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ALGEBRAIC ERROR ESTIMATES FOR THE STOCHASTIC HOMOGENIZATION OF UNIFORMLY PARABOLIC EQUATIONS

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We establish an algebraic error estimate for the stochastic homogenization of fully nonlinear, uniformly parabolic equations in stationary ergodic spatiotemporal media. The approach is similar to that of Armstrong and Smart in the study of quantitative stochastic homogenization of uniformly elliptic equations.

1. Introduction

We study quantitative stochastic homogenization of equations of the form

$$\begin{cases} u_t^\varepsilon + F(D^2u^\varepsilon, x/\varepsilon, t/\varepsilon^2, \omega) = 0 & \text{in } U_T, \\ u^\varepsilon = g & \text{on } \partial_p U_T, \end{cases} \tag{1-1}$$

where F is a random uniformly elliptic operator, determined by an element ω of some probability space, $U_T := U \times (0, T] \subsetneq \mathbb{R}^{d+1}$ is a compact domain, and $\partial_p U_T$ is the parabolic boundary. Lin [2015] showed that, under suitable hypotheses on the environment (namely stationarity and ergodicity of the operator in space and time), $u^\varepsilon(\cdot\,,\cdot\,,\omega)$ converges almost surely to a limiting function u which solves

$$\begin{cases} u_t + \overline{F}(D^2u) = 0 & \text{in } U_T, \\ u = g & \text{on } \partial_p U_T, \end{cases} \tag{1-2}$$

for a uniformly elliptic limiting operator $\overline{F}$ which is independent of ω. Furthermore, a rate of convergence was established under additional quantitative ergodic assumptions. If the environment is strongly mixing with a prescribed logarithmic rate, then the convergence occurs in probability with a logarithmic rate, i.e.,

$$\mathbb{P}\Big[\sup_{U_T} |u^\varepsilon(\cdot\,,\cdot\,,\omega) - u(\cdot\,,\cdot)| \geq f(\varepsilon)\Big] \leq f(\varepsilon) \tag{1-3}$$

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with $f(\varepsilon) \sim |\log \varepsilon|^{-1}$. In this article, we show that, under the assumption of finite range of dependence, the homogenization occurs in probability with an algebraic rate, i.e., $f(\varepsilon) \sim \varepsilon^{\beta}$.

Background and discussion. For nondivergence form equations in the random setting, the pioneering works establishing the qualitative theory of homogenization (the convergence of $u^{\varepsilon} \to u$) include (but are not limited to) the papers of Papanicolaou and Varadhan [1982] and Yurinskiĭ [1982] for linear, nondivergence form, uniformly elliptic equations, and Caffarelli, Souganidis, and Wang [Caffarelli et al. 2005] for fully nonlinear, uniformly elliptic equations. The study of quantitative stochastic homogenization seeks to establish error estimates for this convergence. For linear, uniformly elliptic equations in nondivergence form, the first results were obtained by Yurinskiĭ [1988; 1991]. Assuming that the environment satisfies an algebraic rate of decorrelation, his works present an algebraic rate of convergence for stochastic homogenization in dimensions $d \geq 5$. In dimensions $d = 3, 4$, the same result holds under the additional assumption of small ellipticity contrast, that is, the ratio of ellipticities is close to 1. In dimension $d = 2$, Yurinskiĭ's results yield a logarithmic rate of convergence.

For fully nonlinear equations, the first quantitative stochastic homogenization result appears in [Caffarelli and Souganidis 2010] for elliptic equations, and the parabolic case with spatiotemporal media was considered in [Lin 2015]. Both of these works obtain logarithmic convergence rates from logarithmic mixing conditions. The approach of both papers was to adapt the obstacle problem method of [Caffarelli et al. 2005] to construct approximate correctors, which play the role of correctors in the random setting. The logarithmic rate appears to be the optimal rate attainable with this approach. This left open the question whether an algebraic rate similar to the results of Yurinskiĭ was attainable in the more general setting of fully nonlinear equations, and for problems in lower dimensions.

In the elliptic setting, this was addressed in [Armstrong and Smart 2014b]. They prove algebraic error estimates in all dimensions for the stochastic homogenization of fully nonlinear, uniformly elliptic equations. The main insight of their work was the introduction of a new subadditive quantity that (1) controls the solutions of the equation and (2) can be studied by adapting the regularity theory of Monge–Ampère equations. Their method does not see the presence of correctors and instead controls solutions indirectly via geometric quantities.

The purpose of this article is to adapt the elliptic strategy to the parabolic spatiotemporal setting, which turns out to be subtle. The approach of [Armstrong and Smart 2014b] was to view the convex envelope of a supersolution as an approximate solution of the Monge–Ampère equation

$$\det D^2 w = 1 \tag{1-4}$$

for w convex and to then use ideas from the regularity theory of (1-4) (namely John's lemma) to control the sublevel sets of w. In the parabolic setting, we will show that the *monotone envelope* of a supersolution of (1-1) is an approximate solution of the analogous Monge–Ampère equation

$$-w_t \det D^2 w = 1 \tag{1-5}$$

for w parabolically convex (convex in space and nonincreasing in time). The equation (1-5) was introduced by Krylov [1976], and then it was pointed out by Tso [1985] that this was the most appropriate parabolic analogue of (1-4). Regularity properties of (1-5) have been studied by Gutiérrez and Huang [1998; 2001] and other parabolic Monge–Ampère equations have been studied by Daskalopoulos and Savin [2012]. In spite of this work, the equation (1-5) is still not as well understood as (1-4). In particular, there is no analogue of John's lemma for sublevel sets of parabolically convex functions. This forced us to develop an alternative approach (which can also be used in the elliptic setting), which replaces John's lemma with a compactness argument.

Assumptions, and statement of the main result. We begin by stating the general assumptions on (1-1) and the precise statement of the main result. We work in the stationary ergodic, spatiotemporal setting. We assume there exists an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\Omega := \{F : \mathbb{S}^d \times \mathbb{R}^{d+1} \to \mathbb{R} \text{ satisfies (F1)–(F4)}\},$$

where (F1)–(F4) will be specified below. In particular, we have $F(X, y, s, \omega) = \omega(X, y, s)$. $\mathcal{F}$ is the Borel σ-algebra on Ω, and we assume that Ω is equipped with a set of measurable, measure-preserving transformations $\tau_{(y',s')} : \Omega \to \Omega$ for each $(y', s') \in \mathbb{R}^{d+1}$. We also assume that $\partial_p U_T$ satisfies a uniform exterior cone condition, which allows us to construct global barriers (see [Crandall et al. 1999] for the precise assumption). Our hypotheses can be summarized as follows:

(F1) *Finite range of dependence*: For $A \subseteq \mathbb{R}^{d+1}$, denote

$$\mathcal{B}(A) := \sigma\{F(\,\cdot\,, y, s, \omega) : (y, s) \in A\},$$

the σ-algebra generated by the operators F defined on A. For $(x_1, t_1), (x_2, t_2) \in \mathbb{R}^{d+1}$, let

$$d[(x_1, t_1), (x_2, t_2)] := (|x_1 - x_2|^2 + |t_1 - t_2|)^{1/2}.$$

For $A, B \subseteq \mathbb{R}^{d+1}$, let

$$d[A, B] := \min\big\{d[(x, t), (y, s)] : (x, t) \in A, (y, s) \in B\big\}. \tag{1-6}$$

The finite range of dependence assumption is:

$$\text{For all random variables } \begin{cases} X : \mathcal{B}(A) \to \mathbb{R}, \\ Y : \mathcal{B}(B) \to \mathbb{R}, \end{cases} \text{ with } d[A, B] \geq 1,\ X, Y \text{ are } \mathbb{P}\text{-independent.} \tag{1-7}$$

(F2) *Stationarity*: For every $(M, \omega) \in \mathbb{S}^d \times \Omega$, where $\mathbb{S}^d$ denotes the space of $d \times d$ symmetric matrices with real entries, and for all $(y', s') \in \mathbb{R}^{d+1}$,

$$F(M, y + y', s + s', \omega) = F(M, y, s, \tau_{(y',s')}\omega).$$

In fact, we only use this hypothesis for $(y', s') \in \mathbb{Z}^{d+1}$.

(F3) *Uniform ellipticity*: For a fixed choice of λ, $\Lambda \in \mathbb{R}$ with $0 < \lambda \le \Lambda$, we define Pucci's extremal operators,

$$\mathcal{M}^+(M) = \sup_{\lambda I \le A \le \Lambda I} \{-\operatorname{tr}(AM)\} = -\lambda \sum_{e_i > 0} e_i - \Lambda \sum_{e_i < 0} e_i,$$
$$\mathcal{M}^-(M) = \inf_{\lambda I \le A \le \Lambda I} \{-\operatorname{tr}(AM)\} = -\lambda \sum_{e_i < 0} e_i - \Lambda \sum_{e_i > 0} e_i.$$

We assume that $F(\,\cdot\,, y, s, \omega)$ is uniformly elliptic for each $\omega \in \Omega$, i.e., for all M, $N \in \mathbb{S}^d$ and $(y, s, \omega) \in \mathbb{R}^{d+1} \times \Omega$,

$$\mathcal{M}^-(M - N) \le F(M, y, s, \omega) - F(N, y, s, \omega) \le \mathcal{M}^+(M - N).$$

(F4) *Boundedness and regularity of F*: For every $R > 0$, $\omega \in \Omega$, and $M \in \mathbb{S}^d$ with $|M| \le R$,

$$\{F(M, \cdot\,, \cdot\,, \omega)\} \text{ is uniformly bounded and uniformly equicontinuous on } \mathbb{R}^{d+1},$$

and there exists K_0 such that

$$\operatorname*{ess\,sup}_{\omega \in \Omega} \sup_{(y,s) \in \mathbb{R}^{d+1}} |F(0, y, s, \omega)| < K_0.$$

We also require that there exists a modulus of continuity $\rho[\,\cdot\,]$ and a constant $\sigma > \frac{1}{2}$ such that, for all $(M, y, s, \omega) \in \mathbb{S}^d \times \mathbb{R}^{d+1} \times \Omega$,

$$|F(M, y_1, s_1, \omega) - F(M, y_2, s_2, \omega)| \le \rho\big[(1 + |M|)(|y_1 - y_2| + |s_1 - s_2|)^\sigma\big],$$

where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^d$ and $\mathbb{R}$ respectively. By applying (F4), we have that

$$\operatorname*{ess\,sup}_{\omega \in \Omega} \sup_{(y,s) \in \mathbb{R}^{d+1}} |F(M, y, s, \omega)| \le C + \Lambda |M| \le C(1 + |M|). \tag{1-8}$$

Equipped with these assumptions, we now state the main result:

Theorem 1.1. *Assume (F1)–(F4), and fix a domain U_T and constant M_0. There exists $C = C(\lambda, \Lambda, d, M_0)$ and a random variable $\mathscr{X} : \Omega \to \mathbb{R}$ with $\mathbb{E}[\exp(\mathscr{X}(\omega))] \le C$ such that, if u^ε solves (1-1), u solves (1-2), and*

$$1 + K_0 + \|g\|_{C^{0,1}(\partial_p U_T)} \le M_0,$$

then, for any $p < d + 2$, there exists a $\beta = \beta(\lambda, \Lambda, d, p) > 0$ such that

$$\sup_{U_T} |u(x, t) - u^\varepsilon(x, t, \omega)| \le C[1 + \varepsilon^p \mathscr{X}(\omega)]\varepsilon^\beta. \tag{1-9}$$

The above theorem implies

$$\mathbb{P}\Big[\sup_{U_T} |u(x, t) - u^\varepsilon(x, t, \omega)| > C\varepsilon^\beta\Big] \le C \exp(-\varepsilon^{-p}) \tag{1-10}$$

for $\beta > 0$ independent of the boundary data. It has recently been shown in the elliptic setting [Armstrong and Smart 2014a; Armstrong and Mourrat 2015; Gloria et al. 2014; Fischer and Otto 2015] that quantitative estimates similar to (1-9) lead to a higher regularity theory at large scales. Although we do not discuss higher regularity results in this article, we are motivated by the recent progress in the elliptic setting to state our results in this form.

Notation and conventions. We mention some general notation and conventions used throughout the paper. The letters λ, Λ, K_0, T, U_T will be used exclusively to refer to the constants stated in the assumptions. In the proofs, the letters c, C will constantly be used as a generic constant which depends on these universal quantities, which may vary line by line, but is precisely specified when needed. We will always denote $\mathbb{S}^d$ as the set of symmetric $d \times d$ matrices with real entries and $\mathbb{M}^d$ as the set of $d \times d$ matrices with real entries. We use the notation $|\cdot|$ to denote a norm on a finite-dimensional Euclidean space ($\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{R}^{d+1}$ or $\mathbb{S}^d$) or the Lebesgue measure on $\mathbb{R}^{d+1}$ and we reserve $\|\cdot\|$ to denote a norm on an infinite-dimensional function space.

We choose to employ the parabolic metric

$$d[(x_1, t_1), (x_2, t_2)] = (|x_1 - x_2|^2 + |t_1 - t_2|)^{1/2}.$$

We point out that this equivalent to the metric

$$d_\infty[(x_1, t_1), (x_2, t_2)] = \max\{|x_1 - x_2|, |t_1 - t_2|^{1/2}\}.$$

We say that $f \in C^{0,\alpha}$ if, for any $(x, t), (y, s) \in \mathbb{R}^{d+1}$,

$$|f(x, t) - f(y, s)| \le \|f\|_{C^{0,\alpha}} d[(x, t), (y, s)]^\alpha.$$

For sets, we use the notation $Q \subseteq \mathbb{R}^{d+1}$ to represent an arbitrary space-time domain, i.e., $Q = Q' \times (t_1, t_2]$, where $Q' \subseteq \mathbb{R}^d$. We define the parabolic boundary by

$$\partial_p Q := (Q' \times \{t = t_1\}) \cup (\partial Q' \times [t_1, t_2)).$$

We use the convention that $\overline{Q} = Q \cup \partial_p Q$, and

$$Q(t) := \{x \in \mathbb{R}^d : (x, t) \in Q\}.$$

We use the conventions

$$\begin{aligned} B_r(\bar{x}, \bar{t}) &= B_r(\bar{x}) \times \{t = \bar{t}\}, \\ \mathcal{B}_r(\bar{x}, \bar{t}) &= \left\{(x, t) \in \mathbb{R}^{d+1} : d[(\bar{x}, \bar{t}), (x, t))] < r\right\}, \\ Q_r(\bar{x}, \bar{t}) &= B_r(\bar{x}) \times (\bar{t} - r^2, \bar{t}]. \end{aligned}$$

In general, B_r, $\mathcal{B}(r)$, and Q_r are used to denote $B_r(0, 0)$, $\mathcal{B}_r(0, 0)$, and $Q_r(0, 0)$, respectively. We point out that $\mathcal{B}_r$ and Q_r are nothing more than the open balls generated by $d[\cdot, \cdot]$ and $d_\infty[\cdot, \cdot]$, respectively.

In addition to these sets, we work with a grid of parabolic cubes which partitions $\mathbb{R}^{d+1}$. The grid boxes take the form

$$G_n = \left[-\tfrac{1}{2}3^n, \tfrac{1}{2}3^n\right)^d \times (0, 3^{2n}].$$

For every $(x,t) \in \mathbb{R}^{d+1}$, we identify the cube

$$G_n(x,t) = \left(3^n \lfloor 3^{-n}x + \tfrac{1}{2} \rfloor, 3^{2n} \lfloor 3^{-2n} t \rfloor\right) + G_n.$$

Outline of the method and the paper. In Section 2, we define the appropriate parabolic analogue of the quantity introduced in [Armstrong and Smart 2014b]. We prove the basic properties of this quantity and describe how it controls solutions from one side. In Section 3, we show how the quantity controls the behavior of solutions from the other side, utilizing the connection with the parabolic Monge–Ampère equation. Here our primary innovation beyond [Armstrong and Smart 2014b] appears.

In Section 4, we construct the effective operator $\overline{F}$ using the asymptotic properties of our quantity and we also construct approximate correctors of (1-1). In Section 5, we obtain a rate of decay on the second moments of this quantity, following closely the analysis of [Armstrong and Smart 2014b]. Finally, in Section 6, we show how the rate on the second moments yields a rate of decay on $|u^\varepsilon - u|$ in probability.

2. A subadditive quantity suitable for parabolic equations

Defining $\mu(Q, \omega, \ell, M)$. We now define the quantity which will be used extensively throughout the rest of the paper. This quantity is a functional which measures the amount a function u bends in space and time. We first recall some geometric objects relevant to the study of parabolic equations and we refer the reader to [Krylov 1976; Wang 1992; Imbert and Silvestre 2012; Gutiérrez and Huang 2001] for general references. We consider a subset $Q \subseteq \mathbb{R}^{d+1}$, a fixed environment $\omega \in \Omega$, $\ell \in \mathbb{R}$, and $M \in \mathbb{S}^d$. We then consider the set

$$S(Q, \omega, \ell, M) = \{u \in C(Q) : u_t + F(M + D^2u, x, t, \omega) \geq \ell \text{ in } Q\},$$

where the inequality is satisfied in the viscosity sense [Crandall et al. 1992], and, similarly,

$$S^*(Q, \omega, \ell, M) = \{u \in C(Q) : u_t + F(M + D^2u, x, t, \omega) \leq \ell \text{ in } Q\}.$$

To simplify the notation, we omit parameters when they are assumed to be 0, e.g., $S(Q, \omega)$ refers to the choice $\ell = 0$ and $M = 0$. We say a function u is parabolically convex if $u(\cdot, t)$ is convex for all t and u is nonincreasing in t. For any function u, we define the monotone envelope to be the supremum of all parabolically convex functions lying below u. In particular, Γ^u has the following standard representation formula, which can be taken as the definition:

$$\Gamma^u(x,t) := \sup\{p \cdot x + h : p \cdot y + h \leq u(y,s) \text{ for all } (y,s) \in Q \text{ with } s \leq t\}.$$

We point out that Γ^u depends on the domain Q, however we typically suppress this dependence.

At any point (x_0, t_0), we compute the parabolic subdifferential,

$$\mathcal{P}((x_0,t_0); u) := \left\{(p,h) \subseteq \mathbb{R}^{d+1} : \min_{x \in U, t \leq t_0} u(x,t) - p \cdot x = u(x_0,t_0) - p \cdot x_0 = h\right\},$$

which may be empty.

We then say that, for a domain $Q' \subseteq Q \subseteq \mathbb{R}^{d+1}$,

$$\begin{aligned}\mathcal{P}(Q'; u) &:= \bigcup_{(x_0,t_0)\in Q'} \mathcal{P}((x_0, t_0); u)\\ &= \Big\{(p, h) : \min_{(x,s)\in Q,\, s\le t_0} u(x, s) - p\cdot x = u(x_0, t_0) - p\cdot x_0 = h \text{ for some } (x_0, t_0) \in Q'\Big\}.\end{aligned}$$

We now define the quantity

$$\mu(Q, \omega, \ell, M) := \frac{1}{|Q|} \sup\{|\mathcal{P}(Q; \Gamma^u)| : u \in S(Q, \omega, \ell, M)\}, \tag{2-1}$$

where $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}^{d+1}$.

