ZHONGWEI SHEN

CONVERGENCE RATES AND HÖLDER ESTIMATES IN ALMOST-PERIODIC HOMOGENIZATION OF ELLIPTIC SYSTEMS

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CONVERGENCE RATES AND HÖLDER ESTIMATES
IN ALMOST-PERIODIC HOMOGENIZATION OF ELLIPTIC SYSTEMS

Zhongwei Shen

For a family of second-order elliptic systems in divergence form with rapidly oscillating, almost-periodic coefficients, we obtain estimates for approximate correctors in terms of a function that quantifies the almost periodicity of the coefficients. The results are used to investigate the problem of convergence rates. We also establish uniform Hölder estimates for the Dirichlet problem in a bounded $C^{1,\alpha}$ domain.

1. Introduction and statement of main results

In this paper we consider a family of second-order elliptic operators in divergence form with rapidly oscillating, *almost-periodic* coefficients,

$$\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i}\left(a_{ij}^{\alpha\beta}(x/\varepsilon)\frac{\partial}{\partial x_j}\right), \quad \varepsilon > 0. \tag{1-1}$$

We will assume that $A(y) = (a_{ij}^{\alpha\beta}(y))$ with $1 \le i, j \le d$ and $1 \le \alpha, \beta \le m$ is real and satisfies the ellipticity condition

$$\mu|\xi|^2 \le a_{ij}^{\alpha\beta}(y)\xi_i^\alpha\xi_j^\beta \le \frac{1}{\mu}|\xi|^2 \quad \text{for } y \in \mathbb{R}^d \text{ and } \xi = (\xi_i^\alpha) \in \mathbb{R}^{d\times m}, \tag{1-2}$$

where $\mu > 0$ (the summation convention is used throughout the paper). We further assume that $A = A(y)$ is uniformly almost-periodic in $\mathbb{R}^d$; i.e., A is the uniform limit of a sequence of trigonometric polynomials in $\mathbb{R}^d$.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ be the weak solution of the Dirichlet problem

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \qquad \text{and} \qquad u_\varepsilon = g \quad \text{on } \partial\Omega, \tag{1-3}$$

where $F \in H^{-1}(\Omega; \mathbb{R}^m)$ and $g \in H^{1/2}(\partial\Omega; \mathbb{R}^m)$. Under the ellipticity condition (1-2) and the almost periodicity condition on A, it is known that $u_\varepsilon \rightarrow u_0$ weakly in $H^1(\Omega; \mathbb{R}^m)$ and thus strongly in $L^2(\Omega; \mathbb{R}^m)$ as $\varepsilon \rightarrow 0$. Furthermore, the function u_0 is the solution of

$$\mathcal{L}_0(u_0) = F \quad \text{in } \Omega \qquad \text{and} \qquad u_0 = g \quad \text{on } \partial\Omega, \tag{1-4}$$

where $\mathcal{L}_0 = -\operatorname{div}(\hat{A}\nabla)$ is a second-order elliptic operator with constant coefficients, uniquely determined by $A(y)$. As in the periodic case (see, e.g., [Bensoussan et al. 1978]), the constant matrix $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$ is called the homogenized matrix for A and $\mathcal{L}_0$ the homogenized operator for $\mathcal{L}_\varepsilon$. In this paper we shall be interested in quantitative homogenization results as well as uniform estimates for solutions of (1-3).

Homogenization of elliptic equations with rapidly oscillating, almost-periodic or random coefficients was studied first by S. M. Kozlov [1978; 1979] and by G. C. Papanicolaou and S. R. S. Varadhan [1981]. In particular, the $o(1)$ convergence rate of $u_\varepsilon - u_0$ in $C^\sigma(\overline{\Omega})$ for some $\sigma > 0$ was obtained in [Kozlov 1978] for a scalar second-order elliptic equation in divergence form with almost-periodic coefficients. Under some additional conditions on the frequencies in the spectrum of $A(y)$, the sharp $O(\varepsilon)$ rate in $C(\overline{\Omega})$ was proved in [Kozlov 1978] for operators with sufficiently smooth quasiperiodic coefficients. It is known that, without additional structure conditions on $A(y)$, the $O(\varepsilon)$ rate cannot be expected in general (see [Bondarenko et al. 2005] for some interesting results in the 1-dimensional case).

In contrast to the periodic case, the equation for the exact correctors $\chi(y)$,

$$-\operatorname{div}(A(y)\nabla\chi(y)) = \operatorname{div}(A(y)\nabla P(y)) \quad \text{in } \mathbb{R}^d, \tag{1-5}$$

may not be solvable in the almost-periodic (or random) setting for linear functions $P(y)$. In [Kozlov 1978], solutions $\chi(y)$ of (1-5) with sublinear growth and almost-periodic gradient were constructed and, as a result, homogenization was obtained for operators with trigonometric polynomial coefficients, by a lifting method. The homogenization result for the general case follows by an approximation argument. A different approach, which also gives the homogenization of the second-order elliptic equations with random coefficients, is to formulate and solve an abstract auxiliary equation in a Hilbert space for $\psi(y) = \nabla\chi(y)$. We outline this approach in Section 2 and refer the reader to [Jikov et al. 1994] for a detailed presentation and references.

Another approach to homogenization involves the use of the so-called approximate correctors [Papanicolaou and Varadhan 1981; Kozlov 1979]. Under certain mixing conditions, the approach has been employed successfully to establish quantitative homogenization results for second-order linear elliptic equations and systems in divergence form with random coefficients [Yurinskiĭ 1986; Pozhidaev and Yurinskiĭ 1989; Bourgeat and Piatnitski 2004]. For nonlinear second-order elliptic equations and Hamilton–Jacobi equations, we refer the reader to [Caffarelli and Souganidis 2010; Armstrong et al. 2014; Armstrong and Smart 2014] for recent advances and references on quantitative homogenization results. We point out that the almost-periodic case, which does not satisfy the mixing conditions generally

imposed in the random case, is studied in [Caffarelli and Souganidis 2010; Armstrong et al. 2014]. We also mention that sharp quantitative results were obtained recently in [Gloria and Otto 2011; 2012; Gloria et al. 2014] for stochastic homogenization of discrete linear elliptic equations in divergence form.

In this paper we carry out a quantitative study of the approximate correctors $\chi_T = (\chi_{T,j}^{\beta})$ for $\mathcal{L}_\varepsilon$ in (1-1), where, for $1 \leq j \leq d$ and $1 \leq \beta \leq m$, $u = \chi_{T,j}^{\beta}$ is defined by

$$-\operatorname{div}(A(y)\nabla u) + T^{-2}u = \operatorname{div}(A(y)\nabla P_j^{\beta}(y)) \quad \text{in } \mathbb{R}^d \tag{1-6}$$

and $P_j^{\beta}(y) = y_j(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the β-th position. Among other things, we will prove that, for $T \geq 1$ and $\sigma \in (0, 1)$,

$$T^{-1}\|\chi_T\|_{L^{\infty}(\mathbb{R}^d)} \leq C_\sigma \Theta_\sigma(T), \tag{1-7}$$

$$|\chi_T(x) - \chi_T(y)| \leq C_\sigma T^{1-\sigma}|x - y|^{\sigma} \quad \text{for any } x, y \in \mathbb{R}^d, \tag{1-8}$$

and, for $0 < r \leq T$,

$$\sup_{x \in \mathbb{R}^d} \left(\fint_{B(x,r)} |\nabla \chi_T|^2 \right)^{\frac{1}{2}} \leq C_\sigma \left(\frac{T}{r} \right)^{\sigma}, \tag{1-9}$$

where C_σ depends only on d, m, σ and A. The continuous function $\Theta_\sigma(T)$, which is decreasing and converges to zero as $T \to \infty$, is defined by

$$\Theta_\sigma(T) = \inf_{0 < R \leq T} \left(\rho(R) + \left(\frac{R}{T} \right)^{\sigma} \right), \tag{1-10}$$

where

$$\rho(R) = \sup_{y \in \mathbb{R}^d} \inf_{\substack{z \in \mathbb{R}^d \\ |z| \leq R}} \|A(\cdot + y) - A(\cdot + z)\|_{L^{\infty}(\mathbb{R}^d)} \tag{1-11}$$

is a decreasing and continuous function that quantifies the almost periodicity of A. Indeed, a bounded continuous function A in $\mathbb{R}^d$ is uniformly almost-periodic if and only if $\rho(R) \to 0$ as $R \to \infty$.

With the estimates (1-7), (1-8) and (1-9) at our disposal, we obtain the following theorems on the convergence rates. Our results in Theorems 1.2 and 1.4 are new even in the scalar case $m = 1$.

Theorem 1.1. *Suppose that $A(y) = (a_{ij}^{\alpha\beta}(y))$ satisfies the ellipticity condition* (1-2) *and is uniformly almost-periodic in $\mathbb{R}^d$. Let $p > d$, $\sigma \in (0, 1)$, and Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^d$ for some $\alpha > 0$. Then there exists a modulus $\eta : (0, 1] \to [0, \infty)$, which depends only on A and σ, such that $\lim_{t \to 0} \eta(t) = 0$ and*

$$\|u_\varepsilon - u_0\|_{C^{\sigma}(\bar{\Omega})} \leq C\eta(\varepsilon)\|u_0\|_{W^{2,p}(\Omega)} \tag{1-12}$$

for $\varepsilon \in (0, 1)$ whenever $u_\varepsilon \in H^1(\Omega)$ is the weak solution of (1-3) *and $u_0 \in W^{2,p}(\Omega)$ is the solution of* (1-4). *Furthermore, we have*

$$\|u_\varepsilon - u_0 - \varepsilon \chi_T(x/\varepsilon)\nabla u_0\|_{H^1(\Omega)} \leq C\eta(\varepsilon)\|u_0\|_{W^{2,p}(\Omega)}, \tag{1-13}$$

where $T = \varepsilon^{-1}$ and $\chi_T(y)$ denotes the approximate corrector defined by (1-6). *The constants C in* (1-12) *and* (1-13) *depend only on Ω, p, σ and A.*

The next theorem gives more precise rates of convergence, provided $\rho(R)$ decays fast enough that $\int_1^\infty (\rho(r)/r)\, dr < \infty$.

Theorem 1.2. *Under the same assumptions as in Theorem 1.2,*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\|u_0\|_{W^{2,p}(\Omega)} \left(\int_{1/(2\varepsilon)}^\infty \frac{\Theta_\sigma(r)}{r}\, dr + [\Theta_1(\varepsilon^{-1})]^\sigma \right) \tag{1-14}$$

and

$$\|u_\varepsilon - u_0 - \varepsilon\chi_T(x/\varepsilon)\nabla u_0\|_{H^1(\Omega)} \le C\|u_0\|_{W^{2,p}(\Omega)} \left(\int_{1/(2\varepsilon)}^\infty \frac{\Theta_\sigma(r)}{r}\, dr + [\Theta_1(\varepsilon^{-1})]^{\sigma/2} \right) \tag{1-15}$$

for any $\sigma \in (0,1)$, where $T = \varepsilon^{-1}$ and C depends only on Ω, A, p and σ.

Remark 1.3. By taking $R = \sqrt{T}$ in (1-10), we obtain $\Theta_\sigma(T) \le \rho(\sqrt{T}) + T^{-\sigma/2}$ for $T \ge 1$. It follows that

$$\int_1^\infty \frac{\rho(r)}{r}\, dr < \infty \implies \int_1^\infty \frac{\Theta_\sigma(r)}{r}\, dr < \infty \tag{1-16}$$

for any $\sigma \in (0,1]$. It is not clear whether estimates (1-14) and (1-15) are sharp. However, let us suppose that there exist $\tau > 0$ and $C > 0$ such that

$$\rho(R) \le CR^{-\tau} \quad \text{for all } R \ge 1. \tag{1-17}$$

Then, for $T \ge 1$,

$$\Theta_\sigma(T) \le CT^{-\sigma\tau/(\sigma+\tau)}.$$

It follows from (1-14) that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon^{\sigma\tau/(\sigma+\tau)}\|u_0\|_{W^{2,p}(\Omega)}.$$

Since $\sigma \in (0,1)$ is arbitrary, this gives

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C_\gamma \varepsilon^\gamma \|u_0\|_{W^{2,p}(\Omega)} \quad \text{for any } 0 < \gamma < \frac{\tau}{\tau+1}. \tag{1-18}$$

Similarly, one may deduce from (1-15) that

$$\|u_\varepsilon - u_0 - \varepsilon\chi_T(x/\varepsilon)\nabla u_0\|_{H^1(\Omega)} \le C_\gamma \varepsilon^\gamma \|u_0\|_{W^{2,p}(\Omega)} \tag{1-19}$$

for any $0 < \gamma < \tau/(2(\tau+1))$. It is interesting to note that if A is periodic then $\rho(R) = 0$ for R large and thus the condition (1-17) holds for any $\tau > 1$. Consequently, estimates (1-18) and (1-19) yield convergence rates $O(\varepsilon^{1-\delta})$ and $O(\varepsilon^{1/2-\delta})$ for any $\delta > 0$ in $L^2(\Omega)$ and $H^1(\Omega)$, respectively, which are near optimal. Also note that, under the condition (1-17), our estimate (1-7) gives

$$\|\chi_T\|_{L^\infty} \le C_\delta T^{1/(\tau+1)+\delta} \tag{1-20}$$

for any $\delta > 0$, while one has $\|\chi_T\|_{L^\infty} \le C$ if A is periodic. Section 8 contains some examples of quasiperiodic functions for which condition (1-17) is satisfied.

In this paper we also establish the uniform Hölder estimates for the Dirichlet problem (1-3).

Theorem 1.4. *Suppose that $A(y) = (a_{ij}^{\alpha\beta}(y))$ satisfies the ellipticity condition (1-2) and is uniformly almost-periodic in $\mathbb{R}^d$. Let Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^d$ for some $\alpha > 0$. Let u_ε be a weak solution of*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F + \operatorname{div}(f) \quad in \ \Omega \qquad and \qquad u_\varepsilon = g \quad on \ \partial\Omega. \tag{1-21}$$

Then, for any $\sigma \in (0, 1)$,

$$\|u_\varepsilon\|_{C^\sigma(\overline{\Omega})} \le C\left(\|g\|_{C^\sigma(\partial\Omega)} + \sup_{\substack{x\in\Omega \\ 0<r<r_0}} r^{2-\sigma} \fint_{B(x,r)\cap\Omega} |F| + \sup_{\substack{x\in\Omega \\ 0<r<r_0}} r^{1-\sigma} \left(\fint_{B(x,r)\cap\Omega} |f|^2 \right)^{\frac{1}{2}} \right), \tag{1-22}$$

where $r_0 = \operatorname{diam}(\Omega)$ and C depends only on σ, A and Ω.

We now describe the outline of this paper as well as some of key ideas used in the proof of its main results. In Section 2 we give a brief review of the homogenization of second-order elliptic systems with almost-periodic coefficients, based on an auxiliary equation in $B^2(\mathbb{R}^d)$, the Besicovich space of almost-periodic functions. We also prove a homogenization theorem (Theorem 2.2) for a sequence of operators $\{-\operatorname{div}(B_\ell(x/\varepsilon_\ell)\nabla)\}$, where $\varepsilon_\ell \to 0$ and $\{B_\ell(y)\}$ are obtained from $A(y)$ through rotations and translations. With this theorem, a compactness argument is used in Sections 3 and 4 to establish the uniform interior and boundary Hölder estimates for local solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = F + \operatorname{div}(f)$. The proof of Theorem 1.4 is given in Section 4. We mention that the compactness argument, which originated from the regularity theory in the calculus of variations and minimal surfaces, was introduced to the study of homogenization problems by M. Avellaneda and F. Lin [1987; 1989]. It was used recently in [Kenig et al. 2013] to establish the Lipschitz estimates for the Neumann problem in periodic homogenization. Also see related work in [Shen 2008; Geng et al. 2012; Shen and Geng 2015]. In the almost-periodic setting, the compactness argument was used in [Dungey et al. 2001] to obtain the interior Hölder estimate for operators with complex coefficients. However, we point out that some version of Theorem 2.2 seems to be necessary to ensure that the constants are independent of the centers of balls.

The approximate correctors χ_T are constructed in Section 5, while estimates (1-7), (1-8) and (1-9) are established in Section 6. The proof of (1-8) and (1-9) relies on the uniform Hölder estimates for $\mathcal{L}_\varepsilon$. We will also show that

$$|\chi_T(x) - \chi_T(y)| \le CT\|A(\cdot + x) - A(\cdot + y)\|_{L^\infty} \quad for \ any \ x, y \in \mathbb{R}^d. \tag{1-23}$$

The estimate (1-7) follows from (1-23) and (1-8) in a manner somewhat similar to the case of Hamilton–Jacobi equations in the almost-periodic setting [Ishii 2000; Lions and Souganidis 2005; Armstrong et al. 2014].

