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A POINTWISE INEQUALITY FOR THE FOURTH-ORDER LANE–EMDEN EQUATION

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We prove the pointwise inequality

$$-\Delta u \geq \left(\frac{2}{(p+1)-c_n}\right)^{\frac{1}{2}} |x|^{a/2} u^{(p+1)/2} + \frac{2}{n-4}\frac{|\nabla u|^2}{u} \quad \text{in } \mathbb{R}^n,$$

where $c_n := 8/(n(n-4))$, for positive bounded solutions of the fourth-order Hénon equation, that is,

$$\Delta^2 u = |x|^a u^p \quad \text{in } \mathbb{R}^n$$

for some $a \geq 0$ and $p > 1$. Motivated by Moser's proof of Harnack's inequality as well as Moser iteration-type arguments in the regularity theory, we develop an iteration argument to prove the above pointwise inequality. As far as we know this is the first time that such an argument is applied towards constructing pointwise inequalities for partial differential equations. An interesting point is that the coefficient $2/(n-4)$ also appears in the fourth-order Q-curvature and the Paneitz operator. This, in particular, implies that the scalar curvature of the conformal metric with conformal factor $u^{4/(n-4)}$ is positive.

1. Introduction

We are interested in proving an a priori pointwise estimate for positive solutions of the fourth-order Hénon equation

$$\Delta^2 u = |x|^a u^p \quad \text{in } \mathbb{R}^n, \tag{1-1}$$

where $p > 1$ and $a \geq 0$. Let us first mention that, for the case $a = 0$, it is known that (1-1) only admits $u = 0$ as a nonnegative solution when p is a subcritical exponent, that is, $1 < p < (n+4)/(n-4)$ when $n \geq 5$, and $1 < p$ when $n \leq 4$. Moreover, for the critical case $p = (n+4)/(n-4)$, all entire positive solutions are classified. See [Lin 1998; Wei and Xu 1999]. This is a counterpart of the standard Liouville theorem of Gidas and Spruck [1981a; 1981b] for the second-order Lane–Emden equation

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^n, \tag{1-2}$$

stating that $u = 0$ is the only nonnegative solution for (1-2) when p is a subcritical exponent, that is, $1 < p < (n+2)/(n-2)$ when $n \geq 3$. Note also that, for the fourth-order Hénon equation, it is conjectured that $u = 0$ is the only nonnegative solution of (1-1) when p is a subcritical exponent, that is, when

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$1 < p < (n+4+2a)/(n-4)$ and $n \geq 5$; see [Fazly and Ghoussoub 2014]. Therefore, throughout this note, when we are dealing with (1-1) we assume that $p > (n+4+2a)/(n-4)$ and $n \geq 5$. For more information, see [Fazly and Ghoussoub 2014; Souplet 2009] and references therein.

Pointwise estimates have had tremendous impact on the theory of elliptic partial differential equations. In what follows, we list some of the celebrated pointwise inequalities for certain semilinear elliptic equations and systems. These inequalities have been used to tackle well-known conjectures and open problems. The following inequality has been one of the main techniques to solve De Giorgi's conjecture (1978) for the Allen–Cahn equation and to analyze various semilinear equations and problems.

Theorem 1.1 [Modica 1985]. *Let $F \in C^2(\mathbb{R})$ be a nonnegative function and u be a bounded entire solution of*

$$\Delta u = F'(u) \quad \text{in } \mathbb{R}^n. \tag{1-3}$$

Then

$$|\nabla u|^2 \leq 2F(u) \quad \text{in } \mathbb{R}^n. \tag{1-4}$$

For the specific case $F(u) = \frac{1}{4}(1-u^2)^2$, equation (1-3) is known as the Allen–Cahn equation. Note also that [Caffarelli et al. 1994] extended this inequality to quasilinear equations. We refer interested readers to [Farina and Valdinoci 2010; 2011; 2013; 2014; Castellaneta et al. 2012; Farina et al. 2008] regarding pointwise gradient estimates and certain improvements of (1-4). For the fourth-order counterpart of (1-3) with an arbitrary nonlinearity, a general inequality of the form (1-4) is not known. However, for a particular nonlinearity known as the fourth-order Lane–Emden equation, i.e.,

$$\Delta^2 u = u^p \quad \text{in } \mathbb{R}^n \tag{1-5}$$

it was shown by Wei and Xu [1999, Theorem 3.1] that the negative Laplacian of the positive solutions is nonnegative, that is, $-\Delta u \geq 0$ in $\mathbb{R}^n$. Set $v = -\Delta u$ and, from the fact that $-\Delta u \geq 0$, we can consider (1-5) as a special case (when $q = 1$) of the Lane–Emden system

$$\begin{cases} -\Delta u = v^q & \text{in } \mathbb{R}^n, \\ -\Delta v = u^p & \text{in } \mathbb{R}^n, \end{cases} \tag{1-6}$$

where $p \geq q \geq 1$. Note that there is a significance difference between system (1-6) and equation (1-5), in the sense that this system has Hamiltonian structure while the equation has gradient structure. This system has been of great interest, at least in the past two decades. In particular, the Lane–Emden conjecture, stating that $u = v = 0$ is the only nonnegative solution for this system when $1/(p+1) + 1/(q+1) > (n-2)/n$ has been studied extensively and various methods and techniques have been developed to tackle this conjecture. Among these methods, Souplet [2009] proved the following pointwise inequality for solutions of (1-6) and then used it to prove the Lane–Emden conjecture in four dimensions. Note that the particular case $1 < p < 2$ was done by Phan [2012].

Theorem 1.2 [Souplet 2009]. *Let u and v be nonnegative solutions of* (1-6). *Then*

$$\frac{u^{p+1}}{p+1} \leq \frac{v^{q+1}}{q+1} \quad \text{in } \mathbb{R}^n. \tag{1-7}$$

Applying this theorem, the following pointwise inequality holds for nonnegative solutions of (1-5):

$$-\Delta u \geq \sqrt{\frac{2}{p+1}}u^{(p+1)/2} \quad \text{in } \mathbb{R}^n. \tag{1-8}$$

Note also that Phan [2012], with similar methods to those [Souplet 2009], extended the pointwise inequality (1-7) to nonnegative solutions of the Hénon–Lane–Emden system

$$\begin{cases} -\Delta u = |x|^b v^q & \text{in } \mathbb{R}^n, \\ -\Delta v = |x|^a u^p & \text{in } \mathbb{R}^n, \end{cases} \tag{1-9}$$

where $p \geq q \geq 1$. Suppose that $0 \leq a - b \leq (n-2)(p-q)$; then

$$|x|^a \frac{u^{p+1}}{p+1} \leq |x|^b \frac{v^{q+1}}{q+1} \quad \text{in } \mathbb{R}^n. \tag{1-10}$$

The standard method to prove a pointwise inequality, as is used to prove (1-7) and (1-4), is to derive an appropriate equation — call it an auxiliary equation — for the function that is the difference between the right-hand and left-hand sides of the inequality. Then, whenever we have enough decay estimates on solutions of the auxiliary equation, maximum principles can be applied to prove that the difference function has a fixed sign. So, the key point here is to manipulate a suitable auxiliary equation.

In a more technical framework, to construct an auxiliary equation to prove (1-7) and (1-8), a few positive terms, including a gradient term of the form $|\nabla u|^2 u^{t-2}$ for some number t, are not considered in [Souplet 2009]. To be more explicit, in order to prove (1-8), which is a particular case of (1-7), the difference function $w(x) := \Delta u + \sqrt{2/(p+1)}u^{(p+1)/2}$ is considered. Straightforward calculations show that the following auxiliary equation holds:

$$\left(\sqrt{\frac{2}{p+1}}u^{(1-p)/2}\right)\Delta w = \Delta u + \sqrt{\frac{2}{p+1}}u^{(p+1)/2} + \frac{p-1}{2}\frac{|\nabla u|^2}{u}. \tag{1-11}$$

In order to show that Δw is nonnegative when w is nonnegative, via maximum principles for the above equation, the gradient term $|\nabla u|^2/u$ is not considered in [Souplet 2009]. Note that (1-11) implies, in spirit, that the gradient term $|\nabla u|^2/u$ should have an impact on the inequality, just like the Laplacian operator and the power term $u^{(p+1)/2}$. This is our motivation to attempt to include the gradient term in the inequality (1-8) that gives a lower bound on the Laplacian operator. Let us briefly mention that Modica, in his proof of (1-4), took advantage of similar gradient terms to construct an auxiliary equation. Following ideas provided by Modica [1985] and Souplet [2009], as we shall see in the proof of Proposition 3.1, we manage to keep most of the positive terms when looking for an auxiliary equation.

In this paper, we develop a Moser iteration-type argument to prove a lower bound for the negative Laplacian of positive bounded solutions of (1-1) that involves powers of u and the new term $|\nabla u|^2/u$ with $2/(n-4)$ as the coefficient. The remarkable point is that the coefficient $2/(n-4)$ is exactly what we need in the estimate of the scalar curvature for the conformal metric $g = u^{2/(n-4)}g_0$.

Here is our main result:

Theorem 1.3. *Let u be a bounded positive solution of* (1-1). *Then the following pointwise inequality holds*:

$$-\Delta u \geq \sqrt{\frac{2}{(p+1)-c_n}}|x|^{a/2}u^{(p+1)/2}+\frac{2}{n-4}\frac{|\nabla u|^2}{u} \quad \text{in } \mathbb{R}^n, \tag{1-12}$$

where $c_n := 8/(n(n-4))$ *and* $0 \leq a \leq \inf_{k\geq 0} A_k$ (A_k *is defined in* (4-28)).

Remark 1.4. A natural question here is: what are the best constants in the inequality (1-12)?

Let us now put the inequality (1-12) in a more geometric text. By the conformal change $g = u^{4/(n-4)}g_0$, where g_0 is the usual Euclidean metric, the new scalar curvature becomes

$$S_g = -\frac{4(n-1)}{n-2}u^{-(n+2)/(n-4)}\Delta(u^{(n-2)/(n-4)}).$$

An immediate consequence of (1-12) is that the conformal scalar curvature is positive. Note that this cannot be deduced from the inequality (1-8).

