pi_\varepsilon(x, \xi) \leqslant \frac{C}{d^{n+4s}}$$

for some $C > 0$, where $\Pi_\varepsilon(x, \xi)$ is as defined in (2-18).

Proof. We split the integral into two contributions, one in $B_{d/4}(\xi)$ and one outside such ball.

We can use Proposition 2.4 to obtain that, for $y \in \Omega_\varepsilon \cap B_{d/4}(\xi)$, it holds that $H_\varepsilon(x, y) \leqslant C_o d^{-n-4s}$ and so

$$\pi_1 := \int_{\Omega_\varepsilon \cap B_{d/4}(\xi)} H_\varepsilon(x, y) w_\xi^p(y)\, dy \leqslant C_o d^{-n-4s} \int_{\mathbb{R}^n} w_\xi^p(y)\, dy \leqslant C_1 d^{-n-4s},$$

for some $C_1 > 0$.

Now we consider the case in which $y \in \Omega_\varepsilon \setminus B_{d/4}(\xi)$. We use (1-4) to see that $w_\xi^p(y) \leqslant C_2|y-\xi|^{-p(n+2s)}$ for some $C_2 > 0$. Also, in this case,

$$|y-x| \geqslant |y-\xi| - |x-\xi| \geqslant \frac{d}{4} - \frac{d}{8} = \frac{d}{8};$$

hence, by the maximum principle,

$$H_\varepsilon(x, y) \leqslant \Gamma(x-y) \leqslant \frac{C_3}{|x-y|^{n+2s}} \leqslant \frac{C_4}{d^{n+2s}} \tag{2-19}$$

for some $C_3, C_4 > 0$. As a consequence,

$$\pi_2 := \int_{\Omega_\varepsilon \setminus B_{d/4}(\xi)} H_\varepsilon(x, y) w_\xi^p(y)\, dy \leqslant \frac{C_2 C_4}{d^{n+2s}} \int_{\mathbb{R}^n \setminus B_{d/4}(\xi)} |y - \xi|^{-p(n+2s)}\, dz = \frac{C_5}{d^{2s+p(n+2s)}}$$

for some $C_5 > 0$. In particular, since $d \geqslant 1$ and $p > 1$, we see that $\pi_2 \leqslant C_5 d^{-n-4s}$ and therefore, recalling (2-18), we conclude that $\Pi_\varepsilon(x, \xi) \leqslant \pi_1 + \pi_2 \leqslant (C_1 + C_5) d^{-n-4s}$. □

The function $\mathcal{H}_\varepsilon$ defined in (1-17) will represent the first interesting order in the expansion of the reduced energy functional (see Theorem 4.1 for a precise statement). To show that this reduced energy functional has a local minimum, we will show that $\mathcal{H}_\varepsilon$ (and so the reduced energy functional itself) attains, in a certain domain, values that are smaller than the ones attained at the boundary (concretely, this domain will be given by the subset of Ω_ε with points of distance δ/ε from the boundary for some $\delta \in (0, 1)$ fixed suitably small, possibly dependent on n, s and Ω).

To this extent, a detailed statement will be given in Proposition 2.8 and the necessary bounds on $\mathcal{H}_\varepsilon$ will be given in Corollaries 2.6 and 2.7, which in turn follow from Propositions 2.3 and 2.4, respectively.

Corollary 2.6. *Let $\delta \in (0, 1)$. Let $\xi \in \Omega_\varepsilon$ with*

$$d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \in \left[2, \frac{\delta}{\varepsilon}\right].$$

Then

$$\mathcal{H}_\varepsilon(\xi) \geqslant \frac{c}{d^{n+4s}}$$

for a suitable $c > 0$ as long as δ is sufficiently small.

Proof. Notice that $B_1(\xi) \subseteq B_{d/2}(\xi) \subseteq \Omega_\varepsilon$. So, by Proposition 2.3, $H_\varepsilon(x, y) \geqslant c_o d^{-(n+4s)}$ if $x, y \in B_1(\xi)$ and

$$\mathcal{H}_\varepsilon(\xi) \geqslant \int_{B_1(\xi)} \int_{B_1(\xi)} H_\varepsilon(x, y) w_\xi^p(x) w_\xi^p(y)\, dx\, dy \geqslant c_o d^{-(n+4s)} \left(\int_{B_1} w^p(z)\, dz \right)^2. \qquad \square$$

Corollary 2.7. *Let $\xi \in \Omega_\varepsilon$ with $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 5$. Then*

$$\mathcal{H}_\varepsilon(\xi) \leqslant \frac{C}{d^{n+4s}}$$

for a suitable $C > 0$.

Proof. We split the integral in (1-17) into three contributions: first we treat the case in which $x, y \in B_{d/2}(\xi)$, then the case in which $x, y \in \mathbb{R}^n \setminus B_{d/2}(\xi)$, and finally the case in which $x \in B_{d/2}(\xi)$ and $y \in \mathbb{R}^n \setminus B_{d/2}(\xi)$ (the case in which $y \in B_{d/2}(\xi)$ and $x \in \mathbb{R}^n \setminus B_{d/2}(\xi)$ is, of course, symmetrical to this one).

In the first case, we use Proposition 2.4, obtaining that

$$\begin{aligned} \int_{B_{d/2}(\xi)} dx \int_{B_{d/2}(\xi)} dy\, H_\varepsilon(x, y) w_\xi^p(x) w_\xi^p(y) &\leqslant C_o d^{-(n+4s)} \left(\int_{B_{d/2}} w^p(z)\, dz \right)^2 \\ &\leqslant C_o d^{-(n+4s)} \left(\int_{\mathbb{R}^n} w^p(z)\, dz \right)^2. \end{aligned} \tag{2-20}$$

In the second case, we twice use the decay of w given in (1-4), (2-19) and (2-10), obtaining that

$$
\begin{aligned}
\int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dx \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dy\, H_\varepsilon(x,y) w_\xi^p(x) w_\xi^p(y)
&\leqslant C^{2p} \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dx \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dy\, |x-\xi|^{-p(n+2s)} |y-\xi|^{-p(n+2s)} \Gamma(x-y) \\
&\leqslant C^{2p} (d/2)^{-p(n+2s)} \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dx \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dy\, |x-\xi|^{-p(n+2s)} \Gamma(x-y) \\
&\leqslant C^{2p} (d/2)^{-p(n+2s)} \int_{\mathbb{R}^n\setminus B_{d/2}} d\eta \int_{\mathbb{R}^n} d\theta\, |\eta|^{-p(n+2s)} \Gamma(\theta) \\
&\leqslant C' d^{-2p(n+2s)+n} \\
&\leqslant C' d^{-(n+4s)} \qquad (2\text{-}21)
\end{aligned}
$$

for some $C' > 0$.

As for the third case, we take $x \in B_{d/2}(\xi)$ and $y \in \mathbb{R}^n \setminus B_{d/2}(\xi)$ and we distinguish two subcases: either $|x-y| \leqslant d/6$ or $|x-y| > d/6$.

In the first subcase, we use a translated version of Proposition 2.4. If $x \in B_{d/2}(\xi)$, $y \in \mathbb{R}^n \setminus B_{d/2}(\xi)$ and $|x-y| \leqslant d/6$, we take $\hat{\xi} := (x+y)/2$. Notice that

$$|\xi - y| \leqslant |\xi - x| + |x-y| \leqslant \frac{d}{2} + \frac{d}{6}$$

and therefore

$$2|\hat{\xi} - \xi| = |(x+y) - 2\xi| \leqslant |x-\xi| + |y-\xi| \leqslant \frac{d}{2} + \left(\frac{d}{2} + \frac{d}{6}\right) = \frac{7d}{6}.$$

As a consequence,

$$\hat{d} := \operatorname{dist}(\hat{\xi}, \partial\Omega_\varepsilon) \geqslant \operatorname{dist}(\xi, \partial\Omega_\varepsilon) - |\hat{\xi} - \xi| \geqslant d - \frac{7d}{12} = \frac{5d}{12}. \qquad (2\text{-}22)$$

In particular,

$$\hat{d} \geqslant 2. \qquad (2\text{-}23)$$

Also, by construction, $x - \hat{\xi} = \hat{\xi} - y = (x-y)/2$, and so

$$|x - \hat{\xi}| = |\hat{\xi} - y| = \frac{1}{2}|x-y| \leqslant \frac{d}{12}.$$

This and (2-22) say that

$$x, y \in B_{d/12}(\hat{\xi}) \subseteq B_{\hat{d}/2}(\hat{\xi}). \qquad (2\text{-}24)$$

Thanks to (2-23) and (2-24), we can now use Proposition 2.4 with ξ and d replaced by $\hat{\xi}$ and $\hat{d}$, respectively. So we obtain that, in this case,

$$H_\varepsilon(x,y) \leqslant \frac{C_o}{\hat{d}^{n+4s}} \leqslant \frac{\hat{C}}{d^{n+4s}} \qquad (2\text{-}25)$$

for some $\hat{C} > 0$, where (2-22) was used again in the last inequality.

So, we make use of (1-4) and (2-25) to obtain that

$$\begin{aligned}\int_{B_{d/2}(\xi)} dx \int_{B_{d/6}(x)\setminus B_{d/2}(\xi)} dy\, H_\varepsilon(x,y) w_\xi^p(x) w_\xi^p(y) \\ &\leqslant C^p \hat{C}\, d^{-(n+4s)} \int_{B_{d/2}(\xi)} dx \int_{B_{d/6}(x)\setminus B_{d/2}(\xi)} dy\, w_\xi^p(x)|y-\xi|^{-p(n+2s)} \\ &\leqslant C^p \hat{C}\, d^{-(n+4s)} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n\setminus B_{d/2}} dz\, w_\xi^p(x)|z|^{-p(n+2s)} \\ &\leqslant \tilde{C} d^{-4s-p(n+2s)} \\ &\leqslant \tilde{C} d^{-(n+4s)}. \end{aligned} \tag{2-26}$$

Finally, we consider the subcase in which $x \in B_{d/2}(\xi)$, $y \in \mathbb{R}^n \setminus B_{d/2}(\xi)$ and $|x-y| > d/6$. In this circumstance we use (1-4), (2-19) and (1-12) to conclude that

$$\begin{aligned}\int_{B_{d/2}(\xi)} dx \int_{\substack{\mathbb{R}^n\setminus B_{d/2}(\xi)\\ \{|x-y|>d/6\}}} dy\, H_\varepsilon(x,y) w_\xi^p(x) w_\xi^p(y) \\ &\leqslant C^p \int_{B_{d/2}(\xi)} dx \int_{\substack{\mathbb{R}^n\setminus B_{d/2}(\xi)\\ \{|x-y|>d/6\}}} dy\, \Gamma(x-y) w_\xi^p(x)|y-\xi|^{-p(n+2s)} \\ &\leqslant \underline{C} \int_{B_{d/2}(\xi)} dx \int_{\substack{\mathbb{R}^n\setminus B_{d/2}(\xi)\\ \{|x-y|>d/6\}}} dy\, |x-y|^{-(n+2s)} w_\xi^p(x)|y-\xi|^{-p(n+2s)} \\ &\leqslant \underline{C}(d/6)^{-(n+2s)} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dy\, w_\xi^p(x)|y-\xi|^{-p(n+2s)} \\ &\leqslant \bar{C} d^{-2s-p(n+2s)} \\ &\leqslant \bar{C} d^{-(n+4s)} \end{aligned} \tag{2-27}$$

for suitable $\underline{C}, \bar{C} > 0$. From (2-26) and (2-27) we complete the third case, namely when $x \in B_{d/2}(\xi)$ and $y \in \mathbb{R}^n \setminus B_{d/2}(\xi)$, by obtaining that

$$\int_{B_{d/2}(\xi)} dx \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dy\, H_\varepsilon(x,y) w_\xi^p(x) w_\xi^p(y) \leqslant (\tilde{C}+\bar{C}) d^{-(n+4s)}. \tag{2-28}$$

The desired result follows from (1-17), (2-20), (2-21) and (2-28). □

For concreteness, we summarize the results of Corollaries 2.6 and 2.7 in the following:

Proposition 2.8. *Let $\delta > 0$ be suitably small and*

$$\Omega_{\varepsilon,\delta} := \{x \in \Omega_\varepsilon \mid \operatorname{dist}(x, \partial\Omega_\varepsilon) > \delta/\varepsilon\}. \tag{2-29}$$

Then $\mathcal{H}_\varepsilon$ attains an interior minimum in $\Omega_{\varepsilon,\delta}$, namely there exist $c_1, c_2 > 0$ such that

$$\min_{\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon \leqslant c_1 \varepsilon^{n+4s} < c_2 \left(\frac{\varepsilon}{\delta}\right)^{n+4s} \leqslant \min_{\partial\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon.$$

Proof. Let $\delta_\star$ be the maximal distance that a point of Ω may attain from the boundary of Ω. By scaling, the maximal distance that a point of Ω_ε may attain from the boundary of Ω_ε is $\delta_\star/\varepsilon$. Let $\xi_\star$ be such a point, i.e.,

$$d_\star := \operatorname{dist}(\xi_\star, \partial\Omega_\varepsilon) = \frac{\delta_\star}{\varepsilon}.$$

For δ sufficiently small we have that $\xi_\star \in \Omega_{\varepsilon,\delta}$. So, by Corollary 2.7,

$$\min_{\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon \leqslant \mathcal{H}_\varepsilon(\xi_\star) \leqslant \frac{C}{d_\star^{n+4s}} = \frac{C\varepsilon^{n+4s}}{\delta_\star^{n+4s}} = c_1\varepsilon^{n+4s}$$

for a suitable $c_1 > 0$. On the other hand, by Corollary 2.6,

$$\min_{\partial\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon \geqslant \frac{c\varepsilon^{n+4s}}{\delta^{n+4s}},$$

which implies the desired result for δ appropriately small. □

3. Estimates on $\bar{u}_\xi$ and first approximation of the solution

Now we make some estimates on the function $\bar{u}_\xi$ introduced in (1-9), by using in particular the auxiliary function Π_ε in (2-18). For this, we define, for any $\xi \in \Omega_\varepsilon$,

$$\Lambda_\xi(x) := \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^p(y)\Gamma(x-y)\,dy. \tag{3-1}$$

We have the following estimate for Λ_ξ:

Lemma 3.1. *Let $x, \xi \in \Omega_\varepsilon$. Assume that $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 1$. Then*

$$0 \leqslant \Lambda_\xi(x) \leqslant \frac{C}{d^{(n+2s)p}},$$

where $C > 0$ depends on n, p, s and Ω.

Proof. If $y \in \mathbb{R}^n \setminus \Omega_\varepsilon$ then $|y-\xi| \geqslant \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 1$; therefore, by (1-4),

$$|w_\xi(y)| = |w(y-\xi)| \leqslant C|y-\xi|^{-(n+2s)} \leqslant Cd^{-(n+2s)}.$$

As a consequence of this, and recalling (2-10), we deduce that

$$\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^p(y)\Gamma(x-y)\,dy \leqslant (Cd^{-(n+2s)})^p \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} \Gamma(x-y)\,dy \leqslant (Cd^{-(n+2s)})^p. \qquad \square$$

Lemma 3.2. *Let $x, \xi \in \Omega_\varepsilon$. Assume that $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 1$. Then*

$$\bar{u}_\xi(x) = w_\xi(x) - \Lambda_\xi(x) - \Pi_\varepsilon(x,\xi) \tag{3-2}$$

and

$$0 \leqslant w_\xi(x) - \bar{u}_\xi(x) - \Pi_\varepsilon(x,\xi) \leqslant \frac{C}{d^{(n+2s)p}} \tag{3-3}$$

for a suitable $C > 0$ that depends on n, p, s and Ω.

Proof. First of all, notice that $w = w^p * \Gamma$, since they both satisfy (1-3), thanks to (1-11), and uniqueness holds. As a consequence

$$w_\xi(x) = w(x-\xi) = \int_{\mathbb{R}^n} w^p(x-\xi-y)\Gamma(y)\,dy = \int_{\mathbb{R}^n} w_\xi^p(y)\Gamma(x-y)\,dy. \tag{3-4}$$

Similarly, recalling (1-9), (1-13) and the symmetry of G_ε, we see that

$$\begin{aligned}\bar{u}_\xi(x) &= \int_{\Omega_\varepsilon} \bar{u}_\xi(z)\delta_x(z)\,dz \\ &= \int_{\Omega_\varepsilon} \bar{u}_\xi(z)((-\Delta)^s+1)G_\varepsilon(z,x)\,dz \\ &= \int_{\Omega_\varepsilon} w_\xi^p(z)G_\varepsilon(x,z)\,dz \\ &= \int_{\Omega_\varepsilon} w_\xi^p(z)\Gamma(x-z)\,dz - \int_{\Omega_\varepsilon} w_\xi^p(z)H_\varepsilon(x,z)\,dz \\ &= \int_{\mathbb{R}^n} w_\xi^p(z)\Gamma(x-z)\,dz - \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^p(z)\Gamma(x-z)\,dz - \int_{\Omega_\varepsilon} w_\xi^p(z)H_\varepsilon(x,z)\,dz.\end{aligned}$$

This, (2-18), (3-1) and (3-4) imply (3-2), which, together with Lemma 3.1, implies (3-3). □

Lemma 3.3. *Let $\xi \in \Omega_\varepsilon$. Assume that $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 2$. Then*

$$\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\Lambda_\xi(x)\Pi_\varepsilon(x,\xi)\,dx \leqslant \frac{C}{d^{(n+2s)p+2s}}$$

for a suitable $C > 0$ that depends on n, p, s and Ω.

Proof. First of all, we notice that for $y \in \mathbb{R}^n \setminus \Omega_\varepsilon$ we have $|y-\xi| \geqslant d > 1$, and therefore, thanks to (1-4),

$$|w_\xi(y)| = |w(y-\xi)| \leqslant C|y-\xi|^{-(n+2s)} \leqslant Cd^{-(n+2s)}.$$

Hence, recalling (3-1),

$$\begin{aligned}\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\Lambda_\xi(x)\Pi_\varepsilon(x,\xi)\,dx &= \int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\bigg(\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^p(y)\Gamma(x-y)\,dy\bigg)\Pi_\varepsilon(x,\xi)\,dx \\ &\leqslant Cd^{-(n+2s)p}\int_{\Omega_\varepsilon} dx \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\, w_\xi^{p-1}(x)\Gamma(x-y)\Pi_\varepsilon(x,\xi) \\ &\leqslant Cd^{-(n+2s)p}\int_{\{|x-\xi|\leqslant d/4\}} dx \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\, w_\xi^{p-1}(x)\Gamma(x-y)\Pi_\varepsilon(x,\xi) \\ &\qquad + Cd^{-(n+2s)p}\int_{\{|x-\xi|>d/4\}} dy\, w_\xi^{p-1}(x)\Gamma(x-y)\Pi_\varepsilon(x,\xi) \\ &=: I_1 + I_2.\end{aligned} \tag{3-5}$$

Now, thanks to (3-3), we have that $\Pi_\varepsilon(x,\xi) \leqslant w_\xi(x)$, and so

$$w_\xi^{p-1}(x)\Pi_\varepsilon(x,\xi) \leqslant w_\xi^p(x). \tag{3-6}$$

Therefore, I_1 can be estimated as follows:

$$\begin{aligned} I_1 &\leqslant C d^{-(n+2s)p} \int_{\{|x-\xi|\leqslant d/4\}} dx \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\, w_\xi^p(x)\Gamma(x-y) \\ &\leqslant C d^{-(n+2s)p} \int_{\{|x-\xi|\leqslant d/4\}} dx \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dy\, w_\xi^p(x)\Gamma(x-y). \end{aligned}$$

We notice that, in the above domain,

$$|x-y| \geqslant |y-\xi| - |x-\xi| \geqslant \frac{d}{2} - \frac{d}{4} = \frac{d}{4},$$

hence

$$\Gamma(x-y) \leqslant \frac{C}{|x-y|^{n+2s}}.$$

Now, we can compute in polar coordinates the following integral:

$$\int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} \frac{1}{|x-y|^{n+2s}}\, dy \leqslant \frac{C}{d^{2s}},$$

up to renaming the constant C. This and the fact that w_ξ^p is integrable give

$$\begin{aligned} I_1 \leqslant C_1 d^{-(n+2s)p} \int_{\{|x-\xi|\leqslant d/4\}} dx \int_{\mathbb{R}^n\setminus B_{d/2}(\xi)} dy\, \frac{w_\xi^p(x)}{|x-y|^{n+2s}} &\leqslant C_2 d^{-(n+2s)p} d^{-2s} \int_{\{|x-\xi|\leqslant d/4\}} w_\xi^p(x)\, dx \\ &\leqslant C_3 d^{-(n+2s)p} d^{-2s}, \end{aligned} \tag{3-7}$$

for suitable $C_1, C_2, C_3 > 0$. Now, if $|x-\xi| > d/4$ then, thanks to (1-4),

$$|w_\xi(x)| = |w(x-\xi)| \leqslant C|x-\xi|^{-(n+2s)}.$$

This, together with (3-6) and (2-10), implies that

$$\begin{aligned} I_2 &\leqslant C d^{-(n+2s)p} \int_{\{|x-\xi|> d/4\}} dx \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\, w_\xi^p(x)\Gamma(x-y) \\ &\leqslant C' d^{-(n+2s)p} \int_{\{|x-\xi|> d/4\}} dx \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\, \frac{\Gamma(x-y)}{|x-\xi|^{(n+2s)p}} \\ &\leqslant C'' d^{-(n+2s)p} d^{-(n+2s)p+n} \end{aligned} \tag{3-8}$$

for suitable $C', C'' > 0$, where in the last inequality we have computed the integral in dx in polar coordinates and used (2-10). Putting together (3-7) and (3-8) and recalling (3-5), we get the desired estimate. □

Lemma 3.4. *Let $\xi \in \Omega_\varepsilon$ and $\tilde{p} := \min\{p, 2\}$. Assume that $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 2$. Then*

$$\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\Pi_\varepsilon^2(x,\xi)\, dx \leqslant \frac{C}{d^{\tilde{p}(n+2s)+2s}}$$

for a suitable $C > 0$ that depends on n, p, s and Ω.

Proof. First we observe that

$$\begin{aligned} 2(n+4s) &\geqslant \tilde{p}(n+4s) = \tilde{p}(n+2s+2s) > \tilde{p}(n+2s)+2s \quad \text{if } p \geqslant 2, \\ p(n+4s) &\geqslant \tilde{p}(n+4s) = \tilde{p}(n+2s+2s) > \tilde{p}(n+2s)+2s \quad \text{if } 1 < p < 2, \\ (n+2s)(p+1)-n &= p(n+2s)+2s \geqslant \tilde{p}(n+2s)+2s. \end{aligned} \tag{3-9}$$

Now, we can write the integral that we want to estimate as

$$\begin{aligned} \int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\Pi_\varepsilon^2(x,\xi)\,dx &= \int_{\{|x-\xi|\leqslant d/8\}} w_\xi^{p-1}(x)\Pi_\varepsilon^2(x,\xi)\,dx + \int_{\{|x-\xi|>d/8\}} w_\xi^{p-1}(x)\Pi_\varepsilon^2(x,\xi)\,dx \\ &=: I_1 + I_2. \end{aligned} \tag{3-10}$$

If $p \geqslant 2$, then to estimate I_1 we use Lemma 2.5 together with the fact that w_ξ^{p-1} is integrable to get

$$I_1 \leqslant \frac{C}{d^{2(n+4s)}}. \tag{3-11}$$

If $1 < p < 2$, we notice that, thanks to (3-3), $\Pi_\varepsilon(x,\xi) \leqslant w_\xi(x)$ and so

$$w_\xi^{p-1}(x)\Pi_\varepsilon^2(x,\xi) = w_\xi^{p-1}(x)\Pi_\varepsilon^{2-p}(x,\xi)\Pi_\varepsilon^p(x,\xi) \leqslant w_\xi^{p-1}(x)w_\xi^{2-p}(x)\Pi_\varepsilon^p(x,\xi) = w_\xi(x)\Pi_\varepsilon^p(x,\xi).$$

Therefore, again using Lemma 2.5 and the fact that w_ξ is integrable, we obtain

$$I_1 \leqslant \int_{\{|x-\xi|\leqslant d/8\}} w_\xi(x)\Pi_\varepsilon^p(x,\xi)\,dx \leqslant \frac{C}{d^{p(n+4s)}} \int_{\{|x-\xi|\leqslant d/8\}} w_\xi(x)\,dx \leqslant \frac{C}{d^{p(n+4s)}}. \tag{3-12}$$

To estimate I_2, we use (3-3) to obtain that $\Pi_\varepsilon(x,\xi) \leqslant w_\xi(x)$, and so $w_\xi^{p-1}(x)\Pi_\varepsilon^2(x,\xi) \leqslant w_\xi^{p+1}(x)$. This implies that

$$I_2 \leqslant \int_{\{|x-\xi|>d/8\}} w_\xi^{p+1}(x)\,dx.$$

Since $|x-\xi| > d/8$, thanks to (1-4) we have that

$$|w_\xi(x)| = |w(x-\xi)| \leqslant C|x-\xi|^{-(n+2s)}.$$

Therefore, computing the integral in polar coordinates,

$$I_2 \leqslant \int_{\{|x-\xi|>d/8\}} \frac{C}{|x-\xi|^{(n+2s)(p+1)}}\,dx \leqslant \frac{C}{d^{(n+2s)(p+1)-n}}. \tag{3-13}$$

Putting together (3-11), (3-12) and (3-13) and recalling (3-10), we obtain the result (one can use (3-9) to obtain a simpler common exponent). □

Lemma 3.5. *Let $\delta \in (0,1)$. Let $\xi \in \Omega_\varepsilon$ be such that $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant \delta/\varepsilon$. Then*

$$\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\Lambda_\xi^2(x)\,dx \leqslant C\varepsilon^{2p(n+2s)-n}$$

for a suitable $C > 0$ that depends on n, p, s, δ and Ω.

Proof. We use Lemma 3.1 and the fact that Ω is bounded to obtain that

$$\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\Lambda_\xi^2(x)\,dx \leqslant \frac{C}{d^{2p(n+2s)}}\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\,dx \leqslant \frac{C'}{d^{2p(n+2s)}}|\Omega_\varepsilon| \leqslant \frac{C''}{d^{2p(n+2s)}\varepsilon^n} \leqslant \frac{C''\varepsilon^{2p(n+2s)}}{\delta^{2p(n+2s)}\varepsilon^n}$$

for suitable $C', C'' > 0$. This implies the desired estimate. □

4. Energy estimates and functional expansion in $\bar{u}_\xi$

In this section we make some estimates for the energy functional (1-10). For this, we consider the functional associated to problem (1-3):

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n}((-\Delta)^s u(x)u(x) + u^2(x))\,dx - \frac{1}{p+1}\int_{\mathbb{R}^n} u^{p+1}(x)\,dx, \qquad u \in H^s(\mathbb{R}^n). \tag{4-1}$$

Theorem 4.1. *Fix $\delta \in (0, 1)$ and $\xi \in \Omega_\varepsilon$ such that $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant \delta/\varepsilon$. Then, we have*

$$I_\varepsilon(\bar{u}_\xi) = I(w) + \tfrac{1}{2}\mathcal{H}_\varepsilon(\xi) + o(\varepsilon^{n+4s}) \tag{4-2}$$

as $\varepsilon \to 0$, where I is given by (4-1), w is the solution to (1-3) and $\mathcal{H}_\varepsilon(\xi)$ is defined in (1-17), as long as δ is sufficiently small.

The following simple observation will be used often in the sequel:

Lemma 4.2. *Let $\delta \geqslant 1$ and $q > 1$. Then*

$$\int_{\mathbb{R}^n \setminus B_\delta(\xi)} w_\xi^q(z)\,dz \leqslant \frac{C}{\delta^{n(q-1)+2sq}}$$

for some $C > 0$.

Proof. First of all, we observe that

$$n - 1 - (n+2s)q < n - 1 - (n+2s) = -1 - 2s < -1$$

and therefore

$$\int_\delta^{+\infty} \rho^{n-1-(n+2s)q}\,d\rho = \frac{\delta^{n-(n+2s)q}}{(n+2s)q - n}. \tag{4-3}$$

Now, we use (1-4) to see that

$$\int_{\mathbb{R}^n \setminus B_\delta(\xi)} w_\xi^q(z)\,dz \leqslant \int_{\mathbb{R}^n \setminus B_\delta(\xi)} \frac{C}{|x-\xi|^{(n+2s)q}}\,dz = C'\int_\delta^{+\infty} \rho^{n-1-(n+2s)q}\,d\rho$$

for some $C' > 0$. This and (4-3) imply the desired result. □

Corollary 4.3. *Let $\xi \in \Omega_\varepsilon$, with $d := \operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant 1$. Then*

$$\int_{\mathbb{R}^n \setminus \Omega_\varepsilon} w_\xi^{p+1}(z)\,dz \leqslant \frac{C}{d^{np+2s(p+1)}}$$

for some $C > 0$.

Proof. Notice that $(\mathbb{R}^n \setminus \Omega_\varepsilon) \subseteq (\mathbb{R}^n \setminus B_d(\xi))$ and exploit Lemma 4.2. □

Proof of Theorem 4.1. Using (1-9) and (3-2), we have

$$\begin{aligned} I_\varepsilon(\bar u_\xi) &= \frac{1}{2}\int_{\Omega_\varepsilon}((-\Delta)^s\bar u_\xi(x)+\bar u_\xi(x))\bar u_\xi(x)\,dx - \frac{1}{p+1}\int_{\Omega_\varepsilon}\bar u_\xi^{p+1}(x)\,dx \\ &= \frac{1}{2}\int_{\Omega_\varepsilon} w_\xi^p(x)\bar u_\xi(x)\,dx - \frac{1}{p+1}\int_{\Omega_\varepsilon}\bar u_\xi^{p+1}(x)\,dx \\ &= \frac{1}{2}\int_{\Omega_\varepsilon} w_\xi^p(x)(w_\xi(x)-\Lambda_\xi(x)-\Pi_\varepsilon(x,\xi))\,dx \\ &\qquad\qquad - \frac{1}{p+1}\int_{\Omega_\varepsilon}(w_\xi(x)-\Lambda_\xi(x)-\Pi_\varepsilon(x,\xi))^{p+1}\,dx \\ &= \Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\Omega_\varepsilon} w_\xi^{p+1}(x)\,dx - \frac{1}{2}\int_{\Omega_\varepsilon} w_\xi^p(x)(\Lambda_\xi(x)+\Pi_\varepsilon(x,\xi))\,dx \\ &\qquad\qquad + \frac{1}{p+1}\int_{\Omega_\varepsilon}\big[w_\xi^{p+1}(x)-(w_\xi(x)-\Lambda_\xi(x)-\Pi_\varepsilon(x,\xi))^{p+1}\big]\,dx. \end{aligned} \tag{4-4}$$

We note that the first term in the right-hand side of (4-4) can be written as

$$\begin{aligned} \Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\Omega_\varepsilon} w_\xi^{p+1}(x)\,dx &= \Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^n} w_\xi^{p+1}(x)\,dx - \Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^{p+1}(x)\,dx \\ &= I(w) - \Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^{p+1}(x)\,dx, \end{aligned}$$

since w is a solution to (1-3). Therefore,

$$\begin{aligned} I_\varepsilon(\bar u_\xi) &= I(w) - \Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^{p+1}(x)\,dx - \frac{1}{2}\int_{\Omega_\varepsilon} w_\xi^p(x)(\Lambda_\xi(x)+\Pi_\varepsilon(x,\xi))\,dx \\ &\qquad\qquad + \frac{1}{p+1}\int_{\Omega_\varepsilon}\big[w_\xi^{p+1}(x)-(w_\xi(x)-\Lambda_\xi(x)-\Pi_\varepsilon(x,\xi))^{p+1}\big]\,dx \\ &= I(w) - \Big(\frac{1}{2}-\frac{1}{p+1}\Big)J_1 - \frac{1}{2}J_2 + \frac{1}{p+1}J_3, \end{aligned} \tag{4-5}$$

where

$$\begin{aligned} J_1 &:= \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^{p+1}(x)\,dx, \\ J_2 &:= \int_{\Omega_\varepsilon} w_\xi^p(x)(\Lambda_\xi(x)+\Pi_\varepsilon(x,\xi))\,dx \\ \text{and}\quad J_3 &:= \int_{\Omega_\varepsilon}\big[w_\xi^{p+1}(x)-(w_\xi(x)-\Lambda_\xi(x)-\Pi_\varepsilon(x,\xi))^{p+1}\big]\,dx. \end{aligned}$$

Now, we estimate separately J_1, J_2 and J_3. Thanks to Corollary 4.3, we have that

$$J_1 = \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} w_\xi^{p+1}(x)\,dx \leqslant \frac{C}{d^{np+2s(p+1)}} \leqslant \frac{C}{\delta^{np+2s(p+1)}}\varepsilon^{np+2s(p+1)}. \tag{4-6}$$

Concerning J_2, we write it as

$$J_2 = J_{21} + J_{22},$$

where

$$\begin{aligned} J_{21} &:= \int_{\Omega_\varepsilon} w_\xi^p(x)\Lambda_\xi(x)\,dx, \\ J_{22} &:= \int_{\Omega_\varepsilon} w_\xi^p(x)\Pi_\varepsilon(x,\xi)\,dx. \end{aligned} \tag{4-7}$$

Recalling the definition of Λ_ξ in (3-1) and the estimate in (1-4), we have that

$$\begin{aligned} J_{21} &= \int_{\Omega_\varepsilon} dx \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\, w_\xi^p(x) w_\xi^p(y)\Gamma(x-y) \\ &\leqslant C\int_{\Omega_\varepsilon} dx \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\, w_\xi^p(x)\frac{\Gamma(x-y)}{|y-\xi|^{(n+2s)p}} \\ &\leqslant \frac{C}{d^{(n+2s)p}}\int_{\Omega_\varepsilon} dx \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} dy\, w_\xi^p(x)\Gamma(x-y) \\ &= \frac{C}{d^{(n+2s)p}}\left(\int_{\Omega_\varepsilon} dx \int_{\substack{\mathbb{R}^n\setminus\Omega_\varepsilon \\ \{|x-y|\leqslant d/2\}}} dy\, w_\xi^p(x)\Gamma(x-y) + \int_{\Omega_\varepsilon} dx \int_{\substack{\mathbb{R}^n\setminus\Omega_\varepsilon \\ \{|x-y|> d/2\}}} dy\, w_\xi^p(x)\Gamma(x-y)\right). \end{aligned} \tag{4-8}$$

We notice that, if $x \in \Omega_\varepsilon$ and $y \in \mathbb{R}^n \setminus \Omega_\varepsilon$ with $|x-y| \leqslant d/2$, then

$$|x-\xi| \geqslant |y-\xi| - |x-y| \geqslant d - \frac{d}{2} = \frac{d}{2}.$$

Therefore, using (1-4), (2-10) and the fact that Ω is bounded, we have

$$\begin{aligned} \int_{\Omega_\varepsilon} dx \int_{\substack{\mathbb{R}^n\setminus\Omega_\varepsilon \\ \{|x-y|\leqslant d/2\}}} dy\, w_\xi^p(x)\Gamma(x-y) &\leqslant C'\int_{\Omega_\varepsilon} dx \int_{\substack{\mathbb{R}^n\setminus\Omega_\varepsilon \\ \{|x-y|\leqslant d/2\}}} dy\, \frac{\Gamma(x-y)}{|x-\xi|^{(n+2s)p}} \\ &\leqslant C'\int_{\Omega_\varepsilon} dx \int_{\mathbb{R}^n} d\tilde{y}\, \frac{\Gamma(\tilde{y})}{|x-\xi|^{(n+2s)p}} \\ &\leqslant C''(1/d)^{(n+2s)p}|\Omega_\varepsilon| \\ &\leqslant \frac{C'''}{d^{(n+2s)p}\varepsilon^n} \\ &\leqslant \frac{C'''}{\delta^{(n+2s)p}}\varepsilon^{(n+2s)p-n} \end{aligned} \tag{4-9}$$

for suitable constants $C', C'', C''' > 0$. Moreover, if $|x-y| > d/2$, we use (1-12) to get

$$\int_{\Omega_\varepsilon} dx \int_{\substack{\mathbb{R}^n\setminus\Omega_\varepsilon \\ \{|x-y|> d/2\}}} dy\, w_\xi^p(x)\Gamma(x-y) \leqslant C\int_{\mathbb{R}^n} dx \int_{\substack{\mathbb{R}^n\setminus\Omega_\varepsilon \\ \{|x-y|> d/2\}}} dy\, \frac{w_\xi^p(x)}{|x-y|^{n+2s}} \leqslant \tilde{C}d^{-2s} \leqslant \frac{\tilde{C}}{\delta^{2s}}\varepsilon^{2s} \tag{4-10}$$

for some $\tilde{C} > 0$. Putting together (4-9) and (4-10) and recalling (4-8), we obtain

$$\begin{aligned} J_{21} &\leqslant \frac{C}{d^{(n+2s)p}}\left(\frac{C'''}{\delta^{(n+2s)p}}\varepsilon^{(n+2s)p-n} + \frac{\tilde{C}}{\delta^{2s}}\varepsilon^{2s}\right) \\ &\leqslant \frac{C}{\delta^{(n+2s)p}}\varepsilon^{(n+2s)p}\left(\frac{C'''}{\delta^{(n+2s)p}}\varepsilon^{(n+2s)p-n} + \frac{\tilde{C}}{\delta^{2s}}\varepsilon^{2s}\right) \leqslant \hat{C}\varepsilon^{np+2s(p+1)} \end{aligned} \tag{4-11}$$

for suitable $\hat{C} > 0$. Therefore,

$$J_2 = J_{22} + o(\varepsilon^{n+4s}) = \int_{\Omega_\varepsilon} w_\xi^p(x)\Pi_\varepsilon(x,\xi)\,dx + o(\varepsilon^{n+4s}). \tag{4-12}$$

To estimate J_3 we expand $w_\xi^{p+1}(x)$ in the following way:

$$w_\xi^{p+1}(x) = \bar{u}_\xi^{p+1}(x) + (p+1)w_\xi^p(x)(w_\xi(x) - \bar{u}_\xi(x)) + c_p\alpha_\xi^{p-1}(x)(w_\xi(x) - \bar{u}_\xi(x))^2,$$

where $0 \leqslant \bar{u}_\xi \leqslant \alpha_\xi \leqslant w_\xi$ and c_p is a positive constant depending only on p. Therefore, recalling (3-2) and (4-7),

$$\begin{aligned} J_3 &= \int_{\Omega_\varepsilon} \bigl[w_\xi^{p+1}(x) - (w_\xi(x) - \Lambda_\xi(x) - \Pi_\varepsilon(x,\xi))^{p+1}\bigr]\,dx \\ &= \int_{\Omega_\varepsilon} [w_\xi^{p+1}(x) - \bar{u}_\xi^{p+1}(x)]\,dx \\ &= (p+1)\int_{\Omega_\varepsilon} w_\xi^p(x)(\Lambda_\xi(x) + \Pi_\varepsilon(x,\xi))\,dx + c_p\int_{\Omega_\varepsilon} \alpha_\xi^{p-1}(x)(\Lambda_\xi(x) + \Pi_\varepsilon(x,\xi))^2\,dx \\ &= (p+1)(J_{21} + J_{22}) + c_p\int_{\Omega_\varepsilon} \alpha_\xi^{p-1}(x)(\Lambda_\xi(x) + \Pi_\varepsilon(x,\xi))^2\,dx. \end{aligned} \tag{4-13}$$

Since $\alpha_\xi(x) \leqslant w_\xi(x)$, we have that

$$\begin{aligned} &\int_{\Omega_\varepsilon} \alpha_\xi^{p-1}(x)(\Lambda_\xi(x) + \Pi_\varepsilon(x,\xi))^2\,dx \\ &\qquad \leqslant \int_{\Omega_\varepsilon} w_\xi^{p-1}(x)(\Lambda_\xi(x) + \Pi_\varepsilon(x,\xi))^2\,dx \\ &\qquad = \int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\Lambda_\xi^2(x)\,dx + \int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\Pi_\varepsilon^2(x,\xi)\,dx + 2\int_{\Omega_\varepsilon} w_\xi^{p-1}(x)\Lambda_\xi(x)\Pi_\varepsilon(x,\xi)\,dx. \end{aligned}$$

Hence, from Lemmata 3.3, 3.4 and 3.5 (together with the fact that $d \geqslant \delta/\varepsilon$) we deduce that

$$\int_{\Omega_\varepsilon} \alpha_\xi^{p-1}(x)(\Lambda_\xi(x) + \Pi_\varepsilon(x,\xi))^2\,dx \leqslant C_\delta(\varepsilon^{2p(n+2s)-n} + \varepsilon^{(n+2s)\tilde{p}+2s} + \varepsilon^{(n+2s)p+2s})$$

for some C_δ, where $\tilde{p} = \min\{p, 2\}$. The last estimate, (4-11) and (4-13) give

$$J_3 = (p+1)J_{22} + o(\varepsilon^{n+4s}) = (p+1)\int_{\Omega_\varepsilon} w_\xi^p(x)\Pi_\varepsilon(x,\xi)\,dx + o(\varepsilon^{n+4s}). \tag{4-14}$$

Putting together (4-6), (4-12) and (4-14) and using (4-5), we get

$$I_\varepsilon(\bar{u}_\xi) = I(w) + \frac{1}{2}\int_{\Omega_\varepsilon} w_\xi^p(x)\Pi_\varepsilon(x,\xi)\,dx + o(\varepsilon^{n+4s}).$$

Thus, recalling the definitions of Π_ε and $\mathcal{H}_\varepsilon$ in (2-18) and (1-17), respectively, we obtain (4-2). □

5. Decay of the ground state w

In this section we recall some basic (though not optimal) decay properties of the ground state and of its derivatives.

For this, we start with a general convolution result:

Lemma 5.1. *Let $a, b > n$, $C_a, C_b > 0$ and $f, g \in L^\infty(\mathbb{R}^n)$ with*

$$|f(x)| \leqslant C_a(1+|x|)^{-a} \quad \text{and} \quad |g(x)| \leqslant C_b(1+|x|)^{-b}.$$

Then there exists $C > 0$ such that

$$|(f * g)(x)| \leqslant C(1+|x|)^{-c}$$

with $c := \min\{a, b\}$.

Proof. Fix $x \in \mathbb{R}^n$ and $r := |x|/2$, and observe that if $y \in B_r(x)$ then

$$|y| \geqslant |x| - |x-y| \geqslant |x| - r = \frac{|x|}{2}.$$

As a consequence,

$$\begin{aligned}\int_{B_r(x)} \frac{C_a}{(1+|x-y|)^a}\frac{C_b}{(1+|y|)^b}\,dy &\leqslant \int_{B_r(x)} \frac{C_a}{(1+|x-y|)^a}\frac{C_b}{(1+(|x|/2))^b}\,dy \\ &\leqslant \frac{C}{(1+|x|)^b}\int_{\mathbb{R}^n}\frac{1}{(1+|x-y|)^a}\,dy \leqslant \frac{C}{(1+|x|)^b},\end{aligned} \tag{5-1}$$

up to renaming constants. On the other hand, if $y \in \mathbb{R}^n \setminus B_r(x)$ then $|x-y| \geqslant r = |x|/2$, thus

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_r(x)} \frac{C_a}{(1+|x-y|)^a}\frac{C_b}{(1+|y|)^b}\,dy &\leqslant \int_{\mathbb{R}^n\setminus B_r(x)} \frac{C_a}{(1+(|x|/2))^a}\frac{C_b}{(1+|y|)^b}\,dy \\ &\leqslant \frac{C}{(1+|x|)^a}\int_{\mathbb{R}^n}\frac{1}{(1+|y|)^b}\,dy \leqslant \frac{C}{(1+|x|)^a}.\end{aligned} \tag{5-2}$$

Putting together (5-1) and (5-2), we obtain the desired result. □

Now we fix $\xi \in \Omega_\varepsilon$ and we define, for any $i \in \{1, \dots, n\}$,

$$Z_i := \frac{\partial w_\xi}{\partial x_i}, \tag{5-3}$$

where w_ξ is the ground state solution centered at ξ. Moreover, we denote by $\mathcal{Z}$ the linear space spanned by the functions Z_i.

We prove first the following lemmata:

Lemma 5.2. *There exists a positive constant C such that, for any $i = 1, \dots, n$,*

$$|Z_i| \leqslant C|x-\xi|^{-\nu_1} \quad \text{for any } |x-\xi| \geqslant 1,$$

where $\nu_1 := \min\{(n+2s+1),\, p(n+2s)\}$.

Proof. Given $R > 0$, we take $\Gamma_{1,R} \in C^\infty(\mathbb{R}^n)$, with $0 \leqslant \Gamma_{1,R} \leqslant \Gamma$ in $\mathbb{R}^n$ and $\Gamma_{1,R} = \Gamma$ outside B_R, and we define $\Gamma_{2,R} := \Gamma - \Gamma_{1,R}$. We use (1-11) to write

$$w = \Gamma * w^p = \Gamma_{1,R} * w^p + \Gamma_{2,R} * w^p. \tag{5-4}$$

We assume, up to translation, that $\xi = 0$. Then, our goal is to prove that, for any $k \in \mathbb{N}$, we have that

$$\left| \frac{\partial w}{\partial x_i}(x) \right| \leqslant C_k (1 + |x|)^{-\nu(k)}, \tag{5-5}$$

where

$$\nu(k) := \min\{(n + 2s + 1),\, p(n + 2s),\, k(p - 1)(n + 2s)\} = \min\{\nu_1,\, k(p - 1)(n + 2s)\}$$

for some $C_k > 0$. Indeed, the desired claim would follow from (5-5) simply by taking the smallest k for which $k(p - 1) > p$.

To prove (5-5) we perform an inductive argument. So, we first check (5-5) when $k = 0$. For this, we use the fact that $w \in L^\infty(\mathbb{R}^n)$ and that $\Gamma \in L^1(\mathbb{R}^n)$ to find $R > 0$ sufficiently small that

$$\int_{B_R} \Gamma(y)\, dy \leqslant \frac{1}{2p\|w\|_{L^\infty(\mathbb{R}^n)}}.$$

This fixes R once and for all for the proof of (5-5) when $k = 0$. Hence, we use the sign of $\Gamma_{1,R}$ and the fact that $\Gamma_{2,R} = 0$ outside B_R to obtain that

$$\int_{\mathbb{R}^n} \Gamma_{2,R}(y)\, dy = \int_{B_R} \Gamma_{2,R}(y)\, dy \leqslant \int_{B_R} \Gamma_{1,R}(y) + \Gamma_{2,R}(y)\, dy \leqslant \frac{1}{2p\|w\|_{L^\infty(\mathbb{R}^n)}}. \tag{5-6}$$

Then, for any $t \in (0, 1)$, we define $D_t w(x) := (w(x + te_i) - w(x))/t$ and we infer from (5-4) that

$$D_t w = (D_t \Gamma_{1,R}) * w^p + \Gamma_{2,R} * (D_t w^p). \tag{5-7}$$

Also, from formula (3.2) of [Felmer et al. 2012] we know that

$$|\nabla \Gamma(x)| \leqslant C|x|^{-(n+2s+1)} \quad \text{for any } |x| \geqslant 1. \tag{5-8}$$

As a consequence, if $|x| > 2$ and $\eta \in B_1(x)$, we have that

$$|\eta| \geqslant |x| - |x - \eta| \geqslant \frac{|x|}{2} > 1;$$

hence,

$$|\Gamma(x + te_1) - \Gamma(x)| \leqslant t \sup_{\eta \in B_1(x)} |\nabla \Gamma(\eta)| \leqslant Ct|x|^{-(n+2s+1)},$$

up to renaming C. This gives that $|D_t \Gamma(x)| \leqslant C(1+|x|)^{-(n+2s+1)}$, so $|D_t \Gamma_{1,R}(x)| \leqslant C(1+|x|)^{-(n+2s+1)}$. Accordingly, we have that

$$|(D_t \Gamma_{1,R}) * w^p| \leqslant \|w\|^p_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |D_t \Gamma_{1,R}(y)|\, dy \leqslant C. \tag{5-9}$$

Also,

$$|w^p(x + te_i) - w^p(x)| \leqslant p\|w\|^{p-1}_{L^\infty(\mathbb{R}^n)} |w(x + te_i) - w(x)|.$$

This says that

$$|D_t w^p(x)| \leqslant p\|w\|_{L^\infty(\mathbb{R}^n)}^{p-1}|D_t w(x)|.$$

Moreover,

$$|D_t w(x)| \leqslant \frac{2\|w\|_{L^\infty(\mathbb{R}^n)}}{t};$$

hence we can define

$$M(t) := \sup_{x\in\mathbb{R}^n} |D_t w(x)|,$$

so we obtain that

$$|D_t w^p(x)| \leqslant p\|w\|_{L^\infty(\mathbb{R}^n)}^{p-1} M(t)$$

for every $x \in \mathbb{R}^n$, and thus

$$|\Gamma_{2,R} * (D_t w^p)(x)| \leqslant \int_{\mathbb{R}^n} \Gamma_{2,R}(y)|D_t w^p(x-y)|\,dy \leqslant p\|w\|_{L^\infty(\mathbb{R}^n)}^{p-1} M(t) \int_{\mathbb{R}^n} \Gamma_{2,R}(y)\,dy \leqslant \frac{M(t)}{2},$$

thanks to (5-6). Using this and (5-9) in (5-7), we conclude that

$$D_t w \leqslant C + \frac{M(t)}{2}.$$

By taking the supremum, we obtain that

$$M(t) \leqslant C + \frac{M(t)}{2},$$

and this gives, up to renaming C, that $M(t) \leqslant C$. By sending $t \searrow 0$, we complete the proof of (5-5) when $k = 0$.

Now we suppose that (5-5) holds true for some k and we prove it for $k+1$. The proof is indeed similar to the case $k = 0$: here we take $R := 1$ and use the shorthand notation $\Gamma_1 := \Gamma_{1,R}$ and $\Gamma_2 := \Gamma_{2,R}$. By (5-5) for $k = 0$ and the regularity theory (applied to the equation for $D_t w$), we know that $w \in C^1(\mathbb{R})$; hence, we can differentiate (5-4) and obtain that

$$\frac{\partial w}{\partial x_i} = \frac{\partial \Gamma_1}{\partial x_i} * w^p + \Gamma_2 * \left(p w^{p-1} \frac{\partial w}{\partial x_i}\right). \tag{5-10}$$

So, we use (1-4), (5-8) and Lemma 5.1 to obtain

$$\left|\frac{\partial \Gamma_1}{\partial x_i} * w^p(x)\right| \leqslant C(1+|x|)^{-\min\{(n+2s+1),\,p(n+2s)\}}. \tag{5-11}$$

Moreover, we notice that

$$\begin{aligned}(p-1)(n+2s)+\nu(k) &= \min\{(p-1)(n+2s)+\nu_1,\,(k+1)(p-1)(n+2s)\}\\ &\geqslant \min\{\nu_1,\,(k+1)(p-1)(n+2s)\} = \nu(k+1).\end{aligned}$$

Hence, using (5-5) for k and (1-4) we see that

$$\left|p w^{p-1}\frac{\partial w}{\partial x_i}(x)\right| \leqslant C(1+|x|)^{-(p-1)(n+2s)-\nu(k)} \leqslant C(1+|x|)^{-\nu(k+1)}, \tag{5-12}$$

up to renaming constants (possibly depending on p). Now, we observe that

$$1+|x-y| \geqslant \tfrac{1}{3}(1+|x|) \quad \text{if } x \in \mathbb{R}^n \text{ and } |y|<1. \tag{5-13}$$

Indeed, if $|x| \geqslant 2$ and $|y|<1$, then

$$|x-y| \geqslant |x|-|y| \geqslant \frac{|x|}{2},$$

which implies (5-13) in this case. If instead $|x|<2$ and $|y|<1$, we have that

$$1+|x|<3<3(1+|x-y|),$$

and this finishes the proof of (5-13).