Theorems 1.1 and 1.2 are proved in Section 7. Here we follow an approach for the periodic case by considering

$$w_\varepsilon = u_\varepsilon(x) - u_0(x) - \varepsilon\chi_T(x/\varepsilon)\nabla u_0(x) + v_\varepsilon(x),$$

where $T = \varepsilon^{-1}$ and v_ε is the weak solution of the problem $\mathcal{L}_\varepsilon(v_\varepsilon) = 0$ in Ω and $v_\varepsilon = \varepsilon \chi_T(x/\varepsilon) \nabla u_0(x)$ on $\partial \Omega$. We are able to show that

$$\|w_\varepsilon\|_{H^1(\Omega)} \le C_\sigma \big(\Theta_\sigma(T) + \langle |\psi - \nabla \chi_T| \rangle \big) \|u_0\|_{W^{2,2}(\Omega)} \tag{1-24}$$

for any $\sigma \in (0, 1)$, where ψ is the limit of $\nabla \chi_T$ in $B^2(\mathbb{R}^d)$ as $T \to \infty$. In the periodic case, one of the key steps is to write $\hat{A} - A(y) - A(y) \nabla \chi(y)$ as a divergence of some bounded periodic function. In the almost-periodic setting, this will be replaced by solving the equation

$$-\Delta u + T^{-2} u = B_T - \langle B_T \rangle \quad \text{in } \mathbb{R}^d, \tag{1-25}$$

where $B_T(y) = \hat{A} - A(y) - A(y) \nabla \chi_T(y)$. The same ideas for proving (1-7)–(1-9) are used to obtain the desired estimates for $\|u\|_{L^\infty}$ and $\|\nabla u\|_{L^\infty}$ in terms of the function $\Theta_\sigma(T)$. Finally, in Section 8 we consider the case of quasiperiodic coefficients and provide some sufficient conditions on the frequencies of $A(y)$ for the estimate (1-17) on $\rho(R)$.

Throughout this paper, unless indicated otherwise, we always assume that $A = (a_{ij}^{\alpha\beta})$ satisfies the ellipticity condition (1-2) and is uniformly almost-periodic in $\mathbb{R}^d$. We will use $\fint_E f = (1/|E|) \int_E f$ to denote the L^1 average of f over E, and C to denote constants that depend on $A(y)$, Ω and other relevant parameters, but never on ε or T.

2. Homogenization and compactness

This section contains a brief review of homogenization theory of elliptic systems with almost-periodic coefficients. We refer the reader to [Jikov et al. 1994, pp. 238–242] for a detailed presentation. We also prove a homogenization theorem for a sequence of operators obtained from $\mathcal{L}_\varepsilon$ through translations and rotations.

Let $\mathrm{Trig}(\mathbb{R}^d)$ denote the set of (real) trigonometric polynomials in $\mathbb{R}^d$. A bounded continuous function f in $\mathbb{R}^d$ is said to be uniformly almost-periodic (or almost-periodic in the sense of Bohr) if f is a limit of a sequence of functions in $\mathrm{Trig}(\mathbb{R}^d)$ with respect to the norm $\|f\|_{L^\infty}$. A function f in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ is said to belong to $B^2(\mathbb{R}^d)$ if f is a limit of a sequence of functions in $\mathrm{Trig}(\mathbb{R}^d)$ with respect to the seminorm

$$\|f\|_{B^2} = \limsup_{R \to \infty} \left(\fint_{B(0,R)} |f|^2 \right)^{\frac{1}{2}}. \tag{2-1}$$

Functions in $B^2(\mathbb{R}^d)$ are said to be almost-periodic in the sense of Besicovich. It is not hard to see that, if $f \in B^2(\mathbb{R}^d)$ and g is uniformly almost-periodic, then $fg \in B^2(\mathbb{R}^d)$.

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. A number $\langle f \rangle$ is called the mean value of f if

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} f(x/\varepsilon) \varphi(x) \, dx = \langle f \rangle \int_{\mathbb{R}^d} \varphi \tag{2-2}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^d)$. If $f \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and $\|f\|_{B^2} < \infty$, the existence of $\langle f \rangle$ is equivalent to the condition that, as $\varepsilon \to 0$, $f(x/\varepsilon) \rightharpoonup \langle f \rangle$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$, i.e., $f(x/\varepsilon) \rightharpoonup \langle f \rangle$ weakly in $L^2(B(0, R))$ for

any $R > 1$. In this case, one has

$$\langle f \rangle = \lim_{L \to \infty} \fint_{B(0,L)} f.$$

It is known that if $f, g \in B^2(\mathbb{R}^d)$ then fg has the mean value. Furthermore, under the equivalent relation that $f \sim g$ if $\|f - g\|_{B^2} = 0$, the set $B^2(\mathbb{R}^d)/\sim$ is a Hilbert space with the inner product defined by $(f, g) = \langle fg \rangle$.

A function $f = (f_i^\alpha)$ in $\mathrm{Trig}(\mathbb{R}^d; \mathbb{R}^{d \times m})$ is called potential if there exists $g = (g^\alpha) \in \mathrm{Trig}(\mathbb{R}^d; \mathbb{R}^m)$ such that $f_i^\alpha = \partial g^\alpha / \partial x_i$. A function $f = (f_i^\alpha)$ in $\mathrm{Trig}(\mathbb{R}^d; \mathbb{R}^{d \times m})$ is called solenoidal if $\partial f_i^\alpha / \partial x_i = 0$ for $1 \le \alpha \le m$. Let V_{pot}^2 (resp. V_{sol}^2) denote the closure of potential (resp. solenoidal) trigonometric polynomials with mean value zero in $B^2(\mathbb{R}^d; \mathbb{R}^{d \times m})$. Then

$$B^2(\mathbb{R}^d; \mathbb{R}^{d \times m}) = V_{\mathrm{pot}}^2 \oplus V_{\mathrm{sol}}^2 \oplus \mathbb{R}^{d \times m}. \tag{2-3}$$

By the Lax–Milgram theorem and the ellipticity condition (1-2), for any $1 \le j \le d$ and $1 \le \beta \le m$ there exists a unique $\psi_j^\beta = (\psi_{ij}^{\alpha\beta}) \in V_{\mathrm{pot}}^2$ such that

$$\langle a_{ik}^{\alpha\gamma} \psi_{kj}^{\gamma\beta} \phi_i^\alpha \rangle = -\langle a_{ij}^{\alpha\beta} \phi_i^\alpha \rangle \quad \text{for any } \phi = (\phi_i^\alpha) \in V_{\mathrm{pot}}^2. \tag{2-4}$$

Let

$$\hat{a}_{ij}^{\alpha\beta} = \langle a_{ij}^{\alpha\beta} \rangle + \langle a_{ik}^{\alpha\gamma} \psi_{kj}^{\gamma\beta} \rangle \tag{2-5}$$

and $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$. Then

$$\mu |\xi|^2 \le \hat{a}_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \le \mu_1 |\xi|^2 \tag{2-6}$$

for any $\xi = (\xi_i^\alpha) \in \mathbb{R}^{d \times m}$, where μ_1 depends only on d, m and μ. It is also known that $\widehat{A^*} = (\hat{A})^*$, where A^* denotes the adjoint of A, i.e., $A^* = (b_{ij}^{\alpha\beta})$ with $b_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$.

As the following theorem shows, the homogenized operator for $\mathcal{L}_\varepsilon$ is given by $\mathcal{L}_0 = -\operatorname{div}(\hat{A}\nabla)$.

Theorem 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $F \in H^{-1}(\Omega; \mathbb{R}^m)$. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω. Suppose $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\Omega; \mathbb{R}^m)$. Then $A(x/\varepsilon)\nabla u_\varepsilon \rightharpoonup \hat{A}\nabla u_0$ weakly in $L^2(\Omega; \mathbb{R}^{dm})$. Consequently, if $f \in H^{1/2}(\partial\Omega; \mathbb{R}^m)$ and u_ε is the unique weak solution in $H^1(\Omega; \mathbb{R}^m)$ of the Dirichlet problem $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω and $u_\varepsilon = f$ on $\partial\Omega$, then, as $\varepsilon \to 0$, $u_\varepsilon \to u_0$ weakly in $H^1(\Omega; \mathbb{R}^m)$ and strongly in $L^2(\Omega; \mathbb{R}^m)$, where u_0 is the unique weak solution in $H^1(\Omega; \mathbb{R}^m)$ of the Dirichlet problem $\mathcal{L}_0(u_0) = F$ in Ω and $u_0 = f$ on $\partial\Omega$.*

Proof. See [Jikov et al. 1994] for the single equation case ($m = 1$). The proof for the case $m > 1$ is exactly the same. $\qquad\square$

In Sections 3 and 4 we will use a compactness argument to establish the uniform Hölder estimates for local solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(f) + F$. This requires us to work with a class of operators that are obtained from $\mathcal{L}^A = -\operatorname{div}(A(x)\nabla)$ through translations and rotations of coordinates in $\mathbb{R}^d$. Observe that, if $\mathcal{L}^A(u) = F$ and $x = Oy + z$ for some rotation $O = (O_{ij})$ and $z \in \mathbb{R}^d$, then $\mathcal{L}^B(v) = G$, where

$v(y) = u(Oy + z)$, $B = (b_{ij}^{\alpha\beta}(y))$ with $b_{ij}^{\alpha\beta}(y) = a_{\ell k}^{\alpha\beta}(Oy + z)O_{\ell i}O_{kj}$, and $G(y) = F(Oy + z)$. Thus, for each $A = (a_{ij}^{\alpha\beta})$ fixed, we shall consider the set of matrices

$$\mathcal{A} = \left\{ B = (b_{ij}^{\alpha\beta}(y)) : b_{ij}^{\alpha\beta}(y) = a_{\ell k}^{\alpha\beta}(Oy + z)O_{\ell i}O_{kj} \text{ for some rotation } O = (O_{ij}) \text{ and } z \in \mathbb{R}^d \right\}. \quad (2\text{-}7)$$

Note that, if $B(y) = O^t A(Oy + z)O \in \mathcal{A}$, where O^t denotes the transpose of O, then the homogenized matrix $\hat{B}$ equals $O^t \hat{A} O$.

The proof of Theorems 3.1 and 4.1 relies on the following extension of Theorem 2.1:

Theorem 2.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $F \in H^{-1}(\Omega; \mathbb{R}^m)$. Let $u_\ell \in H^1(\Omega; \mathbb{R}^m)$ be a weak solution of $-\operatorname{div}(A_\ell(x/\varepsilon_\ell)\nabla u_\ell) = F$ in Ω, where $\varepsilon_\ell \to 0$ and $A_\ell \in \mathcal{A}$. Suppose that $u_\ell \rightharpoonup u$ weakly in $H^1(\Omega; \mathbb{R}^m)$. Then u is a weak solution of $-\operatorname{div}(\tilde{A}\nabla u) = F$ in Ω, where $\tilde{A} = O^t \hat{A} O$ for some rotation O in $\mathbb{R}^d$.*

Proof. Suppose that $A_\ell(y) = O_\ell^t A(O_\ell y + z_\ell)O_\ell$ for some rotations O_ℓ and $z_\ell \in \mathbb{R}^d$. By passing to a subsequence we may assume that $O_\ell \to O$ as $\ell \to \infty$. Since $A(y)$ is uniformly almost-periodic, $\{A(y + z_\ell)\}_{\ell=1}^\infty$ is precompact in $C_b(\mathbb{R}^d)$, the set of bounded continuous functions in $\mathbb{R}^d$. Thus, by passing to a subsequence, we may also assume that $A(y + z_\ell)$ converges uniformly in $\mathbb{R}^d$ to an almost-periodic matrix $B(y)$. Consequently, we obtain $A_\ell(y) \to \tilde{B}(y) = O^t B(Oy)O$ uniformly in $\mathbb{R}^d$. Note that $\widehat{\tilde{B}} = O^t \hat{B} O = O^t \hat{A} O$.

Now, let $v_\ell \in H^1(\Omega; \mathbb{R}^m)$ be the weak solution of the Dirichlet problem

$$-\operatorname{div}(\tilde{B}(x/\varepsilon_\ell)\nabla v_\ell) = F \quad \text{in } \Omega \qquad \text{and} \qquad v_\ell = u_\ell \quad \text{on } \partial\Omega.$$

Using $-\operatorname{div}\big(A_\ell(x/\varepsilon_\ell)\nabla(u_\ell - v_\ell)\big) = \operatorname{div}\big((A_\ell(x/\varepsilon_\ell) - \tilde{B}(x/\varepsilon_\ell))\nabla v_\ell\big)$ in Ω and $u_\ell - v_\ell = 0$ on $\partial\Omega$, we may use the energy estimates to deduce that

$$\|u_\ell - v_\ell\|_{H^1(\Omega)} \le C\|A_\ell - \tilde{B}\|_{L^\infty}\|\nabla v_\ell\|_{L^2(\Omega)} \le C\|A_\ell - \tilde{B}\|_{L^\infty}\{\|u_\ell\|_{H^1(\Omega)} + \|F\|_{H^{-1}(\Omega)}\}.$$

It follows that $u_\ell - v_\ell \to 0$ in $H^1(\Omega; \mathbb{R}^m)$ as $\ell \to \infty$.

Finally, since $v_\ell = v_\ell - u_\ell + u_\ell \rightharpoonup u$ weakly in $H^1(\Omega; \mathbb{R}^m)$, it follows from Theorem 2.1 that $\tilde{B}(x/\varepsilon_\ell)\nabla v_\ell \rightharpoonup \tilde{A}\nabla u$ weakly in $H^1(\Omega; \mathbb{R}^{d \times m})$, where $\tilde{A} = \widehat{\tilde{B}} = O^t \hat{A} O$. As a result, we obtain $-\operatorname{div}(\tilde{A}\nabla u) = F$ in Ω. This completes the proof. $\qquad\square$

3. Uniform interior Hölder estimates

The goal of this and the next section is to establish uniform interior and boundary Hölder estimates for solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = f + \operatorname{div}(g)$. We will first use a compactness method to deal with the special case $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$. The results are then used to establish size and Hölder estimates for fundamental solutions and Green functions for $\mathcal{L}_\varepsilon$. The general case follows from the estimates for fundamental solutions and Green functions.

Theorem 3.1. *Let $u_\varepsilon \in H^1(B(x_0, 2r); \mathbb{R}^m)$ be a weak solution of $\mathrm{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(x_0, 2r)$ for some $x_0 \in \mathbb{R}^d$ and $r > 0$. Let $\sigma \in (0, 1)$. Then*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C_\sigma \left(\frac{|x - y|}{r}\right)^\sigma \left(\fint_{B(x_0, 2r)} |u_\varepsilon|^2\right)^{\frac{1}{2}} \tag{3-1}$$

for any $x, y \in B(x_0, r)$, where C_σ depends only on d, m, σ and A (not on ε, x_0 or r).

Theorem 3.1 follows from Theorem 2.2 by a three-step compactness argument, similar to the periodic case in [Avellaneda and Lin 1987].

Lemma 3.2. *Let $0 < \sigma < 1$. Then there exist constants $\varepsilon_0 > 0$ and $\theta \in \left(0, \frac{1}{4}\right)$, depending only on σ and A, such that*

$$\fint_{B(y, \theta)} \left|u_\varepsilon - \fint_{B(y, \theta)} u_\varepsilon\right|^2 \le \theta^{2\sigma} \quad \text{for any } 0 < \varepsilon < \varepsilon_0 \tag{3-2}$$

whenever $u_\varepsilon \in H^1(B(y, 1); \mathbb{R}^m)$ is a weak solution of $\mathrm{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(y, 1)$ for some $y \in \mathbb{R}^d$ and

$$\fint_{B(y, 1)} |u_\varepsilon|^2 \le 1.$$

Proof. If $\mathrm{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(y, 1)$ and $v(x) = u_\varepsilon(x + y)$, then $\mathrm{div}(B(x/\varepsilon)\nabla v) = 0$ in $B(0, 1)$, where $B(x) = A(x + \varepsilon^{-1}y) \in \mathscr{A}$. As a result, it suffices to establish estimate (3-2) for $y = 0$ and for solutions u_ε of $\mathrm{div}(B(x/\varepsilon)\nabla u_) = 0$ in $B(0, 1)$, where $B \in \mathscr{A}$.

To this end, we first note that, if w is a solution of a second-order elliptic system in $B\left(0, \frac{1}{2}\right)$ with constant coefficients satisfying the ellipticity condition (2-6), then

$$\fint_{B(0, \theta)} \left|w - \fint_{B(0, \theta)} w\right|^2 \le C_0 \theta^2 \fint_{B(0, 1/2)} |w|^2 \quad \text{for any } 0 < \theta < \frac{1}{4}, \tag{3-3}$$

where C_0 depends only on d, m and μ. We now choose $\theta \in \left(0, \frac{1}{4}\right)$ so small that

$$2^d C_0 \theta^2 < \theta^{2\sigma}. \tag{3-4}$$

We claim that the estimate (3-2) with $y = 0$ holds for this θ and for some $\varepsilon_0 > 0$, which depends only on A, whenever u_ε is a weak solution of $\mathrm{div}(B(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(0, 1)$ for some $B \in \mathscr{A}$.

Suppose this is not the case. Then there exist $\{\varepsilon_\ell\} \subset \mathbb{R}_+, \{B_\ell\} \subset \mathscr{A}$ and $\{u_\ell\} \subset H^1(B(0, 1); \mathbb{R}^m)$ such that $\varepsilon_\ell \to 0$,

$$\begin{cases} \mathrm{div}(B_\ell(x/\varepsilon_\ell)\nabla u_\ell) = 0 \quad \text{in } B(0, 1), \\ \fint_{B(0, 1)} |u_\ell|^2 \le 1, \end{cases} \tag{3-5}$$

and

$$\fint_{B(0, \theta)} \left|u_\ell - \fint_{B(0, \theta)} u_\ell\right|^2 > \theta^{2\sigma}. \tag{3-6}$$

Since $\{u_\ell\}$ is bounded in $L^2(B(0, 1); \mathbb{R}^m)$, by Cacciopoli's inequality, $\{u_\ell\}$ is bounded in $H^1\left(B\left(0, \frac{1}{2}\right); \mathbb{R}^m\right)$. By passing to a subsequence, we may assume $u_\ell \rightharpoonup u$ weakly in $H^1\left(B\left(0, \frac{1}{2}\right); \mathbb{R}^m\right)$ and in $L^2(B(0, 1); \mathbb{R}^m)$.