The idea of proving a lower bound for the negative of the Laplacian operator is also used in the context of nonlinear eigenvalue problems to prove certain regularity results; see, e.g., [Cowan et al. 2010]. Similar pointwise inequalities are used to prove Liouville theorems in the notion of stability in [Wei et al. 2013; Wei and Ye 2013] and references therein as well. We would like to mention that Gui [2008] proved a very interesting Hamiltonian identity for elliptic systems that may be regarded as a generalization of Modica's inequality. He used this identity to rigorously analyze the structure of level curves of saddle solutions of the Allen–Cahn equation as well as Young's law for the contact angles in triple junction formation. Note also that, as is shown by Farina [2004] for the Ginzburg–Landau system, the analog of Modica's estimate is false for systems in general. We refer interested readers to [Alikakos 2013] for a review of this topic and to [Fazly and Ghoussoub 2013] for De Giorgi-type results for systems.

Here is the organization of the paper. In Section 2, we provide certain standard elliptic estimates that are consequences of Sobolev embeddings and the regularity theory. Then, in Section 3 we develop a Moser iteration-type argument, following ideas provided by Modica [1985] and Souplet [2009]. Finally, in Section 4, we first give a certain maximum principle argument for a quasilinear equation that arises in the Moser iteration process. Then we apply the estimates and methods developed in the earlier sections. We suggest the reader ignores the weight function $|x|^a$ in (1-1) when reading the paper for the first time.

2. Technical elliptic estimates

In this section, we provide some elliptic decay estimates that we use frequently later in the proofs. Deriving the right decay estimates for solutions of (1-1) plays a fundamental role in our proofs. Similar estimates have been also used in the literature to construct Liouville theorems and regularity results. We refer interested readers to [Fazly 2014; Fazly and Ghoussoub 2014; Phan 2012; Souplet 2009; Phan and Souplet 2012]. We start with the following standard estimate:

Lemma 2.1 (L^p-estimate on B_R). *Suppose that u is a nonnegative solution of* (1-1); *then, for any $R > 1$ we have*

$$\int_{B_R} |x|^a u^p \le C R^{n-(4p+a)/(p-1)},$$

where $C = C(n, p, a) > 0$ is independent of R.

Proof. Consider the following test function $\phi_R \in C_c^4(\mathbb{R}^n)$ with $0 \le \phi_R \le 1$:

$$\phi_R(x) = \begin{cases} 1 & \text{if } |x| < R, \\ 0 & \text{if } |x| > 2R, \end{cases}$$

where $\|D^i \phi_R\|_\infty \le C/R^i$ for $1 \le i \le 4$. For fixed $m \ge 2$, we have

$$|\Delta^2 \phi_R^m(x)| \le \begin{cases} 0 & \text{if } |x| < R \text{ or } |x| > 2R, \\ C R^{-4} \phi_R^{m-4} & \text{if } R < |x| < 2R, \end{cases}$$

where $C > 0$ is independent of R. For $m \ge 2$, multiply the equation by ϕ_R^m and integrate to get

$$\int_{B_{2R}} |x|^a u^p \phi_R^m = \int_{B_{2R}} \Delta^2 u \phi_R^m = \int_{B_{2R}} u \Delta^2 \phi_R^m \le C R^{-4} \int_{B_{2R} \setminus B_R} u \phi_R^{m-4}.$$

Applying Hölder's inequality, we get

$$\begin{aligned}\int_{B_{2R}} |x|^a u^p \phi_R^m &\le C R^{-4} \biggl(\int_{B_{2R} \setminus B_R} |x|^{(-a/p)p'} \biggr)^{\frac{1}{p'}} \biggl(\int_{B_{2R} \setminus B_R} |x|^a u^p \phi_R^{(m-4)p} \biggr)^{\frac{1}{p}} \\ &\le C R^{(n-(a/p)p')/p'-4} \biggl(\int_{B_{2R} \setminus B_R} |x|^a u^p \phi_R^{(m-4)p} \biggr)^{\frac{1}{p}},\end{aligned}$$

where $p' = p/(p-1)$. Set $m = (m-4)p$, so that $m = 4p/(p-1)$, to get

$$\int_{B_{2R}} |x|^a u^p \phi_R^m \le C R^{(n-(a/p)p')/p'-4} \biggl(\int_{B_{2R}} |x|^a u^p \phi_R^m \biggr)^{\frac{1}{p}}.$$

Therefore,

$$\int_{B_{2R}} |x|^a u^p \phi_R^m \le C R^{(n-(a/p)p')-4p'}.$$

This finishes the proof. □

From Hölder's inequality we get the following:

Corollary 2.2. *Under the same assumptions as Lemma 2.1,*

$$\int_{B_R \setminus B_{R/2}} u \le C R^{n-(a+4)/(p-1)},$$

where $C = C(n, p, a) > 0$ is independent of R.

We now show that the operator $-\Delta u$ has a sign. Then, we apply this to provide various elliptic estimates for derivatives of u. In addition, later on this helps us to start an iteration argument.

Proposition 2.3. *Let u be a positive solution of* (1-1). *Then, $-\Delta u \ge 0$ in $\mathbb{R}^n$.*

Proof. Let $v = -\Delta u$. Ideas and methods applied in this proof are strongly motivated by the ones given in [Wei and Xu 1999]. Suppose that there is $x_0 \in \mathbb{R}^n$ such that $v(x_0) < 0$. Without loss of generality we take $x_0 = 0$, i.e., if $x_0 \neq 0$ set $\omega(x) = v(x + x_0)$ and apply the same argument. We use the notation $\bar{f}(r) = (1/|\partial B_r|) \int_{\partial B_r} f\, dS$ for the average of a function $f(x)$ on the boundary of B_r. We refer interested readers to [Ni 1982] regarding the average function. Applying Hölder's inequality,

$$\begin{cases} -\Delta_r \bar{u}(r) = \bar{v}(r) & \text{in } \mathbb{R}, \\ -\Delta_r \bar{v}(r) \geq r^a (\bar{u})^p & \text{in } \mathbb{R}, \end{cases} \tag{2-1}$$

where Δ_r is the Laplacian operator in polar coordinates, i.e.,

$$\Delta_r \bar{f}(r) = r^{1-n}(r^{n-1}\bar{f}'(r))'.$$

It is straightforward to see that

$$\bar{v}'(r) = \frac{1}{|\partial B_r|} \int_{B_r} \Delta v = -\frac{1}{|\partial B_r|} \int_{B_r} |x|^a u^p \leq 0.$$

Therefore, $\bar{v}(r) \leq \bar{v}(0) < 0$ for $r > 0$. Similarly, for $\bar{u}'(r)$ we have

$$\bar{u}'(r) = -\frac{1}{|\partial B_r|} \int_{B_r} v = -r^{1-n} \int_0^r s^{n-1} \bar{v}(s)\, ds \geq -\bar{v}(0) r^{1-n} \int_0^r s^{n-1}\, ds = -\frac{\bar{v}(0)}{n} r.$$

From this, for any $r \geq r_0$ we get

$$\bar{u}(r) \geq \alpha r^2, \tag{2-2}$$

where $\alpha = -\bar{v}(0)/(2n) > 0$. We now have a lower bound on $\bar{u}(r)$. Suppose instead that the following more general lower bound holds on $\bar{u}(r)$:

$$\bar{u}(r) \geq \frac{\alpha^{p^k}}{\beta^{s_k}} r^{t_k} \quad \text{for } r \geq r_k, \tag{2-3}$$

where $s_0 := 0$, $t_0 := 2$, $\alpha := -\bar{v}(0)/(2n) > 0$ and $\beta := 2p + a + n + 4 > 0$. Note that (2-1) gives a relation between the two functions $\bar{u}(r)$ and $\bar{v}(r)$. Therefore, the lower bound on $\bar{u}(r)$ forces an upper bound on $\bar{v}(r)$ and vice versa. In the light of this fact, we can construct an iteration argument to improve the bound (2-3). Integrating the second equation of (2-1) over $[r_k, r]$ when $r \geq r_k$, we get

$$\begin{aligned} r^{n-1}\bar{v}'(r) &\leq r_k^{n-1}\bar{v}'(r_k) - \frac{\alpha^{p^{k+1}}}{\beta^{ps_k}} \int_{r_k}^r s^{n-1+a+pt_k}\, ds \\ &\leq -\frac{\alpha^{p^{k+1}}}{\beta^{ps_k}(pt_k + n + a)} (r^{pt_k+n+a} - r_k^{pt_k+n+a}) \quad \text{since } \bar{v}' < 0. \end{aligned}$$

Therefore, $\bar{v}'(r) \leq -\big(\alpha^{p^{k+1}}/(\beta^{ps_k}(pt_k + n + a))\big)(r^{pt_k+a+1} - r_k^{pt_k+a+1})$ for all $r \geq r_k$, that is,

$$\bar{v}'(r) \leq -\frac{\alpha^{p^{k+1}}}{2\beta^{ps_k}(pt_k + n + a)} r^{pt_k+a+1} \quad \text{for all } r \geq 2^{1/(pt_k+a+1)} r_k.$$

Integrating the last inequality over $[2^{1/(pt_k+a+1)}r_k, r]$ when $r \geq 2^{1/(pt_k+a+1)}r_k = \tilde{r}_k$, we obtain

$$\bar{v}(r) \leq \bar{v}(\tilde{r}_k) - \frac{\alpha^{p^{k+1}}}{2\beta^{ps_k}T_{k,n,a,p}}(r^{pt_k+a+2} - \tilde{r}_k^{pt_k+a+2}),$$

where $T_{k,n,a,p} := (pt_k + n + a)(pt_k + 2 + a)$. By similar discussions and by taking r large enough, that is, $r \geq 2^{1/(pt_k+a+1)}2^{1/(pt_k+a+2)}r_k = \tilde{\tilde{r}}_k$, we end up with

$$\bar{v}(r) \leq -\frac{\alpha^{p^{k+1}}}{4\beta^{ps_k}T_{k,n,a,p}}r^{pt_k+a+2}. \tag{2-4}$$

Applying (2-4) and integrating (2-1) again over $[\tilde{\tilde{r}}_k, r]$ when $r \geq \tilde{\tilde{r}}_k$, we have

$$r^{n-1}\bar{u}'(r) = \tilde{r}_k^{n-1}\bar{u}'(\tilde{r}_k) - \int_{\tilde{r}_k}^{r} s^{n-1}\bar{v}(s)\,ds \geq \frac{\alpha^{p^{k+1}}}{4\beta^{ps_k}T_{k,n,a,p}}\int_{\tilde{r}_k}^{r} s^{pt_k+a+n+1}\,ds.$$