Therefore, since Γ_2 vanishes outside B_1, using (5-12) and (5-13) we have

$$\begin{aligned}\left|\Gamma_2 * \left(pw^{p-1}\frac{\partial w}{\partial x_i}\right)(x)\right| &\leqslant C\int_{B_1} \frac{\Gamma_2(y)}{(1+|x-y|)^{\nu(k+1)}}\,dy \\ &\leqslant C\int_{B_1} \frac{\Gamma_2(y)}{(1+|x|)^{\nu(k+1)}}\,dy \leqslant \frac{C}{(1+|x|)^{\nu(k+1)}}\int_{\mathbb{R}^n}\Gamma(y)\,dy = \frac{C}{(1+|x|)^{\nu(k+1)}}.\end{aligned}$$

This and (5-11) establish (5-5) for $k+1$, thus completing the inductive argument. □

Lemma 5.3. *There exists a positive constant C such that, for any $i=1,\dots,n$,*

$$|\nabla Z_i| \leqslant C|x-\xi|^{-\nu_2} \quad \textit{for any } |x-\xi| \geqslant 1,$$

where $\nu_2 := \min\{(n+2s+2),\, p(n+2s)\}$.

Proof. From formula (3.2) of [Felmer et al. 2012], we know that

$$|D^2\Gamma(x)| \leqslant C|x|^{-(n+2s+2)}, \quad |x| \geqslant 1. \tag{5-14}$$

Hence the proof of Lemma 5.3 follows as that of Lemma 5.2, by using (5-14) instead of (5-8). □

Lemma 5.4. *For any $k \in \mathbb{N}$ there exists a positive constant C_k such that, for any $i=1,\dots,n$,*

$$|D^k Z_i| \leqslant C_k|x-\xi|^{-n} \quad \textit{for any } |x-\xi| \geqslant 1.$$

Proof. From Lemma C.1(ii) of [Frank et al. 2015], we have that

$$|D^{k+1}\Gamma(x)| \leqslant C_k|x|^{-n}, \quad |x| \geqslant 1. \tag{5-15}$$

The proof of Lemma 5.4 follows as that of Lemma 5.2 by using (5-15) instead of (5-8). □

We note that

$$\int_{\mathbb{R}^n} Z_i^2\,dx = \int_{\mathbb{R}^n} Z_j^2\,dx \quad \text{for any } i,j=1,\dots,n. \tag{5-16}$$

We set

$$\alpha := \int_{\mathbb{R}^n} Z_1^2\,dx, \tag{5-17}$$

and so, thanks to (5-16), we observe that

$$\int_{\mathbb{R}^n} Z_i^2\,dx = \alpha \quad \text{for any } i = 1, \dots, n. \tag{5-18}$$

Lemma 5.5. *The Z_i satisfy the following condition*:

$$\int_{\mathbb{R}^n} Z_i Z_j\,dx = \alpha\delta_{ij}. \tag{5-19}$$

Also, if $\tau_o \in L^\infty([0,+\infty))$, $\tau(x) := \tau_o(|x-\xi|)$ for any $x \in \mathbb{R}^n$ and $\tilde{Z}_i := \tau Z_i$, then

$$\int_{\mathbb{R}^n} \tilde{Z}_i Z_j\,dx = \tilde{\alpha}\delta_{ij}, \tag{5-20}$$

where[1]

$$\tilde{\alpha} := \int_{\mathbb{R}^n} \tilde{Z}_1 Z_1\,dx. \tag{5-21}$$

Proof. We first observe that the function w is radial (see, for instance, [Felmer et al. 2012]) and therefore, recalling the definition of Z_i in (5-3), we have that

$$Z_i = \frac{\partial w}{\partial x_i}(x-\xi) = w'_\xi(|x-\xi|)\frac{x_i-\xi_i}{|x-\xi|}.$$

Hence, using the change of variable $y = x-\xi$, for any $i, j = 1, \dots, n$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \tilde{Z}_i Z_j\,dx &= \int_{\mathbb{R}^n} \tau_o(|x-\xi|)\big|w'(|x-\xi|)\big|^2 \frac{(x_i-\xi_i)(x_j-\xi_j)}{|x-\xi|^2}\,dx \\ &= \int_{\mathbb{R}^n} \tau_o(|y|)\big|w'(|y|)\big|^2 \frac{y_i y_j}{|y|^2}\,dy.\end{aligned} \tag{5-22}$$

Therefore, if $i \neq j$,

$$\int_{\mathbb{R}^n} \tilde{Z}_i Z_j\,dx = \int_{\mathbb{R}^{n-1}} y_j \left(\int_{\mathbb{R}} \tau_o(|y|)\big|w'(|y|)\big|^2 \frac{y_i}{|y|^2}\,dy_i \right) dy' = 0,$$

since the function $\tau_o(|y|)\big|w'(|y|)\big|^2 y_i/|y|^2$ is odd. This proves (5-20) when $i \neq j$. On the other hand, if $i = j$, formula (5-22) becomes

$$\int_{\mathbb{R}^n} \tilde{Z}_i Z_i\,dx = \int_{\mathbb{R}^n} \tau_o(|y|)\big|w'(|y|)\big|^2 \frac{y_i^2}{|y|^2}\,dy.$$

We observe that the latter integral is invariant under rotation; hence,

$$\int_{\mathbb{R}^n} \tilde{Z}_i Z_i\,dx = \int_{\mathbb{R}^n} \tau_o(|y|)\big|w'(|y|)\big|^2 \frac{y_1^2}{|y|^2}\,dy = \tilde{\alpha}.$$

This establishes (5-20) also when $i = j$. Then, (5-19) follows from (5-20) by choosing $\tau_o := 1$ and comparing (5-18) and (5-21). □

[1] In particular, we note that, if τ_o has a sign and does not vanish identically then $\tilde{\alpha} \neq 0$ (and we will often implicitly assume that this is so in the sequel).

Corollary 5.6. *The Z_i satisfy the condition*

$$\int_{\Omega_\varepsilon} Z_i Z_j \, dx = \alpha \delta_{ij} + O(\varepsilon^\nu)$$

with $\nu > n + 4s$.

Proof. From Lemma 5.5, we have that

$$\int_{\Omega_\varepsilon} Z_i Z_j \, dx = \int_{\mathbb{R}^n} Z_i Z_j \, dx - \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} Z_i Z_j \, dx = \alpha \delta_{ij} - \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} Z_i Z_j \, dx.$$

Moreover, from Lemma 5.2, we obtain that

$$\int_{\mathbb{R}^n \setminus \Omega_\varepsilon} Z_i Z_j \, dx \leqslant C \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} \frac{1}{|x-\xi|^{2\nu_1}} \, dx \leqslant C \varepsilon^{2\nu_1 - n},$$

which implies the desired result. □

6. Some regularity estimates

Here we perform some uniform estimates on the solutions of our differential equations. For this, we introduce some notation: given $\xi \in \Omega_\varepsilon$ with

$$\operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant \frac{c}{\varepsilon} \quad \text{for some } c \in (0,1), \tag{6-1}$$

and $n/2 < \mu < n + 2s$, we define, for any $x \in \mathbb{R}^n$,

$$\rho_\xi(x) := \frac{1}{(1 + |x - \xi|)^\mu}. \tag{6-2}$$

Moreover, we set

$$\|\psi\|_{\star,\xi} := \|\rho_\xi^{-1} \psi\|_{L^\infty(\mathbb{R}^n)}.$$

Lemma 6.1. *Let $g \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and let $\psi \in H^s(\mathbb{R}^n)$ be a solution to the problem*

$$\begin{cases} (-\Delta)^s \psi + \psi + g = 0 & \text{in } \Omega_\varepsilon, \\ \psi = 0 & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon. \end{cases} \tag{6-3}$$

Then, there exists a positive constant C such that

$$\|\psi\|_{L^\infty(\mathbb{R}^n)} + \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|x-y|^s} \leqslant C(\|g\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}).$$

Proof. From Theorem 8.2 in [Dipierro et al. 2014], we have that $\psi \in L^\infty(\mathbb{R}^n)$ and there exists a constant $C > 0$ such that

$$\|\psi\|_{L^\infty(\mathbb{R}^n)} \leqslant C(\|g\|_{L^\infty(\mathbb{R}^n)} + \|\psi\|_{L^2(\mathbb{R}^n)}). \tag{6-4}$$

Now, we show that

$$\|\psi\|_{L^2(\mathbb{R}^n)} \leqslant \|g\|_{L^2(\mathbb{R}^n)}. \tag{6-5}$$

Indeed, we multiply the equation in (6-3) by ψ and we integrate over Ω_ε, obtaining that

$$\int_{\Omega_\varepsilon} (-\Delta)^s \psi\psi + \psi^2 + g\psi \, dx = 0. \tag{6-6}$$

We notice that, thanks to formula (1.5) in [Ros-Oton and Serra 2014b],

$$\int_{\Omega_\varepsilon} (-\Delta)^s \psi\psi \, dx = \int_{\Omega_\varepsilon} |(-\Delta)^s \psi|^2 \, dx \geqslant 0.$$

Hence, from (6-6) we have

$$\int_{\Omega_\varepsilon} \psi^2 \, dx \leqslant \int_{\Omega_\varepsilon} -g\psi \, dx.$$

So, using Hölder's inequality, we get

$$\int_{\Omega_\varepsilon} \psi^2 \, dx \leqslant \left(\int_{\Omega_\varepsilon} g^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\Omega_\varepsilon} \psi^2 \, dx\right)^{\frac{1}{2}},$$

and therefore, dividing by $\left(\int_{\Omega_\varepsilon} \psi^2 \, dx\right)^{1/2}$, we obtain (6-5).

From (6-4) and (6-5), we have that

$$\|\psi\|_{L^\infty(\mathbb{R}^n)} \leqslant C(\|g\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}).$$

Now, since both ψ and g are bounded, from the regularity results in [Silvestre 2007] we have that ψ is C^α in the interior of Ω_ε for some $\alpha \in (0, 2s)$.

It remains to prove that ψ is C^α near the boundary of Ω_ε. For this, we fix a point $p \in \partial\Omega_\varepsilon$ and we look at the equation in the ball $B_1(p)$.

We notice that $|(-\Delta)^s \psi|$ is bounded, since both ψ and g are in $L^\infty(\mathbb{R}^n)$, and therefore we can apply Proposition 3.5 in [Ros-Oton and Serra 2014a], obtaining that, for any $x, y \in B_1(p) \cap \Omega_\varepsilon$,

$$\frac{\psi(x)}{d^s(x)} - \frac{\psi(y)}{d^s(y)} \leqslant C_1(\|\psi\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\mathbb{R}^n)}), \tag{6-7}$$

where $d(x) := \operatorname{dist}(x, \partial\Omega_\varepsilon)$. In particular, we can fix $y \in B_1(p) \cap \Omega_\varepsilon$ such that $d(y) = \frac{1}{2}$. Since ψ is bounded, from (6-7) we have that

$$\frac{\psi(x)}{d^s(x)} \leqslant C_2(\|\psi\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\mathbb{R}^n)}),$$

which gives that

$$\psi(x) \leqslant C_2(\|\psi\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\mathbb{R}^n)}) d^s(x).$$

This implies that ψ is C^s also near the boundary and concludes the proof of the lemma. □

Lemma 6.2. *Let $\xi \in \Omega_\varepsilon$, $\mathcal{B}$ be a bounded subset of $\mathbb{R}^n$, and $R_0 > 0$ be such that*

$$B_{R_0}(\xi) \supseteq \mathcal{B}. \tag{6-8}$$

Let $\mathcal{W} \in L^\infty(\mathbb{R}^n)$ be such that

$$m := \inf_{\mathbb{R}^n \setminus \mathcal{B}} \mathcal{W} > 0. \tag{6-9}$$

Let also $g \in L^2(\mathbb{R}^n)$, with $\|g\|_{\star,\xi} < +\infty$, and let $\psi \in H^s(\mathbb{R}^n)$ be a solution to

$$\begin{cases} (-\Delta)^s \psi + \mathcal{W}\psi + g = 0 & \text{in } \Omega_\varepsilon, \\ \psi = 0 & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon. \end{cases}$$

Then, there exists a positive constant C, possibly depending on m, R_0 and $\|\mathcal{W}\|_{L^\infty(\mathbb{R}^n)}$ (and also on n, s, and Ω), such that[2]

$$\|\psi\|_{\star,\xi} \leqslant C(\|\psi\|_{L^\infty(\mathcal{B})} + \|g\|_{\star,\xi}). \tag{6-10}$$

Proof. We define

$$W := m\chi_{\mathcal{B}} + \mathcal{W}\chi_{\mathbb{R}^n\setminus\mathcal{B}} \quad \text{and} \quad G := (m - \mathcal{W})\chi_{\mathcal{B}}\psi - g. \tag{6-11}$$

We observe that

$$\begin{aligned} \|G\|_{\star,\xi} &\leqslant \sup_{x\in\mathcal{B}} (1+|x-\xi|)^\mu (m + \mathcal{W}(x))\psi(x) + \|g\|_{\star,\xi} \\ &\leqslant 2(1+R_0)^\mu \|\mathcal{W}\|_{L^\infty(\mathbb{R}^n)} \|\psi\|_{L^\infty(\mathcal{B})} + \|g\|_{\star,\xi} \\ &\leqslant C_0 \|\psi\|_{L^\infty(\mathcal{B})} + \|g\|_{\star,\xi} \end{aligned} \tag{6-12}$$

for a suitable $C_0 > 0$ possibly depending on R_0 and $\|\mathcal{W}\|_{L^\infty(\mathbb{R}^n)}$ (notice that (6-8) was used here). Also, ψ is a solution of

$$(-\Delta)^s\psi + W\psi = (W-\mathcal{W})\psi - g = (W - \mathcal{W}\chi_{\mathbb{R}^n\setminus\mathcal{B}} - \mathcal{W}\chi_{\mathcal{B}})\psi - g = (m\chi_{\mathcal{B}} - \mathcal{W}\chi_{\mathcal{B}})\psi - g = G \tag{6-13}$$

and, in virtue of (6-9),

$$W \geqslant m\chi_{\mathcal{B}} + m\chi_{\mathbb{R}^n\setminus\mathcal{B}} = m. \tag{6-14}$$

We let $\rho_0 := (1+|x|)^{-\mu}$ and take $\eta \in H^s(\mathbb{R}^n)$ to be a solution of

$$(-\Delta)^s\eta + m\eta = \rho_0. \tag{6-15}$$

We refer to formula (2.4) in [Dávila et al. 2014] for the existence of such solution and to Lemma 2.2 there for the following estimate:

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^\mu \eta(x) \leqslant C_1 \sup_{x\in\mathbb{R}^n} (1+|x|)^\mu \rho_0(x) = C_1 \tag{6-16}$$

for some $C_1 > 0$, possibly depending on m. Also, by Lemma 2.4 in [Dávila et al. 2014], we have that $\eta \geqslant 0$, and so, recalling (6-14), we obtain that

$$(W(x) - m)\eta(x-\xi) \geqslant 0. \tag{6-17}$$

Now we define $\eta_\xi(x) := \eta(x-\xi)$,

$$C_\star := \|G\|_{\star,\xi} \tag{6-18}$$

and $\omega := C_\star\eta_\xi \pm \psi$. We remark that the quantity $C_\star$ plays a different role from the other constants C_0, C_1 and C_2: indeed, while C_0, C_1 and C_2 depend only on m, R_0 and $\|\mathcal{W}\|_{L^\infty(\mathbb{R}^n)}$ (as well on n, s and Ω), the quantity $C_\star$ also depends on G, and this will be made explicit at the end of the proof.

[2]In (6-10) we use the standard convention that $\|\psi\|_{L^\infty(\mathcal{B})} := 0$ when $\mathcal{B} := \varnothing$ (equivalently, if $\mathcal{B} = \varnothing$, the term $\|\psi\|_{L^\infty(\mathcal{B})}$ can be neglected in the proof of Lemma 6.2, since, in this case, G and g are the same from (6-11) on).

Notice also that $\rho_0(x-\xi)=\rho_\xi(x)$, due to the definition in (6-2), and

$$C_\star\rho_\xi(x)\pm G(x)\geqslant\rho_\xi(x)\big(C_\star-\rho_\xi^{-1}(x)|G(x)|\big)\geqslant\rho_\xi(x)(C_\star-\|G\|_{\star,\xi})\geqslant 0. \tag{6-19}$$

Thus we infer that

$$(-\Delta)^s\omega+W\omega=C_\star((-\Delta)^s\eta_\xi+W\eta_\xi)\pm((-\Delta)^s\psi+W\psi)=C_\star\rho_\xi+C_\star(W-m)\eta_\xi\pm G\geqslant 0 \tag{6-20}$$

in Ω_ε, thanks to (6-13), (6-15), (6-17) and (6-19). Furthermore, in $\mathbb{R}^n\setminus\Omega_\varepsilon$ we have that $\omega=C_\star\eta_\xi\geqslant 0$. As a consequence of this, (6-20) and the maximum principle (see, e.g., Lemma 6 in [Servadei and Valdinoci 2014]), we conclude that $\omega\geqslant 0$ in the whole of $\mathbb{R}^n$.

Accordingly, for any $x\in\mathbb{R}^n$,

$$\begin{aligned}\mp\rho_\xi^{-1}(x)\psi(x)&=\rho_\xi^{-1}(x)(C_\star\eta_\xi(x)-\omega(x))\\&\leqslant C_\star\rho_\xi^{-1}(x)\eta_\xi(x)\\&\leqslant C_\star\sup_{y\in\mathbb{R}^n}\rho_\xi^{-1}(y)\eta_\xi(y)\\&=C_\star\sup_{y\in\mathbb{R}^n}\rho_\xi^{-1}(y+\xi)\eta_\xi(y+\xi)\\&=C_\star\sup_{y\in\mathbb{R}^n}(1+|y|)^\mu\eta(y)\\&\leqslant C_1C_\star,\end{aligned}$$

where (6-16) was used in the last step. Hence, recalling (6-18) and (6-12),

$$|\rho_\xi^{-1}(x)\psi(x)|\leqslant C_1\|G\|_{\star,\xi}\leqslant C_1(C_0\|\psi\|_{L^\infty(\mathcal{B})}+\|g\|_{\star,\xi}),$$

which implies (6-10). □

As a consequence of Lemma 6.2, we obtain the following two corollaries:

Corollary 6.3. *Let $g\in L^2(\mathbb{R}^n)$, with $\|g\|_{\star,\xi}<+\infty$, and let $\psi\in H^s(\mathbb{R}^n)$ be a solution to*

$$\begin{cases}(-\Delta)^s\psi+\psi-pw_\xi^{p-1}\psi+g=0 & \text{in } \Omega_\varepsilon,\\ \psi=0 & \text{in } \mathbb{R}^n\setminus\Omega_\varepsilon.\end{cases}$$

Then, there exist positive constants C and R such that

$$\|\psi\|_{\star,\xi}\leqslant C(\|\psi\|_{L^\infty(B_R(\xi))}+\|g\|_{\star,\xi}). \tag{6-21}$$

Proof. We apply Lemma 6.2 with $\mathcal{W}:=1-pw_\xi^{p-1}$ and $\mathcal{B}:=B_R(\xi)$ (notice that, with this notation, (6-21) would follow from (6-10)). So, we only need to check that (6-9) holds true with a suitable choice of R. For this, we use that w decays at infinity (recall (1-4)); hence we can fix R large enough that

$$pw^{p-1}(x)\leqslant\tfrac12\quad\text{for every } x\in\mathbb{R}^n\setminus B_R.$$

Accordingly, $\mathcal{W}\geqslant 1-\frac12=\frac12$, which establishes (6-9) with $m:=\frac12$. □

Corollary 6.4. *Let $g \in L^2(\mathbb{R}^n)$, with $\|g\|_{\star,\xi} < +\infty$, and let $\psi \in H^s(\mathbb{R}^n)$ be a solution to*

$$\begin{cases} (-\Delta)^s \psi + \psi + g = 0 & \text{in } \Omega_\varepsilon, \\ \psi = 0 & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon. \end{cases}$$

Then, there exists a positive constant C such that

$$\|\psi\|_{\star,\xi} \leqslant C \|g\|_{\star,\xi}.$$

Proof. For this we use Lemma 6.2 with $\mathcal{W} := 1$ and $\mathcal{B} := \varnothing$ (recall the footnote on page 1192). □

7. The Lyapunov–Schmidt reduction

In this section we deal with the linear theory associated to the scaled problem (1-8). For this, we introduce the functional space

$$\Psi := \Big\{ \psi \in H^s(\mathbb{R}^n) \;\Big|\; \psi = 0 \text{ in } \mathbb{R}^n \setminus \Omega_\varepsilon \text{ and } \textstyle\int_{\Omega_\varepsilon} \psi Z_i \, dx = 0 \text{ for any } i = 1, \dots, n \Big\},$$

where the Z_i were introduced in (5-3). We remark that the condition

$$\int_{\Omega_\varepsilon} \psi Z_i \, dx = 0 \quad \text{for any } i = 1, \dots, n$$

means that ψ is orthogonal to the space $\mathcal{Z}$ (that is the space spanned by Z_i) with respect to the scalar product in $L^2(\Omega_\varepsilon)$.

We look for a solution to (1-8) of the form

$$u = u_\xi := \bar{u}_\xi + \psi, \tag{7-1}$$

where $\bar{u}_\xi$ is the solution to (1-9) and ψ is a small function (for ε sufficiently small) which belongs to Ψ.

Inserting u (given in (7-1)) into (1-8) and recalling that $\bar{u}_\xi$ is a solution to (1-9), we have that, in order to obtain a solution to (1-8), ψ must satisfy

$$(-\Delta)^s \psi + \psi - p w_\xi^{p-1} \psi = E(\psi) + N(\psi) \quad \text{in } \Omega_\varepsilon, \tag{7-2}$$

where[3]

$$E(\psi) := (\bar{u}_\xi + \psi)^p - (w_\xi + \psi)^p \quad \text{and} \quad N(\psi) := (w_\xi + \psi)^p - w_\xi^p - p w_\xi^{p-1} \psi. \tag{7-3}$$

Instead of solving (7-2), we will consider a projected version of the problem. Namely we will look for a solution $\psi \in H^s(\mathbb{R}^n)$ of the equation

$$(-\Delta)^s \psi + \psi - p w_\xi^{p-1} \psi = E(\psi) + N(\psi) + \sum_{i=1}^n c_i Z_i \quad \text{in } \Omega_\varepsilon \tag{7-4}$$

[3] As a matter of fact, one should write the positive parts in (7-3), namely set $E(\psi) := (\bar{u}_\xi + \psi)_+^p - (w_\xi + \psi)_+^p$ and $N(\psi) := (w_\xi + \psi)_+^p - w_\xi^p - p w_\xi^{p-1} \psi$, but, a posteriori, this is the same by maximum principle. So, we preferred, with a slight abuse of notation, to drop the positive parts for simplicity of notation.

for some coefficients $c_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$. Moreover, we require that ψ satisfies the conditions

$$\psi = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon \tag{7-5}$$

and

$$\int_{\Omega_\varepsilon} \psi Z_i \, dx = 0 \quad \text{for any } i = 1, \dots, n. \tag{7-6}$$

We will prove that problem (7-4)–(7-6) admits a unique solution, which is small if ε is sufficiently small, and then we will show that the coefficients c_i are equal to zero for every $i \in \{1, \dots, n\}$ for a suitable ξ. This will give us a solution $\psi \in \Psi$ to (7-2), and therefore a solution u of (1-8), thanks to the definition in (7-1).

Linear theory. In this subsection we develop a general theory that will give us the existence result for the linear problem (7-4)–(7-6).

Theorem 7.1. *Let $g \in L^2(\mathbb{R}^n)$ with $\|g\|_{\star,\xi} < +\infty$. If $\varepsilon > 0$ is sufficiently small, there exist a unique $\psi \in \Psi$ and numbers $c_i \in \mathbb{R}$ for any $i \in \{1, \dots, n\}$ such that*

$$(-\Delta)^s \psi + \psi - p w_\xi^{p-1} \psi + g = \sum_{i=1}^{n} c_i Z_i \quad \text{in } \Omega_\varepsilon. \tag{7-7}$$

Moreover, there exists a constant $C > 0$ such that

$$\|\psi\|_{\star,\xi} \leqslant C \|g\|_{\star,\xi}. \tag{7-8}$$

Before proving Theorem 7.1 we need some preliminary lemmata. In the next lemma we show that we can uniquely determine the coefficients c_i in (7-7) in terms of ψ and g. Actually, we will show that the estimate on the c_i holds in a more general case; that is, we do not need the orthogonality condition in (7-6).

Lemma 7.2. *Let $g \in L^2(\mathbb{R}^n)$ with $\|g\|_{\star,\xi} < +\infty$. Suppose that $\psi \in H^s(\mathbb{R}^n)$ satisfies*

$$\begin{cases} (-\Delta)^s \psi + \psi - p w_\xi^{p-1} \psi + g = \sum_{i=1}^{n} c_i Z_i & \text{in } \Omega_\varepsilon, \\ \psi = 0 & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon, \end{cases} \tag{7-9}$$

for some $c_i \in \mathbb{R}$, $i = 1, \dots, n$.

Then, for $\varepsilon > 0$ sufficiently small and for any $i \in \{1, \dots, n\}$, the coefficient c_i is given by

$$c_i = \frac{1}{\alpha} \int_{\mathbb{R}^n} g Z_i \, dx + f_i, \tag{7-10}$$

where α is as defined in (5-17), for a suitable $f_i \in \mathbb{R}$ that satisfies

$$|f_i| \leqslant C \varepsilon^{n/2} (\|\psi\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}) \tag{7-11}$$

for some positive constant C.

Proof. We start with some considerations in Fourier space on a function $T \in C^2(\mathbb{R}^n) \cap H^2(\mathbb{R}^n)$. First of all, for any $j \in \{1, \dots, n\}$,

$$\|\partial_j^2 T\|^2_{L^2(\mathbb{R}^n)} = \|\mathcal{F}(\partial_j^2 T)\|^2_{L^2(\mathbb{R}^n)} = \|\xi_j^2 \hat{T}\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \xi_j^4 |\hat{T}(\xi)|^2 \, d\xi.$$

Moreover, by convexity,

$$|\xi|^4 = \left(\sum_{j=1}^n \xi_j^2\right)^2 \leqslant 2 \sum_{j=1}^n \xi_j^4,$$

and therefore

$$2\|D^2 T\|^2_{L^2(\mathbb{R}^n)} = \sum_{j=1}^n 2\|\partial_j^2 T\|^2_{L^2(\mathbb{R}^n)} \geqslant \int_{\mathbb{R}^n} |\xi|^4 |\hat{T}(\xi)|^2 \, d\xi.$$

As a consequence,

$$\begin{aligned} \|(-\Delta)^s T\|^2_{L^2(\mathbb{R}^n)} &= \|\mathcal{F}((-\Delta)^s T)\|^2_{L^2(\mathbb{R}^n)} = \||\xi|^{2s} \hat{T}\|^2_{L^2(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} |\xi|^{4s} |\hat{T}(\xi)|^2 \, d\xi \leqslant \int_{\mathbb{R}^n} (1 + |\xi|^4) |\hat{T}(\xi)|^2 \, d\xi \\ &\leqslant \|T\|^2_{L^2(\mathbb{R}^n)} + 2\|D^2 T\|^2_{L^2(\mathbb{R}^n)} \leqslant C\|T\|_{H^2(\mathbb{R}^n)} \end{aligned} \tag{7-12}$$

for some $C > 0$.

Now, without loss of generality, we may suppose that

$$B_{c/\varepsilon}(\xi) \subseteq \Omega_\varepsilon \tag{7-13}$$

for some $c > 0$. Fix $\varepsilon > 0$, and choose $\tau_\varepsilon \in C^\infty(\mathbb{R}^n, [0, 1])$ with $\tau_\varepsilon = 1$ in $B_{(c/\varepsilon)-1}(\xi)$, $\tau_\varepsilon = 0$ outside $B_{c/\varepsilon}(\xi)$ and $|\nabla \tau_\varepsilon| \leqslant C$. We set $T_{\varepsilon,j} := Z_j \tau_\varepsilon$. Hence, from (7-12) and Lemmata 5.3 and 5.4,

$$\|(-\Delta)^s T_{\varepsilon,j}\|^2_{L^2(\mathbb{R}^n)} \leqslant C \tag{7-14}$$

for some $C > 0$, independent of ε and j.

Moreover, the function $T_{\varepsilon,j}$ belongs to $H^s(\mathbb{R}^n)$ and vanishes outside $B_{c/\varepsilon}(\xi)$, and so in particular outside Ω_ε, thanks to (7-13).

Thus (see, e.g., formula (1.5) in [Ros-Oton and Serra 2014b]),

$$\int_{\Omega_\varepsilon} (-\Delta)^s \psi T_{\varepsilon,j} \, dx = \int_{\Omega_\varepsilon} (-\Delta)^{s/2} \psi (-\Delta)^{s/2} T_{\varepsilon,j} \, dx = \int_{\Omega_\varepsilon} \psi (-\Delta)^s T_{\varepsilon,j} \, dx. \tag{7-15}$$

As a consequence, recalling (7-14),

$$\left| \int_{\Omega_\varepsilon} (-\Delta)^s \psi T_{\varepsilon,j} \, dx \right| \leqslant \|\psi\|_{L^2(\Omega_\varepsilon)} \|(-\Delta)^s T_{\varepsilon,j}\|_{L^2(\Omega_\varepsilon)} \leqslant C\|\psi\|_{L^2(\mathbb{R}^n)}. \tag{7-16}$$

Now, we fix $j \in \{1, \dots, n\}$, we multiply the equation in (7-9) by $T_{\varepsilon,j}$ and we integrate over Ω_ε. We obtain

$$\sum_{i=1}^n c_i \int_{\Omega_\varepsilon} Z_i T_{\varepsilon,j} \, dx = \int_{\Omega_\varepsilon} T_{\varepsilon,j} ((-\Delta)^s \psi + \psi - p w_\xi^{p-1} \psi + g) \, dx. \tag{7-17}$$

Now, we observe that, thanks to (7-15), we can write

$$\int_{\Omega_\varepsilon} (-\Delta)^s \psi T_{\varepsilon,j}\,dx = \int_{\Omega_\varepsilon} \psi(-\Delta)^s T_{\varepsilon,j}\,dx = \int_{\Omega_\varepsilon} \psi(-\Delta)^s (T_{\varepsilon,j} - Z_j)\,dx + \int_{\Omega_\varepsilon} \psi(-\Delta)^s Z_j\,dx. \tag{7-18}$$

Using Hölder's inequality and (7-12), we have that

$$\begin{aligned}\left|\int_{\Omega_\varepsilon} \psi(-\Delta)^s (T_{\varepsilon,j} - Z_j)\,dx\right| &\leqslant \|\psi\|_{L^2(\mathbb{R}^n)} \|(-\Delta)^s (T_{\varepsilon,j} - Z_j)\|_{L^2(\mathbb{R}^n)} \\ &\leqslant C\|\psi\|_{L^2(\mathbb{R}^n)} \|T_{\varepsilon,j} - Z_j\|_{H^2(\mathbb{R}^n)}.\end{aligned} \tag{7-19}$$

Let us estimate the H^2-norm of $T_{\varepsilon,j} - Z_j$. First, we have that

$$\|T_{\varepsilon,j} - Z_j\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (\tau_\varepsilon - 1)^2 Z_j^2\,dx \leqslant \int_{B^c_{(c/\varepsilon)-1}(\xi)} Z_j^2\,dx,$$

since $\tau_\varepsilon = 1$ in $B_{c/\varepsilon-1}(\xi)$ and takes values in $(0, 1)$. Hence, from Lemma 5.2, we deduce that

$$\|T_{\varepsilon,j} - Z_j\|^2_{L^2(\mathbb{R}^n)} \leqslant C \int_{B^c_{c/\varepsilon-1}(\xi)} \frac{1}{|x-\xi|^{2\nu_1}}\,dx \leqslant C\varepsilon^n,$$

up to renaming C. Therefore,

$$\|T_{\varepsilon,j} - Z_j\|_{L^2(\mathbb{R}^n)} \leqslant C\varepsilon^{n/2}. \tag{7-20}$$

Moreover, we have that

$$\begin{aligned}\|\nabla(T_{\varepsilon,j} - Z_j)\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |(\tau_\varepsilon - 1)\nabla Z_j + \nabla\tau_\varepsilon Z_j|^2\,dx \\ &= \int_{\mathbb{R}^n} (\tau_\varepsilon - 1)^2 |\nabla Z_j|^2 + |\nabla\tau_\varepsilon|^2 Z_j^2 + 2(\tau_\varepsilon - 1) Z_j \nabla Z_j \cdot \nabla\tau_\varepsilon\,dx.\end{aligned}$$

Using the fact that both $\tau_\varepsilon - 1$ and $\nabla\tau_\varepsilon$ have support outside $B_{c/\varepsilon-1}(\xi)$, and Lemmata 5.2 and 5.3, we obtain that

$$\|\nabla(T_{\varepsilon,j} - Z_j)\|_{L^2(\mathbb{R}^n)} \leqslant C\varepsilon^{n/2}. \tag{7-21}$$

Finally, using again the fact that $\tau_\varepsilon - 1$, $\nabla\tau_\varepsilon$ and $D^2\tau_\varepsilon$ have support outside $B_{c/\varepsilon-1}(\xi)$, and Lemmata 5.2, 5.3 and 5.4, we obtain that

$$\|D^2(T_{\varepsilon,j} - Z_j)\|_{L^2(\mathbb{R}^n)} \leqslant C\varepsilon^{n/2}.$$

Using this, (7-20) and (7-21) we have that

$$\|T_{\varepsilon,j} - Z_j\|_{H^2(\mathbb{R}^n)} \leqslant C\varepsilon^{n/2},$$

and so, from (7-19), we obtain

$$\left|\int_{\Omega_\varepsilon} \psi(-\Delta)^s (T_{\varepsilon,j} - Z_j)\,dx\right| \leqslant C\varepsilon^{n/2} \|\psi\|_{L^2(\mathbb{R}^n)}. \tag{7-22}$$

Now, using (7-18), we have that

$$\int_{\Omega_\varepsilon} T_{\varepsilon,j}((-\Delta)^s\psi+\psi-pw_\xi^{p-1}\psi)\,dx \\ = \int_{\Omega_\varepsilon} \psi(-\Delta)^s Z_j + T_{\varepsilon,j}\psi - pw_\xi^{p-1}\psi T_{\varepsilon,j}\,dx + \int_{\Omega_\varepsilon} \psi(-\Delta)^s(T_{\varepsilon,j}-Z_j)\,dx.$$

Since w_ξ is a solution to (1-3), we have that Z_j solves

$$(-\Delta)^s Z_j + Z_j = pw_\xi^{p-1} Z_j,$$

and this implies that

$$\int_{\Omega_\varepsilon} T_{\varepsilon,j}((-\Delta)^s\psi+\psi-pw_\xi^{p-1}\psi)\,dx \\ = \int_{\Omega_\varepsilon} \psi(T_{\varepsilon,j}-Z_j) - pw_\xi^{p-1}\psi(T_{\varepsilon,j}-Z_j)\,dx + \int_{\Omega_\varepsilon} \psi(-\Delta)^s(T_{\varepsilon,j}-Z_j)\,dx.$$

Hence, using the fact that w_ξ is bounded (see (1-4)) and Hölder's inequality, we have that

$$\left|\int_{\Omega_\varepsilon} T_{\varepsilon,j}((-\Delta)^s\psi+\psi-pw_\xi^{p-1}\psi)\,dx\right| \\ \leqslant C\left(\|\psi\|_{L^2(\mathbb{R}^n)}\|T_{\varepsilon,j}-Z_j\|_{L^2(\mathbb{R}^n)} + \left|\int_{\Omega_\varepsilon} \psi(-\Delta)^s(T_{\varepsilon,j}-Z_j)\,dx\right|\right) \\ \leqslant C\varepsilon^{n/2}\|\psi\|_{L^2(\mathbb{R}^n)}, \tag{7-23}$$

where we have used (7-20) and (7-22) in the last step.

Now, we can write

$$\int_{\Omega_\varepsilon} T_{\varepsilon,j} g\,dx = \int_{\Omega_\varepsilon} (T_{\varepsilon,j}-Z_j)g\,dx + \int_{\Omega_\varepsilon} Z_j g\,dx \\ = \int_{\Omega_\varepsilon} (T_{\varepsilon,j}-Z_j)g\,dx + \int_{\mathbb{R}^n} Z_j g\,dx - \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} Z_j g\,dx. \tag{7-24}$$

Using Hölder's inequality and (7-20), we can estimate

$$\left|\int_{\Omega_\varepsilon} (T_{\varepsilon,j}-Z_j)g\,dx\right| \leqslant \|T_{\varepsilon,j}-Z_j\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)} \leqslant C\varepsilon^{n/2}\|g\|_{L^2(\mathbb{R}^n)}.$$

Moreover, from Hölder's inequality and Lemma 5.2 (and recalling that $\mathrm{dist}(\xi,\partial\Omega_\varepsilon)\geqslant c/\varepsilon$), we obtain that

$$\left|\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} Z_j g\,dx\right| \leqslant \|g\|_{L^2(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} \frac{C}{|x-\xi|^{2\nu_1}}\,dx\right)^{\frac{1}{2}} \leqslant C\varepsilon^{n/2}\|g\|_{L^2(\mathbb{R}^n)}.$$

The last two estimates and (7-24) imply that

$$\int_{\Omega_\varepsilon} T_{\varepsilon,j} g\,dx = \int_{\mathbb{R}^n} Z_j g\,dx + \tilde{f}_j,$$

where

$$|\tilde{f}_j| \leqslant C\varepsilon^{n/2}\|g\|_{L^2(\mathbb{R}^n)}.$$

From this, (7-17) and (7-23), we have that

$$\sum_{i=1}^{n} c_i \int_{\Omega_\varepsilon} Z_i T_{\varepsilon,j}\,dx = \int_{\mathbb{R}^n} g Z_j\,dx + \bar{f}_j, \tag{7-25}$$

where

$$|\bar{f}_j| \leqslant C\varepsilon^{n/2}(\|\psi\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}), \tag{7-26}$$

up to renaming the constants.

On the other hand, we can write

$$\int_{\Omega_\varepsilon} Z_i T_{\varepsilon,j}\,dx = \int_{\Omega_\varepsilon} Z_i (T_{\varepsilon,j} - Z_j)\,dx + \int_{\Omega_\varepsilon} Z_i Z_j\,dx. \tag{7-27}$$

From Hölder's inequality, (7-20) and Lemma 5.2, we have that

$$\left|\int_{\Omega_\varepsilon} Z_i (T_{\varepsilon,j} - Z_j)\,dx\right| \leqslant \left(\int_{\Omega_\varepsilon} Z_i^2\,dx\right)^{\frac{1}{2}} \|T_{\varepsilon,j} - Z_j\|_{L^2(\mathbb{R}^n)} \leqslant C\varepsilon^{n/2}.$$

Using this and Corollary 5.6 in (7-27), we obtain that

$$\int_{\Omega_\varepsilon} Z_i T_{\varepsilon,j}\,dx = \alpha\delta_{ij} + O(\varepsilon^{n/2}). \tag{7-28}$$

So, we consider the matrix $A \in \mathrm{Mat}(n \times n)$ defined as

$$A_{ji} := \int_{\Omega_\varepsilon} Z_i T_{\varepsilon,j}\,dx. \tag{7-29}$$

Thanks to (7-28), the matrix $\alpha^{-1}A$ is a perturbation of the identity and so it is invertible for ε sufficiently small, with inverse equal to the identity plus a smaller order term of size $\varepsilon^{n/2}$. Hence, the matrix A is invertible too, with inverse

$$(A^{-1})_{ji} = \alpha^{-1}\delta_{ij} + O(\varepsilon^{n/2}). \tag{7-30}$$

So we consider the vector $d = (d_1, \ldots, d_n)$ defined by

$$d_j := \int_{\mathbb{R}^n} g Z_j\,dx + \bar{f}_j. \tag{7-31}$$

We observe that

$$\left|\int_{\mathbb{R}^n} g Z_j\,dx\right| \leqslant \|g\|_{L^2(\mathbb{R}^n)}\|Z_j\|_{L^2(\mathbb{R}^n)} \leqslant C\|g\|_{L^2(\mathbb{R}^n)},$$

thanks to Lemma 5.2. As a consequence, recalling (7-26), we obtain that

$$|d| \leqslant C(\|\psi\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}), \tag{7-32}$$

up to renaming C.

With the setting above, (7-25) reads

$$\sum_{i=1}^{n} c_i A_{ji} = \int_{\mathbb{R}^n} g Z_j \, dx + \bar{f}_j = d_j;$$

that is, in matrix notation, $Ac = d$. We can invert this relation using (7-30) and write

$$c = A^{-1} d = \alpha^{-1} d + f^{\sharp}$$

with

$$|f^{\sharp}| \leqslant C\varepsilon^{n/2} |d| \leqslant C\varepsilon^{n/2} (\|\psi\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}), \tag{7-33}$$

in virtue of (7-32). So, using (7-31),

$$c_i = \alpha^{-1} d_i + f_i^{\sharp} = \alpha^{-1} \int_{\mathbb{R}^n} g Z_i \, dx + \alpha^{-1} \bar{f}_i + f_i^{\sharp}.$$

This proves (7-10) with

$$f_i := \alpha^{-1} \bar{f}_i + f_i^{\sharp},$$

and then (7-11) follows from (7-26) and (7-33). □

Now, we show that solutions to (7-7) satisfy an a priori estimate. We remark that the result in the following lemma is different from the one in Corollary 6.4, since here also a combination of Z_i for $i = 1, \dots, n$ appears in the equation satisfied by ψ.

Lemma 7.3. *Let $g \in L^2(\mathbb{R}^n)$ with $\|g\|_{\star,\xi} < +\infty$. Let $\psi \in \Psi$ be a solution to (7-7) for some coefficients $c_i \in \mathbb{R}$, $i = 1, \dots, n$, and for ε sufficiently small.*

Then,

$$\|\psi\|_{\star,\xi} \leqslant C \|g\|_{\star,\xi}.$$

Proof. Suppose by contradiction that there exists a sequence $\varepsilon_j \searrow 0$ as $j \to +\infty$ such that, for any $j \in \mathbb{N}$, the function ψ_j satisfies

$$\begin{cases} (-\Delta)^s \psi_j + \psi_j - p w_{\xi_j}^{p-1} \psi_j + g_j = \sum_{i=1}^n c_i^j Z_i^j & \text{in } \Omega_{\varepsilon_j}, \\ \psi_j = 0 & \text{in } \mathbb{R}^n \setminus \Omega_{\varepsilon_j}, \\ \int_{\Omega_{\varepsilon_j}} \psi_j Z_i^j \, dx = 0 & \text{for any } i = 1, \dots, n, \end{cases} \tag{7-34}$$

for suitable $g_j \in L^2(\mathbb{R}^n)$ and $\xi_j \in \Omega_{\varepsilon_j}$, where

$$Z_i^j := \frac{\partial w_{\xi_j}}{\partial x_i}.$$

Moreover,

$$\|\psi_j\|_{\star,\xi_j} = 1 \quad \text{for any } j \in \mathbb{N} \tag{7-35}$$

$$\text{and} \quad \|g_j\|_{\star,\xi_j} \searrow 0 \quad \text{as } j \to +\infty. \tag{7-36}$$

Notice that the fact that the equation in (7-34) is linear with respect to ψ_j, g_j and Z_i^j allows us to take the sequences ψ_j and g_j as in (7-35) and (7-36).

We claim that, for any given $R > 0$,

$$\|\psi_j\|_{L^\infty(B_R(\xi_j))} \to 0 \quad \text{as } j \to +\infty. \tag{7-37}$$

For this, we argue by contradiction and we assume that there exists $\delta > 0$ and $j_0 \in \mathbb{N}$ such that, for any $j \geqslant j_0$, we have that $\|\psi_j\|_{L^\infty(B_R(\xi_j))} \geqslant \delta$.

Thanks to Lemmata 7.2 and 5.2, we have that

$$|c_i^j| \leqslant \frac{C_1}{\alpha_i}\|g_j\|_{\star,\xi_j} + C_2 \varepsilon_j^{n/2}$$

for suitable positive constants C_1 and C_2. Hence, from (7-36), we obtain that

$$c_i^j \searrow 0 \quad \text{as } j \to +\infty \text{ for any } i \in \{1, \dots, n\}. \tag{7-38}$$

Now, from Lemma 6.1, we have that

$$\sup_{x\neq y} \frac{|\psi_j(x) - \psi_j(y)|}{|x-y|^s} \leqslant C\left(\left\| g_j + \sum_{i=1}^n c_i^j Z_i^j + p w_{\xi_j}^{p-1}\psi_j \right\|_{L^\infty(\mathbb{R}^n)} + \left\| g_j + \sum_{i=1}^n c_i^j Z_i^j + p w_{\xi_j}^{p-1}\psi_j \right\|_{L^2(\mathbb{R}^n)} \right). \tag{7-39}$$

We observe that

$$\left\| g_j + \sum_{i=1}^n c_i^j Z_i^j + p w_{\xi_j}^{p-1}\psi_j \right\|_{L^\infty(\mathbb{R}^n)} \leqslant C\left(\|g\|_{\star,\xi_j} + \sum_{i=1}^n |c_i^j| + \|\psi_j\|_{\star,\xi_j} \right),$$

thanks to the decay of Z_i^j in Lemma 5.2 and the fact that $w_{\xi_j}^{p-1}$ is bounded (recall (1-4)). So, from (7-36), (7-38) and (7-35), we obtain that

$$\left\| g_j + \sum_{i=1}^n c_i^j Z_i^j + p w_{\xi_j}^{p-1}\psi_j \right\|_{L^\infty(\mathbb{R}^n)} \leqslant C \tag{7-40}$$

for a suitable constant $C > 0$ independent of j.

We claim that

$$\left\| g_j + \sum_{i=1}^n c_i^j Z_i^j + p w_{\xi_j}^{p-1}\psi_j \right\|_{L^2(\mathbb{R}^n)} \leqslant C, \tag{7-41}$$

where $C > 0$ does not depend on j. Indeed,

$$\|g_j\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} g_j^2 \, dx \right)^{\frac{1}{2}} \leqslant \|g_j\|_{\star,\xi_j} \left(\int_{\mathbb{R}^n} \rho_{\xi_j}^2 \, dx \right)^{\frac{1}{2}} \leqslant \|g_j\|_{\star,\xi_j} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|x-\xi_j|)^{2\mu}} \, dx \right)^{\frac{1}{2}} \leqslant C\|g_j\|_{\star,\xi_j} \leqslant C,$$

since $2\mu > n$ and (7-36) holds. Moreover,

$$\left\| \sum_{i=1}^{n} c_i^j Z_i^j \right\|_{L^2(\mathbb{R}^n)} \leqslant \sum_{i=1}^{n} |c_i^j| \, \|Z_i^j\|_{L^2(\mathbb{R}^n)} \leqslant C,$$

thanks to (7-38) and Lemma 5.2. Finally, using (1-4), the fact that $2\mu > n$, and (7-35), we have that

$$\begin{aligned} \left\| p w_{\xi_j}^{p-1} \psi_j \right\|_{L^2(\mathbb{R}^n)} &\leqslant p \|\psi_j\|_{\star,\xi_j} \left(\int_{\mathbb{R}^n} w_{\xi_j}^{2(p-1)} \rho_{\xi_j}^2 \, dx \right)^{\frac{1}{2}} \\ &\leqslant C \|\psi_j\|_{\star,\xi_j} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|x-\xi_j|)^{2(p-1)(n+2s)+2\mu}} \, dx \right)^{\frac{1}{2}} \leqslant C. \end{aligned}$$

Putting together the above estimates, we obtain (7-41).

Hence, from (7-39), (7-40) and (7-41), we have that the ψ_j are equicontinuous.

For any $j = 1, \dots, n$, we define the function

$$\tilde{\psi}_j(x) := \psi_j(x + \xi_j)$$

and the set

$$\tilde{\Omega}_j := \{ x = y - \xi_j \mid y \in \Omega_{\varepsilon_j} \}.$$

We notice that $\tilde{\psi}_j$ satisfies

$$(-\Delta)^s \tilde{\psi}_j + \tilde{\psi}_j - p w^{p-1} \tilde{\psi}_j + \tilde{g}_j = \sum_{i=1}^{n} c_i^j \tilde{Z}_i \quad \text{in } \tilde{\Omega}_j, \tag{7-42}$$

where $\tilde{g}_j(x) := g(x + \xi_j)$ and $\tilde{Z}_i := \partial w / \partial x_i$. Moreover,

$$\tilde{\psi}_j = 0 \quad \text{in } \mathbb{R}^n \setminus \tilde{\Omega}_j. \tag{7-43}$$

Now, thanks to (6-1), we have that $B_{c/\varepsilon_j}(\xi_j) \subset \Omega_{\varepsilon_j}$. Hence, $B_{c/\varepsilon_j} \subset \tilde{\Omega}_j$, which means that $\tilde{\Omega}_j$ converges to $\mathbb{R}^n$ when $j \to +\infty$.

Furthermore, we have that

$$\|\tilde{\psi}_j\|_{L^\infty(B_R)} \geqslant \delta \quad \text{and} \quad \|(1+|x|)^\mu \tilde{\psi}_j\|_{L^\infty(\mathbb{R}^n)} = 1. \tag{7-44}$$

Now, since the ψ_j are equicontinuous, the $\tilde{\psi}_j$ are equicontinuous too, and therefore there exists a function $\bar{\psi}$ such that, up to subsequences, the $\tilde{\psi}_j$ converge to $\bar{\psi}$ uniformly on compact sets.