It follows from Theorem 2.2 that u is a solution of $\text{div}(\tilde{A}u) = 0$ in $B\left(0, \frac{1}{2}\right)$, where $\tilde{A} = O^t \hat{A} O$ for some rotation O in $\mathbb{R}^d$. Since the matrix $O^t \hat{A} O$ satisfies the ellipticity condition (2-6), estimate (3-3) holds for $w = u$. However, since $u_\ell \to u$ strongly in $L^2\left(B\left(0, \frac{1}{2}\right); \mathbb{R}^m\right)$, we may deduce from (3-6) that

$$\theta^{2\sigma} \leq \fint_{B(0,\theta)} \left| u - \fint_{B(0,\theta)} u \right|^2 \leq C_0 \theta^2 \fint_{B(0,1/2)} |u|^2 \leq 2^d C_0 \theta^2 \fint_{B(0,1)} |u|^2, \tag{3-7}$$

where we have used (3-3) for the second inequality.

Finally, we note that the weak convergence of u_ℓ in $L^2(B(0,1); \mathbb{R}^m)$ and the inequality in (3-5) give

$$\fint_{B(0,1)} |u|^2 \leq 1.$$

In view of (3-7), we obtain $\theta^{2\sigma} \leq 2^d C_0 \theta^2$, which contradicts (3-4). This completes the proof. $\qquad\square$

Lemma 3.3. *Fix $0 < \sigma < 1$. Let ε_0 and θ be the constants given by Lemma 3.2. Let $u_\varepsilon \in H^1(B(y,1); \mathbb{R}^m)$ be a weak solution of $\text{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(y,1)$ for some $y \in \mathbb{R}^d$. Then, if $0 < \varepsilon < \varepsilon_0 \theta^{k-1}$ for some $k \geq 1$,*

$$\fint_{B(y,\theta^k)} \left| u_\varepsilon - \fint_{B(y,\theta^k)} u_\varepsilon \right|^2 \leq \theta^{2k\sigma} \fint_{B(y,1)} |u_\varepsilon|^2. \tag{3-8}$$

Proof. The lemma is proved by an induction argument on k, using Lemma 3.2 and the rescaling property that, if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $B(y,1)$ and $v(x) = u_\varepsilon(\theta^k x)$, then

$$\mathcal{L}_{\varepsilon/\theta^k}(v) = 0 \quad \text{in } B(\theta^{-k} y, \theta^{-k}).$$

See [Avellaneda and Lin 1987] for the periodic case. $\qquad\square$

Proof of Theorem 3.1. By rescaling we may assume that $r = 1$. Suppose that $u_\varepsilon \in H^1(B(y,2); \mathbb{R}^m)$ and $\text{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(y,2)$ for some $y \in \mathbb{R}^d$. We show that

$$\fint_{B(z,t)} \left| u_\varepsilon - \fint_{B(z,t)} u_\varepsilon \right|^2 \leq C t^{2\sigma} \fint_{B(z,1)} |u_\varepsilon|^2 \tag{3-9}$$

for any $0 < t < \theta$ and $z \in B(y,1)$, where $\theta \in \left(0, \frac{1}{4}\right)$ is given by Lemma 3.2. The estimate (3-1) follows from (3-9) by Campanato's characterization of Hölder spaces.

With Lemma 3.3 at our disposal, the proof of (3-9) follows the same line of argument as in the periodic case. We refer the reader to [Avellaneda and Lin 1987] for details. We point out that the classical local Hölder estimates for solutions of elliptic systems in divergence form with continuous coefficients are needed to handle the case $\varepsilon \geq \theta \varepsilon_0$ and $0 < t < \theta$, as well as the case $0 < \varepsilon < \theta \varepsilon_0$ and $0 < t < \varepsilon/\varepsilon_0$. $\qquad\square$

It follows from (3-1) and Cacciopoli's inequality that

$$\fint_{B(y,t)} |\nabla u_\varepsilon|^2 \leq C_\sigma \left(\frac{t}{r}\right)^\sigma \fint_{B(y,r)} |\nabla u_\varepsilon|^2 \quad \text{for any } 0 < t < r \tag{3-10}$$

if $\text{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(y,r)$. Since A^* satisfies the same ellipticity and almost periodicity conditions as A, estimate (3-16) also holds for solutions of $\text{div}(A^*(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(y,r)$. As a result, one may

construct an $m \times m$ matrix of fundamental solutions $\Gamma_\varepsilon(x, y) = (\Gamma_\varepsilon^{\alpha\beta}(x, y))$ such that, for each $y \in \mathbb{R}^d$, $\nabla_x \Gamma_\varepsilon(x, y)$ is locally integrable and

$$\phi^\gamma(y) = \int_{\mathbb{R}^d} a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial}{\partial x_j}(\Gamma_\varepsilon^{\beta\gamma}(x, y)) \frac{\partial \phi^\alpha}{\partial x_i} \, dx \tag{3-11}$$

for any $\phi = (\phi^\alpha) \in C_0^1(\mathbb{R}^d, \mathbb{R}^m)$ (see, e.g., [Hofmann and Kim 2007]). Moreover, if $d \geq 3$, the matrix $\Gamma_\varepsilon(x, y)$ satisfies

$$|\Gamma_\varepsilon(x, y)| \leq C|x - y|^{2-d} \tag{3-12}$$

for any $x, y \in \mathbb{R}^d$ with $x \neq y$, and

$$\begin{aligned}
|\Gamma_\varepsilon(x + h, y) - \Gamma_\varepsilon(x, y)| &\leq \frac{C_\sigma |h|^\sigma}{|x - y|^{d-2+\sigma}}, \\
|\Gamma_\varepsilon(x, y + h) - \Gamma_\varepsilon(x, y)| &\leq \frac{C_\sigma |h|^\sigma}{|x - y|^{d-2+\sigma}},
\end{aligned} \tag{3-13}$$

where $x, y, h \in \mathbb{R}^d$ and $0 < |h| \leq \frac{1}{2}|x - y|$. Since $\mathcal{L}_\varepsilon^*(\Gamma_\varepsilon(x, \cdot)) = 0$ in $\mathbb{R}^d \setminus \{x\}$, using Cacciopoli's inequality and (3-12)–(3-13) we obtain

$$\left(\fint_{R \leq |y-x| \leq 2R} |\nabla_y \Gamma_\varepsilon(x, y)|^2 \, dy \right)^{\frac{1}{2}} \leq \frac{C}{R^{d-1}} \tag{3-14}$$

and

$$\left(\fint_{R \leq |y-x_0| \leq 2R} |\nabla_y \{\Gamma_\varepsilon(x, y) - \Gamma_\varepsilon(z, y)\}|^2 \, dy \right)^{\frac{1}{2}} \leq \frac{C|x - z|^\sigma}{R^{d-1+\sigma}}, \tag{3-15}$$

where $x, z \in B(x_0, r)$ and $R \geq 2r$.

Theorem 3.4. *Let $u_\varepsilon \in H^1(B(x_0, 2r); \mathbb{R}^m)$ be a weak solution of*

$$-\operatorname{div}(A(x/\varepsilon)\nabla u_\varepsilon) = f + \operatorname{div}(g) \quad \text{in } 2B = B(x_0, 2r).$$

Let $0 < \sigma < 1$. Then, for any $x, z \in B = B(x_0, r)$,

$$|u_\varepsilon(x) - u_\varepsilon(z)| \leq C|x - z|^\sigma \left(r^{-\sigma} \left(\fint_{2B} |u_\varepsilon|^2 \right)^{\frac{1}{2}} + \sup_{\substack{y \in B \\ 0 < t < r}} t^{2-\sigma} \left(\fint_{B(y,t)} |f|^2 \right)^{\frac{1}{2}} + \sup_{\substack{y \in B \\ 0 < t < r}} t^{1-\sigma} \left(\fint_{B(y,t)} |g|^2 \right)^{\frac{1}{2}} \right), \tag{3-16}$$

where C depends only on p, σ and A. In particular,

$$\|u_\varepsilon\|_{L^\infty(B)} \leq C \left(\fint_{2B} |u_\varepsilon|^2 \right)^{\frac{1}{2}} + Cr^\sigma \sup_{\substack{y \in B \\ 0 < t < r}} t^{2-\sigma} \left(\fint_{B(y,t)} |f|^2 \right)^{\frac{1}{2}} + Cr^\sigma \sup_{\substack{y \in B \\ 0 < t < r}} t^{1-\sigma} \left(\fint_{B(y,t)} |g|^2 \right)^{\frac{1}{2}}, \tag{3-17}$$

where C depends only on p, σ and A.

Proof. We first note that the L^∞ estimate (3-17) follows easily from (3-16). To see (3-16), we assume $d \geq 3$; the case $d = 2$ follows from the case $d = 3$ by adding a dummy variable (the method of ascending). We

choose a cut-off function $\varphi \in C_0^\infty\left(B\left(x_0, \frac{7}{4}r\right)\right)$ such that $0 \le \varphi \le 1$, $\varphi = 1$ in $B\left(x_0, \frac{3}{2}r\right)$, and $|\nabla\varphi| \le Cr^{-1}$. Since

$$\mathcal{L}_\varepsilon(u_\varepsilon) = f\varphi + \operatorname{div}(g\varphi) - g\nabla\varphi - A(x/\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi - \nabla\{A(x/\varepsilon)u_\varepsilon \cdot \nabla\varphi\},$$

we obtain that, for $x \in B(x_0, r)$,

$$
\begin{aligned}
u_\varepsilon(x) = {}& \int_{\mathbb{R}^d} \Gamma_\varepsilon(x, y) f(y)\varphi(y)\, dy - \int_{\mathbb{R}^d} \nabla_y \Gamma_\varepsilon(x, y) g(y)\varphi(y)\, dy \\
& - \int_{\mathbb{R}^d} \Gamma_\varepsilon(x, y) g(y)\nabla\varphi(y)\, dy - \int_{\mathbb{R}^d} \Gamma_\varepsilon(x, y) A(y/\varepsilon)\nabla u_\varepsilon(y) \cdot \nabla\varphi(y)\, dy \\
& + \int_{\mathbb{R}^d} \nabla_y \Gamma_\varepsilon(x, y) A(y/\varepsilon) u_\varepsilon(y) \nabla\varphi(y)\, dy.
\end{aligned}
\tag{3-18}
$$

It follows that, for any $x, z \in B(x_0, r)$,

$$
\begin{aligned}
|u_\varepsilon(x) - u_\varepsilon(z)| \le {}& C \int_{2B} |\Gamma_\varepsilon(x, y) - \Gamma_\varepsilon(z, y)| |f(y)|\, dy \\
& + C \int_{2B} |\nabla_y\{\Gamma_\varepsilon(x, y) - \Gamma_\varepsilon(z, y)\}| |g(y)|\, dy \\
& + C \int_{2B} |\Gamma_\varepsilon(x, y) - \Gamma_\varepsilon(z, y)| |g(y)| |\nabla\varphi(y)|\, dy \\
& + C \int_{2B} |\Gamma_\varepsilon(x, y) - \Gamma_\varepsilon(z, y)| |\nabla u_\varepsilon(y)| |\nabla\varphi(y)|\, dy \\
& + C \int_{2B} |\nabla_y \Gamma_\varepsilon(x, y) - \nabla_y \Gamma_\varepsilon(z, y)| |u_\varepsilon(y)| |\nabla\varphi(y)|\, dy, \quad \text{(3-19)}
\end{aligned}
$$

where $2B = B(x_0, 2r)$. Since $|\nabla\varphi| = 0$ in $B\left(x_0, \frac{3}{2}r\right)$ and $x, z \in B(x_0, r)$, the last three terms in the right-hand side of (3-19) may be handled easily, using estimate (3-13), Cacciopoli's inequality and (3-15). They are bounded by

$$
C_\sigma \left(\frac{|y - z|}{r}\right)^\sigma \left(\left(\fint_{2B} |u_\varepsilon|^2\right)^{\frac{1}{2}} + r^2\left(\fint_{2B} |f|^2\right)^{\frac{1}{2}} + r\left(\fint_{2B} |g|^2\right)^{\frac{1}{2}}\right)
$$

for any $\sigma \in (0, 1)$.

Next, we use (3-12) and (3-13) to bound the first term in the right-hand side of (3-19) by

$$
C \int_{B(x,4s)} \frac{|f(y)|\, dy}{|x - y|^{d-2}} + C \int_{B(z,5s)} \frac{|f(y)|\, dy}{|z - y|^{d-2}} + Cs^{\sigma_1} \int_{2B \setminus B(x,4s)} \frac{|f(y)|\, dy}{|x - y|^{d-2+\sigma_1}},
\tag{3-20}
$$

where $s = |x - z|$ and $\sigma_1 \in (\sigma, 1)$. By decomposing $B(x, 4s)$ as a union of sets $\{y : |y - x| \sim 2^j s\}$, it is not hard to verify that the first term in (3-20) is bounded by

$$
Cs^\sigma \sup_{\substack{y \in B \\ 0 < t < r}} t^{2-\sigma} \left(\fint_{B(y,t)} |f|^2\right)^{\frac{1}{2}}.
$$

The other two terms in (3-20) may be handled in a similar manner.

Finally, the second term in the right-hand side of (3-19) is bounded by

$$
\int_{B(x,4s)} |\nabla_y \Gamma_\varepsilon(x, y)| |g(y)| \, dy + \int_{B(z,5s)} |\nabla_y \Gamma_\varepsilon(z, y)| |g(y)| \, dy
$$

$$
+ \int_{2B \setminus B(x,4s)} |\nabla_y \{\Gamma_\varepsilon(x, y) - \Gamma_\varepsilon(z, y)\}| |g(y)| \, dy. \quad (3\text{-}21)
$$

By decomposing $2B \setminus B(x, 4s)$ as a union of sets $\{y : |y - x| \sim 2^j s\}$ and using Hölder's inequality and (3-15) (with σ replaced by some $\sigma_1 \in (\sigma, 1)$), we may bound the third term in (3-21) by

$$
Cs^\sigma \sup_{\substack{y \in B \\ 0 < t < r}} t^{1-\sigma} \left(\fint_{B(y,t)} |g|^2 \right)^{\frac{1}{2}}.
$$

The other two terms in (3-21) may be handled in a similar manner. This completes the proof. $\qquad\square$

Remark 3.5. Suppose that $-\operatorname{div}(A(x/\varepsilon)\nabla u_\varepsilon) = f$ in $2B$ and $f \in L^p(2B; \mathbb{R}^m)$ for some $p \geq 2$, where $2B = B(x_0, 2r)$. Assume $d \geq 3$. Using (3-18) and Cacciopoli's inequality, we may obtain that

$$
|u_\varepsilon(x)| \leq C \int_{2B} \frac{|f(y)|}{|x - y|^{d-2}} \, dy + C \left(\fint_{2B} |u_\varepsilon|^2 \right)^{\frac{1}{2}} + Cr^2 \left(\fint_{2B} |f|^2 \right)^{\frac{1}{2}} \quad (3\text{-}22)
$$

for any $x \in B = B(x_0, r)$. By the fractional integral estimates, this gives

$$
\left(\fint_B |u_\varepsilon|^q \right)^{\frac{1}{q}} \leq C \left(\fint_{2B} |u_\varepsilon|^2 \right)^{\frac{1}{2}} + Cr^2 \left(\fint_{2B} |f|^p \right)^{\frac{1}{p}}, \quad (3\text{-}23)
$$

where $0 < 1/p - 1/q \leq 2/d$.

4. Uniform boundary Hölder estimates and proof of Theorem 1.4

For $x_0 \in \partial\Omega$ and $0 < r < r_0 = \operatorname{diam}(\Omega)$, define

$$
\Omega_r(x_0) = B(x_0, r) \cap \Omega \quad \text{and} \quad \Delta_r(x_0) = B(x_0, r) \cap \partial\Omega. \quad (4\text{-}1)
$$

Theorem 4.1. *Let Ω be a bounded $C^{1,\eta}$ domain in $\mathbb{R}^d$ for some $\eta > 0$. Let $u_\varepsilon \in H^1(\Omega_r(x_0); \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $\Omega_r(x_0)$ and $u_\varepsilon = 0$ on $\Delta_r(x_0)$ for some $x_0 \in \partial\Omega$ and $0 < r < r_0$. Then, for any $0 < \sigma < 1$ and $x, y \in \Omega_{r/2}(x_0)$,*

$$
|u_\varepsilon(x) - u_\varepsilon(y)| \leq C \left(\frac{|x - y|}{r} \right)^\sigma \left(\fint_{\Omega_r(x_0)} |u_\varepsilon|^2 \right)^{\frac{1}{2}}, \quad (4\text{-}2)
$$

where C depends only on σ, A and Ω.

Let $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a $C^{1,\eta}$ function such that

$$
\phi(0) = 0, \quad \nabla\phi(0) = 0 \quad \text{and} \quad \|\nabla\phi\|_{C^{0,\eta}(\mathbb{R}^{d-1})} \leq M_0. \quad (4\text{-}3)
$$

Let

$$D(r) = D(r, \phi) = \left\{ (x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } \phi(x') < x_d < \phi(x') + 10(M_0 + 1)r \right\},$$
$$I(r) = I(r, \phi) = \left\{ (x', \phi(x')) \in \mathbb{R}^d : |x'| < r \right\}. \tag{4-4}$$

By translation and rotation, Theorem 4.1 may be reduced to the following:

Theorem 4.2. *Let $u_\varepsilon \in H^1(D(r); \mathbb{R}^m)$ be a weak solution of $\mathrm{div}(B(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $D(r)$ and $u_\varepsilon = 0$ on $I(r)$ for some $r > 0$ and $B \in \mathcal{A}$. Then, for any $0 < \sigma < 1$ and $x, y \in D(r/2)$,*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C \left(\frac{|x - y|}{r} \right)^\sigma \left(\fint_{D_r} |u_\varepsilon|^2 \right)^{\frac{1}{2}}, \tag{4-5}$$

where C depends only on σ, A and (η, M_0) in (4-3).