Therefore, the following new lower bound on $\bar{u}(r)$ holds:

$$\bar{u}(r) \geq \frac{\alpha^{p^{k+1}}}{2^4\beta^{ps_k}\tilde{T}_{k,n,a,p}}r^{pt_k+a+n+4},$$

where

$$r \geq 2^{1/(pt_k+a+3)}2^{1/(pt_k+a+4)}\tilde{\tilde{r}}_k = 2^{\sum_{i=1}^{4} 1/(pt_k+a+i)}r_k$$

and

$$\begin{aligned}\tilde{T}_{k,n,a,p} &= (pt_k + n + a + 2)(pt_k + 4 + a)T_{k,n,a,p}\\ &= (pt_k + n + a)(pt_k + 2 + a)(pt_k + n + a + 2)(pt_k + 4 + a)\\ &\leq (pt_k + n + a + 4)^4.\end{aligned}$$

We now modify this estimate to make the coefficients similar to (2-3). After simplifying, we get

$$\bar{u}(r) \geq \frac{\alpha^{p^{k+1}}}{\beta^{ps_k}M_k}r^{pt_k+a+4} \quad \text{for } r \geq 2^{4/(pt_k+a+1)}r_k, \tag{2-5}$$

where $M_k := 2^4(pt_k + n + a + 4)^4$. In what follows, we put an upper bound on M_k that is expressed as a power of β. Note that

$$\tfrac{1}{2}\sqrt[4]{M_{k+1}} = pt_{k+1}+n+a+4 = p(pt_k+n+a+4)++n+a+4 \leq (pt_k+n+a+4)(p+1) = \tfrac{1}{2}(p+1)\sqrt[4]{M_k}.$$

From this we have $M_{k+1} \leq (p+1)^4 M_k$ and therefore $M_k \leq (p+1)^{4k}M_0$, where $M_0 = 2^4(2p+n+a+4)^4$ because $t_0 = 2$. Since the constant β is defined as $\beta = 2p + n + a + 4$, we get the bound

$$M_k \leq \beta^{4k+4}. \tag{2-6}$$

From this, (2-3) and (2-5), and to complete the iteration process, we set

$$t_{k+1} := pt_k + a + 4 \qquad \text{for } t_0 = 2, \tag{2-7}$$

$$s_{k+1} := ps_k + 4k + 4 \qquad \text{for } s_0 = 0, \tag{2-8}$$

and, therefore,

$$\bar{u}(r) \geq \frac{\alpha^{p^{k+1}}}{\beta^{s_{k+1}}} r^{t_{k+1}} \quad \text{for } r \geq r_{k+1}, \tag{2-9}$$

where $r_{k+1} := 2^{4/(pt_k+a+1)} r_k \geq 2^{\sum_{i=1}^{4} 1/(pt_k+a+i)} r_k$. By direct calculations on these recursive sequences, we get the explicit sequences

$$\begin{aligned} t_k &= \frac{2p^{k+1} + (a+2)p^k - (a+4)}{p-1}, \\ s_k &= \frac{4p^{k+1} - 4p(k+1) + 4k}{(p-1)^2}, \\ r_k &= 2^{\sum_{i=0}^{k-1} 4/(pt_i+a+1)} r_0 \leq 2^{\sum_{i=0}^{\infty} 4/(pt_i+a+1)} r_0 =: r^* < \infty. \end{aligned}$$

Set $R := \beta^{2/(p-1)} M$, where $M = \max\{\alpha^{-1}, m\}$ when $m > 1$ is large enough to ensure $m\beta^{2/(p-1)} \geq r^*$. Therefore, $R \geq r^* \geq r_k$ for any k and we have

$$\bar{u}(R) \geq M^{t_k - p^k} \beta^{2t_k/(p-1) - s_k}.$$

If we take k large enough, e.g., $k \geq (\ln(a+4) - \ln(a+2))/\ln p$, then $t_k > p^k$. The fact that $M > 1$ gives us

$$\bar{u}(R) \geq \beta^{2t_k/(p-1) - s_k} = \beta^{(2(a+2)p^k + 4k(p-1) + 4p - 2(a+4))/(p-1)^2}.$$

Since we have assumed that $a+2 > 0$ and $\beta > 1$, we get $\bar{u}(R) \to \infty$ as $k \to \infty$. Note that $0 < R < \infty$ is independent of k. This finishes the proof. □

We now apply Proposition 2.3 to conclude that $-\Delta u \geq 0$ and therefore we can consider (1-1) as a special case of the Hénon–Lane–Emden equation.

Lemma 2.4 (L^1-estimates on B_R). *Suppose that u is a nonnegative solution of* (1-1); *then, for any $R > 1$ we have*

$$\int_{B_R} |\Delta u| \leq C R^{n-(2p+2+a)/(p-1)},$$

where $C = C(n, p, a) > 0$ is independent of R.

Proof. Set $v = -\Delta u$. From Proposition 2.3 we know that $v \geq 0$. Therefore, the pair (u, v) satisfies the system

$$\begin{cases} -\Delta u = v & \text{in } \mathbb{R}^n, \\ -\Delta v = |x|^a u^p & \text{in } \mathbb{R}^n, \end{cases} \tag{2-10}$$

which is a particular case of the Hénon–Lane–Emden system. From the estimates provided in [Fazly and Ghoussoub 2014, Lemma 2.1], we get the desired result. □

Lemma 2.5 (an interpolation inequality on B_R). *Let $R > 1$ and $z \in W^{2,1}(B_{2R})$. Then*

$$\int_{B_R \setminus B_{R/2}} |Dz| \leq CR \int_{B_{2R} \setminus B_{R/4}} |\Delta z| + CR^{-1} \int_{B_{2R} \setminus B_{R/4}} |z|,$$

where $C = C(n) > 0$ is independent of R.

Corollary 2.6. *Under the same assumptions as Lemma 2.1. The following estimate holds*:

$$\int_{B_R\setminus B_{R/2}} |Du| \le C R^{n-(p+3+a)/(p-1)},$$

where $C = C(n, p, a) > 0$ is independent of R.

Lemma 2.7 (L^τ-estimate on B_R). *Let $1 < \tau < \infty$ and $z \in W^{2,\tau}(B_{2R})$. Then*

$$\int_{B_R\setminus B_{R/2}} |D^2 z|^\tau \le C \int_{B_{2R}\setminus B_{R/4}} |\Delta z|^\tau + C R^{-2\tau} \int_{B_{2R}\setminus B_{R/4}} |z|^\tau,$$

where $C = C(n, \tau) > 0$ does not depend on R.

Lemma 2.8 (L^2-estimates on B_R). *Suppose that u is a bounded nonnegative solution of* (1-1); *then, for any $R > 1$ we have*

$$\int_{B_R} |\Delta u|^2 \le C \int_{B_{2R}} |x|^a u^{p+1} + C R^{-2} \int_{B_{2R}} |\Delta u| + C R^{-4} \int_{B_{2R}\setminus B_R} u, \tag{2-11}$$

where $C = C(n, p, a) > 0$ does not depend on R.

Proof. We proceed in two steps.

Step 1: Multiply both sides of (1-1) by $u\phi^2$, where $\phi \in C_c^\infty(\mathbb{R}^n)\cap[0, 1]$ is a test function. Then, integrating by parts, we get

$$\begin{aligned}\int_{\mathbb{R}^n} |\Delta u|^2\phi^2 &= \int_{\mathbb{R}^n} |x|^a u^{p+1}\phi^2 - 4\int_{\mathbb{R}^n} \Delta u \nabla u\cdot\nabla\phi\phi - \int_{\mathbb{R}^n} u\Delta u(2|\nabla\phi|^2 + 2\phi\Delta\phi)\\ &\le \int_{\mathbb{R}^n} |x|^a u^{p+1}\phi^2 + \delta\int_{\mathbb{R}^n} |\Delta u|^2\phi^2 + C(\delta)\int_{\mathbb{R}^n} |\nabla u|^2|\nabla\phi|^2 + C\int_{\mathbb{R}^n} |\Delta u|(|\nabla\phi|^2 + |\Delta\phi|)\end{aligned}$$

for some constant $C > 0$. Here, we have used Cauchy's inequality for $0 < \delta < 1$. Therefore, if we set ϕ to be the standard test function, that is, $\phi = 1$ in B_R and $\phi = 0$ in $\mathbb{R}^n \setminus B_{2R}$ with $\|D_x^i\phi\|_{L^\infty(B_{2R}\setminus B_R)} \le C R^{-i}$ for $i = 1, 2$, then we get

$$\int_{B_R} |\Delta u|^2 \le \int_{B_{2R}} |x|^a u^{p+1} + C R^{-2}\int_{B_{2R}\setminus B_R} |\nabla u|^2 + C R^{-2}\int_{B_{2R}\setminus B_R} |\Delta u|, \tag{2-12}$$

where $C = C(n, p, a) > 0$ does not depend on R.

Step 2: Multiply both sides of $-\Delta u = v$ by $u\phi^2$, where ϕ is the same test function as in Step 1. Integrating by parts again, we get

$$\int_{\mathbb{R}^n} |\nabla u|^2\phi^2 = \int_{\mathbb{R}^n} uv\phi^2 - 2\int_{\mathbb{R}^n} u\nabla u\cdot\nabla\phi\phi \le \int_{\mathbb{R}^n} uv\phi^2 + \delta\int_{\mathbb{R}^n} |\nabla u|^2\phi^2 + C(\delta)\int_{\mathbb{R}^n} |\nabla\phi|^2 u^2,$$

where we have also used Cauchy's inequality for $0 < \delta < 1$. So,

$$\int_{B_R} |\nabla u|^2 \le C\int_{B_{2R}} |\Delta u| + C R^{-2}\int_{B_{2R}\setminus B_R} u, \tag{2-13}$$

where we have used the boundedness of u. From (2-12) and (2-13) we get

$$\int_{B_R} |\Delta u|^2 \le \int_{B_{2R}} |x|^a u^{p+1} + CR^{-2}\int_{B_{2R}} |\Delta u| + CR^{-4}\int_{B_{2R}\setminus B_R} u. \tag{2-14}$$

This completes the proof. □

We now apply Lemma 2.1, Lemma 2.8 and Corollary 2.2 to get the following:

Corollary 2.9. *Suppose that the assumptions of Lemma 2.1 hold. Moreover, let u be bounded; then*

$$\int_{B_R} |\Delta u|^2 \le CR^{n-(4p+a)/(p-1)}, \tag{2-15}$$

where $C = C(n, p, a) > 0$ is independent of R.