At this time, we point out some properties of $\mu(Q, \omega)$, which are critical for the analysis which follows:

(1) If u is constant in time, then $Q(t)$ is constant in time. The projection of $\mathcal{P}((x_0, t); u)$ into $\mathbb{R}^d$ is precisely the elliptic subdifferential of the convex envelope of u. We denote the elliptic subdifferential by $\partial\Gamma^u[t](\cdot\,;\cdot)$. This shows that, after an appropriate projection and renormalization, μ as defined in (2-1) reduces to the quantity defined in [Armstrong and Smart 2014b].

(2) This quantity respects the scaling on domains with parabolic scaling. For each $u \in S(G_n, \omega)$, let

$$u_n(x, t) := 3^{-2n} u(3^n x, 3^{2n} t) \in S(G_0, \omega).$$

Under this scaling, if $(p, h) \in \mathcal{P}(G_n; u)$, then $(3^{-n} p, 3^{-2n} h) \in \mathcal{P}(G_0; u_n)$. Thus, we have that

$$|\mathcal{P}(G_n; u)| = 3^{n(d+2)} |\mathcal{P}(G_0; u_n)|.$$

This shows us that, in order to prove statements for $\mu(G_n, \omega)$, it is enough to prove statements for $\mu(G_0, \omega)$ and rescale.

(3) If $w \in C^2(Q)$ is parabolically convex, then $\mathcal{P}((x_0, t_0); w)$ reduces to

$$\mathcal{P}((x, t); w) = (Dw(x, t), w(x, t) - Dw(x, t)\cdot x).$$

If we interpret $\mathcal{P}((\cdot\,,\cdot); w)$ as $\mathcal{P}[w](\cdot\,,\cdot) : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$, then by a standard computation,

$$\det \mathcal{D}\mathcal{P}[w] = -w_t \det D^2 w,$$

where $\mathcal{D}\mathcal{P}[w] = D_{t,x}\mathcal{P}[w]$. We point out that the right-hand side is precisely the Monge–Ampère operator first introduced in [Krylov 1976; Tso 1985]. Therefore, by applying the area formula [Evans and Gariepy 1992],

$$\frac{1}{|Q|}|\mathcal{P}(Q; w)| = \frac{1}{|Q|}\int_Q \det \mathcal{D}\mathcal{P}[w]\, dx\, dt = \frac{1}{|Q|}\int_Q -w_t \det D^2 w\, dx\, dt.$$

This shows the formal connection between the quantity $|\mathcal{P}(Q; \Gamma^u)|/|Q|$ and the parabolic Monge–Ampère equation. We will explore this connection further in Section 3.

As introduced in [Armstrong and Smart 2014b], we now define $\mu^*(G_n,\omega)$, which will serve as the analogous quantity corresponding to subsolutions. We define the involution operator $\pi(\omega)=\omega^*$ by

$$F(M,x,t,\omega^*) := -F(-M,x,t,\omega) \quad \text{for } (M,x,t,\omega)\in \mathbb{S}^d\times\mathbb{R}^{d+1}\times\Omega.$$

(Recall we assumed Ω is the space of operators F.) We point out that $\pi:\Omega\to\Omega$ is a bijection and $\omega^{**}=\omega$. Moreover, for $u\in C(\overline{Q})$,

$$u_t + F(-M+D^2u,x,t,\omega^*)\geq -\ell \quad\Longleftrightarrow\quad v:=-u \text{ solves } v_t+F(M+D^2v,x,t,\omega)\leq \ell$$

in the viscosity sense. Therefore, we define

$$\begin{aligned}\mu^*(Q,\omega,\ell,M) &:= \frac{1}{|Q|}\sup\{|\mathcal{P}(Q;\Gamma^u)| : u\in S(Q,\omega^*,-\ell,-M)\} \\ &= \mu(Q,\omega^*,-\ell,-M) \\ &= \frac{1}{|Q|}\sup\{|\mathcal{P}(Q;\Gamma^{-u})| : u\in S^*(Q,\omega,\ell,M)\}.\end{aligned} \tag{2-2}$$

Since $\pi(\omega)=\omega^*$ is an $\mathcal{F}$-measurable function on Ω, we define the pushforward

$$\pi_{\#}\mathbb{P}(E) := \mathbb{P}[\pi^{-1}(E)].$$

This justifies that $\mu^*(Q,\omega)$ enjoys the analogous properties of $\mu(Q,\omega)$ for subsolutions. Throughout the paper, we will focus on showing results for $\mu(Q,\omega)$; the analogous statements hold for $\mu^*(Q,\omega)$.

Regularity properties of $\mu(Q,\omega)$. First, we show that $\mu(Q,\omega)$ controls the behavior of supersolutions on the parabolic boundary from one side.

Lemma 2.1. *There exists a constant $c_1=c_1(d)$ such that, for every $\omega\in\Omega$, $(x,t)\in\mathbb{R}^{d+1}$, $n\in\mathbb{Z}$, and $u\in S(G_n(x,t),\omega)$,*

$$\inf_{\partial_p G_n(x,t)} u \leq \inf_{G_n(x,t)} u + c_1 3^{2n}\mu(G_n(x,t),\omega)^{1/(d+1)}. \tag{2-3}$$

Proof. Without loss of generality, in light of the scaling of $\mu(\,\cdot\,,\omega)$, it is enough to prove the statement for G_0. Moreover, we assume that $a:=\inf_{\partial_p G_0} u - \inf_{G_0} u>0$. Let $(x_0,t_0)\in G_0$ be such that $u(x_0,t_0)=\inf_{G_0} u$. This implies that, for all $|p|\leq a/\sqrt{d}$ and all $(y,s)\in\partial_p G_0$,

$$\begin{aligned}u(x_0,t_0)-p\cdot x_0 = \inf_{\partial_p G_0} u - a - p\cdot x_0 &\leq u(y,s)-p\cdot y + p\cdot(y-x_0)-a \\ &\leq u(y,s)-p\cdot y + a - a = u(y,s)-p\cdot y,\end{aligned}$$

since $|y-x_0|\leq\sqrt{d}$. This implies that the minimum of the map $(x,t)\to u(x,t)-p\cdot x$ occurs in the interior of G_0. Thus, for all $|p|\leq a/\sqrt{d}$, there exists a choice of h such that $(p,h)\in\mathcal{P}(G_0;u)$.

For each fixed p with $|p|\leq a/\sqrt{d}$, we examine which values of h are included in $\mathcal{P}(G_0;u)$. Recall that

$$h=h(t_0)=\min_{(x,t)\in G_0,\,t\leq t_0} u(x,t)-p\cdot x.$$

In particular, for each fixed p, the map $h(\cdot):\mathbb{R}\to\mathbb{R}$ is continuous. This implies that $(p,h)\in\mathcal{P}(G_0;u)$ for all $h\in\big[u(x_0,t_0)-p\cdot x_0, \inf_{\partial_p G_0}(u(x,t)-p\cdot x)\big]$.

Combining these observations, this yields that

$$\left\{(p,h):|p|\le\frac{1}{\sqrt{d}}a,\ \inf_{G_0}u-p\cdot x_0\le h\le\inf_{\partial_p G_0}u-p\cdot x\right\}\subseteq\mathcal{P}(G_0;u). \tag{2-4}$$

The left side of (2-4) contains a hypercone in $\mathbb{R}^{d+1}$ with base radius $a/\sqrt{d}$ and height a.

Therefore, we have that, for $c=c(d)$,

$$ca^{d+1}\le|\mathcal{P}(G_0;u)|.$$

Since $\mathcal{P}(G_0;u)\subseteq\mathcal{P}(G_0;\Gamma^u)$, this yields

$$a\le\left(\frac{1}{c}\right)^{\frac{1}{d+1}}\left(\frac{|\mathcal{P}(G_0;\Gamma^u)|}{|G_0|}\right)^{\frac{1}{d+1}}\le c_1\mu(G_0,\omega)^{1/(d+1)}$$

with $c_1=c_1(d)$. □

We now recall several results regarding the regularity of Γ^u. These results and their proofs can be found in [Krylov 1976; Tso 1985; Wang 1992; Imbert and Silvestre 2012].

It is sometimes useful to use an alternative representation formula for the monotone envelope, in terms of its contact points:

Lemma 2.2 [Imbert and Silvestre 2012, Lemma 4.5]. *Γ^u satisfies the alternative representation formula*

$$\Gamma^u(x,t)=\inf\left\{\sum_{i=1}^{d+1}\lambda_i u(x_i,t_i):\sum_{i=1}^{d+1}\lambda_i x_i=x,\ t_i\in[0,t],\ \sum_{i=1}^{d+1}\lambda_i=1,\ \lambda_i\in[0,1]\right\}.$$

In particular, if

$$\Gamma^u(x^0,t^0)=\sum_{i=1}^{d+1}\lambda_i u(x_i^0,t_i^0)\quad\textit{with}\quad\lambda_i>0,$$

then:

- $\Gamma^u(x_i^0,t_i^0)=u(x_i^0,t_i^0)$ *for* $i=1,\dots,d+1$.
- Γ^u *is constant with respect to t and linear with respect to x in the convex set* $\operatorname{co}\{(x_i^0,t^0),(x_i^0,t_i^0)\}_{i=1}^{d+1}$, *the convex hull of* $\{(x_i^0,t^0),(x_i^0,t_i^0)\}_{i=1}^{d+1}$.

As a consequence of this representation formula, it is natural to expect that Γ^u inherits regularity properties of the function u.

Lemma 2.3 [Imbert and Silvestre 2012, Lemma 4.11]. *Suppose that $u_t+\mathcal{M}^+(D^2u)\ge-1$. The function Γ^u is $C^{1,1}$ with respect to x and Lipschitz continuous with respect to t. In particular, $\mathcal{P}[\Gamma^u]:\mathbb{R}^{d+1}\to\mathbb{R}^{d+1}$ is Lipschitz continuous with respect to (x,t).*

In addition, if u is a supersolution to Pucci's equation, it turns out that Γ^u is actually a supersolution to a linear equation almost everywhere:

Lemma 2.4 [Imbert and Silvestre 2012, Lemma 4.12]. *Suppose that $u_t + \mathcal{M}^+(D^2 u) \geq -1$. The partial derivatives $(\Gamma^u_t, D^2\Gamma^u)$ satisfy, almost everywhere,*

$$\Gamma^u_t - \lambda\Delta\Gamma^u \geq -1 \quad \text{in } Q \cap \{u = \Gamma^u\}.$$

We next establish a lemma which shows that, in fact, $|\mathcal{P}(Q; u)| = |\mathcal{P}(Q; \Gamma^u)|$. As previously mentioned, it is immediate that $\mathcal{P}(Q; u) \subseteq \mathcal{P}(Q; \Gamma^u)$ and, thus, $|\mathcal{P}(Q; u)| \leq |\mathcal{P}(Q; \Gamma^u)|$. In order to conclude, it is enough to show the following lemma, which is the parabolic analogue of Lemma 2.4 of [Armstrong and Smart 2014b].

Lemma 2.5. *Let $Q \subseteq \mathbb{R}^{d+1}$ denote an open subset, with $u \in C(Q)$, $(x_0, t_0) \in Q$, and $r > 0$ such that*

$$Q_r(x_0, t_0) \subseteq \{(x, t) \in Q : \Gamma^u(x, t) < u(x, t)\} = \{\Gamma^u < u\}.$$

Then $|\mathcal{P}(Q_r(x_0, t_0); \Gamma^u)| = 0$.

Proof. Without loss of generality, we may assume that $r < 1$. Moreover, by a covering argument, it is enough to show that $|\mathcal{P}(Q_r(x_0, t_0); \Gamma^u)| = 0$ assuming that $Q_{3r}(x_0, t_0) \subseteq \{\Gamma^u < u\}$.

Suppose for the purposes of contradiction that $|\mathcal{P}(Q_r(x_0, t_0); \Gamma^u)| > 0$. Since the measure is positive, by the Lebesgue density theorem almost every $(p, h) \in \mathcal{P}(Q_r(x_0, t_0); \Gamma^u)$ is a density point. We mention that the density theorem still holds for parabolic cylinders and we refer the reader to the appendix of [Imbert and Silvestre 2012] for a proof. We next have the following claim:

Claim. *There exists $(x', t') \in Q_r(x_0, t_0)$ and $(p, h) \in \mathcal{P}((x', t'); \Gamma^u)$ such that (p, h) is a Lebesgue density point of $\mathcal{P}(Q_r(x_0, t_0); \Gamma^u)$ and, also, $p \in \partial\Gamma^u[t'](x')$ is a Lebesgue density point of $\partial\Gamma^u[t'](B_r(x_0))$.*

This follows from applying the Lebesgue density theorem to both $\mathcal{P}(Q_r(x_0, t_0); \Gamma^u)$ and $\partial\Gamma^u[t'](B_r(x_0))$ for some t' with $|\partial\Gamma^u[t'](B_r(x_0))| > 0$. By adding an affine function in space and translating, we may assume that $x_0 = 0$, $t_0 = 0$, $\Gamma^u(x', t') = 0$, and $(p', h') = (0, 0)$.

Since 0 is a Lebesgue density point of $\partial\Gamma^u[t'](B_r)$, for any $\bar{x} \in \partial B_r$ for r sufficiently small there exists a $\bar{p} \in \partial\Gamma^u[t'](B_r) \setminus 0$ such that

$$\bar{p} \cdot \bar{x} \geq \tfrac{3}{4}|\bar{p}|\,|\bar{x}|.$$

Suppose that $\bar{p} \in \partial\Gamma^u[t'](y)$. Since $\Gamma^u(\,\cdot\,, t') \geq 0$ in B_r, this implies that, for any $\alpha \geq 2$,

$$\Gamma^u(\alpha\bar{x}, t') \geq \Gamma^u(y, t') + \bar{p} \cdot (\alpha\bar{x} - y) \geq \alpha\bar{p} \cdot \bar{x} - \bar{p} \cdot y \geq \tfrac{3}{4}\alpha r|\bar{p}| - r|\bar{p}| > 0.$$

This and the monotonicity of Γ^u allows us to conclude that

$$\Gamma^u > 0 \quad \text{on } \{|x| \geq 2r,\ t \leq t'\}.$$

Moreover, we point out that, since $(0, 0)$ is a Lebesgue point of $\mathcal{P}(Q_r; \Gamma^u)$, for each $|x| \leq r < 1$ there exists $(p_2, h_2) \in \mathcal{P}(Q_r; \Gamma^u) \setminus (0, 0)$ such that

$$p_2 \cdot x + h_2 r^2 > \tfrac{3}{4}|(p_2, h_2)|\,|(x, r^2)| > 0.$$

Let $(p_2, h_2) \in \mathcal{P}((y,s); \Gamma^u)$ for $(y,s) \in Q_r$. This implies that, for all $t \le s$ and all $|x| \le r$, since $h_2 \ge 0$ and $r < 1$,

$$\Gamma^u(x,t) \ge p_2 \cdot x + h_2 = p_2 \cdot x + h_2 r^2 + h_2(1-r^2) > 0.$$

Therefore, for all $t \le -r^2$, we conclude again that $\Gamma^u > 0$. This implies that

$$\Gamma^u > 0 \quad \text{in } (Q \setminus Q_{2r}) \cap \{t \le t'\}.$$

However, since $u > \Gamma^u$ on Q_{3r}, this implies that $u > 0$ on all of $Q \cap \{t \le t'\}$. This contradicts that $\Gamma^u(x', t') = 0$, and hence we have the claim. □

This regularity allows us to establish:

Lemma 2.6. *Assume that $Q \subseteq \mathbb{R}^{d+1}$ is bounded and open, and $u \in C(Q)$ satisfies*

$$u_t + \mathcal{M}^+(D^2 u) \ge -1;$$

then there exists $c_2 = c_2(\lambda, d)$ such that

$$|\mathcal{P}(Q; \Gamma^u)| \le c_2 |\{u = \Gamma^u\} \cap Q|. \tag{2-5}$$

Proof. Given the regularity of Γ^u established by Lemma 2.3, we apply the area formula for Lipschitz functions to conclude that

$$|\mathcal{P}(Q;\Gamma^u)| = \int_Q \det \mathcal{D}\mathcal{P}(\Gamma^u) = \int_{Q\cap\{u=\Gamma^u\}} -\Gamma^u_t \det D^2\Gamma^u = \lambda^{-d} \int_{Q\cap\{u=\Gamma^u\}} -\Gamma^u_t \det D^2 \lambda \Gamma^u.$$

By applying the geometric–arithmetic mean inequality and Lemma 2.4, we have that

$$\begin{aligned}\lambda^{-d} \int_{Q\cap\{u=\Gamma^u\}} -\Gamma^u_t \det D^2\lambda\Gamma^u \, dx\, dt &\le c(\lambda, d) \int_{Q\cap\{u=\Gamma^u\}} [-\Gamma^u_t + \lambda \Delta \Gamma^u]^{d+1} \, dx\, dt \\ &\le c \int_{Q\cap\{u=\Gamma^u\}} 1 \, dx\, dt = c|\{u = \Gamma^u\} \cap Q|,\end{aligned}$$

which yields (2-5). □

We next claim that $\lim_{n\to\infty} \mu(G_n, \omega)$ exists almost surely. This will follow by an application of the subadditive ergodic theorem of [Akcoglu and Krengel 1980] to the quantity

$$\sup_{u \in S(G_n,\omega)} |\mathcal{P}(G_n; \Gamma^u)|.$$

We point out that the result of [Akcoglu and Krengel 1980] also holds for cubes with parabolic scaling. In order to verify the hypotheses, we first show a decomposition property of $\mu(\cdot, \omega)$:

Lemma 2.7. *For each $\omega \in \Omega$, $n \in \mathbb{Z}$, and $m \in \mathbb{N}$,*

$$\mu(G_{n+m}, \omega) \le \fint_{G_{n+m}} \mu(G_n(x,t), \omega) \, dx\, dt. \tag{2-6}$$

Proof. Let $u \in S(G_{n+m}, \omega)$. By applying Lemma 2.6, we have that, for each $(x,t) \in G_{n+m}$,

$$|\mathcal{P}(G_{n+m} \cap \partial_p G_n(x,t); \Gamma^u)| = 0.$$

Therefore,

$$\begin{aligned} |\mathcal{P}(G_{n+m}; \Gamma^u)| \leq \sum_{\{G = G_n(x,t) \subseteq G_{n+m}\}} |\mathcal{P}(G; \Gamma^u)| &= \int_{G_{n+m}} \frac{|\mathcal{P}(G_n(x,t); \Gamma^u)|}{|G_n|} \, dx\, dt \\ &\leq \int_{G_{n+m}} \frac{|\mathcal{P}(G_n(x,t); \Gamma^{\tilde{u}})|}{|G_n|} \, dx\, dt, \end{aligned}$$

where $\tilde{u} = u\restriction_{G_n(x,t)}$ for $(x,t) \in G_{n+m}$. By taking the supremum of both sides, we have (2-6). □

Lemma 2.7 shows that $\mathbb{E}[\mu(G_n, \omega)]$ is nonincreasing in n. We next show universal bounds for μ.