To prove Theorem 4.2, we need a homogenization result for a sequence of matrices in the class $\mathcal{A}$ on a sequence of domains.

Lemma 4.3. *Let $\{B_\ell\}$ be a sequence of matrices in $\mathcal{A}$. Let $\{\phi_\ell\}$ be a sequence of $C^{1,\eta}$ functions satisfying* (4-3). *Suppose that $\mathrm{div}(B_\ell(x/\varepsilon_\ell)\nabla u_\ell) = 0$ in $D(r, \phi_\ell)$ and $u_\ell = 0$ on $I(r, \phi_\ell)$ for some $r > 0$, where $\varepsilon_\ell \to 0$ and $\|u_\ell\|_{H^1(D(r,\phi_\ell))} \le C$. Then there exist subsequences of $\{\phi_\ell\}$ and $\{u_\ell\}$, which we still denote by $\{\phi_\ell\}$ and $\{u_\ell\}$, respectively, a function ϕ satisfying* (4-3) *with $u \in H^1(D(r, \phi); \mathbb{R}^m)$, and a constant matrix $\tilde{B}$, such that*

$$\begin{cases} \phi_\ell \to \phi & \text{in } C^1(|x'| < r), \\ u_\ell(x', x_d - \phi_\ell(x')) \rightharpoonup u(x', x_d - \phi(x')) & \text{weakly in } H^1(D(r, 0); \mathbb{R}^m), \end{cases} \tag{4-6}$$

and

$$\mathrm{div}(\tilde{B}\nabla u) = 0 \quad \text{in } D(r, \phi) \qquad \text{and} \qquad u = 0 \quad \text{on } I(r, \phi). \tag{4-7}$$

Moreover, the matrix $\tilde{B}$, which is given by $O^t \hat{A} O$ for some rotation O in $\mathbb{R}^d$, satisfies the ellipticity condition (2-6).

Proof. Since $\|\nabla\phi_\ell\|_{C^{0,\eta}(\mathbb{R}^{d-1})} \le M_0$ and $\|u_\ell\|_{H^1(D(r,\phi_\ell))} \le C$, (4-6) follows by passing to subsequences. Suppose that $B_\ell(y) = O_\ell^t A(O_\ell y + z_\ell) O_\ell$ for some rotation O_ℓ and $z_\ell \in \mathbb{R}^d$. By passing to a subsequence, we may assume that $O_\ell \to O$. Since $u_\ell \to u$ weakly in $H^1(\Omega; \mathbb{R}^m)$ for any $\Omega \Subset D(r, \phi)$, it follows from Theorem 2.2 that $\mathrm{div}(\tilde{B}u) = 0$ in $D(r, \phi)$, where $\tilde{B} = O^t \hat{A} O$. Finally, since $v_\ell(x', x_d) = u_\ell(x', x_d + \phi_\ell(x')) \rightharpoonup v(x', x_d + \phi(x'))$ weakly in $H^1(D(r, 0))$ and $v_\ell = 0$ on $I(r, 0)$, we may conclude that $v = 0$ on $I(r, 0)$. Hence, $u = 0$ on $I(r, \phi)$. $\qquad\square$

Proof of Theorem 4.2. With Lemma 4.3 at our disposal, Theorem 4.2 follows by the three-step compactness argument, as in the periodic case. We refer the reader to [Avellaneda and Lin 1987] for details. $\qquad\square$

With interior and boundary Hölder estimates in Theorems 3.1 and 4.1, one may construct an $m \times m$ matrix $G_\varepsilon(x, y) = (G_\varepsilon^{\alpha\beta}(x, y))$ of Green functions for $\mathcal{L}_\varepsilon$ for a bounded $C^{1,\eta}$ domain Ω. Moreover, if $d \ge 3$,

$$|G_\varepsilon(x, y)| \le C|x - y|^{2-d} \tag{4-8}$$

for any $x, y \in \Omega$ and

$$|G_\varepsilon(x, y) - G_\varepsilon(z, y)| \le \frac{C_\sigma |x - z|^\sigma}{|x - y|^{d-2+\sigma}} \tag{4-9}$$

for any $x, y, z \in \Omega$ with $|x - z| < \frac{1}{2}|x - y|$ and any $0 < \sigma < 1$. Since $G_\varepsilon(\cdot, y) = 0$ and $G_\varepsilon(y, \cdot) = 0$ on $\partial\Omega$, one also has

$$|G_\varepsilon(x, y)| \le \frac{C(\delta(x))^{\sigma_1}(\delta(y))^{\sigma_2}}{|x - y|^{d-2+\sigma_1+\sigma_2}} \tag{4-10}$$

for any $x, y \in \Omega$ and any $0 \le \sigma_1, \sigma_2 < 1$, where $\delta(x) = \text{dist}(x, \partial\Omega)$ and C depends only on A, Ω, σ_1 and σ_2.

Theorem 4.4. *Let Ω be a bounded $C^{1,\eta}$ domain in $\mathbb{R}^d$ for some $\eta > 0$. Suppose that $\mathscr{L}_\varepsilon(u_\varepsilon) = F$ in Ω and $u_\varepsilon = 0$ on $\partial\Omega$. Then*

$$\|u_\varepsilon\|_{C^\alpha(\overline{\Omega})} \le C_\alpha \sup_{\substack{x \in \Omega \\ 0 < r < r_0}} r^{2-\alpha} \fint_{\Omega(x,r)} |F| \tag{4-11}$$

for any $0 < \alpha < 1$, where $r_0 = \text{diam}(\Omega)$ and C_α depends only on A, Ω and α.

Proof. Since

$$u_\varepsilon(x) = \int_\Omega G_\varepsilon(x, y) F(y)\, dy,$$

it follows that, for any $x, z \in \Omega$,

$$|u_\varepsilon(x) - u_\varepsilon(z)| \le \int_\Omega |G_\varepsilon(x, y) - G_\varepsilon(z, y)| |F(y)|\, dy.$$

Let $t = |x - z|$ and write $\Omega = [\Omega \setminus B(x, 4t)] \cup \Omega(x, 4t)$. We use (4-8) to estimate the integral of $|G_\varepsilon(x, y) - G_\varepsilon(z, y)||F(y)|$ over $\Omega(x, 4t)$. This gives

$$\int_{\Omega(x,4t)} |G_\varepsilon(x, y) - G_\varepsilon(z, y)||F(y)|\, dy \le C \int_{\Omega(x,4t)} \frac{|F(y)|\, dy}{|x - y|^{d-2}} + C \int_{\Omega(z,5t)} \frac{|F(y)|\, dy}{|z - y|^{d-2}}$$

$$\le Ct^\alpha \sup_{\substack{x \in \Omega \\ 0 < r < r_0}} r^{2-\alpha} \fint_{\Omega(x,r)} |F|.$$

For the integral over $\Omega \setminus B(x, 4t)$, we choose $\beta \in (\alpha, 1)$ and use (4-9) to obtain

$$\int_{\Omega \setminus B(x,4t)} |G_\varepsilon(x, y) - G_\varepsilon(z, y)||F(y)|\, dy \le Ct^\beta \int_{\Omega \setminus B(x,4t)} \frac{|F(y)|\, dy}{|x - y|^{d-2+\beta}} \le Ct^\alpha \sup_{\substack{x \in \Omega \\ 0 < r < r_0}} r^{2-\alpha} \fint_{\Omega(x,r)} |F|.$$

Thus we have proved that $|u(x) - u(z)|/|x - z|^\alpha$ is bounded by the right-hand side of (4-11). The remaining estimate for $\|u_\varepsilon\|_{L^\infty(\Omega)}$ is similar. $\square$

Theorem 4.5. *Let Ω be a bounded $C^{1,\eta}$ domain in $\mathbb{R}^d$ for some $\eta > 0$. Suppose that $\mathscr{L}_\varepsilon(u_\varepsilon) = \text{div}(f)$ in Ω and $u_\varepsilon = 0$ on $\partial\Omega$. Then*

$$\|u_\varepsilon\|_{C^\alpha(\overline{\Omega})} \le C_\alpha \sup_{\substack{x \in \Omega \\ 0 < r < r_0}} r^{1-\alpha} \left(\fint_{\Omega(x,r)} |f|^2 \right)^{\frac{1}{2}} \tag{4-12}$$

for any $0 < \alpha < 1$, *where* $r_0 = \mathrm{diam}(\Omega)$ *and* C_α *depends only on* A, Ω *and* α.

Proof. The proof is similar to that of Theorem 4.4, using

$$|u_\varepsilon(x) - u_\varepsilon(z)| \leq \int_\Omega \big|\nabla_y (G_\varepsilon(x, y) - G_\varepsilon(z, y))\big| |f(y)| \, dy.$$

The lack of pointwise estimates for $\nabla_y G_\varepsilon(x, y)$ is overcome by using the following estimates:

$$\int_{r \leq |y-x| \leq 2r} |\nabla_y G_\varepsilon(x, y)|^2 \, dy \leq \frac{C}{r^2} \int_{r/2 \leq |y-x| \leq 3r} |G_\varepsilon(x, y)|^2 \, dy,$$

$$\int_{R \leq |y-x| \leq 2R} |\nabla_y (G_\varepsilon(x, y) - G_\varepsilon(z, y))|^2 \, dy \leq \frac{C}{R^2} \int_{R/2 \leq |y-x| \leq 3R} |G_\varepsilon(x, y) - G_\varepsilon(z, y)|^2 \, dy, \tag{4-13}$$

where $|x - z| < \frac{1}{4}|x - y|$. Estimate (4-13) follows from Cacciopoli's inequality. We omit the rest of the proof. $\qquad\square$

Theorem 4.6. *Let* Ω *be a bounded* $C^{1,\eta}$ *domain in* $\mathbb{R}^d$ *for some* $\eta > 0$. *Suppose that* $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ *in* Ω *and* $u_\varepsilon = g$ *on* $\partial\Omega$. *Then*

$$\|u_\varepsilon\|_{C^\alpha(\bar\Omega)} \leq C_\alpha \|g\|_{C^\alpha(\partial\Omega)} \tag{4-14}$$

for any $0 < \alpha < 1$, *where* C_α *depends only on* A, Ω *and* α.

Proof. Without loss of generality we may assume that $\|g\|_{C^\alpha(\partial\Omega)} = 1$. Let v be the harmonic function in Ω such that $v \in C(\bar\Omega)$ and $v = g$ on $\partial\Omega$. It is well known that $\|v\|_{C^\alpha(\bar\Omega)} \leq C_\alpha \|g\|_{C^\alpha(\partial\Omega)} = C_\alpha$, where C_α depends only on α and Ω. By interior estimates for harmonic functions, one also has

$$|\nabla v(x)| \leq C_\alpha (\delta(x))^{\alpha-1} \tag{4-15}$$

for any $x \in \Omega$. Since $\mathcal{L}_\varepsilon(u_\varepsilon - v) = -\mathcal{L}_\varepsilon(v)$ in Ω and $u_\varepsilon - v = 0$ on $\partial\Omega$, it follows that

$$u_\varepsilon(x) - v(x) = -\int_\Omega \nabla_y G_\varepsilon(x, y) A(y/\varepsilon) \nabla v(y) \, dy.$$

This, together with (4-15), gives

$$|u_\varepsilon(x) - v(x)| \leq C_\alpha \int_\Omega |\nabla_y G_\varepsilon(x, y)| (\delta(y))^{\alpha-1} \, dy. \tag{4-16}$$

We will show that

$$\int_\Omega |\nabla_y G_\varepsilon(x, y)| (\delta(y))^{\alpha-1} \, dy \leq C_\alpha (\delta(x))^\alpha \quad \text{for any } x \in \Omega. \tag{4-17}$$

Assume (4-17) for a moment. Then

$$|u_\varepsilon(x) - v(x)| \leq C_\alpha (\delta(x))^\alpha \quad \text{for any } x \in \Omega. \tag{4-18}$$

It follows that $\|u_\varepsilon\|_{L^\infty(\Omega)} \leq \|v\|_{L^\infty(\Omega)} + C \leq C$. Let $x, y \in \Omega$. To show $|u_\varepsilon(x) - u_\varepsilon(y)| \leq C|x - y|^\alpha$, we consider three cases: (1) $|x - y| < \frac{1}{4}\delta(x)$; (2) $|x - y| < \frac{1}{4}\delta(y)$; (3) $|x - y| \geq \max\big(\frac{1}{4}\delta(x), \frac{1}{4}\delta(y)\big)$. In the first case, since $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω, we may use the interior Hölder estimates in Theorem 3.1 to obtain

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C_\alpha |x - y|^\alpha \|u_\varepsilon\|_{L^\infty(B(x, \delta(x)/2))} \leq C_\alpha |x - y|^\alpha.$$

The second case is handled in the same manner. For the third case we use (4-18) and Hölder estimates for v to see that

$$
\begin{aligned}
|u_\varepsilon(x) - u_\varepsilon(y)| &\le |u_\varepsilon(x) - v(x)| + |v(x) - v(y)| + |v(y) - u_\varepsilon(y)| \\
&\le C(\delta(x))^\alpha + C|x - y|^\alpha + C(\delta(y))^\alpha \\
&\le C_\alpha |x - y|^\alpha.
\end{aligned}
$$

It remains to prove (4-17). To this end we fix $x \in \Omega$ and let $r = \delta(x)/2$. We first note that

$$
\int_{B(x,r)} |\nabla_y G_\varepsilon(x, y)| (\delta(y))^{\alpha-1} \, dy \le Cr^{\alpha-1} \int_{B(x,r)} |\nabla_y G_\varepsilon(x, y)| \, dy \le Cr^\alpha, \tag{4-19}
$$

where the last inequality follows from the first estimate in (4-13) by decomposing $B(x, r) \setminus \{0\}$ as $\bigcup_{j=0}^\infty \{B(x, 2^{-j}r) \setminus B(x, 2^{-j-1}r)\}$. To estimate the integral on $\Omega \setminus B(x, r)$, we observe that, if Q is a cube in $\mathbb{R}^d$ with the property that $3Q \subset \Omega \setminus \{x\}$ and $\ell(Q) \sim \mathrm{dist}(Q, \partial\Omega)$, then

$$
\begin{aligned}
\int_Q |\nabla_y G_\varepsilon(x, y)| (\delta(y))^{\alpha-1} \, dy &\le C(\ell(Q))^{\alpha-1} |Q| \left(\fint_Q |\nabla_y G_\varepsilon(x, y)|^2 \, dy \right)^{\frac{1}{2}} \\
&\le C(\ell(Q))^{\alpha-2} |Q| \left(\fint_{2Q} |G_\varepsilon(x, y)|^2 \, dy \right)^{\frac{1}{2}} \\
&\le Cr^{\alpha_1} (\ell(Q))^{\alpha+\alpha_2-2} |Q| \left(\fint_{2Q} \frac{dy}{|x - y|^{2(d-2+\alpha_1+\alpha_2)}} \right)^{\frac{1}{2}}, \tag{4-20}
\end{aligned}
$$

where $\alpha_1, \alpha_2 \in (0, 1)$. We remark that Caccioppoli's inequality was used for the second inequality above, while the estimate (4-10) was used for the third. Since $3Q \subset \Omega \setminus \{x\}$, we see that $|x - y| \sim |x - z|$ for any $y, z \in 2Q$. As a result, it follows from (4-20) that

$$
\int_Q |\nabla_y G_\varepsilon(x, y)| (\delta(y))^{\alpha-1} \, dy \le Cr^{\alpha_1} \int_Q \frac{(\delta(y))^{\alpha+\alpha_2-2}}{|x - y|^{d-2+\alpha_1+\alpha_2}} \, dy. \tag{4-21}
$$

By decomposing $\Omega \setminus B(x, r)$ as a nonoverlapping union of cubes Q with the said property (a Whitney-type decomposition of Ω), we obtain

$$
\begin{aligned}
\int_{\Omega \setminus B(x,r)} |\nabla_y G_\varepsilon(x, y)| (\delta(y))^{\alpha-1} \, dy &\le Cr^{\alpha_1} \int_\Omega \frac{(\delta(y))^{\alpha+\alpha_2-2}}{(|x - y| + r)^{d-2+\alpha_1+\alpha_2}} \, dy \\
&\le Cr^{\alpha_1} \int_{\mathbb{R}^d_+} \frac{y_d^{\alpha+\alpha_2-2} \, dy}{(|r - y_d| + r + |y'|)^{d-2+\alpha_1+\alpha_2}}. \tag{4-22}
\end{aligned}
$$

Finally, a direct computation shows that the integral on the right-hand side of (4-22) is bounded by $Cr^{\alpha-\alpha_1}$ provided that $\alpha_1 > \alpha$ and $\alpha_2 > 1 - \alpha$. This completes the proof. $\qquad\square$

Proof of Theorem 1.4. This follows from Theorems 4.4, 4.5 and 4.6 by writing $u_\varepsilon = u_\varepsilon^{(1)} + u_\varepsilon^{(2)} + u_\varepsilon^{(3)}$, where $u_\varepsilon^{(1)}$, $u_\varepsilon^{(2)}$ and $u_\varepsilon^{(3)}$ satisfy the conditions in Theorems 4.4, 4.5 and 4.6, respectively. $\qquad\square$

5. Construction of approximate correctors

In this section we construct the approximate correctors $\chi_T = (\chi_{T,j}^{\beta}) = (\chi_{T,j}^{\alpha\beta})$ and obtain some preliminary estimates.