The function $\bar{\psi}$ is in $L^2(\mathbb{R}^n)$. Indeed, by Fatou's Theorem and (7-35), and recalling that $2\mu > n$, we have

$$\int_{\mathbb{R}^n} \bar{\psi}^2 \, dx \leqslant \liminf_{j \to +\infty} \int_{\mathbb{R}^n} \psi_j^2 \, dx \leqslant \liminf_{j \to +\infty} \|\psi_j\|_{\star,\xi_j}^2 \int_{\mathbb{R}^n} \frac{1}{(1+|x-\xi_j|)^{2\mu}} \, dx \leqslant C.$$

Moreover, $\bar{\psi}$ satisfies the conditions

$$\|\bar{\psi}\|_{L^\infty(B_R)} \geqslant \delta \tag{7-45}$$

$$\text{and} \quad \|(1+|x|)^\mu \bar{\psi}\|_{L^\infty(\mathbb{R}^n)} \leqslant 1. \tag{7-46}$$

We prove that $\bar{\psi}$ solves the equation

$$(-\Delta)^s \bar{\psi} + \bar{\psi} = p w^{p-1} \bar{\psi} \quad \text{in } \mathbb{R}^n. \tag{7-47}$$

Indeed, we multiply the equation in (7-42) by a function $\eta \in C_0^\infty(\tilde{\Omega}_j)$ and we integrate over $\mathbb{R}^n$. We notice that both η and $\tilde{\psi}_j$ are equal to zero outside $\tilde{\Omega}_j$ (recall (7-43)), and therefore we can use formula (1.5) in [Ros-Oton and Serra 2014b], and we get

$$\int_{\mathbb{R}^n} ((-\Delta)^s \eta + \eta - p w^{p-1} \eta) \tilde{\psi}_j \, dx + \int_{\mathbb{R}^n} \tilde{g}_j \eta \, dx = \sum_{i=1}^n c_i^j \int_{\mathbb{R}^n} \tilde{Z}_i \eta \, dx. \tag{7-48}$$

Now, we have that

$$\|\tilde{g}_j\|_{\star,0} = \|g_j\|_{\star,\xi_j} \searrow 0 \quad \text{as } j \to +\infty.$$

Moreover,

$$\left| \int_{\mathbb{R}^n} \tilde{g}_j \eta \, dx \right| \leqslant \|\tilde{g}_j\|_{\star,0} \int_{\mathbb{R}^n} \rho_0 \eta \, dx \leqslant C \|\tilde{g}_j\|_{\star,0},$$

since $2\mu > n$, which implies that

$$\int_{\mathbb{R}^n} \tilde{g}_j \eta \, dx \to 0 \quad \text{as } j \to +\infty. \tag{7-49}$$

Also,

$$\left| \sum_{i=1}^n c_i^j \int_{\mathbb{R}^n} \tilde{Z}_i \eta \, dx \right| \leqslant C \sum_{i=1}^n |c_i^j|,$$

and so, thanks to (7-38), we obtain that

$$\sum_{i=1}^n c_i^j \int_{\mathbb{R}^n} \tilde{Z}_i \eta \, dx \to 0 \quad \text{as } j \to +\infty. \tag{7-50}$$

Finally, we fix $r > 0$ and we estimate

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} ((-\Delta)^s \eta + \eta - p w^{p-1} \eta) \tilde{\psi}_j \, dx - \int_{\mathbb{R}^n} ((-\Delta)^s \eta + \eta - p w^{p-1} \eta) \bar{\psi} \, dx \right| \\
&\quad \leqslant \int_{\mathbb{R}^n} |(-\Delta)^s \eta + \eta - p w^{p-1} \eta| \, |\tilde{\psi}_j - \bar{\psi}| \, dx \\
&\quad = \int_{B_r} |(-\Delta)^s \eta + \eta - p w^{p-1} \eta| \, |\tilde{\psi}_j - \bar{\psi}| \, dx + \int_{\mathbb{R}^n \setminus B_r} |(-\Delta)^s \eta + \eta - p w^{p-1} \eta| \, |\tilde{\psi}_j - \bar{\psi}| \, dx.
\end{aligned} \tag{7-51}$$

We define the function

$$\tilde{\eta} := (-\Delta)^s \eta + \eta - p w^{p-1} \eta$$

and we notice that it satisfies the decay

$$|\tilde{\eta}(x)| \leqslant \frac{C_1}{(1 + |x|)^{n+2s}}. \tag{7-52}$$

Hence,

$$\int_{B_r} |(-\Delta)^s\eta+\eta-pw^{p-1}\eta||\tilde{\psi}_j-\bar{\psi}|\,dx \leqslant C_1\|\tilde{\psi}_j-\bar{\psi}\|_{L^\infty(B_r)}\int_{B_r}\frac{1}{(1+|x|)^{n+2s}}\,dx \leqslant C_2\|\tilde{\psi}_j-\bar{\psi}\|_{L^\infty(B_r)},$$

which implies that

$$\int_{B_r} |(-\Delta)^s\eta+\eta-pw^{p-1}\eta||\tilde{\psi}_j-\bar{\psi}|\,dx \searrow 0 \quad \text{as } j\to+\infty \tag{7-53}$$

due to the uniform convergence of $\tilde{\psi}_j$ to $\bar{\psi}$ on compact sets. On the other hand, from (7-44), (7-46) and (7-52), we have that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_r} |(-\Delta)^s\eta+\eta-pw^{p-1}\eta||\tilde{\psi}_j-\bar{\psi}|\,dx &\leqslant 2C_1\int_{\mathbb{R}^n\setminus B_r}\frac{1}{(1+|x|)^{n+2s}}\,dx\\ &\leqslant 2C_1\int_{\mathbb{R}^n\setminus B_r}\frac{1}{|x|^{n+2s}}\,dx\\ &\leqslant C_3 r^{-2s}.\end{aligned}$$

Hence, sending $r\to+\infty$, we obtain that

$$\int_{\mathbb{R}^n\setminus B_r} |(-\Delta)^s\eta+\eta-pw^{p-1}\eta||\tilde{\psi}_j-\bar{\psi}|\,dx \searrow 0. \tag{7-54}$$

Putting together (7-51), (7-53) and (7-54), we obtain that

$$\int_{\mathbb{R}^n}((-\Delta)^s\eta+\eta-pw^{p-1}\eta)\tilde{\psi}_j\,dx \to \int_{\mathbb{R}^n}((-\Delta)^s\eta+\eta-pw^{p-1}\eta)\bar{\psi}\,dx \quad \text{as } j\to+\infty.$$

This, (7-49), (7-50) and (7-48) imply that

$$\int_{\mathbb{R}^n}((-\Delta)^s\eta+\eta-pw^{p-1}\eta)\bar{\psi}\,dx = 0$$

for any $\eta\in C_0^\infty(\mathbb{R}^n)$. This means that $\bar{\psi}$ is a weak solution to (7-47), and so a strong solution, thanks to [Servadei and Valdinoci 2014].

Hence, recalling the nondegeneracy result in [Frank et al. 2015], we have that

$$\bar{\psi} = \sum_{i=1}^n \beta_i\frac{\partial w}{\partial x_i} \tag{7-55}$$

for some coefficients $\beta_i\in\mathbb{R}$.

On the other hand, the orthogonality condition in (7-34) passes to the limit, that is,

$$\int_{\mathbb{R}^n}\bar{\psi}\tilde{Z}_i\,dx = 0 \quad \text{for any } i=1,\dots,n. \tag{7-56}$$

Indeed, we fix $r>0$ and we compute

$$\int_{\mathbb{R}^n}(\bar{\psi}-\tilde{\psi}_j)\tilde{Z}_i\,dx = \int_{B_r}(\bar{\psi}-\tilde{\psi}_j)\tilde{Z}_i\,dx + \int_{\mathbb{R}^n\setminus B_r}(\bar{\psi}-\tilde{\psi}_j)\tilde{Z}_i\,dx. \tag{7-57}$$

Concerning the first term on the right-hand side, we use the uniform convergence of $\tilde{\psi}_j$ to $\bar{\psi}$ on compact sets together with the fact that $\tilde{Z}_i$ is bounded to obtain that

$$\int_{B_r} (\bar{\psi} - \tilde{\psi}_j)\tilde{Z}_i \, dx \to 0 \quad \text{as } j \to +\infty.$$

As for the second term, we use (7-44), (7-46) and Lemma 5.2 and we get

$$\int_{\mathbb{R}^n \setminus B_r} (\bar{\psi} - \tilde{\psi}_j)\tilde{Z}_i \, dx \leqslant C \int_{\mathbb{R}^n \setminus B_r} \frac{1}{|x|^{n+2s}} \, dx \leqslant \bar{C} r^{-2s},$$

which tends to zero as $r \to +\infty$. Using the above two formulas in (7-57) we obtain that

$$0 = \int_{\mathbb{R}^n} \tilde{\psi}_j \tilde{Z}_i \, dx \to \int_{\mathbb{R}^n} \bar{\psi} \tilde{Z}_i \, dx,$$

which implies (7-56).

Therefore, recalling (5-3) also, (7-55) and (7-56) imply that $\bar{\psi} \equiv 0$, thus contradicting (7-45). This proves (7-37).

Now, from Corollary 6.3 (notice that we can take R sufficiently big in order to apply the corollary), we have that

$$\begin{aligned}
\|\psi_j\|_{\star,\xi_j} &\leqslant C\left(\|\psi_j\|_{L^\infty(B_R(\xi_j))} + \left\| g_j + \sum_{i=1}^n c_i^j Z_i^j \right\|_{\star,\xi_j} \right) \\
&\leqslant C\left(\|\psi_j\|_{L^\infty(B_R(\xi_j))} + \|g_j\|_{\star,\xi_j} + \left\| \sum_{i=1}^n |c_i^j| Z_i^j \right\|_{\star,\xi_j} \right) \\
&= C\left(\|\psi_j\|_{L^\infty(B_R(\xi_j))} + \|g_j\|_{\star,\xi_j} + \left\| \sum_{i=1}^n |c_i^j| \rho_{\xi_j}^{-1} Z_i^j \right\|_{L^\infty(\mathbb{R}^n)} \right) \\
&\leqslant C\left(\|\psi_j\|_{L^\infty(B_R(\xi_j))} + \|g_j\|_{\star,\xi_j} + \sum_{i=1}^n |c_i^j| \right),
\end{aligned}$$

up to renaming C, where we have used the decay of Z_i^j (see Lemma 5.2) and the fact that $\mu < n+2s$. Therefore, (7-36), (7-37) and (7-38) imply that

$$\|\psi_j\|_{\star,\xi_j} \to 0 \quad \text{as } j \to +\infty,$$

which contradicts (7-35) and concludes the proof. □

Now we consider an auxiliary problem: we look for a solution $\psi \in \Psi$ of

$$(-\Delta)^s \psi + \psi + g = \sum_{i=1}^n c_i Z_i \quad \text{in } \Omega_\varepsilon, \tag{7-58}$$

and we prove the following:

Proposition 7.4. *Let $g \in L^2(\mathbb{R}^n)$ with $\|g\|_{\star,\xi} < +\infty$. Then, there exists a unique solution $\psi \in \Psi$ to (7-58).*

Furthermore, there exists a constant $C > 0$ *such that*

$$\|\psi\|_{\star,\xi} \leqslant C\|g\|_{\star,\xi}. \tag{7-59}$$

Proof. We first prove the existence of a solution to (7-58).

First of all, we notice that formula (1.5) in [Ros-Oton and Serra 2014b] implies that, for any $\psi, \varphi \in \Psi$,

$$\int_{\mathbb{R}^n} (-\Delta)^s \psi\varphi\, dx = \int_{\mathbb{R}^n} (-\Delta)^{s/2}\psi(-\Delta)^{s/2}\varphi\, dx = \int_{\mathbb{R}^n} \psi(-\Delta)^s\varphi\, dx.$$

Now, given $g \in L^2(\mathbb{R}^n)$, we look for a solution $\psi \in \Psi$ of the problem

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2}\psi(-\Delta)^{s/2}\varphi\, dx + \int_{\mathbb{R}^n} \psi\varphi\, dx + \int_{\mathbb{R}^n} g\varphi\, dx = 0 \tag{7-60}$$

for any $\varphi \in \Psi$. Subsequently we will show that ψ is a solution to the original problem (7-58).

We observe that

$$\langle\psi, \varphi\rangle := \int_{\mathbb{R}^n} (-\Delta)^{s/2}\psi(-\Delta)^{s/2}\varphi\, dx + \int_{\mathbb{R}^n} \psi\varphi\, dx$$

defines an inner product in Ψ, and that

$$F(\varphi) := -\int_{\mathbb{R}^n} g\varphi\, dx$$

is a linear and continuous functional on Ψ. Hence, from Riesz's theorem, we have that there exists a unique function $\psi \in \Psi$ which solves (7-60).

We claim that

$$\psi \text{ is a strong solution to (7-58).} \tag{7-61}$$

For this, we take a radial cutoff $\tau \in C_0^\infty(\Omega_\varepsilon)$ of the form $\tau(x) = \tau_o(|x - \xi|)$ for some smooth and compactly supported real function, and we use Lemma 5.5. So, for any $\phi \in H^s(\mathbb{R}^n)$ such that $\phi = 0$ outside Ω_ε, we define

$$\tilde{\phi} := \phi - \sum_{i=1}^{n} \lambda_i(\phi)\tilde{Z}_i,$$

where

$$\lambda_i(\phi) := \tilde{\alpha}^{-1} \int_{\mathbb{R}^n} \phi Z_i\, dx, \tag{7-62}$$

and $\tilde{Z}_i$ and $\tilde{\alpha}$ are as in Lemma 5.5. We remark that $\tilde{Z}_i$ vanishes outside Ω_ε, hence so does $\tilde{\phi}$. Furthermore, for any $j \in \{1, \dots, n\}$,

$$\begin{aligned}\int_{\Omega_\varepsilon} \tilde{\phi} Z_j\, dx = \int_{\mathbb{R}^n} \tilde{\phi} Z_j\, dx = \int_{\mathbb{R}^n} \phi Z_j\, dx - \sum_{i=1}^{n} \lambda_i(\phi) \int_{\mathbb{R}^n} \tilde{Z}_i Z_j\, dx &= \int_{\mathbb{R}^n} \phi Z_j\, dx - \sum_{i=1}^{n} \lambda_i(\phi)\tilde{\alpha}\delta_{ij} \\ &= \int_{\mathbb{R}^n} \phi Z_j\, dx - \lambda_j(\phi)\tilde{\alpha} \\ &= 0,\end{aligned}$$

thanks to Lemma 5.5 and (7-62). This shows that $\tilde{\phi} \in \Psi$.

As a consequence, we can use $\tilde{\phi}$ as a test function in (7-60) and conclude that

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2}\psi(-\Delta)^{s/2}\Big(\phi - \sum_{i=1}^n \lambda_i(\phi)\tilde{Z}_i\Big)\,dx + \int_{\mathbb{R}^n} \psi\Big(\phi - \sum_{i=1}^n \lambda_i(\phi)\tilde{Z}_i\Big)\,dx + \int_{\mathbb{R}^n} g\Big(\phi - \sum_{i=1}^n \lambda_i(\phi)\tilde{Z}_i\Big)\,dx = 0,$$

that is,

$$\begin{aligned}\int_{\mathbb{R}^n} (-\Delta)^{s/2}\psi(-\Delta)^{s/2}\phi\,dx + \int_{\mathbb{R}^n} \psi\phi\,dx + \int_{\mathbb{R}^n} g\phi\,dx &= \int_{\mathbb{R}^n} (\psi+g)\sum_{i=1}^n \lambda_i(\phi)\tilde{Z}_i\,dx + \int_{\mathbb{R}^n} (-\Delta)^s\psi \sum_{i=1}^n \lambda_i(\phi)\tilde{Z}_i\,dx \\ &= \sum_{i=1}^n \lambda_i(\phi)\int_{\mathbb{R}^n} (\psi + g + (-\Delta)^s\psi)\tilde{Z}_i\,dx. \end{aligned} \tag{7-63}$$

Now, we define

$$b_i := \tilde{\alpha}^{-1}\int_{\mathbb{R}^n} (\psi + g + (-\Delta)^s\psi)\tilde{Z}_i\,dx,$$

we recall (7-62) and we write (7-63) as

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2}\psi(-\Delta)^{s/2}\phi\,dx + \int_{\mathbb{R}^n} \psi\phi\,dx + \int_{\mathbb{R}^n} g\phi\,dx = \sum_{i=1}^n \lambda_i(\phi)\tilde{\alpha}b_i = \sum_{i=1}^n b_i\int_{\mathbb{R}^n} \phi Z_i\,dx.$$

Since ϕ is any test function, this means that ψ is a solution of

$$(-\Delta)^s\psi + \psi + g = \sum_{i=1}^n b_i Z_i$$

in a weak sense, and therefore in a strong sense, thanks to [Servadei and Valdinoci 2014], thus proving (7-61).

Now, we prove the uniqueness of the solution to (7-58). For this, suppose by contradiction that there exist ψ_1 and ψ_2 in Ψ that solve (7-58). We set

$$\tilde{\psi} := \psi_1 - \psi_2,$$

and we observe that $\tilde{\psi}$ is in Ψ and solves

$$(-\Delta)^s\tilde{\psi} + \tilde{\psi} = \sum_{i=1}^n a_i Z_i \quad \text{in } \Omega_\varepsilon \tag{7-64}$$

for suitable coefficients $a_i \in \mathbb{R}$, $i = 1, \dots, n$.

We multiply the equation in (7-64) by $\tilde\psi$ and we integrate over Ω_ε, obtaining that

$$\int_{\Omega_\varepsilon} (-\Delta)^s \tilde\psi \tilde\psi + \tilde\psi^2 \, dx = 0,$$

since $\tilde\psi \in \Psi$ (and so it is orthogonal to Z_i in $L^2(\Omega_\varepsilon)$ for any $i = 1, \dots, n$). Since $\tilde\psi = 0$ outside Ω_ε, we can apply formula (1.5) in [Ros-Oton and Serra 2014b] and we obtain that

$$\int_{\mathbb{R}^n} |(-\Delta)^{s/2} \tilde\psi|^2 + \tilde\psi^2 \, dx = 0,$$

that is,

$$\|\tilde\psi\|_{H^s(\mathbb{R}^n)} = 0,$$

which implies that $\tilde\psi = 0$. Thus $\psi_1 = \psi_2$ and this concludes the proof of the uniqueness.

It remains to establish (7-59). Thanks to (7-58) and Corollary 6.4, we have that

$$\|\psi\|_{\star,\xi} \leqslant C \Big\| g - \sum_{i=1}^n c_i Z_i \Big\|_{\star,\xi} \leqslant C \bigg(\|g\|_{\star,\xi} + \Big\| \sum_{i=1}^n c_i Z_i \Big\|_{\star,\xi} \bigg). \tag{7-65}$$

First, we observe that, for any $i = 1, \dots, n$,

$$\|Z_i\|_{\star,\xi} = \sup_{\mathbb{R}^n} |\rho_\xi^{-1} Z_i| \leqslant C_1,$$

due to Lemma 5.2 and the fact that $\mu < n + 2s$ (recall also (6-2)). Hence,

$$\Big\| \sum_{i=1}^n c_i Z_i \Big\|_{\star,\xi} \leqslant \sum_{i=1}^n |c_i| \|Z_i\|_{\star,\xi} \leqslant C_1 \sum_{i=1}^n |c_i|. \tag{7-66}$$

Now, we claim that

$$\Big\| \sum_{i=1}^n c_i Z_i \Big\|_{\star,\xi} \leqslant C_2 (\|\psi\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}). \tag{7-67}$$

Indeed, we recall Lemma 5.5, we multiply (7-58) by $\tilde Z_j$ for $j \in \{1, \dots, n\}$, and we integrate over $\mathbb{R}^n$, obtaining that

$$\int_{\mathbb{R}^n} (-\Delta)^s \psi \tilde Z_j + \psi \tilde Z_j + g \tilde Z_j \, dx = \tilde\alpha c_j, \tag{7-68}$$

where $\tilde Z_j$ and $\tilde\alpha$ are as in Lemma 5.5. Thanks to formula (1.5) in [Ros-Oton and Serra 2014b], we have that

$$\left| \int_{\mathbb{R}^n} (-\Delta)^s \psi \tilde Z_j \, dx \right| = \left| \int_{\mathbb{R}^n} \psi (-\Delta)^s \tilde Z_j \, dx \right| \leqslant \|(-\Delta)^s \tilde Z_j\|_{L^2(\mathbb{R}^n)} \|\psi\|_{L^2(\mathbb{R}^n)},$$

where we have used Hölder's inequality. Therefore, this and (7-68) give that

$$\tilde\alpha |c_j| \leqslant \|(-\Delta)^s \tilde Z_j\|_{L^2(\mathbb{R}^n)} \|\psi\|_{L^2(\mathbb{R}^n)} + \|\tilde Z_j\|_{L^2(\mathbb{R}^n)} \|\psi\|_{L^2(\mathbb{R}^n)} + \|\tilde Z_j\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)},$$

which, together with (7-66), implies (7-67), since both $\|(-\Delta)^s \tilde Z_j\|_{L^2(\mathbb{R}^n)}$ and $\|\tilde Z_j\|_{L^2(\mathbb{R}^n)}$ are bounded (recall Lemma 5.4).

Now, we observe that

$$\|\psi\|_{L^2(\mathbb{R}^n)} \leqslant \|g\|_{L^2(\mathbb{R}^n)}. \tag{7-69}$$

Indeed, we multiply (7-58) by ψ and we integrate over Ω_ε: we obtain

$$\int_{\Omega_\varepsilon} (-\Delta)^s \psi\psi + \psi^2 + g\psi \, dx = 0,$$

since $\psi \in \Psi$. We notice that the first term in the above formula is quadratic, and so, using Hölder's inequality, we have that

$$\int_{\Omega_\varepsilon} \psi^2 \, dx \leqslant \int_{\Omega_\varepsilon} (-g)\psi \, dx \leqslant \|g\|_{L^2(\mathbb{R}^n)} \|\psi\|_{L^2(\mathbb{R}^n)},$$

which implies (7-69).

Therefore, from (7-67) and (7-69), we deduce that

$$\left\| \sum_{i=1}^n c_i Z_i \right\|_{\star,\xi} \leqslant 2C_3 \|g\|_{L^2(\mathbb{R}^n)}.$$

Moreover,

$$\|g\|_{L^2(\mathbb{R}^n)} \leqslant \|g\|_{\star,\xi} \left(\int_{\mathbb{R}^n} \rho_\xi^2 \, dx \right)^{\frac{1}{2}} = \|g\|_{\star,\xi} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|x-\xi|)^{2\mu}} \, dx \right)^{\frac{1}{2}} \leqslant C\|g\|_{\star,\xi},$$

since $2\mu > n$. The above two formulas give that

$$\left\| \sum_{i=1}^n c_i Z_i \right\|_{\star,\xi} \leqslant C_4 \|g\|_{\star,\xi}.$$

This and (7-65) show (7-59), and conclude the proof. □

Now, for any $g \in L^2(\mathbb{R}^n)$ with $\|g\|_{\star,\xi} < +\infty$, we denote by $\mathcal{A}[g]$ the unique solution to (7-58). We notice that Proposition 7.4 implies that the operator $\mathcal{A}$ is well defined and that

$$\|\mathcal{A}[g]\|_{\star,\xi} \leqslant C\|g\|_{\star,\xi}.$$

We also remark that $\mathcal{A}$ is a linear operator.

We consider the Banach space

$$Y_\star := \left\{ \psi : \mathbb{R}^n \to \mathbb{R} \;\middle|\; \|\psi\|_{\star,\xi} < +\infty \right\} \tag{7-70}$$

endowed with the norm $\|\cdot\|_{\star,\xi}$.

With this notation we can prove the main theorem of the linear theory, Theorem 7.1.

Proof of Theorem 7.1. We notice that solving (7-7) is equivalent to finding a function $\psi \in \Psi$ such that

$$\psi - \mathcal{A}[-pw_\xi^{p-1}\psi] = \mathcal{A}[g]. \tag{7-71}$$

For this, we set

$$\mathcal{B}[\psi] := \mathcal{A}[-pw_\xi^{p-1}\psi]. \tag{7-72}$$

Recalling the definition of $Y_\star$ given in (7-70), we observe that

$$\text{if } \psi \in Y_\star \quad \text{then } \mathcal{B}[\psi] \in Y_\star. \tag{7-73}$$

Indeed, from Proposition 7.4 we deduce that $\mathcal{B}[\psi] \in \Psi$ solves (7-58) with $g := -pw_\xi^{p-1}\psi$, and so

$$\|\mathcal{B}[\psi]\|_{\star,\xi} \leqslant C\| - pw_\xi^{p-1}\psi\|_{\star,\xi} \leqslant \tilde{C}\|\psi\|_{\star,\xi}$$

for some $\tilde{C} > 0$ (recall that w_ξ is bounded thanks to (1-4)), which proves (7-73).

We claim that

$$\mathcal{B} \text{ defines a compact operator in } Y_\star \text{ with respect to the norm } \|\cdot\|_{\star,\xi}. \tag{7-74}$$

Indeed, let $(\psi_j)_j$ a bounded sequence in $Y_\star$ with respect to the norm $\|\cdot\|_{\star,\xi}$. Then, thanks to Lemma 6.1, the fact that w_ξ^{p-1} and Z_i^j are bounded and $w_\xi^{p-1}\rho_\xi$ and Z_i^j belong to $L^2(\mathbb{R}^n)$, and Lemma 7.2, we have that

$$\begin{aligned}
\sup_{x\neq y} & \frac{|\mathcal{B}[\psi_j](x) - \mathcal{B}[\psi_j](y)|}{|x-y|^s} \\
&\leqslant C_1\left(\left\|-pw_\xi^{p-1}\psi_j + \sum_{i=1}^n c_i^j Z_i^j\right\|_{L^\infty(\mathbb{R}^n)} + \left\|-pw_\xi^{p-1}\psi_j + \sum_{i=1}^n c_i^j Z_i^j\right\|_{L^2(\mathbb{R}^n)}\right) \\
&\leqslant C_2\left(\|\psi_j\|_{L^\infty(\mathbb{R}^n)} + \sum_{i=1}^n |c_i^j|\|Z_i^j\|_{L^\infty(\mathbb{R}^n)} + \|\psi_j\|_{\star,\xi}\|w_\xi^{p-1}\rho_\xi\|_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n |c_i^j|\|Z_i^j\|_{L^2(\mathbb{R}^n)}\right) \\
&\leqslant C_3\left(\|\psi_j\|_{\star,\xi} + \sum_{i=1}^n |c_i^j|\right) \\
&\leqslant C_4
\end{aligned}$$

for suitable positive constants C_1, C_2, C_3 and C_4. This gives the equicontinuity of the sequence $\mathcal{B}[\psi_j]$, and so it converges to a function $\bar{b}$ uniformly on compact sets. Hence, for any $R > 0$, we have

$$\|\mathcal{B}[\psi_j] - \bar{b}\|_{L^\infty(B_R(\xi))} \to 0 \quad \text{as } j \to +\infty. \tag{7-75}$$

On the other hand, for any $x \in \mathbb{R}^n \setminus B_R(\xi)$, we have the estimate

$$\begin{aligned}
|w_\xi^{p-1}(x)\psi_j(x)| &\leqslant \|\psi_j\|_{\star,\xi}|w_\xi^{p-1}(x)\rho_\xi(x)| \\
&\leqslant C_5\|\psi_j\|_{\star,\xi}\left|\frac{1}{(1+|x-\xi|)^{(n+2s)(p-1)}}\rho_\xi(x)\right| \\
&\leqslant C_5\|\psi_j\|_{\star,\xi}\rho_\xi^{1+(n+2s)(p-1)/\mu}(x)
\end{aligned}$$

for some $C_5 > 0$, where we have used the decay of w_ξ in (1-4) and the expression of ρ_ξ given in (6-2). This implies that

$$\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)} |\rho_\xi^{-1}w_\xi^{p-1}\psi_j| \leqslant C_5\|\psi_j\|_{\star,\xi} \sup_{x\in\mathbb{R}^n\setminus B_R(\xi)} \rho_\xi^\sigma(x),$$

where

$$\sigma := \frac{(n+2s)(p-1)}{\mu} > 0. \tag{7-76}$$

Hence, since ψ_j is a uniformly bounded sequence with respect to the norm $\|\cdot\|_{\star,\xi}$, we obtain that

$$\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)} |\rho_\xi^{-1}\mathcal{B}[\psi_j]| \leqslant C_6 \sup_{x\in\mathbb{R}^n\setminus B_R(\xi)} \rho_\xi^{\sigma}(x). \tag{7-77}$$

It follows that

$$\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)} |\rho_\xi^{-1}\bar{b}| \leqslant C_6 \sup_{x\in\mathbb{R}^n\setminus B_R(\xi)} \rho_\xi^{\sigma}(x). \tag{7-78}$$

We observe that

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n}\big|\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\big| &= \sup_{x\in\mathbb{R}^n}\big|\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\chi_{B_R(\xi)}+\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\chi_{\mathbb{R}^n\setminus B_R(\xi)}\big| \\
&\leqslant \sup_{x\in\mathbb{R}^n}\Big(\big|\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\chi_{B_R(\xi)}\big|+\big|\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\chi_{\mathbb{R}^n\setminus B_R(\xi)}\big|\Big) \\
&\leqslant \sup_{x\in\mathbb{R}^n}\big|\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\chi_{B_R(\xi)}\big|+\sup_{x\in\mathbb{R}^n}\big|\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\chi_{\mathbb{R}^n\setminus B_R(\xi)}\big| \\
&= \sup_{x\in B_R(\xi)}\big|\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\big|+\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)}\big|\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\big|.
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
\|\mathcal{B}[\psi_j]-\bar{b}\|_{\star,\xi} &= \sup_{x\in\mathbb{R}^n}\big|\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\big| \\
&\leqslant \sup_{x\in B_R(\xi)}\big|\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\big|+\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)}\big|\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\big| \\
&\leqslant \sup_{x\in B_R(\xi)}\big|\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\big|+C_7\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)}\rho_\xi^{\sigma}(x),
\end{aligned} \tag{7-79}$$

where we have also used (7-77) and (7-78). Concerning the first term in the right-hand side, we have

$$\begin{aligned}
\sup_{x\in B_R(\xi)}\big|\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\big| &= \sup_{x\in B_R(\xi)}\big|(1+|x-\xi|)^{\mu}(\mathcal{B}[\psi_j]-\bar{b})\big| \\
&\leqslant (1+R)^{\mu}\|\mathcal{B}[\psi_j]-\bar{b}\|_{L^\infty(B_R(\xi))}.
\end{aligned}$$

Therefore, sending $j\to+\infty$ and recalling (7-75), we obtain that

$$\sup_{x\in B_R(\xi)}\big|\rho_\xi^{-1}(\mathcal{B}[\psi_j]-\bar{b})\big|\to 0 \quad \text{as } j\to+\infty. \tag{7-80}$$

Now, we send $R\to+\infty$ and, recalling (7-76), we get

$$\sup_{x\in\mathbb{R}^n\setminus B_R(\xi)}\rho_\xi^{\sigma}(x)\to 0 \quad \text{as } R\to+\infty. \tag{7-81}$$

Putting together (7-79), (7-80) and (7-81), we obtain that

$$\|\mathcal{B}[\psi_j]-\bar{b}\|_{\star,\xi}\to 0 \text{ as } j\to+\infty,$$

and this shows (7-74).

From (7-59) in Proposition 7.4, we deduce that if $g = 0$ then $\psi = \mathcal{A}[g] = 0$ is the unique solution to (7-58), and so by Fredholm's alternative we obtain that, for any $g \in Y_\star$, there exists a unique ψ that solves (7-71) (recall (7-72) and (7-74)). This gives existence and uniqueness of the solution to (7-7), while the estimate (7-8) follows from Lemma 7.3. This concludes the proof of Theorem 7.1. □

In the next proposition we deal with the differentiability of the solution ψ to (7-7) with respect to the parameter ξ (we recall Theorem 7.1 for the existence and uniqueness of the solution).

For this, we denote by $\mathcal{T}_\xi$ the operator that associates to any $g \in L^2(\mathbb{R}^n)$ with $\|g\|_{\star,\xi} < +\infty$ the solution to (7-7), that is,

$$\psi := \mathcal{T}_\xi[g] \text{ is the unique solution to (7-7) in } Y_\star, \tag{7-82}$$

where $Y_\star$ is as given in (7-70).

We notice that, thanks to Theorem 7.1, $\mathcal{T}_\xi$ is a linear and continuous operator from $Y_\star$ to $Y_\star$ endowed with the norm $\|\cdot\|_{\star,\xi}$, and we will write $\mathcal{T}_\xi \in \mathcal{L}(Y_\star)$.

Proposition 7.5. *The map $\xi \mapsto \mathcal{T}_\xi$ on Ω_ε is continuously differentiable. Moreover, there exists a positive constant C such that*

$$\left\| \frac{\partial \mathcal{T}_\xi[g]}{\partial \xi} \right\|_{\star,\xi} \leqslant C\left(\|g\|_{\star,\xi} + \left\| \frac{\partial g}{\partial \xi} \right\|_{\star,\xi} \right). \tag{7-83}$$

Proof. First, let us prove (7-83) assuming the differentiability of $\xi \mapsto \mathcal{T}_\xi$. Given $\xi \in \Omega_\varepsilon$, $|t| < 1$ with $t \neq 0$ and a function f, we let $\xi_j^t := \xi + te_j$ and

$$D_j^t f := \frac{f(\xi_j^t) - f(\xi)}{t}$$

for any $j = 1, \dots, n$.

Also, we set

$$\varphi_j^t := D_j^t \psi \quad \text{and} \quad d_{i,j}^t := D_j^t c_i. \tag{7-84}$$

Using the fact that ψ is a solution to (7-7), we have that φ_j^t solves

$$(-\Delta)^s \varphi_j^t + \varphi_j^t - p w_\xi^{p-1} \varphi_j^t = p(D_j^t w_\xi^{p-1})\psi - D_j^t g + \sum_{i=1}^n c_i D_j^t Z_i + \sum_{i=1}^n d_{i,j}^t Z_i \quad \text{in } \Omega_\varepsilon. \tag{7-85}$$

Moreover, we have that $\varphi_j^t \in H^s(\mathbb{R}^n)$ and $\varphi_j^t = 0$ outside Ω_ε.

Now, for the fixed index j, for any $i \in \{1, \dots, n\}$ we define

$$\lambda_i(\varphi_j^t) := \tilde{\alpha}^{-1} \int_{\mathbb{R}^n} \varphi_j^t Z_i \, dx, \tag{7-86}$$

where $\tilde{\alpha}$ is as defined in (5-21), and

$$\tilde{\varphi}_j^t := \varphi_j^t - \sum_{i=1}^n \lambda_i(\varphi_j^t) \tilde{Z}_i, \tag{7-87}$$

where the $\tilde{Z}_i$ are the ones in Lemma 5.5. We remark that φ_j^t and $\tilde{Z}_i$ vanish outside Ω_ε by construction. Hence $\tilde{\varphi}_j^t$ vanishes outside Ω_ε as well. Moreover,

$$\begin{aligned}\int_{\mathbb{R}^n} \tilde{\varphi}_j^t Z_k \, dx = \int_{\mathbb{R}^n} \varphi_j^t Z_k \, dx - \sum_{i=1}^n \lambda_i(\varphi_j^t) \int_{\mathbb{R}^n} \tilde{Z}_i Z_k \, dx &= \int_{\mathbb{R}^n} \varphi_j^t Z_k \, dx - \sum_{i=1}^n \lambda_i(\varphi_j^t)\tilde{\alpha}\delta_{ik} \\ &= \int_{\mathbb{R}^n} \varphi_j^t Z_k \, dx - \lambda_k(\varphi_j^t)\tilde{\alpha} \\ &= 0,\end{aligned}$$

thanks to Lemma 5.5 and (7-86). This yields that

$$\tilde{\varphi}_j^t \in \Psi. \tag{7-88}$$

By plugging (7-87) into (7-85), we obtain that

$$(-\Delta)^s \tilde{\varphi}_j^t + \tilde{\varphi}_j^t - p w_\xi^{p-1} \tilde{\varphi}_j^t = \tilde{g}_j + \sum_{i=1}^n d_{i,j}^t Z_i, \tag{7-89}$$

where

$$\tilde{g}_j := -(-\Delta)^s \sum_{i=1}^n \lambda_i(\tilde{\varphi}_j^t)\tilde{Z}_i - \sum_{i=1}^n \lambda_i(\tilde{\varphi}_j^t)\tilde{Z}_i + p w_\xi^{p-1} \sum_{i=1}^n \lambda_i(\tilde{\varphi}_j^t)\tilde{Z}_i + p(D_j^t w_\xi^{p-1})\psi - D_j^t g + \sum_{i=1}^n c_i D_j^t Z_i. \tag{7-90}$$

From (7-89), (7-88) and Lemma 7.3, we obtain that

$$\|\tilde{\varphi}_j^t\|_{\star,\xi} \leqslant C \|\tilde{g}_j\|_{\star,\xi}. \tag{7-91}$$

Now we observe that

$$\Big\| \sum_{i=1}^n \lambda_i(\tilde{\varphi}_j^t)\tilde{Z}_i \Big\|_{\star,\xi} \leqslant C \|g\|_{\star,\xi}. \tag{7-92}$$

To prove this, we notice that the orthogonality condition $\psi \in \Psi$ implies that

$$\int_{\Omega_\varepsilon} \varphi_j^t Z_k \, dx = -\int_{\Omega_\varepsilon} \psi D_j^t Z_k \, dx$$

for any $k \in \{1, \dots, n\}$. Hence, recalling (7-87), (7-88) and Lemma 5.5,

$$\begin{aligned}-\int_{\Omega_\varepsilon} \psi D_j^t Z_k \, dx = \int_{\Omega_\varepsilon} \Big(\tilde{\varphi}_j^t + \sum_{i=1}^n \lambda_i(\tilde{\varphi}_j^t)\tilde{Z}_i \Big) Z_k \, dx &= \sum_{i=1}^n \lambda_i(\tilde{\varphi}_j^t) \int_{\Omega_\varepsilon} \tilde{Z}_i Z_k \, dx \\ &= \sum_{i=1}^n \lambda_i(\tilde{\varphi}_j^t)\tilde{\alpha}\delta_{ik} \\ &= \lambda_k(\tilde{\varphi}_j^t)\tilde{\alpha}.\end{aligned}$$

Therefore,

$$|\lambda_k(\tilde\varphi_j^t)| = |\tilde\alpha^{-1}| \left| \int_{\Omega_\varepsilon} \psi D_j^t Z_k \, dx \right| \leqslant |\tilde\alpha^{-1}| \int_{\Omega_\varepsilon} \rho_\xi^{-1} |\psi| \rho_\xi |D_j^t Z_k| \, dx$$
$$\leqslant |\tilde\alpha^{-1}| \, \|\psi\|_{\star,\xi} \int_{\mathbb{R}^n} \rho_\xi |D_j^t Z_k| \, dx$$
$$\leqslant C \|\psi\|_{\star,\xi},$$

thanks to Lemma 5.3. Using this and Lemma 5.2, and possibly renaming the constants, we obtain that

$$\left\| \sum_{i=1}^n \lambda_i(\tilde\varphi_j^t) \tilde Z_i \right\|_{\star,\xi} \leqslant \sum_{i=1}^n |\lambda_i(\tilde\varphi_j^t)| \|\tilde Z_i\|_{\star,\xi} \leqslant C \sum_{i=1}^n |\lambda_i(\tilde\varphi_j^t)| \leqslant C \|\psi\|_{\star,\xi}.$$

On the other hand, by Lemma 7.3, we have that $\|\psi\|_{\star,\xi} \leqslant C\|g\|_{\star,\xi}$, so the above estimate implies (7-92), as desired.

Now we claim that

$$\left\| (-\Delta)^s \sum_{i=1}^n \lambda_i(\tilde\varphi_j^t) \tilde Z_i \right\|_{\star,\xi} \leqslant C \|g\|_{\star,\xi}. \tag{7-93}$$

Indeed, $\tilde Z_i$ is compactly supported in a neighborhood of ξ, hence $(-\Delta)^s \tilde Z_i$ decays like $|x-\xi|^{-n-2s}$ at infinity. Accordingly, $\|(-\Delta)^s \tilde Z_i\|_{\star,\xi}$ is finite, and then we obtain

$$\left\| (-\Delta)^s \sum_{i=1}^n \lambda_i(\tilde\varphi_j^t) \tilde Z_i \right\|_{\star,\xi} = \left\| \sum_{i=1}^n \lambda_i(\tilde\varphi_j^t) (-\Delta)^s \tilde Z_i \right\|_{\star,\xi}$$
$$\leqslant \sum_{i=1}^n |\lambda_i(\tilde\varphi_j^t)| \, \|(-\Delta)^s \tilde Z_i\|_{\star,\xi}$$
$$\leqslant C \sum_{i=1}^n |\lambda_i(\tilde\varphi_j^t)|$$
$$\leqslant C \|g\|_{\star,\xi},$$

due to (7-92), and this establishes (7-93).

Now we claim that

$$|D_j^t w_\xi^{p-1}| \leqslant C \tag{7-94}$$

with C independent of t. Indeed,

$$D_j^t w_\xi^{p-1}(x) = \frac{1}{t}(w^{p-1}(x-\xi-te_j) - w^{p-1}(x-\xi))$$
$$= \frac{1}{t} \int_0^t \frac{d}{d\tau} w^{p-1}(x-\xi-\tau e_j) \, d\tau$$
$$= \frac{p-1}{t} \int_0^t w^{p-2}(x-\xi-\tau e_j) \frac{d}{d\tau} w(x-\xi-\tau e_j) \, d\tau$$
$$= -\frac{p-1}{t} \int_0^t w^{p-2}(x-\xi-\tau e_j) \nabla w(x-\xi-\tau e_j) \cdot e_j \, d\tau.$$

Also, by formulas (IV.2) and (IV.6) of [Carmona et al. 1990], we know that

$$w(x) \text{ is bounded both from above and from below by a constant times } \frac{1}{1+|x|^{n+2s}}. \tag{7-95}$$

Thus, supposing without loss of generality that $t>0$, and recalling Lemma 5.2, we have that

$$\begin{aligned}
|D_j^t w_\xi^{p-1}(x)| &\leqslant \frac{p-1}{t}\int_0^t w^{p-2}(x-\xi-\tau e_j)|\nabla w(x-\xi-\tau e_j)|\,d\tau \\
&\leqslant \frac{C}{t}\int_0^t (1+|x-\xi-\tau e_j|)^{-(p-2)(n+2s)}(1+|x-\xi-\tau e_j|)^{-(n+2s)}\,d\tau \\
&= \frac{C}{t}\int_0^t (1+|x-\xi-\tau e_j|)^{-(p-1)(n+2s)}\,d\tau \\
&\leqslant \frac{C}{t}\int_0^t 1\,d\tau \\
&= C,
\end{aligned}$$

and this proves (7-94).

From (7-94) and Lemma 7.3 we obtain that

$$\|(D_j^t w_\xi^{p-1})\psi\|_{\star,\xi} \leqslant C\|\psi\|_{\star,\xi} \leqslant C\|g\|_{\star,\xi}. \tag{7-96}$$

Now we use Lemmata 5.3, 7.2 and 7.3 to see that

$$\begin{aligned}
\left\|\sum_{i=1}^n c_i D_j^t Z_i\right\|_{\star,\xi} &\leqslant \sum_{i=1}^n |c_i|\|D_j^t Z_i\|_{\star,\xi} \leqslant C\sum_{i=1}^n |c_i| = C\sum_{i=1}^n\left|\frac{1}{\alpha}\int_{\mathbb{R}^n} gZ_i\,dx + f_i\right| \\
&\leqslant C\biggl(\|g\|_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n |f_i|\biggr) \leqslant C(\|\psi\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}) \\
&\leqslant C(\|\psi\|_{\star,\xi} + \|g\|_{\star,\xi}) \leqslant C\|g\|_{\star,\xi}.
\end{aligned} \tag{7-97}$$

By plugging (7-92), (7-93), (7-96) and (7-97) into (7-90), we obtain that

$$\|\tilde{g}_j\|_{\star,\xi} \leqslant C(\|g\|_{\star,\xi} + \|D_j^t g\|_{\star,\xi}).$$

Therefore, by (7-91),

$$\|\tilde{\varphi}_j^t\|_{\star,\xi} \leqslant C(\|g\|_{\star,\xi} + \|D_j^t g\|_{\star,\xi}).$$

This and (7-92) imply that

$$\|\varphi_j^t\|_{\star,\xi} \leqslant \|\tilde{\varphi}_j^t\|_{\star,\xi} + \left\|\sum_{i=1}^n \lambda_i(\tilde{\varphi}_j^t)\tilde{Z}_i\right\|_{\star,\xi} \leqslant C(\|g\|_{\star,\xi} + \|D_j^t g\|_{\star,\xi}).$$

Hence, we send $t \searrow 0$ and we obtain

$$\left\|\frac{\partial\psi}{\partial\xi}\right\|_{\star,\xi} \leqslant C\biggl(\|g\|_{\star,\xi} + \left\|\frac{\partial g}{\partial\xi}\right\|_{\star,\xi}\biggr),$$

which implies that

$$\left\| \frac{\partial \mathcal{T}_\xi[g]}{\partial \xi} \right\|_{\star,\xi} \leqslant C\left(\|g\|_{\star,\xi} + \left\| \frac{\partial g}{\partial \xi} \right\|_{\star,\xi} \right).$$

Using the previous computation and the implicit function theorem, a standard argument shows that $\xi \mapsto \mathcal{T}_\xi$ is continuously differentiable (see, e.g., Section 2.2.1 in [Ambrosetti and Malchiodi 2006], and in particular Lemma 2.11 there, or [Dávila et al. 2014] below formula (4.20)). □

The nonlinear projected problem. In this subsection we solve the nonlinear projected problem

$$\begin{cases} (-\Delta)^s \psi + \psi - p w_\xi^{p-1} \psi = E(\psi) + N(\psi) + \sum_{i=1}^n c_i Z_i & \text{in } \Omega_\varepsilon, \\ \psi = 0 & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \psi Z_i \, dx = 0 & \text{for any } i = 1, \dots, n, \end{cases} \tag{7-98}$$

where $E(\psi)$ and $N(\psi)$ are as given in (7-3).

Theorem 7.6. *If $\varepsilon > 0$ is sufficiently small, there exists a unique solution $\psi \in H^s(\mathbb{R}^n)$ to (7-98) for suitable real coefficients c_i, $i = 1, \dots, n$, such that there exists a positive constant C such that*

$$\|\psi\|_{\star,\xi} \leqslant C\varepsilon^{n+2s}. \tag{7-99}$$

Before proving Theorem 7.6, we show some estimates for the error terms $E(\psi)$ and $N(\psi)$.

Lemma 7.7. *There exists a positive constant C such that*

$$|\bar{u}_\xi - w_\xi| \leqslant C\varepsilon^{n+2s}. \tag{7-100}$$

Proof. To prove (7-100), we define $\eta_\xi := \bar{u}_\xi - w_\xi$, and we observe that η_ξ satisfies

$$\begin{cases} (-\Delta)^s \eta_\xi + \eta_\xi = 0 & \text{in } \Omega_\varepsilon, \\ \eta_\xi = -w_\xi & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon, \end{cases} \tag{7-101}$$

due to (1-3) and (1-9).

We have

$$|\eta_\xi| = |w_\xi| \leqslant C\varepsilon^{n+2s} \quad \text{outside } \Omega_\varepsilon,$$

thanks to (1-4). Hence, this together with (7-101) and the maximum principle give

$$|\eta_\xi| \leqslant C\varepsilon^{n+2s} \quad \text{in } \mathbb{R}^n,$$

which implies the thesis (recall the definition of η_ξ). □

Moreover, we can prove the following:

Lemma 7.8. *There exists a positive constant C such that*

$$\left| \frac{\partial \bar{u}_\xi}{\partial \xi} - \frac{\partial w_\xi}{\partial \xi} \right| \leqslant C\varepsilon^{\nu_1} \tag{7-102}$$

with $\nu_1 := \min\{(n+2s+1), p(n+2s)\}$.

Proof. We set $\eta_\xi := \bar{u}_\xi - w_\xi$. From (1-3) and (1-9), we have that η_ξ solves

$$(-\Delta)^s \eta_\xi + \eta_\xi = 0 \quad \text{in } \Omega_\varepsilon.$$

Therefore, the derivative of η_ξ with respect to ξ satisfies

$$(-\Delta)^s \frac{\partial \eta_\xi}{\partial \xi} + \frac{\partial \eta_\xi}{\partial \xi} = 0 \quad \text{in } \Omega_\varepsilon. \tag{7-103}$$

Moreover, since $\bar{u}_\xi = 0$ outside Ω_ε, we have that

$$\eta_\xi = \bar{u}_\xi - w_\xi = -w_\xi \quad \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon,$$

which implies

$$\frac{\partial \eta_\xi}{\partial \xi} = -\frac{\partial w_\xi}{\partial \xi} = \frac{\partial w_\xi}{\partial x} \quad \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon.$$

Therefore, from Lemma 5.2 (recall also (5-3)), we have that

$$\left| \frac{\partial \eta_\xi}{\partial \xi} \right| \leqslant C\varepsilon^{\nu_1} \quad \text{outside } \Omega_\varepsilon.$$

From this, (7-103) and the maximum principle, we deduce that

$$\left| \frac{\partial \eta_\xi}{\partial \xi} \right| \leqslant C\varepsilon^{\nu_1} \quad \text{in } \mathbb{R}^n,$$

which gives the desired estimate (recall the definition of η_ξ). □

In the next lemma we estimate the $\star$-norm of the error term $E(\psi)$. For this, we recall the definition of the space $Y_\star$ given in (7-70).

Lemma 7.9. *Let $\psi \in Y_\star$ with $\|\psi\|_{\star,\xi} \leqslant 1$. Then, there exists a positive constant $\bar{C}$ such that*

$$\|E(\psi)\|_{\star,\xi} \leqslant \bar{C}\varepsilon^{n+2s}.$$

Proof. Using (7-100) and Lemma 2.1 in [Dipierro et al. $\geqslant$ 2015] with $a := w_\xi + \psi$ and $b := \bar{u}_\xi - w_\xi$, we obtain that

$$|E(\psi)| = |(\bar{u}_\xi - w_\xi + w_\xi + \psi)^p - (w_\xi + \psi)^p| \leqslant C_1 (w_\xi + \psi)^{p-1} |\bar{u}_\xi - w_\xi| \leqslant C_2 \varepsilon^{n+2s} (w_\xi + \psi)^{p-1}.$$

Hence, since $\|w_\xi\|_{\star,\xi}$ and $\|\psi\|_{\star,\xi}$ are bounded, we have

$$\|E(\psi)\|_{\star,\xi} \leqslant C_3 \varepsilon^{n+2s},$$

which gives the desired result. □

Now, we give a bound for the $\star$-norm of the error term $N(\psi)$.

Lemma 7.10. *Let $\psi \in Y_\star$. Then, there exists a positive constant C such that*

$$\|N(\psi)\|_{\star,\xi} \leqslant C(\|\psi\|^2_{\star,\xi} + \|\psi\|^p_{\star,\xi}).$$

Proof. We take $\psi \in Y_\star$ and we estimate

$$|N(\psi)| = |(w_\xi + \psi)^p - w_\xi^p - p w_\xi^{p-1}\psi| \leqslant C(|\psi|^2 + |\psi|^p)$$

for some positive constant C (see, for instance, Corollary 2.2 in [Dipierro et al. $\geqslant$ 2015], applied here with $a := w_\xi$ and $b := \psi$). Hence,

$$\rho_\xi^{-1}|N(\psi)| \leqslant C\rho_\xi^{-1}(|\psi|^2 + |\psi|^p) \leqslant C(\rho_\xi^{-2}|\psi|^2 + \rho_\xi^{-p}|\psi|^p) \leqslant C(\|\psi\|_{\star,\xi}^2 + \|\psi\|_{\star,\xi}^p),$$

which implies the desired estimate. □

For further reference, we now recall an estimate of elementary nature:

Lemma 7.11. *Fixed $\kappa > 0$, there exists a constant $C_\kappa > 0$ such that, for any $a, b \in [0, \kappa]$, we have*

$$|a^{p-1} - b^{p-1}| \leqslant C_\kappa |a - b|^q, \tag{7-104}$$

where

$$q := \min\{1, p - 1\}. \tag{7-105}$$

Proof. Fixing $\alpha \in (0, 1)$, for any $t > 0$ we define

$$h(t) := \frac{(t+1)^\alpha - 1}{t^\alpha}.$$

Using l'Hospital's rule, we see that

$$\lim_{t\searrow 0} h(t) = \lim_{t\searrow 0} \frac{t^{1-\alpha}}{(t+1)^{1-\alpha}} = 0;$$

hence we can extend h to a continuous function on $[0, +\infty)$ with $h(0) := 0$. Moreover,

$$\lim_{t\to+\infty} h(t) = 1;$$

hence there exists

$$M_0 := \sup_{t\in[0,+\infty)} h(t) < +\infty. \tag{7-106}$$

Now we prove (7-104). For this, we may and do assume that $a > b$. If $p \geqslant 2$, we have that

$$a^{p-1} - b^{p-1} = (p-1)\int_b^a \tau^{p-2}\, d\tau \leqslant (p-1)a^{p-2}(a-b) \leqslant (p-1)\kappa^{p-2}(a-b),$$

that is, (7-104) in this case. On the other hand, if $p \in (1, 2)$ we take $t := a/b - 1 > 0$ and $\alpha := p - 1$, so

$$M_0 \geqslant h(t) = \frac{(a/b)^{p-1} - 1}{(a/b-1)^{p-1}} = \frac{a^{p-1} - b^{p-1}}{(a-b)^{p-1}},$$

thanks to (7-106), and this establishes (7-104) also in this case. □

Now we are ready to complete the proof of Theorem 7.6.