Lemma 2.10 (Sobolev inequalities on the sphere S^{n-1}). *Fix $n \ge 2$, a positive integer i and $1 < t < \tau \le \infty$. For $z \in W^{i,t}(S^{n-1})$,*

$$\|z\|_{L^\tau(S^{n-1})} \le C\|D^i_\theta z\|_{L^t(S^{n-1})} + C\|z\|_{L^1(S^{n-1})},$$

where

$$\begin{cases} \dfrac{1}{\tau} = \dfrac{1}{t} - \dfrac{i}{n-1} & \text{if } it+1 < n, \\ \tau = \infty & \text{if } it+1 > n, \end{cases}$$

and $C = C(i, t, n, \tau) > 0$.

3. Developing the iteration argument

In this section, we develop a counterpart of the Moser iteration argument [1961] for solutions of (1-1). We define a sequence of functions $(w_k)_{k=-1}$ of the form

$$w_k := \Delta u + \alpha_k |\nabla u|^2 (u+\epsilon)^{-1} + \beta_k |x|^{a/2} u^{(p+1)/2},$$

where α_k and β_k are certain nondecreasing sequences of nonnegative numbers with $\alpha_{-1} = \beta_{-1} = 0$.

Assuming that $w_k \le 0$, that is, essentially, a lower bound on the negative Laplacian operator holds, we construct a differential inequality for w_{k+1} with $\alpha_{k+1} \ge \alpha_k$ and $\beta_{k+1} \ge \beta_k$. Then, applying certain maximum principle arguments, we show that $w_{k+1} \le 0$. Note that $w_{k+1} \le 0$ is stronger than $w_k \le 0$, because it forces a stronger lower bound on the negative of the Laplacian operator.

We start by proving that w_{-1}, which is the Laplacian operator of u, is nonpositive; see Proposition 2.3. Then, using this fact and applying (1-9) and (1-10) when $q = 1$ and $b = 0$, we get the following inequality for nonnegative solutions of the fourth-order Hénon equation (1-1):

$$-\Delta u \ge \sqrt{\frac{2}{p+1}}|x|^{a/2}u^{(p+1)/2} \quad \text{in } \mathbb{R}^n, \tag{3-1}$$

where $0 \le a \le (n-2)(p-1)$. Inequality (3-1) is the first step of the iteration argument, meaning that $w_0 \le 0$ for $\alpha_0 = 0$ and $\beta_0 = \sqrt{2/(p+1)}$.

We now perform the iteration argument:

Proposition 3.1. *Let u be a positive classical solution of* (1-1). *Suppose that* $(\alpha_k)_{k=0}$ *and* $(\beta_k)_{k=0}$ *are sequences of numbers. Define the sequence of functions*

$$w_k := \Delta u + \alpha_k |\nabla u|^2 (u+\epsilon)^{-1} + \beta_k |x|^{a/2} u^{(p+1)/2}, \tag{3-2}$$

where $\epsilon = \epsilon(k)$ is a positive constant. Suppose that $w_k \le 0$; then w_{k+1} satisfies the differential inequality

$$\begin{aligned}\Delta w_{k+1} - 2\alpha_{k+1}(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1} + \alpha_{k+1} w_{k+1}(u+\epsilon)^{-2}|\nabla u|^2 - \tfrac{1}{2}\beta_{k+1}(p+1)u^{(p-1)/2}|x|^{a/2} w_{k+1} \\ \ge I^{(1)}_{\epsilon,\beta_k}|x|^a u^p + \alpha_{k+1} I^{(2)}_{\alpha_k}|\nabla u|^4 (u+\epsilon)^{-3} + I^{(4)}_{a,\alpha_k,\beta_k}|x|^{a-2}u^{(p+1)/2} \\ + I^{(3)}_{\epsilon,\alpha_k,\beta_k}|x|^a u^{(p+1)/2}\left|\frac{\nabla u}{u} + \frac{a\beta_{k+1}\left(\frac{1}{2}(p+1) - \alpha_{k+1}u/(u+\epsilon)\right)}{2I^{(3)}_{\epsilon,\alpha_k,\beta_k}}\frac{x}{|x|^2}\right|^2,\end{aligned} \tag{3-3}$$

where

$$\begin{aligned} I^{(1)}_{\epsilon,\alpha_k,\beta_k} &:= 1 - \frac{p+1}{2}\beta^2_{k+1} + \frac{2}{n}\alpha_{k+1}\beta^2_k \frac{u}{u+\epsilon}, \\ I^{(2)}_{\alpha_k} &:= \frac{2}{n}(\alpha_{k+1}+\alpha_k+1)^2 - 2\alpha_{k+1}(\alpha_{k+1}+1) + \alpha_{k+1}, \\ I^{(3)}_{\epsilon,\alpha_k,\beta_k} &:= \frac{4}{n}\alpha_{k+1}\beta_k(\alpha_{k+1}+\alpha_k+1)\frac{u^2}{(u+\epsilon)^2} + \beta_{k+1}\alpha_{k+1}\frac{u^2}{(u+\epsilon)^2} \\ &\qquad - (p+1)\beta_{k+1}\alpha_{k+1}\frac{u}{u+\epsilon} + \frac{p+1}{2}\left(\frac{p-1}{2} - \alpha_{k+1}\frac{u}{u+\epsilon}\right)\beta_{k+1}, \\ I^{(4)}_{a,\epsilon,\alpha_k,\beta_k} &:= \frac{a}{2}\beta_{k+1}\left(n + \frac{a}{2} - 2\right) - \frac{a^2\beta^2_{k+1}\left(\frac{1}{2}(p+1) - \alpha_{k+1}u/(u+\epsilon)\right)^2}{4I^{(3)}_{\epsilon,\alpha_k,\beta_k}}. \end{aligned}$$

Proof. For the sake of simplicity in calculations, set $b := \frac{1}{2}a$ and $q := \frac{1}{2}(p+1)$. From (3-2), the function w_{k+1} is defined as

$$w_{k+1} := \Delta u + \alpha_{k+1}|\nabla u|^2(u+\epsilon)^{-1} + \beta_{k+1}|x|^b u^q.$$

Taking Laplacian of w_{k+1} and using (1-1), we get

$$\Delta w_{k+1} = \Delta^2 u + \alpha_{k+1}\Delta(|\nabla u|^2(u+\epsilon)^{-1}) + \beta_{k+1}\Delta(|x|^b u^q) = |x|^a u^p + I + J, \tag{3-4}$$

where $I := \alpha_{k+1}\Delta(|\nabla u|^2(u+\epsilon)^{-1})$ and $J := \beta_{k+1}\Delta(|x|^b u^q)$. In what follows, we simplify I and J as well as finding lower bounds for these terms. We start with J:

$$\begin{aligned}\frac{J}{\beta_{k+1}} &= \Delta(|x|^b u^q) = \Delta|x|^b u^q + \Delta u^q |x|^b + 2\nabla|x|^b\cdot\nabla u^q \\ &= b(n+b-2)|x|^{b-2}u^q + q(q-1)|x|^b u^{q-2}|\nabla u|^2 + q|x|^b u^{q-1}\Delta u + 2bq|x|^{b-2}u^{q-1}\nabla u\cdot x.\end{aligned}$$

From the definition of w_{k+1}, we have

$$\Delta u = w_{k+1} - \alpha_{k+1}|\nabla u|^2(u+\epsilon)^{-1} - \beta_{k+1}|x|^b u^q. \tag{3-5}$$

Substitute this into the previous equation to simplify J as

$$\frac{J}{\beta_{k+1}} = qu^{q-1}|x|^b w_{k+1} - q\beta_{k+1}u^{2q-1}|x|^{2b} + \Big(q(q-1) - q\alpha_{k+1}\frac{u}{u+\epsilon}\Big)|x|^b u^{q-2}|\nabla u|^2 + b(n+b-2)|x|^{b-2}u^q + 2bq|x|^{b-2}u^{q-1}\nabla u\cdot x. \quad (3\text{-}6)$$

We now simplify I:

$$\begin{aligned}\frac{I}{\alpha_{k+1}} &= \Delta(|\nabla u|^2(u+\epsilon)^{-1}) = \sum_{i,j}\partial_{jj}(u_i^2(u+\epsilon)^{-1})\\ &= 2(u+\epsilon)^{-1}\sum_{i,j}(\partial_{ij}u)^2 + 2(u+\epsilon)^{-1}\nabla u\cdot\nabla\Delta u - 4(u+\epsilon)^{-2}\sum_{i,j}\partial_i u\partial_j u\partial_{ij}u\\ &\qquad - |\nabla u|^2(u+\epsilon)^{-2}\Delta u + 2|\nabla u|^4(u+\epsilon)^{-3}.\end{aligned}$$

Again substituting (3-5) into the term $2(u+\epsilon)^{-1}\nabla u\cdot\nabla\Delta u$ that appears above, we get

$$\begin{aligned}\frac{I}{\alpha_{k+1}} &= 2(u+\epsilon)^{-1}\sum_{i,j}(\partial_{ij}u)^2 - 4(u+\epsilon)^{-2}\sum_{i,j}\partial_i u\partial_j u\partial_{ij}u + 2|\nabla u|^4(u+\epsilon)^{-3} - |\nabla u|^2(u+\epsilon)^{-3}\Delta u\\ &\quad + 2(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1} - 2\alpha_{k+1}(u+\epsilon)^{-1}\nabla u\cdot(|\nabla u|^2(u+\epsilon)^{-1})\\ &\quad - 2\beta_{k+1}(u+\epsilon)^{-1}\nabla u\cdot\nabla(|x|^b u^q).\end{aligned}$$