Proposition 5.1. *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ and $g = (g_1, \ldots, g_d) \in L^2_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^{d \times m})$. Assume that*

$$\sup_{x \in \mathbb{R}^d} \int_{B(x,1)} (|f|^2 + |g|^2) < \infty.$$

Then, for $T > 0$, there exists a unique $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ such that

$$-\operatorname{div}(A(x)\nabla u) + T^{-2}u = f + \operatorname{div}(g) \quad in \ \mathbb{R}^d \tag{5-1}$$

and

$$\sup_{x \in \mathbb{R}^d} \int_{B(x,1)} (|\nabla u|^2 + |u|^2) < \infty. \tag{5-2}$$

Moreover, the solution u satisfies the estimate

$$\sup_{x \in \mathbb{R}^d} \fint_{B(x,T)} (|\nabla u|^2 + T^{-2}|u|^2) \le C \sup_{x \in \mathbb{R}^d} \fint_{B(x,T)} (|g|^2 + T^2|f|^2), \tag{5-3}$$

where C depends only on d, m and μ.

Proof. By rescaling we may assume that $T = 1$. The proof of the existence and estimate (5-3) may be found in [Pozhidaev and Yurinskiĭ 1989]. It uses the fact that, for $f \in L^2(\mathbb{R}^d; \mathbb{R}^m)$ and $g = (g_1, \ldots, g_d) \in L^2(\mathbb{R}^d; \mathbb{R}^{d \times m})$ with compact support, there exists a constant $\lambda > 0$, depending only on d, m and μ, such that the solution of (5-1) in $H^1(\mathbb{R}^d; \mathbb{R}^m)$ satisfies

$$\int_{\mathbb{R}^d} e^{\lambda|x|}(|\nabla u|^2 + |u|^2)\,dx \le C \int_{\mathbb{R}^d} e^{\lambda|x|}(|f|^2 + |g|^2)\,dx.$$

For the uniqueness, assume that $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ satisfies (5-2) and $-\operatorname{div}(A(x)\nabla u) + u = 0$ in $\mathbb{R}^d$. By Cacciopoli's inequality,

$$\int_{B(0,R)} |\nabla u|^2 + \int_{B(0,R)} |u|^2 \le \frac{C}{R^2} \int_{B(0,2R)} |u|^2$$

for any $R \ge 1$. It follows that

$$\int_{B(0,R)} |u|^2 \le \frac{C}{R^{2d}} \int_{B(0,2^d R)} |u|^2$$

for any $R \ge 1$. However, the condition (5-2) implies that $\int_{B(0,2^d R)} |u|^2 \le C_u R^d$. Consequently, we obtain $\int_{B(0,R)} |u|^2 \le C_u R^{-d}$ for any $R \ge 1$ and thus $u \equiv 0$ in $\mathbb{R}^d$. $\qquad\square$

Remark 5.2. The solution u of (5-1), given by Proposition 5.1, in fact satisfies

$$\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,T)}|\nabla u|^p\right)^{\frac{1}{p}} \le C\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,T)}|g|^p\right)^{\frac{1}{p}} + C\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,T)}T^2|f|^2\right)^{\frac{1}{2}}, \tag{5-4}$$

$$\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,T)}T^{-q}|u|^q\right)^{\frac{1}{q}} \le C\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,T)}|g|^p\right)^{\frac{1}{p}} + C\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,T)}T^2|f|^2\right)^{\frac{1}{2}} \tag{5-5}$$

for some $p > 2$, depending only on d, m and μ, where $1/q = 1/p - 1/d$ for $d \ge 3$. If $d = 2$, the left-hand side of (5-5) should be replaced by $T^{-1}\|u\|_{L^\infty}$.

To see (5-4), one uses the weak reverse Hölder estimate: if u is a weak solution of $-\operatorname{div}(A(x)u) = f + \operatorname{div}(g)$ in $B_r = B(x_0, r)$, then

$$\left(\fint_{B_{r/2}}|\nabla u|^p\right)^{\frac{1}{p}} \le \frac{C}{r}\left(\fint_{B_r}|u|^2\right)^{\frac{1}{2}} + C\left(\fint_{B_r}|g|^p\right)^{\frac{1}{p}} + Cr\left(\fint_{B_r}|f|^2\right)^{\frac{1}{2}}$$

for some $p > 2$, depending only on d, m and μ (see, e.g., [Giaquinta 1983]). Estimate (5-5) follows from (5-4) by Sobolev imbedding.

Let $P_j^\beta(x) = x_j e^\beta$, where $1 \le j \le d$, $1 \le \beta \le m$, and $e^\beta = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the β-th position. For $T > 0$, the approximate corrector is defined as $\chi_T = (\chi_{T,j}^{\alpha\beta})$, where, for each $1 \le j \le d$ and $1 \le \beta \le m$, $u = \chi_{T,j}^\beta = (\chi_{T,j}^{1\beta}, \dots, \chi_{T,j}^{m\beta})$ is the weak solution of

$$-\operatorname{div}(A(x)\nabla u) + T^{-2}u = \operatorname{div}(A(x)\nabla P_j^\beta) \quad \text{in } \mathbb{R}^d, \tag{5-6}$$

given by Proposition 5.1. It follows from (5-3) that

$$\sup_{x\in\mathbb{R}^d}\fint_{B(x,T)}(|\nabla \chi_T|^2 + T^{-2}|\chi_T|^2) \le C, \tag{5-7}$$

where C depends only on d, m and μ. Clearly, this gives

$$\sup_{\substack{x\in\mathbb{R}^d \\ L\ge T}}\fint_{B(x,L)}(|\nabla \chi_T|^2 + T^{-2}|\chi_T|^2) \le C, \tag{5-8}$$

where C depends only on d, m and μ.

Lemma 5.3. *Let* x, y, $z \in \mathbb{R}^d$. *Then*

$$\left(\fint_{B(x,T)}|\nabla \chi_T(t+y) - \nabla \chi_T(t+z)|^2\,dt\right)^{\frac{1}{2}} \le C\|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(\mathbb{R}^d)},$$

$$T^{-1}\left(\fint_{B(x,T)}|\chi_T(t+y) - \chi_T(t+z)|^2\,dt\right)^{\frac{1}{2}} \le C\|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(\mathbb{R}^d)}, \tag{5-9}$$

where C *depends only on* d, m *and* μ.

Proof. Fix $y, z \in \mathbb{R}^d$ and $1 \le j \le d$, $1 \le \beta \le m$. Let $u(t) = \chi_{T,j}^{\beta}(t + y)$ and $v(t) = \chi_{T,j}^{\beta}(t + z)$. Then $w = u - v$ is a solution of

$$-\operatorname{div}(A(t + y)\nabla w) + T^{-2}w = \operatorname{div}\left([A(t + y) - A(t + z)]\nabla P_j^{\beta}\right) + \operatorname{div}\left([A(t + y) - A(t + z)]\nabla v\right).$$

In view of Proposition 5.1, we obtain

$$\fint_{B(x,T)} (|\nabla w|^2 + T^{-2}|w|^2)$$

$$\le C \sup_{x \in \mathbb{R}^d} \fint_{B(x,T)} |A(t + y) - A(t + z)|^2 \, dt + C \sup_{x \in \mathbb{R}^d} \fint_{B(x,T)} |A(t + y) - A(t + z)|^2 |\nabla v|^2 \, dt$$

$$\le C\|A(\cdot + y) - A(\cdot + z)\|_{L^\infty}^2 + C\|A(\cdot + y) - A(\cdot + z)\|_{L^\infty}^2 \sup_{x \in \mathbb{R}^d} \fint_{B(x,T)} |\nabla v|^2$$

$$\le C\|A(\cdot + y) - A(\cdot + z)\|_{L^\infty}^2,$$

where we have used (5-7) in the last inequality. This completes the proof. $\qquad\square$

Remark 5.4. For $f \in L_{\mathrm{loc}}^2(\mathbb{R}^d)$, define

$$\|f\|_{W^2} = \limsup_{L \to \infty} \sup_{x \in \mathbb{R}^d} \left(\fint_{B(x,L)} |f|^2 \right)^{\frac{1}{2}}. \tag{5-10}$$

Note that, by (5-7),

$$\|\nabla \chi_T\|_{W^2} + T^{-1}\|\chi_T\|_{W^2} \le C, \tag{5-11}$$

where C depends only on d, m and μ. Moreover, by Lemma 5.3, for any $\tau \in \mathbb{R}^d$,

$$\|\nabla \chi_T(\cdot + \tau) - \nabla \chi_T(\cdot)\|_{W^2} + T^{-1}\|\chi_T(\cdot + \tau) - \chi_T(\cdot)\|_{W^2} \le C\|A(\cdot + \tau) - A(\cdot)\|_{L^\infty}. \tag{5-12}$$

Since A is uniformly almost-periodic, for any $\varepsilon > 0$, the set

$$\{\tau \in \mathbb{R}^d : \|A(\cdot + \tau) - A(\cdot)\|_{L^\infty(\mathbb{R}^d)} < \varepsilon\}$$

is relatively dense in $\mathbb{R}^d$. It follows that, for any $\varepsilon > 0$, the set of τ for which the left-hand side of (5-12) is less than ε is also relatively dense in $\mathbb{R}^d$. By [Besicovitch and Bohr 1931], this implies that $\nabla \chi_T$ and χ_T are limits of sequences of trigonometric polynomials with respect to the seminorm $\|\cdot\|_{W^2}$ in (5-10). In particular, $\nabla \chi_T, \chi_T \in B^2(\mathbb{R}^d)$ for any $T > 0$.

Lemma 5.5. *Let $u_T = \chi_{T,j}^{\beta}$ for some $T > 0$, $1 \le j \le d$ and $1 \le \beta \le m$. Then*

$$\left\langle a_{ik}^{\alpha\gamma} \frac{\partial u_T^{\gamma}}{\partial x_k} \frac{\partial v^{\alpha}}{\partial x_i} \right\rangle + T^{-2}\langle u_T \cdot v \rangle = -\left\langle a_{ij}^{\alpha\beta} \frac{\partial v^{\alpha}}{\partial x_i} \right\rangle, \tag{5-13}$$

where $v = (v^{\alpha}) \in H_{\mathrm{loc}}^1(\mathbb{R}^d; \mathbb{R}^m)$ and $v^{\alpha}, \nabla v^{\alpha} \in B^2(\mathbb{R}^d)$.

Proof. For any $\phi = (\phi^{\alpha}) \in H^1(\mathbb{R}^d; \mathbb{R}^m)$ with compact support, we have

$$\int_{\mathbb{R}^d} a_{ik}^{\alpha\gamma} \frac{\partial u_T^{\gamma}}{\partial x_k} \cdot \frac{\partial \phi^{\alpha}}{\partial x_i} + \frac{1}{T^2} \int_{\mathbb{R}^d} u_T \cdot \phi = -\int_{\mathbb{R}^d} a_{ij}^{\alpha\beta} \frac{\partial \phi^{\alpha}}{\partial x_i}. \tag{5-14}$$

Let $v = (v^\alpha) \in H^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^m)$. Suppose that $v^\alpha \in B^2(\mathbb{R}^d)$ and $\nabla v^\alpha \in B^2(\mathbb{R}^d)$. Choose $\phi(x) = \varphi(\varepsilon x)v(x)$ in (5-14), where $\varphi \in C^\infty_0(\mathbb{R}^d)$. The desired result follows by a simple change of variables $x \mapsto x/\varepsilon$ in (5-14), multiplying both sides of the equation by ε^d, and finally letting $\varepsilon \to 0$. $\square$

Letting v be a constant in (5-13), we see that

$$\langle \chi^\beta_{T,j} \rangle = 0. \tag{5-15}$$

By taking $v = \chi^\beta_{T,j}$, we obtain

$$\langle A \nabla \chi^\beta_{T,j} \cdot \nabla \chi^\beta_{T,j} \rangle + T^{-2} \langle |\chi^\beta_{T,j}|^2 \rangle = -\langle A^* \nabla \chi^\beta_{j,T} \rangle, \tag{5-16}$$

where A^* denotes the adjoint of A. This, in particular, implies that

$$\langle |\nabla \chi_T|^2 \rangle + T^{-2} \langle |\chi_T|^2 \rangle \leq C,$$

where C depends only on d, m and μ.

Lemma 5.6. *Let $\psi = (\psi^{\alpha\beta}_{ij})$ be defined by (2-4). Then, as $T \to \infty$,*

$$\frac{\partial}{\partial x_i}(\chi^{\alpha\beta}_{T,j}) \rightharpoonup \psi^{\alpha\beta}_{ij} \quad \text{weakly in } B^2(\mathbb{R}^d). \tag{5-17}$$

Proof. Fix $1 \leq j \leq d$ and $1 \leq \beta \leq m$. Let $\tilde{\psi}^\beta_j = (\tilde{\psi}^{\alpha\beta}_{ij}) \in B^2(\mathbb{R}^d; \mathbb{R}^{dm})$ be the weak limit in $B^2(\mathbb{R}^d)$ of a subsequence $\nabla \chi^\beta_{T_\ell,j}$, where $T_\ell \to \infty$. Since $\nabla \chi^\beta_{T,j} \in V^2_{\text{pot}}$, we see that $\tilde{\psi}^\beta_j \in V^2_{\text{pot}}$. Moreover, since $T^{-2}\langle |\chi_T|^2 \rangle \leq C$, it follows by letting $T \to \infty$ in (5-13) that

$$\left\langle a^{\alpha\gamma}_{ik} \tilde{\psi}^{\gamma\beta}_{kj} \frac{\partial v^\alpha}{\partial x_i} \right\rangle = -\left\langle a^{\alpha\beta}_{ij} \frac{\partial v^\alpha}{\partial x_i} \right\rangle$$

for any $v = (v^\alpha) \in \text{Trig}(\mathbb{R}^d; \mathbb{R}^m)$. This implies that $\tilde{\psi}^\beta_j$ is a solution of (2-4). By the uniqueness of the solution, we obtain $\tilde{\psi}^\beta_j = \psi^\beta_j$ and hence (5-17). $\square$

Theorem 5.7. *As $T \to \infty$, $T^{-2}\langle |\chi_T|^2 \rangle \to 0$.*

Proof. Note that

$$\mu\langle |\psi - \nabla \chi_T|^2 \rangle \leq \left\langle a^{\alpha\gamma}_{ik}\left(\psi^{\gamma\beta}_{kj} - \frac{\partial}{\partial x_k}(\chi^{\gamma\beta}_{T,j})\right)\left(\psi^{\alpha\beta}_{ij} - \frac{\partial}{\partial x_i}(\chi^{\alpha\beta}_{T,j})\right)\right\rangle$$

$$= \langle a^{\alpha\beta}_{ik}\psi^{\gamma\beta}_{kj}\psi^{\alpha\beta}_{ij} \rangle - \left\langle a^{\alpha\gamma}_{ik}\frac{\partial}{\partial x_k}(\chi^{\gamma\beta}_{T,j})\psi^{\alpha\beta}_{ij}\right\rangle - T^{-2}\langle |\chi_T|^2 \rangle,$$

where we have used equations (2-4) and (5-13). In view of Lemma 5.6, this implies that, as $T \to \infty$, $T^{-2}\langle |\chi_T|^2 \rangle \to 0$ and

$$\|\psi - \nabla \chi_T\|_{B^2} \to 0. \tag{5-18}$$

This concludes the proof. $\square$

Remark 5.8. For $T > 0$, let

$$\hat{a}^{\alpha\beta}_{T,ij} = \langle a^{\alpha\beta}_{ij} \rangle + \left\langle a^{\alpha\gamma}_{ik}\frac{\partial}{\partial x_k}(\chi^{\gamma\beta}_{T,j})\right\rangle \tag{5-19}$$

be the approximate homogenized coefficients. Then

$$\left| \hat{a}_{ij}^{\alpha\beta} - \hat{a}_{T,ij}^{\alpha\beta} \right| = \left| \left\langle a_{ik}^{\alpha\gamma} \left(\psi_{kj}^{\gamma\beta} - \frac{\partial}{\partial x_k}(\chi_{T,j}^{\gamma\beta}) \right) \right\rangle \right| \le C \| \psi - \nabla \chi_T \|_{B^2}. \tag{5-20}$$

6. Estimates of approximate correctors

In this section we will establish sharp estimates for approximate correctors χ_T. The proof relies on the uniform L^∞ and Hölder estimates obtained in Section 3 for solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = f + \mathrm{div}(g)$.

Lemma 6.1. *For* $T \ge 1$,

$$\| \chi_T \|_{L^\infty(\mathbb{R}^d)} \le CT, \tag{6-1}$$

where C is independent of T. Moreover, for any $0 < \sigma < 1$ *and* $|x - y| \le T$,

$$|\chi_T(x) - \chi_T(y)| \le C_\sigma T^{1-\sigma} |x - y|^\sigma, \tag{6-2}$$

where C_σ *depends only on* σ *and A.*

Proof. We consider the case $d \ge 3$. The 2-dimensional case follows by the method of ascending.