Lemma 2.8. *There exists* $c_3 = c_3(\lambda, \Lambda, d) > 0$ *and* $c_4 = c_4(\lambda, \Lambda, d) > 0$ *such that, for every* $\omega \in \Omega$, $n \in \mathbb{Z}$, $M \in \mathbb{S}^d$, *and* $\ell \in \mathbb{R}$,

$$c_3 \inf_{(x,t) \in G_n} (F(M,x,t,\omega) - \ell)_+^{d+1} \leq \mu(G_n, \omega, \ell, M) \leq c_4 \sup_{(x,t) \in G_n} (F(M,x,t,\omega) - \ell)_+^{d+1}. \tag{2-7}$$

Proof. We fix $M \in \mathbb{S}^d$ and, without loss of generality, we assume that $\ell = 0$. By Lemma 2.6, the right inequality holds by scaling and rearranging. To prove the left inequality, we note that, letting

$$\eta := \inf_{(x,t) \in G_n} (F(M,x,t,\omega))_+ \quad \text{and} \quad \varphi(x,t) := -\frac{\eta}{4} t + \frac{\eta}{4d\Lambda} |x|^2,$$

for each $(x,t) \in G_n$ we have

$$\varphi_t + F(M + D^2\varphi, x, t, \omega) \geq \varphi_t + \mathcal{M}^-(D^2\varphi) + F(M,x,t,\omega) = -\frac{\eta}{4} - \frac{\eta}{2} + F(M,x,t,\omega) \geq 0.$$

Therefore, $\varphi \in S(G_n, \omega, M)$, and hence

$$\mu(G_n, \omega, M) \geq \frac{|\mathcal{P}(G_n; \varphi)|}{|G_n|} = \frac{1}{|G_n|} \int -\varphi_t \det D^2\varphi = c_3 \eta^{d+1}. \qquad \square$$

In particular, we mention that (2-8) implies

$$c_3 \inf_{(x,t) \in G_n} (F(M,x,t,\omega) - \ell)_+^{d+1} \leq \mu(G_n, \omega, \ell, M) \leq c_4 [K_0(1 + |M|) - \ell]_+^{d+1}. \tag{2-8}$$

Using the previous two lemmas, we establish:

Corollary 2.9. *The limit* $\lim_{n\to\infty} \mu(G_n, \omega)$ *exists almost surely.*

Proof. We apply the subadditive ergodic theorem to the quantity

$$R(G_n, \omega) := \sup_{u \in S(G_n, \omega)} |\mathcal{P}(G_n; \Gamma^u)|.$$

We note, by the stationarity of $F(\cdot,\cdot,\cdot,\omega)$, it follows that $R(\cdot,\omega)$ is stationary. By Lemma 2.7, Lemma 2.8, and (F4), $R(\cdot,\omega)$ is subadditive on parabolic cubes and bounded almost surely. An application of the subadditive ergodic theorem yields the claim. □

In light of (F1), the limit is a constant almost surely. If $\lim_{n\to\infty} \mu(G_n(x,t),\omega)=0$, then, by (2-3), we obtain a type of comparison principle in the limit. In the next section, we will show that, if the limit is strictly positive, then we obtain control of the growth of an optimizing supersolution.

3. Strict convexity of quasimaximizers

The results in this section are completely deterministic and we suppress all dependencies on the random parameter ω. We show that $|\mathcal{P}(Q;\Gamma^u)|$ yields geometric information about the function $u \in S(Q)$. More specifically, for some $n \le 0$, if $|\mathcal{P}(G_n(x,t);\Gamma^u)|/|G_n| \approx 1$ for all $(x,t) \in G_0$, then the optimizing supersolution for $\mu(G_0)$ is strictly convex. In particular, up to an affine transformation, the optimizing supersolution bends upwards on $\partial_p G_0$.

Formally, if φ is parabolically convex with classical derivatives, then, for n sufficiently small, by the Lebesgue differentiation theorem,

$$-\varphi_t(x,t)\det D^2\varphi(x,t) \approx \fint_{G_n(x,t)} -\varphi_s \det D^2\varphi\, dy\, ds = \frac{|\mathcal{P}(G_n(x,t);\varphi)|}{|G_n|}.$$

Therefore, if $|\mathcal{P}(G_n(x,t);\varphi)|/|G_n| \approx 1$ for all (x,t), this is related to solving the parabolic Monge–Ampère equation $-\varphi_t \det D^2\varphi = 1$. This idea originated in [Armstrong and Smart 2014b], where, given an equivalent measure condition for the elliptic subdifferential of the convex envelope, the authors conclude that the optimizing supersolution is strictly convex.

In this article, we first utilize the regularity properties of $u \in S(G_0)$ to show that the time derivatives and Hessian of $w = \Gamma^u$ are uniformly bounded above almost everywhere. In particular, this bound only depends on the ellipticity constants and dimension. Using the structure of (1-5), we then obtain that the time derivative and Hessian are also strictly positive almost everywhere, which allows us to conclude that the solution must be strictly convex. We mention that this approach can also be applied to the elliptic setting of [Armstrong and Smart 2014b] to produce an alternative argument.

We first show that, by using that $u \in S(G_0)$, the monotone envelope Γ^u satisfies a uniform upper bound on the time derivative and Hessian at its contact points. Recall that, by Lemma 2.3, Γ^u is Lipschitz continuous in time and $C^{1,1}$ in space. Therefore, we may represent $(p,h) \in \mathcal{P}((x_0,t_0);\Gamma^u)$ by $(D\Gamma^u(x_0,t_0), u(x_0,t_0) - D\Gamma^u(x_0,t_0)\cdot x_0) \in \mathcal{P}((x_0,t_0);\Gamma^u)$.

Lemma 3.1. *Let $u \in S(G_0)$ and suppose*

$$\frac{|\mathcal{P}(G_{-2}(x,t);\Gamma^u)|}{|G_{-2}|} \le 2 \quad \textit{for all } (x,t) \in G_0. \tag{3-1}$$

There exists $\gamma = \gamma(\lambda,\Lambda,d)$ such that, for all $(x_0,t_0) \in Q_{1/4}(0,1) \cap \{u = \Gamma^u\}$, we have that, for all $(y,s) \in Q_{1/4}(x_0,t_0)$,

$$\Gamma^u(y,s) \le \Gamma^u(x_0,t_0) + D\Gamma^u(x_0,t_0)\cdot(y-x_0) + \gamma. \tag{3-2}$$

Proof. By the monotonicity of Γ^u, it is enough if we can show that for all $y \in B_{1/4}(x_0)$ where $u(x_0,t_0) = \Gamma^u(x_0,t_0)$,

$$\Gamma^u\big(y, t_0 - \tfrac{1}{16}\big) \le \Gamma^u(x_0,t_0) + D\Gamma^u(x_0,t_0)\cdot(y-x_0) + \gamma. \tag{3-3}$$

We proceed by contradiction. Let $w := \Gamma^u$ be defined in G_0. Assume that there exists a point (x_0, t_0) such that

$$\sup_{B_{1/4}(x_0,t_0)} w\big(\cdot\,, t_0 - \tfrac{1}{16}\big) > w(x_0, t_0) + Dw \cdot (y - x_0) + \gamma, \tag{3-4}$$

with γ to be chosen. Without loss of generality, by adding an affine function, we may assume that $(x_0, t_0) = (0, 1)$ and $\Gamma^u(x_0, t_0) = D\Gamma^u(x_0, t_0) = 0$.

Choose $\bar{y} \in \bar{B}_{1/4}$ so that

$$w\big(\bar{y}, \tfrac{15}{16}\big) := \max_{\bar{B}_{1/4}} w\big(\cdot\,, \tfrac{15}{16}\big).$$

By (3-4),

$$w\big(\bar{y}, \tfrac{15}{16}\big) > \gamma.$$

Since $w\big(\cdot\,, \tfrac{15}{16}\big)$ is convex, and using the definition of $\bar{y}$, this implies that

$$w\big(z, \tfrac{15}{16}\big) > \gamma \quad \text{for all } z \text{ such that } z \cdot \bar{y} \geq |\bar{y}|^2.$$

In particular, let $\Theta := \big\{\big(z, \tfrac{15}{16}\big) : z \in B_{1/2},\ z \cdot \bar{y} \geq |\bar{y}|^2\big\}$.

Let $\mathcal{Q} := B_{1/2} \times \big(\tfrac{15}{16}, 1\big]$. We claim there exists a test function $\varphi \in C^2(\mathcal{Q})$ which satisfies

$$\begin{cases} \varphi_t + \mathcal{M}^-(D^2\varphi) \geq 0 & \text{in } \mathcal{Q}, \\ \varphi \geq -\chi_\Theta & \text{on } \partial_p \mathcal{Q}, \end{cases} \tag{3-5}$$

and $\min \varphi(\cdot\,, 1) \leq -c$ for some universal constant c. First, by approximating $-\chi_\Theta$ by a smooth function from above and applying the Evans–Krylov theorem [Krylov 1982], there exists a supersolution which is C^2 satisfying the boundary conditions of (3-5).

By the strong maximum principle, there exists a nonconstant solution such that $\min \varphi(\cdot\,, 1) \leq -c$. Moreover, by compactness, this c can be chosen universally for all $(x_0, t_0) \in Q_{1/4}(0, 1)$ by a standard covering argument. This implies that $u + \gamma\varphi$ satisfies

$$\begin{cases} (u + \gamma\varphi)_t + F(D^2(u + \gamma\varphi), x, t) \geq 0 & \text{in } \mathcal{Q}, \\ u + \gamma\varphi \geq 0 & \text{on } \partial_p \mathcal{Q}, \\ \min_{\mathcal{Q}} (u + \gamma\varphi)(\cdot\,, 1) \leq -c\gamma. \end{cases}$$

By a similar estimate as in Lemma 2.1, this implies that $|\mathcal{P}(\mathcal{Q})| \geq c\gamma^{d+1}$. Therefore, if we consider covering $\mathcal{Q}$ with a collection of $G_{-2}(x, t) \subseteq G_0$, then

$$c\gamma^{d+1} \leq \sum_{G_{-2}(x,t) \subset G_0} |\mathcal{P}(G_{-2}(x, t))| \leq 2|G_0|.$$

Choosing γ sufficiently large, depending only on λ, Λ, and d, we obtain a contradiction. Therefore, (3-2) holds. □

By rescaling Lemma 3.1, we actually have that if, for all $(x, t) \in G_0$,

$$\frac{|\mathcal{P}(G_n(x, t); u)|}{|G_n|} \leq 2 \quad \text{and} \quad 3^n \leq \tfrac{1}{4} r,$$

then, for any point such that $u(x_0,t_0)=\Gamma^u(x_0,t_0)$, for all $(y,s)\in Q_r(x_0,t_0)$,

$$\Gamma^u(y,s)\le \Gamma^u(x_0,t_0)+D\Gamma^u(x_0,t_0)\cdot(y-x_0)+\gamma r^2. \tag{3-6}$$

By sending $r\to 0$, this implies that $\Gamma^u_t\le\gamma$ and $D^2\Gamma^u\le\gamma\,\mathrm{Id}$ at all contact points where $u=\Gamma^u$. By the construction of the monotone envelope (in particular, Lemma 2.2), this implies that $\Gamma^u_t\le\gamma$ and $D^2\Gamma^u\le\gamma\,\mathrm{Id}$ everywhere in G_0. The proof is identical to the proof of Lemma 2.3, which can be found in [Imbert and Silvestre 2012]. We choose to omit it since it follows verbatim.

We highlight that, unlike Lemma 2.3, the upper bound on the time derivatives and Hessian of Γ^u will be *independent of* K_0. An observation of [Armstrong and Smart 2014b] is that it does not seem feasible to obtain an algebraic rate if these upper bounds depend on K_0. Recall that our goal is to establish an estimate which controls supersolutions from the other side of Lemma 2.1. Since we plan on performing quantitative analysis, it is important that our estimate is *scale-invariant*. If our estimate depended on K_0 then, by (F4), the estimate would depend upon the scaling. In general, the upper bounds on the time derivative and the Hessian are controlled by the quantity $\mu(G_n(x,t))$. In light of (3-1), this is enough to conclude that γ is independent of K_0.

We next show that these upper bounds are actually enough to conclude strict convexity.

Lemma 3.2. *There exists $c_5=c_5(\lambda,\Lambda,d)>0$ such that, for every $\varepsilon>0$, there exists $n_1=n_1(\varepsilon,d)<0$ such that, if $u\in S(G_0)$ and $n\le n_1$ satisfies*

$$1\le\frac{|\mathcal{P}(G_n(x,t);\Gamma^u)|}{|G_n|}\le 2\quad\text{for all }(x,t)\in G_0, \tag{3-7}$$

then, for all $(x_0,t_0)\in Q_{1/4}(0,1)\cap\{u=\Gamma^u\}$ and all $(y,s)\in Q_{1/4}(x_0,t_0)$,

$$\Gamma^u(y,s)\ge\Gamma^u(x_0,t_0)+D\Gamma^u(x_0,t_0)\cdot(y-x_0)+c_5(t_0-s+|y-x_0|^2)-\varepsilon. \tag{3-8}$$

Proof. Fix $\varepsilon>0$. Suppose for the purposes of contradiction that (3-8) does not hold. Therefore, there exists a sequence of $(u_n,\hat y_n,\hat s_n)\in S(G_0)\times G_0$ such that u_n satisfies (3-7) for n and u_n violates (3-8) at $(\hat y_n,\hat s_n)$. Using the convention that $w_n:=\Gamma^{u_n}$ and, without loss of generality, assuming that $w_n\ge 0$ in G_0 and $w_n(0,1)=0$ for each n, this amounts to

$$w_n(\hat y_n,\hat s_n)<c(\hat s_n+|\hat y_n|^2)-\varepsilon \tag{3-9}$$

for c to be chosen.

By (3-6) and (3-2), the family $\{w_n\}$ is equicontinuous and uniformly bounded in $Q_{1/4}(0,1)$. By the Arzelà–Ascoli theorem, this implies that there exists a subsequence converging uniformly to a limiting function w, with w satisfying

$$-w_t\le\gamma\quad\text{and}\quad D^2w\le\gamma\,\mathrm{Id}\quad\text{almost everywhere.}$$

By the Lebesgue differentiation theorem and (3-7), w also satisfies

$$1\le -w_t\det D^2w\le 2\quad\text{almost everywhere.}$$

Therefore, this yields that $-w_t \geq 1/\gamma^d$, and $\det D^2 w \geq (1/\gamma)\,\mathrm{Id}$ almost everywhere. Since $D^2 w \leq \gamma\,\mathrm{Id}$, this yields that there exists a constant $c_\gamma = c(\gamma, d)$ such that $D^2 w \geq c_\gamma\,\mathrm{Id}$.

Consider that, by (3-9), since $(\hat{y}_n, \hat{s}_n) \in G_0$, there exists a subsequence converging to a point $(\hat{y}, \hat{s}) \in G_0$ satisfying

$$w(\hat{y}, \hat{s}) < c(\hat{s} + |\hat{y}|^2) - \varepsilon.$$

However, for c chosen appropriately in terms of γ, this contradicts $-w_t \geq 1/\gamma^d$, $D^2 w \geq (1/\gamma)\,\mathrm{Id}$ almost everywhere. $\square$

Finally, we show that this implies that u will also be strictly convex on the parabolic boundary.

Theorem 3.3. *Let $u \in S(G_1)$. There exist constants $c_6 = c_6(\lambda, \Lambda, d)$ and $n_1 = n_1(d) < 0$ such that, if $n \leq n_1$ satisfies*

$$1 \leq \frac{|\mathcal{P}(G_n(x,t); \Gamma^u)|}{|G_n|} \leq \mu(G_n(x,t)) \leq 1 + 3^{n(d+2)} \quad \textit{for all } (x,t) \in G_1, \tag{3-10}$$

then there exists a point $(x_0, t_0) \in \{u = \Gamma^u\} \cap G_n(0, 9)$ and $(p_0, h_0) \in \mathcal{P}((x_0, t_0); \Gamma^u)$ such that

$$u(x,t) \geq p_0 \cdot x + h_0 + c_6 \quad \textit{for all } \{t \leq t_0\} \cap G_1 \setminus G_0(0,9). \tag{3-11}$$

Proof. In order to prove (3-11), it is enough to obtain a lower bound on $\inf_{\partial_p G_0(0,9)} \Gamma^u(\,\cdot\,, t)$ for $t \leq t_0$. We claim there exists $(x_0, t_0) \in G_n(0,9)$ such that $u(x_0, t_0) = \Gamma^u(x_0, t_0)$. By (3-10), for any $(y,s) \in G_n(0,9)$,

$$\begin{aligned} 1 &\leq \int_{G_0(0,9)} \frac{|\mathcal{P}(G_n(x,t); \Gamma^u)|}{|G_n|}\,dx\,dt \\ &= |\mathcal{P}(G_n(y,s); \Gamma^u)| + \int_{G_0(0,9)\setminus G_n(y,s)} \frac{|\mathcal{P}(G_n(x,t); \Gamma^u)|}{|G_n|}\,dx\,dt \\ &\leq |\mathcal{P}(G_n(y,s); \Gamma^u)| + (1 - 3^{n(d+2)})(1 + 3^{n(d+2)}). \end{aligned}$$

This shows that $|\mathcal{P}(G_n(y,s); \Gamma^u)| > 0$ for any $(y,s) \in G_0$, which implies, by Lemma 2.6, that

$$|G_n(0,9) \cap \{u = \Gamma^u\}| > 0.$$

Let $(x_0, t_0) \in G_n(0,9) \cap \{u = \Gamma^u\}$ and consider $(p_0, h_0) \in \mathcal{P}((x_0, t_0); \Gamma^u)$. Let $\tilde{u}(x,t) = u(x,t) - p_0 \cdot x - h_0$. Then $\tilde{u} \in S(G_0(0,9))$ and $\tilde{u}(x_0, t_0) = \Gamma_{\tilde{u}}(x_0, t_0) = 0$. Moreover, we have that $(0,0) \in \mathcal{P}((x_0, t_0); \Gamma^{\tilde{u}})$ and $\Gamma^{\tilde{u}} \geq 0$ for all $(x,t) \in G_0(0,9) \cap \{t \leq t_0\}$.