Proof of Theorem 7.6. Recalling the definition of the operator $\mathcal{T}_\xi$ in (7-82), we can write

$$\psi = \mathcal{T}_\xi[E(\psi)+N(\psi)].$$

We will prove Theorem 7.6 by a contraction argument. To do this, we set

$$\mathcal{K}_\xi(\psi) := \mathcal{T}_\xi[E(\psi)+N(\psi)]. \tag{7-107}$$

Moreover, we take a constant $C_0 > 0$ and $\varepsilon > 0$ small (we will specify the choice of C_0 and ε in (7-118)), and we define the set

$$B := \{\psi \in Y_\star \mid \|\psi\|_{\star,\xi} \leqslant C_0 \varepsilon^{n+2s}\},$$

where $Y_\star$ was introduced in (7-70).

We claim that

$$\mathcal{K}_\xi \text{ as in (7-107) is a contraction mapping from } B \text{ into itself with respect to the norm } \|\cdot\|_{\star,\xi}. \tag{7-108}$$

First, we prove that

$$\text{if } \psi \in B \quad \text{then } \mathcal{K}_\xi(\psi) \in B. \tag{7-109}$$

Indeed, if $\psi \in B$, we have that

$$\|N(\psi)\|_{\star,\xi} \leqslant C_1(\|\psi\|^2_{\star,\xi} + \|\psi\|^p_{\star,\xi}), \tag{7-110}$$

thanks to Lemma 7.10.

Now, thanks to (7-8), we have that

$$\|\mathcal{K}_\xi(\psi)\|_{\star,\xi} = \|\mathcal{T}_\xi[E(\psi)+N(\psi)]\|_{\star,\xi} \leqslant C\|E(\psi)+N(\psi)\|_{\star,\xi}.$$

This, Lemma 7.9 and (7-110) give that

$$\begin{aligned}
\|\mathcal{K}_\xi(\psi)\|_{\star,\xi} &\leqslant C(\|E(\psi)\|_{\star,\xi} + \|N(\psi)\|_{\star,\xi}) \\
&\leqslant C(\|E(\psi)\|_{\star,\xi} + C_1(\|\psi\|^2_{\star,\xi} + \|\psi\|^p_{\star,\xi})) \\
&\leqslant C(\bar{C}\varepsilon^{n+2s} + C_1 C_0^2 \varepsilon^{2(n+2s)} + C_1 C_0^p \varepsilon^{p(n+2s)}) \\
&= C_0 \varepsilon^{n+2s}\left(\frac{C\bar{C}}{C_0} + CC_1C_0\varepsilon^{n+2s} + CC_1C_0^{p-1}\varepsilon^{(p-1)(n+2s)}\right),
\end{aligned} \tag{7-111}$$

since $\psi \in B$. We assume

$$C_0 > 2C\bar{C} \tag{7-112}$$

and

$$\varepsilon < \varepsilon_1 := \begin{cases} (2CC_1(C_0 + C_0^{p-1}))^{-1/(n+2s)} & \text{if } p \geqslant 2, \\ (2CC_1(C_0 + C_0^{p-1}))^{-1/(p-1)(n+2s)} & \text{if } 1 < p < 2. \end{cases} \tag{7-113}$$

With this choice of C_0 and ε, (7-111) implies that

$$\|\mathcal{K}_\xi(\psi)\|_{\star,\xi} \leqslant C_0 \varepsilon^{n+2s},$$

which proves (7-109).

Now, we take $\psi_1, \psi_2 \in B$. Then,

$$|N(\psi_1)-N(\psi_2)| = |(w_\xi+\psi_1)^p-(w_\xi+\psi_2)^p-pw_\xi^{p-1}(\psi_1-\psi_2)| \\ \leqslant C_2(|\psi_1|+|\psi_2|+|\psi_1|^{p-1}+|\psi_2|^{p-1})|\psi_1-\psi_2|.$$

This and the fact that $\psi_1, \psi_2 \in B$ give that

$$\begin{aligned} \|N(\psi_1)-N(\psi_2)\|_{\star,\xi} &\leqslant C_2(\|\psi_1\|_{\star,\xi}+\|\psi_2\|_{\star,\xi}+\|\psi_1\|_{\star,\xi}^{p-1}+\|\psi_2\|_{\star,\xi}^{p-1})\|\psi_1-\psi_2\|_{\star,\xi} \\ &\leqslant C_2(2C_0\varepsilon^{n+2s}+2C_0^{p-1}\varepsilon^{(p-1)(n+2s)})\|\psi_1-\psi_2\|_{\star,\xi} \\ &\leqslant 2C_2(C_0+C_0^{p-1})\varepsilon^{q(n+2s)}\|\psi_1-\psi_2\|_{\star,\xi}, \end{aligned} \tag{7-114}$$

where q is as defined in (7-105).

We claim that

$$|E(\psi_1)-E(\psi_2)| \leqslant C|\bar u_\xi - w_\xi|^q|\psi_1-\psi_2|, \tag{7-115}$$

where q is as given in (7-105).

Fixing $x \in \Omega_\varepsilon$, given τ in a bounded subset of $\mathbb{R}$ we consider the function

$$e(\tau) := (\bar u_\xi(x)+\tau)^p-(w_\xi(x)+\tau)^p.$$

We have that

$$|e'(\tau)| = p|(\bar u_\xi(x)+\tau)^{p-1}-(w_\xi(x)+\tau)^{p-1}| \leqslant C|\bar u_\xi-w_\xi|^q,$$

where we used (7-104) with $a := \bar u_\xi(x)+\tau$ and $b := w_\xi(x)+\tau$. This gives that

$$|e(\tau_1)-e(\tau_2)| \leqslant C|\bar u_\xi-w_\xi|^q|\tau_1-\tau_2|. \tag{7-116}$$

Now we take $\tau_1 := \psi_1(x)$ and $\tau_2 := \psi_2(x)$; we remark that τ_1 and τ_2 range in a bounded set, by our definition of B, and that $e(\tau_i) = E(\psi_i)$. Thus (7-115) follows from (7-116)

Hence, from (7-115) and (7-100), we obtain that

$$\|E(\psi_1)-E(\psi_2)\|_{\star,\xi} \leqslant \tilde C\varepsilon^{q(n+2s)}\|\psi_1-\psi_2\|_{\star,\xi}.$$

This, (7-114) and (7-8) give that

$$\begin{aligned} \|\mathcal{K}_\xi(\psi_1)-\mathcal{K}_\xi(\psi_2)\|_{\star,\xi} &\leqslant C\big(\|E(\psi_1)-E(\psi_2)\|_{\star,\xi}+\|N(\psi_1)-N(\psi_2)\|_{\star,\xi}\big) \\ &\leqslant C(2C_2(C_0+C_0^{p-1})\varepsilon^{q(n+2s)}+\tilde C\varepsilon^{q(n+2s)})\|\psi_1-\psi_2\|_{\star,\xi}. \end{aligned} \tag{7-117}$$

Now, we let

$$\varepsilon_2 := \left(\frac{1}{C(2C_2(C_0+C_0^{p-1})+\tilde C)}\right)^{\frac{1}{q(n+2s)}}.$$

Therefore, recalling also (7-112) and (7-113), we obtain that if

$$C_0 > 2C\bar C \quad \text{and} \quad \varepsilon < \min\{\varepsilon_1, \varepsilon_2\} \tag{7-118}$$

then, from (7-117), we have that

$$\|\mathcal{K}_\xi(\psi_1) - \mathcal{K}_\xi(\psi_2)\|_{\star,\xi} < \|\psi_1 - \psi_2\|_{\star,\xi},$$

which concludes the proof of (7-108).

From (7-108), we obtain the existence of a unique solution to (7-98) which belongs to B. This shows (7-99) and concludes the proof of Theorem 7.6. □

For any $\xi \in \Omega_\varepsilon$, we say that

$$\Psi(\xi) \text{ is the unique solution to (7-98).} \tag{7-119}$$

Arguing as the proof of Proposition 5.1 in [Dávila et al. 2014], one can also prove the following:

Proposition 7.12. *The map $\xi \mapsto \Psi(\xi)$ is of class C^1, and*

$$\left\|\frac{\partial \Psi(\xi)}{\partial \xi}\right\|_{\star,\xi} \leqslant C\left(\|E(\Psi(\xi))\|_{\star,\xi} + \left\|\frac{\partial E(\Psi(\xi))}{\partial \xi}\right\|_{\star,\xi}\right)$$

for some constant $C > 0$.

Derivative estimates. Here we deal with the derivatives of the solution $\psi = \Psi(\xi)$ to (7-98) with respect to ξ. This will also imply derivative estimates for the error term $\xi \mapsto E(\Psi(\xi))$.

We first show the following:

Lemma 7.13. *Let $\psi \in \Psi$ be a solution[4] to (7-98) with $\|\psi\|_{\star,\xi} \leqslant C\varepsilon^{n+2s}$. Then, there exist positive constants C and γ such that*

$$\left\|\frac{\partial E(\psi)}{\partial \xi}\right\|_{\star,\xi} \leqslant C\left(\varepsilon^{q(n+2s)}\left\|\frac{\partial \psi}{\partial \xi}\right\|_{\star,\xi} + \varepsilon^\gamma\right),$$

where q is as defined in (7-105).

Proof. First of all, we observe that, thanks to Proposition 7.5 (applied here with $g := -(E(\psi) + N(\psi))$), the function $\partial\psi/\partial\xi$ is well defined.

We make the following computations: from (7-3) we have that

$$\begin{aligned}
\frac{\partial E(\psi)}{\partial \xi} &= p(\bar{u}_\xi + \psi)^{p-1}\left(\frac{\partial \bar{u}_\xi}{\partial \xi} + \frac{\partial \psi}{\partial \xi}\right) - p(w_\xi + \psi)^{p-1}\left(\frac{\partial w_\xi}{\partial \xi} + \frac{\partial \psi}{\partial \xi}\right) \\
&= p\frac{\partial \psi}{\partial \xi}\left[(\bar{u}_\xi + \psi)^{p-1} - (w_\xi + \psi)^{p-1}\right] + p(\bar{u}_\xi + \psi)^{p-1}\frac{\partial \bar{u}_\xi}{\partial \xi} - p(w_\xi + \psi)^{p-1}\frac{\partial w_\xi}{\partial \xi} \\
&= p\frac{\partial \psi}{\partial \xi}\left[(\bar{u}_\xi + \psi)^{p-1} - (w_\xi + \psi)^{p-1}\right] + p(\bar{u}_\xi + \psi)^{p-1}\left(\frac{\partial \bar{u}_\xi}{\partial \xi} - \frac{\partial w_\xi}{\partial \xi}\right) \\
&\qquad + p[(\bar{u}_\xi + \psi)^{p-1} - (w_\xi + \psi)^{p-1}]\frac{\partial w_\xi}{\partial \xi}.
\end{aligned}$$

[4] We remark that a solution that fulfils the assumptions of Lemma 7.13 is provided by Theorem 7.6, as long as ε is sufficiently small.

Thus, recalling (7-104), (7-100) and (7-102), we infer that

$$\begin{aligned}\left|\frac{\partial E(\psi)}{\partial \xi}\right| &\leqslant Cp\left|\frac{\partial \psi}{\partial \xi}\right||\bar{u}_\xi - w_\xi|^q + p(|\bar{u}_\xi|+|\psi|)^{p-1}\left|\frac{\partial \bar{u}_\xi}{\partial \xi} - \frac{\partial w_\xi}{\partial \xi}\right| + Cp|\bar{u}_\xi - w_\xi|^q\left|\frac{\partial w_\xi}{\partial \xi}\right| \\ &\leqslant C\left|\frac{\partial \psi}{\partial \xi}\right|\varepsilon^{q(n+2s)} + C(|\bar{u}_\xi|+|\psi|)^{p-1}\varepsilon^{\nu_1} + C\varepsilon^{q(n+2s)}\left|\frac{\partial w_\xi}{\partial \xi}\right| \end{aligned} \tag{7-120}$$

for some $C > 0$. Now, we claim that

$$\sup_{x\in\mathbb{R}^n}(1+|x-\xi|)^\mu|\bar{u}_\xi(x)|^{p-1}\varepsilon^{\nu_1} \leqslant C\varepsilon^\gamma \quad \text{and} \quad \sup_{x\in\mathbb{R}^n}(1+|x-\xi|)^\mu|\psi(x)|^{p-1}\varepsilon^{\nu_1} \leqslant C\varepsilon^\gamma \tag{7-121}$$

for suitable $C > 0$ and $\gamma > 0$. Let us prove the first inequality in (7-121). For this, we use that $\bar{u}_\xi$ vanishes outside Ω_ε, together with (7-100) and (1-4), to see that

$$\begin{aligned}&\sup_{x\in\mathbb{R}^n}(1+|x-\xi|)^\mu|\bar{u}_\xi(x)|^{p-1}\varepsilon^{\nu_1} \\ &\qquad = \sup_{x\in\Omega_\varepsilon}(1+|x-\xi|)^\mu|\bar{u}_\xi(x)|^{p-1}\varepsilon^{\nu_1} \\ &\qquad \leqslant \sup_{x\in\Omega_\varepsilon}(1+|x-\xi|)^\mu\varepsilon^{(p-1)(n+2s)}\varepsilon^{\nu_1} + \sup_{x\in\Omega_\varepsilon}(1+|x-\xi|)^\mu|w_\xi(x)|^{p-1}\varepsilon^{\nu_1} \\ &\qquad \leqslant C\varepsilon^{-\mu}\varepsilon^{(p-1)(n+2s)}\varepsilon^{\nu_1} + \sup_{x\in\Omega_\varepsilon}(1+|x-\xi|)^{\mu(p-1)}|w_\xi(x)|^{p-1}(1+|x-\xi|)^{\mu(2-p)}\varepsilon^{\nu_1} \\ &\qquad \leqslant C\varepsilon^{-\mu}\varepsilon^{(p-1)(n+2s)}\varepsilon^{\nu_1} + \|w_\xi\|_{\star,\xi}^{p-1}\varepsilon^{-\mu(2-p)_+}\varepsilon^{\nu_1} \\ &\qquad \leqslant C\varepsilon^{-\mu}\varepsilon^{(p-1)(n+2s)}\varepsilon^{\nu_1} + C\varepsilon^{-\mu(2-p)_+}\varepsilon^{\nu_1}. \end{aligned} \tag{7-122}$$

Now we observe that

$$\begin{aligned}&-\mu+(p-1)(n+2s)+\nu_1 \\ &\qquad = \min\{-\mu+(p-1)(n+2s)+n+2s+1,\ -\mu+(p-1)(n+2s)+p(n+2s)\} \\ &\qquad > \min\{-(n+2s)+(p-1)(n+2s)+n+2s+1,\ -(n+2s)+(p-1)(n+2s)+p(n+2s)\} \\ &\qquad = \min\{(p-1)(n+2s)+1,\ (2p-2)(n+2s)\} > 0. \end{aligned} \tag{7-123}$$

Moreover, if $p \geqslant 2$, then

$$-\mu(2-p)_+ + \nu_1 = \nu_1 > 0,$$

while, if $1 < p < 2$, then

$$\begin{aligned}-\mu(2-p)_+ + \nu_1 &= \min\{-\mu(2-p)+n+2s+1,\ -\mu(2-p)+p(n+2s)\} \\ &> \min\{-(n+2s)(2-p)+n+2s+1,\ -(n+2s)(2-p)+p(n+2s)\} \\ &= \min\{(p-1)(n+2s)+1,\ (2p-2)(n+2s)\} > 0.\end{aligned}$$

Using this and (7-123) in (7-122), we obtain the first formula in (7-121). Now, we focus on the second inequality: From the assumptions on ψ we have

$$\begin{aligned}\sup_{x\in\mathbb{R}^n}(1+|x-\xi|)^{\mu}|\psi(x)|^{p-1}\varepsilon^{\nu_1} &= \sup_{x\in\Omega_\varepsilon}(1+|x-\xi|)^{\mu}|\psi(x)|^{p-1}\varepsilon^{\nu_1}\\ &= \sup_{x\in\Omega_\varepsilon}(1+|x-\xi|)^{\mu(p-1)}|\psi(x)|^{p-1}(1+|x-\xi|)^{\mu(2-p)}\varepsilon^{\nu_1}\\ &\leqslant \|\psi\|_{\star,\xi}^{p-1}\varepsilon^{-\mu(2-p)_+}\varepsilon^{\nu_1}\\ &\leqslant C\varepsilon^{(p-1)(n+2s)}\varepsilon^{-\mu(2-p)_+}\varepsilon^{\nu_1}. \end{aligned}\tag{7-124}$$

If $p\geqslant 2$ we get the second inequality in (7-121), as desired, hence we focus on the case $1<p<2$. For this, we notice that

$$\begin{aligned}&(p-1)(n+2s)-\mu(2-p)_+ +\nu_1\\ &\quad=\min\{(p-1)(n+2s)-\mu(2-p)+n+2s+1,\ (p-1)(n+2s)-\mu(2-p)+p(n+2s)\}\\ &\quad>\min\{(p-1)(n+2s)-(2-p)(n+2s)+n+2s+1,\ (p-1)(n+2s)-(2-p)(n+2s)+p(n+2s)\}\\ &\quad=\min\{(2p-2)(n+2s)+1,\ (3p-3)(n+2s)\}>0,\end{aligned}$$

and this, together with (7-124), implies the second inequality in (7-121) also in this case. Hence the proof of (7-121) is finished.

Exploiting (7-121) and Lemma 5.2, we infer from (7-120) that

$$\left\|\frac{\partial E(\psi)}{\partial\xi}\right\|_{\star,\xi}\leqslant C\varepsilon^{q(n+2s)}\left\|\frac{\partial\psi}{\partial\xi}\right\|_{\star,\xi}+C\varepsilon^{\gamma}\tag{7-125}$$

for suitable $C>0$ and $\gamma>0$, and this concludes the proof of Lemma 7.13, up to renaming the constants. □

Lemma 7.14. *Let $\psi\in\Psi$ be a solution to (7-98) with $\|\psi\|_{\star,\xi}\leqslant C\varepsilon^{n+2s}$. Then, there exists a positive constant C such that*

$$\left\|\frac{\partial\psi}{\partial\xi}\right\|_{\star,\xi}\leqslant C.$$

Proof. We observe that, thanks to Proposition 7.5 (applied here with $g:=-(E(\psi)+N(\psi))$),

$$\left\|\frac{\partial\psi}{\partial\xi}\right\|_{\star,\xi}\leqslant C\left(\|E(\psi)\|_{\star,\xi}+\|N(\psi)\|_{\star,\xi}+\left\|\frac{\partial E(\psi)}{\partial\xi}\right\|_{\star,\xi}+\left\|\frac{\partial N(\psi)}{\partial\xi}\right\|_{\star,\xi}\right).$$

Therefore, from Lemmata 7.9, 7.10 and 7.13, we obtain that

$$\left\|\frac{\partial\psi}{\partial\xi}\right\|_{\star,\xi}\leqslant C\left(1+\varepsilon^{q(n+2s)}\left\|\frac{\partial\psi}{\partial\xi}\right\|_{\star,\xi}+\left\|\frac{\partial N(\psi)}{\partial\xi}\right\|_{\star,\xi}\right).\tag{7-126}$$

Now we observe that, from (7-3),

$$\begin{aligned}\frac{\partial N(\psi)}{\partial\xi}&=p(w_\xi+\psi)^{p-1}\left(\frac{\partial w_\xi}{\partial\xi}+\frac{\partial\psi}{\partial\xi}\right)-pw_\xi^{p-1}\frac{\partial w_\xi}{\partial\xi}-p(p-1)w_\xi^{p-2}\frac{\partial w_\xi}{\partial\xi}\psi-pw_\xi^{p-1}\frac{\partial\psi}{\partial\xi}\\ &=p[(w_\xi+\psi)^{p-1}-w_\xi^{p-1}]\frac{\partial\psi}{\partial\xi}+p[(w_\xi+\psi)^{p-1}-w_\xi^{p-1}]\frac{\partial w_\xi}{\partial\xi}-p(p-1)w_\xi^{p-2}\frac{\partial w_\xi}{\partial\xi}\psi.\end{aligned}$$

As a consequence, using (7-104) once again,

$$\left|\frac{\partial N(\psi)}{\partial \xi}\right| \leqslant C|\psi|^q\left[\left|\frac{\partial \psi}{\partial \xi}\right|+\left|\frac{\partial w_\xi}{\partial \xi}\right|\right]+Cw_\xi^{p-2}\left|\frac{\partial w_\xi}{\partial \xi}\right||\psi|. \tag{7-127}$$

Now we claim that

$$w_\xi^{p-2}\left|\frac{\partial w_\xi}{\partial \xi}\right| \leqslant C \tag{7-128}$$

for some $C>0$. When $p\geqslant 2$, (7-128) follows from (1-4) and Lemma 5.2, hence we focus on the case $p\in(1,2)$. In this case, we take ν_1 as in Lemma 5.2 and we notice that

$$\begin{aligned}\tilde\nu := \nu_1-(2-p)(n+2s) &= \min\{n+2s+1+(p-2)(n+2s),\,(2p-2)(n+2s)\}\\ &= \min\{(p-1)(n+2s)+1,\,(2p-2)(n+2s)\}>0.\end{aligned}$$

Then we use (7-95) and we obtain that

$$w_\xi^{p-2}\left|\frac{\partial w_\xi}{\partial \xi}\right| \leqslant C|x-\xi|^{(2-p)(n+2s)}|x-\xi|^{-\nu_1} = C|x-\xi|^{-\tilde\nu}.$$

Since w_ξ is positive and smooth in the vicinity of ξ, this proves (7-128).

Now, using (7-128) into (7-127), we obtain that

$$\left|\frac{\partial N(\psi)}{\partial \xi}\right| \leqslant C|\psi|^q\left[\left|\frac{\partial \psi}{\partial \xi}\right|+\left|\frac{\partial w_\xi}{\partial \xi}\right|\right]+C|\psi|. \tag{7-129}$$

We claim that

$$\left\|\frac{\partial N(\psi)}{\partial \xi}\right\|_{\star,\xi} \leqslant C\|\psi\|_{\star,\xi}^q\left[\left\|\frac{\partial \psi}{\partial \xi}\right\|_{\star,\xi}+\left\|\frac{\partial w_\xi}{\partial \xi}\right\|_{\star,\xi}\right]+C\|\psi\|_{\star,\xi}. \tag{7-130}$$

Indeed, the claim plainly follows from (7-129) if $q=1$ (that is, $p\geqslant 2$), hence we focus on the case $q=p-1$ (that is, $1<p<2$). In this case, we observe that

$$\begin{aligned}(1+|x-\xi|)^\mu|\psi|^q\left[\left|\frac{\partial \psi}{\partial \xi}\right|+\left|\frac{\partial w_\xi}{\partial \xi}\right|\right] &= (1+|x-\xi|)^{\mu q}|\psi|^q(1+|x-\xi|)^\mu\left[\left|\frac{\partial \psi}{\partial \xi}\right|+\left|\frac{\partial w_\xi}{\partial \xi}\right|\right](1+|x-\xi|)^{-\mu q}\\ &\leqslant C\|\psi\|_{\star,\xi}^q\left[\left\|\frac{\partial \psi}{\partial \xi}\right\|_{\star,\xi}+\left\|\frac{\partial w_\xi}{\partial \xi}\right\|_{\star,\xi}\right],\end{aligned}$$

and this implies (7-130) also in this case.

Hence, using our assumptions on ψ, we deduce that

$$\left\|\frac{\partial N(\psi)}{\partial \xi}\right\|_{\star,\xi} \leqslant C\varepsilon^{q(n+2s)}\left[\left\|\frac{\partial \psi}{\partial \xi}\right\|_{\star,\xi}+1\right]+C,$$

up to renaming constants. By inserting this into (7-126) we conclude that

$$\left\|\frac{\partial \psi}{\partial \xi}\right\|_{\star,\xi} \leqslant C+\frac{1}{2}\left\|\frac{\partial \psi}{\partial \xi}\right\|_{\star,\xi}$$

as long as ε is sufficiently small. By reabsorbing one term into the left-hand side, we obtain the desired result. □

Lemma 7.15. *Let $\psi \in \Psi$ be a solution to (7-98) with $\|\psi\|_{\star,\xi} \leqslant C\varepsilon^{n+2s}$. Then, there exist positive constants $\tilde{C}$ and γ such that*

$$\left\| \frac{\partial E(\psi)}{\partial \xi} \right\|_{\star,\xi} \leqslant \tilde{C}\varepsilon^{\gamma}.$$

Proof. The proof easily follows from Lemmata 7.13 and 7.14, up to renaming the constants. □

The variational reduction. We seek for solutions to (1-8) of the form (7-1), that is, recalling also (7-119),

$$u_\xi = \bar{u}_\xi + \Psi(\xi). \tag{7-131}$$

We observe that, thanks to (1-9) and (7-98), the function u_ξ satisfies the equation

$$(-\Delta)^s u_\xi + u_\xi = u_\xi^p + \sum_{i=1}^{n} c_i Z_i \quad \text{in } \Omega_\varepsilon. \tag{7-132}$$

Notice that if $c_i = 0$ for any $i = 1, \dots, n$ then we will have a solution to (1-8). Hence, the aim of this subsection is to find a suitable point $\xi \in \Omega_\varepsilon$ such that all the coefficients c_i, $i = 1, \dots, n$, in (7-132) vanish.

In order to do this, we define the functional $J_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ as

$$J_\varepsilon(\xi) := I_\varepsilon(\bar{u}_\xi + \Psi(\xi)) = I_\varepsilon(u_\xi) \quad \text{for any } \xi \in \Omega_\varepsilon, \tag{7-133}$$

where I_ε was introduced in (1-10). We have the following characterization:

Lemma 7.16. *If $\varepsilon > 0$ is sufficiently small, the coefficients c_i, $i = 1, \dots, n$, in (7-132) are equal to zero if and only if ξ satisfies the condition*

$$\frac{\partial J_\varepsilon}{\partial \xi}(\xi) = 0.$$

Proof. We first write $\xi = (\xi_1, \dots, \xi_n)$ and, for any $j = 1, \dots, n$, we take the derivative of u_ξ with respect to ξ_j.

We observe that

$$\frac{\partial u_\xi}{\partial \xi_j} = \frac{\partial \bar{u}_\xi}{\partial \xi_j} + \frac{\partial \Psi(\xi)}{\partial \xi_j}. \tag{7-134}$$

Thanks to (7-102), we have that

$$\frac{\partial \bar{u}_\xi}{\partial \xi_j} = \frac{\partial w_\xi}{\partial \xi_j} + O(\varepsilon^{\nu_1}). \tag{7-135}$$

Moreover, from Proposition 7.12 and Lemmata 7.9 and 7.15, we obtain that

$$\frac{\partial \Psi(\xi)}{\partial \xi_j} = O(\varepsilon^{\gamma}), \tag{7-136}$$

where $\gamma > 0$. Hence, (7-134), (7-135) and (7-136) imply that

$$\frac{\partial u_\xi}{\partial \xi_j} = \frac{\partial w_\xi}{\partial \xi_j} + O(\varepsilon^{\gamma}),$$

which means, recalling (5-3) and using the fact that $\partial w_\xi/\partial\xi_j = -\partial w_\xi/\partial x_j$, that

$$\frac{\partial u_\xi}{\partial \xi_j} = -Z_j + O(\varepsilon^\gamma). \tag{7-137}$$

In particular,

$$\int_{\Omega_\varepsilon} Z_i \frac{\partial u_\xi}{\partial \xi_j}\,dx = \int_{\Omega_\varepsilon} Z_i(-Z_j + O(\varepsilon^\gamma))\,dx = -\int_{\Omega_\varepsilon} Z_i Z_j\,dx + O(\varepsilon^\gamma) \tag{7-138}$$

and, from Lemma 5.2, we deduce that

$$\left|\frac{\partial u_\xi}{\partial \xi_j}\right| \leqslant C_1(|Z_j| + \varepsilon^\gamma) \leqslant C_2. \tag{7-139}$$

With this, we introduce the matrix $M \in \mathrm{Mat}(n\times n)$ whose entries are given by

$$M_{ji} := \int_{\Omega_\varepsilon} Z_i \frac{\partial u_\xi}{\partial \xi_j}\,dx. \tag{7-140}$$

We claim that

$$M \ \text{ is invertible.} \tag{7-141}$$

To prove this, we use (7-138), Corollary 5.6 and the fact that $\alpha > 0$ (recall (5-17)); namely, we compute

$$M_{ji} = -\int_{\Omega_\varepsilon} Z_i Z_j\,dx + O(\varepsilon^\gamma) = -\alpha\delta_{ij} + O(\varepsilon^\gamma).$$

This says that the matrix $-\alpha^{-1}M$ is a perturbation of the identity and therefore it is invertible for ε sufficiently small, hence (7-141) readily follows.

Now, we multiply (7-132) by $\partial u_\xi/\partial\xi$, obtaining that

$$((-\Delta)^s u_\xi + u_\xi - u_\xi^p)\frac{\partial u_\xi}{\partial \xi} = \sum_{i=1}^n c_i Z_i \frac{\partial u_\xi}{\partial \xi} \quad \text{in } \Omega_\varepsilon,$$

and therefore

$$\left|((-\Delta)^s u_\xi + u_\xi - u_\xi^p)\frac{\partial u_\xi}{\partial \xi}\right| \leqslant \sum_{i=1}^n |c_i||Z_i|\left|\frac{\partial u_\xi}{\partial \xi}\right|.$$

This, together with (7-139) and Lemma 5.2, implies that the function $((-\Delta)^s u_\xi + u_\xi - u_\xi^p)\partial u_\xi/\partial\xi$ is in $L^\infty(\Omega_\varepsilon)$, and so in $L^1(\Omega_\varepsilon)$ uniformly with respect to ξ.

This allows us to compute the derivative of J_ε with respect to ξ_j as follows:

$$\begin{aligned}\frac{\partial J_\varepsilon}{\partial \xi_j}(\xi) = \frac{\partial}{\partial \xi_j} I_\varepsilon(u_\xi) &= \frac{\partial}{\partial \xi_j}\left(\int_{\Omega_\varepsilon} \frac{1}{2}(-\Delta)^s u_\xi u_\xi + \frac{1}{2}u_\xi^2 - \frac{1}{p+1}u_\xi^{p+1}\,dx\right)\\ &= \int_{\Omega_\varepsilon} \frac{1}{2}(-\Delta)^s \frac{\partial u_\xi}{\partial \xi_j} u_\xi + \frac{1}{2}(-\Delta)^s u_\xi \frac{\partial u_\xi}{\partial \xi_j} + \frac{\partial u_\xi}{\partial \xi_j} u_\xi - u_\xi^p \frac{\partial u_\xi}{\partial \xi_j}\,dx\\ &= \int_{\Omega_\varepsilon} ((-\Delta)^s u_\xi + u_\xi - u_\xi^p)\frac{\partial u_\xi}{\partial \xi_j}\,dx = \sum_{i=1}^n c_i \int_{\Omega_\varepsilon} Z_i \frac{\partial u_\xi}{\partial \xi_j}\,dx,\end{aligned} \tag{7-142}$$

where we have used (7-132) in the last step. Thus, recalling (7-140), we can write

$$\frac{\partial J_\varepsilon}{\partial \xi_j}(\xi) = \sum_{i=1}^{n} c_i M_{ji}$$

for any $j \in \{1, \dots, n\}$, that is, the vector

$$\frac{\partial J_\varepsilon}{\partial \xi}(\xi) := \left(\frac{\partial J_\varepsilon}{\partial \xi_1}(\xi), \dots, \frac{\partial J_\varepsilon}{\partial \xi_n}(\xi)\right)$$

is equal to the product between the matrix M and the vector $c := (c_1, \dots, c_n)$. From (7-141) we obtain that $\partial J_\varepsilon(\xi)/\partial\xi$ is equal to zero if and only if c is equal to zero, as desired. □

Thanks to Lemma 7.16, the problem of finding a solution to (1-8) reduces to the one of finding critical points of the functional defined in (7-133).To this end, we obtain an expansion of J_ε:

Theorem 7.17. *We have the following expansion of the functional J_ε:*

$$J_\varepsilon(\xi) = I_\varepsilon(\bar{u}_\xi) + o(\varepsilon^{n+4s}).$$

Proof. We know that

$$J_\varepsilon(\xi) = I_\varepsilon(\bar{u}_\xi + \Psi(\xi)).$$

Hence, we can Taylor expand in the vicinity of $\bar{u}_\xi$, obtaining

$$\begin{aligned}
J_\varepsilon(\xi) &= I_\varepsilon(\bar{u}_\xi) + I_\varepsilon'(\bar{u}_\xi)[\Psi(\xi)] + I''(\bar{u}_\xi)[\Psi(\xi), \Psi(\xi)] + O(|\Psi(\xi)|^3) \\
&= I_\varepsilon(\bar{u}_\xi) + \int_{\Omega_\varepsilon} (-\Delta)^s \bar{u}_\xi \Psi(\xi) + \bar{u}_\xi \Psi(\xi) - \bar{u}_\xi^p \Psi(\xi)\, dx \\
&\qquad + \int_{\Omega_\varepsilon} (-\Delta)^s \Psi(\xi)\Psi(\xi) + \Psi^2(\xi) - p\bar{u}_\xi^{p-1}\Psi^2(\xi)\, dx + O(|\Psi(\xi)|^3) \\
&= I_\varepsilon(\bar{u}_\xi) + \int_{\Omega_\varepsilon} ((-\Delta)^s u_\xi + u_\xi - u_\xi^p)\Psi(\xi)\, dx - \int_{\Omega_\varepsilon} ((-\Delta)^s (u_\xi - \bar{u}_\xi) + u_\xi - \bar{u}_\xi - u_\xi^p + \bar{u}_\xi^p)\Psi(\xi)\, dx \\
&\qquad + \int_{\Omega_\varepsilon} (-\Delta)^s \Psi(\xi)\Psi(\xi) + \Psi^2(\xi) - p\bar{u}_\xi^{p-1}\Psi^2(\xi)\, dx + O(|\Psi(\xi)|^3).
\end{aligned}$$

Therefore, using (7-131), we have that

$$J_\varepsilon(\xi) = I_\varepsilon(\bar{u}_\xi) + \int_{\Omega_\varepsilon} ((-\Delta)^s u_\xi + u_\xi - u_\xi^p)\Psi(\xi)\, dx + \int_{\Omega_\varepsilon} (u_\xi^p - \bar{u}_\xi^p - p\bar{u}_\xi^{p-1}\Psi(\xi))\Psi(\xi)\, dx + O(|\Psi(\xi)|^3). \tag{7-143}$$

We notice that

$$\int_{\Omega_\varepsilon} ((-\Delta)^s u_\xi + u_\xi - u_\xi^p)\Psi(\xi)\, dx = 0,$$

thanks to (7-132) and the fact that $\Psi(\xi)$ is orthogonal in $L^2(\Omega_\varepsilon)$ to any function in the space $\mathcal{Z}$. Hence, (7-143) becomes

$$J_\varepsilon(\xi) = I_\varepsilon(\bar{u}_\xi) + \int_{\Omega_\varepsilon} (u_\xi^p - \bar{u}_\xi^p - p\bar{u}_\xi^{p-1}\Psi(\xi))\Psi(\xi)\, dx + O(|\Psi(\xi)|^3). \tag{7-144}$$

Now, we observe that

$$|u_\xi^p - \bar{u}_\xi^p - p\bar{u}_\xi^{p-1}\Psi(\xi)| \leqslant |u_\xi^p - \bar{u}_\xi^p| + p|\bar{u}_\xi^{p-1}\Psi(\xi)| \leqslant C|\bar{u}_\xi^{p-1}\Psi(\xi)|$$

for a positive constant C, and so, also using (7-100), we have

$$\begin{aligned}\left|\int_{\Omega_\varepsilon}(u_\xi^p - \bar{u}_\xi^p - p\bar{u}_\xi^{p-1}\Psi(\xi))\Psi(\xi)\,dx\right| & \\ &\leqslant C\int_{\Omega_\varepsilon}|\bar{u}_\xi|^{p-1}|\Psi(\xi)|^2\,dx \\ &\leqslant C\|\Psi(\xi)\|_{\star,\xi}^2\int_{\Omega_\varepsilon}|\bar{u}_\xi|^{p-1}\rho_\xi^2\,dx \\ &\leqslant C\|\Psi(\xi)\|_{\star,\xi}^2\int_{\Omega_\varepsilon}|w_\xi + O(\varepsilon^{n+2s})|^{p-1}\rho_\xi^2\,dx \\ &\leqslant C\|\Psi(\xi)\|_{\star,\xi}^2\int_{\Omega_\varepsilon}|w_\xi|^{p-1}\rho_\xi^2\,dx + C\varepsilon^{(p-1)(n+2s)}\|\Psi(\xi)\|_{\star,\xi}^2\int_{\Omega_\varepsilon}\rho_\xi^2\,dx. \end{aligned} \tag{7-145}$$

Recalling the definition of ρ_ξ in (6-2) and the fact that $\mu > n/2$, we have that

$$\int_{\Omega_\varepsilon}\rho_\xi^2\,dx \leqslant C_1 \tag{7-146}$$

for a suitable constant $C_1 > 0$. Moreover, thanks to (7-99) (recall also (7-119)), we obtain

$$\varepsilon^{(p-1)(n+2s)}\|\Psi(\xi)\|_{\star,\xi}^2 \leqslant C_2\varepsilon^{(p-1)(n+2s)}\varepsilon^{2(n+2s)} = C_2\varepsilon^{(p+1)(n+2s)},$$

which, together with (7-146), says that

$$C\varepsilon^{(p-1)(n+2s)}\|\Psi(\xi)\|_{\star,\xi}^2\int_{\Omega_\varepsilon}\rho_\xi^2\,dx = o(\varepsilon^{n+4s}). \tag{7-147}$$

Also, using (1-4), we have that

$$\int_{\Omega_\varepsilon}|w_\xi|^{p-1}\rho_\xi^2\,dx \leqslant C_3\int_{\Omega_\varepsilon}\frac{1}{(1+|x-\xi|)^{(p-1)(n+2s)}}\frac{1}{(1+|x-\xi|)^{2\mu}}\,dx \leqslant C_4,$$

and so, using also (7-99) we have that

$$C\|\Psi(\xi)\|_{\star,\xi}^2\int_{\Omega_\varepsilon}|w_\xi|^{p-1}\rho_\xi^2\,dx = o(\varepsilon^{n+4s}).$$

This, (7-144), (7-145) and (7-147) give the desired claim in Theorem 7.17. □

8. Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1. For this, we notice that, thanks to Theorems 4.1 and 7.17, we have that, for any $\xi \in \Omega_\varepsilon$ with $\operatorname{dist}(\xi, \partial\Omega_\varepsilon) \geqslant \delta/\varepsilon$ (for some $\delta \in (0, 1)$),

$$J_\varepsilon(\xi) = I(w) + \tfrac{1}{2}\mathcal{H}_\varepsilon(\xi) + o(\varepsilon^{n+4s}), \tag{8-1}$$

where J_ε and I are as defined in (7-133) and (4-1), respectively (see also (7-119)) and $\mathcal{H}_\varepsilon$ is given by (1-17).

Also, we recall the definition of the set $\Omega_{\varepsilon,\delta}$ given in (2-29), and we claim that J_ε has an interior minimum, namely

$$\text{there exists } \bar{\xi} \in \Omega_{\varepsilon,\delta} \text{ such that } J_\varepsilon(\bar{\xi}) = \min_{\xi \in \overline{\Omega}_{\varepsilon,\delta}} J_\varepsilon(\xi). \tag{8-2}$$

For this, we observe that J_ε is a continuous functional, and therefore

$$J_\varepsilon \text{ admits a minimizer } \bar{\xi} \in \overline{\Omega}_{\varepsilon,\delta}. \tag{8-3}$$

We have that

$$\bar{\xi} \in \Omega_{\varepsilon,\delta}. \tag{8-4}$$

Indeed, suppose by contradiction that $\bar{\xi} \in \partial\Omega_{\varepsilon,\delta}$. Then, from (8-1), we have that

$$J_\varepsilon(\bar{\xi}) = I(w) + \tfrac{1}{2}\mathcal{H}_\varepsilon(\bar{\xi}) + o(\varepsilon^{n+4s}) \geqslant I(w) + \tfrac{1}{2}\min_{\partial\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon + o(\varepsilon^{n+4s}). \tag{8-5}$$

On the other hand, by Proposition 2.8, we know that $\mathcal{H}_\varepsilon$ has a strict interior minimum: more precisely, there exists $\xi_o \in \Omega_{\varepsilon,\delta}$ such that

$$\mathcal{H}_\varepsilon(\xi_o) = \min_{\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon \leqslant c_1 \varepsilon^{n+4s} \tag{8-6}$$

and

$$\min_{\partial\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon \geqslant c_2 \left(\frac{\varepsilon}{\delta}\right)^{n+4s} \tag{8-7}$$

for suitable $c_1, c_2 > 0$. Also, the minimality of $\bar{\xi}$ and (8-1) say that

$$J_\varepsilon(\bar{\xi}) = \min_{\xi \in \overline{\Omega}_{\varepsilon,\delta}} J_\varepsilon(\xi) \leqslant J_\varepsilon(\xi_o) = I(w) + \tfrac{1}{2}\mathcal{H}_\varepsilon(\xi_o) + o(\varepsilon^{n+4s}).$$

By comparing this with (8-5), and using (8-6) and (8-7), we obtain

$$\frac{c_2 \varepsilon^{n+4s}}{2\delta^{n+4s}} + o(\varepsilon^{n+4s}) \leqslant \tfrac{1}{2}\min_{\partial\Omega_{\varepsilon,\delta}} \mathcal{H}_\varepsilon + o(\varepsilon^{n+4s}) \leqslant J_\varepsilon(\bar{\xi}) - I(w) \leqslant \tfrac{1}{2}\mathcal{H}_\varepsilon(\xi_o) + o(\varepsilon^{n+4s}) \leqslant \tfrac{1}{2}c_1\varepsilon^{n+4s} + o(\varepsilon^{n+4s}).$$

So, a division by ε^{n+4s} and a limit argument give that

$$\frac{c_2}{2\delta^{n+4s}} \leqslant \frac{c_1}{2}.$$

This is a contradiction when δ is sufficiently small, thus (8-4) is proved. Hence (8-2) follows from (8-3) and (8-4).

From (8-2), since $\Omega_{\varepsilon,\delta}$ is open, we conclude that

$$\frac{\partial J_\varepsilon}{\partial \xi}(\bar{\xi}) = 0.$$

Therefore, from Lemma 7.16 we obtain the existence of a solution to (1-1) that satisfies (1-5) for ε sufficiently small, and this concludes the proof of Theorem 1.1.

Appendix: Some physical motivation

Equation (1-1) is a particular case of the fractional Schrödinger equation

$$i\hbar\partial_t\psi = \hbar^{2s}(-\Delta)^s\psi + V\psi, \tag{A-1}$$

when the wave function ψ is a standing wave (i.e., $\psi(x,t) = U(x)e^{it/\hbar}$) and the potential V is a suitable power of the density function (i.e., $V = V(|\psi|) = -|\psi|^{p-1}$). As usual, $\hbar$ is the Planck's constant (then we write $\varepsilon := \hbar$ in (1-1)) and $\psi = \psi(x,t)$ is the quantum mechanical probability amplitude for a given particle (of unit mass, for simplicity) to have position x at time t (the corresponding probability density is $|\psi|^2$).

In this setting our Theorem 1.1 describes the confinement of a particle inside a given domain Ω: for small values of $\hbar$ the wave function concentrates to a material particle well inside the domain.

Equation (A-1) is now quite popular (enough to have its own Wikipedia page: see [Wikipedia 2009–2015]) and it is based on the classical Schrödinger equation (corresponding to the case $s = 1$) in which the Brownian motion of the quantum paths is replaced by a Lévy flight. We refer to [Laskin 2000; 2002; 2012] for a throughout physical discussion and detailed motivation of equation (A-1) (see in particular formula (18) in [Laskin 2000]), but here we sketch some heuristics about it.

The idea is that the evolution of the wave function $\psi(x,t)$ from its initial state $\psi_0(x) := \psi(x,0)$ is run by a quantum mechanics kernel (or amplitude) K which produces the forthcoming values of the wave function by integration with the initial state, that is,

$$\psi(x,t) = \int_{\mathbb{R}^n} dy\, K(x,y,t)\psi_0(y). \tag{A-2}$$

The main assumption is that this amplitude $K(x,y,t)$ is modulated by an action functional S_t via the contributions of all the possible paths γ that join x to y in time t, that is,

$$K(x,y,t) = \int_{\mathcal{F}(x,y,t)} d\gamma\, e^{-iS_t(\gamma)/\hbar}. \tag{A-3}$$

The above integral denotes the Feynman path integral over "all possible histories of the system", that is, over "all possible" continuous paths $\gamma : [0,t] \to \mathbb{R}^n$ with $\gamma(0) = y$ and $\gamma(t) = x$; see [Feynman 1948]. We remark that this integral is indeed a functional integral, that is, the domain of integration $\mathcal{F}(x,y,t)$ is not a region of a finite-dimensional space, but a space of functions. The mathematical treatment of Feynman path integrals is by no means trivial; as a matter of fact, the convergence must rely on the highly oscillatory behavior of the system, which produces the necessary cancellations. In some cases, a rigorous justification can be provided by the theory of Wiener spaces, but a complete treatment of this topic is far beyond the scopes of this appendix (see, e.g., [Cameron 1960; Cameron and Storvick 1983; Grosche and Steiner 1998; Albeverio et al. 2008]).

The next structural ansatz we take is that the action functional S_t is the superposition of a (complex) diffusive operator H_0 and a potential term V.

Though the diffusion and the potential operate "simultaneously", with some approximation we may suppose that, at each tiny time step, they operate just one at a time, interchanging their action[5] at a very high frequency. Namely, we discretize a path γ into N adjacent paths of time range t/N, say $\gamma_1, \dots, \gamma_N : [0, t/N] \to \mathbb{R}^n$, with $\gamma_1(0) = y$ and $\gamma_N(t/N) = x$, and we suppose that along each γ_j the action reduces to the subsequent nonoverlapping superpositions of diffusion and potential terms, according to the formula

$$e^{-iS_t(\gamma)/\hbar} = \lim_{N\to+\infty} (e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)})^N. \tag{A-4}$$

Once more, we do not indulge into a rigorous mathematical discussion of such a limit and we just plug (A-3) and (A-4) into (A-2). We obtain

$$\begin{aligned}\psi(x,t) &= \int_{\mathbb{R}^n} dy \int_{\mathcal{F}(x,y,t)} d\gamma\, e^{-iS_t(\gamma)/\hbar}\psi_0(y)\\ &= \lim_{N\to+\infty}\int_{\mathbb{R}^n} dy \int_{\mathcal{F}(x,y,t)} d\gamma\, (e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)})^N \psi_0(y).\end{aligned} \tag{A-5}$$

Therefore, if we formally[6] apply the time derivative to (A-5), we obtain that

[5]In a sense, this is the quantum mechanics version of the Lie–Trotter product formula

$$e^{A+B} = \lim_{N\to+\infty} (e^{A/N} e^{B/N})^N$$

for $A, B \in \mathrm{Mat}(n\times n)$. The procedure of disentangling mixed exponentials is indeed crucial in quantum mechanics computations; see, e.g., [Feynman 1951]. In our computation, a more rigorous approximation scheme lies in explicitly writing $S_t(\gamma)$ as an integral from 0 to t of the Lagrangian along the path γ, then splitting the integral in N time steps of size t/N by supposing that in each of these time steps the Lagrangian is approximately constant. One may also suppose that the Lagrangian involved in the action is a classical one, i.e., it is the sum of a kinetic term and the potential V. Then the effect of taking the integral over all possible paths averages out the kinetic part, reducing it to a diffusive operator. Since here we are not aiming at a rigorous justification of all these delicate procedures (such as infinite-dimensional integrals, limit exchanges, and so on), for simplicity we are just taking H_0 to be a diffusive operator from the beginning. In this spirit, it is also convenient to suppose that the potential is an operator, that is, we identify V with the operation of multiplying a function by V.

[6]The disentangling procedure allows us to take the derivative of the exponentials of the operators "as if they were commuting ones". Namely, the Zassenhaus formula,

$$e^{t(A+B)} = e^{tA}e^{tB}e^{O(t^2)} = e^{tA}e^{tB}(1+O(t^2)),$$

in our case gives

$$e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)} = e^{-it(H_0+V)/(\hbar N)}(1+O(t^2/N^2))$$

and so

$$(e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)})^N = e^{-it(H_0+V)/\hbar}(1+O(t^2/N^2)).$$

Hence,

$$\lim_{N\to+\infty} \partial_t (e^{-itH_0/(\hbar N)} e^{-itV/(\hbar N)})^N = \lim_{N\to+\infty} -\frac{i(H_0+V)}{\hbar} e^{-it(H_0+V)/\hbar}(1+O(t^2/N^2)) + O(t/N^2) = -\frac{i(H_0+V)}{\hbar}.$$

Moreover, we point out that a couple of additional approximations are likely to be hidden in the computation in (A-6). Namely, first of all, we do not differentiate the functional domain of the Feynman integral. This is consistent with the ansatz that the set of the paths joining two points at a macroscopic scale in time t "does not vary much" for small variations of t. Furthermore, we replace the action of H_0 and V along the infinitesimal paths with their effective action after averaging, so that we can take (H_0+V) outside the integral.

$$
\begin{aligned}
i\hbar\partial_t\psi(x,t) &= \lim_{N\to+\infty}\int_{\mathbb{R}^n} dy \int_{\mathcal{F}(x,y,t)} d\gamma\, N\left(\frac{H_0}{N}e^{-itH_0/(\hbar N)}e^{-itV/(\hbar N)}+\frac{V}{N}e^{-itH_0/(\hbar N)}e^{-itV/(\hbar N)}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot(e^{-itH_0/(\hbar N)}e^{-itV/(\hbar N)})^{N-1}\psi_0(y)\\
&= \lim_{N\to+\infty}\int_{\mathbb{R}^n} dy \int_{\mathcal{F}(x,y,t)} d\gamma\, (H_0e^{-itH_0/(\hbar N)}e^{-itV/(\hbar N)}+Ve^{-itH_0/(\hbar N)}e^{-itV/(\hbar N)})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot(e^{-itH_0/(\hbar N)}e^{-itV/(\hbar N)})^{N-1}\psi_0(y)\\
&= (H_0+V)\int_{\mathbb{R}^n} dy \int_{\mathcal{F}(x,y,t)} d\gamma\, e^{-iS_t(\gamma)/\hbar}\psi_0(y)\\
&= (H_0+V)\psi
\end{aligned}
\tag{A-6}
$$

by (A-2), (A-3) and (A-4). The classical Schrödinger equation follows by taking $H_0 := -\hbar^2\Delta$, that is, the Gaussian diffusive process, while (A-1) follows by taking $H_0 := \hbar^{2s}(-\Delta)^s$, that is, the $2s$-stable diffusive process with polynomial tail.

Having given a brief justification of (A-1), we also recall that the fractional Schrödinger case presents interesting differences with respect to the classical one. For instance, the energy of a particle of unit mass is proportional to $|p|^{2s}$ (instead of $|p|^2$; see, e.g., formula (12) in [Laskin 2000]). Also the space/time scaling of the process gives that the fractal dimension of the Lévy paths is $2s$ (differently from the classical Brownian case, in which it is 2); see pages 300–301 of [Laskin 2000].