Let $1 \le j \le d$ and $1 \le \beta \le m$. Fix $z \in \mathbb{R}^d$ and consider the function

$$u(x) = \chi_{T,j}^\beta(x) + P_j^\beta(x - z). \tag{6-3}$$

It follows from (5-7) that

$$\left(\fint_{B(z,4T)} |u|^2 \right)^{\frac{1}{2}} \le CT. \tag{6-4}$$

Since

$$\mathrm{div}(A(x)\nabla u) = T^{-2} \chi_{T,j}^\beta \quad \text{in } \mathbb{R}^d, \tag{6-5}$$

we may apply the estimate (3-23) repeatedly to show that

$$\left(\fint_{B(z,2T)} |u|^p \right)^{\frac{1}{p}} \le C_p T \tag{6-6}$$

for any $2 < p < \infty$, where C_p depends only on p and A. This, together with (3-17), gives

$$\| u \|_{L^\infty(B(z,T))} \le CT.$$

Hence, $|\chi_{T,j}^\beta(z)| \le CT$ for any $z \in \mathbb{R}^d$. Finally, (6-2) follows from (6-1) and the Hölder estimate (3-16). $\square$

Lemma 6.2. *Let* $\sigma_1, \sigma_2 \in (0, 1)$ *and* $2 < p < \infty$. *Then, for any* $1 \le r \le T$,

$$\sup_{x \in \mathbb{R}^d} \left(\fint_{B(x,r)} |\nabla \chi_T|^p \right)^{\frac{1}{p}} \le C T^{\sigma_1} \left(\frac{T}{r} \right)^{\sigma_2}, \tag{6-7}$$

where C depends only on p, σ_1, σ_2 *and A.*

Proof. Let u be the same as in the proof of Lemma 6.1. By Cacciopoli's inequality,

$$\fint_{B(z,r)} |\nabla u|^2 \leq Cr^{-2} \fint_{B(z,2r)} |u - u(z)|^2 + Cr^2 \|T^{-2}\chi_T\|_{L^\infty}^2,$$

where $0 < r \leq T$. In view of (6-1) and (6-2), this gives

$$\sup_{z \in \mathbb{R}^d} \left(\fint_{B(z,r)} |\nabla \chi_T|^2 \right)^{\frac{1}{2}} \leq C_\sigma \left(\frac{T}{r} \right)^\sigma \tag{6-8}$$

for any $\sigma \in (0, 1)$ and $0 < r \leq T$. Since A is uniformly continuous in $\mathbb{R}^d$, by the local $W^{1,p}$ estimates for elliptic systems in divergence form, it follows from (6-5) that

$$\left(\fint_{B(z,1)} |\nabla u|^p \right)^{\frac{1}{p}} \leq C_p \left(\fint_{B(z,2)} |\nabla u|^2 \right)^{\frac{1}{2}} + CT^{-2} \|\chi_T\|_{L^\infty}$$

for any $z \in \mathbb{R}^d$ and $2 < p < \infty$, where C_p depends only on p and A. This, together with (6-8), yields

$$\sup_{z \in \mathbb{R}^d} \left(\fint_{B(z,1)} |\nabla \chi_T|^p \right)^{\frac{1}{p}} \leq C_{p,\sigma} T^\sigma$$

for any $\sigma \in (0, 1)$ and $p \in (2, \infty)$. Consequently, for any $1 \leq r \leq T$ and $\sigma \in (0, 1)$,

$$\sup_{z \in \mathbb{R}^d} \left(\fint_{B(z,r)} |\nabla \chi_T|^p \right)^{\frac{1}{p}} \leq C_{p,\sigma} T^\sigma. \tag{6-9}$$

The desired estimate (6-7) now follows from (6-8) and (6-9) by a simple interpolation of L^p norms. $\square$

Theorem 6.3. *Let $T \geq 1$. The approximate corrector χ_T is uniformly almost-periodic in $\mathbb{R}^d$. Moreover, for any $y, z \in \mathbb{R}^d$,*

$$\|\chi_T(\cdot + y) - \chi_T(\cdot + z)\|_{L^\infty(\mathbb{R}^d)} \leq CT \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(\mathbb{R}^d)}, \tag{6-10}$$

where C is independent of T, y and z.

Proof. We assume $d \geq 3$. The case $d = 2$ follows from the case $d = 3$ by the method of ascending. Fix $y, z \in \mathbb{R}^d$ and $1 \leq j \leq d$, $1 \leq \beta \leq m$. Let

$$u(x) = \chi_{T,j}^\beta(x + y) - \chi_{T,j}^\beta(x + z).$$

Note that

$$-\operatorname{div}(A(x+y)\nabla u) = -T^{-2}u + \operatorname{div}\big((A(x+y) - A(x+z))\nabla P_j^\beta\big) + \operatorname{div}\big((A(x+y) - A(x+z))\nabla v\big), \tag{6-11}$$

where $v(x) = \chi_{T,j}^\beta(x + z)$. Let $B = B(x_0, T)$. As in the proof of Theorem 3.4, we choose a cut-off function $\varphi \in C_0^\infty\big(B\big(x_0, \frac{7}{4}T\big)\big)$ such that $\varphi = 1$ in $B\big(x_0, \frac{3}{2}T\big)$ and $|\nabla \varphi| \leq CT^{-1}$. Using the representation

formula by fundamental solutions and (6-11), we obtain, for any $x \in B$,

$$|u(x)| \le CT^{-2} \int_{2B} |\Gamma^y(x,t)| |u(t)| \, dt + C\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty} \int_{2B} |\nabla_t(\Gamma^y(x,t)\varphi(t))| \, dt$$

$$+ C\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty} \int_{2B} |\nabla v(t)| |\nabla_t(\Gamma^y(x,t)\varphi(t))| \, dt$$

$$+ CT\left(\fint_{2B} |\nabla u|^2\right)^{\frac{1}{2}} + C\left(\fint_{2B} |u|^2\right)^{\frac{1}{2}}, \tag{6-12}$$

where we have used $\Gamma^y(x,t) = \Gamma(x+y, t+y)$ to denote the matrix of fundamental solutions for the operator $-\operatorname{div}(A(\cdot+y)\nabla)$ in $\mathbb{R}^d$. By Lemma 5.3, the last two terms in the right-hand side of (6-12) are bounded by the right-hand side of (6-10). Using the size estimate (3-12) and Cacciopoli's inequality, it is also not hard to see that the second term in the right-hand side of (6-12) is bounded by the right-hand side of (6-10).

To treat the third term in the right-hand side of (6-12), we note that

$$\int_{2B} |\nabla v(t)| |\nabla_t(\Gamma^y(x,t)\varphi(t))| \, dt$$

$$\le C \sum_{\ell=0}^{\infty} \left(\fint_{|t-x|\sim 2^{-\ell}T} |\nabla v(t)|^2 \, dt\right)^{\frac{1}{2}} \left(\fint_{|t-x|\sim 2^{-\ell}T} |\nabla_t(\Gamma^y(x,t)\varphi)|^2 \, dt\right)^{\frac{1}{2}} (2^{-\ell}T)^d$$

$$\le C \sum_{\ell=0}^{\infty} (2^\ell)^\sigma \cdot (2^{-\ell}T)^{1-d} \cdot (2^{-\ell}T)^d$$

$$\le CT,$$

where $\sigma \in (0,1)$ and we have used (6-8) to estimate the integral involving $|\nabla v(t)|^2$ for the second inequality. As a result, we have proved that, for any $x \in B$,

$$|u(x)| \le CT^{-2} \int_{2B} \frac{|u(t)|}{|x-t|^{d-2}} \, dt + CT\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty}. \tag{6-13}$$

By the fractional integral estimates, this implies that

$$\left(\fint_B |u|^q\right)^{\frac{1}{q}} \le C\left(\fint_{2B} |u|^p\right)^{\frac{1}{p}} + CT\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty},$$

where $1 < p < q \le \infty$ and $1/p - 1/q < 2/d$. Since

$$\left(\fint_{2B} |u|^2\right)^{\frac{1}{2}} \le CT\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty}$$

by Lemma 5.3, a simple iteration argument shows that

$$\|u\|_{L^\infty(B)} \le CT\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty}.$$

This completes the proof.   □

Remark 6.4. Let $u(x) = \chi_T(x + y) - \chi_T(x + z)$, as in the proof of Theorem 6.3. Then

$$|u(t) - u(s)| \le C_\sigma \left(\frac{|t - s|}{T} \right)^\sigma T \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty} \tag{6-14}$$

for any $\sigma \in (0, 1)$ and $t, s \in \mathbb{R}^d$, where C_σ depends only on σ and A. This follows from (6-11), (6-10) and (3-16). By Cacciopoli's inequality and (6-14), we may deduce that

$$\sup_{x \in \mathbb{R}^d} \left(\fint_{B(x,r)} |\nabla u|^2 \right)^{\frac{1}{2}} \le C_\sigma \left(\frac{T}{r} \right)^\sigma \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty} \tag{6-15}$$

for any $\sigma \in (0, 1)$.

Theorem 6.5. *Let $T \ge 1$. Then*

$$T^{-1} \|\chi_T\|_{L^\infty(\mathbb{R}^d)} \le C_\sigma \left(\rho(R) + \left(\frac{R}{T} \right)^\sigma \right) \tag{6-16}$$

for any $R > 0$ and $\sigma \in (0, 1)$, where C_σ depends only on σ and A. In particular, $T^{-1} \|\chi_T\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $T \to \infty$.

Proof. Let $y, z \in \mathbb{R}^d$. Suppose $|z| \le R$. Then

$$|\chi_T(y) - \chi_T(0)| \le |\chi_T(y) - \chi_T(z)| + |\chi_T(z) - \chi_T(0)| \le CT \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(\mathbb{R}^d)} + C_\sigma T^{1-\sigma} R^\sigma,$$

where we have used Theorem 6.3 and Lemma 6.1. It follows that

$$\sup_{y \in \mathbb{R}^d} T^{-1} |\chi_T(y) - \chi_T(0)| \le C\rho(R) + C_\sigma \left(\frac{R}{T} \right)^\sigma \tag{6-17}$$

for any $R > 0$.

Finally, we observe that

$$|\chi_T(0)| \le \left| \fint_{B(0,L)} (\chi_T(y) - \chi_T(0))\, dy \right| + \left| \fint_{B(0,L)} \chi_T(y)\, dy \right| \le \sup_{y \in \mathbb{R}^d} |\chi_T(y) - \chi_T(0)| + \left| \fint_{B(0,L)} \chi_T(y)\, dy \right|.$$

Since $\langle \chi_T \rangle = 0$, we may let $L \to \infty$ in the estimate above to obtain

$$|\chi_T(0)| \le \sup_{y \in \mathbb{R}^d} |\chi_T(y) - \chi_T(0)|.$$

This, together with (6-17), yields the estimate (6-16). $\qquad \square$

For $T \ge 1$ and $\sigma > 0$, define

$$\Theta_\sigma(T) = \inf_{0 < R \le T} \left(\rho(R) + \left(\frac{R}{T} \right)^\sigma \right). \tag{6-18}$$

Note that $\Theta_\sigma(T)$ is a decreasing and continuous function of T and $\Theta_\sigma(T) \to 0$ as $T \to \infty$. It follows from Theorem 6.5 that

$$T^{-1} \|\chi_T\|_{L^\infty(\mathbb{R}^d)} \le C_\sigma \Theta_\sigma(T) \quad \text{for any } T \ge 1, \tag{6-19}$$

where $\sigma \in (0, 1)$. By taking $R = T^\alpha$ for some $\alpha \in (0, 1)$ in (6-18), we see that

$$\Theta_\sigma(T) \leq \rho(T^\alpha) + T^{-\sigma(1-\alpha)}. \tag{6-20}$$

This, in particular, implies that

$$\int_1^\infty \frac{\rho(r)}{r} \, dr < \infty \implies \int_1^\infty \frac{\Theta_\sigma(r)}{r} \, dr < \infty.$$

Theorem 6.6. *Let $T \geq 1$. Then*

$$\langle |\psi - \nabla\chi_T|^2\rangle^{1/2} \leq C_\sigma \int_{T/2}^\infty \frac{\Theta_\sigma(r)}{r} \, dr \tag{6-21}$$

for $\sigma \in (0, 1)$, where C_σ depends only on σ and A.

Proof. Fix $1 \leq j \leq d$ and $1 \leq \beta \leq m$. Let $u = \chi_{T,j}^\beta$, $v = \chi_{2T,j}^\beta$ and $w = u - v$. It follows from Lemma 5.5 that

$$\langle A\nabla w \cdot \nabla\varphi\rangle = \frac{1}{4T^2}\langle v \cdot \varphi\rangle - \frac{1}{T^2}\langle u \cdot \varphi\rangle$$

for any $\varphi \in H^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^m)$ with $\varphi, \nabla\varphi \in B^2(\mathbb{R}^d)$. By taking $\varphi = w$, we obtain

$$\langle |\nabla w|^2\rangle \leq CT^{-2}\big(\langle |u|^2\rangle + \langle |v|^2\rangle\big) \leq C_\sigma(\Theta_\sigma(T) + \Theta_\sigma(2T))^2, \tag{6-22}$$

where we have used (6-19) for the second inequality. Hence, we have proved that

$$\langle |\nabla\chi_T - \nabla\chi_{2T}|^2\rangle^{1/2} \leq C_\sigma \int_{T/2}^T \frac{\Theta_\sigma(r)}{r} \, dr,$$

where we have used the fact that $\Theta_\sigma(r)$ is decreasing. Consequently,

$$\sum_{\ell=0}^\infty \langle |\nabla\chi_{2^\ell T} - \nabla\chi_{2^{\ell+1}T}|^2\rangle^{1/2} \leq C_\sigma \int_{T/2}^\infty \frac{\Theta_\sigma(r)}{r} \, dr. \tag{6-23}$$

Recall that, by (5-18), $\langle |\psi - \nabla\chi_T|^2\rangle \to 0$ as $T \to \infty$. The estimate (6-21) now follows from (6-23). $\square$

Remark 6.7. Suppose that there exist $C > 0$ and $\tau > 0$ such that

$$\rho(R) \leq \frac{C}{R^\tau} \quad \text{for } R \geq 1. \tag{6-24}$$

By taking $R = T^{\sigma/(\tau+\sigma)}$ in (6-16), we obtain

$$T^{-1}\|\chi_T\|_{L^\infty} \leq C\Theta_\sigma(T) \leq CT^{-\tau\sigma/(\tau+\sigma)}.$$

Since $\sigma \in (0, 1)$ is arbitrary, this shows that

$$T^{-1}\|\chi_T\|_{L^\infty} \leq C_\delta T^{-\tau/(\tau+1)+\delta} \tag{6-25}$$

for any $\delta \in (0, 1)$, where C_δ depends only on δ and A. Under the condition (6-24), by Theorem 6.6, we also obtain

$$\langle |\psi - \nabla\chi_T|^2\rangle^{1/2} \leq C_\delta T^{-\tau/(\tau+1)+\delta} \quad \text{for any } \delta \in (0, 1). \tag{6-26}$$

7. Convergence rates

In this section we give the proof of Theorems 1.1 and 1.2.

Lemma 7.1. *Let $h \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and $T > 0$. Suppose that there exists $\sigma \in (0, 1)$ such that*

$$\sup_{x \in \mathbb{R}^d} \left(\fint_{B(x,r)} |h|^2 \right)^{\frac{1}{2}} \leq \left(\frac{T}{r} \right)^{1-\sigma} \quad \text{for any } 0 < r \leq T. \tag{7-1}$$

Let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ be the solution of

$$-\Delta u + T^{-2} u = h \quad \text{in } \mathbb{R}^d \tag{7-2}$$

given by Proposition 5.1. Then

$$\|u\|_{L^\infty} \leq C T^2, \quad \|\nabla u\|_{L^\infty} \leq CT, \tag{7-3}$$

and

$$|\nabla u(x) - \nabla u(y)| \leq C T^{1-\sigma} |x - y|^\sigma \quad \text{for any } x, y \in \mathbb{R}^d, \tag{7-4}$$

where C depends only on d and σ. Furthermore, $u \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$\sup_{x \in \mathbb{R}^d} \left(\fint_{B(x,T)} |\nabla^2 u|^2 \right)^{\frac{1}{2}} \leq C. \tag{7-5}$$

Proof. By rescaling we may assume $T = 1$. It follows from Proposition 5.1 and (7-1) that

$$\sup_{x \in \mathbb{R}^d} \left(\fint_{B(x,1)} |u|^2 \right)^{\frac{1}{2}} \leq C \quad \text{and} \quad \sup_{x \in \mathbb{R}^d} \left(\fint_{B(x,1)} |\nabla u|^2 \right)^{\frac{1}{2}} \leq C, \tag{7-6}$$

where C depends only on d. Fix $x_0 \in \mathbb{R}^d$ and let $\phi \in C^\infty_0(B(x_0, 2))$ be a cut-off function such that $\phi = 1$ in $B(x_0, 1)$. By representing $u\phi$ as an integral and using the fundamental solution for $-\Delta$, the desired estimates follow from (7-1) by a standard procedure. We leave the details to the reader. $\square$

Under additional almost periodicity conditions on h, the next lemma gives much sharper estimates for the solution u of (7-2).

Lemma 7.2. *Let $h \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and $T > 0$. Suppose that there exists $\sigma \in (0, 1)$ such that*

$$\sup_{x \in \mathbb{R}^d} \left(\fint_{B(x,r)} |h|^2 \right)^{\frac{1}{2}} \leq C_0 \left(\frac{T}{r} \right)^{1-\sigma},$$

$$\sup_{x \in \mathbb{R}^d} \left(\fint_{B(x,r)} |h(t+y) - h(t+z)|^2 \, dt \right)^{\frac{1}{2}} \leq C_0 \left(\frac{T}{r} \right)^{1-\sigma} \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty} \tag{7-7}$$

for any $0 < r \leq T$ and $y, z \in \mathbb{R}^d$. Let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ be the solution of (7-2), given by Proposition 5.1. Then

$$T^{-2} \|u\|_{L^\infty} \leq C \Theta_1(T) + |\langle h \rangle|,$$

$$T^{-1} \|\nabla u\|_{L^\infty} \leq C \Theta_\sigma(T), \tag{7-8}$$

where $\Theta_\sigma(T)$ is defined by (6-18) and C depends at most on d, σ and C_0.