By Lemma 3.2, letting $\varepsilon = \frac{1}{2} c_5$, since $Q_{1/4}(x_0, t_0) \subset G_0(0,9)$, this implies that, on $\partial_p G_0(0,9)$,

$$u(x,t) \geq \Gamma^u(x,t) \geq \tfrac{1}{2} c_5.$$

Defining $c_6 := \frac{1}{2} c_5$ completes the proof. $\square$

For convenience, we also provide a rescaled version of (3-11) which will be used extensively later in the paper. Let $u \in S(G_{m+n+1})$. Choose $n \leq n_1$ so that

$$\alpha \leq \frac{|\mathcal{P}(G_n(x,t); \Gamma^u)|}{|G_n|} \leq \mu(G_n(x,t)) \leq (1 + 3^{n(d+2)})\alpha \quad \text{for all } (x,t) \in G_{m+n+1}.$$

There exists a point $(x_0, t_0) \in \{u = \Gamma^u\} \cap G_n(0, 3^{2(m+n+1)})$ and $(p_0, h_0) \in \mathcal{P}((x_0, t_0); \Gamma^u)$ such that

$$u(x,t) \geq p_0 \cdot x + h_0 + c_6 \alpha^{1/(d+1)} 3^{2(m+n)} \quad \text{for all } \{t \leq t_0\} \cap G_{m+n+1} \setminus G_{m+n}(0, 3^{2(m+n+1)}). \tag{3-12}$$

4. The construction of $\bar{F}$ and the construction of approximate correctors

We now define the homogenized operator $\bar{F} : \mathbb{S}^d \to \mathbb{R}$. In addition, we show how one can obtain "approximate correctors" as in [Lin 2015] using the quantity μ. For each $M \in \mathbb{S}^d$, we say that w^ε is an approximate corrector of (1-1) if there exists w^ε satisfying

$$\begin{cases} w^\varepsilon_t + F(M + D^2 w^\varepsilon, x, t, \omega) = \bar{F}(M) & \text{in } Q_{1/\varepsilon}, \\ w^\varepsilon = 0 & \text{on } \partial_p Q_{1/\varepsilon}, \end{cases} \tag{4-1}$$

with $\|\varepsilon^2 w^\varepsilon\|_{L^\infty(Q_{1/\varepsilon})} \to 0$ as $\varepsilon \to 0$. Once w^ε exists, the qualitative homogenization (the convergence $u^\varepsilon \to u$ $\mathbb{P}$-a.s.) follows by a standard perturbed test function argument [Evans 1992], as shown in [Lin 2015]. In particular, the uniform ellipticity of $\bar{F}$ follows from the existence of approximate correctors.

Identifying $\bar{F}$. We identify $\bar{F}(M)$ for each fixed $M \in \mathbb{S}^d$. First, we establish a lemma which states that μ is Lipschitz continuous with respect to the right-hand side ℓ.

Lemma 4.1. *There exists $C(\lambda, \Lambda, d, M, K_0) > 0$ such that*

$$0 \geq \mu(Q, \omega, \ell + s, M) - \mu(Q, \omega, \ell, M) \geq -C|Q|s \tag{4-2}$$

for all $s \in [0, 1]$.

Proof. Since $S(Q, \omega, \ell+s, M) \subseteq S(Q, \omega, \ell, M)$, the left inequality follows from the comparison principle for viscosity solutions. To obtain the right inequality, let $u \in S(Q, \omega, \ell, M)$ and define $u^s(x,t) := u(x,t) + st$, which lies in $S(Q, \omega, \ell + s, M)$. Let w^s denote the monotone envelope of u^s. We note that $|w^s_t|, |D^2 w^s| \leq C(K_0, \ell + s, M)$ on the contact set $\{u^s = w^s\}$, by Lemma 2.3 and Lemma 2.6. Therefore, by the area formula, this implies that

$$\begin{aligned} |\mathcal{P}(Q; w^s)| &= \int_{\{u^s = w^s\} \cap Q} -u^s_t \det D^2 u^s \, dx, \\ &\geq \int_{\{u=w\} \cap \{u_t \leq -s\} \cap Q} -u^s_t \det D^2 u^s \, dx, \\ &\geq \int_{\{u=w\} \cap Q} -u_t \det D^2 u - Cs|Q| \\ &= |\mathcal{P}(Q; w)| - Cs|Q|. \end{aligned}$$

By taking the supremum over $u \in S(Q, \omega, \ell, M)$, this yields (4-2). □

Lemma 4.2. *Let $M \in \mathbb{S}^d$. For every $n \in \mathbb{N}$, the map*

$$\ell \to \mathbb{E}[\mu(G_n, \omega, \ell, M)] \quad \textit{is continuous and nonincreasing.}$$

Similarly, the map

$$\ell \to \mathbb{E}[\mu^*(G_n, \omega, \ell, M)] \quad \textit{is continuous and nondecreasing.}$$

In addition, there exists $\bar{\ell}(M) \in \mathbb{R}$ such that, $\mathbb{P}$-a.s. in ω,

$$\lim_{n\to\infty} \mu(G_n, \omega, \bar{\ell}(M), M) = \lim_{n\to\infty} \mathbb{E}[\mu(G_n, \omega, \bar{\ell}(M), M)] = \lim_{n\to\infty} \mathbb{E}[\mu^*(G_n, \omega, \bar{\ell}(M), M)] \\ = \lim_{n\to\infty} \mu^*(G_n, \omega, \bar{\ell}(M), M). \tag{4-3}$$

Proof. The Lipschitz continuity and monotonicity follow from Lemma 4.1. By (2-8), $\mathbb{E}[\mu(G_n, \omega, \ell)] = 0$ for all $\ell \geq K_0(1+|M|)$. In particular, this implies that

$$\lim_{n\to\infty} \mathbb{E}[\mu(G_n, \omega, \ell)] = 0 \quad \text{for all } \ell \geq K_0(1+|M|).$$

Similarly,

$$\lim_{n\to\infty} \mathbb{E}[\mu^*(G_n, \omega, \ell)] = 0 \quad \text{for all } \ell \leq -K_0(1+|M|).$$

Using the monotonicity in ℓ and (2-8), there exists a choice of $\bar{\ell}$ such that $\lim_{n\to\infty} \mathbb{E}[\mu(G_n, \omega, \bar{\ell})] = \lim_{n\to\infty} \mathbb{E}[\mu^*(G_n, \omega, \bar{\ell})]$. The outer equalities of (4-3) hold in light of the ergodicity assumption (F1) and the subadditive ergodic theorem. □

Using Lemma 4.2, we define

$$\bar{F}(M) := \bar{\ell}(M). \tag{4-4}$$

We will now show that $\bar{F}(M)$ agrees with the effective operator constructed in [Lin 2015] and thus the uniqueness follows. To do this, it is enough to show that solutions w^ε of (4-1) exist and satisfy the desired limiting behavior.

A qualitative homogenization argument. The construction of approximate correctors (4-1) follows in two steps. First we show that, for any $M \in \mathbb{S}^d$, it is impossible for $E(\bar{\ell}(M), M) := \lim_{n\to\infty} \mu(G_n, \omega, \bar{\ell}(M), M)$ and $E^*(\bar{\ell}, M) := \lim_{n\to\infty} \mu^*(G_n, \omega, \bar{\ell}(M), M)$ to both be positive. Applying Lemma 2.1 allows us to conclude.

For convenience, we provide a precise statement of the Harnack inequality for parabolic equations, as can be found in [Wang 1992; Imbert and Silvestre 2012]. We will use the notation of this theorem in the future.

Theorem 4.3 (Harnack inequality). *Let u be nonnegative with $-|f| \leq u_t + \mathcal{M}^+(D^2u) \leq |f|$. Then there exists a universal $C = C(\lambda, \Lambda, d)$ such that*

$$\sup_{\tilde{Q}} u \leq C\Big(\inf_{Q_{\rho^2}} u + \|f\|_{L^{d+1}(Q_1)}\Big),$$

where $\tilde{Q} := B_{\rho^2/(2\sqrt{2})} \times \big(-\rho^2 + \frac{3}{8}\rho^4, -\rho^2 + \frac{1}{2}\rho^4\big) \subseteq Q_1$ and $\rho = \rho(\lambda, \Lambda, d)$.

The Harnack inequality implies that E and E^* must vanish when they are equal:

Lemma 4.4. *Fix $M \in \mathbb{S}^d$. If $\ell \in \mathbb{R}$ is such that*

$$\lim_{n\to\infty} \mathbb{E}[\mu(G_n, \omega, \ell, M)] = E(\ell, M) = E^*(\ell, M) = \lim_{n\to\infty} \mathbb{E}[\mu(G_n, \omega^*, -\ell, M)], \tag{4-5}$$

then $E(\ell, M) = E^*(\ell, M) = 0$.

Proof. We drop the dependence on M since it is fixed throughout the proof. Suppose that both $E(\ell) = E^*(\ell) := \alpha > 0$. By the subadditive ergodic theorem, there exists a choice of m sufficiently large such that, for all $(x, t) \in G_{m+n}$, with n large to be chosen,

$$\tfrac{1}{2}\alpha \le \frac{|\mathcal{P}(G_m(x,t); \Gamma^u)|}{|G_m|} \le \mu(G_m, \omega, \ell) \le 2\alpha.$$

Without loss of generality, we assume that $m = 0$. By Theorem 3.3, rescaled, choosing n sufficiently large, and after an affine transformation, there exists a function u such that

$$u_t + F(D^2u, x, t, \omega) = \ell \quad \text{in } G_n(0, 3^{2(n+1)}) \tag{4-6}$$

and $(x_0, t_0) \in G_0(0, 3^{2(n+1)})$ such that

$$u \ge u(x_0, t_0) + C3^{2n}\alpha^{1/(d+1)} \quad \text{on } \partial_p G_n(0, 3^{2(n+1)}) \cap \{t \le t_0\} \tag{4-7}$$

and

$$\inf_{G_n(0,3^{2(n+1)})\cap\{t\le t_0\}} u = \inf_{G_0(0,3^{2(n+1)})\cap\{t\le t_0\}} u = u(x_0, t_0) = 0.$$

This is done by extracting $u' \in S(G_{n+1}, \omega)$ such that (3-11) holds. Upon an affine transformation and solving (4-6) with $u = u'$ on $\partial_p G_n(0, 3^{2(n+1)})$, we have the claim. Similarly, there exists u^* satisfying

$$u^*_t + F(D^2u^*, x, t, \omega^*) = -\ell \quad \text{in } G_n(0, 3^{2(n+1)}) \tag{4-8}$$

and, for some $(x^*_0, t^*_0) \in G_0(0, 3^{2(n+1)})$,

$$u^* \ge u^*(x_0, t_0) + C3^{2n}\alpha^{1/(d+1)} \quad \text{on } \partial_p G_n(0, 3^{2(n+1)}) \cap \{t \le t^*_0\} \tag{4-9}$$

and

$$\inf_{G_n(0,3^{2(n+1)})\cap\{t\le t^*_0\}} u^* = \inf_{G_0(0,3^{2(n+1)})\cap\{t\le t^*_0\}} u^* = u^*(x^*_0, t^*_0) = 0.$$

Let $\bar{t} = \min\{t_0, t^*_0\}$. Notice that $w := u + u^*$ satisfies

$$w_t + \mathcal{M}^+(D^2w) \ge u_t + u^*_t + F(D^2u, x, t, \omega) + F(D^2u^*, x, t, \omega^*) = 0 \quad \text{in } G_n(0, 3^{2(n+1)})$$

and

$$w \ge C3^{2n}\alpha^{1/(d+1)} \quad \text{on } \partial_p G_n(0, 3^{2(n+1)}) \cap \{t \le \bar{t}\}.$$

By the Alexandrov–Backelman–Pucci–Krylov–Tso estimate [Wang 1992; Imbert and Silvestre 2012], this implies that

$$w \ge C3^{2n}\alpha^{1/(d+1)} \quad \text{in } G_n(0, 3^{2(n+1)}) \cap \{t \le \bar{t}\}. \tag{4-10}$$

Let s be defined as the smallest integer such that $\rho^2 3^s \ge \sqrt{d}$, where ρ is defined in the Harnack inequality (Theorem 4.3). We may assume that $s \le n$, by choosing n larger if necessary. We observe that, in $G_s(0, 3^{2(n+1)})$, u and u^* also each satisfy

$$u_t + \mathcal{M}^+(D^2u) \ge -|\ell| - K_0 \quad \text{and} \quad K_0 + |\ell| \ge u_t + \mathcal{M}^-(D^2u),$$
$$u^*_t + \mathcal{M}^+(D^2u^*) \ge -|\ell| - K_0 \quad \text{and} \quad |\ell| + K_0 \ge u^*_t + \mathcal{M}^-(D^2u^*).$$

Since $\inf_{G_0(0,3^{2(n+1)})} u = \inf_{G_0(0,3^{2(n+1)})} u^* = 0$, and

$$G_0(0, 3^{2(n+1)}) \subseteq Q_{\rho^2 3^s}(0, 3^{2(n+1)})$$

by our choice of s, this implies, by the Harnack inequality, that there exists $C = C(\lambda, \Lambda, d, \ell, K_0)$ such that

$$\sup_{\tilde{Q}} u \le C3^{2s} \quad \text{and} \quad \sup_{\tilde{Q}} u^* \le C3^{2s},$$

where $\tilde{Q} \subseteq G_s(0, 3^{2(n+1)})$ is a rescaled version of the $\tilde{Q}$ defined in Theorem 4.3. Thus, there exists $C = C(\lambda, \Lambda, d, \ell, K_0) > 0$ such that

$$w \le C3^{2s} \quad \text{in } \tilde{Q} \subseteq G_s(0, 3^{2(n+1)}).$$

By choosing n sufficiently large, depending on ℓ, K_0, and α, we obtain a contradiction with (4-10). Therefore, $\alpha = 0$. $\square$

We next show that w^ε solving (4-1) has the desired decay with this definition of $\bar{F}(M)$. Letting $\varepsilon = 3^{-n}$, we relabel (4-1) as

$$\begin{cases} w_t^n + F(M + D^2 w^n, x, t, \omega) = \bar{F}(M) & \text{in } G_n, \\ w^n = 0 & \text{on } \partial_p G_n, \end{cases} \tag{4-11}$$

and we want to show that $\|3^{-2n} w^n\|_{L^\infty(G_n)} \to 0$ as $n \to \infty$.

Consider that, since $E(\bar{F}(M), M) = E^*(\bar{F}(M), M) = 0$, this implies that, almost surely,

$$\lim_{n\to\infty} \mu(G_n, \omega) = 0 = \lim_{n\to\infty} \mu^*(G_n, \omega).$$

By Lemma 2.1 and (4-11), this implies that

$$\begin{aligned} &0 \le \inf_{G_n} 3^{-2n} w^n + c_1 \mu(G_n, \omega)^{1/(d+1)} \\ \text{and} \quad &0 \ge \sup_{G_n} 3^{-2n} w^n - c_1 \mu^*(G_n, \omega)^{1/(d+1)}. \end{aligned}$$

Taking $n \to \infty$, this yields

$$\lim_{n\to\infty} \|3^{-2n} w^n\|_{L^\infty(G_n)} \le \lim_{n\to\infty} \max\{\mu(G_n, \omega)^{1/(d+1)}, \mu^*(G_n, \omega)^{1/(d+1)}\} = 0, \tag{4-12}$$

as desired.

5. A rate of decay on the second moments

In this section, we obtain a rate of decay on the second moments of μ. The approach of this section closely follows that of [Armstrong and Smart 2014b]. As before, we suppress the dependence on M. We simplify the notation by adopting the following conventions. Let

$$E_n(\ell) = \mathbb{E}[\mu(G_n, \omega, \ell)] \quad \text{and} \quad E_n^*(\ell) = \mathbb{E}[\mu^*(G_n, \omega, \ell)] = \mathbb{E}[\mu(G_n, \omega^*, -\ell)].$$

Also, let

$$J_n(\ell) = \mathbb{E}[\mu(G_n, \omega, \ell)^2] \quad \text{and} \quad J_n^*(\ell) = \mathbb{E}[\mu^*(G_n, \omega, \ell)^2] = \mathbb{E}[\mu(G_n, \omega^*, -\ell)^2].$$

Our next lemma shows that, if the variance of μ and μ^* are not decaying, then their expectations must be close to zero. The proof resembles the argument for Lemma 4.4, but avoids the dependence on K_0.

Lemma 5.1. *Suppose that there exists $m, n \in \mathbb{N}$ and $\eta, \gamma > 0$ such that*

$$0 < J_m(\ell - \gamma) \le (1+\eta) E^2_{m+n}(\ell - \gamma) \tag{5-1}$$

and

$$0 < J^*_m(-\ell + \gamma) \le (1+\eta) E^{*2}_{m+n}(-\ell + \gamma). \tag{5-2}$$

Then there exists $n_0 = n_0(\lambda, \Lambda, d)$ and $\eta_0 = \eta_0(\lambda, \Lambda, d)$ such that, for all $n \ge n_0$ and all $\eta \le \eta_0$,

$$J_{m+n}(\ell - \gamma) + J^*_{m+n}(-\ell + \gamma) \le C\gamma^{2(d+1)}. \tag{5-3}$$

Proof. Without loss of generality, we assume that $\ell = 0$ and $m = 0$. First, we claim that there exists a choice of environment ω such that $\mu(G_n, \omega)$ and $\mu(G_0(x,t), \omega)$ are approximately constant for all $(x,t) \in G_n$.

Fix $\delta > 0$. There exists $\eta = \eta(\delta)$ such that, if (5-1) and (5-2) hold for this η, there exists an ω such that, for all $(x,t) \in G_n$,

$$(1-\delta) E_n(-\gamma) \le \mu(G_n, \omega, -\gamma) \le \mu(G_0(x,t), \omega, -\gamma) \le (1+\delta) E_n(-\gamma) \tag{5-4}$$

and, similarly for the lower quantity,

$$(1-\delta) E^*_n(\gamma) \le \mu^*(G_n, \omega, \gamma) \le \mu^*(G_0(x,t), \omega, \gamma) \le (1+\delta) E^*_n(\gamma). \tag{5-5}$$

Applying Chebyshev's inequality, we have that, for any $(x,t) \in G_n$,

$$\begin{aligned}
\mathbb{P}\big[\mu(G_0(x,t), \omega, -\gamma) \ge (1+\delta) E_n(-\gamma)\big] &\le \mathbb{P}\big[\mu(G_0(x,t), \omega, -\gamma) - E_n(-\gamma) \ge \delta E_n(-\gamma)\big] \\
&\le \mathbb{P}\big[[\mu(G_0(x,t), \omega, -\gamma) - E_n(-\gamma)]^2 \ge \delta^2 E^2_n(-\gamma)\big] \\
&\le \frac{1}{\delta^2 E^2_n(-\gamma)} \mathbb{E}\big[[\mu(G_0(x,t), \omega, -\gamma) - E_n(-\gamma)]^2\big] \\
&\le \frac{1}{\delta^2 E^2_n(-\gamma)} [J_0(-\gamma) - E^2_n(-\gamma)] \\
&\le \eta\delta^{-2},
\end{aligned}$$

where the last inequality follows from (5-1).