Now, for completeness, we discuss a nonlocal notion of canonical quantization, together with the associated Heisenberg uncertainty principle (see, for example, pages 17–28 of [Giulini 2003] for the classical canonical quantization and related issues).

For this, we introduce the canonical operators, for $k \in \{1, \dots, n\}$,

$$
P_k := -i\hbar^s\partial_k(-\Delta)^{(s-1)/2} \quad\text{and}\quad Q_k := x_k. \tag{A-7}
$$

Notice that Q_k is the classical position operator, namely the multiplication by the k-th space coordinate. On the other hand, P_k is a fractional momentum operator, that reduces[7] to the classical momentum $-i\hbar\partial_k$ when $s = 1$. In this setting, our goal is to check that the commutator

$$
[Q, P] := \sum_{k=1}^{n}[Q_k, P_k]
$$

does not vanish. For this, we suppose $0 < \sigma < n/2$ and use the Riesz potential representation of the inverse of the fractional Laplacian of order σ, that is,

$$
(-\Delta)^{-\sigma}\psi(x) = c(n,s)\int_{\mathbb{R}^n}\frac{\psi(x-y)}{|y|^{n-2\sigma}}\,dy = c(n,s)\int_{\mathbb{R}^n}\frac{\psi(y)}{|x-y|^{n-2\sigma}}\,dy \tag{A-8}
$$

for a suitable $c(n,s) > 0$, see [Landkof 1972].

[7] Of course, the fractional momentum is not a momentum, since it has physical dimension `[Planck constant]`^{s}/`[length]`^{s}, while the classical momentum has physical dimension `[Planck constant]`/`[length]`. Namely, the physical dimension of the fractional momentum is a fractional power of the physical dimension of the classical momentum. Clearly, the same phenomenon occurs for the physical dimension of the fractional Laplace operators in terms of the usual Laplacian.

In our case we use (A-8) with $\sigma := (1-s)/2 \in \big(0, \tfrac{1}{2}\big) \subseteq (0, n/2)$. Then

$$P_k\psi(x) = -c(n,s)i\hbar^s\partial_k \int_{\mathbb{R}^n} \frac{\psi(y)}{|x-y|^{n+s-1}}\,dy = c(n,s)i\hbar^s(n+s-1)\int_{\mathbb{R}^n} \frac{(x_k-y_k)\psi(y)}{|x-y|^{n+s+1}}\,dy$$

and so

$$P_kQ_k\psi(x) = P_k(x_k\psi(x)) = c(n,s)i\hbar^s(n+s-1)\int_{\mathbb{R}^n} \frac{(x_k-y_k)y_k\psi(y)}{|x-y|^{n+s+1}}\,dy.$$

This gives that

$$\begin{aligned} Q_kP_k\psi - P_kQ_k\psi &= c(n,s)i\hbar^s(n+s-1)\bigg[\int_{\mathbb{R}^n} \frac{x_k(x_k-y_k)\psi(y)}{|x-y|^{n+s+1}}\,dy - \int_{\mathbb{R}^n} \frac{(x_k-y_k)y_k\psi(y)}{|x-y|^{n+s+1}}\,dy\bigg] \\ &= c(n,s)i\hbar^s(n+s-1)\int_{\mathbb{R}^n} \frac{(x_k-y_k)^2\psi(y)}{|x-y|^{n+s+1}}\,dy, \end{aligned}$$

and so, by summing up[8] and recalling (A-8), we conclude that

$$[Q,P]\psi = c(n,s)i\hbar^s(n+s-1)\int_{\mathbb{R}^n} \frac{\psi(y)}{|x-y|^{n+s-1}}\,dy = i(n+s-1)\hbar^s(-\Delta)^{(s-1)/2}\psi.$$

Notice that, as $s \to 1$, this formula reduces to the classical Heisenberg uncertainty principle.

We also point out that a similar computation shows that, differently from the local quantum momentum, the k-th fractional quantum momentum does not commute with the m-th spatial coordinates even when $k \neq m$; namely, $[Q_m, P_k]\psi(x)$ is, up to normalizing constants,

$$i\hbar^s \int_{\mathbb{R}^n} \frac{(x_m-y_m)(x_k-y_k)\psi(y)}{|x-y|^{n+s+1}}\,dy.$$

This Heisenberg uncertainty principle is also compatible with (A-1), in the sense that the diffusive operator H_0 is exactly the one obtained by the canonical quantization in (A-7); indeed,

$$\begin{aligned} \sum_{k=1}^n P_k^2 = \sum_{k=1}^n (-i\hbar^s\partial_k(-\Delta)^{(s-1)/2})(-i\hbar^s\partial_k(-\Delta)^{(s-1)/2}) &= -\hbar^{2s}\sum_{k=1}^n \partial_k^2(-\Delta)^{s-1} \\ &= -\hbar^{2s}\Delta(-\Delta)^{s-1} = \hbar^{2s}(-\Delta)^s = H_0. \end{aligned}$$

Moreover, we mention that the fractional Laplace operator also arises naturally in the high energy Hamiltonians of relativistic theories. For further motivation of the fractional Laplacian in modern physics see, for example, [Chen 2004] and the references therein.

---

[8] Alternatively, one can perform the commutator calculation in Fourier space and then reduce to the original variable by an inverse Fourier transform. This computation can be done easily by using the facts that the Fourier transform sends products into convolutions and that (up to constants)

$$\begin{aligned} (\hat{x}_k * g)(\xi) = \mathcal{F}(x_k\mathcal{F}^{-1}g(x))(\xi) &= \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, e^{ix\cdot(y-\xi)}x_kg(y) = i^{-1}\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, \partial_{y_k}e^{ix\cdot(y-\xi)}g(y) \\ &= i\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, e^{ix\cdot(y-\xi)}\partial_kg(y) = i\int_{\mathbb{R}^n} dx\, e^{-ix\cdot\xi}\mathcal{F}^{-1}(\partial_kg)(x) = i\mathcal{F}(\mathcal{F}^{-1}(\partial_kg))(\xi) = i\partial_kg(\xi). \end{aligned}$$

Then we leave to the reader the computation of $\mathcal{F}([Q,P]\psi)(\xi)$.

Acknowledgements

This work has been supported by Millennium Nucleus CAPDE NC130017, Fondecyt grants 1150066, 1130360, Fondo Basal CMM-Chile, EPSRC grant EP/K024566/1 “Monotonicity formula methods for nonlinear PDEs” and ERC grant 277749 “EPSILON Elliptic PDEs and Symmetry of Interfaces and Layers for Odd Nonlinearities”.

References

[Albeverio et al. 2008] S. A. Albeverio, R. J. Høegh-Krohn, and S. Mazzucchi, *Mathematical theory of Feynman path integrals: an introduction*, 2nd ed., Lecture Notes in Mathematics **523**, Springer, Berlin, 2008. MR 2010e:58008 Zbl 1222.46001

[Ambrosetti and Malchiodi 2006] A. Ambrosetti and A. Malchiodi, *Perturbation methods and semilinear elliptic problems on* $\mathbb{R}^n$, Progress in Mathematics **240**, Birkhäuser, Basel, 2006. MR 2007k:35005 Zbl 1115.35004

[Amick and Toland 1991] C. J. Amick and J. F. Toland, “Uniqueness and related analytic properties for the Benjamin–Ono equation—a nonlinear Neumann problem in the plane”, *Acta Math.* **167**:1–2 (1991), 107–126. MR 92i:35099 Zbl 0755.35108

[Cameron 1960] R. H. Cameron, “A family of integrals serving to connect the Wiener and Feynman integrals”, *J. Math. and Phys.* **39** (1960), 126–140. MR 23 #B821 Zbl 0096.06901

[Cameron and Storvick 1983] R. H. Cameron and D. A. Storvick, “A simple definition of the Feynman integral, with applications”, *Mem. Amer. Math. Soc.* **46**:288 (1983), iv+46. MR 86c:81029 Zbl 0527.28015

[Carmona et al. 1990] R. Carmona, W. C. Masters, and B. Simon, “Relativistic Schrödinger operators: asymptotic behavior of the eigenfunctions”, *J. Funct. Anal.* **91**:1 (1990), 117–142. MR 91i:35139 Zbl 0716.35006

[Chen 2004] W. Chen, “Soft matter and fractional mathematics: insights into mesoscopic quantum and time-space structures”, preprint, 2004. arXiv cond-mat/0405345

[Chen and Zheng 2014] G. Chen and Y. Zheng, “Concentration phenomenon for fractional nonlinear Schrödinger equations”, *Commun. Pure Appl. Anal.* **13**:6 (2014), 2359–2376. MR 3248394

[Choi et al. 2014] W. Choi, S. Kim, and K.-A. Lee, “Asymptotic behavior of solutions for nonlinear elliptic problems with the fractional Laplacian”, *J. Funct. Anal.* **266**:11 (2014), 6531–6598. MR 3192461 Zbl 1297.35268

[Dávila et al. 2014] J. Dávila, M. del Pino, and J. Wei, “Concentrating standing waves for the fractional nonlinear Schrödinger equation”, *J. Differential Equations* **256**:2 (2014), 858–892. MR 3121716 Zbl 06447081

[Di Nezza et al. 2012] E. Di Nezza, G. Palatucci, and E. Valdinoci, “Hitchhiker’s guide to the fractional Sobolev spaces”, *Bull. Sci. Math.* **136**:5 (2012), 521–573. MR 2944369 Zbl 1252.46023

[Dipierro et al. 2014] S. Dipierro, A. Figalli, and E. Valdinoci, “Strongly nonlocal dislocation dynamics in crystals”, *Comm. Partial Differential Equations* **39**:12 (2014), 2351–2387. MR 3259559 Zbl 1304.35731

[Dipierro et al. $\geqslant$ 2015] S. Dipierro, F. Mahmoudi, and A. Malchiodi, “Singularly perturbed PDEs and patterns with periodic profiles”, in progress.

[Fall and Valdinoci 2014] M. M. Fall and E. Valdinoci, “Uniqueness and nondegeneracy of positive solutions of $(-\Delta)^s u + u = u^p$ in $\mathbb{R}^N$ when s is close to 1”, *Comm. Math. Phys.* **329**:1 (2014), 383–404. MR 3207007 Zbl 1304.35746

[Felmer et al. 2012] P. Felmer, A. Quaas, and J. Tan, “Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian”, *Proc. Roy. Soc. Edinburgh Sect. A* **142**:6 (2012), 1237–1262. MR 3002595 Zbl 1290.35308

[Feynman 1948] R. P. Feynman, “Space-time approach to non-relativistic quantum mechanics”, *Rev. Modern Physics* **20** (1948), 367–387. MR 10,224b

[Feynman 1951] R. P. Feynman, “An operator calculus having applications in quantum electrodynamics”, *Physical Rev.* (2) **84** (1951), 108–128. MR 13,410e Zbl 0044.23304

[Frank and Lenzmann 2013] R. L. Frank and E. Lenzmann, “Uniqueness of non-linear ground states for fractional Laplacians in $\mathbb{R}$”, *Acta Math.* **210**:2 (2013), 261–318. MR 3070568 Zbl 1307.35315

[Frank et al. 2015] R. L. Frank, E. Lenzmann, and L. Silvestre, “Uniqueness of radial solutions for the fractional Laplacian”, preprint, 2015. To appear in *Comm. Pure Appl. Math.*

[Giulini 2003] D. Giulini, "That strange procedure called quantisation", pp. 17–40 in *Quantum gravity*, edited by D. Giulini et al., Lecture Notes in Phys. **631**, Springer, Berlin, 2003. MR 2206994 Zbl 1052.83019

[Grosche and Steiner 1998] C. Grosche and F. Steiner, *Handbook of Feynman path integrals*, Springer Tracts in Modern Physics **145**, Springer, Berlin, 1998. MR 2000f:81103 Zbl 1029.81045

[Landkof 1972] N. S. Landkof, *Foundations of modern potential theory*, Grundlehren der Math. Wissenschaften **180**, Springer, Heidelberg, 1972. MR 50 #2520 Zbl 0253.31001

[Laskin 2000] N. Laskin, "Fractional quantum mechanics and Lévy path integrals", *Phys. Lett. A* **268**:4–6 (2000), 298–305. MR 2000m:81097 Zbl 0948.81595

[Laskin 2002] N. Laskin, "Fractional Schrödinger equation", *Phys. Rev. E* (3) **66**:5 (2002), 056108, 7. MR 2003k:81043

[Laskin 2012] N. Laskin, "Principles of fractional quantum mechanics", pp. 393–427 in *Fractional dynamics*, edited by J. Klafter et al., World Sci. Publ., Hackensack, NJ, 2012. MR 2932616 Zbl 1297.81070

[Li and Nirenberg 1998] Y. Li and L. Nirenberg, "The Dirichlet problem for singularly perturbed elliptic equations", *Comm. Pure Appl. Math.* **51**:11-12 (1998), 1445–1490. MR 99g:35014 Zbl 0933.35083

[Ni and Wei 1995] W.-M. Ni and J. Wei, "On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems", *Comm. Pure Appl. Math.* **48**:7 (1995), 731–768. MR 96g:35077 Zbl 0838.35009

[del Pino and Felmer 1999] M. del Pino and P. L. Felmer, "Spike-layered solutions of singularly perturbed elliptic problems in a degenerate setting", *Indiana Univ. Math. J.* **48**:3 (1999), 883–898. MR 2001b:35027 Zbl 0932.35080

[del Pino et al. 2000] M. del Pino, P. L. Felmer, and J. Wei, "On the role of distance function in some singular perturbation problems", *Comm. Partial Differential Equations* **25**:1-2 (2000), 155–177. MR 2000m:35017 Zbl 0949.35054

[Ros-Oton and Serra 2014a] X. Ros-Oton and J. Serra, "The Dirichlet problem for the fractional Laplacian: regularity up to the boundary", *J. Math. Pures Appl.* (9) **101**:3 (2014), 275–302. MR 3168912 Zbl 1285.35020

[Ros-Oton and Serra 2014b] X. Ros-Oton and J. Serra, "The Pohozaev identity for the fractional Laplacian", *Arch. Ration. Mech. Anal.* **213**:2 (2014), 587–628. MR 3211861 Zbl 06343915

[Secchi 2013] S. Secchi, "Ground state solutions for nonlinear fractional Schrödinger equations in $\mathbb{R}^N$", *J. Math. Phys.* **54**:3 (2013), 031501, 17. MR 3059423 Zbl 1281.81034

[Servadei and Valdinoci 2014] R. Servadei and E. Valdinoci, "Weak and viscosity solutions of the fractional Laplace equation", *Publ. Mat.* **58**:1 (2014), 133–154. MR 3161511 Zbl 1292.35315

[Silvestre 2005] L. E. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Ph.D. thesis, The University of Texas at Austin, 2005, available at http://search.proquest.com/docview/305384256. MR 2707618

[Silvestre 2007] L. Silvestre, "Regularity of the obstacle problem for a fractional power of the Laplace operator", *Comm. Pure Appl. Math.* **60**:1 (2007), 67–112. MR 2008a:35041 Zbl 1141.49035

[Wikipedia 2009–2015] Wikipedia, "Fractional Schrödinger equation", 2009–2015, available at http://en.wikipedia.org/wiki/Fractional_Schrödinger_equation.

Received 19 Nov 2014. Accepted 30 Apr 2015.

JUAN DÁVILA: jdavila@dim.uchile.cl
Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático, Universidad de Chile, Casilla 170 Correo 3, 8370459 Santiago, Chile

MANUEL DEL PINO: delpino@dim.uchile.cl
Universidad de Chile, 8370459 Santiago, Chile

SERENA DIPIERRO: serena.dipierro@ed.ac.uk
School of Mathematics, University of Edinburgh, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom

ENRICO VALDINOCI: enrico.valdinoci@wias-berlin.de
Weierstrass Institut für Angewandte Analysis und Stochastik, Mohrenstrasse 39, D-10117 Berlin, Germany

ANALYSIS AND PDE
Vol. 8, No. 5, 2015
dx.doi.org/10.2140/apde.2015.8.1237

LOCAL SPECTRAL ASYMPTOTICS FOR METRIC PERTURBATIONS OF THE LANDAU HAMILTONIAN

TOMÁS LUNGENSTRASS AND GEORGI RAIKOV

We consider metric perturbations of the Landau Hamiltonian. We investigate the asymptotic behavior of the discrete spectrum of the perturbed operator near the Landau levels, for perturbations of compact support, and of exponential or power-like decay at infinity.

1. Introduction

Let

$$H_0 := (-i\nabla - A_0)^2$$

with $A_0 = (A_{0,1}, A_{0,2}) := \frac{1}{2}b(-x_2, x_1)$ be the Landau Hamiltonian, self-adjoint in $L^2(\mathbb{R}^2)$, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. In other words, H_0 is the two-dimensional Schrödinger operator with constant scalar magnetic field $b > 0$, that is, the Hamiltonian of a two-dimensional, spinless, nonrelativistic quantum particle subject to a constant magnetic field. As is well known, the spectrum $\sigma(H_0)$ consists of infinitely degenerate eigenvalues $\Lambda_q := b(2q+1)$, $q \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, called *Landau levels* (see, e.g., [Fock 1928; Landau 1930]).

In the present article we consider metric perturbations of H_0. Namely, let

$$m(x) = \{m_{jk}(x)\}_{j,k=1,2}, \quad x \in \mathbb{R}^2,$$

be a Hermitian 2×2 matrix such that $m(x) \geq 0$ for all $x \in \mathbb{R}^2$. Throughout the article we assume that $m_{jk} \in C_{\mathrm{b}}^\infty(\mathbb{R}^2)$, $j, k = 1, 2$, i.e., $m_{jk} \in C^\infty(\mathbb{R}^2)$, and the entries m_{jk} together with all their derivatives are bounded on $\mathbb{R}^2$. Set

$$\Pi_j := -i\frac{\partial}{\partial x_j} - A_{0,j}, \quad j = 1, 2, \tag{1-1}$$

so that $H_0 = \Pi_1^2 + \Pi_2^2$. On Dom H_0, define the operators

$$H_\pm := \sum_{j,k=1,2} \Pi_j(\delta_{jk} \pm m_{jk})\Pi_k = H_0 \pm W,$$

where $W := \sum_{j,k=1,2} \Pi_j m_{jk} \Pi_k$; in the case of H_-, we suppose additionally that $\sup_{x\in\mathbb{R}^2} |m(x)| < 1$. Thus, the matrices $g_\pm(x) = \{g_{jk}^\pm(x)\}_{j,k=1,2}$ with $g_{jk}^\pm := \delta_{jk} \pm m_{jk}$ are positive definite for each $x \in \mathbb{R}^2$. Under these assumptions, the operators $H_\pm$ are self-adjoint in $L^2(\mathbb{R}^2)$, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ (see the Appendix).

MSC2010: 35J10, 35P20, 47G30, 81Q10.
Keywords: Landau Hamiltonian, metric perturbations, position-dependent mass, spectral asymptotics.

From a mathematical physics point of view, the operators $H_\pm$ are special cases of Schrödinger operators with *position-dependent mass*, which have a long history (see, e.g., [Bastard et al. 1975; von Roos 1983]), but have received increased attention during the last decade (see, e.g., [Midya et al. 2010; Gadella and Smolyanov 2008; Killingbeck 2011]). We would like to mention especially [de Souza Dutra and de Oliveira 2009], where the model considered is quite close to the operators $H_\pm$ discussed here.

The operators $H_\pm$ admit also a geometric interpretation, since they are related to the Bochner Laplacians corresponding to connections with constant nonvanishing curvature (see, e.g., [Rosenberg 1997; Colin de Verdière 1986]); we discuss this relation in more detail at the end of Section 2. Further, assume that

$$\lim_{|x|\to\infty} m_{jk}(x) = 0, \qquad j,k = 1,2. \tag{1-2}$$

Thus m models a localized perturbation with respect to a reference medium. Under condition (1-2), the resolvent difference $H_\pm^{-1} - H_0^{-1}$ is a compact operator (see the Appendix), and therefore the essential spectra of $H_\pm$ and H_0 coincide:

$$\sigma_{\text{ess}}(H_\pm) = \sigma_{\text{ess}}(H_0) = \sigma(H_0) = \bigcup_{q=0}^{\infty}\{\Lambda_q\}.$$

The spectrum $\sigma(H_\pm)$ on $\mathbb{R} \setminus \bigcup_{q=0}^{\infty}\{\Lambda_q\}$ may consist of discrete eigenvalues whose only possible accumulation points are the Landau levels. Moreover, taking into account that $W \geq 0$, and applying [Birman and Solomjak 1987, Section 9.4, Theorem 7], we find that the eigenvalues of H_+ (resp. H_-) may accumulate to a given Landau level Λ_q only from above (resp. from below). Fix $q \in \mathbb{Z}_+$. Let $\{\lambda_{k,q}^-\}$ be the eigenvalues of H_- lying on the interval $(\Lambda_{q-1}, \Lambda_q)$ with $\Lambda_{-1} := -\infty$, counted with multiplicities and enumerated in increasing order. Similarly, let $\{\lambda_{k,q}^+\}$ be the eigenvalues of H_+ lying on the interval $(\Lambda_q, \Lambda_{q+1})$, counted with multiplicities and enumerated in decreasing order.

The aim of the article is to investigate the rate of convergence of $\lambda_{k,q}^\pm - \Lambda_q$ as $k \to \infty$, with $q \in \mathbb{Z}_+$ fixed, for perturbations m of compact support, of exponential decay, or of power-like decay at infinity.

The properties of the discrete spectrum generated by perturbative second-order differential operators with decaying coefficients have been considered also in [Alama et al. 1994; Boyarchenko and Levendorskiĭ 1997; Briet et al. 2009; Raikov 2015].

The article is organized as follows. In Section 2 we formulate our main results and briefly comment on them. In Section 3 we reduce our analysis to the study of operators of Berezin–Toeplitz type, and in Section 4 we establish several useful unitary equivalences for these operators. Section 5 contains the proofs of our results in the case of rapid decay, i.e., of compact support or exponential decay, while the proofs for slow, i.e., power-like decay, can be found in Section 6. Finally, in the Appendix we address some standard issues concerning the domain of the operators $H_\pm$ and the compactness of the resolvent difference $H_0^{-1} - H_\pm^{-1}$.

2. Main results

First, we formulate our results concerning perturbations m of compact support. Denote by $m_<(x)$ and $m_>(x)$, with $m_<(x) \leq m_>(x)$, the two eigenvalues of the matrix $m(x)$, $x \in \mathbb{R}^2$.

Theorem 2.1. *Assume that the support of the matrix m is compact, and its smaller eigenvalue $m_<$ does not vanish identically. Fix $q \in \mathbb{Z}_+$. Then we have*

$$\ln(\pm(\lambda_{k,q}^{\pm} - \Lambda_q)) = -k \ln k + O(k), \quad k \to \infty. \tag{2-1}$$

Remarks. (i) Under additional technical hypotheses on $m_{\gtrless}$, we could make the asymptotic relation (2-1) more precise. Namely, assume that there exists a nonincreasing sequence $\{s_j\}_{j\in\mathbb{N}}$ such that $s_j > 0$, $j \in \mathbb{N}$, $\lim_{j\to\infty} s_j = 0$, and the level lines

$$\{x \in \mathbb{R}^2 \mid m_<(x) = s_j\}, \quad j \in \mathbb{N},$$

are bounded Lipschitz curves. In particular, the existence of such a sequence follows from the Sard lemma (see, e.g., [Sternberg 1964, Chapter 2, Theorem 3.1]) if we assume that $m_< \in C^2(\mathbb{R}^2)$. Further, denote by $\mathscr{C}_{\gtrless}$ the *logarithmic capacities* (see, e.g., [Landkof 1972, Chapter II, Section 4]) of $\operatorname{supp} m_{\gtrless}$. Then we have

$$\left(1 + \ln\left(\frac{b\mathscr{C}_<^2}{2}\right)\right)k + o(k) \le \ln(\pm(\lambda_{k,q}^{\pm} - \Lambda_q)) + k\ln k \le \left(1 + \ln\left(\frac{b\mathscr{C}_>^2}{2}\right)\right)k + o(k) \tag{2-2}$$

as $k \to \infty$. We omit the details of the proof of (2-2), which is inspired by [Filonov and Pushnitski 2006].

(ii) For $q \in \mathbb{Z}_+$ and $\lambda > 0$, set

$$\mathcal{N}_q^{\pm}(\lambda) := \#\{k \in \mathbb{Z}_+ \mid \pm(\lambda_{k,q}^{\pm} - \Lambda_q) > \lambda\}. \tag{2-3}$$

Then, a less precise version of (2-1), namely

$$\ln(\pm(\lambda_{k,q}^{\pm} - \Lambda_q)) = -k \ln k\,(1 + o(1)), \quad k \to \infty,$$

is equivalent to

$$\mathcal{N}_q^{\pm}(\lambda) = \frac{|\ln \lambda|}{\ln|\ln \lambda|}(1 + o(1)), \quad \lambda \downarrow 0. \tag{2-4}$$

Further, we state our results concerning perturbations of exponential decay. Assume that there exist constants $\beta > 0$ and $\gamma > 0$ such that

$$\ln m_{\gtrless}(x) = -\gamma|x|^{2\beta} + O(\ln|x|), \quad |x| \to \infty. \tag{2-5}$$

Remark. In (2-5), we suppose that the values of γ and β are the same for $m_<$ and $m_>$. Of course, the remainder $O(\ln|x|)$ could be different for $m_<$ and $m_>$.

Given $\beta > 0$ and $\gamma > 0$, set $\mu := \gamma(2/b)^{\beta}$, $b > 0$ being the constant magnetic field.

Theorem 2.2. *Let $m_{\gtrless}$ satisfy (2-5). Fix $q \in \mathbb{Z}_+$.*

(i) *If $\beta \in (0, 1)$, then there exist constants $f_j = f_j(\beta, \mu)$, $j \in \mathbb{N}$, with $f_1 = \mu$, such that*

$$\ln(\pm(\lambda_{k,q}^{\pm} - \Lambda_q)) = -\sum_{1\le j<1/(1-\beta)} f_j k^{(\beta-1)j+1} + O(\ln k), \quad k \to \infty. \tag{2-6}$$

(ii) *If* $\beta = 1$, *then*

$$\ln(\pm(\lambda_{k,q}^{\pm} - \Lambda_q)) = -(\ln(1+\mu))k + O(\ln k), \quad k \to \infty. \tag{2-7}$$

(iii) *If* $\beta \in (1, \infty)$, *then there exist constants* $g_j = g_j(\beta, \mu)$, $j \in \mathbb{N}$, *such that*

$$\ln(\pm(\lambda_{k,q}^{\pm} - \Lambda_q)) = -\frac{\beta-1}{\beta}k\ln k + \left(\frac{\beta-1-\ln(\mu\beta)}{\beta}\right)k - \sum_{1\le j<\beta/(\beta-1)} g_j k^{(1/\beta-1)j+1} + O(\ln k), \quad k \to \infty. \tag{2-8}$$

Remarks. (i) Let us describe explicitly the coefficients f_j and g_j, $j \in \mathbb{N}$, appearing in (2-6) and (2-8) respectively. Assume first that $\beta \in (0, 1)$. For $s > 0$ and $\epsilon \in \mathbb{R}$, $|\epsilon| \ll 1$, introduce the function

$$F(s;\epsilon) := s - \ln s + \epsilon\mu s^{\beta}. \tag{2-9}$$

Denote by $s_<(\epsilon)$ the unique positive solution of the equation $s = 1 - \epsilon\beta\mu s^{\beta}$, so that $\partial F(s_<(\epsilon);\epsilon)/\partial s = 0$. Set

$$f(\epsilon) := F(s_<(\epsilon);\epsilon). \tag{2-10}$$

Note that f is a real analytic function for small $|\epsilon|$. Then $f_j := (1/j!)\, d^j f(0)/d\epsilon^j$, $j \in \mathbb{N}$.

Let now $\beta \in (1, \infty)$. For $s > 0$ and $\epsilon \in \mathbb{R}$, $|\epsilon| \ll 1$, introduce the function

$$G(s;\epsilon) := \mu s^{\beta} - \ln s + \epsilon s. \tag{2-11}$$

Denote by $s_>(\epsilon)$ the unique positive solution of the equation $\beta\mu s^{\beta} = 1 - \epsilon s$, so that $\partial G(s_>(\epsilon);\epsilon)/\partial s = 0$. Define

$$g(\epsilon) := G(s_>(\epsilon);\epsilon), \tag{2-12}$$

which is a real analytic function for small $|\epsilon|$. Then $g_j := (1/j!)\, d^j g(0)/d\epsilon^j$, $j \in \mathbb{N}$.

(ii) If, instead of (2-5), we assume that

$$\ln m_{\gtrless}(x) = -\gamma|x|^{2\beta}(1+o(1)), \quad |x| \to \infty, \tag{2-13}$$

then we can prove less precise versions of (2-6), (2-7), and (2-8), namely

$$\ln(\pm(\lambda_{k,q}^{\pm} - \Lambda_q)) = \begin{cases} -\mu k^{\beta}(1+o(1)) & \text{if } \beta \in (0,1), \\ -(\ln(1+\mu))k(1+o(1)) & \text{if } \beta = 1, \\ -\dfrac{\beta-1}{\beta}k\ln k(1+o(1)) & \text{if } \beta \in (1,\infty), \end{cases} \quad k \to \infty,$$

which are equivalent to

$$\mathcal{N}_q^{\pm}(\lambda) = \begin{cases} \mu^{-1/\beta}|\ln\lambda|^{1/\beta}(1+o(1)) & \text{if } \beta \in (0,1), \\ \dfrac{1}{\ln(1+\mu)}|\ln\lambda|(1+o(1)) & \text{if } \beta = 1, \\ \dfrac{\beta}{\beta-1}\dfrac{|\ln\lambda|}{\ln|\ln\lambda|}(1+o(1)) & \text{if } \beta \in (1,\infty), \end{cases} \quad \lambda \downarrow 0. \tag{2-14}$$

Note that in (2-13), similarly to (2-5), we assume that the values of γ and β are the same for $m_<$ and $m_>$. However, since the coefficient in (2-14) with $\beta > 1$ does not depend on γ, in this case we could assume different values of $\gamma > 0$ for $m_<$ and $m_>$.

Finally, we consider perturbations m which admit a power-like decay at infinity. For $\rho > 0$ recall the definition of the Hörmander class

$$\mathcal{S}^{-\rho}(\mathbb{R}^2) := \{\psi \in C^\infty(\mathbb{R}^2) \mid |D^\alpha \psi(x)| \le c_\alpha \langle x\rangle^{-\rho-|\alpha|},\ x \in \mathbb{R}^2,\ \alpha \in \mathbb{Z}_+^2\},$$

where $\langle x\rangle := (1+|x|^2)^{1/2}$, $x \in \mathbb{R}^2$. Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ satisfy $\lim_{|x|\to\infty} \psi(x) = 0$. Set

$$\Phi_\psi(\lambda) := |\{x \in \mathbb{R}^2 \mid \psi(x) > \lambda\}|, \quad \lambda > 0, \tag{2-15}$$

where $|\cdot|$ denotes the Lebesgue measure. Fix $q \in \mathbb{Z}_+$, and introduce the function

$$\mathcal{T}_q(x) := \tfrac{1}{2}(\Lambda_q \operatorname{Tr} m(x) - 2b \operatorname{Im} m_{12}(x)), \quad x \in \mathbb{R}^2. \tag{2-16}$$

Note that $\mathcal{T}_q(x) \ge 0$ for any $x \in \mathbb{R}^2$ and $q \in \mathbb{Z}_+$.

Theorem 2.3. *Let $m_{jk} \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$, $j, k = 1, 2$, with $\rho > 0$. Fix $q \in \mathbb{Z}_+$. Suppose that there exists a function $0 < \tau_q \in C^\infty(\mathbb{S}^1)$ such that*

$$\lim_{|x|\to\infty} |x|^\rho \mathcal{T}_q(x) = \tau_q\left(\frac{x}{|x|}\right).$$

Then we have

$$\mathcal{N}_q^\pm(\lambda) = \frac{b}{2\pi}\Phi_{\mathcal{T}_q}(\lambda)(1+o(1)) \asymp \lambda^{-2/\rho}, \quad \lambda \downarrow 0, \tag{2-17}$$

which is equivalent to

$$\lim_{\lambda\downarrow 0} \lambda^{2/\rho}\mathcal{N}_q^\pm(\lambda) = \mathcal{C}_q := \frac{b}{4\pi}\int_0^{2\pi} \tau_q(\cos\theta, \sin\theta)^{2/\rho}\, d\theta, \tag{2-18}$$

or to

$$\pm(\lambda_{k,q}^\pm - \Lambda_q) = \mathcal{C}_q^{\rho/2} k^{-\rho/2}(1+o(1)), \quad k \to \infty. \tag{2-19}$$

Remarks. (i) Relation (2-17) could be regarded as a semiclassical one, although here the semiclassical interpretation is somewhat implicit. In Propositions 4.1 and 4.3 below, we show that the effective Hamiltonian, which governs the asymptotics of $\mathcal{N}_q^\pm(\lambda)$ as $\lambda \downarrow 0$, is a pseudodifferential operator with anti-Wick symbol $w_{q,b} := w_q \circ \mathcal{R}_b$ defined by (4-8) and (4-31). Under the assumptions of Theorem 2.3, $\mathcal{T}_{q,b} := \mathcal{T}_q \circ \mathcal{R}_b$ (see (2-16) and (4-31)) can be considered as the principal part of the symbol $w_{q,b}$, while the difference between the anti-Wick and the Weyl quantization is negligible. Then $(1/2\pi)\Phi_{\mathcal{T}_{q,b}}(\lambda) = (b/2\pi)\Phi_{\mathcal{T}_q}(\lambda)$ is just the main semiclassical asymptotic term for the eigenvalue counting function for a compact pseudodifferential operator with Weyl symbol $\mathcal{T}_{q,b}$.

(ii) There exists an extensive family of alternative sets of assumptions for Theorem 2.3 (see, e.g., [Ivrii 1998; Dauge and Robert 1987]). We have chosen here hypotheses which, for certain, are not the most general ones, but are quite explicit and, hopefully, easy to absorb.

Let us comment briefly on our results. Nowadays, there exists a relatively wide literature on the local spectral asymptotics for various magnetic quantum Hamiltonians. Let us concentrate here on three types of perturbations of H_0 which are considered to be of particular interest (see, e.g., [Ivrii 1998; Mao 2012]):

- Electric perturbations $H_0 + Q$ where $Q : \mathbb{R}^2 \to \mathbb{R}$ plays the role of the perturbative *electric potential*.
- Magnetic perturbations $(-i\nabla - A_0 - A)^2$, where $A = (A_1, A_2)$, and $B := \partial A_2/\partial x_1 - \partial A_1/\partial x_2$ is the perturbative *magnetic field*.
- Metric perturbations $\sum_{j,k=1,2} \Pi_j(\delta_{jk} + m_{jk})\Pi_k$, where $m = \{m_{jk}\}_{j,k=1,2}$ is an appropriate perturbative matrix-valued function.

Typically, the perturbations Q, B, or m are supposed to decay in a suitable sense at infinity. Slowly decaying Q, for example $Q \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$ with $\rho > 0$, were considered in [Raĭkov 1990], and the main asymptotic terms of the corresponding counting functions $\mathcal{N}_q^{\pm}(\lambda)$ as $\lambda \downarrow 0$ were found, utilizing, in particular, anti-Wick pseudodifferential operators. In [Ivrii 1998, Theorem 11.3.17], the case of *combined* electric, magnetic, and metric slowly decaying perturbations was investigated; the main asymptotic terms of $\mathcal{N}_q^{\pm}(\lambda)$ as $\lambda \downarrow 0$, as well as certain remainder estimates were obtained. The semiclassical microlocal analysis applied in [Ivrii 1998] imposed restrictions on the symbols involved, which, in some sense or another, had to decay at infinity less rapidly than their derivatives. These restrictions excluded some rapidly decaying perturbations, e.g., those of compact support, or of exponential decay with $\beta \geq \frac{1}{2}$ (see (2-5)).

Raikov and Warzel [2002] used a different approach based on the spectral analysis of Berezin–Toeplitz operators and obtained the main asymptotic terms of $\mathcal{N}_q^{\pm}(\lambda)$ as $\lambda \downarrow 0$ in the case of potential perturbations Q of exponential decay or of compact support. In particular, in [Raikov and Warzel 2002], formulas of the type (2-4) or (2-14) appeared for the first time. Here, we essentially improve the methods developed in [Raikov and Warzel 2002]. These improvements lead also to more precise results for certain rapidly decaying electric perturbations. Namely, assume that $Q \geq 0$ admits a decay at infinity which is compatible in a suitable sense with the decay of m. Then the results of the article extend quite easily to operators of the form

$$H_{\pm} \pm Q, \tag{2-20}$$

so that $H_{\pm} \pm Q$ are perturbations of H_0 having a definite sign. We do not include these generalizations just in order to avoid an unreasonable increase of the size of the article due to results which do not require any really new arguments.

Combined perturbations of H_0 by compactly supported B and Q were considered in [Rozenblum and Tashchiyan 2008], where the main asymptotic terms of $\mathcal{N}_q^{\pm}(\lambda)$ as $\lambda \downarrow 0$ were found. Note that the magnetic perturbations of H_0 are never of fixed sign, which creates specific difficulties, successfully overcome in [Rozenblum and Tashchiyan 2008].

To our best knowledge, no results on the spectral asymptotics for rapidly decaying *metric* perturbations of H_0 appeared before in the literature. We also included in the article our result on slowly decaying metric perturbations (see Theorem 2.3), since it is coherent with the unified approach of the article and is proved by methods quite different from those in [Ivrii 1998].

Finally, let us discuss briefly the relation of $H_\pm$ to the Bochner Laplacians. Assume that the elements of m are real. In $\mathbb{R}^2$ introduce a Riemannian metric generated by the inverse of $g^\pm$, and the connection 1-form $\sum_{j=1,2} A_{0,j}\,dx_j$. Set $\gamma_\pm := (\det g^\pm)^{-1/2}$. Then the standard positive-definite Bochner Laplacian, self-adjoint in $L^2(\mathbb{R}^2; \gamma_\pm\,dx)$, is written in local coordinates as

$$\mathscr{L}_\pm := \gamma_\pm^{-1} \sum_{j,k=1,2} \Pi_j g^\pm_{jk} \gamma_\pm \Pi_k.$$

Let $U_\pm : L^2(\mathbb{R}^2; \gamma_\pm dx) \to L^2(\mathbb{R}^2; dx)$ be the unitary operator defined by $U_\pm f := \gamma_\pm^{1/2} f$. Then we have

$$U_\pm \mathscr{L}_\pm U_\pm^* = H_\pm + Q_\pm, \tag{2-21}$$

where

$$Q_\pm := \frac{1}{4} \sum_{j,k=1,2} \left(g^\pm_{jk} \frac{\partial \ln \gamma_\pm}{\partial x_k} \frac{\partial \ln \gamma_\pm}{\partial x_j} + 2 \frac{\partial}{\partial x_j} \left(g^\pm_{jk} \frac{\partial \ln \gamma_\pm}{\partial x_k} \right)\right).$$

Generally speaking, the functions $Q_\pm$ do not have a definite sign coinciding with the sign of the operators $H_\pm - H_0$; hence, the operators on the right-hand side of (2-21) are not exactly of the form of (2-20). The fact that the symbol of a Toeplitz operator does not have a definite sign may cause considerable difficulties in the study of the spectral asymptotics of this operator if the symbol decays rapidly, and, in particular, when its support is compact (see, e.g., [Pushnitski and Rozenblum 2011]). Hopefully, we will overcome these difficulties in a future work, where we would consider the local spectral asymptotics of $\mathscr{L}_\pm$.

3. Reduction to Berezin–Toeplitz operators

In this section we reduce the analysis of the functions $\mathcal{N}_q^\pm(\lambda)$ as $\lambda \downarrow 0$ to the spectral asymptotics for certain compact operators of Berezin–Toeplitz type. To this end, we will need some more notations, and several auxiliary results from the abstract theory of compact operators in Hilbert space.

In what follows, we denote by $\mathbb{1}_M$ the characteristic function of the set M. Let T be a self-adjoint operator in a Hilbert space,[1] and $\mathcal{I} \subset \mathbb{R}$ be an interval. Set

$$N_\mathcal{I}(T) := \operatorname{rank} \mathbb{1}_\mathcal{I}(T),$$

where, in accordance with our general notations, $\mathbb{1}_\mathcal{I}(T)$ is the spectral projection of T corresponding to $\mathcal{I}$. Thus, if $\mathcal{I} \cap \sigma_{\text{ess}}(T) = \varnothing$, then $N_\mathcal{I}(T)$ is just the number of the eigenvalues of T lying on $\mathcal{I}$ and counted with their multiplicities. In particular,

$$\mathcal{N}_q^-(\lambda) = N_{(\Lambda_{q-1}, \Lambda_q - \lambda)}(H_-), \quad q \in \mathbb{Z}_+,\ \lambda \in (0, 2b), \tag{3-1}$$

$$\mathcal{N}_q^+(\lambda) = N_{(\Lambda_q + \lambda, \Lambda_{q+1})}(H_+), \quad q \in \mathbb{Z}_+,\ \lambda \in (0, 2b), \tag{3-2}$$

the functions $\mathcal{N}_q^\pm$ being defined in (2-3). Let $T = T^*$ be a linear compact operator in a Hilbert space. For $s > 0$, set

$$n_\pm(s; T) := N_{(s,\infty)}(\pm T);$$

[1] All the Hilbert spaces considered in the article are assumed to be separable.

thus, $n_+(s;T)$ (resp. $n_-(s;T)$) is just the number of the eigenvalues of the operator T larger than s (resp. smaller than $-s$), counted with multiplicities. If $T_j = T_j^*$, $j = 1, 2$, are two linear compact operators acting in a given Hilbert space, then the Weyl inequalities

$$n_\pm(s_1+s_2; T_1+T_2) \le n_\pm(s_1; T_1) + n_\pm(s_2; T_2) \tag{3-3}$$

hold for $s_j > 0$ (see, e.g., [Birman and Solomjak 1987, Section 9.2, Theorem 9]).

Fix $q \in \mathbb{Z}_+$ and denote by P_q the orthogonal projection onto $\operatorname{Ker}(H_0 - \Lambda_q)$. Since the operator $H_0^{-1} W H_0^{-1}$ is compact, the operator $P_q W P_q = \Lambda_q^2 P_q H_0^{-1} W H_0^{-1} P_q$ is compact as well. Similarly, the operators $H_0^{-1} W H_\pm^{-1/2}$ are compact, and hence the operators

$$P_q W H_\pm^{-1} W P_q = \Lambda_q^2 P_q (H_0^{-1} W H_\pm^{-1/2})(H_\pm^{-1/2} W H_0^{-1}) P_q$$

are compact as well.

Proposition 3.1. *Under the general assumptions of the article we have*

$$\begin{aligned} &n_+((1+\varepsilon)\lambda; P_q W P_q \mp P_q W H_\pm^{-1} W P_q) + O(1) \\ &\qquad \le \mathcal{N}_q^\pm(\lambda) \le n_+((1-\varepsilon)\lambda; P_q W P_q \mp P_q W H_\pm^{-1} W P_q) + O(1), \quad \lambda \downarrow 0, \end{aligned} \tag{3-4}$$

for each $\varepsilon \in (0, 1)$.

Proof. The argument is close in spirit to the proof of [Raikov and Warzel 2002, Proposition 4.1], and is based again on the (generalized) Birman–Schwinger principle. However, since the operator $H_0^{-1/2} W H_0^{-1/2}$ is only bounded but not compact, we cannot apply the Birman–Schwinger principle to the operator pair $(H_0, H_\pm)$, and apply it instead to the resolvent pair $(H_0^{-1}, H_\pm^{-1})$. First of all, note that there exist Λ_- and Λ_+ with $\Lambda_- \in (0, \Lambda_0)$ if $q = 0$, $\Lambda_- \in (\Lambda_{q-1}, \Lambda_q)$ if $q \in \mathbb{N}$, and $\Lambda_+ \in (\Lambda_q, \Lambda_{q+1})$ if $q \in \mathbb{Z}_+$, such that

$$\mathcal{N}_q^-(\lambda) = N_{(\Lambda_-, \Lambda_q - \lambda)}(H_-), \quad \lambda \in (0, \Lambda_q - \Lambda_-), \tag{3-5}$$

$$\mathcal{N}_q^+(\lambda) = N_{(\Lambda_q + \lambda, \Lambda_+)}(H_+), \quad \lambda \in (0, \Lambda_+ - \Lambda_q). \tag{3-6}$$

Further, evidently,

$$N_{(\Lambda_-, \Lambda_q - \lambda)}(H_-) = N_{((\Lambda_q-\lambda)^{-1}, \Lambda_-^{-1})}(H_-^{-1}) = N_{((\Lambda_q-\lambda)^{-1}, \Lambda_-^{-1})}(H_0^{-1} + T_-), \tag{3-7}$$

$$N_{(\Lambda_q+\lambda, \Lambda_+)}(H_+) = N_{(\Lambda_+^{-1}, (\Lambda_q+\lambda)^{-1})}(H_+^{-1}) = N_{(\Lambda_+^{-1}, (\Lambda_q+\lambda)^{-1})}(H_0^{-1} - T_+), \tag{3-8}$$

with $T_- := H_-^{-1} - H_0^{-1}$ and $T_+ := H_0^{-1} - H_+^{-1}$. Note that the operators $T_\pm$ are nonnegative and compact. By the generalized Birman–Schwinger principle (see, e.g., [Alama et al. 1989, Theorem 1.3]) we have

$$\begin{aligned} &N_{((\Lambda_q-\lambda)^{-1}, \Lambda_-^{-1})}(H_0^{-1} + T_-) \\ &= n_+\big(1; T_-^{1/2}((\Lambda_q-\lambda)^{-1} - H_0^{-1})^{-1} T_-^{1/2}\big) - n_+(1; T_-^{1/2}(\Lambda_-^{-1} - H_0^{-1})^{-1} T_-^{1/2}) - \dim \operatorname{Ker}(H_- - \Lambda_-), \end{aligned} \tag{3-9}$$

and

$$N_{(\Lambda_+^{-1},(\Lambda_q+\lambda)^{-1})}(H_0^{-1}-T_+)$$
$$=n_+\big(1;\, T_+^{1/2}(H_0^{-1}-(\Lambda_q+\lambda)^{-1})^{-1}T_+^{1/2}\big)-n_+(1;\, T_+^{1/2}(H_0^{-1}-\Lambda_+^{-1})^{-1}T_+^{1/2})-\dim\operatorname{Ker}(H_+-\Lambda_+). \quad (3\text{-}10)$$

Since the operators $T_\pm$ are compact and $\Lambda_\pm \notin \sigma(H_0)$, we find that the two last terms on the right-hand side of (3-9) and (3-10), which are independent of λ, are finite. Next, the Weyl inequalities (3-3) imply

$$\begin{aligned} n_+\big(1+\varepsilon;\, T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}P_qT_-^{1/2}\big)-n_-\big(\varepsilon;\, T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}(I-P_q)T_-^{1/2}\big) \\ \le n_+\big(1;\, T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}T_-^{1/2}\big) \\ \le n_+\big(1-\varepsilon;\, T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}P_qT_-^{1/2}\big) \\ +n_+\big(\varepsilon;\, T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}(I-P_q)T_-^{1/2}\big) \end{aligned} \quad (3\text{-}11)$$

for any $\varepsilon \in (0, 1)$. The operator $T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}(I-P_q)T_-^{1/2}$ tends in norm as $\lambda \downarrow 0$ to the compact operator

$$T_-^{1/2}\Big(\sum_{j\in\mathbb{Z}_+\setminus\{q\}}(\Lambda_q^{-1}-\Lambda_j^{-1})^{-1}P_j\Big)T_-^{1/2}.$$

Therefore,

$$n_\pm(\varepsilon;\, T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}(I-P_q)T_-^{1/2})=O(1), \quad \lambda\downarrow 0, \quad (3\text{-}12)$$

for any $\varepsilon > 0$. Next, for any $s > 0$ we have

$$\begin{aligned} n_+\big(s;\, T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}P_qT_-^{1/2}\big) &= n_+\big(s;\, ((\Lambda_q-\lambda)^{-1}-\Lambda_q^{-1})^{-1}T_-^{1/2}P_qT_-^{1/2}\big) \\ &= n_+(s\lambda(\Lambda_q-\lambda)^{-1}\Lambda_q^{-1};\, P_qT_-P_q). \end{aligned} \quad (3\text{-}13)$$

Hence, (3-9) and (3-11)–(3-13) yield

$$\begin{aligned} n_+((1+\varepsilon)\lambda(\Lambda_q-\lambda)^{-1}\Lambda_q^{-1};\, P_qT_-P_q)+O(1) \\ \le N_{((\Lambda_q-\lambda)^{-1},\Lambda_-^{-1})}(H_0^{-1}+T_-) \le n_+((1-\varepsilon)\lambda(\Lambda_q-\lambda)^{-1}\Lambda_q^{-1};\, P_qT_-P_q)+O(1), \quad \lambda\downarrow 0, \end{aligned} \quad (3\text{-}14)$$

for any $\varepsilon \in (0, 1)$. Similarly, (3-10) and the analogues of (3-11)–(3-13) for positive perturbations imply

$$\begin{aligned} n_+((1+\varepsilon)\lambda(\Lambda_q+\lambda)^{-1}\Lambda_q^{-1};\, P_qT_+P_q)+O(1) \\ \le N_{(\Lambda_+^{-1},(\Lambda_q+\lambda)^{-1})}(H_0^{-1}-T_+) \le n_+((1-\varepsilon)\lambda(\Lambda_q+\lambda)^{-1}\Lambda_q^{-1};\, P_qT_+P_q)+O(1), \quad \lambda\downarrow 0. \end{aligned} \quad (3\text{-}15)$$

By the resolvent identity, we have $T_\pm = H_0^{-1}WH_0^{-1} \mp H_0^{-1}WH_\pm^{-1}WH_0^{-1}$, so that

$$P_qT_\pm P_q = \Lambda_q^{-2}(P_qWP_q \mp P_qWH_\pm^{-1}WP_q).$$

Thus,

$$n_+(s;\, P_qT_\pm P_q) = n_+(s\Lambda_q^2;\, P_qWP_q \mp P_qWH_\pm^{-1}WP_q), \quad s>0. \quad (3\text{-}16)$$

Putting together (3-5)–(3-8) and (3-14)–(3-16), we easily obtain (3-4). □

4. Unitary equivalence for Berezin–Toeplitz operators

Our first goal in this section is to show that, under certain regularity conditions on the matrix m, the operator $P_q W P_q$, $q \in \mathbb{Z}_+$, with domain $P_q L^2(\mathbb{R}^2)$, is unitarily equivalent to $P_0 w_q P_0$ with domain $P_0 L^2(\mathbb{R}^2)$, where w_q is the multiplier by a suitable function $w_q : \mathbb{R}^2 \to \mathbb{C}$. In fact, we will need a slightly more general result, and that is why we introduce first the appropriate notations.