Proof. By applying Lemma 7.1 to the function

$$\frac{u(x+y) - u(x+z)}{C_0 \|A(\cdot+y) - A(\cdot+z)\|_{L^\infty}}$$

with y and z fixed, we obtain

$$
\begin{aligned}
\|u(\cdot+y) - u(\cdot+z)\|_{L^\infty} &\le CT^2 \|A(\cdot+y) - A(\cdot+z)\|_{L^\infty}, \\
\|\nabla u(\cdot+y) - \nabla u(\cdot+z)\|_{L^\infty} &\le CT \|A(\cdot+y) - A(\cdot+z)\|_{L^\infty},
\end{aligned}
\tag{7-9}
$$

where C depends only on d, C_0 and σ. This shows that u and ∇u are uniformly almost periodic. In particular, u and ∇u have mean values and $\langle \nabla u \rangle = 0$. Also, note that condition (7-7) implies that $h \in B^2(\mathbb{R}^d)$ and hence has the mean value $\langle h \rangle$. It is easy to deduce from (7-2) that $\langle u \rangle = T^2 \langle h \rangle$.

Note that, for any $y \in \mathbb{R}^d$ and $z \in \mathbb{R}^d$ with $|z| \le R \le T$,

$$T^{-2}|u(y) - u(0)| \le T^{-2}|u(y) - u(z)| + T^{-2}|u(z) - u(0)| \le C\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty} + CT^{-1}R,$$

where we have used (7-9) and $\|\nabla u\|_{L^\infty} \le CT$ for the second inequality. It follows from the definition of $\rho(R)$ that

$$\sup_{y\in\mathbb{R}^d} T^{-2}|u(y) - u(0)| \le C(\rho(R) + T^{-1}R) \quad \text{for any } 0 < R \le T.$$

By the definition of Θ_1, this gives

$$\sup_{y\in\mathbb{R}^d} T^{-2}|u(y) - u(0)| \le C\Theta_1(T). \tag{7-10}$$

Using

$$|T^{-2}u(0)| \le T^{-2}\left| \fint_{B(0,L)} (u(y) - u(0))\, dy \right| + \left| \fint_{B(0,L)} u(x) \right|$$

for any $L > 0$ and (7-10), we see that, by letting $L \to \infty$,

$$|T^{-2}u(0)| \le C\Theta_1(T) + T^{-2}|\langle u \rangle| = C\Theta_1(T) + |\langle h \rangle|. \tag{7-11}$$

The first inequality in (7-8) now follows from (7-10) and (7-11).

Finally, we point out that the second inequality in (7-8) follows in the same manner, using (7-9) and (7-4) as well as the fact that the mean value of ∇u is zero. □

We are now ready to estimate the rates of convergence of u_ε to u_0.

Theorem 7.3. *Let u_ε ($\varepsilon \ge 0$) be the weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω and $u_\varepsilon = g$ on $\partial\Omega$. Suppose that $u_0 \in W^{2,2}(\Omega)$. Let*

$$w_\varepsilon(x) = u_\varepsilon(x) - u_0(x) - \varepsilon\chi_{T,j}(x/\varepsilon)\frac{\partial u_0}{\partial x_j} + v_\varepsilon, \tag{7-12}$$

where $T = \varepsilon^{-1}$ and $v_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ is the weak solution of the Dirichlet problem

$$\mathcal{L}_\varepsilon(v_\varepsilon) = 0 \quad \text{in } \Omega \qquad \text{and} \qquad v_\varepsilon = \varepsilon\chi_{T,j}(x/\varepsilon)\frac{\partial u_0}{\partial x_j} \quad \text{on } \partial\Omega. \tag{7-13}$$

Then

$$\|w_\varepsilon\|_{H^1(\Omega)} \le C_\sigma \big(\Theta_\sigma(T) + \langle|\psi - \nabla\chi_T|\rangle\big)\|u_0\|_{W^{2,2}(\Omega)} \tag{7-14}$$

for any $\sigma \in (0, 1)$, where C_σ depends only on σ, A and Ω.

Proof. With loss of generality we may assume that

$$\|u_0\|_{W^{2,2}(\Omega)} = 1. \tag{7-15}$$

A direct computation shows that

$$\mathcal{L}_\varepsilon(w_\varepsilon) = -\operatorname{div}(B_T(x/\varepsilon)\nabla u_0) + \varepsilon \operatorname{div}(A(x/\varepsilon)\chi_T(x/\varepsilon)\nabla^2 u_0), \tag{7-16}$$

where $B_T(y) = (b_{T,ij}^{\alpha\beta}(y))$ is given by

$$b_{T,ij}^{\alpha\beta}(y) = \hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(y) - a_{ik}^{\alpha\gamma}(y)\frac{\partial}{\partial y_k}(\chi_{T,j}^{\gamma\beta}(y)). \tag{7-17}$$

Since $w_\varepsilon \in H_0^1(\Omega; \mathbb{R}^m)$, it follows from (7-16) that

$$c \int_\Omega |\nabla w_\varepsilon|^2 \, dx \leq \left| \int_\Omega \operatorname{div}(B_T(x/\varepsilon)\nabla u_0) \cdot w_\varepsilon \, dx \right| + \int_\Omega |\varepsilon \chi_T(x/\varepsilon)| |\nabla^2 u_0| |\nabla w_\varepsilon| \, dx = I_1 + I_2. \tag{7-18}$$

It suffices to show that

$$I_1 + I_2 \leq C_\sigma \big(\Theta_\sigma(T) + \langle |\psi - \nabla \chi_T| \rangle \big) \|w_\varepsilon\|_{H^1(\Omega)} \tag{7-19}$$

for any $\sigma \in (0, 1)$.

First, it is easy to see that

$$I_2 \leq C\varepsilon \|\chi_T\|_{L^\infty} \|\nabla w_\varepsilon\|_{L^2(\Omega)} \leq C\Theta_\sigma(T) \|\nabla w_\varepsilon\|_{L^2(\Omega)} \tag{7-20}$$

for any $\sigma \in (0, 1)$, where we have used (7-15) and (6-19).

Next, to estimate I_1, we let $h(y) = h_T(y) = B_T(y) - \langle B_T \rangle$ and solve (7-2). More precisely, let $h = (h_{ij}^{\alpha\beta})$ and $f = (f_{ij}^{\alpha\beta})$, where $f_{ij}^{\alpha\beta} \in H_{\text{loc}}^2(\mathbb{R}^d)$ solves

$$-\Delta f_{ij}^{\alpha\beta} + T^{-2} f_{ij}^{\alpha\beta} = h_{ij}^{\alpha\beta} \quad \text{in } \mathbb{R}^d. \tag{7-21}$$

By (6-8) and (6-15), the function h satisfies the condition (7-7) for any $\sigma \in (0, 1)$. Since $\langle h \rangle = 0$, it follows from Lemma 7.2 that

$$\begin{aligned} T^{-2} \|f\|_{L^\infty} &\leq C\Theta_1(T), \\ T^{-1} \|\nabla f\|_{L^\infty} &\leq C\Theta_\sigma(T) \end{aligned} \tag{7-22}$$

for any $\sigma \in (0, 1)$. Using (7-21) and integration by parts, we may bound I_1 in (7-18) by

$$\left| \int_\Omega \operatorname{div}(\Delta f(x/\varepsilon)\nabla u_0) \cdot w_\varepsilon \, dx \right| + T^{-2} \int_\Omega |f(x/\varepsilon)| |\nabla u_0| |\nabla w_\varepsilon| \, dx + C\langle |\psi - \nabla \chi_T| \rangle \|w_\varepsilon\|_{L^2(\Omega)}, \tag{7-23}$$

where we have used the fact that $|\langle B_T \rangle| \leq C\langle |\psi - \nabla \chi_T| \rangle$. Note that, by (7-22), the second term in (7-23) is bounded by $C\Theta_1(T) \|\nabla w_\varepsilon\|_{L^2(\Omega)}$.

It remains to estimate the first term in (7-23), which we denote by I_{11}. To this end we write

$$
\begin{aligned}
\operatorname{div}(\Delta f(x/\varepsilon)\nabla u_0) \cdot w_\varepsilon &= \frac{\partial}{\partial x_i}\left(\Delta f_{ij}^{\alpha\beta}(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\right) \cdot w_\varepsilon^\alpha \\
&= \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_k}\left(\frac{\partial f_{ij}^{\alpha\beta}}{\partial x_k} - \frac{\partial f_{kj}^{\alpha\beta}}{\partial x_i}\right)(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\right) \cdot w_\varepsilon^\alpha + \frac{\partial}{\partial x_i}\left(\frac{\partial^2 f_{kj}^{\alpha\beta}}{\partial x_k \partial x_i}(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\right) \cdot w_\varepsilon^\alpha \\
&= -\frac{\partial}{\partial x_i}\left(\varepsilon\left(\frac{\partial f_{ij}^{\alpha\beta}}{\partial x_k} - \frac{\partial f_{kj}^{\alpha\beta}}{\partial x_i}\right)(x/\varepsilon)\frac{\partial^2 u_0^\beta}{\partial x_k \partial x_j}\right) \cdot w_\varepsilon^\alpha + \frac{\partial}{\partial x_i}\left(\frac{\partial^2 f_{kj}^{\alpha\beta}}{\partial x_k \partial x_i}(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\right) \cdot w_\varepsilon^\alpha,
\end{aligned}
$$

where we have used the product rule and the fact that

$$
\frac{\partial^2}{\partial x_i \partial x_k}\left(\left(\frac{\partial f_{ij}^{\alpha\beta}}{\partial x_k} - \frac{\partial f_{kj}^{\alpha\beta}}{\partial x_i}\right)(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\right) = 0.
$$

It then follows from an integration by parts that

$$
I_{11} \le C\varepsilon \int_\Omega |\nabla f(x/\varepsilon)||\nabla^2 u_0||\nabla w_\varepsilon|\,dx + C\sum_{j,\alpha,\beta}\int_\Omega \left|\nabla\frac{\partial f_{kj}^{\alpha\beta}}{\partial x_k}(x/\varepsilon)\right||\nabla u_0||\nabla w_\varepsilon|\,dx = I_{11}^{(1)} + I_{11}^{(2)}. \quad (7\text{-}24)
$$

In view of (7-22), we have

$$
I_{11}^{(1)} \le C\varepsilon\|\nabla f\|_{L^\infty}\|\nabla w_\varepsilon\|_{L^2(\Omega)} \le C\Theta_\sigma(T)\|\nabla w_\varepsilon\|_{L^2(\Omega)} \quad (7\text{-}25)
$$

for any $\sigma \in (0, 1)$.

Finally, to estimate $I_{11}^{(2)}$, we note that, by the definition of χ_T,

$$
\frac{\partial h_{ij}^{\alpha\beta}}{\partial y_i} = \frac{\partial}{\partial y_i}(b_{T,ij}^{\alpha\beta}) = -\frac{1}{T^2}\chi_{T,j}^{\alpha\beta}.
$$

It follows that

$$
-\Delta\frac{\partial f_{ij}^{\alpha\beta}}{\partial y_i} + \frac{1}{T^2}\frac{\partial f_{ij}^{\alpha\beta}}{\partial y_i} = -\frac{1}{T^2}\chi_{T,j}^{\alpha\beta}.
$$

Observe that the function $T^{-1}\chi_T$ satisfies the assumption on h in Lemma 7.2 with $\sigma = 1$. As a result, we obtain

$$
\left\|\nabla\frac{\partial f_{ij}^{\alpha\beta}}{\partial x_i}\right\|_{L^\infty} \le C_\sigma\Theta_\sigma(T)
$$

for any $\sigma \in (0, 1)$. This allows us to bound $I_{11}^{(2)}$ by $C_\sigma\Theta_\sigma(T)\|\nabla w_\varepsilon\|_{L^2(\Omega)}$ and completes the proof. $\square$

The next lemma gives an estimate for the norm of v_ε in $H^1(\Omega)$.

Lemma 7.4. *Let v_ε be the weak solution of* (7-13) *with $T = \varepsilon^{-1}$. Then*

$$
\|v_\varepsilon\|_{H^1(\Omega)} \le C_\sigma(T^{-1}\|\chi_T\|_{L^\infty})^{1/2-\sigma}(\|\nabla u_0\|_{L^\infty(\Omega)} + \|\nabla^2 u_0\|_{L^2(\Omega)}) \quad (7\text{-}26)
$$

for any $\sigma \in \left(0, \frac{1}{2}\right)$, where C_σ depends only on A, Ω and σ.

Proof. We may assume that $\|\nabla u_0\|_{L^\infty(\Omega)}+\|\nabla^2 u_0\|_{L^2(\Omega)}=1$. We may also assume that $\delta=T^{-1}\|\chi_T\|_{L^\infty}>0$ is small, since $\delta\to 0$ as $T\to\infty$. Choose a cut-off function $\eta_\delta\in C_0^\infty(\mathbb{R}^d)$ so that $0\le\eta_\delta\le 1$, $\eta_\delta(x)=1$ if $\mathrm{dist}(x,\partial\Omega)<\delta$, $\eta_\delta(x)=0$ if $\mathrm{dist}(x,\partial\Omega)\ge 2\delta$, and $|\nabla\eta_\delta|\le C\delta^{-1}$. Note that

$$
\begin{aligned}
\|v_\varepsilon\|_{H^1(\Omega)} &\le C\varepsilon\|\chi_T(x/\varepsilon)\nabla u_0\|_{H^{1/2}(\partial\Omega)}\\
&\le C\varepsilon\|\eta_\delta\chi_T(x/\varepsilon)\nabla u_0\|_{H^1(\Omega)}\\
&\le C\left(\|\chi_T\|_{L^\infty}\delta^{-1/2}\varepsilon+\left(\int_{\Omega_\delta}|\nabla\chi_T(x/\varepsilon)|^2\,dx\right)^{\frac{1}{2}}\right),
\end{aligned}
\tag{7-27}
$$

where $\Omega_\delta=\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)\le 2\delta\}$. Since $\|\chi_T\|_{L^\infty}\delta^{-1/2}\varepsilon=\delta^{1/2}$, we only need to estimate the integral of $|\nabla\chi_T(x/\varepsilon)|^2$ over Ω_δ.

To this end, we cover Ω_δ with cubes Q_j of side length δ such that $\sum_j|Q_j|\le C\delta$. It follows that

$$
\begin{aligned}
\int_{\Omega_\delta}|\nabla\chi_T(x/\varepsilon)|^2\,dx &\le\sum_j\int_{Q_j}|\nabla\chi_T(x/\varepsilon)|^2\,dx\le\sum_j|Q_j|\fint_{(1/\varepsilon)Q_j}|\nabla\chi_T|^2\\
&\le C\delta\sup_{\ell(Q)=\delta T}\fint_Q|\nabla\chi_T|^2\le C_\sigma\delta^{1-\sigma}
\end{aligned}
\tag{7-28}
$$

for any $\sigma\in(0,1)$, where we have used the estimate (6-8) in the last inequality. This, together with (7-27), gives (7-26). $\qquad\square$

We are now in a position to give the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. It follows from Theorem 7.3 and Lemma 7.4 that, for any $\sigma\in(0,1)$ and $\delta\in\left(0,\frac{1}{2}\right)$,

$$
\begin{aligned}
\|u_\varepsilon-u_0 &-\varepsilon\chi_T(x/\varepsilon)\nabla u_0\|_{H^1(\Omega)}\\
&\le C\big(\Theta_\sigma(T)+\langle|\psi-\nabla\chi_T|\rangle\big)\|u_0\|_{W^{2,2}(\Omega)}+C(\Theta_\sigma(T))^{1/2-\delta}(\|\nabla u_0\|_{L^\infty(\Omega)}+\|\nabla^2 u_0\|_{L^2(\Omega)})\\
&\le C\big(\langle|\psi-\nabla\chi_T|\rangle+(\Theta_\sigma(T))^{1/2-\delta}\big)\|u_0\|_{W^{2,p}(\Omega)}\\
&\le C\big(\langle|\psi-\nabla\chi_T|\rangle+(\Theta_1(T))^{\sigma(1/2-\delta)}\big)\|u_0\|_{W^{2,p}(\Omega)},
\end{aligned}
\tag{7-29}
$$

where $T=\varepsilon^{-1}$ and we have used the Sobolev imbedding $\|\nabla u_0\|_{L^\infty(\Omega)}\le C\|u_0\|_{W^{2,p}(\Omega)}$ for $p>d$. This implies that

$$
\begin{aligned}
\|u_\varepsilon-u_0\|_{L^2(\Omega)} &\le\|\varepsilon\chi_T(x/\varepsilon)\nabla u_0\|_{L^2(\Omega)}+C\big(\langle|\psi-\nabla\chi_T|\rangle+(\Theta_1(T))^{1/4}\big)\|u_0\|_{W^{2,p}(\Omega)}\\
&\le C\big(\langle|\psi-\nabla\chi_T|\rangle+(\Theta_1(T))^{1/4}\big)\|u_0\|_{W^{2,p}(\Omega)},
\end{aligned}
$$

where C depends only on A and Ω. Since $\langle|\psi-\nabla\chi_T|\rangle+(\Theta_1(T))^{1/4}\to 0$ as $T\to\infty$, one may find a modulus η on $(0,1]$, depending only on A, such that $\eta(0+)=0$ and

$$
\langle|\psi-\nabla\chi_T|\rangle+(\Theta_1(T))^{1/4}\le\eta(T^{-1})
$$

for $T\ge 1$. As a result, we obtain

$$
\|u_\varepsilon-u_0-\varepsilon\chi_T(x/\varepsilon)\nabla u_0\|_{H^1(\Omega)}\le C\eta(\varepsilon)\|u_0\|_{W^{2,p}(\Omega)},
$$

$$
\|u_\varepsilon-u_0\|_{L^2(\Omega)}\le C\eta(\varepsilon)\|u_0\|_{W^{2,p}(\Omega)}.
$$