Similarly,

$$\begin{aligned}
\mathbb{P}\big[\mu(G_n, \omega, -\gamma) < (1-\delta) E_n(-\gamma)\big] &\le \mathbb{P}\big[(\mu(G_n, \omega, -\gamma) - E_n(-\gamma))^2 \ge \delta^2 E_n(-\gamma)^2\big] \\
&\le \frac{1}{\delta^2 E_n(-\gamma)^2} \mathbb{E}\big[(\mu(G_n, \omega, -\gamma) - E_n(-\gamma))^2\big] \\
&\le \frac{1}{\delta^2 E_n(-\gamma)^2} \big(\mathbb{E}[\mu(G_n, \omega, -\gamma)^2] - E_n(-\gamma)^2\big) \\
&\le \eta\delta^{-2}.
\end{aligned}$$

By identical arguments,

$$\mathbb{P}\big[\mu^*(G_0(x,t), \omega, \gamma) \ge (1+\delta) E^*_n(\gamma)\big] \le \eta\delta^{-2} \quad \text{and} \quad \mathbb{P}\big[\mu^*(G_n, \omega, \gamma) < (1-\delta) E^*_n(\gamma)\big] \le \eta\delta^{-2}.$$

By a union bound, this implies that

$$\mathbb{P}\big[\text{(5-4) and (5-5) hold for all } (x,t)\in G_n\big] \geq 1-4\eta\delta^{-2}, \tag{5-6}$$

so, by choosing $\eta \leq \frac{1}{4}\delta^2$, this has positive probability. Let $\omega \in \Omega$ be an element of this set, which implies ω satisfies (5-4) and (5-5) for all $(x,t)\in G_n$. Using this particular ω, we next show that there exist constants c, C, and $s \in \mathbb{N}$ which only depend on λ, Λ, and d such that

$$c(E_n(-\gamma)+E_n^*(\gamma)-C\gamma^{d+1}) \leq (1+\delta)3^{-2(n-s)(d+1)}(E_n(-\gamma)+E_n^*(\gamma)). \tag{5-7}$$

Consider that, by Theorem 3.3, similar to the proof of Lemma 4.4, there exists $n=n(d,\lambda,\Lambda)$ and $u, u^* \in C(G_n(0,3^{2(n+1)}))$ such that

$$u_t + F(D^2u, x, t, \omega) = -\gamma \quad \text{in } G_n(0,3^{2(n+1)})$$

with

$$\inf_{\partial_p G_n(0,3^{2(n+1)})\cap\{t\leq t_0\}} u(x,t) \geq C3^{2n}E_n(-\gamma)^{1/(d+1)} \quad \text{and} \quad \inf_{G_0(0,3^{2(n+1)})} u = \inf_{G_n(0,3^{2(n+1)})} u = 0.$$

Similarly, u^* satisfies

$$u_t^* + F(D^2u^*, x, t, \omega^*) = -\gamma \quad \text{in } G_n(0,3^{2(n+1)}),$$

with

$$\inf_{\partial_p G_n(0,3^{2(n+1)})\cap\{t\leq t_0^*\}} u^*(x,t) \geq C3^{2n}E_n^*(\gamma)^{1/(d+1)} \quad \text{and} \quad \inf_{G_0(0,3^{2(n+1)})} u^* = \inf_{G_n(0,3^{2(n+1)})} u^* = 0.$$

Let $\tilde{t} = \min\{t_0, t_0^*\}$. We note that the function $u+u^*$ satisfies that

$$u+u^* \geq C3^{2n}(E_n(-\gamma)^{1/(d+1)} + E_n^*(\gamma)^{1/(d+1)}) \quad \text{on } \partial_p G_n(0,3^{2(n+1)})\cap\{t\leq\tilde{t}\}$$

and

$$(u+u^*)_t + \mathcal{M}^+(D^2(u+u^*)) \geq -2\gamma \quad \text{in } G_n(0,3^{2(n+1)}).$$

By the Alexandrov–Backelman–Pucci–Krylov–Tso estimate [Wang 1992; Imbert and Silvestre 2012], this implies that

$$u+u^* \geq c3^{2n}[E_n(-\gamma)^{1/(d+1)} + E_n^*(\gamma)^{1/(d+1)}] - C3^{2n}\gamma \quad \text{in } G_n(0,3^{2(n+1)})\cap\{t\leq\tilde{t}\}. \tag{5-8}$$

Next, consider the solutions w, $\tilde{w}$ solving

$$\begin{cases} w_t + F(D^2w, x, t, \omega) = -\gamma & \text{in } G_s(0,3^{2(n+1)}), \\ w = 0 & \text{on } \partial_p G_s(0,3^{2(n+1)}), \end{cases}$$

and

$$\begin{cases} w_t^* + F(D^2w^*, x, t, \omega^*) = -\gamma & \text{in } G_s(0,3^{2(n+1)}), \\ w^* = 0 & \text{on } \partial_p G_s(0,3^{2(n+1)}), \end{cases}$$

with s, to be chosen, such that $s \leq n$.

We have that

$$w + w^* = 0 \quad \text{on } \partial_p G_s(0,3^{2(n+1)})$$

and

$$(w+w^*)_t + \mathcal{M}^-(D^2(w+w^*)) \le -2\gamma \le 0 \quad \text{in } G_s(0, 3^{2(n+1)}).$$

This implies that

$$w + w^* \le 0 \quad \text{in } G_s(0, 3^{2(n+1)}). \tag{5-9}$$

Combining (5-8) and (5-9), we have that, for all $(x,t) \in G_s(0, 3^{2(n+1)}) \cap \{t \le \bar{t}\}$,

$$w(x,t) - u(x,t) + w^*(x,t) - u^*(x,t) \le C3^{2n}\gamma - c3^{2n}(E_n(-\gamma)^{1/(d+1)} + E_n^*(\gamma)^{1/(d+1)}). \tag{5-10}$$

Notice that

$$w - u \le 0 \quad \text{on } \partial_p G_s(0, 3^{2(n+1)})$$

and, in $G_s(0, 3^{2(n+1)})$,

$$(w-u)_t + \mathcal{M}^+(D^2(w-u)) \ge 0 \ge (w-u)_t + \mathcal{M}^-(D^2(w-u)).$$

This implies that $w - u \le 0$ in $G_s(0, 3^{2(n+1)})$. Consider the Harnack inequality (Theorem 4.3) applied to $u - w \ge 0$. By the Harnack inequality, rescaled in $G_s(0, 3^{2(n+1)})$ (where $\tilde{Q}$ corresponds to the rescaled $\tilde{Q}$),

$$\sup_{\tilde{Q}}(u-w) \le C \inf_{Q_{\rho 2^{3^s}}(0, 3^{2(n+1)})}(u-w).$$

This implies that

$$-\sup_{\tilde{Q}}(u-w) \ge -C \inf_{Q_{\rho 2^{3^s}}(0, 3^{2(n+1)})}(u-w),$$

which yields

$$\inf_{\tilde{Q}}(w-u) \ge C \sup_{Q_{\rho 2^{3^s}}(0, 3^{2(n+1)})}(w-u). \tag{5-11}$$

Choose s so that $G_0(0, 3^{2(m+1)}) \subseteq Q_{\rho 2^{3^s}}(0, 3^{2(m+1)})$. Since (5-10) holds for all

$$(x,t) \in G_s(0, 3^{2(n+1)}) \cap \{t \le \tilde{t}\} \quad \text{and} \quad \tilde{Q} \subseteq G_s(0, 3^{2(n+1)}) \cap \{t \le \tilde{t}\},$$

we may assume without loss of generality that

$$\inf_{\tilde{Q}}(w-u) \le \tfrac{1}{2}\big(C3^{2n}\gamma - c3^{2n}(E_n(-\gamma)^{1/(d+1)} + E_n^*(\gamma)^{1/(d+1)})\big).$$

(If not, then we repeat this analysis for $w^* - u^*$.) By (5-11), this implies that, in $Q_{\rho 2^{3^s}}(0, 3^{2(n+1)})$,

$$w - u \le C\big(3^{2n}\gamma - c3^{2n}(E_n(-\gamma)^{1/(d+1)} + E_n^*(\gamma)^{1/(d+1)})\big).$$

In particular, we have that

$$\inf_{Q_{\rho 2^{3^s}}(0, 3^{2(n+1)})} w \le \inf_{Q_{\rho 2^{3^s}}(0, 3^{2(n+1)})} u + c\big(C3^{2n}\gamma - 3^{2n}(E_n(-\gamma)^{1/(d+1)} + E_n^*(\gamma)^{1/(d+1)})\big).$$

Since $(x_0, t_0) \in G_0(0, 3^{2(n+1)}) \subseteq Q_{\rho 2^{3^s}}(0, 3^{2(n+1)})$, this implies that

$$\inf_{Q_{\rho 2^{3^s}}(0, 3^{2(n+1)})} u = 0,$$

which yields

$$\inf_{Q_{\rho 2 3^s}(0,3^{2(n+1)})} w \le c\big(C3^{2n}\gamma - 3^{2n}(E_n(-\gamma)^{1/(d+1)} + E_n^*(\gamma)^{1/(d+1)})\big). \tag{5-12}$$

By Lemma 2.1, since $w = 0$ on $\partial_p G_s(0, 3^{2(n+1)})$,

$$\begin{aligned} 0 &\le \inf_{G_s(0,3^{2(n+1)})} w + c_1 3^{2s}\mu(G_s(0, 3^{2(n+1)}), \omega, -\gamma)^{1/(d+1)} \\ &\le \inf_{Q_{\rho 2 3^s}(0,3^{2(n+1)})} w + c_1 3^{2s}\mu(G_s(0, 3^{2(n+1)}), \omega, -\gamma)^{1/(d+1)}. \end{aligned}$$

By (5-12), this implies

$$\begin{aligned} c3^{2(n-s)(d+1)}(E_n(-\gamma)^{1/(d+1)} + E_n^*(\gamma)^{1/(d+1)} - C\gamma)^{d+1} &\le \mu(G_s(0, 3^{2(n+1)}), \omega, -\gamma) \\ &\le \fint_{G_s(0,3^{2(n+1)})} \mu(G_0(x,t), \omega)\, dx\, dt \\ &\le (1+\delta)E_n(-\gamma) \\ &\le (1+\delta)(E_n(-\gamma) + E_n^*(\gamma)). \end{aligned}$$

This yields

$$3^{2(n-s)(d+1)}c(E_n(-\gamma) + E_n^*(\gamma) - C\gamma^{d+1}) \le (1+\delta)(E_n(-\gamma) + E_n^*(\gamma)),$$

which is equivalent to (5-7).

To conclude, we just need to choose δ, η, and show there is an n sufficiently large to obtain (5-3). Rearranging yields

$$(1 - 3^{-2(n-s)(d+1)} - \delta 3^{-2(n-s)(d+1)})(E_n(-\gamma) + E_n^*(\gamma)) \le C\gamma^{d+1}.$$

Choosing $\delta := 3^{-2s(d+1)}$ and $\eta \le \frac{1}{4}3^{-4s(d+1)}$ yields a choice of $\omega \in \Omega$ such that (5-4) and (5-5) hold, and

$$(1 - 3^{-2(n-s)(d+1)} - 3^{-2n(d+1)})(E_n(-\gamma) + E_n^*(\gamma)) \le C\gamma^{d+1}.$$

For any $n \ge 2s$, we have that

$$E_n(-\gamma) + E_n^*(\gamma) \le C(1 - 3^{-2s(d+1)} - 3^{-4s(d+1)})^{-1}\gamma^{d+1} = C\gamma^{d+1}.$$

This implies that

$$J_n(-\gamma) + J_n^*(\gamma) \le (1+\eta)(E_n(-\gamma)^2 + E_n^*(\gamma)^2) \le C\gamma^{2(d+1)},$$

as asserted. □

We next show how the finite range of dependence assumption (F1) yields a relation between $J_{m+n}(\ell)$ and $J_m(\ell)$ for $n > 0$.

Lemma 5.2. *There exists $c_7 = c_7(d)$ such that, for any ℓ and any $m, n \ge 0$,*

$$J_{m+n}(\ell) \le E_m^2 + \frac{c_7}{3^{n(d+2)}} J_m(\ell). \tag{5-13}$$

Similarly,

$$J^*_{m+n}(-\ell) \le E^{*2}_m + \frac{c_7}{3^{n(d+2)}} J^*_m(-\ell). \tag{5-14}$$

Proof. Since ℓ plays no role, we suppress its dependence. Consider that $G_{m+n} = \bigcup_{i=1}^{3^{n(d+2)}} G^i_m$ for some choice of enumeration of cubes $\{G^i_m\}$. Therefore, for each $u \in S(G_{m+n}, \omega)$,

$$\begin{aligned}
&|\mathcal{P}(G_{m+n}; u)|^2 \\
&\quad = \bigg(\sum_{i=1}^{3^{n(d+2)}} |\mathcal{P}(G^i_m; u)|\bigg)^2 \\
&\quad = \sum_i |\mathcal{P}(G^i_m; u)|^2 + \sum_i \sum_{j \ne i} |\mathcal{P}(G^i_m; u)|\,|\mathcal{P}(G^j_m; u)| \\
&\quad = \sum_{i=1}^{3^{n(d+2)}} |\mathcal{P}(G^i_m; u)|^2 + \sum_{i=1}^{3^{n(d+2)}} \bigg[\sum_{d[G^i_m, G^j_m] > 1} |\mathcal{P}(G^i_m; u)|\,|\mathcal{P}(G^j_m; u)| + \sum_{d[G^i_m, G^j_m] \le 1} |\mathcal{P}(G^i_m; u)|\,|\mathcal{P}(G^j_m; u)| \bigg].
\end{aligned}$$

This implies that

$$\begin{aligned}
\mu(G_{m+n}, \omega)^2 \le{}& \frac{1}{3^{2n(d+2)}} \sum_{i=1}^{3^{n(d+2)}} (\mu(G^i_m, u))^2 \\
&+ \frac{1}{3^{2n(d+2)}} \sum_{i=1}^{3^{n(d+2)}} \bigg[\sum_{d[G^i_m, G^j_m] > 1} \mu(G^i_m, \omega)\mu(G^j_m, \omega) + \sum_{d[G^i_m, G^j_m] \le 1} \mu(G^i_m, \omega)\mu(G^j_m, \omega) \bigg].
\end{aligned}$$

For each i fixed, if $d[G^i_m, G^j_m] > 1$, then, by (1-7), stationarity, and Lemma 2.8,

$$\mathbb{E}[\mu(G^i_m, \omega)\mu(G^j_m, \omega)] = E^2_m.$$

If $d[G^i_m, G^j_m] \le 1$, then, by the Cauchy–Schwarz inequality and stationarity,

$$\mathbb{E}[\mu(G^i_m, \omega)\mu(G^j_m, \omega)] \le \mathbb{E}[\mu(G_m, \omega)^2] = J_m.$$

For any fixed i, the number of cubes such that $d[G^i_m, G^j_m] \le 1$ is at most 3^{d+1}. Therefore, after taking expectation of both sides, summing over $i = 1, \dots, 3^{n(d+2)}$ copies, this yields that

$$J_{m+n} \le \frac{1}{3^{n(d+2)}} (J_m + (3^{n(d+2)} - 3^{d+1})E^2_m + 3^{d+1} J_m) \le E^2_m + \frac{C}{3^{n(d+2)}} J_m. \qquad \square$$

Our next lemma shows that, by perturbing ℓ, we can make E and E^* positive.

Lemma 5.3. *Let ℓ be such that*

$$E(\ell) = \lim_{n\to\infty} \mathbb{E}[\mu(G_n, \omega, \ell)] = \lim_{n\to\infty} \mathbb{E}[\mu^*(G_n, \omega, \ell)] = E^*(\ell).$$

There exists $c_8 = c_8(d, \lambda, \Lambda)$ such that, for any $\gamma > 0$ and any n,

$$\mathbb{E}[\mu(G_n, \omega, \ell - \gamma)] \ge c_8 \gamma^{d+1}. \tag{5-15}$$

Analogously,

$$\mathbb{E}[\mu^*(G_n,\omega,-\ell+\gamma)]=\mathbb{E}[\mu(G_n,\omega^*,\ell-\gamma)]\geq c_8\gamma^{d+1}. \tag{5-16}$$

Proof. First, we observe that, by Lemma 4.4, $E(\ell)=0$. By the subadditive ergodic theorem, we choose $N=N(\delta)$ sufficiently large so that $\mathbb{E}[\mu(G_N,\omega,\ell)]\leq\delta$.

Let w solve

$$\begin{cases} w_t+F(D^2w,x,t,\omega)=\ell & \text{in } G_N,\\ w=0 & \text{on } \partial_p G_N.\end{cases}$$

Since $w\in S(G_N,\omega,\ell)$, by Lemma 2.1 we have

$$0\leq\inf_{G_N} w+c_1 3^{2N}\mu(G_N,\omega,\ell)^{1/(d+1)},$$

which implies that

$$\mathbb{P}\big[w\leq -2^{1/(d+1)}c_1 3^{2N}\delta^{1/(d+1)}\big]\leq\mathbb{P}\big[\mu(G_N,\omega,\ell)\geq 2\delta\big]\leq\tfrac{1}{2}. \tag{5-17}$$

Let $\tilde w:=w-C\gamma\big(\frac{1}{2}|x|^2-3^{2N}\big)+\frac{1}{2}\gamma(3^{2N}-t)$ for C to be chosen. By (5-17),

$$\mathbb{P}\big[\tilde w\geq -2c_1 3^{2N}\delta^{1/(d+1)}+C\gamma 3^{2N}\big]\geq\tfrac{1}{2}.$$

Next we consider that there exists $C=C(d,\lambda)$ such that $\tilde w\in S(G_N,\omega,\ell-\gamma)$. We verify that

$$\begin{aligned}\tilde w_t+F(D^2\tilde w,x,t,\omega)&=w_t-\tfrac{1}{2}\gamma+F(D^2w-C\gamma\,\mathrm{Id},x,t,\omega)\\ &\geq w_t-\tfrac{1}{2}\gamma+F(D^2w,x,t,\omega)+\lambda|C\gamma\,\mathrm{Id}\,|\\ &=\ell-\tfrac{1}{2}\gamma+C\lambda\gamma d\geq\ell-\gamma\end{aligned}$$

for $C=C(\lambda,d)$. Since $\tilde w\geq 0$ on $\partial_p G_N$, by Lemma 2.1 we have

$$\mathbb{P}\big[\mu(G_N,\omega,\ell-\gamma)\geq C\gamma^{d+1}-C\delta\big]\geq\tfrac{1}{2}.$$

Therefore, for all $n\leq N$,

$$\mathbb{E}[\mu(G_n,\omega,\ell+\gamma)]\geq C(\gamma^{d+1}-\delta).$$

Sending $\delta\to 0$, $N(\delta)\to\infty$, and we have the claim by letting $c_8=C$. □

We are now ready to obtain a rate of decay on the second moments of μ.