As usual, for $x = (x_1, x_2) \in \mathbb{R}^2$ we set $z := x_1 + ix_2$ and $\bar{z} := x_1 - ix_2$, so that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

Introduce the magnetic annihilation operator

$$a := -2ie^{-b|x|^2/4}\frac{\partial}{\partial \bar{z}}e^{b|x|^2/4} = -2i\left(\frac{\partial}{\partial \bar{z}} + \frac{bz}{4}\right)$$

and the magnetic creation operator

$$a^* := -2ie^{b|x|^2/4}\frac{\partial}{\partial z}e^{-b|x|^2/4} = -2i\left(\frac{\partial}{\partial z} - \frac{b\bar{z}}{4}\right)$$

with common domain $\operatorname{Dom} a = \operatorname{Dom} a^* = \operatorname{Dom} H_0^{1/2}$. The operators a and a^* are closed and mutually adjoint in $L^2(\mathbb{R}^2)$. On $\operatorname{Dom} H_0$ we have $[a, a^*] = 2b$ and

$$H_0 = a^*a + b = aa^* - b = \frac{1}{2}(aa^* + a^*a). \tag{4-1}$$

Moreover, on $\operatorname{Dom} H_0^{1/2}$ we have

$$\Pi_1 = \frac{1}{2}(a + a^*), \qquad \Pi_2 = \frac{1}{2i}(a - a^*), \tag{4-2}$$

the operators Π_j, $j = 1, 2$, being introduced in (1-1). Next, define the operator $\mathbb{A} : \operatorname{Dom} H_0^{1/2} \to L^2(\mathbb{R}^2; \mathbb{C}^2)$ by

$$\mathbb{A}u := \begin{pmatrix} a^*u \\ au \end{pmatrix}, \qquad u \in \operatorname{Dom} H_0^{1/2}.$$

Then, (4-1) implies that $H_0 = \frac{1}{2}\mathbb{A}^*\mathbb{A}$. Further, introduce the Hermitian matrix-valued function

$$\Omega := \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$$

with $\omega_{jk} \in L^\infty(\mathbb{R}^2)$, $j, k = 1, 2$. Fix $q \in \mathbb{Z}_+$ and define the operator

$$P_q\mathbb{A}^*\Omega\mathbb{A}P_q = \Lambda_q P_q H_0^{-1/2}\mathbb{A}^*\Omega\mathbb{A}H_0^{-1/2}P_q, \tag{4-3}$$

which is bounded and self-adjoint in $P_q L^2(\mathbb{R}^2)$. Utilizing (4-2), we easily find that

$$P_q W P_q = \tfrac{1}{2} P_q \mathbb{A}^* U \mathbb{A} P_q, \tag{4-4}$$

where

$$U := \mathbb{O}^* m \mathbb{O}, \quad \mathbb{O} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \tag{4-5}$$

so that

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \quad \text{with} \quad \begin{aligned} u_{11} &:= \tfrac{1}{2}(\operatorname{Tr} m - 2 \operatorname{Im} m_{12}), \\ u_{22} &:= \tfrac{1}{2}(\operatorname{Tr} m + 2 \operatorname{Im} m_{12}), \\ u_{12} = \bar{u}_{21} &:= \tfrac{1}{2}(m_{11} - m_{22} - 2i \operatorname{Re} m_{12}). \end{aligned}$$

Introduce the Laguerre polynomials

$$\mathrm{L}_q^{(m)} := \sum_{j=0}^{q} \binom{q+m}{q-j} \frac{(-t)^j}{j!}, \quad t \in \mathbb{R}, \; q \in \mathbb{Z}_+, \; m \in \mathbb{Z}_+; \tag{4-6}$$

as usual, we write $\mathrm{L}_q^{(0)} = \mathrm{L}_q$, and for notational convenience we set $q\mathrm{L}_{q-1} = 0$ for $q = 0$. By [Gradshteyn and Ryzhik 1965, Equation 8.974.3] we have

$$\sum_{j=0}^{q} \mathrm{L}_j^{(m)}(t) = \mathrm{L}_q^{(m+1)}(t), \quad t \in \mathbb{R}, \; q \in \mathbb{Z}_+, \; m \in \mathbb{Z}_+. \tag{4-7}$$

Proposition 4.1. *Let Ω be a Hermitian 2×2 matrix-valued function with entries $\omega_{jk} \in C_{\mathrm{b}}^\infty(\mathbb{R}^2)$, $j, k = 1, 2$. Fix $q \in \mathbb{Z}_+$. Then the operator $P_q \mathbb{A}^* \Omega \mathbb{A} P_q$ with domain $P_q L^2(\mathbb{R}^2)$ is unitarily equivalent to the operator $P_0 w_q P_0$ with domain $P_0 L^2(\mathbb{R}^2)$, where*

$$w_q = w_q(\Omega) := \begin{cases} 2b(q+1)\mathrm{L}_{q+1}\left(-\dfrac{\Delta}{2b}\right)\omega_{11} + 2bq\mathrm{L}_{q-1}\left(-\dfrac{\Delta}{2b}\right)\omega_{22} - 8 \operatorname{Re} \mathrm{L}_{q-1}^{(2)}\left(-\dfrac{\Delta}{2b}\right)\dfrac{\partial^2 \omega_{12}}{\partial \bar{z}^2} & \text{if } q \geq 1, \\ 2b\mathrm{L}_1\left(-\dfrac{\Delta}{2b}\right)\omega_{11} & \text{if } q = 0, \end{cases} \tag{4-8}$$

and $\Delta = \sum_{j=1,2} \partial^2/\partial x_j^2$, so that, in accordance with (4-6), $\mathrm{L}_s^{(m)}(-\Delta/(2b))$ with $s \in \mathbb{Z}_+$ and $m \in \mathbb{Z}_+$ is just the differential operator $\sum_{j=0}^{s} \binom{s+m}{s-j} \Delta^j/(j!(2b)^j)$ of order $2s$ with constant coefficients.

Proof. Set

$$\varphi_{0,k}(x) := \sqrt{\frac{b}{2\pi k!}} \left(\frac{b}{2}\right)^{k/2} z^k e^{-b|x|^2/4}, \quad x \in \mathbb{R}^2, \; k \in \mathbb{Z}_+,$$
$$\varphi_{q,k}(x) := \sqrt{\frac{1}{(2b)^q q!}} (a^*)^q \varphi_{0,k}(x), \quad x \in \mathbb{R}^2, \; k \in \mathbb{Z}_+, \; q \in \mathbb{N}.$$

Then $\{\varphi_{q,k}\}_{k \in \mathbb{Z}_+}$ is an orthonormal basis of $P_q L^2(\mathbb{R}^2)$, sometimes called *the angular momentum basis* (see, e.g., [Raikov and Warzel 2002] or [Bruneau et al. 2004, Section 9.1]). Evidently, for $k \in \mathbb{Z}_+$ we have

$$a^* \varphi_{q,k} = \sqrt{2b(q+1)}\varphi_{q+1,k}, \quad q \in \mathbb{Z}_+, \qquad a\varphi_{q,k} = \begin{cases} \sqrt{2bq}\varphi_{q-1,k}, & q \geq 1, \\ 0, & q = 0. \end{cases} \tag{4-9}$$

Define the unitary operator $\mathscr{W} : P_q L^2(\mathbb{R}^2) \to P_0 L^2(\mathbb{R}^2)$ by $\mathscr{W} : u \mapsto v$, where

$$u = \sum_{k\in\mathbb{Z}_+} c_k \varphi_{q,k}, \quad v = \sum_{k\in\mathbb{Z}_+} c_k \varphi_{0,k}, \quad \{c_k\}_{k\in\mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+). \tag{4-10}$$

We will show that

$$P_q \mathbb{A}^* \Omega \mathbb{A} P_q = \mathscr{W}^* P_0 w_q P_0 \mathscr{W}. \tag{4-11}$$

For $V \in C_{\mathrm{b}}^\infty(\mathbb{R}^2)$, $m, s \in \mathbb{Z}_+$, and $k, \ell \in \mathbb{Z}_+$, set

$$\Xi_{m,s}(V; k, \ell) := \langle V\varphi_{m,k}, \varphi_{s,\ell}\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{R}^2)$. Taking into account (4-9) and (4-10), we easily find that

$$\langle P_q \mathbb{A}^* \Omega \mathbb{A} P_q u, u\rangle = 2b \sum_{k\in\mathbb{Z}_+} \sum_{\ell\in\mathbb{Z}_+} ((q+1)\,\Xi_{q+1,q+1}(\omega_{11}; k, \ell) + q\,\Xi_{q-1,q-1}(\omega_{22}; k, \ell)) c_k \bar{c}_\ell$$
$$+ 2b\sqrt{q(q+1)}\,2\,\mathrm{Re} \sum_{k\in\mathbb{Z}_+} \sum_{\ell\in\mathbb{Z}_+} \Xi_{q+1,q-1}(\omega_{21}; k, \ell) c_k \bar{c}_\ell \tag{4-12}$$

if $q \ge 1$, and

$$\langle P_0 \mathbb{A}^* \Omega \mathbb{A} P_0 u, u\rangle = 2b \sum_{k\in\mathbb{Z}_+} \sum_{\ell\in\mathbb{Z}_+} \Xi_{1,1}(\omega_{11}; k, \ell) c_k \bar{c}_\ell. \tag{4-13}$$

Moreover,

$$\langle P_0 w_q P_0 v, v\rangle = \sum_{k\in\mathbb{Z}_+} \sum_{\ell\in\mathbb{Z}_+} \Xi_{0,0}(w_q; k, \ell) c_k \bar{c}_\ell, \quad q \in \mathbb{Z}_+. \tag{4-14}$$

In [Bruneau et al. 2004, Lemma 9.2] (see also the remark after Equation (2.2) in [Bony et al. 2014]), it was shown that

$$\Xi_{m,m}(V; k, \ell) = \Xi_{0,0}\left(\mathrm{L}_m\left(-\frac{\Delta}{2b}\right) V; k, \ell\right), \quad m \in \mathbb{Z}_+. \tag{4-15}$$

Now (4-13), (4-15) with $m = 1$ and $V = \omega_{11}$, and (4-14) with $q = 0$ imply (4-11) in the case $q = 0$. Assume $q \ge 1$. By (4-15), we have

$$\Xi_{q+1,q+1}(\omega_{11}; k, \ell) = \Xi_{0,0}\left(\mathrm{L}_{q+1}\left(-\frac{\Delta}{2b}\right)\omega_{11}; k, \ell\right), \tag{4-16}$$

$$\Xi_{q-1,q-1}(\omega_{22}; k, \ell) = \Xi_{0,0}\left(\mathrm{L}_{q-1}\left(-\frac{\Delta}{2b}\right)\omega_{22}; k, \ell\right). \tag{4-17}$$

Let us now consider the quantity $\Xi_{q+1,q-1}(V; k, \ell)$. Using (4-9), we easily find that, for $q \ge 2$, we have

$$\Xi_{q+1,q-1}(V; k, \ell) = \frac{1}{\sqrt{2b(q+1)}}\,\Xi_{q,q-1}([V, a^*]; k, \ell) + \sqrt{\frac{q-1}{q+1}}\,\Xi_{q,q-2}(V; k, \ell), \tag{4-18}$$

$$\Xi_{q,q-1}([V, a^*]; k, \ell) = \frac{1}{\sqrt{2bq}}\,\Xi_{q-1,q-1}\big([[V, a^*], a^*]; k, \ell\big) + \sqrt{\frac{q-1}{q}}\,\Xi_{q-1,q-2}([V, a^*]; k, \ell). \tag{4-19}$$

Moreover, $[V, a^*] = 2i\,\partial V/\partial z$, and

$$[[V, a^*], a^*] = -4\frac{\partial^2 V}{\partial z^2}. \tag{4-20}$$

Using (4-19), it is not difficult to prove by induction that

$$\Xi_{q,q-1}([V, a^*]; k, \ell) = \frac{1}{\sqrt{2bq}} \sum_{j=0}^{q-1} \Xi_{j,j}\big([[V, a^*], a^*]; k, \ell\big), \quad q \geq 1. \tag{4-21}$$

Now (4-15), (4-20), and (4-7) imply

$$\begin{aligned} \sum_{j=0}^{q-1} \Xi_{j,j}\big([[V, a^*], a^*]; k, \ell\big) &= \sum_{j=0}^{q-1} \Xi_{0,0}\left(-4\mathrm{L}_j\left(-\frac{\Delta}{2b}\right)\frac{\partial^2 V}{\partial z^2}; k, \ell\right) \\ &= \Xi_{0,0}\left(-4\mathrm{L}_{q-1}^{(1)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2 V}{\partial z^2}; k, \ell\right). \end{aligned} \tag{4-22}$$

Setting

$$\mathscr{D}_q := -4\mathrm{L}_{q-1}^{(1)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2}{\partial z^2}, \quad q \in \mathbb{N}, \tag{4-23}$$

we find that (4-21) and (4-22) imply

$$\Xi_{q,q-1}([V, a^*]; k, \ell) = \frac{1}{\sqrt{2bq}} \Xi_{0,0}(\mathscr{D}_q V; k, \ell). \tag{4-24}$$

Bearing in mind (4-18), (4-15), and (4-24), it is not difficult to prove by induction that

$$\Xi_{q+1,q-1}(V; k, \ell) = \frac{1}{2b\sqrt{q(q+1)}} \sum_{s=1}^{q} \Xi_{0,0}(\mathscr{D}_s V; k, \ell). \tag{4-25}$$

Note that (4-7) and (4-25) imply

$$\sum_{s=1}^{q} \mathscr{D}_s = -4\mathrm{L}_{q-1}^{(2)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2}{\partial z^2}. \tag{4-26}$$

Now, (4-25) and (4-26) entail

$$2b\sqrt{q(q+1)}\,\Xi_{q+1,q-1}(\omega_{21}; k, \ell) = \Xi_{0,0}\left(-4\mathrm{L}_{q-1}^{(2)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2 \omega_{21}}{\partial z^2}; k, \ell\right). \tag{4-27}$$

Finally, (4-12) and (4-14) combined with (4-16), (4-17), and (4-27) yield (4-11) with $q \geq 1$. □

In the rest of the section we establish two other suitable representations for the operators $P_q V P_q$, $q \in \mathbb{Z}_+$, with $V : \mathbb{R}^2 \to \mathbb{C}$.

Proposition 4.2. (i) [Fernández and Raikov 2004, Lemma 3.1; Bony et al. 2014, Section 2.3] *Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ satisfy $\lim_{|x|\to\infty} V(x) = 0$. Then, for each $q \in \mathbb{Z}_+$, the operator $P_q V P_q$ is compact.*

(ii) [Raikov and Warzel 2002, Lemma 3.3] *Assume in addition that V is radially symmetric*, *i.e.*, *there exists $v : [0,\infty) \to \mathbb{C}$ such that $V(x) = v(|x|)$, $x \in \mathbb{R}^2$. Then the eigenvalues of the operator $P_q V P_q$ with domain $P_q L^2(\mathbb{R}^2)$*, *counted with multiplicities*, *coincide with the set*

$$\{\langle V\varphi_{q,k}, \varphi_{q,k}\rangle\}_{k\in\mathbb{Z}_+}. \tag{4-28}$$

In particular, *the eigenvalues of $P_0 V P_0$ coincide with*

$$\frac{1}{k!}\int_0^\infty v\Big(\Big(\frac{2t}{b}\Big)^{\frac{1}{2}}\Big)e^{-t}t^k\,dt, \quad k \in \mathbb{Z}_+. \tag{4-29}$$

Remarks. (i) Let us recall that, if f is, say, a bounded function of exponential decay, then

$$(\mathcal{M}f)(z) := \int_0^\infty f(t)t^{z-1}\,dt, \quad z \in \mathbb{C},\ \operatorname{Re} z > 0,$$

is sometimes called *the Mellin transform* of f. Some of the asymptotic properties as $k \to \infty$ of the integrals (4-29), which we will later obtain and use in the proofs of Theorems 2.1 and 2.2, could possibly be deduced from the general theory of the Mellin transform.

(ii) Combining Propositions 4.1 and 4.2, we find that, if the matrix-valued function Ω is radially symmetric and diagonal, then the operator $P_q\mathbb{A}^*\Omega\mathbb{A}P_q$ acting in $P_qL^2(\mathbb{R}^2)$ is unitarily equivalent to a *diagonal* operator in $\ell^2(\mathbb{Z}_+)$. If Ω is just radially symmetric, then $P_q\mathbb{A}^*\Omega\mathbb{A}P_q$ is unitarily equivalent to a *tridiagonal* operator acting in $\ell^2(\mathbb{Z}_+)$.

The last proposition in this section concerns the unitary equivalence between the Berezin–Toeplitz operator P_0WP_0 and a certain Weyl pseudodifferential operator. Let us recall the definition of Weyl pseudodifferential operators acting in $L^2(\mathbb{R})$. Denote by $\Gamma(\mathbb{R}^2)$ the set of functions $\psi : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\|\psi\|_{\Gamma(\mathbb{R}^2)} := \sup_{(y,\eta)\in\mathbb{R}^2}\ \sup_{\ell,m=0,1}\left|\frac{\partial^{\ell+m}\psi(y,\eta)}{\partial y^\ell\partial\eta^m}\right| < \infty.$$

Then the operator $\operatorname{Op}^{\mathrm{w}}(\psi)$, defined initially as a mapping between the Schwartz class $\mathcal{S}(\mathbb{R})$ and its dual class $\mathcal{S}'(\mathbb{R})$ by

$$(\operatorname{Op}^{\mathrm{w}}(\psi)u)(y) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\psi\Big(\frac{y+y'}{2},\eta\Big)e^{i(y-y')\eta}u(y')\,dy'\,d\eta, \quad y \in \mathbb{R},$$

extends uniquely to an operator bounded in $L^2(\mathbb{R})$. Moreover, there exists a constant c such that

$$\|\operatorname{Op}^{\mathrm{w}}(\psi)\| \le c\|\psi\|_{\Gamma(\mathbb{R}^2)} \tag{4-30}$$

(see, e.g., [Boulkhemair 1999, Corollary 2.5(i)]).

Remark. Inequalities of the type (4-30) are known as *Calderón–Vaillancourt* estimates.

Put

$$\mathcal{R}_b := -b^{-1/2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{4-31}$$

and, for $V : \mathbb{R}^2 \to \mathbb{C}$, define

$$V_b(x) := V(\mathcal{R}_b x), \quad x \in \mathbb{R}^2, \ b > 0.$$

Moreover, set $\mathcal{G}(x) := e^{-|x|^2}/\pi$, $x \in \mathbb{R}^2$.

Proposition 4.3 [Pushnitski et al. 2013, Theorem 2.11, Corollary 2.8]. *Let $V \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$. Then the operator $P_0 V P_0$ with domain $P_0 L^2(\mathbb{R}^2)$ is unitarily equivalent to the operator* $\mathrm{Op}^{\mathrm{w}}(V_b * \mathcal{G})$.

Remark. The operator $\mathrm{Op}^{\mathrm{aw}}(\psi) := \mathrm{Op}^{\mathrm{w}}(\psi * \mathcal{G})$ is called a pseudodifferential operator with *anti-Wick symbol* ψ (see, e.g., [Shubin 2001, Section 24]).

5. Proofs of Theorems 2.1 and 2.2

In this section we complete the proofs of Theorems 2.1 and 2.2, concerning perturbations of compact support and of exponential decay.

Let $T = T^*$ be a compact operator in a Hilbert space such that $\operatorname{rank} \mathbb{1}_{(0,\infty)}(T) = \infty$. Denote by $\{\nu_k(T)\}_{k=0}^\infty$ the nonincreasing sequence of the positive eigenvalues of T, counted with multiplicities.

Recall that $m_<(x) \le m_>(x)$ are the eigenvalues of the matrix $m(x)$ for $x \in \mathbb{R}^2$. Since the matrix U (see (4-5)) is unitarily equivalent to m, $m_\gtrless$ are also the eigenvalues of U. Next, we check that Proposition 3.1 implies the following:

Corollary 5.1. *Under the general assumptions of the article, there exist constants $0 < c_<^\pm \le c_>^\pm < \infty$ and $k_0 \in \mathbb{Z}_+$ such that*

$$c_<^\pm \nu_{k+k_0}(P_q \mathbb{A}^* m_< \mathbb{A} P_q) \le \pm(\lambda_{k,q}^\pm - \Lambda_q) \le c_>^\pm \nu_{k-k_0}(P_q \mathbb{A}^* m_> \mathbb{A} P_q) \tag{5-1}$$

for sufficiently large $k \in \mathbb{N}$.

Proof. It is easy to see that

$$0 \le P_q W H_\pm^{-1} W P_q \le c_\pm P_q W P_q \tag{5-2}$$

with

$$c_\pm := \|H_\pm^{-1/2} W H_\pm^{-1/2}\| \le \sup_{x \in \mathbb{R}^2} \left| m(x)(I \pm m(x))^{-1} \right|.$$

Note that $0 \le c_- < \infty$ and $0 \le c_+ < 1$. Moreover, by (4-4) and the mini-max principle,

$$n_+(2s;\, P_q \mathbb{A}^* m_< \mathbb{A} P_q) \le n_+(s;\, P_q W P_q) \le n_+(2s;\, P_q \mathbb{A}^* m_> \mathbb{A} P_q), \quad s > 0. \tag{5-3}$$

Now, (3-4), (5-2), and (5-3) imply that, for any $\varepsilon \in (0, 1)$, we have

$$n_+(2\lambda(1+\varepsilon);\, P_q \mathbb{A}^* m_< \mathbb{A} P_q) + O(1) \le \mathcal{N}_q^-(\lambda) \le n_+(2\lambda(1-\varepsilon);\, (1+c_-) P_q \mathbb{A}^* m_> \mathbb{A} P_q) + O(1), \tag{5-4}$$

$$n_+(2\lambda(1-\varepsilon);\, P_q \mathbb{A}^* m_> \mathbb{A} P_q) + O(1) \ge \mathcal{N}_q^+(\lambda) \ge n_+(2\lambda(1+\varepsilon);\, (1-c_+) P_q \mathbb{A}^* m_< \mathbb{A} P_q) + O(1) \tag{5-5}$$

as $\lambda \downarrow 0$, and estimates (5-4)–(5-5) yield (5-1) with

$$c_<^- = \frac{1}{2(1+\varepsilon)}, \quad c_>^- = \frac{1+c_-}{2(1-\varepsilon)}, \quad c_<^+ = \frac{1-c_+}{2(1+\varepsilon)}, \quad c_>^+ = \frac{1}{2(1-\varepsilon)},$$

and sufficiently large $k_0 \in \mathbb{N}$. □

Let us now complete the proof of Theorem 2.1. Let $\zeta_1 \in C_0^\infty(\mathbb{R}^2)$, $\zeta_1 \geq 0$, $\zeta_1 = 1$ on $\operatorname{supp} m_>$. Set $\zeta_2(x) := (\max_{y\in\mathbb{R}^2} m_>(y))\zeta_1(x)$, $x \in \mathbb{R}^2$. Evidently, $m_> \leq \zeta_2$ on $\mathbb{R}^2$, so that

$$\nu_k(P_q\mathbb{A}^* m_> \mathbb{A} P_q) \leq \nu_k(P_q\mathbb{A}^*\zeta_2\mathbb{A}P_q), \quad k \in \mathbb{Z}_+. \tag{5-6}$$

Further, by Proposition 4.1, the operator $P_q\mathbb{A}^*\zeta_2\mathbb{A}P_q$ is unitarily equivalent to the operator $P_0\zeta_3 P_0$, where

$$\zeta_3 := 2b\biggl((q+1)\mathrm{L}_{q+1}\Bigl(-\frac{\Delta}{2b}\Bigr) + q\mathrm{L}_{q-1}\Bigl(-\frac{\Delta}{2b}\Bigr)\biggr)\zeta_2.$$

Therefore,

$$\nu_k(P_q\mathbb{A}^*\zeta_2\mathbb{A}P_q) = \nu_k(P_0\zeta_3 P_0), \quad k \in \mathbb{Z}_+. \tag{5-7}$$

Let $R_> > 0$ be so large that the disk $B_{R_>}(0)$ of radius $R_>$ centered at the origin contains the support of ζ_3. Then,

$$\nu_k(P_0\zeta_3 P_0) \leq \max_{x\in\mathbb{R}^2} |\zeta_3(x)|\, \nu_k(P_0 \mathbb{1}_{B_{R_>}(0)} P_0), \quad k \in \mathbb{Z}_+. \tag{5-8}$$

Putting together (5-6), (5-7), and (5-8), we find that there exists a constant $K_> < \infty$ such that

$$\nu_k(P_q\mathbb{A}^* m_> \mathbb{A} P_q) \leq K_> \nu_k(P_0 \mathbb{1}_{B_{R_>}(0)} P_0), \quad k \in \mathbb{Z}_+. \tag{5-9}$$

On the other hand,

$$\nu_k(P_q\mathbb{A}^* m_< \mathbb{A} P_q) \geq \nu_k(P_q a m_< a^* P_q). \tag{5-10}$$

Applying (4-9), we easily find that the operators $P_q a m_< a^* P_q$ and $2b(q+1)P_{q+1} m_< P_{q+1}$ are unitarily equivalent. Hence,

$$\nu_k(P_q a m_< a^* P_q) = 2b(q+1)\nu_k(P_{q+1} m_< P_{q+1}), \quad k \in \mathbb{Z}_+. \tag{5-11}$$

Further, since $m_<$ is nonnegative, continuous, and does not vanish identically, there exist $c_0 > 0$, $R_< \in (0, \infty)$, and $x_0 \in \mathbb{R}^2$ such that $m_<(x) \geq c_0 \mathbb{1}_{B_{R_<}(x_0)}(x)$, $x \in \mathbb{R}^2$. Therefore,

$$\nu_k(P_{q+1} m_< P_{q+1}) \geq c_0 \nu_k(P_{q+1} \mathbb{1}_{B_{R_<}(x_0)} P_{q+1}), \quad k \in \mathbb{Z}_+. \tag{5-12}$$

The operators $P_{q+1}\mathbb{1}_{B_{R_<}(x_0)}P_{q+1}$ and $P_{q+1}\mathbb{1}_{B_{R_<}(0)}P_{q+1}$ are unitarily equivalent under the magnetic translation which maps x_0 into 0 (see, e.g., [Raikov and Warzel 2002, Equation (4.21)]). Therefore,

$$\nu_k(P_{q+1} \mathbb{1}_{B_{R_<}(x_0)} P_{q+1}) = \nu_k(P_{q+1} \mathbb{1}_{B_{R_<}(0)} P_{q+1}), \quad k \in \mathbb{Z}_+. \tag{5-13}$$

Combining (5-10)–(5-13), we find that there exists a constant $K_<$ such that

$$K_< \nu_k(P_{q+1} \mathbb{1}_{B_{R_<}(0)} P_{q+1}) \leq \nu_k(P_q\mathbb{A}^* m_< \mathbb{A} P_q), \quad k \in \mathbb{Z}_+. \tag{5-14}$$

By (5-9) and (5-14), it remains to study the asymptotic behavior as $k \to \infty$ of $\nu_k(P_m \mathbb{1}_{B_R(0)} P_m)$, with $m \in \mathbb{Z}_+$ and $R \in (0, \infty)$ fixed. This asymptotic analysis relies on the representation (4-28), and results sufficient for our purposes are available in the literature. Namely, we have:

Lemma 5.2 [Combes et al. 2004, Section 4, Corollary 2]. *Let $m \in \mathbb{Z}_+$, $R \in (0,\infty)$, $b \in (0,\infty)$. Set $\varrho := bR^2/2$. Then*

$$\nu_k(P_m \mathbb{1}_{B_R(0)} P_m) = \frac{e^{-\varrho}\varrho^{-m+1}k^{2m-1}\varrho^k}{m!k!}(1+o(1)), \quad k \to \infty. \tag{5-15}$$

Now, asymptotic relation (2-1) follows from (5-1), (5-9), (5-14), (5-15), and the elementary fact that $\ln k! = k \ln k + O(k)$ as $k \to \infty$.

In the remaining part of this section we prove Theorem 2.2 concerning perturbations m of exponential decay. Assume that m satisfies (2-5). Then there exist $\delta_{\gtrless} \in \mathbb{R}$, $\delta_< \le \delta_>$, and $r > 1$ such that

$$|x|^{\delta_<} e^{-\gamma|x|^{2\beta}} \mathbb{1}_{\mathbb{R}^2 \setminus B_r(0)}(x) \le m_<(x) \le m_>(x) \le |x|^{\delta_>} e^{-\gamma|x|^{2\beta}} \mathbb{1}_{\mathbb{R}^2 \setminus B_r(0)}(x) + \max_{y\in\mathbb{R}^2} m_>(y) \mathbb{1}_{B_r(0)}(x), \quad x \in \mathbb{R}^2. \tag{5-16}$$

Let $\eta_{\gtrless,0} \in C^\infty(\mathbb{R}^2; [0,1])$ be two radially symmetric functions such that $\eta_{<,0} = 1$ on $\mathbb{R}^2 \setminus B_{r+1}(0)$, $\eta_{<,0} = 0$ on $B_r(0)$, and $\eta_{>,0} = 1$ on $\mathbb{R}^2 \setminus B_r(0)$, $\eta_{>,0} = 0$ on $B_{r-1}(0)$. For $x \in \mathbb{R}^2$ set

$$\eta_{<,1}(x) := |x|^{\delta_<} e^{-\gamma|x|^{2\beta}} \eta_{<,0}(x),$$
$$\eta_{>,1}(x) := |x|^{\delta_>} e^{-\gamma|x|^{2\beta}} \eta_{>,0}(x) + \max_{y\in\mathbb{R}^2} m_>(y)(1-\eta_{<,0}(x)).$$

Evidently, $\eta_{\gtrless,1} \in C_{\mathrm{b}}^\infty(\mathbb{R}^2)$, and by (5-16),

$$\eta_{<,1}(x) \le m_<(x), \quad m_>(x) \le \eta_{>,1}(x), \quad x \in \mathbb{R}^2.$$

Therefore, for $k \in \mathbb{Z}_+$, we have

$$\begin{aligned} \nu_k(P_q \mathbb{A}^* m_< \mathbb{A} P_q) &\ge \nu_k(P_q \mathbb{A}^* \eta_{<,1} \mathbb{A} P_q), \\ \nu_k(P_q \mathbb{A}^* m_> \mathbb{A} P_q) &\le \nu_k(P_q \mathbb{A}^* \eta_{>,1} \mathbb{A} P_q). \end{aligned} \tag{5-17}$$

Further, set

$$\eta_{\gtrless,2} := 2b\left((q+1)\mathrm{L}_{q+1}\left(-\frac{\Delta}{2b}\right) + q\mathrm{L}_{q-1}\left(-\frac{\Delta}{2b}\right)\right)\eta_{\gtrless,1}.$$

According to Proposition 4.1, the operators $P_q \mathbb{A}^* \eta_{\gtrless,1} \mathbb{A} P_q$, $q \in \mathbb{Z}_+$, and $P_0 \eta_{\gtrless,2} P_0$ are unitarily equivalent. Therefore,

$$\nu_k(P_q \mathbb{A}^* \eta_{\gtrless,1} \mathbb{A} P_q) = \nu_k(P_0 \eta_{\gtrless,2} P_0), \quad k \in \mathbb{Z}_+. \tag{5-18}$$

Next, a tedious but straightforward calculation shows that

$$\eta_{\gtrless,2}(x) = \eta_{\gtrless,3}(x)(1+o(1)), \quad |x| \to \infty, \tag{5-19}$$

where

$$\eta_{\gtrless,3}(x) := C_{q,\beta}|x|^{\delta_\gtrless} e^{-\gamma|x|^{2\beta}} \begin{cases} 1 & \text{if } \beta \in \left(0, \frac{1}{2}\right], \\ |x|^{2(q+1)(2\beta-1)} & \text{if } \beta \in \left(\frac{1}{2}, \infty\right), \end{cases} \quad x \in \mathbb{R}^2 \setminus \{0\},$$

and $C_{q,\beta} > 0$ are some constants. Even though the exact values of $C_{q,\beta}$ will not play any role in the sequel, we indicate here these values for the sake of the completeness of the exposition:

$$C_{q,\beta} = \begin{cases} 2\Lambda_q & \text{if } \beta \in \left(0, \tfrac{1}{2}\right), \\ 2b\Big((q+1)\mathrm{L}_{q+1}\Big(-\frac{(2\beta\gamma)^2}{2b}\Big) + q\mathrm{L}_{q-1}\Big(-\frac{(2\beta\gamma)^2}{2b}\Big)\Big) & \text{if } \beta = \tfrac{1}{2}, \\ \dfrac{(2\beta\gamma)^{2(q+1)}}{(2b)^q q!} & \text{if } \beta \in \left(\tfrac{1}{2}, \infty\right). \end{cases}$$

Hence, by (5-19), there exists $R \in (0, \infty)$ such that for $x \in \mathbb{R}^2$ we have

$$\eta_{<,2} \ge \tfrac{1}{2}\eta_{<,3}\mathbb{1}_{\mathbb{R}^2\setminus B_R(0)} - c_<\mathbb{1}_{B_R(0)} =: \eta_{<,4}(x), \tag{5-20}$$

$$\eta_{>,2} \le \tfrac{3}{2}\eta_{>,3}\mathbb{1}_{\mathbb{R}^2\setminus B_R(0)} + c_>\mathbb{1}_{B_R(0)} =: \eta_{>,4}(x) \tag{5-21}$$

with $c_{\gtrless} := \max_{y\in\mathbb{R}^2} |\eta_{\gtrless,2}(y)|$. Thus, for any admissible $k \in \mathbb{Z}_+$, we have

$$\nu_k(P_0\eta_{<,2}P_0) \ge \nu_k(P_0\eta_{<,4}P_0), \qquad \nu_k(P_0\eta_{>,2}P_0) \le \nu_k(P_0\eta_{>,4}P_0). \tag{5-22}$$

In order to complete the proof of Theorem 2.2, we need a couple of auxiliary results. For $\beta > 0$, $\mu > 0$, and $\varrho > 0$, set

$$\mathcal{J}_{\beta,\mu}(k) := \int_0^\infty e^{-\mu t^\beta - t} t^k \, dt, \quad \mathcal{E}_\varrho(k) := \int_0^\varrho e^{-t} t^k \, dt, \quad k > -1, \tag{5-23}$$

and, for $\delta \in \mathbb{R}$, $c_0 > 0$ and $c_1 \in \mathbb{R}$, put

$$\mathcal{L}(k) = \mathcal{L}_{\beta,\mu,\varrho,\delta}(k; c_0, c_1) := \frac{c_0\mathcal{J}_{\beta,\mu}(k+\delta) + c_1\mathcal{E}_\varrho(k-\delta_-)}{\Gamma(k+1)}, \quad k > \max\{-1, -\delta-1\},$$

where $\delta_- := \max\{0, -\delta\}$.

Lemma 5.3. *Let $\beta > 0$, $\mu > 0$, $\varrho > 0$, $c_0 > 0$, and $\delta \in \mathbb{R}$, $c_1 \in \mathbb{R}$.*

(i) *The asymptotic relations*

$$\ln \mathcal{L}(k) = \begin{cases} -\sum_{1\le j<1/(1-\beta)} f_j k^{(\beta-1)j+1} + O(\ln k) & \text{if } \beta \in (0,1), \\ -\ln(1+\mu)k + O(\ln k) & \text{if } \beta = 1, \\ -\dfrac{\beta-1}{\beta} k \ln k + k\Big(\dfrac{\beta-1-\ln(\mu\beta)}{\beta}\Big) \\ \qquad\qquad - \sum_{1\le j<\beta/(\beta-1)} g_j k^{(1/\beta-1)j+1} + O(\ln k) & \text{if } \beta \in (1,\infty), \end{cases} \tag{5-24}$$

hold true as $k \to \infty$, the coefficients f_j and g_j being introduced in the statement of Theorem 2.2.

(ii) *We have $\mathcal{L}'(k) < 0$ for sufficiently large k.*

Proof. First, let $\delta = 0$. Assume $\beta \in (0,1)$, $k > 0$, and make the change of variable $t \mapsto ks$ in the first integral in (5-23). Thus we find that

$$\mathcal{J}_{\beta,\mu}(k) = k^{k+1} \int_0^\infty e^{-kF(s;k^{\beta-1})} \, ds. \tag{5-25}$$

The function $F(s;k^{\beta-1})$ defined in (2-9) attains its unique minimum at $s_<(k^{\beta-1})$, and we have $\partial^2 F(s_<(k^{\beta-1});k^{\beta-1})/\partial s^2 = 1+o(1)$, $k\to\infty$. Therefore, applying a standard argument close to the usual Laplace method for asymptotic evaluation of integrals depending on a large parameter, we easily find that

$$\int_0^\infty e^{-kF(s;k^{\beta-1})}\,ds = (2\pi)^{1/2} e^{-kF(s_<(k^{\beta-1});k^{\beta-1})} k^{-1/2}(1+o(1)), \quad k\to\infty. \tag{5-26}$$

Bearing in mind that $F(s_<(k^{\beta-1});k^{\beta-1}) = f(k^{\beta-1})$ (see (2-10)), $f(0)=1$, and

$$\ln\Gamma(k+1) = k\ln k - k + \tfrac{1}{2}\ln k + O(1), \quad k\to\infty, \tag{5-27}$$

(see, e.g., [Abramowitz and Stegun 1964, Equation 6.1.40]), we find that (5-25)–(5-26) imply

$$\begin{aligned}\ln\left(\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)}\right) &= k - kf(k^{\beta-1}) + O(\ln k)\\ &= k - k\sum_{0\le j<1/(1-\beta)} \frac{1}{j!}\frac{d^j f}{d\epsilon^j}(0)k^{(\beta-1)j} + O(\ln k)\\ &= -\sum_{1\le j<1/(1-\beta)} \frac{1}{j!}\frac{d^j f}{d\epsilon^j}(0)k^{(\beta-1)j+1} + O(\ln k)\\ &= -\sum_{1\le j<1/(1-\beta)} f_j k^{(\beta-1)j+1} + O(\ln k), \quad k\to\infty. \end{aligned} \tag{5-28}$$

In the case $\beta = 1$, we simply have

$$\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)} = \frac{1}{\Gamma(k+1)}\int_0^\infty e^{-(\mu+1)t} t^k\,dt = (\mu+1)^{-k-1},$$

that is,

$$\ln\left(\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)}\right) = -(\ln(1+\mu))k + O(1), \quad k\to\infty. \tag{5-29}$$

Now let $\beta\in(1,\infty)$. Making the change of variable $t\mapsto k^{1/\beta}s$ with $k>0$ in (5-23), we find

$$\mathcal{J}_{\beta,\mu}(k) := k^{(k+1)/\beta}\int_0^\infty e^{-kG(s;k^{(1/\beta-1)})}\,ds. \tag{5-30}$$

The function $G(s;k^{1/\beta-1})$ defined in (2-11), attains its unique minimum at $s_>(k^{1/\beta-1})$, and we have

$$\frac{\partial^2 G}{\partial s^2}(s_>(k^{1/\beta-1}),k^{1/\beta-1}) = \beta(\mu\beta)^{2/\beta}(1+o(1)), \quad k\to\infty.$$

Arguing as in the derivation of (5-26), we obtain

$$\int_0^\infty e^{-kG(s;k^{1/\beta-1})}\,ds = \sqrt{2\pi\beta}\,(\mu\beta)^{-1/\beta} e^{-kG(s_>(k^{1/\beta-1});k^{1/\beta-1})} k^{-1/2}(1+o(1)), \quad k\to\infty. \tag{5-31}$$

Bearing in mind that $G(s_>(k^{1/\beta-1});k^{1/\beta-1}) = g(k^{1/\beta-1})$ (see (2-12)), and $g(0) = (1+\ln(\mu\beta))/\beta$, we find that (5-30), (5-31), and (5-27) imply

$$\begin{aligned}\ln\left(\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)}\right) &= -\frac{\beta-1}{\beta}k\ln k + k - kg(k^{1/\beta-1}) + O(\ln k)\\ &= -\frac{\beta-1}{\beta}k\ln k + k - k\sum_{0\le j<\beta/(\beta-1)}\frac{1}{j!}\frac{d^jg}{d\epsilon^j}(0)k^{(1/\beta-1)j} + O(\ln k)\\ &= -\frac{\beta-1}{\beta}k\ln k + k(1-g(0)) - \sum_{1\le j<\beta/(\beta-1)}\frac{1}{j!}\frac{d^jg}{d\epsilon^j}(0)k^{(1/\beta-1)j+1} + O(\ln k)\\ &= -\frac{\beta-1}{\beta}k\ln k + k\left(\frac{\beta-1-\ln(\mu\beta)}{\beta}\right) - \sum_{1\le j<\beta/(\beta-1)} g_jk^{(1/\beta-1)j+1} + O(\ln k), \end{aligned} \tag{5-32}$$

as $k\to\infty$. Let us now consider general $\delta\in\mathbb{R}$. By (5-27),

$$\ln\left(\frac{\Gamma(k+\delta+1)}{\Gamma(k+1)}\right) = \delta\ln k + O(1), \quad k\to\infty. \tag{5-33}$$

Putting together (5-28), (5-29), (5-32), and (5-33), we find that

$$\ln\left(\frac{\mathcal{J}_{\beta,\mu}(k+\delta)}{\Gamma(k+1)}\right) - \ln\left(\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)}\right) = O(\ln k), \quad k\to\infty. \tag{5-34}$$

Finally, by (5-15), we easily find that, for each fixed $\delta\in\mathbb{R}$, we have

$$\frac{\mathscr{E}_\varrho(k-\delta_-)}{\Gamma(k+1)} = o\left(\frac{\mathcal{J}_{\beta,\mu}(k+\delta)}{\Gamma(k+1)}\right), \quad k\to\infty. \tag{5-35}$$

The combination of (5-28), (5-29), (5-32), (5-34), and (5-35) implies (5-24).

For (ii), we have

$$\mathcal{L}'(k) = c_0\left(\frac{\mathcal{J}'_{\beta,\mu}(k+\delta)}{\Gamma(k+1)} - \frac{\Gamma'(k+1)}{\Gamma(k+1)^2}\mathcal{J}_{\beta,\mu}(k+\delta)\right) + c_1\left(\frac{\mathscr{E}'_\varrho(k-\delta_-)}{\Gamma(k+1)} - \frac{\Gamma'(k+1)}{\Gamma(k+1)^2}\mathscr{E}_\varrho(k-\delta_-)\right), \tag{5-36}$$

$$\mathcal{J}'_{\beta,\mu}(k) = \int_0^\infty e^{-\mu t^\beta - t}t^k\ln t\,dt, \quad \mathscr{E}'_\varrho(k) = \int_0^\varrho e^{-t}t^k\ln t\,dt,$$

and

$$\frac{\Gamma'(k+1)}{\Gamma(k+1)} = \ln k + \frac{1}{2k} + O(k^{-2}), \quad k\to\infty,$$

(see, e.g., [Abramowitz and Stegun 1964, Equation 6.3.18]). Performing an asymptotic analysis similar to the one in the proof of the first part of the lemma, we find that there exists a function $\Psi = \Psi_{\beta,\mu,\delta}$ such that $\Psi(k) < 0$ for k large enough, and

$$\frac{\mathcal{J}'_{\beta,\mu}(k+\delta)}{\Gamma(k+1)} - \frac{\Gamma'(k+1)}{\Gamma(k+1)^2}\mathcal{J}_{\beta,\mu}(k+\delta) = \Psi(k)(1+o(1)), \tag{5-37}$$

$$\frac{\mathscr{E}'_\varrho(k-\delta_-)}{\Gamma(k+1)} - \frac{\Gamma'(k+1)}{\Gamma(k+1)^2}\mathscr{E}_\varrho(k-\delta_-) = o(\Psi(k)) \tag{5-38}$$

as $k \to \infty$. Putting together (5-36), (5-37), and (5-38), we conclude that $\mathscr{L}'(k) < 0$ for sufficiently large k. □

Taking into account the definition of the functions $\eta_{\gtrless,4}$ in (5-20)–(5-21), the mini-max principle, representation (4-29), as well as Lemma 5.3(ii), we find that there exist constants $c_{j,\gtrless} > 0$, $j = 0, 1$, $\tilde{\delta}_{\gtrless} \in \mathbb{R}$, and $k_0 \in \mathbb{Z}_+$ such that

$$\begin{aligned} \nu_k(P_0\eta_{<,4}P_0) &\geq \mathscr{L}_{\beta,\mu,\varrho,\tilde{\delta}_<}(k+k_0; c_{0,<}, -c_{1,<}) \\ \nu_k(P_0\eta_{>,4}P_0) &\leq \mathscr{L}_{\beta,\mu,\varrho,\tilde{\delta}_>}(k; c_{0,>}, c_{1,>}), \end{aligned} \tag{5-39}$$

for $\mu = \gamma(2/b)^\beta$, $\varrho = bR^2/2$, and sufficiently large $k \in \mathbb{Z}_+$.

Putting together (5-1), (5-17), (5-18), (5-22), (5-39), and (5-24), we obtain (2-6)–(2-8).

6. Proof of Theorem 2.3

Estimates (3-4) combined with the Weyl inequalities (3-3) and the mini-max principle entail

$$\begin{aligned} &n_+(\lambda(1+\varepsilon); P_qWP_q) + O(1) \\ &\qquad \leq \mathcal{N}_q^-(\lambda) \leq n_+(\lambda(1-\varepsilon)^2; P_qWP_q) + n_+(\lambda\varepsilon(1-\varepsilon); P_qWH_-^{-1}WP_q) + O(1), \end{aligned} \tag{6-1}$$

and

$$\begin{aligned} &n_+(\lambda(1+\varepsilon)^2; P_qWP_q) - n_+(\lambda\varepsilon(1+\varepsilon); P_qWH_+^{-1}WP_q) + O(1) \\ &\qquad \leq \mathcal{N}_q^+(\lambda) \leq n_+(\lambda(1-\varepsilon); P_qWP_q) + O(1) \end{aligned} \tag{6-2}$$

as $\lambda \downarrow 0$. It is easy to check that we have

$$P_qWH_\pm^{-1}WP_q \leq C_{1,\pm}P_q\mathbb{A}^*\langle\cdot\rangle^{-2\rho}\mathbb{A}P_q$$

with

$$C_{1,\pm} := \|H_0^{1/2}H_\pm^{-1/2}\|^2\big(\sup_{x\in\mathbb{R}^2}\langle x\rangle^\rho m_>(x)\big)^2.$$

Therefore, for any $s > 0$,

$$n_+(s; P_qWH_\pm^{-1}WP_q) \leq n_+(s; C_{1,\pm}P_q\mathbb{A}^*\langle\cdot\rangle^{-2\rho}\mathbb{A}P_q). \tag{6-3}$$

Further, by Proposition 4.1, the operator P_qWP_q (resp. $P_q\mathbb{A}^*\langle\cdot\rangle^{-2\rho}\mathbb{A}P_q$) is unitarily equivalent to $\frac{1}{2}P_0w_q(U)P_0$ (resp. $P_0w_q(\langle\cdot\rangle^{-2\rho}I)P_0$). Hence, for any $s > 0$,

$$n_+(s; P_qWP_q) = n_+(2s; P_0w_q(U)P_0), \tag{6-4}$$

$$n_+(s; P_q\mathbb{A}^*\langle\cdot\rangle^{-2\rho}\mathbb{A}P_q) = n_+(s; P_0w_q(\langle\cdot\rangle^{-2\rho}I)P_0) \leq n_+(s; C_2P_0\langle\cdot\rangle^{-2\rho}P_0) \tag{6-5}$$

with $C_2 := \sup_{x\in\mathbb{R}^2}\langle x\rangle^{2\rho}|w_q(\langle x\rangle^{-2\rho}I)|$. Now, write

$$\tfrac{1}{2}w_q(U) = \mathscr{T}_q + \tilde{\mathscr{T}}_q,$$

the symbol $\mathcal{T}_q$ being defined in (2-16), and note the crucial circumstance that $\tilde{\mathcal{T}}_q \in \mathcal{S}^{-\rho-2}(\mathbb{R}^2)$. Then the Weyl inequalities (3-3) entail

$$n_+(s(1+\varepsilon);\, P_0\mathcal{T}_q P_0) - n_-(s\varepsilon;\, P_0\tilde{\mathcal{T}}_q P_0) \le n_+(2s;\, P_0 w_q(U) P_0)$$
$$\le n_+(s(1-\varepsilon);\, P_0\mathcal{T}_q P_0) + n_+(s\varepsilon;\, P_0\tilde{\mathcal{T}}_q P_0) \tag{6-6}$$

for any $s > 0$ and $\varepsilon \in (0, 1)$. Evidently,

$$n_\pm(s;\, P_0\tilde{\mathcal{T}}_q P_0) \le n_+(s;\, C_3 P_0\langle\,\cdot\,\rangle^{-\rho-2} P_0), \quad s > 0, \tag{6-7}$$

with $C_3 := \sup_{x\in\mathbb{R}^2}\langle x\rangle^{\rho+2}|\tilde{\mathcal{T}}_q(x)|$. Recalling Proposition 4.3, we find that we have reduced the asymptotic analysis of $\mathcal{N}_q^\pm(\lambda)$ as $\lambda \downarrow 0$ to the eigenvalue asymptotics for a pseudodifferential operator with elliptic anti-Wick symbol of negative order. The spectral asymptotics for operators of this type has been extensively studied in the literature since the 1970s. In particular, we have the following:

Proposition 6.1. *Let $0 \le \psi \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$, $\rho > 0$. Assume that there exists $0 < \psi_0 \in C^\infty(\mathbb{S}^1)$ such that $\lim_{|x|\to\infty}|x|^\rho\psi(x) = \psi_0(x/|x|)$. Then we have*

$$n_+(\lambda;\, \mathrm{Op}^{\mathrm{aw}}(\psi)) = (2\pi)^{-1}\Phi_\psi(\lambda)(1+o(1)), \quad \lambda \downarrow 0, \tag{6-8}$$

which is equivalent to

$$\lim_{\lambda\downarrow 0}\lambda^{2/\rho} n_+(\lambda;\, \mathrm{Op}^{\mathrm{aw}}(\psi)) = \mathcal{C}(\psi_0) := \frac{1}{4\pi}\int_0^{2\pi}\psi_0(\cos\theta, \sin\theta)^{2/\rho}\, d\theta.$$

Proof. Evidently, for each $\varepsilon \in (0, 1)$ there exist real functions $\psi_{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2)$ such that

$$\psi_{-,\varepsilon}(x) \le \psi(x) \le \psi_{+,\varepsilon}(x), \quad x \in \mathbb{R}^2,$$
$$\psi_{\pm,\varepsilon}(x) = (1\mp\varepsilon)^{-1}|x|^{-\rho}\psi_0\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^2,\ |x| \ge R,$$

for some $R \in (0, \infty)$. Applying the monotonicity of the anti-Wick quantization with respect to the symbol (see, e.g., [Shubin 2001, Proposition 24.1]), the mini-max principle, and the Weyl inequalities, we obtain

$$n_+((1+\varepsilon)\lambda;\, \mathrm{Op}^{\mathrm{w}}(\psi_{-,\varepsilon})) - n_-\big(\varepsilon\lambda;\, (\mathrm{Op}^{\mathrm{aw}}(\psi_{-,\varepsilon}) - \mathrm{Op}^{\mathrm{w}}(\psi_{-,\varepsilon}))\big)$$
$$\le n_+(\lambda;\, \mathrm{Op}^{\mathrm{aw}}(\psi)) \le n_+((1-\varepsilon)\lambda;\, \mathrm{Op}^{\mathrm{w}}(\psi_{+,\varepsilon})) + n_+\big(\varepsilon\lambda;\, (\mathrm{Op}^{\mathrm{aw}}(\psi_{+,\varepsilon}) - \mathrm{Op}^{\mathrm{w}}(\psi_{+,\varepsilon}))\big). \tag{6-9}$$

By [Dauge and Robert 1987], we have the semiclassical result

$$n_+(\lambda;\, \mathrm{Op}^{\mathrm{w}}(\psi_{\pm,\varepsilon})) = (2\pi)^{-1}\Phi_{\psi_{\pm,\varepsilon}}(\lambda)(1+o(1)), \quad \lambda \downarrow 0. \tag{6-10}$$

Further, by [Shubin 2001, Theorem 24.1] the differences $\mathrm{Op}^{\mathrm{aw}}(\psi_{\pm,\varepsilon}) - \mathrm{Op}^{\mathrm{w}}(\psi_{\pm,\varepsilon})$ are pseudodifferential operators of lower order than $\mathrm{Op}^{\mathrm{w}}(\psi_{\pm,\varepsilon})$, so that we easily obtain

$$\lim_{\lambda\downarrow 0}\lambda^{2/\rho} n_\pm\big(\varepsilon\lambda;\, (\mathrm{Op}^{\mathrm{aw}}(\psi_{\pm,\varepsilon}) - \mathrm{Op}^{\mathrm{w}}(\psi_{\pm,\varepsilon}))\big) = 0, \quad \varepsilon > 0. \tag{6-11}$$

Now, (6-9)–(6-11) imply

$$(1+\varepsilon)^{-4/\rho}\mathcal{C}(\psi_0) \le \liminf_{\lambda\downarrow 0}\lambda^{2/\rho} n_+(\lambda;\, \mathrm{Op}^{\mathrm{aw}}(\psi)) \le \limsup_{\lambda\downarrow 0}\lambda^{2/\rho} n_+(\lambda;\, \mathrm{Op}^{\mathrm{aw}}(\psi)) \le (1-\varepsilon)^{-4/\rho}\mathcal{C}(\psi_0)$$

for $\varepsilon \in (0, 1)$. Letting $\varepsilon \downarrow 0$, we obtain (6-8). □

By Propositions 4.3 and 6.1, we have

$$n_+(\lambda;\, P_0 \mathscr{T}_q P_0) = n_+(\lambda;\, \mathrm{Op}^{\mathrm{aw}}(\mathscr{T}_{q,b})) = \frac{1}{2\pi}\Phi_{\mathscr{T}_{q,b}}(\lambda)(1+o(1)) = \frac{b}{2\pi}\Phi_{\mathscr{T}_q}(\lambda)(1+o(1)), \quad \lambda \downarrow 0, \tag{6-12}$$

with $\mathscr{T}_{q,b} = \mathscr{T}_q \circ \mathscr{R}_b$, $\mathscr{R}_b$ being defined in (4-31). Finally, for $\rho_0 > \rho$, we have

$$n_+(\lambda;\, P_0 \langle \cdot \rangle^{-\rho_0} P_0) = O(\lambda^{-2/\rho_0}) = o(\Phi_{\mathscr{T}_q}(\lambda)), \quad \lambda \downarrow 0. \tag{6-13}$$

Now, (2-17) easily follows from (6-1)–(6-8), (6-12), and (6-13). The equivalence of (2-18) and (2-19) can be checked by arguing as in the proof of [Shubin 2001, Proposition 13.1].