Then, collecting similar terms, we obtain

$$\begin{aligned}&\frac{I}{\alpha_{k+1}} - 2(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1}\\ &\quad = 2(u+\epsilon)^{-1}\sum_{i,j}(\partial_{ij}u)^2 - 4(\alpha_{k+1}+1)(u+\epsilon)^{-2}\sum_{i,j}\partial_i u\partial_j u\partial_{ij}u + 2(\alpha_{k+1}+1)|\nabla u|^4(u+\epsilon)^{-3}\\ &\qquad - |\nabla u|^2(u+\epsilon)^{-2}\Delta u - 2\beta_{k+1}b|x|^{b-2}(u+\epsilon)^{-1}u^q\nabla u\cdot x - 2\beta_{k+1}q|x|^b u^{q-1}(u+\epsilon)^{-1}|\nabla u|^2.\end{aligned}$$

Completing the square, we get

$$\begin{aligned}&\frac{I}{\alpha_{k+1}} - 2(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1}\\ &\quad = 2(u+\epsilon)^{-1}\sum_{i,j}(\partial_{ij}u - (\alpha_{k+1}+1)(u+\epsilon)^{-1}\partial_i u\partial_j u)^2 - 2\alpha_{k+1}(\alpha_{k+1}+1)|\nabla u|^4(u+\epsilon)^{-3}\\ &\qquad - |\nabla u|^2(u+\epsilon)^{-2}\Delta u - 2\beta_{k+1}b|x|^{b-2}(u+\epsilon)^{-1}u^q\nabla u\cdot x - 2\beta_{k+1}q|x|^b u^{q-1}(u+\epsilon)^{-1}|\nabla u|^2. \quad (3\text{-}7)\end{aligned}$$

Note that, for any $n\times n$ matrix $A=(a_{i,j})$, the Hilbert–Schmidt norm is defined by $\|A\|_2 = \sqrt{\sum_{i,j}|a_{i,j}|^2} = \sqrt{\mathrm{trace}(AA^*)}$, where A^* denotes the conjugate transpose of A. From the Cauchy–Schwarz inequality, the following inequality holds:

$$|\mathrm{trace}\,A|^2 = |(A,I)|^2 \le \|A\|_2^2\|I\|_2^2 = n\sum_{i,j}|a_{i,j}|^2. \quad (3\text{-}8)$$

Set $a_{i,j} := \partial_{ij}u - (\alpha_{k+1}+1)(u+\epsilon)^{-1}\partial_i u\partial_j u$ in (3-8) to get

$$\sum_{i,j}^{n}(\partial_{ij}u - (\alpha_{k+1}+1)(u+\epsilon)^{-1}\partial_i u\partial_j u)^2 \geq \frac{1}{n}(\Delta u - (\alpha_{k+1}+1)(u+\epsilon)^{-1}|\nabla u|^2)^2.$$

From this lower bound for the Hessian and (3-7), we get

$$\begin{aligned}\frac{I}{\alpha_{k+1}} - 2(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1} \geq \frac{2}{n}(u+\epsilon)^{-1}(\Delta u - (\alpha_{k+1}+1)(u+\epsilon)^{-1}|\nabla u|^2)^2 \\ -2\alpha_{k+1}(\alpha_{k+1}+1)|\nabla u|^4(u+\epsilon)^{-3} - |\nabla u|^2(u+\epsilon)^{-2}\Delta u + T_k,\end{aligned} \tag{3-9}$$

where

$$T_k := -2\beta_{k+1}b|x|^{b-2}(u+\epsilon)^{-1}u^q\nabla u\cdot x - 2\beta_{k+1}q|x|^b u^{q-1}(u+\epsilon)^{-1}|\nabla u|^2.$$

Note also that, from the assumption $w_k \leq 0$, we have the upper bound on the Laplacian operator $\Delta u \leq -\alpha_k|\nabla u|^2(u+\epsilon)^{-1} - \beta_k|x|^b u^q$. Elementary calculations show that, if $t \leq t_* \leq 0$ and $s \geq 0$, then $(t-s)^2 \geq t_*^2 - 2t_*s + s^2$. Set the parameters as $t = \Delta u$, $t_* = -\alpha_k|\nabla u|^2(u+\epsilon)^{-1} - \beta_k|x|^b u^q$ and $s = (\alpha_{k+1}+1)(u+\epsilon)^{-1}|\nabla u|^2$ to get the following lower bound on the square term that appears in (3-9):

$$\begin{aligned}&(\Delta u - (\alpha_{k+1}+1)(u+\epsilon)^{-1}|\nabla u|^2)^2 \\ &\quad\geq (\alpha_k|\nabla u|^2(u+\epsilon)^{-1} + \beta_k|x|^b u^q)^2 + 2(\alpha_k|\nabla u|^2(u+\epsilon)^{-1} + \beta_k|x|^b u^q)(\alpha_{k+1}+1)(u+\epsilon)^{-1}|\nabla u|^2 \\ &\qquad + (\alpha_{k+1}+1)^2(u+\epsilon)^{-2}|\nabla u|^4.\end{aligned} \tag{3-10}$$

Substitute (3-5) into the term $-|\nabla u|^2(u+\epsilon)^{-2}\Delta u$ that appears in (3-9) to eliminate the Laplacian operator. Then, apply inequality (3-10) to simplify (3-9) as

$$\begin{aligned}&\frac{I}{\alpha_{k+1}} - 2(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1} \\ &\quad\geq \frac{2}{n}(u+\epsilon)^{-1}\big((\alpha_{k+1}+\alpha_k+1)^2|\nabla u|^4(u+\epsilon)^{-2} + \beta_k^2|x|^{2b}u^{2q} + 2\beta_k(\alpha_{k+1}+\alpha_k+1)|x|^b u^q(u+\epsilon)^{-1}|\nabla u|^2\big) \\ &\qquad - w_{k+1}(u+\epsilon)^{-2}|\nabla u|^2 - \alpha_{k+1}(2\alpha_{k+1}+1)|\nabla u|^4(u+\epsilon)^{-3} + \beta_{k+1}|x|^b u^q(u+\epsilon)^{-2}|\nabla u|^2 + T_k.\end{aligned}$$

Collecting similar terms and using the value of T_k, we end up with

$$\begin{aligned}&\frac{I}{\alpha_{k+1}} - 2(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1} + w_{k+1}(u+\epsilon)^{-2}|\nabla u|^2 \\ &\quad\geq \frac{2}{n}\beta_k^2|x|^{2b}u^{2q}(u+\epsilon)^{-1} + I_{\alpha_k}^{(2)}|\nabla u|^4(u+\epsilon)^{-3} + S_{\epsilon,\alpha_k,\beta_k}|\nabla u|^2u^{q-2}|x|^b - 2\beta_{k+1}b|x|^{b-2}(u+\epsilon)^{-1}u^q\nabla u\cdot x,\end{aligned}$$

where

$$\begin{aligned}I_{\alpha_k}^{(2)} &:= \frac{2}{n}(\alpha_{k+1}+\alpha_k+1)^2 - 2\alpha_{k+1}(\alpha_{k+1}+1) + \alpha_{k+1}, \\ S_{\epsilon,\alpha_k,\beta_k} &:= \frac{4}{n}\beta_k(\alpha_{k+1}+\alpha_k+1)\frac{u^2}{(u+\epsilon)^2} + \beta_{k+1}\frac{u^2}{(u+\epsilon)^2} - 2\beta_{k+1}q\frac{u}{u+\epsilon}.\end{aligned}$$

Therefore, the following lower bound for I holds:

$$I \geq 2\alpha_{k+1}(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1} - \alpha_{k+1}w_{k+1}(u+\epsilon)^{-2}|\nabla u|^2 + \frac{2}{n}\alpha_{k+1}\beta_k^2|x|^{2b}u^{2q}(u+\epsilon)^{-1} + I_{\alpha_k}|\nabla u|^4(u+\epsilon)^{-3} + S_{\epsilon,\alpha_k,\beta_k}|\nabla u|^2u^{q-2}|x|^b - 2\beta_{k+1}b|x|^{b-2}(u+\epsilon)^{-1}u^q\nabla u\cdot x. \tag{3-11}$$

Finally, applying this lower bound for I and the lower bound given for J in (3-6), from (3-3) we get

$$\begin{aligned}\Delta w_{k+1} - 2\alpha_{k+1}(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1} + \alpha_{k+1}(u+\epsilon)^{-2}|\nabla u|^2 w_{k+1} - \beta_{k+1}qu^{q-1}|x|^b w_{k+1} \\ \geq |x|^a u^p\Big(1 - q\beta_{k+1}^2 + \frac{2}{n}\alpha_{k+1}\beta_k^2\frac{u}{u+\epsilon}\Big) + \alpha_{k+1}I_{\alpha_k}^{(2)}|\nabla u|^4(u+\epsilon)^{-3} \\ + \Big(\alpha_{k+1}S_{\epsilon,\alpha_k,\beta_k} + \Big(q(q-1) - \alpha_{k+1}q\frac{u}{u+\epsilon}\Big)\beta_{k+1}\Big)|\nabla u|^2u^{q-2}|x|^b \\ + 2b\beta_{k+1}\Big(q - \alpha_{k+1}\frac{u}{u+\epsilon}\Big)|x|^{b-2}u^{q-1}\nabla u\cdot x + b\beta_{k+1}(n+b-2)|x|^{b-2}u^q.\end{aligned}$$

Completing the square finishes the proof. □

4. Proof of Theorem 1.3 via iteration arguments

To apply the iteration argument, we need to develop a maximum principle argument for the equation

$$\Delta w - 2\alpha(u+\epsilon)^{-1}\nabla u\cdot\nabla w + \alpha w(u+\epsilon)^{-2}|\nabla u|^2 - \tfrac{1}{2}\beta(p+1)|x|^{a/2}u^{(p-1)/2}w = f(x) \geq 0 \quad \text{in } \mathbb{R}^n \tag{4-1}$$

that appears in Proposition 3.1, where α and β are positive constants, u is a solution of (1-1) and $w,\ f\in C^\infty(\mathbb{R}^n)$.

Lemma 4.1. *Suppose that w is a solution of the differential inequality* (4-1), *where u is a solution of* (1-1) *and*

$$w = \Delta u + \alpha(u+\epsilon)^{-1}|\nabla u|^2 + \beta|x|^{a/2}u^{(p+1)/2} \tag{4-2}$$

for positive constants ϵ, α and β. Then, assuming that $p+1 > 2\alpha$,

$$\Delta\tilde{w} \geq 0 \quad \text{on } \{w\geq 0\}\subset\mathbb{R}^n, \tag{4-3}$$

where $\tilde{w} = (u+\epsilon)^t w$ for $t=-\alpha$.