Theorem 5.4. *There exists $\tau=\tau(\lambda,\Lambda,d)\in(0,1)$ and $c_9=c_9(\lambda,\lambda,d)$ such that, for all $m\in\mathbb{N}$ and each $M\in\mathbb{S}^d$,*

$$J_m(\bar F(M),M)+J_m^*(-\bar F(M),M)\leq c_9(1+|M|)^{2(d+1)}K_0^{2(d+1)}\tau^m. \tag{5-18}$$

Proof. We fix $M\in\mathbb{S}^d$ and drop the dependence on $\bar F(M)$ (although we mention where it is used). In order to prove (5-18), it is enough to prove that there exists an increasing sequence of integers $\{m_k\}$ such that $|m_{k+1}-m_k|\leq C=C(d,\lambda,\Lambda)$ with

$$J_{m_k}(-3^{-k})+J_{m_k}^*(3^{-k})\leq C(1+|M|)^{2(d+1)}K_0^{2(d+1)}3^{-2k(d+1)}. \tag{5-19}$$

Recall that $|\bar{F}(M)| \le CK_0^{d+1}(1+|M|)^{d+1}$. By (2-8) and scaling, it is enough to assume that we work with

$$J_k := \frac{J_k}{C(1+|M|)^{2(d+1)}K_0^{2(d+1)}},$$

so that $|J_k| \le 1$ and then to prove

$$J_{m_k}(-3^{-k}) + J^*_{m_k}(3^{-k}) \le C3^{-2k(d+1)}. \tag{5-20}$$

Let $m_0 = 0$. Suppose that (5-20) holds for the level m_{k-1}. We would like to find m_k satisfying (5-20) such that $m_k - m_{k-1} \le C$. We aim to set up Lemma 5.1, and then choose $\gamma = 3^{-k}$. Given n_0 and η_0 as in Lemma 5.1, we seek m satisfying (5-13).

Consider that, by Lemma 5.2,

$$J_{m-n_0}(-3^{-k}) \le E^2_{m-n_1}(-3^{-k}) + \frac{c_7}{3^{(n_1-n_0)(d+2)}} J_{m-n_1}(-3^{-k}). \tag{5-21}$$

If we can find a choice of m such that, for a fixed n_1 and η_1,

$$E_{m-n_1}(-3^{-k}) \le (1+\eta_1)^{1/2}E_m(-3^{-k}), \quad E^*_{m-n_1}(3^{-k}) \le (1+\eta_1)^{1/2}E^*_m(3^{-k}), \tag{5-22}$$

and

$$J_{m-n_1}(-3^{-k}) \le (1+\eta_1)J_m(-3^{-k}), \quad J^*_{m-n_1}(3^{-k}) \le (1+\eta_1)J^*_m(3^{-k}), \tag{5-23}$$

then, substituting this into (5-21),

$$\begin{aligned} J_{m-n_0}(-3^{-k}) &\le (1+\eta_1)\left[E^2_m(-3^{-k}) + \frac{c_7}{3^{(n_1-n_0)(d+2)}} J_m(-3^{-k})\right] \\ &\le (1+\eta_1)\left[E^2_m(-3^{-k}) + \frac{c_7}{3^{(n_1-n_0)(d+2)}} J_{m-n_0}(-3^{-k})\right], \end{aligned}$$

which implies that

$$\left[1-(1+\eta_1)\frac{c_7}{3^{(n_1-n_0)(d+2)}}\right] J_{m-n_0}(-3^{-k}) \le (1+\eta_1)E^2_m(-3^{-k}).$$

Similarly, by (5-14),

$$\left[1-(1+\eta_1)\frac{c_7}{3^{(n_1-n_0)(d+2)}}\right] J^*_{m-n_0}(3^{-k}) \le (1+\eta_1)E^{*2}_m(3^{-k}).$$

Choosing $n_1(d,\lambda,\Lambda)$, $\eta_1(d,\lambda,\Lambda)$ so that

$$\left[1-(1+\eta_1)\frac{c_7}{3^{(n_1-n_0)(d+2)}}\right]^{-1}(1+\eta_1) \le 1+\eta_0, \tag{5-24}$$

we may apply Lemma 5.1, to conclude that, for m satisfying (5-22) and (5-23),

$$J_m(-3^{-k}) + J^*_m(3^{-k}) \le C3^{-2k(d+1)}.$$

The problem reduces to finding a choice of m satisfying (5-22) and (5-23) such that m is a bounded distance away from m_{k-1}. This is where we will use the inductive hypothesis. We claim that, for given n_1 and η_1, there exists m such that (5-22) and (5-23) hold, and

$$n_1 \le m \le m_{k-1} + C\log\bigl[C(J_{m_{k-1}}(-3^{-(k-1)}) + J^*_{m_{k-1}}(3^{-(k-1)}))\bigr]. \tag{5-25}$$

Consider that, for all m, by Lemma 5.3, since we are solving with right-hand side $\bar{F}(M)$ (and here is the only place where we use that the right-hand side is $\bar{F}(M)$),

$$c_8 3^{-(k-1)(d+1)} \le E_m(-3^{-(k-1)}) \quad \text{and} \quad c_8 3^{-(k-1)(d+1)} \le E^*_m(3^{-(k-1)}).$$

This implies that, for any N,

$$\prod_{j=1}^{N} \frac{J_{m_{k-1}+(j-1)n_1}(-3^{-(k-1)})}{J_{m_{k-1}+jn_1}(-3^{-(k-1)})} \le C\frac{J_{m_{k-1}}(-3^{-(k-1)})}{3^{-2(k-1)(d+1)}},$$

$$\prod_{j=1}^{N} \frac{J^*_{m_{k-1}+(j-1)n_1}(3^{-(k-1)})}{J^*_{m_{k-1}+jn_1}(3^{-(k-1)})} \le C\frac{J^*_{m_{k-1}}(3^{-(k-1)})}{3^{-2(k-1)(d+1)}},$$

$$\prod_{j=1}^{N} \frac{E_{m_{k-1}+(j-1)n_1}(-3^{-(k-1)})}{E_{m_{k-1}+jn_1}(-3^{-(k-1)})} \le C\frac{E_{m_{k-1}}(-3^{-(k-1)})}{3^{-(k-1)(d+1)}},$$

$$\prod_{j=1}^{N} \frac{E^*_{m_{k-1}+(j-1)n_1}(3^{-(k-1)})}{E^*_{m_{k-1}+jn_1}(3^{-(k-1)})} \le C\frac{E^*_{m_{k-1}}(3^{-(k-1)})}{3^{-(k-1)(d+1)}}.$$

Since each individual term in the product is bounded from below by 1, this implies that there exists some element j^i for $i = 1, 2, 3, 4$ such that

$$\frac{J_{m_{k-1}+(j^1-1)n_1}(-3^{-(k-1)})}{J_{m_{k-1}+j^1n_1}(-3^{-(k-1)})} \le C\left(\frac{J_{m_{k-1}}(-3^{-(k-1)})}{3^{-2(k-1)(d+1)}}\right)^{\frac{1}{N}},$$

$$\frac{J^*_{m_{k-1}+(j^2-1)n_1}(3^{-(k-1)})}{J^*_{m_{k-1}+j^2n_1}(3^{-(k-1)})} \le C\left(\frac{J^*_{m_{k-1}}(3^{-(k-1)})}{3^{-2(k-1)(d+1)}}\right)^{\frac{1}{N}},$$

$$\frac{E_{m_{k-1}+(j^3-1)n_1}(-3^{-(k-1)})}{E_{m_{k-1}+j^3n_1}(-3^{-(k-1)})} \le C\left(\frac{J_{m_{k-1}}(-3^{-(k-1)})}{3^{-2(k-1)(d+1)}}\right)^{\frac{1}{2N}},$$

$$\frac{E^*_{m_{k-1}+(j^4-1)n_1}(3^{-(k-1)})}{E^*_{m_{k-1}+j^4n_1}(3^{-(k-1)})} \le C\left(\frac{J^*_{m_{k-1}}(3^{-(k-1)})}{3^{-2(k-1)(d+1)}}\right)^{\frac{1}{2N}}.$$

Let

$$N := \left\lceil C\frac{\log\bigl[3^{2(k-1)(d+1)}(J_{m_{k-1}}(-3^{-(k-1)}) + J^*_{m_{k-1}}(3^{k-1}))\bigr]}{\log(1+\delta_1)}\right\rceil$$

and set $m := m_{k-1} + jn_1$ for $j := \max_i\{j^i\} \le N$. Applying the monotonicity, this choice of m satisfies (5-22) and (5-23). Define $m_k := m$, and this implies, by the inductive hypothesis, that

$$\begin{aligned} m_k &\le m_{k-1} + C\log\bigl[3^{2(k-1)(d+1)}(J_{m_{k-1}}(-3^{-(k-1)}) + J^*_{m_{k-1}}(3^{k-1}))\bigr] \\ &\le m_{k-1} + C\log[C3^{2(k-1)(d+1)}3^{-2(k-1)(d+1)}] \le m_{k-1} + C. \end{aligned}$$

This completes the induction and the proof of (5-19). By the monotonicity in the right-hand side ℓ, this actually yields a sequence $\{m_k\}$ such that $|m_k - m_{k-1}| \le C$ for all k and

$$J_{m_k} + J^*_{m_k} \le C3^{-2k(d+1)}.$$

Using the monotonicity of J_m in m to interpolate between points $m = 3^{m_k}$, we obtain (5-18) for some c_9. $\square$

Using this rate on the decay of the second moments, we apply Chebyshev's inequality to obtain a rate on the decay of μ.

Corollary 5.5. *For every $p < d+2$, there exists $c = c(p, \lambda, \Lambda, d)$ and $\alpha = \alpha(\lambda, \Lambda, p, d)$ such that, for all $m \in \mathbb{N}$ and all $\nu \ge 1$,*

$$\mathbb{P}\bigl[\mu(G_m, \omega, \bar{F}(M), M) \ge (1+|M|)^{d+1}K_0^{d+1}3^{-m\alpha}\nu\bigr] \le \exp(-c\nu 3^{mp}) \tag{5-26}$$

and

$$\mathbb{P}\bigl[\mu^*(G_m, \omega, \bar{F}(M), M) \ge (1+|M|)^{d+1}K_0^{d+1}3^{-m\alpha}\nu\bigr] \le \exp(-c\nu 3^{mp}). \tag{5-27}$$

Proof. We only prove (5-26), since (5-27) follows by identical arguments. Without loss of generality, we assume that $M = 0$ and we drop the dependence on $\bar{F}(0)$.

Fix $m \in \mathbb{N}$ and let $n \in \mathbb{N}$ to be chosen. We consider decomposing $G_{m+n+1} = \bigcup_{i=1}^{3^{d+2}} \mathcal{G}_n^i$, where $\mathcal{G}_n^i = \bigcup_{j=1}^{3^{m(d+2)}} G_n^{ij}$ is a collection of subcubes of size G_n such that each of the subcubes of size G_n is separated by distance at least 1.

By the finite range of dependence assumption (F1), for each i,

$$\mu(G_n^{ij}, \omega) \text{ and } \mu(G_n^{ik}, \omega) \text{ are independent if } j \ne k. \tag{5-28}$$

Using this decomposition yields that

$$\begin{aligned} \log\mathbb{E}\bigl[\exp(\nu 3^{m(d+2)}\mu(G_{m+n+1}, \omega))\bigr] &\le \log\mathbb{E}\Biggl[\prod_{i=1}^{3^{d+2}}\prod_{j=1}^{3^{m(d+2)}}\exp(\nu 3^{-(d+2)}\mu(G_n^{ij}, \omega))\Biggr] \\ &\le 3^{-(d+2)}\sum_{i=1}^{3^{(d+2)}}\log\mathbb{E}\Biggl[\prod_{j=1}^{3^{m(d+2)}}\exp(\nu\mu(G_n^{ij}, \omega))\Biggr] \\ &= 3^{-(d+2)}\sum_{i=1}^{3^{(d+2)}}\log\Biggl(\prod_{j=1}^{3^{m(d+2)}}\mathbb{E}\bigl[\exp(\nu\mu(G_n^{ij}, \omega))\bigr]\Biggr) \\ &= 3^{m(d+2)}\log\mathbb{E}\bigl[\exp(\nu\mu(G_n, \omega))\bigr], \end{aligned}$$

where the last line holds by stationarity. Moreover, if we choose $\nu = CK_0^{-1/(d+1)}$, then $\nu\mu(G_n,\omega) \le 1$ almost surely. Using the elementary inequalities

$$\begin{cases} \exp(s) \le 1+2s & \text{for all } 0 \le s \le 1, \\ \log(1+s) \le s & \text{for all } s \ge 0, \end{cases}$$

yields that, for this choice of ν,

$$\log \mathbb{E}\big[\exp(CK_0^{-(d+1)} 3^{m(d+2)} \mu(G_{m+n+1},\omega))\big] \le 3^{m(d+2)} \mathbb{E}[CK_0^{-(d+1)} \mu(G_n,\omega)] \le C3^{m(d+2)}\tau^n \tag{5-29}$$

by Theorem 5.4.

Therefore, by Chebyshev's inequality and (5-29), this yields that

$$\begin{aligned} \mathbb{P}\big[\mu(G_{m+n+1},\omega) \ge K_0^{d+1}\nu\big] &\le \mathbb{P}\big[\exp(K_0^{-(d+1)} 3^{m(d+2)} \mu(G_{m+n+1},\omega)) \ge \exp(3^{m(d+2)}\nu)\big] \\ &\le C\exp(-3^{m(d+2)}(\nu - \tau^n)). \end{aligned}$$

Letting $\nu = \frac{1}{2}\tau^n\nu$ and using that $\nu \ge 1$, we have that

$$\mathbb{P}\big[\mu(G_{m+n+1},\omega) \ge C\tau^n K_0^{d+1}\nu\big] \le C\exp(-3^{m(d+2)}\tau^n\nu).$$

Choosing $n \sim \lfloor (mp\log 3)/(2(p\log 3 + |\log\tau|))\rfloor \le \frac{1}{2}m$ implies that $c3^{-mp} \le \tau^n \le C3^{-mp}$, which yields that

$$\mathbb{P}\big[\mu(G_{m+n+1},\omega) \ge C3^{-mp} K_0^{d+1}\nu\big] \le C\exp(-3^{m(d+2-p)}\nu).$$

Relabeling $m = m+n+1$ and $p = d+2-p$ yields that there exists $\alpha = \alpha(\lambda,\Lambda,p,d)$ such that

$$\mathbb{P}\big[\mu(G_m,\omega) \ge C3^{-m\alpha} K_0^{d+1}\nu\big] \le C\exp(-3^{mp}\nu). \qquad \square$$

6. The proof of Theorem 1.1

We finally present the rate for homogenization in probability using Theorem 5.4. This follows a general procedure which has been shown in [Caffarelli and Souganidis 2010; Armstrong and Smart 2014b; Lin 2015]. However, for completeness we provide the argument here as well, similar to the approach of [Armstrong and Smart 2014b]. As mentioned in the above references, if the limiting function u is $C^2(\mathbb{R}^{d+1})$ (i.e., $C^2(\mathbb{R}^d) \cap C^1([0,T])$), then obtaining a rate for the homogenization is straightforward. Studying $\lim_{\varepsilon\to 0} w^\varepsilon$, where w^ε solves (4-1), is equivalent to the stochastic homogenization of (1-1) when the limiting function is of the form $u(x,t) = bt + \frac{1}{2}x\cdot Mx$. By (4-12) and Chebyshev's inequality, a rate on the decay of $\mu(G_{1/\varepsilon},\omega)$ immediately yields a rate in probability for the decay of w^ε. If $u \in C^2$, then, by replacing u with its second-order Taylor series expansion with cubic error, we obtain a rate for $u^\varepsilon - u$. In general, since u is not necessarily C^2, we must argue that one can still approximate u by a quadratic expansion. This type of approximation is the motivation for the theory of δ-viscosity solutions, which was introduced in the elliptic setting in [Caffarelli and Souganidis 2010] and generalized to the parabolic setting by Turanova [2015]. The rate in [Lin 2015] was obtained by using this regularization procedure.

For clarity and for a more general approach, we choose to present the argument in terms of a quantified comparison principle as in [Armstrong and Smart 2014b]. We revert to quantifying the traditional "doubling variables" arguments used in the theory of viscosity solutions (see for example [Crandall et al. 1992; Crandall 1997]). We are informed that this is related to forthcoming work by Armstong and Daniel [2015], who generalize this method to finite difference schemes for fully nonlinear, uniformly parabolic equations. The next series of results are entirely deterministic and therefore we suppress the dependence on the random parameter ω.

We first present a result relating the measure of the parabolic subdifferential to the measure of the corresponding touching points in physical space-time.

Proposition 6.1. *Let u and v be such that*

$$u_t + \mathcal{M}^-(D^2u) - R_0 \le 0 \le v_t + \mathcal{M}^+(D^2v) + R_0 \quad \text{in } U_T. \tag{6-1}$$

Assume $\delta > 0$ and let $V = \bar{V} \subseteq U_T \times U_T$ and $W \subseteq \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ be such that, for all $((p, h), (q, k)) \in W$,

$$\Bigg\{(x, t, y, s) : \sup_{U_T \times U_T : \tau \le t, \sigma \le s} u(\xi, \tau) - v(\eta, \sigma) - \frac{1}{2\delta}[|\xi - \eta|^2 + (\tau - \sigma)^2] - p \cdot \xi - q \cdot \eta$$
$$= u(x, t) - v(y, s) - \frac{1}{2\delta}[|x - y|^2 + (t - s)^2] - p \cdot x - q \cdot y,$$
$$h = u(x, t) - \frac{1}{2\delta}[|x - y|^2 + (t - s)^2] - p \cdot x,\ k = -v(y, s) - \frac{1}{2\delta}[|x - y|^2 + (t - s)^2] - q \cdot y\Bigg\} \subseteq V.$$

Then there exists a constant $C = C(\lambda, \Lambda, d, U_T)$ such that

$$|W| \le C(R_0 + \delta^{-1})^{2d+2}|V|. \tag{6-2}$$

Proof. Without loss of generality, we may assume by scaling that $U_T \subseteq Q_1(0, 1)$. As usual, we constantly relabel C for a constant which only depends on λ, Λ, and d. For $i = 1, 2$, let $(x_i, t_i, y_i, s_i, p_i, q_i, h_i, k_i)$ satisfy

$$\sup_{U_T \times U_T, \tau \le t_i, \sigma \le s_i} u(x, \tau) - v(y, \sigma) - \frac{1}{2\delta}(|x - y|^2 + (\tau - \sigma)^2) - p_i \cdot x - q_i \cdot y$$
$$= u(x_i, t_i) - v(y_i, s_i) - \frac{1}{2\delta}(|x_i - y_i|^2 + (t_i - s_i)^2) - p_i \cdot x_i - q_i \cdot y_i$$
$$= h_i + k_i,$$

and let

$$\Delta = (|x_1 - x_2|^2 + |y_1 - y_2|^2 + |t_1 - t_2| + |s_1 - s_2|)^{1/2}. \tag{6-3}$$

We claim that

$$(|p_1 - p_2|^2 + |q_1 - q_2|^2 + |h_1 - h_2|^2 + |k_1 - k_2|^2)^{1/2} \le C(1 + \delta^{-1})\Delta + o(\Delta) \tag{6-4}$$

as $|\Delta| \to 0$.