Appendix: Compactness of the resolvent differences

A priori, the operators H_0 and $H_\pm$, self-adjoint in $L^2(\mathbb{R}^2)$, could be defined as the Friedrichs extensions of the operators $\sum_{j=1,2} \Pi_j^2$ and $\sum_{j,k=1,2} \Pi_j g_{jk}^{\pm} \Pi_k$ defined on $C_0^\infty(\mathbb{R}^2)$. Such a definition implies immediately that

$$\mathrm{Dom}\, H_0^{1/2} = \mathrm{Dom}\, H_\pm^{1/2} = \{u \in L^2(\mathbb{R}^2) \mid \Pi_j u \in L^2(\mathbb{R}^2),\ j = 1, 2\},$$

and that the operators $H_\pm^{1/2} H_0^{-1/2}$ and $H_0^{1/2} H_\pm^{-1/2}$ are bounded. By [Gérard et al. 1991, Proposition A.2], the operators H_0 and $H_\pm$ are essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ and have a common domain

$$\mathrm{Dom}\, H_0 = \mathrm{Dom}\, H_\pm = \{u \in L^2(\mathbb{R}^2) \mid \Pi_j \Pi_k u \in L^2(\mathbb{R}^2),\ j, k = 1, 2\}.$$

Let us now prove the compactness of the operator $H_0^{-1} - H_\pm^{-1}$ in $L^2(\mathbb{R}^2)$. Since we have

$$H_0^{-1} - H_\pm^{-1} = \pm H_0^{-1} W H_\pm^{-1} = \pm H_0^{-1} W H_0^{-1} H_0 H_\pm^{-1},$$

it suffices to prove the compactness of $H_0^{-1} W H_0^{-1}$. The operators $H_0^{-1} W H_0^{-1} = \frac{1}{2} H_0^{-1} \mathbb{A}^* U \mathbb{A} H_0^{-1}$ and $\frac{1}{2} H_0^{-1} \mathbb{A}^* m_> \mathbb{A} H_0^{-1}$ are bounded, self-adjoint, and positive. Moreover,

$$H_0^{-1} \mathbb{A}^* U \mathbb{A} H_0^{-1} \le H_0^{-1} \mathbb{A}^* m_> \mathbb{A} H_0^{-1}. \tag{A-1}$$

On the other hand,

$$H_0^{-1} \mathbb{A}^* m_> \mathbb{A} H_0^{-1} = H_0^{-1} a^* m_> a H_0^{-1} + H_0^{-1} a m_> a^* H_0^{-1}. \tag{A-2}$$

By (A-1) and (A-2), it suffices to prove the compactness of the operator $m_>^{1/2} a^* H_0^{-1}$. We have

$$m_>^{1/2} a^* H_0^{-1} = m_>^{1/2} H_0^{-1/2} (H_0^{-1/2} a^* + 2b H_0^{-1/2} a^* H_0^{-1}).$$

The operator $H_0^{-1/2} a^* + 2b H_0^{-1/2} a^* H_0^{-1}$ is bounded, so that it suffices to prove the compactness of $m_>^{1/2} H_0^{-1/2}$ which follows from $m_> \in L^\infty(\mathbb{R}^2)$, $\lim_{|x|\to\infty} m_>(x) = 0$, and the diamagnetic inequality (see, e.g., [Avron et al. 1978, Theorem 2.5]).

Acknowledgements

The final version of this work has been done during the authors' visit to the Isaac Newton Institute, Cambridge, UK, in January 2015. The authors thank the Newton Institute for financial support and hospitality. The partial support by the Chilean Scientific Foundation *Fondecyt* under Grant 1130591, by *Núcleo Milenio de Física Matemática* RC120002, and by the Faculty of Mathematics, PUC, Santiago de Chile, is gratefully acknowledged as well.

References

[Abramowitz and Stegun 1964] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series **55**, U.S. Government Printing Office, Washington, D.C., 1964. MR 29 #4914 Zbl 0171.38503

[Alama et al. 1989] S. Alama, P. A. Deift, and R. Hempel, "Eigenvalue branches of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$", *Comm. Math. Phys.* **121**:2 (1989), 291–321. MR 90e:35046 Zbl 0676.47032

[Alama et al. 1994] S. Alama, M. Avellaneda, P. A. Deift, and R. Hempel, "On the existence of eigenvalues of a divergence-form operator $A + \lambda B$ in a gap of $\sigma(A)$", *Asymptotic Anal.* **8**:4 (1994), 311–344. MR 95g:47069 Zbl 0806.47042

[Avron et al. 1978] J. Avron, I. Herbst, and B. Simon, "Schrödinger operators with magnetic fields, I: General interactions", *Duke Math. J.* **45**:4 (1978), 847–883. MR 80k:35054 Zbl 0399.35029

[Bastard et al. 1975] G. Bastard, J. K. Furdyna, and J. Mycielski, "Landau levels and cyclotron resonance in graded mixed semiconductors", *Phys. Rev. B* **12** (1975), 4356–4359.

[Birman and Solomjak 1987] M. S. Birman and M. Z. Solomjak, *Spectral theory of selfadjoint operators in Hilbert space*, Reidel, Dordrecht, 1987. MR 93g:47001

[Bony et al. 2014] J.-F. Bony, V. Bruneau, and G. Raikov, "Resonances and spectral shift function singularities for magnetic quantum Hamiltonians", pp. 77–100 in *Spectral and scattering theory and related topics*, edited by F. Hiroshima, RIMS Kôkyûroku Bessatsu **B45**, Res. Inst. Math. Sci. (RIMS), Kyoto, 2014. MR 3287673 Zbl 1301.35074

[Boulkhemair 1999] A. Boulkhemair, "L^2 estimates for Weyl quantization", *J. Functional Anal.* **165**:1 (1999), 173–204. MR 2001f:35454 Zbl 0934.35217

[Boyarchenko and Levendorskiĭ 1997] S. I. Boyarchenko and S. Z. Levendorskiĭ, "An asymptotic formula for the number of eigenvalue branches of a divergence form operator $A + \lambda B$ in a spectral gap of A", *Comm. Partial Differential Equations* **22**:11–12 (1997), 1771–1786. MR 99e:35165 Zbl 0899.35070

[Briet et al. 2009] P. Briet, H. Kovařík, G. Raikov, and E. Soccorsi, "Eigenvalue asymptotics in a twisted waveguide", *Comm. Partial Differential Equations* **34**:7–9 (2009), 818–836. MR 2010j:35089 Zbl 1195.35119

[Bruneau et al. 2004] V. Bruneau, A. Pushnitski, and G. Raikov, "Spectral shift function in strong magnetic fields", *Algebra i Analiz* **16**:1 (2004), 207–238. Reprinted in *St. Petersburg Math. J.* **16** (2005), 181–209. MR 2005h:81118 Zbl 1082.35115

[Colin de Verdière 1986] Y. Colin de Verdière, "L'asymptotique de Weyl pour les bouteilles magnétiques", *Comm. Math. Phys.* **105**:2 (1986), 327–335. MR 87k:58273 Zbl 0612.35102

[Combes et al. 2004] J.-M. Combes, P. D. Hislop, F. Klopp, and G. Raikov, "Global continuity of the integrated density of states for random Landau Hamiltonians", *Comm. Partial Differential Equations* **29**:7–8 (2004), 1187–1213. MR 2005g:81091 Zbl 1085.47050

[Dauge and Robert 1987] M. Dauge and D. Robert, "Weyl's formula for a class of pseudodifferential operators with negative order on $L^2(\mathbb{R}^n)$", pp. 91–122 in *Pseudodifferential operators* (Oberwolfach, 1986), edited by H. O. Cordes et al., Lecture Notes in Math. **1256**, Springer, Berlin, 1987. MR 88k:35150 Zbl 0629.35098

[Fernández and Raikov 2004] C. Fernández and G. Raikov, "On the singularities of the magnetic spectral shift function at the Landau levels", *Ann. Henri Poincaré* **5**:2 (2004), 381–403. MR 2005g:81092 Zbl 1062.81043

[Filonov and Pushnitski 2006] N. Filonov and A. Pushnitski, "Spectral asymptotics of Pauli operators and orthogonal polynomials in complex domains", *Comm. Math. Phys.* **264**:3 (2006), 759–772. MR 2007h:81064 Zbl 1106.81040

[Fock 1928] V. Fock, "Bemerkung zur Quantelung des harmonischen Oszillators im Magnetfeld", *Z. Physik* **47**:5–6 (1928), 446–448. JFM 54.0966.02

[Gadella and Smolyanov 2008] M. Gadella and O. G. Smolyanov, "Feynman formulas for particles with position-dependent mass", *Dokl. Akad. Nauk* **418**:6 (2008), 727–730. In Russian; translated in *Dokl. Math.* **77**:1 (2008), 120–123. MR 2009m:81125 Zbl 1159.35426

[Gérard et al. 1991] C. Gérard, A. Martinez, and J. Sjöstrand, "A mathematical approach to the effective Hamiltonian in perturbed periodic problems", *Comm. Math. Phys.* **142**:2 (1991), 217–244. MR 93a:47079 Zbl 0753.35057

[Gradshteyn and Ryzhik 1965] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Academic Press, New York, 1965. Zbl 1300.65001

[Ivrii 1998] V. Ivrii, *Microlocal analysis and precise spectral asymptotics*, Springer, Berlin, 1998. MR 99e:58193 Zbl 0906.35003

[Killingbeck 2011] J. P. Killingbeck, "The Schrödinger equation with position-dependent mass", *J. Phys. A* **44**:28 (2011), 285208, 6. MR 2012i:81090 Zbl 1222.81163

[Landau 1930] L. Landau, "Diamagnetismus der Metalle", *Z. Physik* **64**:9–10 (1930), 629–637.

[Landkof 1972] N. S. Landkof, *Foundations of modern potential theory*, Grundlehren der Math. Wissenschaften **180**, Springer, New York, 1972. MR 50 #2520 Zbl 0253.31001

[Mao 2012] S. Mao, "Singularities for solutions to Schrödinger equations with asymptotically constant magnetic fields", *J. Math. Phys.* **53**:7 (2012), Art. ID #073707. MR 2985266 Zbl 1276.81054

[Midya et al. 2010] B. Midya, B. Roy, and R. Roychoudhury, "Position dependent mass Schrödinger equation and isospectral potentials: intertwining operator approach", *J. Math. Phys.* **51**:2 (2010), Art. ID #022109. MR 2011g:81089 Zbl 1309.81082

[Pushnitski and Rozenblum 2011] A. Pushnitski and G. Rozenblum, "On the spectrum of Bargmann–Toeplitz operators with symbols of a variable sign", *J. Anal. Math.* **114** (2011), 317–340. MR 2837088 Zbl 1253.47018

[Pushnitski et al. 2013] A. Pushnitski, G. Raikov, and C. Villegas-Blas, "Asymptotic density of eigenvalue clusters for the perturbed Landau Hamiltonian", *Comm. Math. Phys.* **320**:2 (2013), 425–453. MR 3053767 Zbl 1277.47059

[Raĭkov 1990] G. D. Raĭkov, "Eigenvalue asymptotics for the Schrödinger operator with homogeneous magnetic potential and decreasing electric potential, I: Behaviour near the essential spectrum tips", *Comm. Partial Differential Equations* **15**:3 (1990), 407–434. MR 91k:35188 Zbl 0739.35055

[Raikov 2015] G. D. Raikov, "Spectral asymptotics for waveguides with perturbed periodic twisting", preprint, 2015. To appear in *J. Spectral Theory*. arXiv 1407.0757

[Raikov and Warzel 2002] G. D. Raikov and S. Warzel, "Quasi-classical versus non-classical spectral asymptotics for magnetic Schrödinger operators with decreasing electric potentials", *Rev. Math. Phys.* **14**:10 (2002), 1051–1072. MR 2003m:81069 Zbl 1033.81038

[von Roos 1983] O. von Roos, "Position-dependent effective masses in semiconductor theory", *Phys. Rev. B* **27** (1983), 7547–7552.

[Rosenberg 1997] S. Rosenberg, *The Laplacian on a Riemannian manifold: an introduction to analysis on manifolds*, London Mathematical Society Student Texts **31**, Cambridge University Press, 1997. MR 98k:58206 Zbl 0868.58074

[Rozenblum and Tashchiyan 2008] G. Rozenblum and G. Tashchiyan, "On the spectral properties of the perturbed Landau Hamiltonian", *Comm. Partial Differential Equations* **33**:4–6 (2008), 1048–1081. MR 2009e:81073 Zbl 1158.47029

[Shubin 2001] M. A. Shubin, *Pseudodifferential operators and spectral theory*, 2nd ed., Springer, Berlin, 2001. MR 2002d:47073 Zbl 0980.35180

[de Souza Dutra and de Oliveira 2009] A. de Souza Dutra and J. A. de Oliveira, "Two-dimensional position-dependent massive particles in the presence of magnetic fields", *J. Phys. A* **42**:2 (2009), 025304, 13. MR 2010i:82200 Zbl 1154.82030

[Sternberg 1964] S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, N.J., 1964. MR 33 #1797 Zbl 0129.13102

Received 3 Dec 2014. Revised 30 Jan 2015. Accepted 15 Apr 2015.

TOMÁS LUNGENSTRASS: tlungens@mat.puc.cl
Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, 7820436 Santiago, Chile

GEORGI RAIKOV: graikov@mat.puc.cl
Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, 7820436 Santiago, Chile

ANALYSIS AND PDE
Vol. 8, No. 5, 2015
dx.doi.org/10.2140/apde.2015.8.1263

msp

HILBERT TRANSFORM ALONG MEASURABLE VECTOR FIELDS CONSTANT ON LIPSCHITZ CURVES: L^2 BOUNDEDNESS

SHAOMING GUO

We prove the L^2 boundedness of the directional Hilbert transform in the plane relative to measurable vector fields which are constant on suitable Lipschitz curves. One novelty of our proof lies in the definition of the adapted Littlewood–Paley projection (see Definition 3.3). The other novelty is that we will use Jones' beta numbers to control certain commutator in the critical Lipschitz regularity (see Lemma 5.5).

1. Introduction and statement of the main result

On $\mathbb{R}^2$, a direction is given by vector $v_u = (1, u)$, where $u \in \mathbb{R}$. Below, we will suppress the dependence of v upon u. Consider the directional Hilbert transform in the plane defined for a fixed direction $v = (1, u)$ as

$$H_v f(x, y) := \text{p.v.} \int_{\mathbb{R}} f(x - t, y - ut) \frac{dt}{t} \tag{1-1}$$

for any test function f. By the dilation symmetry, the length of the vector v is irrelevant for the value of H_v, which explains our normalization of the first component. By an application of Fubini's theorem and the L^p bounds for the classical Hilbert transform, one obtains a priori L^p bounds for H_v. On the other hand, the corresponding maximal operator $\sup_u |H_v f(x, y)|$ for varying directions is well known to not satisfy any a priori L^p bounds; see the work of Karagulyan [2007].

Bateman and Thiele [2013] proved that

$$\Big\| \sup_{u\in\mathbb{R}} \|H_v f(x, y)\|_{L^p(y)} \Big\|_{L^p(x)} \le C_p \|f\|_p \tag{1-2}$$

for the range $\frac{3}{2} < p < \infty$. Note that the supremum falls between the computation of the norm in y and in x, compared to being completely inside or outside as in the first paragraph. The case $p = 2$ of (1-2) goes back to Coifman and El Kohen (see page 1578 of [Bateman and Thiele 2013] for a detailed discussion), who noticed that a Fourier transform in the y direction makes (1-2) for $p = 2$ equivalent to L^2 bounds for Carleson's operator.

Estimate (1-2) highlights a biparameter structure of the directional Hilbert transform. The biparameter structure arises since the kernel is a tensor product between a Hilbert kernel in direction v and a Dirac delta distribution in the perpendicular direction.

MSC2010: 42B20, 42B25.
Keywords: singular integrals, differentiation theory, Jones' beta numbers, Littlewood–Paley theory on Lipschitz curves, Carleson embedding theorem.

If one considers the linearized maximal operator

$$H_v f(x, y) := \text{p.v.} \int f(x-t, y-u(x, y)t) \, \frac{dt}{t} \tag{1-3}$$

for some function u, then inequality (1-2) can be rephrased as a bound for the linearized maximal operator under the assumption that u is constant on every vertical line $x = x_0$ for all $x_0 \in \mathbb{R}$. Such vector fields v of the form $(1, u(x_0))$ for some measurable function $u : \mathbb{R} \to \mathbb{R}$ are called one-variable vector fields in [Bateman and Thiele 2013].

The purpose of the present paper is to relax this rigid assumption on u, and prove an analogue of (1-2) for vector fields which are constant along suitable families of Lipschitz curves. To formulate such a result, we perturb (1-2) by a bi-Lipschitz horizontal distortion, that is,

$$(x, y) \mapsto (g(x, y), y) \tag{1-4}$$

with

$$(x'-x)/a_0 \le g(x', y) - g(x, y) \le a_0(x'-x) \tag{1-5}$$

for every $x < x'$ and every y, so that the transformation (1-4) maps vertical lines to near vertical Lipschitz curves:

$$|g(x, y) - g(x, y')| \le b_0|y'-y| \tag{1-6}$$

for all x, y, y'. These two conditions can be rephrased as

$$1/a_0 \le \partial_1 g \le a_0 \quad \text{and} \quad |\partial_2 g| \le b_0 \quad \text{a.e.} \tag{1-7}$$

Under these assumptions, L^p norms are distorted boundedly under the transformation (1-4). Namely, (1-5) implies for every y that

$$a_0^{-1}\|f(x, y)\|_{L^p(x)}^p \le \|f(g(x, y), y)\|_{L^p(x)}^p \le a_0\|f(x, y)\|_{L^p(x)}^p \tag{1-8}$$

and we may integrate this in the y direction to obtain equivalence of L^p norms in the plane. Hence the change of measure is not the main point of the following theorem, but rather the effect of the transformation on the linearizing function u, which is now constant along the family of Lipschitz curves which are the images of the lines $x = x_0$ under the map (1-4).

Theorem 1.1 (main theorem). *Let $g : \mathbb{R}^2 \to \mathbb{R}$ satisfy assumption (1-5) for some a_0 and assumption (1-6) for some b_0. Then, for any $c_0 \in (0, 1)$, we have*

$$\Big\| \sup_{|u| \le c_0/b_0} \|H_v f(g(x, y), y)\|_{L^2(y)} \Big\|_{L^2(x)} \le C\|f\|_2. \tag{1-9}$$

Here C is a constant depending only on a_0 and c_0.

Remark 1.2. The constant C is independent of b_0 due to the anisotropic scaling symmetry $x \mapsto x$, $y \mapsto \lambda y$.

In view of the implicit function theorem (see [Azzam and Schul 2012] for recent developments), our result covers a large class of vector fields which are of the critical Lipschitz regularity. Indeed, it implies the following:

Corollary 1.3. *For a measurable unit vector field* $v_0 : \mathbb{R}^2 \to S^1$, *suppose that*:

(i) *there exists a bi-Lipschitz map* $g_0 : \mathbb{R}^2 \to \mathbb{R}^2$ *such that*

$$v_0(g_0(x, y)) \text{ is constant in } y; \tag{1-10}$$

(ii) *there exists* $d_0 > 0$ *such that, for all* $x \in \mathbb{R}$,

$$\angle\big(\partial_2 g_0(x, y), \pm v_0(g_0(x, y))\big) \geq d_0 \quad y\text{-a.e. in } \mathbb{R}. \tag{1-11}$$

Then the associated Hilbert transform, which is defined as

$$H_{v_0} f(x, y) := \int_{\mathbb{R}} f((x, y) - t v_0(x, y)) \,\frac{dt}{t}, \tag{1-12}$$

is bounded in L^2, *with the operator norm depending only on* d_0 *and the bi-Lipschitz norm of* g_0.

Remark 1.4. The structure theorem for Lipschitz functions by Azzam and Schul [2012] states exactly that any Lipschitz function $u : \mathbb{R}^2 \to \mathbb{R}$ (any Lipschitz unit vector field v_0 in our case) can be precomposed with a bi-Lipschitz function $g_0 : \mathbb{R}^2 \to \mathbb{R}^2$ so that $u \circ g_0$ is Lipschitz in the first coordinate and constant in the second coordinate, when restricted to a "large" portion of the domain.

Remark 1.5. Without the assumption that $d_0 > 0$, the operator H_{v_0} might be unbounded in L^p for any $p > 1$. The counterexample is based on the Besicovitch–Kakeya set construction, which will be discussed at the end of the proof of the corollary.

To our knowledge, this is the first result in the context of the directional Hilbert transform with a Lipschitz assumption in the hypothesis. Lipschitz regularity is critical for the directional Hilbert transform, as we will elaborate shortly.

To use the assumption that v is constant along Lipschitz curves, we apply an adapted Littlewood–Paley theory along the level lines of v. This is a refinement of the analysis of Coifman and El Kohen, who use a Fourier transform in the y variable and the analysis of Bateman and Thiele [2013], who use a classical Littlewood–Paley theory in the y variable. This adapted Littlewood–Paley theory is the main novelty of the present paper. It is in the spirit of prior work on the Cauchy integral on Lipschitz curves, for example [Coifman et al. 1989], but it differs from this classical theme in that it is more of biparameter type as it is governed by a whole fibration into Lipschitz curves. We crucially use Jones' beta numbers as a tool to control the adapted Littlewood–Paley theory. To our knowledge this is also the first use of Jones' beta numbers in the context of the directional Hilbert transform.

In this paper we focus on the case L^2, since our goal here is to highlight the use of the adapted Littlewood–Paley theory and Jones' beta numbers in the technically most simple case. We expect to address the more general case L^p with a range of p, as in the Bateman–Thiele theorem, in forthcoming work.

While Coifman and El Kohen use the difficult bounds on Carleson's operator as a black box, Bateman and Thiele [2013] have to unravel this black box following the work of Lacey and Li [2006; 2010] and use time-frequency analysis to prove bounds for a suitable generalization of Carleson's operator. Luckily, in the present work we do not have to delve into time-frequency analysis as we can largely recycle the work of Bateman and Thiele for this aspect of the argument.

An upper bound such as $|u| \le c_0/b_0$ is necessary in our theorem. By a limiting argument we may recover the theorem of Bateman and Thiele, using the scaling to tighten the Lipschitz constant b_0 at the same time as relaxing the condition $|u| \le c_0/b_0$.

An interesting open question remains whether the same holds true for $c_0 = 1$. We do not know of a soft argument to achieve this relaxation. Our estimate of the norms become unbounded as c_0 approaches 1. This question suggests itself for further study.

Part of our motivation is a long history of studies of the linearized maximal operator (1-3) under various assumptions on the linearizing function u. If one truncates (1-3) as

$$H_{v,\epsilon_0} f(x, y) := \text{p.v.} \int_{-\epsilon_0}^{\epsilon_0} f(x-t, y-u(x, y)t)\, \frac{dt}{t},$$

then it is reasonable to ask for pure regularity assumptions on u to obtain boundedness of H_{v,ϵ_0}. It is known that Lipschitz regularity of u is critical, since a counterexample in [Lacey and Li 2010] based on a construction of the Besicovitch–Kakeya set shows that no bounds are possible for C^α regularity with $\alpha < 1$. However, it remains open whether Lipschitz regularity suffices for bounds for H_{v,ϵ_0}. On the regularity scale, the only known result is for real analytic vector fields v by Stein and Street [2012]. A prior partial result in this direction appears in [Christ et al. 1999].

It is our hope that our result corners some of the difficulties of approaching Lipschitz regularity in the classical problem. Further substantial progress (including the case $c_0 = 1$) is likely to use Lipschitz regularity not only of the level curves of u but also of u itself across the level curves. For example, one possibility would be to cut the plane into different pieces by the theorem of Azzam and Schul stated in Remark 1.4, and to analyze each piece separately using Theorem 1.1. We leave this for future study.

Related to the directional Hilbert transform, and thus additional motivation for the present work, is the directional maximal operator

$$M_{v,\epsilon_0} f(x) := \sup_{0<\epsilon<\epsilon_0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |f(x-t, y-u(x, y)t)|\, dt, \tag{1-13}$$

which arises for example in Lebesgue-type differentiation questions and has an even longer history of interest than the directional Hilbert transform. Hilbert transforms and maximal operators share many features; in particular, they have the same scaling and thus share the same potential L^p bounds. The maximal operator is in some ways easier as it is positive and does not have a singular kernel. For example, bounds for the maximal operator under the assumption of real analytic vector fields were proved much earlier by Bourgain [1989].

An instance of bounds satisfied by the maximal operator but not the Hilbert transform arises when one restricts the range of the function u instead of the regularity. For certain sets of directions characterized by Bateman [2009a] there are bounds for the maximal operator (for example for the set of lacunary directions), while Karagulyan [2007] proves that no such bounds are possible for the Hilbert transform.

On the other hand, the Hilbert transform is easier in some other aspects; most notably it is a linear operator. For example, bounds for the bilinear Hilbert transform mapping into L^1 were known [Lacey and Thiele 1997; 1998] before the corresponding maximal operator bounds [Lacey 2000], due to the fact that orthogonality between different tiles is preserved under the Hilbert transform but not the maximal operator. In particular we do not know at the moment whether the analogue of our main theorem holds for the directional maximal operator. This may be an interesting subject for further investigation.

Outline of paper. In Section 2 we will prove Corollary 1.3 by reducing it to the main theorem. The reduction will also be used later in the proof of the main theorem.

In Section 3 we will state the strategy of the proof for the main theorem. As it appears that our result is a Lipschitz perturbation of the one by Bateman and Thiele, this turns out also to be the case for the proof: if we denote by P_k a Littlewood–Paley operator in the y-variable, the main observation in Bateman and Thiele's proof is that H_v commutes with P_k. In our case, this is no longer true. However, we can make use of an adapted version of the Littlewood–Paley projection operator $\tilde{P}_k$ (see Definition 3.3) to partially recover the orthogonality. We split the operator H_v into a main term and a commutator term

$$\sum_{k\in\mathbb{Z}} H_v P_k(f) = \sum_{k\in\mathbb{Z}} (H_v P_k(f) - \tilde{P}_k H_v P_k(f) + \tilde{P}_k H_v P_k(f)). \tag{1-14}$$

The boundedness of the main term $\sum_{k\in\mathbb{Z}} \tilde{P}_k H_v P_k(f)$ is essentially due to Lacey and Li [2010], with conditionality on certain maximal operator estimate. In Section 4 we modify Bateman's argument [2009b; 2013] to the case of vector fields constant on Lipschitz curves and remove the conditionality on that maximal operator.

The main novelty is the boundedness of the commutator term

$$\sum_{k\in\mathbb{Z}} (H_v P_k(f) - \tilde{P}_k H_v P_k(f)), \tag{1-15}$$

which will be presented in Section 5. To achieve this, we will view Lipschitz curves as perturbations of straight lines and use Jones' beta number condition for Lipschitz curves and the Carleson embedding theorem to control the commutator. Here we shall emphasize again that the commutator estimate is free of time-frequency analysis.

Notations. Throughout this paper, we will write $x \ll y$ to mean that $x \le y/10$, $x \lesssim y$ to mean that there exists a universal constant C such that $x \le Cy$, and $x \sim y$ to mean that $x \lesssim y$ and $y \lesssim x$. Lastly, $\mathbb{1}_E$ will always denote the characteristic function of the set E.

2. Proof of Corollary 1.3

In this section we prove Corollary 1.3, by reducing it to the main theorem. The reduction is based on a cutting and pasting argument. Some parts of the reduction will also be used in the proof of the main theorem in the rest of the paper.

We first divide the unit circle S^1 into N arcs of equal length, with the angle of each arc being $2\pi/N$. Choose

$$N > 6\pi/d_0, \tag{2-1}$$

so that $2\pi/N < d_0/3$. Denote these arcs as $\Omega_1, \Omega_2, \ldots, \Omega_N$. For each Ω_i, define

$$H_{v_0,\Omega_i} f(x, y) := \begin{cases} H_{v_0} f(x, y) & \text{if } v_0(x, y) \in \Omega_i, \\ 0 & \text{else.} \end{cases}$$

If we were able to prove that $\|H_{v_0,\Omega_i}\|_{2\to 2}$ is bounded by a constant C which is independent of $i \in \{1, 2, \ldots, N\}$, then we could conclude that

$$\|H_{v_0}\|_{2\to 2} \le CN(d_0). \tag{2-2}$$

Now fix one Ω_i; we want to show the boundedness of H_{v_0,Ω_i}. Choose a new coordinate so that the x-axis passes through Ω_i and bisects it. Then all the vectors in Ω_i form an angle less than $d_0/6$ with the x-axis. As we assume that

$$\angle(\partial_2 g_0, \pm v_0(g_0)) \ge d_0 > 0, \tag{2-3}$$

we see that the vector $\partial_2 g_0$ forms an angle less than $(\pi - d_0)/2$ with the y-axis.

Renormalize the unit vector v_0 so that the first component is 1, and write $v_0 = (1, u_0)$; then, by the fact that v_0 forms an angle less than $d_0/6$ with the x-axis, we obtain

$$|u_0| \le \tan(d_0/6). \tag{2-4}$$

Next we construct the Lipschitz function g in the main theorem from the bi-Lipschitz map g_0, and the coordinate we will use here is still the one associated to Ω_i as above. Under this linear change of variables, we know that g_0 is still bi-Lipschitz. We renormalize the bi-Lipschitz map in such a way that

$$g_0(x, 0) = (x, 0) \quad \text{for all } x \in \mathbb{R}. \tag{2-5}$$

Fix $x \in \mathbb{R}$, the map g_0, when restricted on the vertical line $\{(x, y) : y \in \mathbb{R}\}$, is still bi-Lipschitz. We denote by Γ_x the image of this bi-Lipschitz map, i.e.,

$$\Gamma_x := \{g_0(x, y) : y \in \mathbb{R}\}. \tag{2-6}$$

Define the function g by the relation

$$(g(x, y), y) = g_0(x, y\prime), \tag{2-7}$$

for some $y\prime$. By the fact that g_0 is bi-Lipschitz, we know that such $y\prime$ exists and is unique.

From the above construction and the fact that $\partial_2 g_0$ forms an angle less than $(\pi - d_0)/2$ with the y-axis, we see easily that

$$|g(x, y_1) - g(x, y_2)| \le \cot(d_0/2)|y_1 - y_2| \quad \text{for all } x, y_1, y_2 \in \mathbb{R}. \tag{2-8}$$

Hence, it remains to show that condition (1-5) is also satisfied with a constant a_0 depending only on d_0 and the bi-Lipschitz constant of g_0. One side of the equivalence, $(x_1 - x_2)/a_0 \le g(x_1, y) - g(x_2, y)$, is

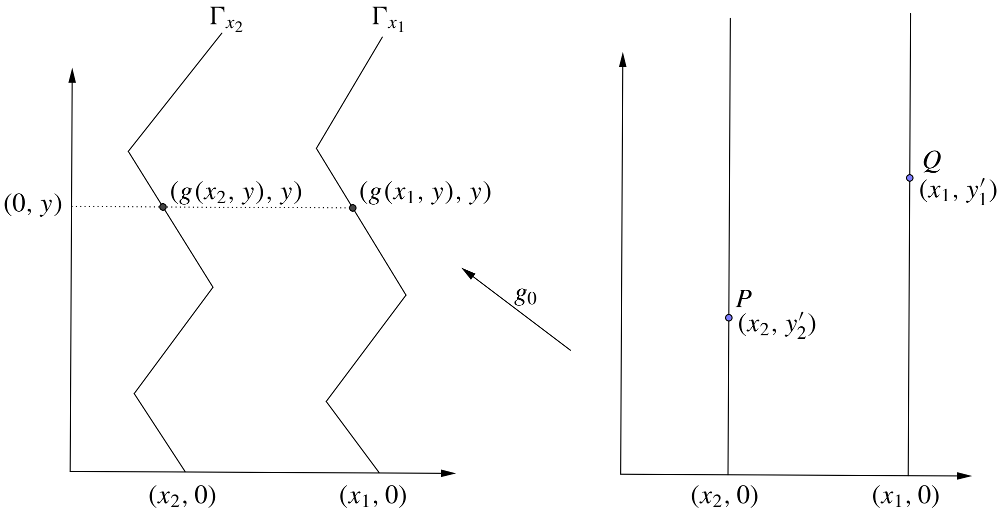

Figure 1. Illustration of the bi-Lipschitz map g_0.

quite clear from Figure 1: the bi-Lipschitz map g_0 sends the points P, Q to $(g(x_1, y), y)$, $(g(x_2, y), y)$ separately, then, by definition of a bi-Lipschitz map, there exists constant a_0 such that

$$g(x_1, y) - g(x_2, y) \geq \frac{1}{a_0}|P - Q| \geq \frac{1}{a_0}(x_1 - x_2). \tag{2-9}$$

For the other side, we argue by contradiction. If, for any $M \in \mathbb{N}$ large, there exists $x_1, x_2, y \in \mathbb{R}$ such that

$$g(x_1, y) - g(x_2, y) \geq M(x_1 - x_2), \tag{2-10}$$

then, together with (2-8), this implies that

$$\operatorname{dist}(K, \Gamma_{x_1}) \geq M \sin(d_0/2)(x_1 - x_2). \tag{2-11}$$

But this is not allowed as, by the definition of the bi-Lipschitz map g_0 and the Lipschitz function g, $\operatorname{dist}(K, \Gamma_{x_1})$ must be comparable to $|x_1 - x_2|$.

So far, we have verified all the conditions in the main theorem with

$$b_0 = \cot(d_0/2) \quad \text{and} \quad c_0 = \frac{\tan(d_0/6)}{\cot(d_0/2)} < 1. \tag{2-12}$$

Hence we can apply the main theorem to obtain the boundedness of H_{v_0,Ω_i}.

In the end, as claimed in the corollary, we still need to show that the operator norm in L^p (for all $p > 1$) blows up without the assumption that $d_0 > 0$. For the range $p \leq 2$, the counterexample is simply a Knapp example: let $B_1(0)$ denote the ball of radius one centered at origin, take the function $f(x) = \mathbb{1}_{B_1(0)}(x)$, let Γ be the upper cone which forms an angle less than $\pi/4$ with the vertical axis. First define the vector field $v(x) = x/|x|$ for $x \in \Gamma \setminus B_1(0)$, then extend the definition to the whole plane properly such that v

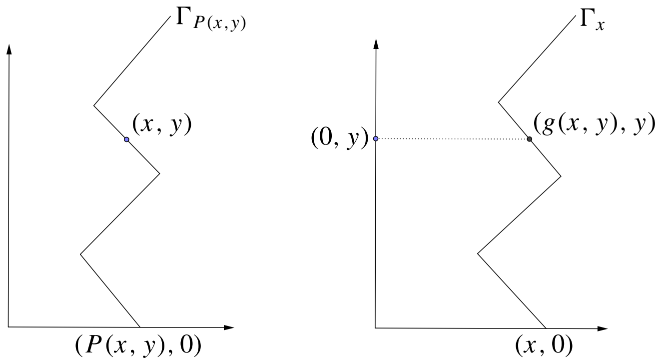

Figure 2. The projection $P(x, y)$.

satisfies the condition (1-10). It is then easy to see that

$$|H_v f(x)| \sim \frac{1}{|x|} \quad \text{for all } x \in \Gamma \setminus B_1(0), \tag{2-13}$$

which does not belong to $L^p(\mathbb{R}^2)$ for $p \le 2$. For the range $p > 2$, the counterexample is given by the standard Besicovitch–Kakeya set construction, which can be found in [Bateman 2013, page 1022] and [Lacey and Li 2010, page 7].

3. Strategy of the proof of the main theorem

If we linearize the maximal operator in the main theorem, what we need to prove turns to be the following

$$\left\| \int_{\mathbb{R}} f(g(x, y) - t, y - tu(x)) \frac{dt}{t} \right\|_2 \lesssim \|f\|_2, \tag{3-1}$$

where $u : \mathbb{R} \to \mathbb{R}$ is a measurable function such that $\|u\|_\infty \le c_0/b_0$. The change of coordinates

$$(x, y) \mapsto (g(x, y), y) \tag{3-2}$$

in (1-4) also changes the measure on the plane. However, we still want to use the original Lebesgue measure for the Littlewood–Paley decomposition. Hence we invert (1-4) and denote the inversion by

$$(x, y) \mapsto (P(x, y), y), \tag{3-3}$$

where "P" stands for "projection". Figure 2 illustrates why we call the map (3-3) a projection.

The change of coordinates in (3-3) turns the estimate (3-1) into the equivalent form

$$\left\| \int_{\mathbb{R}} f(x - t, y - tu(P(x, y))) \frac{dt}{t} \right\|_2 \lesssim \|f\|_2. \tag{3-4}$$

Moreover, we will denote

$$H_v f(x, y) := \int_{\mathbb{R}} f(x - t, y - tu(P(x, y))) \frac{dt}{t}. \tag{3-5}$$

In the rest of the paper, we want to make the convention that whenever H_v appears it denotes the Hilbert transform along the vector field $v(x, y) = (1, u(P(x, y)))$, that is, the above (3-5), to distinguish it from the various H_v that have appeared in the introduction.

To prove the above estimate, we first make several reductions: by the anisotropic scaling

$$x \mapsto x, \quad y \mapsto \lambda y, \tag{3-6}$$

we can without loss of generality assume that $b_0 = 10^{-2}$. By a similar cutting and pasting argument to that in the proof of Corollary 1.3, we can assume that $c_0 \ll 10^{-2}$, that is, the vector field v is of the form $(1, u)$ with $|u| \ll 1$.

Now we start the proof. It was already observed in [Bateman 2013, page 1024] that, under the assumption $|u| \ll 1$, we can without loss of generality assume that supp $\hat{f}$ lies in a two-ended cone which forms an angle less than $\pi/4$ with the vertical axis, as, for functions f with frequency supported on the cone near the horizontal axis, we have that

$$H_v f(x, y) = H_{(1,0)} f(x, y), \tag{3-7}$$

which is the Hilbert transform along the constant vector field $(1, 0)$. But $H_{(1,0)}$ is bounded by Fubini's theorem and the L^2 boundedness of the Hilbert transform.

For the frequencies outside the cone near the horizontal axis, the proof consists of two steps. In the first step we will prove the boundedness of H_v when acting on functions with frequency supported in a single annulus. To be precise, let Γ be the cone which forms an angle less than $\pi/4$ with the vertical axis and Π_Γ be the projection operator on Γ, i.e.,

$$\Pi_\Gamma f := \mathscr{F}^{-1} \mathbb{1}_\Gamma \mathscr{F} f, \tag{3-8}$$

where $\mathscr{F}$ stands for the Fourier transform and $\mathscr{F}^{-1}$ the inverse transform. Let P_k be the k-th Littlewood–Paley projection operator in the vertical direction; as we are always concerned with the frequency in Γ, later for simplicity we will just write P_k instead of $P_k \Pi_\Gamma$ for short. Then what we will prove first is:

Proposition 3.1. *Under the same assumptions as in the main theorem, we have for $p \in (1, \infty)$ that*

$$\|H_v P_k(f)\|_p \lesssim \|P_k(f)\|_p, \tag{3-9}$$

with the constant being independent of $k \in \mathbb{Z}$.

In order to prove the boundedness of H_v, we need to put all the frequency pieces together. In the case of $C^{1+\alpha}$ vector fields for any $\alpha > 0$, Lacey and Li's idea [2010] is to prove the almost orthogonality between different frequency annuli. In the case where the vector field is constant along vertical lines, an important observation in [Bateman and Thiele 2013] is that H_v and P_k commute, which then makes it possible to apply a Littlewood–Paley square function estimate.

In our case, Bateman and Thiele's observation is no longer true. We need to take into account that the vector field is constant along Lipschitz curves, which gives rise to an adapted Littlewood–Paley projection operator (Definition 3.3).

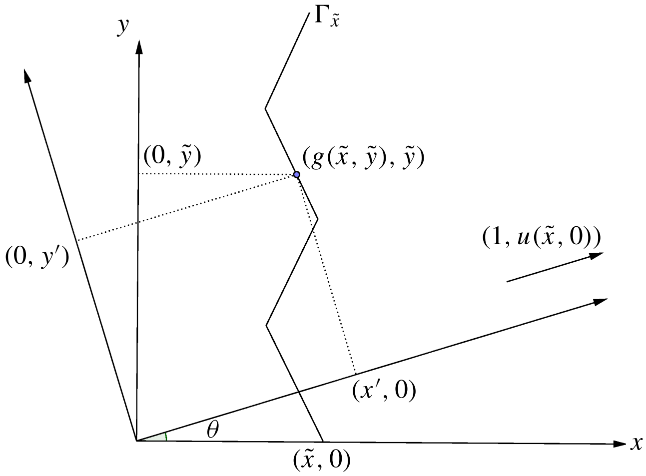

Figure 3. The setting of Lemma 3.2.

Before defining this operator, we first need to make some preparation. Fix one $\tilde{x} \in \mathbb{R}$, take the curve $\Gamma_{\tilde{x}}$ which passes through $(\tilde{x}, 0)$; recall that $\Gamma_{\tilde{x}}$ is given by the set $\{(g(\tilde{x}, \tilde{y}), \tilde{y}) : \tilde{y} \in \mathbb{R}\}$, where g is the Lipschitz function in the main theorem. By the definition of the operator H_v we know that the vector field v is equal to the constant vector $(1, u(\tilde{x}))$ along $\Gamma_{\tilde{x}}$. Change the coordinate so that the horizontal x'-axis is parallel to $(1, u(\tilde{x}))$. The following lemma says that, in the new coordinate, the curve $\Gamma_{\tilde{x}}$ can still be realized as the graph of a Lipschitz function.

Lemma 3.2. *For any fixed $\tilde{x} \in \mathbb{R}$, there exists a Lipschitz function $x' = g_{\tilde{x}}(y')$ such that $\Gamma_{\tilde{x}}$ can be reparametrized as $\{(g_{\tilde{x}}(y'), y') : y' \in \mathbb{R}\}$. Moreover, we have that $\|g_{\tilde{x}}\|_{\mathrm{Lip}} \le (1+b_0)/(1-b_0)$, where b_0 is the constant in the main theorem.*

Proof. Denote by θ the angle between the vector $(1, u(\tilde{x}))$ and the x-axis as in Figure 3.

The new coordinate of the point $(g(\tilde{x}, \tilde{y}), \tilde{y})$ will be given by

$$(x', y') = \left(\tilde{y}\sin\theta + g(\tilde{x}, \tilde{y})\frac{1+\sin^2\theta}{\cos\theta}, \tilde{y}\cos\theta - g(\tilde{x}, \tilde{y})\sin\theta\right). \tag{3-10}$$

Looking at the identity for the second component,

$$y' = \tilde{y}\cos\theta - g(\tilde{x}, \tilde{y})\sin\theta, \tag{3-11}$$

we want to solve $\tilde{y}$ in terms of y' by using the implicit function theorem. As

$$\frac{dy'}{d\tilde{y}} = \cos\theta - \frac{\partial g}{\partial \tilde{y}}\sin\theta, \tag{3-12}$$

by the fact that $|u| \ll 1$ and $|\partial g/\partial \tilde{y}| \le b_0 \le 10^{-2}$ we obtain that

$$\frac{1-b_0}{\sqrt{2}} \le \frac{dy'}{d\tilde{y}} \le \frac{1+b_0}{\sqrt{2}}, \tag{3-13}$$

from which it is clear that the implicit function theorem is applicable.

After solving $\tilde{y}$ in terms of y', we just need to substitute $\tilde{y}$ into the identity for the first component in (3-10), which is

$$x' = \tilde{y}\sin\theta + g(\tilde{x}, \tilde{y})\frac{1+\sin^2\theta}{\cos\theta}, \tag{3-14}$$

to get an implicit expression of x' in terms of y', which we will denote as $x' = g_{\tilde{x}}(y')$.

To estimate the Lipschitz norm of the function $g_{\tilde{x}}$, we just need to observe that, when doing the above change of variables, we have rotated the axis by an angle θ which satisfies $|\theta| \le \pi/4$. Together with the fact that $|\partial g/\partial\tilde{y}| \le b_0$, we can then derive that

$$\left|\frac{\partial g_{\tilde{x}}}{\partial y'}\right| \le \frac{1+b_0}{1-b_0}, \tag{3-15}$$

which finishes the proof of Lemma 3.2. ☐

Definition 3.3 (adapted Littlewood–Paley projection). Select a Schwartz function ψ_0 with support on $\left[\frac{1}{2}, \frac{5}{2}\right] \cup \left[-\frac{5}{2}, -\frac{1}{2}\right]$ such that

$$\sum_{k\in\mathbb{Z}} \psi_0(2^{-k}t) = 1 \quad \text{for all } t \ne 0. \tag{3-16}$$

For $f : \mathbb{R}^2 \to \mathbb{R}$ and for every fixed $\tilde{x} \in \mathbb{R}$, define the adapted (one-dimensional) Littlewood–Paley projection on $\Gamma_{\tilde{x}}$ by

$$\tilde{P}_k(f)(x', y') := \int_{\mathbb{R}} f(g_{\tilde{x}}(z), z)\check{\psi}_k(y'-z)\,dz = P_k(\tilde{f})(y'), \tag{3-17}$$

where $(x', y') = (g_{\tilde{x}}(y'), y')$ denotes one point in $\Gamma_{\tilde{x}}$, $\psi_k(\cdot) := \psi_0(2^{-k}\cdot)$ and we use $\tilde{f}(\cdot)$ to denote the function $f(g_{\tilde{x}}(\cdot), \cdot)$, and P_k the one-dimensional Littlewood–Paley projection operator.

Now it is instructive to regard the Lipschitz curves as perturbations of the straight lines, or, equivalently, to think that $H_v P_k f$ still has frequency supported near the k-th frequency band, which has already been used by Lacey and Li [2010] in their almost orthogonality estimate for $C^{1+\alpha}$ vector fields. We then subtract the term $\tilde{P}_k H_v P_k(f)$ from $H_v P_k(f)$, and estimate the commutator.

To be precise, we first write

$$\sum_k H_v P_k(f) = \sum_k (H_v P_k(f) - \tilde{P}_k H_v P_k(f) + \tilde{P}_k H_v P_k(f)), \tag{3-18}$$

then, by the triangle inequality, we have

$$\left\|\sum_k H_v P_k(f)\right\|_2 \lesssim \left\|\sum_k (H_v P_k(f) - \tilde{P}_k H_v P_k(f))\right\|_2 + \left\|\sum_k \tilde{P}_k H_v P_k(f)\right\|_2. \tag{3-19}$$

We call the second term the main term, and the first term the commutator term. The L^2 boundedness of the main term will follow from an orthogonality argument, which is an adapted Littlewood–Paley theorem:

Lemma 3.4. *For $p \in (1, +\infty)$, we have the following variants of the Littlewood–Paley estimates*:

$$\Big\| \Big(\sum_{k \in \mathbb{Z}} |\tilde{P}_k(f)|^2 \Big)^{\frac{1}{2}} \Big\|_p \sim \|f\|_p, \tag{3-20}$$

$$\Big\| \Big(\sum_{k \in \mathbb{Z}} |\tilde{P}_k^*(f)|^2 \Big)^{\frac{1}{2}} \Big\|_p \sim \|f\|_p, \tag{3-21}$$

with constants depending only on a_0.