Proof. Straightforward calculations show that

$$\Delta\tilde{w} = (u+\epsilon)^t\Delta w + 2t(u+\epsilon)^{t-1}\nabla u\cdot\nabla w + t(u+\epsilon)^{t-1}w\Delta u + t(t-1)w(u+\epsilon)^{t-2}|\nabla u|^2.$$

We now add and subtract two terms, $\frac{1}{2}\beta(p+1)|x|^{a/2}u^{(p-1)/2}(u+\epsilon)^t w$ and $tw(u+\epsilon)^{t-2}|\nabla u|^2$, to the above identity and collect similar terms to get

$$\begin{aligned}\Delta\tilde{w} = (u+\epsilon)^t\big(\Delta w + 2t(u+\epsilon)^{-1}\nabla u\cdot\nabla w - tw(u+\epsilon)^{-2}|\nabla u|^2 - \tfrac{1}{2}\beta(p+1)|x|^{a/2}u^{(p-1)/2}w\big) \\ + \tfrac{1}{2}\beta(p+1)|x|^{a/2}u^{(p-1)/2}(u+\epsilon)^t w + tw(u+\epsilon)^{t-2}|\nabla u|^2 + t(u+\epsilon)^{t-1}w\Delta u \\ + t(t-1)w(u+\epsilon)^{t-2}|\nabla u|^2.\end{aligned}$$

From the fact that $t = -\alpha$ and w satisfies (4-1), we get

$$\Delta\tilde{w} \geq \tfrac{1}{2}\beta(p+1)|x|^{a/2}u^{(p-1)/2}(u+\epsilon)^t w + t(u+\epsilon)^{t-1} w\Delta u + t^2 w(u+\epsilon)^{t-1}\frac{|\nabla u|^2}{u+\epsilon}.$$

Note that we can eliminate the gradient term using (4-2), that is,

$$\alpha(u+\epsilon)^{-1}|\nabla u|^2 = w - \Delta u - \beta|x|^{a/2}u^{(p+1)/2}.$$

Therefore, after collecting similar terms we get

$$\begin{aligned}\Delta\tilde{w} &\geq \frac{t^2}{\alpha}w^2(u+\epsilon)^{t-1} + (u+\epsilon)^{t-1}wt\Big(1-\frac{t}{\alpha}\Big)\Delta u \\ &\quad + \beta(u+\epsilon)^{t-1}|x|^{a/2}u^{(p-1)/2}w\Big(\frac{(p+1)\epsilon}{2} + u\Big(\frac{p+1}{2} - \frac{t^2}{\alpha}\Big)\Big) \\ &=: R_1 + R_2 + R_3.\end{aligned}$$

We claim that the above three terms, R_1, R_2 and R_3, are nonnegative when $w \geq 0$. From the fact that $\alpha > 0$ one can see that R_1 is nonnegative. From the definition of $t = -\alpha < 0$, we have $t(1-t/\alpha) = -2\alpha < 0$. This together with Proposition 2.3, that is, $\Delta u \leq 0$, confirms that R_2 is nonnegative. Positivity of R_3 is an immediate consequence of the assumptions: β is positive and $\frac{1}{2}(p+1) - t^2/\alpha = \frac{1}{2}(p+1) - \alpha$ is also positive. This finishes the proof. □

We now apply Lemma 4.1 to show that any solution w of (4-1) is negative.

Lemma 4.2. *Suppose that $\tilde{w}$ and w as in Lemma 4.1. Let u be a bounded solution of* (1-1); *then $w \leq 0$.*

Proof. The methods and ideas that we apply in the proof are motivated by Souplet [2009]. Multiply (4-3) by $\tilde{w}_+^s$, where $s > 0$ is a parameter that will be determined later. Then, integration by parts over B_R gives us

$$0 \leq \int_{B_R} \Delta\tilde{w}\tilde{w}_+^s = -s\int_{B_R} |\nabla\tilde{w}_+|^2\tilde{w}_+^{s-1} + R^{n-1}\int_{S^{n-1}} \tilde{w}_r\tilde{w}_+^s. \tag{4-4}$$

Therefore,

$$\int_{B_R} |\nabla\tilde{w}_+|^2\tilde{w}_+^{s-1} \leq \frac{1}{s(s+1)}R^{n-1}\int_{S^{n-1}} (\tilde{w}_+^{s+1})_r = C(s)R^{n-1}I'(R), \tag{4-5}$$

where

$$I(R) := \int_{S^{n-1}} \tilde{w}_+^{s+1} = \int_{S^{n-1}} (u+\epsilon)^{-(s+1)\alpha}w_+^{s+1}$$

and $C(s)$ is a constant independent of R. Note that w, given as $w = \Delta u + \alpha|\nabla u|^2(u+\epsilon)^{-1} + \beta|x|^{a/2}u^{(p+1)/2}$, satisfies $w \geq 0$ if and only if $-\Delta u \leq \alpha|\nabla u|^2(u+\epsilon)^{-1} + \beta|x|^{a/2}u^{(p+1)/2}$. Therefore,

$$w_+^{s+1} \leq C|\nabla u|^{2(s+1)}(u+\epsilon)^{-(s+1)} + C|x|^{(s+1)a/2}u^{(s+1)(p+1)/2}, \tag{4-6}$$

where $C = C(\alpha, \beta, s)$. Applying this upper bound for w_+, we can get an upper bound for $I(R)$:

$$
\begin{aligned}
I(R) &\le C \int_{S^{n-1}} (u+\epsilon)^{-(s+1)(\alpha+1)} |\nabla u|^{2(s+1)} + C R^{(s+1)a/2} \int_{S^{n-1}} (u+\epsilon)^{-\alpha(s+1)} u^{(s+1)(p+1)/2} \\
&\le C(\epsilon) \int_{S^{n-1}} |\nabla u|^{2(s+1)} + C(\epsilon) R^{(s+1)a/2} \int_{S^{n-1}} u^{(s+1)(p+1)/2} \\
&=: C(\epsilon)(I_1(R) + I_2(R)).
\end{aligned} \tag{4-7}
$$

In what follows, we show that there is a sequence R such that the two terms $I_1(R)$ and $I_2(R)$ decay to zero for a fixed ϵ. We start with $I_2(R)$, which includes an integral of a positive power of u over the sphere. Due to the boundedness assumption on u, it is straightforward to relate this term to L^p estimates of u over the sphere. As a matter of fact, if $(s+1)(p+1) > 2p$ then, from the boundedness of u, we have

$$
\int_{S^{n-1}} u^{(s+1)(p+1)/2} \le C(n) \|u\|^p_{L^p(S^{n-1})} \tag{4-8}
$$

and for the case $(s+1)(p+1) \le 2p$ we can use Hölder's inequality to get

$$
\int_{S^{n-1}} u^{(s+1)(p+1)/2} \le C(n, p) \|u\|^{(p+1)(s+1)/2}_{L^p(S^{n-1})}. \tag{4-9}
$$

So, to prove a decay estimate for $I_2(R)$ we need to construct a decay estimate for $\|u\|_{L^p(S^{n-1})}$. On the other hand, we apply Lemma 2.10 to get an upper bound for the first term in (4-7), $I_1(R)$. In fact, from Lemma 2.10 with $i = 1$, $\tau = 2(s+1)$ and $t = 2$, we have

$$
\begin{aligned}
\|D_x u\|_{L^{2(s+1)}(S^{n-1})} &\le C \|D_\theta D_x u\|_{L^2(S^{n-1})} + C \|D_x u\|_{L^1(S^{n-1})} \\
&\le C R \|D_x^2 u\|_{L^2(S^{n-1})} + C \|D_x u\|_{L^1(S^{n-1})}
\end{aligned} \tag{4-10}
$$

for $s = 2/(n-3)$. In order to get a decay estimate for $I_1(R)$, we need decay estimates for the two terms in the right-hand side of (4-10), $\|D_x^2 u\|_{L^2(S^{n-1})}$ and $\|D_x u\|_{L^1(S^{n-1})}$.

We now apply the elliptic estimates given in Section 2 to provide decay estimates for $\|u\|_{L^p(S^{n-1})}$, $\|D_x u\|_{L^1(S^{n-1})}$ and $\|D_x^2 u\|_{L^2(S^{n-1})}$. To do so we first find appropriate upper bounds for these terms on the ball of radius R. Then we use certain measure-comparison arguments to construct decay estimates over the sphere. So, from Lemma 2.7 with $\tau = 2$, we get

$$
\int_{R/2}^{R} \|D_x^2 u\|^2_{L^2(S^{n-1})} r^{n-1} \, dr \le C \int_{B_{2R} \setminus B_{R/4}} |\Delta u|^2 + C R^{-4} \int_{B_{2R} \setminus B_{R/4}} u^2. \tag{4-11}
$$

We now apply Corollary 2.9 and Corollary 2.2 to get a decay estimate for the right-hand side of (4-11), namely,

$$
\begin{aligned}
&R^{-4} \int_{B_{2R} \setminus B_{R/4}} u^2 \le C R^{-4} \int_{B_{2R} \setminus B_{R/4}} u \le C R^{-4} R^{n-(a+4)/(p-1)} = C R^{n-(a+4p)/(p-1)}, \\
&\int_{B_{2R} \setminus B_{R/4}} |\Delta u|^2 \le C R^{n-(a+4p)/(p-1)},
\end{aligned}
$$

where C is independent from R. From this and (4-11), we obtain the desired decay estimate on the Hessian operator of u,

$$\int_{R/2}^{R} \|D_x^2 u\|^2_{L^2(S^{n-1})} r^{n-1}\,dr \le C R^{n-(4p+a)/(p-1)}. \tag{4-12}$$

Similarly, from Corollary 2.6 and Lemma 2.1, we have

$$\int_{R/2}^{R} \|D_x u\|_{L^1(S^{n-1})} r^{n-1}\,dr \le C R^{n-(p+3+a)/(p-1)}, \tag{4-13}$$

$$\int_{R/2}^{R} \|u\|^p_{L^p(S^{n-1})} r^{n-1}\,dr \le C R^{n-(a+4)p/(p-1)}. \tag{4-14}$$