Finally, we observe that, by Theorem 1.4, for any $\sigma \in (0, 1)$,

$$\|u_\varepsilon\|_{C^\sigma(\overline{\Omega})} \leq C(\|g\|_{C^\sigma(\partial\Omega)} + \|F\|_{L^d(\Omega)}) \leq C(\|u_0\|_{C^\sigma(\overline{\Omega})} + \|\nabla^2 u_0\|_{L^d(\Omega)}) \leq C\|u_0\|_{W^{2,d}(\Omega)}.$$

It follows by interpolation that, for any $\sigma \in (0, 1)$,

$$\|u_\varepsilon - u_0\|_{C^\sigma(\overline{\Omega})} \leq C\tilde{\eta}(\varepsilon)\|u_0\|_{W^{2,p}(\Omega)},$$

where $\tilde{\eta}$ is a modulus function depending only on A and σ, and $\tilde{\eta}(0+) = 0$. This completes the proof. $\square$

Proof of Theorem 1.2. Estimate (1-15) follows directly from (7-29) and Theorem 6.6. To see (1-14), we use

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq \|u_\varepsilon - u_0 - \varepsilon\chi_T(x/\varepsilon)\nabla u_0 + v_\varepsilon\|_{L^2(\Omega)} + \|v_\varepsilon\|_{L^2(\Omega)}$$
$$\leq C_\sigma\big(\Theta_\sigma(T) + \langle|\psi - \nabla\psi_T|\rangle\big)\|u_0\|_{W^{2,2}(\Omega)} + \|v_\varepsilon\|_{L^2(\Omega)}, \tag{7-30}$$

where v_ε is as defined in Theorem 7.3. By Theorem 1.4 we obtain

$$\|v_\varepsilon\|_{L^2(\Omega)} \leq C\|v_\varepsilon\|_{L^\infty(\Omega)}$$
$$\leq C\|\varepsilon\chi_T(x/\varepsilon)\nabla u_0\|_{C^{\sigma_1}(\partial\Omega)}$$
$$\leq C(\varepsilon^{1-\sigma_1}\|\chi_T\|_{C^{0,\sigma_1}} + \Theta_\sigma(T))\|\nabla u_0\|_{C^{\sigma_1}(\partial\Omega)}$$
$$\leq C(T^{\sigma_1-1}\|\chi_T\|_{C^{0,\sigma_1}} + \Theta_\sigma(T))\|u_0\|_{W^{2,p}(\Omega)},$$

where $p > d$, $\sigma \in (0, 1)$ and $0 < \sigma_1 < 1 - d/p$. Since $T^{-1}\|\chi_T\|_{L^\infty} \leq C_\sigma\Theta_\sigma(T)$ and $|\chi_T(x) - \chi_T(y)| \leq C_\alpha T^{1-\alpha}|x - y|^\alpha$ for any $\alpha \in (0, 1)$, it follows by interpolation that

$$T^{\sigma_1-1}\|\chi_T\|_{C^{0,\sigma_1}} \leq C(\Theta_\sigma(T))^{1-\sigma_2}$$

for any $\sigma_2 > \sigma_1$. Hence,

$$\|v_\varepsilon\|_{L^2(\Omega)} \leq C(\Theta_\sigma(T))^{1-\delta}\|u_0\|_{W^{2,p}(\Omega)} \leq C(\Theta_1(T))^{\sigma(1-\delta)}\|u_0\|_{W^{2,p}(\Omega)}$$

for any $\delta, \sigma \in (0, 1)$ and $p > d$, where C depends only on δ, p, σ, A and Ω. This, together with (7-30) and Theorem 6.6, gives

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\big(\langle|\psi - \nabla\chi_T|\rangle + (\Theta_1(T))^\sigma\big)\|u_0\|_{W^{2,p}(\Omega)}$$
$$\leq C\left(\int_{1/(2\varepsilon)}^\infty \frac{\Theta_\sigma(r)}{r}\,dr + (\Theta_1(\varepsilon^{-1}))^\sigma\right)\|u_0\|_{W^{2,p}(\Omega)}$$

for any $\sigma \in (0, 1)$, and completes the proof. $\square$

8. Quasiperiodic coefficients

In this section we consider the case where $A(x)$ is quasiperiodic and continuous. More precisely, without loss of generality, we will assume that

$$\begin{cases} A(x) = B(j_\lambda(x)), \\ B \text{ is 1-periodic and continuous in } \mathbb{R}^M, \end{cases} \tag{8-1}$$

where $M = m_1 + m_2 + \cdots + m_d$ and, for $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$,

$$j_\lambda(x) = (\lambda_1^1 x_1, \lambda_1^2 x_1, \ldots, \lambda_1^{m_1} x_1, \lambda_2^1 x_2, \ldots, \lambda_2^{m_2} x_2, \ldots, \lambda_d^1 x_d, \ldots, \lambda_d^{m_d} x_d) \in \mathbb{R}^M.$$

Also, for each $i = 1, 2, \ldots, d$, the set $\{\lambda_i^1, \ldots, \lambda_i^{m_i}\}$ is assumed to be linearly independent over $\mathbb{Z}$. Under these conditions, it is known that $A(x)$ is uniformly almost periodic. We shall be interested in conditions on $\lambda = (\lambda_i^j)$ that imply the power decay of $\rho(R)$ as $R \to \infty$. For convenience we consider

$$\rho_1(R) = \sup_{y \in \mathbb{R}^d} \inf_{\substack{z \in \mathbb{R}^d \\ \|z\|_\infty \le R}} \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty}, \tag{8-2}$$

where $\|z\|_\infty = \max(|z_1|, \ldots, |z_d|)$ for $z = (z_1, \ldots, z_d)$. It is easy to see that $\rho_1(\sqrt{d}R) \le \rho(R) \le \rho_1(R)$.

Let

$$\omega(\delta) = \sup\{|B(x) - B(y)| : \|x - y\|_\infty \le \delta\}, \quad \delta > 0,$$

denote the modulus of continuity of $B(x)$. For $x \in \mathbb{R}$, write $x = [x] + \langle x \rangle$, where $[x] \in \mathbb{Z}$ and $\langle x \rangle \in \left[-\frac{1}{2}, \frac{1}{2}\right)$. If $x = (x_1, \ldots, x_M) \in \mathbb{R}^M$, define $[x] = ([x_1], \ldots, [x_N])$ and $\langle x \rangle = (\langle x_1 \rangle, \ldots, \langle x_M \rangle)$. It is easy to see that $\|\langle x \rangle\|_\infty$ gives the distance from x to $\mathbb{Z}^M$ with respect to the norm $\|\cdot\|_\infty$.

Lemma 8.1. *Let $\rho_1(R)$ be defined by (8-2). Then, for any $R > 0$, $\rho_1(R) \le \omega(\theta_\lambda(R))$, where*

$$\theta_\lambda(R) = \sup_{x \in [-1/2, 1/2]^M} \inf_{\substack{z \in \mathbb{R}^d \\ \|z\|_\infty \le R}} \|x - \langle j_\lambda(z) \rangle\|_\infty. \tag{8-3}$$

Proof. Note that, since B is 1-periodic,

$$|B(x) - B(y)| = |B(y + [x - y] + \langle x - y \rangle) - B(y)| = |B(y + \langle x - y \rangle) - B(y)| \le \omega(\|\langle x - y \rangle\|_\infty)$$

for any $x, y \in \mathbb{R}^M$. It follows that

$$|A(x + y) - A(x + z)| \le \omega(\|\langle j_\lambda(y) - j_\lambda(z) \rangle\|_\infty)$$

for any $x, y, z \in \mathbb{R}^d$. This implies that

$$\rho_1(R) \le \sup_{y \in \mathbb{R}^d} \inf_{\substack{z \in \mathbb{R}^d \\ \|z\|_\infty \le R}} \omega(\|\langle j_\lambda(y) - j_\lambda(z) \rangle\|_\infty).$$

Using

$$\|\langle j_\lambda(y) - j_\lambda(z) \rangle\|_\infty = \|\langle \langle j_\lambda(y) \rangle - \langle j_\lambda(z) \rangle \rangle\|_\infty \le \|\langle j_\lambda(y) \rangle - \langle j_\lambda(z) \rangle\|_\infty,$$

we obtain

$$\rho_1(R) \le \sup_{y \in \mathbb{R}^d} \inf_{\substack{z \in \mathbb{R}^d \\ \|z\|_\infty \le R}} \omega(\|\langle j_\lambda(y) \rangle - \langle j_\lambda(z) \rangle\|_\infty) \le \omega(\theta_\lambda(R)),$$

where we have used the continuity of $\omega(\delta)$ for the second inequality. $\qquad \square$

Let $\lambda_i = (\lambda_i^1, \lambda_i^2, \ldots, \lambda_i^{m_i}) \in \mathbb{R}^{m_i}$ for each $1 \le i \le d$ and

$$j_{\lambda_i}(t) = (\lambda_i^1 t, \lambda_i^2 t, \ldots, \lambda_i^{m_i} t) \in \mathbb{R}^{m_i} \quad \text{for } t \in \mathbb{R}.$$

Thus, for $z = (z_1, z_2, \ldots, z_d) \in \mathbb{R}^d$,

$$j_\lambda(z) = (j_{\lambda_1}(z_1), j_{\lambda_2}(z_2), \ldots, j_{\lambda_d}(z_d)).$$

It follows that

$$\|x - <j_\lambda(z)>\|_\infty = \max_{1 \le i \le d} \|x_i - <j_{\lambda_i}(z_i)>\|_\infty,$$

where $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^M$ and $x_i \in \mathbb{R}^{m_i}$. This implies that

$$\theta_\lambda(R) = \max_{1 \le i \le d} \theta_{\lambda_i}(R), \tag{8-4}$$

where

$$\theta_{\lambda_i}(R) = \sup_{x \in [-1/2, 1/2]^{m_i}} \inf_{\substack{t \in \mathbb{R} \\ |t| \le R}} \|x - <j_{\lambda_i}(t)>\|_\infty. \tag{8-5}$$

Note that if $m_i = 1$ then $\theta_{\lambda_i}(R) = 0$ for R large. We will use the Erdős–Turán–Koksma inequality in the discrepancy theory to estimate the function $\theta_{\lambda_i}(R)$, defined by (8-5), for $m_i \ge 2$.

Let $P = P_N = \{x_1, x_2, \ldots, x_N\}$ be a finite subset of $\left[-\frac{1}{2}, \frac{1}{2}\right]^m$. The discrepancy of P is defined as

$$D_N(P) = \sup_B \left| \frac{A(B; P)}{N} - |B| \right|,$$

where the supremum is taken over all rectangular boxes $B = [a_1, b_1] \times \cdots \times [a_m, b_m] \subset \left[-\frac{1}{2}, \frac{1}{2}\right]^m$ and $A(B; P)$ denotes the number of elements of P in B. It follows from the Erdős–Turán–Koksma inequality that

$$D_N(P) \le C \left\{ \frac{1}{H} + \sum_{\substack{n \in \mathbb{Z}^m \\ 0 < \|n\|_\infty \le H}} \frac{1}{(1 + |n_1|) \cdots (1 + |n_m|)} \left| \frac{1}{N} \sum_{x \in P} e^{2\pi i (n \cdot x)} \right| \right\} \tag{8-6}$$

for any $H \ge 1$, where C depends only on m (see, e.g., [Drmota and Tichy 1997, p. 15]). It is not hard to see that

$$\max_{y \in [-1/2, 1/2]^m} \min_{z \in P_N} \|y - z\|_\infty \le \frac{1}{2} [D_N(P_N)]^{1/m}. \tag{8-7}$$

Lemma 8.2. *Let $R \ge 2$ and $\ell \ge 2$ be two positive integers. We divide the interval $[-R, R]$ into $2R\ell$ subintervals of length $1/\ell$. Let $N = 2R\ell$ and*

$$P_N = \left\{ x = <j_\lambda(t)> \in \left[-\frac{1}{2}, \frac{1}{2}\right]^m : t = j + \frac{k}{\ell}, \ -R \le j \le R - 1 \ and \ 0 \le k \le \ell - 1 \right\},$$

where $\lambda = (\lambda^1, \ldots, \lambda^m) \in \mathbb{R}^m$ and $m \ge 2$. Suppose that there exist $c_0 > 0$ and $\tau > 0$ such that

$$|n \cdot \lambda| \ge c_0 |n|^{-\tau} \quad for \ any \ n \in \mathbb{Z}^m \setminus \{0\}. \tag{8-8}$$

Then

$$D_N(P_N) \le C(R^{-1/(\tau+1)} (\log R)^{m-1} + N^{-1} R^{1+1/(\tau+1)} (\log R)^{m-1}), \tag{8-9}$$

where C depends only on $m, c_0, |\lambda|$ and τ.

Proof. Let $f(t) = e^{2\pi i (n \cdot \lambda) t}$ and

$$I_n = \frac{1}{N} \sum_{x \in P_N} e^{2\pi i (n \cdot x)} = \frac{1}{N} \sum_{j,k} f(t_{jk}), \tag{8-10}$$

where $n \in \mathbb{Z}^m \setminus \{0\}$, $j = -R, \ldots, R-1$, $k = 0, \ldots, \ell-1$ and $t_{jk} = j + k/\ell$. Using

$$\left| \frac{1}{2R} \int_{-R}^{R} f(t) \, dt - \frac{1}{N} \sum_{j,k} f(t_{jk}) \right| \leq C \ell^{-1} \|f'\|_\infty,$$

we obtain

$$|I_n| \leq C \ell^{-1} \|f'\|_\infty + \left| \frac{1}{2R} \int_{-R}^{R} f(t) \, dt \right| \leq C \ell^{-1} |n \cdot \lambda| + \frac{C}{R |n \cdot \lambda|} \leq C(\ell^{-1} |n| + R^{-1} |n|^\tau),$$

where we have used the assumption (8-8). In view of (8-6), we obtain

$$\begin{aligned}
D_N(P_N) &\leq C \left(\frac{1}{H} + \sum_{\substack{n \in \mathbb{Z}^m \\ 0 < \|z\|_\infty \leq H}} \frac{\ell^{-1} |n| + R^{-1} |n|^\tau}{(1 + |n_1|) \cdots (1 + |n_m|)} \right) \\
&\leq C \left(\frac{1}{H} + \int_{|x| \leq CH} \frac{\ell^{-1} |x| + R^{-1} |x|^\tau}{(1 + |x_1|) \cdots (1 + |x_m|)} \, dx \right) \\
&\leq C \left(\frac{1}{H} + R N^{-1} H (\log H)^{m-1} + R^{-1} H^\tau (\log H)^{m-1} \right)
\end{aligned}$$

for any $H \geq 2$. By taking $H = R^{1/(\tau+1)}$, we obtain the estimate (8-9). $\qquad \square$

Theorem 8.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ with $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^{m_i}) \in \mathbb{R}^{m_i}$ for $1 \leq i \leq d$. Suppose that there exist $c_0 > 0$ and $\tau > 0$ such that, for each $1 \leq i \leq d$ with $m_i \geq 2$,*

$$|n \cdot \lambda_i| \geq c_0 |n|^{-\tau} \quad \text{for any } n \in \mathbb{Z}^{m_i} \setminus \{0\}. \tag{8-11}$$

Then, for any $R \geq 2$,

$$\theta_\lambda(R) \leq C R^{-1/(\tilde{m}(\tau+1))} (\log R)^{1-1/\tilde{m}}, \tag{8-12}$$

where $\tilde{m} = \max\{m_1, \ldots, m_d\}$ and C depends only on $d, \tilde{m}, c_0$ and τ.

Proof. Suppose $m_i \geq 2$. Let $P = P_N$ be same as in Lemma 8.2. It follows from (8-7) and Lemma 8.2 that

$$\theta_{\lambda_i}(R) \leq C \left(R^{-1/(\tau+1)} (\log R)^{m_i-1} + N^{-1} R^{1+1/(\tau+1)} (\log R)^{m_i-1} \right)^{1/m_i} \leq C R^{-1/(m_i(\tau+1))} (\log R)^{1-1/m_i},$$

where we have taken $N = C R^{1+2/(\tau+1)}$. This, together with (8-4), gives (8-12). $\qquad \square$

Remark 8.4. Suppose that $A(x) = B(j_\lambda(x))$ and $B(y)$ is 1-periodic. Also assume that λ satisfies the condition (8-11) and $B(y)$ is Hölder continuous of order α for some $\alpha \in (0, 1]$. It follows from Lemma 8.1 and Theorem 8.3 that

$$\rho(R) \leq C R^{-\alpha/(\tilde{m}(\tau+1))} (\log R)^{\alpha(1-1/\tilde{m})} \tag{8-13}$$

for $R \geq 1$. In view of Remark 1.3, this leads to

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C_\gamma \varepsilon^\gamma \|u_0\|_{W^{2,p}(\Omega)}$$

for any $0 < \gamma < \alpha/(\alpha + \tilde{m}(\tau + 1))$. We point out that, for $A(y)$ that satisfies the condition (8-11) and is sufficiently smooth, the sharp estimate $\|u_\varepsilon - u_0\|_{L^2(\Omega)} = O(\varepsilon)$ was obtained in [Kozlov 1978].

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