If (6-4) holds, then one can obtain (6-2) using standard measure-theoretic arguments. A priori, this may not be apparent since the left-hand side of (6-4) corresponds to the Euclidean distance between points in $\mathbb{R}^{d+1}$, whereas Δ corresponds to the parabolic distance under the metric $d[\cdot, \cdot]$. However, the parabolic cylinders have the appropriate doubling property with respect to Lebesgue measure, and thus standard measure-theoretic arguments apply.

We prove a series of claims, using standard techniques in the method of doubling variables.

Claim. *For each i,*

$$|t_i - s_i| \le \delta R_0 + C. \tag{6-5}$$

Consider that the map

$$(x,t) \to u(x,t) - \frac{1}{2\delta}[|x-y_1|^2 + (t-s_1)^2] - p_1 \cdot x$$

achieves its maximum over $U \times (0, t_1]$ at (x_1, t_1). Therefore, by (6-1),

$$\frac{1}{\delta}(t_1 - s_1) + \mathcal{M}^-(\delta^{-1}\,\mathrm{Id}) \le R_0,$$

implying that

$$t_1 - s_1 \le \delta[R_0 - (-C\delta^{-1})] = \delta R_0 + C. \tag{6-6}$$

Similarly, the map

$$(y,s) \to v(y,s) + \frac{1}{2\delta}[|x_1-y|^2 + (t_1-s)^2] + q_1 \cdot y$$

achieves its minimum over $U \times (0, s_1]$ at (y_1, s_1). By (6-1),

$$t_1 - s_1 \ge \delta(-R_0 - C\delta^{-1}) = -\delta R_0 - C. \tag{6-7}$$

Combining (6-6) and (6-7) yields (6-5).

Claim. *Let $u_t + \mathcal{M}^+(D^2u) \ge -1$ in Q_1. Let $(p_1, h_1) \in \mathcal{P}((x_1,t_1);u)$ and $(p_2,h_2) \in \mathcal{P}((x_2,t_2);u)$. Then*

$$|p_1 - p_2|^2 + |h_1 - h_2|^2 \le C(|x_1-x_2|^2 + |t_1-t_2|^2 + |x_1-x_2|^4 + |t_1-t_2|). \tag{6-8}$$

Without loss of generality, by subtracting a plane and translating, we may assume that $(p_2, h_2) = (0,0)$ and $(x_2,t_2) = (0,0)$. The claim will follow from the regularity of Γ^u (Lemma 2.3). Since (x_1,t_1), $(0,0) \in \{u = \Gamma^u\}$ and $D\Gamma^u$ is Lipschitz continuous, this implies that

$$|p_1| \le C(|x_1|^2 + |t_1|)^{1/2}.$$

To estimate $|h_1|$, we again apply the regularity of Γ^u and the bound on $|p_1|$ to conclude that

$$|h_1| = |h_1 - h_2| = |u(x_1,t_1) - p_1 \cdot x_1 - u(x_2,t_2)| \le C(|x_1|^2 + |t_1|)^{1/2}(1 + |x_1|).$$

Therefore,

$$|h_1|^2 \le C^2(|x_1|^2 + |t_1|)(1+|x_1|)^2 \le C(|x_1|^2 + |t_1|^2 + |x_1|^4 + |t_1|).$$

Combining these observations yields (6-8).

Next, we apply these observations to the parabolic subdifferentials. For simplicity, we adopt some notation. Without loss of generality, assume that $s_1 \geq s_2$. Let $T_{\min} := \min\{t_1, t_2, s_2\}$ and $T_{\max} := \max\{t_1, t_2, s_1\}$. Notice that, by (6-5), $T_{\max} - T_{\min} \leq \delta R_0 + C + \Delta^2 := \gamma^2$. Therefore, $(x_1, t_1), (x_2, t_2) \in Q_\gamma(x_1, T_{\max})$. Let

$$\tilde{u}(x,t) := -u(x,t) + \frac{1}{2\delta}[|x - y_1|^2 + (t - s_1)^2].$$

This implies that

$$\begin{aligned}
\tilde{u}_t + \mathcal{M}^+(D^2\tilde{u}) &= -u_t + \delta^{-1}(t - s_1) + \mathcal{M}^+(-D^2u + \delta^{-1}\,\mathrm{Id}) \\
&\geq -u_t + \delta^{-1}(t - s_1) - \mathcal{M}^-(D^2u) - \delta^{-1}C \\
&\geq -R_0 - C(1 + \delta R_0 + \Delta^2)\delta^{-1} \\
&\geq -C(R_0 + \delta^{-1}(1 + \Delta^2)) \quad \text{in } Q_\gamma(x_1, T_M).
\end{aligned} \tag{6-9}$$

We next find elements in the parabolic subdifferential of $\tilde{u}$.

Claim. *We have*

$$(-p_1, \tilde{u}(x_1, t_1) + p_1 \cdot x_1) \in \mathcal{P}((x_1, t_1); \tilde{u}). \tag{6-10}$$

Since

$$u(x_1, t_1) - \frac{1}{2\delta}[|x_1 - y_1|^2 + (t_1 - s_1)^2] - p_1 \cdot x_1 \geq u(x,t) - \frac{1}{2\delta}[|x - y_1|^2 + (t - s_1)^2] - p_1 \cdot x$$

for all $t \leq t_1$ and $x \in U$, this implies that

$$\tilde{u}(x_1, t_1) - (-p_1 \cdot x_1) = -u(x_1, t_1) + \frac{1}{2\delta}(|x_1 - y_1|^2 + (t_1 - s_1)^2) + p_1 \cdot x_1 \leq \tilde{u}(x,t) - (-p_1 \cdot x)$$

for all $t \leq t_1$ and $x \in U$. This yields (6-10).

Claim. *We have*

$$\left(-p_2 + \frac{y_2 - y_1}{\delta}, \tilde{u}(x_2, t_2) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x_2\right) \in \mathcal{P}((x_2, t_2); \tilde{u}). \tag{6-11}$$

Since

$$\begin{aligned}
&-u(x,t) + \frac{1}{2\delta}[|x - y_2|^2 + (t - s_2)^2] + p_2 \cdot x \\
&\qquad = \tilde{u}(x,t) + \frac{1}{2\delta}[|x - y_2|^2 + (t - s_2)^2 - |x - y_1|^2 - (t - s_1)^2] + p_2 \cdot x \\
&\qquad = \tilde{u}(x,t) + \left(\frac{1}{\delta}(-y_2 + y_1) + p_2\right) \cdot x + \frac{1}{2\delta}[(t - s_2)^2 - (t - s_1)^2 + |y_2^2| - |y_1|^2],
\end{aligned}$$

we obtain that

$$\begin{aligned}
&\tilde{u}(x_2, t_2) + \left(\frac{1}{\delta}(-y_2 + y_1) + p_2\right) \cdot x_2 + \frac{1}{2\delta}[(t_2 - s_2)^2 - (t_2 - s_1)^2] \\
&\qquad \leq \tilde{u}(x,t) + \left(\frac{1}{\delta}(-y_2 + y_1) + p_2\right) \cdot x + \frac{1}{2\delta}[(t - s_2)^2 - (t - s_1)^2].
\end{aligned}$$

Simplifying yields that

$$\tilde{u}(x_2, t_2) + \left(\frac{1}{\delta}(-y_2 + y_1) + p_2\right) \cdot x_2 + \frac{1}{\delta}[-(t_2 - t)(s_2 - s_1)] \leq \tilde{u}(x,t) + \left(\frac{1}{\delta}(-y_2 + y_1) + p_2\right) \cdot x.$$

Therefore, for $t \leq t_2$, since $s_1 \geq s_2$,

$$\tilde{u}(x_2, t_2) + \Big(\frac{1}{\delta}(-y_2 + y_1) + p_2\Big) \cdot x_2 \leq \tilde{u}(x, t) + \Big(\frac{1}{\delta}(-y_2 + y_1) + p_2\Big) \cdot x,$$

which yields the claim.

By combining (6-8), (6-9), (6-10), and (6-11),

$$\begin{aligned}\Big|p_1 - p_2 + \frac{1}{\delta}(y_2 - y_1)\Big|^2 + \Big|\tilde{u}(x_1, t_1) + p_1 \cdot x_1 - \tilde{u}(x_2, t_2) - \Big(p_2 - \frac{1}{\delta}(y_2 - y_1)\Big) \cdot x_2\Big|^2 \\ \leq C[R_0 + \delta^{-1}(1 + \Delta^2)]^2(|x_1 - x_2|^2 + |t_1 - t_2|^2 + |x_1 - x_2|^4 + |t_1 - t_2|).\end{aligned}$$

Recall that

$$-\tilde{u}(x_1, t_1) - p_1 \cdot x_1 = h_1$$

and

$$\begin{aligned}-\tilde{u}(x_2, t_2) - \Big(p_2 - \frac{1}{\delta}(y_2 - y_1)\Big) \cdot x_2 &= h_2 + \frac{1}{2\delta}(|y_2|^2 - |y_1|^2) + \frac{1}{2\delta}[(t_2 - s_2)^2 - (t_2 - s_1)^2] \\ &= h_2 + \frac{1}{2\delta}[|y_2|^2 - |y_1|^2 + s_2^2 - s_1^2 - 2t_2(s_2 - s_1)].\end{aligned}$$

Collecting terms yields that

$$\begin{aligned}|p_1 - p_2|^2 + |h_1 - h_2|^2 &\leq C[R_0 + \delta^{-1}(1 + \Delta^2)]^2[|x_1 - x_2|^2 + |t_1 - t_2|^2 + |x_1 - x_2|^4 + |t_1 - t_2|] \\ &\qquad + \frac{1}{\delta^2}|y_2 - y_1|^2 + \frac{1}{4\delta^2}[|y_2|^2 - |y_1|^2 + s_2^2 - s_1^2 - 2t_2(s_2 - s_1)]^2 \\ &\leq C[R_0 + \delta^{-1}(1 + \Delta^2)]^2\Delta^2 + \frac{1}{\delta^2}o(\Delta^2) \\ &\leq C[R_0 + \delta^{-1}]^2\Delta^2 + o(\Delta^2),\end{aligned}$$

which implies that

$$(|p_1 - p_2|^2 + |h_1 - h_2|^2)^{1/2} \leq C(R_0 + \delta^{-1})\Delta + o(\Delta).$$

An analogous argument yields that

$$(|q_1 - q_2|^2 + |k_1 - k_2|^2)^{1/2} \leq C(R_0 + \delta^{-1})\Delta + o(\Delta).$$

Combined, this yields (6-4). □

Next, we show that, if $|u - u^\varepsilon|$ is large somewhere, then we can find a matrix M^* and a parabolic cube G^* such that $\mu(G^*, \bar{F}(M^*), M^*)$ is very large. We mention that both M^* and G^* come from a countable family of matrices and cubes. In order to select M^* and G^*, we must construct the appropriate approximation of u to argue that u is close to a quadratic expansion. We will employ the $W^{3,\alpha}$ estimate proven in [Daniel 2015], which yields an estimate on the measure of points which can be well-approximated by a quadratic expansion. We state the result slightly differently than it appears in [Daniel 2015], in order to readily apply it for our purposes.

Theorem 6.2 [Daniel 2015, Theorem 1.2]. *Let $u_t + F(D^2u) = 0$ in Q_1, $u = g$ on $\partial_p Q_1$, with F uniformly parabolic. Let $Q \subseteq Q_1$. For each $\kappa > 0$, let*

$$\Sigma_\kappa := \big\{(x,t) \in Q_1 : \exists (M,\xi,b) \in \mathbb{S}^d \times \mathbb{R}^d \times \mathbb{R} \text{ with } |M| \le \kappa \text{ such that, for all } (y,s) \in Q_1 \text{ with } s \le t,$$
$$\big|u(y,s) - u(x,t) - b(s-t) - \xi\cdot(y-x) - \tfrac{1}{2}(y-x)\cdot M(y-x)\big| \le \tfrac{1}{6}\kappa(|x-y|^3 + |s-t|^{3/2})\big\}.$$

There exists $C = C(\lambda,\Lambda,d)$ and $\alpha = \alpha(\lambda,\Lambda,d)$ such that, for every $\kappa > 0$,

$$\big|Q_1 \setminus \big(\Sigma_\kappa \cap Q_{1/2}(0,-\tfrac{1}{4})\big)\big| \le C\left(\frac{\kappa}{\sup_{Q_1}\big([|u| + |F(0,\cdot,\cdot)|] + \|g\|_{C^{0,1}(\partial_p Q_1)}\big)}\right)^{-\alpha}.$$

We note that Σ_κ corresponds to the set of points which can be touched monotonically in time by a quadratic expansion with controllable error. Moreover, the points in Σ_κ are touched from above and below by polynomials. We are now ready to show the existence of M^* and G^*. For simplicity, we say that a function $\Phi : U_T \times U_T$ achieves a monotone maximum at (x_0,t_0,y_0,s_0) if $\Phi(x_0,t_0,y_0,s_0) \ge \Phi(x,t,y,s)$ for all $x, y \in U$ and all $t \le t_0$, $s \le s_0$.

Proposition 6.3. *Let u and v satisfy*

$$\begin{cases} u_t + \bar{F}(D^2u) = f(x,t) = v_t + F(D^2v,x,t) & \text{in } U_T, \\ u = v = g(x,t) & \text{on } \partial_p U_T, \end{cases}$$

such that

$$\|\bar{F}(0)\|_{L^\infty(U_T)} + \sup\|F(0,\cdot,\cdot)\|_{L^\infty(U_T)} + \|g\|_{C^{0,1}(\partial_p U_T)} + \|f\|_{C^{0,1}(U_T)} \le R_0 < +\infty.$$

There exists an exponent $\sigma = \sigma(\lambda,\Lambda,d) \in (0,1)$ and constants $c = c(\lambda,\Lambda,d,U_T)$, $C = C(\lambda,\Lambda,d,U_T)$ such that, for any $l \le \eta$, if

$$A := \sup_{U_T}(u - v) \ge C R_0 \eta^\sigma > 0, \tag{6-12}$$

then there exists $M^* \in \mathbb{S}^d$, $(y^*,s^*) \in U_T$ such that:

- $|M^*| \le \eta^{\sigma-1}$,
- *$l^{-1}M^*$, $\eta^{-1}y^*$, and $\eta^{-2}s^*$ have integer entries,*
- $\mu((y^*,s^*) + \eta G_0, \bar{F}(M^*), M^*) \ge cA^{d+1}$,

where $\eta G_0 = \big(-\frac{1}{2}\eta, \frac{1}{2}\eta\big]^d \times (-\eta^2, 0]$.

Proof. As usual, c and C will denote constants which depend on universal quantities, which will vary line by line. We first point out some simplifications which we take without loss of generality. We assume that $R_0 = 1$ and $U_T \subseteq Q_1(0,1)$, and appropriately renormalize.

Next, we claim that we may replace v by $\tilde{v}$ solving

$$\begin{cases} \tilde{v}_t + F(D^2\tilde{v},x,t) = f(x,t) + cA & \text{in } U_T, \\ \tilde{v} = v & \text{on } \partial_p U_T. \end{cases} \tag{6-13}$$

The Alexandrov–Backelman–Pucci–Krylov–Tso estimate [Wang 1992; Imbert and Silvestre 2012] yields that

$$\tilde{v} - v \le CA \quad \text{in } U_T,$$

so, by adjusting the constant in (6-12), we may take the replacement at no cost.

Finally, we point out that, by the Krylov–Safonov estimates [Wang 1992; Imbert and Silvestre 2012], u and v are Hölder continuous and, since $R_0 \le 1$, there exists $\alpha(\lambda, \Lambda, d) \in (0, 1)$ such that

$$\|u\|_{C^{0,\alpha}(\bar{U}_T)} + \|v\|_{C^{0,\alpha}(\bar{U}_T)} \le C. \tag{6-14}$$

Without loss of generality, assume that $\alpha \le \frac{1}{2}$. Since $u = v$ on $\partial_p U_T$, this implies that, for all $(x, t), (y, s) \in U_T$,

$$|u(x, t) - v(y, s)| \le C\big(d[(x, t), \partial_p U_T]^\alpha + d[(y, s), \partial_p U_T]^\alpha + d[(x, t), (y, s)]^\alpha\big).$$

Consider the function

$$\Phi(x, t, y, s, p, q) = u(x, t) - v(y, s) - \frac{1}{2\delta}[|x - y|^2 + (t - s)^2)] - p \cdot x - q \cdot y.$$

Suppose there exists a point (x_0, t_0) such that $u(x_0, t_0) - v(x_0, t_0) \ge \frac{3}{4}A$. This implies that

$$\Phi(x_0, t_0, x_0, t_0, 0, 0) \ge \tfrac{3}{4}A.$$

Let

$$U_T(\rho) := \{(x, t) \in U_T \times U_T : d[(x, t), \partial_p U_T] \ge \rho\}.$$

Let $p, q \in B_r$, where we define $r := \frac{1}{8}A$. We would like to show that $\Phi(\cdot, \cdot, \cdot, \cdot, p, q)$ achieves it monotone maximum in $U_T(\rho) \times U_T(\rho)$ for some choice of ρ.

We note that

$$\begin{aligned}
&\Phi(x, t, y, s, p, q)\\
&\quad = u(x, t) - v(y, s) - \frac{1}{2\delta}[|x - y|^2 + (t - s)^2] - p \cdot x - q \cdot y\\
&\quad \le C\big(d[(x, t), \partial_p U_T]^\alpha + d[(y, s), \partial_p U_T]^\alpha + d[(x, t), (y, s)]^\alpha\big) - \frac{1}{2\delta}[|x - y|^2 + (t - s)^2] + 2r.
\end{aligned}$$

By Young's inequality,

$$|x - y|^\alpha = A^{(2-\alpha)/2}[A^{-(2-\alpha)/\alpha}|x - y|^2]^{\alpha/2} \le \frac{1}{8C}A + CA^{-(2-\alpha)/\alpha}|x - y|^2$$

and

$$|t - s|^{\alpha/2} = A^{(4-\alpha)/4}[A^{-(4-\alpha)/\alpha}|t - s|^2]^{\alpha/4} \le \frac{1}{8C}A + CA^{-(4-\alpha)/\alpha}(t - s)^2.$$

Assume $A \le 1$. This implies that $A^{-(2-\alpha)/\alpha} \le A^{-(4-\alpha)/\alpha}$. Therefore,

$$\begin{aligned}
&\Phi(x, y, t, s, p, q)\\
&\quad \le Cd[(x, t), \partial_p U_T]^\alpha + Cd[(y, s), \partial_p U_T]^\alpha + \tfrac{1}{4}A + \tfrac{1}{4}A + C\Big(A^{-(4-\alpha)/\alpha} - \frac{1}{2\delta}\Big)[|x - y|^2 + (t - s)^2].
\end{aligned}$$

By letting

$$\delta \leq \tfrac{1}{2} A^{(4-\alpha)/\alpha}, \tag{6-15}$$

we have that

$$\Phi(x, y, t, s, p, q) \leq Cd[(x,t), \partial_p U_T]^\alpha + C[d(y,s), \partial_p U_T]^\alpha + \tfrac{1}{2}A.$$

Therefore, letting $\rho := CA^{1/\alpha}$ yields that, for any $p, q \in B_r$, Φ achieves its monotone maximum in $U_T(\rho) \times U_T(\rho)$.