Now let's define the following sets. These sets are meant to facilitate our arguments towards construction of decay estimates for $\|u\|_{L^p(S^{n-1})}$, $\|D_x u\|_{L^1(S^{n-1})}$ and $\|D_x^2 u\|_{L^2(S^{n-1})}$. For a large number M, which will be determined later, define

$$\begin{aligned}
\Gamma_1(R) &:= \{r \in (R/2, R) : \|u\|^p_{L^p(S^{n-1})} > M R^{-(a+4)p/(p-1)}\},\\
\Gamma_2(R) &:= \{r \in (R/2, R) : \|D_x u\|_{L^1(S^{n-1})} > M R^{-(p+3+a)/(p-1)}\},\\
\Gamma_3(R) &:= \{r \in (R/2, R) : \|D_x^2 u\|^2_{L^2(S^{n-1})} > M R^{-(a+4p)/(p-1)}\}.
\end{aligned}$$

We claim that $|\Gamma_i(R)| \le R/4$ for $1 \le i \le 3$: Using (4-12), we get

$$\begin{aligned}
C &\ge R^{-n+(a+4p)/(p-1)} \int_{R/2}^{R} \|D_x^2 u\|^2_{L^2(S^{n-1})} r^{n-1}\,dr\\
&\ge N R^{-n+(a+4p)/(p-1)} R^{n-1} \int_{R/2}^{R} \|D_x^2 u\|^2_{L^2(S^{n-1})}\,dr\\
&\ge N M R^{-n+(a+4p)/(p-1)} R^{n-1} \int_{|\Gamma_3(R)|} R^{-(a+4p)/(p-1)}\,dr\\
&\ge N M R^{-n+(a+4p)/(p-1)} R^{n-1} |\Gamma_3(R)| R^{-(a+4p)/(p-1)} = N M |\Gamma_3(R)| R^{-1},
\end{aligned}$$

where $N = \left(\frac{1}{2}\right)^{n-1}$. Therefore, $|\Gamma_3(R)| \le CR/NM$. Now, choosing M to be large enough, that is, $M > 4C/N$, we get $|\Gamma_3(R)| \le R/4$. Similarly, applying (4-13) and (4-14), one can show that $|\Gamma_i(R)| \le R/4$ for $i = 1, 2$. Hence, $|\Gamma_i(R)| \le R/4$ for $1 \le i \le 3$ while $\Gamma_i(R) \subset (R/2, R)$. So, we can find a sequence of $\tilde{R}$ such that

$$\tilde{R} \in (R/2, R) \setminus \bigcup_{i=1}^{i=3} \Gamma_i(R) \ne \varnothing. \tag{4-15}$$

Therefore, for the sequence $\tilde{R}$, we obtain

$$\|u\|^p_{L^p(S^{n-1})} \le M R^{-(a+4)p/(p-1)}, \tag{4-16}$$

$$\|D_x u\|_{L^1(S^{n-1})} \le M R^{-(p+3+a)/(p-1)}, \tag{4-17}$$

$$\|D_x^2 u\|^2_{L^2(S^{n-1})} \le M R^{-(a+4p)/(p-1)}. \tag{4-18}$$

Substituting (4-16) into (4-8) and (4-9), we get the decay estimate on $I_2(R)$

$$\begin{aligned} I_2(R) &\le C\chi\{(s+1)(p+1) > 2p\}R^{(s+1)/2a-(a+4)p/(p-1)} \\ &\quad + C\chi\{(s+1)(p+1) \le 2p\}R^{(s+1)a/2-(a+4)(p+1)(s+1)/(2(p-1))} \\ &= C\chi\{(s+1)(p+1) > 2p\}R^{-\eta_1} + C\chi\{(s+1)(p+1) > 2p\}R^{-\eta_2}, \end{aligned} \tag{4-19}$$

where χ is the characteristic function, $\eta_1 := a\bigl(p/(p-1) - \frac{1}{2}(s+1)\bigr) + 4p/(p-1) > 0$ and $\eta_2 := (s+1)(ap+2(p+1))/(p+1) > 0$. Note that we have used the fact that $p/(p-1) - \frac{1}{2}(s+1) > 0$ because $0 < s = 2/(n-3) \le 1$ when $n \ge 5$. On the other hand, substituting (4-17) and (4-18) into the Sobolev embedding (4-10), we get

$$\|D_x u\|_{L^{2(s+1)}(S^{n-1})} \le CR^{1-(a+4p)/(p-1)} + CR^{-(p+3+a)/(p-1)} = 2CR^{-(p+3+a)/(p-1)}. \tag{4-20}$$

From this and the definition of $I_1(R)$, we end up with the decay estimate on $I_1(R)$

$$I_1(R) = \int_{S^{n-1}} |\nabla u|^{2(s+1)} \le CR^{-2(p+3+a)(s+1)/(p-1)} = CR^{-\eta_3}, \tag{4-21}$$

where $\eta_3 := 2(p+3+a)(s+1)/(p-1) > 0$. Finally, from (4-21) and (4-19), we observe that

$$I(R) \le CR^{-\eta} \quad \text{for all } R > 1,$$

where $\eta := \min\{\eta_1, \eta_2, \eta_3\} > 0$. So, $I(R) \to 0$ as $R \to \infty$. Note that $\tilde{R} \to \infty$ as $R \to \infty$. Since $I(R)$ is a positive function and converges to zero, there is a sequence such that the functional $I'(R)$ is nonpositive. Therefore, (4-5) yields

$$\int_{B_R} |\nabla \tilde{w}_+|^2 \tilde{w}_+^{s-1} \le 0. \tag{4-22}$$

Hence, $\tilde{w}_+$ has to be a constant. From the continuity of $\tilde{w}$, we have $\tilde{w} \equiv C$. Note that the constant C cannot be strictly positive. So, $\tilde{w}_+ = 0$ and therefore $w_+ = 0$. This finishes the proof. □

Note that Lemma 4.1 and Lemma 4.2 imply an iteration argument for the sequence of functions, for $k \ge -1$,

$$w_k = \Delta u + \alpha_k (u+\epsilon)^{-1}|\nabla u|^2 + \beta_k |x|^{a/2} u^{(p+1)/2} \tag{4-23}$$

as long as the right-hand side of (3-3) stays nonnegative. For the rest of this section, we construct sequences $\{\alpha_k\}_{k=-1}$ and $\{\beta_k\}_{k=-1}$ such that the right-hand side of (3-3) is nonnegative.

Constructing sequences α_k and β_k. In this part, we define sequences α_k and β_k needed for the iteration argument.

Lemma 4.3. *Suppose $\alpha_0 = 0$ and define*

$$\alpha_{k+1} := \frac{4(\alpha_k+1) - n + \sqrt{n(16\alpha_k^2 + 24\alpha_k + n + 8)}}{4(n-1)}. \tag{4-24}$$

Then $(\alpha_k)_k$ is a positive, bounded and increasing sequence that converges to $\alpha := 2/(n-4)$ provided $n > 4$ and $p > 1$. Moreover, for this choice of $(\alpha_k)_k$, the sequence $I_{\alpha_k}^{(2)}$ of coefficients defined in Proposition 3.1 equals zero.

Proof. It is straightforward to show that $\alpha_k > 0$ for any $k \geq 0$. Also, direct calculations show that $\alpha_k \to \alpha := 2/(n-4)$ provided α_k is convergent. Note that $\alpha_1 = (4-n+\sqrt{n^2+8n})/(4n-4) < 2/(n-4)$ and, by induction, one can see that $\alpha_k \leq \alpha$ for all $k \geq 0$. Lastly, we show that α_k is an increasing sequence: For any k,

$$\begin{aligned}\alpha_{k+1} - \alpha_k &= \frac{\sqrt{n(16\alpha_k^2 + 24\alpha_k + n + 8)} - ((n-4) + 4a_k(n-2))}{4(n-1)} \\ &= \frac{8(n-1)(n-4)(2\alpha_k + 1)}{S_{n,k}} \left(\frac{2}{n-4} - \alpha_k \right),\end{aligned}$$

where $S_{n,k} = \sqrt{n(16\alpha_k^2 + 24\alpha_k + n + 8)} + (n-4) + 4a_k(n-2) > 0$. Therefore, from the fact that $\alpha_k \leq \alpha = 2/(n-4)$, we get the desired result. □

Similarly, we provide an explicit formula for the sequence β_k:

Lemma 4.4. *Suppose $\beta_0 = \sqrt{2/(p+1)}$ and define*

$$\beta_{k+1} := \sqrt{\frac{2}{p+1} + \frac{4}{(p+1)n}\alpha_k \beta_k^2}, \tag{4-25}$$

where $(\alpha_k)_k$ is as in Lemma 4.3. Then $(\beta_k)_k$ is a positive, bounded and increasing sequence that converges to $\beta := \sqrt{2/((p+1) - c_n)}$, where $c_n = 8/(n(n-4))$ provided that $n > 4$ and $p > 1$. Moreover, for this choice of $(\alpha_k)_k$ and $(\beta_k)_k$, the sequence $I^{(1)}_{0,\alpha_k,\beta_k}$ of coefficients defined in Proposition 3.1 is strictly positive.

Proof. The sequence $(\beta_k)_k$ for all $k \geq 0$ is positive. Note that boundedness of the sequence $(\alpha_k)_k$ forces the boundedness of the $(\beta_k)_k$, meaning that $\beta_{k+1} \leq \sqrt{2/(p+1) + (4\alpha/((p+1)n))\beta_k^2}$ for any k. By straightforward calculations we get

$$\beta_{k+1}^2 \leq \frac{2}{p+1} \sum_{i=0}^{k+1} \left(\frac{4\alpha}{n(p+1)} \right)^i.$$

Note that $4\alpha/(n(p+1)) = 8/(n(n-4)(p+1)) < 1$ provided that $n > 4$ and $p > 1$. Therefore, $\sum_{i=0}^{\infty} \big(4\alpha/(n(p+1))\big)^i < \infty$. This proves the boundedness of $(\beta_k)_k$.