Proof. In (1-8) from the introduction, we have already explained the coarea formula

$$\int_{\mathbb{R}^2} |f(x, y)|\, dx\, dy \sim \int_{\mathbb{R}} \Big[\int_{\Gamma_{\tilde{x}}} |f|\, ds_{\tilde{x}} \Big] d\tilde{x}. \tag{3-22}$$

We apply this formula to the left-hand side of (3-20) to obtain

$$\Big\| \Big(\sum_{k \in \mathbb{Z}} |\tilde{P}_k(f)|^2 \Big)^{\frac{1}{2}} \Big\|_p^p \sim \int_{\mathbb{R}} \int_{\Gamma_{\tilde{x}}} \Big(\sum_{k \in \mathbb{Z}} |\tilde{P}_k(f)|^2 \Big)^{\frac{p}{2}} ds_{\tilde{x}}\, d\tilde{x}. \tag{3-23}$$

For every fixed $\tilde{x}$, by Definition 3.3, the right-hand side of (3-23) becomes

$$\int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \Big(\sum_k |P_k(\tilde{f}_{\tilde{x}})(y')|^2 \Big)^{\frac{p}{2}} dy' \Big] d\tilde{x}, \tag{3-24}$$

where $\tilde{f}_{\tilde{x}}(y') = f(g_{\tilde{x}}(y'), y')$. Then the classical Littlewood–Paley theory applies and we can bound the last expression by

$$\int_{\mathbb{R}} \|f\|_{L^p(\Gamma_{\tilde{x}})}^p\, d\tilde{x} \lesssim \|f\|_{L^p}^p. \tag{3-25}$$

For the boundedness of the adjoint operator, it suffices to prove that

$$\sum_{k \in \mathbb{Z}} \langle \tilde{P}_k^*(f), f_k \rangle \lesssim \|f\|_{L^p} \Big\| \Big(\sum_{k \in \mathbb{Z}} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p'}}. \tag{3-26}$$

First, by linearity and Hölder's inequality, we derive

$$\sum_{k \in \mathbb{Z}} \langle \tilde{P}_k^*(f), f_k \rangle = \Big\langle f, \sum_{k \in \mathbb{Z}} \tilde{P}_k(f_k) \Big\rangle \lesssim \|f\|_{L^p} \Big\| \sum_{k \in \mathbb{Z}} \tilde{P}_k(f_k) \Big\|_{L^{p'}}. \tag{3-27}$$

Applying the coarea formula (3-22), we obtain

$$\Big\| \sum_{k \in \mathbb{Z}} \tilde{P}_k(f_k) \Big\|_{L^{p'}} \sim \Big(\int_{\mathbb{R}} \Big(\int_{\Gamma_{\tilde{x}}} \Big| \sum_{k \in \mathbb{Z}} \tilde{P}_k(f_k) \Big|^{p'} ds_{\tilde{x}} \Big) d\tilde{x} \Big)^{\frac{1}{p'}}. \tag{3-28}$$

By Definition 3.3, for every fixed $\tilde{x} \in \mathbb{R}$, the inner integration in the last expression becomes

$$\int_{\mathbb{R}} \Big| \sum_{k \in \mathbb{Z}} P_k(\tilde{f}_{k,\tilde{x}})(y') \Big|^{p'} dy', \tag{3-29}$$

where $\tilde{f}_{k,\tilde{x}}(y') := f_k(g_{\tilde{x}}(y'), y')$. Now the classical Littlewood–Paley theory applies and we bound the term in (3-29) by

$$\int_{\mathbb{R}} \Big(\sum_{k\in\mathbb{Z}} |\tilde{f}_{k,\tilde{x}}(y')|^2 \Big)^{\frac{p'}{2}} dy' \lesssim \int_{\Gamma_{\tilde{x}}} \Big(\sum_{k\in\mathbb{Z}} |f_k|^2 \Big)^{\frac{p'}{2}} ds_{\tilde{x}} \lesssim \Big\| \Big(\sum_{k\in\mathbb{Z}} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|^{p'}_{L^{p'}(\Gamma_{\tilde{x}})}. \tag{3-30}$$

Then, to prove (3-26), we just need to integrate $d\tilde{x}$ in (3-30) and apply the coarea formula (3-22) to derive

$$\Big\| \sum_{k\in\mathbb{Z}} \tilde{P}_k(f_k) \Big\|_{L^{p'}} \lesssim \Big(\int_{\mathbb{R}} \Big\| \Big(\sum_{k\in\mathbb{Z}} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|^{p'}_{L^{p'}(\Gamma_{\tilde{x}})} d\tilde{x} \Big)^{\frac{1}{p'}} \lesssim \Big\| \Big(\sum_{k\in\mathbb{Z}} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p'}}.$$

Thus we have finished the proof of Lemma 3.4. □

Now we will show how to prove the L^2 boundedness of the main term using Lemma 3.4 and Proposition 3.1: first by duality, we have

$$\Big\| \sum_k \tilde{P}_k H_v P_k(f) \Big\|_2 = \sup_{\|g\|_2=1} \Big| \Big\langle \sum_k \tilde{P}_k H_v P_k(f), g \Big\rangle \Big| = \sup_{\|g\|_2=1} \Big| \Big\langle \sum_k H_v P_k(f), \tilde{P}_k^*(g) \Big\rangle \Big|.$$

Applying the Cauchy–Schwartz inequality and Hölder's inequality, we can bound the last term by

$$\sup_{\|g\|_2=1} \Big\| \Big(\sum_k |H_v P_k(f)|^2 \Big)^{\frac{1}{2}} \Big\|_2 \Big\| \Big(\sum_k |\tilde{P}_k^*(g)|^2 \Big)^{\frac{1}{2}} \Big\|_2. \tag{3-31}$$

For the former term, Proposition 3.1 implies that

$$\Big\| \Big(\sum_k |H_v P_k(f)|^2 \Big)^{\frac{1}{2}} \Big\|_2 \le \Big(\sum_{k\in Z} \|H_v P_k(f)\|_2^2 \Big)^{\frac{1}{2}} \lesssim \Big(\sum_{k\in\mathbb{Z}} \|P_k(f)\|_2^2 \Big)^{\frac{1}{2}} \lesssim \|f\|_2.$$

For the latter term, Lemma 3.4 implies that

$$\Big\| \Big(\sum_k |\tilde{P}_k^*(g)|^2 \Big)^{\frac{1}{2}} \Big\|_2 \lesssim \|g\|_2. \tag{3-32}$$

Thus we have proved the L^2 boundedness the main term, modulo Proposition 3.1.

As the second step, we will prove the L^2 boundedness of the commutator, which is

$$\Big\| \sum_k (H_v P_k(f) - \tilde{P}_k H_v P_k(f)) \Big\|_2 \lesssim \|f\|_2. \tag{3-33}$$

To do this, we first split the operator H_v into a dyadic sum: Select a Schwartz function ψ_0 such that ψ_0 is supported on $\big[\frac{1}{2}, \frac{5}{2}\big]$, let

$$\psi_l(t) := \psi_0(2^{-l}t); \tag{3-34}$$

by choosing ψ_0 properly, we can construct a partition of unity for $\mathbb{R}^+$, i.e.,

$$\mathbb{1}_{(0,\infty)} = \sum_{l\in\mathbb{Z}} \psi_l. \tag{3-35}$$

Let

$$H_l h(x, y) := \int \check{\psi}_l(t) h(x-t, y-tu(P(x, y)))\, dt; \tag{3-36}$$

then the operator H_v can be decomposed into the sum

$$H_v = -\mathbb{1} + 2\sum_{l\in\mathbb{Z}} H_l. \tag{3-37}$$

Hence, to bound the commutator, it is equivalent to bound

$$\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}(H_l P_k f - \tilde{P}_k H_l P_k f). \tag{3-38}$$

Notice that, by definition, $H_l P_k f$ vanishes for $l > k$, which simplifies the last expression to

$$\sum_{l\geq 0}\sum_{k\in\mathbb{Z}}(H_{k-l} P_k f - \tilde{P}_k H_{k-l} P_k f). \tag{3-39}$$

By the triangle inequality, it suffices to prove:

Proposition 3.5. *Under the same assumption as in the main theorem, there exists $\gamma > 0$ such that*

$$\Big\|\sum_{k\in\mathbb{Z}}(H_{k-l} P_k f - \tilde{P}_k H_{k-l} P_k f)\Big\|_2 \lesssim 2^{-\gamma l}\|f\|_2, \tag{3-40}$$

with the constant independent of $l \in \mathbb{N}$.

So far, we have reduced the proof of the main theorem to that of Proposition 3.1 and Proposition 3.5, which we will present separately in the following sections.

4. Boundedness of the Lipschitz–Kakeya maximal function and proof of Proposition 3.1

In their prominent work, Lacey and Li [2010] have reduced the L^2 boundedness of the operator H_{v,ϵ_0} to the boundedness of an operator they introduced, the so called Lipschitz–Kakeya maximal operator. As soon as this operator is bounded, we can then repeat the argument in Chapter 4 of [Lacey and Li 2010] to obtain Proposition 3.1 as a corollary.

Here we follow [Bateman 2013], where a slightly different version of the Lipschitz–Kakeya maximal operator is used; see Lemma 4.3. The only place in [Bateman 2013] where the one-variable vector field plays a special role is Lemma 6.2 on page 1037. Hence, to prove Proposition 3.1, we just need to replace this lemma by Lemma 4.3, and leave the rest of the argument unchanged.

In this section, we make an observation that both the boundedness of the Lipschitz–Kakeya maximal operator (Corollary 4.4) and its variant (Lemma 4.3) can be proved by adapting Bateman's argument [2009b] to our case, where the vector fields are constant only on Lipschitz curves.

Before defining the Lipschitz–Kakeya maximal operator, we first need to introduce several definitions.

Definition 4.1 (popularity). For a rectangle $R \subset \mathbb{R}^2$, with $l(R)$ its length and $w(R)$ its width, we define its uncertainty interval $EX(R) \subset \mathbb{R}$ to be the interval of width $w(R)/l(R)$ and centered at slope(R). Then the popularity of the rectangle R is defined to be

$$\mathrm{pop}_R := \left|\left\{(x, y) \in \mathbb{R}^2 : u(P(x, y)) \in EX(R)\right\}\right| \,/\, |R|. \tag{4-1}$$

Definition 4.2. Given two rectangles R_1 and R_2 in $\mathbb{R}^2$, we write $R_1 \le R_2$ whenever $R_1 \subset CR_2$ and $EX(R_2) \subset EX(R_1)$, where C is some properly chosen large constant and CR_2 is the rectangle with the same center as R_2 but dilated by the factor C.

Denote $\mathcal{R}_{\delta,\omega} := \{R \in \mathcal{R} : \mathrm{slope}(R) \in [-1, 1], \mathrm{pop}_R \ge \delta, w(R) = \omega\}$, where $\mathcal{R}$ is the collection of all the rectangles in $\mathbb{R}^2$. Then the Lipschitz–Kakeya maximal function is defined as

$$M_{\mathcal{R}_{\delta,\omega}}(f)(x) := \sup_{x \in R \in \mathcal{R}_{\delta,\omega}} \frac{1}{|R|} \int_R |f|. \tag{4-2}$$

Lemma 4.3. *Let u and P be the functions given in the definition of the operator H_v in (3-5). Suppose $\mathcal{R}_0$ is a collection of pairwise incomparable* (*under "$\le$"*) *rectangles of uniform width such that, for each $R \in \mathcal{R}_0$, we have*

$$\frac{|(u \circ P)^{-1}(EX(R)) \cap R|}{R} \ge \delta \tag{4-3}$$

(*i.e.*, $\mathrm{pop}_R \ge \delta$) *and*

$$\frac{1}{|R|} \int_R \mathbb{1}_F \ge \lambda. \tag{4-4}$$

Then, for each $p > 1$,

$$\sum_{R \in \mathcal{R}_0} |R| \lesssim \frac{|F|}{\delta \lambda^p}. \tag{4-5}$$

The same covering lemma argument as in Lemma 3.1 of [Bateman 2009b] shows the boundedness of Lacey and Li's Lipschitz–Kakeya maximal operator as a corollary of Lemma 4.3.

Corollary 4.4. *For all $p \in (1, \infty)$ we have the bound*

$$\|M_{\mathcal{R}_{\delta,\omega}}\|_{L^p \to L^p} \le C(p, a_0) \frac{1}{\delta}. \tag{4-6}$$

Proof of Lemma 4.3. The proof is essentially due to Bateman [2009b], with just one minor modification in order to adapt to the family of Lipschitz curves on with the vector field is constant.

Definition 4.5 (rectangles adapted to the vector field). For a rectangle $R \in \mathcal{R}_{\delta,\omega}$, with its two long sides lying on the parallel lines $y = kx + b_1$ and $y = kx + b_2$ for some $k \in [-1, 1]$ and $b_1, b_2 \in \mathbb{R}$, define $\tilde{R}$ to be the adapted version of R, which is given by the set

$$\{(x, y) : P(x, y) \in P(R)\} \cap \{(x, kx + b) : x \in \mathbb{R}, b \in [b_1, b_2]\}, \tag{4-7}$$

where P is the projection operator in (3-3).

What we need to do is just to replace the rectangles R in [Bateman 2009b] by $\tilde{R}$, and observe that the two key quantities — length and popularity of rectangles — are both preserved under the projection operator P up to a constant depending on the constant a_0 in the main theorem. Hence, we leave out the details. □

5. Proof of Proposition 3.5

This section consists of two subsections. In Section 5A we will introduce some notations, most of which we adopt from Bateman [2013] with minor changes for our purpose. In Section 5B we will use Jones' beta numbers and the Carleson embedding theorem to prove Proposition 3.5.

5A. ***Discretization.*** The content of this subsection is basically taken from Bateman [2013], with minor changes as we are now dealing with all frequencies instead of a single frequency annulus.

Discretizing the functions. Fix $l \geq 0$; we write $\mathcal{D}_l$ as the collection of the dyadic intervals of length 2^{-l} contained in $[-2, 2]$. Fix a smooth positive function $\beta : \mathbb{R} \to \mathbb{R}$ such that

$$\beta(x) = 1 \quad \text{for all } |x| \leq 1 \qquad \text{and} \qquad \beta(x) = 0 \quad \text{for all } |x| \geq 2. \tag{5-1}$$

Also choose β so that $\sqrt{\beta}$ is a smooth function. Then fix an integer c (whose exact value is unimportant), and, for each $\omega \in \mathcal{D}_l$, define

$$\beta_\omega(x) = \beta(2^{l+c}(x - c_{\omega_1})), \tag{5-2}$$

where ω_1 is the right half of ω and c_{ω_1} is its center.

Define

$$\beta_l(x) = \sum_{\omega \in \mathcal{D}_l} \beta_\omega(x); \tag{5-3}$$

note that

$$\beta_l(x + 2^{-l}) = \beta_l(x) \quad \text{for all } x \in [-2, 2 - 2^{-l}]. \tag{5-4}$$

Define

$$\gamma_l = \frac{1}{2} \int_{-1}^{1} \beta_l(x + t)\, dt; \tag{5-5}$$

because of the above periodicity, we know that γ_l is constant for $x \in [-1, 1]$, independent of l. Say $\gamma_l(x) = \delta > 0$; hence,

$$\frac{1}{\delta} \gamma_l(x) \mathbb{1}_{[-1,1]}(x) = \mathbb{1}_{[-1,1]}(x). \tag{5-6}$$

Define another multiplier $\tilde{\beta} : \mathbb{R} \to \mathbb{R}$ with support in $\left[\frac{1}{2}, \frac{5}{2}\right]$ and $\tilde{\beta}(x) = 1$ for $x \in [1, 2]$. We define the corresponding multiplier on $\mathbb{R}^2$,

$$\hat{m}_{k,\omega}(\xi, \eta) = \tilde{\beta}(2^{-k}\eta)\beta_\omega\left(\frac{\xi}{\eta}\right) \hat{m}_{k,l,t}(\xi, \eta) = \tilde{\beta}(2^{-k}\eta)\beta_l\left(t + \frac{\xi}{\eta}\right) \hat{m}_{k,l}(\xi, \eta) = \tilde{\beta}(2^{-k}\eta)\gamma_l\left(\frac{\xi}{\eta}\right).$$

Then what we need to bound can be written as

$$\left\| \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} H_l P_k(f) \right\|_p = \left\| \int_{-1}^{1} \sum_{k\in\mathbb{Z}} \sum_{l\geq 0} H_{k-l}\left(\frac{1}{\delta} m_{k,l} * f\right) dt \right\|_p \leq \int_{-1}^{1} \left\| \sum_{k\in\mathbb{Z}} \sum_{l\geq 0} H_{k-l}\left(\frac{1}{\delta} m_{k,l,t} * f\right) \right\|_p dt,$$

where the terms $H_l P_k$ for $l > k$ in the sum vanish as explained before.

So it suffices to prove a uniform bound on $t \in [-1, 1]$; without loss of generality we will just consider the case $t = 0$, which is

$$\sum_{k\in\mathbb{Z}} \sum_{l\geq 0} H_{k-l}(m_{k,l,0} * f) = \sum_{k\in\mathbb{Z}} \sum_{l\geq 0} H_{k-l}\left(\left[\tilde{\beta}(2^{-k}\eta)\beta_l\left(\frac{\xi}{\eta}\right)\right] * f\right). \tag{5-7}$$

Constructing the tiles. For each $k \in \mathbb{Z}$ and $\omega \in \mathcal{D}_l$ with $l \geq 0$, let $\mathcal{U}_{k,\omega}$ be a partition of $\mathbb{R}^2$ by rectangles of width 2^{-k} and length 2^{-k+l} whose long sides have slope θ, where $\tan\theta = -c(\omega)$, which is the center of the interval ω. If $s \in \mathcal{U}_{k,\omega}$, we will write $\omega_s := \omega$, and $\omega_{s,1}$ to be the right half of ω and $\omega_{s,2}$ the left half.

An element of $\mathcal{U}_{k,\omega}$ for some $\omega \in \mathcal{D}_l$ is called a "tile". Choose $\varphi_{k,\omega}$ such that

$$|\hat{\varphi}_{k,\omega}|^2 = \hat{m}_{k,\omega}; \tag{5-8}$$

then $\varphi_{k,\omega}$ is smooth by our assumption on β mentioned above.

For a tile $s \in \mathcal{U}_{k,\omega}$, define

$$\varphi_s(p) := \sqrt{|s|}\varphi_{k,\omega}(p - c(s)), \tag{5-9}$$

where $c(s)$ is the center of s. Notice that

$$\|\varphi_s\|_2^2 = \int_{\mathbb{R}^2} |s|\varphi_{k,\omega}^2 = |s| \int_{\mathbb{R}^2} \hat{m}_{k,\omega} = 1, \tag{5-10}$$

i.e., φ_s is L^2 normalized.

The purpose of constructing of the tiles above, by the uncertainty principle, is to localize the function further in space, which is realized through:

Lemma 5.1 [Bateman 2013, page 1030]. *Under the above notations, for the frequency-localized function $f * m_{k,\omega}$, we have the representation*

$$f * m_{k,\omega}(x) = \lim_{N\to\infty} \frac{1}{4N^2} \int_{[-N,N]^2} \sum_{s\in\mathcal{U}_{k,\omega}} \langle f, \varphi_s(p+\cdot)\rangle \varphi_s(p+x)\, dp \tag{5-11}$$

The above lemma allows us to pass to the model sum

$$\sum_{k\in\mathbb{Z}} \sum_{l\geq 0} H_{k-l}(f * m_{k,l,0}) = \sum_{k\in\mathbb{Z}} \sum_{l\geq 0} \sum_{\omega\in\mathcal{D}_l} \sum_{s\in\mathcal{U}_{k,\omega}} \langle f, \varphi_s\rangle H_{k-l}(\varphi_s),$$

define

$$\psi_s = \psi_{-\log(\mathrm{length}(s))}, \tag{5-12}$$

and

$$\phi_s(x, y) := \int \check{\psi}_s(t)\varphi_s(x - t, y - tu(P(x, y)))\, dt; \tag{5-13}$$

then the model sum becomes

$$\sum_{k\in\mathbb{Z}} \sum_{l\geq 0} \sum_{\omega\in\mathcal{D}_l} \sum_{s\in\mathcal{U}_{k,\omega}} \langle f, \varphi_s\rangle \phi_s. \tag{5-14}$$

Lemma 5.2. *We have that $\phi_s(x, y) = 0$ unless $-u(P(x, y)) \in \omega_{s,2}$.*

The proof of Lemma 5.2 is by the Plancherel theorem; we just need to observe that the frequency support of ψ_s and $\hat{\varphi}_s$ will be disjoint at the point (x, y) unless $-u(P(x, y)) \in \omega_{s,2}$.

5B. *Boundedness of the commutator and proof of Proposition 3.5.* This subsection is devoted to the proof of Proposition 3.5, which is largely motivated by the proof of the $T(b)$ theorem and the boundedness of the paraproduct; see [Auscher et al. 2002; Coifman et al. 1989], for example.

In our case, unlike Bateman and Thiele's proof for the one-variable vector fields, it's no longer true that $H_v P_k f$ still has frequency in the k-th annulus. In order to get enough orthogonality for the term $H_v P_k f$ to apply the Littlewood–Paley theory, we need to subtract the term $H_v P_k f - \tilde{P}_k H_v P_k f$, which should be viewed as a family of paraproducts.

We proceed with the details of the proof. If we expand the summation on the left-hand side of Proposition 3.5 with (5-14), what we need to bound can be rewritten as

$$\Big\| \sum_k \sum_{\omega \in \mathfrak{D}_l} \sum_{s \in \mathfrak{U}_{k,\omega}} \langle f, \varphi_s \rangle (\phi_s - \tilde{P}_k \phi_s) \Big\|_2 \lesssim 2^{-\gamma l} \|f\|_2. \tag{5-15}$$

In order to use the orthogonality of different wave packets, we will prove the L^2 bound for the dual operator, which is

$$\sum_k \sum_{\omega \in \mathfrak{D}_l} \sum_{s \in \mathfrak{U}_{k,\omega}} \langle h, \phi_s - \tilde{P}_k \phi_s \rangle \varphi_s. \tag{5-16}$$

Notice that, for $s_1 \in \mathfrak{U}_{k_1,\omega_1}$ and $s_2 \in \mathfrak{U}_{k_2,\omega_2}$ with $(k_1, \omega_1) \neq (k_1, \omega_2)$, we have

$$\langle \varphi_{s_1}, \varphi_{s_2} \rangle = 0 \tag{5-17}$$

by the definition of the wavelet function φ_s in (5-9). Also, if we know that s_1 and s_2 are in the same $\mathfrak{U}_{k,\omega}$ for some k and ω, then we can find $m_0, n_0 \in \mathbb{Z}$ such that

$$c(s_2) = c(s_1) + (m_0 \cdot l(s_1), n_0 \cdot w(s_1)), \tag{5-18}$$

where $c(s)$ is the center of the tile s, $l(s)$ its length and $w(s)$ its width. Then, by the nonstationary phase method, for any $N \in \mathbb{N}$, there exists a constant C_N depending only on N such that

$$|\langle \varphi_{s_1}, \varphi_{s_2} \rangle| \le \frac{C_N}{(|m_0| + |n_0| + 1)^N}. \tag{5-19}$$

Here we want to make a remark that the exact value of N is not important, it just denotes some large number which might vary from line to line if we use the same notation later.

Applying the above two estimates, (5-17) and (5-19), we obtain

$$\Big\| \sum_k \sum_{\omega \in \mathfrak{D}_l} \sum_{s \in \mathfrak{U}_{k,\omega}} \langle h, \phi_s - \tilde{P}_k \phi_s \rangle \varphi_s \Big\|_2^2 = \sum_k \sum_{\omega \in \mathfrak{D}_l} \sum_{s_1 \in \mathfrak{U}_{k,\omega}} \sum_{s_2 \in \mathfrak{U}_{k,\omega}} \langle h, \phi_{s_1} - \tilde{P}_k \phi_{s_1} \rangle \langle \varphi_{s_1}, \varphi_{s_2} \rangle \langle h, \phi_{s_2} - \tilde{P}_k \phi_{s_2} \rangle.$$

For any $s_1, s_2 \in \mathfrak{U}_{k,\omega}$ there exist $m_0, n_0 \in \mathbb{Z}$ such that

$$c(s_2) = c(s_1) + (m_0 \cdot l(s_1), n_0 \cdot w(s_1)), \tag{5-20}$$

so the above sum can be rewritten as

$$\sum_{m_0,n_0\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{\omega\in\mathfrak{D}_l}\sum_{s_1\in\mathfrak{U}_{k,\omega}}\langle h,\phi_{s_1}-\tilde{P}_k\phi_{s_1}\rangle\langle\varphi_{s_1},\varphi_{s_2}\rangle\langle h,\phi_{s_2}-\tilde{P}_k\phi_{s_2}\rangle \tag{5-21}$$

with s_1, s_2 satisfying the relation (5-20).

Now fix $m_0, n_0\in\mathbb{Z}$; by the estimate in (5-19), we know that

$$\sum_k\sum_{\omega\in\mathfrak{D}_l}\sum_{s_1\in\mathfrak{U}_{k,\omega}}|\langle h,\phi_{s_1}-\tilde{P}_k\phi_{s_1}\rangle\langle\varphi_{s_1},\varphi_{s_2}\rangle\langle h,\phi_{s_2}-\tilde{P}_k\phi_{s_2}\rangle| \\ \lesssim\frac{1}{(|m_0|+|n_0|+1)^N}\sum_k\sum_{\omega\in\mathfrak{D}_l}\sum_{s_1\in\mathfrak{U}_{k,\omega}}|\langle h,\phi_{s_1}-\tilde{P}_k\phi_{s_1}\rangle\langle h,\phi_{s_2}-\tilde{P}_k\phi_{s_2}\rangle|,$$

and, by the Cauchy–Schwarz inequality, the last term is bounded by

$$\frac{1}{(|m_0|+|n_0|+1)^N}\sum_k\sum_{\omega\in\mathfrak{D}_l}\sum_{s\in\mathfrak{U}_{k,\omega}}|\langle h,\phi_s-\tilde{P}_k\phi_s\rangle|^2, \tag{5-22}$$

so it suffices to prove that

$$\sum_k\sum_{\omega\in\mathfrak{D}_l}\sum_{s\in\mathfrak{U}_{k,\omega}}\langle h,\phi_s-\tilde{P}_k\phi_s\rangle^2\lesssim 2^{-\gamma l}\|h\|_2^2. \tag{5-23}$$

First we estimate every single term $\langle h,\phi_s-\tilde{P}_k\phi_s\rangle$ for a fixed tile s: let $s_{m,n}$ be the shift of s by (m,n) units, that is,

$$s_{m,n}:=\{(x,y)\in\mathbb{R}^2:(x-m\cdot l(s),\,y-n\cdot w(s))\in s\}; \tag{5-24}$$

then, by the triangle inequality, we know that

$$|\langle h,\phi_s-\tilde{P}_k\phi_s\rangle|\le\sum_{m,n\in\mathbb{Z}}\left|\int_{s_{m,n}}h\cdot(\phi_s-\tilde{P}_k\phi_s)\,dy\,dx\right|. \tag{5-25}$$

Recall that in Definition 4.5 we use $\tilde{R}$ to denote the adapted version of the rectangle R to the family of Lipschitz curves; then clearly $\tilde{s}_{m,n}\supset s_{m,n}$. Thus

$$|\langle h,\phi_s-\tilde{P}_k\phi_s\rangle|\le\sum_{m,n\in\mathbb{Z}}\left|\int_{\tilde{s}_{m,n}}h\cdot(\phi_s-\tilde{P}_k\phi_s)\,dy\,dx\right|. \tag{5-26}$$

By the coarea formula (3-22), we obtain

$$\begin{aligned}|\langle h,\phi_s-\tilde{P}_k\phi_s\rangle|&\le\sum_{m,n\in\mathbb{Z}}|\int_{\tilde{s}_{m,n}}h\cdot(\phi_s-\tilde{P}_k\phi_s)\,dy\,dx|\\&\lesssim\sum_{m,n\in\mathbb{Z}}\int_{P(s_{m,n})}\int_{\Gamma_x\cap\tilde{s}_{m,n}}|h\cdot(\phi_s-\tilde{P}_k\phi_s)|\,ds_x\,dx,\end{aligned}$$

where ds_x stands for the arc-length measure of the Lipschitz curve Γ_x.

Now, for the inner integration along the curve Γ_x, we do the same change of coordinates and the same parametrization of Γ_x as in Definition 3.3, i.e., we choose coordinates such that the horizontal axis is parallel to $(1, u(x))$, and represent the curve Γ_x by the Lipschitz function $g_x(\cdot)$. If we let $J(x, s_{m,n})$ denote the projection of $\Gamma_x \cap \tilde{s}_{m,n}$ on the new vertical axis, the last expression becomes

$$\sum_{m,n\in\mathbb{Z}} \int_{P(s_{m,n})} \int_{J(x,s_{m,n})} \big|h(g_x(y), y)\big(\phi_s(g_x(y), y) - P_k[\phi_s(g_x(y), y)]\big)\big|\, dy\, dx. \tag{5-27}$$

To bound the above term, Jones' beta number will play a crucial role.

Definition 5.3 [Jones 1989]. For a Lipschitz function $A:\mathbb{R}\to\mathbb{R}$, we first take the Calderón decomposition of $a(x) = A'(x)$, which yields the representation

$$a(x) = \sum_{I \text{ dyadic}} a_I \psi_I(x), \tag{5-28}$$

where ψ_I is some mean-zero function supported on $3I$ with $|\psi_I'(x)| \le |I|^{-1}$. For each dyadic interval I, let

$$\alpha_I = \sum_{|J|\ge|I|} a_I \psi_J(c_I) \tag{5-29}$$

denote the "average slope" of the Lipschitz curve near I, where c_I stands for the center of I, and define the beta number

$$\beta_0(I) := \sup_{x\in 3I} \frac{|A(x) - A(c_I) - \alpha_I(x - c_I)|}{|I|}, \tag{5-30}$$

and the j_0-th beta number

$$\beta_{j_0}(I) := \sup_{x\in 3j_0 I} \frac{|A(x) - A(c_I) - \alpha_I(x - c_I)|}{|I|}. \tag{5-31}$$

For beta numbers, we have the following Carleson condition:

Lemma 5.4 [Jones 1989]. *For any Lipschitz function A, we have*

$$\sup_J \frac{1}{|J|} \sum_{I\subset J} \beta_0^2(I)|I| \lesssim \|A\|_{\mathrm{Lip}}^2, \tag{5-32}$$

and also, for any $j_0 \in \mathbb{N}$,

$$\sup_J \frac{1}{|J|} \sum_{I\subset J} \beta_{j_0}^2(I)|I| \lesssim j_0^3 \|A\|_{\mathrm{Lip}}^2. \tag{5-33}$$

After introducing Jones' beta number, we are ready to state:

Lemma 5.5. *for $x \in P(s_{m,n})$, we have the estimate*

$$\int_{J(x,s_{m,n})} \big|h(g_x(y), y)\big(\phi_s(g_x(y), y) - P_k[\phi_s(g_x(y), y)]\big)\big|\, dy$$
$$\lesssim \sum_{j_0\in\mathbb{N}} \frac{2^{-3l/2}}{(|j_0| + |m| + |n| + 1)^N} \beta_{j_0}(x, s_{m,n})[h]_{x,s_{m,n}} \mathbb{1}_{\{-u(x)\in\omega_{s,2}\}}(x),$$

where $\beta_{j_0}(x, s_{m,n})$ *is the* j_0*-th beta number for the Lipschitz curve* $g_x(\cdot)$ *on the interval* $J(x, s_{m,n})$ *and* $[h]_{x,s_{m,n}}$ *is the average of the function h on the interval* $J(x, s_{m,n})$,

$$[h]_{x,s_{m,n}} := \frac{1}{w(s)} \int_{J(x,s_{m,n})} |h(g_x(y), y)|\, dy. \tag{5-34}$$

The proof of Lemma 5.5 will be postponed to the end. Substitute the estimate in Lemma 5.5 into the estimate for the term $\langle h, \phi_s - \tilde{P}_k\phi_s\rangle$; we then have that

$$\begin{aligned}|\langle h, \phi_s - \tilde{P}_k\phi_s\rangle| &\lesssim \sum_{m,n} \int_{P(s_{m,n})} \int_{J(x,s_{m,n})} \bigl|h(g_x(y), y)\bigl(\phi_s(g_x(y), y) - P_k[\phi_s(g_x(y), y)]\bigr)\bigr|\, dy\, dx \\ &\lesssim \sum_{m,n} \int_{P(s_{m,n})} \sum_{j_0\in\mathbb{N}} \frac{2^{-3l/2}}{(|j_0|+|m|+|n|+1)^N} \beta_{j_0}(x, s_{m,n})[h]_{x,s_{m,n}} \mathbb{1}_{\{-u(x)\in\omega_{s,2}\}}(x)\, dx,\end{aligned}$$

hence

$$\begin{aligned}&\sum_k \sum_{\omega\in\mathfrak{D}_l} \sum_{s\in\mathfrak{U}_{k,\omega}} |\langle h, \phi_s - \tilde{P}_k\phi_s\rangle|^2 \\ &\quad\lesssim \sum_k \sum_{\omega\in\mathfrak{D}_l} \sum_{s\in\mathfrak{U}_{k,\omega}} \sum_{m,n,j_0} \frac{2^{-3l}}{(|j_0|+|m|+|n|+1)^N} \left|\int_{P(s_{m,n})} \beta_{j_0}(x, s_{m,n})[h]_{x,s_{m,n}} \mathbb{1}_{\{-u(x)\in\omega_{s,2}\}}(x)\, dx\right|^2 \\ &\quad\lesssim \sum_{m,n,j_0} \frac{2^{-2l}}{(|j_0|+|m|+|n|+1)^N} \sum_k \sum_{\omega\in\mathfrak{D}_l} \sum_{s\in\mathfrak{U}_{k,\omega}} w(s) \int_{P(s_{m,n})} \beta^2_{j_0}(x, s_{m,n})[h]^2_{x,s_{m,n}} \mathbb{1}_{\{-u(x)\in\omega_{s,2}\}}(x)\, dx.\end{aligned}$$

Lemma 5.6. *For any fixed* x, m, n, j_0,

$$\sum_k \sum_{\omega\in\mathfrak{D}_l} \sum_{s\in\mathfrak{U}_{k,\omega}} w(s)\mathbb{1}_{P(s_{m,n})}(x)\beta^2_{j_0}(x, s_{m,n})[h]^2_{x,s_{m,n}} \mathbb{1}_{\{-u(x)\in\omega_{s,2}\}}(x) \lesssim j_0^3 \|h\|^2_{L^2(\Gamma_x)}. \tag{5-35}$$

Proof. This lemma is akin to the Carleson embedding theorem, as we have the Carleson-type condition

$$\sup_{s_{m,n}} \frac{1}{|J(x, s_{m,n})|} \sum_{s'_{m,n}:J(x,s'_{m,n})\subset J(x,s_{m,n})} \beta^2_{j_0}(J(x, s'_{m,n}))w(s'_{m,n}) \lesssim j_0^3 \operatorname{Lip}^2(\Gamma_x), \tag{5-36}$$

where the term $\mathbb{1}_{\{-u(x)\in\omega_{s,2}\}}$ has the following purpose: originally there are 2^l groups of dyadic rectangles

$$\bigcup_k \bigcup_{\omega\in\mathfrak{D}_l} \bigcup_{s\in\mathfrak{U}_{k,\omega}} \{s_{m,n}\} \tag{5-37}$$

in the summation $\sum_k \sum_{\omega\in\mathfrak{D}_l} \sum_{s\in\mathfrak{U}_{k,\omega}}$, which means that there are also 2^l groups of dyadic intervals

$$\bigcup_k \bigcup_{\omega\in\mathfrak{D}_l} \bigcup_{s\in\mathfrak{U}_{k,\omega}} \{J(x, s_{m,n})\} \tag{5-38}$$

which are the projections of the intersection of the dyadic rectangles with Γ_x on the vertical axis; the term $\mathbb{1}_{\{-u(x)\in\omega_{s,2}\}}$ just guarantees that there is just one such collection that contributes, i.e., which has the right orientation in the sense of Lemma 5.2.

Then the desired estimate will just follow from the Carleson embedding theorem, for which we refer to Lemma 5.1 in [Auscher et al. 2002]. □

Continuing the calculation before the above lemma,

$$\sum_k \sum_{\omega\in\mathfrak{D}_l} \sum_{s\in\mathfrak{U}_{k,\omega}} |\langle h, \phi_s - \tilde{P}_k\phi_s\rangle|^2 \lesssim \sum_{m,n,j_0} \frac{2^{-2l} j_0^3}{(|j_0|+|m|+|n|+1)^N} \int_{\mathbb{R}} \|h\|^2_{L^2(\Gamma_x)}\, dx \lesssim 2^{-2l}\|h\|_2^2.$$

This finishes the proof for (5-23) and then Proposition 3.5, modulo Lemma 5.5, which we will present now.

Proof of Lemma 5.5. We assume that $-u(x) \in \omega_{s,2}$, which means the vector $(1, u(x))$ is roughly parallel to the long side of $s_{m,n}$, otherwise the left-hand side in Lemma 5.5 will also vanish due to Lemma 5.2. After the change of variables in (5-27), the vector $(1, u(x))$ becomes $(1, 0)$.

Proof by ignoring the tails. In order to explain how Jones' beta number appears, we first sketch the proof by ignoring the tails of the wavelet functions and the tail of the kernel of the Littlewood–Paley projection operator P_k.

By the above simplification, we only need to consider the case $m = n = 0$. What we need to "prove" becomes

$$\int_{J(x,s)} \big|h(g_x(y), y)\big(\phi_s(g_x(y), y) - P_k[\phi_s(g_x(y), y)]\big)\big|\, dy \lesssim 2^{-3l/2}\beta_0(J(x,s))[h]_{x,s}. \tag{5-39}$$

For fixed x, we denote by $\tau_{x,s}y + b$ the line of "average slope" we picked in the definition of the beta number for the Lipschitz curve $g_x(\cdot)$ on the interval $J(x,s)$; for the sake of simplicity we assume $b = 0$. Moreover, as both x and s are fixed, we will also just write τ instead of $\tau_{x,s}$. Then we make the crucial observation that

$$P_k[\phi_s^x(\tau y, y)] = \phi_s^x(\tau y, y), \tag{5-40}$$

where

$$\phi_s^x(\tau y, y) := \int_{\mathbb{R}} \check{\psi}_s(t)\varphi_s(\tau y - t, y)\, dt, \tag{5-41}$$

due to the fact that, for any function φ_s with frequency supported on the k-th annulus, if we restrict the function to a straight line, it will still have frequency supported on the k-th annulus (with one dimension less).

In comparison with the definition of ϕ_s in (5-13), $\phi_s^x(\tau y, y)$ is defined as the Hilbert transform along the vector $(1, u(x))$ (which is $(1, 0)$ after the change of the variables we made in Lemma 3.2 and in the expression (5-27)) instead of the direction of the vector field v at the point $(\tau y, y)$.

Hence, from the identity in (5-40), we obtain

$$\begin{aligned}\phi_s(g_x(y), y) - P_k[\phi_s(g_x(y), y)] &= \phi_s(g_x(y), y) - P_k[\phi_s(g_x(y), y) - \phi_s^x(\tau y, y) + \phi_s^x(\tau y, y)] \\ &= \phi_s(g_x(y), y) - \phi_s^x(\tau y, y) - P_k[\phi_s(g_x(y), y) - \phi_s^x(\tau y, y)]. \end{aligned} \tag{5-42}$$

As we have also ignored the tails of the kernel of P_k, it is easy to see that the former and the latter terms in the last expression can essentially be handled in the same way. Hence in the following we will only consider the former term, which corresponds to the term

$$\int_{J(x,s)} \big|h(g_x(y), y)\big(\phi_s(g_x(y), y) - \phi_s^x(\tau y, y)\big)\big|\, dy. \tag{5-43}$$

By the definitions of ϕ_s and ϕ_s^x, we have

$$\begin{aligned}|\phi_s(g_x(y), y) - \phi_s^x(\tau y, y)| &= \left|\int_{\mathbb{R}} \check{\psi}_{k-l}(t)\varphi_s(g_x(y) - t, y)\, dt - \int_{\mathbb{R}} \check{\psi}_{k-l}(t)\varphi_s(\tau y - t, y)\, dt\right| \\ &= 2^{k-l}\left|\int_{\mathbb{R}} \check{\psi}_0(2^{k-l}t)\varphi_s(g_x(y) - t, z)\, dt - \int_{\mathbb{R}} \check{\psi}_0(2^{k-l}t)\varphi_s(\tau y - t, y)\, dt\right| \\ &= 2^{k-l}\left|\int_{\mathbb{R}} \big[\check{\psi}_0(2^{k-l}(t + g_x(y) - \tau y)) - \check{\psi}_0(2^{k-l}t)\big]\varphi_s(\tau y - t, z)\, dt\right|. \end{aligned} \tag{5-44}$$

By the definition of the beta numbers, we have that

$$|g_x(y) - \tau y| \lesssim \beta_0(x, s)2^{-k}, \tag{5-45}$$

which implies that

$$|\check{\psi}_0(2^{k-l}(t + g_x(y) - \tau y)) - \check{\psi}_0(2^{k-l}t)| \lesssim 2^{-l}\beta_0(x, s) \tag{5-46}$$

by the fundamental theorem. In the end, by substituting the above estimate into (5-44) and (5-43), we obtain the desired estimate (5-39).

The full proof. The main idea is still the same, and the difference is that we need to be more careful with the tails of the wavelet functions and the kernel of P_k.

For fixed x, m and n, denote by $\tau(x, s_{m,n})y + b$ the line of "average slope" for the Lipschitz curve $g_x(\cdot)$ on the interval $J(x, s_{m,n})$; for the sake of simplicity we assume $b = 0$. Then the crucial observation (5-40) becomes

$$P_k[\phi_s^x(\tau(x, s_{m,n})y, y)] = \phi_s^x(\tau(x, s_{m,n})y, y). \tag{5-47}$$

Hence, similar to (5-42), we obtain from (5-47) that

$$\begin{aligned}&\phi_s(g_x(y), y) - P_k[\phi_s(g_x(y), y)] \\ &\qquad = \phi_s(g_x(y), y) - \phi_s^x(\tau(x, s_{m,n})y, y) - P_k[\phi_s(g_x(y), y) - \phi_s^x(\tau(x, s_{m,n})y, y)].\end{aligned}$$

Denote

$$I_{s_{m,n}} = \left|\int_{J(x,s_{m,n})} h(g_x(y), y) \cdot \big(\phi_s(g_x(y), y) - \phi_s^x(\tau(x, s_{m,n})y, y)\big)\, dy\right| \tag{5-48}$$

and also

$$II_{s_{m,n}} = \left|\int_{J(x,s_{m,n})} h(g_x(y), y) \cdot P_k\big[\phi_s(g_x(y), y) - \phi_s^x(\tau(x, s_{m,n})y, y)\big]\, dy\right|. \tag{5-49}$$

Lemma 5.7. *Under the above notations, for $z \in J(x, s_{m,n}) + j_0 2^{-k}$ with $j_0 \in \mathbb{Z}$, we have the pointwise estimate*

$$|\phi_s(g_x(z), z) - \phi_s^x(\tau(x, s_{m,n})z, z)| \lesssim \frac{\beta_{|j_0|}(x, s_{m,n})2^k 2^{-3l/2}}{(\min\{|m| + |n|, |m| + |n| - |j_0|\} + 1)^N}. \tag{5-50}$$

Let us first complete the proof of Lemma 5.5: For the first term $I_{s_{m,n}}$, we take j_0 in Lemma 5.7 to be zero, then

$$|\phi_s(g_x(z),z)-\phi_s^x(\tau(x,s_{m,n})z,z)|\lesssim\frac{\beta_0(x,s_{m,n})2^k2^{-3l/2}}{(|m|+|n|+1)^N},\tag{5-51}$$

which implies that

$$I_{s_{m,n}}\lesssim\frac{2^{-3l/2}}{(|m|+|n|+1)^N}\beta_0(x,s_{m,n})[h]_{x,s_{m,n}}.\tag{5-52}$$

For the second term $II_{s_{m,n}}$, by the definition of P_k,

$$\begin{aligned}\big|P_k[\phi_s(g_x(y),y)-\phi_s^x(\tau(x,s_{m,n})y,y)]\big|&=\bigg|\int_{\mathbb{R}}\big(\phi_s(g_x(z),z)-\phi_s^x(\tau(x,s_{m,n})z,z)\big)2^k\check{\psi}_0(2^k(y-z))\,dz\bigg|\\&\le\bigg|\sum_{j_0\in\mathbb{Z}}\int_{J(x,s_{m,n})+j_02^{-k}}\big(\phi_s(g_x(z),z)-\phi_s^x(\tau(x,s_{m,n})z,z)\big)2^k\check{\psi}_0(2^k(y-z))\,dz\bigg|.\end{aligned}$$

For $y\in J(x,s_{m,n})$ and $z\in J(x,s_{m,n})+j_02^{-k}$, by the nonstationary phase method we have that

$$|\check{\psi}_0(2^k(y-z))|\lesssim\frac{1}{(j_0+1)^N}.\tag{5-53}$$

Together with the estimate in Lemma 5.7, we arrive at

$$\begin{aligned}\big|P_k[\phi_s(g_x(y),y)-\phi_s^x(\tau(x,s_{m,n})y,y)]\big|&\lesssim\sum_{j_0\in\mathbb{Z}}\frac{\beta_{|j_0|}(x,s_{m,n})2^k2^{-3l/2}}{(\min\{|m|+|n|,|m|+|n|-|j_0|\}+1)^N}\frac{1}{(j_0+1)^N}\\&\lesssim\sum_{j_0\in\mathbb{Z}}\frac{\beta_{|j_0|}(x,s_{m,n})2^k2^{-3l/2}}{(|m|+|n|+|j_0|+1)^N}.\end{aligned}$$

Substituting the last expression into the estimate for $II_{s_{m,n}}$, we get the desired estimate.

So far we have finished the proof of Lemma 5.5 except for Lemma 5.7, which we will do now.

Proof of Lemma 5.7. As x and $s_{m,n}$ are fixed now, for simplicity we will just write τ instead of $\tau_{x,s_{m,n}}$. Notice that in the new coordinate we chose for Γ_x, the vector field along Γ_x points in the direction of $(1,0)$. Then, by the definition of ϕ_s and ϕ_s^x, we have

$$|\phi_s(g_x(z),z)-\phi_s^x(\tau z,z)|=2^{k-l}\bigg|\int_{\mathbb{R}}\big[\check{\psi}_0(2^{k-l}(t+g_x(z)-\tau z))-\check{\psi}_0(2^{k-l}t)\big]\varphi_s(\tau z-t,z)\,dt\bigg|.$$

By the definition of the beta numbers, we have that

$$|g_x(z)-\tau z|\lesssim\beta_{|j_0|}(x,s_{m,n})2^{-k},\tag{5-54}$$

which implies that

$$|\check{\psi}_0(2^{k-l}(t+g_x(z)-\tau z))-\check{\psi}_0(2^{k-l}t)|\lesssim 2^{-l}\beta_{|j_0|}(x,s_{m,n})\tag{5-55}$$

by the fundamental theorem of calculus. The nonstationary phase method leads to the final estimate:

$$2^{k-l}\left|\int_{\mathbb{R}}[\check{\psi}_0(2^{k-l}(t+g_x(z)-\tau z))-\check{\psi}_0(2^{k-l}t)]\varphi_s(\tau z-t,z)\,dt\right| \lesssim \frac{2^{-l}\beta_{|j_0|}(x,s_{m,n})2^{k/2}2^{(k-l)/2}}{(\min\{|m|+|n|,|m|+|n|-|j_0|\}+1)^N}.$$

Thus we have finished the proof of Lemma 5.7, and hence Lemma 5.5. □

Acknowledgements

This work was done under the supervision of Prof. Christoph Thiele, to whom the author would like to express his most sincere gratitude for suggesting to him such an interesting topic, for sharing with him his deep insight into this problem, and for numerous suggestions on the exposition of this paper. The author would also like to thank the anonymous referee for many helpful comments.

References

[Auscher et al. 2002] P. Auscher, S. Hofmann, C. Muscalu, T. Tao, and C. Thiele, "Carleson measures, trees, extrapolation, and $T(b)$ theorems", *Publ. Mat.* **46**:2 (2002), 257–325. MR 2003f:42019 Zbl 1027.42009

[Azzam and Schul 2012] J. Azzam and R. Schul, "Hard Sard: quantitative implicit function and extension theorems for Lipschitz maps", *Geom. Funct. Anal.* **22**:5 (2012), 1062–1123. MR 2989430 Zbl 1271.26004

[Bateman 2009a] M. Bateman, "Kakeya sets and directional maximal operators in the plane", *Duke Math. J.* **147**:1 (2009), 55–77. MR 2009m:42029 Zbl 1165.42005

[Bateman 2009b] M. Bateman, "L^p estimates for maximal averages along one-variable vector fields in $\mathbb{R}^2$", *Proc. Amer. Math. Soc.* **137**:3 (2009), 955–963. MR 2010e:42023 Zbl 1173.42008

[Bateman 2013] M. Bateman, "Single annulus L^p estimates for Hilbert transforms along vector fields", *Rev. Mat. Iberoam.* **29**:3 (2013), 1021–1069. MR 3090145 Zbl 1283.42019

[Bateman and Thiele 2013] M. Bateman and C. Thiele, "L^p estimates for the Hilbert transforms along a one-variable vector field", *Anal. PDE* **6**:7 (2013), 1577–1600. MR 3148061 Zbl 1285.42014

[Bourgain 1989] J. Bourgain, "A remark on the maximal function associated to an analytic vector field", pp. 111–132 in *Analysis at Urbana, I* (Urbana, IL, 1986–1987), edited by E. Berkson and T. Peck, London Math. Soc. Lecture Note Ser. **137**, Cambridge Univ. Press, Cambridge, 1989. MR 90h:42028 Zbl 0692.42006

[Christ et al. 1999] M. Christ, A. Nagel, E. M. Stein, and S. Wainger, "Singular and maximal Radon transforms: analysis and geometry", *Ann. of Math.* (2) **150**:2 (1999), 489–577. MR 2000j:42023 Zbl 1285.42014

[Coifman et al. 1989] R. R. Coifman, P. W. Jones, and S. Semmes, "Two elementary proofs of the L^2 boundedness of Cauchy integrals on Lipschitz curves", *J. Amer. Math. Soc.* **2**:3 (1989), 553–564. MR 90k:42017 Zbl 0713.42010

[Jones 1989] P. W. Jones, "Square functions, Cauchy integrals, analytic capacity, and harmonic measure", pp. 24–68 in *Harmonic analysis and partial differential equations* (El Escorial, 1987), edited by J. García-Cuerva, Lecture Notes in Math. **1384**, Springer, Berlin, 1989. MR 91b:42032 Zbl 0675.30029

[Karagulyan 2007] G. A. Karagulyan, "On unboundedness of maximal operators for directional Hilbert transforms", *Proc. Amer. Math. Soc.* **135**:10 (2007), 3133–3141. MR 2008e:42044 Zbl 1162.42009

[Lacey 2000] M. T. Lacey, "The bilinear maximal functions map into L^p for $\frac{2}{3} < p \le 1$", *Ann. of Math.* (2) **151**:1 (2000), 35–57. MR 2001b:42015 Zbl 0967.47031

[Lacey and Li 2006] M. T. Lacey and X. Li, "Maximal theorems for the directional Hilbert transform on the plane", *Trans. Amer. Math. Soc.* **358**:9 (2006), 4099–4117. MR 2006k:42018 Zbl 1095.42010

[Lacey and Li 2010] M. Lacey and X. Li, *On a conjecture of E. M. Stein on the Hilbert transform on vector fields*, vol. 205, Mem. Amer. Math. Soc. **965**, American Mathematical Society, Providence, RI, 2010. MR 2011c:42019 Zbl 1190.42005

[Lacey and Thiele 1997] M. Lacey and C. Thiele, “L^p estimates for the bilinear Hilbert transform”, *Proc. Nat. Acad. Sci. U.S.A.* **94**:1 (1997), 33–35. MR 98e:44001 Zbl 1027.42009

[Lacey and Thiele 1998] M. T. Lacey and C. M. Thiele, “On Calderón’s conjecture for the bilinear Hilbert transform”, *Proc. Natl. Acad. Sci. USA* **95**:9 (1998), 4828–4830. MR 99e:42013 Zbl 0915.42011

[Stein and Street 2012] E. M. Stein and B. Street, “Multi-parameter singular Radon transforms, III: Real analytic surfaces”, *Adv. Math.* **229**:4 (2012), 2210–2238. MR 2880220 Zbl 1242.42010

Received 10 Jan 2015. Revised 8 Mar 2015. Accepted 15 Apr 2015.

SHAOMING GUO: shaoming@math.uni-bonn.de
Mathematisches Institut, University of Bonn, Endenicher Allee 60, D-53115 Bonn, Germany

Guidelines for Authors

Authors may submit manuscripts in PDF format on-line at the Submission page at msp.org/apde.

Originality. Submission of a manuscript acknowledges that the manuscript is original and and is not, in whole or in part, published or under consideration for publication elsewhere. It is understood also that the manuscript will not be submitted elsewhere while under consideration for publication in this journal.

Language. Articles in APDE are usually in English, but articles written in other languages are welcome.

Required items. A brief abstract of about 150 words or less must be included. It should be self-contained and not make any reference to the bibliography. If the article is not in English, two versions of the abstract must be included, one in the language of the article and one in English. Also required are keywords and subject classifications for the article, and, for each author, postal address, affiliation (if appropriate), and email address.

Format. Authors are encouraged to use LaTeX but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages (Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc.) allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@msp.org with details about how your graphics were generated.

White space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Proofs. Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

ANALYSIS & PDE

Volume 8 No. 5 2015