Using the language of Proposition 6.1, we choose $W \subseteq \mathbb{R}^{d+1}$ such that $Q_r \times Q_r \subseteq W$. This yields that

$$\begin{aligned} V := \bigl\{&(x,t,y,s) \in U_T \times U_T : \text{for some } (p,q) \in B_r \times B_r,\\ &\Phi(\cdot,\cdot,\cdot,\cdot,p,q) \text{ achieves its monotone maximum at } (x,t,y,s) \text{ for appropriate } (h,k) \in \mathbb{R}^2\bigr\}\\ &\subseteq U_T(\rho) \times U_T(\rho). \end{aligned}$$

By Proposition 6.1, this implies that

$$|V| \geq C(1+\delta^{-1})^{-2d-2} r^{2d+2} \geq C(1+A^{-(4-\alpha)/\alpha})^{-2d-2} A^{2d+2} \geq CA^{(8d+8)/\alpha}.$$

If we define the projection $\pi : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ by $\pi((A,B)) = A$, we have that

$$\pi(V) \geq |U_T|^{-1}|V| \geq |Q_1|^{-1}|V| \geq CA^{(8d+8)/\alpha}. \tag{6-16}$$

Finally, we note that, for every $((x,t),(y,s)) \in V$, since $\Phi(x,t,y,s,p,q) \geq 0$ for some $p, q \in B_r \subseteq B_1$, $\alpha \leq \frac{1}{2}$, and $A \leq 1$, this implies that

$$|x-y|^2 + |t-s|^2 \leq C\delta \leq CA^{(4-\alpha)/\alpha} \leq CA^6. \tag{6-17}$$

Next, we use (6-16) to show that there are points in $\pi(V)$ where u can be approximated by a quadratic expansion. Let Σ_κ as in the $W^{3,\alpha}$ estimate (Theorem 6.2).

By the $W^{3,\alpha}$ estimate, assuming that $U_T \subseteq Q_1$,

$$|U_T \setminus \Sigma_\kappa(U_T)| \leq \bigl|Q_1 \setminus \Sigma_\kappa(U_T) \cap Q_{1/2}\bigl(0, -\tfrac{1}{4}\bigr)\bigr| \leq C\kappa^{-\alpha}. \tag{6-18}$$

Although a priori the two α's in (6-16) and (6-18) are not necessarily the same, we can assume without loss of generality they are the same by taking the minimum of the two.

Thus, if we let $\kappa \geq CA^{-4(d+2)/\alpha^2}$, then

$$|U_T \setminus \Sigma_\kappa(U_T)| < |\pi(V)|,$$

which implies that $\pi(V) \cap \Sigma_\kappa \neq \varnothing$. This implies that there are points of $\pi(V)$ where u can be touched monotonically in time by a quadratic expansion with controllable error, and the function Φ achieves it monotone maximum there.

Finally, we show that there exist M^*, y^*, s^*, and G^* which satisfy the conclusion of the proposition. By the previous step, there exists $(x_1, t_1, y_1, s_1) \in V$ with $(x_1, t_1) \in \Sigma_\kappa$. In other words, there exist $p, q \in B_r$ such that

$$\Phi(x_1, t_1, y_1, s_1, p, q) = \sup_{U_T(\rho) \times U_T(\rho), \tau \leq t_1, \sigma \leq s_1} \Phi(x, \tau, y, \sigma, p, q),$$

and (M, ξ, b) such that $|M| \le \kappa$ and, for all $(x,t) \in U_T$, $t \le t_1$,

$$\left|u(x,t) - u(x_1,t_1) - b(t-t_1) - \xi \cdot (x-x_1) - \tfrac{1}{2}(x-x_1)\cdot M(x-x_1)\right| \le \tfrac{1}{6}\kappa(|x-x_1|^3 + |t-t_1|^{3/2}).$$

Notice that, since $u_t + \bar{F}(D^2u) = f(x,t)$ in U_T and u is touched from above and below at (x_1,t_1) by polynomials with Hessians equal to M, this implies that $b + \bar{F}(M) = f(x_1,t_1)$. Therefore, defining

$$\phi(x,t) := u(x_1,t_1) + b(t-t_1) + (\xi - p)\cdot(x-x_1) + \tfrac{1}{2}(x-x_1)\cdot M(x-x_1) - \tfrac{1}{6}\kappa(|x-x_1|^3 + |t-t_1|^{3/2}),$$

we have

$$\begin{aligned} u(x_1,t_1) - v(y_1,s_1) &- \frac{1}{2\delta}[|x_1-y_1|^2 + (t_1-s_1)^2] \\ &\ge \sup_{U_T\times U_T,\, t\le t_1,\, s\le s_1} \left\{\phi(x,t) - v(y,s) - \frac{1}{2\delta}[|x-y|^2 + (t-s)^2] - q\cdot(y-y_1)\right\}. \end{aligned} \tag{6-19}$$

To control the right-hand side from below, we consider that, for any $(y,s) \in U_T$ with $s \le s_1$, letting $x = x_1 + y - y_1$ and $t = t_1 + s - s_1 \le t_1$,

$$\begin{aligned} &\sup_{(x,t)\in U_T,\, t\le t_1} \left\{\phi(x,t) - \frac{1}{2\delta}[|x-y|^2 + (t-s)^2]\right\} \\ &\quad \ge \phi(x_1+y-y_1, t_1+s-s_1) - 12[|x_1-y_1|^2 + (t_1-s_1)^2] \\ &\quad = u(x_1,t_1) + b(s-s_1) + (\xi-p)\cdot(y-y_1) + \tfrac{1}{2}(y-y_1)\cdot M(y-y_1) \\ &\qquad\qquad - \tfrac{1}{6}\kappa(|y-y_1|^3 + |s-s_1|^{3/2}) - \frac{1}{2\delta}[|x_1-y_1|^2 + (t_1-s_1)^2]. \end{aligned} \tag{6-20}$$

Combining (6-19) and (6-20) yields that

$$\begin{aligned} &u(x_1,t_1) - v(y_1,s_1) - \frac{1}{2\delta}[|x_1-y_1|^2 + (t_1-s_1)^2] \\ &\quad \ge \sup_{(y,s)\in U_T,\, s\le s_1} \Big\{u(x_1,t_1) + b(s-s_1) + (\xi-p)\cdot(y-y_1) + \tfrac{1}{2}(y-y_1)\cdot M(y-y_1) - \tfrac{1}{6}\kappa(|y-y_1|^3 + |s-s_1|^{3/2}) \\ &\qquad\qquad - \frac{1}{2\delta}[|x_1-y_1|^2 + (t_1-s_1)^2] - v(y,s) - q\cdot(y-y_1)\Big\}. \end{aligned}$$

This implies that

$$\begin{aligned} v(y_1,s_1) \le \inf_{(y,s)\in U_T,\, s\le s_1} \Big\{&v(y,s) - b(s-s_1) - (\xi-p-q)\cdot(y-y_1) \\ &- \tfrac{1}{2}(y-y_1)\cdot M(y-y_1) + \tfrac{1}{6}\kappa(|y-y_1|^3 + |s-s_1|^{3/2})\Big\}. \end{aligned} \tag{6-21}$$

Since $l \le \eta$, choose $M^* \in \mathbb{S}^d$ so that $M \le M^* \le M + C\eta^\sigma\,\mathrm{Id}$ and $l^{-1}M^*$ has integer entries. Using that $\bar{F}$ is uniformly elliptic, $\bar{F}(M^*) \le \bar{F}(M) = f(x_1,t_1) - b$. Let

$$\begin{aligned} \Theta(y,s) := v(y,s) &- b(s-s_1) - (\xi-p-q)\cdot(y-y_1) \\ &- \tfrac{1}{2}(y-y_1)\cdot(M - C\eta^\sigma\,\mathrm{Id})(y-y_1) + \tfrac{1}{6}\kappa(|y-y_1|^3 + |s-s_1|^{3/2}). \end{aligned}$$

By (6-13),

$$
\begin{aligned}
&\Theta_s + F(M^* + D^2\Theta, y, s) \\
&\qquad = v_s - b + \tfrac{1}{4}\kappa|s-s_1|^{1/2} \\
&\qquad\qquad + F\Big(M^* + D^2v - M + C\eta^\sigma \operatorname{Id} + \tfrac{1}{2}\kappa|y-y_1| \operatorname{Id} + \tfrac{1}{2}\kappa \frac{(y-y_1)\otimes(y-y_1)}{|y-y_1|}, y, s\Big) \\
&\qquad \geq v_s - b + F(D^2v, y, s) - C\big(M^* - M + C\eta^\sigma \operatorname{Id} + C\tfrac{1}{2}\kappa|y-y_1| \operatorname{Id}\big) \\
&\qquad \geq f(y,s) + cA - b - C\eta^\sigma - C\tfrac{1}{2}\kappa|y-y_1| \\
&\qquad \geq f(y,s) + cA - b - C\eta^\sigma - C\tfrac{1}{2}(\kappa+1)|y-y_1| \\
&\qquad \geq \bar{F}(M) - CA^6 + cA - C\eta^\sigma - C\tfrac{1}{2}(\kappa+1)|y-y_1|,
\end{aligned}
$$

where the last line holds by (6-17), using that $\bar{F}(M) = f(x_1, t_1) - b$.

This implies that, in $Q_{cA(\kappa+1)^{-1}}(y_1, s_1)$,

$$\Theta_s + F(M^* + D^2\Theta, y, s) \geq \bar{F}(M) - CA^6 + cA - C\eta^\sigma.$$

In addition, comparing (6-21) and the definition of Θ,

$$\Theta(y_1, s_1) \leq \inf_{(y,s)\in U_T, s\leq s_1} (\Theta - C\eta^\sigma |y-y_1|^2). \tag{6-22}$$

Let (y^*, s^*) be such that $(\eta^{-1}y^*, \eta^{-2}s^*) \in \mathbb{Z}^{d+1}$ and $d[(y^*, s^*), (y_1, s_1)] \leq \sqrt{d}\eta$.

Let

$$G^* := (y^*, s^*) - \eta G_0.$$

Since $(y_1, s_1) \in U_T(\rho)$, we have $d[(y^*, s^*), \partial_p U_T] \geq \rho - \sqrt{d}\eta \geq \sqrt{d}\eta$ so long as $\rho := CA^{1/\alpha} \geq C\eta$ (which is satisfied if $\sigma \leq \alpha$). This implies that $G^* \subseteq U_T$.

We next claim that $G^* \subseteq Q_{cA(\kappa+1)^{-1}}(y_1, s_1)$ for an appropriate choice of κ. Let $\kappa := \eta^{\sigma-1}$ with $\sigma := \big((1+4(d+2))/\alpha^2\big)^{-1} \leq \alpha$. Since $A \geq C\eta^\sigma$, we may choose the constants so that $cA(\kappa+1)^{-1} \geq \sqrt{d}\eta$. This yields that $G^* \subseteq Q_{cA(\kappa+1)^{-1}}(y_1, s_1)$, as asserted.

Therefore,

$$\Theta_s + F(M^* + D^2\Theta, y, s) \geq \bar{F}(M^*) \quad \text{in } G^*. \tag{6-23}$$

By (6-22), we conclude that

$$\inf_{G^*} \Theta \leq \inf_{\partial_p G^*} \Theta - C\eta^\sigma. \tag{6-24}$$

This implies, by Lemma 2.1 and (6-24), that

$$\mu(G^*, \bar{F}(M^*), M^*) \geq cA^{d+1}$$

and this completes the proof. □

Finally, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We prove a rate in probability for the decay of $u - u^\varepsilon$. Fix M_0 and U_T such that $U_T \subset Q_1$ and

$$1 + K_0 + \|g\|_{C^{0,1}(\partial_p U_T)} \leq M_0.$$

We will show that there exists $\beta > 0$ and a random variable $\mathscr{X} : \Omega \to \mathbb{R}$ such that

$$\sup_{U_T}\{u(x,t) - u^\varepsilon(x,t,\omega)\} \leq C[1 + \varepsilon^p \mathscr{X}(\omega)]\varepsilon^\beta.$$

We mention that a rate on $u^\varepsilon - u$ follows by a completely analogous argument for μ^*, so we choose to omit it.

Fix $\varepsilon \in (0, 1)$ and $p < d+2$, and let σ be as in Proposition 6.3. Let α be the α associated with p as in Corollary 5.5 and let $q := \frac{1}{4}p$. Choose m such that

$$\max\{3^{-m/4}, 3^{-m\alpha/(d+1)}\} \leq \varepsilon. \tag{6-25}$$

In the language of Proposition 6.3, let $\eta := 3^{-m\alpha/2(d+1)}$ and choose $l := 3^{-m\alpha/2d}$. Notice that we have that $l \leq \eta \leq \varepsilon^{1/2}$. This implies that, for any $A \geq C\eta^\sigma$,

$$\begin{aligned}\Big\{\omega : \sup_{(x,t)\in U_T} u(x,t) - u^\varepsilon(x,t,\omega) \geq A\Big\}& \\ &\subseteq \bigcup_{(y,s,M)\in\mathcal{I}(A)} \big\{\omega : \mu((y/\varepsilon, s/\varepsilon^2) + \eta\varepsilon^{-1}G_0, \omega, \bar{F}(M), M) \geq cA^{d+1}\big\} \\ &= \bigcup_{(y,s,M)\in\mathcal{I}(A)} \big\{\omega : \mu((y/\varepsilon, s/\varepsilon^2) + G_m, \omega, \bar{F}(M), M) \geq cA^{d+1}\big\},\end{aligned}$$

where

$$\mathcal{I}(A) := \big\{(y, s, M) : (y,s) \in Q_1, (\eta^{-1}y, \eta^{-2}s) \in \mathbb{Z}^{d+1}, |M| \leq 3^{m\alpha/2(d+1)}\big\}.$$

This is possible since $\eta < 1$ and Proposition 6.3 yields that $\sigma < 1$, which implies that $|M| \leq \eta^{\sigma-1} \leq \eta^{-1} \leq 3^{m\alpha/2(d+1)}$. We mention also that $l^{-1}M \in \mathbb{Z}^{d^2} \cap \mathbb{S}^d$.

This implies that

$$\sup_{(x,t)\in U_T}\{u(x,t) - u^\varepsilon(x,t,\omega)\} \leq cA^{d+1} + \mathcal{Y}_m(\omega), \tag{6-26}$$

where

$$\mathcal{Y}_m(\omega) := \big\{\sup \mu((z,r) + G_m, \omega, \bar{F}(M), M) : (z\varepsilon^{-1}, r\varepsilon^{-2}, M) \in \mathcal{I}(A)\big\}. \tag{6-27}$$

To find the number of elements in $\mathcal{I}(A)$, consider that, since $\eta^{-1}z \in \mathbb{Z}^d \cap Q_{1/\varepsilon}$ and $\eta^{-2}s \in \mathbb{Z} \cap [0, 1/\varepsilon^2]$, there are $(\varepsilon\eta)^{-(d+2)}$ choices for (z, s). This implies that there are at most $3^{3m\alpha}$ choices. For the matrices, consider that, since $3^{m\alpha/2d}M \in \mathbb{Z}^{d^2} \cap \mathbb{S}^d$ and $|M| \leq 3^{m\alpha/2(d+1)}$, this implies that there are at most $3^{m\alpha(d+1)}$ terms. In total, there are $3^{m\alpha(d+4)}$ combinations to choose from in $\mathcal{I}(A)$.

By Corollary 5.5, for each $(z, r, M) \in \mathcal{I}(A)$,

$$\mathbb{P}\big[(z,r) + \mu(G_m, \omega, \bar{F}(M), M) \geq (1+|M|)^{d+1}3^{-m\alpha}\tau\big] \leq C\exp(-c3^{mp}\tau).$$

Since $|M|^{d+1} \leq 3^{m\alpha/2}$, this implies that

$$\mathbb{P}\big[(z,r)+\mu(G_m,\omega,\overline{F}(M),M) \geq 3^{-m\alpha/2}\tau\big] \leq \exp(-c3^{mp}\tau).$$

Using a union bound and summing over all of the terms in $\mathcal{I}(A)$,

$$\mathbb{P}\big[\mathcal{Y}_m(\omega) \geq 3^{-m\alpha/2}\tau\big] \leq C3^{m\alpha(d+4)}\exp(-c3^{mp}\tau) \leq C\exp(-c3^{mp}\tau).$$

Replacing τ by $\tau+1$, we have that, for all $\tau \geq 0$,

$$\mathbb{P}\big[(3^{m\alpha/2}\mathcal{Y}_m(\omega)-1)_+ \geq \tau\big] \leq C\exp(-c3^{mp}\tau).$$

Replacing again $\tau \to 3^{-mq}\tau$ yields that

$$\mathbb{P}\big[3^{mq}3^{m\alpha/2}(\mathcal{Y}_m(\omega)-1)_+ \geq \tau\big] \leq C\exp(-c3^{m(p-q)}\tau).$$

Summing over m and using that $p > q$, this implies that, for all $\tau \geq 0$,

$$\mathbb{P}\big[\sup_m\{3^{mq}3^{m\alpha/2}(\mathcal{Y}_m(\omega)-1)_+\} \geq \tau\big] \leq \sum_m \mathbb{P}\big[3^{mq}3^{m\alpha/2}(\mathcal{Y}_m(\omega)-1)_+ \geq \tau\big] \leq C\exp(-c\tau). \tag{6-28}$$

Letting

$$\mathcal{X}(\omega) := \sup_m\{3^{mq}(3^{m\alpha/2}\mathcal{Y}_m(\omega)-1)_+\} \tag{6-29}$$

and integrating (6-28) in τ yields that

$$\mathbb{E}\big[\exp(\mathcal{X}(\omega))\big] \leq C. \tag{6-30}$$

This implies that

$$\sup_{(x,t)\in U_T}\{u(x,t)-u^\varepsilon(x,t,\omega)\} \leq C\eta^{\sigma(d+1)} + C(3^{-mq}\mathcal{X}(\omega)+1)3^{-m\alpha/2} \leq C(1+\varepsilon^p\mathcal{X}(\omega))\varepsilon^\beta$$

for some choice of β, where $\beta(\lambda,\Lambda,d,p)$. □

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