Since $(\alpha_k)_{k=0}$ is an increasing sequence, the sequence $(\beta_k)_{k=0}$ will be nondecreasing by induction. Note that

$$\beta_1 = \beta_0 \quad \text{and} \quad \beta_2 = \sqrt{\frac{2}{p+1} + \frac{8}{(p+1)^2 n} \frac{4-n+\sqrt{n^2+8n}}{4n-4}} > \beta_1 = \sqrt{\frac{2}{p+1}}.$$

Suppose that $\beta_{k-1} \leq \beta_k$ for a certain index $k \geq 2$; then we apply the fact that $\alpha_k \geq \alpha_{k-1}$ to show $\beta_k \leq \beta_{k+1}$. This can be found as a consequence of

$$\beta_{k+1} - \beta_k = \frac{\beta_{k+1}^2 - \beta_k^2}{\beta_{k+1} + \beta_k} = \frac{4}{(p+1)n(\beta_{k+1} + \beta_k)} (\beta_k^2 \alpha_k - \beta_{k-1}^2 \alpha_{k-1}) \geq \frac{4\alpha_{k-1}(\beta_k + \beta_{k-1})}{(p+1)n(\beta_{k+1} + \beta_k)} (\beta_k - \beta_{k-1}).$$

So, $(\beta_k)_k$ is convergent and converges to $\beta := \sqrt{2n(n-4)/((p+1)(n-4)n - 8)}$. Since $(p+1)n(n-4) > 8$ for $p > 1$ and $n > 4$, β is well-defined. □

Note that, based on the definition of the sequences $\{\alpha_k\}_{k=-1}$ and $\{\beta_k\}_{k=-1}$, we concluded that $I^{(1)}_{0,\alpha_k,\beta_k}>0$ and $I^{(2)}_{\alpha_k}=0$. In the next two lemmata we investigate the positivity of $I^{(3)}_{\epsilon,\alpha_k,\beta_k}$ and $I^{(4)}_{a,\epsilon,\alpha_k,\beta_k}$, the sequences that appeared in (3-3) in Proposition 3.1.

Lemma 4.5. *Set $\epsilon=0$ in $I^{(3)}_{\epsilon,\alpha_k,\beta_k}$, which is defined in Proposition 3.1. Then*

$$I^{(3)}_{0,\alpha_k,\beta_k}\to I^{(3)}_{0,\alpha,\beta}:=\frac{4}{n}\alpha\beta(2\alpha+1)+\alpha\beta+\beta q(q-3\alpha-1) \tag{4-26}$$

as $k\to\infty$. The constant $I^{(3)}_{0,\alpha,\beta}$ is positive provided $p>(n+4)/(n-4)$ and $n>4$.

Proof. Note that when $p>(n+4)/(n-4)$ and $n>4$, we have $\frac{1}{2}(p+1)>n/(n-4)$. As $k\to\infty$, from Lemma 4.3 and Lemma 4.4, the sequences $\alpha_k\to\alpha:=2/(n-4)$ and $\beta_k\to\beta:=\sqrt{2/((p+1)-c_n)}$. Therefore,

$$\begin{aligned}\frac{I^{(3)}_{0,\alpha,\beta}}{\beta}&=\frac{4}{n}\Big(\frac{2}{n-4}\Big)\Big(\frac{4}{n-4}+1\Big)+\frac{2}{n-4}+\frac{p+1}{2}\Big(\frac{p-1}{2}-\frac{6}{n-4}\Big)\\&=\Big(\frac{p+1}{2}\Big)^2-\Big(\frac{p+1}{2}\Big)\Big(\frac{n+2}{n-4}\Big)+\frac{2n}{(n-4)^2}\\&=\Big(\frac{p+1}{2}-\frac{n}{n-4}\Big)\Big(\frac{p+1}{2}-\frac{2}{n-4}\Big)>0.\end{aligned}$$

□

Note that $I^{(4)}_{a,\epsilon,\alpha_k,\beta_k}$ appears in (3-3) mainly because of the weight function $|x|^a$. In other words, we have $I^{(4)}_{0,\epsilon,\alpha_k,\beta_k}=0$ in the case of $a=0$.

Lemma 4.6. *For any $k\ge 0$,*

$$I^{(3)}_{0,\alpha_k,\beta_k}<\beta_{k+1}\big(\tfrac{1}{2}(p+1)-\alpha_{k+1}\big)^2 \tag{4-27}$$

provided $p>(n+4)/(n-4)$ and $n>4$. Therefore, for any $a\ge 0$ that satisfies the upper bound

$$a\le A_k:=\frac{2(n-2)I^{(3)}_{0,\alpha_k,\beta_k}}{\beta_{k+1}\big(\frac{1}{2}(p+1)-\alpha_{k+1}\big)^2-I^{(3)}_{0,\alpha_k,\beta_k}}, \tag{4-28}$$

the sequence $I^{(4)}_{a,0,\alpha_k,\beta_k}$ is positive for any k.

Proof. Basic calculations show that

$$\begin{aligned}&\beta_{k+1}\Big(\frac{p+1}{2}-\alpha_{k+1}\Big)^2-I^{(3)}_{0,\alpha_k,\beta_k}\\&\quad=\beta_{k+1}\Big(\frac{p+1}{2}-\alpha_{k+1}\Big)^2-\frac{4}{n}\alpha_{k+1}\beta_k(\alpha_{k+1}+\alpha_k+1)-\alpha_{k+1}\beta_{k+1}-\beta_{k+1}\frac{p+1}{2}\Big(\frac{p+1}{2}-3\alpha_{k+1}-1\Big)\\&\quad\ge\beta_{k+1}\Big(\Big(\frac{p+1}{2}-\alpha_{k+1}\Big)^2-\frac{4}{n}\alpha_{k+1}(\alpha_{k+1}+\alpha_k+1)-\alpha_{k+1}-\frac{p+1}{2}\Big(\frac{p+1}{2}-3\alpha_{k+1}-1\Big)\Big)\\&\quad=\beta_{k+1}\Big(\frac{n-4}{n}\alpha^2_{k+1}-\frac{4}{n}\alpha^2_{k+1}-\frac{4}{n}\alpha_{k+1}+\frac{p-1}{2}\alpha_{k+1}+\frac{p+1}{2}\Big),\end{aligned}$$

where we have used the fact that β_k and α_k are increasing sequences in the first and the second inequality, respectively. Therefore,

$$\begin{aligned}\beta_{k+1}\Big(\frac{p+1}{2}-\alpha_{k+1}\Big)^2 - I^{(3)}_{0,\alpha_k,\beta_k} &\geq \beta_{k+1}\Big(\frac{n-4}{n}\alpha_{k+1}^2+\alpha_{k+1}\Big(\frac{p-1}{2}-\frac{4}{n}\alpha_{k+1}\Big)+\frac{p+1}{2}-\frac{4}{n}\alpha_{k+1}\Big)\\ &\geq \beta_{k+1}\Big(\frac{n-4}{n}\alpha_{k+1}^2+(\alpha_{k+1}+1)\Big(\frac{p-1}{2}-\frac{4}{n}\alpha\Big)\Big)\\ &> 0.\end{aligned}$$

Note that in the last inequality we have used the fact that

$$\frac{p-1}{2}-\frac{4}{n}\alpha = \frac{p-1}{2}-\frac{4}{n}\frac{2}{n-4} > \frac{4}{(n-4)n}(n-2) > 0,$$

since $p > (n+4)/(n-4)$ and $n > 4$. □

Remark 4.7. It would be interesting if a counterpart of (1-12) could be proved for bounded solutions of the fourth-order semilinear equation $\Delta^2 u = f(u)$ under certain assumptions on the arbitrary nonlinearity $f \in C^1(\mathbb{R})$. We expect that such an inequality could be established for some convex nonlinearity f.

Appendix

We would like to mention that given the estimates in Lemma 2.1 and Lemma 2.4, one can provide a somewhat simpler proof of Proposition 2.3, as follows.

Second proof of Proposition 2.3. From Lemma 2.1, we have $\int_{\mathbb{R}^n} |x|^{2-n+a} u^p\, dx < \infty$. Hence, we define the function

$$w(x) = \frac{1}{n(n-2)\omega_n}\int_{\mathbb{R}^n}\frac{|y|^a u^p(y)}{|x-y|^{n-2}}\, dy.$$

It is clear that $w(x) \geq 0$ and $\Delta w = -|x|^a u^p$. This implies that, for a solution u of (1-1), the function $h(x) := w(x) + \Delta u(x)$ is a well-defined harmonic function on $\mathbb{R}^n$. Thus, for any $x_0 \in \mathbb{R}^n$ and any $R > 0$, by the mean value theorem for harmonic functions we will have

$$h(x_0) := \int_{\partial B_R(x_0)} h\, d\sigma = \int_{\partial B_R(x_0)} (w+\Delta u)\, d\sigma \leq \int_{\partial B_R(x_0)} w\, d\sigma + \int_{\partial B_R(x_0)} |\Delta u|\, d\sigma. \tag{A-1}$$

Since $w(x_0) < \infty$, through Tonelli's theorem, we can change the order of the integrations to see that the first integral on the right-hand side of (A-1) tends to zero as $R \to \infty$ for all R. To be more precise, notice that, up to a constant multiple, the first integral can be written as

$$\int_{\mathbb{R}^n}\int_{\partial B_R(x_0)} \frac{d\sigma_x}{|x-y|^{n-2}} |y|^a u^p(y)\, dy.$$

Then we use the fact that $\int_{\partial B_R(x_0)} 1/|x-y|^{n-2}\, d\sigma_x = |y-x_0|^{2-n}$ if $|y-x_0| > R$ and equals R^{2-n} if $|y-x_0| < R$. Thus the integral will split into two parts. The outside part tends to zero as $R \to \infty$ due to the fact that $w(x_0) < \infty$, while the inside part tends to zero due to the fact that, by Lemma 2.1,

$$R^{2-n}\int_{B_R(x_0)} |y|^a u^p(y)\, dy \leq R^{2-n}\int_{B_{R+|x_0|}(0)} |y|^a u^p\, dy \leq CR^{2-n}(R+|x_0|)^{n-(4p+a)/(p-1)}$$

tends to zero as $R \to \infty$. The second integral will tend to zero for some sequence of R by Lemma 2.4 again. Apply the above inequality to this sequence to see that $h(x_0) \leq 0$. Since x_0 is arbitrary, we have $-\Delta u \geq 0$. □

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